# STUDY OF SOFT SET THEORY, ITS ALGEBRA AND APPLICATION

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# STUDY OF SOFT SET THEORY, ITS ALGEBRA AND APPLICATION


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# DECLARATION

I hereby declare that the work in this dissertation entitled " STUDY OF SOFT SET THEORY, ITS ALGEBRA AND APPLICATION" was performed by me in the Department of Mathematics under the Supervision of Professor D. Singh, Dr. A. M. Ibrahim and Professor S.B Junaidu.

The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this work has been presented for another degree or diploma at any institution.

**ONYEOZILI, IJEOMA ABIGAIL**

| Name of student | Signature | Date |
|---|---|---|

# CERTIFICATION

This dissertation entitled " STUDY OF SOFT SET THEORY, ITS ALGEBRA AND APPLICATION" by ONYEOZILI, Ijeoma Abigail meets the regulations governing the award of the degree of Doctor of Philosophy of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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# DEDICATION

This dissertation is dedicated to the glory of God Almighty for seeing me through to the end of my programme and also to the memory of my beloved late sister Onyebuchi, Ogochukwu Grace and my late nephew U'Peters Nickson Ogechukwu.

# ACKNOWLEDGEMENT

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# ABSTRACT

This dissertation presents a systematic and critical study of the fundamentals of soft set theory which includes soft set operations, soft set relations and functions, soft matrices and their properties. In course of studying operations on soft sets, the complementation operation is strengthened, some new results on distributive and absorption properties with respect to various soft set operations are obtained,and soft set operations and their corresponding soft matrix operations are shown to be equivalent. An up-to-date study of soft algebraic structures such as soft groups, softrings, soft semirings and soft lattices is presented. In particular,new concepts are introduced, which helped developing certain monoids, semirings and lattices of soft subsets and partitions of a soft set with respect to various operations on soft sets. Finally techniques of applications of soft set theory in decision making are investigated and exemplified. It was observed that *Soft Choice Matrix* technique can solve problems involving a huge number of decision makers in a much economical way.

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# CHAPTER 1

# GENERAL INTRODUCTION

## 1.1 Background of the Study

The theory of sets is fundamental and also an indispensable mathematical tool. Among the various paradigmatic issues associated with set is the concept of uncertainty and vagueness.

According to traditional view, mathematics should strive for certainty (i.e., precision, perfection, exactness, consistency, specificity etc.,) in all its manifestations. Hence uncertainty (i.e., imprecision, vagueness, inconsistency etc.,) is regarded as non-mathematical. In modern view, uncertainty in mathematics is considered inevitable especially in real life. This uncertainty or the representation of imperfect knowledge has been a problem for a long time for philosophers, logicians and mathematicians.

Later, the emergence of computer technology and its rapidly growing power made it possible to deal with increasingly complex problems. Initially, it was commonly believed by many mathematicians that the level of complexity that can be handled basically depends on the level of computational power at our disposal. It was later

realized that there is a limit to which computer technology can overcome. Thus, the fundamental question is, how can we handle these problems of uncertainty if no computational power (computer technology) alone is sufficient?

For this reason, complicated problems in many disciplines such as economics, engineering, environmental sciences, sociology, medical science and many other fields, which are highly dependent on the task of modelling uncertainties cannot be solved using classical or modern theories. Such modern theories include probability theory, fuzzy set theory, rough set theory, theory of interval mathematics, theory of vague set, etc., which can be considered as mathematical tools for dealing with uncertainties. However, all these theories have their inherent difficulties, possibly due to inadequacy of the parameterization tools of the theories as pointed out in [Molodtsov, 1999]. For instance, probability theory can only deal with stochastically stable systems, where a limit of the sample mean should exit in a long series of trials. The method of interval mathematics is not sufficiently adaptable for problems with different sorts of uncertainties, while rough set theory approach can handle problems that involve uncertainties caused by indiscernible elements with different values in decision attributes. The fuzzy set theory approach is found most appropriate for dealing with uncertainties. However, it is short of providing a mechanism on how to set a membership function, because the nature of the membership function is extremely individualistic.

Consequently, in 1999, a Russian researcher (D. Molodtsov, 1999) initiated the concept of soft set as a complete new mathematical tool for modelling uncertainties where a soft set is associated with adequate set of parameters and thus is free from the difficulties aforementioned. Unlike classical mathematics, where exact solution

of a mathematical model is needed, soft set theory instead requires an approximate description of an object as its initial point. The choice of adequate parameterization tools such as words, real number, functions etc., make soft set theory very convenient and easy to apply in practice.

During the last few years, studies on fundamentals and applications of soft sets in various fields have been perceptible. These include (Maji *et al.*, 2003) who initiated theoretical study of soft set theory,( Ali *et al.,* 2009) introduced new operations on soft sets, (Maji *et al.,* 2002) who described application of soft set to decision making problem.

Apart from the rich application of soft set theory, its algebraic structures have also been studied extensively by many researchers. These include, the works of (Aktas and Cagman, 2007) on soft groups and their basic properties,( Feng *et al.,* 2008) who introduced the notion of soft semirings, (Acar *et al.*, 2010) discussed the concept of soft rings and soft ideals of soft rings. Other are, (Jun, 2008), (Jun and Park, 2008), (Ali *et al.,* 2011) etc.

In recent times, many authors ( Maji *et al.*, 2001,

Alkhazaleh *et al.*, 2011, Pinaki , 2012, Singh *et al.*, 2014) etc., have combined soft sets with other sets such as fuzzy set, rough set, multiset, intuitionistic fuzzy set, etc., to generate hybrid structures like fuzzy soft sets, rough soft sets, Multi soft sets, intuitionistic fuzzy soft sets, among others.

Applications of these hybrids in areas like decision making, medical diagnosis, forecasting, etc., have also been shown by many researchers (Roy and Maji, 2007), (Das and Borgohain,2010) (Tridiv and Dusmanta,2011B),(Zhi *et al.,* 2011), (Rajarajeswari and Dhanalakshimi,2013) etc.

## 1.2 Statement of the Problem

In recent times, studies on fundamentals and various algebraic structures of soft set theory have been conducted. However, some fundamental properties related to soft set operations and some algebraic structures have not been studied.

## 1.3 Justification

In classical mathematics, the notion of exact solution of a mathematical model is defined. However, in general, mathematical models are quite complicated and it becomes an arduous task to define exact solution of these models. As a result, the notion of approximate solutions is introduced.Such introduction included the emergence of soft set theory where an approximate description of the object is provided. In fact, in soft set theory, there is no restriction on the parameterization tools which makes it very convenient and easily applicable in real life. Thus, soft set approach has come to be recognised as fundamentally important.

Algebraic structures play a prominent role in mathematics with wide ranging applications in many disciplines such as computer science, information science, topological spaces and so on. This provides sufficient motivation to researchers to review various concepts and results on algebraic structures and in the broader frame work of soft set setting. Soft set theory has a rich potential for application in many directions, few of which had been shown by Molodtsov(1999) in his pioneer work. These include smoothness of functions, game theory, operations research, Reimann and Perron integrations, probability theory and measure theory.

Recently, various other applications of soft set theory especially in decision making

by some researchers such as Maji *et al.*, (2003), Cagman and Enginglu,(2010A,2010B), Majumdar and Samanta,(2010) etc., have been developed. Thus, the study of soft set theory, its applications and the underlying algebraic structures become imperative.

## 1.4 Aim and Objectives

The aim of this dissertation is to study the theory of soft sets, its algebraic structures and application in decision making.

In order to achieve this aim, the following objectives are proposed to be accomplished:

A. To conduct a systematic and critical study of the concept and fundamentals of soft sets and obtain some new results:

(i) Clarify some conceptual misunderstandings of the operations of *Not* set of a set and complement of a set.

(ii) Establish and investigate, with suitable examples, some distributive and absorption properties with respect to various soft set operations.

(iii) Compare soft set operations with their corresponding soft matrix operations.

B. To study some algebraic structures of soft sets and formulate some new algebraic structures.

(i) Study soft groups, soft rings, soft semirings, soft lattices.

(ii) Formulate certain monoids, semirings, and lattices of soft subsets and partitions of a soft set.

C. To investigate various techniques used in the application of soft sets to decision making problems and exemplify them. and set

## 1.5 Methodology

In order to achieve the proposed objectives outlined above, an up-to-date review of the existing literature on the concept and fundamentals, soft algebraic structures and techniques for application of soft set theory in decision making problems was conducted. In the sequel, new concepts were introduced and illustrated, which helped developing some new algebraic structures.

## 1.6 Organization of the Dissertation

The dissertation is organized as follows:

Besides chapter one, an up-to date literature survey of soft set theory is presented in chapter two.

In chapter three, the fundamentals of soft set theory are discussed. Some new results on distributive and absorption properties of operations of soft set theory are also presented.

Chapter four presents various algebraic structures of soft set theory and their basic properties. New result on monoids, semirings and lattices of soft subsets and partitions of a soft set is also presented.

In chapter five, some techniques in the applications of soft set theory in decision making problems are presented. In chapter six, the summary, conclusion and recommendations for future study is presented. A list of references cited in the dissertation is presented at the end.

# CHAPTER 2

# LITERATURE REVIEW

In this chapter, an up-to-date literature survey of the concept and fundamentals of soft set theory, a systematization of various algebraic structures of soft sets and finally, some applications of soft set theory in decision making are presented.

## 2.1 Concept and Fundamentals of Soft Set Theory

The concept and development of soft set theory whose origin could be traced to the work of (Pawlak, 1993) was initiated by (Molodtsov, 1999). Molodtsov proposed that soft set theory is a general mathematical tool for solving complicated problems dealing with vagueness and uncertainties which classical methods and some modern mathematical methods, such as probability theory (Dempster, 1958), fuzzy set theory (Zadeh, 1965), rough set theory (Pawlak, 1982), interval mathematics theory (Gorzalzany, 1987), vague theory (Gau and Buchrer, 1993) etc., cannot successfully solve due to inadequacy of their parameterization tools.

Molodtsov (1999) pointed out that soft set theory provides enough parameter and

as a result accommodates initial approximate descriptions of an object. This, makes soft set theory free from the above difficulty and becomes very convenient and easily applicable in practice. Molodtsov (1999) therefore defined a soft set as a parameterized family of subsets of a universe set, where each element is considered as the set of approximate elements of the soft set. He also successfully applied soft set theory in areas such as *smoothness of functions* (where he compared *smoothness of functions* as being similar to *continuity of functions* in the classical case), *game theory*, *operations research*, *Rieman and Perron integrations, probability theory* and *measurement theory,* and introduced the basic notions of the theory of soft sets.

Based on the work of( Molodtsov,1999), (Maji *et al.*, 2003) initiated the theoretical study of the soft set theory. This includes, the definition *of soft subset, soft superset, equality of soft sets, complement of a soft set* among others, with some illustrative examples. Soft binary operations such as *AND, OR, union* and *intersection operations* were also defined. Verification of De Morgan's law and a number of results on soft set theory were presented. For the purpose of storing a soft set in a computer, they represented a soft set in the form of a table.

Pei and Miao (2005) discussed the relationship between soft sets and information systems and showed that soft sets are a class of special information systems.

Yang (2008) was the first to point out error in the work of (Maji *et al., 2003*) by giving a counter example.

Ali *et al.,* (2009) also pointed out several assertions in (Maji *et al.*, 2003) that are

not true in general by counter examples Some new operations such as *restricted union, restricted intersection, restricted difference* and *extended intersection* were further introduced. Moreover the notion of complement of a soft set was improved and that certain De Morgan's laws hold in soft set theory with respect to these new operations. Ali *et al.,* (2009) also remarked that the incorrectness of the assertions mentioned above may be as a result of the way and manner some of the related notions were defined.

Gong *et al.*, (2010) proposed the concept of *bijective soft set* and some of its operations such as the *restricted AND* and the *relaxed AND operations.* Dependency between two bijective soft sets and the bijective soft decision system was also defined. Finally an application of bijective soft set in decision making problem was given.

Xiao *et al.*, (2010), reviewed the notions of soft sets and bijective soft sets and proposed the notion of *exclusive disjunctive soft sets.* Some studies on operations of exclusive disjunctive soft sets , such as restricted/ relaxed AND- operations and the dependency between exclusive disjunctive soft sets and bijective soft sets were also established. An application of exclusive disjunctive soft set to attribute reduction of incomplete information system was demonstrated. It was pointed out that potential studies could be focusing on how to extend it to fuzzy soft set as well as applying it in solving decision- making problems.

Kharal and Ahmad (2011) introduced the notion of *mapping on soft classes* and

studied several properties of images and inverse images of soft sets supported with examples and counter examples. An application of these notions to the problem of *medical diagnosis* in medical expect systems, where a patient's complaints or symptoms may easily be encoded into a soft set, was finally given.

In line with the work of (Kharal and Ahmad, 2011), (Majumdar and Samanta, 2010) introduced the idea of *soft mapping* and studied some of its properties with the observation that fuzzy mapping is a special case of soft mapping. Majumdar *et al.,* (2010) also defined the image and the respective inverse image of a soft set under a soft mapping and studied some of their properties. They also applied the soft mapping in medical diagnosis problem and remarked that the model used in the diagnosis was very preliminary and may be improved by incorporating detailed disease- symptom information and clinical results. For future research, a work on the theory of fuzzy soft mapping and the intuitionistic fuzzy soft mapping was also proposed.

Babitha and Sunil (2010) introduced the concept of *soft set relation* as a subsoft set of the Cartesian product of soft sets. Besides, many related concepts such as *equivalence soft set relation, partition of soft set*, *soft set function, and composition of soft set functions* with related results were proposed. In their concluding remarks, a future research on the theoretical aspect of these generalized concepts in relation to topology generated by soft set relation was proposed.

In continuation of their work, (Babitha and Sunil, 2011) introduced the concept of

*anti- symmetric relation* and *transitive closure of a soft set relation* and proposed an analogue of Warshall's algorithm for calculating the transitive closure of a soft set relation. *Ordering on a soft set* was also defined and the relationship between different orderings was established and supported with examples.

In an attempt to broaden the theoretical aspect of soft set relation, (Yang and Guo , 2011) introduced the notions of *anti- reflexive kernel, symmetric kernel; reflexive closure* and *symmetric closure of a soft set relation* and obtained with proofs some results involving them. The notions of soft set relation mapping and its inverse were also proposed and some of their related properties discussed.

Ge and Yang (2011), further investigated the operational rules given by (Maji *et al.*, 2003) and (Ali *et al.*, 2009) and obtained some *necessary and sufficient* conditions for the rules to hold. It was specifically noted that several assertions in relation to null soft set and absolute soft set are incorrect, due to the notation of the related definitions in (Maji *et al.*, 2003).

Fu Li (2011) pointed out and corrected some errors in (Maji *et al., 2003*) and also discussed the distributive laws with respect to certain operations of union and intersection introduced in (Ali *et al., 2009*) using De Morgan laws.

Sezgin and Atagun (2011B) proved that certain De Morgan's laws with respect to different operations on soft set defined by (Maji *et al.*, 2003), (Pei and Miao 2005) and( Ali *et al.*, 2009) hold. Further discussions on various basic properties of

operations on soft set such as union, restricted union, extended and restricted intersections, restricted difference and their interconnection between them were made. Finally the notion of *restricted symmetric difference* of soft sets was defined and its properties investigated with examples.

Tridiv and Dusmanta (2011A) reintroduced the concept of complement of soft set and showed that the laws of exclusion and contradiction, involution, De Morgan's inclusion and De Morgan's laws are valid for soft sets with respect to this new definition of complement.

Won Keon Min (2012) introduced and studied the concept of similarity between soft sets which is an extension of the equality for soft sets and investigated some properties. The concept of *conjunctive* $(\propto \wedge \beta)$ and *disjunctive* $(\propto \vee \beta)$ of ordered pair parameter $(\propto, \beta)$ for soft set theory were introduced. Some operations of soft set theory introduced by( Maji *et al.*, 2003) were modified and investigated.

Zue and Wen (2013) redefined the intersection, complement and differences of soft sets and investigated the algebraic properties of these operations along with the known union operation. It was found that these new operations inherit all basic properties of operations on classical sets.

## 2.2 Algebraic Structures Of Soft Sets

The algebraic structures of soft set theory have also been studied extensively. Aktas and Cagman (2007) introduced the basic concepts of *soft groups, soft subgroups, normal soft subgroups and soft homomorphism* and discussed their basic properties.

Feng *et al.,* (2008) considered the algebraic structure of semiring and introduced the notion of *soft semiring.* Some basic algebraic properties of soft semiring and some related notions such as *soft ideals, idealistic soft semirings* and *soft semiring homomorphism* were defined and investigated with illustrative examples

Jun (2008) applied the notion of soft sets to the theory of BCK/BCI- algebras and introduced the notion of *soft BCK/ BCI- algebras* and *soft subalgebras* and then derived their basic properties.

Jun and Park (2008) introduced the notion of soft ideals and idealistic soft BCK/BCI-algebras and established their basic properties.

Jun *et al.*, (2008) applied the notion of soft sets by Molodtsov to commutative ideals of BCK-algebras, *commutative soft ideals* and commutative idealistic soft BCK-algebras and investigated their basic properties.

Sun *et al.,* (2008) presented the definition of *soft set modules* and constructed some basic properties using modules.

Jun and Park (2009) introduced the concept of *soft Hilbert algebra*, soft Hilbert abysmal algebra and soft Hilbert deductive algebra and investigated their properties. ( Jun *et al.*, 2009) also in another paper, introduced the notion of *soft p- ideals and p- idealistic soft BCI- algebras* and discussed their basic properties.

Based on the work of( Feng *et al.*, 2008),( Acar *et al.*, 2010) introduced the basic notions of *soft rings* as a parameterized family of subrings of a ring over a ring with some illustrative examples. The notions of *soft subrings, soft ideal of a soft ring, idealistic soft rings* and *soft ring homomorphism* were introduced with some corresponding examples.

Qin and Hong (2010) made a theoretical study of the algebraic structures of soft sets such as *lattice structures* and introduced the concept of *soft equality relation* and also discussed its related properties. It was proved that soft equality relation is a congruence relation with respect to some operations. Finally, the quotient algebra with respect to restricted union and extended intersection was established.

Fu Li (2010) defined the notion of *soft lattice,* and the operations on soft lattice with several examples.

In relation to the work of( Acar *et al.,* 2010), (Celik *et al.*, 2011) introduced some new operations on soft ring such as *extended sum, restricted sum, extended product, restricted product* and established some of their basic properties.

Ali *et al.,* (2011) defined some algebraic structures such as *semigroups, semirings and lattices associated with soft sets,* and completely described the distributive and absorption laws on operations of soft sets. MV- algebras and BCK-algebras associated with soft set, with a fixed set of parameters were also studied.

Atagun and Sezgin (2011A) introduced and studied some sub structures such as *soft subrings and soft ideals of a ring, soft subfield of a field* and *soft sub module of a module* with several illustrative examples. Some related properties on operations of restricted intersection, product and sum for these soft sub- structures were established and investigated with examples.

In another work, (Sezgin and Atagun, 2011B), corrected some assertions in (*Soft set and soft group by (Aktas and Cagman, 2007)* and further extended the theoretical aspects of soft group defined in (Aktas and Cagman,2007) by introducing the concept of *normalistic soft group, normalistic soft group homomorphism,* and establishing that the normalistic soft group isomorphism is an equivalence relation on normalistic soft groups.

Jun *et al.*, (2011) discussed the notion of *positive implicative ideals* of BCK-algebras based on soft set theory and their basic properties.

Nagaranjan and Meenambigar(2011) further defined soft lattices, *soft distributive lattices, soft modular lattices, soft lattice ideals* and *soft lattice hormomorphism* and investigated several related properties and some characterization theorems.

Lee *et al.,* (2011) and Jun *et al.,* (2011) introduced the notions of *implicative, positive implicative* and *commutative soft idea*ls in BCK-algebras. The notion of *implicative idealistic soft BCK- algebra* with some illustrative examples was introduced and their basic properties derived.

Ozturk and Inan (2011) presented *soft-rings* and *idealistic soft rings.* Pal & Mondal (2013) defined soft matrices based on soft set. Operations of soft matrices such as AND, OR, union, intersection etc., for soft matrices are also defined. Commutative, associative, distributive, De Morgan's and convergent properties were investigated.

Karaaslan *et al.,* (2012) also defined soft lattices and their substructures different from the authors (Fu Li 2010 and Nagaranjan and Meenambigar ,2011).

Turkmen and Pancer (2013) introduced the notions of *sum and direct sum of soft submodules*, small soft sub modules and radical of a soft module and obtained their basic properties.

Feng and Li (2013) studied several types of soft subsets and *soft product operations* and characterized them, while (Burak , 2013) introduced and investigated the basic concepts of soft set theory and soft group theory and gave some examples.

Jayanta, (2014) studied the algebraic structure of soft sets defined on a fixed set of parameters and investigated its properties. It was shown that this structure formed

a Boolean algebra.

Lee *et al.,* (2014) discussed the generalizations of *int-soft subsemigroups.* Also the notion *of $\theta$-generalized int-soft subsemigroup* in semigroups was introduced and several properties investigated.

## 2.3 Applications Of Soft Set Theory

Various applications of soft set theory were made by several authors. (Maji *et al.*, 2002), gave a practical application of soft sets in decision making problem using the notion of knowledge *reduction in rough set theory of Pawlak* (1982).

Chen *et al.,* (2005) pointed out that the result of *soft set reductions* proposed by ( Maji *et al.,* 2002) was incorrect and explained with example that the algorithms used to first compute the *reduct -soft set* and then the *choice value* for selecting the *optimal objects* in the decision problem were unreasonable, and as a result, proposed a new definition of *parameter reduction* for the improvement of (Maji *et al.*, 2002) and concluded that the idea of reduct under rough set theory generally cannot be applied directly in reduct under soft set theory, with the observation that the parameter reduction may well play an important role in some knowledge discovery problems.

Mushrif *et al.*, (2006) studied the *texture classification* via soft set theory based on a classification Algorithm.

Kong *et al.*, (2008) analyzed that the problems of *sub- optimal choice* was not addressed by (Chen *et al.*, 2005) and so introduced the notion of *normal parameter reduction* in soft set theory to overcome the problems in Chen's model. Two new definitions viz,-the parameter important degree and the soft decision partition were described and used to analyze the algorithm of normal parameter reduction. They noted that with this approach, the *optimal choices* are still preserved.

Zou and Xiao (2008) proposed a new technique for decision making of soft set theory under incomplete information systems which is based on the calculation of weighted –average of all possible choice values of objects.

Herewan *et al.*, (2010) also analyzed the existing works on *attributes* and *parameterization reduction of soft sets* and remarked that all the techniques were based on Boolean- valued information system. An alternative approach for attribute reduction which was based on the notion of multi-soft sets, constructed from a multi-valued information system was proposed, and a result which turned out to be equivalent to Pawlak's rough set reduction was obtained.

Cagman and Enginolu (2010A) redefined some operations of soft set theory in order to make them more functional for improving several new results. Four products of soft sets viz-*AND* ($\wedge$) *- product, OR* ($\vee$)*- product AND-Not* ( $\bar{\wedge}$)*- product and OR-NOT* ( $\underline{\vee}$ )*-product* were also defined and their properties discussed. Using AND ($\wedge$)-product, they constructed an *uni-int (union- intersection) soft decision making*

*method* which select a set of optimum elements from the alternatives without using the rough sets and fuzzy soft sets. Finally the method was applied to a decision making problem, remarking that apart from the uni-int soft decision method, similar types of decision methods using *int-uni (intersection-union), uni-uni (union-union) and int-int (intersection-intersection can be constructed for* $\vee$*-product,* $\bar{\wedge}$*-product and* $\underline{\vee}$*-product* respectively in a similar way though with some modifications.

In a similar work (Cagman and Enginoglu, 2010B), introduced *soft matrices* which are matrix representation of soft sets and defined *AND* ($\wedge$)*, OR* ($\vee$)*, AND-NOT* ($\bar{\wedge}$) *and OR-NOT* ($\underline{\vee}$) *products of soft matrices.* Also, using the AND-product of soft matrices, a *soft max-min (maximum-minimum) decision making method* was constructed and applied to a real estate problem, noting that similarly, other products $\vee$, $\bar{\wedge}$ and $\underline{\vee}$ of soft matrices could be used for other convenient or similar problems.

Other works on applications of soft set theory include (Gong *et al.*, 2010), (Majundar and Samanta, (2010), among others.

Mitra *et al.*, (2012) defined different types of soft matrices and introduced some new operations on these matrices. Moreover, using these matrices, a new and efficient solution procedure to solve *soft set based real life decision making problem involving more than one decision maker* was proposed.

Samsiah and Mohamad (2012) proposed algorithm that generalized (Cagman and

Enginoglu 2012B) soft max-min decision making method for problems involving more than two decision makers and applied it to a house selection problem.

In line with Cagman and Enginolu (2010B), (Vijayabalaji and Ramesh, 2013) constructed a *soft Max-Max decision making method* and applied it to decision making problems involving two and three decision makers.

# CHAPTER 3

# FUNDAMENTALS OF SOFT SET THEORY

In this chapter, a systematic and critical study of the fundamentals of soft set theory, which include operations on soft sets and their properties, soft set relations and functions, and matrix representation of soft sets are presented. Some new results on properties of soft set operations are also presented.

## 3.1 Basic Definitions and Results

In this section, some basic definitions and results on soft sets with suitable examples, much of which were introduced in (Molodtsov, 1999, Maji *et al.*, 2003, Pie and Mio, 2005, Ali *et al.*, 2009) are given.

Molodtsov (1999) in his pioneer work therefore defined soft set as follows:

**Definition 3.1.1: Soft set (Molodtsov, 1999)**

Let $U$ be an initial *universe set* and E a set of *parameters* with respect to $U$. Let P ($U$) denote the power set of $U$ and $A \subseteq E$. A pair (F, $A$) is called a *soft set* over $U$, where F is a mapping given by $F : A \rightarrow$ P(U). In other words, a soft set over a universe $U$ is a parameterized family of subsets of the universe $U$. For e$\in$A, F(e)

may be considered as the set of *e- elements* or *e- approximate elements* of the soft set (F, A). Thus,

(F, A) =$\{$F(e)$\in$ P $(U)$: e$\in A \subseteq E\}$.

As an illustration, let us consider the following example:

**Example 3.1.1** Suppose a universe $U$ is the set of six houses under consideration, given by

$U$ = { $h_1$, $h_2$, $h_3$ $h_4$, $h_5$, $h_6$}, the parameter set E={ $e_1$, $e_2$, $e_3$,$e_4$,$e_5$} where each parameter $e_i$, $i$ =1, 2, 3, 4, 5 stands for *expensive, beautiful, cheap, modern, wooden* respectively, and A= { $e_1$, $e_2$, $e_3$}$\subset E$.

Let a soft set (F, A) describe the attractiveness of the houses which say Mr. X wants to buy. In this case, to define the soft set (F, A) means to point out the houses for each parameter, i.e., point out *expensive houses, beautiful houses, cheap houses*, etc. Now consider the mapping F, where F: A$\longrightarrow$ P($U$) is given by

F($e_1$)={ $h_2$, $h_4$}, F($e_2$)= { $h_1$, $h_3$ $h_5$ }, F($e_3$)= {$h_1$, $h_3$, $h_6$}.

Then the soft set (F, A) is a parameterized family {F($e_i$), i= 1,2,3} of subsets of the universe $U$ given by

(F, A)= {{$h_2$, $h_4$}, {$h_1$, $h_3$, $h_5$}, {$h_1$, $h_3$, $h_6$}}

where for example F($e_1$) means *house (expensive),*whose *functional value,* called the $e_1$- *approximate value set,* is the set {$h_2$, $h_4$}. Thus, we can view the soft set (F, A) as consisting of a collection of approximations given by

(F, A) = {F($e_1$)= {$h_2$, $h_4$}, F($e_2$)= {$h_1$, $h_3$, $h_5$}, F($e_3$)= {$h_1$, $h_3$, $h_6$}},

where each approximation has two parts:

1. *a predicate* F($e_1$), or F($e_2$) or F($e_3$) and

2. *the approximate set* {$h_2$, $h_4$} or {$h_1$, $h_3$, $h_5$} or {$h_1$, $h_3$, $h_6$}, respectively.

Table 3.1: **Tabular Representation of (F, A)**

| $U$ | $e_1$ | $e_2$ | $e_3$ | Choice value |
|---|---|---|---|---|
| $h_1$ | 0 | 1 | 1 | 2 |
| $h_2$ | 1 | 0 | 0 | 1 |
| $h_3$ | 0 | 1 | 1 | 2 |
| $h_4$ | 1 | 0 | 0 | 1 |
| $h_5$ | 0 | 1 | 0 | 1 |
| $h_6$ | 0 | 0 | 1 | 1 |

The soft set (F, A) can also be represented by the set of ordered pairs given by

(F, A)= $\{(e_1, F(e_1)), (e_2, F(e_2)), (e_3, F(e_3))\}$ i.e.,

(F, A)= $\{(e_1,\{h_2, h_4\}), (e_2, \{h_1, h_3, h_5,\}), (e_3,\{h_1, h_3, h_6\})\}$.

