EXISTENCE AND LINEAR STABILITY OF EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM WITH POYNTING-ROBERTSON DRAG AND OBLATENESS

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DECLARATION

I declare that the work in this thesis entitled **"EXISTENCE AND LINEAR STABILITY OF EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM WITH POYNTING-ROBERTSON DRAG AND OBLATENESS"** has been carried out by me in the Department of Mathematics under the supervision of Prof. Jagadish Singh. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

Tajudeen Oluwafemi AMUDA

___________________	________________	_________________
Name of Student	Signature	Date

CERTIFICATION

This thesis entitled **''EXISTENCE AND LINEAR STABILITY OF EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL RESTRICTED THREE-BODY PROBLEM WITH POYNTING-ROBERTSON DRAG AND OBLATENESS''** by Tajudeen Oluwafemi AMUDA meets the regulations governing the award of the degree of Master of Science (M.Sc.) of the Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

Prof. Jagadish Singh

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External Examiner	Signature	Date

Prof. A. Joshua

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Dean, School of Postgraduate Studies	Signature	Date

DEDICATION

This research work is dedicated to Almighty ALLAH, the creator of the universe. Also to my late mother, Alhaja Asiawu Amuda, may Allah forgive her (Ameen), and to all members of my family.

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ABSTRACT

This research work investigates the existence and linear stability of a test particle of infinitesimal mass around the equilibrium points in the photogravitational restricted three-body problem with Poynting-Robertson (P-R) drag and oblateness. The primaries are modelled as an oblate spheroid and a radiating mass. The equations of motion are model such that the equilibrium points and linear stability can be study. It is seen that three points lying on the line joining the primaries (collinear equilibrium points) exist and depend only on the radiation pressure force of the smaller primary, oblateness of the bigger primary and the mass parameter of the system. Aside the collinear points, a pair of equilibrium points (triangular equilibrium points) forming triangles with the line joining the primaries, exist and are defined by the mass parameter, oblateness of the bigger primary, radiation pressure and P-R drag of the smaller primary. Further, the equilibrium points lying out of the orbital plane of motion (out-of-plane equilibrium points) was found. The linear stability of the equilibrium points is studied. The collinear equilibrium points are unstable due to a positive root of the governing characteristic equation. The triangular and the out-of-plane equilibrium points are also unstable due to positive real part of the complex roots and a positive root. In general, all these forces; that is, the oblateness of the bigger primary, radiation and P-R drag, are destabilizing forces. Further, the numerical explorations are performed in order to give precise and accurate results about the positions of the equilibrium points and their stability for different systems.

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NOMENCLATURE

$q =$ Radiation factor (Mass reduction factor)

$\mu =$ Mass ratio

$v =$ Velocity

$\mu_c =$ Critical mass ratio

$r_1 =$ Distance of infinitesimal mass from the bigger primary

$r_2 =$ Distance of infinitesimal mass from the smaller primary

$r_{12} =$ Distance between the primaries

$\Omega =$ Force function

$G =$ Gravitational constant

$F_g =$ Gravitational force of attraction

$m =$ Infinitesimal mass

$n =$ Mean motion

$A =$ Oblateness coefficient

$L_{1,2,3} =$ Collinear equilibrium points

$L_{4,5} =$ Triangular equilibrium points

$L_{6,7} =$ Out-of-plane equilibrium points

$W_2 =$ Poynting-Robertson effect

$F_p =$ Radiation pressure force

$c_d =$ Velocity of light

CHAPTER 1

GENERAL INTRODUCTION

1.1 INTRODUCTION

The most celebrated problem of space dynamics is the problem of three bodies (3BP). The 3BP is defined in terms of three bodies with arbitrary masses, attracting one another according to the Newtonian law of gravitation, and are free to move in space. The degenerated case of the 3BP, is the restricted three-body problem (R3BP), which describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, moving in circular orbits, around their common center of mass, on account of their mutual gravitational attraction and the infinitesimal mass not influencing the motion of the primaries. The R3BP is one of the most widely studied area in space dynamics as well as in celestial mechanics; and very significant results have been produced by well known mathematicians and scientists, in an attempt to understand and predict the motion of natural bodies.

1.2 STATEMENT OF RESEARCH PROBLEM

Generalized problem of the classical R3BP, in which the bigger primary is modeled as an oblate spheroid, while the smaller primary is a radiation emitter, having a Poynting-Robertson (P-R) drag force. The third body is assumed to be a test particle having infinitesimal mass and influenced by the gravitational attraction of the primaries. Since the motion of the primaries moves in circular orbits and are determined by the equations of motion of the two-body problem. The motion of a test particle, positions of the equilibrium points and their linear stability are investigated.

1.3 SIGNIFICANCE/JUSTIFICATION OF THE STUDY

The study of the restricted three-body problem over the last 200 years has produced significant results and is the backdrop for space technology. Man-made satellites are modeled and built as test particles in orbits of celestial bodies. Results considering the shape of the Earth and radiation effect of the Sun, in the Sun-Earth-Satellites system are examples. The equilibrium points of the R3BP are of enormous importance to space applications in the past, present and future. The equilibrium solutions of this system are widely used in many branches of astronomy, both for constant and variable masses as in the case of the Roche model for binary star system (Luk'yanov 1989).

Firstly, the collinear equilibrium points given their location and accessibility are good spot for space-based observatories and they provide easy access to orbits in the case of lunar and earth orbits; though all of them are unstable, which means a spacecraft to be kept at or orbiting around them will require correction maneuvers typically to be performed at the expense of propellant mass.

Secondly, the triangular equilibrium points are also very interesting from the point of view of astronomical objects: possible location for interplanetary dust and asteroids, hence, they have been suggested as convenient sites to locate future space colonies. Since then (Late O'Neill, 1974) scientists and astronomers have been devoted to set up a colony at one of the two triangular points of the Earth-Moon system.

Finally, the P-R effect is important in the study of stability of the zodiacal cloud, orbital evolution of cometary meteor streams, asteroidal particles and dust rings around planets. This research work have enormous applications in various astronomical problems as it relates to understanding stability of the zodiacal cloud, orbital evolution of cometary meteor streams, asteroidal particles and dust rings around planets.

Poynting (1903) pointed out the effect of absorption and subsequent re-emission of sunlight by small isolated particles in the solar system. His work was modified by Robertson (1937), using relativistic treatments of the first order in the ratio of the velocity of the particle to that of light. This effect was termed "Poynting-Robertson (P-R)" drag. After this breakthrough Chernikov (1970), Schuerman (1980), Ishwar and Kushvah (2006), Kushvah, (2008) and Das et al. (2009) have carried out studies under different assumptions.

Also, in reality, celestial bodies are not in general spherical. The Earth, Saturn, Jupiter and several stars in our stellar systems are sufficiently oblate. The lack of sphericity, of the celestial bodies can cause large perturbations from a two-body orbit. The motions of artificial Earth Satellites are examples of this. Several studies that have considered one or both primaries as an oblate spheroid, include among many; SubbaRao and Sharma (1976), Sharma (1987), Khanna and Bhatnagar (1999), Singh and Ishwar (1999), Douskos and Markellos (2006), AbdulRaheem and Singh (2006), Kushvah (2008), Singh and Begha (2011), Singh and Leke (2012), Singh and Laraba (2012) and Singh and Umar (2012) and Singh and Leke (2013).

Hence, in connection to the above previous studies, it is reasonable to study the case when, the bigger primary is an oblate spheroid, the smaller one a source of radiation when its P-R drag is included.

1.4 AIM AND OBJECTIVES OF THE STUDY

The aim of this study in to investigate and analysis the motion of a test particle when the bigger primary is modeled as an oblate spheroid, the smaller primary is radiating emitter having its Poynting-Robertson drag.

In line with this idea and the stated problem, the objectives of the study are to treat:

1. The equations of motion of the infinitesimal mass, when the bigger primary is an oblate spheroid and the smaller is a radiation source whose P-R drag effect is taken into account.
2. Study the possible equilibrium points, and search further if oblateness of the bigger primary, radiation pressure and P-R drag of the smaller primary would yield additional equilibrium points.
3. Investigate the linear stability of the equilibrium points obtained.

1.5 PRELIMINARY IDEAS

In this section, some terminologies and ideas upon which this thesis is built are discussed for better understanding, as the case may be.

1.5.1 Circular Restricted Three-Body Problem

The restricted three-body is defined in the introduction section. It said, as studying the motion of a test particle of infinitesimal mass in the neighborhood of the main masses. This statement, can be put in the form: consider an isolated dynamical system consisting of three gravitationally interacting point masses, m_1, m_2 and m_3. Suppose, however, that m_3 (the third body or the infinitesimal mass), is so much smaller than the other two such that its effect on their motion is negligible. Suppose, that the first two masses, m_1 and m_2 (called the primaries), execute circular orbits about their common center of mass. This simplified problem of the motion of the infinitesimal mass is general and known as the *circular restricted three-body problem (CR3BP).*

1.5.2 Equilibrium/Lagrangian/Stationary/Libration points.

The equilibrium points, which are also known as Lagrangian or stationary points, are the solutions of the equations of motion of the infinitesimal mass, under the gravitational effects of the primaries. Because, of the complexity in solving the equations of motion, the method, is often to resort, to the particular solutions of the system by setting the acceleration and velocity of the infinitesimal mass to zero.

The CR3BP possesses five equilibrium points. Three of these points are on a straight line, the $(x-axis)$, and are called the collinear equilibrium points; while the other two form equilateral triangles with the primaries, and are called triangular equilibrium points. The collinear equilibrium points were found by Euler (1767) while the triangular equilibrium points were worked out by Lagrange (1772). Figure 1.1 below illustrates the five equilibrium points in the Sun-Earth-Moon system.

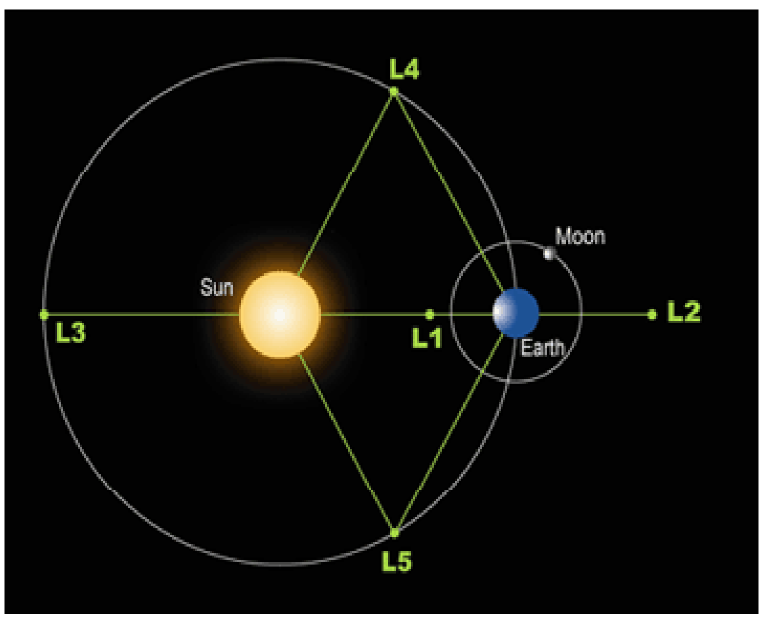

Fig.1. 1: The five Lagrangian points in CR3BP (Sun-Earth-Moon system)

Aside the five equilibrium points of the classical restricted three-body problem, further studies have revealed other types of equilibrium points. Examples are the out-of-plane equilibrium points, which are referred to as coplanar points. These equilibrium points have been found in the studies of the restricted three-body problem with variable masses (see Singh and Leke 2010, 2012, 2013a, b, c, d). In the case of problem of constant masses, these points are found by expressing the equations of motion in three-dimensional form, when radiation or oblateness is involved (see Radzievsky 1950, 1953; Douskos and Markellos 2006; Shankaran et al. 2011; Singh and Umar 2013 and Singh 2013).

1.5.3 Equations of Motion of the Two-Body Problem

In the restricted three-body problem, the motion of the primaries is governed by the equations of motion of the two-body problem. We shall pour some light in this direction.

Now, let 0 be a fixed point in the space of motion, and suppose m_1 and m_2 denote the masses of the two bodies while $\vec{r}_1$ and $\vec{r}_2$ are the vectors from 0 to the respective masses. Let $\vec{r}_1$ be the radius vector between the bodies. Then, the force of attraction on the particle of mass m_1 due to the particle of mass m_2 is given by the Newton's law of gravitation, as

$$\vec{F}_1 = G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \tag{1.1}$$

while the force acting on the body of mass m_2 due to the body of mass m_1 is

$$\vec{F}_2 = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \tag{1.2}$$

Here, G is the gravitational constant.

By Newton's second law,

$$\vec{F}_1 = \frac{d^2\vec{r}_1}{dt^2}m_1 \tag{1.3}$$

$$\vec{F}_2 = \frac{d^2\vec{r}_2}{dt^2} m_2$$

Substituting equations of system (1.3) in Eq. (1.1) and Eq. (1.2), we have

$$\frac{d^2\vec{r}_1}{dt^2} m_1 = G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r}, \qquad \frac{d^2\vec{r}_2}{dt^2} m_2 = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r},$$

so that

$$\ddot{\vec{r}}_1 = G\frac{m_2}{r^2}\frac{\vec{r}}{r}, \quad \ddot{\vec{r}}_2 = -G\frac{m_1}{r^2}\frac{\vec{r}}{r}, \tag{1.4}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$.

Equations (1.4) are the vectorial equations of motion of m_2 about m_1.

1.5.3.1 Equations of Relative Motion

Let $\vec{v}_1$ and $\vec{v}_2$ be the velocities of the bodies m_1 and m_2 respectively. Let $\vec{v}_R$ be the velocity vector of the body m_2 with respect to m_1.

Then,

$$\vec{v}_R = \vec{v}_2 - \vec{v}_1 \tag{1.5}$$

Now, the vectorial equations of system (1.4) take the form

$$\frac{d\vec{v}_1}{dt} = G\frac{m_2}{r^2}\frac{\vec{r}}{r}, \tag{1.6}$$

$$\frac{d\vec{v}_2}{dt} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r} \tag{1.7}$$

Subtracting Eq. (1.6) into Eq. (1.7) we have

$$\frac{d\vec{v}_2}{dt} - \frac{d\vec{v}_1}{dt} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r} - G\frac{m_2}{r^2}\frac{\vec{r}}{r}$$

$$= -G\frac{(m_1 + m_2)}{r^2}\frac{\vec{r}}{r}$$

or $$\frac{d}{dt}(\vec{v}_2 - \vec{v}_1) = -G\frac{(m_1+m_2)}{r^2}\frac{\vec{r}}{r}.$$

Using Eq. (1.5), we immediately have

$$\frac{d\vec{v}_R}{dt} = -G\frac{(m_1+m_2)}{r^2}\frac{\vec{r}}{r}. \tag{1.8}$$

Now, if we take $\mu = -G(m_1+m_2)$, then, Eq. (1.8) takes the form

$$\frac{d\vec{v}_R}{dt} = \frac{\mu}{r^2}\frac{\vec{r}}{r}.$$

Dropping the suffix, we get

$$\frac{d\vec{v}}{dt} = \frac{\mu}{r^2}\frac{\vec{r}}{r}. \tag{1.9}$$

Equation (1.9) is called the relative motion of the body of mass m_2 with respect to the body m_1, where r is the distance between the bodies.

1.6 THEORETICAL FRAMEWORK

Here, the literature some terms and relational definitions are discussed.

1.6.1. Radiation Pressure

The classical restricted three-body problem did not discuss the case when one or both primary bodies are radiation source. The problem according to Radzievsky (1950, 1953) is called the photogravitational problem. For example, when a star acts upon a particle in a cloud of gas and dust, the dominant factor is by no means gravity, but the repulsive force of the radiation pressure. Bodies radiate thermal energy according to their temperature. The emissions are electromagnetic radiation, and therefore have the properties of energy and momentum. The energy leaving a body tends to reduce its temperature. The momentum of the radiation causes a reactive force, expressed as a pressure across the radiating surface. Radiation pressure implies an interaction between electromagnetic radiation and bodies of various types, including clouds of particles or gases. The interactions can be absorption, reflection, or some of both (the common case). Bodies also emit radiation and thereby experience a resulting pressure. Radiation from a material or body is the transfer of heat energy in the form of electromagnetic wave which does not require an electric medium or the emitting of energy. Any hot object radiate heat energy in form of electromagnetic wave of all wavelengths. However, the heat emitted by surface, depends on the nature of the surface, temperature of the body and the surface area. Example of radiating body is the sun in the solar system. The forces generated by radiation pressure are generally too small to be detected under everyday circumstances; however, they do play a crucial role in some settings, particularly astronomy and astrodynamics. For example, had the effects of the sun's radiation pressure on the spacecraft of the "Viking program" been ignored, the spacecraft would have missed Mars orbit by about 15,000 kilometers (Space science note book).

Now, since the solar radiation pressure force F_p changes with distance by the same law of gravitational attraction F_g and acts in opposite direction to it, it is possible that the force lead to reduction of the effective mass of the body. Thus the resultant force on the particle is

$$F = F_g - F_p = F_g\left(1 - \frac{F_p}{F_g}\right)$$

$$= qF_g .$$

Where $q = F_g\left(1 - \frac{F_p}{F_g}\right)$ is the mass reduction factor and the force of the body is given by

$F_p = (1-q)F_g$, such that $0 < (1-q) << 1$.

We note that:

if $q = 1,$ the radiation pressure has no effect.

if $0 < q < 1,$ the gravitational force is greater than the radiational one.

if $q = 0,$ the radiation force balances the gravitational one.

if $q < 0,$ the radiation pressure overrides the gravitational attraction.

Following this discovery, several studies Hamilton & Burns (1992), Singh & Ishwar (1999), Kunitsyn (2000), Singh (2009), Singh & Leke (2010), Singh and Umar (2012), Singh and Leke (2013), Singh and Taura (2013) have produced significant results in the study of the restricted three-body problem under different assumptions, by taking into account the radiation pressure forces.

1.6.2 Poynting–Robertson (P-R) effect

In estimating the light radiation force, all the above studies of photogravitational R3BP took into account just one of the three components of the light pressure field, which is due to the central force: the gravitation and the radiation pressure. The other two components are arising from the Doppler Shift and the absorption and subsequent re-emission of the incident radiation. These last two components constitute the so-called Poynting-Robertson (P-R) effect.

The Poynting–Robertson effect, also known as Poynting–Robertson drag, named after John Henry Poynting and Howard Percy Robertson, is a process by which solar radiation causes a dust grain in the Solar system to spiral slowly into the Sun. The drag is essentially a component of radiation pressure tangential to the grain's motion. Poynting (1903) gave a description of the effect based a theory which was superseded by the theories of relativity. Robertson (1937) described the effect in terms of general relativity. The Poynting–

Robertson drag can be understood as an effective force that opposes the direction of the dust grain's orbital motion, leading to a drop in the grain's angular momentum. It should be mentioned that while the dust grain thus spirals slowly into the Sun, its orbital speed increases continuously.

Robertson (1937) showed that, the term $\frac{v}{c}$, the total radiation force on a particle P, due to radiating body S, is

$$F = F_p\left(\frac{r}{r} - \frac{(v.r)\,r}{cr^2} - \frac{v}{c}\right),$$

where, F_p is the measure of the radiation pressure force, r is the position vector of p with respect to s, v is the corresponding velocity and c is the velocity of light. The three terms composing of the force in the above equation, express:

1. the radiation pressure, a force opposite to the gravitational attraction;
2. the Doppler shift of the incident radiation, which opposes the radiation pressure and
3. the Poynting-Robertson drag which is opposite to the velocity vector.

The latter two components are caused by the absorption and subsequent re-emission of the radiation and constitute the Poynting-Robertson effect. The P-R effect is important in the study of stability of the zodiacal cloud, orbital evolution of cometary meteor streams, asteroidal particles and dust rings around planets. Given the importance of the problem, several authors (Chernikov 1970; Schuerman 1980; Murray 1994; Ragos & Zafiropoulos 1995; Kushvah 2008) have done their studies taking P-R effect into account.

1.6.3 Doppler Effect

The Doppler effect (or Doppler shift), named after the Austrian physicist Christian Doppler, who proposed it in 1842 in Prague, the change in frequency of a wave (or other periodic event) for an observer moving relative to its source. For waves that propagate in a medium, such as, sound waves, the velocity of the observer and of the source is relative to the medium in which the waves are transmitted. The total Doppler Effect may therefore result from motion of the source, motion of the observer, or motion of the medium. Each of these effects is analyzed separately. For waves which do not require a medium, such as light or gravity in general relativity, only the relative difference in velocity between the observer and the source needs to be considered.

1.6.4 Oblate Spheroid

An oblate spheroid is a rotationally symmetric ellipsoid having a polar axis shorter than the diameter of the equatorial circle whose plane bisects it. Oblate spheroids are contracted along a line, whereas prolate spheroids are elongated.

It can be formed by rotating an ellipse about its minor axis, forming an equator with the end points of the major axis. As with all ellipsoids, it can also be described by the lengths of three mutually perpendicular principal axes, which are in this case two arbitrary equatorial semi-major axes and one semi-minor axes.

The R3BP assumes that the masses concerned are spherically symmetrical in homogenous layers, but it is found that celestial bodies, such as Saturn and Jupiter (see table 1.1), are sufficiently oblate (Beatty et al. 1999). The minor planets (e.g., Ceres) and meteoroids have irregular shapes (Millis et al. 1978; Norton & Chitwood 2008). The oblateness or triaxiality of a body can produce perturbations-deviation from the two-body motion. The orbit of the fifth satellite of Jupiter Amalthea is one of the most striking examples of perturbations arising from oblateness in the solar system (Moulton 1914).

Rotation in stars produces an equatorial bulge due to centrifugal force, as a result due to the rapid rotation after formation of Neutron stars, white and black dwarfs, they may be considered oblate. A neutron star on formation can rotate at rate of nearly a thousand rotations per second (Du et al. 2009). The millisecond pulsar PSRB1937+21, spinning about 642 times a second and the pulsar PSRJ1748-422ad, spinning 716 times a second are some of the swiftest spinning pulsar (Hessels et al. 2006). This motivated several researchers SubbaRao & Sharma (1975), Elipe & Ferrer (1985), Singh & Ishwar (1999), Sharma et al. (2001), AbdulRaheem & Singh (2006), Vishnu et al. (2008), Mital et al. (2009), Singh (2011,2012); Singh & Umar (2012a,b), Singh & Leke (2013a,b), Singh & Umar (2013) to include oblateness of one or both primaries in their studies.

Planet	Oblateness. (A_1)6km
Mercury	0.0
Venus	0.0
Earth	0.0033529
Mars	0.0074
Jupiter	0.06487
Saturn	0.09796
Uranus	0.02293
Neptune	0.0171

Table 1.1: Oblateness of the planet

1.6.5 Linear Stability

The stability of linear systems of ordinary differential equations is determined completely by the Eigenvalue of the coefficient matrix. Due to perturbations, the position of the infinitesimal body would be displaced a little from the equilibrium point. If the resultant motion of the infinitesimal body is a rapid departure from the vicinity of the point, we can call such a position of equilibrium point an “unstable one”, if however the body merely oscillates about the equilibrium point, it is said to be a “stable position” (in the sense of Lyapunov).

To examine the stability of the orbits in the vicinity of the libration points, we adopt or apply this small displacement method shifting the origin or coordinates of the infinitesimal mass and linearizing the equations of motion around the coordinates of the equilibrium point. The variational equations of motion corresponding to the dynamical system are derived, which in turn through the trial solutions is transformed to a matrix form and a characteristic equation of the variational equations of the dynamical system is obtained. The roots of the characteristic equation are found and the stability of the solutions depends on the nature of the characteristic roots. For complex roots of the characteristic equation, the equilibrium point is asymptotically stable when all roots have negative real parts, and unstable when some or all roots have positive real part; multiple complex roots can either be stable or unstable. For pure imaginary characteristic roots, the motion is oscillatory and the equilibrium point is stable, though not asymptotically stable. If there are multiple roots, the solution contains mixed terms (periodic and secular terms) and the equilibrium point is unstable. If all the roots are both real and negative, the equilibrium point is stable; if however, any of the roots are positive, the equilibrium point is unstable. This statement is also true for multiple roots.

CHAPTER 2

LITERATURE REVIEW

The gravitational problem of three bodies in its traditional sense dates in substance from 1687, when Sir Isaac Newton (1642-1727) published his 'Principia' (Philosophy Naturalis Principia Mathematica), where he took the first step in defining and studying the problem of the movements of three massive bodies subject to their mutually perturbing gravitational attraction. He also applied his result to the motion of moon under the Sun-Earth system. Other pioneers of the 3BP are Euler, Laplace, Lagrange, Jacobi, Hamilton, Birkoff and Poincare; Tarbera (2007). There is an enormous literature devoted to works that have been done in the field of the restricted three-body problem. This includes both analytic and numerical developments. A review of some published works as related to the study is given below in this chapter.

The R3BP is of fundamental importance in mechanics, with significant applications to astrodynamics. Because no general solution in the CR3BP is available, particular solutions are sought to obtain insight into the problem. These particular solution referred to as the equilibrium or libration points are five for the classical restricted problem; two triangular and three collinear. L. Euler (1767) determined a set of three collinear libration points and J.L Lagrange (1772) found the triangular points.

The condition for linear stability of the triangular points was established by Routh (1875). When this condition is satisfied, all roots of the characteristic equation are pure imaginary, which leads to pure oscillatory solution. This solution was later referred to as "critical solution" by Lyapunov (1892). The collinear points were found to be unstable in the linear and nonlinear sense, as Routh original result shows, when interpreted in the light of Lyapunov's work.

The effect of a small perturbation of the Coriolis force on the stability of the equilibrium points, keeping the centrifugal force constant, was studied by Szebehely (1967b). He maintained that the collinear points remain unstable and obtained for the stability of the triangular points a relation between the critical value of the mass parameter μ_c and the change $\in$ in the Coriolis force:

$$\mu_c = \mu_0 + \frac{16 \in}{3\sqrt{69}},$$

and concluded that the Coriolis force is a stabilizing force.

SubbaRao and Sharma (1975) examined the same problem as Szebehely (1967b) with the consideration that the primaries are oblate spheroids and its equatorial plane coinciding with the plane of motion. They showed that the oblateness of the primaries resulted in an increase in both the Coriolis and the centrifugal forces and further establish that the range of linear stability of the triangular points decreases, thereby concluding that the Coriolis force is not always a stabilizing force.

Schuerman (1980) carried out the investigations of the classical restricted three-body problem to include the force of radiation and the Poynting-Robertson effect. The position of the Lagrangian points L_4 and L_5 were found as functions of the ratio of radiation to gravitational forces. The Poynting-Robertson effect renders the points unstable on a time scale long compare to the period of rotation of the massive bodies. Further, implications for space colonization and a mechanism for producing Azimuthal asymmetries in the interplanetary dust complex were discussed.

Sharma (1982) extended this work by considering the linear stability of triangular libration points of the R3BP when the more massive primary is a source of radiation and an oblate spheroid as well. He found that the eccentricity of the conditional retrograde elliptical periodic orbits around the triangular points at the critical mass μ_c increases with the increase in the oblateness coefficient and the radiation force; and becomes unity when μ_c is zero.

Elipe and Ferrer (1985) studied solution in the circular planar restricted problem of three rigid bodies. They obtained explicit equations which allow collinear and triangular solutions which in general are not equilateral; also, they established numerically that the influence of non-sphericity of the primaries brings about a significant shift in location of the libration points when compared to the classical case. El-Shaboury (1989) and El-Shaboury *et al.* (1991) also established, numerically, a shift in the location of libration points due to non-sphericity (triaxiality) and rotation of the primaries, when compared to the classical restricted problem. Also, El-Shaboury (1990) gave a possibility of nine libration points for small values of oblateness in the photogravitational R3BP when the infinitesimal mass is an axisymmetric body and one of the finite masses is a spherical luminous body, while the other is assumed to be an axisymmetric non-luminous body.

Ragos and Zagouras (1993) studied the existence of the out-of-plane equilibrium points in the photogravitational restricted three-body problem. They noted that indeed these equilibrium points exist. The location and stability of the five Lagrangian equilibrium points in the planar, circular restricted three-body problem are investigated when the third body is acted on by a variety of drag forces, was examined by Murray (1994). The approximate locations of the displaced equilibrium points were calculated for small mass ratios and a simple criterion for their linear stability was derived.

A different kind of the restricted three-body problem formulated by Robe (1977) was studied by Giordano et al. (1996) when linear drag forces are present. They discussed, in particular, the stability of the model's equilibrium points discussed by Robe and when fluid body is assumed as a Roche's ellipsoid. Singh and Ishwar (1999) generalized the R3BP by considering both primaries to be source of radiation and as well as oblate spheroids. They observe that the equations of motion and locations of the equilibrium points are affected by the radiation pressure forces and oblateness of the primaries. They found that the triangular points are stable for $0 < \mu \le \mu_c$ and unstable for $\mu_c < \mu \le \frac{1}{2}$, and further established that the range of stability depends upon the radiating and oblateness coefficients. That same year, Khanna and Bhatnagar (1999) investigated the R3BP by modeling the bigger primary as an oblate spheroid and the smaller primary as a triaxial rigid body; they maintain the unstable nature of the collinear points and the stable nature of the triangular points for some values of mass parameter. They further observed that the triangular points have long or short periodic elliptical orbits in the same range of the mass parameter.

Sharma *et al.* (2001) further studied the existence and stability of libration points in the R3BP, considering the case where both primaries are triaxial rigid bodies. They found three collinear points which are unstable and two triangular points which are stable for certain values of mass parameter. The stability of equilibrium points in the Relativistic restricted three-body problem was carried out by Douskos and Perdios (2001). They observed that the stability of the triangular points so determined are contrary to other findings as a region of linear stability in the parameter space is obtained. The positions of the collinear points are approximated by series expansion and their stability is similarly determined. Further, they found that these are always unstable.

The linear stability of triangular equilibrium points in the generalized photogravitational restricted three-body problem with Poynting-Robertson drag was studied by Ishwar and Kushvah (2006). Here, they considered the smaller primary to be an oblate spheroid while the bigger one is a radiating body. Obviously, the equations of motion depend on the radiation pressure force, oblateness and P-R drag. They verified all classical results involving photogravitational and oblateness in restricted three body problem and concluded with the help of the roots of the characteristic equation, that triangular equilibrium points are unstable.

The combined effect of perturbations, radiation and oblateness on the stability of equilibrium points in the R3BP was studied by AbdulRaheem and Singh (2006). The model was studied under the influence of small perturbations in the Coriolis and centrifugal forces, together with the effects of oblateness and radiation pressure of the primaries. They found that the collinear points remain unstable, while the triangular points are stable for $0 < \mu \leq \mu_c$ and unstable for $\mu_c < \mu \leq \frac{1}{2}$. They observed further that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation and oblateness of the primaries have destabilizing effects; the presence of any one or more of the latter makes weak the stabilizing ability of the former, consequently the overall effect is that the range of stability of the triangular points decreases. Douskos and Markellos (2006) studied the out-of-plane equilibrium points in the R3BP with oblateness, and they found that these points exist when the equations of motion are studied under three-dimension, their existence has significant effects on the topology of the zero velocity curves and numerical evidence indicates that these points are unstable.

Kushvah (2008) examined the existence of equilibrium points and the effect of radiation pressure numerically. They generalized the problem by considering bigger primary as a source of radiation and small primary as an oblate spheroid and discussed the Poynting-Robertson (P-R) effect which is caused due to radiation pressure. They observe that the collinear points deviate from the axis joining the two primaries, while the triangular points are not symmetrical due to radiation pressure. The collinear points are linearly unstable while the triangular points are conditionally stable in the sense of Lyapunov when P-R effect is not considered while P-R effect induces instability in the sense of Lyapunov.

Das et al. (2009) studied the out-of-plane equilibrium points $L_i (i = 6,7,8,9)$ of passive micron size particle and their stability, in the field of radiating-binary stellar systems, taking into account the P-R drag. They proved that the points $L_{6,7}$ are stable in the absence of P-R drag and binary systems like (Kruger 60 and RW-Monocerotis), while they are all unstable in the presence of P-R drag. The combined effects of oblateness and radiation of the primaries on the nonlinear stability of the triangular libration points was studied by Singh (2009). It was found that, in the nonlinear sense, the triangular point is stable for all mass ratios in the range of linear stability except for three mass ratios at which the Moser (1961) theorem fails. Tilemahos (2009) examined new aspects of the photo-gravitational Copenhagen problem. Here, they dealt with some new aspects of the photogravitational Copenhagen case of the restricted three-body problem. More particularly, the distribution and the attracting domains of the stationary solutions of small particles that move in the neighborhood of two major bodies with equal masses when one or both primaries are radiation sources with constant luminosity. Under these conditions, each particle is subjected not only to gravitational forces but to the radiation emitted from the primaries as well.

