THE STUDY OF ITO-CLIFFORD STOCHASTIC INTEGRALS AND STOCHASTIC DIFFERENTIAL EQUATIONS

BY

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DEDICATION

I dedicate this thesis to my beloved late wife Surrayya Yusuf.

May her soul rest in perfect peace,(amen).

DECLARATION

I hereby declare that this thesis has been carried out by me and that to the best of my knowledge, it has never been submitted to Ahmadu Bello University or any other institution of higher learning for the award of any degree. All resources used has been duly acknowledged in the form of references.

.... 05/07/2010 ....

Yusuf, Ahmed Omeiza Date

CERTIFICATION

This thesis "THE STUDY OF ITÔ-CLIFFORD STOCHASTIC INTEGRALS AND STOCHASTIC DIFFERENTIAL EQUATIONS" by Yusuf Ahmed Omeiza meets the regulation governing the award of the degree of Masters of Science (M. Sc.) of A.B.U. Zaria and is approved for its contributions to knowledge and literacy presentations.

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ABSTRACT

In this thesis we review an integral of anti-commuting elements analogous to the Ito-intcgral for Brownian motion.

We will also extend the stochastic integral to a wider class of integrands. The extension was achieved by using the inequalities in Remark 4.11 of chapter four.

Similarly we will also shown that a stochastic differential equation of the form $dX_t = F(X_t,t)dW_t + dW_t G(X_t,t) + H(X_t,t)dt$ has a unique solution in the L^2-space of Clifford algebra for any initial condition provided that F,G,H satisfy a Lipschitz condition.

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CHAPTER ONE

PRELIMINARIES

1.0 Introduction

In this chapter we give the definition of von Neumann algebras and also give the classification of von Neumann algebras. We have also introduced the Tomita Takesaki theorem.

1.1 Definition

Let H be a complex Hilbert space with the inner product $\langle .,. \rangle$. B(H) the algebra of all bounded linear operators on H. A von Neumann algebra is a subalgebra M of B(H) which is self-adjoint, contains the identity operator and is closed in the weak operator topology(for the definition of the weak operator topology see below).

1.2 Topologies on B(H)

(i) The strong operator topology: This is the locally convex topology induced by the family of seminorms $\{p_x : x \in H\}$,defined by, B(H): a $\to p_x(a) = \|ax\|, x \in H$.

In the strong operator topology on B(H), has a base of neighborhoods of an operator $a_0 \in B(H)$ consists of sets of the form; $\{a \in B(H) : \|(a - a_0)x_i\| < \varepsilon, i = 1,2,..n\}$ where x_1, x_2,.....x_n are in H and $\varepsilon \geq 0$. A net $\{a_i\}$ of operators in B(H) converges to $a \in (H)$ with respect to strong operator topology iff $\|(a_i - a)x\| \to 0$ as $i \uparrow \forall x \in H$

(ii) The weak operator topology on B(H): This is the locally convex topology induced by the family seminorms $\{p_{x,y} : x, y \in H\}$defined by $B(H) : a \to p_{x,y}(a) = |\langle ax, y\rangle|, x, y \in H$

The family of sets of the form $\{a \in B(H) : |\langle a - a_0\rangle x_i, y_i| < \varepsilon, i = 1,2,....,n\}$ where $\varepsilon > 0$ and $x_1, x_2, ...x_n, y_1, y_2, ...y_n \in H$ forms a base of neighborhoods of $a_0 \in B(H)$ in the weak operator topology on B(H).

A net (a_i) of operators in B(H) converges to a with respect to weak operator topology iff $|\langle a_i - a\rangle x, y| \to 0$ as $i \uparrow \forall x, y \in H$.

(iii) The ultral weak(or $\sigma - weak$) operator topology on B(H): This is th locally convex topology induced by the family of seminorms $\{p_{x_n y_n} : x_n, y_n \in H\}$, with (x_n), $(y_n) \subset H$, $\sum \|x_n\|^2 < \infty, \sum \|y_n\|^2 < \infty$, defined by; $B(H) : a \to p_{(x_n)(y_n)}(a) = \sum |\langle ax_n, y_n\rangle|$

A net $\{a_i\}_{i\in I}$ of operators in B(H) converges to a with respect to $\sigma-$weak operator topology iff $\sum|\langle(a_i-a)x_n,y_n\rangle|\to 0$ as $i\uparrow$ for all (x_n), $(y_n)\subset H$ with $\sum\|x_n\|^2<\infty,\sum\|y_n\|^2<\infty$.

(iv) The ultra strong(or $\sigma-$strong) operator topology on B(H): This is the locally convex topology induced by the family of seminorms $\{p_{x_n};x_n\in H\}$with $(x_n)\subset H,\sum\|x_n\|^2<\infty$ defined by ; $B(H):a\to\left(\sum\|ax_n\|^2\right)^{1/2}$.

A net $\{a_i\}_{i\in I}$ of operators in B(H) converges to $a\in B(H)$ with respect to $\sigma-strong$ operator topology iff $\sum\|(a_i-a)x_n\|^2\to 0$ as $i\uparrow$ with $\sum\|x_n\|^2<\infty,(x_n)\subset H$.

(v) The uniform (or norm) operator topology: This is the topology given by the norm $\|\|$, $\|a\|=\sup_{x\in H}\{\|ax\|:x\in H,\|x\|\le 1\}$

A net $\{a_i\}_{i\in I}$ of operators in B(H) converges to $a\in B(H)$ iff $\|a_i-a\|\to 0$ as $i\uparrow$.

1.3 Definition

An operator p in B(H) is said to be a projection if $p^*=p$ and $p^2=p$.

If p is a projection , then its orthogonal complement $(1-p)=p^\perp$ is also a projection.

Two projections p and q in a von Neumann algebra M are said to be equivalent (written $p \sim q$) if there exists a partial isometry $U \in M$ such that $U^*U = p$ and $UU^* = q$.

A projection p in a von Neumann algebra M is said to be finite if whenever $p \sim q \leq p$ for a projection q in M , it follows that $p = q$.

A projection p in M is called a central projection if it belongs to the centre z of M.

The set of all central projections $p \in Z$ such that $pa = a$, $a \in M$ is called the central support of a and is denoted by C(a).

A projection p in a von Neumann algebra M is said to be properly infinite if there has no non-zero finite central projection.

A projection p in a von Neumann algebra M is said to be purely infinite if there has no non-zero finite projection.

A projection p on M is said to be faithful if no non-zero central projection of M annihilate p, or if $C(p) = 1$.

1.4 Classification of von Neumann algebras:

Let M be a von Neumann algebra on H. Then,

(i) M is said be finite, properly infinite, and purely infinite if the identity projection I is finite, properly infinite, or purely infinite.

(ii) M is said to be semi finite if any non-zero central projection contains a non-zero finite projection.

(iii) M is said to be type I if any non-zero central projection contain a non-zero abelian projection.

(iv) M is said to be type I_∞ if it is properly infinite and of type I.

(v) M is said to be type II if it is semi finite and does not contain any non-zero abelian projection.

(vi) M is said to be type II_1 if it is both finite and of type II.

(vii) M is said to be type II_∞ if it is properly infinite and of type II.

(viii) M is said to be type III if it does not contain any non-zero finite projection.

1.5 Definition

A weight on a von Neumann algebra M is a function

$\phi : M_+ \to [0,\infty]$ satisfying the following conditions;

(i) $\phi(x+y) = \phi(x) + \phi(y), x, y \in M_+$

(ii) $\phi(\lambda x) = \lambda\phi(x), x \in M_+, \lambda \geq 0.$

The weight ϕ is said to be

(iii) faithful if $\phi(x) = 0 \Rightarrow x = 0 \ \forall x \in M_+$

(iv) normal if $\phi(\sup x_i) = \sup \phi(x_i)$ whenever (x_i) is the monotone, bounded increasing net $\{x_i\} \subset M_+$.

(v) semi finite if every non-zero $x \in M_+$ majorises some non-zero $y \geq 0$ with $\phi(y) < +\infty$.

If a weight ϕ satisfies (iii), (iv) and (v) above, it is called a faithful normal semi finite weight on M.

1.6 Definition

A trace τ on a von Neumann algebra M is a weight which satisfies an additional property that $\tau(x^*x) = \tau(xx^*), x \in M$. The trace τ is said to be faithful, normal, semifinite if it satisfies (iii), (iv) and (v) above respectively.

1.7 Definition

A linear functional ϕ over a von Neumann algebra M is called to be positive if $\phi(x^*x) \geq 0$ for all $x \in M$.

1.8 Definition

A state ϕ on a von Neumann algebra M is a positive linear functional which satisfies an additional property that $\phi(I) = 1$

1.9 Definition

The predual M_* of a von Neumann algebra M is the space of all $\sigma - weakly$ continuous linear functionals on M.

1.10 Definition

The commutant M' of any subset M of B(H) is the set of all bounded linear operators on H commuting with every operator in M (written $M' = \{x \in B(H) : xy = yx, \forall y \in M\}$)

$M'' = (M')'$ is called the bicommutant of M.

1.11 Theorem(von Neumann double commutant theorem)

Let M be a * - subalgebra of B(H), then the following conditions are equivalent.

(i) $M = M''$

(ii) M is weakly closed

(iii) M is strongly closed

1.12 Definition

A * -homomorphism between two von Neumann algebras M and N is a mapping $\pi : x \in M \mapsto \pi(x) \in N$ defined for all $x \in M$ s.t.

(i) $\pi(\alpha x + \beta y) = \alpha\pi(x) + \beta\pi(y), \forall x, y \in M, \alpha, \beta \in R$

(ii) $\pi(xy)=\pi(x)\pi(y), \forall x,y \in M$

(iii) $\pi(x^*)=\pi(x)^*, \quad \forall x \in M$

1.13 Lemma

Let M and N be von Neumann algebras and π a * -homomorphism of M onto N. It follows that

(i) π is positivity preserving (written $x \geq 0 \Rightarrow \pi(x) \geq 0$)

(ii) π is continuous and $\|\pi(x)\| \leq \|x\|$ *for all* $x \in M$.

Proof

(i) If $x \geq 0$ then $x = y^* y$ for some $x \in M$. Thus $\pi(x)=\pi(y^* y)=\pi(y)^* \pi(y) \geq 0$

(ii) One has, $0 \leq (x^* x)^2 \leq x^* x \|x^* x\|$. Therefore part (i) implies that

$0 \leq \pi(x^* x)^2 \leq \pi(x^* x)\|x^* x\|$.

Finally, one has $\|\pi(x)\|^4 = \|\pi(x^* x)\|^2 \leq \|\pi(x^* x)\|\|x^* x\| = \|\pi(x)\|^2 \|x\|^2$, which is equivalent to

$\|\pi(x)\| \leq \|x\|$

1.14 Definition

A * -homomorphism π of M into N is a * - isomorphism if it is one-to-one and onto i.e a *- homomorphism π of a von Neumann algebra M into a von Neumann algebra N is a *- isomorphism iff, $\ker\pi = \{0\}$.

