STUDY ON SOME FIXED POINT THEOREMS FOR BREGMAN NONEXPANSIVE TYPE MAPPING IN BANACH SPACES

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LAWAN, Mohammed Suleman

DEPARTMENT OF MATHEMATICS

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LAWAN, Mohammed Suleman

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DECLARATION

I declare that the work in this dissertation entitled "STUDY ON SOME FIXED POINT THEOREMS FOR BREGMAN NONEXPANSIVE TYPE MAPPING IN BANACH SPACES" has been carried out by me in the Department of Mathematics under the supervision of Dr. Bashir Ali and Prof. Babangida Sani. The information derived from the literature has been duly acknowledged in the text and the list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

LAWAN, Mohammed Suleman

Name of student	Signature	Date

CERTIFICATION

This dissertation entitled " STUDY ON SOME FIXED POINT THEOREMS FOR BREGMAN NONEXPANSIVE TYPE MAPPING IN BANACH SPACES" by LAWAN, Mohammed Suleman (MSC/SCI/34373/2012-2013), meets the regulations governing the award of the degree of Master of Science of the Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

Dr. B. Ali	Date
Chairman Supervisory Committee	
Prof. B. Sani	Date
Member Supervisory Committee	
Prof. E. U. Ofeodu	Date
External Examiner	
Prof. A. A. Tijjani	Date
Head of Department	
Prof. Kabir Bala	Date
Dean, Postgraduate School	

DEDICATION

This research work is dedicated to my parents, Alh. Suleman Mohammed and Hajia Hauwa'u Shuaibu.

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In the name of Allah, the most gracious the most merciful, thanks be to Allah lord of the universe, master of the day of judgment, who gave me the strength, opportunity and endurance to pass through this M.Sc degree program, despite all odds. For Allah nothing is impossible.

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ABSTRACT

In this dissertation, an iterative sequence for approximation of common fixed points of finite family quasi-Bregman total asymptotically nonexpansive mappings is constructed. Necessary and sufficient conditions for the convergence of the scheme to a common fixed point of the mappings are given. Furthermore, sufficient condition for convergence of the iteration process to a common fixed point of the mappings under the same setting is established.Moreover a new iterative sequence by hybrid method is also introduced and a strong convergence of the sequence to a common element in the set of fixed points of finite family of closed quasi-Bregman total asymptotically strictly pseudocontractive mapping and common solution to a system of equilibrium problems in reflexive Banach spaces is proved. Our results extend important recent results announced by several authors.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Background of the Study

The study of nonlinear functional analysis has been in existence since early twentieth century, with investigation into the existence properties of solutions of certain boundary value problems, arising in nonlinear partial differential equations. The study of the approximation of fixed point theory of nonexpansive mappings, is applied to solutions of diverse problems such as solving variational inequality, equilibrium problems and finding solutions of certain evolution equations. Some authors have proposed several methods for approximation of fixed points of generalizations of nonexpansive mappings. For example Qin and Su (2007) established strong convergence theorems for relatively nonexpansive mappings in Banach spaces. Again, a new hybrid iterative scheme called a shrinking projection method for nonexpansive mappings was also introduced by (Takahashi et al., 2008).

1.2 Statement of the Problem

Chidume and Ofoedu (2007), constructed an iterative sequence for approximation of common fixed points of finite families of total asymptotically nonexpansive mappings. Then the question arises, can approximation of common fixed points of finite families of quasi-Bregman total asymptotically nonexpansive mappings be established? In this dissertation we answer this question by constructing an iterative sequence for approximation of common fixed points, of quasi-Bregman total asymptotically nonexpansive mappings.

Secondly, Ugwunnadi et al. (2014), proved a strong convergence theorem for quasi-Bregman strictly pseudocontractive mappings and equilibrium problems in reflexive Banach space. Again the question arises can strong convergence be established for quasi-Bregman total strictly pseudocontractive mappings and equilibrium problems in reflexive Banach space? We also answered this question by developing a new iterative scheme, by hybrid method and proved a strong convergence theorem for quasi-Bregman total asymptotically strictly pseudocontractive mappings and equilibrium problems in reflexive Banach spaces.

1.3 Aim and Objectives

The aim of this research is to establish some fixed point theorems for Bregman nonexpansive mapping in Banach spaces.

The aim will be achieved through the following objectives:

(i) Construction of an iterative sequence for approximation of common fixed points of quasi-Bregman total asymptotically nonexpansive mappings.

(ii) Development of a new hybrid iterative scheme and establishment of strong convergence theorem for quasi-Bregman total asymptotically strictly pseudocontractive mappings and equilibrium problems in reflexive Banach spaces.

1.4 Scope and Limitations

This research covers asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings and total asymptotically strictly pseudocontrac-

tive mappings using Bregman distance. The mappings must be uniformly L_i-Lipschitzian and uniformly asymptotically regular.

1.5 Research Methodology

The method used in this research, is by reviewing relevant and necessary articles, in literature; on fixed point theory, nonexpansive mappings, asymptotically nonexpansive mappings, total asymptotically nonexpansive mappings and total asymptotically strictly pseudocontractive mappings. These articles are reviewed extensively to cover researches done for Bregman distance in arbitrary Banach spaces.

1.6 Outline of the Thesis

The dissertation contains four other chapters, apart from this introductory chapter. The outline of the remaining chapters is as follows:
Chapter Two presents a survey of the necessary and relevant literature on fixed point theory, nonexpansive mappings, asymptotically nonexpansive mappings, strictly pseudocontractive mappings and Bregman distance.
Chapter Three review all relevant Lemmmas and Theorem that are necessary for proving our result.
Chapter Four discusses the construction of an iterative sequence for approximation of common fixed points of quasi-Bregman total asymptotically nonexpansive mappings ; necessary and sufficient conditions for the convergence of the scheme to a common fixed point of the mappings are given. Also a new iterative scheme by hybrid method was introduced and a strong convergence theorem of a common element in the set of fixed point of finite family

of closed quasi-Bregman total asymptotically strictly pseudocontractive mapping. A common solution to a system of equilibrium problems in reflexive Banach spaces is also established.
Chapter Five gives a summary of the results obtained in the research along with some directions for further research.

1.7 Preliminaries

In this section, some basic definitions and concepts which are useful and related to the context of our results will e given.

Let E be a real Banach space with dual E^*. The normalised duality map from E to 2^{E^*} denoted by J is defined by $J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2, \|x\| = \|f\|\}$.

Let C be a nonempty subset of E and $T : C \to C$ be a map, a point $x \in C$ is called a fixed point of T if and only if $Tx = x$, and the set of all fixed points of T is denoted by $F(T)$. The map T is called L-Lipschitzian or simply Lipschitz if there exists $L > 0$, such that $\|Tx - Ty\| \leq L\|x - y\|$, $\forall x, y \in C$. If $L = 1$, then the map T is called nonexpansive. T is called asymptotically nonexpansive if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, \infty)$ with $\lim_{n\to\infty} \mu_n = 0$, such that for all $x, y \in C$,

$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\|$ for all $n \geq 1$, Gobel and Kirk (1972). Let $g : C \times C \to R$ be a bifunction. The equilibrium problem with respect to g is to find $z \in C$ such that $g(z, y) \geq 0, \quad \forall\, y \in C$.

The set of solutions of equilibrium problem is denoted by EP(g). Thus

EP(g)=:$\{z \in C : g(z, y) \geq 0,\ \forall\, y \in C\}$.

Let E be a reflexive real Banach space with norm $\|\cdot\|$ and E^* the dual space of E. Throughout this dissertation, we shall assume $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\mathrm{dom} f := \{x \in E : f(x) < +\infty\}$ the domain of f.

Let $x \in \mathrm{intdom} f$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle y - x, x^* \rangle \leq f(y),\ \forall\, y \in E\},$$

The fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x, x^* \rangle - f(x) : x \in E\}.$$

That the young-Fenchel inequality holds:

$$\langle x, x^* \rangle \leq f(x) + f^*(x^*), \quad \forall x \in E.$$

A function f on E is coercive, Hiriart-Urruty and Lemarchal (1993), if and only if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

A function f on E is said to be strongly coercive, Zalinescu (2002), if

$$\lim_{\|x\| \to +\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in \mathrm{intdom} f$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^\circ(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}.$$

The function f is said to be Gateaux differentiable at x if and only if $\lim_{t \to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any $y \in E$. In this case, $f^\circ(x, y)$ coincides with $\bigtriangledown f(x)$, the value of the gradient $\bigtriangledown f$ of f at x. The function f is said to be Gateaux differentiable

if and only if it is Gateaux differentiable for any $x \in \text{int}\text{dom}f$. The function f is said to be Frechet differentiable at x if and only if this limit is attained uniformly in. Finally, f is said to be Gateaux differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$. It is well known that if f is Gateaux differentiable (resp. Frechet differentiable) on $\text{int}\text{dom}f$, then f is continuous and its Gateaux differentiable $\bigtriangledown f$ is norm-to-weak* continuous (resp. continuous) on $\text{int}\text{dom}f$ (see Asplund and Rockafellar (1969); Bonnans and Shapiro (2000)). We will need the following results.

Definition 1.7.1 *(Bauschke et al., 2001) A function f is said to be:*

(i) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $\text{dom}f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 1.7.1 *Let E be a reflexive Banach space, then:*

(i) f is essentially smooth if and only if f^* is essentially strictly convex (Bauschke et al. (2001));

(ii) $(\partial f)^{-1} = \partial f^*$ (Bonnans and Shapiro (2000));

(iii) f is Legendre if and only if f^* is Legendre (Bauschke et al. (2001));

(iv) If f is Legendre, then $\bigtriangledown f$ is a bijection satifying $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$, $ran \bigtriangledown f = dom \bigtriangledown f^* = int dom f^*$ and $ran \bigtriangledown f^* = dom f = int dom f$ (Bauschke et al. (2001)).

Examples of Legendre functions were given in Bauschke et al. (2001); Bauschke and Borwein (1997) .One important and interesting Legendre is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$

when E is a smooth and strictly convex Banach space. In this case, the gradient $\bigtriangledown f$ of f is coincident with the generalized duality mapping of E, i.e, $\bigtriangledown f = J_p$ $(1 < p < \infty)$. In particular, $\bigtriangledown f = I$ the identity mapping in Hilbert spaces. In the rest of this dissertation, we always assume that $f : E \to (-\infty, +\infty]$ is a Legendre.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The function
$D_f : \text{dom}f \times \text{intdom}f \to (-\infty, +\infty]$, defined as

$$D_f(x, y) := f(x) - f(y) - \langle x - y, \bigtriangledown f(y)\rangle, \tag{1.7.1}$$

is called the Bregman distance with respect to f, (Lent and Censor, 1981). It is obvious from the definition of D_f that

$$D_f(z, x) := D_f(z, y) + D_f(y, x) + \langle z - y, \bigtriangledown f(y) - \bigtriangledown f(x)\rangle. \tag{1.7.2}$$

Recall that the Bregman projection, (Bregman, 1967), of $x \in \text{intdom}f$ onto nonempty, closed and convex set $C \subset \text{dom}f$ is necessarily the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Let $f : E \to (-\infty, +\infty]$ be a Gateaux differentiable function. The modulus of total convexity of f at $x \in \text{intdom}f$ is the function $v_f(x, \cdot) : [0, +\infty]$ defined by

$v_f(x, t) := \inf\{D_f(x, y) : y \in \text{dom}f,\ \|y - x\| = t\}$

The function f is called totally convex at x if $v_f(x, t) > 0$ whenever $t > 0$. The

function f is called convex if it is totally convex at any point $x \in \text{intdom}f$ and is said to be totally convex on bounded set if $v_f(B,t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of the total convexity of the function f on the set B is the function $v_f : \text{intdom}f \times [0,+\infty) \to [0,+\infty)$ defined by

$$v_f(B,t) := \inf\{v_f(x,t) : x \in B \cap \text{ dom}f\}.$$

Recall that the function f is called sequentially consistent if for any two sequence $\{x_n\}$ and $\{y_n\}$ in E such that $\{x_n\}_{n=1}^{\infty}$ is bounded

$$\lim_{n\to+\infty} D_f(y_n, x_n) = 0 \qquad \Rightarrow \qquad \lim_{n\to+\infty} \|y_n - x_n\| = 0.$$

The following definition is slightly different from that given by Butnariu and Lusem (2000).

