STUDY ON BETA-KUMARASWAMY DISTRIBUTION

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JUNE, 2018

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BY

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P14SCMT8007

A THESIS SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD

OF MASTER OF SCIENCE DEGREE IN STATISTICS

DEPARTMENT OF STATISTICS,

FACULTY OF PHYSICAL SCIENCES

AHMADU BELLO UNIVERSITY,

ZARIA, NIGERIA

JUNE, 2018

Declaration

I declare that the work in this dissertation titled "study on Beta-kumaraswamy distribution" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

……………………..	……………………….	……………………………..
Sani Musa Makama	signature	Date

Certification

This dissertation titled “STUDY ON BETA-KUMARASWAMY DISTRIBUTION”: by Sani Musa MAKAMA meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University and is approved for its contribution to knowledge and literary presentation.

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Dedication

To my late father ,ALHAJI SANI ISMAILA HAFIZI (makaman ubandoma zauzau) , may his soul rest in peace, ameen.

ACKNOWLEDGEMENT

All praises and thanks are to the Giver of knowledge, Master of the world for giving me the opportunity giving to me for the completion of this work. My sincere appreciation goes to my father late Alhaji Sani Makama and my lovely mother, Hajiya Sarah Abdurrahman for their guidance, prayers and moral support, may Almighty Allah grant you the best in your final abode, Amen. This work would not have been possible without the professional guidance, comments and suggestions of my supervisors, Dr. B.B Alhaji and Dr. Tukkur Dahiru for their patience and encouragement despite their tight schedule.

My special thanks go to Dr.Yahaya Abubakar, the Postgraduate Coordinator, to my friend and brother Abu-Salma for their continuous support and encouragement- throughout the time of the research. I wish to thank all those that contributed to the success of this research in one way or the other especially Dr.MusaGarba Abdullah (the ProvostShehu Idris Collage of Health Sciences and Technology, Makarfi,Kaduna State), Mr. NnamaniChibuike, all the lecturers of the Statistics Department, and my M.Sc. Colleagues.

Furthermore, my profound gratitude to my family, especially my sister Maimuna Makama (Asabe),Muhammad (Daha), Abdulrahaman, Abdullahi ,Fatima,Amina,Ismail and fatima.

Abstract

In recent years a new generalized family of continuous distribution known as beta-G Distribution was proposed. This Base-Line Distribution has been used to generate new distributions. We proposed therefore in this research an extension of kumaraswamy distribution using the logit of the beta random variable and study some of its properties. The Probability Density Function (PDF) and Cumulative Density Function (CDF)of the Beta -kumaraswamy are defined. Hazard Rate Function, Survival Function and Asymptotic Behavior of the distribution are obtained.The method of maximum likelihood was used to estimate the parameters of the newdistribution. The performance of the distribution was evaluated using both simulated and real-lifedatasets. The new distribution is recommended for modeling positively skewed datasets with higher peaks and tails.it was alsoquite flexible when subjected to some of the existing distribution by fitting it to two real datasets.

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CHAPTER ONE

INTRODUCTION

1.1Background to The Study

The Kumaraswamy Distribution describes a distribution in which the outcomes are limited to a specific range; the probability density function within this range is being characterized by two shape parameters. It is similar to the Beta Distribution but possibly easier to use because it has simpler analytical expressions for its Probability Density Function and Cumulative Distribution function. The Kumaraswamy Probability Distribution was originally proposed by Poondi Kumaraswamy (1980) for double bounded random processes for hydrological applications. The Kumaraswamy double bounded distribution denoted bykw(a,b) distribution is a family of continuous probability distributions defined on the interval [0,1].For $0 < x < 1$, where $a \geq 0$ and $b \geq 0$ are the shape parameters.

The Kumaraswamy probability density function has the same basic properties as the beta distribution. According to Jones (2009) and Cordeiro *et al*. (2010, 2012) it behave in the same way as the beta distribution on the values of its parameters: it is unimodal for $a \geq 1$ and $b \geq 1$; decreasing for $a \leq 1$ and $b > 1$ and constant for $a = b = 1$. Jones (2009) investigated not only properties of the Kumaraswamy distribution but also some similarities and differences between the beta andKumaraswamy distributions. According to Jones (2009) the Kumaraswamy distribution has several advantages over the beta distribution, such as a simple normalizing constant, simple explicit formulae for the distribution and quantile functions which do not involve any special functions, a simple formula for random variable generation, explicit formulae for L-moments and simpler formulae for moments of order statistics. On the other hand, the beta distribution has simpler formulae for moments and the moment generating function, a one-parameter sub-family of symmetric distributions, simpler moment estimation and more ways of generating the distribution through physical processes. The Kumaraswamy distribution is applicable to a number of hydrological problems and many natural phenomena whose process values are bounded on both sides.

Cordeiro and de Castro (2009) studied a new class of the Kumaraswamy generalized distributions (denoted by the Kw–G distribution) based on the Kumaraswamy distribution (denoted by Kw distribution). They derived almost all formulas for the probability characteristics of the Kw–G distribution In hydrology and related areas, the Kumaraswamydistribution has received considerable interest, see Cordeiro *et al.* (2010, 2012), Fletcher and Ponnambalam (1996),Ganji (2006), Ponnambalam*et al*. (2001),and Seifi*et al.* (2000). According to Nadarajah (2008), many papers in the hydrological literature have used this distribution because it is deemed to be a "better alternative" to the beta distribution; see, for example, Koutsoyiannis and Xanthopoulos (1989).

one of the most used generalized distribution generator is the beta-G. The earliest of the beta extended distributions is the class of distributions generated from the logit of a beta random variable with cumulative distribution function that involve employing two parameters whose role is to introduce skewness and to vary tail weights (Eugene*et al.*, 2002). Jones (2004) discusses general beta family influenced by its order statistics and shows that it has beautiful distributional properties and potential for interesting statistical applications. The generalization method used is the logit of beta distribution.

The beta distribution is a continuous probability distribution with two positive shape parameters α and β. It is the conjugate prior of the binomial distribution. It is also a natural extension of the uniform distribution. The beta distribution models events which are constrained to take place within an interval defined by a minimum and maximum value. One of its main uses is to describe certainty or random variation. The other main use of the beta distribution is that one can rescale and shift it to create distributions with a wide range of shapes and over any finite range and as a result it has been used for a wide variety of other applications.

1.2 Statement of the Problem

The general problem in distribution theory today is that a lot of available dataset today do not follow any of the existing and well known standard probability distributions (models) and hence cannot be model appropriately.

Besides the uses and importance of the Kumaraswamy distribution, there has been no research on the Beta-Kumaraswamy distribution since the introduction of the Betafamily of compound distributions, that is, no researcher has combined the two distributions together. Therefore, we provide a compound distribution with better performance.

1.3 Aim and objectives of the study

The aim of this work is to extend a Kumaraswamy distribution using the Beta generalized family (Beta-G) by Jones (2004), derive its properties and evaluate its performance using two real life dataset.

The Specific Objectives To:

i. Define the PDF and CDF of Beta-Kumaraswamy distribution.

ii. Derive some statistical properties of the new distribution such as the moments, the moment generating function, the characteristics function, the survival function, hazard function, quantile function and distribution of order statistics.

iii. Obtain the non linear system of equation partial derivative of the log likehood function with respect to each parameter.

iv. Evaluate the performance of the distribution using simulated data and real-life data set.

v. Evaluate the performance of the distribution and compare it to other distribution.

1.4 Significance of the Study

The study of the proposed distribution, its properties and the parameter estimates has increase the flexibility of the kumaraswamy distribution which will model various dataset that cannot be properly fitted by the existing generalized kumaraswamy distribution using three information criteria. Akaike information criterion, Consistent Akaike Information criterion and Bayesian Information Criterion to identify the most fitted model under two real life data set.

1.5 Motivation

From the work of jones *et al.* (2004) proposed a general family (Beta G family) a two parameter generator that can be used to generalize any continuous distribution, taking advantage of this to generalize the kumaraswamy distribution this will increase flexibility, study some properties of the new generalized distribution and apply it to some real data set. many authors generalize Beta-generated distribution, using different forms of G(x)(the base line distribution). These distributions include the normal, Cauchy, Laplace, Exponential, Pareto, Gumbel and beta-generalized Weibull distribution. Among others.

Statistically, modeling of real life scenario help us to better understand and explain random events when they occur, thereby enabling us to reproduce such scenario either on large or on simplified scale aimed at describing only critical parts of the phenomenon. These real life phenomena are captured by means of statistical learned directly from data gathered about them. Every distribution model has a set of parameters that needs to be estimated. These parameters specify the model and provide a mechanism for efficient and accurate use of data.

1.6Scopeof The Study

In this dissertation, we shall only introduce a new distribution, derive some mathematical expressionssuch as, moments, moment generating function, characteristics function, reliability analysis, density function, orderstatistics, estimate the model parameters using the method of maximum likelihood estimation MLE.Finally accessed the performance of the proposed distribution compared to other extensions of Kumaraswamy distribution.

CHAPTER TWO

LITERATURE REVIEW

There are several generalizations of the Kumaraswamy distributions some of which have considered the base distribution or the others like Nadarajah and Kotz (2004)extended the logit of betadistribution. The Kumaraswamy distribution has also being used both as a generalized distribution and as a generator. The essence of this review is to provide an understanding of the tools and techniques often employed in generalizations of continuous probability distributions so as to enable the researcher add some parameters to the well-known Kumaraswamy distribution. These among others include the following:

Jones*et al.,*(2004) introduced a general family (Beta-Generalized family) that prompted other authors to consider generalized distributions with some statistical applications like mhmoudi (2011) introduced the beta generalized pareto distribution. Thereafter, some authors have defined and studied various forms of beta-compounded distributions. These among others include;

Nadarajah and Kotz (2004) were motivated by the work of Eugene *et al*. (2002) to generalize the Gumbel distribution thereby introducing the beta-Gumbel distribution (BGD). It was generated from the logit of a beta random variable. A generalization of the Gumbel distribution was obtained with the hope of attracting broad application in engineering.

The Gumbel distribution is widely used in statistical distribution for problems mainly in engineering. They found the generalizations to have allowed greater flexibility of its tail, because most problems studied using Gumbel distribution are concerned with the tail behavior of one or more variables.

The PDF of the beta Gumbel distribution exhibited a single mode and the hazard rate function which is an important quantity that characterizes life phenomena. An expression for the nth moment of the distribution was also the variance, skewness, and kurtosis measures were computed.the beta Gumbel distribution was more flexible than the Gumbel distribution. The parameters of the distribution were estimated by method of Maximum Likelihood.

Nadarajah and Kotz (2004) introduced the beta Frechet(BFr) distribution by taking *G(x)* to be the Frechet distribution ,derived the analytical shapes of the density and hazard rate functions and obtained the asymptotic distribution of the extreme order statistics.

Furthermore, Nadaraja and Kotz (2006) introduced the beta exponential distribution (BED) by generalization of the exponential distribution, using the logit of a beta random variable The distribution was extended with the goal that it will attract wider applications in reliability analysis.Statisticalproperties of the distribution was define.exponentiated exponential distribution is a special case of BED (Gupta and Kundu, 2003). The exponential distribution was another special case of BED the shapes of the PDF and the hazard rate function of the BED depend only on the parameter α (shape parameter). The PDF was noted to monotonically decrease if $\alpha < 1$. It was also noted if $\alpha > 1$ then the PDF had a unique mode at $x = x_0$ with the PDF increasing for $x > x_0$ and the PDF decreasing for $x > x_0$. It was noted that if $\alpha < 1$, then the hazard rate function monotonically decreases with x. while, if $\alpha > 1$, and then the hazard rate function monotonically increases with *x*.

The moment generating function and the characteristics function were derived for the BED. The nth moment of the BED was obtained from the moment generating function and obtained the first four moments. The first three central moments, Swekness and the kurtosis of the distribution were also obtained. It was noted that the Swekness and the kurtosis depend only on α and β (Nadaraja and Kotz, 2006).

The beta-Weibull distribution (BWD) with four parameters was proposed by Famoye *et al.* (2005). Like the previously mentioned generalized distributions, it was also generated from the logit of a beta random variable. It was shown that the beta-Weibull distribution is unimodal and some results on the non-central moments were obtained. Lee *et al*. (2007) gave some properties of the hazard function, entropies and an application to censored data for the BWD. Additional mathematical properties of this distribution were derived by Carrasco *et al*. (2008) such as expressions for the moments. The parameters were estimated using the maximum likelihood technique and a likelihood ratio test was derived for the beta-Weibull distribution.

The Weibull distribution is known to have wide applications in many disciplines. Different generalizations of the Weibull distribution have also been studied. The Weibull-exponential distribution was proposed by Oguntunde *et al*. (2015). Mudholkar and Kollia (1994) introduced a generalized Weibull distribution with an additional shape parameter. There was later a paper on the application of the generalized Weibull distribution to model survival data by Mudholkar and Hutson (1996). They showed that the graph of the pdf of the distribution may be increasing, decreasing or bathtub depending on the values of the parameters. The hazard function was further shown to be unimodal. An application of the exponentiated Weibull distribution on extreme value data was also done. The exponentiated exponential distribution was also shown to be a special case of the exponentiated Weibull family.

It was also noted that the biases of the maximum log likelihood estimates from beta-Weibull distribution were smaller than those from exponentiated Weibull model with comparable standard errors when $\beta < 1$.

Akinsete *et al.*(2008) defined and studied a four-parameter distribution called beta-Pareto distribution (BPD), just like the previous distributions studied it, therefore, it was generated from the logit of a beta random variable. The Pareto distribution is mainly used in modeling the heavy-tailed distributions that include data on income distribution, city population size and size of firms. Some other quantities measured in the physical, biological, technological and social systems of various kinds.