1.15 Definition

A representation of a von Neumann algebra M is defined to be a pair (H, π) where H is a complex Hilbert space and π is a *- homomorphism of M into B(H)

The representation (H, π) is said to be faithful iff π is a * - isomorphism between M and $\pi(M)$. H is called the representation space and π is a representation of M on H.

1.16 Proposition

Let (H, π) be a representation of the von Neumann algebra M. The representation is faithful if and only if it satisfies each of the following equivalent conditions

(i) $\ker\pi = \{0\}$

(ii) $\|\pi(x)\| = \|x\|$ for all $x \in M$

(iii) $\pi(x) > 0$ for all $x > 0$

Proof

The equivalence of condition (i) and faithfulness is by definition. We now proof $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$.

$(i) \Rightarrow (ii)$ As $\ker \pi = \{0\}$ we can define a homomorphism π^{-1} from the range of π into M by $\pi^{-1}(\pi(x)) = x$ and then applying Lemma 1.14 to π^{-1} and π successively one has

$$\|x\| = \|\pi^{-1}(\pi(x))\| \leq \|\pi(x)\| \leq \|x\|.$$

$(ii) \Rightarrow (iii)$ If x > 0 then $\|x\| > 0$ and hence $\|\pi(x)\| > 0, or\, \pi(x) \neq 0$. but $\pi(x) \geq 0$ by Lemma 1.11. and therefore $\pi(x) > 0$.

$(iii) \Rightarrow (i)$ If condition (i) is false then there is a $y \in \ker \pi$ with $y \neq 0$ and $\pi(y^* y) = 0$. But $\|y^* y\| \geq 0$ and as $\|y^* y\| = \|y\|^2$ one has $y^* y > 0$. Thus condition (iii) is false.

1.17 Definition

A *- automorphism π of a von Neumann algebra M is defined to be a *-isomorphism of M into itself.

1.18 Proposition

Let (H,π) be a representation of a von Neumann algebra M. The following statements are equivalent.

(i) The closed subspace $[\pi(M)H]$ spanned by $\pi(x)\xi, x \in M, \xi \in H$ coincide with the whole space

(ii) For any nonzero $\xi \in H, \exists$ an element $x \in M$ with $\pi(x)\xi \neq 0$.

Suppose that (i) holds, *and that* $\pi(a)\xi = 0$ *for every* $a \in M$ *For any* $\eta \in H$, we *have,*

$(\pi(a)\eta/\xi) = (\eta/\pi(a)^*\xi) = (\eta/\pi(a^*)\xi) = 0.$

Hence ξ *is orthogonal to* $[\pi(A)H]$ *By assumption, this means* $\xi = 0$. *Hence* (ii) *follows. Conversely,* suppose *that* (ii) *holds. Let* ξ *be a vector of* H *orthogonal to* $[\pi(M)H]$ *We then have,*

$$0 = (\xi / \pi(a^* a)\xi) = (\xi / \pi(a^*)\pi(a)\xi) = (\pi(a)\xi / \pi(a)\xi), a \in M$$

So that $\|\pi(a)\xi\|^2 = 0$ *for every* $a \in M$. *By assumption,* $\xi = 0$. *Thus* (i) *follows.*

1.19 Definition

A representation (H, π) of M is said to be proper or nondegenerate if either (i) or (ii) of the previous proposition holds. Otherwise, the closed subspace $(H, \pi(M))$ is called the essential space of π and denoted by $H(\pi)$.

1.20 Definition

A vector Ω in a Hilbert space H is said to be cyclic for a set of bounded operators M if the set $\{x\Omega : x \in M\}$ is dense in H.

1.21 Definition

A cyclic representation of a von Neumann algebra M is a triple (H, π, Ω) where (H, π) is a representation of M and Ω is a vector in H which is cyclic for M in H.

1.22 Definition

A set M of bounded operators on the Hilbert space H is irreducible if the only closed subspaces of H which are invariant under the action of M are the trivial subspaces $\{0\}$ and H.

1.23 Definition

A representation (H,π) of a von Neumann algebra M is defined to be irreducible if the set $\pi(M)$ is irreducible on H.

If (H,π) is a nondegenerate representation of a von Neumann algebra M and Ω is a cyclic vector in H with $\|\Omega\|=1$ then ω_Ω defined by

$\omega_\Omega(x)=(\Omega,\pi(x)\Omega)$ is a state over M.

Proof

(i) ω_Ω is positive since

$$\omega_\Omega(x^*x)=(\Omega,\pi(x^*x)\Omega)=(\Omega,\pi(x^*)\pi(x)\Omega)$$

$$=(\pi(x)\Omega,\pi(x)\Omega)=\|\pi(x)\Omega\|^2\geq 0.$$

(ii) ω_Ω is linear since ,

$$\omega_\Omega(\alpha x) = (\Omega, \pi(\alpha x)\Omega) = (\Omega, \alpha\pi(x)\Omega) = \bar{\alpha}(\Omega, \pi(x)\Omega) = \bar{\alpha}\omega_\Omega(x) = \alpha\omega_\Omega(x), \alpha \in \mathbb{R}.$$

and (iii) $\omega_\Omega(I) = 1, \sin ce$

$$\omega_\Omega(I) = (\Omega, \pi(I)\Omega) = (\Omega, \Omega) = \|\Omega\|^2 = 1$$

Hence, the linear functional, $\omega_\Omega(x) = (\Omega, \pi(x)\Omega)$ is a state over M, this type of state is called a vector state.

Now we want to prove the converse. Every state is a vector state for some nondegenerate representation. Thus starting from a state ω, we must construct a representation (H_ω, π_ω) of M and a vector $\Omega_\omega \in H_\omega$ such that ω is identified as the vector state ω_{Ω_ω} such that ,

$$\omega(x) = (\Omega_\omega, \pi_\omega(x)\Omega_\omega) \text{ for all } x \in M.$$

The algebra M which is a Banach space and with the aid of the state ω it may be converted into a pre-Hilbert space by introduction of the positive semidefinite scalar product $\langle x, y\rangle = \omega(x^* y)$. Next, we defined J_Ω by,

$J_{\Omega} = \{x : x \in A, \omega(x^{*}x) = 0\}$.

The set J_{ω} is a left ideal of M because $I \in J_{\omega}$ and $x \in M$ implies that $0 \le \omega((xI)^{*}xI) \le \|x\|^{2}\omega(I^{*}I) = 0$. Next, we define equivalence classes ψ_x, ψ_y by

$\psi_x = \{\hat{x} : \hat{x} = x + I, I \in J_{\omega}\}$. These equivalence classes form a complex vector space when equipped with the operations inherited from M, $\psi_x + \psi_y = \psi_{x+y}, \alpha\psi_x = \psi_{\alpha x}$. This later space is a strict pre-Hilbert space with respect to the scalar product

$$(\psi_x, \psi_y) = \langle x, y \rangle = \omega(x^{*}y)$$

(ψ_x, ψ_y) is independent of the particular class representatives used in its definition because $\omega((x + I_1)^{*}(y + I_2)) = \omega(x^{*}y) + \overline{\omega}(y^{*}I_1) + \omega(x^{*}I_2) + \omega(I_1^{*}I_2) = \omega(x^{*}y)$ whenever $I_1, I_2 \in J_{\omega}$.

A strict pre-Hilbert space may be completed (written linearly embedded as a dense subspace of a Hilbert space in a manner which preserves the scalar product and the completion of this space is defined as the representation space H_{ω}.)

Next, we consider the definition of the representatives $\pi_{\omega}(x)$. First we specify their action on the dense subspace of the H_{ω} formed by the vectors $\psi_y, y \in M$ by definition

$$\pi_\omega(x)\psi_{y+I} = \psi_{xy}$$

This relation is again independent of the representatives used for the class ψ_y because

$$\pi_\omega(x)\psi_{y+I} = \psi_{xy} + xI = \psi_{xy} = \pi_\omega(x)\psi_y \text{ for } I \in J_\omega \quad .$$

Each $\pi_\omega(x)$ is a linear operator because

$$\pi_\omega(x)(\lambda\psi_y + \psi_z) = \pi_\omega(x)\psi_{\lambda y+z} = \psi_{\lambda xy+xz} = \lambda\psi_{xy} + \psi_{xz}$$
$$= \lambda\pi_\omega(x)\psi_y + \pi_\omega(x)\psi_z.$$

Finally, from $|\omega(x^*yx)| \le \omega(x^*x)\|y\|$ we have, $\|\pi_\omega(x)\psi_y\|^2 =$

$(\psi_{xy}, \psi_{xy}) = \omega(y^*x^*xy) \le \|x\|^2\|\psi_y\|^2$

Hence $\lambda_\omega(x)$ has a bounded closure, also denoted by $\pi_\omega(x)$. The algebraic properties of the π_ω follow easily, $\pi_\omega(x_1)\pi_\omega(x_2)\psi_y = \psi_{x_1x_2y} = \lambda_{\omega(x_1x_2)}\psi_y$

and hence $\pi_\omega(x_1)\pi_\omega(x_2) = \pi_\omega(x_1x_2)$

Thus we have now constructed the representation (H_ω, π_ω)

Next, we specify the vector Ω_ω, we define Ω_ω by $\Omega_\omega = \psi_1$.

This gives the correct identification of ω , $(\Omega_\omega, \pi_\omega(x)\Omega_\omega) = (\psi_1, \psi_x) = \omega(x)$

The set $\{\pi_\omega(x)\Omega_\omega : x \in M\}$ is exactly the dense set of equivalence classes $\{\psi_x : x \in A\}$ and hence Ω_ω is cyclic for (H_ω, π_ω).

1.24 Definition

Let the operators S_0 and F_0 be defined by $S_0 A\Omega = A^*\Omega$ for $A \in M$ and

$F_0 A'\Omega = A'^*\Omega$ for $A' \in M'$.

Then S_0 and F_0 are both well defined with domains $M\Omega$ and $M'\Omega$ respectively.

They are antilinear and are closable.Let $S_0^* = \overline{F}_0 = F$ and $F_0^* = \overline{S}_0 = S$ be their closures.

Let Δ be the unique positive self-adjoint operator which is invertible in the sense of unbounded operators (written Δ is 1-1) and J the unique self-adjoint antiunitary operator occurring in the polar decomposition of $S = J\Delta^{1/2}$.

Since $S^* = F$ then $F = J\Delta^{-1/2}$

Δ is called the modular operator associated with the pair $\{M, \Omega\}$ and J is called the modular conjugation.

