AN ALGEBRAIC STUDY OF NON-COMMUTATIVE GENERALLINEAR RHOTRIX GROUP

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AHMADU BELLO UNIVERSITY, ZARIA

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DECLARATION

I declare that the work in this dissertation titled "**AN ALGEBRAIC STUDY OF NON-COMMUTATIVE GENERAL LINEAR RHOTRIX GROUP**" has been performed by me in the Department of Mathematics under the supervision of Dr. A.Mohammed and Prof. B Sani. The information derived from literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any Institution.

OKON, UbongEne	__________________	__________________
Name of Student	Signature	Date

CERTIFICATION

This dissertation titled "**AN ALGEBRAIC STUDY OF NON-COMMUTATIVE GENERALLINEAR RHOTRIX GROUP**" by OKON,UbongEne meets the regulations governing the award of the degree of Master of Science in Mathematics of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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Dr. A. Mohammed Date

Chairman, Supervisory Committee

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External Examiner Date

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Prof. B. Sani Date

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DEDICATION

This thesis is dedicated to the Almighty God. Also to my wonderful parents; Mr. and Mrs. EneOkon, my lovely wife;Vivian UbongEneOkon and my children; Nsikakabasi, Mfonabasi and Ekomabasi.

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ABSTRACT

This dissertation presents a studyof a certain algebraic system,termed: “Non-commutative general linear rhotrix group”. The groupis considered to be analogous to the well-known General Linear Group. The Non-commutative General Linear Rhotrix Group consists of all invertible rhotrices of size n with entries from an arbitrary field F and it has beenshown to possessnon-commutative rhotrix groups as its subgroups. Certain subgroups of non-commutative general linear rhotrix group have also beenidentified and then shown to be embedded in some subgroups of the general linear group. Furthermore, some finite non-commutative groups of rhotrices as well as their subgroups areconstructed and shown as concrete examples. It is of interest that this study will go to a large extent in simplification of teaching and learning of group theory in Mathematical discipline.

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LIST OF NOTATIONS AND SYMBOLS

R_n	-A rhotrix R of size n
$h(R)$	- Heart of rhotrix R
$R_n(F)$	- Therhotrix set of size n over a field F
$GR_n(F)$	- The set of all invertible rhotrices of size n over a field F
$(GR_n(F),\circ)$	-The non-commutative general linear rhotrix group of size n over a field F
$\langle x^{ni} \rangle$	- A rhotrix row vector of size n
$\langle x^{nj} \rangle$	-A rhotrix column vector of size n
$M_n(F)$	-The set of all $n \times n$ matrices over a field F
$GL_n(F)$	-The set of all invertible $n \times n$ matrices over a field F
$(GL_n(F),\cdot)$	-The general linear group of size n over a field F
$FGR_n(Z_p)$	- The set of all invertible rhotrices of size n over a finite field Z_p
$(FGR_n(Z_p),\circ)$	-The non-commutative general rhotrix group of size n over a finite field
$\lvert (FGR_n(Z_p),\circ) \rvert$	-The order of a finite group $(FGR_n(Z_n),\circ)$

Rn	-A particular rhotrix ‘ Rn ’
$\circ(Rn)$	-The order of a particular rhotrix ‘ Rn ’
Z_p	-The field of integers modulo p, (p is a positive prime number)
$2Z^+ +1$	-The set of all positive odd integers

CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

This chapter presents the general introduction of the dissertation, showing the background of the research, aim and objectives of the study, methodology for carrying out the research, definition of termsand outline of the dissertation.

1.2 Background of Research

Mathematics is significant and indispensable for any individual regardless of culture or age. Mathematics provides a powerful universal language and intellectual toolkit for abstraction, generalization and synthesis. It is the language of science and technology which enables us to probe the natural universe and develop new technologies that have helped us control and master our environment, change societal expectations and standards of living. Mathematical skills are highly valued and sought after. Mathematical training disciplines the mind, develops logical and critical reasoning, and develops analytical and problemsolving skills to a high degree. Smith (2004)

Rhotrix theory is a relatively new area of mathematical discipline which deals with algebra and analysis of array of numbers in mathematical rhomboid form. It beganwith the work of Ajibade (2003), wherethe concept, algebra, and analysis of rhotrices which was motivated by the ideas on matrix-tersions and matrix-noitrets proposed by Atanassov and Shannon (1998). Ajibade (2003)gave the initial definition of rhotrix of size 3as a mathematical array that is in some way, between two-dimensional vectors and 2×2 dimensional matrices. Since then, many authors (Mohammed (2009), Tudunkaya *etal* (2010b), Aminu (2010c), Mohammed *etal* (2011), Mohammed *etal* (2012), etc.) have shown interest in the usage of rhotrix set, as an underlying set, for construction of algebraic structures.

Sani (2004) proposed an alternative method for multiplication of rhotrices of size three, based on their rows and columns, as comparable to matrix multiplication. This was in an attempt to answer the question of 'how can we transform a rhotrix to a matrix and vice versa'posed by Ajibade (2003) in the concluding remarks of his article. This method of rhotrix multiplication is now referred to as 'row-column-based method for rhotrix multiplication'. Unlike Ajibade's method of multiplication, that is both commutative and associative, Sani's method of rhotrix multiplication is non-commutative but associative.The alternative method for multiplication of base rhotrices proposed by Sani (2004) was later generalized to include rhotrices ofsize *n* in Sani (2007).

It was also shown in Sani (2007) that there exists an isomorphic relationship between the group of all invertible rhotrices of size *n*and the group of all invertible $w\times w$dimensional matrices, where $w=\frac{n+1}{2}$and $n\in 2Z^{+}+1$.

Thus, two methods for multiplication of rhotrices are presently available in the literature of rhotrix theory. In this work, we shall refer to the method for multiplication of rhotrices defined in Ajibade (2003) as "commutative method for rhotrix multiplication" and the alternative method for multiplication of rhotrices defined in Sani (2004, 2007) as "non-commutative method for rhotrix multiplication".

Mohammed (2007a) adopted the commutative method for rhotrix multiplication to propose classification of rhotrices and their expression as algebraic structures of groups, semigroups, monoids, rings and Boolean algebras.

Based on non-commutative method for rhotrix multiplication, the search for a transformation of a rhotrix to a matrix and vice-versa was completely achieved in Sani (2008), where he proposed a method of converting rhotrix to a special form of matrix called '*coupled matrix'*. This coupled matrix was used to solve two different systems of linear equations simultaneously, where one is an $n \times n$ system while the other one is an $(n-1)\times(n-1)$ system. Following this idea, Sani (2009) presented the solution of two coupled matrices by extending the idea of a coupled matrix presented in his earlier work to a general case involving $(m \times n)$ and $(m-1)\times(n-1)$ matrices.

It is noteworthy to mention that any research work by interested author(s) in the literature of rhotrix theory is based on either commutative method or non-commutative method for rhotrix multiplication. In the presentation of thisstudy on the non-commutative general linear rhotrix group, it is obvious that we adopt the non-commutative multiplication technique proposed in Sani (2004, 2007). The reason behind the choice is that an algebraically non-commutative group offers an exciting platform for carrying out mathematical research in group theory and number theory. Above all, this area is still fertile and beckons on researchers to explore the rich properties analogous to matrices in the study of rhotrices.

One of the well-known areas of Mathematics is Group theory. In mathematics a group is defined as a non empty set having a binary operation defined on it and satisfying four axioms,as follows: closure, associativity, existence of an identity element and existence of inverses.Lloyd and Frank (2004). A rhotix group is a group having rhotrix set as the underlying set.

Group theory has been well developed by researchers before the twentieth century. General linear group has played a vital role in the study of group representation theory. Many concepts in rhotrix theory were analogous to matrix theory, but the concept of rhotrices has two different multiplication methods, providing two parallel study areas for research. This makes Rhotrix Theory a well-deserved area of research.

This research work is dealing with the study of generalized rhotrix groups. We consider a rhotrix set, $R_n(F)$ of size n over a field F, together with the binary operation ofnon-commutative method for rhotrix multiplication proposed by Sani (2007),in order to construct certain algebraic systems, which we term as '*Rhotrix Groups*'. We identify certain subgroups of these groups using Lagrange's theorem as a check. Furthermore, we identified certain finite groups of rhotrices of size 3. We also showed isomorphic relationship between certain groups and certain subgroups within our constructions.

1.3Aim and Objectives of the Study

The aim of this dissertation is to present an algebraic study of the development of non-commutative general linear rhotrix group. In particular, the following are the research objectives:

(i) To develop the basic fundamentals necessary for the algebraic study of the concept of 'non-commutative general rhotrix group' as a new paradigm of science.

(ii) To identify and study the properties of General Linear Rhotrix Group as analogous to the well-known General Linear Group in the literature.

(iii) To dissect the General Rhotrix Group in order to uncover its subgroups.

(iv) To establish the embedment of a particular subgroup of General Linear Rhotrix Group in a particular subgroup of General Linear Group.

(v) To construct some finite non-commutative groups of rhotrices and identify their subgroups.

1.4 Justification of Research

Since its inception, so much has been done on the algebra of rhotrices but, from existing literature available to us, nothing has been done to advance group theory by using rhotrices as an underlying set for construction of groups. We intend to fill this gap by presenting an algebraic study of non-commutative general linear rhotrix group.We shall also consider finite non-commutative rhotrix groups with entries taken from the set of integers modulo p (Z_p).

1.5Research Methodology

The method adopted in this dissertation is by consulting all necessary and relevant papers in the literature on fundamentals of Rhotrix theory, Matrix theory and Group theory in order to obtain background information for developing the theory of non-commutative groups of rhotrices.

In line with the work of Mohammed and Balarabe (2014) on the review of developments in rhotrix theory, an algebraic study of non-commutative general rhotrix group will be presented as an extension to the development of non-commutative rhotrix theory. This will be achieved through our adoption of row-column-based method for rhotrix multiplication defined in Sani (2004, 2007) as the group binary operation.

Next, a dissection of non-commutative general rhotrix group would be made, so as to uncover its subgroups and then establish an isomorphic relationshipbetween them. Furthermore, an introduction of the study of finite groups of rhotrices with entries from the field of integer modulo p, Z_p will be considered.

1.6 Scope and Limitation

This study will be limited to the Algebraic properties of non-commutative general linear rhotrix group. The Analysis, Topology and Representation of rhotrix group will not be considered.

1.7 DEFINITION OF TERMS

The following definitions will be useful in carrying out the work:

Definition 1.7.1 Matrix set

A matrix set $M_{m\times n}(C)$ is a collection of rectangular arrays, called $m\times n$ dimensional matrices with entries from the field of complex numbers. Thus,

$$M_{m\times n}(C)=\left\{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & \dots & \dots & \dots & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & \dots & \dots & \dots & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & a_{mn} \end{bmatrix} : a_{11}, a_{12}, \dots, a_{mn}, \dots \in C\right\} \tag{1.1}$$

Definition 1.7.2 Matrix-tertion and matrix-noitret

Matrix-tertion and Matrix-noitret can be defined as mathematical arrays that are in some way between 2-dimensional vectors and 2×2 -dimensional matrices introduced by Atanassov and Shannon in (1998). Matrix-tertion and Matrix-noitret are denoted by T and N respectively and defined as

$$T=\left\{\begin{array}{ccc} a & & b \\ & \diagdown\ \diagup & \\ & c & \end{array} : a, b, c \in C\right\} \tag{1.2}$$

and

$$N = \left\{ \begin{array}{ccc} & a & \\ b & & c \end{array} : a, b, c \in C \right\} \qquad (1.3)$$

Definition 1.7.3 Rhotrix

A rhotrix R of size n is a rhomboidal array with entries from a field F which can be expressed as a couple of two square matrices A and C of sizes $(t \times t)$ and $(t-1)\times(t-1)$, where $t = \frac{n+1}{2}$ and $n \in 2Z^{+} + 1$. This can be represented according to Sani (2008) as:

$$R_n = \langle A_{t\times t}, C_{(t-1)(t-1)} \rangle = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \dots & \dots\dots & \dots & \dots & \dots & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & a_{1t} \\ & \dots & \dots & \dots & \dots & \dots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle = \left\langle \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1(t-1)} & a_{1t} \\ a_{21} & a_{22} & \dots & a_{2(t-1)} & a_{2t} \\ \dots & \dots & \dots & \dots & \dots \\ a_{(t-1)1} & a_{(t-1)2} & \dots & a_{(t-1)(t-1)} & a_{(t-1)t} \\ a_{t1} & a_{t2} & \dots & a_{t(t-1)} & a_{tt} \end{bmatrix}, \begin{bmatrix} c_{11} & \dots & c_{1(t-1)} \\ \dots & \dots & \dots \\ c_{(t-1)1} & \dots & c_{(t-1)(t-1)} \end{bmatrix} \right\rangle \qquad (1.4)$$

Examples: a rhotix of size 5 can be written as:

$$R_5 = \left\langle \begin{array}{ccccc} & & a & & \\ & b & c & d & \\ e & f & g & h & j \\ & k & l & m & \\ & & n & & \end{array} \right\rangle = \left\langle \begin{bmatrix} a & d & j \\ b & g & m \\ e & k & n \end{bmatrix}, \begin{bmatrix} c & h \\ f & l \end{bmatrix} \right\rangle \qquad (1.5)$$

Also, a rhotrix of size 7 can be writtenas

$$R_7 = \left\langle \begin{array}{ccccccc} & & & a & & & \\ & & b & c & d & & \\ & e & f & g & h & j & \\ k & l & m & p & q & r & w \\ & s & t & u & v & x & \\ & & y & \varepsilon & \sigma & & \\ & & & n & & & \end{array} \right\rangle = \left\langle \begin{bmatrix} a & d & j & w \\ b & g & q & x \\ e & m & u & \sigma \\ k & s & y & n \end{bmatrix}, \begin{bmatrix} c & h & r \\ f & p & v \\ l & t & \varepsilon \end{bmatrix} \right\rangle \qquad (1.6)$$

Definition 1.7.4 A set of all rhotrices of size *n*

The set of all rhotrices of size *n* with entries from a field F is a collection of all rhotrices of size *n*, defined as:

$$R_n(F)=\left\{\left\langle\begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle : a_{ij}\in F, c_{lk}\in F\right\}, \qquad (1.7)$$

$$where\, 1\le i, j\le t,\quad 1\le l,k\le t-1;\ t=\frac{n+1}{2}\ and\ n\in 2Z^{+}+1$$

Definition 1.7.5 Determinant of a rhotrix of size *n*

$$If\ R_n=\langle a_{ij},c_{lk}\rangle=\left\langle\begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array}\right\rangle,$$

then the determinant of R_n written as $\det(R_n)=\det(A_t)\det(C_{t-1})$ where A_t and C_{t-1} are the two square matrices of dimension $(t\times t)$ and $(t-1)\times(t-1)$ respectively which make up the rhotrix R_n with $t=\dfrac{n+1}{2}\, and\ n\in 2Z^{+}+1.$

Definition 1.7.6 The inverse of a rhotrix of size n

The inverse of a rhotrix $R_n = \langle a_{ij}, c_{lk} \rangle$, is the rhotrix $R_n^{-1} = \langle q_{ij}, r_{lk} \rangle$ such that

$R_n \circ R_n^{-1} = \langle a_{ij}, c_{lk} \rangle \circ \langle q_{ij}, r_{lk} \rangle = \langle I_{ij}, I_{lk} \rangle$ where $\left[q_{ij} \right]_{t\times t}$ and $\left[r_{ij} \right]_{t-1\times t-1}$ are the inverses of the two square matrices $\left[a_{ij} \right]_{t\times t}$ and $\left[c_{ij} \right]_{t-1\times t-1}$ respectively,which make up the rhotrix R_n with $t = \frac{n+1}{2}$ *and* $n \in 2Z^{+} + 1$.

Remark: A rhotrix R_n is said to be invertible or non- singular if the determinant is non-zero. That is R_n is invertible *iff* $\det(R_n) \neq 0$.

Definition 1.7.7 A set of all invertible rhotrices of size n

A collection of all invertible rhotrices of size $n$with entries from an arbitrary field F, denoted by $GR_n(F)$ is called a set of all invertible rhotrices of size*n*.

Definition 1.7.8Operations of addition and multiplication on rhotrices of size n

Let A_n and B_n be two rhotrices of size *n*. The operation of addition of A_n and B_n is defined according to Mohammed (2011) as:

$$A_n + B_n = \left\langle \begin{array}{ccccccc} & & & p_{11} & & & \\ & & p_{21} & q_{11} & p_{12} & & \\ & & \cdots & \cdots & \cdots & & \\ p_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & p_{1t} \\ & & \cdots & \cdots & \cdots & & \\ & & p_{t(t-1)} & q_{(t-1)(t-1)} & p_{(t-1)t} & & \\ & & & p_{tt} & & & \end{array} \right\rangle + \left\langle \begin{array}{ccccccc} & & & u_{11} & & & \\ & & u_{21} & v_{11} & u_{12} & & \\ & & \cdots & \cdots & \cdots & & \\ u_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & u_{1t} \\ & & \cdots & \cdots & \cdots & & \\ & & u_{t(t-1)} & v_{(t-1)(t-1)} & u_{(t-1)t} & & \\ & & & u_{tt} & & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccccc} & & & p_{11}+u_{11} & & & \\ & & p_{21}+u_{21} & q_{11}+v_{11} & p_{12}+u_{12} & & \\ & & \cdots & \cdots & \cdots & & \\ p_{t1}+u_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & p_{1t}+u_{1t} \\ & & \cdots & \cdots & \cdots & & \\ & & p_{t(t-1)}+u_{t(t-1)} & p_{(t-1)(t-1)}+v_{(t-1)(t-1)} & p_{(t-1)t}+u_{(t-1)t} & & \\ & & & p_{tt}+u_{tt} & & & \end{array} \right\rangle \qquad (1.8)$$

and the operation of multiplication of A_n and B_n is defined according to Sani (2008) as

$$\begin{aligned} A_n \circ B_n &= \left\langle p_{i_1 j_1}, q_{l_1 k_1} \right\rangle \circ \left\langle u_{i_2 j_2}, v_{l_2 k_2} \right\rangle \\ &= \left\langle \sum_{i_2 j_1 = 1}^{t} (p_{i_1 j_1} \cdot u_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (q_{l_1 k_1} \cdot v_{l_2 k_2}) \right\rangle \end{aligned} \qquad (1.9)$$

Definition 1.7.9 Group

A group $(GR_n(F), \cdot)$ is a non-empty set $GR_n(F)$ together with a binary operation $(\cdot)$ satisfying the following properties:

$(GR_n(F), \cdot)1$: Closure: for all $x, y \in (GR_n(F), \cdot)$, $x \cdot y \in (GR_n(F), \cdot)$.