Definition 1.7.2 *(Kohsaka and Takahashi, 2005) Let E be a Banach space. The function $f : E \to \mathbb{R}$ is said to be a Bregman function if and only if the following conditions are satisfied:*

(i) f is continuous, strictly convex and Gâteaux differentiable;

(ii) the set $\{y \in E : D_f(x,y) \leq r\}$ is bounded for all $x \in E$ and $r > 0$.

Definition 1.7.3 *(Reich and Sabach, 2011) A map $T : C \to C$ is said to be Bregman firmly nonexpansive , if for all $x, y \in C$.*

$$\langle \bigtriangledown f(Tx) - \bigtriangledown f(Ty), Tx - Ty\rangle \leq \langle \bigtriangledown f(x) - \bigtriangledown f(y), Tx - Ty\rangle$$

or equivalently,

$$D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).$$

.

Let $T : C \to C$ be a map, a point $p \in C$ is said to be asymptotic fixed point of T if and only if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim_{n\to+\infty} \|x_n - Tx_n\| = 0$.

We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T.

A point $p \in C$ is said to be strong asymptotic fixed point of a map $T : C \to C$, if and only if there exists a sequence $\{x_n\}$ in C which converges strongly to p such that $\lim_{n\to+\infty} \|x_n - Tx_n\| = 0$.

We denote by $\tilde{F}(T)$ the set of strong asymptotic fixed points of T.

Definition 1.7.4 *(Matsushita and Takahashi, 2004) A map $T : C \to C$ is said to be Bregman relatively nonexpansive, if $F(T) \neq \emptyset$, $\hat{F}(T) = F(T)$ and $D_f(p, Tx) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$.*

The map $T : C \to C$ is said to be Bregman weak relatively nonexpansive if $F(T) \neq \emptyset$, $\tilde{F}(T) = F(T)$ and $D_f(p, Tx) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be quasi-Bregman relatively nonexpansive if $F(T) \neq \emptyset$ and $D_f(p, Tx) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 1.7.2 *In the work of Reich and Sabach (2010a) quasi-Bregman relatively nonexpansive mapping is called left quasi-Bregman relatively nonexpansive.*

Definition 1.7.5 *(Martin-Marquez et al., 2012) A map $T : C \to C$ is called right quasi-Bregman relatively nonexpansive if $F(T) \neq \emptyset$ and $D_f(Tx, p) \leq D_f(x, p)$ for all $x \in C$ and $p \in F(T)$,.*

Definition 1.7.6 *(Ugwunnadi et al., 2014) The map T is said to be quasi-Bregman strictly pseudocontractive if $F(T) \neq \emptyset$ and there exists a constant $\lambda \in [0, 1)$ such that $D_f(p, Tx) \leq D_f(p, x) + \lambda D_f(x, Tx)$ for all $x \in C$ and $p \in F(T)$.*

T is said to be Bregman quasi-asymptotically nonexpansive if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, \infty)$ with $\lim_{n\to\infty} \mu_n = 0$ such that for all $x \in C$ and $p \in F(T) \quad D_f(T^n x, p) \leq (1 + \mu_n) D_f(x, p)$ for all $n \geq 1$. The map T is said to be quasi-Bregman asymptotically nonexpansive in the intermediate sence if $F(T)) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{x\in C, p\in F(T)} \big(D_f(T^n x, p) - D_f(x, p)\big) \leq 0, (Tomizawa,\ 2014) \quad (1.7.3)$$

Put

$$\sigma_n = max\left\{0, \sup_{x\in C, p\in F(T)} \big(D_f(T^n x, p) - D_f(x, p)\big)\right\}$$

then $\sigma_n \to 0$ as $n \to \infty$ and (1.7.3) reduces to

$$D_f(T^n x, p) \leq D_f(x, p) + \sigma_n \quad (1.7.4)$$

Definition 1.7.7 *A mapping $T : C \to C$ is said to be quasi-Bregman total asymptotically nonexpansive if and only if there exists nonnegative real sequences $\{\mu_n\}\{l_n\} \subset [0, \infty)$, $n \geq 1$ with $\mu_n \to 0, l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in C$ and $p \in F(T)$,*

$$D_f(T^n x, p) \ \leq \ D_f(x, p) + \mu_n \phi\big(D_f(x, p)\big) + l_n, \quad n \geq 1. \quad (1.7.5)$$

Chang et al. (2014), gave an example of Bregman quasi-total asymptotically nonexpansive mapping.

Remark 1.7.3 *If $\phi(\lambda) = \lambda$, then (1.7.5) reduces to*

$$D_f(T^n x, p) \leq (1 + \mu_n) D_f(x, p) + l_n, \quad n \geq 1$$

In addition, if $l_n = 0$ for all $n \geq 1$, then quasi-Bregman total asymptotically nonexpansive mappings coincide with quasi-Bregman asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (1.7.5) the class of mappings that include the class of quasi-Bregman nonexpansive mappings. If $\mu_n = 0$ and $l_n = \sigma_n = max\{0, a_n\}$, where $a_n := \sup_{x \in C, p \in F(T)} \big(D_f(p, T^n x) - D_f(p, x)\big)$ for all $n \geq 1$, then (1.7.5) reduces to (1.7.4) which has been studied as quasi-Bregman asymptotically nonexpansive mappings in the intermediate sense.

Remark 1.7.4 *From these definitions,it is obvious that if $\hat{F}(T) = F(T) \neq \emptyset$, then a Bregman strongly nonexpansive mapping is a Bregman relatively nonexpansive mapping; A Bregman relatively nonexpansive mapping is a quasi-Bregman nonexpansive mapping. A quasi-Bregman nonexpansive mappings is a quasi-Bregman asymptotically nonexpansive mapping, but the converse in all cases is not true.*

Definition 1.7.8 *A mapping $T : C \to C$ is said to be quasi-Bregman asymptotically strictly pseudocontractive if and only if there exists $\lambda \in [0, 1)$, $F(T) \neq \emptyset$ and a sequence $\{\mu_n\}_{n \geq 1} \subset [0, \infty)$ with $\lim_{n \to \infty} \mu_n = 0$ such that*

$$D_f(p, T^n x) \leq (1 + \mu_n) D_f(p, x) + \lambda D_f(x, Tx) \quad \forall\, x \in C,\, p \in F(T).$$

Definition 1.7.9 *A mapping $T : C \to C$ is said to be quasi-Bregman total asymptotically strictly pseudocontractive if and only if there exists $\lambda \in [0, 1)$, $F(T) \neq$*

$\emptyset$ *and nonnegative real sequences* $\{\mu_n\}$ *and* $\{l_n\}, n \geq 1$ *with* $\mu_n \to 0, l_n \to 0$ *as* $n \to \infty$ *and strictly increasing continuous function* $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ *with* $\phi(0) = 0$ *such that*

$$D_f(p, T^n x) \leq D_f(p, x) + \lambda D_f(x, T^n x) + \mu_n \phi(D_f(p, x)) + l_n \quad \forall x \in C,\, p \in F(T). \tag{1.7.6}$$

Remark 1.7.5 *If* $\phi(\lambda) = \lambda$, *then (1.7.6) reduces to*

$$D_f(p, T^n x) \leq (1 + \mu_n) D_f(p, x) + \lambda D_f(x, Tx) + l_n \quad \forall\, x \in C,\, p \in F(T).$$

In addition, if $l_n = 0$ for all $n \geq 1$, then quasi-Bregman total asymptotically strictly pseudocontractive mappings coincide with quasi-Bregman asymptotically strictly pseudocontractive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (1.7.6) the class of mappings that include the class of quasi-Bregman strictly pseudocontractive mappings.

Remark 1.7.6 *From definitions 1.7.8 and 1.7.9 it is obvious that if* $\hat{F}(T) = F(T) \neq \emptyset$, *then a quasi-Bergman nonexpansive mapping is a quasi-Bregman strictly pseudocontractive mapping; A quasi-Bregman strictly pseudocontractive mapping is a quasi-Bregman asymptotically strictly pseudocontractive mapping but the converse in all cases is not true.*

CHAPTER TWO

LITERATURE REVIEW

2.1 Asymptotically Nonexpansive Mappings

Gobel and Kirk (1972) intoroduced the class of asymptotically nonexpansive mappings as a generalization of the class of nonexpansive mappings. If C is a nonempty closed convex bounded subset of a uniformly convex Banach space E and T is a self mapping of C which is asymptotically nonexpansive in the intermediate sense, then T has a fixed point, (Kirk, 1974). However the class of mappings which are asymptotically nonexpansive in the intermediate sense contains the class of asymptotically nonexpansive mappings.

A modified Mann iteration to approximate fixed points of asymptotically nonexpansive mappings in uniformly convex Banach spaces was introduced by (Schu, 1991) Osilike and Aniagbosor (2000) and Shahzad and Udomene (2006) obtained weak and strong convergence theorem for finding a fixed point of asymptotically nonexpansive mappings.

A more general class of mappings called total asymptotically nonexpansive mappings was introduced by Albert et al. (2006) and studied method of approximation of fixed points of mappings belonging to this class. Several authors are constructing iterative sequences for finding the fixed point of total asymptotically nonexpansive mappings.(such as Chidume and Ofoedu (2007) and Yolacan and Kizitunc (2012))

Chidume and Ofoedu (2007) constructed the system (2.1.1) for the approxima-

tion of common fixed points of finite families of total asymptotically nonexpansive mappings, and gave necessary and sufficient conditions for the convergence of the scheme to common fixed points of the mappings in arbitrary real Banach spaces. A sufficient condition for convergence of the iteration process to a common fixed point of mappings under the same setting was also established in real uniformly convex Banach spaces.

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + T_1^n x_n, \quad \text{if } m = 1,\ n \geq 1, \\ x_1 \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1^n y_{1n}, \\ y_{1n} = (1-\alpha_n)x_n + \alpha_n T_1^n y_{2n}, \\ \quad \vdots \\ y(m-2)_n = (1-\alpha_n)x_n + \alpha_n T_{m-1}^n y(m-1)_n, \\ y(m-1)_n = (1-\alpha_n)x_n + \alpha_n T_m^n x_n, \quad \text{if } m \geq 2,\ n \geq 1, \end{cases} \tag{2.1.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequences in [0,1] bounded away from 0 and 1.

Inchan (2008) introduced a new hybrid iterative scheme by using the shrinking projection method with the modified Mann iteration for asymptotically nonexpansive mappings.