1.25 Proposition

The following relations hold:

$$\Delta = FS, \Delta^{-1} = SF$$
$$S = J\Delta^{1/2}, F = J\Delta^{-1/2}$$
$$J = J^*, J^2 = 1$$
$$\Delta^{-1/2} = J\Delta^{1/2}J$$

1.26 (Tomita Takesaki) Theorems

Let M be a von Neumann algebra with cyclic and separating vector Ω and let Δ be the associated modular operator and J the associated modular conjugation. It follows that,

$$JMJ = M'$$

and moreover,

$$\Delta^{it} M \Delta^{-it} = M \text{ for all } t \in \mathbb{R}$$

Where $\left\{\Delta^{it} : t \in \mathbb{R}\right\}$ is the one parameter group of strongly continuous unitary operators.

The operators $\Delta^{it}, t \in \mathbb{R}$, define a one parameter group of *- automorphisms $\{\sigma_t : t \in \mathbb{R}\}$ of M by $\sigma_t(x) = \Delta^{it} x \Delta^{it}$. The group $t \to \sigma_t$ is called the modular automorphism group associated with the pair (M, Ω).

Let M be a von Neumann algebra, ω a faithful normal state on M, $(H_\omega, \pi_\omega, \Omega_\omega)$ the corresponding cyclic representation and Δ the modular operator associated with the pair $(\pi_\omega(M), \Omega)$. The Tomita- Takesaki theorem establishes the existence of a $\sigma - \textit{weakly}$ continuous one parameter group $t \to \sigma_t$ of *-*automorphisms* of M through the definition.

1.27 Definition

A gauge space Γ is a triple (H, M, τ) composed of a Hilbert space H, a von Neumann algebra M and a faithful normal semifinite trace τ .

τ is said to be completely additive in case for any set M_{proj} of mutually orthogonal projections in M with least upper bound (l.u.b.) projection, $\tau(p) = \sum \tau(Q) \{Q \in M_{proj}\}$

τ is unitarily invariant in case for every unitary operator $m \in M$ and projection $p \in M, \tau(m * pm) = \tau(p)$, and Γ is said to be (metrically) finite in case $\tau(I)$ is finite , where I denote the identity operator.

1.28 Definition

An operator T on a Hilbert space H is said to be affiliated with a von Neumann algebra M on H(written, $T\eta M$) if it commutes with every unitary operator in the commutant M' of M.

1.29 Definition

A linear set D in a Hilbert space H is said to be strongly dense in H with respect to a von Neumann algebra M on H if

(i) $D\eta M$

(ii) there exists a sequence $\{R_n\}$ of closed linear manifolds associated with M such that $R_n \subset D, R_n^{\perp}$ is a-finite and $R_n^{\perp} \downarrow 0. D$ is then said to be defined by $\{R_n\}$.

An operator T on H is called measurable with respect to M if

(i) $T\eta M$

(ii) T has a strongly dense domain

(iii) T is closed.

An operator T is essentially measurable with respect to M if ;

(i) $T\eta M$

(ii) There exists a sequence $\{R_n\}$ of closed linear manifold associated with M such that $R_n \subset D(T)$, the contraction of T to R_n is a-finite, and $R_n^{\perp} \downarrow 0$.;

(iii) The closure of T exists. In either case T is said to be strongly defined on $\{R_n\}$

1.30 Theorem

If S and T are essentially measurable w.r.t. a von Neumann algebra M on a Hilbert space H, then so are S^*, S + T, and ST.

1.31 Definition

If S and T are operators on H then S + T is the operator with domain $D = D(S) \cap D(T)$ and ST is the operator with the domain $\{x \in D(T) : Tx \in D(S)\}$

1.32 Theorem

If S and T are essentially measurable with respect to a von Neumann algebra on a Hilbert space H and agree on a strongly dense domain (with respect to that von Neumann) then they have identical closures.

1.33 Definition

If S and T are measurable operators with respect to a von Neumann algebra on a Hilbert space, the closures of S+T and ST are called the strong sum and strong product of S and T resp. (w.r.t. the von Neumann algebra).

1.34 Remark

In the case of abelian von Neumann algebras, the concept of measurable operator just introduced is essentially equivalent to the concept of measurable function.

1.35 Definition

A sequence $\{T_n\}$ of operators on a Hilbert space H each of which is measurable with respect to the von Neumann M on H, is said to converge nearly everywhere(n.e.)(relative to M) to a measurable operator T, if for every positive ε there exists a sequence$\{P_n\}$ of projections in M such that $P_n \uparrow I$ as $n \uparrow \infty$, $\|(T_n - T)P_n\| < \varepsilon (n = 1,2,\ldots)$ and $I - P_n$ is a-finite .If R is any projection in M, $\{T_n\}$ is said to converges n.e. on RH if $\{RT_nR\}$ converges to RTR in the sense just define.

1.36 Definition

Let T be an operator on a Hilbert space H which is associated with a von Neumann algebra M on H and τ be a faithful normal semifinite trace on M. The (metric) rank of T, with respect to the gauge (H,M,τ) is defined as the gauge of the closure of the range of T. T is said to be elementary or elop, if it is everywhere define and its rank is finite, and metrically-equivalent to zero, or n.e. zero ($T\approx 0$) in case its rank is zero.

1.37 Definition

The L_1-norm designated $\|T\|_1$ of an elop T, with respect to a gauge space (H,M,τ) is define as l.u.b. $|\tau(ST)|\{S\in M,\|S\|\le 1\}$ The bound of an operator S in M will be denoted as $\|S\|_\infty$ in a situation where $\|S\|$ might be interpreted as the L_1-norm.

1.38 Definition

A sequence $\{T_n\}$ of measurable operators converges metrically(n.e.) to a measurable operator T, with respect to a gauge space (H,M,τ), if it converges n.e. to T on the carrier of M.

A measurable operator T on the gauge space (H,M,τ) is called integrable if it is the limit m.n.e. of a sequence $\{T_n\}$ of elops that is cauchy in L_1.

The integral or trace of T, denoted $\tau^1(T)$ is defined as $\lim_n \tau(T_n)$.

1.39 Theorem

The integral of an integrable operator is unique.

1.40 Corollary

If S and T are bounded measurable and integrable operators respectively, then S.T and TS are integrable and $\tau^1(S.T) = \tau^1(ST)$ Moreover T^* and $|T|$ are integrable, and $\tau^1(T^*) = \tau^1(T)$

If S and T are integrable operators with $S \geq T$ then $\tau(S) \geq \tau(T)$

Note: The integral of an operator T over a gauge space (H, M, τ) will be designated simply as $\tau(T)$

1.41 Definition

An operator T is said to be a square-integrable with respect to a gauge space Γ if it is measurable and if T = R+ *i*S with R and S symmetric and R^2 and S^2 integrable. The collection of all square-integrable operators is designated $L_2(\Gamma)$

1.42 Corollary

For any square-integrable operators S and T, S. T is integrable, $|\tau(S^*T)|^2 \leq \tau(S^*S)\tau(T^*T)$

1.43 Corollary

A measurable operator T is square-integrable iff T*T is integrable.

1.44 Definition

The L_2-norm of a measurable operator T is defined as $(\tau(T^*T))^{1/2}$, and designated $\|T\|_2$,

and the $L_2(M)$ *is the completion of* M *with respect to this norm.*

1.45 Corollary

The L_2-norm of the operators square-integrable over a gauge space has actually the properties of a norm.

1.46 Corollary

For any integrable operator T, $\|T\|_1 = \tau(|T|)$ *and the* $L_1(M)$ *is the completion of* M *with respect to this norm.*

For any integrable operator T the p-norm of T is defined as $\|T\|_p = \left(\tau\left(|T|^p\right)\right)^{1/p}$ *and the* $L_p(M)$ *is the completion of M with respect to this norm.*

If N is a subspace of M then we can regard $L_p(N)$ *as a subspace of*

1.47 Definition

Let $N \subseteq M$ be von Neumann algebras on H. Then by a projection of norm one from M onto N we mean a projection mapping whose norm is one.

1.48 Remark

Note that If M is a von Neumann algebra and N its von Neumann subalgebra. If π is a projection of norm one from M to N, then it satisfies the following properties

(i) π *is order preserving*

(ii) $\pi(axb) = a\pi(x)b \qquad \forall a,b \in N, and \forall x \in M$

(iii) π *is* $*$*-preserving i.e* $\pi(x)^* = \pi(x^*)$

1.49 Remark

Shuh- Teh C. Moy has discussed the characteristic properties of the conditional expectation as a linear transformation of the space of all extended real-valued measurable functions on a probability space into itself.

1.50 Remark

Let M be a W^*- algebra acting on a Hilbert space H, with a (faithful) normal trace τ with $\tau(I) = 1$ and let N be an arbitrary (but fixed) W^*-subalgebra of M.

The mapping $x \to E(x)$ from $L_1(M)$ onto $L_1(N)$ satisfying the following conditions for any $x, y \in L_1(M)$ and any complex numbers α, β,

(i) $E(\alpha x + \beta y) = \alpha E(x) + \beta E(y), \textit{linearity conditions}$

(ii) $E(x^*) = E(x)^*, * - \textit{preserving conditions}$

(iii) $x \geq 0 \Rightarrow E(x) \geq 0, \textit{order preserving}$

(iv) $x \geq 0 \textit{ and } E(x) = 0 \Rightarrow x = 0, \textit{ faithful conditions.}$

(v) $E(Z) = Z \textit{ for } Z \in L_1(M) \; (\textit{fixed})$

Moreover the mapping $x \to E(x)$ transforms M onto N satisfying $\|E(x)\| \leq \|x\|$, contractive conditions.

(vi) $E(E(x)y) = E(xE(y)) = E(x)E(y)$ *for* $x \in L_1(M), y \in M, or, y \in L_1(M), x \notin \in M$.

(vii) $x_\gamma \uparrow x$ implies $E(x_\gamma) \uparrow E(x)$ for $x_\gamma, x \in M$, *continuity conditions.*

(viii) $x \to E(x)$ is strongly and weakly continuous on the unit sphere of M.

(ix) $E(xy) = E(yx), for, x \in L_1(M) and, y \in M_1' \cap M$.

(x) $\mu(|E(x)|) \leq \mu(|x|)$, *where* μ *is a faithful normal tracial state on* M

is called a conditional exp*ectation from* M *onto* N.

1.51 Remark

From the discussion above we can see that every faithful normal projection of norm one is a conditional expectation. The theory developed by Umegaki depends on a finite von Neumann algebra with faithful normal tracial state. This theory is not applicable to von Neumann of type III. The question then arise; can conditional expectation exists always on any von Neumann algebra? The question was answered by Masamichi Takesaki.