Singh and Leke (2010) studied the Stability of the photogravitational restricted three-body problem with variable masses in which the masses of the luminous primaries vary isotropically in accordance with the unified Meshcherskii law, and their motion takes place within the framework of the Gylden–Meshcherskii problem. For the system with constant masses the collinear and coplanar points are unstable, while the triangular points are conditionally stable, while the stability of equilibrium points varying with time are unstable using the Lyapunov Characteristic Numbers (LCN). Shankaran *et al.* (2011) also examined the out-of-plane points in the generalized photogravitational non-planar R3BP, when the smaller primary is an oblate spheroid and both primaries are radiating, while Singh and Begha (2011), generalized the RTBP to include the case when the more massive primary as a triaxial rigid body while the less massive one is an oblate spheroid under the influence of small perturbation in the Coriolis and centrifugal forces. They asserted that the position of the triangular and collinear equilibrium points was affected by the non-sphericity of the primaries and the change in the centrifugal force. The stability of the triangular point was investigated and it was seen that triangular points are stable for $0 < \mu \leq \mu_c$ and unstable for $\mu_c < \mu \leq \frac{1}{2}$, where μ_c is the critical value of the mass parameter, which depends on the joint effect of perturbations, oblateness and triaxiality. The collinear points were found to be unstable in the linear approximation. The nonlinear stability of the problem considered by AbdulRaheem and Singh (2006) was examined by Singh (2011). The study was restricted to the triangular equilibrium points which are stable for all mass ratios in the range of linear stability expect for three masses ratio depending upon perturbations, oblateness coefficients and mass reduction factors.

The R3BP when the masses of the primaries are not constant was generalized by Singh and Leke (2012) to include the variation in oblateness of the first primary when the second one is luminous; under effects of small perturbations in the Coriolis and the centrifugal forces. They found that the collinear equilibrium points are stable and the triangular points are conditionally stable, while the out-of-plane points are generally unstable. Motion in the photogravitational elliptic restricted three-body problem under an oblate primary.

Singh and Umar (2012a) looked at motion of an infinitesimal mass around seven equilibrium points in the framework, of the elliptic restricted three-body problem under the assumption that the primary of the system is non- luminous, oblate spheroid and the secondary is luminous. A practical application of the problem is the study of the dynamical evolution of dust particles in orbits around a binary system with a dark degenerated primary and a secondary stellar companion. The conditional stability of the motion around the triangular points depends on the mass ratio and the critical mass ratio of the system. Singh (2012) examined the out-of-plane equilibrium points $L_{6,7}$ by considering effect of a small change in Coriolis and centrifugal forces, when the primaries are both radiating and oblate spheroids, while Singh and Umar (2012b) followed up immediately by studying these equilibrium points in the elliptic restricted three-body problem with radiating and oblate primaries, and applying their study to Gamma Leporis and Altair.

The equilibrium points and zero velocity surfaces in the restricted four body- problem with solar wind drag, was carried out by Kumari and Kushvah (2012). Here, forces which govern the motion are mutual gravitational attractions of the primaries, radiation pressure force and solar wind drag. The equations of motion, Jacobi integral, zero velocity surfaces and particular solutions of the system were found. The stability property of the system is examined with the help of Poincare´s surface of section (PSS) and Lyapunov characteristic exponents (LCEs) and reveals that, in the presence of drag forces LCE is negative for a specific initial condition; and thereby concluded that corresponding trajectory is regular whereas regular islands in the PSS are expanded.

Singh and Taura (2013) investigated motion in the generalized restricted three-body problem when both primaries are radiating, oblate bodies, together with the effect of gravitational potential from a belt. They found the positions of the equilibrium points and examined their linear stability. They found that, in addition to the usual five equilibrium points, there appear two new collinear points due to the potential from the belt, and in the presence of all these perturbations. The collinear equilibrium points remain; however, remain unstable, while the triangular points are stable, under some conditions defined by the mass ratio and the critical mass ratio of the system. They claimed that a practical application of their model could be the study of the motion of a dust particle near the oblate, radiating binary stars systems surrounded by a belt.

Recently, Singh and Leke (2013a) modified the Robe's circular restricted three-body problem to include the effect of mass variations of the primaries in accordance with the unified Meshcherskii law, when the motion of the primaries is determined by Gylden-Meshcherskii problem. The position and linear stability of the equilibrium points of the Robe's circular restricted three-body problem were discussed. They examined the linear stability of the equilibrium points and observed that the equilibrium points at the center of the shell of the autonomized system is stable; the triangular equilibrium points is unstable, while the non- autonomized system are also unstable.

The motion of a test particle around the equilibrium points under the Robe's Circular Restricted problem of three variable mass bodies in which the masses of the three bodies vary arbitrarily with time at the same rate has been studied by Singh and Leke (2013b). The first primary was assumed to be fluid in the shape of a sphere whose density also varies with time. Two collinear equilibrium points exist, with one positioned at the center of the fluid while the other exist for mass ratio and density parameter provided the density parameter assumes value greater than one. Circular equilibrium points and pairs of out-of-plane equilibrium points forming triangles with the centers of the primaries were also found. The linear stability of the equilibrium points is studied and it is seen that the circular and out-of-plane equilibrium points are unstable while the collinear equilibrium points are stable under some condition.

Also, the periodic orbits around triangular points in the range of linear stability of the restricted three body problem, when the smaller primary and the test particle have the shape of an oblate spheroid and larger primary is radiation emitter with allowance for the gravitational potential from the belt, was studied by Singh and Leke (2013c). They worked out the orbits to be an ellipse having long and short periodic orbits. The period, orientation, eccentricities, the semi-major and semi-minor axis of the elliptic orbits were established and further gave some numerical examples in case of the Sun-Earth and Sun-Jupiter systems.

CHAPTER 3

EQUATIONS OF MOTION

3.1 INTRODUCTION

In this chapter, we give the equations of motion of the infinitesimal mass when it moves in the gravitational field of two primaries, which moves in circular orbits, around their center of mass. Because the bigger primary is an oblate spheroid, and the smaller one is a radiation emitter, with effects of the Poynting-Robertson drag; the dynamical system governing motion of the infinitesimal mass will differ from those of the classical restricted three-body problem. Firstly, the unperturbed mean motion will now be affected by oblateness of the bigger primary and the potential between the primaries will also now, depend on the oblateness, radiation and P-R drag parameters.

Here, we have also given the three-dimensional equations of motion for our problem, which provide us an avenue to investigate the out-of-plane equilibrium points.

3.2 MATHEMATICAL MODEL

Following the lead of Chernikov (1970) in which the relativistic treatment to first order in $\frac{\vec{v}}{c}$ is given by

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 \tag{3.1}$$

where

$$\vec{F}_1 = F_p \frac{\vec{R}}{R}, \quad \vec{F}_2 = -F_p \frac{\vec{v}.\vec{R}}{cR} \frac{\vec{R}}{R}, \quad \vec{F}_3 = -F_p \frac{\vec{v}}{c}.$$

Here, $F_p = \frac{3Lm}{16\pi R^2 \rho s c}$ denotes the measure of the radiation pressure force, $\vec{R}$ the position vector of a particle p with respect to radiation source, $\vec{v}$ the corresponding velocity vector and c the velocity of light. In the expression given by F_p; L is luminosity of the radiating body, while m, ρ and s are, the mass, density and cross-section of the particle respectively.

The first term in equation (3.1) expresses the radiation pressure, the second one represents the Doppler shift of the incident radiation and the third one is due to the absorption and subsequent re-emission of the incident radiation. These last two terms taken together are called the Poynting-Robertson effect.

We consider the barycentric rotating co-ordinate system $Oxyz$ relative to an inertial system with angular velocity ω and common z -axis. We take a line joining the primaries as the x -axis. Let m_1, m_2 be the masses of the bigger and smaller primaries, respectively and r_{12} the distance between them. Let r_e, r_p be the equatorial and polar radii of m_1 respectively. Let (x, y, z) be the coordinates of an infinitesimal mass m. We take units such that the sum of the masses and distance between primaries be unity, the unit of time i.e. time period of m_2 about m_1 consists of 2π units such that the Gaussian constant of gravitational $K^2 = 1$. Then perturbed mean motion n of the primaries is given by $n^2 = 1 + \frac{3A}{2}$, where $A = \frac{r_e^2 - r_\rho^2}{5r_{12}^2}$ is the oblateness coefficient of m_1. Let $\mu = \frac{m_2}{m_1 + m_2}$ be the mass parameter, then $1 - \mu = \frac{m_1}{m_1 + m_2}$ with $m_1 > m_2$. The coordinates of m_1 and m_2 are then $(-\mu\ , 0)$ and $(1-\mu, 0)$ respectively. Also, considering the dimensionless velocity of light as c_d, which depends on the physical masses of the two primaries and the distance between them. In this study,

$c_d = 299792458$ is for all numerical results (Kushvah 2008). In the above mentioned reference system, the total acceleration on the mass m is given as follows:

$$\vec{a} + 2\vec{w}\times\vec{v} + \vec{w}\times(\vec{w}\times\vec{r}) = -\frac{(1-\mu)\vec{r}_1}{r_1^3} - \frac{3(1-\mu)A\vec{r}_1}{2r_1^5} - \frac{\mu q\,\vec{r}_2}{r_2^3}$$
$$+\frac{\mu(1-q)}{r_2^2}\left[\frac{\vec{r}_2}{r_2} - \{\frac{(\dot{\vec{r}}_2 + w\times\vec{r}_2).\vec{r}_2}{c_d r_2}\}\frac{\vec{r}_2}{r_2} - \frac{(\dot{\vec{r}}_2 + w\times\vec{r}_2)}{c_d}\right], \tag{3.2}$$

where

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}, \quad \vec{v} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}, \quad \vec{a} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}, \quad \vec{w} = n\vec{k},$$

$$\vec{r}_1 = (x+\mu)\vec{i} + y\vec{j} + z\vec{k}, \quad \vec{r}_2 = (x+\mu-1)\vec{i} + y\vec{j} + z\vec{k},$$

$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \quad r_2^2 = (x+\mu-1)^2 + y^2 + z^2,$$

$$\vec{w}\times\vec{v} = -n(\dot{y}\vec{i} - \dot{x}\vec{j}), \qquad \vec{w}\times(\vec{w}\times\vec{r}) = -n^2(x\vec{i} + y\vec{j}),$$

$$\dot{\vec{r}}_2 + \vec{w}\times\vec{r}_2 = (\dot{x} - ny)\vec{i} + [\dot{y} + n(x+\mu-1)]\,\vec{j} + \dot{z}\vec{k},$$

$$(\dot{\vec{r}}_2 + \vec{w}\times\vec{r}_2).\vec{r}_2 = [(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}].$$

Substituting the above values in the L.H.S of equations (3.2), we have

$$\vec{a} + 2\vec{w}\times\vec{v} + \vec{w}\times(\vec{w}\times\vec{r}) = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} - 2n(\dot{y}\vec{i} - \dot{x}\vec{j}) - n^2(x\vec{i} + y\vec{j}),$$

$$= \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} - 2n\dot{y}\vec{i} + 2n\dot{x}\vec{j} - n^2x\vec{i} - n^2y\vec{j},$$

$$= \ddot{x}\vec{i} - 2n\dot{y}\vec{i} - n^2x\vec{i} + \ddot{y}\vec{j} + 2n\dot{x}\vec{j} - n^2y\vec{j} + \ddot{z}\vec{k},$$

$$\vec{a} + 2\vec{w}\times\vec{v} + \vec{w}\times(\vec{w}\times\vec{r}) = (\ddot{x} - 2n\dot{y} - n^2x)\vec{i} + (\ddot{y} + 2n\dot{x} - n^2y)\,\vec{j} + \ddot{z}\vec{k}. \tag{3.3}$$

Now, simplifying the R.H.S, we have

$$\text{R.H.S}= -\frac{(1-\mu)\vec{r}_1}{r_1^3}-\frac{3(1-\mu)A\vec{r}_1}{2r_1^5}-\frac{\mu q\,\vec{r}_2}{r_2^3}+\frac{\mu(1-q)}{r_2^2}\left[\frac{\vec{r}_2}{r_2}-\{\frac{(\dot{\vec{r}}_2+w\times\vec{r}_2).\vec{r}_2}{c_d r_2}\}\frac{\vec{r}_2}{r_2}-\frac{(\dot{\vec{r}}_2+w\times\vec{r}_2)}{c_d}\right]$$

Substituting the values of $\vec{r}_1$ and $\vec{r}_2$ in the above equation yields

R.H.S=

$$-\frac{(1-\mu)}{r_1^3}[(x+\mu)\vec{i}+y\vec{j}+z\vec{k}]-\frac{3(1-\mu)A}{2r_1^5}[(x+\mu)\vec{i}+y\vec{j}+z\vec{k}]-\frac{\mu q}{r_2^3}[(x+\mu-1)\vec{i}+y\vec{j}+z\vec{k}] \quad (3.4)$$

$$+\frac{\mu(1-q)}{c_d r_2^2}\left[\frac{[(x+\mu-1)\dot{x}+y\dot{y}+z\dot{z}][(x+\mu-1)\vec{i}+y\vec{j}+z\vec{k}]}{r_2^2}+(\dot{x}-ny)\vec{i}+[\dot{y}+n(x+\mu-1)]\vec{j}+z\vec{k}\right]$$

Equating equations (3.3) and (3.4), we at once have

$$(\ddot{x}-2n\dot{y}-n^2x)\vec{i}+(\ddot{y}+2n\dot{x}-n^2y)\vec{j}+\ddot{z}\vec{k}=-\frac{(1-\mu)}{r_1^3}[(x+\mu)\vec{i}+y\vec{j}+z\vec{k}]$$

$$-\frac{3(1-\mu)A}{2r_1^5}[(x+\mu)\vec{i}+y\vec{j}+z\vec{k}]-\frac{\mu q}{r_2^3}[(x+\mu-1)\vec{i}+y\vec{j}+z\vec{k}] \quad (3.5)$$

$$+\frac{\mu(1-q)}{c_d r_2^2}\left[\frac{[(x+\mu-1)\dot{x}+y\dot{y}+z\dot{z}][(x+\mu-1)\vec{i}+y\vec{j}+z\vec{k}]}{r_2^2}+(\dot{x}-ny)\vec{i}+[\dot{y}+n(x+\mu-1)]\vec{j}+z\vec{k}\right]$$

Now, setting $W_2=\dfrac{\mu(1-q)}{c_d}$ in equation (3.5), we get

$$(\ddot{x}-2n\dot{y}-n^2x)\vec{i}+(\ddot{y}+2n\dot{x}-n^2y)\vec{j}+\ddot{z}\vec{k}=-\frac{(1-\mu)}{r_1^3}[(x+\mu)\vec{i}+y\vec{j}+z\vec{k}]$$

$$-\frac{3(1-\mu)A}{2r_1^5}[(x+\mu)\vec{i}+y\vec{j}+z\vec{k}]-\frac{\mu q}{r_2^3}[(x+\mu-1)\vec{i}+y\vec{j}+z\vec{k}] \quad (3.6)$$

$$+\frac{W_2}{r_2^2}\left[\frac{[(x+\mu-1)\dot{x}+y\dot{y}+z\dot{z}][(x+\mu-1)\vec{i}+y\vec{j}+z\vec{k}]}{r_2^2}+(\dot{x}-ny)\vec{i}+[\dot{y}+n(x+\mu-1)]\vec{j}+z\vec{k}\right]$$

The coefficients $\vec{i}$, $\vec{j}$ and $\vec{k}$ on both sides of equation (3.6) give the following system of equations:

$$\ddot{x} - 2n\dot{y} = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} - \frac{\mu q(x+\mu-1)}{r_2^3}$$
$$-\frac{W_2}{r_2^2}\left[\frac{(x+\mu-1)}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{x} - ny\right], \tag{3.7}$$

$$\ddot{y} + 2n\dot{x} = n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{3(1-\mu)Ay}{2r_1^5} - \frac{\mu q y}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{y}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{y} + n(x+\mu-1)\right],$$

$$\ddot{z} = -\frac{(1-\mu)z}{r_1^3} - \frac{3(1-\mu)Az}{2r_1^5} - \frac{\mu q z}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{z}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{z}\right].$$

Hence, the motion of the infinitesimal mass in the $xy-$orbital plane, is described by the equations:

$$\ddot{x} - 2n\dot{y} = \Omega_x, \qquad \ddot{y} + 2n\dot{x} = \Omega_y \tag{3.8}$$

where

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} - \frac{\mu q(x+\mu-1)}{r_2^3}$$
$$-\frac{W_2}{r_2^2}\left[\frac{(x+\mu-1)}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y}\} + \dot{x} - ny\right],$$
$$\Omega_y = n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{3(1-\mu)Ay}{2r_1^5} - \frac{\mu q y}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{y}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y}\} + \dot{y} + n(x+\mu-1)\right],$$

$$r_1^2 = (x+\mu)^2 + y^2, \qquad r_2^2 = (x+\mu-1)^2 + y^2, \tag{3.9}$$

$$n^2 = 1 + \frac{3A}{2}. \tag{3.10}$$

Here r_1 and r_2 are the distances of the third body from the primaries; μ is the mass ratio, defined as, the ratio of the mass of the smaller primary to the total mass of the primaries and $0 < \mu \le \frac{1}{2}$; A denotes the oblateness coefficient of the bigger primary and is such that $0 < A << 1,$ while q is the radiation pressure force of the smaller primary. W_2 is the parameter representing the P-R drag and defined by, the mass ratio $\mu,$ the radiation pressure of smaller primary and the velocity of light c_d. The mean motion n of the primaries, and the dot denote differentiation with respect to time t. Equations (3.8) define the motion of the infinitesimal mass body in the barycentric coordinate system, when the bigger primary is an oblate spheroid, the smaller being a radiation source with its Poynting-Robertson-drag effects. Obviously, the dynamical system is affected by the presence of these parameters.

3.2.1 Equations of motion in three-dimensional due to oblateness of the bigger primary

Having stated the equations of motion (3.8), we follow previous works SubbaRao and Sharma (1976), Douskos and Markellos (2006) and Singh (2013), and write these equations as a result of oblateness of the bigger primary in three-dimensional form while accommodating the system parameters; in the following forms:

$$\ddot{x} - 2n\dot{y} = \Omega_x,$$

$$\ddot{y} + 2n\dot{x} = \Omega_y, \qquad (3.11)$$

$$\ddot{z} = \Omega_z,$$

where

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} + \frac{15(1-\mu)A(x+\mu)z^2}{2r_1^7} - \frac{\mu q(x+\mu-1)}{r_2^3}$$
$$- \frac{W_2}{r_2^2}\left[\frac{(x+\mu-1)}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{x} - ny\right],$$

$$\Omega_y = n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{3(1-\mu)Ay}{2r_1^5} + \frac{15(1-\mu)Ayz^2}{2r_1^7} - \frac{\mu q y}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{y}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{y} + n(x+\mu-1)\right],$$

$$\Omega_z = -\frac{(1-\mu)z}{r_1^3} - \frac{9(1-\mu)Az}{2r_1^5} + \frac{15(1-\mu)Az^3}{2r_1^7} - \frac{\mu q z}{r_2^3} - \frac{W_2}{r_2^2}\left[\frac{z}{r_2^2}\{(x+\mu-1)\dot{x} + y\dot{y} + z\dot{z}\} + \dot{z}\right],$$

$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \quad r_2^2 = (x+\mu-1)^2 + y^2 + z^2, \tag{3.12}$$

$$W_2 = \frac{\mu(1-q)}{c_d}, \quad n^2 = 1 + \frac{3A}{2}.$$

Evidently, when $z = 0$, the equations (3.11) immediately reduce to equations (3.8). Both equations are affected by the oblateness of the bigger primary and the radiation pressure force and its Poynting-Robertson drag effects. While it becomes almost impossible to verify the existence of the out-of-plane equilibrium points, using equations (3.8), we resort to using equations (3.11) to achieve our aim.

The equations of motion of a passively gravitating test particle is derived, having infinitesimal mass in a barycentric coordinate system under the assumptions that the bigger primary is an oblate-shaped body, while the smaller one is a radiation source, whose Doppler shift of the incident radiation and the absorption and subsequent re-emission of the incident radiation are considered. The equations are affected by the oblateness, radiation and P-R drag parameters.

The first derived equations (3.8) clearly won't permit the existence of solutions on the xz-plane; so we rewrite these equations in three-dimensional form, so that the possibilities of finding solutions, which are regarded as "out of the orbital plane of motion solutions", can be achieved. Both systems of equation are however, influenced by the parameters representing, the oblateness of the bigger primary, the mass ratio, the radiation pressure force and P-R drag of the smaller primary. Evidently, when $z = 0$, the equations (3.11) immediately reduce to equations (3.8). It becomes almost impossible to verify the existence of the out-of-plane equilibrium points, using equations (3.8), we resort to using equations (3.11) to achieve the aim of this study. The equations of motion (3.11) are analogous but not exact to those of Douskos & Markellos (2006), Shankaran et al. (2011) and Singh (2012).

CHAPTER 4

LOCATION OF EQUILIBRIUM POINTS

4.1 INTRODUCTION

The equilibrium points are those points at which the velocity and acceleration of the infinitesimal mass are zero. The classical R3BP has five equilibrium points, three of which are called the collinear points, and the remaining two are called, triangular equilibrium points. The collinear equilibrium point are those points which are located on the line joining the primaries, while the triangular equilibrium points form two equilateral triangles with the primaries. Further investigations of the restricted three-body problem, which is the case when one or both primaries are radiation source, have revealed the existence of out of the orbital plane equilibrium points. Studies such as, Radzievsky (1950, 1953) have pointed out this.

In this chapter, the possible equilibrium points of the problem is investigate, taking into account, the radiation pressure and Poynting-Robertson (P-R) drag of the smaller primary when the bigger primary is an oblate spheroid. Hence, we shall seek the equilibrium points lying on the line joining the primaries (collinear points), the equilibrium points which make triangles with the primaries (triangular points) and the out-of-plane equilibrium points. The out-of-plane equilibrium points shall be obtained by expressing the equations of motion in three-dimensional form. This is achieved due to the oblateness of the bigger primary.

4.1.1 Position collinear points

The collinear equilibrium points are found by solving equation (3.2), that is

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} - \frac{\mu q(x+\mu-1)}{r_2^3}$$
$$-\frac{W_2}{r_2^2}\left[\frac{(x+\mu-1)}{r_2^2}\{(x+\mu-1)\dot{x}+y\dot{y}\}+\dot{x}-ny\right]=0,$$

with $y=0,\ \dot{x}=\dot{y}=\ddot{x}=\ddot{y}=0.$

Therefore, we have

$$n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} - \frac{\mu q(x+\mu-1)}{r_2^3} = 0, \tag{4.1}$$

where $r_1 = |x+\mu|$ and $r_2 = |x+\mu-1|$ (4.2)

Now, equation (4.1) obviously does not contain the parameter representing the P-R drag of the smaller primary. Hence, the collinear points of our problem reduces to the case of the restricted three-body problem with an oblate bigger primary and a smaller radiating body, otherwise these equilibrium points do not exist.

Let us now denote the L.H.S of equation (4.1) by $f(x)$, that is

$$f(x) = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} - \frac{\mu q(x+\mu-1)}{r_2^3}$$

The abscissas of the collinear points are the roots of the equation $f(x)=0$. To locate these points, we observe the behaviors of the function $f(x)$.

Now,

$$\frac{df}{dx} = n^2 + \frac{2(1-\mu)}{|x+\mu|^3} + \frac{6(1-\mu)A}{|x+\mu|^5} + \frac{2\mu q}{|x+\mu-1|^3}$$

Therefore, since, $0<\mu\le\frac{1}{2}$, $A<<1$ and $q\approx 1$, we see that $\frac{df}{dx}>0.$

Also, for $x = \pm\infty$, we get $\dfrac{df(x)}{dx} = n^2$;

for $x = -\mu$, we get $\dfrac{df(x)}{dx} = n^2 + \infty - 2\mu q + \infty = \infty$,

and, for $x = 1-\mu$, we have $\dfrac{df(x)}{dx} = n^2 + 2(1-\mu) + \infty + 6(1-\mu)A = \infty$.

Hence, since $\dfrac{df}{dx} > 0$, in each of the open interval $(-\infty, -\mu)$, $(-\mu, 1-\mu)$, $(1-\mu, \infty)$, it fellows

that $f(x)$ is strictly increasing in each of them.

Also,

$$f(-\mu-1) = n^2(-\mu-1) - \frac{(1-\mu)}{\left|-\mu-1+\mu\right|^2} - \frac{3(1-\mu)A}{2\left|-\mu-1+\mu\right|^4} - \frac{\mu q}{\left|-\mu-1+\mu-1\right|^2}$$

or $$f(-\mu-1) = n^2(-\mu-1) + (1-\mu) - \frac{3(1-\mu)A}{2} - \frac{\mu q}{4}$$

or $$= \left(1 + \frac{3A}{2}\right)(-\mu-1) + (1-\mu) - \frac{3(1-\mu)A}{2} - \frac{\mu q}{4}$$

or $$= -\mu - \frac{3\mu A}{2} - 1 - \frac{3A}{2} + 1 - \mu + \frac{3A}{2} + \frac{3\mu A}{2} - \frac{\mu q}{4}$$

or $$= -2\mu - 3\mu A + \frac{\mu q}{4}.$$

So that

$$f(-\mu-1) = \mu\left(-2 - 3A + \frac{q}{4}\right)$$

Since $0 < \mu \le \frac{1}{2}$, $A << 1$ and $q \approx 1$, we must have $f(-\mu-1) < 0$.

Next,

$$f(0) = -\frac{(1-\mu)\mu}{|\mu|^3} - \frac{3(1-\mu)\mu A}{2|\mu|^5} - \frac{\mu q(\mu-1)}{|\mu-1|^3}$$

$$= -\frac{(1-\mu)}{|\mu|^2} - \frac{3(1-\mu)A}{2|\mu|^4} - \frac{\mu q}{|\mu-1|^2}$$

Therefore,

$$f(0) = -\frac{(1-\mu)}{(\mu)^2} - \frac{3(1-\mu)A}{2(\mu)^4} + \frac{\mu q}{(1+\mu)^2}.$$

Using same analysis as done above, we see that $f(0) > 0$.

In order to observe the behavior of the function in the other intervals, we have

$$f(2-\mu) = n^2(2-\mu) - \frac{(1-\mu)(2-\mu+\mu)}{|2-\mu+\mu|^3} - \frac{3(1-\mu)(2-\mu+\mu)A}{2|2-\mu+\mu|^5} - \frac{\mu q(2-\mu+\mu-1)}{|2-\mu+\mu-1|^3},$$

$$= n^2(2-\mu) - \frac{(1-\mu)}{|2-\mu+\mu|^2} - \frac{3(1-\mu)A}{2|2-\mu+\mu|^4} - \frac{\mu q}{|2-\mu+\mu-1|^2},$$

$$= n^2(2-\mu) - \frac{(1-\mu)}{4} - \frac{3(1-\mu)A}{32} - \mu q,$$

$$= \left(1+\frac{3A}{2}\right)(2-\mu) - \frac{(1-\mu)}{4} - \frac{3(1-\mu)A}{32} - \mu q,$$

$$= 2 + 6A - \mu - 3\mu A - \frac{1}{4} + \frac{\mu}{4} - \frac{3A}{32} - \frac{3\mu A}{32} - \mu q.$$

Hence, $f(2-\mu) = \frac{7}{4} - \frac{3\mu}{4} + \frac{185A}{32} - \frac{93\mu A}{32} - \mu q > 0.$

There are only three real roots of the equation $f(x) = 0$ with one lying in each of the open interval $(-\mu-1, -\mu)$, $(-\mu, 0)$, and $(1-\mu, 2-\mu,)$. These roots correspond to the three collinear points and are denoted by L_1, L_2 and L_3.

3.1.1.1 Computation of the position of collinear point L_1.

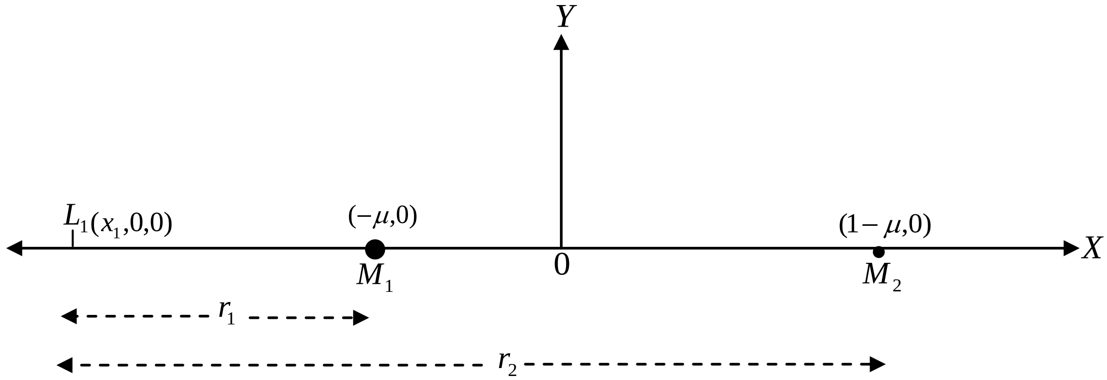

Fig. 4.1: Position of the collinear point L_1

Here, L_1 is located to the left of the bigger primary on the negative x-axis, as illustrated in Figure 4.1. From figure 4.1, we have $r_1 = -x_1 - \mu,\ r_2 = 1-\mu - x_1$ and substituting (4.2) into equation (4.1), we have

$$n^2 x_1 - \frac{(1-\mu)}{|x_1+\mu|^2} - \frac{3(1-\mu)A}{2|x_1+\mu|^4} - \frac{\mu q}{|x_1+\mu-1|^2} = 0. \tag{4.3}$$

Now, let ε_1 be the distance of L_1 from the primary m_1, then we have $r_1 = \varepsilon_1$ and

$$x_1 = -\mu - \varepsilon_1. \tag{4.4}$$

Using equation (4.4) in (4.1), we get

$$n^2(-\mu-\varepsilon_1) - \frac{(1-\mu)(-\mu-\varepsilon_1+\mu)}{\left|(-\mu-\varepsilon_1+\mu)^2\right|^{\frac{3}{2}}} - \frac{3(1-\mu)A(-\mu-\varepsilon_1+\mu)}{2\left|(-\mu-\varepsilon_1+\mu)^2\right|^{\frac{5}{2}}} - \frac{\mu q(-\mu-\varepsilon_1+\mu-1)}{\left|(-\mu-\varepsilon_1+\mu-1)^2\right|^{\frac{3}{2}}} = 0,$$

such that

$$n^2(-\mu-\varepsilon_1) + \frac{(1-\mu)}{\varepsilon_1^{\ 2}} + \frac{3(1-\mu)A}{2\varepsilon_1^{\ 4}} + \frac{\mu q}{(1+\varepsilon_1)^2} = 0. \tag{4.5}$$

Multiplying throughout by $(1+\varepsilon_1)^2\ \varepsilon_1^{\ 4}$, we get

$$n^2(-\mu-\varepsilon_1)(1+\varepsilon_1)^2\varepsilon_1^{\ 4}+(1-\mu)(1+\varepsilon_1)^2\varepsilon_1^{\ 2}+\frac{3}{2}(1-\mu)A(1+\varepsilon_1)^2+\mu q\varepsilon_1^{\ 4}=0,$$

or

$$\left(-n^2\varepsilon_1^{\ 4}\mu-n^2\varepsilon_1^{\ 5}\right)(1+\varepsilon_1)^2+\left(\varepsilon_1^{\ 2}-\mu\varepsilon_1^{\ 2}\right)(1+\varepsilon_1)^2+\left(\frac{3}{2}A+\frac{3}{2}\mu A\right)(1+\varepsilon_1)^2+\mu q\varepsilon_1^{\ 4}=0,$$

or

$$\left(-n^2\varepsilon_1^{\ 4}\mu-n^2\varepsilon_1^{\ 5}\right)\left(1+2\varepsilon_1+\varepsilon_1^{\ 2}\right)+\left(\varepsilon_1^{\ 2}-\mu\varepsilon_1^{\ 2}\right)\left(1+2\varepsilon_1+\varepsilon_1^{\ 2}\right)+\left(\frac{3}{2}A+\frac{3}{2}\mu A\right)\left(1+2\varepsilon_1+\varepsilon_1^{\ 2}\right)$$
$$+\mu q\varepsilon_1^{\ 4}=0,$$

or

$$-n^2\varepsilon_1^{\ 4}\mu-n^2\varepsilon_1^{\ 5}-2n^2\varepsilon_1^{\ 5}\mu-2n^2\varepsilon_1^{\ 6}-n^2\varepsilon_1^{\ 6}\mu-n^2\varepsilon_1^{\ 7}+\varepsilon_1^{\ 2}-\mu\varepsilon_1^{\ 2}+2\varepsilon_1^{\ 3}-2\mu\varepsilon_1^{\ 3}$$
$$+\varepsilon_1^{\ 4}-\mu\varepsilon_1^{\ 4}+\frac{3}{2}A-\frac{3}{2}\mu A+3A\varepsilon_1-3\mu A\varepsilon_1+\frac{3}{2}A\varepsilon_1^{\ 2}-\frac{3}{2}\mu A\varepsilon_1^{\ 2}+\mu q\varepsilon_1^{\ 4}=0.$$

So that

$$\varepsilon_1^{\ 7}(-n^2)+\varepsilon_1^{\ 6}(-2n^2-n^2\mu)+\varepsilon_1^{\ 5}(-n^2-2n^2\mu)+\varepsilon_1^{\ 4}(1-\mu-n^2\mu+\mu q)$$
$$+\varepsilon_1^{\ 3}(2-2\mu)+\varepsilon_1^{\ 2}\left(1-\mu+\frac{3}{2}A-\frac{3}{2}\mu A\right)+\varepsilon_1(3A-3\mu A)+\frac{3}{2}A-\frac{3}{2}\mu A=0.$$

Multiplying through by $\left(-\frac{1}{n^2}\right)$, we get

$$\varepsilon_1^{\ 7}+\varepsilon_1^{\ 6}(2+\mu)+\varepsilon_1^{\ 5}(1+2\mu)+\varepsilon_1^{\ 4}\left(-\frac{1}{n^2}+\frac{\mu}{n^2}+\mu-\frac{\mu q}{n^2}\right)+\varepsilon_1^{\ 3}\left(-\frac{2}{n^2}+\frac{2\mu}{n^2}\right)$$
$$+\varepsilon_1^{\ 2}\left(-\frac{1}{n^2}+\frac{\mu}{n^2}-\frac{3}{2n^2}A+\frac{3}{2n^2}\mu A\right)+\varepsilon_1\left(-\frac{3}{n^2}A+\frac{3}{n^2}\mu A\right)-\frac{3}{2n^2}A+\frac{3}{2n^2}\mu A=0. \tag{4.6}$$

Equation (4.6) is an algebraic equation of seventh degree of ε_1, there is only one change in sign. According to Descarte's sign rule, there exists only one positive root.