It is worth noting that the sets F(e), e∈A may be arbitrary, maybe empty or may have non- empty intersection. Also the soft set (F, A) can be denoted by $F_A$ or $(f_A$,E)

In order to store a soft set in a computer, a two- dimensional table is used to represent it. Table 1.1 below shows the tabular representation of the soft set (F, A), such that if $h_i \in F(e_j)$, then $h_{ij}= 1$, otherwise $h_{ij}= 0$, where $h_{ij}$ are the entries in the table below.

Suppose that Mr. X is interested in buying a house on the basis of his *choice parameters expensive, beautiful* and *cheap*. Then, according to the *highest choice value*, Mr. X can choose $h_1$ or $h_3$.

It is interesting to observe that Zadeh's fuzzy(1965) set may be considered as a special case of the soft set as follows:

Let $U$ be a universe and A, a fuzzy set on the universe $U$, characterized by its membership function $\mu_A$, such that $\mu_A$: $U \longrightarrow [0, 1]$, A $\subseteq U$.

Thus the fuzzy set A can be completely defined as a set of ordered pairs given by $A = \{(x, \mu_A(x)) : x \in U\}$ and $\mu_A(x) \in [0, 1]$.

Now let us consider the family of $\alpha$-level sets for $\mu_A$, given by

F($\alpha$)= $\{x \in U : \mu_A(x) \geq \alpha\}$, $\alpha \in [0, 1]$, such that given F, we can find $\mu_A(x)$ by the formula

$$\mu_A(x) = \sup\{\alpha \in [0, 1] : x \in F(\alpha)\}.$$

Then with adequate parameterization of soft set, Zadeh's fuzzy set may be considered as a special case of soft set (F, [0, 1]).

In order to have a comparative view of soft set with other existing sets, we briefly define the following:

Let $U$ be an initial universe, E a set of parameters, P($U$) the power set of $U$ and $A \subseteq E$.

(i) A *classical (or crisp) set* C $\subseteq U$ is a set characterized by the function $\chi_C$, called the *characteristic function* of C, where $\chi_C$: $U \longrightarrow \{0, 1\}$ and C is defined by

$$\text{C} = \{\text{x} \in U : \chi_C(\text{x}) = \begin{cases} 1, if\ x \in C \\ \\ 0, if\ x \notin C \end{cases} \}$$

(ii) A *multi set* M $\subseteq U$ is a set characterized by a *numeric– valued function* $\alpha_M$ : $U \longrightarrow \{\mathbb{N} \cup 0\}$ where M is defined by

$$\text{M} = \{\text{x} \in U : \alpha_M(\text{x}) = \begin{cases} n \in \mathbb{N}, if\ x \in M \\ \\ 0, if\ x \notin M \end{cases} \}.$$

(iii) A *fuzzy set* F over $U$ is a set characterized by a *membership function* $\mu_F$ of F where $\mu_F$: $U \longrightarrow [0, 1]$ and F is represented as;

F = $\{(\text{x}, \mu_F\ (\text{x})) : \text{x} \in U, \mu_F\ (\text{x}) \in [0, 1]\}$.

(iv) Let R be an equivalence relation on $U$, and for x $\in$ $U$, let $[x]_R$ denote the equivalence class of R determined by x. Let X $\subseteq$ $U$, X is characterized with respect to R by a pair of crisp sets called;

(a) the *lower approximation* of X, denoted by $R_*X$ and defined as;

$R_*X = \{x \in U : [x]_R \subseteq X\}$ which consist of the set of all objects that *certainly* belong to X.

(b) the *upper approximation* of X denoted by $R^*X$ defined as;

$R^*X = \{x \in U: [x]_R \cap X \neq \emptyset\}$ consisting of the set of all objects which *possibly* belong to X.

Then the set X is called a *rough set* if;

$R^*X$ - $R_*X \neq \emptyset$ and X is crisp if $R^*X$ - $R_*X = \emptyset$.

(v) A *vague set* V is a set characterized by two membership functions viz, a *true membership function* $t_v : U \longrightarrow [0, 1]$ and a *false membership function* $f_v : U \longrightarrow [0, 1]$, with $0 \leq t_v(x)+f_v(x) \leq 1$ and V is represented as; $V = \{(x, [t_v(x), f_v(x)]: x \in U \}$, where the interval $[t_v(x), 1\text{-}f_v(x)]$ is called the vague value of x in V.

(vi) A *soft set* (F, A), $F_A$ or ($f_A$, E) over $U$ is characterized by an *approximate set valued function*; $f_A$: E$\longrightarrow$ P($U$) and (F, A) is represented as a set of ordered pairs defined as; (F, A) = $\{(x, f_A(x))$: $f_A(x) \in P(U)$ and $f_A(x) = \emptyset$, if x $\notin$ A$\}$.

$$(F, A) = \{F(e) \in P(U) : e \in E,\ F(e) = \emptyset \text{ if } e \notin A\}.$$

.

Subsequent to ( Molodtsov, 1999), (Maji *et al.*, 2003) gave the following definitions:

**Definition 3.1.2: Soft subset**

Let $(F,A)$ and $(G,B)$ be two soft sets over a common universe $U$.

(a) $(F,A)$ is a **soft subset** of $(G,B)$, denoted $(F,A)\tilde{\subseteq}(G,B)$ , if

(i) $A \subseteq B$, and

(ii) $\forall\ e \in A$ , $F(e)$ and $G(e)$are identical approximations.

(b) $(F,A)$ is **soft equal** to $(G,B)$, denoted $(F,\ A) = (G,\ B)$, if $(F,\ A)\tilde{\subseteq}(G,\ B)$ and $(G,\ B)\tilde{\subseteq}(F,\ A)$.

Pei and Miao (2005) pointed out that generally in (a) (ii) $F(e)$ and $G(e)$ may not be identical and modified the definition of soft subset in the following way

**Definition 3.1.3: Soft subset redefined (Pei and Miao, 2005)**

For two soft sets $(F,A)$ and $(G,B)$ over a universe $U$, $(F,\ A)\tilde{\subseteq}(G,\ B)$, if

(i) $A \subseteq B$, and

(ii) $\forall\ e \in A,\ F(e)\ \subseteq G(e)$.

**Example 3.1.2**

Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a universe set and $E = \{e_1, e_2, e_3, e_4, e_5\}$ be a set of parameters. Let$A = \{e_1, e_2, e_5\} \subset E$ and $B = \{e_1, e_2, e_3, e_5\} \subset E$.

Suppose $(F,A)$ and $(G,B)$ are two soft sets over $U$, where

$F\left(e_1\right) = \{u_2, u_4\}, F\left(e_2\right) = \{u_3, u_4, u_5\}, F\left(e_5\right) = \{u_1\}$ and $G\left(e_1\right) = \{u_2, u_4\}$,

$G\left(e_2\right) = \{u_1, u_3, u_4, u_5\}, G\left(e_3\right) = \{u_3, u_4, u_5\}, G\left(e_5\right) = \{u_1, u_4\}$. Then $(F, A)\tilde{\subset}(G, B)$, since $A \subset B$and $F(e) \subset G(e)$ , $\forall\ e \in A$ . But $(G, B)\tilde{\not\subset}(F, A)$. Hence$(F, A) \neq (G, B)$.

**Remark 3.1.1** Let $(F,A)$ and $(G,B)$ be soft sets over a common universe $U$. $(F, A)\tilde{\subset}(G, B)$ does not imply that every element of $(F,A)$ is an element of $(G,B)$.

Therefore, the definition of classical subset does not hold for soft subset.

For example, let $U = \{u_1, u_2, u_3, u_4\}$ be a universe and $E = \{e_1, e_2, e_3\}$ be a set of parameters such that if $A = \{e_1\}$, $B = \{e_1, e_3\}$ and $(F, A) = \{(e_1, \{u_2, u_4\})\}$, $(G, B) = \{(e_1, \{u_2, u_3, u_4\}), (e_3, \{u_1, u_5\})\}$, then, $\forall\ e \in A,\ F(e) \subset G(e)$ and $A \subset B$. Hence $(F, A)\tilde{\subset}(G, B)$. Clearly $(e_1, F(e_1)) \in (F, A)$ but $(e_1, F(e_1)) \notin (G, B)$. Maji *et al.*,( 2003) gave the next definition and came up with Proposition 3.1.1 below

**Definition 3.1.4**

Let $E = \{e_1, e_2, e_3, \ldots, e_n\}$be a set of parameters. The '**Not set** of $E$', denoted $\neg E$,is defined by $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \ldots, \neg e_n\}$, where $\neg e_i$means not $e_i$, $\forall\ i = 1, 2, 3, \ldots, n$.

**Proposition 3.1.1** Let $E$ be a universal parameter set, and $A$ , $B \subset E$, then

(i) $\neg(\neg A) = A$

(ii) $\neg(A \cup B) = \neg A \cup \neg B$

(iii) $\neg(A \cap B) = \neg A \cap \neg B$

Fu Li, (2011) misconstrued definition 3.1.4 and came up with the following proposition.

**Proposition 3.1.2**

(i) $\neg(A \cup B) = \neg A \cap \neg B$ (De Morgan's Law)

(ii) $\neg(A \cap B) = \neg A \cup \neg B$ (De Morgan's law)

Singh and Onyeozili,( 2012C) clarified the conceptual misunderstanding of (Fu Li, 2011) and (Ge and Yang, 2011) in the following finding:

### 3.1.1 Clarification of some conceptual misunderstandings

appearing in [Fu Li,2011] and [Ge and Yang, 2011]

The counter example to Proposition 2.1 (1 – 3) of [Maji *et al.*, 2003], provided in [Fu Li, 2011], is misconstrued. The latter has not taken into account that the two unary operations ¬ (defined on a set of parameters) and ¬ (defined on an individual parameter) are different ( [Maji *et al.*,2003], P.557) and goes on to proving its false claim using De Morgan's laws for sets, along with numerous apparent errors appearing throughout. Moreover, Proposition 2.5 of [Fu Li, 2011] has been already proved incorrect in [Ali *et al.*, 2011], and our own analytic proof in section 2.2 of [Singh and Onyeozili, 2012C].

Let us consider the counter example used in ( [Fu Li, 2011], P. 205):

Let X = {a, b, c, d},A={a,b,c}, B = {b, d}, A∪B = {a, b, c, d}. Then ¬A (as defined in [Maji *et al.*, 2003]) is { ¬a, ¬b, ¬c} and not the set {d} (as in [Fu Li, 2011]). Accordingly, for example, ¬A ∪ ¬B = { ¬a, ¬b, ¬c, ¬d} = ¬(A ∪ B) is correct. Similar observations apply to all other results of Proposition 2.1 [Maji *et al.*, 2003] to conclude their correctness. In fact, the counter example considered in [Fu Li, 2011] is not in place at all. At its best, given X = {a, b, c, d}, the set of all parameters, and A = {a, b, c}, the set {d} = $A^C$ in X, the relative complement of A, defined in classical set theory in which ¬A is not a well formed formula. We need to emphasize that ¬A (the *Not set* of *the set* A of parameters) is different from the complement of the soft set (F, A), denoted $(F, A)^C = (F^C, \neg A)$, where

$F^C$: ¬A → P(U) is a mapping given by

$F^C(\alpha)$ = U – F(¬$\alpha$), for all $\alpha \in \neg A$.

For illustration, let us consider the example 2.1 of [Maji *et al.*, 2003]:

Let U = {$h_1$, $h_2$, $h_3$, $h_4$, $h_5$, $h_6$} be the set of all houses for consideration, and E = {$e_1$, $e_2$, $e_3$, $e_4$, $e_5$} be the set of all possible parameters with respect to U.

Let A = {$e_1$}, where $e_1$ denotes the predicate name *expensive houses* which are $h_2$ and $h_4$. Then, F($e_1$) = {houses (expensive)} = {$h_2$, $h_4$}, the *functional – value* corresponding to the predicate name *expensive houses*, and the soft set (F, A) = {expensive houses = {$h_2$, $h_4$}}.Also, ¬A = { ¬$e_1$} ⊂ ¬E, and $F^C$( ¬$e_1$) = U – F ( ¬( ¬$e_1$)) = U – F($e_1$)= U – {$h_2$, $h_4$}= {$h_1$, $h_3$, $h_5$, $h_6$} ⊂ U. Clearly, F( ¬$e_1$) is not defined, rather it is $F^C$ ( ¬$e_1$) which is defined. Note that ¬- operator is defined on a set of parameters but not on soft sets. In fact, it is the *complementation* which is defined on soft sets. Further, in view of the intended interpretation of *Not set* discussed above, it follows that the *Assumption*: *Not – set of each set of parameters is a subset of E* in [Ge and Yang, 2011] is untenable. In the example taken above, where ¬A={¬$e_1$},

¬E = { ¬$e_1$, ¬$e_2$, ¬$e_3$, ¬$e_4$, ¬$e_5$}, ¬A ⊂ ¬E and so ¬A ⊄ E.

**Remark 3.1.2**: It is important to note that ¬A is different from $A^C$ though related

**Definition 3.1.5**

Let *U be* a universe, *E* a set of parameters and A ⊆ *E*.

(i) *(F, A)* is called a **relative null soft set** with respect to *A,* denoted $\tilde{\Phi}_A$, if *F(e)* = ∅ , ∀ *e* ∈ *A*.

(ii) *(F , A)* is called a **relative whole soft set** or *A*-universal with respect to *A,* denoted $\tilde{U}_A$, if *F(e) = U* , ∀ *e* ∈ *A*.

(iii) The relative whole soft set with respect to *E*, denoted $\tilde{U}_E$, is called the **absolute soft set** over *U*.

(iv) The relative null soft set with respect to E, denoted $\tilde{\Phi}_E$,is called the **null soft set** over $U$.

(v) The unique soft set (F.A) is called the **empty soft set** over $U$, denoted $\tilde{\Phi}_{\emptyset}$, with empty parameter set.

(vi) The **suppor**t of (F,A) written supp(F,A) is the set defined as:

$supp(F, A) = \{x \in A : F(x) \neq \emptyset\}$.

(vii) (F,A) is called a **non-null** soft set if supp(F,A)$\neq\emptyset$.

**Remark 3.1.3:** $\tilde{\Phi}_{\emptyset} \tilde{\subseteq} \tilde{\Phi}_A \tilde{\subseteq} \tilde{\Phi}_E$ *and (F, A)* $\tilde{\subseteq} \tilde{U}_A \tilde{\subseteq} \tilde{U}_E$.

**Example 3.1.3**

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$be a universe and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters. If $A = \{e_2, e_3, e_4\}$ such that $(F, A) = \{(e_2, \{u_2, u_4\}), (e_3, \{\ \}), (e_4, U)\}$ and $B = \{e_1, e_3\}$ such that $(G, B) = \{(e_1, \emptyset), (e_3, \emptyset)\}$, then

(i) The soft set (*G,B*) is a relative null soft set, i.e., $(G, B) = \tilde{\Phi}_B$.

(ii) If $C = \{e_1, e_2\}$ such that $(H, C) = \{(e_1, U), (e_2, U)\}$, then (*H,C*) is a relative whole soft set, i.e., $(H, C) = \tilde{U}_C$.

(iii) If $D = E$ such that *F(* $e_i$*)* $= U$ , $\forall$ *ei* $\in$ E, *i = 1,2,3,4,* then, the soft set *(F,D)* $= \tilde{U}_E$ is an absolute soft set.

(iv) The support of (F, A) = $\{e_2, e_4\}$

(v) (F, A) is non-null since its support is non-empty.

**Proposition 3.1.3** Let $U$ be a universe, $E$ a set of parameters, $A$ , $B$ , $C \subset E$. If *(F,A)* , *(G,B)* and *(H,C)* are soft sets over $U$ , Then

(i) *(F,A)* $\tilde{\subset} \tilde{U}_A$.

(ii) $\tilde{\Phi}_A \tilde{\subset}$ *(F,A)*

(ii) *(F, A)* $\tilde{\subset}$ *(F,A)*

(iv) *(F, A)* $\tilde{\subset}$ *(G,B)* and *(G,B)* $\tilde{\subset}$ *(H,C)* $\Longrightarrow$ *(F,A)* $\tilde{\subset}$ *(H,C)*

(v) *(F,A)* = *(G,B)* and *(G,B)* = *(H,C)* $\Longleftrightarrow$ *(F,A)* = *(H,C)*

**Proof**:[see Maji *et al.*, 2003 and Ali *et al.*, 2009 ]

**Definition 3.1.6** The **complement** of a soft set (*F,A*), denoted , $(F, A)^C$ is defined as $(F, A)^C = \left(F^C : \neg A\right)$ $F^C : \neg A \to P(U)$ is a mapping given by $F^C(\alpha) = U - F(\neg\alpha)\,, \forall\, \alpha \in \neg A.$

Later Ali *et al.*, (2009) introduced a new notion of complement called *relative complement*, which is defined in the next definition.

**Definition 3.1.7** The **relative complement** of a soft set (*F,A*), denoted $(F, A)^r$, is defined by $(F, A)^r = (F^r, A)$ where $F^r : A \to P(U)$ is a mapping given by $F^r(\alpha) = U - F(\alpha),\ \forall\ \alpha \in A$. In view of the above discussion, we present the following example:

**Example 3.1.4** Consider the soft set described in Example 3.1.3

(i) $(F, A)^C = \{(\neg e_2, \{u_1, u_3, u_5\})\,, (\neg e_3, U)\,, \}\,.$

(ii) $(F, A)^r = \{(e_2, \{u_1, u_3, u_5\})\,, (e_3, U)\,, \}\,.$

**Proposition 3.1.4**

Let *(F,A)* be a soft set over a universe *U*. Then

(i) $((F,A)^c)^c$ = *(F,A)*

(ii) $((F,A)^r)^r$ = *(F,A)*

(iii) $\tilde{U}_A^C = \tilde{\Phi}_A = \tilde{U}_A^r$

(iv) $\tilde{\Phi}_A^C = \tilde{U}_{\ A} = \tilde{\Phi}_A^r$

**Proof**: [see Ali *et al.*, 2009 ]

## 3.2 Soft Set Operations

In this section, the operations of soft sets and their basic properties are discussed and illustrated.

**Definition 3.2.1**

Let $(F,A)$ and $(G,B)$ be two soft sets over a common universe $U$. Then:

(i) The **union** of $(F,A)$ and $(G,B)$, denoted $(F,A)\tilde{\cup}(G,B)$, is a soft set $(H,C)$, where $C = A \cup B$ and $\forall\ e \in C$,

$$H(e) = \begin{cases} F(e), & e \in A - B \\ G(e), & e \in B - A \\ F(e) \cup G(e), & e \in A \cap B. \end{cases}$$

(ii) The **intersection** of $(F,A)$ and $(G,B)$, denoted $(F,A)\tilde{\cap}(G,B)$, is a soft set (H,C), where $C = A \cap B$ and, $\forall$e$\in$ C, $H(e) = F(e)$ or $G(e)$ (as both are same set).

(iii) The **AND-operation** of $(F,A)$ and $(G,B)$, denoted $(F,A)$ AND $(G,B)$ or $(F,A) \wedge (G,B)$, is a soft set defined by $(F,A) \wedge (G,B) = (H, A \times B)$, where $H(a,b) = F(a) \cap G(b),\ \forall\ (a,b) \in A \times B$.

(iv) The **OR-operation** of $(F,A)$ and $(G,B)$, denoted $(F,A)$ OR $(G,B)$, or $(F,A) \vee (G,B)$, is a soft set defined by $(F,A) \vee (G,B) = (H, A \times B)$,where $H(a,b) = F(a) \cup G(b),\ \forall\ (a,b) \in A \times B$.

Pei and Miao (2005) pointed out that in Definition 3.2.1 (ii), $F(e)$ and $G(e)$ may not be the same set and thus revised the definition as follows:

**Definition 3.2.2: Intersection redefined (Pei and Miao, 2005)**

Let $(F,A)$ and $(G,B)$ be two soft sets over $U$. The intersection (also called bi-intersection in Feng *et al.* (2008)) of $(F,A)$ and $(G,B)$, denoted $(F,A)\tilde{\cap}(G,B)$, is a soft set $(H,C)$ where $C = A \cap B$ and, $\forall\ e \in C,\ H(e) = F(e) \cap G(e)$.

Ahmad and Kharal (2009) pointed out that in the above Definition 3.2.2,$A \cap B$must be non-empty in order to avoid the degenerate cases. The modified definition is as follows:

**Definition 3.2.3: Intersection redefined (Ahmad and Kharal, 2009)**

Let $(F,A)$ and $(G,B)$ be two soft sets over $U$ with $A \cap B \neq \emptyset$. The intersection of $(F,A)$ and $(G,B)$, denoted $(F,A)\tilde{\cap}(G,B)$, is a soft set $(H,C)$ where $C = A \cap B$and $\forall\ e \in C,\ H(e) = F(e) \cap G(e)$

In the sequel,Ali *et al.* (2009) introduced the following new operations:

**Definition 3.2.4**

Let $(F,A)$ and $(G,B)$ be two soft sets over $U$.

(i) The **extended intersection** of $(F,A)$ and $(G,B)$, denoted $(F,A) \cap_{\varepsilon} (G,B)$, is a soft set $(H,C)$, where $C = A \cup B$ and, $\forall\ e \in C$,

$$H(e) = \begin{cases} F(e), \text{ if } e \in A - B \\ G(e), \text{ if } e \in B - A \\ F(e) \cap G(e), \text{ if } e \in A \cap B. \end{cases}$$

(ii) The **restricted intersection** (also called intersection in Pei and Miao(2005) and bi-intersection in Feng *et al.*,(2008)) of $(F,A)$ and $(G,B)$, denoted $(F,A)\circledcirc(G,B)$, is a soft set $(H,C)$, where $C = A \cap B$ and, $\forall\ e \in C,\ H(e) = F(e) \cap G(e)$. If

$A \cap B = \emptyset$ then $(F,A) \Cap (G,B) = \tilde{\Phi}_{\emptyset}$.

(iii) The **restricted union** of $(F,A)$ and $(G,B)$, denoted $(F,A) \cup_R (G,B)$, is a soft set $(H,C)$, where $C = A \cap B$ and $\forall\ e \in C$, $H(e) = F(e) \cup G(e)$. If $A \cap B = \emptyset$ then, $(F,A) \cup_R (G,B) = \tilde{\Phi}_{\emptyset}$ .

(iv) The **restricted difference** of $(F,A)$ and $(G,B)$ denoted $(F,A) \smile_R (G,B)$ is a soft set $(H,C)$ where $C = A \cap B$ and, $\forall\ e \in C,\ H(e) = F(e) - G(e)$.

Sezgin and Atagun (2011), defined the following operation;

**Definition 3.2.5 (Restricted symmetric difference) Sezgin and Atagun , (2011)**

The restricted symmetric difference of $(F,A)$ and $(G,B)$ denoted $(F,A)\tilde{\Delta}(G,B)$ is a soft set defined by $(F,A)\tilde{\Delta}(G,B) = ((F,A) \cup_R (G,B)) \smile_R ((F,A) \Cap (G,B))$ or by $(F,A)\tilde{\Delta}(G,B) = ((F,A) \smile_R (G,B)) \cup_R ((G,B) \smile_R (F,A))$.

This can also be defined as follows:

**Definition 3.2.6**

The restricted symmetric difference of $(F,A)$ and $(G,B)$, denoted $(F,A)\tilde{\Delta}(G,B)$, is a soft set $(H,C)$ where $C = A \cap B$ and , $\forall\ e \in C,\ H(e) = F(e) \Delta G(e)$ (the symmetric difference of $F(e)$ and $G(e)$).

**Example 3.2.1**

Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6\}$ and, $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the parameter set with respect to $U$. Let $A = \{e_1, e_2, e_3\} \subset E$, and the soft set $(F,A)$ over $U$ be given by $(F,A) = \{(e_1, \{h_2, h_4\}), (e_2, \{h_1, h_3, h_5\}), (e_3, \{h_3, h_4, h_5\})\}$.

Let $B = \{e_3, e_4, e_5\}$ and $(G,B)$ over U be

$(G,B) = \{(e_3, \{h_1, h_2, h_3\}), (e_4, \{h_2, h_3, h_6\}), (e_5, \{h_2, h_3, h_4\})\}$. Then

(i) $(F,A)\tilde{\cup}(G,B) = \{(e_1,\{h_2,h_4\}),(e_2,\{h_1,h_3,h_5\}),(e_3,\{h_1,h_2,h_3,h_4,h_5\})$

$(e_4,\{h_2,h_3,h_6\}),(e_5,\{h_2,h_3,h_4\})\},$

(ii) $(F,A)\cup_R(G,B) = \{(e_3,\{h_1,h_2,h_3,h_4,h_5\})\},$

(iii) $(F,A)\Cap(G,B) = \{(e_3,\{h_3\})\},$

(iv) $(F,A)\cap_\varepsilon(G,B) = \{(e_1,\{h_2,h_4\}),(e_2,\{h_1,h_3,h_5\}),(e_3,\{h_3\})$

$(e_4,\{h_2,h_3,h_6\}),(e_5,\{h_2,h_3,h_4\})\},$

(v) $(F,A)\smile_R(G,B) = \{(e_3,\{h_3,h_5\})\},$

(vi) $(F,A)\tilde{\Delta}(G,B) = \{(e_3,\{h_1,h_2,h_4,h_5\})\},$

(vii) $(F,A)\wedge(G,B) = \{((e_1,e_3),\{h_2\}),((e_1,e_4),\{h_2\}),((e_1,e_5),\{h_4\})$

$((e_2,e_3),\{h_1,h_3\}),((e_2,e_4),\{h_3\}),((e_2,e_5),\{h_5\})$

$((e_3,e_3),\{h_3\}),((e_3,e_4),\{h_3\}),((e_3,e_5),\{h_3,h_4\})\},$ *and*

(viii) $(F,A)\vee(G,B) = \{((e_1,e_3),\{h_1,h_2,h_3,h_4\}),$

$((e_1,e_4),\{h_2,h_3,h_4,h_5\}),$

$((e_1,e_5),\{h_2,h_3,h_4\}),$

$((e_2,e_3),\{h_1,h_2,h_3,h_5\}),$

$((e_2,e_4),\{h_1,h_2,h_3,h_5,h_6\}),$

$((e_2,e_5),\{h_1,h_2,h_3,h_4,h_5\}),$

$((e_3,e_3),\{h_1,h_2,h_3,h_4,h_5\}),$

$((e_3,e_4),\{h_2,h_3,h_4,h_5,h_6\}),$

$((e_3,e_5),\{h_2,h_3,h_4,h_5\})\}.$

### 3.2.1 Properties of soft set operations

Most properties in the proposition below are drawn from various sources e.g., [ Maji *et al.,* 2003, Ali *et al.*, 2009, Sezgin and Atagun, 2011, Singh and Onyeozili,2012A 2012B 2012C, and many others not mentioned here ]

**Proposition 3.2.1**

Let $(F,A)$, $(G,B)$ and $(H,C)$ be soft sets over a common universe. Then the following properties hold:

1. **Idempotent properties**

(i) $(F,A)\tilde{\cup}(F,A) = (F,A) = (F,A) \cup_R (F,A)$

(ii) $(F,A)\tilde{\cap}(F,A) = (F,A) = (F,A) \cap_\varepsilon (F,A)$

2. **Identity Properties**

(i) $(F,A)\tilde{\cup}\tilde{\Phi} = (F,A) = (F,A) \cup_R \tilde{\Phi}$

(ii) $(F,A)\tilde{\cap}\tilde{U} = (F,A) = (F,A) \cap_\varepsilon \tilde{U}$

(iii) $(F,A) \smile_R \tilde{\Phi} = (F,A) = (F,A)\tilde{\Delta}\tilde{\Phi}$

(iv) $(F,A) \smile_R (F,A) = \tilde{\Phi} = (F,A)\tilde{\Delta}(F,A)$

3. **Domination Properties**

(i) $(F,A)\tilde{\cup}\tilde{U} = \tilde{U} = (F,A) \cup_R \tilde{U}$

(ii) $(F,A)\tilde{\cap}\tilde{\Phi} = \tilde{\Phi} = (F,A) \cap_\varepsilon \tilde{\Phi}$

4. **Complementation Properties**

(i) $\tilde{\Phi}^C = \tilde{U} = \tilde{\Phi}^r$

(ii) $\tilde{U}^C = \tilde{\Phi} = \tilde{U}^r$

5. **Double Complementation (Involution) Property**

$\left((F,A)^C\right)^C = (F,A) = ((F,A)^r)^r$

6. **Exclusion Properties**

$(F,A)\tilde{\cup}(F,A)^r = \tilde{U} = (F,A) \cup_R (F,A)^r$

7. **Contradiction Properties**

$(F,A)\tilde{\cap}(F,A)^r = \tilde{\Phi} = (F,A) \cap_\varepsilon (F,A)^r$

**Remark 3.2.1**

Exclusion and contradiction properties do not hold with respect to complementation introduced in Definition 3.1.6 [For details see Tridiv and Dusmanta, 2011A].