1.52 Theorem

Let M be a von Neumann algebra and μ a faithful normal semifinite weight on M_+. Let N be a von Neumann subalgebra of M on which μ is semifinite. Then the following two statements are equivalent:

(I) N is invariant under the modular automorphism group σ_t^{μ} associated with μ $\left(i.e.\sigma_t^{\mu}(N) \subseteq N\right)$

(ii) There exist a faithful normal projection of norm one E from M onto N such that

$\mu = \mu \circ E$

1.53 Remark

If N is a von Neumann subalgebra of a von Neumann algebra M and if there exists a conditional expectation E from M onto N, then for every faithful normal semifinite weight μ on N, the inequality $\mu(x) = \mu \circ E(x), x \in M_+$, defines a faithful normal semifinite weight on M_+ with respect to which E is a conditional expectation of M onto N. Hence N is invariant under the modular automorphism group σ_t^{μ} associated with μ.

(i) $B(0) = 0$;

(ii) for any $0 \le t_0 < t_1 < \dots < t_n$, the random variables $B(t_k) - B(t_{k-1})(1 \le k \le n)$ are independent;

(iii) if $0 \le s < t, B(t) - B(s)$ is normally distributed with

$$E(B(t) - B(s)) = (t - s)\mu,$$
$$E(B(t) - B(s))^2 = (t - s)\sigma^2$$

where μ, σ are real constants, $\sigma \ne 0$.

μ is called the drift and σ^2 is called the variance.

Property (ii) implies that B(t) – B(s) is independent of

$\mathcal{F}(B(\lambda), \lambda \le s$ *or, more generally, that* $\mathcal{F}(B(\mu) - B(s), \mu \ge s$ *is independen of*

$\mathcal{F}(B(\lambda), \lambda \le s)$.

If $B(t)$*is a Brownlan motion with drift* μ *and variance* δ^2,

and if $0 \le t_0 < t_1 < \dots < t_n$, *then*

A Brownian motion or Wiener process is a stochastic process $B(t), t > 0$ satisfying;

$$\Gamma(t_j,t_k) = E(B(t_j) - \mu t_j)(B(t_k) - \mu t_k) = \delta^2 \min(t_j,t_k)$$

Indeed, if $t_j > t_k$,

$$\Gamma(t_j,t_k) = E[B(t_j) - B(t_k) - \mu(t_j - t_k) + B(t_k) - \mu t_k][B(t_k) - \mu t_k]$$

$$= E(B(t_k) - \mu t_k)^2 = t_k\delta^2.$$

If $\mu = 0,\ \delta^2 = 1$, then we speak of normalized Brownian motion.

If $B(t)$ is a Brownian motion and

$\mathcal{F}_t = \mathcal{F}(B(\lambda),\ 0 \le \lambda \le t)$, *then*

$E[(B(t) - B(s))/\mathcal{F}_s] = 0,$ ………….(i)

$E[(B(t) - B(s))^2/\mathcal{F}_s] = t - s$ ………(ii)

a.s. for any $0 \le s < t$. Note that (i) and (ii) hold if and only if $B(t)$ and $B^2(t) - t$ are martingales.

1.55 Theorem

Let $B(t), t \geq 0$ be a continuous process and let $\mathcal{F}_t (t \geq 0)$ be an increasing family of $\sigma - fields$ such that $B(t)$ is $\mathcal{F}_t - measurable$ and (i) and (ii) hold a.s. for all $0 \leq s < t$. Then $B(t)$ is a Brownian motion.

Let $0 \leq t < \infty$. A stochastic process $f(t)$ defined for $0 \leq t < \infty$ is called a *nonanticipative* function with respect to $\mathcal{F}_t$ if :

(i) $f(t)$ is a separable process;

(ii) $f(t)$ is a measurable process,

(iii) for each $t \in [0,t]$, $f(t)$ is $\mathcal{F}_t - measurable$.

When (iii) holds we say that $f(t)$ is adapted to $\mathcal{F}_t$. We denote by $Lp(\mathcal{F}) (1 \leq p \leq \infty)$ the class of all *nonaticipative* functions $f(t)$ satisfying:

$$p\left\{\int_0^t |f(t)|^p \, dt < \infty\right\} = 1 \qquad \left(p\left\{ess \sup_{0 \leq t < \infty} |f(t)| < \infty\right\} = 1 if p = \infty\right)$$

1.56 Definition

A stochastic process $f(t)$ defined on $[0,t]$ is called a *step function* if there exists a partition $0 = t_0 < t_1 < ..... < t_r = t$ of $[0,t]$ such that

$f(t) = f(t_i)$ if $t_i \le t < t_{i+1}, 0 \le i \le r-1$.

1.57 Definition

Let $f(t)$ be a step function in $L_B{}^2(\Omega, \mathcal{F}_t, \mathrm{P})$, say $f(t) = f_i$ if $t_i \le t < t_{i+1}, 0 \le i \le n-1$ where $0 = t_0 < t_1 < .....t_n = t$ The random variable

$\sum_{k=0}^{n-1} f(t_k)[B(t_{k+1}) - B(t_k)]$ is denoted by $\int_0^t f(t)dB(t)$ and is called the *stochastic* integral of f with respect to the Brownian motion B; it is also called *Itô* – integral.

1.58 Lemma

Let f_1, f_2 be two step functions in $L_B{}^2(\Omega, \mathcal{F}_t, \mathrm{P})$ and let λ_1, λ_2 be real numbers. Then $\lambda_1 f_1 + \lambda_2 f_2$ is

in $L_B{}^2(\Omega, \mathcal{F}_t, \mathrm{P})$ and

$$\int_0^t [\lambda_1 f_1(t) + \lambda_2 f_2(t)]dB(t) = \lambda_1 \int_0^t f_1(t)dB(t) + \lambda_2 \int_0^t f_2(t)dB(t) \text{ --------- } (1)$$

1.59 Lemma

If f is a step functions in $L_B^2(\Omega, \mathcal{F}_t, P)$ then

$$E\int_0^t f(t)dB(t) = 0 \text{ --------- } (2)$$

$$E\left|\int_0^t f(t)dB(t)\right|^2 = E\int_0^t f^2(t)dt. \text{ ----- } (3)$$

Proof

Since

$E\int_0^t f^2(t)dt. = \sum_{i=0}^{r-1} Ef^2(t_i)(t_{i+1} - t_i)$ ------- (4) is finite, by assumption, we deduce that

$Ef^2(t_i) < \infty$. In particular, $E|f(t_i)| < \infty$. *Also,* $E|B(t_{i+1}) - B(t_i)| < \infty$.*But Since* $f(t_i)$ *is* $\mathcal{F}_t$ *– measurable whereas* $B(t_{i+1}) - B(t_i)$ *is independent of* $\mathcal{F}_t$.

$Ef(t_i)(B(t_{i+1}) - B(t_i)) = Ef(t_i)E(B(t_{i+1}) - B(t_i)) = 0.$

Summing over i (2) follows.

Next, Since $f^2(t_i)$ *and* $(B(t_{i+1})-B(t_i))^2$ *are independent and have finite expectation, also* $f^2(t_i)(B(t_{i+1})-B(t_i))^2$ *has finite expectation. By Schwartz's inequality it follows that*

$$E|f(t_k)f(t_i)(B(t_{i+1})-B(t_i))|<\infty.$$

If $k>i$, *then* $B(t_{k+1})-B(t_k)$ *is independent of* $f(t_k)f(t_i)(B(t_{i+1})-B(t_i))$. *In view of the last inequality*

and the finiteness of $E|B(t_{k+1})-B(t_k)|$, *we deduce that*

$$Ef(t_k)f(t_i)(B(t_{i+1})-B(t_i))(B(t_{k+1})-B(t_k))=0.$$

Hence

$$E\left|\int_0^t f(t)dB(t)\right|^2=\sum_{i=0}^{n-1}Ef^2(t_i)(B(t_{i+1})-B(t_i))^2$$

$$=\sum_{i=0}^{n-1}Ef^2(t_i)E(B(t_{i+1})-B(t_i))^2$$

$$=\sum_{i=0}^{n-1}Ef^2(t_i)(t_{i+1}-t_i)=E\int_0^t f^2(t)dt$$

By (4), and (3) is proved.

1.60 Lemma

For any step function f in $L_B^2(\Omega, \mathcal{F}_t, P)$ and for any $\varepsilon > 0, N > 0$,

$$P\left\{\left|\int_0^t f(t)dB(t)\right| > \varepsilon\right\} \le P\left\{\int_0^t f^2(t)dt > N\right\} + \frac{N}{\varepsilon^2}. \text{---------} (5)$$

Proof : *Let*

$$\phi_N(t) = \begin{cases} f(t) & \text{if} \quad t_k \le t < t_{k+1} \quad \text{and} \quad \sum_{i=0}^{k} f^2(t_i)(t_{i+1} - t_i) \le N, \\ 0 & \text{if} \quad t_k \le t < t_{k+1} \quad \text{and} \quad \sum_{i=0}^{k} f^2(t_i)(t_{i+1} - t_i) > N, \end{cases}$$

where $f(t) = f(t_i)$ *if* $t_i \le t < t_{i+1}$; $t_0 = \alpha < t_1 < \dots < t_r = \beta$.*Then* $\phi_N \in L_B^2(\Omega, \mathcal{F}_t, P)$

and

$$\int_\alpha^\beta \phi_N^2(t)dt = \sum_{i=0}^{v} f^2(t_i)(t_{i+1} - t_i)$$

Where v is the largest integer such that

$$\sum_{i=0}^{\nu} f^2(t_i)(t_{i+1} - t_i) \le N, \ \ \nu \le r-1.$$

Hence

$$E\int_0^t \phi_N{}^2(t)dt \le N.$$

Further, $f(t) - \phi_N(t) = 0$ *for all* $t \in [\alpha, \beta)$ *if* $\int_0^t f^2(t)dt < N$*.Therefore*

$$P\left\{\left|\int_0^t f(t)dB(t)\right| > \varepsilon\right\} \le P\left\{\left|\int_0^t \phi_N(t)dB(t)\right| > \varepsilon\right\} + P\left\{\int_0^t f^2(t)dt > N\right\}.$$

Since, by Chebyshev's inequality, the first integral on the right is bounded by

$$\frac{1}{\varepsilon^2}E\left|\int_0^t \phi_N(t)dB(t)\right|^2 \le \frac{N}{\varepsilon^2}, \text{the assertion (5) follows.}$$

We shall now proceed to define the stochastic integral for any function

f *in* $L_B{}^2(\Omega, \mathcal{F}_t, \mathrm{P})$. By a Lemma, .say,

Let $f \in L_B{}^2(\Omega, \mathcal{F}_t, \mathrm{P})$ *then :*

(i) *there exists a sequence of continuous functions* g_n *in* $L_B^2(\Omega, \mathcal{F}_t, P)$ *such that*

$$\lim_{n\to\infty}\int_0^t |f(t)-g_n(t)|^2 dt = 0 \quad a.s.;------(1)$$

(ii) *there exists a sequence of step functions* f_n *in* $L_B^2(\Omega, \mathcal{F}_t, P)$ *such that*

$$\lim_{n\to\infty}\int_0^t |f(t)-f_n(t)|^2 dt = 0 \quad a.s.;------(2)$$

There is a sequence of step functions f_n *in* $L_B^2(\Omega, \mathcal{F}_t, P)$ *such that*

$$\int_0^t |f_n(t)-f(t)|^2 dt \to 0 \text{ if } n\to\infty..---------(6)$$

Hence

$\lim\limits_{n,m\to\infty} P\int_0^t |f_n(t)-f_m(t)|^2 dt \xrightarrow{P} 0.$ *By Lemma* (1.60), *for any* $\varepsilon > 0, \rho > 0,$

$$P\left\{\left|\int_0^t f_n(t)dB(t)-\int_0^t f_m(t)dB(t)\right| > \varepsilon\right\} \le \rho + P\left\{\int_0^t |f_n(t)-f_m(t)|^2 dt > \varepsilon^2\rho\right\}.$$

It follows that the sequence

$$\left\{\int_0^t f_n(t)dB(t)\right\}$$

is convergent in probability. We denote the limit by

$\int_0^t f(t)dB(t)$ *and call it the stochastic integral or the Itô – integral of* $f(t)$ *with respect to the Brownian motion* $B(t)$.