2.2 Bregman Nonexpansive Mappings

Iterative methods for approximating fixed points of nonexpansive, quasi nonexpansive mappings and their generalizations have been studied by various authors such as Browder (1967) and Halpern (1967), in Hilbert spaces. But extending this theory to Banach spaces encountered some difficulties because the useful examples of nonexpansive operators in Hilbert spaces are no longer nonexpansive in Banach spaces (e.g., the resolvent $R_A = (I \equiv rA)^{-1}$, for $r > 0$, of a monotone mapping $A : C \to 2^H$ and the metric projection P_C onto a nonempty, closed, and convex subset C of H).

To overcome these difficulties, Bregman (1967) discovered an effective technique using the so called Bregman distance function $D_f(.,.)$ in the process of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique was applied in various ways in order to design and analyze iterative algorithms for solving not only feasibility and optimization problems, but also algorithms for solving variational inequality problems, equilibrium problems, fixed point problems for nonlinear mappings and so on [see (Bauschke and Borwein, 1997), (Butnariu and Resmerita, 2006) and (Zegeye and Shahzad, 2011)].

In recent years, several authors are constructing iterative sequences for finding the fixed point of nonlinear operators by using Bregman distance and the Bregman projection,

Zegeye and Shahzad (2014a) constructed an iterative sequence and proved a strong convergence theorem for a finite family of Bregman strongly nonexpansive mappings in the framework of a real reflexive Banach space. Furthermore

they approximated a common zero of a finite family of maximal monotone mappings and solution of a finite convex feasibility problems in reflexive real Banach spaces. Alghamdi et al. (2014) introduced an iterative sequence and studied strong convergence for a common fixed point of a finite family of quasi-Bregman nonexpansive mappings in the framework of a real reflexive Banach space. They proved a strong convergence of the iterative algorithms for finding a common solution of finite family of equilibrium problems and common zero of a finite family of maximal monotone mappings. Panc et al. (2014) proved weak convergence theorems for Bregman relatively nonexpansive mappings. They proved a strong convergence of the iterative algorithms for finding a common solution of finite family of equilibrium problems and common zero of a finite family of maximal monotone mappings. Shahzad and Zegeye (2014) constructed an iterative sequence for approximation of a common fixed point of a finite family of multivalued Bregman relatively nonexpansive mappings. They proved that, the sequence generated converges strongly to a common fixed point of a finite family of multivalued Bregman nonexpansive mappings in reflexive real Banach spaces. Zegeye and Shahzad (2014b) proved a strong convergence theorem for a finite family of Bregman weak relatively nonexpansive mappings. Tomizawa (2014) introduced a new class of Bregman asymptotically quasi-nonexpansive mappings in the intermediate sense. They established a strong convergence theorem of the shrinking method with the modified Mann iteration, to find fixed points of the mappings in reflexive Banach spaces.

Li and Liu (2014) established some strong convergence theorems for hybrid Halpern's iteration for a countable family of Bregman totally quasi-asymptotically nonexpansive multi-valued mappings in the framework of reflexive Banach spaces. Ugwunnadi et al. (2014), proved a strong convergence theorem of a common element in the set of fixed points of a finite family of closed quasi-

Bregman strictly pseudocontractive mappings and common solutions of a system of equilibrium problems in reflexive Banach space. They studied the strong convergence of the following scheme:

$$\begin{cases} x_1 \in C \\ y_n = \bigtriangledown f^*(\alpha_n \bigtriangledown f(x_n) + (1-\alpha_n)\bigtriangledown f(T_k^n x_n) \\ u_{jn} = Res_{g_j}^f \, y_n \\ w_n = \bigtriangledown f^*(\sum_{j=1}^m \, B_{jn} \bigtriangledown f(u_{jn})) \\ c_{n+1} = \{w \in c_n : D_f(x_n, w_n) \leq \frac{1}{1-\lambda}\langle \bigtriangledown f(x_n) \\ \qquad -\bigtriangledown f(T_n x_n), x_n - w\rangle + \langle \bigtriangledown f(T_n x_n) - \bigtriangledown f(w_n), x_n - w\rangle\} \\ x_{n+1} = P_{c_{n+1}}^f(x) \qquad n \in N \end{cases}$$

They proved that the sequence $\{x_n\}$ converges strongly to $P_F^f(x)$

Yongfu and Yongchun (2015) proved a new hybrid shrinking projection algorithm for common fixed points of a family of countable quasi-Bregman strictly pseudocontractive mappings with equilibrium and variational inequality and optimization problems. Wang (2015) proved a strong convergence theorem for Bregman quasi-strict pseudo-contractions in reflexive Banach spaces with applications. Ni and Yao (2015) introduced a new modified Ishikawa iterative algorithm with errors by a shirinking projection method for generalized mixed equilibrium problems and a countable family of uniformly Bregman totally quasi-D-asymptotically nonexpansive mappings in reflexive Banach spaces. They established strong convergence of the sequence generated by the proposed algorithms under some suitable assumptions.

2.3 Strictly Asymptotically Pseudocontractive Mappings

The class of strictly asymptotically pseudocontractive mappings was first introduced in Hilbert spaces by Liu (1996). The convergence problem of iterative sequences for strictly asymptotically pseudocontractive mappings has been studied by several authors Osilike et al. (2007) obtained weak and strong convergence for k-strictly asymptotically pseudocontractive mappings. Liu and sen Chang (2012) obtained strong convergence for λ-strict asymptotically pseudocontractive mapping in a uniformly convex and 2-uniformly smooth Banach spaces. Chang et al. (2012)) propose an algorithm for solving the split common fixed point problems for total asymptotically strictly pseudocontractive mappings in infinite-dimensional Hilbert spaces.

2.4 Equilibrium and Variational Inequality Problems

Numerous problems in optimization, economics and physics reduce to finding a solution of an equilibrium problem. Some methods have been proposed to solve the equilibrium problems in Hilbert spaces. For example Oettli and Blum (1994), Combettes and Hirstoaga (2005); Tada and Takahashi (2007); Takahashi and Zembayashi (2009); Takahashi and Takahashi (2003) obtained weak and strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and a set of fixed points of nonexpansive mappings in a Hilbert space. In particular, Takahashi and Zembayashi (2009) established a strong convergence theorem for finding a common element of the set of solutions of an equilibrium problem and a set of fixed points of

nonexpansive mappings by using the hybrid method introduced in Nakajo and Takahashi (2003). They also proved such a strong convergence theorem in a uniformly convex and a uniformly smooth Banach space.

Reich and Sabach (2012) and Kassay et al. (2011) proved some convergence theorems for the solutions of some equilibrium and variational inequality problems in the setting of reflexive Banach spaces.

Motivated and inspired by the above mentioned results, in our first result, we extend and generalize the result of Chidume and Ofoedu (2007) by using the Bregman distance for the approximation of common fixed points of finite families of quasi-total asymptotically nonexpansive mappings; and gave necessary and sufficient conditions for the convergence of the scheme to common fixed points of the mappings in arbitrary real Banach spaces. A sufficient condition for convergence of the the iteration process to a common fixed point of mapping under the same setting is also established in real Banach spaces.

In our second result, a new hybrid iterative scheme is introduce and a strong convergence theorem of a common element in the set of fixed points of finite family of closed quasi-Bregman total asymptotically strictly pseudocontractive mappings and common solution to a system of equilibrium problems in reflexive Banach spaces is proved.

CHAPTER THREE

THEORY OF METHODS

Lemma 3.0.1 *(Reich and Sabach, 2009) If $f : E \to \mathbb{R}$ is uniformly Frechet differentiable and bounded on a bounded subsets of E, then $\bigtriangledown f$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Concerning the Bregman projection, the following results are well known.

Lemma 3.0.2 *(Butnariu and Resmerita, 2006) Let C be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function and let $x \in E$. then:*

(a) $z = P_C^f(x)$ if and only if $\langle y - z, \bigtriangledown f(x) - \bigtriangledown f(z)\rangle \leq 0, \quad \forall\, y \in C$;

(b) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), x) \leq D_f(y, x), \quad \forall\, x \in E,\ y \in C.$

Lemma 3.0.3 *(Resmerita, 2004) If $x \in domf$, then the following statements are equivalent:*

(i) The function f is totally convex at x;

(ii) for any sequence $\{y_n\} \subset domf$.

$$\lim_{n\to+\infty} D_f(y_n, x) = 0 \qquad \Rightarrow \qquad \lim_{n\to+\infty} \|y_n - x\| = 0.$$

Lemma 3.0.4 *(Naraghirad and Yao, 2013) Let E be a Banach space and $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subset of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then*

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} ||x_n - y_n|| = 0.$$

Lemma 3.0.5 *(Butnariu and Lusem, 2000) The function f is totally convex on a bounded set if and only if the function f is sequentially consistent.*

Lemma 3.0.6 *(Reich and Sabach, 2010a) Let $f : E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function. If $x_\circ \in E$ and the sequence $\{D_f(x_n, x_\circ)\}$ is bounded then the sequence $\{x_n\}$ is bounded too.*

Lemma 3.0.7 *(Reich and Sabach, 2010a) Let $f : E \to \mathbb{R}$ be a Gateaux differentiable and totally convex function, $x_\circ \in E$ and let C be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_\circ) \leq D_f(P_C^f(x_\circ), x_\circ)$ for any $n \in \mathbb{R}$, then $\{x_n\}$ converges strongly to $P_C^f(x_\circ)$.*

The following lemma follows from Butnariu and Lusem (2000) and Zǎ Zalinescu (2002).

Lemma 3.0.8 *Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then*

(i) $\nabla f : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;

(ii) $\langle x - y, \nabla f(x) - \nabla f(y) \rangle = 0$ if and only if $x = y$;

(iii) $\{x \in E : D_f(x, y) \leq r\}$ is bounded for all $y \in E$ and $r > 0$;

(iv) dom $f^* = E^*$, f^* is Gâteaux differentiable and $\nabla f^* = (\nabla f)^{-1}$.

The following two results are well known; see Zalinescu (2002)

Theorem 3.0.9 *Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:*

(1) f is strongly coercive and uniformly convex on bounded subsets of E;

(2) $domf^* = E^*$, f^* *is bounded on bounded subsets and uniformly smooth on bounded subsets of* E^**;*

(3) $domf^* = E^*, f^*$ *is Frechet differentiable and* ∇f *is uniformly norm-to-norm continuous on bounded subsets of* E^**.*

Theorem 3.0.10 *Let* E *be a reflexive Banach space and let* $f : E \to \mathbb{R}$ *be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:*

(1) f *is bounded on bounded subsets and uniformly smooth on bounded subsets of* E*;*

(2) f^* *is Frechet differentiable and* f^* *is uniformly norm-to-norm continuous on bounded subsets of* E^**;*

(3) $domf^* = E^*, f^*$ *is strongly coercive and uniformly convex on bounded subsets of* E^**.*

The following result was proved by Butnariu and Resmerita (2006).