8. **De Morgan's Properties**

(i) $((F,A)\tilde{\cup}(G,B))^C = (F,A)^C \cap_\varepsilon (G,B)^C$

(ii) $((F,A) \cap_\varepsilon (G,B))^C = (F,A)^C\tilde{\cup}(G,B)^C$

(iii) $((F,A) \cup_R (G,B))^r = (F,A)^r\tilde{\cap}(G,B)^r$

(iv) $((F,A)\tilde{\cap}(G,B))^r = (F,A)^r \cup_R (G,B)^r$

(v) $((F,A) \wedge (G,B))^C = (F,A)^C \vee (G,B)^C$

(vi) $((F,A) \vee (G,B))^C = (F,A)^C \wedge (G,B)^C$

(vii) $((F,A) \wedge (G,B))^r = (F,A)^r \vee (G,B)^r$

(viii) $((F,A) \vee (G,B))^r = (F,A)^r \wedge (G,B)^r$

(ix) $((F,A)\tilde{\cup}(G,B))^r = (F,A)^r \cap_\varepsilon (G,B)^r$

(x) $((F,A) \cap_\varepsilon (G,B))^r = (F,A)^r \tilde{\cup} (G,B)^r$

**Remark 3.2.2**

De Morgan's Properties do not hold for restricted union and restricted intersection with respect to complement in Definition 3.1.6[see Ali *et al.*, 2009].

i.e. (i) $((F,A) \cup_R (G,B))^C \neq (F,A)^C \tilde{\cap} (G,B)^C$

(ii) $((F,A)\tilde{\cap}(G,B))^C \neq (F,A)^C \cup_R (G,B)^C$

9. **Absorption Properties(Singh and Onyeozili,2012A)**

(i) $(F,A)\tilde{\cup}((F,A)\tilde{\cap}(G,B)) = (F,A)$

(ii) $(F,A)\tilde{\cap}((F,A)\tilde{\cup}(G,B)) = (F,A)$

(iii) $(F,A) \cup_R ((F,A) \cap_\varepsilon (G,B)) = (F,A)$

(iv) $(F,A) \cap_\varepsilon ((F,A) \cup_R (G,B)) = (F,A)$

***Proof.***

(i) Let $(F,A) \cup_R ((F,A) \cap_\varepsilon (G,B))$

$$= (F,A) \cup_R (K, A \cup B)$$

$$= (L, A \cap (A \cup B))$$

$$= (L,A)$$

where $\forall\ \alpha \in A \cup B$,

$$K(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - B; \\ G(\alpha),\ \alpha \in B - A; \text{ and} \\ F(\alpha) \cap G(\alpha),\ \alpha \in A \cap B. \end{cases}$$

and $\forall\ \alpha \in A$,

$$L(\alpha) = F(\alpha) \cup K(\alpha).$$

$$L(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - B; \\ F(\alpha) \cup G(\alpha),\ \alpha \in A \cap (B - A) = \emptyset; \text{ and} \\ F(\alpha) \cup \{F(\alpha) \cap G(\alpha)\},\ \alpha \in A \cap B. \end{cases}$$

$$= \begin{cases} F(\alpha),\ \alpha \in A - B,\ \text{and} \\ F(\alpha),\ \alpha \in A \cap B. \end{cases}$$

Hence, $L(\alpha) = F(\alpha),\ \forall\ \alpha \in A$;

i.e., $(L, A) = (F, A)$ which proves (i).

Similarly (ii) can be proved.

(iii) Let $(F, A) \cap_\varepsilon ((F, A) \cup_R (G, B))$

$$= (F, A) \cap_\varepsilon (K, A \cap B)$$

$$= (J, A \cup (A \cap B)) = (J, A)$$

where $\forall\ \alpha \in A \cap B$,

$$K(\alpha) = F(\alpha) \cup G(\alpha)$$

and $\forall\ \alpha \in A$,

$$J(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - (A \cap B); \\ K(\alpha),\ \alpha \in (A \cap B) - A = \emptyset;\ \text{and} \\ F(\alpha) \cap K(\alpha) = F(\alpha) \cap \{F(\alpha) \cup G(\alpha)\},\ \alpha \in A \cap (A \cap B). \end{cases}$$

$$= \begin{cases} F(\alpha),\ \alpha \in A - B;\ \text{and} \\ F(\alpha),\ \alpha \in A \cap B. \end{cases}$$

Implying, $J(\alpha) = F(\alpha)\ \forall\ \alpha \in A$.

Hence $(J,A) = (F,A)$ proving (iii).

Similarly (iv) can be proved.

Let us illustrate property 9(i) in the following example.

**Example 3.2.2**

Let $U = \{u_1, u_2, u_3, u_4, u_5, u_6\}$ be a universe set consisting of six houses under consideration and let $E = \{e_1, e_2, e_3, e_4, u_5\}$ be the parameter set with respect to $U$, where $e_i,\ i = 1, 2, 3, 4, 5$ represent *expensive*, *beautiful modern*, '*in a good location*', *spacious*, respectively.

Let $A$ and $B$ be subsets of $E$ given by $A = \{e_2, e_3, e_5\}$and $B = \{e_1, e_2, e_3\}$.

Let $(F,A)$ and $(G,B)$ be two soft sets over the universe $U$ such that:

$(F, A) = \{(e_2, \{u_5\}),\ (e_3, \{u_4\}),\ (e_5, \{u_3, u_5, u_6\})\}$; and

$(G, B) = \{(e_1, \{u_1, u_2, u_6\}), (e_2, \{u_3, u_4, u_5\}),\ (e_3, \{u_2, u_3, u_4\})\}$;

Let $(F, A) \cap_\varepsilon (G, B) = (K, A \cup B)$, where $\forall\ \alpha \in A \cup B = \{e_1, e_2, e_3, e_5\}$,

$$K(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - B = \{e_5\}; \\ G(\alpha),\ \alpha \in B - A = \{e_1\};\ \text{and} \\ F(\alpha) \cap G(\alpha),\ \alpha \in A \cap B = \{e_2, e_3\}. \end{cases}$$

Thus $(K, A \cup B) = \{(e_1 \{u_1, u_2, u_6\}), (e_2, \{u_5\}), (e_3 \{u_4\}), (e_5, \{u_3, u_5, u_6\})\}$.

Let $(F, A) \cup_R (K, A \cup B) = (L, A \cap (A \cup B)) = (L, A)$ where $\forall\ \alpha \in A = \{e_2, e_3, e_5\}$,

$$\begin{aligned} L(\alpha) &= F(\alpha) \cup K(\alpha); \\ &= \begin{cases} F(\alpha),\ \alpha \in A - B = \{e_5\}; \\ F(\alpha) \cup G(\alpha),\ \alpha \in \emptyset;\ \text{and} \\ F(\alpha) \cup \{F(\alpha) \cap G(\alpha)\},\ \alpha \in A \cap B = \{e_2, e_3\}. \end{cases} \\ &= \begin{cases} F(\alpha),\ \alpha \in \{e_5\};\ \text{and} \\ F(\alpha),\ \alpha \in \{e_2, e_3\}. \end{cases} \end{aligned}$$

Hence,

$$\begin{aligned} (L, A) &= \{(e_2, \{u_5\}), (e_3, \{u_4\}), (e_5, \{u_3, u_5, u_6\})\} \\ &= (F, A), \text{ establishing (i).} \end{aligned}$$

**Remark 3.2.3 (See Proposition 3.2.3)**

(i) $\tilde{\cup}$ and $\cap_\varepsilon$ do not absorb over each other.

(ii) $\cup_R$ and $\tilde{\cap}$ do not absorb over each other.

10. **Commutative Properties**

(i) $(F, A)\tilde{\cup}(G, B) = (G, B)\tilde{\cup}(F, A)$

(ii) $(F, A) \cup_R (G, B) = (G, B) \cup_R (F, A)$

(iii) $(F, A)\tilde{\cap}(G, B) = (G, B)\tilde{\cap}(F, A)$

(iv) $(F, A) \cap_\varepsilon (G, B) = (G, B) \cap_\varepsilon (F, A)$

(v) $(F, A)\tilde{\Delta}(G, B) = (G, B)\tilde{\Delta}(F, A)$(**Singh and Onyeozili, 2012B**)

**Remark 3.2.4**

$\wedge$ and $\vee$do not commute[see Maji *et al.*, 2003].

11. **Associative Properties**

(i) $(F,A)\tilde{\cup}\left((G,B)\tilde{\cup}(H,C)\right) = \left((F,A)\tilde{\cup}(G,B)\right)\tilde{\cup}(H,C)$

(ii) $(F,A)\cup_R\left((G,B)\cup_R(H,C)\right) = \left((F,A)\cup_R(G,B)\right)\cup_R(H,C)$

(iii) $(F,A)\tilde{\cap}\left((G,B)\tilde{\cap}(H,C)\right) = \left((F,A)\tilde{\cap}(G,B)\right)\tilde{\cap}(H,C)$

(iv) $(F,A)\cap_\varepsilon\left((G,B)\cap_\varepsilon(H,C)\right) = \left((F,A)\cap_\varepsilon(G,B)\right)\cap_\varepsilon(H,C)$

(v) $(F,A)\wedge\left((G,B)\wedge(H,C)\right) = \left((F,A)\wedge(G,B)\right)\wedge(H,C)$

(vi) $(F,A)\vee\left((G,B)\vee(H,C)\right) = \left((F,A)\vee(G,B)\right)\vee(H,C)$

12. **Distributive Properties**

(i) $(F,A)\tilde{\cup}\left((G,B)\tilde{\cap}(H,C)\right) = \left((F,A)\tilde{\cup}(G,B)\right)\tilde{\cap}\left((F,A)\tilde{\cup}(H,C)\right)$

(ii) $(F,A)\tilde{\cap}\left((G,B)\tilde{\cup}(H,C)\right) = \left((F,A)\tilde{\cap}(G,B)\right)\tilde{\cup}\left((F,A)\tilde{\cap}(H,C)\right)$

(iii) $(F,A)\cup_R\left((G,B)\cap_\varepsilon(H,C)\right) = \left((F,A)\cup_R(G,B)\right)\cap_\varepsilon\left((F,A)\cup_R(H,C)\right)$

(iv) $(F,A)\cap_\varepsilon\left((G,B)\cup_R(H,C)\right) = \left((F,A)\cap_\varepsilon(G,B)\right)\cup_R\left((F,A)\cap_\varepsilon(H,C)\right)$

(v) $(F,A)\cup_R\left((G,B)\tilde{\cap}(H,C)\right) = \left((F,A)\cup_R(G,B)\right)\tilde{\cap}\left((F,A)\cup_R(H,C)\right)$

(vi) $(F,A)\tilde{\cap}\left((G,B)\cup_R(H,C)\right) = \left((F,A)\tilde{\cap}(G,B)\right)\cup_R\left((F,A)\tilde{\cap}(H,C)\right)$

(vii) $(F,A)\smile_R\left((G,B)\tilde{\cup}(H,C)\right) = \left((F,A)\smile_R(G,B)\right)\tilde{\cup}\left((F,A)\smile_R(H,C)\right)$

(viii) $(F,A)\smile_R\left((G,B)\cap_\varepsilon(H,C)\right) = \left((F,A)\smile_R(G,B)\right)\cap_\varepsilon\left((F,A)\smile_R(H,C)\right)$

(ix) $(F,A)\smile_R\left((G,B)\cup_R(H,C)\right) = \left((F,A)\smile_R(G,B)\right)\cup_R\left((F,A)\smile_R(H,C)\right)$

(x) $(F,A) \smile_R ((G,B)\tilde{\cap}(H,C)) = ((F,A) \smile_R (G,B)) \tilde{\cap} ((F,A) \smile_R (H,C))$

(xi) $(F,A) \tilde{\cap} ((G,B) \smile_R (H,C)) = ((F,A) \tilde{\cap}(G,B)) \smile_R ((F,A)\tilde{\cap}(H,C))$

(xii) $(F,A)\tilde{\cap}\Big((G,B)\tilde{\Delta}(H,C)\Big) = ((F,A)\tilde{\cap}(G,B)) \,\tilde{\Delta}\, ((F,A)\tilde{\cap}(H,C))$

**Remark 3.2.5 (See Proposition 3.2.2)**

(i) $\tilde{\cup}$ and $\cap_\varepsilon$do not distribute over each other

(ii) $\vee$ and $\wedge$ do not distribute over each other

(iii) $\tilde{\cup}, \cup_R$ and $\cap_\varepsilon$ do not distribute over $\smile_R$

(iv) $\tilde{\cup}, \cup_R$, $\cap_\varepsilon, \tilde{\cap}$ and $\smile_R$ do not distribute over $\wedge$ and $\vee$

(v) $\cup_R$ distributes over $\tilde{\cup}$ but the converse is false

(vi) $\tilde{\cap}$ distributes over $\cap_\varepsilon$ but the converse is false

(vii) $\wedge$ distributes over $\smile_R$ but $\vee$ does not.

**Proposition 3.2.2 (Singh and Onyeozili, 2012A, 2012B, 2012C)**

Let *(F,A) , (G,B) and (H,C)* be soft sets over a common universe. Then,

(i) $(F,A)\tilde{\cup}((G,B) \cap_\varepsilon (H,C)) \neq ((F,A)\tilde{\cup}(G,B)) \cap_\varepsilon ((F,A)\tilde{\cup}(H,C))$

(ii) $(F,A) \cap_\varepsilon ((G,B)\tilde{\cup}(H,C)) \neq ((F,A) \cap_\varepsilon (G,B)) \,\tilde{\cup}\, ((F,A) \cap_\varepsilon (H,C))$

(iii) $(F,A) \cup_R ((G,B) \smile_R (H,C)) \neq ((F,A) \cup_R (G,B)) \smile_R ((F,A) \cup_R (H,C))$

(iv) $(F,A) \cap_\varepsilon ((G,B) \smile_R (H,C)) \neq ((F,A) \cap_\varepsilon (G,B)) \smile_R ((F,A) \cap_\varepsilon (H,C))$

(v) $(F,A)\tilde{\cup}((G,B) \smile_R (H,C)) \neq ((F,A)\tilde{\cup}(G,B)) \smile_R ((F,A)\tilde{\cup}(H,C))$

(vi) $(F,A) \cup_R ((G,B)\tilde{\cup}(H,C)) = ((F,A) \cup_R (G,B)) \,\tilde{\cup}\, ((F,A) \cup_R (H,C))$

(vii) $(F,A)\tilde{\cap}\left((G,B)\cap_{\varepsilon}(H,C)\right)=\left((F,A)\tilde{\cap}(G,B)\right)\cap_{\varepsilon}\left((F,A)\tilde{\cap}(H,C)\right)$

(viii) $(F,A)\tilde{\cup}\left((G,B)\cup_{R}(H,C)\right)\neq\left((F,A)\tilde{\cup}(G,B)\right)\cup_{R}\left((F,A)\tilde{\cup}(H,C)\right)$

(ix) $(F,A)\cap_{\varepsilon}\left((G,B)\tilde{\cap}(H,C)\right)\neq\left((F,A)\cap_{\varepsilon}(G,B)\right)\tilde{\cap}\left((F,A)\cap_{\varepsilon}(H,C)\right)$

(x) $(F,A)\wedge\left((G,B)\smile_{R}(H,C)\right)=\left((F,A)\wedge(G,B)\right)\smile_{R}\left((F,A)\wedge(H,C)\right)$

(xi) $(F,A)\vee\left((G,B)\smile_{R}(H,C)\right)\neq\left((F,A)\vee(G,B)\right)\smile_{R}\left((F,A)\vee(H,C)\right)$

**Proof:**

(i) Let $(G,\,B)\cap_{\in}(H,\,C)=(L,\,D)$, where D = B∪C and, ∀d∈D,

$$L(d)=\begin{cases} G(d)\ \ ,\ \ if\quad d\in B-C; \\ H(d)\ \ ,\ \ \ \ if\ \ d\in C-B; \\ G(d)\cap H(d),\ if\ d\in B\cap C. \end{cases}$$

$$Let\,(F,\,A)\,\widetilde{\cup}\,(L,\,D)=(M,\,K)=(M,\ A\cup(B\cup C))\,where,\,\forall k\in K,$$

$$M(k)=\begin{cases} F(k),\ if\,k\in A-\,(B\cup C) \\ \qquad =\ (A-B)\cap(A-C); \\ L(k),\ if\,k\in\,(B\cup C)-A \\ \qquad =(B-A)\cup(C-A); \\ F(k)\cup L(k),\ if\,k\in A\cap(B\cup C) \\ \qquad =(A\cap B)\cup(A\cap C). \end{cases}$$

Let $(F,\,A)\,\widetilde{\cup}\,(G,\,B)=(H_1,\,A\cup B)$

and $(F,\,A)\,\widetilde{\cup}\,(H,\,C)=(H_2,\,A\cup C)$.

Then, $\big((F,\,A)\,\widetilde{\cup}\,(G,\,B)\big)\cap_{\in}\big((\,F,\,A)\,\widetilde{\cup}\,(H,\,C)\big)$

$$=(H_1,\,A\cup B)\cap_{\in}(H_2,\,A\cup C).$$

$$Let\,(H_1,\,A\cup B)\cap_{\in}\,(H_2,\,A\cup C)$$

$$=(H_3,\,(A\cup B)\cup(A\cup C))$$

$$=(H_3\,,A\cup B\cup C)\,.$$

Then, we have, $\forall$ k $\in$ A$\cup$B,

$$H_1\,(k)=\begin{cases} F\,(k)\,,\;\; if\,k\in\,A-B;\\ G\,(k)\,,\;\; if\,k\in\,B-A;\\ F\,(k)\,\cup G\,(k)\,,\;\; if\,k\in\,A\cap B. \end{cases}$$

$\forall$k$\in$ A$\cup$C,

$$H_2\,(k)=\begin{cases} F\,(k)\,,\;\; if\,k\in\,A-C;\\ H\,(k)\,,\;\; if\,k\in\,B-A;\\ F\,(k)\,\cup H\,(k)\,,\;\; if\,k\in\,A\cap C. \end{cases}$$

$\forall$k$\in$ A$\cup$B$\cup$C,

$$H_3\,(k)=\begin{cases} H_1\,(k)\,,\;\; if\,k\in(A\cup B)-(A\cup C)\,;\\ H_2\,(k)\,,\;\; if\,k\in\,(A\cup C)-(A\cup B)\,;\\ H_1\,(k)\cap H_2\,(k)\,,\;\; if\,k\in(A\cup B)\cap(A\cup C)\\ \qquad\qquad =A\cup(B\cap C)\,. \end{cases}$$

Hence, M(k) $\neq$ $H_3$ (k), $\forall$ k $\in$ A $\cup$ B $\cup$ C, asserting (i). Similarly, it can be proved that (ii) holds.

(iii) Let $(G,B)\backsim_R(H,C)=(K,B\cap C)$ where,$\forall\;\alpha\in B\cap C$,

$$K(\alpha)=G(\alpha)-H(\alpha).$$

Let $(F,A)\cup_R(K,B\cap C)=(J,A\cap(B\cap C))$ where,$\forall\;\alpha\in A\cap(B\cap C)$,

$$\begin{aligned} J(\alpha) &= F(\alpha)\cup K(\alpha)\\ &= F(\alpha)\cup\{G(\alpha)-H(\alpha)\}\,. \end{aligned}$$

Let $(F, A) \cup_R (G, B) = (L, A \cap B)$ where, $\forall\ \alpha \in A \cap B$,

$$L(\alpha) = F(\alpha) \cup G(\alpha).$$

Let $(F, A) \cup_R (H, C) = (M, A \cap C)$ where, $\forall\ \alpha \in A \cap C$,

$$M(\alpha) = F(\alpha) \cup H(\alpha).$$

Let $(L, A \cap B) \smile_R (M, A \cap C) = (N, (A \cap B) \cap (A \cap C))$

where, $\forall\ \alpha \in (A \cap B) \cap (A \cap C) = A \cap (B \cap C)$,

$$\begin{aligned} N(\alpha) &= L(\alpha) - M(\alpha) \\ &= \{F(\alpha) \cup G(\alpha)\} - \{F(\alpha) \cup H(\alpha)\} \\ &\neq F(\alpha) \cup \{G(\alpha) - H(\alpha)\}. \end{aligned}$$

Hence, $J \neq N$ on $A \cap (B \cap C)$, which completes the proof of (iii)

(iv) $(G, B) \smile_R (H, C) = (K, B \cap C)$ where, $\forall\ \alpha \in B \cap C$,

$$K(\alpha) = G(\alpha) - H(\alpha)$$

Let $(F, A) \cap_\varepsilon (K, B \cap C) = (I, A \cup (B \cap C))$ where, $\forall\ \alpha \in A \cup (B \cap C)$,

$$\begin{aligned} I(\alpha) &= \begin{cases} F(\alpha),\ \alpha \in A - (B \cap C); \\ K(\alpha),\ \alpha \in (B \cap C) - A;\ \text{and} \\ F(\alpha) \cap K(\alpha),\ \alpha \in A \cap (B \cap C) \end{cases} \\ &= \begin{cases} F(\alpha),\ \alpha \in A - (B \cap C); \\ G(\alpha) - H(\alpha),\ \alpha \in (B \cap C) - A;\ \text{and} \\ F(\alpha) \cap \{G(\alpha) - H(\alpha)\},\ \alpha \in A \cap (B \cap C). \end{cases} \end{aligned}$$

Let $(F, A) \cap_\varepsilon (G, B) = (R, A \cup B)$ where, $\forall\ \alpha \in A \cup B$,

$$R(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - B; \\ G(\alpha),\ \alpha \in B - A;\ \text{and} \\ F(\alpha) \cap G(\alpha),\ \alpha \in A \cap B. \end{cases}$$

Let $(F,A)\cap_{\varepsilon}(H,C)=(L,A\cup C)$ where, $\forall\ \alpha\in A\cup C$,

$$L(\alpha)=\begin{cases} F(\alpha),\ \alpha\in A-C; \\ H(\alpha),\ \alpha\in C-A;\ \text{and} \\ F(\alpha)\cap H(\alpha),\ \alpha\in A\cap C. \end{cases}$$

Let $(R,A\cup B)\smile_{R}(L,A\cup C)=(M,(A\cup B)\cap(A\cup C))$ where, $\forall\ \alpha\in(A\cup B)\cap(A\cup C)=A\cup(B\cap C)$, $M(\alpha)=R(\alpha)-L(\alpha)$

$$=\begin{cases} \emptyset,\ \alpha\in A-(B\cup C); \\ G(\alpha)-F(\alpha),\ \alpha\in(A\cap B)-C; \\ G(\alpha)-\{F(\alpha)\cap H(\alpha)\},\ \alpha\in A\cap B\cap C; \\ G\alpha)-H(\alpha),\ \alpha\in(B\cap C)-A;\ \text{and} \\ F(\alpha)-\{F(\alpha)\cap H(\alpha)\},\ \alpha\in(A\cap C)-B. \end{cases}$$

Hence, mappings $I$ and $M$ are not equal on $A\cup(B\cap C)$.This completes the proof of (iv)

Similarly, (v) can be proved.

(vi): Let $(G,B)\tilde{\cup}(H,C)=(K,B\cup C)\text{where},\forall\ \ \alpha\in(B\cup C)$,

$$K(\alpha)=\begin{cases} G(\alpha),\ \text{if}\ \alpha\in B-C; \\ H(\alpha),\ \text{if}\ \alpha\in C-B;\ \ \text{and} \\ G(\alpha)\cup H(\alpha),\ \text{if}\ \alpha\in B\cap C. \end{cases}$$

Let $(F,A)\cup_{R}(K,B\cup C)=(J,A\cap(B\cup C))$ where, $\forall\ \alpha\in A\cap(B\cup C)$ ,

$$J(\alpha)=F(\alpha)\cup K(\alpha).$$

$$=\begin{cases} F(\alpha)\cup G(\alpha),\ \alpha\in A\cap(B-C); \\ F(\alpha)\cup H(\alpha),\ \alpha\in A\cap(C-B);\ \text{and} \\ F(\alpha)\cup G(\alpha)\cup H(\alpha),\ \alpha\in A\cap(B\cap C). \end{cases}$$

Let $(F,A)\cup_{R}(G,B)=(L,A\cap B)\text{where},\ \forall\ \alpha\in A\cap B,\ L(\alpha)=F(\alpha)\cup G(\alpha).$

Let $(F,A) \cup_R (H,C) = (M, A \cap B)$where ,$\forall\ \alpha \in A \cap C$, $M(\alpha) = F(\alpha) \cup H(\alpha)$. Let

$(L, A \cap B)\tilde{\cup}(M, A \cap C) = (I, (A \cap B) \cup (A \cap C))$ where, $\forall\ \alpha \in A \cap (B \cup C)$,

$$I(\alpha) = \begin{cases} L(\alpha),\ \alpha \in (A \cap B) - (A \cap C); \\ M(\alpha),\ \alpha \in (A \cap C) - (A \cap B);\ \text{and} \\ L(\alpha) \cup M(\alpha),\ \alpha \in A \cap (B \cap C). \end{cases}$$

$$= \begin{cases} F(\alpha) \cup G(\alpha),\ \alpha \in A \cap (B - C); \\ F(\alpha) \cup H(\alpha),\ \alpha \in A \cap (C - B);\ \text{and} \\ [F(\alpha) \cup G(\alpha)] \cup [F(\alpha) \cup H(\alpha)] = F(\alpha) \cup \{G(\alpha) \cup H(\alpha)\},\ \alpha \in A \cap (B \cap C). \end{cases}$$

Hence, $J(\alpha) = I(\alpha)\ \forall\ \alpha \in A \cap (B \cup C)$; i.e., (vi) holds.

Similarly, (vii) can be proved.

( viii) Let $(G,B) \cup_R (H,C) = (K, B \cap C)$ where, $\forall\ \alpha \in B \cap C$,

$$K(\alpha) = G(\alpha) \cup H(\alpha).$$

Let $(F,A)\tilde{\cup}(K, B \cap C) = (I, A \cup (B \cap C))$ where, $\forall\ \alpha \in A \cup (B \cap C)$,

$$I(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - (B \cap B); \\ K(\alpha),\ \alpha \in (B \cap C) - A;\ \text{and} \\ F(\alpha) \cup K(\alpha),\ \alpha \in A \cap (B \cap C). \end{cases}$$

$$= \begin{cases} F(\alpha),\ \alpha \in A - (B \cap C); \\ G(\alpha) \cup H(\alpha),\ \alpha \in (B \cap C) - A;\ \text{and} \\ F(\alpha) \cup \{G(\alpha) \cup H(\alpha)\},\ \alpha \in A \cap (B \cap C). \end{cases}$$

Also, let $(F,A)\tilde{\cup}(G,B) = (L, A \cup B)$ where, $\forall\ \alpha \in A \cup B$,

$$L(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - B; \\ G(\alpha),\ \alpha \in B - A;\ \text{and} \\ F(\alpha) \cup G(\alpha),\ \alpha \in A \cap B. \end{cases}$$

Let $(F, A)\tilde{\cup}(H, C) = (M, A \cup C)$ where, $\forall\ \alpha \in A \cup C$,

$$M(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - C; \\ H(\alpha),\ \alpha \in C - A;\ \text{and} \\ F(\alpha) \cup H(\alpha),\ \alpha \in A \cap C. \end{cases}$$

Let $(L, A \cup B) \cup_R (M, A \cup C) = (N, (A \cup B) \cap (A \cup C))$ where, $\forall\ \alpha \in A \cup (B \cap C)$,

$$N(\alpha) = L(\alpha) \cup M(\alpha). = \begin{cases} F(\alpha),\ \alpha\imath(A - B) \cap (A - C) = A - (B \cup C); \\ F(\alpha) \cup G(\alpha),\ \alpha \in (A \cap B) - C; \\ F(\alpha) \cup H(\alpha),\ \alpha \in (A \cap C) - B; \\ G(\alpha) \cup H(\alpha),\ \alpha \in (B \cap C) - A;\ \text{and} \\ F(\alpha) \cup \{G(\alpha) \cup H(\alpha)\},\ \alpha \in A \cap B \cap C. \end{cases}$$

Thus, $I \neq N$. Hence (viii) holds. Similarly, (ix) can be proved.