$(GR_n(F), \cdot)2$: Associativity: for all $x, y, z \in (GR_n(F), \cdot)$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

$(GR_n(F),\cdot)3$: Existence of identity: $\exists\ e \in (GR_n(F),\cdot) : x \cdot e = e \cdot x\ \forall\ x \in (GR_n(F),\cdot)$

$(GR_n(F),\cdot)4$:Existence of inverse: each element of $(GR_n(F),\cdot)$ possesses an inverse. That is, $\exists\ x^{-1} \in (GR_n(F),\cdot) : x \cdot x^{-1} = x^{-1} \cdot x = e$, $\forall\ x \in (GR_n(F),\cdot)$

Definition 1.7.10 Subgroup

A non-empty subset $(H_n(F),\cdot)$ of a group $(GR_n(F),\cdot)$ is called a subgroup of $(GR_n(F),\cdot)$ if $(H_n(F),\cdot)$ is a group under the binary operation $'\cdot'$ defined on $(GR_n(F),\cdot)$.

Definition 1.7.11 Rhotrix group

A rhotrix group is a group having rhotrix set as an underlying set.

Definition 1.7.12 Rhotrix subgroup

A non-empty subset $H_n(F)$ of a rhotrix group $(GR_n(F),\cdot)$ is said to be a rhotrix subgroup of $(GR_n(F),\cdot)$ if the composition in $(GR_n(F),\cdot)$ is also a composition in $(H_n(F),\cdot)$ and for this composition, $(H_n(F),\cdot)$ is itself a rhotrix group.

1.8 OUTLINE OF THE DISSERTATION

The outline of the dissertationis as follows:

Chapter one presents the general introduction of the dissertation, the aim and objectives of the study, the methodology for carrying out the research and definition of terms.

Chapter two focuses on a review of developments in the literature of rhotrix theory, starting from the year 2003, when the concept of rhotrix was introduced up to the end of 2014.

Chapterthree considers the rhotrix set $GR_n(F)$ consisting of all invertible rhotrices of size*n*with entries from a given field F and together with row-column-based method for rhotrix multiplication, in order toinitiate the concept of *non-commutative general linear rhotrix group*. We identify and characterize its subgroups. Furthermore, we investigate some relationships between the non-commutative general linear rhotrix group and the well-known general linear group.

Chapter four introduces concrete constructions of finite non-commutative groups of base rhotrices. Ideas in the study of permutations and number theory are employed to ascertain the number of elements in each group, bearing in mind the axiomatic requirements for a rhotrix group.

Chapter five gives the summary, conclusion and recommendations for future researches.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

This chapter undertakes a review of existing literature in the theory of rhotrices starting from inception, 2003 up to the time when this dissertation was written. It considers the work of various researchers in the development of the theory of rhotrices. It reviews journal articles in both commutative and non-commutative rhotrix theories. It also considers the general linear group (group of matrices) as it will be seen later in chapter three to be analogues to the non-commutative general rhotrix group.

2.1 Rhotrix Theory

Rhotrix theory was initiated by Ajibade (2003) who defined rhotrix as a rhomboidal form of representing array of numbers. The concept is an extension of ideas on matrix-tension and matrix-noitret proposed by Atanassov and Shannon (1998). Ajibade (2003) presented the initial concept, analysis and algebra of rhotrices, where he defined an operation of multiplication of rhotrices of size three. This operation of multiplication is known as *heart-based multiplication* and it satisfies the commutative property of binary operators.

Sani (2004) proposed an alternative method for multiplication of rhotrices of size three and later generalized the idea to rhotrices of size n in Sani (2007). This alternative method for rhotrix multiplication is known as *row-column-based method for rhotrix multiplication* and it is known to be non-commutative but associative.

Therefore, in the literature of rhotrix theory, two methods for multiplication of rhotrices having the same size are currently available. We have the *heart-based method for rhotrix multiplication* given by Ajibade (2003). This was followed by the *row-column-based method for rhotrix multiplication* proposed in Sani (2004), in an attempt to answer the question posed in Ajibade (2003) in the concluding section of his article. However, each of the two methods provides enabling environment to explore the usefulness of rhotrices as tools for carrying out mathematical research.

This chapter presents a comprehensive literature review of related articles in rhotrix theory and also gives associated literature on matrix theory. To achieve this, a classification of all the articles in the literature of rhotrix theory into two classes in line with the review of rhotrix theory carried out by Mohammed and Balarabe (2014) will be adopted. In their work, one class of the articles in the literature of rhotrix theory was termed as *commutative rhotrix theory,*while the other class was termed as *non-commutative rhotrix theory.* The reason behind their classification was due to the fact that contributory author(s) / researcher(s) in a single article, either adopted Ajibade (2003) heart-based method for multiplication of rhotrices or Sani (2004, 2007) row-column-based method for multiplication of rhotrices. Therefore, articles in literature adopting the heart-based method for rhotrix multiplication belong to *the class ofcommutative rhotrix theory,* while those articles in the literature adopting Sani's row-column-based method for rhotrix multiplication belong to *theclass of non-commutative rhotrix theory*.

2.2 Commutative Rhotrix Theory

Ajibade (2003) introduced the initial concept, algebra, and analysis of rhotrix of size *3* as an extension of ideas of matrix-tersion and matrix-noitret proposed by Atanassov and Shannon (1998). A set of all rhotrices of size *3* was defined by Ajibade (2003) as follows with modification:

$$R_3(\Re) = \left\{ \left\langle \begin{array}{ccc} & \mathrm{a} & \\ \mathrm{b} & \mathrm{c} & d \\ & e & \end{array} \right\rangle : a,b,c,d,e \in \Re \right\}, \qquad (2.1)$$

where $c = h(R)$ is called the heart of any rhotrix $R \in R_3(\Re)$. Extension of size of $R_3(\Re)$ from *3* to *n* was also considered possible. Thus, for a rhotrix of size *n* denoted by $R(n)$ or R_n, we mean a rhomboidal array having $\frac{1}{2}(n^2+1)$ entries and of size $n \in 2Z^+ + 1$.

The operation of addition ($+$), scalar multiplication (m) and multiplication $(\circ)$ were also defined in Ajibade (2003) and is recorded as below:

Let $R = \left\langle \begin{array}{ccc} & \mathrm{a} & \\ \mathrm{b} & \mathrm{h(R)} & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & \mathrm{f} & \\ \mathrm{g} & \mathrm{h(Q)} & i \\ & j & \end{array} \right\rangle$ be any two rhotrices of size three and *m* a scalar, then

$$R+Q = \left\langle \begin{array}{ccc} & \mathrm{a} & \\ \mathrm{b} & \mathrm{h(R)} & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & \mathrm{f} & \\ \mathrm{g} & \mathrm{h(Q)} & i \\ & j & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \mathrm{a+f} & \\ \mathrm{b+g} & \mathrm{h(R)+h(Q)} & d+\mathrm{i} \\ & e+\mathrm{j} & \end{array} \right\rangle, \qquad (2.2)$$

$$mR = m\left\langle \begin{array}{ccc} & \mathrm{a} & \\ \mathrm{b} & \mathrm{h(R)} & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & \mathrm{ma} & \\ \mathrm{mb} & \mathrm{mh(R)} & md \\ & me & \end{array} \right\rangle, \qquad (2.3)$$

and

$$R \circ Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & i \\ & j & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+ih(R) \\ & eh(Q)+jh(R) & \end{array} \right\rangle. \quad (2.4)$$

Remark 2.2.1

The above operation of multiplication is commutative and $(R_3(\Re),+)$ is an Abelian group that is a commutative group with zero rhotrix:

$$0 = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle. \quad (2.5)$$

Ajibade (2003) also determined the identity and inverse of the rhotrix $R \in R_3(\Re)$ as

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle \quad (2.6)$$

and

$$R^{-1} = \frac{-1}{(h(R))^2} \left\langle \begin{array}{ccc} & a & \\ b & -h(R) & d \\ & e & \end{array} \right\rangle \quad (2.7)$$

respectively, where $h(R) \neq 0$.

Ajibade (2003) also established certain relationships between a rhotrix and its heart recorded as follows:

Theorem 2.2.2

A rhotrix R is invertible if and only if $h(R) \neq 0.$

Theorem 2.2.3

For any rhotrix $R \neq 0$, $R^2 = 0,$ if and only if, $h(R) = 0.$ where 0 is the zero rhotrix.

The set of rhotrices of size *3* and the operation of rhotrix multiplication defined in Ajibade(2003) were later extended to rhotrices of size *n* by Mohammed (2011) recorded as follows:

$$R_n(\Re) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & r_1 & & & & \\ & & & r_2 & r_3 & r_4 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ r_{\left\{\frac{\frac{1}{2}(n^2+1)+1}{2}\right\}-n\backslash 2} & \cdots & \cdots & \cdots & r_{\left\{\frac{\frac{1}{2}(n^2+1)+1}{2}\right\}} & \cdots & \cdots & \cdots & r_{\left\{\frac{\frac{1}{2}(n^2+1)+1}{2}\right\}+n\backslash 2} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & r_{\frac{1}{2}(n^2-5)} & r_{\frac{1}{2}(n^2-3)} & r_{\frac{1}{2}(n^2-1)} & & & \\ & & & & r_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle : r_1, \ldots, r_{\frac{1}{2}(n^2+1)} \in \Re \right\}, \qquad (2.8)$$

Where $h(R) = r_{\frac{\frac{1}{2}(n^2+1)+1}{2}}$ is the heart of the rhotrix, and $n \backslash 2$ is the integer value obtained on division of n by 2.

Thus,for $n = 3$, we get $R(3) = \left\langle \begin{array}{ccc} & r_1 & \\ r_2 & h(R) & r_4 \\ & r_5 & \end{array} \right\rangle$ and so on.

Furthermore, if A and $B$are any two rhotrices of the same size n, then $A\ o\ B$ is the resultant rhotrix C defined as:

$$C(n)=A(n)\circ B(n)=\left\langle \begin{array}{ccccccccccc} & & & & & a_1 & & & & & \\ & & & & a_2 & a_3 & a_4 & & & & \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ a_{\left\{\frac{w+1}{2}\right\}-n\backslash 2} & \cdots & \cdots & \cdots & & a_{\left\{\frac{w+1}{2}\right\}} & & \cdots & \cdots & \cdots & r_{\left\{\frac{w+1}{2}\right\}+n\backslash 2} \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ & & & & a_{w-3} & a_{w-2} & a_{w-1} & & & & \\ & & & & & a_w & & & & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccccccccc} & & & & & b_1 & & & & & \\ & & & & b_2 & b_3 & b_4 & & & & \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ b_{\left\{\frac{w+1}{2}\right\}-n\backslash 2} & \cdots & \cdots & \cdots & & b_{\left\{\frac{w+1}{2}\right\}} & & \cdots & \cdots & \cdots & b_{\left\{\frac{w+1}{2}\right\}+n\backslash 2} \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ & & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ & & & & b_{w-3} & b_{w-2} & b_{w-1} & & & & \\ & & & & & b_w & & & & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccccc} & & & a_1h(B)+b_1h(A) & & & \\ & & r_2h(B)+b_2h(A) & r_3h(B)+b_3h(A) & r_4h(B)+b_4h(A) & & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots & \\ a_{\left\{\frac{w+1}{2}\right\}-n\backslash 2}h(B)+b_{\left\{\frac{w+1}{2}\right\}-n\backslash 2}h(A) & \cdots\ \cdots & \cdots & h(A)h(B) & \cdots & \cdots\ \cdots & a_{\left\{\frac{w+1}{2}\right\}+n\backslash 2}h(B)+b_{\left\{\frac{w+1}{2}\right\}+n\backslash 2}h(A) \\ & \vdots\ \vdots & \vdots & \vdots & \vdots & \vdots\ \vdots & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & & a_{w-3}h(B)+b_{w-3}h(A) & a_{w-2}h(B)+a_{w-2}h(A) & a_{w-1}h(B)+b_{w-1}h(A) & & \\ & & & a_wh(B)+b_wh(A) & & & \end{array} \right\rangle \quad (2.9)$$

where $n \in 2Z^{+}+1$, $w=\frac{1}{2}(n^2+1)$, $h(A)=a_{\left\{\frac{t+1}{2}\right\}}$, $h(B)=b_{\left\{\frac{w+1}{2}\right\}}$ and $n\backslash 2$ is the integer value obtained on division of n by 2.

Mohammed (2007a) adopted the rhotrix operations defined in Ajibade (2003) for presentation of various imaginations of rhomboidal arrays forming algebraic structures such as groups, semigroups, monoids and rings using rhotrix set as underlying set. Also, in his work, various special types of rhotrices such as symmetric rhotrix, diagonal rhotrix, lower and upper triangular rhotrix, zero heart rhotrix, unit heart rhotrix, odd and even heart rhotrix, nonzero heart rhotrix, odd and even rhotrix, and hearty rhotrix were presented.

The theorem on rhotrix exponent rule was first given without proof in Mohammed (2007a),further, Mohammed (2007b) established and characterized the theorem on rhotrix exponent rule and extended the result to systematization of expressing special series and polynomial equations over rhotrices.

We record the following theorem from Mohammed (2007b)

Theorem 2.2.4

Let $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ be any rhotrix of size *3*, then for any integer value *m*,

$R^m = (h(R))^{m-1} \left\langle \begin{array}{ccc} & ma & \\ mb & h(R) & md \\ & me & \end{array} \right\rangle$. In particular, R^0 and R^{-1} are the identity and inverse of R respectively, provided *h(R)* is non-zero.

Remark 2.2.5

For the general case, $R(n), \ni \ n \in 2Z^+ + 1$, it was noted in Mohammed (2011) that

$$R^m(n) = (h(R))^{m-1} \left\langle \begin{array}{ccccccccccc} & & & & & mr_1 & & & & & \\ & & & & mr_2 & mr_3 & mr_4 & & & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ mr_{\left\{\frac{w+1}{2}\right\}-n\backslash 2} & & \cdots & \cdots & \cdots & h(R) & \cdots & \cdots & \cdots & & mr_{\left\{\frac{w+1}{2}\right\}+n\backslash 2} \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & & mr_{r_{w-3}} & mr_{r_{w-2}} & mr_{r_{w-1}} & & & & \\ & & & & & mr_{r_w} & & & & & \end{array} \right\rangle, \qquad (2.10)$$

where $n \in 2Z^+ + 1$, $w = \frac{1}{2}(n^2+1)$, $h(R) = r_{\left\{\frac{w+1}{2}\right\}}$.

Mohammed (2009) presented a remark on classification of rhotrices as abstract structures. In the work, rhotrix ring, rhotrix integral domain and rhotrix field were constructed with certain conditions.

Aminu (2009) initiated the study of linear systems over rhotrices. In the work, he investigated the system of linear equations arising from the rhotrix equation $A \circ B = C$, where *A, B and C* are rhotrices of the same size.

We record the following proposition in Aminu (2009), establishing a necessary and sufficient condition for the solvability of the system of linear equations arising from the rhotrix equation

$A \circ B = C$.

Proposition 2.2.6

Let A, B and C be three rhotrices of the same size with entries in $\Re$, then the system of linear equations resulting from $A \circ B = C$ has

(i) a unique solution, if and only if, $h(A) \neq 0$ and $h(C) \neq 0$

(ii) an infinite solution, if and only if, $h(A) = h(C) = 0$

(iii)no solution, if and only if, $h(A) = 0$ and $h(C) \neq 0$

In line with the work of Aminu (2009) on linear system of equations arising from rhotrix equation $A \circ B = C$, where one of the equations was treated and a number of solvability conditions were suggested, Aminu (2012a) extended the problem to the case when all the systems were considered to be solved simultaneously.

Usaini and Tudunkaya (2012a) extended the work in Mohammed (2009) to construct certain field of fractions over rhotrices. The construction was done step by step, where at each step, a particular algebraic property was shown. But it was later discovered by Usaini and Tudunkaya (2012b) that the rhotrix field proposed inMohammed (2009) can only be possible if the underlying rhotrix set is a set of all hearty rhotrices of the same size defined in Mohammed (2007a).

As an extension to Mohammed (2011), where the generalization of heart-based method for multiplication of rhotrices of size n was presented, Absalom *et al.*(2011a) presented an algorithmic implementation for the generalized heart-based method for multiplication of rhotrices of size n. Thereafter, Absalom *et al.*(2011b) proposed a simplified version of the rhotrix expression generalization proposed by Mohammed (2011) for heart-based rhotrices of size *n*.

Mohammed and Sani (2011) extended the concept of graph theory to rhotrix theory through their introduction of rhomtrees of order $w = \frac{1}{2}(n^2 + 1)$ as a graphical representation of rhotrices of size *n*, where $n \in 2Z^+ + 1$. The rhomtrees were shown in their work to have relationship with certain real world problems such as topology of computing network, methane compound and certain product of sets.

Mohammed and Tijjani (2011) defined metric or distance function from a rhotrix set to the set of all real numbers. They extended their work to construction of metric topological spaces over rhotrices.

Tudunkaya and Makanjuola (2010) presented a method of constructing finite fields over rhotrices. The cardinality of these finite fields was also given through concrete examples.

Tudunkaya and Makanjuola (2012) proposed certain quadratic extension, as an extension to the work of Mohammed (2007b), where a note on rhotrix exponent rule and its applications to special series and polynomial equations defined over rhotrices was presented. Thereafter, rhotrix polynomial and polynomial rhotrices were proposed by Tudunkaya (2013) as another further extension to the work of Mohammed (2007b).

Usaini and Tudunkaya (2011), presented properties of certain finite fields constructed over rhotrices, an extension to the work in Tudunkaya and Makanjuola (2010a).

Mohammed and Tella (2012) presented rhotrix sets and rhotrix spaces, categorized over real and complex fields. They further extended their idea to stimulate systematization of characterizing rhotrices over the fields of numbers and their expression as rhotrix set spaces.

Aminu and Michael(2014) introduced a structure called *paraletrix* which is a generalization of rhotrix whose number of rows and columns are not necessarily the same, and discussed a number of its properties based on the heart-based and row-column-basedmultiplications. In their work, a paraletrix was defined as below:

$$P_{m\times n} = \langle a_{ij}, c_{lk} \rangle$$

$$= \left\langle \begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & c_{11} & a_{12} & & & \\
 & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
a_{\frac{m+1}{2}1} & c_{\frac{m-1}{2}1} & \cdots & \cdots & \cdots & \cdots & \cdots & c_{1\frac{n-1}{2}} & a_{1\frac{n+1}{2}} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & & a_{\frac{m+1}{2}\frac{n-3}{2}} & c_{\frac{m-1}{2}\frac{m-3}{2}} & a_{\frac{m-1}{2}\frac{n-1}{2}} & c_{\frac{m-1}{2}\frac{m-1}{2}} & a_{\frac{m-3}{2}\frac{n+1}{2}} & & \\
 & & & a_{\frac{m+1}{2}\frac{n-1}{2}} & c_{\frac{m-1}{2}\frac{m-1}{2}} & a_{\frac{m-1}{2}\frac{n+1}{2}} & & & \\
 & & & & a_{\frac{m+1}{2}\frac{n+1}{2}} & & & &
\end{array} \right\rangle \qquad (2.11)$$

where $a_{ij}, c_{lk} \in \Re \ for \ i = 1, 2, \ldots \frac{m+1}{2}; \ j = 1, 2, \ldots \frac{n+1}{2} \ and \ l = 1, 2, \ldots \frac{m-1}{2}; k = 1, 2, \ldots \frac{n-1}{2}$

For example, $P_{3\times5} = \left\langle \begin{array}{ccccc} & a_{11} & & & \\ a_{21} & c_{11} & a_{12} & & \\ & a_{22} & c_{12} & a_{13} & \\ & & a_{23} & & \end{array} \right\rangle$

is a paraletrix with three rows and five columns.