Akinsete*et al*.,(2008) noted that when $\alpha = \beta = 1$(where α, β are shape parameters) the BPD reduces to the Pareto distribution with parameters k and θ as shape and scale parameters respectively. It was also shown that when $\alpha = 1$, the BPD reduces to the Pareto distribution with parameters $k\beta = c$ and θ. The hazard rate of the BPD was derived, and it was noted that this reduces to the hazard rate of the Pareto distribution when $\alpha = \beta = 1$. The BPD was noted to have unimodal failure rate when $\alpha > 1$ and a decreasing failure rate when $0 < \alpha \leq 1$. The moments of the BPD were obtained using the beta integral. Using this, the mean, variance, skewness and kurtosis were also obtained. The mean deviation from the mean, the mean deviation from the median, the Renyi entropy and the Shannon entropy were obtained. The method of maximum likelihood was also used to estimate the parameters.

Bourguignon *et al*., (2012), proposed the Kumaraswamy Pareto distribution. They defined and expressed the pdf of the distribution as a linear combination of the Pareto densities, which allowed them to derive some of its mathematical properties like the moments, generating function and they provided the density function of the order statistics and obtain their moments. The method of maximum likelihood was used for estimating the model parameters and the observed information matrix was derived.

Santana *et al*., (2012) introduced the Kumaraswamy-log-logistic distribution based on the log-logistic and Kumaraswamy distributions. The new distribution was found to contain several important distributions discussed in the literature as sub-models such as the log-logistic, exponentiated log-logistic and Burr XII distributions, among several others. The beauty and importance of the new distribution lies in its ability to model non-monotonic failure rate functions, which are quite common in lifetime data analysis and reliability. Some of its structural properties were studied the method of maximum likelihood was used to estimate the model parameters. Two real data sets were used to illustrate the importance and flexibility of the proposed models.

Shittu and Adepoju (2013) used existing two parameters Nakagami distribution to extend and introduce two extra shape parameters- thus defining the beta-Nakagami distribution which has a better shape, broader tails and a class of hazard rate functions depending on the parameters. In the paper, they studied the statistical properties of the proposed distribution which include moments and moment generating function among others. The parameters of the defined model were estimated by method of maximum likelihood, to enable the derivation of fisher information matrix. Application to real-life indicates that Beta-Nakagami distribution apart from being more flexible has better representation of data than Nakagami distribution.

El-sherpieny and Ahmed (2014) proposed a new distribution called the Kumaraswamy-Kumaraswamy (KK) distribution, as a Special model from the class of Kumaraswamy Generalized (KW-G) distributions. The probability density function (pdf), the cumulative distribution function (cdf), moments, quantiles, the median, the mode, the mean deviation, the entropy, order statistics, L-moments and parameters estimation based on maximum likelihood were obtained. A numerical illustration was used to study the properties of the parameters. Finally, it was found that the KK distribution provides a rather flexible mechanism of fitting a wide spectrum of data sets.

Gamze *et al*., (2014) introduce the Kumaraswamy generalized exponential Weibull distribution beauty and importance of this distribution lies in its ability to model monotone and non-monotone failure rate functions which are quite common in the environmental studies. The new distribution has a number of well-known lifetime special sub-models such as exponential Weibulland Rayleigh distribution. Some statistical properties of the proposed distribution were studied.

Bourguignon *et al*., (2014) also developed a new wider family of distributions called the Weibull generalized (Weibull-G) family of distributions. They derived general mathematical properties of the new wider Weibull family of distributions. They provided some of this structural properties such as the ordinary and incomplete moments, Quantile function and order statistics. This study also proposed that for any Weibull-G family distributions, the estimation of model parameters should be done by method of maximum likelihood estimation. They also proposed that for any baseline distribution G, their methodology should be adopted.

Cordeiro *et al*., (2014) proposed a new class of distributions called the Lomax generator with two extra positive parameters to generalize any continuous baseline distribution. Some special models such as the Lomax-normal, Lomax–Weibull, Lomax-log-logistic and Lomax–Pareto distributions were discussed. Some mathematical properties of the new generator including ordinary and incomplete moments, quantile and generating functions mean and median deviations, distribution of the order statistics and some entropy measures were presented. They discussed the estimation of the model parameters using the method of maximum likelihood. They also proposed a minification process based on the marginal Lomax-exponential distribution. They also defined a log-Lomax–Weibull regression model for censored data. The importance of the new generator was illustrated by means of three real data sets.

Adepoju and Chukwu (2014) used existing two parameter power exponential distribution to extend it by two extra shape parameters to give rise to the beta-Power Exponential Distribution (BPED) which has a better shape, broader tails and a class of hazard rate functions depending on the parameters. They studied in detail some of the statistical properties of the proposed distribution which include moments and moment generating function among others. The parameters of the model were estimated by method of maximum likelihood which actually paves way for the derivation of Fisher information matrix. Real-life application indicates that Beta-Power Exponential Distribution apart from its flexibility has better representation of data than exponentiated exponential distribution.

Chukwu and Ogunde (2015) introduced and studied the beta Gompertz-Mekaham (BGM) distribution. In the paper, the mortality model was derived. Various properties of the distribution were discussed in the paper; these include the asymptotic behaviors, Cumulative density function and hazard functions among others. Also, they derived the estimation of parameters by the method of maximum likelihood. Real-life application indicates that new distribution apart from being more flexible has better representation of data than its sub-modelGompertz-Mekaham distribution.

Bourguignon *et al*., (2014) also developed a new wider family of distributions called the Weibull generalized (Weibull-G) family of distributions. They derived general mathematical properties of the new wider Weibull family of distributions. They provided some of this structural properties such as the ordinary and incomplete moments, Quantile function and order statistics. This study also proposed that for any Weibull-G family distributions, the estimation of model parameters should be done by method of maximum likelihood estimation. They also proposed that for any baseline distribution G, their methodology should be adopted.

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Cordeiro *et al.*, (2015), proposed a new generalized Weibull family of distributions with two extra positive parameters, the family can be express as a mixture of linear combination of exponentiated G densities function. This family was able to generate any continuous probability distributions in full generality. This family also provide explicit expressions for the ordinary and incomplete moments, generating function, mean deviations, probability weighted moments, entropies, reliability and distributions of order statistics. The work proposed that, the parameters of this distributions family can be estimated by the method of maximum likelihood estimation (MLEs).

Merovci and Elbatal (2015) used the Methodology by Bourguignon *et al*., (2014) to define the Weibull-Rayleigh (WR) distribution. They derived expressions for the moments and the moments generating functions. Estimation of parameters of their model was done by the method of maximum likelihood estimation. Their model performance was compared to some baseline distributions such as Beta-Weibull, Exponentiated-Weibull and the standard Weibull distribution. The results of this comparison using the likelihood ratio statistics showed that the Weibull-Rayleigh distribution can be used quite effectively to provide better fit than the Weibull distribution.

Javanshiri*et al*., (2015) proposed and studied exp-kumaraswamy distribution. Some of its properties were derived, including the density function, hazard rate function, quantile function, moments, skewness and kurtosis. A data set was used to illustrate an application of the proposed distribution. The New distribution is an alternative to skew-normal distribution, Basic properties of this new distribution, such as moment generating function, moments, skewness, kurtosis and maximum likelihood estimation were studied. Its applicability was also illustrated by means of two real data sets.

Oguntunde *et al*. (2015) introduced the Kumaraswamy Power distribution a generalization of the power distribution. The proposed model serves as a generalization of the two-parameter Power distribution using the Kumaraswamy Generalized family of distributions. They investigated some of its statistical properties. The Generalized Power distribution, Exponentiated Power distribution and the Power distribution were found to be sub-models of the proposed distribution. The method of maximum likelihood estimation was proposed in estimating the parameters of the model.

Gupta *et al.*,(2015) introduced a new distribution constructed by composition of the cumulative density functions (*cdf*) of Frechet probability distribution and Lomax probability distribution which was named as the Lomax-Frechet distribution. With wider applicability in engineering,They derived expressions for its moments, characteristic function, hazard rate function and survival function. They plot some graphs for its probability density function (pdf) using the software 'Mathematical' and also investigated the variation of the skewness and kurtosis.

Khan *et al.*, (2016) proposed a generalization of Kumaraswamy distribution which is referred to as the transmuted Kumaraswamy (TK) distribution. The new transmuted distribution was developed using the quadratic rank transmutation map studied by Shaw and Buckley (2007). A comprehensive account of the mathematical properties of the new distribution was provided. Explicit expressions were derived for the moments, moment generating function, entropy, mean deviation, Bonferroni and Lorenz curves, and formulated moments for order statistics. The TKdistribution parameters were estimated by using the method of maximum likelihood. Monte Carlo simulation was performed in order to investigate the performance of the MLEs. A flood data and HIV/ AIDS data applications were used to illustrate the usefulness of the proposed model.

Mandouh (2016) did a Bayesian analysis for the Kumaraswamy-Weibull (Kum-W) distribution. Approximate Bayes estimates were obtained under the assumptions of non-informative priors using the Gibbs sampling procedure.This procedure allowed for generating samples from posterior distributions. Also, using Bayesian framework, the predictive density for a single future response, a bivariate future response, and several future responses were derived. The predictive means, standard deviations, highest predictive density (HPD) intervals, and the shape characteristics for a single future response were also determined. Finally, applications to real data sets were utilized to illustrate the potentiality of the Bayesian analysis and the predictive results.

Adepoju*et al*. (2016) proposed the Kumaraswamy-F (KUMAF) distribution which is a generalization of the conventional Fisher Snedecor (F-distribution). The new distribution can be used even when one or more of the regular assumptions are violated. It was obtained with the addition of two shape parameters to a continuous F-distribution which is commonly used to test the null hypothesis in the Analysis of Variance (ANOVA test). The statistical properties of the proposed distribution such as moments, moment generating function, the asymptotic behavior among others were investigated. The method of maximum likelihood was used to estimate the model parameters and the observed information matrix was derived. The distribution was found to be more flexible and robust to regular assumptions of the conventional F-distribution. They stated that in future research, the flexibility of this distribution as well as its robustness using a real data set will be examined. The new distribution was recommended for use in most applications where the assumption underlying the use of conventional F distribution for one-way analysis of variance are violated such as homogeneity of variance or normality assumption probably as a result of the presence of outlier(s). It is instructive to note that the new distribution preserves the originality of the data without transformation.

(Domma and Condino, 2016) by constructing beta generated Kumaraswamy –G distribution. expansions of the *pdf* and the *cdf*of the proposed family were derivedand seen as infinite mixtures of the Kumaraswamy–G distribution. Order statistics, Probability weighted moments, moment generating function, Rényi entropies, quantile power series, random sample generation, asymptotes and shapes were also investigated. Two methods of parameter estimation werepresented. Suitability of the proposed model in comparisons to its sub models was carried out considering two real life data sets. Finally, some new classes of beta generated families were proposed for future investigations.

Afify*et al*., (2016), introduced a four parameter distribution named the Weibull-Frechet distribution using the family by Bourguignon *et al*.,(2014). They studied some of its mathematical and statistical properties. The applications of the new distribution with some baseline distributions showed that it is more fitted compared to kumaraswamy-Frechet (*KFr*), exponentiated-Frechet (*EFr*), Beta-Frechet (*BFr*), gamma extended Frechet (*GEFr*), transmitted marshalOlkin-Frechet (*TMOFr*) and Frechet (*Fr*) distributions

CHAPTER THREE

3.0METHODOLOGY

3.1.1 Defining the Beta distribution

Beta distribution is a continuous probability distribution with two positive shape parameters α and β and it is the conjugate prior of the binomial distribution.

The probability density function (PDF) of the beta distribution is given by equation (3.1):

$$f(x) = \frac{(1-x)^{\beta-1} x^{\alpha-1}}{B(\alpha,\beta)}, \quad 0 < x < 1, \ \alpha > 0 \ , \beta > 0 \tag{3.1}$$

where $B(\alpha,\beta)$ is the beta function defined as

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \tag{3.2}$$

The $B(\alpha,\beta)$ can also be defined in terms of gamma function by equation (3.3):

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3.3}$$

where $\Gamma(.)$ is the gamma function which is defined by:

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-\alpha y} dy = (\alpha-1)! \tag{3.4}$$

Integral that defines the beta function can be generalized by changing the interval of integration from (*0,1*) to (*0,x*) where x ∈ (0,1),this give incomplete beta functionwhich is defined by (3.5):

$$B(t;\alpha,\beta) = \int_0^\infty t^{\alpha-1}(1-t)^{\beta-1} dt \tag{3.5}$$

where

$$t \in (0,1)$$

The cumulative distribution function (CDF) of the beta distribution is given by

$$F(x) = \frac{1}{B(\alpha,\beta)} \int_0^x t^{\alpha-1} (1-t)^{\beta-1} dt \tag{3.6}$$

where,

$$0, < \alpha, \beta < \infty$$

F(x) is sometimes known as the regularized incomplete beta function. it can be expressed as by (3.7)

$$F(x) = \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)}, \quad 0 < x < 1 \tag{3.7}$$

In some cases, the distribution function *F(x)* and its inverse$F^{-1}(x)$quartile function can be computed in closed form, without resorting to special functions.

The beta distribution has two parameters with a rich variety of shapes. Qualitatively, the first order properties of beta distribution depend on whether each parameter is less than, equal to, or greater than 1. Beta distribution is very flexible and can be used to analyze different types of data sets. Many of the baseline distributions encountered in practice can easily be transmuted into the standard beta distribution. In econometric, quite often the data are analyzed by using baseline distributions (Dila, 2015).

Kumaraswamy Distribution

A random variable X with parameters a and b is said to follow the Kumaraswamy distribution if its probability density function and cumulative distribution function are respectively given by equations (3.8) and (3.9)

$$g(x) = \begin{cases} abx^{a-1}\left(1-x^{a}\right)^{b-1} & 0 \leq \mathrm{x} \leq 1 \\ 0 & elsewhere \end{cases} \tag{3.8}$$

$$G\left(x\right)=\left[1-\left(1-x^{a}\right)^{b}\right] \tag{3.9}$$

where$a$ and b are the shape parameters of the distribution and $\mathrm{a,b} > 0$

The location parameter is used to determine the position of the data distribution along the x-axis. The location is either estimated from the data or specified based on historical process knowledge. The location parameter defines the shift of the data. A positive location value shifts your distribution to the right hand side, while a negative location value shifts your data distribution to the left hand side.