Solving equation (4.6) for ε_1 small parameter method, we get

$$\varepsilon_1^{\ 7} + 2\varepsilon_1^{\ 6} + \varepsilon_1^{\ 5} - \varepsilon_1^{\ 4}\left(\frac{1}{n^2}\right) - \varepsilon_1^{\ 3}\left(\frac{2}{n^2}\right) - \varepsilon_1^{\ 2}\left(\frac{1}{n^2} + \frac{3}{2n^2}A\right) - \varepsilon_1\left(\frac{3}{n^2}A\right) - \frac{3}{2n^2}A = 0. \qquad (4.7)$$

This, shows that there is at least one real root $\varepsilon_1 = 0$, now, equation (4.5) can be expressed

$$n^2(-\mu - \varepsilon_1 - \mu\varepsilon_1 + \mu\varepsilon_1) + (1-\mu)\varepsilon_1^{\ -2} + \frac{3}{2}(1-\mu)A\varepsilon_1^{\ -4} + \mu q(1+\varepsilon_1)^{-2} = 0.$$

Factorization yields

$$n^2\left[-(1-\mu)\varepsilon_1 - \mu - \mu\varepsilon_1\right] + (1-\mu)\varepsilon_1^{\ -2} + \frac{3}{2}(1-\mu)A\varepsilon_1^{\ -4} + \mu q(1+\varepsilon_1)^{-2} = 0,$$

or

$$-n^2(1-\mu)\varepsilon_1 - \mu n^2 - \mu n^2\varepsilon_1 + (1-\mu)\varepsilon_1^{\ -2} + \frac{3}{2}(1-\mu)A\varepsilon_1^{\ -4} + \mu q(1+\varepsilon_1)^{-2} = 0,$$

or

$$-n^2(1-\mu)\varepsilon_1 + (1-\mu)\varepsilon_1^{\ -2} + \frac{3}{2}(1-\mu)A\varepsilon_1^{\ -4} = \mu n^2 + \mu n^2\varepsilon_1 - \mu q(1+\varepsilon_1)^{-2}.$$

So that $\quad (1-\mu)\left[-n^2\varepsilon_1 + \varepsilon_1^{\ -2} + \frac{3}{2}A\varepsilon_1^{\ -4}\right] = \mu\left[n^2 + n^2\varepsilon_1 - q(1+\varepsilon_1)^{-2}\right],$

or $$\frac{\mu}{(1-\mu)} = \frac{\left[-n^2\varepsilon_1 + \varepsilon_1^{\ -2} + \frac{3}{2}A\varepsilon_1^{\ -4}\right]}{n^2 + n^2\varepsilon_1 - q(1+\varepsilon_1)^{-2}},$$

or $$\upsilon = \frac{\left[-n^2\varepsilon_1^{\ 5} + \varepsilon_1^{\ 2} + \frac{3}{2}A\right]\varepsilon_1^{\ -4}}{\left[n^2(1+\varepsilon_1)^2\varepsilon_1^{\ -1} + n^2(1+\varepsilon_1)^2 - q\varepsilon_1^{\ -1}\right](1+\varepsilon_1)^{-2}\varepsilon_1},$$

or $$\upsilon = \frac{\left[-n^2\varepsilon_1^{\ 5} + \varepsilon_1^{\ 2} + \frac{3}{2}A\right](1+\varepsilon_1)^2}{\left[n^2(1+\varepsilon_1)^2\varepsilon_1^{\ -1} + n^2(1+\varepsilon_1)^2 - q\varepsilon_1^{\ -1}\right]\varepsilon_1^{\ 4}\varepsilon_1},$$

or $$\upsilon\left[n^2(1+\varepsilon_1)^2\varepsilon_1^{\ -1} + n^2(1+\varepsilon_1)^2 - q\varepsilon_1^{\ -1}\right]\varepsilon_1^{\ 5} = \left[-n^2\varepsilon_1^{\ 5} + \varepsilon_1^{\ 2} + \frac{3}{2}A\right](1+\varepsilon_1)^2,$$

or $$\upsilon\left[n^2(1+\varepsilon_1)^2\varepsilon_1^{-1}+n^2(1+\varepsilon_1)^2-q\varepsilon_1^{-1}\right]\varepsilon_1^{5}-\left[-n^2\varepsilon_1^{5}+\varepsilon_1^{2}+\frac{3}{2}A\right](1+\varepsilon_1)^2=0\ ,$$

or $$\upsilon\left[n^2(1+\varepsilon_1)^2\varepsilon_1^{4}+n^2(1+\varepsilon_1)^2\varepsilon_1^{5}-q\varepsilon_1^{4}\right]-\left[-n^2\varepsilon_1^{5}+\varepsilon_1^{2}+\frac{3}{2}A\right](1+\varepsilon_1)^2=0\ ,$$

or

$$\upsilon\left[n^2\left(1+2\varepsilon_1+\varepsilon_1^{2}\right)\varepsilon_1^{4}+n^2\left(1+2\varepsilon_1+\varepsilon_1^{2}\right)\varepsilon_1^{5}-q\varepsilon_1^{4}\right]-\left[-n^2\varepsilon_1^{5}+\varepsilon_1^{2}+\frac{3}{2}A\right]\left(1+2\varepsilon_1+\varepsilon_1^{2}\right)=0,$$

or

$$\upsilon\left[n^2\varepsilon_1^{4}+2n^2\varepsilon_1^{5}+n^2\varepsilon_1^{6}+n^2\varepsilon_1^{5}+2n^2\varepsilon_1^{6}+n^2\varepsilon_1^{7}-q\varepsilon_1^{4}\right]+n^2\varepsilon_1^{5}-\varepsilon_1^{2}-\frac{3}{2}A+2n^2\varepsilon_1^{6}-2\varepsilon_1^{3}-3A\varepsilon_1$$
$$+n^2\varepsilon_1^{7}-\varepsilon_1^{4}-\frac{3}{2}A\varepsilon_1^{2}=0.$$

Factorizing, we get

$$\varepsilon_1^{7}\left(n^2+n^2\upsilon\right)+\varepsilon_1^{6}\left(2n^2+2n^2\upsilon+n^2\upsilon\right)+\varepsilon_1^{5}\left(n^2+2n^2\upsilon+n^2\upsilon\right)+\varepsilon_1^{4}\left(-1+n^2\upsilon-q\upsilon\right)-2\varepsilon_1^{3}$$
$$-\varepsilon_1^{2}\left(1+\frac{3}{2}A\right)-3A\varepsilon_1-\frac{3}{2}A=0,$$

so that

$$\varepsilon_1^{7}(1+\upsilon)+\varepsilon_1^{6}(2+2\upsilon+\upsilon)+\varepsilon_1^{5}(1+2\upsilon+\upsilon)+\varepsilon_1^{4}\left(-\frac{1}{n^2}+\upsilon-\frac{q\upsilon}{n^2}\right)-\varepsilon_1^{3}\left(\frac{2}{n^2}\right)$$
$$-\varepsilon_1^{2}\left(\frac{1}{n^2}+\frac{3}{2n^2}A\right)-A\varepsilon_1\left(\frac{3}{n^2}\right)-\frac{3}{2n^2}A=0.$$

Dividing through by $(1+\upsilon)$, we get

$$\varepsilon_1^{7}+\varepsilon_1^{6}\left(\frac{2+3\upsilon}{1+\upsilon}\right)+\frac{\varepsilon_1^{5}}{1+\upsilon}(1+3\upsilon)+\frac{\varepsilon_1^{4}}{1+\upsilon}\left(-\frac{1}{n^2}+\upsilon-\frac{q\upsilon}{n^2}\right)-\frac{\varepsilon_1^{3}}{1+\upsilon}\left(\frac{2}{n^2}\right)$$
$$-\frac{\varepsilon_1^{2}}{1+\upsilon}\left(\frac{1}{n^2}+\frac{3}{2n^2}A\right)-\frac{\varepsilon_1}{1+\upsilon}\left(\frac{3}{n^2}A\right)-\frac{1}{1+\upsilon}\left(\frac{3}{2n^2}A\right)=0.$$

Hence,

$$\varepsilon_1^{7}+a\varepsilon_1^{6}+b\varepsilon_1^{5}+c\varepsilon_1^{4}-d\varepsilon_1^{3}-e\varepsilon_1^{2}-f\varepsilon_1-g=0, \qquad (4.8)$$

where

$$\upsilon = \frac{\mu}{(1-\mu)},$$

$$a = \left(\frac{2+3\upsilon}{1+\upsilon}\right),\; b = \frac{1}{1+\upsilon}(1+3\upsilon),\; c = \frac{1}{1+\upsilon}\left(-\frac{1}{n^2}+\upsilon-\frac{q\upsilon}{n^2}\right),\; d = \frac{1}{1+\upsilon}\left(\frac{2}{n^2}\right),$$

$$e = \frac{1}{1+\upsilon}\left(\frac{1}{n^2}+\frac{3}{2n^2}A\right),\; f = \frac{1}{1+\upsilon}\left(\frac{3}{n^2}A\right) \text{ and } g = \frac{1}{1+\upsilon}\left(\frac{3}{2n^2}A\right).$$

3.1.1.2 Computation of the position of collinear point L_2.

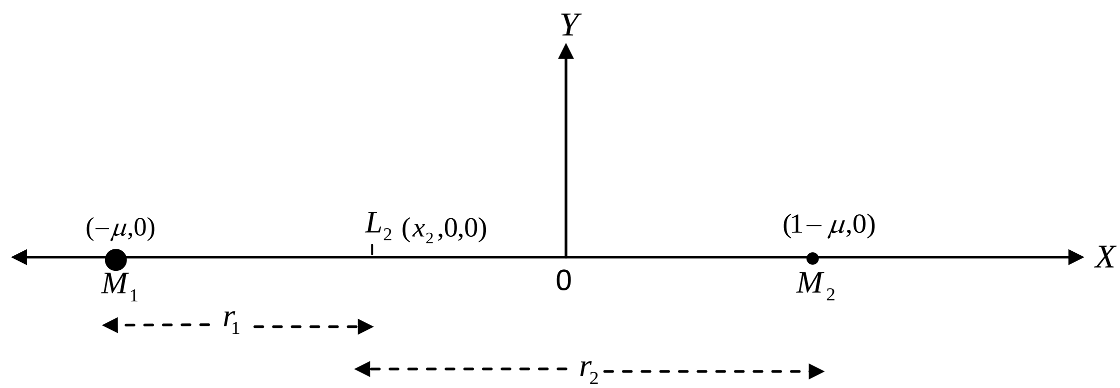

Fig. 4.2: Position of the collinear point L_2

Here, the point L_2 lies to the left of the radiating smaller primary m_2, on the negative x-axis (see figure 4.2). From figure 4.1, we have $r_1 = x_2+\mu,\; r_2 = 1-\mu-x_2$ and substituting (4.2) into equation (4.1), we have

$$n^2 x_2 - \frac{(1-\mu)}{|x_2+\mu|^2} - \frac{3(1-\mu)A}{2|x_2+\mu|^4} - \frac{\mu q}{|x_2+\mu-1|^2} = 0. \qquad (4.9)$$

Now, let ε_2 be the distance of L_2 from m_2, then

$$x_2 = 1-\mu-\varepsilon_2. \qquad (4.10)$$

Substituting equation (4.10) in (4.1), we have

$$n^2(1-\mu-\varepsilon_2)-\frac{(1-\mu)(1-\mu-\varepsilon_2+\mu)}{\left|(1-\mu-\varepsilon_2+\mu)^2\right|^{\frac{3}{2}}}-\frac{3(1-\mu)A(1-\mu-\varepsilon_2+\mu)}{2\left|(1-\mu-\varepsilon_2+\mu)^2\right|^{\frac{5}{2}}}-\frac{\mu q(1-\mu-\varepsilon_2+\mu-1)}{\left|(1-\mu-\varepsilon_2+\mu-1)^2\right|^{\frac{3}{2}}}=0,$$

such that

$$n^2(1-\mu-\varepsilon_2)-\frac{(1-\mu)}{(1-\varepsilon_2)^2}-\frac{3(1-\mu)A}{2(1-\varepsilon_2)^4}+\frac{\mu q}{\varepsilon_2{}^2}=0. \qquad (4.11)$$

Multiplying throughout by $(1-\varepsilon_2)^4\varepsilon_2{}^2$, we get

$$n^2(1-\mu-\varepsilon_2)(1-\varepsilon_2)^4\varepsilon_2{}^2-(1-\mu)(1-\varepsilon_2)^2\varepsilon_2{}^2-\frac{3}{2}(1-\mu)A\varepsilon_2{}^2+\mu q(1-\varepsilon_2)^4=0$$

or

$$\left(n^2\varepsilon_2{}^2-n^2\mu\varepsilon_2{}^2-n^2\varepsilon_2{}^3\right)(1-\varepsilon_2)^4-\left(\varepsilon_2{}^2-\varepsilon_2{}^2\mu\right)(1-\varepsilon_2)^2-\frac{3}{2}A\varepsilon_2{}^2+\frac{3}{2}\mu A\varepsilon_2{}^2+\mu q(1-\varepsilon_2)^4=0$$

or

$$\left(n^2\varepsilon_2{}^2-n^2\mu\varepsilon_2{}^2-n^2\varepsilon_2{}^3\right)\left(1-4\varepsilon_2+6\varepsilon_2{}^2-4\varepsilon_2{}^3+\varepsilon_2{}^4\right)+\left(-\varepsilon_2{}^2+\varepsilon_2{}^2\mu\right)\left(1-2\varepsilon_2+\varepsilon_2{}^2\right)$$
$$-\frac{3}{2}A\varepsilon_2{}^2+\frac{3}{2}\mu A\varepsilon_2{}^2+\mu q\left(1-4\varepsilon_2+6\varepsilon_2{}^2-4\varepsilon_2{}^3+\varepsilon_2{}^4\right)=0$$

or

$$n^2\varepsilon_2{}^2-n^2\mu\varepsilon_2{}^2-n^2\varepsilon_2{}^3-4n^2\varepsilon_2{}^3+4n^2\mu\varepsilon_2{}^3+4n^2\varepsilon_2{}^4+6n^2\varepsilon_2{}^4-6n^2\mu\varepsilon_2{}^4-6n^2\varepsilon_2{}^5$$
$$-4n^2\varepsilon_2{}^5+4n^2\mu\varepsilon_2{}^5+4n^2\varepsilon_2{}^6+n^2\varepsilon_2{}^6-n^2\mu\varepsilon_2{}^6-n^2\varepsilon_2{}^7-\varepsilon_2{}^2+\mu\varepsilon_2{}^2+2\varepsilon_2{}^3-2\mu\varepsilon_2{}^3$$
$$-\varepsilon_2{}^4+\mu\varepsilon_2{}^4+\mu q-4\mu q\varepsilon_2+6\mu q\varepsilon_2{}^2-4\mu q\varepsilon_2{}^3+\mu q\varepsilon_2{}^4-\frac{3}{2}A\varepsilon_2{}^2+\frac{3}{2}\mu A\varepsilon_2{}^2=0.$$

Factorizing, yields

$$\varepsilon_1{}^7\left(-n^2\right)+\varepsilon_2{}^6\left(4n^2+n^2-n^2\mu\right)+\varepsilon_2{}^5\left(-6n^2-4n^2+4n^2\mu\right)+\varepsilon_2{}^4\left(4n^2+6n^2-6n^2\mu-1+\mu+\mu q\right)$$
$$+\varepsilon_2{}^3\left(-n^2-4n^2-4n^2\mu+2-2\mu-4\mu q\right)+\varepsilon_2{}^2\left(n^2-n^2\mu-1+\mu+6\mu q-\frac{3}{2}A+\frac{3}{2}\mu A\right)$$
$$+\varepsilon_2\left(-4\mu q\right)+\mu q=0.$$

Multiplying throughout by $\left(-\dfrac{1}{n^2}\right)$, we at once have

$$\varepsilon_1^{\ 7}+\varepsilon_2^{\ 6}(-5+\mu)+\varepsilon_2^{\ 5}(10-4\mu)+\varepsilon_2^{\ 4}\left(-10+6\mu+\frac{1}{n^2}-\frac{\mu}{n^2}-\frac{\mu q}{n^2}\right)$$
$$+\varepsilon_2^{\ 3}\left(5-4\mu-\frac{2}{n^2}(1-\mu)+\frac{4\mu q}{n^2}\right)+\varepsilon_2^{\ 2}\left(-1+\mu+\frac{1}{n^2}(1-\mu)-\frac{6\mu q}{n^2}+\frac{3}{2n^2}A-\frac{3}{2n^2}\mu A\right) \quad (4.12)$$
$$+\varepsilon_2\frac{(4\mu q)}{n^2}-\frac{\mu q}{n^2}=0.$$

Equation (4.12) is an algebraic equation of seventh degree in ε_2 and there is only one change in sign. Hence, there exists only one positive root according to Descarte's sign rule. Solving equation (4.12) for small parameter method, we get

$$\varepsilon_2^{\ 7}-5\varepsilon_2^{\ 6}+10\varepsilon_2^{\ 5}+\varepsilon_2^{\ 4}\left(-10+\frac{1}{n^2}\right)+\varepsilon_2^{\ 3}\left(5-\frac{2}{n^2}\right)+\varepsilon_2^{\ 2}\left(-1+\frac{1}{n^2}+\frac{3}{2n^2}A\right)=0.$$

This, shows that there is at least one real root $\varepsilon_2=0$.

Now, equations (4.11) can be expressed in the form

$$n^2(1-\mu-\varepsilon_2+\mu\varepsilon_2-\mu\varepsilon_2)-(1-\mu)(1-\varepsilon_2)^{-2}-\frac{3}{2}(1-\mu)A(1-\varepsilon_2)^{-4}+\mu q\varepsilon_2^{\ -2}=0, \qquad (4.13)$$

or $\quad n^2\left[(1-\mu)-(1-\mu)\varepsilon_2-\mu\varepsilon_2\right]-(1-\mu)(1-\varepsilon_2)^{-2}-\dfrac{3}{2}(1-\mu)A(1-\varepsilon_2)^{-4}+\mu q\varepsilon_2^{\ -2}=0,$

or $\quad n^2(1-\mu)(1-\varepsilon_2)-n^2\mu\varepsilon_2-(1-\mu)(1-\varepsilon_2)^{-2}-\dfrac{3}{2}(1-\mu)A(1-\varepsilon_2)^{-4}+\mu q\varepsilon_2^{\ -2}=0,$

so that

$$n^2(1-\mu)(1-\varepsilon_2)-(1-\mu)(1-\varepsilon_2)^{-2}-\frac{3}{2}(1-\mu)A(1-\varepsilon_2)^{-4}=n^2\mu\varepsilon_2-\mu q\varepsilon_2^{\ -2}.$$

Factorizing, we get

$$(1-\mu)\left[n^2(1-\varepsilon_2)-(1-\varepsilon_2)^{-2}-\frac{3}{2}A(1-\varepsilon_2)^{-4}\right]=\mu\left[n^2\varepsilon_2-q\varepsilon_2{}^{-2}\right],$$

or $$\frac{\mu}{(1-\mu)}=\frac{n^2(1-\varepsilon_2)-(1-\varepsilon_2)^{-2}-\frac{3}{2}A(1-\varepsilon_2)^{-4}}{n^2\varepsilon_2-q\varepsilon_2{}^{-2}},$$

or $$\upsilon=\frac{\left[n^2(1-\varepsilon_2)^5-(1-\varepsilon_2)^2-\frac{3}{2}A\right](1-\varepsilon_2)^{-4}}{\left(n^2\varepsilon_2{}^3-q\right)\varepsilon_2{}^{-2}},$$

or $$\upsilon=\frac{\left[n^2(1-\varepsilon_2)^5-(1-\varepsilon_2)^2-\frac{3}{2}A\right]\varepsilon_1{}^2}{\left(n^2\varepsilon_2{}^3-q\right)(1-\varepsilon_2)^4},$$

or $$\upsilon\left(n^2\varepsilon_2{}^3-q\right)(1-\varepsilon_2)^4=\left[n^2(1-\varepsilon_2)^5-(1-\varepsilon_2)^2-\frac{3}{2}A\right]\varepsilon_2{}^2,$$

or $$\upsilon\left(n^2\varepsilon_2{}^3-q\right)(1-\varepsilon_2)^4-\left[n^2(1-\varepsilon_2)^5-(1-\varepsilon_2)^2-\frac{3}{2}A\right]\varepsilon_2{}^2=0.$$

Expanding, we have

$$\upsilon\left(n^2\varepsilon_2{}^3-q\right)\left(1-4\varepsilon_2+6\varepsilon_2{}^2-4\varepsilon_2{}^3+\varepsilon_2{}^4\right)+\left[-n^2\left(1-5\varepsilon_2+10\varepsilon_2{}^2-10\varepsilon_2{}^3+5\varepsilon_2{}^4-\varepsilon_2{}^5\right)\right.$$
$$\left.+\left(1-2\varepsilon_2+\varepsilon_2{}^2\right)+\frac{3}{2}A\right]\varepsilon_2{}^2=0,$$

or

$$\upsilon\left[n^2\varepsilon_2{}^3-q-4n^2\varepsilon_2{}^4+4q\varepsilon_2+6n^2\varepsilon_2{}^5-6q\varepsilon_2{}^2-4n^2\varepsilon_2{}^6+4q\varepsilon_2{}^3+n^2\varepsilon_2{}^7-q\varepsilon_2{}^4\right]$$
$$+\left[-n^2+5n^2\varepsilon_2-10n^2\varepsilon_2{}^2+10n^2\varepsilon_2{}^3-5n^2\varepsilon_2{}^4+n^2\varepsilon_2{}^5+1-2\varepsilon_2+\varepsilon_2{}^2+\frac{3}{2}A\right]\varepsilon_2{}^2=0,$$

or

$$\upsilon\left[n^2\varepsilon_2{}^3-q-4n^2\varepsilon_2{}^4+4q\varepsilon_2+6n^2\varepsilon_2{}^5-6q\varepsilon_2{}^2-4n^2\varepsilon_2{}^6+4q\varepsilon_2{}^3+n^2\varepsilon_2{}^7-q\varepsilon_2{}^4\right]$$
$$+\left[-n^2\varepsilon_2{}^2+5n^2\varepsilon_2{}^3-10n^2\varepsilon_2{}^4+10n^2\varepsilon_2{}^5-5n^2\varepsilon_2{}^6+n^2\varepsilon_2{}^7+\varepsilon_2{}^2-2\varepsilon_2{}^3+\varepsilon_2{}^4+\frac{3}{2}A\varepsilon_2{}^2\right]=0,$$

or

$$\varepsilon_2{}^7\left(n^2+n^2\upsilon\right)+\varepsilon_2{}^6\left(-5n^2-4n^2\upsilon\right)+\varepsilon_2{}^5\left(10n^2+6n^2\upsilon\right)+\varepsilon_2{}^4\left(1-10n^2-q\upsilon-4n^2\upsilon\right)$$
$$+\varepsilon_2{}^3\left(-2+5n^2+4q\upsilon+n^2\upsilon\right)+\varepsilon_2{}^2\left(1-n^2-6q\upsilon+\frac{3}{2}A\right)+\varepsilon_2\left(4q\upsilon\right)-q\upsilon=0.$$

So that

$$\varepsilon_2{}^7\left(1+\upsilon\right)+\varepsilon_2{}^6\left(-5-4\upsilon\right)+\varepsilon_2{}^5\left(10+6\upsilon\right)+\varepsilon_2{}^4\left(\frac{1}{n^2}-10-\frac{q\upsilon}{n^2}-4\upsilon\right)$$
$$+\varepsilon_2{}^3\left(-\frac{2}{n^2}+5+\frac{4q\upsilon}{n^2}+\upsilon\right)+\varepsilon_2{}^2\left(\frac{1}{n^2}-1-\frac{6q\upsilon}{n^2}+\frac{3}{2n^2}A\right)+\frac{\varepsilon_2}{n^2}\left(4q\upsilon\right)-\frac{q\upsilon}{n^2}=0.$$

Dividing through by $\left(1+\upsilon\right)$, we have

$$\varepsilon_2{}^7-\frac{\varepsilon_2{}^6}{1+\upsilon}\left(5+4\upsilon\right)+\frac{\varepsilon_2{}^5}{1+\upsilon}\left(10+6\upsilon\right)-\frac{\varepsilon_2{}^4}{1+\upsilon}\left(-\frac{1}{n^2}+10+\frac{q\upsilon}{n^2}+4\upsilon\right)$$
$$+\frac{\varepsilon_2{}^3}{1+\upsilon}\left(-\frac{2}{n^2}+5+\frac{4q\upsilon}{n^2}+\upsilon\right)-\frac{\varepsilon_2{}^2}{1+\upsilon}\left(-\frac{1}{n^2}+1+\frac{6q\upsilon}{n^2}-\frac{3}{2n^2}A\right)+\frac{\varepsilon_2}{(1+\upsilon)n^2}\left(4q\upsilon\right)-\frac{q\upsilon}{(1+\upsilon)n^2}=0,$$

we get

$$\varepsilon_2{}^7-a\varepsilon_2{}^6+b\varepsilon_2{}^5-c\varepsilon_2{}^4+d\varepsilon_2{}^3-e\varepsilon_2{}^2+f\varepsilon_2-g=0, \qquad (4.14)$$

where

$$\upsilon=\frac{\mu}{(1-\mu)},$$

$$a=\frac{1}{1+\upsilon}\left(5+4\upsilon\right),\quad b=\frac{1}{1+\upsilon}\left(10+6\upsilon\right),\quad c=\frac{1}{1+\upsilon}\left(-\frac{1}{n^2}+10+\frac{q\upsilon}{n^2}+4\upsilon\right),$$

$$d=\frac{1}{1+\upsilon}\left(-\frac{2}{n^2}+5+\frac{4q\upsilon}{n^2}+\upsilon\right),\ e=\frac{1}{1+\upsilon}\left(-\frac{1}{n^2}+1+\frac{6q\upsilon}{n^2}-\frac{3}{2n^2}A\right),$$

$$f=\frac{1}{(1+\upsilon)n^2}\left(4q\upsilon\right)\ \text{and}\ g=\frac{q\upsilon}{(1+\upsilon)n^2}.$$

3.1.1.3 Computation of the position of collinear point L_3.

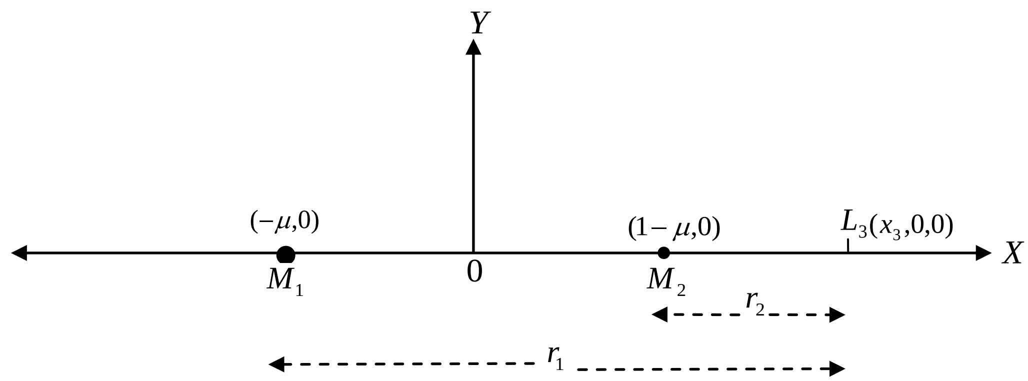

Fig. 4.3: Position of the collinear point L_3

In this region, L_3 is located to the right of the radiating smaller primary away from the center of mass, on the positive x-axis (see figure 4.3).

Here, we have $r_1 = x_3 + \mu,\ r_2 = x_3 + \mu - 1$ and substituting (4.2) into equation (4.1), we get

$$n^2 x_3 - \frac{(1-\mu)}{|x_3+\mu|^2} - \frac{3(1-\mu)A}{2|x_3+\mu|^4} - \frac{\mu q}{|x_3+\mu-1|^2} = 0. \qquad (4.15)$$

Now, let ε_3 be the distance of L_3 from the smaller primary m_2, then we have $r_3 = \varepsilon_3$ and

$$x_3 = 1 - \mu + \varepsilon_3. \qquad (4.16)$$

Using equation (4.16) and (4.1), we get

$$n^2(1-\mu+\varepsilon_3) - \frac{(1-\mu)(1-\mu+\varepsilon_3+\mu)}{\left|(1-\mu+\varepsilon_3+\mu)^2\right|^{\frac{3}{2}}} - \frac{3(1-\mu)A(1-\mu+\varepsilon_3+\mu)}{2\left|(1-\mu+\varepsilon_3+\mu)^2\right|^{\frac{5}{2}}} - \frac{\mu q(1-\mu+\varepsilon_3+\mu-1)}{\left|(1-\mu+\varepsilon_3+\mu-1)^2\right|^{\frac{3}{2}}} = 0,$$

or $$n^2(1-\mu+\varepsilon_3) - \frac{(1-\mu)}{(1+\varepsilon_3)^2} - \frac{3(1-\mu)A}{2(1+\varepsilon_3)^4} - \frac{\mu q}{\varepsilon_3^{\ 2}} = 0. \qquad (4.17)$$

Multiplying throughout by $(1+\varepsilon_3)^4\ \varepsilon_3{}^2$, we get

$$n^2(1-\mu+\varepsilon_3)(1+\varepsilon_3)^4\varepsilon_3{}^2-(1-\mu)(1+\varepsilon_3)^2\varepsilon_3{}^2-\frac{3}{2}(1-\mu)A\varepsilon_3{}^2-\mu q(1+\varepsilon_3)^4=0,$$

or

$$\left(n^2\varepsilon_3{}^2-n^2\mu\varepsilon_3{}^2+n^2\varepsilon_3{}^3\right)(1+\varepsilon_3)^4-\left(\varepsilon_3{}^2-\mu\varepsilon_3{}^2\right)(1+\varepsilon_3)^2-\frac{3}{2}A\varepsilon_3{}^2+\frac{3}{2}\mu A\varepsilon_3{}^2-\mu q(1+\varepsilon_3)^4=0,$$

or

$$\left(n^2\varepsilon_3{}^2-n^2\mu\varepsilon_3{}^2+n^2\varepsilon_3{}^3\right)\left(1+4\varepsilon_3+6\varepsilon_3{}^2+4\varepsilon_3{}^3+\varepsilon_3{}^4\right)-\left(\varepsilon_3{}^2-\mu\varepsilon_3{}^2\right)\left(1+2\varepsilon_3+\varepsilon_3{}^2\right)$$
$$-\frac{3}{2}A\varepsilon_3{}^2+\frac{3}{2}\mu A\varepsilon_3{}^2-\mu q\left(1+4\varepsilon_3+6\varepsilon_3{}^2+4\varepsilon_3{}^3+\varepsilon_3{}^4\right)=0,$$

or

$$n^2\varepsilon_3{}^2-n^2\mu\varepsilon_3{}^2+n^2\varepsilon_3{}^3+4n^2\varepsilon_3{}^3-4n^2\mu\varepsilon_3{}^3+4n^2\varepsilon_3{}^4+6n^2\varepsilon_3{}^4-6n^2\mu\varepsilon_3{}^4+6n^2\varepsilon_3{}^5$$
$$+4n^2\varepsilon_3{}^5-4n^2\mu\varepsilon_3{}^5+4n^2\varepsilon_3{}^6+n^2\varepsilon_3{}^6-n^2\mu\varepsilon_3{}^6+n^2\varepsilon_3{}^7-\varepsilon_3{}^2+\mu\varepsilon_3{}^2-2\varepsilon_3{}^3+2\mu\varepsilon_3{}^3$$
$$-\varepsilon_3{}^4+\mu\varepsilon_3{}^4-\frac{3}{2}A\varepsilon_3{}^2+\frac{3}{2}\mu A\varepsilon_3{}^2-\mu q-4\mu q\varepsilon_3-6\mu q\varepsilon_3{}^2-4\mu q\varepsilon_3{}^3-\mu q\varepsilon_3{}^4=0.$$

Factorization yields

$$n^2\varepsilon_3{}^7+\varepsilon_3{}^6\left(4n^2+n^2-n^2\mu\right)+\varepsilon_3{}^5\left(6n^2+4n^2-4n^2\mu\right)+\varepsilon_3{}^4\left(-1+4n^2+6n^2-6n^2\mu+\mu-\mu q\right)$$
$$+\varepsilon_3{}^3\left(-2+2\mu-4\mu q+n^2+4n^2-4n^2\mu\right)+\varepsilon_3{}^2\left(-1+\mu+n^2-n^2\mu-6\mu q-\frac{3}{2}A+\frac{3\mu}{2}A\right)$$
$$-\varepsilon_3(4\mu q)-\mu q=0.$$

Multiplying throughout by $\left(\dfrac{1}{n^2}\right)$, we at once have

$$\varepsilon_3{}^7+\varepsilon_3{}^6(5-\mu)+\varepsilon_3{}^5(10-4\mu)+\varepsilon_3{}^4\left(10-\frac{1}{n^2}-6\mu+\frac{\mu}{n^2}-\frac{\mu q}{n^2}\right)$$
$$+\varepsilon_3{}^3\left(5-4\mu-\frac{2}{n^2}(1-\mu)-\frac{4\mu q}{n^2}\right)+\varepsilon_3{}^2\left(1-\mu-\frac{1}{n^2}(1-\mu)-\frac{6\mu q}{n^2}-\frac{3}{2n^2}A+\frac{3\mu}{2n^2}A\right) \tag{4.18}$$
$$-\varepsilon_3\frac{(4\mu q)}{n^2}-\frac{\mu q}{n^2}=0.$$

Equation (4.18) is an algebraic equation of seventh degree of ε_3, there is only one change in sign. According to Descarte's sign rule, there exists only one positive root.