The above definition is independent of the particular sequence $\{f_n\}$.

For if $\{g_n\}$ *is another sequence of step functions* f_n *in* $L_B^{\,2}(\Omega, \mathcal{F}_t, P)$ *converging to* f *in the sense that*

$$\int_0^t |g_n(t) - f(t)|^2 dt \xrightarrow{p} 0,$$

then the sequence $\{h_n\}$ *where* $h_{2n} = f_n, h_{2n+1} = g_n$ *is also convergent to* f *in the same sense. But then, by what we have proved, the sequence*

$$\left\{\int_0^t h_n(t)dB(t)\right\}$$

is convergent in probability.It follows that the lim*its in* (*probability*) *of* $\int_0^t f_n dB$ *and* $\int_0^t g_n dB$ *are equal a.s. Lemma* 1.58 – 1.60 *extend to any function from* $L_B^2(\Omega, \mathcal{F}_t, P)$

1.61 Theorem

Let f_1, f_2 *be functions from* $L_B^2(\Omega, \mathcal{F}_t, P)$ *and let* λ_1, λ_2 *be real numbers. Then* $\lambda_1 f_1 + \lambda_2 f_2$ *is in* $L_B^2(\Omega, \mathcal{F}_t, P)$ *and*

$$\int_0^t [\lambda_1 f_1(t) + \lambda_2 f_2(t)]dB(t) = \lambda_1 \int_0^t f_1(t)dB(t) + \lambda_2 \int_0^t f_2(t)dB(t). ------(7)$$

1.62. Theorem

If f *is a function in* $M_\omega^2(\Omega, \mathcal{F}_t, P)$, *then*

$$E\int_0^t f(t)dB(t) = 0, ---------(8)$$

$$E\left|\int_0^t f(t)dB(t)\right|^2 = E\int_0^t f^2(t)dt. --------(9)$$

1.63. Theorem

If f is a function in $L_B^2(\Omega, \mathcal{F}_1, P)$, then, for for any $\varepsilon > 0, N > 0,$

$$P\left\{\left|\int_0^t f(t)dB(t)\right| > \varepsilon\right\} \leq P\left\{\int_0^t f^2(t)dt > N\right\} + \frac{N}{\varepsilon^2}. ------(10)$$

Proof of Theorem 1.61

By Lemma 1.58 there exists a sequence of step function

f_n *in* $L_B^2(\Omega, \mathcal{F}_t, P)$ *such that*

$$E\int_0^t |f_n(t) - f(t)|^2 dt \to 0 \text{ if } n \to \infty.$$

This implies that

$$E\int_0^t f_n^2(t)dt \to E\int_0^t f^2(t)dt. ------(11)$$

By Lemma1.58,

$$E\int_0^t f_n(t)dB(t) = 0, --------(12)$$

$$E\left|\int_0^t f_n(t)dB(t)\right|^2 = E\int_0^t f_n^2(t)dt, \text{---------} (13)$$

$$E\left|\int_0^t f_n(t)dB(t) - \int_0^t f_m(t)dB(t)\right|^2 = E\int_0^t |f_n(t) - f_m(t)|^2 dt \to 0 \text{ if } n,m \to \infty. \text{------} (14)$$

From the definition of the stochastic integral,

$$\int_0^t f_n(t)dB(t) \xrightarrow{P} \int_0^t f(t)dB(t)$$

Using (14) we conclude that actually

$$\int_0^t f_n(t)dB(t) \to \int_0^t f(t)dB(t) \qquad in\ L^2(\Omega, \mathcal{F}_t, \mathrm{P})$$

Hence, in particular,

$$E\int_0^t f(t)dB(t) = \lim_{n\to\infty} E\int_0^t f_n(t)dB(t),$$

$$E\left|\int_0^t f(t)dB(t)\right|^2 = \lim_{n\to\infty} E\left|\int_0^t f_n(t)dB(t)\right|^2,$$

and using (12) and (13), (11), the assertion (8), (9) follows.

Proof of Theorem 1.62

By Lemma 1.58 there exists a

sequence of step functions f_n *in* $L_B^{\,2}(\Omega, \mathcal{F}_1, \mathrm{P})$ *such that*

$$\int_0^t |f_n(t) - f(t)|^2 \, dt \xrightarrow{P} 0. \text{-----------} (15)$$

By definition of stochastic integral,

$$\int_0^t f_n(t) dB(t) \xrightarrow{P} \int_0^t f(t) dB(t) \text{----------} (16)$$

Applying Lemma 1.60.to f_n we have

$$P\left\{ \left| \int_0^t f_n(t) dB(t) \right| > \varepsilon' \right\} \le P\left\{ \int_0^t f_n^{\,2}(t) dt > N' \right\} + \frac{N'}{(\varepsilon')^2}.$$ *Taking* $n \to \infty$ *and*

Using (15), (16), we get

$$P\left\{\left|\int_0^t f(t)dB(t)\right| > \varepsilon\right\} \le P\left\{\int_0^t f^2(t)dt > N\right\} + \frac{N'}{(\varepsilon')^2}$$

for any $\varepsilon > \varepsilon', N > N'$*.Taking* $\varepsilon' \uparrow \varepsilon, N' \downarrow N,(10)$ *follows.*

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

In this chapter we review the literature relevant to the study of Ito-Clifford Stochastic Integral. This will enable us to put the research in proper perspective, by establishing what has been done and what need to be accomplished.

2.1 Literature review

The Theory of von Neumann algebras was first introduced by J. von Neumann in 1929 with his grand aim of giving a sound foundation to quantum mechanics.

The foundation of this new field of Mathematics, operator algebras, was laid down by J. von Neumann and F. J.Murray in 1930s and early 1940s.

Since then various related fields of mathematics emerged and a number of topics branched out to independent fields.

Umegaki (1954) showed the existence of a conditional expectation from a finite von Neumann algebra onto its von Neumann subalgebra which occurred as a non commutative extension of the classical conditional expectation when the von Neumann algebra has a faithful normal semi finite trace τ *with* $\tau(I)=1$. The theory of conditional expectation developed by Umegaki depends on the existence of traces and this theory is not applicable to von Neumann algebras of types III.

Tomiyama (1957) showed that each projection of norm one of a von Neumann algebra onto its von Neumann subagebra enjoys most of the properties of conditional expectations. He demonstrated that a projection of norm one from a von Neumann algebra M onto its von Neumann *subalgebras N is a mapping* $\pi: M \to N$ *such that*

(i) π *is order preserving*

(ii) $\pi(axb)=a\pi(x)b, \forall a,b \in N, and\ \forall x \in M.$

(iii) $\pi(x)^{*}\pi(x) \leq \pi(x^{*}x), \forall x \in M$

(iv) $\pi(x^{*})=\pi(x)^{*}, \forall x \in M.$

Takesaki (1972) gave the necessary and sufficient conditions for the existence of conditional expectations as a projection of norm one from a von Neumann algebra onto its von Neumann subalgebra and is stated as:

Let M be a von Neumann algebra and φ a faithful normal semi finite weight on M_+, let N be its von Neumann subalgebra on which φ is semi finite then the following are equivalent.

(i) *N is invariant under the modular automorphism group* σ_t *associated with* φ

(ii) *There exists a* σ *– weakly continuous faithful projection*

of norm one E from M onto N such that $\varphi = \varphi o E$.

C. Barnett, R.F. Streater and I.F. Wilde (1982) described how a stochastic integral of non commutative process can be constructed. This was achieved via a Doob-Meyer decomposition for the square of self-adjoint L^2 – Martingale.

They also introduced a "condition D", derived from the 'class D' of stochastic process theory, and showed that if the square of a self-adjoint L^2- martingale satisfies this condition then it has a decomposition of the Doob-Meyer type. The class D process is define as follows:

Let $(X_\alpha) \subseteq L^1(\mathcal{A}_\infty)$ *be a process we say that* (X_α) *is of class D if and only if*

$S(X_\alpha) \stackrel{def}{=} \left\{ \sum_{i=1}^{n} E_{\alpha_{i-1}}\left(X_{\alpha_i} - X_{\alpha_{i-1}}\right); 0 \le \alpha_1 < \alpha_2 < \ldots < \alpha_n < \infty, n \in N \right\}$ *is weakly relatively compact in* $L^1(\mathcal{A}, m)$.

Also C. Barnett, R. F. Streater and I. F. Wilde(1983) have shown that a stochastic differential equation of the form $dX_t = F(X_t, t)dW_t + dW_t G(X_t, t) + H(X_t, t)dt$ has a unique solution in the L_p – space of Clifford algebra provided F, G,H satisfy Lipschitz condition

CHAPTER THREE

Ito- Clifford Stochastic Integral

3.0 Introduction

In this chapter we review an integral of anti-commuting elements analogous to the Ito-integral for Brownian motion.

3.1 Definition

Let H be a Hilbert space and let J be a conjugation on H. The anti-symmetric Fock space over H is the Hilbert space $\Lambda(H)=\oplus^{\infty}_{n=0}\Lambda_n(H)$, where $\Lambda_0(H)=\mathbb{C}$ and $\Lambda_n(H)$ is the Hilbert space anti- symmetric n-fold tensor product of H with itself.

For each $z\in H$, the creation operator $a(z)$ is defined by

$a(z):\Lambda_n(H)\rightarrow\Lambda_{n+1}(H), u\mapsto(n+1)^{1/2}p(z\otimes u)$, where p is the anti-symmetrization

projection. By linearity and continuity, $a(z)$ defines a bounded operator on $\Lambda(H)$ with norm $\|a(z)\|=\|z\|$.