3.1.2 Generalization of the beta distribution

Generalized Beta Distributions have been widely studied in statistics and a lot of researchers have come up with various classes of these distributions, The main use of the beta distribution as stated earlier has being used in the generalization of different distributions, which has been introduced and studied by different authors in literature such as: Nadarajah and Gupta, (2004).Let X be a random variable with parameter vector $\Theta=\left(\alpha,\beta,\xi\right)$ where ξ is also a vector of parameters of the baseline distribution, the random variable X is said to follow the beta-G distribution if its CDF is defined by (3.10)

$$F\left(x,\alpha,\beta,\xi\right)=\frac{1}{B\left(\alpha,\beta\right)}\int_0^{G(x,\xi)} y^{\alpha-1}\left(1-y\right)^{\beta-1}dy \tag{3.10}$$

where $\alpha,\ \beta>0$

while the corresponding probability density function $g\left(x\right)=G^{'}\left(x\right)$ is defined by (3.11)

$$f\left(x,\ \alpha,\ \beta,\ \xi\right)=\frac{1}{B\left(\alpha,\beta\right)}\left(G(x;\xi)\right)^{\alpha-1}\left(1-G\left(x;\xi\right)\right)^{\beta-1}g\left(x;\xi\right) \tag{3.11}$$

where $G^{'}\left(x:\xi\right)=g\left(x:\xi\right)$ and $G\left(x:\xi\right)$ are PDF andCDFof any continuous distribution to be generalized respectively and α>0 and β>0 are the two new additional shape parameters of the distribution.

3.1.3 The Proposed distribution

Suppose that $f\left(x,\ \alpha,\ \beta,\ \xi\right)$ denotes the probability density function(PDF) of a random variable*X*.The PDF of a generalized class of distributionfor the random variable *X*is defined by equation (3.11). Substituting the pdf and cdf of the Kumaraswamy distribution defined by equations (3.8) (3.9) respectively yields:

$$f(x,\alpha,\beta,a,b)=\frac{1}{B\left(\alpha,\beta\right)}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}\left[\left(1-x^{a}\right)^{b}\right]^{\beta-1}abx^{a-1}\left(1-x^{a}\right)^{b-1}$$

$$=\frac{abx^{a-1}}{B(\alpha,\beta)}\left(1-x^{a}\right)^{b(\beta-1)+b-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}$$

$$f(x)=\frac{abx^{a-1}}{B(\alpha,\beta)}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}$$

Hence the proposed distribution is given by equation (3.12)

$$f(x)=\frac{abx^{a-1}}{B(\alpha,\beta)}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1} \qquad \alpha,\beta,a,b>0 \tag{3.12}$$

3.1.4 Model validity check

To check the validity of the proposed model we use the equation below.

Proof

A given PDF $f(x)$ is said to be valid if $\int_0^\infty f(x)dx=1$.

Proof

$$f(x)=\int_0^1\frac{abx^{a-1}}{B(\alpha,\beta)}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}dx$$

$$let\ p=1-x^{a}\Rightarrow x=\left(1-p\right)^{\frac{1}{a}}$$

$$ax^{a-1}dx=-dp\Rightarrow dx=\frac{-dp}{a\left(1-p\right)^{\frac{a-1}{a}}}$$

as x→ 0, p→ 1 and as x→ 1, p→0

$$\int_0^1\frac{ab\left(1-p\right)^{\frac{a-1}{a}}}{B\left(\alpha,\beta\right)}p^{b\beta-1}\left[1-p^{b}\right]^{\alpha-1}\frac{dp}{a\left(1-p\right)^{\frac{a-1}{a}}}$$

$$=\frac{b}{B\left(\alpha,\beta\right)}\int_0^1 p^{b\beta-1}\left(1-p^{b}\right)^{\alpha-1}dp$$

Let $m=1-p^{b}\Rightarrow p=\left(1-m\right)^{\frac{1}{b}}$, and $bp^{b-1}dp=-dm\Rightarrow dp=\dfrac{-dm}{b\left(1-m\right)^{\frac{b-1}{b}}}$

$as\ p\rightarrow 0, m\rightarrow 1$ and

$as\ p\rightarrow 1, m\rightarrow 0$

$$\int_0^1 \frac{b}{B(\alpha,\beta)}(1-m)^{\frac{b\beta-1}{b}} m^{\alpha-1} \frac{dm}{b(1-m)^{\frac{b-1}{b}}}$$

$$\frac{1}{B(\alpha,\beta)}\int_0^1 m^{\alpha-1}(1-m)^{\beta-1}\,dm$$

$$=\frac{B(\alpha,\beta)}{B(\alpha,\beta)}=1$$

Therefore, $f(x)$ is indeed a proper probability density function of a continuous distribution.

3.1.5 Expansion of the probability Density Function

Using the binomial series expansion $(1-x)^n=\sum_{r=0}^{\infty}(-1)^r\binom{n}{r}x^r$ on the binomial series in the last term $\left[1-(1-x^a)^b\right]^{\alpha-1}$ in equation (3.12)we have,

$$\left[1-(1-x^a)^b\right]^{\alpha-1}=\sum_{r=0}^{\infty}(-1)^r\binom{\alpha-1}{r}(1-x^a)^{br}$$

Hence,

$$f(x)=\frac{abx^{a-1}}{B(\alpha,\beta)}\sum_{r=0}^{\infty}(-1)^r\binom{\alpha-1}{r}(1-x^a)^{br+b\beta-1}$$

$$f(x)=\frac{1}{B(\alpha,\beta)}\sum_{r=0}^{\infty}(-1)^r\binom{\alpha-1}{r}\frac{ab(r+\beta)}{r+\beta}x^{a-1}(1-x^a)^{b(r+\beta)-1}$$

$$f(x)=\sum_{r=0}^{\infty}w_r g(x;a,b(r+\beta)) \tag{3.13}$$

where $w_r=\dfrac{(-1)\binom{\alpha-1}{r}}{(r+\beta)\beta(\alpha,\beta)}$

g(x:a,b(r+β)) is the pdf of kumaraswamy distribution with parameters a and b(r+β).equation (3.13)shows that the beta-Kumaraswamy distribution is an infinite linear combination of the Kumaraswamy distribution .if α is an integer, the index r stop at $\alpha - 1$.

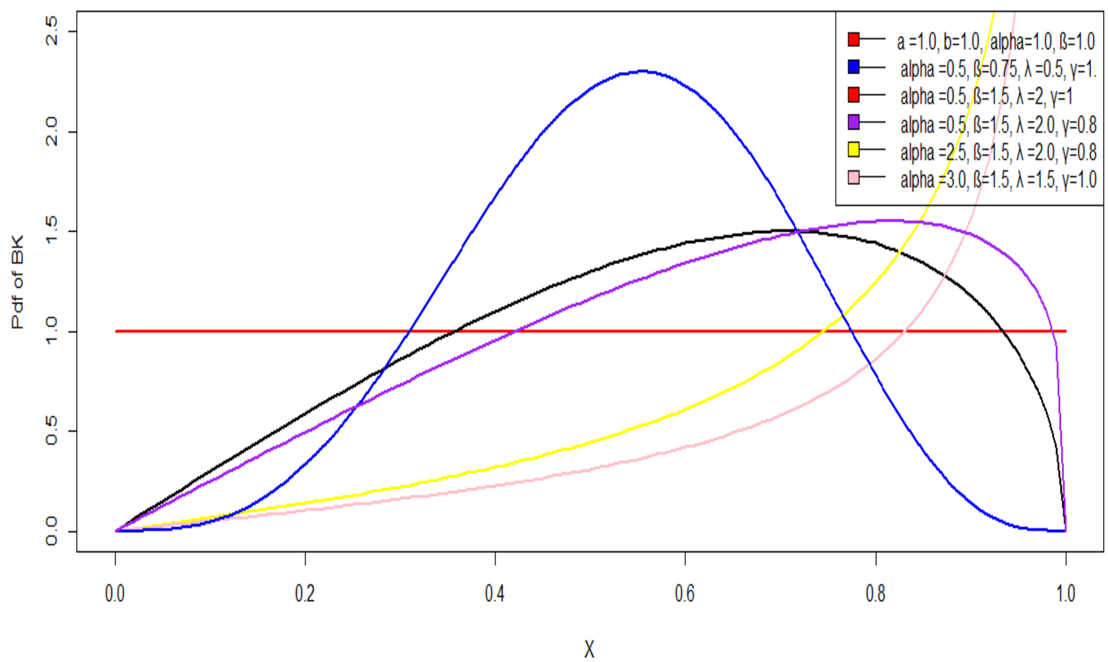

Figure 3.1: The graph of the pdf of beta-Kumaraswamy distribution for various parameter values.

The graph of the pdf of the distribution is skewed right keeping all $a = b = \alpha$ constant,j shape or bathtubbase on the values of the parameters. Keeping all the parameters constant the distribution tendto uniform distribution.As $a = b = \alpha = \beta = 1$

3.1.6 Asymptotic Behavior

To determine the performance of the beta-Kumaraswamy distribution, we study the asymptotic behavior of the distribution as X tends to zero and as X tend to 1.

$$f(x) = \frac{abx^{a-1}}{B(\alpha,\beta)}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}$$

$$\lim_{x\to 0} f(x) = 0 \quad \text{Since} \quad \lim_{x\to 0} x^{a-1} = 0$$

$$\frac{ab}{\beta(\alpha,\beta)}\lim_{x\to 0}x^{a-1}\ \lim_{x\to 0}\left(1-x^{a}\right)^{b\beta-1}.\lim_{x\to 0}\left(1-\left(1-x^{a}\right)\right)$$

$$\lim_{x\to 1}f(x)=0 \qquad \lim_{x\to 1}\left(1-x^{a}\right)^{b\beta-1}=0$$

$$\frac{ab}{\beta(\alpha,\beta)}\ \lim_{x\to 1}x^{a-1}\ \lim_{x\to 1}\left(1-x^{a}\right)^{b\beta-1}.\lim_{x\to 1}\left(1-\left(1-x^{a}\right)\right)$$

Hence, since $\lim_{x\to 0}f(x)=\lim_{x\to 1}f(x)=0,$ we conclude that the beta-Kumaraswamy distribution is having at least one mode.

3.1.7 Cumulative Density Function OF Beta-KumaraswamyDistribution

The Cumulative distribution function of the beta-kumaraswamy distribution is derived asfollows:

$$F(Y)=\int_{0}^{x}\frac{aby^{a-1}}{B(\alpha,\beta)}\left(1-y^{a}\right)^{b\beta-1}\left[1-\left(1-y^{a}\right)^{b}\right]^{\alpha-1}dy$$

Let $p=1-y^{a}\Rightarrow y=\left(1-p\right)^{\frac{1}{a}}$, and $ay^{a-1}dy=-dp\Rightarrow dy=\frac{-dp}{a\left(1-p\right)^{\frac{a-1}{a}}}$. Then,

$as\ \ y\Rightarrow 0, p\Rightarrow 1$ and

$as\ \ y\Rightarrow x, p\Rightarrow 1-y^{a}$

Therefore,

$$F\left(Y\right)=\int_{1}^{1-y^{a}}\frac{ab}{B(\alpha,\beta)}\left(1-p\right)^{\frac{a-1}{a}}\ p^{b\beta-1}\left(1-p^{b}\right)^{\alpha-1}\frac{dp}{a\left(1-p\right)^{\frac{a-1}{a}}}$$

$$=\frac{b}{B(\alpha,\beta)}\int_{1}^{1-y^{a}}p^{b\beta-1}\left(1-p^{b}\right)^{\alpha-1}dp$$

$$let\, m=1-p^{b}\rightarrow p^{b}=1-m\rightarrow p=\left(1-m\right)^{\frac{1}{b}}$$

$$bp^{b-1}dp = \frac{-dm}{b(1-m)\frac{b-1}{b}}$$

$$as\ p \to 1, m \to 0$$

$$as\ p \to 1-y^a, m \to 1-\left(1-y^a\right)^b$$

$$F(x) = \frac{b}{B(\alpha,\beta)} \int_0^{1-\left(1-y^a\right)^b} (1-m)^{\frac{b\beta-1}{b}} m^{\alpha-1} \frac{dm}{b(1-m)^{\frac{b-1}{b}}}$$

$$\frac{1}{B(\alpha,\beta)} \int_0^{1-\left(1-y^a\right)^b} m^{\alpha-1}(1-m)^{\beta-1}\,dm$$

Therefore,

$$F(x) = \frac{B\left[1-\left(1-y^a\right)^b, \alpha, \beta\right]}{B(\alpha,\beta)} \tag{3.14}$$

where $B\left[1-\left(1-y^a\right)^b, \alpha, \beta\right]$ Is an incomplete Beta function.