Solving equation (4.18) for ε_3 small parameter method, we get

$$\varepsilon_3^{\ 7} + 5\varepsilon_3^{\ 6} + 10\varepsilon_3^{\ 5} + \varepsilon_3^{\ 4}\left(10 - \frac{1}{n^2}\right) + \varepsilon_3^{\ 3}\left(5 - \frac{2}{n^2}\right) + \varepsilon_3^{\ 2}\left(1 - \frac{1}{n^2} - \frac{3}{2n^2}A\right) = 0.$$

This shows that there is at least one real root for $\varepsilon_3 = 0$. Now, equation (4.17) can be expressed as:

$$n^2(1-\mu+\varepsilon_3-\mu\varepsilon_3+\mu\varepsilon_3)-(1-\mu)(1+\varepsilon_3)^{-2}-\frac{3}{2}(1-\mu)A(1+\varepsilon_3)^{-4}-\mu q\varepsilon_3^{\ -2}=0,$$

or

$$n^2\left[(1-\mu)+(1-\mu)\varepsilon_3+\mu\varepsilon_3\right]-(1-\mu)(1+\varepsilon_3)^{-2}-\frac{3}{2}(1-\mu)A(1+\varepsilon_3)^{-4}-\mu q\varepsilon_3^{\ -2}=0,$$

or

$$n^2(1-\mu)(1+\varepsilon_3)+n^2\mu\varepsilon_3-(1-\mu)(1+\varepsilon_3)^{-2}-\frac{3}{2}(1-\mu)A(1+\varepsilon_3)^{-4}-\mu q\varepsilon_3^{\ -2}=0.$$

So that

$$n^2(1-\mu)(1+\varepsilon_3)-(1-\mu)(1+\varepsilon_3)^{-2}-\frac{3}{2}(1-\mu)A(1+\varepsilon_3)^{-4}=-n^2\mu\varepsilon_3+\mu q\varepsilon_3^{\ -2}.$$

Factorizing, we get

$$(1-\mu)\left[n^2(1+\varepsilon_3)-(1+\varepsilon_3)^{-2}-\frac{3}{2}A(1+\varepsilon_3)^{-4}\right]=-\mu\left[n^2\varepsilon_3-q\varepsilon_3^{\ -2}\right],$$

or $$\frac{\mu}{(1-\mu)}=\frac{n^2(1+\varepsilon_3)-(1+\varepsilon_3)^{-2}-\frac{3}{2}A(1+\varepsilon_3)^{-4}}{-n^2\varepsilon_3+q\varepsilon_3^{\ -2}},$$

or $$\upsilon = \frac{n^2(1+\varepsilon_3) - (1+\varepsilon_3)^{-2} - \frac{3}{2}A(1+\varepsilon_3)^{-4}}{-n^2\varepsilon_2 + q\varepsilon_3^{-2}},$$

or $$\upsilon = \frac{\left[n^2(1+\varepsilon_3)^5 - (1+\varepsilon_3)^2 - \frac{3}{2}A\right](1+\varepsilon_3)^{-4}}{-\left(n^2\varepsilon_3^{\ 3} - q\right)\varepsilon_3^{-2}},$$

or $$\upsilon = \frac{\left[n^2(1+\varepsilon_3)^5 - (1+\varepsilon_3)^2 - \frac{3}{2}A\right]\varepsilon_3^{\ 2}}{-\left(n^2\varepsilon_3^{\ 3} - q\right)(1+\varepsilon_3)^4},$$

or $$\upsilon\left(-n^2\varepsilon_3^{\ 3} + q\right)(1+\varepsilon_3)^4 = \left[n^2(1+\varepsilon_3)^5 - (1+\varepsilon_3)^2 - \frac{3}{2}A\right]\varepsilon_3^{\ 2},$$

or $$\upsilon\left(-n^2\varepsilon_3^{\ 3} + q\right)(1+\varepsilon_3)^4 - \left[n^2(1+\varepsilon_3)^5 - (1+\varepsilon_3)^2 - \frac{3}{2}A\right]\varepsilon_2^{\ 2} = 0.$$

Expanding, we have

$$\upsilon\left(-n^2\varepsilon_3^{\ 3} + q\right)\left(1 + 4\varepsilon_3 + 6\varepsilon_3^{\ 2} + 4\varepsilon_3^{\ 3} + \varepsilon_3^{\ 4}\right) -$$
$$\left[n^2\left(1 + 5\varepsilon_3 + 10\varepsilon_3^{\ 2} + 10\varepsilon_3^{\ 3} + 5\varepsilon_3^{\ 4} + \varepsilon_3^{\ 5}\right) - \left(1 + 2\varepsilon_3 + \varepsilon_3^{\ 2}\right) - \frac{3}{2}A\right]\varepsilon_3^{\ 2} = 0,$$

or

$$\upsilon\left[-n^2\varepsilon_3^{\ 3} + q - 4n^2\varepsilon_3^{\ 4} + 4q\varepsilon_3 - 6n^2\varepsilon_3^{\ 5} + 6q\varepsilon_3^{\ 2} - 4n^2\varepsilon_3^{\ 6} + 4q\varepsilon_3^{\ 3} - n^2\varepsilon_3^{\ 7} + q\varepsilon_2^{\ 4}\right]$$
$$-\left[n^2 + 5n^2\varepsilon_3 + 10n^2\varepsilon_3^{\ 2} + 10n^2\varepsilon_3^{\ 3} + 5n^2\varepsilon_3^{\ 4} + n^2\varepsilon_3^{\ 5} - 1 - 2\varepsilon_3 - \varepsilon_3^{\ 2} - \frac{3}{2}A\right]\varepsilon_3^{\ 2} = 0,$$

or

$$\upsilon\left[-n^2\varepsilon_3^{\ 3} + q - 4n^2\varepsilon_3^{\ 4} + 4q\varepsilon_3 - 6n^2\varepsilon_3^{\ 5} + 6q\varepsilon_3^{\ 2} - 4n^2\varepsilon_3^{\ 6} + 4q\varepsilon_3^{\ 3} - n^2\varepsilon_3^{\ 7} + q\varepsilon_3^{\ 4}\right]$$
$$+\left[-n^2\varepsilon_3^{\ 2} - 5n^2\varepsilon_3^{\ 3} - 10n^2\varepsilon_3^{\ 4} - 10n^2\varepsilon_3^{\ 5} - 5n^2\varepsilon_3^{\ 6} - n^2\varepsilon_3^{\ 7} + \varepsilon_3^{\ 2} + 2\varepsilon_3^{\ 3} + \varepsilon_3^{\ 4} + \frac{3}{2}A\varepsilon_3^{\ 2}\right] = 0.$$

Factorizing, we get

$$\varepsilon_3{}^7\left(-n^2-\upsilon n^2\right)+\varepsilon_3{}^6\left(-5n^2-4\upsilon n^2\right)+\varepsilon_3{}^5\left(-10n^2-6\upsilon n^2\right)+\varepsilon_3{}^4\left(1-10n^2-4n^2\upsilon+q\upsilon\right)$$
$$+\varepsilon_3{}^3\left(2-5n^2-n^2\upsilon+4q\upsilon\right)+\varepsilon_3{}^2\left(1-n^2+6q\upsilon+\frac{3}{2}A\right)+4q\upsilon\varepsilon_3+q\upsilon=0,$$

so that

$$\varepsilon_3{}^7(1+\upsilon)+\varepsilon_3{}^6(5+4\upsilon)+\varepsilon_3{}^5(10+6\upsilon)+\varepsilon_3{}^4\left(10-\frac{1}{n^2}+4\upsilon-\frac{q\upsilon}{n^2}\right)$$
$$+\varepsilon_3{}^3\left(5-\frac{2}{n^2}+\upsilon-\frac{4q\upsilon}{n^2}\right)+\varepsilon_3{}^2\left(1-\frac{1}{n^2}-\frac{6q\upsilon}{n^2}-\frac{3}{2n^2}A\right)-\frac{4q\upsilon}{n^2}\varepsilon_3-\frac{q\upsilon}{n^2}=0.$$

Dividing through by $(1+\upsilon)$, we get

$$\varepsilon_3{}^7+\varepsilon_3{}^6\frac{(5+4\upsilon)}{1+\upsilon}+\varepsilon_3{}^5\frac{(10+6\upsilon)}{1+\upsilon}+\frac{\varepsilon_3{}^4}{1+\upsilon}\left(10-\frac{1}{n^2}+4\upsilon-\frac{q\upsilon}{n^2}\right)$$
$$+\frac{\varepsilon_3{}^3}{1+\upsilon}\left(5-\frac{2}{n^2}+\upsilon-\frac{4q\upsilon}{n^2}\right)+\frac{\varepsilon_3{}^2}{1+\upsilon}\left(1-\frac{1}{n^2}-\frac{6q\upsilon}{n^2}-\frac{3}{2n^2}A\right)-\frac{4q\upsilon}{n^2(1+\upsilon)}\varepsilon_3$$
$$-\frac{q\upsilon}{n^2(1+\upsilon)}=0.$$

Hence,

$$\varepsilon_3{}^7+a\varepsilon_3{}^6+b\varepsilon_3{}^5+c\varepsilon_3{}^4+d\varepsilon_3{}^3+e\varepsilon_3{}^2-f\varepsilon_3-g=0, \qquad (4.19)$$

where

$$\upsilon=\frac{\mu}{(1-\mu)},$$

$$a=\frac{(5+4\upsilon)}{1+\upsilon},\ b=\frac{(10+6\upsilon)}{1+\upsilon},\ c=\frac{1}{1+\upsilon}\left(10-\frac{1}{n^2}+4\upsilon-\frac{q\upsilon}{n^2}\right),$$

$$d=\frac{1}{1+\upsilon}\left(5-\frac{2}{n^2}+\upsilon-\frac{4q\upsilon}{n^2}\right),\ e=\frac{1}{1+\upsilon}\left(1-\frac{1}{n^2}-\frac{6q\upsilon}{n^2}-\frac{3}{2n^2}A\right),$$

$$f=\frac{4q\upsilon}{n^2(1+\upsilon)}\ \text{and}\ g=\frac{q\upsilon}{n^2(1+\upsilon)}.$$

Figures 4.1, 4.2 and 4.3 give the positions of the three collinear points. However, the figures are not drawn to scale. Using the mathematical software, we give the numerical computations of the collinear equilibria L_1, L_2, and L_3 as in tables 4.1 and 4.2 below for varying oblateness of the bigger primary and radiation pressure force of the smaller primary for, $\mu = 0.000003$, and $\mu = 0.0967$ respectively. The classical case of the restricted three-body problem of Szebehely (1967) is verified for $A = 0$ and $q = 1$. Hence, these points draw nearer to or away from the primaries; depend on the system parameters.

Cases	A	q	xL_1	xL_2	xL_3
1	0	1 0.9985	-1.00001 -1.00001	0.990030 0.990035	1.00996 1.00995
2	0.001	1 0.9985	-1.00001 -1.00001	0.990039 0.990044	1.00995 1.00995
3	0.002	1 0.9985	-1.00001 -1.00001	0.990047 0.990052	1.00994 1.00994
4	0.01	1 0.9985	-1.00001 -1.00001	0.990113 0.990118	1.00988 1.00987
5	0.02	1 0.9985	-1.00001 -1.00001	0.990192 0.990197	1.00980 1.00979
6	0.03	1 0.9985	-1.00001 -1.00001	0.990269 0.990274	1.00972 1.00972
7	0.05	1 0.9985	-1.00001 -1.00001	0.990416 0.990421	1.00957 1.00957

Table 4. 1: computations of the collinear equilibrium points, for varying oblateness and radiation for $\mu = 0.000003$.

Cases	A	q	xL_1	xL_2	xL_3
1	0	1 0.9985	-1.04024 -1.04023	0.615350 0.615474	1.08528 1.08526
2	0.001	1 0.9985	-1.04027 -1.04029	0.615657 0.615781	1.08524 1.08522
3	0.002	1 0.9985	-1.04033 -1.04032	0.615963 0.616087	1.08520 1.08517
4	0.01	1 0.9985	-1.04070 -1.04069	0.618347 0.618469	1.08486 1.08484
5	0.02	1 0.9985	-1.04113 -1.04112	0.621183 0.621304	1.08444 1.08442
6	0.03	1 0.9985	-1.04154 -1.04153	0.623876 0.623996	1.08403 1.08401
7	0.05	1 0.9985	-1.04231 -1.04230	0.628884 0.629001	1.08323 1.08320

Table 4. 2: Computations of the collinear equilibrium points, for varying oblateness and radiation for $\mu = 0.0967$.

4.1.2 Position of triangular equilibrium points

In this section, the positions of the triangular points is investigate, under the joint action of, oblateness of the bigger primary, radiation pressure force and P-R drag of the smaller primary.

Now, the coordinates of triangular equilibrium points are the solutions of the equations $\Omega_x = 0,\ \Omega_y = 0,\ y \neq 0$ with $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0$. That is, they are the solutions of the equations:

$$n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} - \frac{\mu q(x+\mu-1)}{r_2^3} + \frac{W_2 n y}{r_2^2} = 0$$

$$\left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5} - \frac{\mu q}{r_2^3}\right] y - \frac{W_2 n(x+\mu-1)}{r_2^2} = 0 \tag{4.20}$$

Now, when the P-R drag effect are ignored and the first primary is a sphere; equations (4.20) are such that

$$x-\frac{(1-\mu)(x+\mu)}{r_1^3}-\frac{\mu q(x+\mu-1)}{r_2^3}=0$$

$$\left[1-\frac{(1-\mu)}{r_1^3}-\frac{\mu q}{r_2^3}\right]y=0 \qquad (4.21)$$

Now, from the second equation of (4.21), since $y \neq 0$, we must have

$$1-\frac{(1-\mu)}{r_1^3}-\frac{\mu q}{r_2^3}=0 \qquad (4.22)$$

Also, from the first equation of equation (4.21), we get

$$x\left[1-\frac{(1-\mu)}{r_1^3}-\frac{\mu q}{r_2^3}\right]-(1-\mu)\mu\left[\frac{1}{r_1^3}-\frac{q}{r_2^3}\right]=0.$$

Substitution equation (4.22) in the above equation, we get

$$-(1-\mu)\mu\left[\frac{1}{r_1^3}-\frac{q}{r_2^3}\right]=0$$

Since, $-(1-\mu)\mu \neq 0$, we must have

$$\frac{1}{r_1^3}-\frac{q}{r_2^3}=0.$$

So that

$$\frac{1}{r_1^3}=\frac{q}{r_2^3}. \qquad (4.23)$$

Now, for equation (4.22), that is

$$1-\frac{(1-\mu)}{r_1^3}-\frac{\mu q}{r_2^3}=0.$$

With the help of equation (4.23), we get

$$1-\frac{1}{r_1^3}+\frac{\mu}{r_1^3}-\frac{\mu}{r_1^3}=0.$$

So that

$$1-\frac{1}{r_1^3}=0.$$

Therefore,

$$r_1=1 \tag{4.24}$$

Substitute for r_1 in equation (4.23), to get

$$r_2=q^{\frac{1}{3}} \tag{4.25}$$

Knowing that equations (4.24) and (4.25) are the solutions when the second primary is radiating, we can with their help, assume the solutions when P-R drag and oblateness of the first primary are present (i.e. $A \neq 0,\ W_2 \neq 0$) to be

$$r_1=1+\varepsilon_1, \qquad r_2=q^{\frac{1}{3}}+\varepsilon_2; \qquad |\varepsilon_1| << 1,\ |\varepsilon_2| << 1 \tag{4.26}$$

Now, the exact x-coordinate of the triangular point is found by subtracting the second equation of (3.9) from the first to get,

$$r_1^2-r_2^2=(x+\mu)^2-(x+\mu-1)^2,$$

$$r_1^2-r_2^2=2(x+\mu)-1,$$

$$x=\frac{1}{2}-\mu+\frac{r_1^2-r_2^2}{2}. \tag{4.27}$$

Substituting the values of r_1 and r_2 in equations (4.26) into equation (4.27), we have

$$x+\mu = \frac{2+2\varepsilon_1 - \left(q^{\frac{2}{3}} + 2q^{\frac{1}{3}}\varepsilon_2\right)}{2},$$

or $$x+\mu = \frac{2+2\varepsilon_1 - q^{\frac{2}{3}} - 2q^{\frac{1}{3}}\varepsilon_2}{2},$$

or $$x = 1-\mu - \frac{q^{\frac{2}{3}}}{2} + \varepsilon_1 - q^{\frac{1}{3}}\varepsilon_2.$$

So that $$x = x_0 + \varepsilon_1 - q^{\frac{1}{3}}\varepsilon_2, \qquad (4.28)$$

where $$x_0 = 1-\mu - \frac{q^{\frac{2}{3}}}{2}.$$

Now, from $y^2 = r_1^2 - (x+\mu)^2$, we get

$$y^2 = 1+2\varepsilon_1 - \left[1 - q^{\frac{2}{3}} + \frac{q^{\frac{4}{3}}}{4} + \left(2 - q^{\frac{2}{3}}\right)\varepsilon_1 - \left(2 - q^{\frac{2}{3}}\right)q^{\frac{1}{3}}\varepsilon_2\right],$$

or $$y^2 = q^{\frac{2}{3}} - \frac{q^{\frac{4}{3}}}{4} + q^{\frac{2}{3}}\varepsilon_1 + \left(2 - q^{\frac{2}{3}}\right)q^{\frac{1}{3}}\varepsilon_2,$$

let $$y_0^2 = q^{\frac{2}{3}} - \frac{q^{\frac{4}{3}}}{4}, \qquad (4.29)$$

we now have

$$y^2 = y_0^2 + q^{\frac{2}{3}}\varepsilon_1 + \left(2 - q^{\frac{2}{3}}\right)q^{\frac{1}{3}}\varepsilon_2,$$

or $$y^2 = y_0^2\left[1+\frac{q^{\frac{2}{3}}\varepsilon_1}{y_0^2}+\frac{\left(2-q^{\frac{2}{3}}\right)q^{\frac{1}{3}}\varepsilon_2}{y_0^2}\right],$$

or $$y = y_0\left[1+\frac{q^{\frac{2}{3}}\varepsilon_1}{y_0^2}+\frac{\left(2-q^{\frac{2}{3}}\right)q^{\frac{1}{3}}\varepsilon_2}{y_0^2}\right]^{\frac{1}{2}},$$

so that $$y = \pm\, y_0\left[1+\frac{q^{\frac{2}{3}}\varepsilon_1}{2y_0^2}+\frac{\left(2-q^{\frac{2}{3}}\right)q^{\frac{1}{3}}\varepsilon_2}{2y_0^2}\right], \tag{4.30}$$

where $y_0 = \frac{q^{\frac{1}{3}}}{2}\left[4-q^{\frac{2}{3}}\right]^{\frac{1}{2}}$.

Expressing $q = 1-\delta$ when δ is very small and substituting in x_0 and y_0, we have

$$x_0 = \frac{1}{2}-\mu+\frac{1}{3}\delta$$

and (4.31)

$$y_0 = \frac{\sqrt{3}}{2}\left(1-\frac{2}{9}\delta\right)$$

Now, substituting equations (3.10), (4.26), (4.28) and (4.30) into equations (4.20), neglecting higher order terms of small quantities, we get the respective equations:

$$a_1\varepsilon_1 + b_1\varepsilon_2 = c_1 \tag{4.32}$$

$$a_2\varepsilon_1 + b_2\varepsilon_2 = c_2 \tag{4.33}$$

where

$$a_1 = 3x_0 + 3\mu - 3\mu x_0 - 3\mu^2 + \frac{15}{2}x_0 A - \frac{15}{2}\mu x_0 A + 9\mu A - \frac{15}{2}\mu^2 A + \frac{1}{2y_0}W_2$$

$$b_1 = -3\mu q^{\frac{-1}{3}} + 3\mu^2 q^{\frac{-1}{3}} + 3\mu q^{\frac{-1}{3}} x_0 - \frac{3}{2}\mu q^{\frac{1}{3}} A - 2q^{-1} y_0 W_2 + \frac{\left(2 - q^{\frac{2}{3}}\right) q^{\frac{-1}{3}}}{2y_0} W_2$$

$$c_1 = \frac{3}{2}\mu A - \frac{3}{2}\mu^2 A - \frac{3}{2}\mu x_0 A - q^{\frac{-2}{3}} y_0 W_2$$

$$a_2 = 3y_0 - 3\mu y_0 + \frac{15}{2}y_0 A - \frac{15}{2}\mu y_0 A + \frac{3q^{\frac{2}{3}}}{4y_0}\mu A - q^{\frac{-2}{3}} W_2$$

$$b_2 = 3\mu y_0 q^{\frac{-1}{3}} + \frac{3\left(2 - q^{\frac{2}{3}}\right) q^{\frac{1}{3}}}{4y_0}\mu A + 2q^{-1} x_0 W_2 + q^{\frac{-1}{3}} W_2 + 2\mu q^{-1} W_2 - 2q^{-1} W_2$$

$$c_2 = -\frac{3}{2}\mu y_0 A + q^{\frac{-2}{3}} x_0 W_2 + \mu q^{\frac{-2}{3}} W_2 - q^{\frac{-2}{3}} W_2$$

The solutions of ε_1 and ε_2 can be found as follows using these relations

$$\varepsilon_1 = \frac{b_2 c_1 - b_1 c_2}{a_1 b_2 - a_2 b_1}, \qquad \varepsilon_2 = \frac{a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$$

to get

$$\varepsilon_1 = -\frac{2W_2}{3\sqrt{3}(1-\mu)},\ \varepsilon_2 = -\frac{A}{2} + \frac{W_2}{3\mu\sqrt{3}} \tag{4.34}$$

Putting the values of ε_1 and ε_2 above and equations (4.31) into the equations (4.28) and (4.30), we at once have

$$x_4 = \frac{1}{2} - \mu + \frac{1}{3}\delta + \frac{A}{2} - \frac{W_2(1+\mu)}{3\mu\sqrt{3}(1-\mu)}$$

$$y_4 = \pm\left[\frac{\sqrt{3}}{2}\left(1 - \frac{2}{9}\delta - \frac{A}{3}\right) + \frac{W_2(1-3\mu)}{9\mu(1-\mu)}\right] \tag{4.35}$$

Equations (4.35) give the coordinates of the triangular points of the system under investigation. Since $r_1 \neq r_2$, each of the two points defined by (4.35) form a scalene triangle with the primaries and are denoted by $L_{4,5}(x_4, \pm y_4)$ and called the triangular equilibrium points by virtue of the two triangles they form with lines joining the primaries. The positions of these points depend on the mass ratio, oblateness of the first primary, radiation pressure force and P-R drag effects of the second primary. These are shown graphically in $3D$ plots to display the effects of the oblateness of the bigger primary, radiation pressure of the smaller primary together with its P-R drag. Tables 4.3 and 4.4 below give the numerical positions of these points for varying oblateness of the first primary and radiation pressure of the second one respectively. To show the validity of our analysis [formulas (4.35)], we have also solved system (4.20) numerically and have listed them in the tables. Figures 1 and 2 below show the position for x_4 and y_4 with $\mu = 0.00003$, $c_d = 299792458$, and $0 < q \leq 1$ and $0 \leq A \leq 0.2$, while figures 3, 4, 5 and 6 show the effect of mass ratio μ on the positions.

	System (4.35)		System (4.20)	
A	x_4	$\pm y_4$	x_4	$\pm y_4$
0.00	0.50047	0.86573673	0.50047013	0.86573646
0.01	0.50547	0.86284998	0.50540351	0.86286556
0.02	0.51047	0.85996323	0.51021685	0.86002799
0.03	0.51547	0.85707647	0.51491477	0.85722336
0.04	0.52047	0.85418972	0.51950163	0.85445122
0.05	0.52547	0.85130297	0.52398156	0.85171115
0.06	0.53047	0.84841622	0.52835850	0.84900270
0.07	0.53547	0.84552947	0.53263619	0.84632543
0.08	0.54047	0.84264272	0.53681819	0.84367886
0.09	0.54547	0.83975597	0.54090786	0.84106256
0.10	0.55047	0.83686922	0.54490843	0.83847606

Table 4. 3: Coordinates of triangular points for $\mu = 0.00003$,

$c_d = 299792458$, $\delta = 0.0015$ and $0 \leq A \leq 0.2$

	System (4.35)		System (4.20)	
δ	x_4	$\pm y_4$	x_4	$\pm y_4$
0.000	0.50997000	0.86025190	0.50972649	0.86031873
0.001	0.51030333	0.86005945	0.51005337	0.86012496
0.002	0.51063667	0.85986700	0.51038036	0.85993096
0.003	0.51097000	0.85967455	0.51070747	0.85973673
0.004	0.51130333	0.85948210	0.51103468	0.85954226
0.010	0.51330333	0.85832740	0.51300025	0.85837053
0.020	0.51663667	0.85640290	0.51628505	0.85639872
0.030	0.51997000	0.85447840	0.51958105	0.85440293
0.040	0.52330333	0.85255390	0.52288839	0.85238275
0.050	0.52663667	0.85062940	0.52620723	0.85033780
0.080	0.53663667	0.84485589	0.53623443	0.84405003

Table 4. 4: Coordinates of triangular points for $\mu = 0.00003$,

$c_d = 299792458$, $\delta = 0.0015$ and $0 \leq A \leq 0.2$

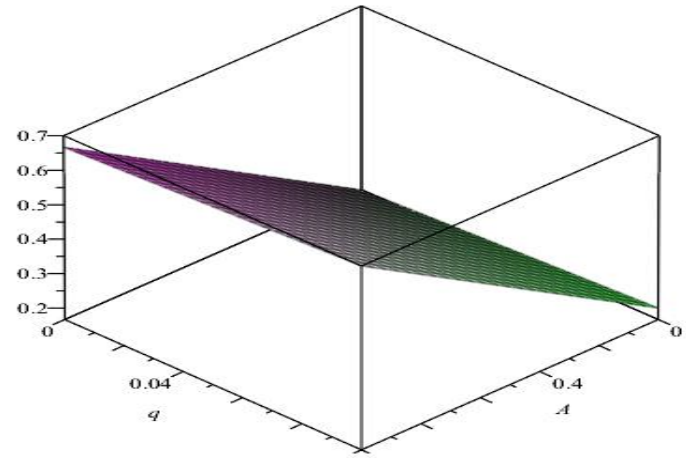

Fig.4. 4: Position of x_4, for $\mu = 0.00003$, $c_d = 299792458$, and $0 < q \leq 1$ and $0 \leq A \leq 0.2$.

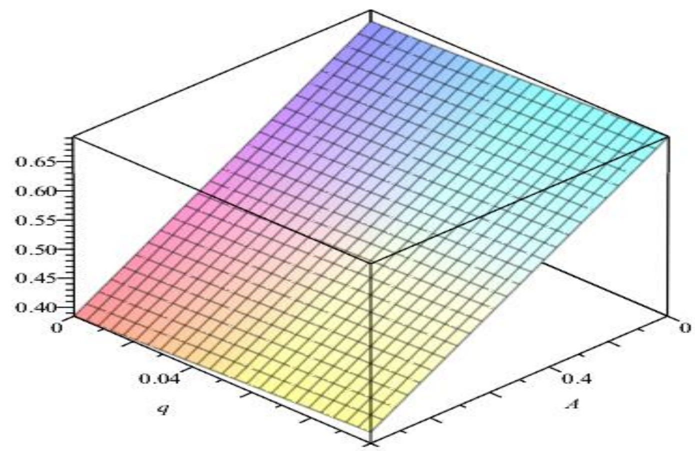

Fig.4. 5: Position of y_4, for $\mu = 0.00003$, $c_d = 299792458$, and $0 < q \leq 1$ and $0 \leq A \leq 0.2$.

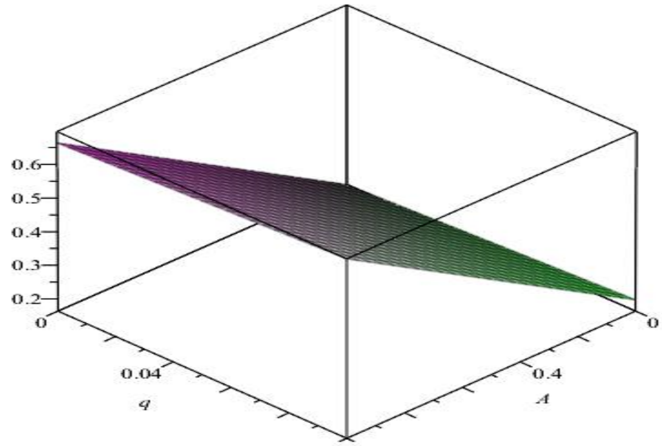

Fig.4.6: Position of x_4, for $\mu = 0.02$, $c_d = 299792458$, and $0 < q \leq 1$ and $0 \leq A \leq 0.2$.

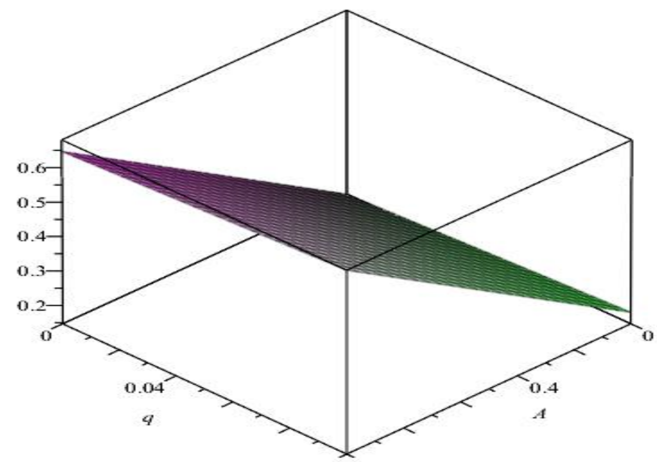

Fig.4. 7: Position of x_4, for $\mu = 0.02$, $c_d = 299792458$, and $0 < q \leq 1$ and $0 \leq A \leq 0.2$.

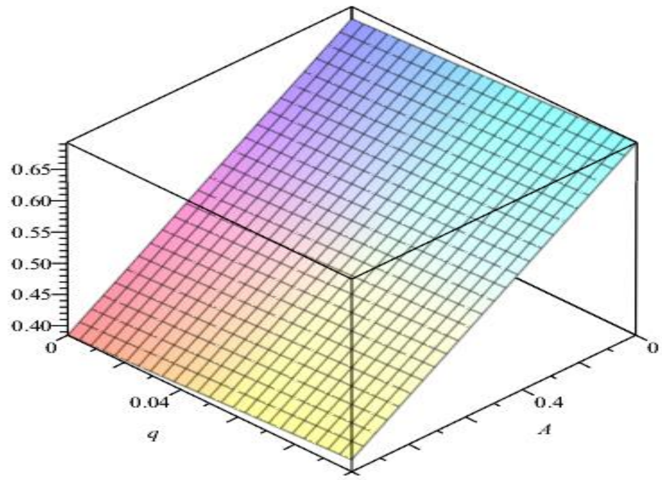

Fig.4. 8: Position of y_4, for $\mu = 0.03$, $c_d = 299792458$, and $0 < q \leq 1$ and $0 \leq A \leq 0.2$.

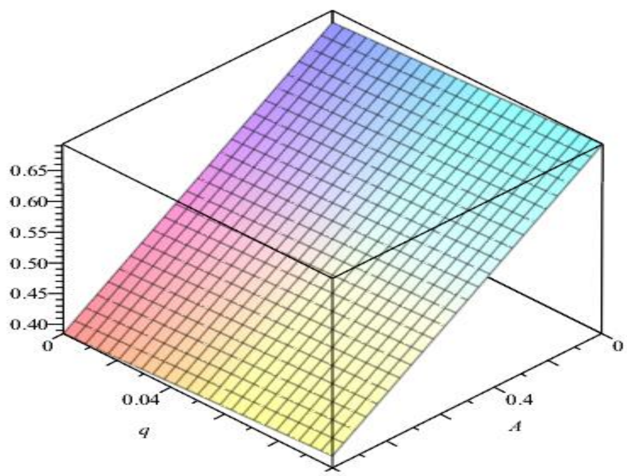

Fig.4. 9: Position of y_4, for $\mu = 0.03$, $c_d = 299792458$, and $0 < q \leq 1$ and $0 \leq A \leq 0.2$.