The annihilation operator $a(z)^*$ is the adjoint of $a(z)$. The fermion field $\psi(z)$ is defined on $\Lambda(H)$ by $\psi(z) = a(z) + a(z)^*$. Evidently, $\psi(): H \to B(\Lambda(H))$ is linear, and the anti-commutation relations hold;

$$\{\psi(z), \psi(z')\} \equiv \psi(z)\psi(z') + \psi(z')\psi(z) = 2(Jz', z) \text{ for all } z, \quad \in H$$

Furthermore, if z is $J - real(Jz = z)$ *then* $\psi(z)$ is self-adjoint.

3.2 Definition

Let $\mathscr{C}$ denote the W^* – algebra generated by the bounded operators $\{\psi(z) : z \in H\}$. $\mathscr{C}$ is called the weakly-closed Clifford operator-algebra of (H, J).

The Fock vacuum (" no-particle vector") is the vector $\Omega = 1 \in \Lambda_0(H) \subset \Lambda(H)$.

It is well-known that Ω is cyclic for $\mathscr{C}$ and $\tau(.) = (\Omega, \Omega)$ is a [illegible]ful, central state on $\mathscr{C}$, and so ($\Lambda(H)$, $\mathscr{C}$, τ) [illegible] regul[illegible]bability gauge space.

For $1 \le p < \infty$, $L_p(\mathscr{C})$ [illegible] the completion of $\mathscr{C}$ with respect to the norm $\|u\|_p = \tau(|u|^p)^{1/p} = (\Omega, (u^*u)^{p/2}\Omega)^{1/p}$. The elements of $L_p(\mathscr{C})$ can be identified with (possible unbounded) operators on $\Lambda(H)$.

$L_\infty(\mathscr{C})$ is , by definition, $\mathscr{C}$ equipped with its operator norm.

The map $u \mapsto u\Omega$ from $\mathscr{C}$ into $\Lambda(H)$ extends to a unitary operator

$D: L^2(\mathscr{C}) \to \Lambda(H)$, and is called the duality transform. Under D, the action of $\mathscr{C}$ on $\Lambda(H)$ becomes left multiplication on $L^2(\mathscr{C})$ so that $(\Lambda(H), C, \tau)$ is standard.

If B is a W*-subalgebra of $\mathscr{C}$, then $L^p(B)$ is the completion of B with respect to $\|\cdot\|_p$ and so can be considered as a closed subspace of $L_p(\mathscr{C})$.

3.3 Theorem

For any $1 \leq p < \infty$, there is a unique map $L^p(\mathscr{C}) \to L^p(B), u \mapsto \hat{u}$ satisfying

$\tau(\hat{u}v) = \tau(uv)$ for all $u \in L^p(\mathscr{C}), v \in L^{p'}(B), with\ \ p' = p/(p-1)$.

$\hat{u}$ is called the conditional expectation of u with respect to B and is denoted by $\tau(u/B)$.

$\tau(u/B)$ enjoys the following usual properties;

(i) $\tau(u/B)$ is a contraction from $L^p(\mathscr{C}) onto L^p(B)$ for all $1 \leq p \leq \infty$;

(ii) $\tau(u/B)$ is positive preserving;

(iii) $\tau(vu/B) = v\tau(u/B)$ for all $u \in L^p(\mathscr{C}), v \in L^{p'}(B)$,

(iv) if $B_1 \subseteq B_2$, *then* $\tau(\tau(./B_2)/B_1) = \tau(./B_1)$.

$\tau(./B)$ on $L^2(C)$ is the projection onto the subspace $L^2(B)$.

Let $H = L^2(\Re_+, ds)$ *and let* J *be the complex conjugation on* $L^2(\Re_+, ds)$

For given $0 \le s \le t$, *defined* $\mathscr{C}_t$ *to be* W^* – *subalgebra of* $\mathscr{C}$ *generated by the field* $\psi(u)$ *for* $u \in L^2(\Re_+, ds)$ *with* $\operatorname{ess\,sup} u \subseteq [0,t]$ *Clearly* $\mathscr{C}_s \subseteq \mathscr{C}_t$ *for* $0 \le s \le t$, *and* $\mathscr{C}$ *is generated by the* $\mathscr{C}_t, t \in \Re_+$. *We shall denote* $\tau(./C_t)$ *by* $E_s(.), s \in \Re_+$.

If $S \in B(H)$ *with* $\|S\| \le 1$, *there is a bounded operator* $\Gamma(S)$ *on* $\Lambda(H)$ *whose action on* $\Lambda_n(H)$ *is given by* $S \quad ...\otimes S$ ctors) for $n \ge 1$, and is the identity on $\Lambda_0(H) = \mathscr{C}$ *If* e_t *denote the projection on* H *given by* $e_t u = \chi_{[0,t]}u$, then it is easy to see that $D^{-1}\Gamma(e_t)D$ is equal to the conditional expectation map $E_t()$ on $L^2(\mathscr{C})$.

This follows from the easily established facts:

$\Gamma(p_t)\Lambda(H) = \Lambda(p_t H)$, D $L^2(\mathscr{C}_t)$ L^2 and $E()$ is the projection of $L^2(\mathscr{C})$ onto $L^2(\mathscr{C}_t)$

3.4 Definition

Let $0 \le t_0 \le t$. An adapted process on $[t_0,t]$ is a map $h:[t_0,t] \to L^1(\mathscr{C})$ such that $h(s) \in L^1(\mathscr{C}_s)$ for all $s \in [t_0,t]$. An L^p – process is a process h such that $h(s) \in L^p(\mathscr{C}_s)$ for all s.

3.5 Definition

A process h on $[t_0,t]$ is said to be simple if it can be expressed as

$$h = \sum_{k=1}^{n} h_{k-1}\chi_{[t_{k-1},t_k)} \text{ on } [t_0,t), \text{ for some } t_0 \le t_1 \le \dots \le t_n = t \text{ and } h_k \in L^1(\mathscr{C}_{t_k}), 1 \le k \le n-1.$$

Since h is a process, we see that $h(s) = h_{k-1} \in L^1(\mathscr{C}_s)$ *for all* $t_{k-1} \le s < t_k$, *i.e.,* $h_{k-1} \in L^1$ $L^1(\mathscr{C}_{t_{k-1}})$ *Let* $u \in L_{loc}^2(\Re_+, ds)$ *be real – valued. We shall denote* $\psi(u\chi_{[0,t]})$ *by* ψ_t. *Then* ψ_t *is self – adjoint and belong to* $L^\infty(\mathscr{C}_t)$ *for all* $t \in \Re_+$.

For h a simple process on $[t_0, t]$, *the Itô – Clifford stochastic integral of h over* $[t_0, t]$ *is*

$\int_{t_0}^{t} h(s) d\psi_s = \sum_{k=1}^{n} h_{k-1}(\psi_{t_k} - \psi_{t_{k-1}})$ *Clearly* $\int_{t_0}^{t} h(s) d\psi_s \in L^r(\mathcal{C}_s)$ *and* is independent of the decomposition of h as a sum of step-functions. For notational convenience we will sometimes write $I(h)$ *for* $\int_{t_0}^{t} h(s) d\psi_s$ *and* $\Delta\psi_k$ *for* $\psi_{t_k} - \psi_{t_{k-1}}$.

3.6 Lemma

Let $\{X_t : t \in \Re_+\}$ *be an* L^p *– martingale, and suppose that* $0 \le s \le t$. *Then*

$\tau(f(X_t - X_s)g) = 0$ *for any* $h \in L^q(\mathcal{C}_s)$, $g \in L^r(\mathcal{C}_s)$ *with* $1/p + 1/q + 1/r = 1$. *In particular,* $\tau(h(\psi_t - \psi_s)g) = 0$ *for any* $h \in L^q(\mathcal{C}_s)$, $g \in L^r(\mathcal{C}_s)$ *with* $1/q + 1/r = 1$.

Proof

We have

$$\tau(h(X_t - X_s)g) = \tau(gh(X_t - X_s)) = \tau(ghE_s(X_t - X_s)) = 0$$

The last part follows since $\psi_t \in L^\infty$ *for all t.* ■

3.7 Theorem

The Itô-Clifford Stochastic Integral satisfies the following:

(i) $\int_{t_0}^{t} (\alpha h(s) + \beta g(s)) d\psi_s = \alpha \int_{t_0}^{t} h(s) d\psi_s + \beta \int_{t_0}^{t} g(s) d\psi_s$ *for simple processes* h, g *and* $\alpha, \beta \in \mathbb{C}$.

(ii) $\tau\left(\int_{t_0}^{t} h(s) d\psi_s\right) = 0,$ *for simple* h.

(iii) *If* h *is a simple* L^2 *– process on* $[t_0 - t]$, *written* $h(s) \in L^2(\mathcal{C})$ *for all* $t_0 \leq s \leq t$,

then $\int_{t_0}^{t} h(s) d\psi_s \in L^2(_t)$ *and* $\left\| \int_{t_0}^{t} h(s) d\psi_s \right\|_2^{\,2} = \int_{t_0}^{t} \|h(s)\|^2{}_2 |u(s)|^2 ds.$

Proof

(i) This is clear

(ii) We have

$$\tau\left(\int_{t_0}^{t} h\,d\psi\right)=\sum_{k=1}^{n}\tau\left(h_{k-1}\left(\psi_{t_k}-\psi_{t_{k-1}}\right)\right) \text{ for suitable } t_0 \le t_1 \le \ldots \le t_n = t \quad \text{and } h_{k-1}\in L^1\left(\mathcal{C}_{t_{k-1}}\right)$$

$$=0$$

By lemma 3.6,

(iii) $$\left\|\int_{t_0}^{t} h\,d\psi\right\|_2^2=\tau\left(\left\{\sum_{k=1}^{n}h_{k-1}\Delta\psi_k\right\}^{*}\left\{\sum_{j=1}^{n}h_{j-1}\Delta\psi_j\right\}\right)=\sum_{k,j}\tau\left(\Delta\psi_k h^{*}_{k-1}h_{j-1}\Delta\psi_j\right)$$

By Lemma above we see that off-diagonal ($\text{written } k\neq j$) terms all vanish, so we need only consider the terms with $k=j$. However,

$$\tau\left(\Delta\psi_k h^{*}_{k-1}h_{k-1}\Delta\psi_k\right)=\tau\left(h^{*}_{k-1}h_{k-1}\left(\Delta\psi_k\right)^2\right)$$
$$=\tau\left(h^{*}_{k-1}h_{k-1}\right)\int_{t_{k-1}}^{t_k}\left|u(s)\right|^2 ds=\int_{t_0}^{t}\tau\left(h^{*}_{k-1}h_{k-1}\right)\chi_{[t_{k-1},t_k]}(s)\left|u(s)\right|^2 ds.$$

Summing over k gives the result. ■

We shall called property (iii) the isometry property of $It\hat{o}-Clifford$ stochastic integral. It is this property which will allow us to define the stochastic integral for processes other than simple ones. Let $d\mu$ *denote the measure* $\left|\mu(s)\right|^2 ds$ *on* $\Re_+$.