Lemma 3.0.11 *Let* E *be a reflexive Banach space, let* $f : E \to \mathbb{R}$ *be a strongly coercive Bregman function and let* V *be the function defined by*

$$V(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*),\ x \in E,\ x^* \in E^*.$$

Then the following assertions hold:

(1) $D_f(x, \nabla f(x^*)) = V(x, x^*)$ *for all* $x \in E$ *and* $x^* \in E^*$*.*

(2) $V(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$ *for all* $x \in E$ *and* $x^*, y^* \in E^*$*.*

Lemma 3.0.12 *Naraghirad and Yao (2013) Let* E *be a Banach space and* $f : E \to \mathbb{R}$ *be a convex function which is uniformly convex on bounded subsets of* E*. Let* $r > 0$ *be a constant and* ρ_r *be the gauge of uniform convexity of* f *.*

Then

(i) For any $x, y \in B_r$ and $\alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) - \alpha(1-\alpha)\rho_r(||x-y||).$$

(ii) For any $x, y \in B_r$,

$$\rho_r(||x-y||) \leq D_f(x,y)$$

(iii) If, in addition, f is bounded on bounded subsets and uniformly convex on bounded subsets of E then, for any $x \in E, y^*, z^* \in B_r$ and $\alpha \in (0,1)$,

$$V_f(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_f(x, y^*) + (1-\alpha)V_f(x, z^*) - \alpha(1-\alpha)\rho_r^*(||y^* - x^*||).$$

Lemma 3.0.13 *(Reich and Sabach, 2010a) Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let C be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to C. If $D_f(x_n, x_0) \leq D_f(P_C^f(x_0), x_0)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $P_C^f(x_0)$.*

Lemma 3.0.14 *(Reich and Sabach, 2009). If $f : E \to \mathbb{R}$ is uniformly Frechet differentiable and bounded on a bounded subset of E, then $\bigtriangledown f$ is uniformly continuous on a bounded subset of E from the strong topology of E to the strong topology of E^*.*

Lemma 3.0.15 *Chidume and Ofoedu (2007). Let $\{a_n\}, \{\alpha_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers such that $a_{n+1} \leq (1+\alpha_n)a_n + b_n$.*

Suppose also that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then a_n is bounded and $\lim_{n\to\infty} a_n$ exists. Moreover, if in addition,
$\liminf_{n\to\infty} = 0, then \lim_{n\to\infty} a_n = 0$

Lemma 3.0.16 *Phelps (1993). Let E be a real reflexive Banach space, $f : E \to -(\infty, +\infty]$ a proper lower semi-continuous function, then $f : E \to -(\infty, +\infty]$ is a proper weak* lower semi-continuous and a convex function. Thus, for all $z \in E$, we have*

$$D_f\left(z, \bigtriangledown f^*\left(\sum_{i=1}^{N} t_i \bigtriangledown f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{3.0.1}$$

In order to solve the equilibrium problem, let us assume that a bifunction $g : C \times C \to \mathbb{R}$ satisfies the following conditions

(A1)$g(x, x) = 0$, $\forall x \in C$;

(A2) g is monotone, *i.e.*, $g(x, y) + g(y, x) \leq 0 \quad \forall x, y \in C$;

(A3) $\limsup_{t \to o} g(x + t(z - x), y) \leq g(x, y), \quad \forall x, y, z \in C$;

(A4) the function $y \mapsto g(x, y)$ is convex and lower semi-continuous; Oettli and Blum (1994).

The resolvent of a bifunction g is the operator $\text{Res}_g^f : E \to 2^C$ defined by

$$Res_g^f(x) = \left\{z \in g(z, y) + \langle \bigtriangledown f(z) - \bigtriangledown f(x), y - z\rangle \geq 0 \quad \forall y \in C\right\}. \tag{3.0.2}$$

Combettes and Hirstoaga (2005)

From Reich and Sabach (2010b)), if $f : (-\infty, +\infty] \to \mathbb{R}$ is strongly and Gateaux differentiable function, and g satisfies condition (A1)-(A4) above, then $\text{dom}(\text{Res}_g^f) = E$. The following lemma gives some characterization of the resolvent Res_g^f.

Lemma 3.0.17 *(Reich and Sabach, 2010b) Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E. Let $f : E \to (-\infty, +\infty]$ be a function. If the bifunction $g : C \times C \to \mathbb{R}$ satisfies the condition (A1)-(A4) above, then the following hold:*

(i) Res_g^f is singled-valued;

(ii) Res_g^f is a Bregman firmly nonexpansive operator

(iii) $F(\text{Res}_g^f)$ =EP(g);

(iv) EP(g) is closed and convex subset of C; for all $x \in E$ and for all $q \in F(\text{Res}_g^f)$

$$D_f(q, Res_g^f(x)) + D_f(Res_g^f(x), x) \leq D_f(q, x). \tag{3.0.3}$$

Lemma 3.0.18 *Let $F : E \to \mathbb{R}$ be a Legendre function which is uniformly Fréchet differentiable and bounded on subset of E and let $T : C \to C$ be a Bregman quasi-total asymptotically strictly-pseudocontractive mapping with respect to F. Then for any $x \in C, p \in F(T)$ and $\lambda \in [0, 1)$ the following holds:*

$$D_f(x, T^n x) \leq \frac{1}{1-\lambda}\left[\mu_n \phi(D_f(p, x)) + l_n + \langle \bigtriangledown f(x) - \bigtriangledown f(T^n x), x - p\rangle\right] \tag{3.0.4}$$

Proof. Let $x \in C, p \in F(T)$ and $\lambda \in [0, 1)$ by the definition of T we have

$$D_f(p, T^n x) \leq D_f(p, x) + \lambda D_f(x, T^n x) + \mu_n \phi(D_f(p, x)) + l_n$$

and from (1.7.2)

$$\begin{aligned} D_f(p, x) + D_f(x, T^n x) + \langle \bigtriangledown f(x) - \bigtriangledown f(T^n x), p - x\rangle &\leq D_f(p, x) + \lambda D_f(x, T^n x) \\ &+ \mu_n \phi(D_f(p, x)) + l_n. \end{aligned}$$

Therefore

$$(1-\lambda) D_f(x, T^n x) \leq \mu_n \phi(D_f(p, x)) + l_n + \langle \bigtriangledown f(x) - \bigtriangledown f(T^n x), x - p\rangle.$$

Thus,

$$D_f(x, T^n x) \leq \frac{1}{1-\lambda}[\mu_n \phi(D_f(p, x)) + l_n + \langle \bigtriangledown f(x) - \bigtriangledown f(T^n x), x - p\rangle].$$

This complete the proof.

□

Lemma 3.0.19 *Let $f : E \to \mathbb{R}$ be a legendre function which is uniformly Frechet differentialble and bounded on subset of E. Let C be a nonempty closed and convex subset E and let $T : C \to C$ be a closed and Bregman quasi-total asymptotically strictly-pseudocontractive mapping with respect to f with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ $n \geq 1$ and mapping $\phi_i : [0, \infty) \to [0, \infty)$ such that $\mu_n \to 0, l_n \to 0$ as $n \to \infty$. Suppose that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^*\lambda$ for all $\lambda \geq M$. Then $F(T)$ is closed and convex.*

Proof. Let $F(T)$ be a nonempty set. First we show that $F(T)$ is closed. Letting $\{z_n\}$ be a sequence in $F(T)$ with $z_n \to z$ (as $n \to \infty$), we prove that $z \in F(T)$. In fact, since $\{z_n\} \subset F(T)$ and $z_n \to z$, we have $Tz_n = z_n \to z$ (as $n \to \infty$). By the closeness of T , we have $Tz = z$. This implies that $F(T)$ is closed.

Next we show that $F(T)$ is convex, let $z_1, z_2 \in F(T)$ for any $t \in (0, 1)$ putting $z = tz_1 + (1 - t)z_2$ we need to show $z \in F(T)$, from Lemma 3.0.18 we obtain respectively

$$\begin{aligned} D_f(z, T^n z) &\leq \frac{1}{1-\lambda}\big[\mu_n\phi(D_f(z_1, z)) + l_n \\ &\quad + \langle \bigtriangledown f(z) - \bigtriangledown f(T^n z), z - z_1\rangle\big] \end{aligned} \tag{3.0.5}$$

and

$$\begin{aligned} D_f(z, T^n z) &\leq \frac{1}{1-\lambda}\big[\mu_n\phi(D_f(z_2, z)) + l_n \\ &\quad + \langle \bigtriangledown f(z) - \bigtriangledown f(T^n z), z - z_2\rangle\big]. \end{aligned} \tag{3.0.6}$$

Since ϕ is an increasing function,by hypothesis $\phi(\lambda) \leq M^*\lambda$, if $\lambda \geq M$. that is

$$\phi(D_f(z_i, z)) \leq M^* D_f(z_i, z), \quad i = 1, 2. \tag{3.0.7}$$

Using (3.0.7) in (3.0.5) and (3.0.6) and multiplying the result respectively by t and $(1-t)$ and adding the result we have

$$\begin{aligned} D_f(z, T^n z) \;\leq\; & \frac{1}{1-\lambda}\big[\mu_n M^*(D_f(z, z)) + l_n \\ & + \langle \bigtriangledown f(z) - \bigtriangledown f(T^n z), z - z\rangle\big] \end{aligned} \tag{3.0.8}$$

which implies $D_f(z, T^n z) \to 0$ as $n \to \infty$. Since f is totally convex on bounded subsets of E; f is sequentially consistent we have $T^n z \to z$. This implies that $TT^n z = T^{n+1} z \to z$. Since T is closed, we have $z = Tz$, i.e., $z \in F(T)$. This completes the proof. □

CHAPTER FOUR

MAIN RESULTS

4.1 Finite Families of Quasi-Bregman Total Asymptotically Nonexpansive Mappings

Let C be a nonempty closed convex subset of a real Banach space E. Let $T_1, T_2, \cdots, T_m : C \to C$ be m Bregman quasi-total asymptotically nonexpansive mappings. We define the iterative sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n x_n)\big), \quad \text{if } m = 1,\ n \geq 1, \\ x_1 \in C, \\ x_{n+1} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n y_{1n})\big), \\ y_{1n} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n y_{2n})\big), \\ \quad \vdots \\ y_{(m-2)n} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_{m-1}^n y(m-1)_n)\big), \\ y_{(m-1)n} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_m^n x_n)\big), \quad \text{if } m \geq 2,\ n \geq 1, \end{cases} \tag{4.1.1}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequences in [0,1]. The following result is obtained.

Theorem 4.1.1 *Let E be a real Banach space, C be a nonempty closed convex subset of E and $f : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $T_i : C \to C \quad i = 1, 2, \ldots, m$ be Bregman total asymptotically quasi-nonexpansive mappings with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ $n \geq 1$ and mapping $\phi_i : [0, \infty) \to [0, \infty) \quad i = 1, 2, \ldots, m$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty, \quad \sum_{n=1}^{\infty} l_{in} < \infty, \quad i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$.*

Let $\{x_n\}$ be defined as in (4.1.1) and suppose there exists $M_i,\ M_i^* > 0$ such that $\phi(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i \quad i = 1, 2, \ldots, m$ then the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} D_f(x_n, p)$ exists, where $p \in F$.

Proof. For $m = 1$ we have from (4.1.1) that $x_1 \in C$, and

$$x_{n+1} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n x_n)\big),$$

then

$$\begin{aligned} D_f(x_{n+1}, p) &= D_f\big(\bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n x_n)\big), p\big) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n D_f(T_1^n x_n, p) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n \big[D_f(x_n, p) + \mu_{1n}\phi_1\big(D_f(x_n, p)\big) + l_{1n}\big] \\ &= D_f(x_n, p) + \alpha_n \mu_{1n}\phi_1\big(D_f(x_n, p)\big) + \alpha_n l_{1n}. \end{aligned}$$

Since ϕ_1 is an increasing function, it follows that $\phi_1(\lambda_1) \leq \phi_1(M_1)$ whenever $\lambda_1 \leq M_1$ and by hypothesis $\phi_1(\lambda_1) \leq M_1^* \lambda_1$, if $\lambda_1 \geq M_1$. In either case, we have

$$\phi_1\big(D_f(x_n, p)\big) \leq \phi_1(M_1) + M_1^* D_f(x_n, p)$$

for some $M_1 > 0,\ M_1^* > 0$. Thus,

$$\begin{aligned} D_f(x_{n+1}, p) &\leq D_f(x_n, p) + \alpha_n \mu_{1n}\phi_1(M_1) + \alpha_n \mu_{1n} M_1^* D_f(x_n, p) + \alpha_n l_{1n} \\ &= (1 + \mu_{1n} Q_1) D_f(x_n, p) + (\mu_{1n} + l_{1n}) Q_1, \end{aligned}$$

for some constant $Q_1 \geq 1$.