4.1.3 Position of the out-of-plane equilibrium points

The coordinates equilibrium points are the solutions of the equations $\Omega_x = 0$, $\Omega_y = 0$, $\Omega_z = 0$, with $\dot{x} = \dot{y} = \dot{z} = 0$, $x \neq 0$, $y \neq 0$, and $z \neq 0$. That is, they are the solutions of the equations:

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} + \frac{15(1-\mu)A(x+\mu)z^2}{2r_1^7} - \frac{\mu q(x+\mu-1)}{r_2^3}$$
$$+\frac{W_2 ny}{r_2^2},$$

$$\Omega_y = n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{3(1-\mu)Ay}{2r_1^5} + \frac{15(1-\mu)Ayz^2}{2r_1^7} - \frac{\mu qy}{r_2^3} - \frac{W_2 n(x+\mu-1)}{r_2^2},$$

$$\Omega_z = -\frac{(1-\mu)z}{r_1^3} - \frac{9(1-\mu)Az}{2r_1^5} + \frac{15(1-\mu)Az^3}{2r_1^7} - \frac{\mu qz}{r_2^3},$$

$$r_1^2 = (x+\mu)^2 + y^2 + z^2, \qquad r_2^2 = (x+\mu-1)^2 + y^2 + z^2,$$

$$W_2 = \frac{\mu(1-q)}{c_d},$$

$$\Omega_x = x\left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)Az^2}{2r_1^7} - \frac{\mu q}{r_2^3}\right] + \frac{\mu q}{r_2^3} + \frac{W_2 ny}{r_2^2} + \mu\left[-\frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5}\right.$$
$$\left. + \frac{15(1-\mu)Az^2}{2r_1^7} - \frac{\mu q}{r_2^3}\right] = 0, \tag{4.36}$$

$$\Omega_y = y\left[n^2 - \frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)Az^2}{2r_1^7} - \frac{\mu q}{r_2^3}\right] - \frac{W_2 n(x+\mu-1)}{r_2^2} = 0, \tag{4.37}$$

$$\Omega_z = z\left[-\frac{(1-\mu)}{r_1^3} - \frac{9(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)Az^2}{2r_1^7} - \frac{\mu q}{r_2^3}\right] = 0. \tag{4.38}$$

Because $z \neq 0$, equation (4.36) becomes

$$n^2 x + \frac{3(1-\mu)Ax}{r_1^5} + \frac{3\mu(1-\mu)A}{r_1^5} + \frac{\mu q}{r_2^3} + \frac{W_2 ny}{r_2^2} = 0, \tag{4.39}$$

Equation (4.37) becomes

$$n^2 y + \frac{3(1-\mu)Ay}{r_1^5} - \frac{W_2 n(x+\mu-1)}{r_2^2} = 0, \tag{4.40}$$

Equation (4.38) becomes

$$\frac{1-\mu}{r_1^3}+\frac{9(1-\mu)A}{2r_1^5}-\frac{15(1-\mu)Az^2}{2r_1^7}+\frac{\mu q}{r_2^3}=0. \tag{4.41}$$

From equation (4.40)

$$y=\frac{W_2(x+\mu-1)}{r_2^2}. \tag{4.42}$$

Substitute the value of y into equation (4.39), we have

$$n^2x+\frac{3(1-\mu)Ax}{r_1^5}+\frac{3\mu(1-\mu)A}{r_1^5}+\frac{\mu q}{r_2^3}+\frac{W_2 n}{r_2^2}\left[\frac{W_2(x+\mu-1)}{r_2^2}\right]=0,$$

or $$x=-\frac{n^2\mu q}{r_2^3}-\frac{3n^2\mu(1-\mu)A}{r_1^5}+\frac{3\mu q(1-\mu)A}{r_1^5 r_2^3}.$$

So that

$$x_0=-\frac{\mu q}{r_{20}^3}-\frac{3\mu(1-\mu)A}{r_{10}^5}+\frac{3\mu qA}{2r_{20}^3}+\frac{3\mu q(1-\mu)A}{r_{10}^5 r_{20}^3}. \tag{4.43}$$

Substituting (4.43) in (4.22), we at once have

$$y_0=\frac{W_2(x_0+\mu-1)}{r_{20}^2}. \tag{4.44}$$

From equation (4.41), we get

$$z_0^2=\frac{\dfrac{1-\mu}{r_1^3}+\dfrac{9(1-\mu)A}{2r_1^5}+\dfrac{\mu q}{r_{20}^3}}{\dfrac{15(1-\mu)A}{2r_{10}^7}},$$

so that

$$z_0^2=\frac{2r_{10}^4}{15A}+\frac{3r_{10}^2}{5A}+\frac{2uqr_{10}^7}{15(1-\mu)r_{20}^3A}. \tag{4.45}$$

Now, we proceed by choosing initial approximations for the solutions. Hence, we choose $x_0 = 1-\mu, \quad y_0 = 0, \; z_0 = \sqrt{3A}$. The initials condition for y_0 is obtained by substituting those for x_0 and z_0 in equation (4.44). This allows us to compute the out-of-plane equilibrium points for $y_0 \in [0,0.3]$.

Using the software package Mathematica, we are able to determine the position of points $x_0, y_0, \pm z_0$ denoted by $L_{6,} L_7$ and L_8 which we express in power series form to second order terms in A from equations (4.43, 4.44, and 4.45) as:

$$x_0 = -\frac{q\mu}{3\sqrt{3}A^{\frac{3}{2}}}, \; z_0 = \infty,$$

when $y_0 = 0$; (4.46)

$$x_0 = \mu\left[-1000q + A\{-2.92629 + q(454426 - 2926.29\mu) + 2.92629\mu\} \right.$$
$$\left. A^2\{21.7299(1-\mu) + q(-1.70764\times10^8 + 1.33856\times10^6\mu)\}\right],$$

$$z_0 = 1.014 + 60691.1q\mu(\mu-1) + \frac{0.00134667 + q(138.059 - 138.059\mu)\mu}{A}, \tag{4.47}$$

when $y_0 = 0.1$;

$$x_0 = \mu\left[-125q - A\{2.71981(1-\mu) - q(14590 - 339.976\mu)\} \right.$$
$$\left. A^2\{19.614(1-\mu) + q(-1.38015\times10^6 + 40699\mu)\}\right],$$

$$z_0 = 1.056 + q\mu(-1957.86 + 1957.86\mu) + \frac{0.00554667 + 19.119q(1-\mu)\mu}{A}, \tag{4.48}$$

when $y_0 = 0.2$.

These points lie on the $x_0, y_0, \pm z_0$ plane of the primaries and are denoted by $L_{6,7,8}$. They are symmetrical w.r.t. the orbital plane and belong to the family of the out-of-plane equilibrium points. These points depend on the mass ratio, oblateness of the bigger primary; radiation pressure force and P-R drag of the smaller primary. The coordinates are shown numerically and graphically in Tables (4.5-4.10), and Figures (4.7-4.12), respectively. The graphs have been plotted using parametric plots when $\mu = 0.000003, c_d = 299792458$, for a wide range of the radiation repulsive force of the smaller primary and oblateness of the bigger primary.

A	x_0	$\pm z_0$
0	-0.00299550	Infinite remote solution
0.001	-0.00214580	1.610100
0.005	-0.00897743	1.088230
0.010	-0.04053560	1.004100
0.015	-0.09766990	0.974448
0.020	-0.18038000	0.959276
0.030	-0.42253000	0.943861
0.040	-0.76698400	0.936059
0.050	-1.21374000	0.931346
0.060	-1.76281000	0.928190
0.070	-2.41417000	0.925930
0.080	-3.16785000	0.924231
0.090	-4.02382000	0.922908

Table 4. 5: Coordinates of out of plane equilibrium points for $y_0 = 0.1$,

$\mu = 0.000003, c_d = 299792458$, $\delta = 0.0015$ and $0 < A \leq 0.09$

δ	x_0	$\pm z_0$
0	-0.18065100	0.959150
0.001	-0.18047100	0.959234
0.002	-0.18029000	0.959318
0.003	-0.18010900	0.959403
0.004	-0.17992900	0.959487
0.010	-0.17884500	0.959991
0.020	-0.17703800	0.960831
0.030	-0.17523200	0.961671
0.040	-0.13425000	0.962509
0.050	-0.17161900	0.963347
0.080	-0.16619900	0.965856
0.090	-0.16439300	0.966691

Table 4. 6: Coordinates of out of plane equilibrium points for $y_0 = 0.1$,

$\mu = 0.000003, c_d = 299792458$, $0 < q \leq 1$ and $A = 0.02$

A	x_0	$\pm z_0$
0	-0.000374438	Infinite remote solution
0.001	-0.000334875	2.5795500
0.005	-0.000259312	1.4734100
0.010	-0.000350895	1.2690700
0.015	-0.000649187	1.1932000
0.020	-0.001154190	1.1534000
0.030	-0.002784320	1.1121800
0.040	-0.005241290	1.0909800
0.050	-0.008525090	1.0780600
0.060	-0.012635700	1.0693600
0.070	-0.017573200	1.0631000
0.080	-0.023337500	1.0583900
0.090	-0.029928700	1.0547000

Table 4. 7: Coordinates of out of plane equilibrium points for $y_0 = 0.2$,

$\mu = 0.000003, c_d = 299792458$, $\delta = 0.0015$ and $0 < A \leq 0.09$

δ	x_0	$\pm z_0$
0	-0.00115592	1.15340000
0.001	-0.00115476	1.15340000
0.002	-0.00115361	1.15340000
0.003	-0.00115245	1.15340000
0.004	-0.00115130	1.15340000
0.010	-0.00114436	1.15341000
0.020	-0.00113280	1.15342000
0.030	-0.00112125	1.15344000
0.040	-0.00110969	1.15345000
0.050	-0.00109813	1.15346000
0.080	-0.00106346	1.15350000
0.090	-0.00105190	1.15352000

Table 4. 8: Coordinates of out of plane equilibrium points for $y_0 = 0.2$,

$\mu = 0.000003, c_d = 299792458$, $0 < q \leq 1$ and $A = 0.02$

A	x_0	$\pm z_0$
0	-0.0001109440	Infinite remote solution
0.001	-0.0001052240	3.7716300
0.005	-0.0000874376	1.9352500
0.010	-0.0000766641	1.5605100
0.015	-0.0000786241	1.4136900
0.020	-0.0000933174	1.3342400
0.030	-0.0001609040	1.2497400
0.040	-0.0002794240	1.2052800
0.050	-0.0004488780	1.1777900
0.060	-0.0006692650	1.1591100
0.070	-0.0009405850	1.1455700
0.080	-0.0012628400	1.1353200
0.090	-0.0016360300	1.1272700

Table 4. 9: Coordinates of out of plane equilibrium points for $y_0 = 0.3$,

$\mu = 0.000003, \ c_d = 299792458$, $\delta = 0.0015$ and $0 < A \leq 0.09$

δ	x_0	$\pm z_0$
0	-0.0000934574	1.3342400000
0.001	-0.0000933641	1.3342400000
0.002	-0.0000932707	1.3342400000
0.003	-0.0000931774	1.3342400000
0.004	-0.0000930841	1.3342400000
0.010	-0.0000925241	1.3342400000
0.020	-0.0000915908	1.3342400000
0.030	-0.0000906574	1.3342400000
0.040	-0.0000897241	1.3342400000
0.050	-0.0000887908	1.3342400000
0.080	-0.0000859908	1.3342300000
0.090	-0.0000850575	1.3342300000

Table 4.10: Coordinates of out of plane equilibrium points for $y_0 = 0.3$,

$\mu = 0.000003, \ c_d = 299792458$, $0 < q \leq 1$ and $A = 0.02$

Tables 4.5 and 4.6; Tables 4.7 and 4.8; Tables 4.9 and 4.10, represent the cases when we have set $y_0 = 0.1$, $y_0 = 0.2$ and $y_0 = 0.3$ for varying oblateness of the bigger primary and radiation of the smaller one, respectively. The numerical computations of the coordinates when $y_0 = 0$ does not suffice because the solutions turns out to be infinity on the $z-axis$. This perhaps occurs because, for y_0 be zero, either $W_2 = 0$ or $z \to \infty$. In this case, we get

$x = -0.003313798$ which reduce to a collinear equilibrium point, since it lies on the line joining the primaries. We note that when $y_0 < 0.1$, these points do not appear realistic. For instance, when $y_0 = 0.01$, we get $x = -1.13547 \times 10^7$ and z will be imaginary.

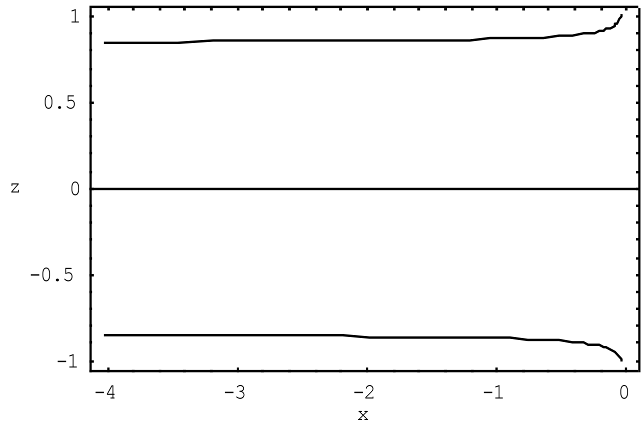

Fig.4. 10: Coordinates of out of plane equilibrium points for $y_0 = 0.1$,

$\mu = 0.000003$, $c_d = 299792458$, $\delta = 0.0015$ and $0 < A \leq 0.09$

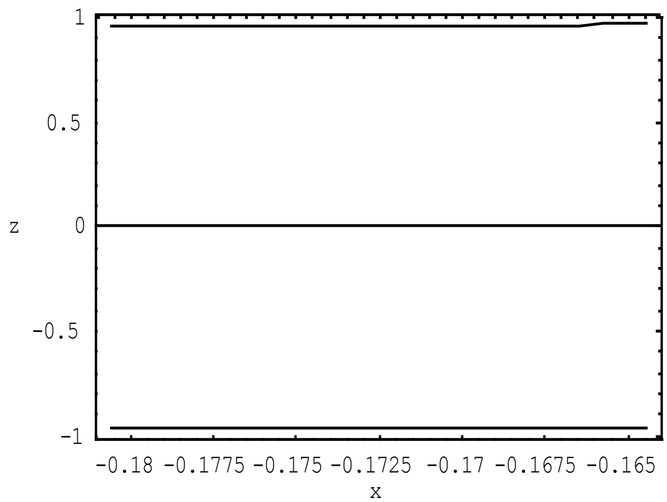

Fig.4. 11: Coordinates of out of plane equilibrium points for $y_0 = 0.1$,

$\mu = 0.000003$, $c_d = 299792458$, $0 < q \leq 1$ and $A = 0.02$

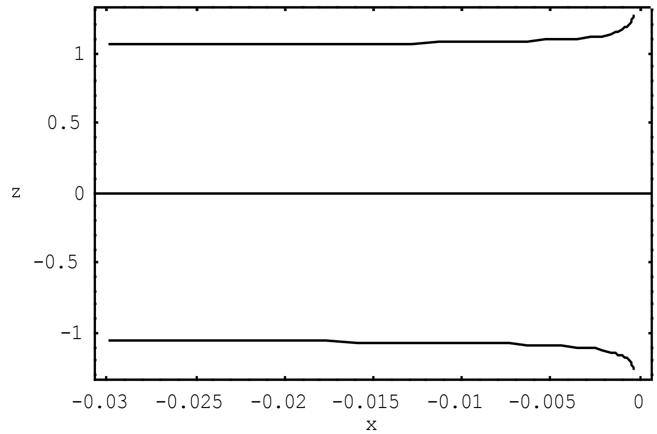

Fig.4. 12: Coordinates of out of plane equilibrium points for $y_0 = 0.2$,

$\mu = 0.000003$, $c_d = 299792458$, $\delta = 0.0015$ and $0 < A \leq 0.09$

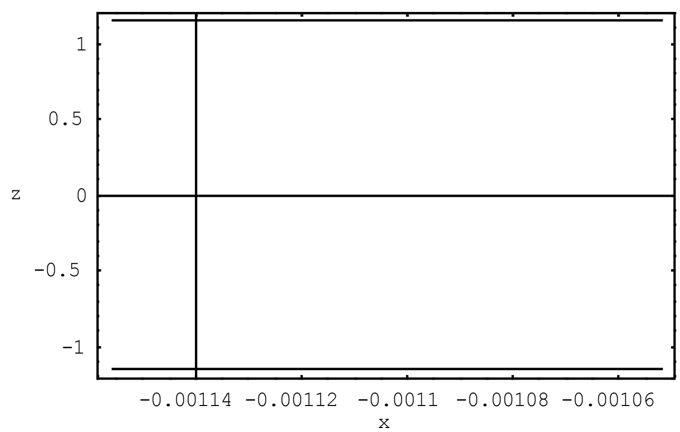

Fig.4. 1: Coordinates of out of plane equilibrium points for $y_0 = 0.2$,

$\mu = 0.000003$, $c_d = 299792458$, $0 < q \leq 1$ and $A = 0.02$

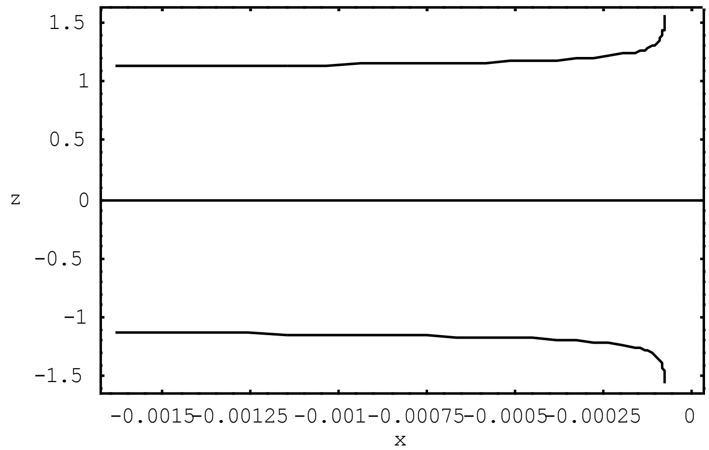

Fig.4. 2: Coordinates of out of plane equilibrium points for $y_0 = 0.3$,

$\mu = 0.000003$, $c_d = 299792458$, $\delta = 0.0015$ and $0 < A \leq 0.09$

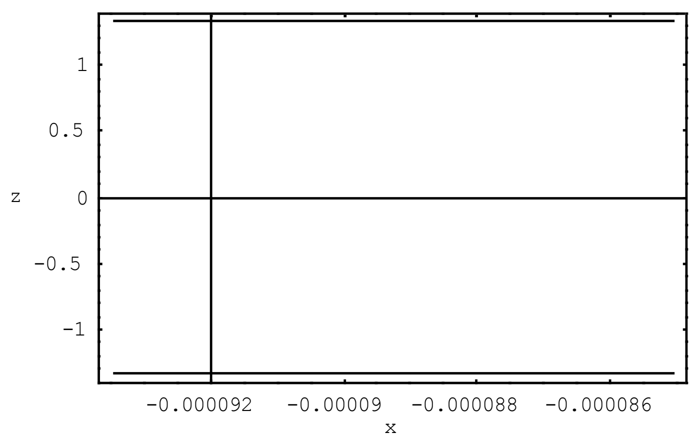

Fig.4. 3: Coordinates of out of plane equilibrium points for $y_0 = 0.3$,

$\mu = 0.000003$, $c_d = 299792458$, $0 < q \leq 1$ and $A = 0.02$

We have performed an analytical and numerical study of nine equilibrium points, of a test particle, having infinitesimal mass, under the gravitational attraction of two massive bodies. The bigger being an oblate spheroid while the smaller is a source of radiation and has a Poynting-Robertson drag influence.

There are three collinear points, a pair of triangular points and four-coordinate points lying on the out of the orbital plane of motion.

The three collinear equilibrium points depends on the oblateness of the bigger primary, the mass ratio and radiation pressure of the smaller primary. However, they are not influenced by the P-R drag of the smaller primary. The numerical computations of the collinear equilibria are given in tables (4.1) and (4.2) for $\mu = 0.00003$and $\mu = 0.0967$ respectively, together with some chosen parametric values, such that $q = 0.9985,\ A = 0.002,$ $c_d = 299792458$, using the mathematical software. We observe that the positions of the collinear equilibrium points are affected by the presence of these parameters and shift closer or away from the center of mass of the primaries.

The triangular points of the problem have been found by adopting a perturbation method of solutions. We have first considered when the bigger primary is not oblate, the smaller not radiating and no P-R drag influence. Then, we assume that when these parameters are present, there will only be slight change in the solutions. It is observed that the triangular equilibrium points given by equations (4.35) unlike the collinear points are influenced by the P-R drag, radiation pressure of the smaller primary and oblateness of the bigger one. These are different from the triangular solutions of the classical restricted three-body problem (Szebehely 1967).

The equations of motion given in equations (3.11), are the three-dimensional dynamical system governing the motion of the infinitesimal mass, and are achieved due to the non-sphericity of the bigger primary. With the help of these equations, we have found the out-of-plane equilibrium points, under the combine effects of oblateness of the bigger primary, radiation force of the smaller primary and its P-R drag. We observe that the three-dimensional equations of motion are affected by the radiation and P-R drag of the smaller primary and oblateness of the bigger primary. These points are different from those found in previous studies, because we have included the Poynting-Robertson (P-R) drag occurring due to the radiation of the smaller primary, and contained in the expression of y_0 coordinate. We also observe that equation (7) does not solve for $y = 0$, so, we solve the remaining two equations, (6) and (8) numerically, using power series, and taking $y_0 \in [0, 0.3]$. We observe that the numerical computations of the coordinates, when $y_0 = 0$ does not suffice, because the solutions on the $z-axis$, results to infinity, while for $y_0 = 0$, $y_0 < 0.1$, these points do not appear realistic. For $y_0 = 0.01$, we get $x = -1.13547 \times 10^7$ and z will be imaginary.

CHAPTER 5

STABILITY OF THE EQUILIBRIUM POINTS

5.1 INTRODUCTION

Generally speaking, the stability of differential equations refers to the global effects of local perturbations. Stability of dynamical systems is define in different ways in different context, but the notion of stability must change depending upon the physical system under consideration. Lyapunov, A. M. (1857-1918) concentrated on the stability of equilibrium points, the motion of the mechanical system and stability of a uniformly rotating fluid and devised methods of approximation. Lyapunov's method introduced in 1899 provides ways of determining the stability of sets of ordinary differential equations.

In this chapter, we shall study the linear stability of the equilibrium found in this thesis. We shall devote sections 5.2, 5.3 and 5.4 to the stability analysis of the collinear, triangular and the out-of-plane equilibrium points, respectively. To study the stability of any equilibrium configuration, that is, its ability to restrain the body motion in its vicinity, we displace the third body a little from an equilibrium point with a small velocity. If its motion is a rapid departure from the vicinity of the points, we call such a position an unstable one. However, if the body merely oscillates about the points, it is said to be a stable position.

5.1.1 Variational equations

Let the location of an equilibrium point be denoted by (a_0, b_0) and consider a small displacement (ξ, η) from the point such that $x = a_0 + \xi$ and $y = b_0 + \eta.$ Substituting these values in (3.8), we obtain the variational equations, when motion is considered on the xy-plane:

Now, taking the derivative of equations of $x = a_0 + \xi$ and $y = b_0 + \eta$, we have

$$\dot{x} = \dot{\xi}, \qquad \dot{y} = \dot{\eta}$$

$$\ddot{x} = \ddot{\xi}, \qquad \ddot{y} = \ddot{\eta} \tag{5.1}$$

Substituting these derivatives in the equation of motion (3.8), we have

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial \Omega}{\partial x}$$

$$(5.2)$$

$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial \Omega}{\partial y}$$

where

$$\frac{\partial \Omega}{\partial x} = \Omega_x = \Omega_x(x, y) = \Omega_x(a_0 + \xi, b_0 + \eta).$$

Expanding (5.2) about Ω_x and Ω_y in Taylor series, we get

$$\Omega_x = \Omega_x(a_0 + \xi, b_0 + \eta)$$

so that

$$\Omega_0(a_0, b_0) = \xi(\Omega_{xx})^0 + \eta(\Omega_{xy})^0 + \xi(\Omega_{x\dot{x}})^0 + \eta(\Omega_{x\dot{y}})^0$$

Similarly

$$\Omega_y = \Omega_y(a_0 + \xi, b_0 + \eta)$$

so that

$$\Omega_y(a_0, b_0) = \xi(\Omega_{yx})^0 + \eta(\Omega_{yy})^0 + \xi(\Omega_{y\dot{x}})^0 + \eta(\Omega_{y\dot{y}})^0.$$

Now, since equilibrium points $\Omega_x(a_0, b_0) = 0$, $\Omega_y(a_0, b_0) = 0$,

then $\quad \Omega_x = \xi(\Omega_{xx})^0 + \eta(\Omega_{xy})^0 + \xi(\Omega_{x\dot{x}})^0 + \eta(\Omega_{x\dot{y}})^0.$

$$\Omega_y = \xi(\Omega_{yx})^0 + \eta(\Omega_{yy})^0 + \xi(\Omega_{y\dot{x}})^0 + \eta(\Omega_{y\dot{y}})^0.$$

Therefore

$$\ddot{\xi} - 2n\dot{\eta} = \xi(\Omega_{xx})^0 + \eta(\Omega_{xy})^0 + \xi(\Omega_{x\dot{x}})^0 + \eta(\Omega_{x\dot{y}})^0$$
$$\ddot{\eta} + 2n\dot{\xi} = \xi(\Omega_{yx})^0 + \eta(\Omega_{yy})^0 + \xi(\Omega_{y\dot{x}})^0 + \eta(\Omega_{y\dot{y}})^0 \tag{5.3}$$

Equation (5.3) is the variational equation of motion of the test particle of infinitesimal mass. Here, only linear terms in ξ and η have been considered, the second partial derivative of Ω are denoted by subscripts and indicate that the derivatives have been evaluated at the equilibrium points.

5.1.2 Characteristic equation

Now, we assume trial solutions as

$$\xi = Ae^{\lambda t}, \qquad \eta = Be^{\lambda t}$$
$$\dot{\xi} = A\lambda e^{\lambda t}, \qquad \dot{\eta} = B\lambda e^{\lambda t} \tag{5.4}$$
$$\ddot{\xi} = A\lambda^2 e^{\lambda t}, \qquad \ddot{\eta} = B\lambda^2 e^{\lambda t}.$$

Substituting equations (5.4), in the variational equations (5.3), we get

$$A\lambda^2 e^{\lambda t} - 2nB\lambda e^{\lambda t} = \Omega^0_{x\dot{x}} A\lambda e^{\lambda t} + \Omega^0_{y\dot{x}} B\lambda e^{\lambda t} + \Omega^0_{xx} Ae^{\lambda t} + \Omega^0_{xy} Be^{\lambda t}$$

$$B\lambda^2 e^{\lambda t} + 2nA\lambda e^{\lambda t} = \Omega^0_{y\dot{x}} A\lambda e^{\lambda t} + \Omega^0_{y\dot{y}} B\lambda e^{\lambda t} + \Omega^0_{yx} Ae^{\lambda t} + \Omega^0_{yy} Be^{\lambda t}$$

or

$$A\lambda^2 e^{\lambda t} - 2nB\lambda e^{\lambda t} - \Omega^0_{x\dot{x}} A\lambda e^{\lambda t} - \Omega^0_{y\dot{x}} B\lambda e^{\lambda t} - \Omega^0_{xx} Ae^{\lambda t} - \Omega^0_{xy} Be^{\lambda t} = 0$$

$$B\lambda^2 e^{\lambda t} + 2nA\lambda e^{\lambda t} - \Omega^0_{y\dot{x}} A\lambda e^{\lambda t} - \Omega^0_{y\dot{y}} B\lambda e^{\lambda t} - \Omega^0_{yx} Ae^{\lambda t} - \Omega^0_{yy} Be^{\lambda t} = 0$$

or

$$A\left[\lambda^2 - \lambda\Omega^0_{x\dot{x}} - \lambda\Omega^0_{xx}\right]e^{\lambda t} + B\left[-2n\lambda - \lambda\Omega^0_{x\dot{y}} - \Omega^0_{xy}\right]e^{\lambda t} = 0$$

$$A\left[2n\lambda - \lambda\Omega^0_{y\dot{x}} - \Omega^0_{yx}\right]e^{\lambda t} + B\left[\lambda^2 - \lambda\Omega^0_{y\dot{y}} - \lambda\Omega^0_{yy}\right]e^{\lambda t} = 0$$

or

$$A\left[\lambda^2 - \lambda\Omega^0_{x\dot{x}} - \lambda\Omega^0_{xx}\right] + B\left[-2n\lambda - \lambda\Omega^0_{x\dot{y}} - \Omega^0_{xy}\right] = 0$$

$$A\left[2n\lambda - \lambda\Omega^0_{y\dot{x}} - \Omega^0_{yx}\right] + B\left[\lambda^2 - \lambda\Omega^0_{y\dot{y}} - \lambda\Omega^0_{yy}\right] = 0 \qquad (5.5)$$

Eliminating A and B from above, we have that the determinant equation (5.5) is:

$$\begin{vmatrix} \lambda^2 - \lambda\Omega^0{}_{x\dot{x}} - \Omega^0{}_{xx} & -2n\lambda - \lambda\Omega^0{}_{x\dot{y}} - \Omega^0{}_{xy} \\ 2n\lambda - \lambda\Omega^0{}_{y\dot{x}} - \Omega^0{}_{yx} & \lambda^2 - \lambda\Omega^0{}_{y\dot{y}} - \Omega^0{}_{yy} \end{vmatrix} = 0$$

Expanding, we have

$$\left(\lambda^2 - \lambda\Omega^0_{x\dot{x}} - \Omega^0_{xx}\right)\left(\lambda^2 - \lambda\Omega^0_{y\dot{y}} - \Omega^0_{yy}\right) - \left\{\left(-2n\lambda - \lambda\Omega^0_{x\dot{y}} - \Omega^0_{xy}\right)\left(2n\lambda - \lambda\Omega^0_{y\dot{x}} - \Omega^0_{yx}\right)\right\} = 0$$

or

$$\lambda^4 - \lambda^3\Omega^0_{y\dot{y}} - \lambda^2\Omega^0_{yy} - \lambda^3\Omega^0_{x\dot{x}} + \lambda^2\Omega^0_{x\dot{x}}\Omega^0_{y\dot{y}} + \lambda\Omega^0_{x\dot{x}}\Omega^0_{yy} - \lambda^2\Omega^0_{xx} + \lambda\Omega^0_{xx}\Omega^0_{y\dot{y}} + \Omega^0_{xx}\Omega^0_{yy}$$
$$4n^2\lambda^2 + 2n\lambda^2\Omega^0_{x\dot{y}} + 2n\lambda\Omega^0_{xy} - 2n\lambda^2\Omega^0_{y\dot{x}} - \lambda^2\Omega^0_{y\dot{x}} - \lambda\Omega^0_{y\dot{x}}\Omega^0_{xy} - 2n\lambda\Omega^0_{yx} - \lambda\Omega^0_{yx}\Omega^0_{x\dot{y}} - \Omega^0_{yx}\Omega^0_{xy} = 0$$

or

$$\lambda^4 - \lambda^3\Omega^0_{y\dot{y}} - \lambda^2\Omega^0_{yy} - \lambda^3\Omega^0_{x\dot{x}} + \lambda^2\Omega^0_{x\dot{x}}\Omega^0_{y\dot{y}} + \lambda\Omega^0_{x\dot{x}}\Omega^0_{yy} - \lambda^2\Omega^0_{xx} + \lambda\Omega^0_{xx}\Omega^0_{y\dot{y}} + \Omega^0_{xx}\Omega^0_{yy}$$
$$4n^2\lambda^2 + 2n\lambda\Omega^0_{xy} - \lambda^2\Omega^0_{y\dot{x}} - \lambda\Omega^0_{y\dot{x}}\Omega^0_{xy} - 2n\lambda\Omega^0_{yx} - \lambda\Omega^0_{yx}\Omega^0_{x\dot{y}} - \Omega^0_{yx}\Omega^0_{xy} = 0$$

or

$$\lambda^4 + \left[-\Omega^0_{y\dot{y}} - \Omega^0_{x\dot{x}}\right]\lambda^3 + \left[-\Omega^0_{yy} + \Omega^0_{x\dot{x}}\Omega^0_{y\dot{y}} - \Omega^0_{xx} + 4n^2 - \left(\Omega^0_{x\dot{y}}\right)^2\right]\lambda^2 + \left[\Omega^0_{x\dot{x}}\Omega^0_{yy} + \Omega^0_{xx}\Omega^0_{y\dot{y}} + 2n\Omega^0_{xy}\right.$$

$$\left. - \Omega^0_{y\dot{x}}\Omega^0_{xy} - 2n\Omega^0_{yx} - \Omega^0_{yx}\Omega^0_{x\dot{y}}\right]\lambda + \Omega^0_{xx}\Omega^0_{yy} - \Omega^0_{yx}\Omega^0_{xy} = 0$$

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0 \qquad (5.6)$$

where

$$a = -\left(\Omega^0{}_{y\dot{y}} + \Omega^0{}_{x\dot{x}}\right)$$

$$b = 4n^2 + \Omega^0{}_{x\dot{x}}\Omega^0{}_{y\dot{y}} - \Omega^0{}_{yy} - \left(\Omega^0{}_{x\dot{y}}\right)^2$$

$$c = \Omega^0{}_{x\dot{x}}\Omega^0{}_{yy} + \Omega^0{}_{xx}\Omega^0{}_{y\dot{y}} + 2n\Omega^0{}_{xy} - 2n\Omega^0{}_{yx} - \Omega^0{}_{y\dot{x}}\Omega^0{}_{xy} - \Omega^0{}_{yx}\Omega^0{}_{x\dot{y}}$$

$$d = \Omega^0{}_{xx}\Omega^0{}_{yy} - \Omega^0{}_{yx}\Omega^0{}_{xy}$$

Equation (5.6) is the characteristic equation corresponding to the variational equation (5.5). Here, the superscript indicates that the partial derivatives have been computed at the equilibrium points.