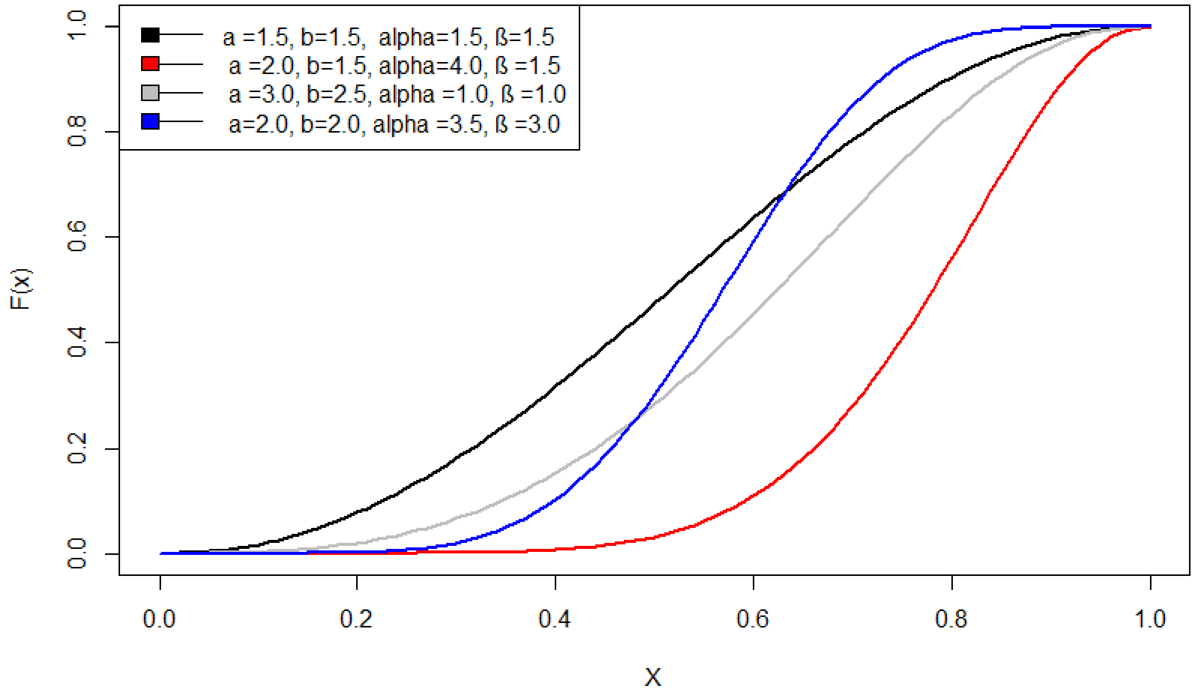

Figure 3.2: The graph of the cdf of beta-kumaraswamy distribution for some selected parameter values.

We can see from figure 3.2 that the value of the CDF function increases and slowly drops as *X* increases. This means that the *BKWD* may be appropriate for modeling time dependent events.

3.1.7 Some properties of the proposed distribution

3.1.8 Moments

Let *X* denotes a continuous random variable, the q^{th} moment of a random variable*X* is defined as:

$$E(X) = \int_{-\infty}^{\infty} x^q f(x) dx$$

Hence, the qth moment of our proposed distribution is obtained as follows.

$$E\left(x^q\right) = \int_0^1 \frac{x^q abx^{a-1}\left(1-x^a\right)^{b\beta-1}\left[1-\left(1-x^a\right)^b\right]^{\alpha-1}}{B(\alpha,\beta)} dx \tag{3.15}$$

Using the binomial series expansion

$$\left(1-p\right)^{n}=\sum_{r=0}^{\infty}\left(-1\right)\frac{n!}{\left(n-r\right)!r!}p^{r}\ \ for\ \ \left|p\right|<1 \tag{3.16}$$

Hence, the term $\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}$ can be written in the form

$$\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}=\sum_{r=0}^{\infty}\frac{\left(-1\right)^{r}\left(\alpha-1\right)!}{\left(\alpha-r-1\right)!r!}\left(1-x^{a}\right)^{br}$$

Therefore, equation (3.16) becomes:

$$E\left(X^{q}\right)=\int_{0}^{1}\frac{ab}{B\left(\alpha,\beta\right)}x^{a+q-1}\left(1-x^{a}\right)^{b\beta-1}\sum_{r=0}^{\infty}\frac{\left(-1\right)^{r}\left(\alpha-1\right)!}{\left(\alpha-r-1\right)!r!}\left(1-x^{a}\right)^{br}dx$$

$$=\frac{ab}{B\left(\alpha,\beta\right)}\int_{0}^{1}x^{a+q-1}\sum_{r=0}^{\infty}\frac{\left(-1\right)^{r}\left(\alpha-1\right)!}{\left(\alpha-r-1\right)!r!}\left(1-x^{a}\right)^{b\beta+br-1}dx$$

$$\frac{ab}{B\left(\alpha,\beta\right)}\sum_{r=0}^{\infty}\frac{\left(-1\right)^{r}\left(\alpha-1\right)!}{\left(\alpha-r-1\right)!r!}\int_{0}^{1}x^{a+q-1}\left(1-x^{a}\right)^{b\beta+br-1}dx$$

$$E\left(x^{q}\right)=\frac{ab}{B\left(\alpha,\beta\right)}\sum_{r=0}^{\infty}\frac{\left(-1\right)^{r}\left(\alpha-1\right)!}{\left(\alpha-r-1\right)!r!}\int_{0}^{1}x^{a+q-1}\left(1-x^{a}\right)^{b\beta+br-1}dx \tag{3.17}$$

taking the integral part in equation (3.17)

$$let\ \ m=1-x^{a}\ \Rightarrow x^{a}=1-m\rightarrow x=\left(1-m\right)^{\frac{1}{a}}$$

$$ax^{a-1}dx=-dm\Rightarrow dx=\frac{-dm}{a\left(1-m\right)^{\frac{a-1}{a}}}$$

$$\int_{0}^{1}\left(1-m\right)^{\frac{b\beta+br-1}{a}}m^{b\beta+br-1}\frac{dm}{a\left(1-m\right)^{\frac{a-1}{a}}}$$

$$\frac{1}{a}\int_{0}^{1}m^{b\beta+br-1}\left(1-m\right)^{\frac{q}{a}}dm$$

$$E\left(X^{q}\right)=\frac{1}{a}B\left(b\beta+bi,\frac{q}{a}+1\right) \tag{3.18}$$

Substituting (3.18) in (3.17) yield

$$E\left(X^{q}\right)=b\sum_{i=0}^{\infty}\frac{(-1)^{i}(\alpha-1)!}{(\alpha-i-1)!i!}\;\frac{B\left(b\beta+bi,\frac{q}{a}+1\right)}{B(\alpha,\beta)} \tag{3.19}$$

3.1.8Moment generating function

Let $X \sim Bkw$ (a,b,α, β,) the mgf of X denoted by $M_X(t)=E\left(e^{tx}\right)$ is given by:

$$M_X(t)=\sum_{p=0}^{\infty}\frac{t^{p}}{p!}E(X^{K})$$

where $E(X^{K})$ follow from equation (3.19)

3.1.9 Skewness and Kurtosis

The n^{th} central moment and n^{th} cumulant can easily be obtained from equation (3.19) by (3.20) and (3.21) respectively

$$\mu_n=\sum_{z=1}^{n}(-1)^{z}\binom{n}{z}\left(\mu_1^{'}\right)^{z}\mu_{n-z} \tag{3.20}$$

$$\kappa_n=\mu_n^{'}-\sum_{z=0}^{n-1}(-1)^{z}\binom{n-1}{z-1}\kappa_z\mu_{n-1}^{'} \tag{3.21}$$

Hence, to obtained the coefficient of skewness and kurtosis, we obtain second, third and fourth cumulant which are

$$\kappa_2 = \mu_2' - \left(\mu_1'\right)^2, \quad \kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3, \quad \kappa_4 = \mu_4' - 4\mu_3'\mu_1' + 12\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_2'\right)^2 - 6\left(\mu_1'\right)^4$$

respectively. Therefore the coefficient of skewnessand kurtosis are given as follows: (3.22)and (3.23):

$$coefficient\ of\ skewness = \frac{\kappa_3}{\left(\kappa_2\right)^{3/2}} = \frac{\mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3}{\left[\mu_2' - \left(\mu_1'\right)^2\right]^{3/2}} \tag{3.22}$$

$$coefficient\ of\ kurtosis = \frac{\kappa_4}{\kappa_2^2} = \frac{\mu_4' - 4\mu_3'\mu_1' + 12\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_2'\right)^2 - 6\left(\mu_1'\right)^4}{\left(\mu_2' - \left(\mu_1'\right)^2\right)^2} \tag{3.23}$$

3.1.10Hazard Rate Function

The hazard rate function of a random variable *X* with probability density function *f(x)* and a cumulative distribution function *F(x)* denoted by $h(x)$ is defined as the ratio of the density function *f(x)* to its survival function(*S(x)=1- F(x)*). therefore, the hazard rate function *h(x)* of the four-parameter Beta-kumaraswamy distribution is obtained by substituting equations (3.12) and (3.14) in equation (3.24)

$$h(x) = \frac{f(x)}{1 - F(X)} \tag{3.24}$$

$$h(x) = \frac{\dfrac{abx^{a-1}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)\right]^{a-1}}{B(\alpha,\beta)}}{1-\dfrac{B\left[1-\left(1-x^{a}\right)^{b},\alpha,\beta\right]}{B(\alpha,\beta)}}$$

$$h(x)=\frac{abx^{a-1}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}}{B(\alpha,\beta)-B\left[1-\left(1-x^{a}\right)^{b},\alpha,\beta\right]}$$

Hence, the hazard rate function of the beta-Kumaraswamy is given by (3.25)

$$h(x)=\frac{abx^{a-1}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{a-1}}{B(\alpha,\beta)-B\left[1-\left(1-x^{a}\right)^{b},\alpha,\beta\right]} \tag{3.25}$$

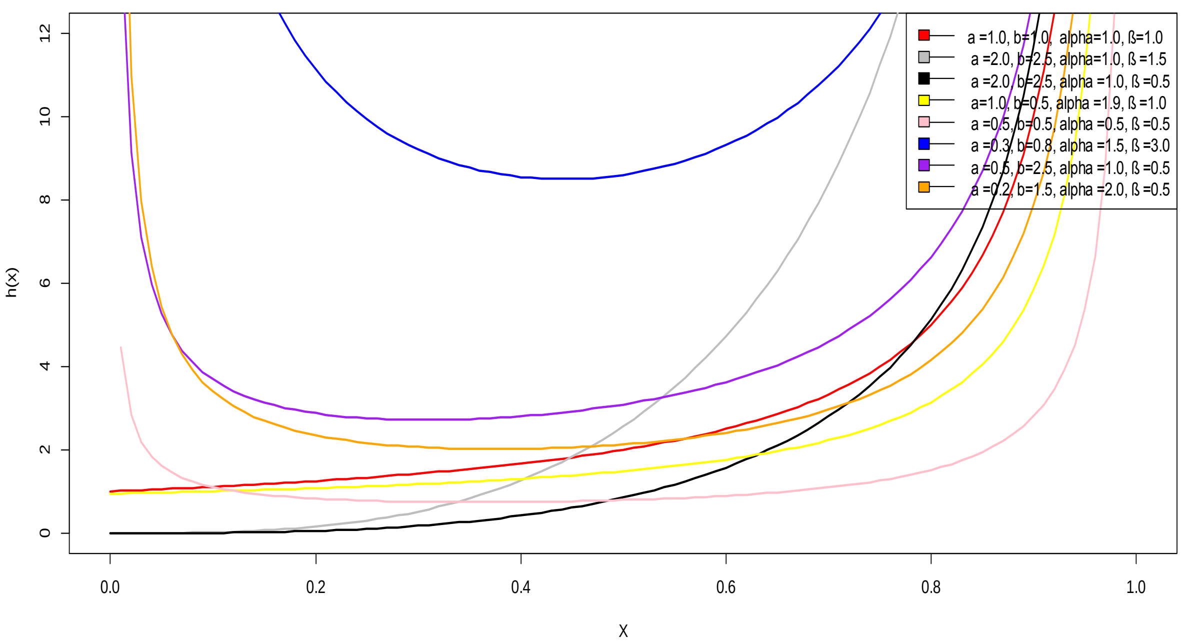

Figure 3.3: The graph of hazard rate function of beta-Kumaraswamy distribution for some selected parameter values

from figure 3.3 that the hazard function is U shape as X increases.

The hazard rate function is also known as failure rate or force of mortality. The hazard function is not a density or a probability function. However, it is thought of as the probability of failure in an infinitesimally small time period $t+\partial t$ giventhat the subject has survived up to time t .The hazardfunction is a measure of risk.The greater the hazard betweentimes t_1 and t_2 , the greater the risk of failure in this timeinterval.

3.1.11Survival Function

The survival function is the probability that a patient, device, or object of interest will survive beyond a specified period of time. In medicine the term survival function is used while in engineering it is referred to as reliability function.