Next, for $m = 2$, we obtain from (4.1.1) that

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n y_{1n})\big), \\ y_{1n} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_2^n x_n)\big), \end{cases}$$

from this, we have

$$\begin{aligned} D_f(x_{n+1}, p) &= D_f\big(\bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n y_{1n})\big), p\big) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n D_f(T_1^n y_{1n}, p) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n \big[D_f(y_{1n}, p) + \mu_{1n}\phi_1\big(D_f(y_{1n}, p)\big) + l_{1n}\big] \end{aligned}$$

and

$$\begin{aligned} D_f(y_{1n}, p) &= D_f\big(\bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_2^n x_n)\big), p\big) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n D_f(T_2^n x_n, p) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n \Big[D_f(x_n, p) \\ &\quad + \mu_{2n}\phi_2\big(D_f(x_n, p)\big) + l_{2n}\Big] \end{aligned} \tag{4.1.2}$$

Again, since ϕ_i is an increasing function for $i = 1, 2$, it follows that

$\phi_i(\lambda_i) \leq \phi_i(M_i) + M_i^*\lambda_i$ for some $M_i > 0,\ M_i^* > 0,\ i = 1, 2.$

Hence,

$$\begin{aligned}
D_f(x_{n+1},p) &\leq (1-\alpha_n)D_f(x_n,p)+\alpha_n D_f(y_{1n},p)+\alpha_n\mu_{1n}\phi_1(M_1)\\
&\quad+\alpha_n\mu_{1n}M_1^*D_f(y_{1n},p)+\alpha_n l_i n\\
&\leq (1-\alpha_n)D_f(x_n,p)+\alpha_n\big[D_f(x_n,p)+\alpha_n\mu_{2n}\phi_2(M_2)\\
&\quad+\alpha_n\mu_{2n}M_2^*D_f(x_n,p)+\alpha_n l_{2n}\big]+\alpha_n\mu_{1n}\phi_1(M_1)\\
&\quad+\alpha_n\mu_{1n}M_1^*\big[D_f(x_n,p)+\alpha_n\mu_{2n}\phi_2(M_2)\\
&\quad+\alpha_n\mu_{2n}M_2^*D_f(x_n,p)+\alpha_n l_2 n\big]+\alpha_n l_{1n}\\
&\leq D_f(x_n,p)+\alpha_n\mu_{2n}\phi_2(M_2)+\alpha_n\mu_{2n}M_2^*D_f(x_n,p)\\
&\quad+\alpha_n l_{2n}+\alpha_n\mu_{1n}\phi_1(M_1)+\alpha_n\mu_{1n}M_1^*D_f(x_n,p)\\
&\quad+\alpha_n\mu_{1n}\mu_{2n}\phi_2(M_2)M_1^*+\alpha_n\mu_{1n}\mu_{2n}M_1^*M_2^*D_f(x_n,p)\\
&\quad+\alpha_n\mu_{1n}l_{2n}M_1^*+\alpha_n l_{1n}\\
&\leq \big(1+(\mu_{1n}+\mu_{2n})Q_2\big)D_f(x_n,p)+(\mu_{1n}+\mu_{2n}+l_{1n}+l_{2n})Q_2,
\end{aligned}$$

for some constant $Q_2>0$.

Following the computation above, we obtain for $m\geq 2$ that

$$D_f(x_{n+1},p)\leq\left(1+Q\sum_{j=1}^{m}\mu_{jn}\right)D_f(x_n,p)+Q\sum_{j=1}^{m}(\mu_{jn}+l_{jn})$$

for some $Q>0$.

Hence,

$$D_f(x_{n+1},p)\leq(1+\delta_n)D_f(x_n,p)+\gamma_n,\quad n\geq 1, \tag{4.1.3}$$

where $\delta_n=Q\sum_{n=1}^{\infty}\mu_{jn}$ and $\gamma_n=Q\sum_{n=1}^{\infty}(\mu_{jn}+l_{jn})$. Observed that $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}\gamma_n<\infty$. It follows from (4.1.3) and Lemma 3.0.15 that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}D_f(x_n,p)$ exists for each $p\in F$. □

4.1.1 Necessary and sufficient conditions for convergence

Theorem 4.1.2 *Let E be a real Banach space, C be a nonempty closed convex subset of E and $f : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $T_i : C \to C \quad i = 1, 2, \ldots, m$ be Bregman quasi-total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ and mapping $\phi_i : [0, \infty) \to [0, \infty) \quad n \geq 1 \quad i = 1, 2, \ldots, m$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty, \quad \sum_{n=1}^{\infty} l_{in} < \infty, \quad i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined as in (4.1.1) and suppose there exists M_i, $M_i^* > 0$ such that $\phi(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i \quad i = 1, 2, \ldots, m$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_i, $i = 1, 2, \ldots, m$ if and only if $\liminf_{n\to\infty} \left(\inf_{y \in F} D_f(x_n, y) \right) = 0, \ n \geq 1$.*

Proof. The necessity is trivial. We prove the sufficiency. Let $\liminf_{n\to\infty} \left(\inf_{y\in F} D_f(x_n, y) \right) = 0$ we show $\{x_n\}$ is Cauchy sequence in E. Since $\lim_{n\to\infty} D_f(x_n, p)$ exists $\forall p \in F$ from Lemma 3.0.15 we obtain $\lim_{n\to\infty} D_f(x_n, p) = 0$ for some $p \in F$. Let there be any other $p^* \in F$ then $\lim_{n\to\infty} D_f(x_n, p^*) = 0$, in view of Lemma 3.0.4, we have $\lim\limits_{n\to\infty} \|x_n - p^*\| = 0$. Thus given $\epsilon > 0$ there exists a positive integer N_0 such that $\forall n \geq N_0$, $\|x_n - p^*\| < \frac{\epsilon}{2}$. Then for any $k \in N$, we have for $n \geq N_0$

$$\begin{aligned} \|x_{n+k} - x_n\| &\leq \|x_{n+k} - p^*\| + \|p^* - x_n\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

and so $\{x_n\}$ is Cauchy sequence. Let $x_n \to p$, we need to show that $p \in F$. Let $T_i \in \{T_1, T_2, \ldots, T_m\}$. Since $\lim_{n\to\infty} D_f(x_n, p) = 0$, for all $p \in F$, which implies that $\lim_{n\to\infty} \|x_n - p\| = 0$, then there exists $N \in \mathbb{N}$ sufficiently large and $p^* \in F$ such that for all $n \geq \mathbb{N}$ implies that $\|x_n - p\| < \frac{\epsilon}{6(1+\omega_1)}$ and

$\|p^* - x_n\| < \frac{\epsilon}{6(1+\omega_1)}$ then

$$\|p^* - p\| \leq \|p^* - x_n\| + \|x_n - p\| < \frac{\epsilon}{3(1+\omega_1)}.$$

Thus we have the following estimates for $n \geq \mathbb{N}$ and arbitrary $T_i, i = 1, 2, \cdots, m$.

$$\begin{aligned}\|p - T_1 p\| &\leq \|p - x_n\| + \|x_n - p^*\| + \|p^* - T_1 p\| \\ &\leq \|p - x_n\| + \|x_n - p^*\| + (1+\omega_1)\|p^* - p\| \\ &\leq \frac{\epsilon}{6(1+\omega_1)} + \frac{\epsilon}{6(1+\omega_1)} + \frac{\epsilon}{3}.\end{aligned}$$

This shows that $p \in F(T_i)$ for each $i \in \{1, 2, \cdots, m\}$. Thus $p \in F$. This complete the proof.

□

Lemma 4.1.3 *Let E be a real uniformly convex Banach space, C a nonempty closed convex subset of E and $f : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $T_i : C \to C,\ i = 1, 2, \ldots, m$ be m uniformly continuous Bregman quasi-total asymptotically nonexpansive mappings and uniformly L_i-Lipschitzian which is also uniformly asymptotically regular, with sequences $\{\mu_{in}\}, \{l_{in}\}, \subset [0, \infty)$ $n \geq 1$ and mapping $\phi_i : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty,\ \sum_{n=1}^{\infty} l_{in} < \infty,\ i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{\alpha_n\} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$,. From arbitrary $x_1 \in E$ define the sequence $\{x_n\}$ by (4.1.1). Suppose that there exist $M_i,\ M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ whenever $\lambda_i \geq M_i \quad i = 1, 2, \ldots, m$ then $\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0,\ i = 1, 2, \ldots, m$.*

Proof. Let $p \in F$ then by Theorem 4.1.1, $\lim_{n\to\infty} D_f(x_n, p)$ exists. Let $\lim_{n\to\infty} D_f(x_n, p) = r$. If $r = 0$ then by continuity of $T_i,\ i = 1, 2, \ldots, m$ we are done. Now suppose $r > 0$, we show $\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0,\ i = 1, 2, \ldots, m$.

In Theorem 4.1.2 we have shown that $\{x_n\}$ is Cauchy. In particular Now, for $m = 1$, we get from (4.1.1) that

$$x_1 \in K \quad x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)).$$

Using this and Lemma 3.0.12 we have for some constant $Q_1 > 0$ that

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n))) \\
&= V(p, (1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)) \\
&= f(p) - \langle p, (1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)\rangle \\
&\quad + f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)) \\
&\leq (1-\alpha_n) f(p) + \alpha_n f(p) - (1-\alpha_n)\langle p, \nabla f(x_n)\rangle \\
&\quad - \alpha_n \langle p, \nabla f(T_1^n x_n) \\
&\quad + (1-\alpha_n) f^*(\nabla f(x_n)) + \alpha_n f^*(\nabla f(T_1^n x_n)) \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\
&= (1-\alpha_n) V(p, \nabla f(x_n)) + \alpha_n V(p, \nabla f(T_1^n x_n)) \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\
&= (1-\alpha_n) D_f(p, x_n) + \alpha_n D_f(p, T_1^n x_n) \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||)
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n) D_f(p, x_n) + \alpha_n [D_f(p, x_n) \\
&\quad + \mu_{1n}\phi_1(D_f(p, x_n)) + l_{1n}] \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\
&\leq D_f(p, x_n) + (\mu_{1n} + l_{1n}) Q_1 \\
&\quad - \alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||).
\end{aligned}$$

Thus,

$$\epsilon^2\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \leq D_f(p, x_n) - D_f(p, x_{n+1}) + (\mu_{1n} + l_{1n})Q_1$$

which implies,

$$\epsilon^2 \sum_{n=1}^{\infty} \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \leq D_f(p, x_1) + Q_1 \sum_{n=1}^{\infty}(\mu_{1n} + l_{1n}).$$

Hence $\epsilon^2 \sum_{n=1}^{\infty} \rho_s^*(||\nabla f(x_n)-\nabla f(T_1^n x_n)||) < \infty$. So, $\lim_{n\to\infty} \rho_s^*(||\nabla f(x_n)-\nabla f(T_1^n x_n)||) = 0$; and properties of ρ_s^* imply $\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(T_1^n x_n)|| = 0$. Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty} ||x_n - T_1^n x_n|| = 0. \tag{4.1.4}$$