The work concluded that if the number of rows and columns of the paraletrix are equal, then the resultingstructure is the well-known rhotrix.

Mohammed (2014) presented a new technique for expressing rhotrices in a generalized form using multiple array indexes.It was proposed in the work that if $\hat{C}(n)$ is the set of all rhotrices of size *n* over a field *F*(real or complex), since any rhotrix $B(n) \in \hat{C}(n)$ has total number of entries as $\frac{1}{2}(n^2+1)$ where $n \in 2Z^+ + 1$, then a technique for expressing the set of all rhotrices of the same size n in a generalized form is given by:

$$\hat{C}(n) = \left\{ \left\langle \begin{array}{cccccccccc} & & & & c_{1,\frac{n+1}{2}} & & & & & \\ & & & c_{2,\frac{n-1}{2}} & c_{2,\frac{n+1}{2}} & c_{1,\frac{n+3}{2}} & & & & \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ c_{\frac{n+1}{2},1} & c_{\frac{n+1}{2},2} & \cdots & \cdots & c_{\frac{n+1}{2}}, c_{\frac{n+1}{2}} & \cdots & \cdots & c_{\frac{n+1}{2},n-1} & c_{\frac{n+1}{2},n} \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \\ & & \vdots & \vdots & \vdots & \vdots & \vdots & & & \\ & & & c_{n-1,\frac{n-1}{2}} & c_{n-1,\frac{n+1}{2}} & c_{n-1,\frac{n+3}{2}} & & & & \\ & & & & c_{n,\frac{n+1}{2}} & & & & & \end{array} \right\rangle : c_{i,j} \in F \right\} \quad (2.12)$$

where $c_{i,j}$ are entries from a field of real or complex numbers F. It is clear that any entry $b_{i,j}$ in $B(n)$ must beindicated by i^{th} row and j^{th} column. This method of expressing rhotrices in a general form is analogous tothat of matrices because any entry $c_{i,j}$ within a rhotrix is indicated out by its row i and column j, which isanalogous to any entry $x_{i,j}$ in a matrix set X of all $m \times n$ dimensional matrices, denoted by

$$\hat{X}(n) = \left\{ \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix} : x_{11}, x_{12}, \cdots, x_{mn} \in F \right\} \qquad (2.13)$$

This method promises exactness, efficiency and convenience in presentation of research results in rhotrix theory especially when the operations over the new expression are algorithmatized for computingmachines.

2.3Non-commutative Rhotrix Theory

The non-commutative rhotrix theory was revisited by Mohammed and Balarabe (2014) as a collection of articles using row-column-based method of rhotrix multiplication. A review of articles on non-commutative rhotrix theory is presented as follows:

Sani (2004) extended the concept of row-column multiplication of two dimensional matrices to propose an alternative method for multiplication of rhotrices of size three. This was in an attempt to answer the question of ‘how can we transform a rhotrix to matrix and vice versa' askedin Ajibade (2003). In Sani (2004) multiplication of two rhotrices was defined as:

$$R \circ Q = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & f & \\ g & h(Q) & i \\ & j & \end{matrix} \right\rangle = \left\langle \begin{matrix} & af + dg & \\ bf + eg & h(R)h(Q) & ai + dj \\ & bi + ej & \end{matrix} \right\rangle \qquad (2.14)$$

This multiplication is non-commutative, but it is associative in property. The method of rhotrix multiplication was used to establish some relationship between rhotrices of size *3* and matrices of dimension *2*.

Sani (2004) further determined the identity, inverse, determinant and transpose of the rhotrix *R(3)* respectively as follows:

$$I = \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \qquad (2.15)$$

$$R^{-1} = \left\langle \begin{matrix} & \frac{e}{ae - bd} & \\ \frac{-b}{ae - bd} & \frac{1}{h(R)} & \frac{-d}{ae - bd} \\ & \frac{a}{ae - bd} & \end{matrix} \right\rangle, \qquad (2.16)$$

$$\det(R) = h(R)(ae - bd),$$

and

$$R^{T} = \left\langle \begin{matrix} & a & \\ d & h(R) & b \\ & e & \end{matrix} \right\rangle \qquad (2.17)$$

An extension of row-column method for rhotrix multiplication in Sani (2004) to rhotrices of size *n* was given in Sani(2007) as follows:

$$\begin{aligned} \mathrm{R}(n)\circ \mathrm{S}(n) &= \left\langle a_{ij}, c_{kl}\right\rangle \circ \left\langle b_{ij}, d_{kl}\right\rangle \\ &= \left\langle \sum_{i,j=1}^{t}(a_{i,j}\cdot b_{i,j}), \sum_{k,l=1}^{t-1}(c_{k,l}\cdot d_{k,l})\right\rangle, \\ & where\ \ t=\frac{1}{2}(n^2+1) \end{aligned} \tag{2.18}$$

Thus, $\mathrm{R}(n)$ and $\mathrm{S}(n)$ can be expressed as in Equation (2.19) and (2.20)respectively.

$$\mathrm{R}(n)=\left\langle a_{i,j}, c_{k,l}\right\rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{1,1} & & & & \\ & & & a_{2,1} & c_{1,1} & a_{1,2} & & & \\ & & a_{3,1} & c_{2,1} & a_{2,1} & c_{1,2} & a_{1,3} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1,t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t,t-2} & c_{t-1,t-2} & a_{t-1,t-1} & c_{t-2,t-1} & a_{t-2,t} & & \\ & & & a_{t.t-1} & c_{t-1,t-1} & a_{t-2,t} & & & \\ & & & & a_{t,t} & & & & \end{array} \right\rangle \tag{2.19}$$

and

$$S(n)=\left\langle b_{i,j}, d_{k,l}\right\rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{1,1} & & & & \\ & & & b_{2,1} & d_{1,1} & b_{1,2} & & & \\ & & b_{31} & d_{2,1} & b_{2,1} & d_{1,2} & b_{1,3} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{t,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_{1,t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & b_{t,t-2} & d_{t-1,t-2} & b_{t-1,t-1} & d_{t-2,t-1} & b_{t-2,t} & & \\ & & & b_{t.t-1} & d_{t-1,t-1} & b_{t-2,t} & & & \\ & & & & b_{t,t} & & & & \end{array} \right\rangle. \tag{2.20}$$

The elements $a_{i,j}(i, j = 1, 2, ..., t)$ and $c_{k,l}(k, l = 1, 2, \ldots, t-1)$ are called the major and minor entries of $R(n)$ respectively. Similarly, The elements $b_{i,j}(i, j = 1, 2, ..., t)$ and $d_{k,l}(k, l = 1, 2, ..., t-1)$ are the major and minor entries of $S(n)$ respectively.

Also Sani (2007), generalized the definition of the transpose, determinant, identity and inverse of rhotrix $R(n)$ of size *n*, (provided $R(n) \neq 0)$. Sani (2007) further established some interesting relationships between invertible *n*-size rhotrices and invertible $t \times t$ dimensional matrices, where $t = \frac{1}{2}(n+1),\ n \in 2Z^{+} + 1$.

Kaurangini and Sani (2007) presented the concept of Hilbert matrix and its relationship with a special rhotrix, where they constructed a special form of rhotrix, termed as *'Hilbert rhotrix'* of size *5* coupling two Hilbert matrices of dimensions 3×3 and 2×2.

The question of transforming rhotrix to matrix and vice-versa posed by Ajibade (2003) was completely resolved by Sani (2008), when he proposed a method of converting rhotrix to a special form of matrix called '*coupled matrix'*. This is done by rotating the rhotrix R of size $n \in 2Z^{+} + 1$ through 45^{0} in anti-clockwise direction. This is a special form of matrix with missing values. For example, the coupled matrix of rhotrix $R(5)$ is as follows:

$$R^{T/2}(n) = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle^{T/2} = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{12} & \\ a_{21} & & a_{22} & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix}, \qquad (2.21)$$

where $T/2$ indicates a rotation through 45° in anti-clockwise direction. The special matrix in (2.21) is a coupling of 3×3 matrix with a 2×2 matrix, hence, the name 'a coupled matrix'.

Therefore, in general; we have:

$$R^{T/2}(n) = \left\langle a_{ij}, c_{kl} \right\rangle^{T/2} = \left[a_{ij}, c_{kl} \right] = [Ac]_n . \qquad (2.22)$$

That rotation results into a coupled matrix, consisting of two matrices of dimensions $t\times t$ and $(t-1)\times(t-1)$.

Two coupled matrices $[Ac]_n$ and $[Bd]_n$ can be multiplied together by simply filling the missing spaces with zeros, after the multiplication, we remove the zeros in order to have the result in filled coupled matrix form.

We record the following result from Sani (2008):

Theorem 2.3.1

If a coupled matrix $[Ac]_n$ is completed with zeros, then its determinant is the product of the determinants of the matrices $A_{t\times t}$ and $c_{(t-1)\times(t-1)}$, where $t = \frac{1}{2}(n+1)$.

Remark 2.3.2

The determinant of a coupled matrix $[Ac]_n$ can be obtained in the same way as the determinant of a rhotrix of size *n*, that is, $\det[Ac]_n = \det[A]_{t\times t}\det[c]_{(t-1)\times(t-1)}$.

It is noteworthy. that the idea of a coupled matrix can be used to solve problems involving two different systems simultaneously, where one is a $t\times t$ system, $\mathbf{A}X = b$ while, the other is a $(t-1)\times(t-1)$ system, $cY = d$. The two systems $\mathbf{A}X = b$ and $cY = d$ can be coupled together as:

$$\begin{bmatrix} a_{1,1} & & a_{1,2} & & \cdots & \cdots & \cdots & & a_{1,t} \\ & c_{1,1} & & c_{1,2} & \cdots & \cdots & \cdots & c_{1,t-1} & \\ a_{2,1} & & a_{2,2} & & \cdots & \cdots & \cdots & & a_{2,t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{t-1,1} & & a_{t-1,2} & & \cdots & \cdots & \cdots & & a_{t-1,t} \\ & c_{t-1,1} & & c_{t-1,2} & \cdots & \cdots & \cdots & c_{t-1,t-1} & \\ a_{t,1} & & a_{t,2} & & \cdots & \cdots & \cdots & & a_{t,t} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ \cdots \\ \cdots \\ \cdots \\ x_{t-1} \\ y_{t-1} \\ x_t \end{bmatrix} = \begin{bmatrix} b_1 \\ d_1 \\ y_2 \\ \cdots \\ \cdots \\ \cdots \\ b_{t-1} \\ d_{t-1} \\ b_t \end{bmatrix} \qquad (2.23)$$

If we fill the missing spaces of Equation 2.12 with zeros, we get $n\times n$ matrix which can then be solved to get solution of the two systems simultaneously.

Sani (2009) presented the solution of two coupled matrices by extending the idea of a coupled matrix presented in his earlier work to a generalized case involving $m\times n$ and $(m-1)\times(n-1)$ matrices as follows:

Suppose $A=[a_{ij}]$ is $m\times n$ matrix and $b=[b_{lk}]$ is $(m-1)\times(n-1)$ matrix then these two matrices can be coupled together to form the matrix:

$$\begin{bmatrix} a_{11} & & a_{12} & & \cdots & \cdots & \cdots & & a_{1n} \\ & b_{11} & & b_{12} & \cdots & \cdots & \cdots & b_{1n-1} & \\ a_{21} & & a_{22} & & \cdots & \cdots & \cdots & & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m-11} & & a_{m-12} & & \cdots & \cdots & \cdots & & a_{m-1n} \\ & b_{m-11} & & b_{m-12} & \cdots & \cdots & \cdots & b_{m-1n-1} & \\ a_{m1} & & a_{m2} & & \cdots & \cdots & \cdots & & a_{mn} \end{bmatrix} = \left[a_{ij}, b_{kl} \right] = \left[Ab \right]_{(2m-1)\times(2n-1)} \tag{2.24}$$

Completing the above Equation (2.24) with zeros, we obtain a $(2m-1)\times(2n-1)$ dimensional matrix whose properties could be deduced from those of the two separate matrices. This concept is used to solve problems involving $m\times n$ and $(m-1)\times(n-1)$ matrices simultaneously.

The next theorem recorded from Sani (2009) shows that, the two systems above could be coupled together which will lead to a method of solving the two systems simultaneously.

Theorem 2.3.3

If $\mathbf{A}X = C$ is a $s\times t$ system of linear equations, whose solution is the vector X and $bY = d$ is an $(s-1)\times(t-1)$ system whose solution is the vector Y, then the following coupled system gives the solution of the two systems

$$\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \dots & \dots & \dots & 0 & a_{1t} \\ 0 & b_{11} & 0 & b_{12} & \dots & \dots & \dots & b_{1t-1} & 0 \\ a_{21} & 0 & a_{22} & 0 & \dots & \dots & \dots & 0 & a_{2t} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{s-11} & 0 & a_{s-12} & 0 & \dots & \dots & \dots & 0 & a_{s-1t} \\ 0 & b_{s-11} & 0 & b_{s-12} & \dots & \dots & \dots & b_{s-1t-1} & 0 \\ a_{s1} & 0 & a_{s2} & 0 & \dots & \dots & \dots & 0 & a_{st} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ \dots \\ \dots \\ \dots \\ x_{t-1} \\ y_{t-1} \\ x_t \end{bmatrix} = \begin{bmatrix} c_1 \\ d_1 \\ c_2 \\ \dots \\ \dots \\ \dots \\ c_{t-1} \\ d_{s-1} \\ c_s \end{bmatrix} \qquad (2.25)$$

Aminu (2010a) adopted the concept of rows and columns of rhotrix suggested inSani (2007) to define the concept of rhotrix row and column vectors of size *n*as

$$\left\langle \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & 0 & 0 & & & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ a_{t1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\ & & a_{tt-2} & 0 & 0 & 0 & 0 & & \\ & & & a_{tt-1} & 0 & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle \qquad (2.26)$$

and

$$
\left\langle
\begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & 0 & 0 & & & \\
 & & a_{31} & 0 & 0 & 0 & 0 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & 0 & 0 & 0 & 0 & 0 & & \\
 & & & 0 & 0 & 0 & & & \\
 & & & & 0 & & & &
\end{array}
\right\rangle \tag{2.27}
$$

respectively.

The difference between $t \times t$ dimensional matrix vectors and n-size rhotrix vectors is that there is a unique representation of any $t \times t$ dimensional matrix vector, while any n-size rhotrix vector can be represented in t different ways, where $t = \frac{1}{2}(n+1),\ n \in 2Z^{+} + 1.$

The representations of the main rhotrix column and row vectors are respectively

$$\left\langle x^{nj} \right\rangle \text{and} \left\langle x^{in} \right\rangle. \tag{2.28}$$

Similarly, other columns and rows which are minor as

$$\left\langle x^{n-1k} \right\rangle \text{and} \left\langle x^{\ln-1} \right\rangle, \tag{2.29}$$

where $i, j = 1, 2, \ldots, t$, $k, l = 1, 2, \ldots, t-1$ and $t = \frac{1}{2}(n+1)$.

Aminu (2010a) adopted the alternative method for multiplication of rhotrices by Sani (2004, 2007) to introduce the concept of rhotrix vector spaces. This is given as follows:

'a rhotrix vector space $\langle v \rangle$ *with entries from the set of all real numbers is a non-empty set of rhotrix vectors with two operations, addition and scalar multiplication which satisfy the axioms of vector space'*.

The properties of these vector spaces were discussed. Furthermore, it was shown that the set of all rhotrices of size-*n* forms a vector space.

The following theorem recorded from Aminu (2010a) presents properties of rhotrix vectors space.

Theorem 2.3.4

Let $A(n)$, $B(n)$ and $C(n)$ be *n*-dimensional rhotrix vectors with the same representation. If α and β are scalars, then:

(i) $A(n) + O = A(n)$

(ii) $0\,A(n) = O$

(iii) $A(n) + B(n) = B(n) + A(n)$

(iv) $(A(n) + B(n)) + C(n) = A(n) + (B(n)) + C(n))$

(v) $\alpha\,(A(n) + B(n)) = \alpha\,A(n) + \alpha\,B(n)$

(vi) $(\alpha + \beta)\,A(n) = \alpha\,A(n) + \beta\,A(n)$

(vii) $(\alpha\,\beta)\,A(n) = \alpha\,(\beta\,A(n))$

Note that *0* and O denotes the usual zero and zero rhotrix respectively. *O* is the identity element under addition, and for convenience, we use O to denote any rhotrix vector having every entry as 0.

Aminu (2010b) further constructed and presented one-sided system identity of the form

$$R_n(X) = b, \tag{2.30}$$

where R_n is a*n*-dimensional rhotrix, X the unknown *n*-dimensional rhotrix vector and b the right hand side rhotrix vector. The necessary and sufficient condition for the solvability of the system of rhotrix equation $R_n\langle x^{ni}\rangle = \langle b^{nj}\rangle$ was recorded.

Any system of the form in Equation (2.30) is called a system of *n* rhotrix equations. The following theorem was recorded from Aminu (2010b).

Theorem 2.3.5

Let $R_n = \langle a_{ij}, c_{kl}\rangle$ be an *n*-dimensional rhotrix. A necessary and sufficient condition for the solvability of the system $R_n\langle x^{nj}\rangle = \langle b^{nj}\rangle$ is that, the corresponding system of equations, $Ax^{tj} = b^{tj}$, is solvable, where $A = (a_{ij}) \in \Re^{t\times t}$, $x^{tj}, b^{tj} \in \Re^{t\times t}$ and $t = \frac{1}{2}(n+1)$.

Aminu (2010b) alsointroduced the concepts of rhotrix eigenvector and eigenvalue problems.

Given $R_n = \langle a_{ij}, c_{kl}\rangle$, find all $\lambda \in \Re$ (eigenvalues) and an n-dimensional rhotrix column vector $\langle x^{nj}\rangle$, $\langle x^{nj}\rangle \neq 0$ (eigenvectors) such that

$$R_n\langle x^{nj}\rangle = \langle b^{nj}\rangle. \tag{2.31}$$

The following resultswere recorded from Aminu, (2010b) to solve Rhotrix Eigenvalue Problem (REP).

Theorem 2.3.6

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an *n*-dimensional rhotrix, then, $\lambda \in \Re$, is a rhotrix eigenvalue of R_n if and only if $\det(A - \lambda I) = 0$, where $A = a_{ij} \in \Re^{t \times t}$ and $t = \frac{1}{2}(n+1)$.

Corollary 2.3.7

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an *n*-dimensional rhotrix and $A = a_{ij} \in \Re^{t \times t}$ be matrix generated from $R(n)$ where $t = \frac{1}{2}(n+1)$, then $\langle v^{nj} \rangle$ is the rhotrix eigenvector corresponding to the eigenvalue λ if the system $(A - \lambda I) = 0$.