(x) Let $(F, A) \wedge ((G, B) \smile_R (H, C))$

$$= (F, A) \wedge (K,\ B \cap C)$$

$$= (L,\ A \times (B \cap C))$$

where, $\forall\ \alpha \in B \cap C$, $K(\alpha) = G(\alpha) - H(\alpha)$;

and, $\forall\ (\alpha, \beta) \in A \times (B \cap C)$, $L(\alpha, \beta) = F(\alpha) \cap K(\beta)$

$$= F(\alpha) \cap \{G(\beta) - H(\beta)\}.$$

Again, let $((F, A) \wedge (G, B)) \smile_R ((F, A) \wedge (H, C))$

$$= (M,\ A \times B) \smile_R (N,\ A \times C)$$

$$= (P, (A \times B) \cap (A \times C))$$

$$= (P, A \times (B \cap C))$$

where, $\forall\ (\alpha, \beta) \in A \times B$, $M(\alpha, \beta) = F(\alpha) \cap G(\beta)$;

$$\forall\ (\alpha, \beta) \in A \times C,\ N(\alpha, \beta) = F(\alpha) \cap H(\beta);$$

and, $\forall\ (\alpha, \beta) \in A \times (B \cap C)$,

$$\begin{aligned} P(\alpha,\beta) &= M(\alpha,\beta) - N(\alpha,\beta) \\ &= [F(\alpha) \cap G(\beta)] - [F(\alpha) \cap H(\beta)] \\ &= F(\alpha) \cap \{G(\beta) - H(\beta)\}. \end{aligned}$$

Hence, $L(\alpha,\beta) = P(\alpha,\beta)$.Thus, (x) holds.

(xi) Let $(F,A) \vee ((G,B) \smile_R (H,C))$

$$= (F,A) \vee (L,\ B \cap C)$$

$$= (K,\ A \times (B \cap C))$$

where, $\forall\ \alpha \in B \cap C, \quad L(\alpha) = G(\alpha) - H(\alpha)$;

and,$\forall\ (\alpha,\beta) \in A \times (B \cap C)$,

$$\begin{aligned} K(\alpha,\beta) &= F(\alpha) \cup L(\beta) \\ &= F(\alpha) \cup [G(\beta) - H(\beta)] \end{aligned}$$

Let $((F,A) \vee (G,B)) \smile_R ((F,A) \vee (H,C))$

$$= (I,\ A \times B) \smile_R (J,\ A \times C)$$

$$= (R, (A \times B) \cap (A \times C))$$

$$= (R, A \times (B \cap C))$$

where, $\forall\ (\alpha,\beta) \in A \times B$,

$$I(\alpha,\beta) = F(\alpha) \cup G(\beta);$$

$$\forall\ (\alpha,\beta) \in A \times C,$$

$$J(\alpha,\beta) = F(\alpha) \cup H(\beta),$$

and, $\forall\ (\alpha,\beta) \in A \times (B \cup C)$,

$$\begin{aligned} R(\alpha,\beta) &= I(\alpha,\beta) - J(\alpha,\beta) \\ &= [F(\alpha) \cup G(\beta)] - [F(\alpha) \cup H(\beta)] \\ &\neq F(\alpha) \cup [G(\beta) - H(\beta)]. \end{aligned}$$

Hence, $(K,\ A \times (B \cap C)) \neq (R,\ A \times (B \cap C))$, this proves (xi).

**Proposition 3.2.3: Absorption inclusion (Singh and Onyeozili, 2012A)**

Let *(F,A)* and *(G,B)* be two soft sets over a common universe, then

(i) $(F,A) \cup_R ((F,A)\tilde{\cap}(G,B))\, \tilde{\subset}(F,A)$,

(ii) $(F,A)\tilde{\cap}\,((F,A) \cup_R (G,B))\, \tilde{\subset}(F,A)$,

(iii) $(F,A)\tilde{\cup}\,((F,A) \cap_\varepsilon (G,B))\, \tilde{\supset}(F,A)$, and

(iv) $(F,A) \cap_\varepsilon ((F,A)\tilde{\cup}(G,B))\, \tilde{\supset}(F,A)$.

**Proof:**

(i) Let $(F,A) \cup_R ((F,A)\tilde{\cap}(G,B))$

$$= (F,A) \cup_R (K,(A \cap B))$$

where,$\forall\ \alpha \in A \cap B$, $= (P, A \cap (A \cap B))$

$$= (P, A \cap B)$$

$$\begin{aligned} K(\alpha) &= F(\alpha) \cap G(\alpha) \\ P(\alpha) &= F(\alpha) \cup K(\alpha) \\ &= F(\alpha) \cup \{F(\alpha) \cap G(\alpha)\} \\ &= F(\alpha). \end{aligned}$$

Thus, $P(\alpha) = F(\alpha),\ \forall\ \alpha \in A \cap B \subset A$

Hence, $(P, A \cap B)\tilde{\subset}(F,A)$, which completes the proof.

Using similar arguments, (ii) can be proved.

(iii) Let $(F,A)\tilde{\cup}\,((F,A) \cap_\varepsilon (G,B))$

$$= (F,A)\tilde{\cup}(K, A \cup B)$$

$$= (O, A \cup (A \cup B))$$

$$= (O, A \cup B)$$

where, $\forall\ \alpha \in A \cup B$,

$$K(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - B; \\ G(\alpha),\ \alpha \in B - A; \\ F(\alpha) \cap G(\alpha),\ \alpha \in A \cap B; \end{cases}$$

and

$$\begin{aligned} O(\alpha) &= \begin{cases} F(\alpha),\ \alpha \in A - (A \cup B) = \emptyset; \\ K(\alpha),\ \alpha \in (A \cup B) - A = B - A;\ \text{and} \\ F(\alpha) \cup K(\alpha) = F(\alpha) \cup \{F(\alpha) \cap G(\alpha) = F(\alpha)\},\ \alpha \in A \cap (A \cup B) = A; \end{cases} \\ &= \begin{cases} G(\alpha),\ \alpha \in B - A;\ \text{and} \\ F(\alpha),\ \alpha \in A. \end{cases} \end{aligned}$$

It implies, $O(\alpha) = F(\alpha),\ \forall\ \alpha \in A \subset A \cup B$.

Hence, $(O, A \cup B)\tilde{\supset}(F, A)$, this proves (iii).

Similarly, we can prove (iv). We illustrate (ii) of proposition 3.2.3 in the following using soft sets in Example 3.2.2.

**Example 3.2.3**

Let $(F, A) \cup_R (G, B) = (K, A \cap B)$ where $\forall\ \alpha \in A \cap B = \{e_2, e_3\}$,

$$K(\alpha) = F(\alpha) \cup G(\alpha).$$

Thus, $K(e_2) = F(e_2) \cup G(e_2) = \{u_2, u_3, u_5\}$; and

$$K(e_3) = F(e_3) \cup G(e_3) = \{u_2, u_3, u_4\}.$$

Let $(F, A)\tilde{\cap}(K, A \cap B) = (L, A \cap (A \cap B)) = (L, A \cap B)$

where $\forall\ \alpha \in A \cap B = \{e_2, e_3\}$,

$$L(\alpha) = F(\alpha) \cap K(\alpha).$$

Then, $L(e_2) = F(e_2) \cap K(e_2) = \{u_5\}$ and $L(e_3) = F(e_3) \cap K(e_3) = \{u_4\}$.

Hence,

$$(L, A \cap B) = \{(e_2, \{u_5\}), (e_3, \{u_4\})\} \tilde{\subset} (F, A)$$

proving (ii)

Similarly, we can also illustrate (iv) in example 3.2.4, using the same argument in example 3.2.3 as follows:

**Example 3.2.4**

Let $(F, A)\tilde{\cup}(G, B) = (K, A \cup B)$ where $\forall\ \alpha \in A \cup B = \{e_1, e_2, e_3, e_5\}$,

$$K(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - B = \{e_5\}; \\ G(\alpha),\ \alpha \in B - A = \{e_1\};\ \text{and} \\ F(\alpha) \cup G(\alpha),\ \alpha \in A \cap B = \{e_2, e_3\}. \end{cases}$$

Thus $K(e_1) = \{u_1, u_2, u_6\}$;

$$K(e_2) = \{u_2, u_3, u_5\};$$

$K(e_3) = \{u_2, u_3, u_4\}$; and

$$K(e_5) = \{u_3, u_5, u_6\}.$$

Let $(F, A) \cap_\varepsilon (K, A \cup B) = (L, A \cup (A \cup B)) = (L, A \cup B)$ where $\forall\ \alpha \in A \cup B = \{e_1, e_2, e_3, e_5\}$,

$$L(\alpha) = \begin{cases} F(\alpha),\ \alpha \in A - (A \cup B) = \emptyset; \\ G(\alpha),\ \alpha \in (A \cup B) - A = B - A = \{e_1\};\ \text{and} \\ F(\alpha) \cap K(\alpha),\ \alpha \in A \cap (A \cup B) = A = \{e_2, e_3, e_5\}. \end{cases}$$

We have $L(e_1) = \{u_1, u_2, u_6\}$;

$$L(e_2) = \{u_5\};$$

$L(e_3) = \{u_4\}$; and

$$L(e_5) = \{u_3, u_5, u_6\}.$$

Hence,

$$(L, A \cup B) = \{(e_1, \{u_1, u_2, u_6\}), (e_2, \{u_5\}), (e_3\{u_4\}), (e_5, \{lu_3, u_6\})\} \tilde{\supset} (F, A)$$

Thus (iv) holds.

## 3.3 Soft Set Relations and Functions

This section presents soft set relations and functions and their basic properties. More details about this section can be found in (Babitha and Sunil, 2011).

**Definition 3.3.1**

Let $(F,A)$ and $(G,B)$ be two soft sets over a common universe $U$. Then the

**Cartesian product** of $(F,A)$ and $(G,B)$, denoted $(F, A) \times (G, B)$,is a soft set $(H, A \times B)$where $H : A \times B \to P(U \times U)$ such that $H(a, b) = F(a) \times G(b)$, $\forall$ $(a, b) \in A \times B$ i.e.,$H(a, b) = \{(h_i, h_j) : h_i \in F(a)$ and $h_j \in G(b)\}$.

**Definition 3.3.2: Soft set relation(Babitha and Sunil)**

Let $(F,\ A)$ and $(G,B)$ be two soft sets over a common universe $U$. Then a relation from $(F,A)$ to $(G,B)$, called a **soft set relation,** denoted $(R,C)$ or simply $R$, is a soft subset of $(F, A) \times (G, B)$ where $C \subseteq A \times B$ and, $\forall$ $(a, b) \in C$, $R(a,b) = H(a,b)$ where $(H, A \times B) = (F, A) \times (G, B)$.

A soft set relation on $(F,A)$ is a soft subset of $(F, A) \times (F, A)$. Equivalently, we can define a relation $R$ on a soft set $(F,A)$ in the parameterized form as follows:

If $(F, A) = \{F(a), F(b), \ldots\}$, then, $F(a)RF(b)$ iff $F(a) \times F(b) \in R$.

**Definition 3.3.3**

Let $R$ be a soft set relation from $(F,A)$ to $(G,B)$, such that$(F, A) \times (G, B) = (H, A \times B)$. Then

(i) **the domain of $R$ (dom$R$)** is a soft set $(D, A_1)\, \tilde{\subset}(F, A)$ where, $A_1 \subset A$ and $A_1 = \{a \in A : H(a, b) \in R\,, \text{for some } b \in B\}$and $D\,(a_1) = F\,(a_1)\ \ \forall\ a_1 \in A_1$,

(ii) **the range of $R$ (ran $R$)** is a soft set $(E, B_1)\, \tilde{\subset}(G, B)$ where, $B_1 \subset B$ and $B_1 = \{b \in B : H(a, b) \in R, \text{ for some } a \in A\}$ and $E\,(b_1) = G\,(b_1)\ , \forall\ b_1 \in B_1$

(iii) **the inverse of $R$,** denoted $R^{-1}$, is a soft set relation from $(G, B)to(F, A)$defined by $R^{-1} = \{G(b) \times F(a) : F(a)RG(b)\}$.

**Example 3.3.1**

Let $U$ be a set of ten students in an institution, given by

$U = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\}$.

Let $A$ be a set four courses offered by the institution, given by

$A = \{\text{Mathematics, Medicine, Engineering, Education}\}$ represented by $A = \{a_1, a_2, a_3, a_4\}$, respectively.

Let $B$ be a set of qualified students given by

$B = \{\text{B.Sc., B.Tech., MBBS, M.Sc.}\}$ represented by $B = \{b_1, b_2, b_3, b_4\}$, respectively.

Then, the soft set $(F,A)$ given by

$$(F, A) = \{F\,(a_1) = \{s_1, s_2\}\,, F\,(a_2) = \{s_4, s_5\}\,, F\,(a_3) = \{s_7, s_9\}\,,$$
$$F\,(a_4) = \{s_3, s_4, s_7\}\}$$

describes students offering different courses, while the soft set *(G,B)* given by

$$(G,B) = \{G(b_1) = \{s_1, s_2, s_6, s_8, s_{10}\}, G(b_2) = \{s_3, s_6, s_7, s_9\}, G(b_3) = \{s_3, s_4, s_5, s_8\},$$
$$G(b_4) = \{s_1, s_2, s_3, s_8\}\}$$

describes students to obtain different qualifications.

Now, if we define a relation $R$ from $(F,A)$ to $(G,B)$ by

$F(a)RG(b)$ iff $F(a) \subset G(b)$, then we have the following:

(i) $R = \{F(a_1) \times G(b_1), F(a_2) \times G(b_3), F(a_3) \times G(b_2), F(a_1) \times G(b_4)\}$

(ii) dom $R = (D, A_1)$, where $A_1 = \{a_1, a_2, a_3\} \subset A$ and $D(a) = F(a)$, $\forall\ a \in A_1$

(iii) ran $R = (E, B_1)$, where $B_1 = \{b_1, b_2, b_3, b_4\}$ and $E(b) = G(b)$, $\forall\ b \in B_1$

(iv) $R^{-1} = \{G(b_1) \times F(a_1), G(b_2) \times F(a_3), G(b_3) \times F(a_2), G(b_4) \times F(a_1)\}$.

**Definition 3.3.4**

Let $R$, $Q$ be two soft set relations on a soft set $(F,A)$.

(i) $R$ **is a subset of** $Q$**,** denoted $R \subset Q$, if $\forall\ a, b \in A$, $F(a) \times F(b) \in R \Rightarrow F(a) \times F(b) \in Q$.

(ii) The **complement of** $\boldsymbol{R}$**,** denoted $R^C$, is defined by

$$R^C = \{F(a) \times F(b) : F(a) \times F(b) \notin R, a, b \in A\}.$$

(iii) The **union of** $\boldsymbol{R}$ **and** $\boldsymbol{Q}$, denoted $R \cup Q$,is defined by

$$R \cup Q = \{F(a) \times F(b) : F(a) \times F(b) \in R \text{ or } F(a) \times F(b) \in Q\}.$$

(iv) The **intersection of** $\boldsymbol{R}$ **and** $\boldsymbol{Q}$**,** denoted $R \cap Q$, is defined by $R \cap Q = \{F(a) \times F(b) : F(a) \times F(b) \in R \text{ and } F(a) \times F(b) \in Q\}$.

**Example 3.3.2** Consider a soft set $(F,A)$ over $U$ where $U = \{u_1, u_2, u_3, u_4\}$, $A = \{a_1, a_2\}$ , and $F(a_1) = \{u_1, u_2\}, F(a_2) = \{u_2, u_3, u_4\}$. If a soft set relation $R$ on $(F,A)$ is defined as $R = \{F(a_1) \times F(a_1), F(a_2) \times F(a_1)\}$ ,then

$R^C = \{F(a_1) \times F(a_2), F(a_2) \times F(a_2)\}$. If another soft set relation $Q$ on $(F,A)$ is defined as $Q = \{F(a_1) \times F(a_1), F(a_2) \times F(a_2)\}$ ,then

$R \cup Q = \{F(a_1) \times F(a_1), F(a_2) \times F(a_1), F(a_2) \times F(a_2)\}$, *and*

$$R \cap Q = \{F(a_1) \times F(a_1)\}.$$

It can be verified that the union and the intersection of soft set relations satisfy commutative, associative and distributive properties.

**Definition 3.3.5** Let $R$ be a soft set relation on $(F,A)$.

(i) $R$ is **reflexive** if $F(a) \times F(a) \in R$, $\forall\ a \in A$.

(ii) $R$ is **symmetric** if $F(a) \times F(b) \in R \Rightarrow F(b) \times F(a) \in R$, $\forall\ (a,b) \in A \times A$.

(iii) $R$ is **transitive** if $F(a) \times F(b) \in R$ and $F(b) \times F(c) \in R \Rightarrow F(a) \times F(c) \in R$, $\forall\ a, b, c \in A$.

(iv) $R$ is an **equivalence** if it is reflexive, symmetric and transitive.

(v) $R$ is an **identity** if $a \neq b, F(a) \times F(b) \in R$ but $F(a) \times F(b) \notin R\ \forall\ a, b \in A$.

That is, $F(a) \times F(b) \in R \Rightarrow a = b$, $\forall\ a, b \in A$.

$R = \{F(a) \times F(a), F(b) \times F(b), F(c) \times F(c)\}$,

$\forall\ a, b, c \in A$.

**Example 3.3.3**

Consider a soft set $(F,A)$ over $U$ where $A = \{a_1, a_2\}$. If a relation $R$ on $(F,A)$ is defined by

$R = \{F(a_1) \times F(a_2), F(a_2) \times F(a_1), F(a_1) \times F(a_1), F(a_2) \times F(a_2)\}$, then

$R$ is a soft set equivalence relation. In other words,$R = (F, A) \times (F, A)$.

**Definition 3.3.6**

Let $(F,A)$, $(G,B)$ and $(H,C)$ be three soft sets over a common universe. Let $R$ be a soft set relation from $(F,\ A)$ to $(G,B)$ and $S$ be another soft set relation from $(G,B)$ to $(H,C)$. Then, a new soft set relation from $(F,A)$ to $(H,C)$, called the **composition of *R* and *S*,** denoted $SoR$,is defined as follows:

If $F(a) \in (F, A)$ and $H(c) \in (H, C)$, then $F(a) \times H(c) \in SoR$, where

$F(a) \times G(b) \in R$ and $G(b) \times H(c) \in S$,for some $G(b) \in (G, B)$.

**Example 3.3.4**

Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$. Let $R$ and $S$ be soft set relations over a common universe defined, respectively, from $(F,A)$ to $(G,B)$ and (G,B) to $(H,C)$ such that $R = \{F(a_1) \times G(b_1), F(a_2) \times G(b_2), F(a_3) \times G(b_2)\}$, and $S = \{G(b_1) \times H(c_1), G(b_2) \times H(c_2)\}$.

Then $SoR = \{F(a_1) \times H(c_1), F(a_2) \times H(c_2), F(a_3) \times H(c_2)\}$.

Now, let$A = B = C = \{e_1, e_2, e_3\}$ such that

$R = \{F(e_1) \times F(e_2), F(e_2) \times F(e_3), F(e_3) \times F(e_1)\}$, and

$S = \{F(e_1) \times F(e_3), F(e_2) \times F(e_2), F(e_3) \times F(e_3)\}$.

Then, we have

$SoR = \{F(e_1) \times F(e_2), F(e_2) \times F(e_3), F(e_3) \times F(e_3)\}$, and

$RoS = \{F(e_1) \times F(e_1), F(e_2) \times F(e_3), F(e_3) \times F(e_1)\}$ which are not equal.

Note that, akin to set relations, $SoR \neq RoS$ in general.

**Definition 3.3.7**

Let $(F,A)$ and $(G,B)$ be two non-empty soft sets over $U$. Then a soft set relation $f$

from (*F*,*A*) to (*G*,*B*) written $f : (F, A) \to (G, B)$ is called a **soft set function** if every element in the domain of $f$ has a unique element in the range of *f*. If $F(a)$ $f$ $G(b)$, i.e., $F(a) \times G(b) \in f$, then we write $f\left(F(a)\right) = G(b)$.

**Example 3.3.5**

Let $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2\}$. Consider the soft sets (*F*,*A*) and (*G*,*B*) over a universe U. Then, a soft set function $f$ from (F,A) to (G,B) can be given by

(i) $f = \{F\left(a_1\right) \times G\left(b_1\right), F\left(a_2\right) \times G\left(b_1\right), F\left(a_3\right) \times G\left(b_3\right), F\left(a_4\right) \times G\left(b_2\right)\},$

(ii) $f = \{F\left(a_1\right) \times G\left(b_1\right), F\left(a_2\right) \times G\left(b_1\right), F\left(a_3\right) \times G\left(b_1\right), F\left(a_4\right) \times G\left(b_1\right)\}.$

But$\{F\left(a_1\right) \times G\left(b_1\right), F\left(a_1\right) \times G\left(b_2\right), F\left(a_2\right) \times G\left(b_2\right), F\left(a_3\right) \times G\left(b_1\right)\}$is not a soft set function.

**Definition 3.3.8** A soft set function $f$ from (*F*,*A*) to (*G*,*B*) is called

(i) **Injective (one-to-one)** if $F(a) \neq F(b) \Rightarrow f\left(F(a)\right) \neq f\left(F(b)\right)$,

(ii) **Surjective (onto)** if range = (*G*,*B*),

(iii) **Bijective (one- to-one and onto)** if $f$ is both injective and surjective.

**Example 3.3.6** Consider the function $f$ in Example 3.3.5, (i) is onto but (ii) is not.

**Definition 3.3.9** The **identity soft set function** $I$ on a soft set (F,A) is defined by $I : (F, A) \to (F, A)$such that $I\left(F(a)\right) = F(a)\ \forall\ F(a) \in (F, A)$.

**Definition 3.3.10**

Let *(F, A)* and *(G, B)* be two soft sets over $U_1$and $U_2$ respectively. Let $f$: $U_1 \longrightarrow U_2$, and $g$: A$\longrightarrow$ B be two functions. Then the pair *(f,g)* is called a **soft function** from (F, A) to (G, B), denoted *(f,g)*: (F, A)$\longrightarrow$(G,B), if $f$(F(x)) = G$(g$(x)) for all $x \in A$.

If $f$ and $g$ are injective (resp., surjective, bijective) then

($f$, $g$) is said to be injective (resp., surjective, bijective).

Note that soft set function (Definition 3.3.7) is different from soft function (Definition3.3.10).

## 3.4 Matrix Representation of Soft Sets (Soft Matrices)

Matrix representation of soft sets, their basic operations and properties are presented in this section. Most of the definitions and results discuss in this section can be found in (Cagman and Enginoglu, 2010B, Mitra et al., 2012, Singh and Onyeozili, 2012D)

**Definition 3.4.1: Soft matrix**

Let $U$ be a universe, $E$ a set of parameters with respect to $U$ and $A \subseteq E$. Let $(f_A, E)$ be a soft set over $U$. Then a subset $R_A$ of $U \times E$,uniquely defined as $R_A = \{(u, e) : e \in A,\, u \in f_A(e)\}$, is called a *relation form* of the soft set $(f_A, E)$ .The characteristic function $\chi_{R_A}$ of $R_A$ is defined as

$\chi_{RA} : U \times E \to \{0, 1\}$, where

$$\chi_{R_A}(u, e) = \begin{cases} 1, & (u, e) \in R_A; \\ 0, & (u, e) \notin R_A. \end{cases}$$

Now if $U = \{u_1, u_2, \ldots, u_m\}$ *and* $E = \{e_1, e_2, \ldots, e_n\}$ then the soft set *($f_A$,E)* can be represented by a matrix $_{[a_{ij}]}$ called an $m\times$ $n$ **soft matrix** of the soft set *($f_A$,E)* over

$U$ as follows

$$[a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ where } a_{ij} = \chi_{R_A}(u_i, e_j)$$

In other words, a soft set is uniquely represented by its corresponding soft matrix.

**Example 3.4.1**

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$ be a universe set, and

$E = \{e_1, e_2, e_3, e_4\}$ be a set of parameters with respect to $U$.

Let $A = \{e_1, e_3, e_4\}$, such that $f_A(e_1) = \{u_3, u_4\}$, $f_A(e_3) = \emptyset$, and

$f_A(e_4) = \{u_1, u_3, u_5\}$.

Then the soft set $(f_A, E)$ is given by

$$(f_A, E) = \{(e_1, \{u_3, u_4\}), (e_4, \{u_1, u_3, u_5\})\}.$$

The relation form $R_A$ of $(f_A, E)$ is given by

$$R_A = \{(u_3, e_1), (u_4, e_1), (u_1, e_4), (u_3, e_4), (u_5, e_4)\}$$

. Hence the soft matrix $[a_{ij}]$ of the soft set $(f_A, E)$ is given by

$$[a_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad i = 1, 2, \ldots, 5; \ j = 1, 2, \ldots, 4.$$

As noted earlier, $f_A(e_3) = \emptyset$, since there is no element in $U$ related to the parameter $e_3 \in A$, so it does not appear in the aforesaid description of the soft set $(f_A, E)$.

**Definition 3.4.2: Special Soft Matrices**

Let the set of all $m \times n$ soft matrices over $U$, be denoted $SM(U)_{m\times n}$, or just $SM(U)$.

Let $[a_{ij}] \in SM(U)$. Then $[a_{ij}]$ is called

(a) A ***zero soft matrix***, denoted $\left[\widetilde{0}\right]$, if $a_{ij} = 0$ , $\forall$ $i$ and $j$;

(b) An ***A-universal soft matrix***, denoted$[\tilde{a}_{ij}]$, if $a_{ij} = 1$ , $\forall$ $j \in I_A = \{j : e_j \in A\}$ and $i$.

(Note that it is so called, since $a_{ij} = 1$ only for the parameters appearing in the set $A \subset E$); and

(c) A ***universal soft matrix*** denoted $\left[\tilde{I}\right]$, if $a_{ij} = 1$ , $\forall$ $i$ and $j$

**Example 3.4.2**

Let $U = \{u_1, u_2, u_3, u_4\}$, $E = \{e_1, e_2, e_3, e_4\}$ and $[a_{ij}]$, $[b_{ij}]$, $[c_{ij}] \in SM(U)_{4\times4}$. If $A = \{e_1, e_2, e_3\}$, $f_A(e_1) = f_A(e_2) = f_A(e_3) = \emptyset$ then $[a_{ij}] = \left[\tilde{0}\right]$ is a zero soft matrix given by

$$\left[\tilde{0}\right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $B = \{e_2, e_4\}$, $f_B(e_2) = U = f_B(e_4)$, then $[b_{ij}]$ is a $B$-universal soft matrix given by

$$\left[\tilde{b}_{ij}\right] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

If $C = E, f_e(e_i) = U$ for each $i$, then $[\tilde{c}_{ij}] = \left[\tilde{I}\right]$ is a universal soft matrix given by

$$\left[\tilde{I}\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

**Definition 3.4.3(Soft Submatrices)**

Let $M = [a_{ij}]$, $N = [b_{ij}] \in SM(U)$. Then we define the following:

(i) M is a ***soft submatrix*** of $N$, denoted $M\tilde{\subseteq}N$ if $a_{ij} \leq b_{ij}$ for each $i$ and $j$.

In this case, we also say that $M$ is ***dominated*** by $N$ or $N$ *dominates* $M$. Note that similar to $R^k(k > 1)$, the $k$-dimensional real space, $\leq$ holds without the holding of either $<$ or $=$.

We define $M$ and $N$ ***comparable***, denoted $M \| N$, iff $M\tilde{\subseteq}N$ *or* $N\tilde{\subseteq}M$;

(ii) $M$ is a ***proper soft submatrix*** of $N$, denoted $M\tilde{\subset}N$, if $[a_{ij}]\,\tilde{\subset}\,[b_{ij}]$ and for at least one term $a_{ij} < b_{ij}$ for all $i$ and $j$. In this case, we say that $M$ is ***properly dominated*** by $N$.

(iii) $M$ is a ***strictly proper soft submatrix*** of $N$, denoted $M \underset{\neq}{\tilde{\subset}} N$, if $M\tilde{\subset}N$ and $a_{ij} < b_{ij}$ , for each $i$ and $j$. In this case we say that $M$ is ***strictly dominated*** by $N$.

(vi) $M$ and $N$ are ***soft equal matrices*** denoted $M\tilde{=}N$ if $a_{ij} = b_{ij}$ for each $i$ and $j$. Equivalently, if $M\tilde{\subseteq}N$ and $N\tilde{\subseteq}M$, then

$M \tilde{=} N$. It is immediate to see that $\tilde{\subseteq}$ is a partial ordering (reflexive, antisymmetric and transitive) on the class of soft matrices.

### 3.4.1 Operations on soft matrices

The operations of union, intersection complement, difference and products of soft matrices and their basic properties are discussed in this sub section.