Let*X*be a continuous random variable with cumulative distribution function*F*(*x*) on the interval (0, ∞). Thesurvival function or reliability function of *X*is defined by (3.26):

$$S(x) = P(X > x) = \int_x^{\infty} f(u)\,du = 1 - F(x) \tag{3.26}$$

This gives the probability that a device or item or object will survival beyond time t. One of the uses of the survivorship function is to predict quartiles of the survival time.So, thesurvival functionof Beta-Kumaraswamy distribution is given as (3.27):

$$S(x) = 1 - F(x)$$

$$= 1 - \frac{B\left[1-\left(1-x^{a}\right)^{b}, \alpha, \beta\right]}{B(\alpha, \beta)} \tag{3.27}$$

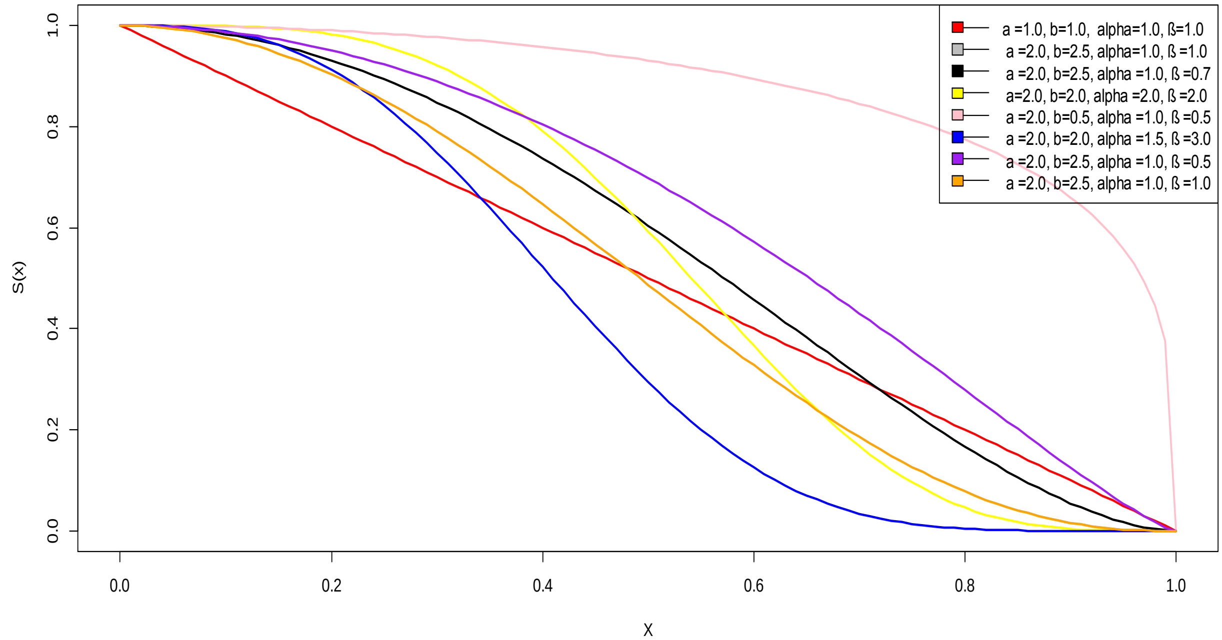

Figure 3.4: The graph of survival function of beta-Kumaraswamy distribution for some selected parameters values.

The graph in fig.3**.**shows that the value of the survival function equals one (1) at initial time or early age and it decreases as*X* increases and equals zero (0) as *X* becomes larger.

3.1.12 Maximum Likelihood Estimation of the Parameters of the Proposed Model

One of the most widely used methods for the estimation of parameters of distribution- the maximum likelihood estimation method (MLE).

3.1.13 Maximum Likelihood Estimation

If $x_1, x_2, ..., x_n$ is a random sample from a given population with pdf, $f(x,\xi)$ where ξ is a parameter vector. Then, the likelihood function of the random sample denoted by $L(X,\xi)$ is defined by;

$$L(X,\xi) = \prod_{i=1}^{n} f(x,\xi)$$

Hence, if $x_{1,} x_{2,...,} x_n$ is a random sample from a Beta-Kumaraswamy distribution with parameters a, b, α, β is defined by

$$L\left(x,a,b,\alpha,\beta\right)=\prod_{i=1}^{n}\frac{abx_i^{a-1}}{B\left(\alpha,\beta\right)}\left(1-x^a{}_i\right)^{b\beta-1}\left[1-\left(1-x_i^a\right)^b\right]^{\alpha-1}$$

$$L\left(x,a,b,\alpha,\beta\right)=a^nb^n\left(\frac{1}{B(\alpha,\beta)}\right)^n\prod_{i=1}^{n}x_i^{a-1}\left(1-x^a{}_i\right)^{b\beta-1}\left[1-\left(1-x_i^a\right)^b\right]^{\alpha-1}$$

Recall that $B\left(\alpha,\beta\right)=\dfrac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)}$

$$L\left(x,a,b,\alpha,\beta\right)=a^nb^n\left(\frac{\Gamma\alpha\,\Gamma\beta}{\Gamma(\alpha+\beta)}\right)^n\prod_{i=1}^{n}x_i^{a-1}\left(1-x_i^a\right)^{b\beta-1}\left[1-\left(1-x_i^a\right)^b\right]^{\alpha-1} \tag{3.28}$$

Taking the natural logarithm ofthe likelihood function in (3.28) gives:

$$\begin{aligned}\ell\left(x,a,b,\alpha,\beta\right)&=n\log a+n\log b+n\log\Gamma(\alpha)+n\log\Gamma(\beta)-n\log\Gamma(\alpha)-\\&n\log\Gamma(\beta)+\left(a-1\right)\sum_{i=1}^{n}\log\left(x_i\right)+\left(b\beta-1\right)\sum_{i=1}^{n}\log\left(1-x_i^a\right)+\\&\left(\alpha-1\right)\sum_{1=a}^{n}\log\left[1-\left(1-x_i^a\right)^b\right]\end{aligned}$$

(3.28)

To obtained the MLE of the parameters, we differentiate (3.28) partially with respect to*a*,*b*, α,β and equate to zero. This gives:

$$\frac{\partial\ell}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ell n\left(x_i\right)-\left(b\beta-1\right)\sum_{i=1}^{n}\frac{x_i^a\ell n\left(x\right)}{1-x_i^a}+ab\left(\alpha-1\right)\sum_{i=1}^{n}\ell n\frac{x_i^{a-1}\left(1-x_i^a\right)^{b-1}}{1-\left(1-x_i^a\right)^b} \tag{3.29}$$

$$\frac{\partial\ell}{\partial b}=\frac{n}{b}+\beta\sum_{i=1}^{n}\left(1-x^a{}_i\right)-\left(\alpha-1\right)\sum_{i=1}^{n}\frac{\left(1-x_i^a\right)^b\ell n\left(1-x_i^a\right)}{1-\left(1-x_i^a\right)^b} \tag{3.30}$$

$$\frac{\partial \ell}{\partial \alpha}=n\psi\left(\alpha+\beta\right)-n\psi\left(\alpha\right)+\sum_{i=1}^{n}\ell n\left[1-\left(1-x_i^{\ a}\right)^b\right] \tag{3.31}$$

$$\frac{\partial \ell}{\partial \beta}=n\psi\left(\alpha+\beta\right)-n\psi\left(\beta\right)+b\sum_{i=1}^{n}\ell n\left(1-x_i^{\ a}\right) \tag{3.32}$$

To obtain the estimate of a,b, α,β ,we equate equation (3.29) to (3.32) to zero and we solve for the respective parameters. The resulting equations (3.33) to (3.36) cannot be solve algebraically due to complexity in the derivatives except numerically using suitable statistical software like python, SAS, e.t.c with available data sets.

$$\frac{\partial \ell}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\ell n\left(x_i\right)-\left(b\beta-1\right)\sum_{i=1}^{n}\frac{x_i^{\ a-1}\ell n\left(x\right)}{1-x_i^{\ a}}+ab\left(\alpha-1\right)\sum_{i=1}^{n}\frac{x_i^{\ a-1}\left(1-x_i^{\ a}\right)^{b-1}}{1-\left(1-x_i^{\ a}\right)^b}=0 \tag{3.33}$$

$$\frac{\partial \ell}{\partial b}=\frac{n}{b}+\beta\sum_{i=1}^{n}\ell n\left(1-x^a{}_i\right)-\left(\alpha-1\right)\sum_{i=1}^{n}\frac{\left(1-x_i^{\ a}\right)^b\ell n\left(1-x_i^{\ a}\right)}{1-\left(1-x_i^{\ a}\right)^b}=0 \tag{3.34}$$

$$\frac{\partial \ell}{\partial \alpha}=n\psi\left(\alpha+\beta\right)-n\psi\left(\alpha\right)+\sum_{i=1}^{n}\ell n\left[1-\left(1-x_i^{\ a}\right)^b\right]=0 \tag{3.35}$$

$$\frac{\partial \ell}{\partial \beta}=n\psi\left(\alpha+\beta\right)-n\psi\left(\beta\right)+b\sum_{i=1}^{n}\ell n\left(1-x_i^{\ a}\right)=0 \tag{3.36}$$

3.1.14 Order Statistics and Rényi Entropy

In this section an explicit expression for the pdf of the i^{th} order $fi;n(x)$ of a random sample $x_1,x_2,...x_n$ from the beta-kumaraswamy distribution is defined by:

$$f_{i:n}\ (x)=\frac{1}{B(i,n-i+1)}f(x)F(x)^{i-1}\left[1-F(x)\right]^{n-i}$$

for $i = 1, 2, \cdots, n$.

Applying binomial series expansion in the last term of equation(3.12),we have:

$$f_{i:n}(x)=\sum_{p=0}^{n-i}\frac{(-1)^p\binom{n-i}{p}}{B(i,n-i+1)}f(x)F(x)^{i+p-1} \tag{3.37}$$

Substituting (3.12) and (3.14) in (3.37),the pdf of the i^{th} order statistics is:

$$f_{i:n}(x)=\sum_{p=0}^{n-i}\frac{(-1)^p\binom{n-i}{p}}{B(i,n-i+1)}\frac{ab}{B(\alpha,\beta)}\sum_{r=0}^{\infty}(-1)^r\binom{\alpha-1}{r}x^{a-1}\left(1-x^a\right)^{b(r+\beta)-1}\frac{B\left[1-\left(1-x^a\right)^b,\alpha,\beta\right]}{B(\alpha,\beta)}$$

$$f_{i:n}\left(x\right)=\sum_{p=0}^{n-i}\frac{\left(-1\right)^p\binom{n-i}{p}}{B\left(i,n-i+1\right)}\frac{ab}{\left(B\left(\alpha,\beta\right)\right)^2}\sum_{r=0}^{\infty}\left(-1\right)^r\binom{\alpha-1}{r}x^{a-1}\left(1-x^a\right)^{b(r+\beta)-1}\frac{B\left[1-\left(1-x^a\right)^b,\alpha,\beta\right]}{B\left(\alpha,\beta\right)}$$

$$f_{i:n}\left(x\right)=\sum_{p=0}^{n-i}\sum_{r=0}^{\infty}\frac{\left(-1\right)^{p+r}\binom{n-i}{p}\binom{\alpha-1}{r}abB\left[1-\left(1-x^a\right)^b,\alpha,\beta\right]}{B\left(i,n-i+1\right)\left(B\left(\alpha,\beta\right)\right)^2}x^{a-1}\left(1-x^a\right)^{b(r+\beta)-1}$$

$$f_{i:n}\left(x\right)=\sum_{p=0}^{n-i}\sum_{r=0}^{\infty}\frac{\left(-1\right)^{p+r}\binom{n-i}{p}\binom{\alpha-1}{r}B\left[1-\left(1-x^a\right)^b,\alpha,\beta\right]}{B\left(i,n-i+1\right)\left(B\left(\alpha,\beta\right)\right)^2\left(r+\beta\right)}ab\left(r+\beta\right)x^{a-1}\left(1-x^a\right)^{b(r+\beta)-1}$$

$$f_{i:n}\left(x\right)=\sum_{p=0}^{n-i}\sum_{r=0}^{\infty}w_{p,r}\,g\left(x,a,b\left(r+\beta\right)\right)$$

where

$$w_{p,r}=\frac{\left(-1\right)^{p+r}\binom{n-i}{p}\binom{\alpha-1}{r}B\left[1-\left(1-x^a\right)^b,\alpha,\beta\right]}{B\left(i,n-i+1\right)\left(B\left(\alpha,\beta\right)\right)^2\left(r+\beta\right)}$$

and $g(x;a,b(r+b))$ is the pdf of kumaraswamy distribution with parameters a and b(r+β). Thus, the distribution of the n^{th} order statistics is a linear combination of the kumaraswamy distribution.

3.1.15 Renyi Entropy

Renyi entropy of a random variable variable X_n with pdf $f(.)$ it is defined by:

$$H_R(\rho)=\frac{\rho}{1-\rho}\ell\left[\int f(x)^{\rho}dx\right] \tag{3.38}$$

$\rho>0$ and $\rho\neq 1$

The Renyi entropy is a special case of Shannon entropy where $\rho\to 1$

$$f(x)^{\rho}=\left\{\frac{abx^{a-1}}{b(\alpha,\beta)}\left(1-x^{a}\right)^{b\beta-1}\left[1-\left(1-x^{a}\right)^{b}\right]^{\alpha-1}\right\}^{\rho}$$

$$\left(\frac{ab}{B(\alpha,\beta)}\right)^{\rho}x^{\rho(a-1)}\left(1-x^{a}\right)^{\rho(b\beta-1)}\left[1-\left(1-x^{a}\right)^{b}\right]^{\rho(\alpha-1)}$$

Appling the binomial series expansion in the last term ofequation (3.12) we have:

$$f(x)^{\rho}=\left(\frac{ab}{B(\alpha,\beta)}\right)^{\rho}x^{\rho(a-1)}\sum_{r=0}^{\infty}(-1)^{r}\binom{\rho(\alpha-1)}{r}\left(1-x^{a}\right)^{br+\rho(b\beta-1)}$$

Substituting this expansion in (3.38) gives:

$$H_R(\rho)=\frac{\rho}{1-\rho}\ell n\left[\left(\frac{ab}{B(\alpha,\beta)}\right)^{\rho}\sum_{r=0}^{\infty}(-1)^{r}\binom{\rho(\alpha-1)}{r}\right]\int_0^1 x^{\rho(a-1)}\left(1-x^{a}\right)^{br+\rho(b\beta-1)}dx$$

Taking the integral part and letting $p=x^{a}$, we have:

$$\int_0^1 x^{\rho(a-1)}\left(1-x^{a}\right)^{br+\rho(b\beta-1)}dx=\frac{1}{a}\int_0^1 p^{\frac{(a-1)(\rho-1)}{a}}\left(1-p\right)^{br+\rho(b\beta-1)}dp$$

$$=\frac{1}{a}B\left[\frac{(a-1)(\rho-1)}{a}+1,\ br+\rho(b\beta-1)+1\right]$$

Hence, the measure of uncertainty of the proposed distribution is as follow

$$H_R(\rho)=\frac{\rho}{1-\rho}\ell n\left[\left(\frac{ab}{B(\alpha,\beta)}\right)^{\rho}\sum_{r=0}^{\infty}(-1)^r\binom{\rho(\alpha-1)}{r}\right]\frac{1}{a}B\left[\frac{(a-1)(\rho-1)}{a}+1,\ br+\rho(b\beta-1)+1\right]$$

CHAPTER FOUR

4.0 RESULTS AND DISCUSSION

This chapter, a real data set used to fit the new distribution and compared it with other Beta-G distributions.