For $m = 2$, (4.1.1) becomes

$$\begin{cases} x_1 \in C \\ x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \nabla f(T_1^n y_{1n})) \\ y_{1n} = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)) \end{cases} \tag{4.1.5}$$

using (4.1.2), (4.1.5) and Lemma 3.0.12, we obtain, for some $Q_3 > 0$ that

$$\begin{aligned} D_f(p, x_{n+1}) &\leq (1-\alpha_n)D_f(p, x_n) + \alpha_n D_f(p, T_1^n y_{1n}) \\ &\quad -\alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||) \\ &\leq (1-\alpha_n)D_f(p, x_n) + \alpha_n[D_f(p, y_{1n}) + \mu_{1n}\phi_1(D_f(p, y_{1n})) + l_{1n}] \\ &\quad -\alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||) \\ &\leq D_f(p, x_n) + (\mu_{2n} + l_{2n} + \mu_{1n} + l_{1n})Q_3 \\ &\quad -\alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||). \end{aligned}$$

Thus,

$$\epsilon^2\rho_s^*(||\nabla f(x_n)-\nabla f(T_1^n y_{1n})||) \leq D_f(p,x_n)-D_f(p,x_{n+1})+(\mu_{2n}+l_{2n}+\mu_{1n}+l_{1n})Q_3$$

which implies ,

$$\epsilon^2\sum_{n=1}^{\infty}\rho_s^*(||\nabla f(x_n)-\nabla f(T_1^n y_{1n})||) \leq D_f(p,x_1)+Q_3\sum_{n=1}^{\infty}(\mu_{2n}+l_{2n}+\mu_{1n}+l_{1n}) < \infty.$$

So, $\lim_{n\to\infty}\rho_s^*||\nabla f(x_n)-\nabla f(T_1^n y_{1n})|| = 0$, and properties of ρ_s yield $\lim_{n\to\infty}||\nabla f(x_n)-\nabla f(T_1^n y_{1n})|| = 0$. Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty}||x_n - T_1^n y_{1n}|| = 0. \tag{4.1.6}$$

Thus from Lemma 3.0.4, we obtain

$$\lim_{n\to\infty} D_f(x_n, T_1^n y_{1n}) = 0. \tag{4.1.7}$$

Also, from (1.7.2) and (4.1.7), we have

$$\begin{aligned} D_f(p,x_n) &\leq D_f(p,T_1^n y_{1n}) + D_f(T_1^n y_{1n},x_n) + \langle \nabla f(T_1^n y_{1n}) - \nabla f(x_n), p - T_1^n y_{1n}\rangle \\ &\leq D_f(p,T_1^n y_{1n}) + D_f(T_1^n y_{1n},x_n) + ||\nabla f(T_1^n y_{1n}) - \nabla f(x_n)||||p - T_1^n y_{1n}|| \\ &\leq D_f(p,y_{1n}) + \mu_{1n}M + l_{1n} + D_f(T_1^n y_{1n},x_n) \\ &\quad + ||\nabla f(T_1^n y_{1n}) - \nabla f(x_n)||||p - T_1^n y_{1n}|| \end{aligned}$$

for some constant $M > 0$. Hence, we deduce from this that

$$r \leq \liminf_{n\to\infty} D_f(p,y_{1n}).$$

Also, since

$$D_f(p, y_{1n}) \leq (1 + \mu_{2n}Q_2)D_f(p, x_n) + (\mu_{2n} + l_{2n})Q_2,$$

this gives

$$\limsup_{n\to\infty} D_f(p, y_{1n}) \leq r.$$

Thus

$$\lim_{n\to\infty} D_f(p, y_{1n}) = r.$$

Also from (4.1.5) and Lemma 3.0.12, we obtain, for some $Q_2 > 0$ that

$$\begin{aligned} D_f(p, y_{1n}) &\leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n D_f(p, T_2^n x_n) \\ &\quad -\alpha_n(1 - \alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\ &\leq (1 + \alpha_n\mu_{2n}Q_2)D_f(p, x_n) + (\mu_{2n} + l_{2n})Q_2 \\ &\quad -\alpha_n(1 - \alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||). \end{aligned}$$

Therefore

$$\epsilon^2\rho_s^*(||\nabla f(x_x)-\nabla f(T_1^n x_n)||) \leq (1+\alpha_n\mu_{2n}Q_2)D_f(p, x_n)-D_f(p, y_{1n})+(\mu_{2n}+l_{2n})Q_2$$

So, $\lim_{n\to\infty} \rho_s^*||\nabla f(x_n)-\nabla f(T_2^n x_n)|| = 0$, and properties of ρ_s^* yield $\lim_{n\to\infty} ||\nabla f(x_n)-\nabla f(T_2^n x_n)|| = 0$. Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} ||x_n - T_2^n x_n|| = 0. \tag{4.1.8}$$

Hence, from Lemma 3.0.4, we obtain

$$\lim_{n\to\infty} D_f(x_n, T_2^n x_n) = 0. \tag{4.1.9}$$

So, from (4.1.5) and (4.1.9), we obtain

$$\begin{aligned} D_f(x_n, y_{1n}) &\leq (1-\alpha_n) D_f(x_n, x_n) \\ &+ \alpha_n D_f(x_n, T_2^n x_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.1.10}$$

which implies from Lemma 3.0.4, that

$$\lim_{n\to\infty} ||x_n - y_{1n}|| = 0. \tag{4.1.11}$$

Also,

$$\begin{aligned} ||T_1^n x_n - x_n|| &\leq ||T_1^n x_n - T_1^n y_{1n}|| + ||T_1^n y_{1n} - x_n|| \\ &\leq L_1 ||x_n - y_{1n}|| + ||T_1^n y_{1n} - x_n||. \end{aligned}$$

Hence, from (4.1.6) and (4.1.11), we obtain

$$\lim_{n\to\infty} ||T_1^n x_n - x_n|| = 0.$$

Hence

$$\lim_{n\to\infty} ||T_1^n x_n - x_n|| = \lim_{n\to\infty} ||T_2^n x_n - x_n|| = 0.$$

Continuing, we get

$$\lim_{n\to\infty} ||T_i^n x_n - x_n|| = 0, \quad i = 1, 2, \cdots, m. \tag{4.1.12}$$

Furthermore

$$\begin{aligned} ||T_i x_n - x_n|| &\leq ||T_i x_n - T_i(T_i^n x_n)|| + ||T_i(T_i^n x_n) - T_i^n x_n|| + ||T_i^n x_n - x_n|| \\ &\leq (L_i + 1)||x_n - T_i^n x_n|| + ||T_i^{n+1} x_n - T_i^n x_n||. \end{aligned}$$

From (4.1.12) and asymptotically regularity of T_i, for each $i \in \{1, 2. \cdots, m\}$, we obtain

$$\lim_{n\to\infty} ||T_i x_n - x_n|| = 0. \tag{4.1.13}$$

This complete the proof. □

Theorem 4.1.4 *Let E be a real uniformly convex Banach space, C a nonempty closed convex subset of E and $f : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $T_i : C \to C,\ i = 1, 2, \ldots, m$ be m Bregman quasi-total asymptotically nonexpansive mappings and uniformly L_i-Lipschitzian which is also uniformly asymptotically regular with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ such that with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ $n \geq 1$ and mapping $\phi_i : [0, \infty) \to [0, \infty), i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{\alpha_{in}\} \subset [\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$,. From arbitrary $x_1 \in E$ define the sequence $\{x_n\}$ by (4.1.1). Suppose that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ whenever $\lambda_i \geq M_i \quad i = 1, 2, \ldots, m$ and that one of $T_1, T_2, \ldots, T_m$ is semi-compact, then $\{x_n\}$ converges strongly to some $x^* \in F$.*

Proof. Without lost of generality, let T_1 be semi-compact. Since T_1 is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for some $x^* \in E,\ T_1 x_{n_k} \to x^*$ as $k \to \infty$. This implies that $T_1^{n_k} x_{n_k} \to x^*$ as $k \to \infty$. Thus $T_1^{n_{k+1}} x_{n_k} \to T_1 x^*$ as $k \to \infty$ and from (4.1.12) we have $\lim_{k\to\infty} x_{n_k} = x^*$. Also from (4.1.12) $T_2^{n_k} x_{n_k} \to x^*,\ T_3^{n_k} x_{n_k} \to x^*, \ldots, T_m^{n_k} x_{n_k} \to x^*$ as $k \to \infty$. Thus $T_2^{n_{k+1}} x_{n_k} \to T_2 x^*,\ T_3^{n_{k+1}} x_{n_k} \to T_3 x^*, \ldots, T_m^{n_{k+1}} x_{n_k} \to T_m x^*$ as $k \to \infty$. Now $\|x_{n_{k+1}} - x_{n_k}\| \leq \|T_1^{n_k} y_{1n_k} - x_{n_k}\|$ from (4.1.6), it follows that $x_{n_{k+1}} \to x^*$

as $k \to \infty$. Next, we show that $x^* \in F$. Observed that

$$\begin{aligned}\|x^* - T_1 x^*\| &\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_1^{n_{k+1}} x_{n_{k+1}}\| \\ &\quad + \|T_1^{n_{k+1}} x_{n_{k+1}} - T_1^{n_{k+1}} x_{n_k}\| + \|T_1^{n_{k+1}} x_{n_k} - T_1 x^*\|.\end{aligned}$$

Taking the limit as $k \to \infty$ and using the fact that T_1 is uniformly asymptotically regular we have that $x^* = T_1 x^*$ and so $x^* \in F(T_1)$. Also,

$$\begin{aligned}\|x^* - T_2 x^*\| &\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_2^{n_{k+1}} x_{n_{k+1}}\| \\ &\quad + \|T_2^{n_{k+1}} x_{n_{k+1}} - T_2^{n_{k+1}} x_{n_k}\| + \|T_2^{n_{k+1}} x_{n_k} - T_2 x^*\|.\end{aligned}$$

Taking the limit as $k \to \infty$ and using the fact that T_2 is uniformly asymptotically regular we obtain that $x^* = T_2 x^*$ and so $x^* \in F(T_2)$. Again,

$$\begin{aligned}\|x^* - T_3 x^*\| &\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_3^{n_{k+1}} x_{n_{k+1}}\| \\ &\quad + \|T_3^{n_{k+1}} x_{n_{k+1}} - T_3^{n_{k+1}} x_{n_k}\| + \|T_3^{n_{k+1}} x_{n_k} - T_3 x^*\|.\end{aligned}$$

As $k \to \infty$, we have that $x^* \in F(T_3)$. Eventually, we have that, $x^* \in F$. But by Theorem 4.1.1 $\lim_{n\to\infty} D_f(x_n, x^*)$ exists, $x^* \in F$. Hence x_n converges strongly to $x^* \in F$. This complete the proof. □

Our result extends the result of Chidume and Ofoedu (2007) from total asymptotically nonexpansive mapping to quasi-Bregman total asymptotically nonexpansive mapping.