**Definition 3.4.4: Soft Matrix Operations**

Let $M = [a_{ij}]$ , $N = [b_{ij}] \in SM(U)$. Then, a soft matrix $P = [c_{ij}] \in SM(U)$is called the

(i) ***union*** of $M$ and $N$, denoted $M \tilde{\cup} N$, if

$$c_{ij} = \max\{a_{ij}, b_{ij}\} \text{ for all } i \text{ and } j;$$

(ii) ***intersection*** of $M$ and $N$, denoted $M \tilde{\cap} N$, if

$$c_{ij} = \min\{a_{ij}, b_{ij}\} \text{ for all } i \text{ and } j;$$

(iii) ***complement*** of $M$, denoted $M^0$, if

$$c_{ij} = 1 - a_{ij} \text{ for all } i \text{ and } j;$$

(iv) ***difference*** of $N$ from $M$, also called the ***relative complement*** of $N$ in $M$, denoted $M - N$ or $M \backslash N$ if $P = M \tilde{\cap} N^0$.

In view of the (ii) above, $M$ and $N$ are said to be ***disjoint*** if $M \tilde{\cap} N = \left[\tilde{0}\right]$.

**Example 3.4.3**

Let

$$M = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } N = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then,

(i) $M \tilde{\cup} N = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$;

(ii) $M \tilde{\cap} N = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \left[\tilde{0}\right]$ which implies that $M$ and $N$ are disjoint;

(iii) $M^0 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ and $M \tilde{\cup} M^0 = \left[\tilde{I}\right]$,

$M \tilde{\cap} M^0 = \left[\tilde{0}\right]$;

(iv) $N^0 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$;

(v) $M - N = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = M\tilde{\cap}N^0$; and

(vi) $N - M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = N\tilde{\cap}M^0$.

### 3.4.2 Properties of soft matrix operations

**Proposition 3.4.1**

Let $M = [a_{ij}]$, $N = [b_{ij}]$, $P = [c_{ij}] \in SM(U)$.Then the following hold:

(i) $M\tilde{\cup}M = M$; $M\tilde{\cap}M = M$ (Idempotent laws)

(ii) $M\tilde{\cup}\left[\tilde{0}\right] = M$; $M\tilde{\cap}\left[\tilde{I}\right] = M$ (Identity laws)

(iii) $M\tilde{\cup}\left[\tilde{I}\right]=\left[\tilde{I}\right]$ ; $M\tilde{\cap}\left[\tilde{0}\right]=\left[\tilde{0}\right]$ (Domination laws)

(iv) $\left[\tilde{0}\right]^0=\left[\tilde{I}\right]$ ; $\left[\tilde{I}\right]^0=\left[\tilde{0}\right]$ (De Morgan′s laws)

(v) $M\tilde{\cup}M^0=\left[\tilde{I}\right]$ ; $M\tilde{\cap}M^0=\left[\tilde{0}\right]$ (De Morgan′s laws)

(vi) $(M\tilde{\cup}N)^0=M^0\cap N^0$ ; $(M\tilde{\cap}N)^0=M^0\tilde{\cup}N^0$
(De Morgan′s laws)

(vii) $(M^0)^0=M$ for all $M$ (Involution law)

(viii) $M\tilde{\cup}N=N\tilde{\cup}M$ ; $M\tilde{\cap}N=N\tilde{\cap}M$(Commutative laws)

(ix) $M\tilde{\cup}(N\tilde{\cup}P)=(M\tilde{\cup}N)\tilde{\cup}P$ ;
$M\tilde{\cap}(N\tilde{\cap}P)=(M\tilde{\cap}N)\tilde{\cap}P$(Associative laws)

(x) $M\tilde{\cup}(N\tilde{\cap}P)=(M\tilde{\cup}N)\tilde{\cap}(M\tilde{\cup}P)$ ;

$M\tilde{\cap}(N\tilde{\cup}P)=(M\tilde{\cap}N)\tilde{\cup}(M\tilde{\cap}P)$.(Distributive Laws)

**Proof:** Most of the proofs follow from definitions.

Let us, for example, prove the first parts of (vi), (ix) and (x).

(vi) For each $i$ and $j$,

$$\begin{aligned}(M\tilde{\cup}N)^0 &= ([a_{ij}]\,\tilde{\cup}\,[b_{ij}])^0\\ &= [max\{a_{ij},b_{ij}\}]^0\\ &= [1-max\{a_{ij},b_{ij}\}\\ &= [min\{1-a_{ij},1-b_{ij}\}]\\ &= [a_{ij}]^0\tilde{\cap}[b_{ij}]^0\\ &= M^0\tilde{\cap}N^0\end{aligned}$$

(ix)

$$\begin{aligned} M\tilde{\cup}(N\tilde{\cup}P) &= [a_{ij}]\,\tilde{\cup}\,([b_{ij}]\,\tilde{\cup}\,[c_{ij}]) \\ &= [\max\{a_{ij},\,\max(b_{ij},\,c_{ij})\}] \\ &= [\max\{\max(a_{ij},\,b_{ij}),\,c_{ij}\}] \\ &= ([a_{ij}]\,\tilde{\cup}\,[b_{ij}])\,\tilde{\cup}\,[c_{ij}] \\ &= (M\tilde{\cup}N)\,\tilde{\cup}P. \end{aligned}$$

(x)

$$\begin{aligned} M\tilde{\cup}(N\tilde{\cap}P) &= [a_{ij}]\,\tilde{\cup}\,([b_{ij}]\,\tilde{\cap}\,[c_{ij}]) \\ &= [\max\{a_{ij},\,\min(b_{ij},\,c_{ij})\}] \\ &= \left[\min\left\{\max\{a_{ij},\,b_{ij}\},\,\max\left\{a_{ij,\,c_{ij}}\right\}\right\}\right] \\ &= ([a_{ij}]\,\tilde{\cup}\,[b_{ij}])\,\tilde{\cap}\,([a_{ij}]\,\tilde{\cup}\,[c_{ij}]) \\ &= (M\tilde{\cup}N)\,\tilde{\cap}\,(M\tilde{\cup}P). \end{aligned}$$

**Definitions 3.4.5 ( Product of Soft Matrices)**

Let $M = [a_{ij}]$, $N = [b_{ik}] \in SM(U)_{m\times n}$. Then

(i) ***AND-product*** of $M$ and $N$, denoted $M \wedge N$, is defined

$\wedge : SM(U)_{m\times n} \times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that$[a_{ij}] \wedge [b_{ik}] = [c_{ip}]$,

where $c_{ip} = \min\{a_{ij},\, b_{ik}\}$and $P = n(j-1)+k$.

(ii) ***OR-product*** of $M$ and $N$, denoted $M \vee N$,is defined $\vee : SM(U)_{m\times n} \times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that $[a_{ij}] \vee [b_{ik}] = [c_{ip}]$,

where $c_{ip} = \max\{a_{ij},\, b_{ik}\}$ and $P = n(j-1)+k$.

(iii) ***AND-NOT-product*** of $M$ and $N$, denoted $M\bar{\wedge}N$,is defined $\bar{\wedge} : SM(U)_{m\times n}\times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that $[a_{ij}]\,\bar{\wedge}\,[b_{ik}] = [c_{ip}]$,

where $c_{ip} = \min\{a_{ij},\, 1-b_{ik}\}$and $P = n(j-1)+k$.

(iv) ***OR-NOT-product*** of $M$ and $N$, denoted $M \underline{\vee} N$,is defined $\underline{\vee} : SM(U)_{m\times n} \times SM(U)_{m\times n} \to SM(U)_{m\times n^2}$ such that $[a_{ij}] \underline{\vee} [b_{ik}] = [c_{ip}]$,

where $c_{ip} = \max\{a_{ij},\, 1 - b_{ik}\}$and $P = n(j-1) + k$.

**Note:** Products of soft matrices hold if the two matrices are of the same order or have the same number of rows.

**Proposition 3.4.2: Properties of Product of Soft Matrices**

Let $M = [a_{ij}]\,,\ N = [b_{ik}] \in SM(U)$. Then the following hold:

(i) $(M \vee N)^0 = M^0 \wedge N^0;\ \ (M \wedge N)^0 = M^0 \vee N^0$ (De Morgan's laws)

(ii) $(M \underline{\vee} N)^0 = M^0 \bar{\wedge} N^0;\ \ (M \bar{\wedge} N)^0 = M^0 \underline{\vee} N^0$ (De Morgan's laws)

***Proof*** *:* The proofs follow from definitions.

**Example 3.4.4**

Let $M = [a_{ij}]\,,\ N = [b_{ik}] \in SM(U)_{4\times 4}$ be given by

$$M = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Then

$$M \bar{\wedge} N = [a_{ij}] \bar{\wedge} [b_{ik}] = [c_{ip}]_{4\times 16} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Similarly, the other products $M \vee N,\ M \bar{\wedge} N$ and $M \underline{\vee} N$can be found.

Also,

$$(M \wedge N)^{\circ} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$M^{\circ} \vee N^{\circ} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Thus,$(M \wedge N)^0 = M^0 \vee N^0$.

Note that the commutative laws are not valid for products of soft matrices.

## 3.5 Comparing Soft Sets Operations with their corresponding Soft Matrices Operations

(**Singh and Onyeozili, 2012D**)

We recall from section 3.4 that a soft set can be uniquely characterized by a soft matrix and vice versa. Thus a soft set $(f_A, E)$over $U$ is formally equal to its soft matrix $[a_{ij}]_{m\times n}$where $m = |U|$and $n = |E|$, the cardinality of $U$ and $E$ respectively.

In order to show that the operations on soft sets and that on their corresponding soft matrices are equivalent, we first recall the following definitions on soft sets.

**Definitions 3.5.1**

Let $(f_A, E)$and $(f_B, E)$be two soft sets over $U$. We define the

(i) *union* of $(f_A, E)$and $(f_B, E)$, denoted $(f_A, E)\,\tilde{\cup}\,(f_B, E)$, as a soft set $(f_C, E)$, defined by the approximate function $f_C(e) = f_A(e) \cup f_B(e),\ \forall\ e \in E$.

(ii) *intersection* of $(f_A, E)$and $(f_B, E)$, denoted $(f_A, E)\,\tilde{\cap}\,(f_B, E)$, as a soft set $(f_D, E)$,
defined by the approximate function $f_D(e) = f_A(e) \cap f_B(e),\ \forall\ e \in E$.

(iii) *complement* of $(f_A, E)$ denoted $(f_A, E)^C = \left(f_A^C, E\right)$ or $(f_{A^C}, E)$, as a soft set $(f_{A^C}, E)$,defined by the approximate function $f_{A^C}(e) = U - f_A(e),\ \forall\ e \in E$.

(iv) *difference* of $(f_A, E)$and $(f_B, E)$, denoted $(f_A, E) \setminus (f_B, E)$, as a soft set $(f_K, E)$, defined by the approximate function $f_K(e) = f_A(e) - f_B(e) = f_A(e) \cap f_{B^C}(e)$
$\forall\ e \in E$.

(v) *AND-product* of $(f_A, E)$and $(f_B, E)$, denoted $(f_A, E) \wedge (f_B, E)$, as a soft set $(f_G, E \times E)$,defined by the function $f_G(x, y) = f_A(x) \cap f_B(y)\ \forall\ (x, y) \in E \times E$.

(vi) *OR-product* of $(f_A, E)$and $(f_B, E)$, denoted $(f_A, E) \vee (f_B, E)$,as a soft set $(f_H, E \times E)$,defined by the function $f_H(x, y) = f_A(x) \cup f_B(y)\ \forall\ (x, y) \in E \times E$.

We illustrate these operations as follows:

**Example 3.5.1**

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$be the universe set and $E = \{e_1, e_2, e_3, e_4\}$ be the set of parameters.

Let $A = \{e_1, e_2\}$and $B = \{e_2, e_3, e_4\}$. Suppose that the soft sets $(f_A, E)$ and $(f_B, E)$ are given by

$(f_A, E) = \{(e_1, \{u_2, u_4\}), (e_2, \{u_1, u_2\})\}$and

$(f_B, E) = \{(e_2, \{u_1, u_2\}), (e_3, \{u_1, u_4\}), (e_4, U)\}$, respectively.

Then, their soft matrices $[a_{ij}]$ and $[b_{ij}]$ are respectively represented by

$$[a_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } [b_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

From Definition 3.5.1:

(i) $(f_A, E) \tilde{\cup} (f_B, E) = (f_C, E)$, where $f_C(e) = f_A(e) \cup f_B(e), \ \forall \ e \in E$.

Then $(f_C, E) = \{(e_1, \{u_2, u_4\}), (e_2, \{u_1, u_2, u_3\}), (e_3 \{u_1, u_4\}), (e_4, U)\}$ and its soft matrix $[c_{ij}]$ is given by

$$[c_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [a_{ij}] \tilde{\cup} [b_{ij}].$$

(ii) $(f_A, E) \tilde{\cap} (f_B, E) = (f_D, E)$, where $f_D(e) = f_A(e) \cap f_B(e), \ \forall \ e \in E$

Then $(f_D, E) = \{(e_2, \{u_1\})\}$ and its soft matrix $[d_{ij}]$ is given by

$$[d_{ij}] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [a_{ij}] \tilde{\cap} [b_{ij}] .$$

(iii) $(f_A, E)^C = (f_{A^C}, E)$, where $f_{A^C}(e) = U - f_A(e), \ \forall \ e \in E$.

Then $(f_{A^C}, E) = \{(e_1, \{u_1, u_3\}), (e_2, \{u_2, u_4, u_5\}), (e_3, U), (e_4, U)\}$ and its soft matrix $\left[a_{ijC}\right]$ is given by

$$\left[a_{ijC}\right] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = [a_{ij}]^0 .$$

(iv) $(f_A, E) \setminus (f_B, E) = (f_K, E)$, where $f_K(e) = f_A(e) - f_B(e) = f_A(e) \cap f_{B^C}(e), \ \forall \ e \in E$.

Then $(f_K, E) = \{(e_1, \{u_2, u_4\}), (e_2, \{u_3\})\}$ and its soft matrix $[k_{ij}]$ is given by

$$[k_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [a_{ij}] \tilde{\cap} [b_{ij}]^0 .$$

$(f_A, E) \wedge (f_B, E) = (f_G, E \times E)$, where $f_G(x, y) = f_A(x) \cap f_B(y), \ \forall \ (x, y) \in E \times E$.

Then

$$(f_G, E \times E) = \{((e_1, e_2),\ \{u_2\}), ((e_1, e_3),\ \{u_4\}), ((e_1, e_4),\ \{u_2, u_4\})$$
$$((e_2, e_2),\ \{u_1\}),\ ((e_2, e_3),\ \{u_1\}),\ ((e_2, e_4),\ \{u_1, u_3\})\},$$

and its soft matrix $[g_{ip}]$ is given by

$$[g_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = [a_{ij}] \wedge [b_{ik}].$$

(v) $(f_A, E) \vee (f_B, E) = (f_H, E \times E)$, where $f_H(x, y) = f_A(x) \cup f_B(y)\ \forall\ (x, y) \in E \times E$.

Then

$$(f_H, E \times E) = \{((e_1, e_1),\ \{u_2, u_4\}), ((e_1, e_2),\ \{u_1, u_2, u_4\}),$$
$$((e_1, e_3), \{u_1, u_2, u_4\}),\ ((e_1, e_4),\ U),$$
$$((e_2, e_1),\ \{u_1, u_3\}),\ ((e_2, e_2),\ \{u_1, u_2, u_3\}),$$
$$((e_2, e_3),\ \{u_1, u_3, u_4\}),\ ((e_2, e_4),\ U),$$
$$((e_3, e_2), \{u_1, u_2\}), ((e_3, e_3), \{u_1, u_4\}), ((e_3, e_4), U)$$
$$((e_4, e_2),\ \{u_1, u_2\}),\ ((e_4, e_3),\ \{u_1, u_4\}),\ ((e_4, e_4),\ U)\},$$

and its soft matrix $[h_{ip}]$ is given by

$$[h_{ip}] = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} = [a_{ij}] \vee [b_{ik}].$$

Similarly,the other products $(f_A, E) \bar{\wedge} (f_B, E) = [a_{ij}] \bar{\wedge} [b_{ik}]$ and

$(f_A, E) \underline{\vee} (f_B, E) = [a_{ij}] \underline{\vee} [b_{ik}]$ can also be found.

# CHAPTER 4

# ALGEBRA OF SOFT SETS

In this chapter, soft groups, soft rings, soft semi-rings, and soft lattices are presented. Finally, monoids, semirings and lattices of soft subsets and partitions of a soft set are developed.

## 4.1 Soft Groups

Soft groups and their substructures, such as soft subgroup, soft normal subgroup, and soft group homomorphism and isomorphism were introduced in (Aktas and Cagman, 2007) and further studied in (Sezgin and Atagun, 2011). The concept of soft set over a group given in the following is the basic tool to define a soft group.

**Definition 4.1.1**

Let G be a group and A be a non empty set. Let F be a set –valued function from A to G, defined as: $F(x) = \{y \in G: (x, y) \in R, x \in A \text{ and } y \in G\}$, where $R \subseteq A \times G$. Then, (F,A) is called a **soft set over a group** G**.** It may be observed that the function $F: A \longrightarrow P(G)$, in fact, defines the relation $R: A \longrightarrow G$ given by

$R = \{ (x, y) \in A \times G, x \in A \text{ and } y \in F(x)\}$.

**Definition 4.1.2 ( Soft group)**

Let (F, A) be a soft set over a group G. Then, (F, A) is called a **soft group** over G if and only if F(x) is a subgroup of G, denoted $F(x) <_g G$, $\forall x \in A$. In other words, the soft group (F, A) can be viewed as the collection of subgroups of G.

**Example 4.1.1**

Let G = {1, -1, i, -i} be the cyclic group of order 4 with the multiplication operation table 4.1 given below.

**Table 4.1: Multiplication Table of cyclic group G of order 4**

| . | 1 | -1 | i | -i |
|---|---|---|---|---|
| 1 | 1 | -1 | i | -i |
| -1 | -1 | 1 | -i | i |
| i | i | -i | -1 | 1 |
| -i | -i | i | 1 | -1 |

Let G=A and (F,A) be a soft set over G, where F:A $\longrightarrow$ P(G) is a set-valued function defined by F(x) = {y $\in$ G: xRy $\Leftrightarrow$ y = $x^n$for some n$\in \mathbb{N}$}.Then (F, A) is a soft group consisting of a parameterized family {F(x): x ? A} of subgroups of G, given by

(F, A)={( F(1) = {1}), F(-1) = {1,-1}) , (F(i) = F(-i) = {1, -1, i, -i} = G)}.

Table 4.2 below shows the tabular representation of the soft group (F,A), where

$$(x, y) = \begin{cases} 1, \text{if } y = x^n , \\ 0, \text{ if } y \neq x^n, \text{ for some } n \in \mathbb{N}. \end{cases}$$

**Definition 4.1.3**

Let (F, A) be a soft group over G. Then,

(1) (F, A) is said to be an **identity soft group** over G, if F(x)={e}, $\forall$ x $\in$ A, where e is the identity element of G.

Table 4.1: **Tabular Representation of soft group (F, A)**

| . | 1 | -1 | i | -i |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| -1 | 0 | 1 | 1 | 1 |
| i | 0 | 0 | 1 | 1 |
| -i | 0 | 0 | 1 | 1 |

(2) (F, A) is said to be an **absolute soft group** over G, if F(x)=G, $\forall$ x $\in$ A.

**Theorem 4.1.1**

Let (F,A) and (H,B) be two soft groups over G. Then, the following hold.

(i) (F,A)$\tilde{\cap}$ (H,B) is a soft group over G.

(ii) (F,A)$\cap_{\varepsilon}$(H,B) is a soft group over G.

(iii) (F,A)$\Cap$(H,B) is a soft group over G.

(iv) (F,A)$\tilde{\wedge}$ (H,B) is a soft group over G.

(v) (F,A)$\tilde{\cup}$ (H,B) is a soft group over G, if A$\cap$B=$\emptyset$.

Sezgin and Atagun,(2011) pointed out ,by providing counter examples, that some assertions in Theorem 4.1.1 (Aktas and Cagman,2007) above were incorrect, due to the consequence of the definition of intersection by Maji *et al.*, 2003, and as a result, came up with the following theorem.

**Theorem 4.1.2**

Let (F,A) and (H,B) be two soft groups over G. Then, the following hold.

(i) (F,A)$\cap_{\varepsilon}$(H,B) is a soft group over G, if it is non- null.

(ii) (F,A)$\Cap$(H,B) is a soft group over G, if it is non-null.

(iii) (F,A)$\tilde{\wedge}$ (H,B) is a soft group over G, if it is non-null.

(iv) (F,A)$\tilde{\cup}$ (H,B) is a soft group over G, if it is non-null, and if A$\cap$B=$\emptyset$.

**Proof:**

(i) Let (F,A)$\cap_{\varepsilon}$ (H,B)=(U,D),where D=A$\cup$B and $\forall$ x$\in$ supp(U,D),

$$U(x) = \begin{cases} F(x) \neq \emptyset, x \in A - B, \\ H(x) \neq \emptyset, x \in B - A, \\ F(x) \cap H(x) \neq \emptyset, x \in A \cap B. \end{cases}$$

Since F(x) $<_g$ G and H(x) $<_g$ G, we have F(x) $\cap$ H(x) $<_g$G.

Thus, U(x) $<_g$ G. Hence, (U, D) is a soft group over G.

(ii) follows from (i).

(iii) Let (F,A)$\tilde{\wedge}$ (H,B)=(K,C), where C=A$\times$B and ,$\forall$(x,y)$\in$ C; K(x,y)=F(x) $\cap$H(y).

Since F( x) $<_g$ G and H(y) $<_g$ G, we have F(x) $\cap$ H(y) $<_g$ G.

Thus, K(x,y) $<_g$G$\Rightarrow$ (K,C) is a soft group of G.

(iv) Let (F, A) $\tilde{\cup}$ (H, B) = (L, C), where C= A$\cup$B. Now, $\forall$ x $\in$ supp (L, C),

$$L(x) = \begin{cases} F(x) \neq \emptyset, x \in A - B, \\ H(x) \neq \emptyset, x \in B - A, \\ \emptyset, \text{ since } A \cap B = \emptyset. \end{cases}$$

Since F(x) $<_g$ G and H(x) $<_g$G, we have L(x) $<_g$ G.

Therefore, (L, C) in a soft group over G.

**Definition 4.1.4** Let (F, A) and (H, B) be two soft groups over G. Then (F, A) is a **soft subgroup** of (H, B), written $(F, A)\tilde{<}g(H, B)$, if

(i) A $\subseteq$ B,

(ii) F(x) is a sub group of H(x), written F(x) $<_g$ H(x), $\forall$ x $\in$ A.

**Example 4.1.2**

Let G = {0, a, b, c} be the Klein-4 group with the operation table 4.3 given below

Table 4.2: **Table of Klein-4 group G**

| . | 1 | a | b | c |
|---|---|---|---|---|
| 1 | 1 | a | b | c |
| a | a | 1 | c | b |
| b | b | c | 1 | a |
| c | c | b | a | 1 |

Let G=B and let (H, B) be a soft set over G, where H:B $\longrightarrow$ P(G) is a set-valued function defined by H(x) = {y $\in$ G: (x, y) $\Leftrightarrow$ y = $x^n$, n $\in N$}. Then, H(1)={1}, H(a)={1,a}, H(b)={1,a,b,c}=H(c) are all subgroups of G. Hence, (H B) is a soft group over G.

Let A ={1, a, b}$\subset$ B and the soft set (F ,A) over G where F:A$\longrightarrow$ P(G) is defined by F(x) = {1}$\cup${y $\in$ G: (x, y) $\Leftrightarrow$ y = $x^2$, }. Then, F(1)={1}= F(a) , F(b)={1,a} are all subgroups of G. Thus, (F, A) is a soft group over G. Hence, (F, A) $\tilde{<}_g$ (H, B), since A $\subset$ B and F(x) $<_g$ H(x), $\forall$ x $\in$ A.

**Definition 4.1.5**

Let (F, A) be a soft group over G and (H, B) be a soft subgroup of (F, A). Then we say that (H, B) is a **normal soft subgroup** of (F, A), written (H, B)$\tilde{\triangleleft}_g$(F, A), if H(x) is a normal subgroup of F(x), written H(x)$\triangleleft g$F(x), $\forall$ x $\in$ B.

**Example 4.1.3** In Example 4.1.2 above, since F(x)$\triangleleft g$H(x), $\forall$ x $\in$ A, then (F, A)$\tilde{\triangleleft}_g$(H, B)

**Definition 4.1.6: Normalistic soft group**

Let G be a group and (F, A) be a non-null soft set over G. Then (F, A) is called a **normalistic soft group** over G, if F(x) $\lhd g$ G, $\forall$ x $\in$ supp (F, A).

**Example 4.1.4**

Let G =A= $A_3$ = {e, (123), (132)} be the alternating group of $S_3$ = {e, (12), (13), (23), (123), (132)} and (F, A) be a soft set over G, where F: A$\longrightarrow$ P(G) be defined by

F(x) = {y $\in$ $A_3$ = G: x R y $\Leftrightarrow$ y $\in$ $\langle x \rangle$} $\forall$ x $\in$ A. Then F(e) = {e}, F(123) = F(132) = {e, (123), (132)}. Since F(x) $\lhd g$ G, $\forall$ x $\in$ supp (F, A), then, (F, A) is a normalistic soft group over G.

**Definition 4.1.7**

Let (F, A) be a normalistic soft group over G. Then we have that:

(a). (F, A) is said to be **trivial normalistic soft group** if F(x) = {$e_G$}, $\forall$ x $\in$ supp (F, A).

(b). (F, A) is said to be **whole normalistic soft group** if F(x) = G, $\forall$ x $\in$ supp (F, A).

**Example 4.1.5** Let G = {1, 3, 5, 7} be as given in the klein-4 group with the operation table 4.3 of Example 4.1.1.

Let A = {1, 3} and (F, A) the soft set over G, where F:A $\longrightarrow$ P(G) is a set-valued function defined by

F(x) = {y $\in$ G: xRy $\Leftrightarrow$ y = $x^2$}.Then

F(1) = F(3) = {1} = {$e_G$},$\forall$ x $\in$ supp (F, A).

Therefore (F, A) is a trivial normalistic soft group over G.

Let B = {5. 7}, and (H, B) the soft set over G, where H:B $\longrightarrow$ P(G) is a set valued

function defined by

H(x) = {y ∈ G: xRy ⇔ y = $x^n$ for some n ∈ $N$}. Then

H(5) = H(7) = {1, 3, 5,7} = G.

Hence, (H, B) is a whole normalistic soft group over G.

**Definition 4.1.8**

Let (F, A) and (H, B) be two soft groups over $G_1$ and $G_2$,respectively, and let *(f, g)* be a soft function from (F, A) to (H, B). Then;

(A) *(f, g)* is a **soft group homomorphism,** and that (F, A) is **soft group homomorphic** to (H,B)

written (F,A) $\sim_g$(H,B) if the following conditions are satisfied;

(i) $f$ is a homomorphism from $G_1$ onto $G_2$.

(ii) $g$ is a mapping from A onto B.

(B) *(f, g)* is called a **soft group isomorphism,** and that (F, A) is **soft group isomorphic** to (H,B) written (F, A)$\simeq_g$ (H, B) if the following conditions are satisfied;

(i) $f$ is an isomorphism from $G_1$to $G_2$.

(ii) $g$ is a 1-1 mapping from A onto B.

**Example 4.1.6**

Consider the groups;

($Z$, +)={....-3,-2,-1,0,1,2,3... with operation +}i.e., the set of integers with respect to addition, and

($Z_m$, ⊕)={0,1,2,3,....,m-1, with operation ⊕},i.e., the set of positive integers modulo m with addition operation

Let a homomorphism$f$ from ($Z$, +) onto ($Z_{m,}$⊕) be defined by

$f(k) = \bar{k}$ for k ∈ $(Z, +)$ and a mapping $g$ from $Z^+$ onto $Z_m$be defined by $g(k) = \bar{k}$ for k ∈ $Z^+$.

Let F: $Z^+ \rightarrow$ P$(Z, +)$ and H: $Z_m \rightarrow$ P$(Z_m \oplus)$ be defined by

F(x)= {y ∈$(Z, +)$: y=5kx, k ∈ $Z$} and H(x)={y ∈$(Z_m \oplus)$: y=xk, k ∈ $5Z$}.

Then F(x)=5x$Z$ and H(u)=$\bar{k}u$: k ∈ $5Z$.

Thus, (F, $Z^+$) and (H, $Z_m$) are soft groups over $(Z, +)$ and $(Z_m, \oplus)$, respectively, since F(x) and H(x) are subgroups of $(Z,+)$ and $(Z_m, \oplus)$, respectively.

Also $f$ (F(x)) = {$5\bar{k}x$: k ∈ $Z$} and H(g(x))={$\bar{x}k$: k ∈ $5Z$}={$5\bar{x}k$: k ∈ $Z$}

Thus, $f$ (F(x))=H(g(x)). Hence, ($f$, g) is a soft group homomorphism, and (F, $Z^+$) is soft group homomorphic to (H, $Z_m$), i.e (F,$Z^+$)$\sim_g$(H, $Z_m$)

## 4.2 Soft Rings and Soft Semirings

In this section, the definitions, examples and basic properties of soft rings, soft semi rings, their sub structures and soft ring and soft semiring homomorphism and isomorphism, much of which were introduced in(Feng *et al.*, 2008, Acar *et al.*, 2010, Celik *et al.*, 2011) are presented.