Aminu (2010c) extended the concept of linear mapping to rhotrices, where he considered the linear mapping $T : U \rightarrow V$, such that U and V are rhotrix vector spaces and present its properties. It was also shown in his work that the proposed method of converting a rhotrix to a special matrix called 'coupled matrix' suggested by Sani (2009) is a linear mapping.

A note on relationship between invertible rhotrices and associated invertible matrices was presented by Sharma and Kanwar (2011).

Ezugwu *et al.* (2011) presented an algorithm design for the implementation of row-column-based (non-commutative) method for multiplication of rhotrices of size n.

Usaini (2012a) presented elementary row operations on rhotrix, due to the vital roles they played in matrix theory. These operations can be used to determine rhotrix inverses and solve a system of *n* rhotrix equations.

Aminu (2012b) extended the concept of determinant method (one of the well-known methods that is formulated and proved in linear algebra on matrices), to the concept of rhotrix. Here, a rhotrix system of linear equations was solved using determinant method (otherwise known as Cramer's rule).

The concept of involutory matrix was also extended to rhotrices. That is a matrix which is its own inverse. Such matrices are of great importance in matrix theory and algebraic cryptography. Usaini (2012b) extended this concept of involution to rhotrices and presented their properties.

A method of constructing involutory rhotrices was also indicated in the work. Thereafter, extension on the concept of involutory rhotrix was considered by Sharma and Kanwar (2013) to give certain theorems of involution in the context of rhotrices. Also, the concept of Pascal rhotrix and its related properties were presented.

Aminu (2012c) extended the concept of Cayley-Hamilton, one of the well-known theorems that is formulated and proved in linear algebra on matrices to the concept of rhotrix and also present some properties that are inherent to it. This extension was also considered by Sharma and Kanwar (2012a).

Following the row-column-based method for multiplication of rhotrices defined by Sani (2004, 2007), Chinedu (2012) identified various methods of representing an arbitrary rhotrix. One of the methods - the row-wise method – was chosen as it was observed to be flexible in analyzing rhotrices for mathematical enrichment. A relationship between the location of the heart of a rhotrix and the dimension of the rhotrix and also a relationship between the location of the heart of a rhotrix and the order of the principal matrix of the rhotrix were determined. The flexibility of the representation paved way for two formulae, one for row-column multiplication of arbitrary rhotrices and the other for heart-based multiplication of arbitrary rhotrices. Chinedu (2012) further gave some examples as a way of demonstrating the application of the proposed formulae.

Mohammed *etal*.(2012) considered the rank of a rhotrix and characterizes its properties. The necessary and sufficient condition under which a linear map can be represented over rhotrices was also presented.

Sharma and Kanwar (2012b) presented the concept of adjoint of rhotrix and its basic properties. They described adjoint of a rhotrix, and also proved some related analogous results of matrices in the context of rhotrices. Sharma and Kanwar (2012c) introduced the concept of inner product and bilinear forms over real rhotrices.

Aminu (2013) presented the concept of minimal polynomial of a rhotrix and illustrated the procedure for determining the minimal polynomial of a rhotrix.

Usaini and Mohammed(2014) extended the work of Aminu (2010a) by presenting some properties of rhotrix eigenvalues and eigenvectors considering the numerous applications of matrix eigenvector andeigenvalue problems in areas of Applied Mathematics. They also introduced the diagonalization problem in terms of rhotrices (RDP).

Sharma *etal.* (2013) introduced the concept of Hadamard rhotrix over finite field.

Sharma*et al*. (2014)introduced Hadamard codes using Hadamard rhotrices and applied Hadamard rhotrices to cryptography and coding theory.

A review carried out by Mohammed and Balarabe (2014) can be consulted for more information on Non-Commutative rhotrix theory.

2.4 General Linear Group

2.4.1 Definition of general linear group

The set of all invertible matrices with the binary operation of matrix multiplication forms a group called the General Linear group denoted by $(GL_n(F),\cdot)$ for any given field F where *n*denotes the order of the matrix.

If F is a finite field, then $(GL_n(F),\cdot)$ has only finitely many elements.

The number of elements in $(GL_n(F_q),\cdot)$ is $\prod_{i=1}^{n}(q^n - q^{i-1})$, where *n* is the size of the matrices and *q* is the number of elements in F_q

2.5 Subgroups of General Linear Group

2.5.1 Special linear group

The Special Linear group, denoted by $(SL_n(F),\cdot)$, is asubgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all matrices having determinant as 1.

2.5.2 Diagonal linear group

The Diagonal Linear group, denoted by $(DL_n(F),\cdot)$,is a subgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all non-singular diagonal matrices.

2.5.3 Scalar linear group

TheScalar Linear group, denoted by $(KL_n(F),\cdot)$, is a subgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all non-singular scalar matrices.These are matrices with diagonal entries being non-zero equal scalars.

2.5.4 Upper triangular linear group

TheUpper Triangular Linear group, denoted by $(UTL_n(F),\cdot)$,is a subgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all non-singular upper triangular matrices.

2.5.5 Special upper triangular linear group

TheSpecial Upper Triangular Linear group, denoted by $(SUTL_n(F),\cdot)$,is a subgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all non-singular upper triangular matrices having determinant as 1.

2.5.6 Lower triangular linear subgroup

TheLower Triangular Linear group, denoted by $(LTL_n(F),\cdot)$,is a subgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all non-singular lower triangular matrices.

2.5.7 Special lower triangular linear group

TheSpecial Lower Triangular Linear group, denoted by $(SLTL_n(F),\cdot)$,is a subgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all non-singular lower triangular matrices having determinant as 1.

2.5.8 Orthogonal linear group

TheOrthogonal Linear group, denoted by $(OL_n(F),\cdot)$,is a subgroup of the general linear group $(GL_n(F),\cdot)$ consisting of all non-singular matrices satisfying $A^{-1} = A^T$.

2.5.9 Special Orthogonal linear subgroup

TheSpecial Orthogonal Linear group, denoted by $(SOL_n(F),\cdot)$,is a subgroup of the general linear group $(GL_n(F),\cdot)$ and it consists of all non-singular orthogonal matrices having determinant as 1.

CHAPTER THREE

THE NON-COMMUTATIVE RHOTRIX GROUPS

3.1Introduction

This chapter considers the pair $(GR_n(F),\circ)$ consisting of the set of all invertible rhotrices of size *n* over a field *F*; together with the binary operation of row-column-based method for rhotrix multiplication; $'\circ'$, in order tointroduce it as the concept of "*non-commutative general linear rhotrix group"*. We identify the subgroups of the $(GR_n(F),\circ)$ andshow that any of its particular subgroups is embedded in a particular subgroup of the general linear group. Furthermore, an investigation of some isomorphic relationships between the subgroups in $(GR_n(F),\circ)$ is made.

3.2 Non-commutative General Linear Rhotrix Group

In Sani (2007), it was stated as a remark (without proof) that the set of all invertible rhotrices of the same size with entries from the set of real numbers is a group with respect to row-column (non-commutative) method for rhotrix multiplication. In the following theorem, a generalization of non-commutative groups of rhotrices having the same size *n* with entries from an arbitrary field *F* is proposed.

Theorem 3.2.1 (A Generalization of Non-CommutativeRhotrix Groups)

Let $GR_n(F)$ be the set of all invertible rhotrices with entries from an arbitrary field F and let $\circ$ be the non-commutative (row-column) method for rhotrix multiplication. Then, the pair $(GR_n(F), \circ)$ is a non-commutative general linear rhotrix group.

Proof

We shall show that the pair $(GR_n(F), \circ)$ is a group under the binary operation of row-column multiplication of rhotrices. That is, we shall show that the following group axioms are satisfied:

(i) Closure: for any two rhotrices A_n , $B_n \in GR_n(F)$, we have $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$, so

$A_n \circ B_n \in GR_n(F)$, since $\det(A_n \circ B_n) = \det(A_n) \times \det(B_n) \neq 0.$

Thus, $GR_n(F)$ is closed under the group binary operation.

(ii) Associativity:

$\forall \; A_n, B_n \; and \; C_n \in GR_n(F),$ we are to show that

$$A_n \circ (B_n \circ C_n) = (A_n \circ B_n) \circ C_n \qquad 3.1$$

Now let $A_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle, B_n = \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle, C_n = \langle f_{i_3 j_3}, g_{l_3 k_3} \rangle,$

then, from the left hand side of Equation 3.1,

$$A_n \circ (B_n \circ C_n) = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \; \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle \circ \langle f_{i_3 j_3}, g_{l_3 k_3} \rangle$$

$$= \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \left(\left\langle \sum_{i_3 j_2 = 1}^{t} (b_{i_2 j_2} f_{i_3 j_3}), \sum_{l_3 k_2 = 1}^{t-1} (d_{l_2 k_2} g_{l_3 k_3}) \right\rangle \right)$$

$$=\left\langle \sum_{i_2 j_1=1}^{t} a_{i_1 j_1}\left(\sum_{i_3 j_2=1}^{t}(b_{i_2 j_2} f_{i_3 j_3})\right), \sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1}\left(\sum_{l_3 k_2=1}^{t-1}(d_{l_2 k_2} g_{l_3 k_3})\right)\right\rangle$$

$$=\left\langle \sum_{i_2 j_1=1}^{t}\sum_{i_3 j_2=1}^{t} a_{i_1 j_1}(b_{i_2 j_2} f_{i_3 j_3}), \sum_{l_2 k_1=1}^{t-1}\sum_{l_3 k_2=1}^{t-1} c_{l_1 k_1}(d_{l_2 k_2} g_{l_3 k_3})\right\rangle$$

$$=\left\langle \sum_{i_2 j_1=1}^{t}\sum_{i_3 j_2=1}^{t} a_{i_1 j_1} b_{i_2 j_2} f_{i_3 j_3}, \sum_{l_2 k_1=1}^{t-1}\sum_{l_3 k_2=1}^{t-1} c_{l_1 k_1} d_{l_2 k_2} g_{l_3 k_3}\right\rangle$$

By associativity of field property, we have

$$=\left\langle \sum_{i_3 j_2=1}^{t}\sum_{i_2 j_1=1}^{t} (a_{i_1 j_1} b_{i_2 j_2}) f_{i_3 j_3}, \sum_{l_3 k_2=1}^{t-1}\sum_{l_2 k_1=1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) g_{l_3 k_3}\right\rangle$$

$$=\left\langle \sum_{i_3 j_2=1}^{t} f_{i_3 j_3}\left(\sum_{i_2 j_1=1}^{t} a_{i_1 j_1} b_{i_2 j_2}\right), \sum_{l_3 k_2=1}^{t-1} g_{l_3 k_3}\left(\sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1} d_{l_2 k_2}\right)\right\rangle$$

$$=\left\langle \sum_{i_2 j_1=1}^{t} a_{i_1 j_1} b_{i_2 j_2}, \sum_{l_2 k_1=1}^{t-1} c_{l_1 k_1} d_{l_2 k_2}\right\rangle \circ \left\langle f_{i_3 j_3}, g_{l_3 k_3}\right\rangle$$

$$=(A_n \circ B_n)\circ C_n$$

(iii) Existenceof identity:

$$\text{The rhotrix } I_n = \left\langle I_{t\times t}, I_{(t-1)\times(t-1)}\right\rangle = \left\langle \begin{matrix} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \ldots & \ldots. & \ldots & \ldots & \ldots & \\ 0 & \ldots & \ldots & 1 & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{matrix}\right\rangle \in GR_n(F), \quad (3.2)$$

since $\det(I_n) \neq 0$.

Furthermore, for all $A_n \in GR_n(F)$, we have

$I_n \circ A_n = A_n \circ I_n = A_n$, implying that I_n is an identity rhotrix in $GR_n(F)$.

(iv) Existence of inverse.

For each $A_n \in GR_n(F)$, $\exists\ A_n^{-1} \in GR_n(F)$ since $\det(A_n) \neq 0$.

$A_n \circ A_n^{-1} = A_n^{-1} \circ A_n = I_n \in GR_n(F)$, implying that A_n^{-1} is the inverse of A_n.

Hence, the proof is complete.

Corollary 3.2.2

Let $GR_n(\Re)$ be the set of all invertible rhotrices of size n over $\Re$. Let $\circ$ be the row-column multiplication of rhotrices; then the pair $(GR_n(\Re), \circ)$ is anon-commutative general linear rhotrix group of size n over real numbers.

Proof

By substituting $F = \Re$ in theorem 3.2.1 above, the result follows.

A particular case:

The pair $(GR_3(\Re), \circ)$ is a group of all invertible rhotrices of size 3.

Proof:

$$GR_3(\Re) \neq \varnothing \text{ since } I_3 \in GR_3(\Re) \text{ as } \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle \tag{3.3}$$

Let A_3, B_3, C_3 be any elementsin $GR_3(\Re)$ given as

$$A_3 = \left\langle \begin{array}{ccc} & a & \\ b & h(A) & d \\ & e & \end{array} \right\rangle, B_3 = \left\langle \begin{array}{ccc} & f & \\ g & h(B) & j \\ & k & \end{array} \right\rangle \text{ and } C_3 = \left\langle \begin{array}{ccc} & l & \\ m & h(C) & n \\ & p & \end{array} \right\rangle \text{ then}$$

$A_3 \circ B_3 = \left\langle \begin{array}{ccc} & af + dg & \\ bf + eg & \text{h(A)h(B)} & aj + dk \\ & bj + ek & \end{array} \right\rangle \in GR_3(\Re)$ hence the set $GR_3(\Re)$ is closed under the binary operation of row-column multiplication of rhotrices.

$$(A_3 \circ B_3) \circ C_3 = \left\langle \begin{array}{ccc} & (af + dg)l + (aj + dk)m & \\ (bf + eg)l + (bj + ek)m & h(A)h(B)h(C) & (af + dg)n + (aj + dk)p \\ & (bf + eg)n + (bj + ek)p & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & afl + ajm + dgl + dkm & \\ bfl + bjm + egl + ekm & h(A)h(B)h(C) & afn + ajp + dgn + dkp \\ & bfn + bjp + egn + ekp & \end{array} \right\rangle \tag{3.4}$$

Also, $B_3 \circ C_3 = \left\langle \begin{array}{ccc} & fl + jm & \\ gl + km & h(B)h(C) & fn + jp \\ & gn + kp & \end{array} \right\rangle$

$$A_3 \circ (B_3 \circ C_3) = \left\langle \begin{array}{ccc} & a(fl + jm) + d(gl + km) & \\ b(fl + jm) + e(gl + km) & h(A)h(B)h(C) & a(fn + pj) + d(gn + kp) \\ & b(fn + pj) + e(gn + kp) & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & afl + ajm + dgl + dkm & \\ bfl + bjm + egl + ekm & h(A)h(B)h(C) & afn + ajp + dgn + dkp \\ & bfn + bjp + egn + ekp & \end{array} \right\rangle \quad (3.5)$$

then $A_3 \circ B_3 \circ C_3 = A_3 \circ B_3 \circ C_3$, therefore $(GR_3(\Re), \circ)$ is associative.

Next: for each $A_3 = \left\langle \begin{array}{ccc} & a & \\ b & h(A) & d \\ & e & \end{array} \right\rangle \in GR_3(\Re), \exists\, A_3^{-1} \in GR_3(\Re)$ such that

$A_3 \circ A_3^{-1} = A_3^{-1} \circ A_3 = I_3$.

In particular,

$$A_3^{-1} = \frac{1}{ae - bd} \left\langle \begin{array}{ccc} & e & \\ -b & \dfrac{ae - bd}{h(A)} & -d \\ & a & \end{array} \right\rangle \quad (3.6)$$

With these axioms satisfied, the pair $(GR_3(\Re), \circ)$ of all invertible rhotrices of size *3* over row- column multiplication of rhotrices is a group of real rhotrices thus completing the proof.

Theorem 3.2.4

The non-commutative general linear rhotrix group $(GR_n(F),\circ)$ is embedded in the general linear group $GL_n(F),\cdot$.

Proof

Let $(GR_n(F),\circ)$ be the group of all invertible rhotrices of size *n* and let $GL_n(F),\cdot$ be the group of all*n*-dimensionalinvertible matrices. We define the mapping

$\theta:\ (GR_n(F),\circ)\ \rightarrow\ GL_n(F),\cdot$ by:

$$\theta\left(\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle\right)=\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{bmatrix}$$

where θ maps each R_n in $GR_n(F)$ to its corresponding filled coupled matrix M_n in $GL_n(F),\cdot$; $\circ$ and $\cdot$ are respectively the row-column-based method for multiplication of rhotrices and the usual matrix multiplication.

Clearly, θ is well defined and it is a $1-1$ mapping since $\forall$ R1, R2 $\in GR_n(F),\ \theta(R1)=\theta(R2)\Rightarrow R1=R2$

Obviously, no two different rhotrices will have the same filled coupled matrix.

To show that θ is a homomorphism, we need to show that if $R_n = \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle$ and $Q_n = \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle$ are any two rhotrices in $(GR_n(F), \circ)$, then $\theta\ R_n \circ Q_n\ = \theta(R_n) \cdot \theta(Q_n)$

$$\theta\ R_n \circ Q_n\ = \theta\ \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle$$

$$= \theta\left(\left\langle \sum_{i_2 j_1 = 1}^{t} (a_{i_1 j_1} b_{i_2 j_2}), \sum_{l_2 k_1 = 1}^{t-1} (c_{l_1 k_1} d_{l_2 k_2}) \right\rangle \right)$$

$= \ ab, cd\ _n$, the corresponding filled coupled matrix.

On the other hand,

$$\theta(R_n) \cdot \theta(Q_n) = \theta\ \langle a_{i_1 j_1}, c_{l_1 k_1} \rangle\ \cdot \theta\ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle$$

$= \ ac\ _n \cdot\ bd\ _n$, product of the corresponding filled coupled matrices.

$= \ ab, cd\ _n$

Hence the proof.

3.3 Subgroups of Non-commutative General Linear Rhotrix Group

3.3.1Definition of Unitary Rhotrices

A rhotrix R_n is called a unitary rhotrix if the determinant of R_n is equal to 1.