4.2 Quasi-Bregman Total Asymptotically Strictly Pseudocontractive Mappings and Equilibrium Problems

Theorem 4.2.1 *Let C be a nonempty closed and convex subset of a reflexive Banach space E and $f : E \to \mathbb{R}$ a strongly coercive legendre function which is bounded uniformly Fréchet differentiable and totally convex on bounded subset of E.For each $j = 1, 2, \ldots, m$ let g_j be a function from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4) and let $\{T\}_{i=1}^{k}$ be a finite family of L_i-lipchitzian $i = 1, 2, \cdots, k$ closed and Bregman quasi-total asymptotically strictly pseudocontractive self mapping of C such that $F := (\cap_{j=1}^{m} EP(g_j)) \cap (\cap_{i=1}^{N} F(T_i)) \neq \emptyset$ with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ $n \geq 1$ and mapping $\phi_i : [0, \infty) \to [0, \infty)$ such that $\mu_{in} \to 0, l_{in} \to 0$ as $n \to \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 = x \in C \quad C_1 = C$ and*

$$\begin{cases} x_1 \in C \\ y_n = \bigtriangledown f^*(\alpha_{n,0} \bigtriangledown f(x_n) + \sum_{i=1}^{N} \alpha_{n,i} \bigtriangledown f(T_i^n x_n)) \\ u_{jn} = Res_{g_j}^{f}\, y_n \\ C_{n+1} = \{w \in C_n : D_f(w, u_{jn}) \leq D_f(w, y_n) \leq D_f(w, x_n) + R(\mu_n + \mu_n D_f(w, x_n) + l_n) \\ \qquad + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{n,i} \langle \bigtriangledown f(x_n) - \bigtriangledown f(T_i^n x_n), x_n - w \rangle\} \\ x_{n+1} = P_{C_{n+1}}^{f}(x) \qquad n \in \mathbb{N} \end{cases} \tag{4.2.1}$$

where $\lambda \in [0, 1)$ for $i = 1, 2, \ldots, k$; T_i is uniformly continuous. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_{jn}\}_{n=1}^{\infty} \quad j = 1, 2, \ldots, m$ are sequences in $(0, 1)$ such that (i) $\lim_{n\to\infty}(1 - \alpha_n) > 0 \quad (ii) \sum_{j=1}^{\infty} \beta_{jn} = 1 \quad n \geq 1$ also suppose that there exist $M, M^* > 0$ such that $\phi(\lambda) \leq M^* \lambda$ for all $\lambda \geq M$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $P_F^f(x)$; where $P_F^f(x)$ is the Bregman projection of E onto F.

Proof. The proof is divided into six steps.

Step I. Show that $F := (\cap_{j=1}^{m} EP(g_j)) \cap (\cap_{i=1}^{N} F(T_i))$ is closed and convex. From Lemma 3.0.19, $\cap_{i=1}^{N} F(T_i)$ is closed and convex and from (iv) of Lemma 3.0.17, $\cap_{j=1}^{m} EP(g_j)$ is closed and convex. So $F := (\cap_{j=1}^{m} EP(g_j)) \cap (\cap_{i=1}^{N} F(T_i))$ is closed and convex.

Step II. Show that C_n is closed and convex for all $n \geq 1$, $C_1 = C$ is closed and convex. Let

$$\begin{aligned} Q_n &= \Big\{ w \in E : D_f(w, y_n) \leq D_f(w, x_n) + R(\mu_n + \mu_n D_f(w, x_n) + l_n) \\ &\quad + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{n,i} \left\langle \bigtriangledown f(x_n) - \bigtriangledown f(T_i^n x_n), x_n - w \right\rangle \Big\} \\ A_{j,n} &= \big\{ w \in E : D_f(w, u_{j,n}) \leq D_f(w, y_n) \big\} \qquad j = 1, 2 \ldots, m, \end{aligned}$$

such that

$$C_{n+1} = C \cap C_n \cap Q_n \cap \big(\cap_{j=1}^{m} A_{j,n} \big)$$

For all $n \geq 1$ we need to proved that Q_n and $A_{j,n}$ are closed and convex. First we show Q_n is closed and convex for all $n \geq 1$.

$$
\begin{aligned}
Q_n &= \Big\{ w \in E : D_f(w, y_n) \leq D_f(w, x_n) + R(\mu_n + \mu_n D_f(w, x_n) + l_n) \\
&\quad + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{n,i} \left\langle \bigtriangledown f(x_n) - \bigtriangledown f(T_i^n x_n), x_n - w \right\rangle \Big\} \\
&= \Big\{ w \in E : D_f(w, y_n) - D_f(w, x_n) \leq R(\mu_n + \mu_n D_f(w, x_n) + l_n) \\
&\quad + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{n,i} \left\langle \bigtriangledown f(x_n) - \bigtriangledown f(T_i^n x_n), x_n - w \right\rangle \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big\{ w \in E : f(x_n) - f(y_n) - \langle \bigtriangledown f(y_n), w - y_n \rangle \\
&\quad + \langle \bigtriangledown f(x_n), w - x_n \rangle \leq R(\mu_n + \mu_n D_f(w, x_n) + l_n) \\
&\quad + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{n,i} \left\langle \bigtriangledown f(x_n) - \bigtriangledown f(T_i^n x_n), x_n - w \right\rangle \Big\} \\
&= \Big\{ w \in E : \langle \bigtriangledown f(x_n), w - x_n \rangle - \langle \bigtriangledown f(y_n), w - y_n \rangle \leq f(y_n) - f(x_n) \\
&\quad + R(\mu_n + \mu_n D_f(w, x_n) + l_n) + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{n,i} \Big[\langle \bigtriangledown f(x_n), x_n - w \rangle \\
&\quad - \langle \bigtriangledown f(T_i^n x_n), x_n - w \rangle \Big] \Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \Big\{ w \in E : \langle \bigtriangledown f(x_n), w\rangle + \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\, \langle \bigtriangledown f(x_n), w\rangle \\
&\quad - \langle \bigtriangledown f(y_n), w\rangle - \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\, \langle \bigtriangledown f(T_i^n x_n), w\rangle \\
&\quad \leq f(y_n) - f(x_n) + R(\mu_n + \mu_n D_f(w, x_n) + l_n) + \langle \bigtriangledown f(x_n), x_n\rangle \\
&\quad + \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\, \langle \bigtriangledown f(x_n), x_n\rangle - \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\, \langle \bigtriangledown f(T_i^n x_n), x_n\rangle \Big\} \\
&= \Big\{ w \in E : \Big\langle \Big(1 + \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\Big) \bigtriangledown f(x_n) - \bigtriangledown f(y_n) \\
&\quad - \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\, \bigtriangledown f(T_i^n x_n), w \Big\rangle \leq f(y_n) - f(x_n) \\
&\quad + R(\mu_n + \mu_n D_f(w, x_n) + l_n) + \Big\langle \Big(1 + \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\Big) \bigtriangledown f(x_n), x_n \Big\rangle \\
&\quad - \langle \bigtriangledown f(y_n), y_n\rangle - \Big\langle \frac{\lambda}{1-\lambda}\sum_{i=1}^{N} \alpha_{n,i}\, \bigtriangledown f(T_i^n x_n), x_n \Big\rangle \Big\}.
\end{aligned}
$$

This shows that Q_n is closed and convex for all $n \geq 1$.

Next we show that $A_{j,n}$ is closed and convex for all $n \geq 1$ and $j = 1, 2, \ldots, m$.

$$
\begin{aligned}
A_{j,n} &= \big\{ w \in E : D_f(w, u_{j,n}) \leq D_f(w, y_n) \big\} \\
&= \big\{ w \in E : f(y_n) - f(u_{j,n}) + \langle \bigtriangledown f(y_n), w - y_n\rangle \\
&\quad - \langle \bigtriangledown f(u_{j,n}), w - u_{j,n}\rangle \leq 0 \big\} \\
&= \big\{ w \in E : f(y_n) - f(u_{j,n}) + \langle \bigtriangledown f(u_{j,n}), u_{j,n}\rangle - \langle \bigtriangledown f(y_n), y_n\rangle \\
&\quad \leq \langle \bigtriangledown f(u_{j,n}) - \bigtriangledown f(y_n), w\rangle \big\}.
\end{aligned}
$$

This shows $A_{j,n}$ is closed and convex for all $n \geq 1$ and $j = 1, 2, \ldots, m$. So for all $n \geq 1$, C_n is closed and convex.

Step III. We show that $F \subset C_n$ for every $n \geq 1$. Note that $F \subset C_1 = C$.

Supposed $F \subset C_h$, for $h \geq 1$, then for all $w \in F \subset C_h$, since $u_{jh} = Res^f_{g_j}(y_h)$ for each $j = 1, 2, \ldots, m$ from Lemma 3.0.17, we have

$$D_f(w, u_{j,h}) = D_f(w, Res^f_{g_j}(y_h)) \leq D_f(w, y_h). \tag{4.2.2}$$

And

$$\begin{aligned} D_f(w, y_h) &= D_f(w, \bigtriangledown f^*(\alpha_{h,0} \bigtriangledown f(x_h) + \sum_{i=1}^{N} \alpha_{h,i} \bigtriangledown f(T_i^n x_h)) \\ &\leq \alpha_{h,0} D_f(w, x_h) + \sum_{i=1}^{N} \alpha_{h,i} D_f(w, T_i^h x_h) \end{aligned}$$

$$\begin{aligned} &\leq \alpha_{h,0} D_f(w, x_h) + \sum_{i=1}^{N} \alpha_{h,i} \Big[D_f(w, x_h) + \lambda D_f(x_h, T_i^h x_h) \\ &+ \mu_h \phi(D_f(w, x_h)) + l_h \Big]. \end{aligned}$$

Since ϕ is an increasing function, it follows that $\phi(\lambda) \leq \phi(m)$ whenever $\lambda \leq m$ and by hypothesis $\phi(\lambda) \leq m^* \lambda$ if $\lambda \geq m$. In either case we have

$\phi(D_f(w, x_h)) \leq \phi(m) + m^* D_f(w, x_h)$ for some $m > 0, \quad m^* > 0$. Thus we have

$$\begin{aligned} D_f(w, y_h) &\leq \alpha_{h,0} D_f(w, x_h) + \sum_{i=1}^{N} \alpha_{h,i} \Big[D_f(w, x_h) + \lambda D_f(x_h, T_i^h x_h) \\ &\quad + \mu_h \phi(m) + \mu_h m^* D_f(w, x_h) + l_h \Big] \\ &\leq D_f(w, x_h) + \lambda \sum_{i=1}^{N} \alpha_{h,i} \, D_f(x_h, T_i^h x_h) + \sum_{i=1}^{N} \alpha_{h,i} \, \mu_h \phi(m) \\ &\quad + \sum_{i=1}^{N} \alpha_{h,i} \, \mu_h m^* D_f(w, x_h) + \sum_{i=1}^{N} \alpha_{h,i} \, l_h \\ &\leq D_f(w, x_h) + \lambda \sum_{i=1}^{N} \alpha_{h,i} \, D_f(x_h, T_i^h x_h) + \mu_h R_1 \\ &\quad + \mu_h R_1 D_f(w, x_h) + R_1 l_h \end{aligned}$$

$$
\begin{aligned}
&\leq D_f(w, x_h) + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{h,i} \Big[\mu_h \phi(D_f(w, x_h)) + l_h \\
&\quad + \langle \bigtriangledown f(x_h) - \bigtriangledown f(T_i^h x_h), x_h - w \rangle \Big] \\
&\quad + \mu_h R_1 + \mu_h R_1 D_f(w, x_h) + R_1 l_h \\
&= D_f(w, x_h) + R_2(\mu_h + \mu_h D_f(w, x_h) + l_h) + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{h,i} \langle \bigtriangledown f(x_h) \\
&\quad - \bigtriangledown f(T_i^h x_h), x_h - w \rangle + R_1(\mu_h + \mu_h D_f(w, x_h) + l_h).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
D_f(w, y_h) \leq\ & D_f(w, x_h) + R(\mu_h + \mu_h D_f(w, x_h) + l_h) + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{h,i} \langle \bigtriangledown f(x_h) \\
& - \bigtriangledown f(T_i^h x_h), x_h - w \rangle. \qquad (4.2.3)
\end{aligned}
$$

for some $R_1 > 0$, $R_2 > 0$ and $R > 0$.