**Definition 4.2.1: Soft ring ideal**

Let R be a ring and let (F, A) be a non-null soft set over R. Then (F, A) is called a **soft ring** over R, if F(x) is a subring of R, denoted F(x) $<_r$R, ∀ x∈A.

**Example 4.2.1**

Let R=$Z_6$={0,1,2,3,4,5}, and A={0,1,2,3}.Consider the soft set (F, A) over R, where F: A→ P (R) is a set valued function defined by F(x)={y ∈R : x.y=0}. Then, F(0)=R, F(1)={0} F(2)={0,3}, F(3)={0,2,4}, which are all subrings of R. Hence, (F, A) is a soft ring over R._

**Theorem 4.2.1**

Let (F, A) and (G, B) be soft rings over R. Then:

(1) (F, A) $\tilde{\wedge}$(G, B) is a soft ring over R, if it is non-null.

(2) (F, A) $\Cap$ (G, B) is a soft ring over R, if it is non-null.

(3) (F, A) $\tilde{\cup}$(G, B) is a soft ring over R if A$\cap$B=$\emptyset$.

**Proof:**

(1) Let (F, A) $\tilde{\wedge}$ (G, B) = (H, C) where C= A$\times$B and H(x, y) =F(x) $\cap$G(y), $\forall$(x, y) $\in$ C.

Since (H, C) is non-null, we have that H(x, y) =F(x) $\cap$G(y) $\neq \emptyset$, and since F(x) and G(y) are subrings of R, H(x, y) is a subring of R. Hence, (H, C) is a soft ring over R.

(2) Let (F, A) $\Cap$ (G, B) = (H, C) where C=A$\cap$B $\neq \emptyset$ and H(x) =F(x) $\cap$G(x), $\forall$ x $\in$ C. H(x) =F(x) $\cap$G(x) $\neq \emptyset$, $\forall$ x $\in$ supp (H, C), and F(x) and G(x) are subrings of R. Therefore, H(x) $<_r$R. Hence, (H, C) is a soft ring over R.

(3) Let (F, A) $\tilde{\cup}$(G, B) = (H, C) where C= A$\cup$B and, $\forall$ x$\in$ C

$$H(x) = \begin{cases} F(x), & x \in A - B, \\ G(x), & x \in B - A, \\ \emptyset, & x \in A \cap B = \emptyset. \end{cases}$$

Now, $\forall$ x $\in$ supp (H, C), H(x) $\neq \emptyset$. Since F(x)$<_r$R and G(x)$<_r$R, we have H(x)$<_r$R. Therefore, (H, C) is a soft ring over R.

**Definition 4.2.2: Soft sub ring**

Let (F, A) and (G, B) be soft rings over R. Then, (G, B) is called a **soft subring** of (F, A), denoted (F,A)$\tilde{<}_r$(G,B), if it satisfies the following:

(i) B$\subset$A,

(ii) G(x) $<_r$ F(x), $\forall$ x $\in$ supp (G, B).

**Example 4.2.2**

Let R=A=$Z$ and B=2$Z$ $\subset$ A. Consider the set-valued functions F:A $\rightarrow$P(R) and G: B$\rightarrow$ P(R) given by F(x)=$\{$y $\in$ R: y=nx, n$\in Z\}$ and G(x)=$\{$y $\in$R: y=3nx, n$\in Z\}$. Clearly, G(x) =3x$Z$ is a subring of F(x) =x$Z$, $\forall$ x$\in$B.

Hence, (G, B) is a soft subring of (F, A).

**Definition 4.2.3 : Soft ring ideal**

Let (F, A) be a soft ring over R. A non-null soft set (I, B) over R is called a **soft ring ideal** of (F, A), denoted (I, B) $\tilde{\triangleleft}_R$ (F, A), if it satisfies the following conditions:

(1) B$\subset$A,)

(2) I(x) is an ideal of F(x), written I(x) $\triangleleft$ F(x), $\forall$ x $\in$ supp (I, B).

**Example 4.2.3**

Consider the soft set (F,A) over R in Example 4.2.1. Now let B=$\{0,1,2\}\subset$A.

Consider the soft set (I, B) over R, where the function I:B$\rightarrow$ P(R) is a set-valued function given by I(x)=$\{$y $\in$ R$|$ x. y $\in\{0,3\}$. Then, I(0)=R $\triangleleft$ F(0), I(1)=$\{0\}\triangleleft$ F(1)=$\{0,2\}$ and I(2)=$\{0\}\triangleleft$ F(2)=$\{0,3\}$. Hence, (I, B) $\tilde{\triangleleft}R$ (F, A).

**Definition 4.2.4: Idealistic soft ring**

Let (F, A) be a non-null soft set over R. Then (F, A) is called an **idealistic (or ideal) soft ring** over R, if F(x) is an ideal of R, $\forall$ x $\in$ supp (F, A).

**Example 4.2.4** In Example 4.2.3, (F, A) and (I, B) are idealistic soft ring over R, $\forall$ x $\in$ A.

**Theorem 4.2.2**

Let (F, A) and (G, B) be idealistic soft rings over R. Then

(1) (F, A) $\Cap$ (G, B) is an idealistic soft rings over R, if it is non-null.

(2) (F, A)$\tilde{\wedge}$ (G, B) is an idealistic soft ring over R, if it is non-null.

(3) (F, A)$\tilde{\cup}$(G, B) is an idealistic soft ring over R if A$\cap$B=$\emptyset$.

**Proof**:_Similar to proof of Theorem 4.2.1 in view of Definition 4.2.4 and the fact that every idealistic soft ring is a soft ring.

**Definition 4.2.5**

Let (F, A) and (G, B) be two soft rings over $R_1$ and $R_2$respectively. Let f: $R_1 \rightarrow R_2$ and g:A$\rightarrow$B. Then the pair (f,g) is called a **soft ring homomorphism** if:

(a) f is an epimorphism of rings

(b) g is a surjective mapping,

(c) f(F(x))=G(g(x)), $\forall$ x $\in$ A.

If there exist a soft ring homomorphism between (F, A) and (G, B), then (F, A) is said to be **soft ring homomorphic** to (G, B), denoted (F, A) $\sim_R$ (G, B). If f is an isomorphism of rings and g is bijective, then, (f,g) is said to be a **soft ring isomorphism,** and that (F, A) is **soft ring isomorphic** to (G, B), denoted (F, A) $\simeq_R$(G, B).

**Definition 4.2.6**

Let (F, A) be a non-null soft set over a semiring S. Then (F, A) is called a **soft semiring** over S, if F(x) is a subsemiring of S, written F(x) $<_s$ S, $\forall$ x $\in$ supp (F, A).

**Example 4.2.5**

Let S=$Z_6$=$\{0,1,2,3,4,5\}$ be the semiring of integers modulo 6. Let (F, A) be a soft set over $Z_6$, where A=$Z_6$ and F: A$\rightarrow$ P($Z_6$) is a set-valued function defined by F(x)=$\{y \in Z_6 : xRy \Leftrightarrow x.y \in \{0,2,4\}$.Then, $\forall$ x $\in$ A, F(0)=$Z_6$, F(1)=$\{0,2,4\}$, F(2)=$Z_6$, F(3)=$\{0,2,4\}$, F(4)=$Z_6$, F(5)=$\{0,2,4\}$ are all subsemirings of $Z_6$. Hence, (F, A) is a soft semiring over S.

**Definition 4.2.7**

Let (F, A) and (G, B) be soft semirings over a semiring S. Then (F, A) is called a **soft subsemiring** of (G, B), if it satisfies.

(1) A$\subset$B,

(2) F(x) $<_s$ G(x), $\forall$ x $\in$ supp (G, B).

**Theorem 4.2.3**

Let (F, A) and (G, B) be soft semirings over S. Then

(1) (F, A) $\tilde{\wedge}$(G, B) be soft semiring over S, if it is non-null.

(2) (F, A) $\Cap$ (G, B) is a soft semiring over S, if it is non-null.

(3) (F, A)$\tilde{\cup}$ (G, B) is a soft semiring over S if A$\cap$B=$\emptyset$.

**Proof:** This can be obtained using the same arguments in Theorem 4.2.1 in view of Definitions 4.2.6 and the fact that every soft ring is a soft semiring.

**Definition 4.2.8**

Let (F, A) be a soft semiring over a semiring S.

A non-null soft set (G, B) over S is called a **soft semiring ideal** of (F, A), denoted (G, B)$\tilde{\lhd}_S$(F, A), if it satisfies the following conditions

(i) B$\subset$A,

(ii) G(x) is an ideal of F(x), $\forall$ x $\in$ supp (G, B).

**Example 4.2.6**

Let (F, A) be a soft set over S in Example 4.2.5. Let (G, B) be a soft set over $Z_6$, where B=$\{0,1,2\}$ and G: B$\rightarrow$ P($Z_6$) is a set-valued function defined by G(x)=$\{$y$\in$ $Z_6$ : xRy $\Leftrightarrow$ xy=0$\}$, $\forall$ x $\in$ B, then G(0)=$Z_6\lhd$ F(0), G(1)=$\{0,2,4\}$= F(1) and G2)=$\{0,3\}$ $\lhd$ $Z_6$=F(2). Hence, (G, B) $\tilde{\lhd}S$ (F, A).

**Remark 4.2.1:**Since every ideal of a semiring is a subsemiring, It is clear that

every soft ideal of a soft semiring over a semiring S is a soft subsemiring of the soft semiring, but the converse is not generally true.

**Definition 4.2.9**

Let (F, A) be a non-null soft set over S. Then (F, A) is called an **idealistic soft semiring** over S, if F(x) is an ideal of S, $\forall$ x $\in$ supp (F, A).

**Example 4.2.7** Consider the product semiring S=R$\times$ $Z$, where R is a semiring with zero and $Z$ is the semiring of integers.

Let F: S $\rightarrow$P(S) be a set-valued function defined by

$$F(x, n)= \begin{cases} R \times n\mathbb{Z}, \text{ if } n \in \mathbb{N}, \text{ the set of all non} - \text{negative integers} \\ \{(0,\ 0)\} \text{ otherwise } \forall (x,\ n) \in S. \end{cases}$$

Then (F, S) is an idealistic soft semiring over S since F(x, n) is an ideal for every (x, n) $\in$ S.

## 4.3 Soft Lattices

This section introduces soft lattice, soft sub lattice, soft lattice ideal, soft distributive lattices, soft modular lattices, soft lattice homomorphism and their basic algebraic properties. Much of the definitions and facts in this section can be found in

( Nagarajan and Meenanbigai, 2011)

**Definition 4.3.1**

Let L be a lattice and let (F, A) be a non null soft set over L. Then (F, A) is said to be a **soft lattice** over L, if F(x) is a sublattice of L, denoted F(x) $<_L$ L, $\forall$ x $\in$ A.

**Example 4.3.1**

Consider the lattice L={0,1,a,b,c,d,e,f; $\wedge$ }, whose Cayley Table is shown in Tables 4.4 below.

Table 4.3: **Cayley Table of (L,∧)**

| ∧ | 0 | 1 | a | b | c | d | e | f |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | a | b | c | d | e | f |
| a | 0 | a | a | 0 | 0 | a | a | 0 |
| b | 0 | b | 0 | b | 0 | b | 0 | b |
| c | 0 | c | 0 | 0 | c | o | c | c |
| d | 0 | d | a | b | 0 | d | a | b |
| e | 0 | e | a | 0 | c | a | e | c |
| f | 0 | f | 0 | b | c | b | c | f |

Let A = {0, a, b, d}. Define the set-valued function F: A→ P(L) by F(x)=(y ∈ L: xRy⇔ x∧y=0,∀ x ∈ A .Then, F(0)=L, F(a)={0,b,c,f}, F(b)={0,a,c,e}, F(d)={0, c}. Therefore, F(x) $<_L$ L, ∀ x ∈ A. Hence, (F, A) is a soft lattice over L.

**Proposition 4.3.1**

Let (F, A) and (G, B) be soft lattices over a lattice L. Then the following hold.

(1) If $A \cap B \neq \emptyset$ and $F(e) \cap G(e) \neq \emptyset$ for all e ∈A∩B, then (F, A) ⋒ (G, B) is a soft lattice over L.

(2) If $A \cap B = \emptyset$, then (F, A)$\tilde{\cup}$ (G, B) is a soft lattice over L.

(3) If $F(x) \cap G(y) \neq \emptyset$ for all (x, y) ∈ A×B, then (F, B)$\tilde{\wedge}$(G, B) is a soft lattice over L.

**Proof**:This is similar to the proof of Theorem 4.2.1 in view of Definition 4.3.1

**Definition 4.3.2**

Let (F, A) and (G, B) be two soft lattices. Then (G, B) is called a **soft sub lattice** of (F, A), denoted (G,B) $\tilde{<}_L$(F,A), if

(1) B $\subseteq$ A,

(2) G(x) is a sub lattice of F(x), $\forall$ x $\in$ B.

**Example 4.3.2**

Let us consider the lattice L={0,1,a,b,c,d,e; $\vee$, }, whose Cayley Table is shown in Table 4.5 below:

Let A={0, a, b, c, d, 1) and B={0, b, c, d}. Define the set-valued function

Table 4.4: **Cayley Table of (L, $\vee$)**

| $\vee$ | 0 | 1 | a | b | c | d | e |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | a | b | c | d | e |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| a | a | 1 | a | e | 1 | 1 | e |
| b | b | 1 | e | b | d | d | e |
| c | c | 1 | 1 | d | c | d | 1 |
| d | d | 1 | 1 | d | d | d | 1 |
| e | e | 1 | e | e | 1 | 1 | e |

$F : A \to P(L)$ by F(x)={y $\in$ L : xRy $\Leftrightarrow$ x$\vee$y=x}, then F(0)={0}, F(a)={0, a}, F(b) ={0, b}, F(c)={0, c}, F(d)={0, b, c, d}, F(1)={0, a, b, c, d, 1}. Define the set-valued function G:B$\to$ P(L) by G(x) = {y $\in$ L : xRy $\Leftrightarrow$ x $\vee$ y=x}, then G(0)={0} G(b)={0, b}, G(c)={0, c}, G(d)={0, b, c, d}. Therefore, F(x) is a sublattice of L, $\forall$ x $\in$ A and G(x) is a sub-lattice of L, $\forall$ x $\in$ B. Hence, (F, A) and (G, B) are soft lattices over L. Here B$\subset$A and G(x) is a sub lattice of F(x), $\forall$ x $\in$ B. Hence, (G, B) is a soft sub lattice of (F, A).

**Definition 4.3.3**

Let (F, A) be a soft set over L. Then (F, A) is said to be a **soft distributive lattice**

over L, if F(x) is a distributive sub lattice of L, denoted F(x) $< _d$L, $\forall$ x $\in$ A.

**Example 4.3.3** Consider the Lattice L={0,1,a,b,c; $\wedge$} shown in table 4.6 below;

Table 4.5: **Cayley Table of (L, $\wedge$)**

| $\wedge$ | 0 | 1 | a | b | c |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | a | b | c |
| a | 0 | a | a | a | a |
| b | 0 | b | a | b | b |
| c | 0 | c | a | b | c |

Let A= {a, b, c}. Define the set-valued function F: A $\rightarrow$ P(L) by F(x)={y $\in$ L: xRy$\Longleftrightarrow$ x$\wedge$y = y} . Then, F(a)={0, a}, F(b)={0, a, b}, F(c)={0, a, b, c}. Thus, F(x) is a distributive sub lattice of L, $\forall$ x $\in$ A. Hence, (F, A) is a soft distributive lattice over L.

**Definition 4.3.4**

Let (F, A) be a soft set over L. Then (F, A) said to be a **soft modular lattice** over L, if F(x) is a modular sub lattice of L denoted F(x) $< _m$L, $\forall$ x $\in$ A.

**Example 4.3.4**

Let us consider the lattice L, shown in Example 4.3.1Table 4.4 above. Let A= {0, a, b, c, 1}. Define the set-valued function F: A$\rightarrow$ P(L) by F(x) ={y $\in$ L: xRy $\Longleftrightarrow$ x$\wedge$y=0. Then, F(0)=L, F(a)={0, b, c, f}, F(b)={0, a, c, e} F(c)={0, a, b, d}. Thus, F(x) $<_m$ L, $\forall$ x $\in$ A. Hence, (F, A) is a soft modular lattice over L.

**Proposition 4.3.2**

Let L be a modular (respectively distributive) lattice. Then every soft lattice over L is a soft modular (respectively distributive) lattice over L.

**Proof:**Let (F, A) be a soft lattice over L. Then, F(x) $<_m$ L(resp. F(x) $<_d$L), $\forall$ x $\in$ A. Since L is modular (resp. distributive) and every sublattice of modular (resp.distributive) lattice is modular(resp.distributive), then, F(x) $<_m$ L(resp. F(x) $<_d$L), $\forall$ x $\in$ A. Hence, (F, A) is a soft modular(resp.distributive) lattice over L.

**Definition 4.3.5**

Let (F, A) be a soft lattice over L. A soft set (G, B) over L is called a **soft lattice ideal** of (F, A), denoted (G, B)$\tilde{\lhd}_L$(F, A), if

(1) B$\subseteq$A,

(2) G(x) is an ideal of F(x), $\forall$ x $\in$ B. Let us illustrate this definition using the following example:

**Example 4.3.5** Consider the lattice L shown in Table 4.7 below:

Table 4.6: **Cayley Table of (L, $\vee$)**

| $\vee$ | 0 | 1 | a | b | c |
|---|---|---|---|---|---|
| 0 | 0 | 1 | a | b | c |
| 1 | 1 | 1 | 1 | 1 | 1 |
| a | a | 1 | a | b | 1 |
| b | b | 1 | b | b | 1 |
| c | c | 1 | 1 | 1 | c |

Let A= {0, 1, a, b, c, }and the set-valued function F:A$\rightarrow$ P(L) be defined by $F(x) = \{y \in L : xRy \Leftrightarrow x \vee y = x\}$. Then, F(0)= {0}, F(a)={0, a}, F(b)={0, a, b}, F (c)={0, c}, F(1)={0, b, c, 1}. Hence, (F, A) is a soft lattice over L. Let B= {a, b, c}and let I:B$\rightarrow$ P(L) be defined by I(x) ={y $\in$ L: xRy$\Leftrightarrow$ x$\vee$y=x}. Then, I(a)={0, a}, I(b)={0, a, b}, I(c)={0, c}, I(b)={0, a, b}, I(c)={0, c}. Thus ,(I, B) is a soft

set over L. Since B⊆A and I(x) is an ideal of F(x), ∀ x ∈ B, hence, (I, B) is a soft lattice ideal of (F, A).

**Definition 4.3.6**

Let (F, A) and (G, B) be two soft lattices over $L_1$ and $L_2$ respectively. Let f:$L_1 \to L_2$ and g: A → B. Then 1. (f,g) is said to be a **soft lattice homomorphism** if:

(a) f is a lattice epimorphism,

(b) g is a surjective mapping,

(c) f(F(x))=G(g(x)), ∀ x ∈ A.

2. (F, A) is said to be **soft lattice homomorphic** to (G, B), denoted (F, A)$\tilde{}_L$(G, B). If f is a lattice isomorphism and g is bijective, then, (f, g) is said to be a **soft lattice isomorphism,** and that (F, A) is **soft lattice isomorphic** to (G, B), denoted (F, A) $\simeq_L$(G, B).

**Example 4.3.6** Let $L_1$ and $L_2$ be two lattices shown in Tables 4.8 and 4.9 below respectively.

Table 4.7: **Cayley Table of (L, ∨)**

| ∨ | 0 | 1 | a | b |
|---|---|---|---|---|
| 0 | 0 | 1 | a | b |
| 1 | 1 | 1 | 1 | 1 |
| a | a | 1 | a | 1 |
| b | b | 1 | 1 | b |

Table 4.8: **Cayley Table of (L, ∧)**

| ∧ | $0^/$ | $1^/$ |
|---|---|---|
| $0^/$ | $0^/$ | $1^/$ |
| $1^/$ | $1^/$ | $1^/$ |

Let A= {0, 1, a, b}, and the set-valued function F:A $\rightarrow$ P($L_1$) be defined by F(x)={y $\in$ $L_1$:xRy$\Leftrightarrow$ x$\vee$y=x}, $\forall$ x $\in$ A. Then, F(0)= {0}, F(a)={0, a}, F(b)={0, b}, F (c)={0, c}, F(1)={0,a, b, 1}. Hence, (F, A) is a soft lattice over $L_1$.

Let B={$0^{/}$,$1^{/}$} and the set-valued function G:B $\rightarrow$ P($L_2$) be defined by G(x)={y $\in$ L: xRy$\Longleftrightarrow$ x$\wedge$y = y}. Then, G($0^{/}$)={$0^{/}$}, G($1^{/}$)={$0^{/}$,$1^{/}$}. Hence, (G, B) is a soft lattice over $L_2$.

Let f:$L_1$ $\rightarrow$ $L_2$ be defined by f(0)= $0^{/}$ , f(a)= $0^{/}$ , f(b)= $1^{/}$ , f(1)= $1^{/}$, and g:A $\rightarrow$ B be defined by g(0)= $0^{/}$ , g(a)= $0^{/}$ , g(b)= $1^{/}$ , g(1)= $1^{/}$ .Then, f is a lattice epimorphism and g is a surjective mapping.Also, f(F(x))=G(g(x)), $\forall$ x $\in$ A. Hence, (f, g) is a soft lattice homomorphism and (F, A) is soft lattice homomorphic to (G, B).

## 4.4 Monoids, Semirings and Lattices of Soft subsets and Partitions of a soft set

In this section, monoids, semirings and lattices of soft subsets and partitions of a soft set, with respect to various operations, are formulated. This is our main contribution to the study of algebraic structures of soft sets.

Let us recall the following definitions:

**Definition 4.4.1**

A collection $\tilde{P}$ of non-empty pair wise disjoint soft subsets of a soft set (F, A) whose union

is (F,A), is called a **partition** of (F,A). In other words, a collection $\tilde{P}$ = {($F_i$, $A_i$), i $\in$ I} is called a partition of (F, A), if

(i) (F, A) $=\bigcup$i$\in$I ($F_i$, $A_i$), and

(ii) $A_i \cap A_j = \emptyset$ for i $\neq$j.

The elements of $\tilde{P}$ are called the *blocks* or *cells* of $\tilde{P}$.

**Example 4.4.1**

Let $U = \{u_1, u_2, u_3, u_4, u_5\}$, and $A = \{a_1,\ a_2,\ a_3,\ a_4\}$

Let (F, A) = {F($a_1$), F($a_2$), F($a_3$), F($a_4$)} such that

$F(a_1) = \{u_1, u_3\}$,

$F(a_2) = \{u_3\}$,

$F(a_3) = \{u_3, u_4\}$,

and $F(a_4) = \{u_4, u_5\}$.

Now let $A_1 = \{a_1,\ a_2\}$ *and* $A_2 = \{a_3,\ a_4\}$ such that

$(F_1,\ A_1) = \{F_1(a_1)\ ,\ F_1(a_2)\}$ *and*

$(F_2,\ A_2) = \{F_2(a_3)\ ,\ F_2\ (a_4)\}$ *, where*

$F_i(a) = F(a)\ \forall\ i = 1, 2.$

Then, $(F,\ A) = (F_1,\ A_1)\ \tilde{\cup}\ (F_2,\ A_2)$ *and* $A_1 \cap A_2 = \emptyset$.

Thus, {($F_1$, $A_1$), ($F_2$, $A_2$)} is a soft set partition of (F, A), where ($F_1$, $A_1$) and ($F_2$, $A_2$) are the blocks of $\tilde{P}$.

**Definitions 4.4.2** An algebraic structure (S, $*$) consisting of a non-empty set S with an associative binary operation $*$ is called a **semigroup**. If there exists an element e in (S, $*$) such that e$*$x = x$*$e = x for all x $\in$ S, then S is called a **monoid** with e as the identity element. If for every element x $\in$ S, x$*$x = x, then S is called **idempotent**.

An algebraic structure (R, +,.) consisting of a non-empty set R together with two binary operations, usually called addition (+) and multiplication (.), is called a

**semiring** if the following conditions are satisfied:

1. (R, +) is a commutative monoid,

2. (R, .) is a monoid, and

3. Multiplication distributes over addition from both sides i.e., a(b + c ) = ab + ac and (b + c)a = ba + ca.

A **lattice** (L,$\vee, \wedge$) is a non-empty set L, with two binary operations $\vee$ and $\wedge$ such that

(i) (L , $\vee$) is a commutative, idempotent semigroup.

(ii) (L , $\wedge$) is a commutative, idempotent semigroup.

(iii) Absorption properties hold.

If a lattice has identity elements with respect to both the operations, then it is called **bounded**.

If distributive properties hold in a lattice, then it is called a **distributive lattice**.

### 4.4.1 Commutative, idempotent monoids and semirings of Soft Subsets of a Soft Set

Let U be an initial universe, A, a parameter set with respect to U, and (F, A) a soft set over U.

Let SS(F, A) denote the collection of all soft subsets of a soft set (F, A).

It is immediate to see from Proposition 3.2.1: 1, 10 and 11 that (SS(F, A), $*$) are commutative, idempotent semigroups for $* \in \{\tilde{\cup}, \cup_R, \tilde{\cap}, \cap_R\}$.

**A. Monoids**

1. (SS(F,A), $* \in \{\tilde{\cup},\tilde{\cap}\}$) are commutative, idempotent monoids with $\tilde{\Phi}_{\emptyset}$ as the identity element.

2. (SS(F,A), $\cup_R$) is a commutative, idempotent monoid with $\tilde{\Phi}_A$ as the identity element.

3. (SS(F,A), $\cap_R$) is a commutative, idempotent monoid with (F, A) as the identity element.

**B. Semirings**

1. (SS(F,A), $\tilde{\cup}$,$\cup_R$) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$ .

2. (SS(F,A), $\cap_R$, $\tilde{\cup}$) is a commutative, idempotent semiring with identity element (F,A).

3. (SS(F,A), $\cup_R$,$\tilde{\cap}$) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_A$ .

4. (SS(F,A), $\tilde{\cap}$,$\cup_R$ ) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$ .

5. (SS(F,A), $\cup_R$, $\cap_R$) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_A$ .

6. (SS(F,A), $\cap_R$,$\cup_R$ } ) is a commutative, idempotent semiring with identity element (F,A).

7. (SS(F,A), $\tilde{\cap}$, $\cap_R$ ) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$ .

8. (SS(F, A), $\tilde{\cup}$, $\cap_R$} ) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$ .

[Proofs for A and B follow directly from definitions and properties of the respective operations given in Proposition 3.2.1]

**Remark 4.4.1**

1. (SS(F, A),$\tilde{\cap}$,$\tilde{\cup}$) and (SS(F,A), $\tilde{\cup}$ ,$\tilde{\cap}$) are not semirings since $\tilde{\cap}$ and $\tilde{\cup}$ do not distribute over each other( see Proposition 3.2 2(i and ii))

2. (SS(F, A), $\cup_R$,$\tilde{\cup}$) and (SS(F,A), $\cap_R$ ,$\tilde{\cap}$) are not semirings since $\tilde{\cap}$ and $\tilde{\cup}$ do not distribute over $\cap_R$ and $\cup_R$, respectively (see Proposition 3.2.2(vii and ix)).

### 4.4.2 Bounded, distributive lattices of soft subsets of a soft set

Let SS(F, A) be as defined in 4.4.1 above.

**Proposition 4.4.1**

(i) (SS(F, A), $\tilde{\cap}$,$\cup_R$ ) and (SS(F, A), $\cup_R$,$\tilde{\cap}$), and

(ii) (SS(F, A), $\cap_R$, $\tilde{\cup}$) and (SS(F, A), $\tilde{\cup}$, $\cap_R$)

are bounded, distributive lattices.

**Proof**

(i) (SS(F, A), $\tilde{\cap}$) and (SS(F, A), $\cup_R$ ) are commutative, idempotent semi groups with $\tilde{\Phi}_{\emptyset}$ and $\tilde{\Phi}_A$ as the identity elements, respectively(see 4.4.1 above). Moreover $\tilde{\cap}$ and $\cup_R$ distribute and absorb over each other [see Proposition 3.2.1:12 (iii) (iv) and 9(iii),(iv), respectively]. Hence, (i) holds

(ii)Similarly, (SS(F, A), $\tilde{\cup}$) and (SS(F, A), $\cap_R$) are commutative, idempotent semi groups with $\tilde{\Phi}_{\emptyset}$ and (F,A) as the identity elements respectively. Also $\tilde{\cup}$ and $\cap_R$ distribute and absorb over each other [see Proposition 3.2.1:12 (i) (ii) and 9(i) (ii), respectively]. Hence, (ii) holds.