5.2 STABILITY OF COLLINEAR POINTS

To study the stability of the collinear equilibrium points, we note that, because these points are not affected by the P-R drag effect, the variational equations and the characteristic equation here will be different. However, they can be obtained analogously from those presented in equations (5.5) and (5.6) respectively by substituting $\dot{x} = \dot{y} = 0$ in them.

Therefore, the variational equations corresponding to the collinear points, when motion is considered on the x -plane and when P-R drag are not effective, gives

$$\ddot{\xi} - 2n\dot{\eta} = \xi(\Omega_{xx})^0 + \eta(\Omega_{xy})^0$$
$$\ddot{\eta} + 2n\dot{\xi} = \xi(\Omega_{xy})^0 + \eta(\Omega_{yy})^0 \tag{5.7}$$

Following same routine as done above, we get the characteristic equation:

$$\lambda^4 - (\Omega^0{}_{xx} + \Omega^0{}_{yy} - 4n^2)\lambda^2 + \Omega^0{}_{xx}\Omega^0{}_{yy} - (\Omega^0{}_{xy})^2 = 0, \tag{5.8}$$

where the superscripts indicate that, the partial derivatives have been calculated at the collinear equilibrium points.

Clearly, from equations (3.8), when $W_2 = 0, y \neq 0$, we get the equations

$$\Omega_x = n^2 x - \frac{(1-\mu)(x+\mu)}{r_1^3} - \frac{3(1-\mu)A(x+\mu)}{2r_1^5} - \frac{\mu q(x+\mu-1)}{r_2^3}, \tag{5.9}$$

$$\Omega_y = n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{3(1-\mu)Ay}{2r_1^5} - \frac{\mu q y}{r_2^3}, \tag{5.10}$$

where $r_1^2 = (x+\mu)^2 + y^2$, $\qquad r_2^2 = (x+\mu-1)^2 + y^2$.

Therefore, from equations (5.9) and (5.10), we get

$$\Omega_{xx} = n^2 - \frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)(x+\mu)^2}{r_1^5} - \frac{3(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)A(x+\mu)^2}{2r_1^7} - \frac{\mu q}{r_2^3} + \frac{3\mu q(x+\mu-1)^2}{r_2^5}, \tag{5.11}$$

$$\Omega_{yy} = n^2 - \frac{(1-\mu)}{r_1^3} + \frac{3(1-\mu)y^2}{r_1^5} - \frac{3(1-\mu)A}{2r_1^5} + \frac{15(1-\mu)Ay^2}{2r_1^7} - \frac{\mu q}{r_2^3} + \frac{3\mu q y^2}{r_2^5}, \tag{5.12}$$

$$\Omega_{xy} = \Omega_{yx} = \frac{3(1-\mu)(x+\mu)y}{r_1^5} + \frac{15(1-\mu)A(x+\mu)A}{2r_1^7} + \frac{3\mu q(x+\mu-1)^2}{r_2^5}, \tag{5.13}$$

Now, substituting $x_0 = 1-\mu-\varepsilon_1$ in equation (5.11) to get

$$\Omega_{xx}^0 = n^2 - \frac{(1-\mu)}{|x_0+\mu|^3} + \frac{3(1-\mu)(x_0+\mu)^2}{|x_0+\mu|^5} - \frac{3(1-\mu)A}{2|x_0+\mu|^5} + \frac{15(1-\mu)A(x_0+\mu)^2}{2|x_0+\mu|^7}$$
$$- \frac{\mu q}{|x_0+\mu-1|^3} + \frac{3\mu q(x_0+\mu-1)^2}{|x_0+\mu-1|^5},$$

or

$$\Omega_{xx}^0 = n^2 - \frac{(1-\mu)}{|1-\mu-\varepsilon_1+\mu|^3} + \frac{3(1-\mu)}{|1-\mu-\varepsilon_1+\mu|^3} - \frac{3(1-\mu)A}{2|1-\mu-\varepsilon_1+\mu|^5} + \frac{15(1-\mu)A}{2|1-\mu-\varepsilon_1+\mu|^5}$$
$$- \frac{\mu q}{|1-\mu-\varepsilon_1+\mu-1|^3} + \frac{3\mu q}{|1-\mu-\varepsilon_1+\mu-1|^3},$$

or

$$\Omega_{xx}^0 = n^2 - \frac{(1-\mu)}{|1-\varepsilon_1|^3} + \frac{3(1-\mu)}{|1-\varepsilon_1|^3} - \frac{3(1-\mu)A}{2|1-\varepsilon_1|^5} + \frac{15(1-\mu)A}{2|1-\varepsilon_1|^5} - \frac{\mu q}{|-\varepsilon_1|^3} + \frac{3\mu q}{|-\varepsilon_1|^3},$$

or

$$\Omega_{xx}^0 = n^2 - \frac{(1-\mu)}{(1-\varepsilon_1)^3} + \frac{3(1-\mu)}{(1-\varepsilon_1)^3} - \frac{3(1-\mu)A}{2(1-\varepsilon_1)^5} + \frac{15(1-\mu)A}{2(1-\varepsilon_1)^5} - \frac{\mu q}{(-\varepsilon_1)^3} + \frac{3\mu q}{(-\varepsilon_1)^3},$$

or $$\Omega_{xx}^0 = n^2 + \frac{2(1-\mu)}{(1-\varepsilon_1)^3} + \frac{6(1-\mu)A}{(1-\varepsilon_1)^5} + \frac{2\mu q}{(-\varepsilon_1)^3},$$

or $$\Omega_{xx}^0 = 1 + \frac{3A}{2} + 2\left[\frac{(1-\mu)}{(1-\varepsilon_1)^3} + \frac{3(1-\mu)A}{(1-\varepsilon_1)^5} + \frac{\mu q}{(-\varepsilon_1)^3}\right],$$

or $$\Omega_{xx}^0 = 1 + 2\left[\frac{3A}{4} + \frac{(1-\mu)}{(1-\varepsilon_1)^3} + \frac{3(1-\mu)A}{(1-\varepsilon_1)^5} + \frac{\mu q}{(-\varepsilon_1)^3}\right].$$

Hence,

$$\Omega_{xx}^{0} = 1 + 2f \,, \tag{5.14}$$

where

$$f = \frac{3A}{4} + \frac{(1-\mu)}{(1-\varepsilon_1)^3} + \frac{3(1-\mu)A}{(1-\varepsilon_1)^5} + \frac{\mu q}{(-\varepsilon_1)^3}.$$

Similarly, from equation (5.13), we get

$$\Omega_{yy}^{0} = n^2 - \frac{(1-\mu)}{r_1^3} - \frac{3(1-\mu)A}{2r_1^5} - \frac{\mu q}{r_2^3},$$

$$\Omega_{yy}^{0} = n^2 - \frac{(1-\mu)}{|x_0+\mu|^3} - \frac{3(1-\mu)A}{2|x_0+\mu|^5} - \frac{\mu q}{|x_0+\mu-1|^3},$$

$$\Omega_{yy}^{0} = n^2 - \frac{(1-\mu)}{|1-\mu-\varepsilon_1+\mu|^3} - \frac{3(1-\mu)A}{2|1-\mu-\varepsilon_1+\mu|^5} - \frac{\mu q}{|1-\mu-\varepsilon_1+\mu-1|^3},$$

$$\Omega_{yy}^{0} = n^2 - \frac{(1-\mu)}{|1-\varepsilon_1|^3} - \frac{3(1-\mu)A}{2|1-\varepsilon_1|^5} - \frac{\mu q}{|-\varepsilon_1|^3},$$

$$\Omega_{yy}^{0} = n^2 - \frac{(1-\mu)}{(1-\varepsilon_1)^3} - \frac{3(1-\mu)A}{2(1-\varepsilon_1)^5} + \frac{\mu q}{(\varepsilon_1)^3},$$

$$\Omega_{yy}^{0} = 1 + \frac{3A}{2} - \left[\frac{(1-\mu)}{(1-\varepsilon_1)^3} + \frac{3(1-\mu)A}{2(1-\varepsilon_1)^5} - \frac{\mu q}{(\varepsilon_1)^3}\right],$$

$$\Omega_{yy}^{0} = 1 - \frac{3}{2}\left[-A + \frac{2(1-\mu)}{3(1-\varepsilon_1)^3} + \frac{(1-\mu)A}{(1-\varepsilon_1)^5} - \frac{2\mu q}{3(\varepsilon_1)^3}\right].$$

Therefore,

$$\Omega_{yy}^{0} = 1 - f \,, \tag{5.15}$$

where $f = \dfrac{3}{2}\left[-A + \dfrac{2(1-\mu)}{3(1-\varepsilon_1)^3} + \dfrac{(1-\mu)A}{(1-\varepsilon_1)^5} - \dfrac{2\mu q}{3(\varepsilon_1)^3} \right]$

$$f = \frac{3}{2}\left[-A + \frac{2(1-\mu)}{3(1-\varepsilon_1)^3} + \frac{(1-\mu)A}{(1-\varepsilon_1)^5} - \frac{2\mu q}{3(\varepsilon_1)^3} \right]$$

And lastly, from equation (5.13), we get

$$\Omega_{xy}^{0} = \Omega_{yx}^{0} = 0. \tag{5.16}$$

Now, since $\varepsilon_i > 0\,(i = 1,2,3)$, $(1-q) << 1$, $0 < \mu \le \dfrac{1}{2}$, therefore, for $A << 1$, we have that $\Omega_{xx}^{0} > 0$, while Ω_{yy}^{0} is negative.

Now, substituting equations (5.14), (5.15) and (5.16) in the characteristic equation (5.8), we get

$$\lambda^4 - \left[1 + 2f + 1 - f - 4\left(1 + \frac{3A}{2} \right) \right]\lambda^2 + (1+2f)(1-f) - 0 = 0, \tag{5.17}$$

$$\lambda^4 - \left[2 + f - 4 - 6A \right]\lambda^2 + 1 + 2f - f - 2f^2 = 0,$$

$$\lambda^4 - \left[-2 + f - 6A \right]\lambda^2 + 1 + f - 2f^2 = 0,$$

$$\lambda^4 + \left[2 - f + 6A \right]\lambda^2 + 1 + f - 2f^2 = 0.$$

So that

$$\lambda^4 + p\lambda^2 + q = 0,$$

where

$$p = [2 - f + 6A]$$

$$q = 1 + f - 2f^2.$$

The roots of the characteristic equation (5.8) are given by:

$$\lambda_{1,2}^2 = -\frac{p \pm \sqrt{p^2 - 4q}}{2},$$

where $\Delta = p^2 - 4q$ is the discriminant.

Now, since $\Omega_{xx}^0 > 0, \Omega_{yy}^0 < 0$ therefore $\Omega_{xx}^0 \Omega_{yy}^0 < 0.$

This implies that $q = \Omega_{xx}^0 \Omega_{yy}^0 < 0.$ If this happen, the discriminant $\Delta = p^2 - 4q$ will be positive since $\lambda_1 = s_1$, $\lambda_2 = -s_1$, $\lambda_2 = s_2$, and $\lambda_4 = -s_2$ where $s_i\,(i = 1,2)$ are real. So the motion around the collinear points is unstable.

The same stability analysis can be done for the collinear equilibrium points L_2 and L_3. Hence, these points remain unstable as those of the classical restricted three-body problem.

5.3 STABILITY OF TRIANGULAR EQUILIBRIUM POINTS

The characteristic in the case of the triangular points is given by:

$$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0$$

Evaluating the second partial derivatives at the equilibrium points, we obtain

$$\Omega^0_{\dot{x}\dot{x}} = -(x_0 + \mu - 1)^2 W_2 - W_2$$

$$\Omega^0_{\dot{x}\dot{y}} = -(x_0 + \mu - 1) y_0 W_2 = \Omega^0_{\dot{y}\dot{x}} \text{ Say}$$

$$\Omega^0_{\dot{y}\dot{y}} = -(y_0^2 + 1) W_2$$

$$\begin{aligned}\Omega^0_{xx} &= 3x_0^2 - 3\mu^2 + 3\mu + \left(2\mu - 4\mu^2 + 2\mu^3 - 4\mu x_0 + 2\mu x_0^2 + 4\mu^2 x_0\right)\delta \\ &+ \frac{A}{2}(3\mu + 15\mu^2 - 15\mu^3 + 15x_0^2 + 30\mu x_0 - 15\mu x_0^2 - 30\mu^2 x_0) \\ &+ \left(3 - 3\mu - 15\mu^2 + 15\mu^3 + 6x_0 - 15x_0^2 - 30\mu x_0 + 15\mu x_0^2 + 30\mu^2 x_0\right)\varepsilon_1 \\ &+ \left(-12\mu + 30\mu^2 - 15\mu^3 - 6x_0 - 15\mu x_0^2 + 30\mu x_0 - 30\mu^2 x_0\right)\varepsilon_2 - \left(2x_0 y_0 + 2\mu y_0 - 2y_0\right)W_2\end{aligned}$$

$$\begin{aligned}\Omega^0_{xy} &= 3x_0 y_0 + \left(2\mu x_0 y_0 + 2\mu^2 y_0 - 2\mu y_0\right)\delta + \frac{15A}{2}\left(x_0 y_0 + \mu y_0 - \mu x_0 y_0 - \mu^2 y_0\right) \\ &+ \left(3y_0 + \frac{3x_0}{2y_0} - 15x_0 y_0 - 15\mu y_0 + 15\mu x_0 y_0 + 15\mu^2 y_0\right)\varepsilon_1 + \left(-3y_0 + \frac{3x_0}{2y_0} + 15\mu y_0\right. \\ &\left. - 15\mu x_0 y_0 - 15\mu^2 y_0\right)\varepsilon_2 + \left(1 - 2y_0^2\right)W_2\end{aligned}$$

$$\begin{aligned}\Omega^0_{yx} &= 3x_0 y_0 + \left(2\mu x_0 y_0 + 2\mu^2 y_0 - 2\mu y_0\right)\delta + \frac{15A}{2}\left(x_0 y_0 + \mu y_0 - \mu x_0 y_0 - \mu^2 y_0\right) \\ &+ \left(3y_0 + \frac{3x_0}{2y_0} - 15x_0 y_0 - 15\mu y_0 + 15\mu x_0 y_0 + 15\mu^2 y_0\right)\varepsilon_1 + \left(-3y_0 + \frac{3x_0}{2y_0} + 15\mu y_0\right. \\ &\left. - 15\mu x_0 y_0 - 15\mu^2 y_0\right)\varepsilon_2 + 2\left(x_0 + \mu - 1\right)^2 W_2 - W_2\end{aligned}$$

$$\begin{aligned}\Omega^0_{yy} &= 3y_0^2 + 2\mu y_0^2\delta + \frac{A}{2}\left(3\mu + 15y_0^2 - 15\mu y_0^2\right) + \left(6 - 3\mu - 15y_0^2 + 15\mu y_0^2\right)\varepsilon_1 + \left(3 + 3\mu - 15\mu y_0^2\right)\varepsilon_2 \\ &+ \left(2x_0 y_0 + 2\mu y_0 - 2y_0\right)W_2\end{aligned}$$

Substituting for $\Omega^0_{\dot{x}\dot{x}}, \Omega^0_{\dot{x}\dot{y}}, \Omega^0_{\dot{y}\dot{y}}, \Omega^0_{xx}, \Omega^0_{xy}, \Omega^0_{yy}$, and Ω^0_{yx}, in the characteristic equation (5.7), we get

$a = 3W_2$

$b = 1 - \frac{3A}{2}(1-2\mu) + \frac{W_2}{\sqrt{3}}$

$c = \frac{W_2}{4}[-21 + 9\mu]$

$d = \frac{27}{4}\mu(1-\mu) + \frac{3}{2}\mu(1-\mu)\delta + \frac{117}{4}\mu(1-\mu)A + \frac{W_2}{4\sqrt{3}(1-\mu)}\left[54 - 135\mu^2 + 81\mu^3\right]$

The six roots of equation (5.6) are:

$$\lambda_1 \to -\frac{a}{4} - \frac{1}{2}\sqrt{L} - \frac{1}{2}\left[\frac{a^2}{2} - \frac{4b}{3} - \frac{G}{3}\left(\frac{2}{H}\right)^{1/3} - \left(\frac{H}{32}\right)^{1/3} - \frac{a^3 + 4ab - 8c}{4\sqrt{L}}\right]^{1/2}$$

$$\lambda_2 \to -\frac{a}{4} - \frac{1}{2}\sqrt{L} + \frac{1}{2}\left[\frac{a^2}{2} - \frac{4b}{3} - \frac{G}{3}\left(\frac{2}{H}\right)^{1/3} - \left(\frac{H}{32}\right)^{1/3} - \frac{a^3 + 4ab - 8c}{4\sqrt{L}}\right]^{1/2} \qquad (5.18)$$

$$\lambda_3 \to -\frac{a}{4} + \frac{1}{2}\sqrt{L} - \frac{1}{2}\left[\frac{a^2}{2} - \frac{4b}{3} - \frac{G}{3}\left(\frac{2}{H}\right)^{1/3} - \left(\frac{H}{32}\right)^{1/3} - \frac{a^3 + 4ab - 8c}{4\sqrt{L}}\right]^{1/2}$$

$$\lambda_4 \to -\frac{a}{4} + \frac{1}{2}\sqrt{L} + \frac{1}{2}\left[\frac{a^2}{2} - \frac{4b}{3} - \frac{G}{3}\left(\frac{2}{H}\right)^{1/3} - \left(\frac{H}{32}\right)^{1/3} - \frac{a^3 + 4ab - 8c}{4\sqrt{L}}\right]^{1/2}$$

where,

$L = \frac{a^2}{4} - \frac{2b}{3} + \frac{G}{3}\left(\frac{2}{H}\right)^{1/3} + \left(\frac{H}{32}\right)^{1/3}$

$F = 2b^3 - 9abc + 27\left(c^2 + a^2 d\right) - 72bd$,

$G = b^2 - 3ac + 12d$

$H = F + \sqrt{-4G^3 + F^2}$

We now compute the values of these roots numerically, in table 3 and 4, for motion of a test particle in the neighborhood of a binary star in table 3 and 4 below. In particular, the system $RXJ\ 0450.1-5856$, which has a mass of, $M_1 = 1.4M_{sun}$ and red dwarf companion with a mass we shall assume to be $M_2 = 0.15M_{sun}$ fits the model of the problem under consideration (Singh and Umar 2012). Consequently, we get $\mu = 0.0967,$ and take $q = 0.9965$ while oblateness of the bigger primary is varied.

A	w_1	w_2
0	1.19018	$1.16940\times10^{-12}+0.64538i$
0.0001	1.19020	$1.16961\times10^{-12}+0.64552\ i$
0.0005	1.19027	$1.16346\times10^{-12}+0.646033i$
0.0010	1.19037	$1.16201\times10^{-12}+0.64668\ i$
0.0050	1.19114	$1.15054\times10^{-12}+0.65181\ i$
0.0100	1.19209	$1.13639\times10^{-12}+0.65816i$
0.0150	1.19303	$1.12243\times10^{-12}+0.66443i$
0.0200	1.19396	$1.08680\times10^{-12}+0.67062i$
0.0300	1.19580	$1.08175\times10^{-12}+0.68281i$
0.0500	1.19937	$1.03006\times10^{-12}+0.70638i$

5. 1: Stability for $RXJ\ 0450.1-5856$ with varying oblateness and P-R drag for $\mu = 0.0967,\ \delta = 0.0035$ and $c_d = 299792458$.

A	w_1	w_2
0	1.19000	$0.64521i$
0.0001	1.19010	$0.64534i$
0.0005	1.19018	$0.64586i$
0.0010	1.19028	$0.64651i$
0.0050	1.19104	$0.65164i$
0.0100	1.19199	$0.65799i$
0.0150	1.19294	$0.66426i$
0.0200	1.19387	$0.67046i$
0.0300	1.19570	$0.68265i$
0.0500	1.19928	$0.70623i$

5. 2:Stability for $RXJ\ 0450.1-5856$ with varying oblateness and without P-R drag for $\mu = 0.0967,\ \delta = 0.0035$ and $c_d = 299792458$.

The stability of the triangular points $L_{4,5}$ under the joint effects of oblateness of the bigger primary, radiation force of the smaller and the force due to the P-R effect are determined by the roots of the characteristic equation (5.6). Our numerical exploration in the computations of these roots reveals the existence of at least a positive root or/and a positive real part of the complex roots. Hence, we conclude that the triangular equilibrium points are unstable due to a positive root of equation (5.6).

5.4 STABILITY OF THREE-DIMENSIONAL OUT-OF-PLANE EQUILIBRIUM POINTS

We now examine the stability of the out-of-plane equilibrium points in an analogous way as that done for the triangular points. However, here, we let the location of the out of plane points be denoted by (a_0, b_0, c_0) and (ξ, η, ζ) the small displacement. Then, the body will be displaced to the points $x = a_0 + \xi,\ y = b_0 + \eta$ and $z = c_0 + \zeta$. Substituting these in equations (3.11), we obtain the variational equations:

$$\begin{aligned}
\ddot{\xi} - 2n\dot{\eta} &= (\Omega_{xx})^0 \xi + (\Omega_{xy})^0 \eta + (\Omega_{xz})^0 \zeta + (\Omega_{x\dot{x}})^0 \dot{\xi} + (\Omega_{x\dot{y}})^0 \dot{\eta} + (\Omega_{x\dot{z}})^0 \dot{\zeta} \\
\ddot{\eta} + 2n\dot{\xi} &= (\Omega_{yx})^0 \xi + (\Omega_{yy})^0 \eta + (\Omega_{yz})^0 \zeta + (\Omega_{y\dot{x}})^0 \dot{\xi} + (\Omega_{y\dot{y}})^0 \dot{\eta} + (\Omega_{y\dot{z}})^0 \dot{\zeta} \\
\ddot{\zeta} &= (\Omega_{zx})^0 \xi + (\Omega_{zy})^0 \eta + (\Omega_{zz})^0 \zeta + (\Omega_{z\dot{x}})^0 \dot{\xi} + (\Omega_{z\dot{y}})^0 \dot{\eta} + (\Omega_{z\dot{y}})^0 \dot{\zeta}
\end{aligned} \qquad (5.19)$$

Here, only linear terms in ξ, η and ζ have been taken. The second partial derivatives of Ω are denoted by subscripts. The superscript 0, indicates that the derivatives have been computed at the equilibrium points (a_0, b_0, c_0).

Substituting the trial solution in the variational equation, we get the determinant equation:

$$\begin{vmatrix} \lambda^2 - \lambda\Omega^0{}_{x\dot{x}} - \Omega^0{}_{xx} & -2n\lambda - \lambda\Omega^0{}_{x\dot{y}} - \Omega^0{}_{xy} & -\lambda\Omega^0_{x\dot{z}} - \Omega^0_{xz} \\ 2n\lambda - \lambda\Omega^0{}_{y\dot{x}} - \Omega^0{}_{yx} & \lambda^2 - \lambda\Omega^0{}_{y\dot{y}} - \Omega^0{}_{yy} & -\lambda\Omega^0_{y\dot{z}} - \Omega^0_{yz} \\ -\lambda\Omega^0{}_{z\dot{x}} - \Omega^0{}_{zx} & -\lambda\Omega^0_{z\dot{y}} - \Omega^0_{zy} & \lambda^2 - \lambda\Omega^0_{z\dot{z}} - \Omega^0_{zz} \end{vmatrix} = 0.$$

Expanding the determinant, we have

$$\begin{aligned} &\lambda^2 - \lambda\Omega^0{}_{x\dot{x}} - \Omega^0{}_{xx}\Big[\lambda^4 - \lambda^2\Omega^0{}_{zz} - \lambda^3\Omega^0{}_{z\dot{z}} - \lambda^2\Omega^0{}_{yy} + \Omega^0{}_{yy}\Omega^0{}_{zz} + \lambda\Omega^0{}_{yy}\Omega^0{}_{z\dot{z}} - \lambda^3\Omega^0{}_{y\dot{y}} \\ &+ \lambda\Omega^0{}_{zz}\Omega^0{}_{y\dot{y}} + \lambda^2\Omega^0{}_{y\dot{y}}\Omega^0{}_{z\dot{z}} - \Omega^0{}_{yz}\Omega^0{}_{zy} - \lambda\Omega^0{}_{yz}\Omega^0{}_{z\dot{y}} - \lambda\Omega^0{}_{zy}\Omega^0{}_{y\dot{z}} - \lambda^2\Omega^0{}_{yz}\Omega^0{}_{z\dot{y}}\Big] \\ &+ 2n\lambda + \lambda\Omega^0{}_{x\dot{y}} + \Omega^0{}_{xy}\Big[2n\lambda^3 - 2n\lambda\Omega^0{}_{zz} - 2n\lambda^2\Omega^0{}_{z\dot{z}} - \lambda^2\Omega^0{}_{yx} + \Omega^0{}_{yx}\Omega^0{}_{zz} + \lambda\Omega^0{}_{yx}\Omega^0{}_{z\dot{z}} \\ &- \lambda^3\Omega^0{}_{y\dot{x}} + \lambda\Omega^0{}_{y\dot{x}}\Omega^0{}_{zz} + \lambda^2\Omega^0{}_{y\dot{x}}\Omega^0{}_{z\dot{z}} - \Omega^0{}_{yz}\Omega^0{}_{zx} - \lambda\Omega^0{}_{yz}\Omega^0{}_{z\dot{x}} + \lambda\Omega^0{}_{y\dot{z}}\Omega^0{}_{zx} - \lambda^2\Omega^0{}_{y\dot{z}}\Omega^0{}_{z\dot{x}}\Big] \\ &- \lambda\Omega^0_{x\dot{z}} - \Omega^0_{xz}\Big[-2n\lambda\Omega^0{}_{yz} - 2n\lambda^2\Omega^0{}_{z\dot{y}} + \Omega^0{}_{yx}\Omega^0{}_{zy} + \lambda\Omega^0{}_{yx}\Omega^0{}_{z\dot{y}} + \lambda\Omega^0{}_{y\dot{x}}\Omega^0{}_{zy} + \lambda^2\Omega^0{}_{y\dot{x}}\Omega^0{}_{z\dot{y}} \\ &+ \lambda^2\Omega^0{}_{zx} + \lambda^3\Omega^0{}_{z\dot{x}} - \Omega^0{}_{yy}\Omega^0{}_{zx} - \lambda\Omega^0{}_{yy}\Omega^0{}_{z\dot{x}} - \lambda\Omega^0{}_{y\dot{y}}\Omega^0{}_{z\dot{x}} - \lambda^2\Omega^0{}_{y\dot{y}}\Omega^0{}_{z\dot{x}}\Big] = 0 \end{aligned}$$

or

$$
\begin{aligned}
&\lambda^6 - \lambda^4\Omega^0_{zz} - \lambda^5\Omega^0_{z\dot z} - \lambda^4\Omega^0_{yy} + \lambda^2\Omega^0_{yy}\Omega^0_{zz} + \lambda^3\Omega^0_{yy}\Omega^0_{z\dot z} + \lambda^5\Omega^0_{y\dot y} + \lambda^3\Omega^0_{zz}\Omega^0_{y\dot y} + \lambda^4\Omega^0_{y\dot y}\Omega^0_{z\dot z} \\
&- \lambda^2\Omega^0_{yz}\Omega^0_{zy} - \lambda^3\Omega^0_{yz}\Omega^0_{z\dot y} - \lambda^3\Omega^0_{zy}\Omega^0_{y\dot z} - \lambda^4\Omega^0_{y\dot z}\Omega^0_{z\dot y} - \lambda^4\Omega^0_{xx} + \lambda^2\Omega^0_{xx}\Omega^0_{zz} + \lambda^3\Omega^0_{xx}\Omega^0_{z\dot z} + \lambda^2\Omega^0_{xx}\Omega^0_{yy} \\
&- \Omega^0_{xx}\Omega^0_{yy}\Omega^0_{zz} - \lambda\Omega^0_{xx}\Omega^0_{yy}\Omega^0_{z\dot z} + \lambda^3\Omega^0_{xx}\Omega^0_{y\dot y} - \lambda\Omega^0_{xx}\Omega^0_{zz}\Omega^0_{y\dot y} - \lambda\Omega^0_{xx}\Omega^0_{y\dot y}\Omega^0_{z\dot z} + \Omega^0_{xx}\Omega^0_{yz}\Omega^0_{zy} \\
&+ \lambda\Omega^0_{xx}\Omega^0_{yz}\Omega^0_{z\dot y} + \lambda\Omega^0_{xx}\Omega^0_{zy}\Omega^0_{y\dot z} + \lambda^2\Omega^0_{xx}\Omega^0_{y\dot z}\Omega^0_{z\dot y} - \lambda^5\Omega^0_{x\dot x} + \lambda^3\Omega^0_{x\dot x}\Omega^0_{zz} + \lambda^4\Omega^0_{x\dot x}\Omega^0_{z\dot z} + \lambda^3\Omega^0_{x\dot x}\Omega^0_{yy} \\
&- \lambda\Omega^0_{x\dot x}\Omega^0_{yy}\Omega^0_{zz} - \lambda^2\Omega^0_{x\dot x}\Omega^0_{yy}\Omega^0_{z\dot z} + \lambda^4\Omega^0_{x\dot x}\Omega^0_{y\dot y} - \lambda^2\Omega^0_{x\dot x}\Omega^0_{zz}\Omega^0_{y\dot y} - \lambda^3\Omega^0_{x\dot x}\Omega^0_{y\dot y}\Omega^0_{z\dot z} + \lambda\Omega^0_{x\dot x}\Omega^0_{yz}\Omega^0_{zy} \\
&+ \lambda^2\Omega^0_{x\dot x}\Omega^0_{yz}\Omega^0_{z\dot y} + \lambda^2\Omega^0_{x\dot x}\Omega^0_{zy}\Omega^0_{yz} + \lambda^3\Omega^0_{x\dot x}\Omega^0_{y\dot z}\Omega^0_{z\dot y} + 4n^2\lambda^4 - 4n^2\lambda^2\Omega^0_{zz} - 4n^2\lambda^3\Omega^0_{z\dot z} - 2n\lambda^3\Omega^0_{yx} \\
&+ 2n\lambda\Omega^0_{yx}\Omega^0_{zz} + 2n\lambda^2\Omega^0_{yx}\Omega^0_{z\dot z} - 2n^2\lambda^4\Omega^0_{y\dot x} + 2n\lambda^2\Omega^0_{y\dot x}\Omega^0_{zz} + 2n\lambda^4\Omega^0_{y\dot x}\Omega^0_{z\dot z} - 2n\lambda\Omega^0_{yz}\Omega^0_{zx} \\
&- 2n\lambda^2\Omega^0_{yz}\Omega^0_{z\dot x} - 2n\lambda^2\Omega^0_{y\dot z}\Omega^0_{zx} - 2n\lambda^3\Omega^0_{y\dot z}\Omega^0_{z\dot x} + 2n\lambda^3\Omega^0_{xy} - 2n\lambda\Omega^0_{zz}\Omega^0_{xy} - 2n\lambda^2\Omega^0_{z\dot z}\Omega^0_{xy} \\
&- \lambda^2\Omega^0_{yx}\Omega^0_{xy} + \Omega^0_{yx}\Omega^0_{zz}\Omega^0_{xy} + \lambda\Omega^0_{yx}\Omega^0_{z\dot z}\Omega^0_{xy} - \lambda^3\Omega^0_{y\dot x}\Omega^0_{xy} + \lambda\Omega^0_{y\dot x}\Omega^0_{zz}\Omega^0_{xy} + \lambda^2\Omega^0_{y\dot x}\Omega^0_{z\dot z}\Omega^0_{xy} \\
&- \Omega^0_{yz}\Omega^0_{zx}\Omega^0_{xy} - \lambda\Omega^0_{yz}\Omega^0_{z\dot x}\Omega^0_{xy} - \lambda\Omega^0_{y\dot z}\Omega^0_{zx}\Omega^0_{xy} - \lambda^2\Omega^0_{y\dot z}\Omega^0_{z\dot x}\Omega^0_{xy} + 2n\lambda^4\Omega^0_{x\dot y} - 2n\lambda^2\Omega^0_{zz}\Omega^0_{x\dot y} \\
&- 2n\lambda^3\Omega^0_{z\dot z}\Omega^0_{x\dot y} - \lambda^3\Omega^0_{yx}\Omega^0_{x\dot y} + \lambda\Omega^0_{yx}\Omega^0_{zz}\Omega^0_{x\dot y} + \lambda^2\Omega^0_{yx}\Omega^0_{z\dot z}\Omega^0_{x\dot y} - \lambda^4\Omega^0_{y\dot x}\Omega^0_{x\dot y} + \lambda^2\Omega^0_{y\dot x}\Omega^0_{x\dot y}\Omega^0_{zz} \\
&+ \lambda^3\Omega^0_{y\dot x}\Omega^0_{z\dot z}\Omega^0_{x\dot y} - \lambda\Omega^0_{yz}\Omega^0_{zx}\Omega^0_{x\dot y} - \lambda^2\Omega^0_{yz}\Omega^0_{z\dot x}\Omega^0_{x\dot y} - \lambda^2\Omega^0_{y\dot z}\Omega^0_{zx}\Omega^0_{x\dot y} - \lambda^3\Omega^0_{y\dot z}\Omega^0_{z\dot x}\Omega^0_{x\dot y} + 2n\lambda\Omega^0_{xz}\Omega^0_{zy} \\
&+ 2n\lambda^2\Omega^0_{xz}\Omega^0_{z\dot y} - \Omega^0_{xz}\Omega^0_{yx}\Omega^0_{zy} - \lambda\Omega^0_{xz}\Omega^0_{yx}\Omega^0_{z\dot y} - \lambda\Omega^0_{xz}\Omega^0_{y\dot x}\Omega^0_{zy} - \lambda^2\Omega^0_{xz}\Omega^0_{y\dot x}\Omega^0_{z\dot y} \\
&- \lambda^2\Omega^0_{xz}\Omega^0_{zx} - \lambda^3\Omega^0_{xz}\Omega^0_{z\dot x} + \Omega^0_{xz}\Omega^0_{yy}\Omega^0_{zx} + \lambda\Omega^0_{xz}\Omega^0_{yy}\Omega^0_{z\dot x} + \lambda\Omega^0_{xz}\Omega^0_{y\dot y}\Omega^0_{zx} + \lambda^2\Omega^0_{xz}\Omega^0_{y\dot y}\Omega^0_{z\dot x} \\
&+ 2n\lambda^2\Omega^0_{x\dot z}\Omega^0_{zy} + 2n\lambda^3\Omega^0_{x\dot z}\Omega^0_{z\dot y} - \lambda\Omega^0_{x\dot z}\Omega^0_{yx}\Omega^0_{zy} - \lambda^2\Omega^0_{x\dot z}\Omega^0_{yx}\Omega^0_{z\dot y} - \lambda^2\Omega^0_{x\dot z}\Omega^0_{y\dot x}\Omega^0_{zy} - \lambda^3\Omega^0_{x\dot z}\Omega^0_{y\dot x}\Omega^0_{z\dot y} \\
&- \lambda^3\Omega^0_{x\dot z}\Omega^0_{zx} - \lambda^4\Omega^0_{x\dot z}\Omega^0_{z\dot x} + \lambda\Omega^0_{x\dot z}\Omega^0_{yy}\Omega^0_{zx} + \lambda^2\Omega^0_{x\dot z}\Omega^0_{yy}\Omega^0_{zx} + \lambda^2\Omega^0_{x\dot z}\Omega^0_{y\dot y}\Omega^0_{zx} + \lambda^3\Omega^0_{x\dot z}\Omega^0_{y\dot y}\Omega^0_{z\dot x} = 0
\end{aligned}
$$