We denote the set of all such unitary rhotrices of size n with entries from field F as $SR_n(F)$.

$$\text{Thus, } SR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : \begin{array}{l} \\ a_{ij}, c_{lk} \in F, \det(a_{ij}) \neq 0, \det(c_{lk}) \neq 0 \ and \det\left\langle a_{ij}, c_{lk} \right\rangle = 1 \end{array} \right\} \quad (3.7)$$

Proposition 3.3.1

If R_n is a rhotrix of size $n$with $\det(R_n) \neq 0$, then it possesses an inverse denoted by R_n^{-1} and the determinant of the inverse is $\det(R_n^{-1}) = \dfrac{1}{\det(R_n)}$

Proof

Let $R_n = \left\langle a_{ij}, c_{lk} \right\rangle$, be any rhotrix and $\det(R_n) \neq 0$, then by Definition 1.4.6, R_n possesses an inverse, $R_n^{-1} = \left\langle q_{ij}, r_{lk} \right\rangle$ such that $R_n \circ R_n^{-1} = \left\langle a_{ij}, c_{lk} \right\rangle \circ \left\langle q_{ij}, r_{lk} \right\rangle = \left\langle I_{ij}, e_{lk} \right\rangle$ where $\left[q_{ij} \right]_{t \times t}$

and $r_{lk} \; {}_{t-1\times t-1}$ are the inverses of the two square matrices $\left[a_{ij}\right]_{t\times t}$ and $c_{lk} \; {}_{t-1\times t-1}$ respectively, which make up the rhotrix R_n with $t = \frac{n+1}{2} \, and \; n \in 2Z^{+} + 1.$

By Definition 1.4.5, $\det(R_n) = \det \left[a_{ij}\right]_{t\times t} \cdot \det \; c_{lk} \; {}_{t-1\times t-1}$

and $\det(R_n^{-1}) = \det \left[q_{ij}\right]_{t\times t} \cdot \det \; r_{lk} \; {}_{t-1\times t-1}$

In Larson *et al.* (2009), if **A** is a matrix such that $\det(A) \neq 0$, then $\det(A^{-1}) = \frac{1}{\det(A)}$

Since $\left[q_{ij}\right]_{t\times t}$ and $r_{lk} \; {}_{t-1\times t-1}$ are square matrices satisfying $\det \left[q_{ij}\right]_{t\times t} \neq 0$ and $\det \left[r_{ij}\right]_{t-1\times t-1} \neq 0$, then, $\det(R_n^{-1}) = \det \left[a_{ij}\right]_{t\times t}^{-1} \cdot \det \; c_{lk} \; {}_{t-1\times t-1}^{-1}$

$$= \frac{1}{\det \left[a_{ij}\right]_{t\times t}} \cdot \frac{1}{\det \; c_{lk} \; {}_{t-1\times t-1}}$$

$$= \frac{1}{\det \left[a_{ij}\right]_{t\times t} \cdot \det \; c_{lk} \; {}_{t-1\times t-1}}$$

$$= \frac{1}{\det(R_n)}$$

Theorem 3.3.2

The pair $(SR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$.

Proof

Since $I_n \in SR_n(F),$ then $SR_n(F) \neq \varnothing$.

Now, Let A_n and $B_n \in SR_n(F)$,

Then it follows that, det(A_n) $=1 \neq 0$ and det(B_n) $=1 \neq 0$ respectively. This implies that for each A_n and $B_n \in SR_n(F),\ \exists\ A_n^{-1}$ and $B_n^{-1} \in SR_n(F) \ni A_n \circ B_n^{-1} \in SR_n(F)$. This is because by Proposition 3.3.1, since B_n is invertible then $\det(B_n^{-1}) = \dfrac{1}{\det(B_n)} = \dfrac{1}{1} = 1$ and

$\det(A_n \circ B_n^{-1}) = \det(A_n) \circ \det(B_n^{-1}) = 1 \circ 1$

Hence $(SR_n(F),\circ)$ is a subgroup of $(GR_n(F),\circ)$

Remark: The subgroup $(SR_n(F),\circ)$ is termed the special rhotrix subgroup since it possesses entries with determinant equal to one.

Theorem 3.3.3

The special rhotrix subgroup $(SR_n(F),\circ)$ of $(GR_n(F),\circ)$ is embedded in the special linear subgroup $(SL_n(F),\cdot)$ of $(GL_n(F),\cdot)$.

Proof

We define a mapping $\theta : (SR_n(F),\circ) \rightarrow (SL_n(F),\cdot)$ by

$$\theta\left(\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle\right)=\left[\begin{array}{ccccccccc} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{array}\right]$$

where θ maps each R_n in $SR_n(F)$ to its corresponding filled coupled matrix M_n in

$SL_n(F),\cdot$. It is clear to see that θ is an injective homomorphism (See proof of Theorem 3.2.4)

Also,

$$\det\left(\left\langle\begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array}\right\rangle\right)=\det\left(\left[\begin{array}{ccccccccc} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1(t-1)} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{11} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & & \cdots & \cdots & \cdots & 0 & a_{(t-1)t} \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{array}\right]\right)=1$$

hence the result follows.

3.3.2 Definition of Diagonal Rhotrix

A rhotrix R_n is called a diagonal rhotrix if all the elements in the vertical diagonal are non-zero, while others are zero(s). We denote the set of all invertible diagonal rhotrices of size *n* as $DR_n(F)$.

$$\text{Thus, } DR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & 0 & & & \\ & & 0 & 0 & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : \begin{array}{l} \\ a_{ij}, c_{lk} \in F, \det(a_{ij}) \neq 0, \det(c_{lk}) \neq 0. \end{array} \right\} \qquad (3.8)$$

Theorem 3.3.4

The pair $(DR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$.

Proof

$$DR_n(F) \neq \varnothing \text{ since } I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in DR_n(F). \qquad (3.9)$$

$$\text{Next, let } A_n = (p_{ij}, q_{lk}) = \left\langle \begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & q_{11} & 0 & & & \\ & & 0 & 0 & p_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} & & & & \end{array} \right\rangle \in DR_n(F) \quad (3.10)$$

and

$$B_n = (r_{ij}, s_{lk}) = \left\langle \begin{array}{ccccccccc} & & & & r_{11} & & & & \\ & & & 0 & s_{11} & 0 & & & \\ & & 0 & 0 & r_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & r_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & s_{(t-1)(t-1)} & 0 & & & \\ & & & & r_{tt} & & & & \end{array} \right\rangle \in DR_n(F) \tag{3.11}$$

it follows that $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$ respectively. Implying that A_n^{-1} and B_n^{-1} exist in $DR_n(F)$.

So,

$$A_n \circ B_n^{-1} = \left\langle \begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & q_{11} & 0 & & & \\ & & 0 & 0 & p_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & q_{(t-1)(t-1)} & 0 & & & \\ & & & & p_{tt} & & & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccccccc} & & & & \frac{1}{r_{11}} & & & & \\ & & & 0 & \frac{1}{s_{11}} & 0 & & & \\ & & 0 & 0 & \frac{1}{r_{22}} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & \frac{1}{r_{(t-1)(t-1)}} & 0 & 0 & & \\ & & & 0 & \frac{1}{s_{(t-1)(t-1)}} & 0 & & & \\ & & & & \frac{1}{r_{tt}} & & & & \end{array} \right\rangle \tag{3.12}$$

$$
= \left\langle \begin{array}{ccccccccc}
 & & & & \frac{p_{11}}{r_{11}} & & & & \\
 & & & 0 & \frac{q_{11}}{s_{11}} & 0 & & & \\
 & & 0 & 0 & \frac{p_{22}}{r_{221}} & 0 & 0 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\
 & & 0 & 0 & \frac{p_{(t-1)(t-1)}}{r_{(t-1)(t-1)}} & 0 & 0 & & \\
 & & & 0 & \frac{q_{(t-1)(t-1)}}{s_{(t-1)(t-1)}} & 0 & & & \\
 & & & & \frac{p_{tt}}{r_{tt}} & & & &
\end{array} \right\rangle \in DR_n(F) \qquad (3.13)
$$

Hence $(DR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$

Theorem 3.3.5

The Diagonal rhotrix subgroup $(DR_n(F),\circ)$ of $(GR_n(F),\circ)$ is embedded in the diagonal linear subgroup $(DL_n(F),\cdot)$ of $(GL_n(F),\cdot)$

Proof

We define a mapping $\theta : (DR_n(F),\circ) \to (DL_n(F),\cdot)$ by

$$
\theta\left(\left\langle \begin{array}{c} a_{11} \\ 0 \quad c_{11} \quad 0 \\ 0 \quad 0 \quad a_{22} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ 0 \quad c_{(t-1)(t-1)} \quad 0 \\ a_{tt} \end{array} \right\rangle\right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & c_{(t-1)(t-1)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & a_{tt} \end{bmatrix}
$$

Where θ mapped each rhotrix R_n in $DR_n(F)$, to its filled coupled matrix M_n in $DL_n(F)$, Clearly, θ is awell-defined, 1 – 1 mapping since no two different rhotrices will have the same filled coupled matrix.

θ is a homomorphism because $\theta(A_n \circ B_n) = \theta(A_n) \cdot \theta(B_n)\ \forall\ A_n, B_n \in DR_n(F)$. (See proof of Theorem 3.2.4), hence the diagonal linear rhotrix group is embedded in the diagonal linear group.

This completes the proof.

3.3.3 Definition of Scalar Rhotrix

A rhotrix R_n is called a scalar rhotrix if it is a diagonal rhotrix in which all the elements in the vertical diagonal are non-zero equal scalars, while others are zero(s). Scalar rhotrices are rhotrices of the form kI_n where I_n is the identity rhotrix and k is a non-zero constant.

We denote the set of all invertible scalar rhotrices of size *n*as $KR_n(F)$.

Thus,

$$KR_n(F)=\left\{\left\langle\begin{array}{ccccccccc} & & & & k_{11} & & & & \\ & & & 0 & \kappa_{11} & 0 & & & \\ & & 0 & 0 & k_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & k_{(t-1)(t-1)} & 0 & 0 & & \\ & & & 0 & \kappa_{(t-1)(t-1)} & 0 & & & \\ & & & & k_{tt} & & & & \end{array}\right\rangle : \begin{array}{c} \\ \end{array} k_{ij},\kappa_{lk}\in F,\det(k_{ij})\neq 0,\det(\kappa_{lk})\neq 0; k_{ij}=\kappa_{lk}=k\right\} \tag{3.14}$$

Theorem 3.3.6

The pair $(KR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$

Proof

$$KR_n(F)\neq\varnothing \text{ since } I_n=\left\langle\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array}\right\rangle\in KR_n(F). \tag{3.15}$$

$$\text{Next, let } A_n=\langle p_{ij},p_{lk}\rangle=pI_n=\left\langle\begin{array}{ccccccccc} & & & & p_{11} & & & & \\ & & & 0 & p_{11} & 0 & & & \\ & & 0 & 0 & p_{11} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & p_{11} & 0 & 0 & & \\ & & & 0 & p_{11} & 0 & & & \\ & & & & p_{11} & & & & \end{array}\right\rangle\in KR_n(F) \tag{3.16}$$

and

$$B_n = \langle r_{ij}, r_{lk} \rangle = rI_n = \left\langle \begin{array}{c} r_{11} \\ 0 \quad r_{11} \quad 0 \\ 0 \quad 0 \quad r_{11} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad r_{11} \quad 0 \quad 0 \\ 0 \quad r_{11} \quad 0 \\ r_{11} \end{array} \right\rangle \in KR_n(F) \qquad (3.17)$$

it follows that $\det(A_n) \neq 0$ and $\det(B_n) \neq 0$ respectively. Implying that A_n^{-1} and B_n^{-1} exist in $KR_n(F)$.

So,

$$A_n \circ B_n^{-1} = pI_n \circ \frac{1}{r} I_n$$

$$= \left\langle \begin{array}{c} p_{11} \\ 0 \quad p_{11} \quad 0 \\ 0 \quad 0 \quad p_{11} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad p_{11} \quad 0 \quad 0 \\ 0 \quad p_{11} \quad 0 \\ p_{11} \end{array} \right\rangle \circ \left\langle \begin{array}{c} \frac{1}{r_{11}} \\ 0 \quad \frac{1}{r_{11}} \quad 0 \\ 0 \quad 0 \quad \frac{1}{r_{11}} \quad 0 \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad 0 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ 0 \quad 0 \quad \frac{1}{r_{11}} \quad 0 \quad 0 \\ 0 \quad \frac{1}{r_{11}} \quad 0 \\ \frac{1}{r_{11}} \end{array} \right\rangle$$

$$
=\left\langle \begin{array}{ccccccccccc}
 & & & & & \frac{p_{11}}{r_{11}} & & & & & \\
 & & & & 0 & \frac{p_{11}}{r_{11}} & 0 & & & & \\
 & & & 0 & 0 & \frac{p_{11}}{r_{11}} & 0 & 0 & & & \\
 & \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots & & & \\
0 & \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots & \ldots & 0 & \\
 & \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots & & & \\
 & & & 0 & 0 & \frac{p_{11}}{r_{11}} & 0 & 0 & & & \\
 & & & & 0 & \frac{p_{11}}{r_{11}} & 0 & & & & \\
 & & & & & \frac{p_{11}}{r_{11}} & & & & &
\end{array} \right\rangle = \frac{p}{r} I_n \in KR_n(F) \qquad (3.18)
$$

Hence $(KR_n(F),\circ)$ is a rhotrix subgroup of $(GR_n(F),\circ)$

Theorem 3.3.7

The scalar rhotrix subgroup $(KR_n(F),\circ)$ of $(GR_n(F),\circ)$ is embedded in the Scalar linear subgroup $(KL_n(F),\cdot)$ of $(GL_n(F),\cdot)$

Proof

We define a mapping $\theta:(KR_n(F),\circ)\rightarrow(KL_n(F),\cdot)$ by

$$
\theta\left(\left\langle \begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & 0 & a_{11} & 0 & & & \\
 & & 0 & 0 & a_{11} & 0 & 0 & & \\
 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\
0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\
 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\
 & & 0 & 0 & 0 & 0 & 0 & & \\
 & & & 0 & a_{11} & 0 & & & \\
 & & & & a_{11} & & & &
\end{array} \right\rangle \right) = \begin{bmatrix}
a_{11} & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\
0 & a_{11} & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\
0 & 0 & a_{11} & 0 & \ldots & \ldots & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots & \ldots & \ldots \\
0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & a_{11} & 0 \\
0 & 0 & 0 & 0 & \ldots & \ldots & \ldots & 0 & a_{11}
\end{bmatrix}
$$

Where Θ maps each rhotrix R_n in $KR_n(F)$, to its filled coupled matrix M_n in $KL_n(F)$, clearly, θ is well defined and a 1 – 1 mapping since no two different rhotrices will have the same filled coupled matrix.

Also: $\theta(A_n \circ B_n) = \theta(A_n) \cdot \theta(B_n) \ \forall \ A_n, B_n \in KR_n(F)$ (See Proof of Theorem 3.2.4).

Hence, θ is a homomorphism.

This completes the proof.

3.3.4 Definition of Left Triangular Rhotrix

A rhotrix R_n is called a left triangular rhotrix if all the elements in the right of the vertical diagonal are all zero.

We denote the set of all invertible left triangular rhotrices of size *n*as $LTR_n(F)$.

Thus,

$$LTR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) \neq 0 \neq \det(c_{lk}) \right\},$$

$$where \ \ a_{ij} = 0 \ \ if \ \ i < j \quad and \quad c_{lk} = 0 \ if \ \ l < k$$

.

Proposition 3.3.8

If A_n and B_n are left triangular rhotrices, then their product $A_n \circ B_n$, is a left triangular rhotrix.

Proof

Suppose that $A_n = \langle a_{ij}, c_{lk} \rangle$ is a left triangular rhotrix then,

$$A_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & a_{11} & & & & & \\ & & a_{21} & c_{11} & 0 & & & & \\ & a_{31} & c_{21} & a_{22} & 0 & 0 & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & & & \\ a_{t1} \quad \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & & & \\ & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & & \\ & & & a_{tt} & & & & & \end{array} \right\rangle = \left\langle \begin{bmatrix} a_{11} & 0 & \dots & 0 & 0 \\ a_{21} & a_{22} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{(t-1)1} & a_{(t-1)2} & \dots & 0 & 0 \\ a_{t1} & a_{t2} & \dots & a_{t(t-1)} & a_{tt} \end{bmatrix}, \begin{bmatrix} c_{11} & \dots & 0 \\ \dots & \dots & \dots \\ c_{(t-1)1} & \dots & c_{(t-1)(t-1)} \end{bmatrix} \right\rangle$$

Observe that all $A_n = \langle a_{ij}, c_{lk} \rangle$ can be expressed as a couple of two square matrices $\left[a_{ij}\right]_{t\times t}$ and $c_{lk}\ {}_{t-1\times t-1}$, which make up the rhotrix A_n with $t = \dfrac{n+1}{2}\, and\ n \in 2Z^{+} + 1.$ In particular, $\left[a_{ij}\right]_{t\times t}$ and $c_{lk}\ {}_{t-1\times t-1}$ are lower triangular matrices. InAbadir *etal.*(2005), the product of two lower triangular matrices is itself a lower triangular matrix, therefore, for all $A_n, B_n \in LTR_n(F)$, $A_n \circ B_n \in LTR_n(F)$, hence the proof.

Theorem 3.3.9

The pair $(LTR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$.

Proof

Since $I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in LTR_n(F),\ then\ LTR_n(F) \neq \varnothing.$

Let

$$A_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$$

and $$B_n = \langle b_{ij}, d_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_{21} & d_{11} & 0 & & & \\ & & b_{31} & d_{21} & b_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ b_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & b_{t(t-2)} & d_{(t-1)(t-2)} & b_{(t-1)(t-1)} & 0 & 0 & & \\ & & & b_{t(t-1)} & d_{(t-1)(t-1)} & 0 & & & \\ & & & & b_{tt} & & & & \end{array} \right\rangle$$

be two rhotrices of size n in $LTR_n(F)$, it follows that $(A_n \circ B_n) \in LTR_n(F)$ from Proposition 3.3.8.

So the set $LTR_n(F)$ is closed under the operation of rhotrix multiplication.

Next, for any $A_n \in LTR_n(F)$, $\det(A_n) \neq 0$ and by Definition 1.4.6, $\exists A_n^{-1}$.

Observe that $A_n = \langle a_{ij}, c_{lk} \rangle$ where $[a_{ij}]_{t\times t}$ and $[c_{ij}]_{t-1\times t-1}$ respectively are the two invertible lower triangular matrices which make up the rhotrix A_n with

$t = \dfrac{n+1}{2}$ *and* $n \in 2Z^{+}+1$. According to Abadir*etal.*(2005), the inverse of every non-singular lower triangular matrix is a lower triangular matrix, then $A_n^{-1} \in LTR_n(F)$

Also,by Proposition 3.3.8, $\forall\ A_n, B_n \in LTR_n(F)$,we have $(A_n \circ B_n^{-1}) \in LTR_n(F)$

Hence $(LTR_n(F),\circ)$ is a subgroup of $(GR_n(F),\circ)$

Theorem 3.3.10

The subgroup $(LTR_n(F),\circ)$ of $(GR_n(F),\circ)$ is embedded in $(LTM_n(F),\cdot)$,the lower triangular linear subgroup.

Proof

We define a mapping $\theta : (LTR_n(F),\circ) \rightarrow (LTM_n(F),\cdot)$ by

$$\theta\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle\right) = \begin{bmatrix} a_{11} & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & c_{11} & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(t-1)1} & 0 & a_{(t-1)2} & 0 & \cdots & \cdots & a_{(t-1)(t-1)} & 0 & 0 \\ 0 & c_{(t-1)1} & 0 & c_{(t-1)2} & \cdots & \cdots & 0 & c_{(t-1)(t-1)} & 0 \\ a_{t1} & 0 & a_{t2} & 0 & \cdots & \cdots & a_{t(t-1)} & 0 & a_{tt} \end{bmatrix}$$

where θ maps every left triangular rhotrix to its corresponding filled coupled lower triangular matrix. We observe that θ is a well-defined and a 1 – 1 mapping (See Proof of Theorem 3.2.4). Therefore, θ is a homomorphism.