3.8 Theorem

Let g be a continuous L^2 *– process on* $[t_0,t]$ *i.e.,* $s \mapsto g(s)$ is continuous from $[t_0,t]$ into $L^2(\mathscr{C})$. *Then, for given* $\varepsilon > 0$, *there is a simple* L^2 *– process h on* $[t_0,t]$ *such that*

$$\int_{t_0}^{t} \|g(s)-h(s)\|_2^{\,2}\, d\mu(s) < \varepsilon.$$

Proof

Let $\varepsilon' > 0$ *be given. Since g is continuous, it is uniformly continuous on* $[t_0,t]$ *and so there is* $t_0 \le t_1 \le \ldots\ldots \le t_n = t$ *such that*

$$\|g(s)-g(t_{k-1})\|_2^{\,2} < \varepsilon' \text{ whenever } t_{k-1} \le s \le t_k.$$

putting $h = \sum_{k=1}^{n} g(t_{k-1})\chi_{[t_{k-1},t_k)}$, *we see that* $\|g(s)-h(s)\|_2^{\,2} < \varepsilon'$ *for all* $t_0 \le s < t$, *and so*

$$\int_{t_0}^{t}\|g(s)-h(s)\|_2^{\,2}\,d\mu(s)<\varepsilon'\int_{t_0}^{t}d\mu(s).$$ ■

3.9 Lemma

For fixed $X\in L^2(\mathcal{C})$, the map $s\mapsto E_s(X)$ is continuous from $\Re_+$ into $L^2(\mathcal{C})$.

Proof

The map $s\mapsto e_s$ *is strongly continuous on* $H=L^2(\Re_+,dx)$ *and so* $\Gamma(e_s)$ *is strongly continuous on* $\Lambda(H)$. *Therefore* $s\mapsto E_s(X)=D^{-1}\Gamma(e_s)DX$ *is continuous from* $\Re_+$ into $D^{-1}\Lambda(H)=L^2(\mathcal{C})$ ■

Let $L^2([t_0,t],d\mu,L^2(\mathcal{C}))$, the complex Hilbert space of $L^2(\mathcal{C})$ *– valued* martingale maps on $[t_0,t]$ square-integrable with respect to $d\mu$, is isomorphic to the Hilbert- space tensor product $L^2([t_0,t],d\mu)\otimes L^2(\mathcal{C})$ This, in turn, is the completion of the algebraic tensor product $C([t_0,t])\otimes L^2(\mathcal{C})$ *where* $C([t_0,t])$ is the space of complex-valued continuous functions on $[t_0,t]$.

We will consider elements of $L^2([t_0,t],d\mu,L^2(\mathcal{C}))$ *as maps* : $[t_0,t]\to L^2(\mathcal{C})$ *define* μ almost everywhere rather than equivalence classes of maps. Thus, $h\in L^2([t_0,t],d\mu,L^2(\mathcal{C}))$, is a process if $h(s)\in L^2(\mathcal{C}_s)$ μ *a.e.*

Let $\mathfrak{H}[t_0,t]$ denote the set of processes in $L^2([t_0,t],d\mu;L^2(\mathfrak{H}))$

3.10 Proposition

$\mathfrak{H}[t_0,t]$ is a closed subspace of $L^2([t_0,t],d\mu,L^2(\mathcal{C}))$, i.e., $\mathfrak{H}[t_0,t]$ is a Hilbert-space.

Proof

Let (h_n) be a sequence in $\mathfrak{H}[t_0,t]$ such that $h_n\to h$ *in* $L^2([t_0,t],d\mu,L^2(\mathcal{C}))$. By passing to a subsequence, we may suppose that $h_n(s)\to h(s)\mu$ *a.e. in* $L^2(\mathcal{C})$. *But* $h_n(s)\in L^2(\mathcal{C}_s)$ μ *a.e.and so* $h(s)\in L^2(\mathcal{C}_s)$ μ *a.e. Hence* $h\in\mathfrak{H}[t_0,t]$. ■

3.11 Theorem

Let $h\in\mathfrak{H}[t_0,t]$. Then h can be approximated arbitrarily closely in $[t_0,t]$ by simple processes in $\mathfrak{H}[t_0,t]$.

Proof

For given $\varepsilon > 0$ *there is* $n \in \mathbb{N}$ *and* $\phi_j \in C([t_0,t]), X_j \in L^2(\mathcal{C}), 1 \le j \le n$ *such that*

$$\int_{t_0}^{t} \left\| h(s) - \sum_{j=1}^{n} \phi_j(s) X_j \right\|_2^2 d\mu(s) < \varepsilon^2.$$

Sinces h *is a process and* $E_s : L^2(\mathcal{C}) \to L^2(\mathcal{C}_s)$ *is a contraction, we have*

$$\int_{t_0}^{t} \left\| h(s) - \sum_{j=1}^{n} \phi_j(s) E_s(X_j) \right\|_2^2 d\mu(s)$$

$$= \int_{t_0}^{t} \left\| E_s\left(h(s) - \sum_{j=1}^{n} \phi_j(s) X_j \right) \right\|_2^2 d\mu(s) \le \varepsilon^2.$$

By Lemma 3.9, each $\phi_j(.)E.(X_j)$ *is a continuous* L^2 *– process* (and so belongs to $\mathfrak{H}[t_0,t]$) and can therefore be approximated by simple processes in $\mathfrak{H}[t_0,t]$, by Theorem 3.8. The result now follows since a finite linear combination of simple processes is a simple process. ∎

3.12 Theorem

For $h \in \mathfrak{H}[t_0, t]$, *there is a sequence* (h_n) *of* simple processes converging to h *in* $\mathfrak{H}[t_0, t]$, and there is (h_n) such that $\int_{t_0}^{t} h_n(s) d\psi_s$ *converging to* $I(h)$ *in* $L^2(\mathcal{C}_t)$. *Moreover,* $I(h)$ is independent of the particular sequence (h_n) converging to h.

Proof

The existence of (h_n) *converging to* h *in* $\mathfrak{H}[t_0, t]$ is guaranteed by Theorem 3.8. In particular, (h_n) is a Cauchy sequence in $\mathfrak{H}[t_0, t]$, and therefore, by the isometry property, $\left(\int_{t_0}^{t} h_n d\psi\right)$ is a Cauchy sequence in $L^2(\mathcal{C}_t)$. The existence of the required $I(h)$ follows from *the completeness of* $L^2(\mathcal{C}_t)$.

It is easy to see that $I(h)$ *is independent of the particular sequence* (h_n). ■

3.13 Definition

For $h \in \mathfrak{H}[t_0, t]$, *the Itô – Clifford stochastic integral of* h *is* $\int_{t_0}^{t} h(s) d\psi_s = I(h)$, *where* $I(h) \in L^2(\mathcal{C}_t)$ *is as given by Theorem* 3.13.

If $0 \le t_0 \le t_1 \le t_2$ and $h \in \mathfrak{H}[t_0,t_2]$, then the restriction of h to $[t_0,t]$ and $[t_1,t_2]$ belong to $\mathfrak{H}[t_0,t_1]$ and $\mathfrak{H}[t_1,t_2]$, respectively, and

$$\int_{t_0}^{t_2} h\,d\psi_s = \int_{t_0}^{t_1} h\,d\psi_s + \int_{t_1}^{t_2} h\,d\psi_s.$$

3.14 Theorem

(i) $\int_{t_0}^{t} (\alpha h + \beta g)\,d\psi_s = \alpha \int_{t_0}^{t} h\,d\psi_s + \beta \int_{t_0}^{t} g\,d\psi_s$, *for any* h, g in $\mathfrak{H}[t_0,t]$ $\alpha, \beta \in \mathbb{C}$.

(ii) $\left(\int_{t_0}^{t} h\,d\psi_s \right) = 0$ *for any* $h \in \mathfrak{H}[t_0,t_2]$.

(iii) $\left\| \int_{t_0}^{t} h\,d\psi_s \right\|_2 = \|h\|_A$, *for any* $h \in \mathfrak{H}[t_0,t_2]$.

(iv) $E_{t_0}\left(\int_{t_0}^{t} h\,d\psi_s \right) = 0$, *for any* $h \in \mathfrak{H}[t_0,t_2]$.

(v) *If* $h \in \mathfrak{H}[0,t]$ *for all* $t \in \Re_+$, *then* $\left\{ \int_0^t h\,d\psi_s : t \in \Re_+ \right\}$ *is an* L^2 *–martingale adapted to* $\{\mathfrak{C}_t, t \in \Re_+\}$.

Proof

parts (i), (ii), and (iii) follows immediately from Theorem 3.7

(iv) *Let* $h=\sum_{k=1}^{n} h_{k-1}\chi_{[t_{k-1},t_k)}$ with $t_0 \le t_1 \le \ldots \le t_n = t, h_{k-1} \in L^2(\mathcal{C}_{t_{k-1}})$ be simple process in $\mathfrak{H}[t_0,t]$. Then, for any $g \in L^2(\mathcal{C}_{t_0})$ we have

$$\tau\left(g\int_{t_0}^{t} h\,d\psi_s\right)=\sum_{k=1}^{n}\tau\left(gh_{k-1}\Delta\psi_k\right)=0$$ by lemma 3.6

It follows that $E_{t_0}\left(\int_{t_0}^{t} h\,d\psi_s\right)=0$ *for all simple processes h in* $\mathfrak{H}[t_0,t]$, *and hence all* $h \in \mathfrak{H}[t_0,t]$.