The practical application of the proposed methods is considered in this section by fitting four-parameters distribution to the well know data set on maximum flood level from Antle and Dumonceaux (1973) several authors include Hirose (1995), Nagatsuka (2014)and Etienne Ouindllassida (2017) have used this data set to evaluate the performance.

The data below to fit our proposed distribution (Bkw) and some other distributions which include: kumaraswamy, beta andKumaraswamy-Kumaraswamy.

0.265 0.269 0.297 0.315 0.324 0.338 0.379 0.379 0.392 0.402 0.412 0.416 0.418 0.423 0.449 0.484 0.494 0.613 0.654 0.740

The goodness-of-fit test is the common and good criteria used to determine which of the model fits the dataset best. Thecriteria employed to choose the best model amongst the various distributions that will be compared in this work include: Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC) and Consistent Akaike Information Criterion (CAIC).

The most popular information criterion isAkaike information criterion (AIC) which compensates for the number of parameters in the model to encourage parsimony:

$$AIC = -2\ln\left(L\right) + 2P$$

where:

$P$is the number of parameters in the model.

ln(L) =log-likelihood function for the statistical model (deviance statistic for a given model).

When comparing two models, the model with the lowest AIC value would be selected as themodel demonstrating better fit than the model it is compared with. additional information criteria, which haveextended the AIC to give room for both modelcomplexity and sample size is the Bayesian Information Criteria (BIC) and Consistent Akaike information criterion (CAIC) which are defined as:

$$BIC = -2\ln\left(L\right) + p\log(n)$$

$$CAIC = -2\ln\left(L\right) + \frac{2Pn}{(n-p-1)}$$

where

$P$is the number of parameters in the model.

ln(L) log-likelihood function for the statistical model

n is the sample size.

If any two or more estimated models, the model with the lower value of AIC, BIC and CAIC is preferred. A lower value of AIC, BIC and CAIC implies either fewer explanatory variables, better fit, or both.

Table4.1: performance of the distribution using AIC,BIC and CAIC values of the models base on data set 1.

Model	**Parameter estimates**	**-ll(Log likelihood value)**	**AIC**	**BIC**	**CAIC**	**Ranks of models performance**
Beta-kumaraswamy	$\hat{a}$ =2.2473 $\hat{b}$ =76.4776 $\hat{c}$ =33.4167 $\hat{d}$ =0.0959	-15.9356	-23.8112	-31.8712	-21.2045	2
Beta	$\hat{a}$ =6.7593 $\hat{b}$ =9.1138	-14.06513	-23.1303	-28.1303	-23.4244	3
Kumaraswamy	$\hat{a}$ =3.3635 $\hat{b}$ =11.7906	-12.86855	-21.7371	-25.7371	-21.0312	5
Kumaraswamy -kumaraswamy	$\hat{a}$ =0.7618 $\hat{b}$ =0.05713 $\hat{c}$ =143.951 $\hat{d}$ =22.9249	-104.1742	-197.682	-208.348	-197.682	1
Kumaraswamy -Normal	$\hat{a}$ =0.94415 $\hat{b}$ =1.148925 $\hat{\mu}$ =0.44422 $\hat{\sigma}$ =0.12614	-13.54361	-19.0872	-21.8831	-24.08722	4

Table 4.1 gives the MLE, AIC, BIC and CAIC for flood data obtained from Dumonceaux and Antle (1973).The table indicates that the kw-kw distribution has the lowest values for the AIC,BIC, and CAIC statistics among the fitted models for this dataset follow by BKW.This shows that the proposed distribution is a strong competitor. Additionally, it is evident that the worst fit model was KW distributions according to the data .The required numerical evaluations are implemented using the R software.

The second Data set

The second data set are Measurements of 48 rock samples from a petroleum reservoir. This data was extracted from Bp research image analysis by Ronit (1992) and it is given below:

0.0903296,0.189651,0.228595,0.200071,0.280887,0.311646,0.176969,0.464125,0.148622,0.164127,0.231623,0.144810,0.179455,0.276016,0.438712, 0.420477, 0.183312, 0.203654, 0.172567, 0.113852, 0.191802, 0.19753, 0.163586, 0.200744, 0.117063, 0.162394, 0.153481, 0.291029, 0.133083, 0.326635, 0.253832, 0.262651, 0.122417, 0.150944, 0.204314, 0.240077, 0.225214, 0.154192, 0.328641, 0.182453, 0.167045, 0.148141, 0.262727, 0.161865, 0.341273, 0.276016, 0.230081, 0.200447.

Table 4.2: performance of the distribution using AIC,BIC and CAIC values of the models base on data set 2.

Model	**Parameter estimates**	**-ll (Log likelihood value)**	**AIC**	**BIC**	**CAIC**
Beta-kumaraswamy	$\hat{a}$ =0.879045 $\hat{b}$ =20.8605 $\hat{c}$ =53.6535 $\hat{d}$ =0.4129	-58.47007	-108.94	-116.94	-108.01
Beta	$\hat{a}$ =5.9422 $\hat{b}$ =21.2077	-55.60024	-107.2	-111.2	-106.934
Kumaraswamy	$\hat{a}$ =2.788 $\hat{b}$ =44.667	-52.4914	-100.983	-104.983	-100.716
Kumaraswamy-kumaraswamy	$\hat{a}$ =001.242 $\hat{b}$ =00001.142 $\hat{c}$ =0009.054 $\hat{d}$ =0002.884	-470.7315	-933.463	-941.463	-932.533
Kumaraswamy-Normal	$\hat{a}$ =0.792506 $\hat{b}$ =1.392262 $\hat{\mu}$ =0.263631 $\hat{\sigma}$ =0.087053	-50.5611	-93.1222	-94.3972	-98.1222

Table 4.2, gives the MLE, AIC, BIC and CAIC of BKW, BD,KW, KW-KW and KWN distributions. It is observed that BKW distribution is better than BD, KW and KWN distributions since it is having smaller value of AIC, BIC and CAIC values. Hence, BKW distribution is a good competitor in modeling real life dataset.

4.2 Simulation:

A simulation study is conducted to evaluate the performance of our methodology using maximum likelihood estimation method.

A random sample is drawn from the mixture model of Beta and kumaraswamy distribution with parameter a, b ,α and β to ascertain the validity of the methodology.

For each sample size 1000 iteration is done. The simulation is partitioned in scenario with small sample size consider in the simulation for n= 50.

The mixture model is given by

$$y = w_1 f_1(x \backslash \theta_1) + w_2 f_2(x \backslash \theta_2) \qquad (4.1)$$

where

$W = \left(w_{1,} w_2\right)$ isthe weight of the distribution such that $\sum_{i=1}^{n} w_i = 1$, $f_1(x \backslash \theta_1)$ and $f_2(x \backslash \theta_2)$ follow Beta and Kumaraswamy distribution respectively.

4.1.1 Scenario 1

The true parameter chosen for the two distribution are a=2, b = 2,α=5 and β=10 n=50 to test for small sample size the estimate of the parameter are shown below.

The model is give as .0.9 *rbeta(n,2,2)+ 0.1 *rkumar(n,5,10)

Table;4.3 .simulation study

a b α β	$\hat{a}$	$\hat{b}$	$\hat{c}$	$\hat{d}$
2, 2, 5, 10	1.2950	1.1072	4.4215	7.8956
1, 2, 10, 10	0.8481	1.0558	8.3347	8.2416
2, 10, 6, 1	1.5761	9.5098	5.8052	0.6110
1, 2, 8, 6	0.9822	1.4102	7.0402	5.8904

The evident that the true parameter are closer to the estimate.

CHAPTER FIVE

5.0 SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1Summary

The goal of this dissertation as stated in the objectives was to propose a continuous distribution which we called the beta-kumaraswamy distribution; it generalizes the kumaraswamy distribution by employing the logit of beta distribution introduced by Jones(2004). The CDF, PDF, hazard function were derived. Subsequently, some of its mathematical and statistical properties were also provided. The model parameters were estimated by using the method of maximum likelihood estimation.

Finally, we subjected the distribution to real-life data set to assess the performance of the model by comparing it to three other distributions KWD,BD ,and KWKWD.Criteria like log likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian information criterion (BIC) was appliedin determining the best model.

5.2 Conclusion

In this work, we studied the beta-kumaraswamy distribution as a generalization of the kumaraswamy distribution. We provided a good foundation for some mathematical properties for this distribution, including the derivation of the hazard rate function and order statistics. The model parameters were estimated by the method of maximum likelihood. An application of the BKW distribution to a real data set indicates that this distribution outperforms both the KW and other generalized distributions. We hope that the proposed model may attract wider applications in statistics.

5.3 Recommendation

Based on the model constructed, we recommend that;

1. When it is required to use a distribution that will fits better with both shape and location parameters and proved to be parsimonious, the BKW will be good alternative.

2. The proposed distribution model can be used in modeling a continuous data set.

5.4 Contribution to Knowledge

In this research, we were able to contribute to the body of knowledge in so many ways, which include the following:

i. We proposed some formulae for the extended kumaraswamy distribution after conjoining it with the beta distribution to build beta-kumaraswamy.

ii. We provide the estimates of the parameters of the proposed model in this research.

iii. We proved that the estimators of the BKW are more flexible and fit in better than other distributions.

5.5 Areas of Further Research

i. There are other forms of family of distribution yet to be explored; these include exponential exponentiated kumaraswamy distribution, and kumaraswamy –inverse exponentiated distribution.

ii. Further research can be undertaken to determine some mathematical and statistical properties of the proposed distribution such as other classical methods of estimation and non-classical/ Bayesian method of estimation.

REFERENCES

Adepoju, K. A. and Chukwu, A. (2014). The Beta Power Exponential Distribution. *Journal of Statistical Science and Application*, 9:37-46.

Afify, M. Z., Yousof, H. M., Cordeiro, G. M., Ortega, E. M. M. and Nofal, Z. M. (2016).The WeibullFrechet Distribution and Its Applications. *Journal ofApplied Statistics* and operation Research ,10:435-454

Akinsete, A. and Lowe, C. (2009). Beta-Rayleigh distribution in reliability measure. *Proc Joint Stat Meet Phys Eng Science*,18: 3103-3107.

Akinsete, A. Famoye, F. and Lee, C. (2008). The beta-pareto Distribution. *Statistics*, 42 (6), 547-563.

Barreto-Souza, W., Cordeiro, G. and Simas, A. (2011). Some results for beta Fréchet distribution. *Communications in Statistics- Theory and Methods*, 81, 798-811.

Bourguignion, M., Silva, R. and Cordeiro, G. (2014). The Weibull G family of probability distributions (12). *Journal of Data Science*, 2:53-68.

Carrasco, J., Ortega, E. and Cordeiro, G. (2008). A generalized modified Weibull distribution for lifetime modeling. *Computational Statistics and Data Analysis*, 53, 450-462.

Cordeiro, G. M., De Castro, M. (2009). A new family of generalized distributions, *Journal of Statistical Computation and Simulation*,81(7),883-898(2011)

Cordeiro, G. M., Ortega, E. M. M., Nadarajah, S., (2010). The Kumaraswamy Weibull distribution with application to failure data. *Journal of Frank Institute*, 47, 1399–1429.

Cordeiro, G. M., Nadarajah, S., Ortega, E. M. M., (2012). The Kumaraswamy Gumbel distribution, *Statistical Methods and Applications*, 21: 139–168.

El-Sherpieny, E. E. and Ahmed, M. A. (2014). On the Kumaraswamy-Kumaraswamy distribution. *International Journal of Basic and Applied Sciences,* 3(4): 372-381

Eugene, N., Lee, C. and Famoye, F. (2004). beta-Normal Distribution and its Applications. *Communications in Statistics- Theory and Methods*, 31(4), 497-512.

Famoye, F., Lee, C. and Olumolade, O. (2005). The beta-Weibull distribution. *Journal of Statistics and Theoritical Applications* 4(2), 121-136.

Fischer, M. and Vaughan, D. (2010). The beta-hyperbolic secant distribution. *Austrian Journal of Statistics (39)*, 245-258.

Ganji, K. Ponnambalam, D. Khalili, and M. Karamouz. Grain yield reliability analysis with crop water demand uncertainty. *Stochastic Environmental Research and Risk Assessment*, 20, 259–277,2006.

Gupta A. K. and Nadarajah S. (2004). On the moments of the beta normal distribution. *Communications in Statistics Theory and Methods*, 33(1):1-13.

Gupta, V., Bhatt, M. and Gupta, J. (2015). The Lomax-Frechet distribution. *Journal of Rajasthan Academy of Physical Sciences*, 14(1): 25-43

Javanshiri, Z., Habibi Rad, A. and Arghami, N. R. (2015). Exp-Kumaraswamy distributions: Some properties and Applications. *Journal of Sciences, Islamic Republic of Iran,* 26(1): 57-69.

Jones, M. C., (2009). Kumaraswamy's distribution: a beta-type distribution with some tractability advantages. *Statistics and Methodology*, 6: 70–91.

Johnson, N. L., Kotz, S. and Balakrishnan, N. (1995). *Continuous Univariate Distributions* (2nd Edition ed., Vol. vol. 2). New York: Wiley.

Jones, M. C. (2004). Families of distributions arising from distributions of order statistics. *Sociedad de Estadistica e Investigacion Operativa Test*, 13(1), 1-43.
Javanshiri, Z., Habibi Rad, A. and Arghami, N. R. (2015). Exp-Kumaraswamy distributions: Some properties and Applications. *Journal of Sciences, Islamic Republic of Iran,* 26(1): 57-69.
Khan,M.S.,king,R. and hudson,I.L(2016).Transmuted kumaraswamy distribution .statistics in transition ,17(2):183-210

Kong, L., Lee, C. and Sepanski, J. (2007). On the properties of beta-gamma distribution. *Journal of Modern Applied Statistical Methods (6)*, 187-211.