Hence, the left triangular rhotrix subgroup is embedded in the Lower triangular matrix group.

3.3.5 Definition of Special Left Triangular Rhotrix

A rhotrix R_n is called a special left triangular rhotrix if all the elements in the right of the vertical diagonal are all zero and $\det(R_n) = 1$.

We denote the set of all special left triangular rhotrices of size *n*as $LTR_n^*(F)$.

$$\text{Thus, } LTR_n^*(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & 0 & & & \\ & & a_{31} & c_{21} & a_{22} & 0 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & a_{t(t-2)} & c_{(t-1)(t-2)} & a_{(t-1)(t-1)} & 0 & 0 & & \\ & & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) = 1 = \det(c_{lk}) \right\},$$

$$where\ \ a_{ij} = 0 \ \ if\ \ i < j \quad and \quad c_{lk} = 0 \ if\ \ l < k$$

Theorem3.3.11

Let $(LTR_n^*(F), \circ)$ be the special right triangular rhotrix subgroup of $(GR_n(F), \circ)$ and let $(SR_n(F), \circ)$ be the special rhotrix subgroup of $(GR_n(F), \circ)$, then the pair $(LTR_n^*(F), \circ)$ is a rhotrix subgroup of $(SR_n(F), \circ)$.

Proof

Since $I_n \in LTR_n^*(F)$ then $RTR_n^*(F) \neq \varnothing$.

Now, Let A_n and $B_n \in$ $LTR_n^*(F)$,

Then det(A_n) =1 $\neq 0$ det(B_n) =1 $\neq 0$ respectively. This implies that for each $A_n, B_n \in LTR_n^*(F)$ $\exists$ A_n^{-1} and B_n^{-1} and by Theorem 3.3.9, A_n^{-1} and B_n^{-1} $\in LTR_n^*(F)$. Observe that $\det(A_n \circ B_n^{-1}) = \det(A_n) \circ \det(B_n^{-1}) = 1 \circ 1$ (See proof of Theorem 3.3.2)

$\therefore \forall$ $A_n, B_n \in LTR_n^*(F)$, $A_n \circ B_n^{-1} \in$ $LTR_n^*(F)$

Hence $LTR_n^*(F)$ is a subgroup of $(SR_n(F), \circ)$

3.3.6 Definition of Right Triangular Rhotrix

A rhotrix R_n is called a right triangular rhotrix if all the elements in the left of the vertical diagonal are all zero.

We denote the set of all invertible right triangular rhotrices of size *n* as $RTR_n(F)$.

$$Thus, RTR_n(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) \neq 0, \det(c_{lk}) \neq 0 \right\},$$

$$where \quad a_{ij} = 0 \;\; if \;\; i > j \;\; and \;\; c_{lk} = 0 \; if \;\; l > k$$

Proposition 3.3.12

If A_n and B_n are right triangular rhotrices, then their product $A_n \circ B_n$, is a right triangular rhotrix.

Proof

Suppose that $A_n = \langle a_{ij}, c_{lk} \rangle$ is any right triangular rhotrix of size *n*, then,

$$A_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ 0 & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1t} \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle = \left\langle \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1(t-1)} & a_{1t} \\ 0 & a_{22} & \ldots & a_{2(t-1)} & a_{2t} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_{(t-1)(t-1)} & a_{(t-1)t} \\ 0 & 0 & \ldots & 0 & a_{tt} \end{bmatrix}, \begin{bmatrix} c_{11} & \ldots & c_{1(t-1)} \\ \ldots & \ldots & \ldots \\ 0 & \ldots & c_{(t-1)(t-1)} \end{bmatrix} \right\rangle$$

Observe that all $A_n = \langle a_{ij}, c_{lk} \rangle$ can be expressed as a couple of two square matrices

$\left[a_{ij} \right]_{t \times t}$ and $c_{lk} \; {}_{t-1 \times t-1}$, which make up the rhotrix A_n with $t = \frac{n+1}{2}$ *and* $n \in 2Z^+ + 1.$ In particular, $\left[a_{ij} \right]_{t \times t}$ and $c_{lk} \; {}_{t-1 \times t-1}$ are upper triangular matrices. In Abadir *etal.*(2005), the product of two upper triangular matrices is itself an upper triangular matrix, therefore, for all $A_n, B_n \in LTR_n(F)$, $A_n \circ B_n \in LTR_n(F)$, hence the proof.

Theorem 3.3.13

The pair $(RTR_n(F), \circ)$ is a rhotrix subgroup of $(GR_n(F), \circ)$.

Proof

Since $I_n = \left\langle \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & 1 & 0 & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & 1 & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & 1 & 0 & & \\ & & & 1 & & & \end{array} \right\rangle \in RTR_n(F),\ then\ RTR_n(F) \neq \varnothing.$

Let $A_n = \langle a_{ij}, c_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$

and

$B_n = \langle b_{ij}, d_{lk} \rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & 0 & d_{11} & b_{12} & & & \\ & & 0 & 0 & b_{22} & d_{12} & b_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & b_{(t-1)(t-1)} & d_{(t-2)(t-1)} & b_{(t-2)t} & & \\ & & & 0 & d_{(t-1)(t-1)} & b_{(t-1)t} & & & \\ & & & & b_{tt} & & & & \end{array} \right\rangle$

be two rhotrices of size n in $RTR_n(F)$, it follows that $(A_n \circ B_n) \in RTR_n(F)$ from Proposition 3.3.12.

So the set $RTR_n(F)$ is closed under the operation of rhotrix multiplication.

Next, for any $A_n \in RTR_n(F), \det(A_n) \neq 0$ and by Definition 1.4.6, $\exists\ A_n^{-1}$.

Observe that $A_n = \langle a_{ij}, c_{lk} \rangle$ where $\left[a_{ij}\right]_{t\times t}$ and $\left[c_{ij}\right]_{t-1\times t-1}$ respectively are the two invertible upper triangular matrices which make up the rhotrix A_n with $t = \dfrac{n+1}{2}$ *and* $n \in 2Z^{+}+1$. According to Abadir *etal.*(2005), the inverse of every non-singular upper triangular matrix is an upper triangular matrix,

then $A_n^{-1} \in RTR_n(F)$

Also,by Proposition 3.3.12 $\forall\ A_n, B_n \in RTR_n(F)$, we have $(A_n \circ B_n^{-1}) \in RTR_n(F)$

Hence $(RTR_n(F), \circ)$ is a subgroup of $(GR_n(F), \circ)$

Theorem 3.3.14

The subgroup $(RTR_n(F), \circ)$ is embedded in $(UTM_n(F), \cdot)$, the upper triangular matrix group.

Proof

We define a mapping $\theta : (RTR_n(F), \circ) \to (UTM_n(F), \cdot)$ by

$$\theta\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle\right) = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & a_{1(t-1)} & 0 & a_{1t} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & 0 & c_{1(t-1)} & 0 \\ 0 & 0 & a_{22} & 0 & \cdots & \cdots & a_{2(t-1)} & 0 & a_{2t} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots. & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & a_{(t-1)(t-1)} & 0 & a_{(t-1)t} \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & c_{(t-1)(t-1)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & a_{tt} \end{bmatrix}$$

Where θ maps every right triangular rhotrix to its corresponding filled coupled upper triangular matrix. We observe that θ is a well-defined and a 1 – 1 mapping (See Proof of Theorem 3.2.4). Therefore, θ is a homomorphism, hence, the right triangular rhotrix group is embedded in the upper triangular matrix group.

3.3.7 Definition of Special Right Triangular Rhotrix

A rhotrix R_n is called a special right triangular rhotrix if all the elements in the left of the vertical diagonal are all zero and $\det(R_n) = 1$.

We denote the set of all special right triangular rhotrices of size n as $RTR_n^*(F)$.

$$\text{Thus, } RTR_n^*(F) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & 0 & c_{11} & a_{12} & & & \\ & & 0 & 0 & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & 0 & 0 & a_{(t-1)(t-1)} & c_{(t-2)(t-1)} & a_{(t-2)t} & & \\ & & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle : a_{ij}, c_{lk} \in F, \det(a_{ij}) = 1, \det(c_{lk}) = 1 \right\},$$

$$where \quad a_{ij} = 0 \ \ if \ \ i > j \ \ and \ \ c_{lk} = 0 \ if \ \ l > k$$

Theorem3.3.15

Let $(RTR_n^*(F),\circ)$ be the special right triangular rhotrix subgroup of $(GR_n(F),\circ)$ and let $(SR_n(F),\circ)$ be the special rhotrix subgroup of $(GR_n(F),\circ)$, then the pair $(RTR_n^*(F),\circ)$ is a rhotrix subgroup of $(SR_n(F),\circ)$

Proof

Since $I_n \in RTR_n^*(F)$ then, $RTR_n^*(F) \neq \varnothing$.

Now, Let A_n and $B_n \in$ $RTR_n^*(F)$,

Then,det(A_n) =1 $\neq 0$, det(B_n) =1 $\neq 0$ respectively. This implies that for each $A_n, B_n \in RTR_n^*(F)$ $\exists\, A_n^{-1}$ and B_n^{-1} and by Theorem 3.3.13, A_n^{-1} and B_n^{-1} $\in RTR_n^*(F)$. Observe that $\det(A_n \circ B_n^{-1}) = \det(A_n) \circ \det(B_n^{-1}) = 1 \circ 1^{-1}$ (See proof of Theorem 3.3.2)

$\therefore \forall\ A_n, B_n \in RTR_n^*(F),\ A_n \circ B_n^{-1} \in RTR_n^*(F)$

Hence $RTR_n^*(F)$ is a subgroup of $(SR_n(F),\circ)$.

3.4 Isomorphism between Some Subgroups OfNon-commutative General Linear Rhotrix Group

In this section, we introduce isomorphic relationship between particular subgroups of the non-commutative General Linear Rhotrix Group.

Theorem 3.4.1

Let φ be a mapping from $(LTR_n(F),\circ)$ to $(RTR_n(F),\circ)$ defined by:

$$\varphi\left(\left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & 0 & & \\ & \cdots & \cdots\cdot & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{t(t-1)} & c_{(t-1)(t-1)} & 0 & & \\ & & & a_{tt} & & & \end{array} \right\rangle\right) = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & 0 & c_{11} & a_{12} & & \\ & \cdots & \cdots\cdot & \cdots & \cdots & \cdots & \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & 0 & c_{(t-1)(t-1)} & a_{(t-1)t} & & \\ & & & a_{tt} & & & \end{array} \right\rangle$$

Then the mapping φ is an isomorphism.

Proof

From the hypothesis of the theorem, φ is defined as

$$\varphi:(LTR_n(F),\circ)\to(RTR_n(F),\circ)$$

where

$$\varphi(R_n)=\varphi\ \langle a_{ij},c_{lk}\rangle\ =\langle a_{ji},c_{kl}\rangle$$

This is a homomorphism since, if $R_n = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle$ and $Q_n = \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$ then

$\varphi(R_n \circ Q_n) = \varphi \ \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$

$$= \varphi\left(\left\langle \sum_{i_2 j_1 = 1}^{t} a_{i_1 j_1} b_{i_2 j_2}, \sum_{l_2 k_1 = 1}^{t-1} c_{l_1 k_1} d_{l_2 k_2} \right\rangle\right)$$

$$= \left\langle \sum_{i_2 j_1 = 1}^{t} a_{j_1 i_1} b_{j_2 i_2}, \sum_{l_2 k_1 = 1}^{t-1} c_{k_1 l_1} d_{k_2 l_2} \right\rangle$$

$$= \left\langle a_{j_1 i_1}, c_{k_1 l_1} \right\rangle \circ \left\langle b_{j_2 i_2}, d_{k_2 l_2} \right\rangle$$

$$= \varphi \ \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \ \circ \varphi \ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$$

$$= \varphi \ R_n \ \circ \varphi \ Q_n$$

Next, φ is an injection since $\ker(\varphi) = \ I_n \in (LTR_n(F), \circ) : \varphi \ I_n \ = I_n^T \in (RTR_n(F), \circ)$.

Also, it is not hard to see that φ is an onto mapping because corresponding to any arbitrary rhotrix $R_n = \left\langle a_{ji}, c_{kl} \right\rangle \in RTR_n(F)$ there is a rhotrix $Q_n = \left\langle a_{ij}, c_{lk} \right\rangle \in LTR_n(F)$ such that $= \varphi \ Q_n \ = R_n$.

Corollary 3.4.1

Let φ be a mapping from $(LTR_5(F), \circ)$ to $(RTR_5(F), \circ)$ defined by

$$
\varphi\left(\left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & 0 & \\ a_{31} & c_{21} & a_{22} & 0 & 0 \\ & a_{32} & c_{22} & 0 & \\ & & a_{33} & & \end{array} \right\rangle\right) = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & 0 & c_{11} & a_{12} & \\ 0 & 0 & a_{22} & c_{12} & a_{13} \\ & 0 & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle
$$

The mapping φ is an isomorphism and $(LTR_5(F),\circ) \cong (RTR_5(F),\circ)$

Proof

Putting $n = 5$ in Theorem 3.4.1 above, the result follows.

CHAPTER FOUR

CONSTRUCTION OF SOME FINITE NON-COMMUTATIVERHOTRIX GROUPS

4.1 Introduction

In this chapter, we introduce concrete constructions of finite non-commutative rhotrix groups having entries from set of integers modulo *p*,where *p* is a positive prime.

4.2 The Non-commutative Rhotrix Group Over Finite Field

Theorem 4.2.1

Let *p* be a prime positive integer, let Z_p be the field of integers modulo *p*.

Let $$FGR_3(Z_p) = \left\{ \left\langle \begin{matrix} & a & \\ b & d & e \\ & c & \end{matrix} \right\rangle : a,b,c,d,e \in 0,1,2...(p-1) \quad and \ \det\left(\left\langle \begin{matrix} & a & \\ b & d & e \\ & c & \end{matrix} \right\rangle \right) \neq 0 \right\} \quad (4.1)$$

Let '$\circ$' be the binary operation of non-commutative rhotrix multiplication, then the pair $FGR_n(Z_p),\circ$ is the Non-Commutive General Linear Rhotrix Group of size *n* over the finitefield Z_p.

Proof

It is simple to show that the pair $FGR_n(Z_p),\circ$ satisfies all the axioms stated in Vashishta (2002) for finite groups.

4.3 Finite non-commutative rhotrix group of size 3 taking entries from Z_2

Let $R_3(Z_2)$ denote the set of all rhotrices of size 3 with entries from Z_2

$$R_3(Z_2) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & d & e \\ & c & \end{array} \right\rangle : a,b,c,d,e \in 0,1 \right\} \tag{4.2}$$

By the rule of permutation, the rhotrix set $R_3(Z_2)$ has cardinality 2^5 rhotrices. In tabular form of a set, we have

$$R_3(Z_2) = \left\{ \begin{array}{l}
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{array} \right\rangle
\end{array} \right\} \tag{4.3}$$

Now, our interest is to construct a rhotrix group consisting of all invertible rhotrices in $R_3(Z_2)$ and together with non-commutative method of rhotrix multiplication, and denote it as

$FGR_3(Z_2), \circ$.

To achieve this objective, we start by defining the set $FGR_3(Z_2)$ as follows:

$$FGR_3(Z_2) = \left\{ \left\langle \begin{matrix} & a & \\ b & d & e \\ & c & \end{matrix} \right\rangle : a,b,c,d,e \in 0,1 \quad and \ \det\left(\left\langle \begin{matrix} & a & \\ b & d & e \\ & c & \end{matrix} \right\rangle \right) \neq 0 \right\}. \tag{4.4}$$

Implying that $FGR_3(Z_2)$ is a collection of all rhotrices in $R_3(Z_2)$ satisfying the condition that the sub-matrices which make up such rhotrices in $R_3(Z_2)$ must be non-singular. Thus, in tabular form, we have

$$FGR_3(Z_2) = \left\{ \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle \right\}$$

.

The following result is a corollary to Theorem 4.2.1

Corollary 4.3.1

The pair $FGR_3(Z_2), \circ$ is a finite non-commutative rhotrix group of order 6.

Let us denote the elements in $FGR_3(Z_2), \circ$ as follows:

$$R1 = \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, R2 = \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, R3 = \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, R4 = \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, R5 = \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, R6 = \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle$$

.The multiplication table for $FGR_3(Z_2), \circ$ is given below by Table 4.1

Table 4.1: Multiplication table for $FGR_3(Z_2),\circ$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$
$R1$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$
$R2$	$R2$	$R4$	$R6$	$R1$	$R3$	$R5$
$R3$	$R3$	$R5$	$R1$	$R6$	$R2$	$R4$
$R4$	$R4$	$R1$	$R5$	$R2$	$R6$	$R3$
$R5$	$R5$	$R6$	$R4$	$R3$	$R1$	$R2$
$R6$	$R6$	$R3$	$R2$	$R5$	$R4$	$R1$

4.3.2 Subgroups of $FGR_3(Z_2),\circ$.

We investigate the subgroups of $FGR_3(Z_2),\circ$. Observe that there exist at least five subgroups of $FGR_3(Z_2),\circ$ and then the group itself. The subgroups are given in the following lemmas:

Lemma 4.3.1

Let $S1FGR_3(Z_2)$ be a subset of $FGR_3(Z_2)$ defined by

$$S1FGR_3(Z_2) = R1, R6$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S1FGR_3(Z_2),\circ$ is a subgroup of $FGR_3(Z_2),\circ$. In particular, $S1FGR_3(Z_2),\circ$ is the Orthogonal rhotrix subgroup of $FGR_3(Z_2),\circ$ and $\left| S1FGR_3(Z_2),\circ \right| = 2$

The multiplication table for $S1FGR_3(Z_2),\circ$ is given by Table 4.2

Table 4.2: Multiplication table for $S1FGR_3(Z_2),\circ$

$\circ$	$R1$	$R6$
$R1$	$R1$	$R6$
$R6$	$R6$	$R1$

Lemma 4.3.2

Let $S2FGR_3(Z_2)$ be a subset of $FGR_3(Z_2)$ defined by

$$S2FGR_3(Z_2) = R1, R3$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S2FGR_3(Z_2),\circ$ is a subgroup of $FGR_3(Z_2),\circ$. In particular, $S2FGR_3(Z_2),\circ$ is the right triangular rhotrix subgroup of $FGR_3(Z_2),\circ$ and $\left| S1FGR_3(Z_2),\circ \right| = 2$

The multiplication table for $S2FGR_3(Z_2),\circ$ is given by Table 4.3

Table 4.3: Multiplication table for $S2FGR_3(Z_2),\circ$

$\circ$	$R1$	$R3$
$R1$	$R1$	$R3$
$R3$	$R3$	$R1$

Lemma 4.3.3

Let $S3FGR_3(Z_2)$ be a subset of $FGR_3(Z_2)$ defined by

$$S3FGR_3(Z_2) = R1, R5$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then

$S3FGR_3(Z_2),\circ$ is a subgroup of $FGR_3(Z_2),\circ$. In particular, $S3FGR_3(Z_2),\circ$ is the left

triangular rhotrix subgroup of $FGR_3(Z_2),\circ$ and $\left| S3FGR_3(Z_2),\circ \right| = 2$

The multiplication table for $S3FGR_3(Z_2),\circ$ is given by Table 4.4

Table 4.4: Multiplication table for $S3FGR_3(Z_2),\circ$

$\circ$	$R1$	$R5$
$R1$	$R1$	$R5$
$R5$	$R5$	$R1$

Lemma 4.3.4

Let $S4FGR_3(Z_2)$ be a subset of $FGR_3(Z_2)$ defined by

$$S4FGR_3(Z_2) = R1, R2, R4$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then

$S4FGR_3(Z_2),\circ$ is a subgroup of $FGR_3(Z_2),\circ$. In particular, $S4FGR_3(Z_2),\circ$ is the

Symmetric rhotrix subgroup of $FGR_3(Z_2),\circ$ and $\left| S4FGR_3(Z_2),\circ \right| = 3$

The multiplication table for $S4FGR_3(Z_2),\circ$ is given by Table 4.5

Table 4.5: Multiplication table for $S3FGR_3(Z_2),\circ$

$\circ$	$R1$	$R2$	$R4$
$R1$	$R1$	$R2$	$R4$
$R2$	$R2$	$R4$	$R1$
$R4$	$R4$	$R1$	$R2$

Lemma 4.3.5

Let $S5FGR_3(Z_2)$ be a subset of $FGR_3(Z_2)$ defined by

$$S5FGR_3(Z_2) = \quad R1, R2, R3, R4, R5, R6$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then

$S5FGR_3(Z_2), \circ$ is a subgroup of $FGR_3(Z_2), \circ$. In particular, $S5FGR_3(Z_2), \circ$ is the

Special rhotrix subgroup $SFR_3(Z_2), \circ$ of $FGR_3(Z_2), \circ$ and $\left| \; S5FGR_3(Z_2), \circ \; \right| = 6$

The multiplication table for $S5FGR_3(Z_2), \circ$ is given by Table 4.1 above.