**Remark 4.4.2**

1. (SS(F, A),$\tilde{\cap}$,$\tilde{\cup}$) and (SS(F, A),$\tilde{\cup}$ ,$\tilde{\cap}$) are not lattices since $\tilde{\cap}$ and $\tilde{\cup}$do not distribute and absorb over each other[ see Proposition 3.2.2(i, ii) and 3.2.3(iii, iv) ].

2. (SS(F, A), $\cup_R$, $\cap_R$) and (SS(F,A), $\cap_R$ ,$\cup_R$) are not lattices since $\cup_R$, and $\cap_R$ do not absorb over each [see Propositions 3.2.3(i ,ii)].

### 4.4.3 Commmutative, idempotent monoids and semirings of Partitions of a Soft Set

Let $\tilde{P} = \{(F_i, A_i)\ i \in I\}$ be a partition of a soft set (F, A) and let $\overline{P}_k$ denote the block of a partition.

Let $\wp$ (F, A) denote the collection of all partitions of (F, A).

We define a binary operation $\sqcap$ on $\wp$ (F, A) as follows:

Given $\tilde{P}_1$,$\tilde{P}_2 \in \wp$(F, A), let $\tilde{P}_1 \sqcap \tilde{P}_2$ be the soft set consisting of non-empty restricted intersection

($\cap_R$) of every block of $\tilde{P}_1$ with every block of $\tilde{P}_2$.

Clearly, the operation $\sqcap$ on $\wp$(F, A) is both associative and commutative, since the operation $\cap_R$ on (F, A) is both associative and commutative and also every element $\tilde{P}_k \in \wp$ (F, A) is idempotent with respect to $\sqcap$ i.e. $\tilde{P}_k \sqcap \tilde{P}_k = \tilde{P}_k$.

Moreover, the identity element with respect to $\sqcap$ is the partition with the single block.

Thus ( $\wp$(F, A), $\sqcap$) is a commutative, idempotent monoid.

Similarly, if we define a binary operation $\sqcup$ on $\wp$(F, A) such that every resulting soft set consists of non-empty restricted union $\cup_R$ of every block of $\tilde{P}_1$ with every block of $\tilde{P}_2$, for all $\tilde{P}_1$, $\tilde{P}_2 \in \wp$(F, A), then ($\wp$(F, A) $\sqcup$) also gives rise to a commutative,

idempotent monoid with the partition consisting of singleton blocks as the identity element.

Furthermore the following collections give rise to commutative, idempotent semirings:

1. ($\wp$(F, A), $\sqcup$, $\sqcap$) with the partition consisting of the single block as the identity element.

2. ($\wp$(F, A), $\sqcap$,$\sqcup$) with the partition consisting of the singleton blocks as the identity element.

**Example 4.4.2**

Let $U = \{u_1, u_2, u_3, u_4\}$ and A = {$a_1$, $a_2$, $a_3$}. Let (F, A) be a soft set over $U$ given by

(F, A) = { F($a_1$) = $\{u_1, u_2\}$, F($a_2$) = $\{u_3\}$, F($a_3$) = $\{u_3, u_4\}$}; Now let $A_1$={$a_1$}, $A_2$={$a_2$}, $A_3$={$a_3$};

($F_1$,$A_1$)={$F_1(a_1)$}, ($F_2$,$A_2$)={$F_2(a_2)$}, ($F_3$,$A_3$)={$F_3(a_3)$} , where $F_i$ is the restriction of F to $A_i$ , i=1,2,3.

It is easy to verify that $\left\{\overline{(F_1, A_1)}, \overline{(F_2, A_2)}, \overline{(F_3, A_3)}\right\}$, is a partition of (F,A). In other words, $\left\{\overline{F(a1)}, \overline{F(a2)}, \overline{F(a3)}\right\}$ is a partition of (F,A).

Similarly, let $B_1$= {$a_1$,$a_2$} ,$B_2$={$a_3$};

($F_1$,$B_1$) ={$F_1(a_1)$ ,$F_1(a_2)$}, ($F_2$,$B_2$) = {$F_2(a_3)$} where $F_i$ is the restriction of F to $B_i$ , i=1,2.

Then $\left\{\overline{F(a1),\ F(a2)},\ \overline{F(a3)}\right\}$ is another partition of (F, A).

Thus the collection $\wp$(F, A) of all partitions of (F, A) can be given as

$\wp$ (F, A) =$\left\{\tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4, \tilde{P}_5\right\}$, where

$$\tilde{P}_1 = \left\{\overline{F(a1)}, \overline{F(a2)}, \overline{F(a3)}\right\},$$

$$\tilde{P}_2 = \left\{\overline{F(a1),\ F(a2)},\ \overline{F(a3)}\right\},$$

$$\tilde{P}_3 = \left\{\overline{F(a1)},\ \overline{F(a2), F(a3)}\right\},$$

$\tilde{P}_4 = \left\{\overline{F(a1)\ F(a3)},\ \overline{F(a2)}\right\}$, and

$$\tilde{P}_5 = \overline{\{F(a1),\ F(a2),\ F(a3)\}}.$$

It is not hard to see that $\tilde{P}_1 \sqcap\ \tilde{P}_2 = \left\{\overline{F(a1)}, \overline{F(a2)}, \overline{F(a3)}\right\}$ ,

$\tilde{P}_2 \sqcap \tilde{P}_2 = \left\{\overline{F(a1),\ F(a2)},\ \overline{F(a3)}\right\} = \tilde{P}_2$, and $\tilde{P}_1 \sqcap\ \tilde{P}_2 = \tilde{P}_2 \sqcap \tilde{P}_1$. As an illustration, let us consider Example 3.1:

Let $\tilde{P}_1 = \left\{\overline{(F_1, A_1)}\ , \overline{(F_2, A_2)},\ \overline{(F_3, A_3)}\right\} = \left\{\overline{(F, A_1)}\ , \overline{(F, A_2)}\ ,\ \overline{(F, A_3)}\right\}$, and

$\tilde{P}_2 = \left\{\overline{(F_1, B_1)}\ , \overline{(F_2, B_2)}\right\} = \left\{\overline{(F, B_1)}\ , \overline{(F, B_2)}\right\}$.

Then, $\tilde{P}_1 \sqcap \tilde{P}_2 =$ $\{(F,\ A_1) \cap_R (F, B_1)\ , (F, A_1) \cap_R (F, B_2)\ ,\ (F, A_2) \cap_R (F, B_1)\ ,$
$(F, A_2) \cap_R (F, B_2)\ ,\ (F, A_3) \cap_R (F, B_1)\ ,\ (F, A_3) \cap_R\ (F, B_2)\}\ .$

$$= \left\{\overline{F(a1)}, \overline{F(a2)}, \overline{F(a3)}\right\}.$$

Similarly, $\tilde{P}_2\ \sqcap\ \tilde{P}_1 = \left\{\overline{F(a1)}, \overline{F(a2)}, \overline{F(a3)}\right\}$ .

Also, $\tilde{P}_2 \sqcap\ \tilde{P}_2 = \{(F, B_1) \cap_R (F, B_1)\ ,\ (F, B_1) \cap_R (F, B_2)\ ,\ (F, B_2) \cap_R (F, B_1)\ ,$

$(F, B_2) \cap_R (F, B_2)\}$

$$= \left\{\overline{F(a1),\ F(a2)},\ \overline{F(a3)}\right\} = \tilde{P}_2.$$

Similar results hold for all other combinations.

Thus ($\wp$ (F, A), $\sqcap$) is a commutative, idempotent monoid with $\overline{\{F(a1),\ F(a2),\ F(a3)\}}$ as the identity element. Similarly, the other structures described above can be illustrated.

### 4.4.4 Bounded, distributive lattices of partitions of a soft set

Let $\wp$ (F, A) be as defined in 4.4.3 above.

We define a binary operation $\sqcap_R$ on $\wp$ (F, A) as follows:

Given $\tilde{P}_1,\ \tilde{P}_2 \in \wp$(F, A), let $\tilde{P}_1 \sqcap_R \tilde{P}_2$ be the soft set consisting of non-empty restricted intersections of every block of $\tilde{P}_1$ with every block of $\tilde{P}_2$.

Similarly, let the operation $\sqcup_E$ on $\wp$(F, A) be defined as: $\tilde{P}_1 \sqcup_E \tilde{P}_2$ be the soft set consisting of union of every block of $\tilde{P}_1$ with every block of $\tilde{P}_2$, for all $\tilde{P}_1, \tilde{P}_2 \in \wp$(F, A).

**Proposition 4.4.2**

(i) ($\wp$(F, A), $\sqcup_E$, $\sqcap_R$),

(ii) ($\wp$(F, A), $\sqcap_R$, $\sqcup_E$),

(iii) ($\wp$(F, A), $\sqcup$, $\sqcap$) and (iv) ($\wp$(F, A), $\sqcap$ , $\sqcup$) are bounded , distributive lattices.

**Proof**

Clearly, the operations $\sqcap_R$and $\sqcup_E$ on $\wp$(F, A) are both associative and commutative, since the operations $\cap_R$ and $\tilde{\cup}$ on (F, A) are both associative and commutative [see Proposition 3.2.1:10(i ,iv and 11(ii, iv), respectively]. Also, every element $\tilde{P}_k \in \wp$ (F, A) is idempotent with respect to $\sqcap_R$ and $\sqcup_E$ i.e., for example $\tilde{P}_k \sqcap_R \tilde{P}_k = \tilde{P}_k$, since both $\cap_R$ and $\tilde{\cup}$ are idempotent [see Propositions 3.2.1 :1(i, ii)].Hence, ($\wp$(F, A), $\sqcap_R$) and($\wp$(F, A), $\sqcup_E$) are commutative, idempotent semigroups. Also $\sqcap_R$and $\sqcup_E$ absorb over each other [see Propositions 3.2.1:9(i, ii) ]. Thus ($\wp$(F, A), $\sqcup_E$, $\sqcap_R$) and ($\wp$(F, A), $\sqcap_R$, $\sqcup_E$) give rise to lattices. Moreover, the identity elements with respect to $\sqcap_R$ and $\sqcup_E$ are the partition consisting of the single block and the partition consisting of the singleton blocks, respectively. Furthermore, $\sqcap_R$ and $\sqcup_E$ distribute over each

other [see Proposition 3.2.1:12(i, ii)]. Hence, (i) and (ii) hold.

Similarly, (iii) and (iv) can be proved.

# CHAPTER 5

# APPLICATIONS OF SOFT SET THEORY IN DECISION MAKING

A number of researchers [Maji *et al.*,(2002), Chen *et al.*, (2005), Cagman and Enginoglu,(2010), Mitra et al.,(2012), Samsiah and Mohamad,(2012)] etc., have applied the concepts of soft set theory to real life problems involving uncertainties. Such problems involve decision making, medical diagnosis, forecasting, among others. In this chapter, some typical applications using different approaches, especially in decision making, is discussed, and compared. The parameter reduction in soft set theory which was employed by some researchers is first described.

## 5.1 Parameter /Attribute Reduction in Soft Set Theory

Various techniques for parameter reduction in soft set theory have been formulated by some researchers [Chen *et al.*,(2005), Herawan *et al.*,(2010),Kong *et al.*,(2008)]etc.

In this section, the parameterization reduction of soft sets in (Chen *et al.*,2005) and the attribute reduction of soft sets using rough set approach in (Pawlak,1982) is described.

### 5.1.1 Parameterization reduction of soft set

**in [Chen *et al.*, 2008]**

Suppose $U=\{h_1,h_2,\ldots,h_n\}$ is a universe set, $E=\{e_1,e_2,\ldots,e_m\}$, a set of parameters with respect to $U$, and $(F,E)$ a soft set with tabular representation defined by $f_E(h_i)=\sum_j h_{ij}$ where $h_{ij}$ are the entries in the table of $(F,E)$. Denote $M_E$ as the collection of objects in $U$ which takes the max value of $f_E$. For every $A\subset E$, if $M_{E-A}=M_E$, then $A$ is called a *dispensable* set in $E$, otherwise $A$ is called an indispensable set in $E$. Roughly speaking, $A\subset E$ is dispensable means that the difference among all objects according to the parameters in $A$ does not influence the final decision. The parameter set $E$ is called *independent* if every proper subset of $E$ is indispensable, otherwise $E$ is dependent. $B\subseteq E$ is called a *reduction* of $E$ if $B$ is independent and $M_B=M_E$, i.e., $B$ is the minimal subset of $E$ that keeps the optimal choice objects invariant. Clearly, after the reduction of the parameter set in $E$, we have less parameters and the optimal choice objects have not been changed.

**Example 5.1.1:**

Consider a soft set *(F,E)* with the tabular representation displayed in Table5.1

Table 5.1: **Reduct - Soft Set (F, E)**

| $\bigcup$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | Choice value $f_E(.)$ |
|---|---|---|---|---|---|---|
| $h_1$ | 1 | 1 | 1 | 1 | 1 | 5 |
| $h_2$ | 1 | 1 | 1 | 1 | 0 | 4 |
| $h_3$ | 1 | 0 | 1 | 1 | 1 | 4 |
| $h_4$ | 1 | 0 | 1 | 1 | 0 | 3 |
| $h_5$ | 1 | 0 | 1 | 0 | 0 | 2 |
| $h_6$ | 1 | 1 | 1 | 1 | 1 | 5 |

Clearly, $f_E(h_1)= f_E(h_6)=5$ are the maximum choice value. Thus $h_1$ and $h_6$ are the optimal choice objects. Now if we delete $e_1$, $e_3$ and $e_4$ from $E$, then the optimal choice objects are unchanged. That is $M_E=M_{(E-\{e1,\ e3,\ e4\})}$

Hence the subset $B=\{ e_2, e_5\} \subset E$ is the reduct of $E$ which is the minimal parameter set that keeps the optimal choice objects invariant.

### 5.1.2 Attribute reduction of soft sets using rough set approach

**Definition 5.1.1**

A pair *(U , A)* is called a **knowledge representation system** where $U$ is a non-empty finite set of objects and A is a non-empty finite set of attributes.

Each subset of attribute $B \subseteq A$ determines a binary relation denoted *IND(B)* called

an **indiscernibility relation**, defined by *IND(B)*={(x,y) $\in$ $U \times U$ : a(x)=a(y), $\forall$ $a \in B$}

Obviously, *IND(B)* is an equivalence relation and so induces a partition of $U$ which is denoted by $U/B$.

A soft set can be expressed as a knowledge representation system, where the word attribute is replaced by the word parameter.

**Definition 5.1.2**

Suppose $U$=$\{h_1,h_2,\ldots,h_n\}$ is a universe set, $E$=$\{e_1,e_2,\ldots,e_m\}$, a set of parameters with respect to $U$, and $(F,E)$ a soft set with tabular representation defined by $f_E(h_i)$=$\sum_j h_{ij}$ where $h_{ij}$ are the entries in the table of $(F,E)$. With every subset of parameters $B \subseteq E$, an indiscernibility relation $IND(B)$ is defined by $IND(B)$=$\{(h_i,h_j) \in U \times U : f_B(h_i)=f_B(h_j)\}$.A subset $P \subset E$ is said to be **dispensable** in $E$ if $IND(E)=IND(E-\{P\})$,otherwise it is indispensable in $E$.

A subset $Q \subset E$ is called the **reduct** of $E$ if

(a) $IND(Q)=IND(E)$,

(b) $U/Q = U/E$, that is to say, $Q$ is the minimal subset of $E$ that keeps the classification ability of the indiscernibility relation.

**Example 5.1.2:**

Let us consider the soft set in Table5.1 above. The partition $U/E$ induced by *IND (E),* is given by $U/E = \{\{h_1,h_6\}, \{h_2\},\{h_3\}, \{h_4\}, \{h_5\}\}$.

Now, if we delete $\{e_1, e_3\}$ from $E$,then the indiscernibility relation and the induced partition are unchanged. That is $U/E = U/(E-\{e_1, e_3\})$. So both $e_1$ and $e_3$ are dispensable in $E$ by Definition 5.1.2. Also, if we delete one of the remaining set $\{e_2,e_4, e_5\}$ from $E$, then the indiscernibility relation and the induced partition would

be changed. Thus all of these three parameters are indispensable. Hence, the subset $Q$= {$e_2$,$e_4$, $e_5$} is the reduct of $E$, which is the minimal parameter (attribute) set that retains the classification ability of the indiscernibility relation.

**Remark 5.1.1** Examples 5.1.1 and 5.1.2 above confirm that the reduction of parameters in soft set theory and the reduction of attributes in rough set theory are different tools for different purposes.

## 5.2 Reduct - Soft Set in Decision making using Rough Set Approach

Maji *et al.,* (2002) presented an application of soft set theory in a decision making problem with the help of rough set approach in the following way:

Let U = { $h_1$, $h_2$, $h_3$, $h_4$, $h_5$, $h_6$ } be a universal set consisting of six houses and E={expensive, beautiful, wooden cheap, in good surrounding, modern, in good repairs, in bad repairs} be a set of parameters.

Let the soft set (F, E) over U, which describes the *attractiveness of the houses,* be given by (F, E)= {expensive houses =$\emptyset$, beautiful houses =U, wooden houses ={$h_1$, $h_2$, $h_6$}, modern houses ={$h_1$, $h_2$, $h_3$, $h_4$, $h_6$}, in bad repairs ={$h_2$, $h_4$, $h_5$}, cheap houses =U, in good repairs ={$h_1$, $h_3$, $h_6$}, in green surrounding ={$h_1$, $h_2$, $h_3$, $h_4$, $h_6$}}.

Let Mr. X(say), wish to buy a house on the basis of his choice parameters P= {beautiful ($e_1$), wooden ($e_2$), cheap($e_3$), in the green surrounding($e_4$),

in good repairs($e_5$)} $\subset$E. The problem here is to select the house which is most suitable for Mr. X according to his choice parameters.

The ensuing soft set (F, P) is presented in Table 5.2 below:

Table 5.2: **Soft set (F,P)**

| $\bigcup$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | choice value($c_i$) |
|---|---|---|---|---|---|---|
| $h_1$ | 1 | 1 | 1 | 0 | 1 | $c_1$=4 |
| $h_2$ | 1 | 1 | 1 | 0 | 0 | $c_2$=3 |
| $h_3$ | 1 | 0 | 1 | 1 | 0 | $c_3$=3 |
| $h_4$ | 1 | 0 | 1 | 1 | 0 | $c_4$=3 |
| $h_5$ | 1 | 0 | 1 | 0 | 0 | $c_5$=2 |
| $h_6$ | 1 | 0 | 1 | 1 | 1 | $c_6$=4 |

where,

$h_{ij}= \begin{cases} 1, if\ h_i \in F\ (e_j) \\ o, otherwise \end{cases}$ and, the choice value of an object $h_i \in U$ is $c_i$ given by $c_i = \sum h_{ij}$

This is how soft set can be viewed as a knowledge representation system where the set of attributes is replaced by a set of parameters.

Let us consider the aforesaid tabular representation of the soft set (F,P). If Q is a reduct of P, then the soft set (F,Q) is called the reduct-soft set of (F,P). To select the optimal choice object (house) $h_k$ by Mr X, the following algorithm was put to work:

**Step1**: Compute the soft set (F, P)

**Step2**: Compute all reducts of P

**Step3:** Choose one reduct,say Q, and compute reduct soft set (F,Q)

**Step4:** Find k for which $c_k = \max c_i$

**Step5:** Select the optimal choice object $h_k$(. If k has more than one value, then choose anyone).

**Example 5.2.1:**

On applying this algorithm to solve the above described problem,we have:

**Step 1**: The soft set (F,P) is computed in table 5.2 above.

**Step 2**: The reducts of P are Q={$e_1$, $e_2$, $e_4$, $e_5$} and R={ $e_2$,$e_3$,$e_4$,$e_5$}.

**Step 3**: The reduct soft set (F,Q) is represented with the choice value in Table 5.3 below

Table 5.3: **Reduct -Soft Set (F, Q)**

| $\bigcup$ | $e_1$ | $e_2$ | $e_4$ | $e_5$ | Choice value($c_i$) |
|---|---|---|---|---|---|
| $h_1$ | 1 | 1 | 0 | 1 | $c_1$=3 |
| $h_2$ | 1 | 1 | 0 | 0 | $c_2$=2 |
| $h_3$ | 1 | 0 | 1 | 0 | $c_3$=2 |
| $h_4$ | 1 | 0 | 1 | 0 | $c_4$=2 |
| $h_5$ | 1 | 0 | 0 | 0 | $c_5$=1 |
| $h_6$ | 1 | 0 | 1 | 1 | $c_6$=3 |

**Step 4:** $c_1$or $c_6$= max $c_i$

**Step 5**: Optimal choice is $h_1$or $h_6$

Hence, Mr. X can buy either house $h_1$or $h_6$

It may happen that in buying a house, all the parameters in P are not of equal importance to Mr X. In such a situation, he would attach weight to his choice parameters according to their importance for him. Formally, for each parameter $p_i \in$ P, there is a weight $w_i \in$ (0,1], where the weighted choice value $c_i$ of an object $h_i \in$ U, is given by

$c_i = \sum d_{ij}$, when $d_{ij} = w_i \times h_{ij}$

For example, let the weights $w_1$=0.9, $w_2$=0.4, $w_4$=0.8, $w_5$=0.8 be attached to the

parameters of the set Q.Then, the weighted table of the soft set (F,Q) is shown in Table 5.4 below:

Table 5.4: **Weighted Reduct- soft set (F, Q)**

| $\bigcup$ | $e_1\times w_1$ | $e_2\times w_2$ | $e_4\times w_4$ | $e_5 \times w_5$ | $\sum w_1 \times h_{ij}$ |
|---|---|---|---|---|---|
| $h_1$ | 0.9 | 0.4 | 0 | 0.8 | $c_{1=}2.1$ |
| $h_2$ | 0.9 | 0.4 | 0 | 0 | $c_{2=}1.3$ |
| $h_3$ | 0.9 | 0 | 0.8 | 0 | $c_{3=}1.7$ |
| $h_4$ | 0.9 | 0 | 0.8 | 0 | $c_{4=}1.7$ |
| $h_5$ | 0.9 | 0 | 0 | 0 | $c_{5=}$ 0.9 |
| $h_6$ | 0.9 | 0 | 0.8 | 0.8 | $c_6=$ 2.5 |

Thus for Mr X, house $h_6$would be the best choice.

**Remark 5.2.1**: Note that the unique choice obtained in the above described weighted reduct-soft set (F, Q) is not always so. It would depend on how the weights are attached.

## 5.3 Parameterization Reduction of Soft Set in Decision making

Chen *et al.*(2005) pointed out an error in parameter reduction technique described in (Maji *et al.*,2002) .They also clarified that *parameterization reduction* in soft set theory and the *attributes reduction* in rough set theory described in (Pawlak,1982) are quite different tools for different purposes, and thereby proposed a new parameterization reduction in soft set theory, which can be visualized as follows:

Let us consider the soft set (F, E) as described in table 5.5 below:

Table 5.5: **Soft set (F, E)**

| $\bigcup$ | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $e_6$ | $e_7$ | choice value ($c_i$) |
|---|---|---|---|---|---|---|---|---|
| $h_1$ | 1 | 0 | 1 | 1 | 1 | 0 | 0 | $c_{1=}4$ |
| $h_2$ | 0 | 1 | 1 | 1 | 0 | 1 | 1 | $c_{2=}5$ |
| $h_3$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | $c_{3=}3$ |
| $h_4$ | 1 | 0 | 1 | 1 | 0 | 0 | 0 | $c_{4=}3$ |
| $h_5$ | 1 | 0 | 1 | 0 | 0 | 1 | 0 | $c_{5=}3$ |
| $h_6$ | 1 | 1 | 1 | 1 | 1 | 0 | 0 | $c_{6=}4$ |

Clearly, the optimal choice object is $h_2$, since $c_2$ =5 is the highest choice value.

In Chen *et al.*,2005), the parameter reduction of the set E is defined as the minimal subset of E, called the key parameter set that keeps the optimal choice object invariant. In the example taken in Table 5.5, the subsets $\{e_{2,}\ e_6\}$ and $\{e_{6,}\ e_7\}$ of E are two *parameterization reductions* of E, since they agree on the optimal choice object as $h_2$. Note however, that they are not the *attributes reductions* of E, since they induce different partitions of U. On the other hand, the attributes reduction set$\{e_2,\ e_{4,}\ e_5\}$ of E is not the parameterization reduction set of E, since it presents another optimal choice object $h_6$, though with the same partition of U.

## 5.4 Soft Set Uni-Int Decision making Method

Cagman and Enginoglu (2010A) constructed an Uni-Int (Union-Intersection) soft decision making method which reduces a large set of alternatives to smaller subsets according to the choice parameters of decision makers. This method is illustrated

in the following example:

Suppose there are 48 candidates who applied for admission in a department of a university, and there are two decision makers who want to select a smaller set of candidates to be interviewed according to their choice parameters.

Let the set of candidates be U= $\{u_1, u_2, ..., u_{48}\}$ and the set of parameters be E= $\{e_1, e_2, ..., e_7\}$ where each $e_i$, i= 1,2,...,7 stands for $e_1$=School Cert. result, $e_2$= Catchment area, $e_3$=Jamb. Result, $e_4$= Age, $e_5$=Post UME Result, $e_6$=Educationally backward area and $e_7$=Computer knowledge. In order to get a set of short-listed candidates, the Uni-Int soft decision making method is applied in steps as follows:

**Step 1:** Assume that the decision makers' choice of parameters are A= $\{e_1, e_2, e_4, e_7\} \subset$ E, and B= $\{e_1, e_2, e_5\} \subset$ E.

**Step 2:** Assume that the following soft sets $F_A$ *and* $F_B$ are obtained according to their choice parameters:

$F_A = \{(e_1, \{u_4, u_7, u_{13}, u_{21}, u_{28}, u_{31}, u_{32}, u_{36}, u_{39}, u_{41}, u_{43}, u_{44}, u_{48}\}),$

$(e_2, \{u_1, u_3, u_{13}, u_{18}, u_{19}, u_{21}, u_{22}, u_{24}, u_{28}, u_{32}, u_{36}, u_{42}, u_{44}, u_{46}\}),$

$(e_4, \{u_2, u_3, u_{13}, u_{15}, u_{18}, u_{23}, u_{25}, u_{28}, u_{30}, u_{33}, u_{36}, u_{38}, u_{42}, u_{43}\}),$

$(e_7, \{u_1, u_5, u_{12}, u_{13}, u_{17}, u_{20}, u_{24}, u_{28}, u_{29}, u_{34}, u_{36}, u_{41}, u_{45}, u_{47}\})\},$

$F_B = \{(e_1, \{u_3, u_4, u_5, u_8, u_{14}, u_{21}, u_{22}, u_{26}, u_{27}, u_{34}, u_{35}, u_{37}, u_{40}, u_{42}, u_{46}\}),$

$(e_2, \{u_1, u_4, u_7, u_{10}, u_{11}, u_{13}, u_{15}, u_{21}, u_{29}, u_{30}, u_{32}, u_{36}, u_{42}, u_{43}, u_{45}\}),$

$(e_5, \{u_2, u_4, u_8, u_9, u_{12}, u_{13}, u_{14}, u_{16}, u_{17}, u_{21}, u_{23}, u_{28}, u_{36}, u_{42}, u_{44}\})\}.$

**Step 3:** *Compute the* $\wedge$ *- product* $F_A \wedge F_B$*:*

$F_A \wedge F_B$= {*(*$(e_1,\ e_1)$*,* $\{u_4, u_{21}\}$*),* *(*$(e_1, e_2)$*,* $\{u_4,\ u_7,\ u_{13},\ u_{21},\ u_{32},\ u_{36},\ u_{43}\}$*),*

*(*$(e_1,\ e_5)$*,* $\{u_4,\ u_{13},\ u_{21},\ u_{28},\ u_{36},\ u_{44}\}$*),* *(*$(e_2,\ e_1)$*,* $\{u_3,\ u_{21},\ u_{22},\ u_{42},\ u_{46}\}$*),*

*(*$(e_2,\ e_2)$*,* $\{u_1,\ u_{13},\ u_{21},\ u_{32},\ u_{36},\ u_{42}\}$*),* *(*$(e_2,\ e_5)$*,* $\{u_{13},\ u_{21},\ u_{28},\ u_{36},\ u_{42},\ u_{44}\}$*),*

*(*$(e_4,\ e_1)$*,* $\{u_3,\ u_{42}\}$*),* *(*$(e_4,\ e_2)$*,* $\{u_{13},\ u_{15},\ u_{30},\ u_{36},\ u_{42},\ u_{43}\}$*),*

*(*$(e_4,\ e_5)$*,* $\{u_2,\ u_{13},\ u_{23},\ u_{28},\ u_{36},\ u_{42}\}$*),* *(*$(e_7,\ e_1)$*,* $\{u_5,\ u_{34}\}$*),*

*(*$(e_7,\ e_2)$*,* $\{u_1,\ u_{13},\ u_{29},\ u_{36},\ u_{45}\}$*),* *(*$(e_7,\ e_5)$*,* $\{u_{12},\ u_{13},\ u_{17},\ u_{28},\ u_{36}\}$*)*}.