or

$$
\begin{aligned}
&\lambda^6 - \left(\Omega^0_{z\dot{z}} + \Omega^0_{y\dot{y}} + \Omega^0_{x\dot{x}}\right)\lambda^5 + \left(-\Omega^0_{zz} - \Omega^0_{yy} + \Omega^0_{y\dot{y}}\Omega^0_{z\dot{z}} - \Omega^0_{y\dot{z}}\Omega^0_{z\dot{y}} - \Omega^0_{xx} + \Omega^0_{x\dot{x}}\Omega^0_{z\dot{z}}\right.\\
&\left.+ \Omega^0_{x\dot{x}}\Omega^0_{y\dot{y}} + 4n^2 - 2n\Omega^0_{y\dot{x}} + 2n\Omega^0_{y\dot{x}}\Omega^0_{z\dot{z}} + 2n\Omega^0_{x\dot{y}} - \Omega^0_{y\dot{x}}\Omega^0_{x\dot{y}} - \Omega^0_{x\dot{z}}\Omega^0_{z\dot{x}}\right)\lambda^4\\
&+ \left(\Omega^0_{yy}\Omega^0_{z\dot{z}} + \Omega^0_{zz}\Omega^0_{y\dot{y}} - \Omega^0_{yz}\Omega^0_{z\dot{y}} - \Omega^0_{zy}\Omega^0_{y\dot{z}} + \Omega^0_{xx}\Omega^0_{z\dot{z}} + \Omega^0_{xx}\Omega^0_{y\dot{y}} + \Omega^0_{x\dot{x}}\Omega^0_{zz} + \Omega^0_{x\dot{x}}\Omega^0_{yy}\right.\\
&- \Omega^0_{x\dot{x}}\Omega^0_{y\dot{y}}\Omega^0_{z\dot{z}} + \Omega^0_{x\dot{x}}\Omega^0_{y\dot{z}}\Omega^0_{z\dot{y}} - 4n^2\Omega^0_{z\dot{z}} - 2n\Omega^0_{yx} - 2n\Omega^0_{y\dot{z}}\Omega^0_{z\dot{x}} + 2n\Omega^0_{xy} - \Omega^0_{y\dot{x}}\Omega^0_{xy}\\
&- 2n\Omega^0_{z\dot{z}}\Omega^0_{x\dot{y}} - \Omega^0_{yx}\Omega^0_{x\dot{y}} + \Omega^0_{y\dot{x}}\Omega^0_{z\dot{z}}\Omega^0_{x\dot{y}} - \Omega^0_{y\dot{z}}\Omega^0_{z\dot{x}}\Omega^0_{x\dot{y}} - \Omega^0_{xz}\Omega^0_{z\dot{x}} + 2n\Omega^0_{x\dot{z}}\Omega^0_{z\dot{y}}\\
&\left.- \Omega^0_{x\dot{z}}\Omega^0_{y\dot{x}}\Omega^0_{z\dot{y}} - \Omega^0_{x\dot{z}}\Omega^0_{zx} + \Omega^0_{x\dot{z}}\Omega^0_{y\dot{y}}\Omega^0_{z\dot{x}}\right)\lambda^3 + \left(\Omega^0_{yy}\Omega^0_{zz} - \Omega^0_{yz}\Omega^0_{zy} + \Omega^0_{xx}\Omega^0_{zz} + \Omega^0_{xx}\Omega^0_{yy}\right.\\
&- \Omega^0_{xx}\Omega^0_{y\dot{y}}\Omega^0_{z\dot{z}} + \Omega^0_{xx}\Omega^0_{y\dot{z}}\Omega^0_{z\dot{y}} - \Omega^0_{x\dot{x}}\Omega^0_{yy}\Omega^0_{z\dot{z}} - \Omega^0_{x\dot{x}}\Omega^0_{zz}\Omega^0_{y\dot{y}} + \Omega^0_{x\dot{x}}\Omega^0_{yz}\Omega^0_{z\dot{y}}\\
&+ \Omega^0_{x\dot{x}}\Omega^0_{zy}\Omega^0_{z\dot{y}} + \Omega^0_{x\dot{x}}\Omega^0_{zy}\Omega^0_{y\dot{z}} - 4n^2\Omega^0_{zz} + 2n\Omega^0_{yx}\Omega^0_{z\dot{z}} + 2n\Omega^0_{y\dot{x}}\Omega^0_{zz}\\
&- 2n\Omega^0_{yz}\Omega^0_{z\dot{x}} - 2n\Omega^0_{y\dot{z}}\Omega^0_{zx} - 2n\Omega^0_{z\dot{z}}\Omega^0_{xy} - \Omega^0_{yx}\Omega^0_{xy} + \Omega^0_{y\dot{x}}\Omega^0_{z\dot{z}}\Omega^0_{xy} - \Omega^0_{y\dot{z}}\Omega^0_{z\dot{x}}\Omega^0_{xy}\\
&- 2n\Omega^0_{zz}\Omega^0_{x\dot{y}} + \Omega^0_{yx}\Omega^0_{z\dot{z}}\Omega^0_{x\dot{y}} + \Omega^0_{y\dot{x}}\Omega^0_{x\dot{y}}\Omega^0_{zz} - \Omega^0_{yz}\Omega^0_{z\dot{x}}\Omega^0_{x\dot{y}} - \Omega^0_{y\dot{z}}\Omega^0_{zx}\Omega^0_{x\dot{y}} + 2n\Omega^0_{xz}\Omega^0_{z\dot{y}}\\
&- \Omega^0_{xz}\Omega^0_{y\dot{x}}\Omega^0_{z\dot{y}} - \Omega^0_{xz}\Omega^0_{zx} + \Omega^0_{xz}\Omega^0_{y\dot{y}}\Omega^0_{z\dot{x}} + 2n\Omega^0_{x\dot{z}}\Omega^0_{zy} - \Omega^0_{x\dot{z}}\Omega^0_{yx}\Omega^0_{z\dot{y}} - \Omega^0_{x\dot{z}}\Omega^0_{y\dot{x}}\Omega^0_{zy}\\
&\left.+ \Omega^0_{yy}\Omega^0_{x\dot{z}}\Omega^0_{z\dot{x}} + \Omega^0_{x\dot{z}}\Omega^0_{y\dot{y}}\Omega^0_{zx}\right)\lambda^2 + \left(-\Omega^0_{xx}\Omega^0_{yy}\Omega^0_{z\dot{z}} - \Omega^0_{xx}\Omega^0_{zz}\Omega^0_{y\dot{y}} + \Omega^0_{yz}\Omega^0_{xx}\Omega^0_{z\dot{y}}\right.\\
&+ \Omega^0_{xx}\Omega^0_{zy}\Omega^0_{y\dot{z}} - \Omega^0_{x\dot{x}}\Omega^0_{yy}\Omega^0_{zz} + \Omega^0_{x\dot{x}}\Omega^0_{yz}\Omega^0_{zy}\, 2n\Omega^0_{yx}\Omega^0_{zz} - 2n\Omega^0_{yz}\Omega^0_{zx} - 2n\Omega^0_{zz}\Omega^0_{xy}\\
&+ \Omega^0_{yx}\Omega^0_{z\dot{z}}\Omega^0_{xy} + \Omega^0_{y\dot{x}}\Omega^0_{zz}\Omega^0_{xy} - \Omega^0_{yz}\Omega^0_{z\dot{x}}\Omega^0_{xy} - \Omega^0_{y\dot{z}}\Omega^0_{zx}\Omega^0_{xy} + \Omega^0_{yx}\Omega^0_{zz}\Omega^0_{x\dot{y}}\\
&- \Omega^0_{yz}\Omega^0_{zx}\Omega^0_{x\dot{y}} + 2n\Omega^0_{xz}\Omega^0_{zy} - \Omega^0_{xz}\Omega^0_{yx}\Omega^0_{z\dot{y}} - \Omega^0_{xz}\Omega^0_{y\dot{x}}\Omega^0_{zy} + \Omega^0_{xz}\Omega^0_{yy}\Omega^0_{z\dot{x}}\\
&\left.+ \Omega^0_{xz}\Omega^0_{y\dot{y}}\Omega^0_{zx} - \Omega^0_{x\dot{z}}\Omega^0_{yx}\Omega^0_{zy} + \Omega^0_{x\dot{z}}\Omega^0_{yy}\Omega^0_{zx}\right)\lambda - \Omega^0_{xx}\Omega^0_{yy}\Omega^0_{zz} + \Omega^0_{xx}\Omega^0_{yz}\Omega^0_{zy}\\
&+ \Omega^0_{yx}\Omega^0_{zz}\Omega^0_{xy} - \Omega^0_{yz}\Omega^0_{zx}\Omega^0_{xy} - \Omega^0_{xz}\Omega^0_{yx}\Omega^0_{zy} + \Omega^0_{xz}\Omega^0_{yy}\Omega^0_{zx} = 0.
\end{aligned}
$$

So that,

$$\lambda^6 + a\lambda^5 + b\lambda^4 + c\lambda^3 + d\lambda^2 + e\lambda + f = 0 \qquad (5.20)$$

where

$$a = -(\Omega^0_{z\dot{z}} + \Omega^0_{y\dot{y}} + \Omega^0_{x\dot{x}})$$

$$
\begin{aligned}
b = &-\Omega^0_{zz} - \Omega^0_{yy} + \Omega^0_{y\dot{y}}\Omega^0_{z\dot{z}} - \Omega^0_{y\dot{z}}\Omega^0_{z\dot{y}} - \Omega^0_{xx} + \Omega^0_{x\dot{x}}\Omega^0_{z\dot{z}} + \Omega^0_{x\dot{x}}\Omega^0_{y\dot{y}} + 4n^2 - 2n\Omega^0_{y\dot{x}}\\
&+ 2n\Omega^0_{y\dot{x}}\Omega^0_{z\dot{z}} + 2n\Omega^0_{x\dot{y}} - \Omega^0_{y\dot{x}}\Omega^0_{x\dot{y}} - \Omega^0_{x\dot{z}}\Omega^0_{z\dot{x}}
\end{aligned}
$$

$$
\begin{aligned}
c = {} & \Omega^{0}_{yy}\Omega^{0}_{z\dot{z}} + \Omega^{0}_{zz}\Omega^{0}_{y\dot{y}} - \Omega^{0}_{yz}\Omega^{0}_{z\dot{y}} - \Omega^{0}_{zy}\Omega^{0}_{y\dot{z}} + \Omega^{0}_{xx}\Omega^{0}_{z\dot{z}} + \Omega^{0}_{xx}\Omega^{0}_{y\dot{y}} \\
& + \Omega^{0}_{x\dot{x}}\Omega^{0}_{zz} + \Omega^{0}_{x\dot{x}}\Omega^{0}_{yy} - \Omega^{0}_{x\dot{x}}\Omega^{0}_{y\dot{y}}\Omega^{0}_{z\dot{z}} + \Omega^{0}_{x\dot{x}}\Omega^{0}_{y\dot{z}}\Omega^{0}_{z\dot{y}} - 4n^{2}\Omega^{0}_{z\dot{z}} - 2n\Omega^{0}_{yx} \\
& - 2n\Omega^{0}_{y\dot{z}}\Omega^{0}_{z\dot{x}} + 2n\Omega^{0}_{xy} - \Omega^{0}_{y\dot{x}}\Omega^{0}_{xy} - 2n\Omega^{0}_{z\dot{z}}\Omega^{0}_{x\dot{y}} - \Omega^{0}_{yx}\Omega^{0}_{x\dot{y}} + \Omega^{0}_{y\dot{x}}\Omega^{0}_{z\dot{z}}\Omega^{0}_{x\dot{y}} \\
& - \Omega^{0}_{y\dot{z}}\Omega^{0}_{z\dot{x}}\Omega^{0}_{x\dot{y}} - \Omega^{0}_{xz}\Omega^{0}_{z\dot{x}} + 2n\Omega^{0}_{x\dot{z}}\Omega^{0}_{z\dot{y}} - \Omega^{0}_{x\dot{z}}\Omega^{0}_{y\dot{x}}\Omega^{0}_{z\dot{y}} - \Omega^{0}_{x\dot{z}}\Omega^{0}_{zx} + \Omega^{0}_{x\dot{z}}\Omega^{0}_{y\dot{y}}\Omega^{0}_{z\dot{x}}
\end{aligned}
$$

$$
\begin{aligned}
d = {} & \Omega^{0}_{yy}\Omega^{0}_{zz} - \Omega^{0}_{yz}\Omega^{0}_{zy} + \Omega^{0}_{xx}\Omega^{0}_{zz} + \Omega^{0}_{xx}\Omega^{0}_{yy} - \Omega^{0}_{xx}\Omega^{0}_{y\dot{y}}\Omega^{0}_{z\dot{z}} + \Omega^{0}_{xx}\Omega^{0}_{y\dot{z}}\Omega^{0}_{z\dot{y}} \\
& - \Omega^{0}_{x\dot{x}}\Omega^{0}_{yy}\Omega^{0}_{z\dot{z}} - \Omega^{0}_{x\dot{x}}\Omega^{0}_{zz}\Omega^{0}_{y\dot{y}} + \Omega^{0}_{x\dot{x}}\Omega^{0}_{yz}\Omega^{0}_{z\dot{y}} + \Omega^{0}_{x\dot{x}}\Omega^{0}_{zy}\Omega^{0}_{z\dot{y}} + \Omega^{0}_{x\dot{x}}\Omega^{0}_{zy}\Omega^{0}_{y\dot{z}} \\
& - 4n^{2}\Omega^{0}_{zz} + 2n\Omega^{0}_{yx}\Omega^{0}_{z\dot{z}} + 2n\Omega^{0}_{y\dot{x}}\Omega^{0}_{zz} - 2n\Omega^{0}_{yz}\Omega^{0}_{z\dot{x}} - 2n\Omega^{0}_{y\dot{z}}\Omega^{0}_{zx} - 2n\Omega^{0}_{z\dot{z}}\Omega^{0}_{xy} \\
& - \Omega^{0}_{yx}\Omega^{0}_{xy} + \Omega^{0}_{y\dot{x}}\Omega^{0}_{z\dot{z}}\Omega^{0}_{xy} - \Omega^{0}_{y\dot{z}}\Omega^{0}_{z\dot{x}}\Omega^{0}_{xy} - 2n\Omega^{0}_{zz}\Omega^{0}_{x\dot{y}} + \Omega^{0}_{yx}\Omega^{0}_{z\dot{z}}\Omega^{0}_{x\dot{y}} + \Omega^{0}_{y\dot{x}}\Omega^{0}_{x\dot{y}}\Omega^{0}_{zz} \\
& - \Omega^{0}_{yz}\Omega^{0}_{z\dot{x}}\Omega^{0}_{x\dot{y}} - \Omega^{0}_{y\dot{z}}\Omega^{0}_{zx}\Omega^{0}_{x\dot{y}} + 2n\Omega^{0}_{xz}\Omega^{0}_{z\dot{y}} - \Omega^{0}_{xz}\Omega^{0}_{y\dot{x}}\Omega^{0}_{z\dot{y}} - \Omega^{0}_{xz}\Omega^{0}_{zx} \\
& + \Omega^{0}_{xz}\Omega^{0}_{y\dot{y}}\Omega^{0}_{z\dot{x}} + 2n\Omega^{0}_{x\dot{z}}\Omega^{0}_{zy} - \Omega^{0}_{x\dot{z}}\Omega^{0}_{yx}\Omega^{0}_{z\dot{y}} - \Omega^{0}_{x\dot{z}}\Omega^{0}_{y\dot{x}}\Omega^{0}_{zy} + \Omega^{0}_{yy}\Omega^{0}_{x\dot{z}}\Omega^{0}_{z\dot{x}} \\
& + \Omega^{0}_{x\dot{z}}\Omega^{0}_{y\dot{y}}\Omega^{0}_{zx}
\end{aligned}
$$

$$
\begin{aligned}
e = {} & -\Omega^{0}_{xx}\Omega^{0}_{yy}\Omega^{0}_{z\dot{z}} - \Omega^{0}_{xx}\Omega^{0}_{zz}\Omega^{0}_{y\dot{y}} + \Omega^{0}_{yz}\Omega^{0}_{xx}\Omega^{0}_{z\dot{y}} + \Omega^{0}_{xx}\Omega^{0}_{zy}\Omega^{0}_{y\dot{z}} - \Omega^{0}_{x\dot{x}}\Omega^{0}_{yy}\Omega^{0}_{zz} \\
& + \Omega^{0}_{x\dot{x}}\Omega^{0}_{yz}\Omega^{0}_{zy}\, 2n\Omega^{0}_{yx}\Omega^{0}_{zz} - 2n\Omega^{0}_{yz}\Omega^{0}_{zx} - 2n\Omega^{0}_{zz}\Omega^{0}_{xy} + \Omega^{0}_{yx}\Omega^{0}_{z\dot{z}}\Omega^{0}_{xy} + \Omega^{0}_{y\dot{x}}\Omega^{0}_{zz}\Omega^{0}_{xy} \\
& - \Omega^{0}_{yz}\Omega^{0}_{z\dot{x}}\Omega^{0}_{xy} - \Omega^{0}_{y\dot{z}}\Omega^{0}_{zx}\Omega^{0}_{xy} + \Omega^{0}_{yx}\Omega^{0}_{zz}\Omega^{0}_{x\dot{y}} - \Omega^{0}_{yz}\Omega^{0}_{zx}\Omega^{0}_{x\dot{y}} + 2n\Omega^{0}_{xz}\Omega^{0}_{zy} - \Omega^{0}_{xz}\Omega^{0}_{yx}\Omega^{0}_{z\dot{y}} \\
& - \Omega^{0}_{xz}\Omega^{0}_{y\dot{x}}\Omega^{0}_{zy} + \Omega^{0}_{xz}\Omega^{0}_{yy}\Omega^{0}_{z\dot{x}} + \Omega^{0}_{xz}\Omega^{0}_{y\dot{y}}\Omega^{0}_{zx} - \Omega^{0}_{x\dot{z}}\Omega^{0}_{yx}\Omega^{0}_{zy} + \Omega^{0}_{x\dot{z}}\Omega^{0}_{yy}\Omega^{0}_{zx}
\end{aligned}
$$

$$
\begin{aligned}
f = {} & -\Omega^{0}_{xx}\Omega^{0}_{yy}\Omega^{0}_{zz} + \Omega^{0}_{xx}\Omega^{0}_{yz}\Omega^{0}_{zy} + \Omega^{0}_{yx}\Omega^{0}_{zz}\Omega^{0}_{xy} - \Omega^{0}_{yz}\Omega^{0}_{zx}\Omega^{0}_{xy} - \Omega^{0}_{xz}\Omega^{0}_{yx}\Omega^{0}_{zy} \\
& + \Omega^{0}_{xz}\Omega^{0}_{yy}\Omega^{0}_{zx}
\end{aligned}
$$

Now, the partial derivatives after ignoring products and higher order terms of very small parameters are:

$$
\begin{aligned}
\Omega^{0}_{xx} = {} & n^{2} - \frac{(1-\mu)}{r_{10}^{3}} + \frac{3(1-\mu)(x_{0}+\mu)^{2}}{r_{10}^{5}} - \frac{3(1-\mu)A}{2r_{10}^{5}} + \frac{15(1-\mu)A(x_{0}+\mu)^{2}}{2r_{10}^{7}} + \frac{15(1-\mu)Az_{0}^{2}}{2r_{10}^{7}} \\
& - \frac{15(1-\mu)A(x_{0}+\mu)^{2}z_{0}^{2}}{2r_{10}^{9}} - \frac{\mu q}{r_{20}^{3}} + \frac{3\mu q(x_{0}+\mu-1)^{2}}{r_{20}^{5}}
\end{aligned}
$$

$$
\Omega^{0}_{x\dot{x}} = -\frac{(x_{0}+\mu-1)^{2}}{r_{20}^{4}}W_{2} - \frac{1}{r_{20}^{2}}W_{2}
$$

$$\Omega_{xy}^0 = \frac{3(1-\mu)(x_0+\mu)y_0}{r_{10}^5} + \frac{3\mu q(x_0+\mu-1)y_0}{r_{20}^5} + \frac{n}{r_{20}^5}W_2$$

$$\Omega_{yy}^0 = n^2 - \frac{(1-\mu)}{r_{10}^3} - \frac{3(1-\mu)A}{2r_{10}^3} + \frac{15(1-\mu)Az_0^2}{2r_{10}^7} - \frac{\mu q}{r_{20}^3}$$

$$\Omega_{y\dot{y}}^0 = -\frac{1}{r_{20}^2}W_2$$

$$\Omega_{yx}^0 = \frac{3(1-\mu)(x_0+\mu)y_0}{r_{10}^5} + \frac{3\mu q(x_0+\mu-1)y_0}{r_{20}^5} + \frac{n}{r_{20}^5}W_2 + \frac{3n(x_0+\mu-1)}{r_{20}^4}W_2$$

$$\Omega_{zz}^0 = -\frac{(1-\mu)}{r_{10}^3} + \frac{3(1-\mu)z_0^2}{r_{10}^5} - \frac{9(1-\mu)A}{2r_{10}^5} + \frac{45(1-\mu)Az_0^2}{r_{10}^7} - \frac{105(1-\mu)Az_0^4}{2r_{10}^7} - \frac{\mu q}{r_{20}^3} + \frac{3\mu qz_0^2}{r_{20}^5}$$

$$\Omega_{z\dot{z}}^0 = -\frac{z_0^2}{r_{20}^4}W_2 - \frac{1}{r_{20}^4}W_2$$

$$\Omega_{z\dot{x}}^0 = -\frac{(x_0+\mu-1)z_0}{r_{20}^4}W_2 \qquad (5.21)$$

$$\Omega_{zx}^0 = -\frac{3(1-\mu)(x_0+\mu)z_0}{r_{10}^5} + \frac{45(1-\mu)(x_0+\mu)z_0A}{2r_{10}^7} - \frac{105(1-\mu)(x_0+\mu)z_0^3A}{2r_{10}^9} + \frac{3\mu q(x_0+\mu-1)z_0}{r_{20}^5}$$

$$\Omega_{zy}^0 = -\frac{3(1-\mu)y_0z_0}{r_{10}^5} + \frac{3\mu qy_0z_0}{r_{20}^5}$$

$$\Omega_{xz}^0 = \frac{3(1-\mu)(x_0+\mu)z_0}{r_{10}^5} + \frac{15(1-\mu)(x_0+\mu)z_0A}{2r_{10}^7} + \frac{15(1-\mu)(x_0+\mu)z_0A}{r_{10}^7}$$
$$- \frac{105(1-\mu)(x_0+\mu)z_0^3A}{2r_{10}^9} + \frac{3\mu q(x_0+\mu-1)z_0}{r_{20}^5}$$

$$\Omega_{yz}^0 = -\frac{3(1-\mu)y_0z_0}{r_{10}^5} + \frac{3\mu qy_0z_0}{r_{20}^5} + \frac{2n(x_0+\mu-1)z_0}{r_{20}^5}W_2$$

$$\Omega_{x\dot{y}}^0 = 0,\ \Omega_{y\dot{x}}^0 = 0,\ \Omega_{y\dot{z}}^0 = 0,\ \Omega_{z\dot{y}}^0 = 0.$$

In Simplified form with $Q_1 = (1-\mu),\ Q_2 = \mu q$ can be written as

$$a_5 = \frac{W_2}{r_{20}^2}\left[3 + \frac{(x_0+\mu-1)^2}{r_{20}^2} + \frac{z_0^2}{r_{20}^2}\right]$$

$$\begin{aligned} a_4 = {} & 2 + 3A + \frac{3Q_1}{r_{10}^3} + \frac{3Q_2}{r_{20}^3} - \frac{3Q_1 z_0^2}{r_{10}^5} + \frac{9Q_1 A}{2r_{10}^5} - \frac{45Q_1 z_0^2 A}{r_{10}^7} + \frac{105Q_1 z_0^4 A}{2r_{10}^7} - \frac{3Q_2 z_0^2}{r_{20}^5} + \frac{3Q_1 A}{r_{10}^5} - \frac{15Q_1 z_0^2 A}{r_{10}^7} \\ & - \frac{3Q_1(x_0+\mu)^2}{r_{10}^5} - \frac{15Q_1(x_0+\mu)^2}{2r_{10}^7} + \frac{105Q_1(x_0+\mu)^2 z_0^2 A}{2r_{10}^9} - \frac{3Q_2(x_0+\mu-1)^2}{r_{20}^5} \end{aligned}$$

$$\begin{aligned} a_3 = {} & -\frac{2z_0^2}{r_{20}^4}W_2 - \frac{4}{r_{20}^2}W_2 + \frac{2Q_1 z_0^2}{r_{10}^3\, r_{20}^4}W_2 + \frac{6Q_1}{r_{10}^3\, r_{20}^2}W_2 - \frac{4Q_2 z_0^2}{r_{20}^7}W_2 + \frac{6Q_2}{r_{20}^5}W_2 - \frac{6Q_1 z_0^2}{r_{10}^5\, r_{20}^2}W_2 \\ & - \frac{3Q_1(x_0+\mu)^2 z_0^2}{r_{10}^5\, r_{20}^4}W_2 - \frac{6Q_1(x_0+\mu)^2}{r_{10}^5\, r_{20}^2}W_2 - \frac{4Q_2(x_0+\mu-1)^2}{r_{20}^7}W_2 + \frac{2Q_1(x_0+\mu-1)^2}{r_{10}^3\, r_{20}^4}W_2 \\ & - \frac{3Q_1(x_0+\mu-1)^2 z_0^2}{r_{10}^5\, r_{20}^4}W_2 - \frac{(x_0+\mu-1)^2}{r_{20}^4}W_2\ \frac{6Q_1(x_0+\mu-1)(x_0+\mu)z_0^2}{r_{10}^5\, r_{20}^4}W_2 + \frac{8n^2}{r_{20}^2}W_2 \\ & + \frac{4n^2 z_0^2}{r_{20}^2}W_2 - \frac{4n^2(x_0+\mu-1)^2}{r_{20}^4}W_2 \end{aligned}$$

$$
\begin{aligned}
a_2 =& -\frac{6n^2 Q_1 z_0^2}{r_{20}^4} + 9n^2 Q_2 A + \frac{165 n^2 Q_1 z_0^2 A}{r_{20}^7} + 105 n^2 Q_1 z_0^4 A - \frac{6n^2 Q_2 z_0^2}{r_{20}^5} + \frac{3Q_1^2}{r_{10}^6} - \frac{6Q_1^2 z_0^2}{r_{10}^8} \\
& + \frac{3Q_1 Q_2 A}{r_{10}^8} - \frac{114 Q_1^2 z_0^2 A}{r_{10}^{10}} + \frac{105 Q_1^2 z_0^4 A}{r_{10}^{10}} + \frac{6Q_1 Q_2}{r_{10}^3 r_{20}^3} - \frac{6Q_1 Q_2 z_0^2}{r_{10}^3 r_{20}^5} - \frac{6Q_1 Q_2 z_0^2}{r_{10}^5 r_{20}^3} + \frac{6Q_1^2 A}{r_{10}^8} \\
& + \frac{6Q_1 Q_2 A}{r_{10}^5 r_{20}^3} - \frac{9Q_1 Q_2 z_0^2 A}{r_{10}^5 r_{20}^5} + \frac{45 Q_1^2 z_0^4 A}{2r_{10}^{12}} - \frac{105 Q_1 Q_2 z_0^2 A}{r_{10}^7 r_{20}^3} + \frac{45 Q_1 Q_2 z_0^4 A}{2r_{10}^7 r_{20}^5} + \frac{9Q_2^2 A}{r_{10}^5 r_{20}^3} \\
& + \frac{105 Q_1 Q_2 z_0^4 A}{r_{10}^7 r_{20}^3} + \frac{3Q_2^2}{r_{20}^6} - \frac{6Q_2^2 z_0^2}{r_{20}^8} - \frac{6Q_1^2 (x_0+\mu)^2}{r_{10}^8} - \frac{27 Q_1 Q_2 (x_0+\mu)^2 A}{2\, r_{10}^{10}} \\
& + \frac{150 Q_1^2 (x_0+\mu)^2 z_0^2 A}{r_{10}^{12}} - \frac{315 Q_1^2 (x_0+\mu)^2 z_0^2 A}{2r_{10}^{12}} - \frac{6Q_1 Q_2 (x_0+\mu)^2}{r_{10}^5 r_{20}^3} + \frac{9Q_1 Q_2 (x_0+\mu)^2 z_0^2}{r_{10}^5 r_{20}^5} \\
& - \frac{33 Q_1^2 (x_0+\mu)^2 A}{2r_{10}^{10} r_{20}^3} - \frac{15 Q_1 Q_2 (x_0+\mu)^2 A}{r_{10}^7 r_{20}^3} + \frac{45 Q_1 Q_2 (x_0+\mu)^2 z_0^2 A}{2r_{10}^7 r_{20}^5} + \frac{315 Q_1^2 (x_0+\mu)^2 z_0^4 A}{2r_{10}^{14}} \\
& + \frac{105 Q_1 Q_2 (x_0+\mu)^2 z_0^2 A}{r_{10}^9 r_{20}^3} \frac{315 Q_1 Q_2 (x_0+\mu)^2 z_0^4 A}{2r_{10}^9 r_{20}^5} - \frac{6Q_1 Q_2 (x_0+\mu-1)^2}{r_{10}^3 r_{20}^5} - \frac{9Q_1 Q_2 (x_0+\mu-1)^2 z_0^2}{r_{10}^5 r_{20}^5} \\
& - \frac{27 Q_2^2 (x_0+\mu-1)^2 A}{2r_{10}^5 r_{20}^5} + \frac{105 Q_1 Q_2 (x_0+\mu-1)^2 z_0^2 A}{r_{10}^7 r_{20}^5} - \frac{315 Q_1 Q_2 (x_0+\mu-1)^2 z_0^4 A}{2r_{10}^7 r_{20}^5} \\
& - \frac{6Q_2^2 (x_0+\mu-1)^2}{r_{20}^8} + n^4 - \frac{3n^2 Q_1 A}{r_{10}^5} - \frac{3n^2 Q_1 (x_0+\mu)^2}{r_{10}^5} + \frac{15 n^2 Q_1 (x_0+\mu)^2 A}{2r_{10}^7} \\
& - \frac{105 n^2 Q_1 (x_0+\mu)^2 A z_0^2}{2r_{10}^9} + \frac{3n^2 Q_2 (x_0+\mu-1)^2}{r_{20}^5} - \frac{9Q_1 Q_2 (x_0+\mu-1)^2 A}{2r_{10}^5 r_{10}^5} + \frac{45 Q_1 Q_2 (x_0+\mu-1)^2 z_0^2 A}{2r_{10}^7 r_{20}^5} \\
& + \frac{315 Q_1 Q_2 (x_0+\mu)(x_0+\mu-1)^2 z_0^4 A}{2r_{10}^9 r_{20}^5}
\end{aligned}
$$