$S5FGR_3(Z_2), \circ$ is an improper subgroup of $FGR_3(Z_2), \circ$.

$S6FGR_3(Z_2), \circ \; = \quad R1 \;\; , \circ$ is the trivial subgroup of $FGR_3(Z_2), \circ$

This is in perfect harmony with Lagrange's Theorem on subgroups of finite groups in Vashishtha (2002).

Note that:

The order of each of the element of $FGR_3(Z_2), \circ$ is given below:

$\circ(R1) = 1, \circ(R2) = 3, \circ(R3) = 2, \circ(R4) = 3, \circ(R5) = 2, \circ(R6) = 2.$

4.4 Finite non-commutative rhotrix group of size 3 taking entries from Z_3

Let $R_3(Z_3)$ denote the set of all rhotrices of size 3 with entries from Z_3

$$R_3(Z_3) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & d & e \\ & c & \end{array} \right\rangle : a, b, c, d, e \in 0,1,2 \right\} \tag{4.5}$$

By the rule of permutation, the rhotrix set $R_3(Z_3)$ has cardinality 3^5 rhotrices. In tabular form of a set, the set $R_3(Z_3)$ is given by:

$$R_3(Z_3) = \left\{ \begin{array}{l}
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 2 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 2 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 0 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 2 \\ & 2 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 2 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 1 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 0 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 2 \\ & 0 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 2 & 0 & 2 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 2 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 2 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 1 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 2 \\ & 1 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 1 & \\ 0 & 0 & 0 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 2 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 2 \\ & 0 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 0 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 1 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 2 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 1 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 2 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 1 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 0 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 1 \\ & 1 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 1 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 2 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 2 & 0 & 2 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 2 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 0 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 2 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 2 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 2 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 2 & \\ 1 & 0 & 1 \\ & 1 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 2 & \\ 1 & 0 & 0 \\ & 1 & \end{array} \right\rangle
\end{array} \right\}$$

$R_3(Z_3)$ Continued…

$$\left\{\begin{array}{l}
\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 0 & 2 \\ & 1 & \end{smallmatrix}\right\rangle, \\
\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 1 \\ & 1 & \end{smallmatrix}\right\rangle, \\
\left\langle\begin{smallmatrix} & 2 & \\ 0 & 0 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 2 & 1 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 2 & 1 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 2 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 1 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle \\
\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 2 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 1 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 0 \\ & 2 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 1 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 0 \\ & 1 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 1 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 2 \\ & 0 & \end{smallmatrix}\right\rangle,\left\langle\begin{smallmatrix} & 2 & \\ 2 & 1 & 0 \\ & 0 & \end{smallmatrix}\right\rangle
\end{array}\right\}$$

$R_3(Z_3)$ Continued…

$$\Bigg\{ \left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 1 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 0 \\ & 2 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 0 \\ & 1 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 2 \\ & 2 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 2 \\ & 1 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 1 \\ & 2 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 0 \\ & 0 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 0 & \\ 0 & 2 & 0 \\ & 2 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 2 \\ & 2 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 2 \\ & 0 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 1 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 2 \\ & 1 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 0 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 0 & 2 \\ & 1 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 0 & 0 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 0 & 2 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 0 & 0 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 0 & 1 \\ & 2 & \end{array}\right\rangle$$

$$\left\langle\begin{array}{ccc} & 2 & \\ 1 & 0 & 2 \\ & 2 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 1 & 0 & 1 \\ & 0 & \end{array}\right\rangle, \left\langle\begin{array}{ccc} & 2 & \\ 2 & 0 & 2 \\ & 2 & \end{array}\right\rangle \Bigg\}$$

Now, our interest is to construct a rhotrix group consisting of all invertible rhotrices in $R_3(Z_3)$ and together with non-commutative method of rhotrix multiplication, and denote it as $FGR_3(Z_3),\circ$.

To achieve this objective, we start by defining the set $FGR_3(Z_3)$ as follows:

$$FGR_3(Z_3)=\left\{\left\langle \begin{array}{ccc} & a & \\ b & d & e \\ & c & \end{array}\right\rangle : a,b,c,d,e\in \ 0,1,2 \quad and \ \det\left(\left\langle \begin{array}{ccc} & a & \\ b & d & e \\ & c & \end{array}\right\rangle\right)\neq 0\right\}. \qquad (4.6)$$

Implying that $FGR_3(Z_3)$ is a collection of all rhotrices in $R_3(Z_3)$ satisfying the condition that the sub-matrices which make up such rhotrices in $R_3(Z_3)$ are all non-singular. Thus, in tabular form, we have $FGR_3(Z_3)$ given by the set below:

$$FGR_3(Z_3)=\left\{\begin{array}{l}
R1=\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array}\right\rangle, R2=\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 1 & \end{array}\right\rangle, R3=\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 2 & \end{array}\right\rangle, R4=\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 2 & \end{array}\right\rangle, R5=\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{array}\right\rangle \\
R6=\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 1 & \end{array}\right\rangle, R7=\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array}\right\rangle, R8=\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 1 & \end{array}\right\rangle, R9=\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 2 & \end{array}\right\rangle, R10=\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 2 & \end{array}\right\rangle \\
R11=\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 2 & \end{array}\right\rangle, R12=\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 2 & \end{array}\right\rangle, R13=\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{array}\right\rangle, R14=\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 0 \\ & 1 & \end{array}\right\rangle, R15=\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 2 & \end{array}\right\rangle \\
R16=\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 0 \\ & 2 & \end{array}\right\rangle, R17=\left\langle \begin{array}{ccc} & 1 & \\ 2 & 1 & 0 \\ & 2 & \end{array}\right\rangle, R18=\left\langle \begin{array}{ccc} & 2 & \\ 2 & 1 & 0 \\ & 2 & \end{array}\right\rangle, R19=\left\langle \begin{array}{ccc} & 1 & \\ 2 & 1 & 0 \\ & 1 & \end{array}\right\rangle, R20=\left\langle \begin{array}{ccc} & 2 & \\ 2 & 1 & 0 \\ & 1 & \end{array}\right\rangle \\
R21=\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array}\right\rangle, R22\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 2 \\ & 1 & \end{array}\right\rangle, R23=\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 2 \\ & 2 & \end{array}\right\rangle, R24=\left\langle \begin{array}{ccc} & 0 & \\ 2 & 1 & 2 \\ & 2 & \end{array}\right\rangle, R25=\left\langle \begin{array}{ccc} & 0 & \\ 2 & 1 & 1 \\ & 2 & \end{array}\right\rangle \\
R26=\left\langle \begin{array}{ccc} & 0 & \\ 2 & 1 & 2 \\ & 1 & \end{array}\right\rangle, R27=\left\langle \begin{array}{ccc} & 0 & \\ 2 & 1 & 1 \\ & 1 & \end{array}\right\rangle, R28=\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 2 & \end{array}\right\rangle, R29=\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{array}\right\rangle, R30=\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 1 \\ & 0 & \end{array}\right\rangle
\end{array}\right\}$$

$FGR_3(Z_3)$ Continued…

$$\left\{\begin{array}{l}
R31=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ & 0 & \end{array}\right\rangle, R32=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 2 \\ & 0 & \end{array}\right\rangle, R33=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 2 \\ & 0 & \end{array}\right\rangle, R34=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 2 \\ & 0 & \end{array}\right\rangle, R35=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 1 \\ & 0 & \end{array}\right\rangle, R36=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 1 \\ & 0 & \end{array}\right\rangle \\
R37=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 1 \\ & 1 & \end{array}\right\rangle, R38=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ & 1 & \end{array}\right\rangle, R39=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 2 \\ & 2 & \end{array}\right\rangle, R40=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 2 \\ & 2 & \end{array}\right\rangle, R41=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 1 \\ & 2 & \end{array}\right\rangle, R42=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 2 \\ & 1 & \end{array}\right\rangle \\
R43=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 1 \\ & 1 & \end{array}\right\rangle, R44=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 2 & \end{array}\right\rangle, R45=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 1 \\ & 0 & \end{array}\right\rangle, R46=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 1 & 2 \\ & 0 & \end{array}\right\rangle, R47=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{array}\right\rangle, R48=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 2 \\ & 0 & \end{array}\right\rangle, \\
R49=\left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 0 \\ & 1 & \end{array}\right\rangle, R50=\left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 0 \\ & 1 & \end{array}\right\rangle, R51=\left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 0 \\ & 2 & \end{array}\right\rangle, R52=\left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 0 \\ & 2 & \end{array}\right\rangle, R53=\left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 1 \\ & 1 & \end{array}\right\rangle, R54=\left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 1 \\ & 1 & \end{array}\right\rangle, \\
R55=\left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 2 \\ & 1 & \end{array}\right\rangle, R56=\left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 2 \\ & 1 & \end{array}\right\rangle, R57=\left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 1 \\ & 2 & \end{array}\right\rangle, R58=\left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 1 \\ & 2 & \end{array}\right\rangle, R59=\left\langle\begin{array}{ccc} & 1 & \\ 0 & 2 & 2 \\ & 2 & \end{array}\right\rangle, R60=\left\langle\begin{array}{ccc} & 2 & \\ 0 & 2 & 2 \\ & 2 & \end{array}\right\rangle \\
R61=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 0 \\ & 1 & \end{array}\right\rangle, R62=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 0 \\ & 1 & \end{array}\right\rangle, R63=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 0 \\ & 2 & \end{array}\right\rangle, R64=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 0 \\ & 2 & \end{array}\right\rangle, R65=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 0 \\ & 2 & \end{array}\right\rangle, R66=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 0 \\ & 2 & \end{array}\right\rangle \\
R67=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 0 \\ & 1 & \end{array}\right\rangle, R68=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 0 \\ & 1 & \end{array}\right\rangle, R69=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 1 & \end{array}\right\rangle, R70=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 2 \\ & 1 & \end{array}\right\rangle, R71=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 2 \\ & 2 & \end{array}\right\rangle, R72=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 2 \\ & 2 & \end{array}\right\rangle \\
R73=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 1 \\ & 2 & \end{array}\right\rangle, R74=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 2 \\ & 1 & \end{array}\right\rangle, R75=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 1 \\ & 1 & \end{array}\right\rangle, R76=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 2 & \end{array}\right\rangle, R77=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 0 & \end{array}\right\rangle, R78=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 1 \\ & 0 & \end{array}\right\rangle \\
R79=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 2 \\ & 0 & \end{array}\right\rangle, R80=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 2 \\ & 0 & \end{array}\right\rangle, R81=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 2 \\ & 0 & \end{array}\right\rangle, R82=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 2 \\ & 0 & \end{array}\right\rangle, R83=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 1 \\ & 0 & \end{array}\right\rangle, R84=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 1 \\ & 0 & \end{array}\right\rangle \\
R85=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 1 \\ & 1 & \end{array}\right\rangle, R86=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 2 \\ & 1 & \end{array}\right\rangle, R87=\left\langle\begin{array}{ccc} & 2 & \\ 1 & 2 & 2 \\ & 2 & \end{array}\right\rangle, R88=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 2 \\ & 2 & \end{array}\right\rangle, R89=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 1 \\ & 2 & \end{array}\right\rangle, R90=\left\langle\begin{array}{ccc} & 2 & \\ 2 & 2 & 2 \\ & 1 & \end{array}\right\rangle \\
R91=\left\langle\begin{array}{ccc} & 1 & \\ 2 & 2 & 1 \\ & 1 & \end{array}\right\rangle, R92=\left\langle\begin{array}{ccc} & 1 & \\ 1 & 2 & 1 \\ & 2 & \end{array}\right\rangle, R93=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 1 \\ & 0 & \end{array}\right\rangle, R94=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 2 \\ & 0 & \end{array}\right\rangle, R95=\left\langle\begin{array}{ccc} & 0 & \\ 1 & 2 & 1 \\ & 0 & \end{array}\right\rangle, R96=\left\langle\begin{array}{ccc} & 0 & \\ 2 & 2 & 2 \\ & 0 & \end{array}\right\rangle
\end{array}\right\}$$

The following result is a corollary to theorem 4.2.1

Corollary 4.4.1

Let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication and let $FGR_3(Z_3)$ be the set of all invertible rhotrices of size 3 with entries from Z_3, Then the pair $FGR_3(Z_3),\circ$ is a finite non-commutative rhotrix group of order 96.

4.4.2 SUBGROUPS OF $FGR_3(Z_3),\circ$.

An investigation of the subgroups of $FGR_3(Z_3),\circ$ is considered. Observe that there exist at least 17 proper subgroups of $FGR_3(Z_3),\circ$ and the group itself. The subgroups are given by the following lemmas:

Lemma 4.4.2

Let $S1FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S1FGR_3(Z_3) = R1, R52$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S1FGR_3(Z_3),\circ$ is a subgroup of $FGR_3(Z_3),\circ$. In particular, $S1FGR_3(Z_3),\circ$ is a scalar rhotrix subgroup of $FGR_3(Z_3),\circ$ and $\left| S1FGR_3(Z_3),\circ \right| = 2$.

The multiplication table for $S1FGR_3(Z_3),\circ$ is given by Table 4.6

Table 4.6: Multiplication table for $S1FGR_3(Z_3), \circ$

$\circ$	$R1$	$R52$
$R1$	$R1$	$R52$
$R52$	$R52$	$R1$

Lemma 4.4.3

Let $S2FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S2FGR_3(Z_3) = R1, R2, R3, R4$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S2FGR_3(Z_3), \circ$ is a subgroup of $FGR_3(Z_3), \circ$. In particular, $S2FGR_3(Z_3), \circ$ is a diagonal rhotrix subgroup of $FGR_3(Z_3), \circ$ with unit heart and $\left| S2FGR_3(Z_3), \circ \right| = 4$.

The multiplication table for $S2FGR_3(Z_3), \circ$ is given by Table 4.7

Table 4.7: Multiplication table for $S2FGR_3(Z_3), \circ$

$\circ$	$R1$	$R2$	$R3$	$R4$
$R1$	$R1$	$R2$	$R3$	$R4$
$R2$	$R2$	$R1$	$R4$	$R3$
$R3$	$R3$	$R4$	$R1$	$R2$
$R4$	$R4$	$R3$	$R2$	$R1$

Lemma 4.4.4

Let $S3FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S3FGR_3(Z_3) = R1, R4, R50, R51$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S3FGR_3(Z_3), \circ$ is a subgroup of $FGR_3(Z_3), \circ$. In particular, $S3FGR_3(Z_3), \circ$ is a special diagonal rhotrix subgroup of $FGR_3(Z_3), \circ$ and $\left| S3FGR_3(Z_3), \circ \right| = 4$.

The multiplication table for $S3FGR_3(Z_3), \circ$ is given by Table 4.8

Table 4.8: Multiplication table for $S3FGR_3(Z_3), \circ$

$\circ$	$R1$	$R4$	$R50$	$R51$
$R1$	$R1$	$R4$	$R50$	$R51$
$R4$	$R4$	$R1$	$R51$	$R50$
$R50$	$R50$	$R51$	$R1$	$R4$
$R51$	$R51$	$R50$	$R4$	$R1$

Lemma 4.4.5

Let $S4FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S4FGR_3(Z_3) = R1, R2, R3, R4, R49, R50, R51, R52$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S4FGR_3(Z_3), \circ$ is a subgroup of $FGR_3(Z_3), \circ$. In particular, $S4FGR_3(Z_3), \circ$ is a diagonal rhotrix subgroup of $FGR_3(Z_3), \circ$ and $\left| S4FGR_3(Z_3), \circ \right| = 8$.

The multiplication table for $S4FGR_3(Z_3),\circ$ is given by Table 4.9.

Table 4.9: Multiplication table for $S4FGR_3(Z_3),\circ$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R49$	$R50$	$R51$	$R52$
$R1$	$R1$	$R2$	$R3$	$R4$	$R49$	$R50$	$R51$	$R52$
$R2$	$R2$	$R1$	$R4$	$R3$	$R50$	$R49$	$R52$	$R51$
$R3$	$R3$	$R4$	$R1$	$R2$	$R51$	$R52$	$R49$	$R50$
$R4$	$R4$	$R3$	$R2$	$R1$	$R52$	$R51$	$R50$	$R49$
$R49$	$R49$	$R50$	$R51$	$R52$	$R1$	$R2$	$R3$	$R4$
$R50$	$R50$	$R49$	$R52$	$R51$	$R2$	$R1$	$R4$	$R3$
$R51$	$R51$	$R52$	$R49$	$R50$	$R3$	$R4$	$R1$	$R2$
$R52$	$R52$	$R51$	$R50$	$R49$	$R4$	$R3$	$R2$	$R1$

Lemma 4.4.6

Let $S5FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S5FGR_3(Z_3)=\begin{Bmatrix} R1,R4,R5,R7,R10,R12,R13,R16,R18,R19,R22,R23, \\ R25,R27,R31,R32,R35,R36,R37,R40,R42,R44,R45,R46 \end{Bmatrix}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S5FGR_3(Z_3),\circ$ is a subgroup of $FGR_3(Z_3),\circ$. In particular, $S5FGR_3(Z_3),\circ$ is a Special rhotrix subgroup of $FGR_3(Z_3),\circ$ with unit heart.