It follows from (4.2.2) and (4.2.3) that $w \in C_{h+1}$, which implies $F \subset C_n$ for every $n > 1$

Step IV. We show that $\lim_{n\to\infty} D_f(x_n, x)$ exists. From (4.2.1), $x_n = P_{C_n}^f x$, which from (a) of Lemma 3.0.2 implies

$$\langle \bigtriangledown f(x) - \bigtriangledown f(x_n), y - x_n \rangle \leq 0 \quad \forall\, y \in C.$$

Since $F \in C_n$ we have

$$\langle \bigtriangledown f(x) - \bigtriangledown f(x_n), w - x_n \rangle \leq 0 \quad \forall\, w \in F. \qquad (4.2.4)$$

From (b) of Lemma 3.0.2, we have

$$\begin{aligned} D_f(x_n, x) &= D_f(P^f_{C_n}x, x) \leq D_f(w, x) - D_f(w, P^f_{C_n}x) \\ &\leq D_f(w, x) \quad \forall\, n \geq 1, \quad w \in F. \end{aligned}$$

This implies that $\{D_f(x_n, x)\}$ is bounded. It follows from Lemma 2.4, $\{x_n\}$ is bounded. By the construction of C_n we have $x_m \in C_m \subset C_n$ and $x_n = P^f_{C_n}x$, for any positive integer $m \geq n$. Then we obtain

$$\begin{aligned} D_f(x_m, x_n) &= D_f(x_m, P^f_{C_n}x) \leq D_f(x_m, x) - D_f(P^f_{C_n}x, x) \\ &= D_f(x_m, x) - D_f(x_n, x). \end{aligned} \tag{4.2.5}$$

In particular,

$$D_f(x_{n+1}, x_n) \leq D_f(x_{n+1}, x) - D_f(x_n, x).$$

Since $x_n = P^f_{C_n}x$ and $x_{n+1} = P^f_{C_{n+1}}x \in C_{n+1} \subset C_n$, we obtain $D_f(x_n, x) \leq D_f(x_{n+1}, x)$, $\forall\, n \geq 1$.

This shows that $\{D_f(x_n, x)\}$ is nondecreasing and hence the $\lim_{n\to\infty} D_f(x_n, x)$ exists. Thus from(4.2.5), taking the limit as $m, n \to \infty$, we obtain $\lim_{n\to\infty} D_f(x_m, x_n) = 0$. Since f is totally convex on bounded subsets of E, f is sequentially consistent, then it follows that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence $\{x_n\}$ is Cauchy sequence in C. As $\{x_n\}$ is Cauchy in a complete space E, there exists $p \in E$ such that $x_n \to p$ as $n \to \infty$. Clearly $p \in C$. Since $D_f(x_m, x_n) \to 0$ as $m, n \to \infty$, we have in particular

$$\lim_{n\to\infty} D_f(x_{n+1}, x_n) = 0 \tag{4.2.6}$$

and this implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.2.7}$$

Step V. Next we show that $p \in F$.

Since $x_{n+1} = P^f_{C_{n+1}} x \in C_{n+1}$, we have from (4.2.1)

$$\begin{aligned} D_f(x_{n+1}, y_n) &\leq D_f(x_{n+1}, x_n) + R(\mu_n + \mu_n D_f(x_{n+1}, x_n) \\ &\quad + l_n) + \frac{\lambda}{1-\lambda} \sum_{i=1}^{N} \alpha_{n,i} \langle \bigtriangledown f(x_n) \\ &\quad - \bigtriangledown f(T_i^n x_n), x_n - x_{n+1} \rangle, \end{aligned} \tag{4.2.8}$$

which implies that $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$. Since f is totally convex on bounded subsets of E, f is sequentially consistent (see Butnariu and Lusem (2000)). It follows that

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0. \tag{4.2.9}$$

From (4.2.7) and (4.2.9), we have

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{4.2.10}$$

Since f is uniformly Frechet differentiable, it follows from Lemma 1.1 that $\bigtriangledown f$ is uniformly continuous and f is uniformly continuous on bounded subsets of E (see Prodi and Ambrosetti (1993),Theorem 1.8). Hence

$$\lim_{n\to\infty} \|\bigtriangledown f(x_n) - \bigtriangledown f(y_n)\| = 0. \tag{4.2.11}$$

From (4.2.1), we have

$$\|\bigtriangledown f(T_i^n x_n) - \bigtriangledown f(x_n)\| = \frac{1}{\sum_{i=1}^{N} \alpha_{n,i}} \|\bigtriangledown f(x_n) - \bigtriangledown f(y_n)\|$$

which implies from (4.2.11) that

$$\lim_{n\to\infty} \|\bigtriangledown f(T_i^n x_n) - \bigtriangledown f(x_n)\| = 0. \tag{4.2.12}$$

Since f is strongly coercive and uniformly convex on bounded subsets of E, f^* is uniformly Frechet differentiable on bounded sets. Moreover, f^* is bounded on bounded sets, and from (4.2.12) we obtain

$$\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0. \tag{4.2.13}$$

On the other hand, we see that

$$\begin{aligned} \|x_n - T_{i+l}^n x_n\| &\leq \|x_n - x_{i+l}\| + \|x_{i+l} - T_{i+l}^n x_{n+l}\| \\ &+ \|T_{i+l}^n x_{n+l} - T_{i+l}^n x_n\| \\ &\leq (1+L)\|x_n - x_{i+l}\| + \|x_{i+l} - T_{i+l}^n x_{n+l}\| \end{aligned}$$

for all $l \in \{1, 2, \ldots, N\}$, where $L := \sup_{1\leq i\leq N} L_i$. it follows from (4.2.7) and (4.2.13) that

$$\lim_{n\to\infty} \|x_n - T_{i+l}^n x_n\| = 0.$$

for all $l \in \{1, 2, \ldots, N\}$. Thus,

$$\lim_{n\to\infty} \|x_n - T_l^n x_n\| = 0 \tag{4.2.14}$$

$\forall\ l \in \{1, 2, \ldots, N\}$. Since $x_n \to p$ as $n \to \infty$ by the closedness of T_l for each $l \in \{1, 2, \ldots, N\}$ we obtain $p \in \cap_{l=1}^k F(T_l)$.

Next, we show that $p \in EP(g_j)$, for each $j = 1, 2, \ldots, m$. Since $x_{n+1} \in C_{n+1}$ we have from (4.2.1)

$$D_f(x_{n+1}, u_{j,n}) \leq D_f(x_{n+1}, y_n), \qquad j = 1, 2, \ldots, m.$$

Using the fact that $\lim_{n\to\infty} D_f(x_{n+1}, y_n) = 0$, we have

$$\lim_{n\to\infty} D_f(x_{n+1}, u_{j,n}) = 0, \qquad j = 1, 2, \ldots, m.$$

and this implies that

$$\lim_{n\to\infty} \|x_{n+1} - u_{j,n}\| = 0, \qquad j = 1, 2, \ldots, m. \tag{4.2.15}$$

From (4.2.7) and (4.2.15), we have

$$\lim_{n\to\infty} \|x_n - u_{j,n}\| = 0, \qquad j = 1, 2, \ldots, m. \tag{4.2.16}$$

Also from (4.2.10) and (4.2.16), we have

$$\lim_{n\to\infty} \|y_n - u_{j,n}\| = 0, \qquad j = 1, 2, \ldots, m.$$

From the uniform continuity of $\bigtriangledown f$, for each $j = 1, 2, \ldots, m$ we have

$$\lim_{n\to\infty} \|\bigtriangledown f(u_{jn}) - \bigtriangledown f(y_n)\| = 0 \tag{4.2.17}$$

From(3.0.2), we have, for $j = 1, 2, \ldots, m$

$$g_j(u_{jn}, y) + \langle \bigtriangledown f(u_{jn} - \bigtriangledown f(y_n), y - u_{jn} \rangle \geq 0, \quad \forall y \in C.$$

Furthermore, using (A2) in the inequality above, we obtain

$$\langle \bigtriangledown f(u_{jn} - \bigtriangledown f(y_n), y - u_{jn} \rangle \geq g_j(y, u_{jn}), \quad \forall y \in C.$$

By (A4),(4.2.17),and $u_{jn} \to p$ as $n \to \infty$, we have

$$g_j(y, p) \leq 0 \quad \forall y \in C.$$

Let $z_t = ty + (1 - t)p$ for $t \in (0, 1]$ and $y \in C$. This implies that $z_t \in C$. This yields $g_j(z_t, p) \leq 0$. It follows from (A1) and (A4) that

$$0 = g_j(z_t, z_t) \leq tg_j(z_t, y) + (1-t)g_j(z_t, p) \leq tg_j(z_t, y)$$

This implies

$$0 \leq g_j(z_t, y).$$

From condition (A3), we obtain

$$g_j(p, y) \geq 0, \quad \forall y \in C \text{ and } \forall j \in \{1, 2, \ldots, m\}.$$

This implies that $p \in EP(g_j)$, for each $j = 1, 2, \ldots, m$. Thus, $p \in \cap_{j=1}^{m} EP(g_j)$. Hence we have $p \in F := (\cap_{j=1}^{m} EP(g_j)) \cap (\cap_{i=1}^{k} F(T_i))$.

Step VI. Finally, we show that $p = P_F^f x$. Setting $n \to \infty$ in (4.2.4), we obtain

$$\langle \bigtriangledown f(x) - \bigtriangledown f(p), w - p \rangle \leq 0, \quad \forall\, w \in F.$$

By (a) of Lemma 3.0.2 we have $p = P_F^f x$. This complete the proof.

□

The above result, extends the result of Ugwunnadi et al. (2014) from quasi-Bregman strictly pseudocontractive mapping to quasi-Bregman total asymptotically strictly pseudocontractive mapping.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this dissertation, various classes of mappings, namely, asymptotically nonexpansive, total asymptotically nonexpansive mappings and total asymptotically strictly pseudocontractive mappings were studied.

In chapter three, we reviewed all relevant Lemmas and Theorem that are necessary for establishment of our results, while in chapter four, an iterative sequence for approximation of common fixed point (assuming existence) of quasi-Bregman total asymptotically nonexpansive mapping was constructed. Necessary and sufficient conditions for the convergence of the scheme to a common fixed point of the mappings were given. Furthermore, sufficient condition for convergence of the iteration process to a common fixed point of the mappings was established. Secondly a new iterative scheme by hybrid method was introduced and a strong convergence theorem for finding a common element in the set of fixed points of finite family of closed quasi-Bregman total asymptotically strictly pseudocontractive mapping and common solution to a system of equilibrium problems in reflexive Banach spaces were established.

5.2 Conclusions

In chapter Three. Theorems 4.1.1, 4.1.2 and 4.1.4 extends the result of Chidume and Ofoedu (2007) from total asymptotically nonexpansive mapping to quasi-

Bregman total asymptotically nonexpansive mapping. In chapter four. Theorem 4.2.1 extends the result of Ugwunnadi et al. (2014) from quasi-Bregman strictly pseudocontractive mapping to quasi-Bregman total asymptotically strictly pseudocontractive mapping.

5.3 Recommendations

Various classes of mappings have been introduced, to enable researchers have access to further research. In particular, in chapter Four, we recommend that our results should be established using different iterative scheme.

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