Lee, C., Famoye, F. and Olumolade, O. (2007). Beta-Weibull distribution: Some properties and applications to censored data. *Journal of Modern Applied Statistical Methods*, 173-186.

Lemonte, A. J. and Cordeiro, G. M. (2011). The exponentiated generalized inverse Gaussian distribution. *Statistics and Probability Letters*, 81, 506-517.

Merovci, F., Khaleel, M. A., Ibrahim, N. A. and Shitan, M. (2016). The beta Burr type X distribution. *SpringerPlus,* 697.

Mhmoudi,E.(2011).Beta generalized pareto distribution (BGPD),mathematics and computer in simulation ,81;2414-2430

Mudholkar, G. and Hutson, A. (1996). The exponentiated Weibull family: Some properties and a flood data application. *Communications in Statistics- Theory and Methods*, 25, 3059-3083.

Nadaraja, S. and Kotz. (2006). The beta exponential distribution. *Reliability Engineering and System Safety*, 91, 689-697.

Nadarajah, S. and Kotz, S. (2004). The beta Gumbel distribution. *Mathematical Probabillty Eng*, 10, 323-332.

Nadarajah, S. and Gupta, A. (2004). The beta Fréchet distribution. *Far East Journal of Theoritical Statistics* 14, 15-24.

Nadarajah, S., (2008). On the distribution of Kumaraswamy. *Journal of Hydrology*, 48: 568–569.

Oguntunde, P., Balogun, O., Okagbue, H. and Bishop, S. (2015). The Weibull-Exponential Distribution: Its Properties and Applications. *Journal of Applied Sciences*, 5:1305-1311.

Ramos, M.W., marinho ,P.R.D ., cordeiro G.M., and Hamedani G. (2015).the kumaraswamy -G poisson family of distribution .journal of statistical theory and application .14(3):222-239

Paranaíba, P., Ortega, E., Cordeiro, G. and Pescim, R. (2011). The beta Burr XII distribution with application to lifetime data. *Comput Statist Data Anal*, 45:1118-1136.

Santana,T.V,ortega E.M.M., cordeiro, G.M., silva,R.V. and Hameedani, G.(2012).the kumaraswamy log-logistics distribution .journal of statistical theory and Application,11(3):265-291

Rodriguez, R. N. (1977). A guide to the Burr Type XII distributions. *Biometrika, 64* , 129-134.

Shaw, W. T. and Buckley, I. R. (2007). The alchemy of probability distributions: beyond Gram-Charlier expansions and a skew-kurtotic-normal distribution from a rank transmutation map. *Technical Report.* http://arxiv.org/abs/0901.0434

Shittu, O. I. and Adepoju, K. A. (2013). On the beta-Nakagami distribution. *Progress in Applied Mathematics*, 5(1), 49-58.

APPENDIX

R COMMANDS USED IN THIS DESATATION

PDF.

```
BK=function(x, a, b, c, d){a*b*x^(a-1) *(1-x^a)^(b*d-1) *(1-(1-x^a)^b)^(c-1)/beta(c,d)}

plot(x,BK(x, 0.6,0.9,0.5,0.2),type="n",lwd=2,ylim=c(0,2.5), xlab="X",ylab="Pdf of BK")

lines(x, BK (x,1.0,1.0,1.0,1.0),lwd=2,lty=1, col="red")

lines(x, BK (x,2.0,2.5,1.0,0.5),lwd=2,lty=1, col="grey")

lines(x, BK (x,2.0,1.5,1.0,1.0),lwd=2,lty=1, col="black")

lines(x, BK (x,1.0,0.5,1.9,1.0),lwd=2,lty=1, col="yellow")

lines(x, BK (x,2.0,0.5,1.0,0.5),lwd=2,lty=1, col="pink")

lines(x, BK (x,2.0,2.0,2.0,2.0),lwd=2,lty=1, col="blue")

lines(x, BK (x,2.0,2.5,1.0,0.5),lwd=2,lty=1, col="purple")

legend("topright", c("a =1.0, b=1.0,  alpha=1.0, β=1.0", " a =2.0, b=2.5, alpha=1.0, β =0.5", " a =2.0, b=1.5, alpha =1.0, β =1.0", " a=1.0, b=0.5, alpha =1.9, β =1.0", " a =2.0, b=0.5, alpha =1.0, β =0.5", " a =2.0, b=2.0, alpha =2.0, β =2.0"," a =2.0, b=2.5, alpha =1.0, β =0.5"), fill=c("red", "grey", "black", "yellow", " pink", "blue", " purple"), lwd=1)
```

CDF

```
### Codes for the graph of the cdf

x=seq(0, 1, .01)

BKCDF=function(x, a, b, c, d){(Ibeta((1-(1-x^a)^b), c, d))/beta(c, d) }

plot(x,BKCDF(x, 0.6,0.9,0.5,0.2),type="n",lwd=2,ylim=c(0,1.0), xlab="X", ylab="F(x)")

lines(x, BKCDF(x,1.5,1.5,1.5,1.5),lwd=2,lty=1, col="black")

lines(x, BKCDF(x,2.0,1.5,4.0,1.5),lwd=2,lty=1, col="red")

lines(x, BKCDF(x,3.0,2.5,1.0,1.0),lwd=2,lty=1, col="grey")

lines(x, BKCDF(x,2.0,2.0,3.5,3.0),lwd=2,lty=1, col="blue")

legend("topright", c("a =1.5, b=1.5,  alpha=1.5, β=1.5", " a =2.0, b=1.5, alpha=4.0, β=1.5", " a =3.0, b=2.5, alpha =1.0, β =1.0", " a=2.0, b=2.0, alpha =3.5, β=3.0"), fill=c("black", "red ", "grey","blue"), lwd=1)
```

HARZAD FUNCTION

```
### Codes for the graph of the hazard function

BKH=function(x, a, b, c, d){(a*b*x^(a-1)*(1-x^a)^(b*d-1)*(1-(1-x^a)^b)^(a-1)/(beta(c, d)-Ibeta((1-(1-x^a)^b), c, d))) }

plot(x,BKH(x, 0.6,0.9,0.5,0.2),type="n",lwd=2,ylim=c(0,12.0), xlab="X", ylab="h(x)")

lines(x, BKH (x,1.0,1.0,1.0,1.0),lwd=2,lty=1, col="red")

lines(x, BKH (x,2.0,2.5,1.0,1.5),lwd=2,lty=1, col="grey")

lines(x, BKH (x,2.0,2.5,1.0,0.5),lwd=2,lty=1, col="black")

lines(x, BKH (x,1.0,0.5,1.9,1.0),lwd=2,lty=1, col="yellow")

lines(x, BKH (x,.50,0.5,.50,0.5),lwd=2,lty=1, col="pink")

lines(x, BKH (x,.3,.8,1.5,3.0),lwd=2,lty=1, col="blue")

lines(x, BKH (x,.5,2.5,1.0,0.5),lwd=2,lty=1, col="purple")

lines(x, BKH (x,.2,1.5,2.0,.50),lwd=2,lty=1, col="orange")

legend("top right", c("a =1.0, b=1.0,  alpha=1.0, β=1.0", " a =2.0, b=2.5, alpha=1.0, β=1.5", " a =2.0, b=2.5, alpha =1.0, β=0.5", " a=1.0, b=0.5, alpha =1.9, β =1.0", " a =0.5, b=0.5, alpha =0.5, β =0.5", " a =0.3, b=0.8, alpha =1.5, β =3.0"," a =0.5, b=2.5, alpha =1.0, β =0.5", " a =0.2, b=1.5, alpha =2.0, β =0.5"), fill=c("red", "grey", "black", "yellow", " pink", "blue", " purple", " orange"), lwd=1)
```

SURVIVAL FUNCTION

```
### Codes for the graph of the survival function

BKSUF=function(x, a, b, c, d){1-(Ibeta((1-(1-x^a)^b), c, d))/beta(c, d) }

plot(x,BKSUF (x, 0.6,0.9,0.5,0.2),type="n",lwd=2,ylim=c(0,1.0), xlab="X", ylab="S(x)")

lines(x, BKSUF (x,1.0,1.0,1.0,1.0),lwd=2,lty=1, col="red")

lines(x, BKSUF (x,2.0,2.5,1.0,1.0),lwd=2,lty=1, col="grey")

lines(x, BKSUF (x,2.0,2.5,1.0,0.7),lwd=2,lty=1, col="black")

lines(x, BKSUF (x,2.0,2.0,2.0,2.0),lwd=2,lty=1, col="yellow")

lines(x, BKSUF (x,2.0,0.5,1.0,0.5),lwd=2,lty=1, col="pink")

lines(x, BKSUF (x,2.0,2.0,1.5,3.0),lwd=2,lty=1, col="blue")
```

lines(x, BKSUF (x,2.0,2.5,1.0,0.5),lwd=2,lty=1, col="purple")

lines(x, BKSUF (x,2.0,2.5,1.0,1.0),lwd=2,lty=1, col="orange")

legend("topright", c("a =1.0, b=1.0, alpha=1.0, β=1.0", " a =2.0, b=2.5, alpha=1.0, β=1.0", " a =2.0, b=2.5, alpha =1.0, β=0.7", " a=2.0, b=2.0, alpha =2.0, β=2.0", " a =2.0, b=0.5, alpha =1.0, β =0.5", " a =2.0, b=2.0, alpha =1.5, β =3.0"," a =2.0, b=2.5, alpha =1.0, β =0.5", " a =2.0, b=2.5, alpha =1.0, β=1.0"), fill=c("red", "grey", "black", "yellow", " pink", "blue", " purple", " orange"), lwd=1)

ANALYSIS FOR DATA SET I

>x=c(0.265,0.269,0.297,0.315,0.324,0.338,0.379,0.379,0.392,0.402,0.412,0.416,0.418,0.423,0.449,0.484, 0.494,0.613,0.654,0.740)

BETA-KUMARASWAMY ANALYSIS

> set.seed(111)

> loglikBK=function(p){-sum(log(p[1]*p[2]*gamma(p[3]+p[4])*(gamma(p[3])*gamma(p[4]))^(-1)*x^(p[1]-1)*(1-x^p[1])^(p[2]*p[4]-1)*(1-(1-x^p[1])^p[2])^(p[3]-1)))}

> a=optim(c(1,5,3,2),fn=loglikBK,hessian=T)

> vca=solve(a$hessian)

> sea=sqrt(diag(vca))

Warning message:

In sqrt(diag(vca)) : NaNs produced

> a$convergence

[1] 1

> a

$par

[1] 2.24731149 76.47756801 33.41670021 0.09591677

$value

[1] -15.93561

$counts

function gradient

```
    501      NA

$convergence

[1] 1

$message

NULL

$hessian

           [,1]        [,2]        [,3]         [,4]

[1,]   63.6326143 -0.707988584  0.276861079 -225.0992481

[2,]   -0.7079886  0.008915121 -0.002765311    3.7663933

[3,]    0.2768611 -0.002765311  0.001765059   -0.6057827

[4,] -225.0992481  3.766393293 -0.605782660 2202.5925375

> vca

           [,1]         [,2]        [,3]        [,4]


[1,]  -0.54028403    -81.025900  -15.0107595  0.079208758

[2,] -81.02589996 -10494.394983 -458.3843701  9.538517935

[3,] -15.01075946   -458.384370 2148.2335173 -0.159399540

[4,]   0.07920876      9.538518   -0.1593995 -0.007805592

> sea

[1]      NaN      NaN 46.34904      NaN
```

BETA ANALYSIS

```
> set.seed(111)

> loglikbeta=function(p){-sum(log(x^(p[1]-1)*(1-x)^(p[2]-
1)/(gamma(p[1])*gamma(p[2])/gamma(p[1]+p[2]))))}

> b=optim(c(3,8),fn=loglikbeta,hessian=T)

> vcb=solve(b$hessian)
```

```
> seb=sqrt(diag(vcb))

> b$convergence

[1] 0

> b

$par

[1] 6.759302 9.113861

$value

[1] -14.06513

$counts

function gradient

      91       NA

$convergence

[1] 0

$message

NULL

$hessian

          [,1]      [,2]

[1,]  1.887997 -1.300510

[2,] -1.300510  1.018733

> vcb

         [,1]     [,2]

[1,] 4.390366 5.604722

[2,] 5.604722 8.136575

> seb

[1] 2.095320 2.852468
```

KUMARASWAMY ANALYSIS

```
>> set.seed(111)

> loglikkum=function(p){-sum(log(p[1]*p[2]*x^(p[1]-1)*(1-x^p[1])^(p[2]-1)))}

> c=optim(c(0.5,5),fn=loglikkum,hessian=T)

> vcc=solve(c$hessian)

> sec=sqrt(diag(vcc))

> c$convergence

[1] 0

> c

$par

[1]  3.363578 11.790568

$value

[1] -12.86855

$counts

function gradient

     91       NA

$convergence

[1] 0

$message

NULL

$hessian

         [,1]       [,2]

[1,] 11.354516 -1.1128055

[2,] -1.112805  0.1438668

> vcc

         [,1]      [,2]

[1,] 0.3640338  2.815791
```

```
[2,] 2.8157907 28.730934

> sec

[1] 0.6033521 5.3601245
```