**Step 4:** Compute the decision set Uni-Int *(* $F_A \wedge F_B$*)* as follows:

Uni-Int *(*$F_A \wedge F_B$*)* = Uni$_x$-Int$_y$ *(*$F_A \wedge F_B$ *)* $\bigcup$ Uni$_y$-Int$_x$*(*$F_A \wedge F_B$*)* . Now, Uni$_x$-Int$_y$ ($F_A \wedge F_B$ ) $= \bigcup_{x\in A}\left\{\bigcap_{y\in B}(fA \wedge B(x,y))\right\}$

$$= \bigcup \begin{cases} \bigcap\{\{u4,u21\},\{u4,u7,u13,u21,u32,u36,u43\},\{u4,u13,u21,u28,u36,u44\}\} \\ \bigcap\{\{u3,u21,u22,u42,u46\},\{u1,u13,u21,u32,u36,u42\}, \\ \{u13,u15,u30,u36,u42,u44\}\} \\ \bigcap\{\{u3,u42\},\{u13,u15,u30,u36,u42,u43\},\{u2,u13,u23,u29,u36,u42\}\} \\ \bigcap\{\{u5,u34\},\{u1,u13,u29,u36,u45\},\{u12,u13,u17,u28,u36\}\} \end{cases}$$

$$= \bigcup\{\{u4,u21\},\{u42\},\{u42\},\emptyset\} = \{u4,u21,u42\}\,, and$$

and Uni$_y$-Int$_x$ *(*$F_A \wedge F_B$ *)* =

$$= \bigcup \begin{cases} \bigcap\{\{u4,u21\},\{u3,u21,u22,u42,u46\},\{u3,u42\},\{u5,u34\}\} \\ \bigcap\{\{u4,u7,u13,u21,u32,u36,u43\},\{u1,u13,u21,u32,u36,u42\}, \\ \{u13,u15,u30,u36,u42,u43\},\{u1,u13,u29,u36,u45\}\} \\ \bigcap\{\{u4,u13,u21,u28,u36,u44\},\{u13,u21,u28,u36,u42,u44\}, \\ \{u2,u13,u23,u28,u36,u42\},\{u12,u13,u17,u28,u36\}\} \end{cases}$$

$$= \bigcup\{\emptyset,\{u13,u36\},\{u13,u36\}\} = \{u13,u36\}\,.$$

Thus, *Uni-Int* $(F_A \wedge F_B) = \{u_4, u_{21}, u_{42}\} \cup \{u_{13}, u_{36}\} = \{u_4, u_{13}, u_{21}, u_{36}, u_{42}\}$,which is the list of objectically short-listed set of candidates to be finally interviewed.

## 5.5 Soft Matrix Max- Min Decision making Method

Cagman and Enginoglu,( 2010B) constructed a soft max-min decision making method ( SMmDMM) which selects optimum alternatives from a set of alternatives according to decision makers‘ choice of parameters. This method is demonstrated by way of taking an example of decision making problem.

Suppose a man and his wife want to purchase a car from a dealer who has a set of four cars U=$\{c_1, c_2, c_3, c_4\}$, which are characterized by a set of parameters E=$\{e_1, e_2, e_3, e_4\}$,where $e_i$, i= 1, 2, 3, 4, stand for cheap ($e_1$), fuel efficient ($e_2$) modern ($e_3$), durable ($e_4$). The SMmDMM runs in steps as follows:

**Step 1:** Assume that the man and his wife choose the sets of parameters A= $\{e_2, e_3, e_4\}$ and B= $\{e_1, e_3, e_4\}$, respectively.

**Step 2**: Assume that the following soft matrices are obtained according to the parameters:

$$[a_{ij}] = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \text{ and } [b_{il}] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

**Step 3**: Find the $\wedge$ product $[c_{ip}]$ of $[a_{ij}]$ and $[b_{il}]$ (since the choices of both the man and his wife are to be considered conjunctively) as follows:

$$[a_{ij}] \wedge [b_{il}] = [c_{ip}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**Step 4**: Find a max-min one- column decision soft matrix Mm ($[c_{ip}]$) of $[c_{ip}]$ ,defined by

$$[d_{i1}] = \mathrm{Mm}([c_{ip}]) = \left[\max_{k \in I}\{tik\}\right], where\ tik = \begin{cases} \min\limits_{p \in Iik}\{cip\},\ Iik \neq \emptyset \\ o \qquad\qquad , Iik = \emptyset \end{cases}$$

and $I_{ik}$= {p : ∃i, $c_{ip} \neq 0$, (k-1)n< $p \leq kn$} for all k∈I= {1,2,..., n}.

Thus $[d_{i1}] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ .

**Step 5**: Finally, the optimum set of U with respect to $[d_{i1}]$ is determined as:

Opt $_{[di1]}$ (U) = $\{c_2\}$

Hence $c_2$is identified as the car of optimum choice on which both the man and his wife opted to agree to purchase.

**Remark 5.5.1** Similar to the observation made in Remark 5.2.1, the optimum set of U may contain more than one element. It may also be observed that the soft set uni-int decision making method described in 5.4 aims at reducing a large number of alternatives to a smaller one after which the soft matrix max-min decision making method in 5.5 is applied in order to obtain the optimum set.

## 5.6 Generalised Soft Max-Min Decision making Method

Samsiah and Mohamad, (2012) extended the soft max-min decision making method described in 5.5 to deal with problems involving more than two decision makers .We describe this method by taking an example as follows:

Let Mr. X, his wife and his daughter be equally involved in arriving at a decision to buy a house for the family.

Suppose that there are six houses available for selection, and there are nine parameters: aesthetics ($e_1$), Safety ($e_2$), Exterior ($e_3$), Interior ($e_4$), System ($e_5$), School ($e_6$), Government ($e_7$), Social ($e_8$) and Entertainment ($e_9$) to be considered. Let U=$\{h_1,h_2,h_3,h_4,h_5,h_6\}$ denote the set of housesand E=$\{e_1,e_2,e_3,...,e_9\}$, the set of parameters. The proposed method consists of the following steps:

**Step 1**: Let Mr. X, his wife, and his daughter choose their set of parameters, respectively as

A = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, B = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, and

C =$\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$.

**Step 2:** Assume that the soft matrices constructed according to their parameters are as follows:

$$[a_{ij}] = \begin{bmatrix} 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}, [b_{ik}] = \begin{bmatrix} 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}, \text{ and}$$

$$[c_{il}] = \begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}.$$

**Step 3:** The $\bigwedge$-product of [ $a_{ij}$] and [$b_{ik}$] is:

$$[a_{ij}] \bigwedge [b_{ik}] = \begin{bmatrix} 1\ 1\ 1\,1\ 1\,1\,1\ 1\ 0 & 1\ 1\,1\,1\,1\,1\ 1\ 1\ 0 & 1\ 1\,1\,1\,1\,1\ 1\ 1\ 0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,1\,1\,1\,1\,1\ 0\ 0 & 0\,1\,1\,1\,1\,1\,1\ 0\ 0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 1\,1\,0\,1\,1\,0\,1\,0\ 0 & 1\,1\,0\,1\,1\,0\,1\,0\ 0 \\ 0\,1\ 1\ 0\,1\,1\,1\ 1\,0 & 0\,1\ 1\ 0\,1\,1\,1\ 1\,0 & 0\,0\,0\,0\,0\,0\,0\ 0\ 0 \\ 1\ 1\,1\ 1\,1\ 1\,0\ 1\,0 & 1\ 1\,1\ 1\,0\ 1\,0\ 1\,0 & 1\ 1\,1\ 1\,0\ 1\,0\ 1\,0 \\ 0\,1\,1\ 0\ 1\,1\,0\,0\,0 & 1\,1\,1\ 0\ 1\,0\,0\,0\,0 & 1\,1\,1\ 0\ 1\,0\,0\,0\,0 \end{bmatrix}$$

$$\begin{bmatrix} 1\ 1\,1\,1\,1\,1\ 1\ 1\ 0 & 1\ 1\,1\,1\ 1\ 1\ 1\ 1\ 0 & 1\,1\,1\,1\ 1\ 1\ 1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\ 0\ 0 & 0\,1\,1\,1\,1\,1\ 1\ 0\ 0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\ 0 & 1\,1\,0\,1\,1\,0\ 1\ 0\ 0 & 1\,1\,0\,1\,1\,0\,1\,0\,0 \\ 0\,1\,1\,0\,1\,1\,1\ 1\ 0 & 0\,1\,1\,0\,1\,1\ 1\ 1\ 0 & 0\,1\,1\,0\,1\,1\,1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\ 0\ 0 & 1\,1\,1\,1\,0\,1\ 0\ 1\ 0 & 1\,1\,1\,1\,0\,1\,0\,1\,0 \\ 1\,1\,1\,0\,1\,0\,0\ 0\ 0 & 1\,1\,1\,0\,1\ 0\ 0\ 0\,0 & 1\,1\,1\,0\,1\,0\,0\,0\,0 \end{bmatrix}$$

$$\begin{bmatrix} 1\,1\,1\,1\;\;1\;\;1\;\;1\,1\,0 & 1\,1\,1\,1\;\;1\;\;1\;\;1\,1\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,1\,1\,1\,1\,1\,1\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,0\,1\,1\,0\,1\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,1\,1\,0\,1\,1\,1\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,1\,1\,0\,1\,0\,1\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 1\,1\,1\,0\,1\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix} .$$

**Step 4:** The minimum of $[a_{ij}] \bigwedge [b_{IK}]$ is

$$\text{Min}([a_{ij}] \bigwedge [b_{IK}]) = \begin{bmatrix} 1\,1\,1\,1\;\;1\;\;1\;\;1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix} = [d_{ip}] .$$

**Step 5:** The $\bigwedge$- product of $[d_{ip}]$ and $[c_{il}]$ is

$$[d_{ip}] \bigwedge [c_{il}] = \begin{bmatrix} 1\,1\,0\,1\;1\;1\;1\,1\,0 & 1\,1\,0\,1\;1\;1\;1\,1\,0 & 1\,1\,0\,1\;1\;1\;1\,1\,0 & 1\,1\,0\,1\;1\;1\;1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}$$

$$\begin{bmatrix} 1\,1\,0\,1\;1\;1\;1\,1\,0 & 1\,1\,0\,1\;1\;1\;1\,1\,0 & 1\,1\,0\,1\;1\;1\;1\,1\,0 & 1\,1\,0\,1\,1\,1\,1\,1\;0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 & 0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}$$

$$\begin{bmatrix} 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \end{bmatrix}$$

**Step 6**: The minimum of $[d_{ip}] \wedge [c_{il}]$ is

$$\text{Min}([d_{ip}]\wedge[c_{il}])=\begin{bmatrix} 1\,1\,1\,1 & 1 & 1 & 1\,1\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0\,0\,0\,0 \\ \end{bmatrix} = [e_{im}]\ .$$

**Step 7**: The $\max([e_{im}])=\text{Mm}([a_{ij}])\wedge[b_{il}]\wedge[c_{il}])=[1\ 0\ 0\ 0\ 0\ 0]=[u_{il}]$

**Step 8**: Finally, the optimum set of U is $\text{opt}_{[di1]}$ (U)=$h_1$ .Hence, $h_1$ is the optimal choice.

**Remark 5.6.1**: The observation made in Remark 5.2.1 applies to this method as well. Moreover as can be seen from above description, the computation aspect becomes unwieldy as the number of decision makers becomes large. In fact, micro soft excel would be required even for a moderate number.

## 5.7 Soft Choice Matrix in Decision making Method

Mitra *et al.*,(2012) proposed the concept of choice matrices representing the choice parameters of the decision makers which are used to develop a new method for solving soft set based decision making problems involving more than two decision makers.

This method is described by considering the following problem:

Supposes three friends Mr A, Mr B and Mr C together want to buy a dress out of four dresses for their common friend Mr D. Let U = {$d_1$, $d_2$, $d_3$, $d_4$} be the set of four dresses and E ={cheap ($e_1$), beautiful ($e_2$), comfortable ($e_3$), gorgeous($e_4$)} a

set of parameters. Let A= $\{e_1, e_3\}$, B=$\{e_2, e_3\}$, C=$\{e_1, e_4,\}$ respectively, be their sets of choice parameter. Let (F,A), (G, B), and (H,C) be the soft sets describing the attractiveness of the dresses respectively.

The method described in ( Mitra *et al.*,2012) is illustrated in the following steps:

**Step1**. Let the soft matrices representing the soft sets (F,A), (G,B) and (H,C),respectively be

$$[a_{ij}]=\begin{bmatrix} 1\,0 & 1 & 0 \\ 1\,0 & 0\,0 & \\ 1 & 0\,1 & 0 \\ 0\,0\,0\,0 & & \end{bmatrix}, [b_{ik}]=\begin{bmatrix} 0\,0 & 0\,0 & \\ 0\,1 & 1 & 0 \\ 0\,0 & 1 & 0 \\ 0\,1\,0 & 0 & \end{bmatrix} \text{ and } [c_{il}]=\begin{bmatrix} 1\,0 & 0 & 0 \\ 0 & 0 & 0\,1 \\ 1 & 0 & 0\,1 \\ 0 & 0 & 0\,1 \end{bmatrix}.$$

**Step 2**. The combined choice matrices with respect to the choice parameters of the decision makers in different forms are constructed as follows:

$$\begin{pmatrix} \xi ij \\ (A,B\wedge C) \end{pmatrix} = e_A \begin{bmatrix} e_B \Lambda c \\ 0\ 0\ 0\,0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \end{bmatrix}, \text{ B } \Lambda\text{C}=\emptyset, \text{ A}= \{e_1, e_3\} ;$$

$$\begin{pmatrix} \eta ij \\ (B,A\wedge C) \end{pmatrix} = e_B \begin{bmatrix} e_A \Lambda c \\ 0\ 0\ 0\,0 \\ 1\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \end{bmatrix}, \text{ A}\Lambda\text{C}=\{e_1\} , \text{ B}=\{e_2, e_3\};$$

and $\left(\underset{(C,A\wedge B)}{\omega ij}\right) = \mathrm{e}_C \begin{bmatrix} eA\Lambda B \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 1\ 0 \end{bmatrix}$ AΛB={e$_3$}, C={e$_1$,e$_4$}.

**Step3.** The corresponding product soft matrices called the weighted soft matrices, using rule of multiplication $\bigotimes$ of soft matrices discussed in (Mitra *et al.*,2012), are as follows:

$$[\mathrm{a}_{ij}]\bigotimes(\varepsilon_{ij}) = \begin{bmatrix} 1\,0 & 1\ 0 \\ 1\,0 & 0\ 0 \\ 1 & 0\ 1\ 0 \\ 0\ 0\ 0\ 0 \end{bmatrix} \bigotimes \begin{bmatrix} 0\ 0 & 0\,0 \\ 0\ 0 & 0\ 0 \\ 0\ 0 & 0\ 0 \\ 0\ 0 & 0\ 0 \end{bmatrix} = \begin{bmatrix} 0\ 0 & 0\,0 \\ 0\ 0 & 0\ 0 \\ 0\ 0 & 0\ 0 \\ 0\ 0 & 0\ 0 \end{bmatrix}$$

$$[\mathrm{b}_{ik}]\bigotimes(\eta_{ij}) = \begin{bmatrix} 1\,0 & 0\ 0 \\ 0\ 0 & 0\ 1 \\ 1\ 0 & 0\ 1 \\ 0\ 1 & 0\ 0 \end{bmatrix} \bigotimes \begin{bmatrix} 0\ 0 & 0\,0 \\ 1\ 0 & 0\ 0 \\ 1\ 0 & 0\ 0 \\ 0\ 0 & 0\,0 \end{bmatrix} = \begin{bmatrix} 0\ 0 & 0\,0 \\ 2\ 0 & 0\ 0 \\ 1\ 0 & 0\ 0 \\ 1\ 0 & 0\ 0 \end{bmatrix}$$

$$[\mathrm{C}_{il}]\bigotimes(\omega_{ij}) = \begin{bmatrix} 1\,0 & 0\ 0 \\ 0\ 0 & 0\ 1 \\ 1\ 0 & 0\ 1 \\ 0\ 0 & 0\ 1 \end{bmatrix} \bigotimes \begin{bmatrix} 0\ 0 & 1\,0 \\ 0\ 0 & 0\ 1 \\ 0\ 0 & 0\ 0 \\ 0\ 0 & 1\,0 \end{bmatrix} = \begin{bmatrix} 0\ 0 & 1\,0 \\ 0\ 0 & 1\ 0 \\ 0\ 0 & 2\ 0 \\ 0\ 0 & 1\ 0 \end{bmatrix}$$

**Step 4.** The sum of these product soft matrices using the rule of addition $\oplus$ of

soft matrices described in (Mitra *et al.*,2012), is as follows:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} = Rs.$$

**Step 5.** Now, the resultant one- column weighted soft matrice

$$R_{w=}\begin{bmatrix} \\ 1 \\ 3 \\ 3 \\ 2 \\ \\ \end{bmatrix}.$$

**Step 6.** Finally, the optimum choice object(s) corresponding to the row with the highest weight are to be identified. In the example considered here, $d_2$ or $d_3$ is the required object. That is Mr. A, Mr. B and Mr. C collectively may buy any of the dresses $d_2$ or $d_3$.

**Remark 5.7.**1: The advantage of this method over the one described in( Cagman and Enginoglu, 2010B) and in (Samsiah and Mohamad , 2012) is that it can solve soft set based decision making problems involving a huge number of decision makers in a much economical way.

**Concluding Remark** In real world decision making problems, meaningful and better choices could be made by expert decision groups instead of only single decision maker as the saying goes "Two heads are better than one." Based on this fact and on the remarks made on the techniques studied, the Mitra *et al.*, 2012 technique is most recommended.

# CHAPTER 6

### SUMMARY, CONCLUSION AND RECOMMENDATIONS

## 6.1 Summary

This dissertation, consisting of six chapters, presents a systematic study of soft set theory, its algebras and applications in decision making.

In chapter one, a general introduction of the dissertation, which includes the background of the study, motivation and justification of the study, the aim and objectives amongst others, is presented.

An up- to- date literature survey of the development of soft set theory, in respect of fundamentals, algebraic structures and applications in decision making, is presented in chapter two.

Chapter three presents a comprehensive and critical study of the fundamentals of soft set theory: soft set operations, soft set relations and functions, soft matrices and their properties and some new results, mainly on the distributive and absorption properties, with respect to various soft set operations are presented. In addition, soft set operations are shown to be equivalent to their corresponding soft matrix operations. This chapter forms a stepping stone to the study of the underlying algebraic structures and applications in the forthcoming chapters.

A study of various soft set algebras such as soft groups, soft rings, soft semirings, and soft lattices, which have been developed so far in the literature, is presented in chapter four. The main contribution in the part of the dissertation is formulation and exemplification of monoids, semirings and lattices of soft subsets and partitions of a soft set.

In chapter five, techniques of applications of soft set theory in decision making are presented.

Finally, conclusion and recommendations of the dissertation are presented.

## 6.2 Conclusion

In this dissertation, fundamentals of soft set theory, various algebraic structures of soft sets and their applications in decision making have been critically studied and some new results obtained.

The following new results are the main contributions of this dissertation:

Let *(F,A) , (G,B) and (H,C)* be soft sets over a common universe. Then,

1. **Distributive Properties-Proposition 3.2.2 (Singh and Onyeozili, 2012A, 2012B, 2012C)**

   (i) $(F,A)\tilde{\cup}\left((G,B)\cap_{\varepsilon}(H,C)\right) \neq \left((F,A)\tilde{\cup}(G,B)\right)\cap_{\varepsilon}\left((F,A)\tilde{\cup}(H,C)\right)$

   (ii) $(F,A)\cap_{\varepsilon}\left((G,B)\tilde{\cup}(H,C)\right) \neq \left((F,A)\cap_{\varepsilon}(G,B)\right)\tilde{\cup}\left((F,A)\cap_{\varepsilon}(H,C)\right)$

   (iii) $(F,A)\cup_{R}\left((G,B)\smile_{R}(H,C)\right) \neq \left((F,A)\cup_{R}(G,B)\right)\smile_{R}\left((F,A)\cup_{R}(H,C)\right)$

   (iv) $(F,A)\cap_{\varepsilon}\left((G,B)\smile_{R}(H,C)\right) \neq \left((F,A)\cap_{\varepsilon}(G,B)\right)\smile_{R}\left((F,A)\cap_{\varepsilon}(H,C)\right)$

   (v) $(F,A)\tilde{\cup}\left((G,B)\smile_{R}(H,C)\right) \neq \left((F,A)\tilde{\cup}(G,B)\right)\smile_{R}\left((F,A)\tilde{\cup}(H,C)\right)$

(vi) $(F,A) \cup_R ((G,B)\tilde{\cup}(H,C)) = ((F,A) \cup_R (G,B)) \,\tilde{\cup}\, ((F,A) \cup_R (H,C))$

(vii) $(F,A)\tilde{\cap} ((G,B) \cap_\varepsilon (H,C)) = ((F,A)\tilde{\cap}(G,B)) \cap_\varepsilon ((F,A)\tilde{\cap}(H,C))$

(viii) $(F,A)\tilde{\cup} ((G,B) \cup_R (H,C)) \neq ((F,A)\tilde{\cup}(G,B)) \cup_R ((F,A)\tilde{\cup}(H,C))$

(ix) $(F,A) \cap_\varepsilon ((G,B)\tilde{\cap}(H,C)) \neq ((F,A) \cap_\varepsilon (G,B)) \,\tilde{\cap}\, ((F,A) \cap_\varepsilon (H,C))$

(x) $(F,A) \wedge ((G,B) \backsim_R (H,C)) = ((F,A)\,\tilde{\cap}(G,B)) \backsim_R ((F,A)\tilde{\cap}(H,C))$

(xi) $(F,A) \vee ((G,B) \backsim_R (H,C)) \neq ((F,A)\,\tilde{\cap}(G,B)) \backsim_R ((F,A)\tilde{\cap}(H,C))$

[Proofs, see page 44]

## 2. Absorption Properties and Absorption Inclusions (Singh and Onyeozili,2012A)

### 2.1 Absorption Properties-Proposition 3.2.1

(i) $(F,A)\tilde{\cup} ((F,A)\tilde{\cap}(G,B)) = (F,A)$

(ii) $(F,A)\tilde{\cap} ((F,A)\tilde{\cup}(G,B)) = (F,A)$

(iii) $(F,A) \cup_R ((F,A) \cap_\varepsilon (G,B)) = (F,A)$

(iv) $(F,A) \cap_\varepsilon ((F,A) \cup_R (G,B)) = (F,A)$

[Proofs, see page 38 ]

### 2.2 Absorption Inclusions-Proposition 3.2.3

(i) $(F,A) \cup_R ((F,A)\tilde{\cap}(G,B)) \,\tilde{\subset}(F,A)$,

(ii) $(F,A)\tilde{\cap} ((F,A) \cup_R (G,B)) \,\tilde{\subset}(F,A)$,

(iii) $(F,A)\tilde{\cup} ((F,A) \cap_\varepsilon (G,B)) \,\tilde{\supset}(F,A)$, and

(iv) $(F,A) \cap_\varepsilon ((F,A)\tilde{\cup}(G,B)) \,\tilde{\supset}(F,A)$.

[Proofs see page 51 ]

# 3. Formulation of Monoids, Semirings and Lattices of Soft subsets and Partitions of a soft set

## 3.1 Commutative, Idempotent Monoids and Semirings of Soft Subsets of a Soft Set

### A. Monoids

1. (SS(F,A), $* \in \{\tilde{\cup},\tilde{\cap}\}$) are commutative, idempotent monoids with $\tilde{\Phi}_{\emptyset}$ as the identity element.

2. (SS(F,A), $\cup_R$) is a commutative, idempotent monoid with $\tilde{\Phi}_A$ as the identity element.

3. (SS(F,A), $\cap_R$) is a commutative, idempotent monoid with (F, A) as the identity element.

### B. Semirings

1. (SS(F,A), $\tilde{\cup},\cup_R$) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$ .

2. (SS(F,A), $\cap_R$, $\tilde{\cup}$) is a commutative, idempotent semiring with identity element (F,A).

3. (SS(F,A), $\cup_R,\tilde{\cap}$) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_A$ .

4. (SS(F,A), $\tilde{\cap},\cup_R$ ) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$ .

5. (SS(F,A), $\cup_R$, $\cap_R$) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_A$ .

6. (SS(F,A), $\cap_R$,$\cup_R$ } ) is a commutative, idempotent semiring with identity element (F,A).

7. (SS(F,A), $\tilde{\cap}$, $\cap_R$ ) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$ .

8. (SS(F, A), $\tilde{\cup}$, $\cap_R$} ) is a commutative, idempotent semiring with identity element $\tilde{\Phi}_{\emptyset}$.

[Proofs, see page 96 ]

## 3.2 Bounded, Distributive Lattices of Soft Subsets of a Soft Set

(i) (SS(F, A), $\tilde{\cap}$,$\cup_R$ ) and (SS(F, A), $\cup_R$,$\tilde{\cap}$), and

(ii) (SS(F, A), $\cap_R$, $\tilde{\cup}$) and (SS(F, A), $\tilde{\cup}$, $\cap_R$)

are bounded, distributive lattices.

[Proofs, see page 98 ]

## 3.3 Commmutative, Idempotent Monoids and Semirings of Partitions of a Soft Set

### A. Monoids

1 ( $\wp$(F, A), $\sqcap$) is a commutative, idempotent monoid.

2 ($\wp$(F, A) $\sqcup$) is a commutative, idempotent monoid.

### B. Semirings

1. ($\wp$(F, A), $\sqcup$, $\sqcap$) with the partition consisting of the single block as the identity element.

2. ($\wp$(F, A), $\sqcap$,$\sqcup$) with the partition consisting of the singleton blocks as the identity element.

[Proofs see page 99 ]

3.4 **Bounded, Distributive Lattices of Partitions of a Soft Set**

(i) ($\wp$(F, A), $\sqcup_E$, $\sqcap_R$),

(ii) ($\wp$(F, A), $\sqcap_R$, $\sqcup_E$), and

(iii) ($\wp$(F, A), $\sqcup$, $\sqcap$) and (iv) ($\wp$(F, A), $\sqcap$ , $\sqcup$)

are bounded , distributive lattices.

[Proofs, see page 102 ]

## 6.3 Recommendations

Since soft set theory is a relatively new area in mathematics and also has been recognized to be important in applications to real life decision making problems involving uncertainties, the following recommendations may be taken into consideration:

In view of the techniques of soft set theory being relatively simpler and widely applicable, researches in various hybrid theories like fuzzy soft sets, multi soft sets, soft multi sets, etc., are promising(Maji *et al.*, 2001,

Alkhazaleh *et al.*, 2011, Pinaki and Samanta, 2012, Singh *et al.*, 2014, are some specific references).

Also, as studies in the area of soft topological analysis are appearing recently (Cagman, *et al.*, 2011, Shabir and Naz, 2011, Tariq and Salma, 2014, are some representative references), it may be a promising research area to obtain some deep mathematical results.

In view of the aforesaid facts, soft set theory could be inculcated into the curricula of studies, both at undergraduate and postgraduate levels.

# LIST OF PUBLICATIONS

## Journal Papers

1. Singh, D., Onyeozili, I.A. (2012). Some Conceptual Misunderstandings of the Fundamental of Soft Set Theory. *ARPN Journal of Systems and Software*, 2(9):251-254.

2. Singh, D., Onyeozili, I.A. (2012). Notes on Soft Matrices Operations. *ARPN Journal of Science and Technology,* 2(9):861-869.

3. Singh, D., Onyeozili, I.A. (2012). Some Results On Distributive and Absorption Properties of Soft Operations. *IOSR Journal of Mathematics*, 4(2):18-30.

4. Singh, D., Onyeozili, I.A. (2012). On Some New Properties of Soft Set Operations. *International Journal of Computer Applications*, 59(4):39-44.

5. Singh, D., Onyeozili, I.A. (2013). Matrix Representation of Soft Sets and Its Application to Decision Making Problem. *The International Journal of Engineering And Science*, 2(2):48-56.

6. Singh, D., Onyeozili, I.A., Alkali, A. J. (2014). Notes on Multi Soft Matrices. *Journal of Emerging Trends in Computing and Information Sciences*, 5(5):421-427.

## Communicated Papers

1. Singh, D., Onyeozili, I.A., Monoids and Semirings of Soft subsets and Partitions of a Soft set.*Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis*(under review).

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