Note that

$$\left| S5FGR_3(Z_3),\circ \right| = 24$$

Lemma 4.4.7

Let $S6FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S6FGR_3(Z_3) = \begin{Bmatrix} R1, R4, R5, R7, R10, R12, R13, R16, R18, R19, R22, R23, \\ R25, R27, R31, R32, R35, R36, R37, R40, R42, R44, R45, R46, \\ R50, R51, R54, R56, R57, R59, R62, R63, R65, R68, R69, R72, \\ R74, R76, R77, R78, R81, R82, R86, R87, R89, R91, R95, R96 \end{Bmatrix}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S6FGR_3(Z_3),\circ$ is a subgroup of $FGR_3(Z_3),\circ$. In particular, $S6FGR_3(Z_3),\circ$ is the Special rhotrix subgroup of $FGR_3(Z_3),\circ$.

Note that

$$\left| S6FGR_3(Z_3),\circ \right| = 48$$

Lemma 4.4.8

Let $S7FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S7FGR_3(Z_3) = R1, R2, R3, R4, R13, R14, R15, R16, R17, R18, R19, R20$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S7FGR_3(Z_3),\circ$ is a subgroup of $FGR_3(Z_3),\circ$. In particular, $S7FGR_3(Z_3),\circ$ is a Left Triangular rhotrix subgroup of $FGR_3(Z_3),\circ$ with unit heart.

The multiplication table for $S7FGR_3(Z_3),\circ$ is given by Table 4.10

Table 4.10: The multiplication table for $S7FGR_3(Z_3),\circ$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R13$	$R14$	$R15$	$R16$	$R17$	$R18$	$R19$	$R20$
$R1$	$R1$	$R2$	$R3$	$R4$	$R13$	$R14$	$R15$	$R16$	$R17$	$R18$	$R19$	$R20$
$R2$	$R2$	$R1$	$R4$	$R3$	$R14$	$R13$	$R16$	$R15$	$R18$	$R17$	$R20$	$R19$

$R3$	$R3$	$R4$	$R1$	$R2$	$R17$	$R18$	$R19$	$R20$	$R13$	$R14$	$R15$	$R16$
$R4$	$R4$	$R3$	$R2$	$R1$	$R18$	$R17$	$R20$	$R19$	$R14$	$R13$	$R16$	$R15$
$R13$	$R13$	$R20$	$R15$	$R18$	$R19$	$R2$	$R17$	$R4$	$R3$	$R16$	$R1$	$R14$
$R14$	$R14$	$R19$	$R16$	$R17$	$R20$	$R1$	$R18$	$R3$	$R4$	$R15$	$R2$	$R13$
$R15$	$R15$	$R18$	$R13$	$R20$	$R3$	$R16$	$R1$	$R14$	$R19$	$R2$	$R17$	$R4$
$R16$	$R16$	$R17$	$R14$	$R19$	$R4$	$R15$	$R2$	$R13$	$R20$	$R1$	$R18$	$R3$
$R17$	$R17$	$R16$	$R19$	$R14$	$R15$	$R4$	$R13$	$R2$	$R1$	$R20$	$R3$	$R18$
$R18$	$R18$	$R15$	$R20$	$R13$	$R16$	$R3$	$R14$	$R1$	$R2$	$R19$	$R4$	$R17$
$R19$	$R19$	$R14$	$R17$	$R16$	$R1$	$R20$	$R3$	$R18$	$R15$	$R4$	$R13$	$R2$
$R20$	$R20$	$R13$	$R18$	$R15$	$R2$	$R19$	$R4$	$R17$	$R16$	$R3$	$R14$	$R1$

Note that: $\left| S7FGR_3(Z_3), \circ \right| = 12$

Lemma 4.4.9

Let $S8FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S8FGR_3(Z_3) = \begin{Bmatrix} R1, R2, R3, R4, R13, R14, R15, R16, R17, R18, R19, R20 \\ R49, R50, R51, R52, R61, R62, R63, R64, R65, R66, R67, R68 \end{Bmatrix}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S8FGR_3(Z_3), \circ$ is a subgroup of $FGR_3(Z_3), \circ$. In particular, $S8FGR_3(Z_3), \circ$ is a left triangular rhotrix subgroup of $FGR_3(Z_3), \circ$.

Note that: $\left| S8FGR_3(Z_3), \circ \right| = 24$

Lemma 4.4.10

Let $S9FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S9FGR_3(Z_3) = R1, R2, R3, R4, R5, R6, R7, R8, R9, R10, R11, R12$$

and let '∘' be a binary operation of non-commutative method of rhotrix multiplication, then $S9FGR_3(Z_3),\circ$ is a subgroup of $FGR_3(Z_3),\circ$. In particular, $S9FGR_3(Z_3),\circ$ is a right triangular rhotrix subgroup of $FGR_3(Z_3),\circ$ with unit heart.

The multiplication table for $S9FGR_3(Z_3),\circ$ is given by Table 4.11

Table 4.11: Multiplication table for $S9FGR_3(Z_3),\circ$

$\circ$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$	$R7$	$R8$	$R9$	$R10$	$R11$	$R12$
$R1$	$R1$	$R2$	$R3$	$R4$	$R5$	$R6$	$R7$	$R8$	$R9$	$R10$	$R11$	$R12$
$R2$	$R2$	$R1$	$R4$	$R3$	$R8$	$R7$	$R6$	$R5$	$R12$	$R11$	$R10$	$R9$
$R3$	$R3$	$R4$	$R1$	$R2$	$R9$	$R10$	$R11$	$R12$	$R5$	$R6$	$R7$	$R8$
$R4$	$R4$	$R3$	$R2$	$R1$	$R7$	$R11$	$R10$	$R9$	$R8$	$R7$	$R6$	$R5$
$R5$	$R5$	$R6$	$R11$	$R12$	$R12$	$R8$	$R1$	$R2$	$R3$	$R4$	$R9$	$R10$
$R6$	$R6$	$R5$	$R12$	$R11$	$R2$	$R1$	$R8$	$R7$	$R10$	$R9$	$R4$	$R3$
$R7$	$R7$	$R8$	$R9$	$R10$	$R1$	$R2$	$R5$	$R6$	$R11$	$R12$	$R3$	$R4$
$R8$	$R8$	$R7$	$R10$	$R9$	$R6$	$R5$	$R2$	$R1$	$R4$	$R3$	$R12$	$R11$
$R9$	$R9$	$R10$	$R7$	$R8$	$R11$	$R12$	$R3$	$R4$	$R1$	$R2$	$R5$	$R6$
$R10$	$R10$	$R9$	$R8$	$R7$	$R4$	$R3$	$R12$	$R11$	$R6$	$R5$	$R2$	$R1$
$R11$	$R11$	$R12$	$R5$	$R6$	$R3$	$R4$	$R9$	$R10$	$R7$	$R8$	$R1$	$R2$
$R12$	$R12$	$R11$	$R6$	$R5$	$R10$	$R9$	$R4$	$R3$	$R2$	$R1$	$R8$	$R7$

Note that: $\left| S9FGR_3(Z_3),\circ \right| = 12$

Lemma 4.4.11

Let $S10FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S10FGR_3(Z_3) = \left\{ \begin{array}{l} R1, R2, R3, R4, R5, R6, R7, R8, R9, R10, R11, R12, \\ R49, R50, R51, R52, R53, R54, R55, R56, R57, R58, R59, R60 \end{array} \right\}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S10FGR_3(Z_3),\circ$ is a subgroup of $FGR_3(Z_3),\circ$. In particular, $S10FGR_3(Z_3),\circ$ is a right triangular rhotrix subgroup of $FGR_3(Z_3),\circ$.

Note that: $\left| S10FGR_3(Z_3),\circ \right| = 24$

Lemma 4.4.12

Let $S11FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S11FGR_3(Z_3) = R1, R4, R13, R16, R18, R19$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S11FGR_3(Z_3),\circ$ is a subgroup of $FGR_3(Z_3),\circ$. In particular, $S11FGR_3(Z_3),\circ$ is a special left triangular rhotrix subgroup of $FGR_3(Z_3),\circ$ with unit heart.

The multiplication table for $S11FGR_3(Z_3),\circ$ is given Table 4.12

Table 4.12: Multiplication table for $S11FGR_3(Z_3),\circ$

$\circ$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$
$R1$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$
$R4$	$R4$	$R1$	$R18$	$R19$	$R13$	$R16$
$R13$	$R13$	$R18$	$R19$	$R4$	$R16$	$R1$

$R16$	$R16$	$R19$	$R4$	$R13$	$R1$	$R18$
$R18$	$R18$	$R13$	$R16$	$R1$	$R19$	$R4$
$R19$	$R19$	$R16$	$R1$	$R18$	$R4$	$R13$

Note that: $\left| S11FGR_3(Z_3), \circ \right| = 6$

Lemma 4.4.13

Let $S12FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S12FGR_3(Z_3) = \begin{Bmatrix} R1, R4, R13, R16, R18, R19, \\ R50, R51, R62, R63, R65, R68 \end{Bmatrix}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S12FGR_3(Z_3), \circ$ is a subgroup of $FGR_3(Z_3), \circ$. In particular, $S12FGR_3(Z_3), \circ$ is a special left triangular rhotrix subgroup of $FGR_3(Z_3), \circ$.

The multiplication table for $S12FGR_3(Z_3), \circ$ is given by Table 4.13

Table 4.13: The multiplication table for $S12FGR_3(Z_3), \circ$

$\circ$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$	$R50$	$R51$	$R62$	$R63$	$R65$	$R68$
$R1$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$	$R50$	$R51$	$R62$	$R63$	$R65$	$R68$
$R4$	$R4$	$R1$	$R18$	$R19$	$R13$	$R16$	$R51$	$R50$	$R65$	$R68$	$R62$	$R63$
$R13$	$R13$	$R18$	$R19$	$R4$	$R16$	$R1$	$R68$	$R63$	$R50$	$R65$	$R51$	$R62$

$R16$	$R16$	$R19$	$R4$	$R13$	$R1$	$R18$	$R65$	$R62$	$R63$	$R50$	$R68$	$R51$
$R18$	$R18$	$R13$	$R16$	$R1$	$R19$	$R4$	$R63$	$R68$	$R51$	$R62$	$R50$	$R65$
$R19$	$R19$	$R16$	$R1$	$R18$	$R4$	$R13$	$R62$	$R65$	$R68$	$R51$	$R63$	$R50$
$R50$	$R50$	$R51$	$R62$	$R63$	$R65$	$R68$	$R1$	$R4$	$R13$	$R16$	$R18$	$R19$
$R51$	$R51$	$R50$	$R65$	$R68$	$R62$	$R63$	$R4$	$R1$	$R18$	$R19$	$R13$	$R16$
$R62$	$R62$	$R65$	$R68$	$R51$	$R63$	$R50$	$R19$	$R16$	$R1$	$R18$	$R4$	$R13$
$R63$	$R63$	$R68$	$R51$	$R62$	$R50$	$R65$	$R18$	$R13$	$R16$	$R1$	$R19$	$R4$
$R65$	$R65$	$R62$	$R63$	$R50$	$R68$	$R51$	$R16$	$R19$	$R4$	$R13$	$R1$	$R18$
$R68$	$R68$	$R63$	$R50$	$R65$	$R51$	$R62$	$R13$	$R18$	$R19$	$R4$	$R16$	$R1$

Note that: $\left| S12FGR_3(Z_3),\circ \right| = 12$

Lemma 4.4.14

Let $S13FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S13FGR_3(Z_3),\circ = R1, R4, R5, R7, R10, R12$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $S13FGR_3(Z_3),\circ$ is a subgroup of $(FGR_3(Z_3),\circ)$. In particular, $(S13FGR_3(Z_3),\circ)$ is a special right triangular rhotrix subgroup of $(FGR_3(Z_3),\circ)$ with unit heart.

The multiplication table for $(S13FGR_3(Z_3),\circ)$ is given by Table 4.14

Table 4.14: Multiplication table for $(S13FGR_3(Z_3),\circ)$

$\circ$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$
$R1$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$
$R4$	$R4$	$R1$	$R12$	$R10$	$R7$	$R5$
$R5$	$R5$	$R12$	$R7$	$R1$	$R4$	$R10$

$R7$	$R7$	$R10$	$R1$	$R5$	$R12$	$R4$
$R10$	$R10$	$R7$	$R4$	$R12$	$R5$	$R1$
$R12$	$R12$	$R5$	$R10$	$R4$	$R1$	$R7$

Note that: $\left| S13FGR_3(Z_3),\circ \right| = 6$

Lemma 4.4.15

Let $S14FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S14FGR_3(Z_3),\circ = R1, R4, R5, R7, R10, R12, R50, R51, R54, R56, R57, R59$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $(S14FGR_3(Z_3),\circ)$ is a subgroup of $(FGR_3(Z_3),\circ)$. In particular, $(S14FGR_3(Z_3),\circ)$ is a special right triangular rhotrix subgroup of $(FGR_3(Z_3),\circ)$.

The multiplication table for $(S14FGR_3(Z_3),\circ)$ is given by Table 4.15

Table 4.15: Multiplication table for $R5$ $(S14FGR_3(Z_3),\circ)$

$\circ$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$	$R50$	$R51$	$R54$	$R56$	$R57$	$R59$
$R1$	$R1$	$R4$	$R5$	$R7$	$R10$	$R12$	$R50$	$R51$	$R54$	$R56$	$R57$	$R59$
$R4$	$R4$	$R1$	$R12$	$R10$	$R7$	$R5$	$R51$	$R50$	$R59$	$R57$	$R56$	$R54$

$R5$	$R5$	$R12$	$R7$	$R1$	$R4$	$R10$	$R54$	$R59$	$R56$	$R50$	$R51$	$R57$
$R7$	$R7$	$R10$	$R1$	$R5$	$R12$	$R4$	$R56$	$R57$	$R50$	$R54$	$R59$	$R51$
$R10$	$R10$	$R7$	$R4$	$R12$	$R5$	$R1$	$R57$	$R56$	$R51$	$R59$	$R54$	$R50$
$R12$	$R12$	$R5$	$R10$	$R4$	$R1$	$R7$	$R59$	$R54$	$R57$	$R51$	$R50$	$R56$
$R50$	$R50$	$R51$	$R56$	$R54$	$R59$	$R57$	$R1$	$R4$	$R7$	$R5$	$R12$	$R10$
$R51$	$R51$	$R50$	$R57$	$R59$	$R54$	$R56$	$R4$	$R1$	$R10$	$R12$	$R5$	$R7$
$R54$	$R54$	$R59$	$R50$	$R56$	$R57$	$R51$	$R5$	$R12$	$R1$	$R7$	$R10$	$R4$
$R56$	$R56$	$R57$	$R54$	$R50$	$R51$	$R59$	$R7$	$R10$	$R5$	$R1$	$R4$	$R12$
$R57$	$R57$	$R56$	$R59$	$R51$	$R50$	$R54$	$R10$	$R7$	$R12$	$R4$	$R1$	$R5$
$R59$	$R59$	$R54$	$R51$	$R57$	$R56$	$R50$	$R12$	$R5$	$R4$	$R10$	$R7$	$R1$

Note that: $\left| S14FGR_3(Z_3),\circ \right| = 12$

Lemma 4.4.16

Let $S15FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S15FGR_3(Z_3) = \{R1, R4, R47, R48\}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $(S15FGR_3(Z_3),\circ)$ is a subgroup of $(FGR_3(Z_3),\circ)$. In particular, $(S15FGR_3(Z_3),\circ)$ is a Symmetric rhotrix subgroup of $(FGR_3(Z_3),\circ)$ and $\left|(S15FGR_3(Z_3),\circ)\right| = 4$

The multiplication table for $S15FGR_3(Z_3),\circ$ is given by Table 4.16

Table 4.16: Multiplication table for $(S15FGR_3(Z_3),\circ)$

$\circ$	$R1$	$R4$	$R47$	$R48$
$R1$	$R1$	$R4$	$R47$	$R48$
$R4$	$R4$	$R1$	$R48$	$R47$

$R47$	$R47$	$R48$	$R4$	$R1$
$R48$	$R48$	$R47$	$R1$	$R4$

Lemma 4.4.17

Let $S16FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S16FGR_3(Z_3) = \{R1\}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $(S16FGR_3(Z_3), \circ)$ is a subgroup of $(FGR_3(Z_3), \circ)$.

Lemma 4.4.18

Let $S17FGR_3(Z_3)$ be a subset of $FGR_3(Z_3)$ defined by

$$S17FGR_3(Z_3) = \left\{ \begin{array}{l} R1, R2, R3, R4, R5, R6, R7, R8, R9, R10, R11, R12, \\ R13, R14, R15, R16, R17, R18, R19, R20, R21, R22, R23, R24, \\ R25, R26, R27, R28, R29, R30, R31, R32, R33, R34, R35, R36, \\ R37, R38, R39, R40, R41, R42, R43, R44, R45, R46, R47, R48 \end{array} \right\}$$

and let '$\circ$' be the binary operation of non-commutative method of rhotrix multiplication, then $(S17FGR_3(Z_3), \circ)$ is a subgroup of $(FGR_3(Z_3), \circ)$ .In particular $(S17FGR_3(Z_3), \circ)$ is a rhotrix subgroup of $(FGR_3(Z_3), \circ)$ with unit heart.

Note that: $\left| S17FGR_3(Z_3), \circ \right| = 48$

Remarks 4.4.19

The pair $(FGR_3(Z_3), \circ)$, is an improper subgroup of itself.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

We have considered an algebraic study of non-commutative rhotrix groups using rhotrix sets as underlying sets. In the process, a review of the progress made so far in the literature of rhotrix theory, starting from Ajibade (2003), when the concept was introduced, up to 2014 was made. A construction of non-commutative general linear rhotrix group considered to be analogous to the General Linear Group was presented.

The non-commutative general linear rhotrix group consists of all invertible rhotrices of size n with entries from an arbitrary field F and it possesses all non-commutative rhotrix groups as its subgroups. Certain subgroups of non-commutative general linear rhotrix group were identified and then shown to be embedded in certain subgroups of the general linear group. Furthermore, some finite non-commutative groups of rhotrices as well as their subgroups were constructed and schematized.

5.2 Conclusion

In conclusion, we have presented new algebraic systems termed as Non-commutative General Linear Rhotrix Groups. Some finite and infinite non commutative rhotrix groups and their generalization were considered. A number of theorems had also been developed. It is our hope that this study will go to a large extent in simplification of teaching and learning of group theory in Mathematical discipline.

5.3 Recommendations

We recommend that the theory of rhotrix groups being a relatively new paradigm of Algebra be applied in a number of areas as follows:

1. Computing non-commutative finite groups of rhotrices of larger size.
2. Development of non-commutative finite cyclic groups of rhotrices.
3. Construction and development of composition series for non-commutative finite group of rhotrices.
4. Extension of Sylow theorems to non-commutative finite groups of rhotrices.
5. Construction and development of non-commutative Polynomial groups of rhotrices.

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