TRANSMUTED KUMARASWAMY ANALYSIS

```
>> set.seed(111)

> logliktkum=function(p){-sum(log(p[1]*p[2]*x^(p[1]-1)*(1-x^p[1])^(p[2]-1)*(1+p[3]-2*p[3]*(1-(1-
x^p[1])^p[2]))))}

> d=optim(c(1,1,0.5),fn=logliktkum,hessian=T)

There were 30 warnings (use warnings() to see them)

> vcd=solve(d$hessian)

> sed=sqrt(diag(vcd))

Warning message:

In sqrt(diag(vcd)) : NaNs produced

> d$convergence

[1] 1

> d

$par

[1] 1.801258e+00 7.019666e-01 1.319097e+05

$value

[1] -217.8288

$counts

function gradient

    502       NA

$convergence

[1] 1

$message
```

```
NULL

$hessian

          [,1]          [,2]          [,3]

[1,]  31.54057 -6.494179e+01  0.000000e+00

[2,] -64.94179  1.651902e+02  7.105427e-09

[3,]   0.00000  7.105427e-09 -7.105427e-09

> vcd

           [,1]       [,2]          [,3]

[1,] 0.16639513 0.06541547  6.541547e-02

[2,] 0.06541547 0.03177063  3.177063e-02

[3,] 0.06541547 0.03177063 -1.407375e+08

> sed

[1] 0.4079156 0.1782432       NaN
```

KUMARASWAMY-KUMARASWAMY ANALYSIS

```
>> set.seed(111)

> loglikkkum=function(p){-sum(log(p[1]*p[2]*p[3]*p[4]*x^(p[1]-1)*(1-x^p[1])^(p[4]-1)*(1-(1-
x^p[1])^p[4])^(p[1]-1)*(1-(1-(1-x^p[1])^p[4])^p[1])^(p[2]-1)))}

>> e=optim(c(1,3.0,5.5,0.5),fn=loglikkkum,hessian=T)

Warning messages:

1: In log(p[1] * p[2] * p[3] * p[4] * x^(p[1] - 1) * (1 - x^p[1])^(p[4] -  :

  NaNs produced

2: In log(p[1] * p[2] * p[3] * p[4] * x^(p[1] - 1) * (1 - x^p[1])^(p[4] -  :

  NaNs produced

> vce=solve(e$hessian)

> see=sqrt(diag(vce))

> e$convergence
```

```
[1] 0
> e
$par
[1]   0.76181071   0.05713646 143.95182208  22.92487567
$value
[1] -104.1742
$counts
function gradient
     463       NA
$convergence
[1] 0
$message
NULL
$hessian
             [,1]      [,2]         [,3]         [,4]
[1,] 163901.65074 -766.1672 0.0000000000     30.75306
[2,]   -766.16724 6130.1270 0.0000000000    359.83349
[3,]      0.00000    0.0000 0.0009651515      0.00000
[4,]     30.75306  359.8335 0.0000000000 163833.77954
> vce
              [,1]          [,2]     [,3]          [,4]
[1,]  6.104788e-06  7.631659e-07    0.000 -2.822089e-09
[2,]  7.631659e-07  1.632452e-04    0.000 -3.586840e-07
[3,]  0.000000e+00  0.000000e+00 1036.107  0.000000e+00
[4,] -2.822089e-09 -3.586840e-07    0.000  6.104536e-06
> see
```

```
[1]  0.002470787  0.012776744 32.188612138  0.002470736
```

KUMARASWAMY NORMAL DISTRIBUTION

```
loglike4=function(p,x){-sum(log(p[1]*p[2]/p[4]*dnorm((x-p[3])/p[4])*((pnorm((x-p[3])/p[4]))^(p[1]-1))*((1-(pnorm((x-p[3])/p[4]))^p[1])^(p[2]-1))))}

result4=optim(c(0.8,0.4,0.7,0.9),method="CG",fn=loglike4,hessian=T,x=x)

There were 50 or more warnings (use warnings() to see the first 50)

result4$convergence

[1] 1

result4

$par

[1] 0.9441595 1.1489252 0.4442270 0.1261485

$value

[1] -13.54361

$counts

function gradient

    348      101

$convergence

[1] 1

$message

NULL


$hessian

          [,1]       [,2]       [,3]       [,4]

[1,]   25.40259  -12.76458  167.2679 -108.02620

[2,]  -12.76458   15.15120 -127.5166  -76.61068

[3,]  167.26786 -127.51663 1315.7083 -162.46240[4,] -108.02620  -76.61068 -162.4624 2551.62535
```

ANALYSIS FOR DATA SET II

>x=c(0.0903296,0.189651,0.228595,0.200071,0.280887,0.311646,0.176969,0.464125,0.148622,0.164127,0.231623,0.144810,0.179455,0.276016,0.438712,0.420477,0.183312,0.203654,0.172567,0.113852,0.191802,0.19753,0.163586,0.200744,0.117063,0.162394,0.153481,0.291029,0.133083,0.326635,0.253832,0.262651,0.122417,0.150944,0.204314,0.240077,0.225214,0.154192,0.328641,0.182453,0.167045,0.148141,0.262727,0.161865,0.341273,0.276016,0.230081,0.200447)

BETA-KUMARASWAMY ANALYSIS

> loglikBK=function(p){-sum(log(p[1]*p[2]*gamma(p[3]+p[4])*(gamma(p[3])*gamma(p[4]))^(-1)*x^(p[1]-1)*(1-x^p[1])^(p[2]*p[4]-1)*(1-(1-x^p[1])^p[2])^(p[3]-1)))}

> a=optim(c(1,2.5,7,0.2),fn=loglikBK,hessian=T)

> vca=solve(a$hessian)

> sea=sqrt(diag(vca))

> a$convergence

[1] 1

> a

$par

[1] 0.8790457 20.8605291 53.6534758 0.4128924

$value

[1] -58.47007

$counts

function gradient

    501   NA

$convergence

[1] 1

$message

NULL

$hessian

```
          [,1]        [,2]       [,3]        [,4]
[1,] 1613.674197 -37.48498919  3.27266921 -531.1080219
[2,]  -37.484989   0.91319802 -0.07272423   14.8388655
[3,]    3.272669  -0.07272423  0.00696037   -0.8960581
[4,] -531.108022  14.83886550 -0.89605805  329.2772573
> vca
           [,1]        [,2]       [,3]       [,4]
[1,]    0.9839656    35.94338  -140.5457  -0.4151640
[2,]   35.9433794  1389.04176 -4589.9685 -17.1128275
[3,] -140.5457266 -4589.96851 24187.4162  45.9743005
[4,]   -0.4151640   -17.11283    45.9743   0.2296959
> sea
[1]   0.9919504  37.2698505 155.5230407   0.4792660
>> set.seed(111)
> loglikBK=function(p){-sum(log(p[1]*p[2]*gamma(p[3]+p[4])*(gamma(p[3])*gamma(p[4]))^(-1)*x^(p[1]-
1)*(1-x^p[1])^(p[2]*p[4]-1)*(1-(1-x^p[1])^p[2])^(p[3]-1)))}
> a=optim(c(1,2.5,7,0.2),fn=loglikBK,hessian=T)
> vca=solve(a$hessian)
> sea=sqrt(diag(vca))
> a$convergence
[1] 1
> a
$par
[1]  0.8790457 20.8605291 53.6534758  0.4128924
$value
[1] -58.47007
```

```
$counts
function gradient
     501       NA
$convergence
[1] 1
$message
NULL
$hessian
            [,1]        [,2]        [,3]         [,4]
[1,] 1613.674197 -37.48498919  3.27266921 -531.1080219
[2,]  -37.484989   0.91319802 -0.07272423   14.8388655
[3,]    3.272669  -0.07272423  0.00696037   -0.8960581
[4,] -531.108022  14.83886550 -0.89605805  329.2772573
> vca
             [,1]        [,2]        [,3]        [,4]
[1,]    0.9839656    35.94338  -140.5457  -0.4151640
[2,]   35.9433794  1389.04176 -4589.9685 -17.1128275
[3,] -140.5457266 -4589.96851 24187.4162  45.9743005
[4,]   -0.4151640   -17.11283    45.9743   0.2296959
> sea
[1]   0.9919504  37.2698505 155.5230407   0.4792660
```

BETA ANALYSIS

```
>> set.seed(111)
> loglikbeta=function(p){-sum(log(x^(p[1]-1)*(1-x)^(p[2]-
1)/(gamma(p[1])*gamma(p[2])/gamma(p[1]+p[2]))))}
> b=optim(c(3,8),fn=loglikbeta,hessian=T)
```

```
> vcb=solve(b$hessian)

> seb=sqrt(diag(vcb))

> b$convergence

[1] 0

> b

$par

[1]  5.942273 21.207778

$value

[1] -55.60024

$counts

function gradient

      65       NA

$convergence

[1] 0

$message

NULL

$hessian

          [,1]       [,2]

[1,]  6.994404 -1.8009112

[2,] -1.800911  0.5166085

> vcb

         [,1]      [,2]

[1,] 1.395910  4.866179

[2,] 4.866179 18.899335

> seb

[1] 1.181486 4.347337
```

KUMARASWAMY ANALYSIS

```
>> set.seed(111)

> loglikkum=function(p){-sum(log(p[1]*p[2]*x^(p[1]-1)*(1-x^p[1])^(p[2]-1)))}

> c=optim(c(0.5,5),fn=loglikkum,hessian=T)

> vcc=solve(c$hessian)

> sec=sqrt(diag(vcc))

> c$convergence

[1] 0

> c

$par

[1]  2.718819 44.666462

$value

[1] -52.49137

$counts

function gradient

      79       NA

$convergence

[1] 0

$message

NULL

$hessian

          [,1]        [,2]

[1,] 86.291982 -1.34047073

[2,] -1.340471  0.02405903

> vcc

          [,1]       [,2]
```

```
[1,] 0.08615905   4.800429

[2,] 4.80042919 309.024686

> sec

[1]  0.2935286 17.5790980
```

TRANSMUTED KUMARASWAMY ANALYSIS

```
>> set.seed(111)

> logliktkum=function(p){-sum(log(p[1]*p[2]*x^(p[1]-1)*(1-x^p[1])^(p[2]-1)*(1+p[3]-2*p[3]*(1-(1-
x^p[1])^p[2]))))}

> d=optim(c(1,1,0.5),fn=logliktkum,hessian=T)

There were 29 warnings (use warnings() to see them)

> vcd=solve(d$hessian)

> sed=sqrt(diag(vcd))

Warning message:

In sqrt(diag(vcd)) : NaNs produced

> d$convergence

[1] 1

> d

$par

[1]   10.79911 2734.55380 6665.68285

$value

[1] -161.9987

$counts

function gradient

    502       NA

$convergence

[1] 1
```

```
$message
NULL
$hessian
              [,1]          [,2]          [,3]
[1,]  6.970217e+03 -3.296703e+00 -2.926406e-04
[2,] -3.296703e+00  1.551051e-03  1.456613e-07
[3,] -2.926406e-04  1.456613e-07  1.087130e-06
> vcd
             [,1]          [,2]          [,3]
[1,]  -0.02716894     -57.74666 4.237902e-01
[2,] -57.74665511 -122093.79730 8.143659e+02
[3,]   0.42379024     814.36589 9.198578e+05
> sed
[1]      NaN      NaN 959.0922
```

KUMARASWAMY-KUMARASWAMY ANALYSIS

```
>> set.seed(111)
> loglikkkum=function(p){-sum(log(p[1]*p[2]*p[3]*p[4]*x^(p[1]-1)*(1-x^p[1])^(p[4]-1)*(1-(1-
x^p[1])^p[4])^(p[1]-1)*(1-(1-(1-x^p[1])^p[4])^p[1])^(p[2]-1)))}
> e=optim(c(3,2,2,2),fn=loglikkkum,hessian=T)
> vce=solve(e$hessian)
> see=sqrt(diag(vce))
> e$convergence
[1] 1
> e
$par
[1] 1.242899e+00 1.142637e+04 9.054754e+03 2.884283e-03
```

$value

[1] -470.7315

$counts

function gradient

    501   NA

$convergence

[1] 1

$message

NULL

$hessian

```
              [,1]          [,2]          [,3]          [,4]
[1,]  3.914367e+03 -3.602219e-02  0.000000e+00 -1.789165e+05
[2,] -3.602219e-02  3.552714e-07  0.000000e+00  1.663087e+00
[3,]  0.000000e+00  0.000000e+00  5.684342e-07 -1.415401e-08
[4,] -1.789165e+05  1.663087e+00 -1.415401e-08  1.148366e+07
```

> vce

```
             [,1]          [,2]          [,3]          [,4]
[1,] 4.285921e-03  3.787417e+02  2.969300e-07  1.192490e-05
[2,] 3.787417e+02  4.220871e+07 -5.276811e-03 -2.119201e-01
[3,] 2.969300e-07 -5.276811e-03  1.759219e+06  7.558694e-09
[4,] 1.192490e-05 -2.119201e-01  7.558694e-09  3.035620e-07
```

> see

[1] 6.546695e-02 6.496823e+03 1.326355e+03 5.509646e-04

KUMARASWAMY NORMAL

```
> loglike4=function(p,x){-sum(log(p[1]*p[2]/p[4]*dnorm((x-p[3])/p[4])*((pnorm((x-p[3])/p[4]))^(p[1]-1))*((1-(pnorm((x-p[3])/p[4]))^p[1])^(p[2]-1))))}
```

```
> result4=optim(c(0.8,0.4,0.7,0.9),method="CG",fn=loglike4,hessian=T,x=x)

There were 17 warnings (use warnings() to see them)

> There were 50 or more warnings (use warnings() to see the first 50)

Error: unexpected symbol in "There were"

> result4$convergence

[1] 1

> [1] 1

Error: unexpected '[' in "["

> result4

$par

[1] 0.79250692 1.39226378 0.26363188 0.08705305

$value

[1] -50.5611

$counts

function gradient

   358      101

$convergence

[1] 1

$message

NULL

$hessian

          [,1]       [,2]       [,3]        [,4]

[1,]  102.37642  -33.85739   760.4076  -709.55929

[2,]  -33.85739   24.76274  -366.8818   -98.46491

[3,]  760.40759 -366.88182  6592.6814 -2822.55178

[4,] -709.55929  -98.46491 -2822.5518 14091.27061
```

```
> $par
```