$$
\begin{aligned}
a_1 = {} & \frac{n^4 z_0^2}{r_{20}^4} W_2 + \frac{n^4}{r_{20}^2} W_2 - \frac{2n^2 Q_1 z_0^2}{r_{10}^3\, r_{20}^4} W_2 - \frac{8n^2 Q_2 z_0^2}{r_{20}^7} W_2 + \frac{Q_1^2 z_0^2}{r_{10}^6\, r_{20}^4} W_2 + \frac{3Q_1^2}{r_{10}^6\, r_{20}^2} W_2 - \frac{4Q_1 Q_2 z_0^2}{r_{10}^3\, r_{20}^7} W_2 \\
& + \frac{6Q_1 Q_2}{r_{10}^3\, r_{20}^5} W_2 + \frac{3n^2 Q_1 (x_0+\mu)^2 z_0^2}{r_{10}^5\, r_{20}^4} W_2 + \frac{3n^2 Q_1 (x_0+\mu)^2}{r_{10}^5\, r_{20}^2} W_2 - \frac{3Q_1^2 (x_0+\mu)^2 z_0^2}{r_{10}^8\, r_{20}^4} W_2 \\
& - \frac{3Q_1^2 (x_0+\mu)^2}{r_{10}^8\, r_{20}^4} W_2 - \frac{3Q_1 Q_2 (x_0+\mu)^2}{r_{10}^5\, r_{20}^5} W_2 + \frac{5Q_2^2 z_0^2}{r_{20}^{10}} W_2 + \frac{3Q_2^2}{r_{20}^8} W_2 + \frac{3n^2 Q_1 (x_0+\mu-1)^2 z_0^2}{r_{20}^9} W_2 \\
& + \frac{3n^2 Q_1 (x_0+\mu-1)^2}{r_{20}^7} W_2 - \frac{5Q_1 Q_2 (x_0+\mu-1)^2}{r_{10}^3\, r_{20}^7} W_2 - \frac{5(x_0+\mu-1)^2}{r_{20}^{10}} W_2 - \frac{6n^2 Q_1 z_0^2}{r_{10}^5\, r_{20}^2} W_2 \\
& + \frac{2Q_1^2}{r_{10}^6\, r_{20}^2} W_2 - \frac{6Q_1^2 z_0^2}{r_{10}^8\, r_{20}^2} W_2 - \frac{3Q_1^2 (x_0+\mu)^2 z_0^2}{r_{10}^8\, r_{20}^2} W_2 - \frac{3Q_1 Q_2 (x_0+\mu)^2}{r_{10}^5\, r_{20}^5} W_2 - \frac{6Q_1 Q_2 z_0^2}{r_{10}^5\, r_{20}^5} W_2 \\
& - \frac{6Q_1 Q_2 (x_0+\mu-1)^2}{r_{10}^5\, r_{20}^7} W_2 - \frac{3Q_1 Q_2 (x_0+\mu-1)^2}{r_{20}^{12}} W_2 - \frac{5n^2 Q_1 (x_0+\mu-1)^2}{r_{10}^3\, r_{20}^4} W_2 + \frac{Q_1^2 (x_0+\mu)^2}{r_{10}^6\, r_{20}^4} W_2 \\
& + \frac{15n^2 Q_1 (x_0+\mu-1)^2 z_0^2}{r_{10}^5\, r_{20}^4} W_2 + \frac{3Q_1^2 (x_0+\mu-1)^2 z_0^2}{r_{10}^8\, r_{20}^4} W_2 - \frac{5n^2 Q_2 (x_0+\mu-1)^2 z_0^2}{r_{10}^5\, r_{20}^4} W_2 \\
& + \frac{Q_1 Q_2 (x_0+\mu-1)^2}{r_{10}^3\, r_{20}^7} W_2 - \frac{3Q_1 Q_2 (x_0+\mu-1)^2 z_0^2}{r_{10}^3\, r_{20}^9} W_2 - \frac{6nQ_1 Q_2 (x_0+\mu)^2 y_0}{r_{10}^5\, r_{20}^2} \\
& - \frac{15n^2 Q_1 (x_0+\mu)(x_0+\mu-1)^2 z_0^2}{r_{10}^5\, r_{20}^4} W_2 - \frac{18nQ_1^2 (x_0+\mu)^2 y_0 z_0^2}{r_{10}^5\, r_{20}^2} - \frac{18nQ_1 Q_2 (x_0+\mu-1)^2 y_0 z_0^2}{r_{10}^5\, r_{20}^5} \\
& + \frac{6nQ_1 Q_2 (x_0+\mu)^2 y_0}{r_{10}^5\, r_{20}^3} - \frac{18nQ_1 Q_2 (x_0+\mu)^2 y_0 z_0^2}{r_{10}^5\, r_{20}^5} - \frac{3n^2 Q_1 (x_0+\mu)(x_0+\mu-1) z_0^3}{r_{10}^5\, r_{20}^4} W_2 \\
& + \frac{3Q_1^2 (x_0+\mu)(x_0+\mu-1) z_0^3}{r_{10}^8\, r_{20}^4} W_2 + \frac{3Q_1 Q_2 (x_0+\mu-1) z_0^3}{r_{10}^5\, r_{20}^7} W_2 - \frac{3n^2 Q_2 (x_0+\mu-1)^2 z_0^3}{r_{20}^9} W_2 \\
& + \frac{3Q_1 Q_2 (x_0+\mu-1)^2 z_0^3}{r_{10}^3\, r_{20}^9} W_2 + \frac{3Q_2^2 (x_0+\mu-1)^2 z_0^3}{r_{20}^{12}} W_2 - \frac{15Q_1 Q_2 (x_0+\mu)(x_0+\mu-1) z_0^2}{r_{10}^5\, r_{20}^7} W_2 \\
& + \frac{3Q_1^2 (x_0+\mu)(x_0+\mu-1) z_0^2}{r_{10}^8\, r_{20}^4} W_2
\end{aligned}
$$

$$
\begin{aligned}
a_0 = {} & \frac{n^4 Q_1}{r_{10}^3} - \frac{2n^2 Q_1^2}{r_{10}^6} - \frac{15n^2 Q_1^2 A}{2r_{10}^8} + \frac{106n^2 Q_1^2 z_0^2 A}{r_{10}^{10}} - \frac{4n^2 Q_1 Q_2}{r_{10}^3\, r_{20}^3} + \frac{15Q_1^3 A}{2r_{10}^{11}} + \frac{114Q_1^2 Q_2 z_0^2 A}{r_{10}^{10}\, r_{20}^3} \\
& + \frac{Q_1^3}{r_{10}^9} - \frac{71Q_1^3 A}{r_{10}^{13}} + \frac{3Q_1^2 Q_2}{r_{10}^6\, r_{20}^3} + \frac{3n^2 Q_1 (x_0+\mu)^2}{r_{10}^8} - \frac{3Q_1^3 (x_0+\mu)^2}{r_{10}^{11}} - \frac{51Q_1^3 (x_0+\mu)^2 A}{2r_{10}^{13}} \\
& + \frac{195Q_1^3 (x_0+\mu)^2 z_0^2 A}{2r_{10}^{15}} - \frac{3Q_1^2 Q_2 (x_0+\mu)^2}{r_{10}^8\, r_{10}^2} + \frac{15Q_1^2 Q_2 A}{r_{10}^8\, r_{10}^3} + \frac{15n^2 Q_1^2 (x_0+\mu)^2 A}{2r_{10}^{10}} \\
& - \frac{15Q_1 Q_2 (x_0+\mu)^2 A}{2r_{10}^{10}\, r_{20}^3} + \frac{114Q_1^2 Q_2 z_0^2 A}{r_{10}^{10}\, r_{20}^{13}} - \frac{75n^2 Q_1^2 (x_0+\mu)^2 z_0^2 A}{r_{10}^{12}} + \frac{75Q_1^2 Q_2 (x_0+\mu)^2 z_0^2 A}{r_{10}^{12}\, r_0^3} \\
& + \frac{3Q_1 Q_2^2}{r_{10}^3\, r_{20}^6} + \frac{3n^2 Q_1 Q_2 (x_0+\mu-1)^2}{2r_{10}^{10}\, r_{20}^3} - \frac{3Q_1^2 Q_2 (x_0+\mu-1)^2}{r_{10}^6\, r_{20}^5} - \frac{18Q_1^2 Q_2 (x_0+\mu-1)^2 A}{r_{10}^8\, r_{20}^5} \\
& \frac{36Q_1^2 Q_2 (x_0+\mu-1)^2 z_0^2 A}{r_{10}^{10}\, r_{20}^5} - \frac{6Q_1^2 Q_2 (x_0+\mu-1)^2}{r_{10}^3\, r_{20}^8} - \frac{3n^4 Q_1 z_0^2}{r_{10}^5} + \frac{6n^2 Q_1^2 z_0^2}{r_{10}^8} - \frac{45n^2 Q_1 z_0^4 A}{r_{10}^{12}} \\
& + \frac{6n^2 Q_1 Q_2 z_0^4}{r_{10}^3\, r_{20}^5} - \frac{3Q_1^3 z_0^2}{r_{10}^{11}} + \frac{45Q_1^3 z_0^4 A}{r_{10}^{15}} - \frac{6Q_1^2 Q_2 z_0^4}{r_{10}^8\, r_{20}^3} - \frac{9n^2 Q_1 (x_0+\mu)^2 z_0^2}{r_{10}^{10}} - \frac{105Q_1^3 (x_0+\mu)^2 z_0^3 A}{2r_{10}^{17}} \\
& - \frac{9Q_1^2 Q_2 (x_0+\mu)^2 z_0^2}{r_{10}^{10}\, r_{20}^2} + \frac{45Q_1^2 Q_2 z_0^4 A}{r_{10}^{10}\, r_{20}^2} - \frac{315n^2 Q_1^2 (x_0+\mu)^2 z_0^4 A}{2r_{10}^{14}} + \frac{315Q_1^3 (x_0+\mu)^2 z_0^4 A}{r_{10}^{17}} \\
& + \frac{6n^2 Q_1 Q_2 z_0^2}{r_{10}^5\, r_{20}^3} + \frac{315Q_1^2 Q_2 (x_0+\mu)^2 z_0^4 A}{2r_{10}^{14}\, r_{20}^3} \frac{45Q_1^2 Q_2 z_0^4 A}{2r_{10}^{12}\, r_{20}^3} - \frac{3Q_1^2 Q_2 z_0^2}{r_{10}^5\, r_{20}^6} - \frac{9n^2 Q_1 Q_2 (x_0+\mu-1)^2 z_0^2}{r_{10}^5\, r_{10}^5} \\
& + \frac{9Q_1^2 Q_2 (x_0+\mu-1)^2 z_0^2}{r_{10}^8\, r_{10}^5} - \frac{135Q_1^2 Q_2 (x_0+\mu-1)^2 z_0^4}{2r_{10}^{12}\, r_{10}^5} + \frac{9Q_1 Q_2^2 (x_0+\mu-1)^2 z_0^2}{r_{10}^5\, r_{20}^8} + \frac{9n^4 Q_1 A}{2r_{10}^5} \\
& - \frac{9n^2 Q_1 Q_2 A}{2r_{10}^5\, r_{20}^2} - \frac{9n^2 Q_1^2 A}{2r_{10}^5} + \frac{27n^2 Q_1 (x_0+\mu)^2 A}{2r_{10}^{10}} - \frac{27Q_1^2 Q_2 (x_0+\mu)^2 A}{2r_{10}^{10}\, r_{20}^2} - \frac{15n^2 Q_1 Q_2 A}{2r_{10}^5\, r_{20}^3} \\
& + \frac{15Q_1 Q_2 A}{2r_{10}^5\, r_{20}^6} + \frac{27n^2 Q_1 Q_2 (x_0+\mu-1)^2 A}{2r_{10}^5\, r_{20}^5} - \frac{18n^2 Q_1 Q_2^2 (x_0+\mu-1)^2 A}{r_{10}^5\, r_{20}^8} - \frac{45n^4 Q_1 z_0^2 A}{r_{10}^7} \\
& + \frac{195n^2 Q_1 Q_2 z_0^2 A}{r_{10}^7\, r_{20}^3} + \frac{135Q_1^2 Q_2 (x_0+\mu)^2 z_0^4 A}{r_{10}^{12}\, r_{20}^2} - \frac{45Q_1^2 Q_2^2 z_0^2 A}{r_{10}^7\, r_{20}^6} - \frac{135n^2 Q_1 Q_2 (x_0+\mu-1)^2 z_0^2 A}{r_{10}^7\, r_{20}^5} \\
& + \frac{135Q_1^2 Q_2 (x_0+\mu-1)^2 z_0^2 A}{r_{10}^{10}\, r_{20}^5} + \frac{215Q_1 Q_2^2 (x_0+\mu-1)^2 z_0^2 A}{2r_{10}^7\, r_{20}^8} + \frac{105n^4 Q_1 z_0^4 A}{2r_{10}^7} - \frac{105n^2 Q_1 z_0^4 A}{r_{10}^{10}} \\
& - \frac{105n^2 Q_1 z_0^4 A}{2r_{10}^{10}} \frac{105Q_1^3 z_0^4 A}{2r_{10}^{13}} - \frac{105n^2 Q_1 Q_2 z_0^4 A}{r_{10}^7\, r_{10}^3} + \frac{315n^2 Q_1 (x_0+\mu)^2 z_0^4 A}{2r_{10}^{12}} - \frac{315Q_1^3 (x_0+\mu)^2 z_0^4 A}{2r_{10}^{15}} \\
& - \frac{315Q_1^2 Q_2 (x_0+\mu)^2 z_0^4 A}{2r_{10}^{12}\, r_{20}^2} \frac{105Q_1^2 Q_2 z_0^4 A}{2r_{10}^{10}\, r_{20}^3} + \frac{105Q_1^2 Q_2^2 z_0^4 A}{2r_{10}^7\, r_{20}^6} + \frac{315n^2 Q_1 Q_2 (x_0+\mu-1)^2 z_0^4 A}{2r_{10}^7\, r_{20}^5} \\
& - \frac{315Q_1^2 Q_2 (x_0+\mu-1)^2 z_0^4 A}{2r_{10}^{10}\, r_{20}^5} - \frac{315Q_1 Q_2^2 (x_0+\mu-1)^2 z_0^4 A}{2r_{10}^7\, r_{20}^8} + \frac{n^4 Q_2}{r_{20}^3} + \frac{15n^2 Q_1 Q_2 z_0^2 A}{2r_{10}^7\, r_{20}^3}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{2n^2Q_2^2}{r_{20}^6}+\frac{3n^2Q_2(x_0+\mu)^2}{r_{10}^5r_{20}^3}-\frac{3Q_1^2Q_2(x_0+\mu)^2}{r_{10}^8r_{20}^3}-\frac{9Q_1^2Q_2(x_0+\mu)^2A}{2r_{10}^{10}r_{20}^3}+\frac{45Q_1^2Q_2(x_0+\mu)^2z_0A}{2r_{10}^{12}r_{20}^3}\\
&-\frac{3Q_1Q_2^2(x_0+\mu)^2}{r_{10}^5r_{20}^5}+\frac{15n^2Q_1Q_2(x_0+\mu)^2A}{2r_{10}^7r_{20}^3}-\frac{15Q_1^2Q_2(x_0+\mu)^2A}{2r_{10}^7r_{20}^3}-\frac{15Q_1Q_2^2(x_0+\mu)^2A}{2r_{10}^7r_{20}^6}-\frac{15Q_1Q_2^2z_0^2A}{2r_{10}^7r_{20}^6}\\
&-\frac{105n^2Q_1Q_2(x_0+\mu)^2z_0^2A}{2r_{10}^9r_{20}^3}-\frac{165Q_1^2Q_2(x_0+\mu)^2z_0^2A}{2r_{10}^{12}r_{20}^3}+\frac{105Q_1Q_2^2(x_0+\mu)^2z_0^2A}{2r_{10}^9r_{20}^6}-\frac{15Q_1Q_2^2z_0^2A}{2r_{10}^7r_{20}^6}\\
&+\frac{Q_2^3}{r_{20}^9}+\frac{3n^2Q_2^2(x_0+\mu-1)^2}{r_{20}^8}-\frac{3Q_2^3(x_0+\mu-1)^2}{r_{20}^{11}}-\frac{3n^4Q_2z_0^2}{r_{20}^8}+\frac{9n^2Q_1Q_2z_0^2A}{r_{10}^5r_{20}^5}-\frac{45n^2Q_1Q_2z_0^4A}{r_{10}^7r_{20}^5}\\
&+\frac{6n^2Q_2^2z_0^2}{r_{20}^8}-\frac{3Q_1^2Q_2z_0^2}{r_{10}^6r_{20}^5}-\frac{9Q_1^2Q_2z_0^2A}{r_{10}^8r_{20}^5}+\frac{45Q_1^2Q_2z_0^4A}{r_{10}^{10}r_{20}^5}-\frac{4Q_1Q_2^2z_0^2A}{r_{10}^3r_{20}^8}-\frac{9n^2Q_2(x_0+\mu)^2z_0^2}{r_{10}^5r_{20}^5}\\
&+\frac{9Q_1^2Q_2(x_0+\mu)^2z_0^2}{r_{10}^8r_{20}^5}+\frac{36Q_1^2Q_2(x_0+\mu)^2z_0^2A}{r_{10}^{10}r_{20}^5}-\frac{135Q_1^2Q_2(x_0+\mu)^2z_0^3A}{2r_{10}^{12}r_{20}^5}+\frac{9Q_1Q_2^2(x_0+\mu)^2z_0^2}{r_{10}^5r_{20}^7}\\
&-\frac{9Q_1Q_2^2z_0^2A}{2r_{10}^5r_{20}^8}-\frac{45n^2Q_1Q_2(x_0+\mu)^2z_0^2A}{2r_{10}^7r_{20}^5}+\frac{45Q_1Q_2^2(x_0+\mu)^2z_0^2A}{2r_{10}^7r_{20}^8}+\frac{45Q_1Q_2^2z_0^4A}{2r_{10}^7r_{20}^8}\\
&+\frac{105Q_1Q_2^2(x_0+\mu)^2z_0^2A}{2r_{10}^9r_{20}^6}-\frac{15Q_1Q_2^2z_0^2A}{2r_{10}^7r_{20}^6}+\frac{Q_2^3}{r_{20}^9}+\frac{3n^2Q_2^2(x_0+\mu-1)^2}{r_{20}^8}-\frac{3Q_2^3(x_0+\mu-1)^2}{r_{20}^{11}}\\
&-\frac{3n^4Q_2z_0^2}{r_{20}^8}+\frac{9n^2Q_1Q_2z_0^2A}{r_{10}^5r_{20}^5}-\frac{45n^2Q_1Q_2z_0^4A}{r_{10}^7r_{20}^5}+\frac{6n^2Q_2^2z_0^2}{r_{20}^8}-\frac{3Q_1^2Q_2z_0^2}{r_{10}^6r_{20}^5}-\frac{9Q_1^2Q_2z_0^2A}{r_{10}^8r_{20}^5}+\frac{45Q_1^2Q_2z_0^4A}{r_{10}^{10}r_{20}^5}\\
&-\frac{4Q_1Q_2^2z_0^2A}{r_{10}^3r_{20}^8}-\frac{9n^2Q_2(x_0+\mu)^2z_0^2}{r_{10}^5r_{20}^5}+\frac{9Q_1^2Q_2(x_0+\mu)^2z_0^2}{r_{10}^8r_{20}^5}+\frac{36Q_1^2Q_2(x_0+\mu)^2z_0^2A}{r_{10}^{10}r_{20}^5}\\
&-\frac{135Q_1^2Q_2(x_0+\mu)^2z_0^3A}{2r_{10}^{12}r_{20}^5}+\frac{9Q_1Q_2^2(x_0+\mu)^2z_0^2}{r_{10}^5r_{20}^7}-\frac{9Q_1Q_2^2z_0^2A}{2r_{10}^5r_{20}^8}-\frac{45n^2Q_1Q_2(x_0+\mu)^2z_0^2A}{2r_{10}^7r_{20}^5}\\
&+\frac{45Q_1Q_2^2(x_0+\mu)^2z_0^2A}{2r_{10}^7r_{20}^8}+\frac{45Q_1Q_2^2z_0^4A}{2r_{10}^7r_{20}^8}+\frac{105n^2Q_1Q_2(x_0+\mu)^2z_0^4A}{2r_{10}^9r_{20}^5}-\frac{105Q_1^2Q_2(x_0+\mu)^2z_0^4A}{2r_{10}^{12}r_{20}^5}\\
&-\frac{105Q_1Q_2^2(x_0+\mu)^2z_0^4A}{2r_{10}^9r_{20}^8}-\frac{9Q_1Q_2^2z_0^2A}{2r_{10}^5r_{20}^8}+\frac{45Q_1Q_2^2z_0^4A}{2r_{10}^7r_{20}^8}-\frac{3Q_2^3z_0^2}{r_{20}^{11}}+\frac{9Q_1Q_2^2(x_0+\mu-1)^2z_0^2}{r_{10}^3r_{20}^{10}}\\
&+\frac{27Q_1Q_2^2(x_0+\mu-1)^2z_0^2A}{2r_{10}^5r_{20}^5}+\frac{9n^2Q_1^2(x_0+\mu)^2z_0^2}{r_{10}^{10}}+\frac{18n^2Q_1Q_2(x_0+\mu)(x_0+\mu-1)z_0^4A}{2r_{10}^5r_{20}^5}\\
&+\frac{135n^2Q_1Q_2(x_0+\mu)(x_0+\mu-1)z_0^4A}{r_{10}^7r_{20}^5}-\frac{315n^2Q_1Q_2(x_0+\mu)(x_0+\mu-1)z_0^4A}{r_{10}^9r_{20}^5}\\
&-\frac{18Q_1^2Q_2(x_0+\mu)(x_0+\mu-1)z_0^2}{r_{10}^8r_{20}^5}-\frac{297Q_1^2Q_2(x_0+\mu)(x_0+\mu-1)z_0^2A}{2r_{10}^{10}r_{20}^5}\\
&+\frac{225Q_1^2Q_2(x_0+\mu)(x_0+\mu-1)z_0^4A}{r_{10}^{12}r_{20}^5}-\frac{9Q_1Q_2^2(x_0+\mu-1)^2z_0^2}{r_{10}^3r_{20}^{10}}
\end{aligned}
$$

$$-\frac{27Q_1^2Q_2(x_0+\mu)(x_0+\mu-1)z_0^2A}{2r_{10}^{15}r_{20}^5}-\frac{27Q_1Q_2^2(x_0+\mu-1)^2z_0^2A}{2r_{10}^5r_{20}^{10}}$$
$$+\frac{135Q_1^2Q_2(x_0+\mu)(x_0+\mu-1)z_0^4A}{2r_{10}^{12}r_{20}^5}-\frac{9Q_1^2Q_2(x_0+\mu)^2z_0^2}{r_{10}^{10}r_{20}^3}-\frac{18Q_1Q_2^2(x_0+\mu)(x_0+\mu-1)z_0^2}{r_{10}^5r_{20}^8}$$
$$-\frac{135Q_1Q_2^2(x_0+\mu)(x_0+\mu-1)z_0^2A}{r_{10}^7r_{20}^8}+\frac{315Q_1Q_2^2(x_0+\mu)(x_0+\mu-1)z_0^4A}{r_{10}^9r_{20}^8}$$

To make our problem simplified, we compute these partial derivatives and roots of the characteristic equation (5.20) numerically in tables (5.3) and (5.4), for $\mu=0.000003$, $c_d=299792458$, $\delta=0.0015$ and $A=0.02$ respectively.

A	a	b	c	d	e	f
0	Complex infinity	Complex infinity	Complex infinity	Complex infinity	Complex infinity	Complex infinity
0.001	1.32462×10^{-18}	-14.33550	-1.14793×10^{-16}	134.572	0.0008290420	-12.67460
0.005	1.04177×10^{-17}	0.600769	1.19956×10^{-17}	4.94634	0.0000684243	-1.130940
0.010	1.64744×10^{-17}	1.760990	1.90767×10^{-17}	4.19649	0.0000510158	-0.580974
0.015	1.95045×10^{-17}	2.099620	2.12028×10^{-17}	4.72635	0.0000738433	-0.445336
0.020	2.12753×10^{-17}	2.267370	2.23391×10^{-17}	5.41514	0.0001147920	-0.391451
0.030	2.32214×10^{-17}	2.448540	2.37266×10^{-17}	6.91784	0.0002402500	-0.356126
0.040	2.42377×10^{-17}	2.557880	2.46970×10^{-17}	8.49509	0.0004204830	-0.354322
0.050	2.48321×10^{-17}	2.640050	2.55010×10^{-17}	10.1238	0.0006551370	-0.365282
0.060	2.51943×10^{-17}	2.709060	2.62160×10^{-17}	11.7982	0.0009448970	-0.382524
0.070	2.54109×10^{-17}	2.770700	2.68731×10^{-17}	13.5166	0.0012908300	-0.403351
0.080	2.55271×10^{-17}	2.827800	2.74871×10^{-17}	15.2781	0.0016942500	-0.426425
0.090	2.55689×10^{-17}	2.881900	2.80654×10^{-17}	17.0822	0.0021566400	-0.450991

5. 3: Numerical values of the coefficient of the characteristic equation (5.21), for $\mu=0.000003$, $c_d=299792458$, $\delta=0.0015$, and $A=0.02$

A	$\lambda_{1,2}$	λ_3	λ_4	$\lambda_{5,6}$
0	Complex infinity	Complex infinity	Complex infinity	Complex infinity
0.001	-3.05468+1.486910 i	-0.308455	0.308448	3.05468 +1.486910 i
0.005	-0.962735+1.15652 i	-0.469649	0.469636	0.962742 +1.15652 i
0.010	-0.762287+1.23570 i	-0.361582	0.361571	0.762292 +1.23570 i
0.015	-0.749687+1.28727 i	-0.300730	0.300716	0.749695 +1.28727 i
0.020	-0.772412+1.32868 i	-0.264893	0.264873	0.772422 +1.32868 i
0.030	-0.838007+1.39708 i	-0.224863	0.224830	0.838024 +1.39707 i
0.040	-0.903704+1.45474 i	-0.202977	0.202929	0.903728 +1.45472 i
0.050	-0.964061+1.50578 i	-0.189092	0.189029	0.964093 +1.50576 i
0.060	-1.01906+1.552150 i	-0.179432	0.179353	1.019100 +1.55213 i
0.070	-1.06946+1.594990 i	-0.172265	0.172170	1.069510 +1.59497 i
0.080	-1.11601+1.635040 i	-0.166689	0.166579	1.116060 +1.63502 i
0.090	-1.15932+1.672790 i	-0.162185	0.162060	1.159390 +1.67277 i

5. 4: Roots of the characteristic equation (5.21) for $\mu = 0.000003, c_d = 299792458$, $\delta = 0.0015,$ and $A = 0.02$

The software package *Mathematica* is used to compute the coefficients and the six characteristic roots of equation (5.20), and are presented numerically in Tables 5.3 and 5.4 respectively, for $\mu = 0.000003, c_d = 299792458$, $\delta = 0.0015,$ and $A = 0.02$. It is note that, for the chosen values of the free parameters, there are one negative, one positive and four complex conjugate roots. Therefore, the case where all six roots are purely imaginary quantities or complex values with negative real parts, do not arise and so the solutions are not bounded and the motion is unstable. Hence, we conclude that the out-of-plane equilibrium points are unstable equilibrium points due to a positive root and positive real part in the complex roots. Our result agrees with previous results of Douskos and Markellos (2006), Shankaran et al. (2011), Singh (2012), Singh and Umar (2012b), and Singh and Leke (2013c).

The linear stability of the equilibrium points is discussed in this research work. We have, in all cases linearized the equations of motion governing the systems. Trial solutions are used and the variational equations are solved to get the characteristic equations in all the cases of the different equilibrium points. We observe that the characteristic equation for the collinear equilibrium points, given by equation (5.8), is different from those of the triangular equilibrium points and the out of plane points, which are given in equations (5.6) and (5.20), respectively. The partial derivatives are computed at the equilibrium points, and the roots of these equations are found.

Firstly, in the case of the collinear equilibrium points, two of the four roots are real and opposite in sign while the remaining two are imaginary roots of opposite signs too. In this case, the collinear points are unstable due to a positive root.

Secondly, for the triangular point, the roots of the characteristics equations computed numerically in both cases when P-R drag is present (see table 5.1) and when it is not present (see table 5.2); shows the existence of at least a positive real part of the complex conjugates and one positive root, respectively. Hence, the motion is also unbounded and we conclude that, the triangular points are unstable.

Lastly, the stability of the out-of-plane equilibrium points is also completely determined by the roots of the characteristic equation (5.20). Because of the ambiguity in expressing the partial derivatives analytically, we decided to study the stability of the system using a numerical approach. Our numerical exploration in the computations of these roots reveals the existence of at least a positive root and/or a positive real part. Consequently, motion is unbounded and thus unstable due to the presence of a positive root and positive real parts of the complex roots.

The result concerning the stability of the collinear equilibrium points are in agreement with those of Szebehely (1967), Singh and Leke (2010), Singh and Begha (2011), Singh (2013), while result of the triangular points agrees with those of Ishwar and Kushvah (2005), Kushvah (2008) among many. For the out of plane points, our assertion that the points are unstable are in agreement with results of, Douskos and Markellos (2006), Shankaran et al (2011), Singh (2012), Singh and Umar (2012b) and Singh and Leke (2013c). However, it is not check whether the outcome of the stability analysis will turn out different when the P-R drag effect is ignored.

CHAPTER 6

SUMMARY, CONCLUSION AND AREAS FOR FURTHER RESEARCH

6.1 INTRODUCTION

In this chapter, we shall summarize the foci points of this thesis in section 6.2. The conclusion of this thesis is drawn in section 6.3, while the recommendations are stated in section 6.4.

6.2 SUMMARY

The restricted three-body problem, which is defined as the study of motion of an infinitesimal mass in the gravitational field of the finite masses, is modified to include the case when the bigger primary is an oblate spheroid and the smaller one is an intense emitter of radiation with its Poynting-Robertson drag force. We have represented these forces, with appropriate parameters and have done the mathematical derivations of the equations of motion in chapter 3. The equations of motion presented in (3.8) and (3.11) are the dynamical systems which govern the motion of the infinitesimal mass.

In chapter 4, we use these equations to examine the positions of the equilibrium points; three collinear equilibrium points, two triangular points and the out-of-plane equilibrium points, are found.

In chapter 5, we have studied the linear stability of these equilibrium points, by linearizing the equations of motion around the equilibrium points. The characteristic equations are obtained analogously for each of the equilibrium points. The collinear and triangular equilibrium points are unstable due to the appearance of at least a positive root of the characteristic equations governing the system, while the out-of-plane equilibrium points are unstable due to positive real parts of the complex roots and/or a positive root. Hence, all the equilibrium points studied in this research work are unstable.

6.3 CONCLUSION

The generalized problem of the classical restricted three-body problem is studied. The model is such that the bigger primary is an oblate spheroid and the smaller one is a radiating emitter having Poynting-Robertson (P-R) drag. It is seen that three points lying on the line joining the primaries, called collinear equilibrium points, they exist and depend on the radiation pressure force of the smaller primary, oblateness of the bigger primary and the mass parameter of the system. Aside these points, a pair of equilibrium points, called triangular equilibrium points forming triangles with the line joining the primaries exist. They are defined by the mass parameter, oblateness of the bigger primary, radiation pressure and P-R drag of the smaller primary. Also, the equilibrium points lying out of the orbital plane of motion (out-of-plane equilibrium points) with its points lying on the three coordinate axes were found. The linear stability of the equilibrium points is studied. It is seen that the collinear equilibrium points are unstable due to a positive root of the governing characteristic equation. The triangular and the out-of-plane equilibrium points are also unstable due to positive real part of the complex roots and a positive root. The numerical explorations were performed in order to give precise and accurate result about the positions of the equilibrium points and their stability for different systems.

6.4 AREAS FOR FURTHER RESEARCH

The photogravitational restricted three-body problem has been modeled to include the non-sphericity of the bigger primary. The inclusion of the oblateness and the Poynting Robertson (P-R) drag, have immense effects on the dynamics of the infinitesimal mass. Hence, we recommend studies of the following problems:

(i). the case when both bodies are oblate spheroid and are source of radiation with Poynting-Robertson (P-R) effects;

(ii). the inclusion of small perturbations in the Coriolis and centrifugal forces in (i) above;

(iii). the non-linear stability of the problem is still an open problem.

(iv). the problem can be extended to the study of R3BP with variable masses, which is still an open problem.

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