(v) *Let* $0 \le t_0 \le t$. *Then*

$$E_{t_0}\left(\int_0^t h\,d\psi\right)=E_{t_0}\left(\int_0^{t_0} h\,d\psi+\int_{t_0}^{t_1} h\,d\psi\right)=E_{t_0}\left(\int_0^{t_0} h\,d\psi\right)=\int_0^{t} h\,d\psi \quad by(iv).$$ ■

3.15 Remark

Let $\mathfrak{H}_{loc}[0,\infty]$ denote the set of processes f such that $h \in \mathfrak{H}[t_0,t]$ for all

$t \in \Re_+$. Then we have seen that $\left\{\int_0^t h d\psi_s : t \in \Re_+\right\}$ is a centred L^2 *− martingale*. Let $h \in L^1([t_0, t], d\mu, L^2(\mathcal{C}))$, the Banach space of measurable maps h from $[t_0, t]$ into $L^1(\mathcal{C}_t)$ *such that* $\|h(.)\|_1$ *is integrable over* $[t_0, t]$ *with respect to* $d\mu$. *Then* $\int_{t_0}^t h(s) d\mu(s)$ *is a* well-defined element of $L^1(\mathcal{C})$. Indeed, it is uniquely determined by the formula

$$\tau\left(g \int_{t_0}^t h(s) d\mu(s)\right) = \int_{t_0}^t \tau(gh(s)) d\mu(s) \quad \text{for all } g \in L^\infty(\mathcal{C}).$$

Furthermore, for any $h \in L^1([t_0, t], d\mu, L^2(\mathcal{C}))$, *and* $\alpha \in \Re_+$, *we have*

$$\int_{t_0}^t E_\alpha(h(s)) d\mu(s) = E_\alpha\left(\int_{t_0}^t h(s) d\mu(s)\right).$$ *In particular it follows that*

$$\int_{t_0}^t E_{t_0}(h(s)) d\mu(s) \in L^2(\mathcal{C}_{t_0})$$

3.16 Definition

An element $h \in L^2(\mathcal{C})$ *is said to be even* (*resp. odd*) *if* $\beta h = h$ (*resp.* $-h$), *where* β : $L^2(\mathcal{C}) \to L^2(\mathcal{C})$ *denotes the self − adjoint unitary* $D^{-1}\Gamma(-1)D$. *We say that h has definite parity if h is either even or odd. If* $h : [t_0, t] \to L^2(\mathcal{C})$ *we say that h is even*

$(resp.odd)$ *if* $\beta h(s) = h(s)$ $(resp. -h(s))$ *for all* $s \in [t_0, t]$ *we say that* h *has definite parity if* h *is either even or odd. Written* $h \in L^2(\mathcal{C})$ *as* $h = \frac{1}{2}(h + \beta h) + \frac{1}{2}(h - \beta h)$, *we* see that h is (uniquely) the sum of an even element and an odd element of $L^2(\mathcal{C})$ (defining 0 as both even and odd).

Furthermore, $\beta h = \pm h$ *iff* $\Gamma(-1)h\Omega = \pm h\Omega$, *and so* h *is even iff* $h\Omega \in \bigoplus_{n=0}^{\infty} \Lambda_{2n}(H)$, *and* h *is odd iff* $h\Omega \in \bigoplus_{n=0}^{\infty} \Lambda_{2n+1}(H)$ *it follows that if* h *is even* $(resp.odd)$ *there is a sequence* (g_n) *of even* $(resp.odd)$ *polynomials in* $\mathcal{C}^0$ the algebraic span of the fields $\psi(z), z \in H$, such that $g_n\Omega \to h\Omega$ *in* $\Lambda(H)$, *i.e.,* $g_n \to h$ *in* $L^2(\mathcal{C})$. *Evendently, for any* $z \in H, \psi(z)$ *is odd. We also note that if* $h \in L^2(\mathcal{C}_t)$ *then* $\beta h \in L^2(\mathcal{C}_t)$ L^2 *and if* $g \in L^\infty(\mathcal{C}_t)$ *then* $\beta h \in L^\infty(\mathcal{C}_t)$.

3.17 Lemma

(i) *Let* $h' \in L^2(\mathcal{C})$ *be even, and* $h'' \in L^2(\mathcal{C})$ *be odd. Then* $\tau(h' * h'') = 0$.

(ii) *Let* $h \in L^2(\mathcal{C})$, *and* $g \in L^\infty(\mathcal{C})$ *have definite parity. Then* hg *is even if* h *and* g *have the same parity, otherwise* hg *is odd.*

Proof

(i) $\tau(h' * h'') = (h'\Omega, h''\Omega) = 0$, *Since* $h'\Omega$ *and* $h''\Omega$ lie in orthogonal subspaces of $\Lambda(H)$. (ii) Suppose $h \in L^2(\wp), g \in L^\infty(\wp)$ are both even. Then there is a sequence (g_n) of even polynomials *in* C^0 *such that* $g_n \to h$ *in* $L^2(\mathcal{C})$, *and therefore* $g_n g \to hg$ *in* $L^2(\mathcal{C})$. But $\Gamma(-1)g_n g\Omega = g_n g\Omega$ for all n, and so $\Gamma(-1)hg\Omega = hg\Omega; i.e., \beta(hg) = hg$.

The other cases are similar. ■

3.18 Lemma

Let $0 \le t_0 \le s \le t$, *and* suppose $g \in L^2(\mathcal{C}_{t_0})$ L^2 has definite parity. Then

$(\psi_t - \psi_s)g = \pm g(\psi_t - \psi_s)$ depending on whether g is even or odd.

Proof

Suppose g is odd. Then there is a sequence (g_n) of odd polynomials in $\hat{\mathfrak{C}}$ the self-adjoint subalgebra of $\hat{\mathfrak{C}}$ generated by the fields $\psi(z)$ *with* $z \in H$ *with* $\operatorname{ess\,sup} z \subseteq [0, t_0]$, *such that* $g_n \to g$ *in* $L^2(\hat{\mathfrak{C}}_{t_0})$.

Now, $\psi_t - \psi_s = \psi(u\chi_{[s,t]})$ *and* $u\chi_{[s,t]}$ *is orthogonal in* $H = L^2(\Re_+, ds)$ *to all* z *with* $\operatorname{ess\,sup} z \subseteq [0, t_0]$. It follows from the anti-commutation relations that $(\psi_t - \psi_s)g_n = -g_n(\psi_t - \psi_s)$. *The result follows by letting* $n \to \infty$. *For* g even the proof is similar. ■

3.19 Theorem

Let $h', h'' \in \mathfrak{H}[t_0, t]$, both have definite parity. Then

$$E_{t_0}\left(\left(\int_{t_0}^{t} h' d\psi\right)^* \left(\int_{t_0}^{t} h'' d\psi\right)\right) = \pm \int_{t_0}^{t} E_{t_0}\left(h'(s)^* h''(s)\right) d\mu(s)$$ *depending on whether* h' *and* h'' *have equal or opposite parity.*

Proof

First we note that if $h \in \mathfrak{H}[t_0,t]$, then $|h()|^2 \in L^1([t_0,t],d\mu,L^2(\mathcal{C}))$, and so, by polarization, we see that $h'()^* h''()$ is (a process) in $L^1([t_0,t],d\mu,L^2(\mathcal{C}))$. Hence $\int_{t_0}^{t} E_{t_0}(h'(s)^* h''(s))d\mu(s) \in L^1(\mathcal{C}_{t_0})$

We shall prove the theorem for simple processes, and then use a limiting argument. Let $h',h'' \in \mathfrak{H}[t_0,t]$ be simple processes of definite parity. Then there is a partition $t_0 \leq t_1 \leq \ldots\ldots \leq t_n = t$ such that

$$h' = \sum_{k=1}^{n} h'_{k-1}\chi_{[t_{k-1},t_k)} \text{ and } h'' = \sum_{k=1}^{n} h''_{k-1}\chi_{[t_{k-1},t_k)} \text{ on } [t_0,t) \text{ for } h'_{k-1},h''_{k-1} \in L^2(\mathcal{C}_{t_{k-1}})$$

For any $g \in L^2(\mathcal{C}_{t_0})$, we have

$$\tau\left(E_{t_0}\left(I(h')^* I(h'')\right)g\right) = \tau\left(I(h')^* I(h'')g\right) = \sum_{k=1}^{n}\sum_{j=1}^{n} \tau\left(\Delta\psi_k h'_{k-1} h''_{j-1} \Delta\psi_j g\right)$$

By Lemma 3.6, we see that the off-diagonal terms all vanish, and if h' and h'' have opposite parity and g is even then, by Lemma 3.15, the diagonal terms also vanish. For g odd, we apply Definition 3.16, to obtain

$$-\sum_{k=1}^{n} \tau\left(h'_{k-1}{}^{*} h''_{k-1} g (\Delta \psi_k)^2\right) = -\sum_{k=1}^{n} \tau\left(h'^{*}_{k-1} h''_{k-1} g\right) \int_{t_{k-1}}^{t_k} d\mu.$$

Thus, for h', h'' of opposite parity and g odd, we have

$$\tau\left(I(h')^{*} I(h'') g\right) = -\int_{t_0}^{t} \tau\left(h'(s)^{*} h''(s) g\right) d\mu(s).$$

However, this remains valid for g even since then both sides vanish. Therefore it holds for all $g \in L^{\infty}(\mathfrak{C}_{t_0})$ *hence, for* h', h'' of opposite parity,

$$E_{t_0}\left(I(h')^{*} I(h'')\right) = -\int_{t_0}^{t} E_{t_0}\left(h'(s)^{*} h''(s)\right) d\mu(s).$$

For h', h'' of equal parity the proof is similar.

Now let $h', h'' \in \mathfrak{H}[t_0, t]$ be of definite parity, and let $(h'_n), (h''_n)$ be sequences of simple processes converging to h', h'', *respectively, in* $\mathfrak{H}[t_0, t]$.

By considering $\frac{1}{2}(h'_n \pm \beta h'_n)$ *and* $\frac{1}{2}(h''_n \pm \beta h''_n)$ as necessary, we may suppose that h'_n *and* h''_n have the same parity as h' *and* h'', *respectively.*

We have $I(h_n') \to I(f')$ *and* $I(h_n'') \to I(f'')$ *in* $L^2(\mathcal{C})$ *and so* $I(h_n')^* I(h_n'') \to I(f')^* I(f'')$ *in* $L^2(\mathcal{C})$

Hence $E_{t_0}\left(I(h_n')^* I(h_n'')\right) \to E_{t_0}\left(I(h')^* I(h'')\right)$ *in* $L^2(\mathcal{C})$

Now let $g \in L^\infty(\mathcal{C}_{t_0})$. *Then*

$$\left|\int_{t_0}^{t} \tau\left(\left(h'(s)^* h''(s) - h_n'(s)^* h_n''(s)\right)g\right) d\mu(s)\right| =$$

$$\left|\int_{t_0}^{t} \tau\left(h'(s)^* (h''(s) - h_n''(s))g + \left(h'(s)^* - h_n'(s)^*\right) h_n''(s) g\right) d\mu\right|$$

$$\leq \int_{t_0}^{t} \left\|g h'(s)^*\right\|_2 \left\|h''(s) - h_n''(s)\right\|_2 d\mu + \int_{t_0}^{t} \left\|h'(s)^* - h_n'(s)^*\right\|_2 \left\|h_n''(s) g\right\|_2 d\mu$$

$$\leq \left\{\int_{t_0}^{t} \left\|h'(s)^*\right\|_2^2 d\mu \int_{t_0}^{t} \left\|h''(s) - h_n''(s)\right\|_2^2 d\mu\right\}^{1/2} \|g\|_\infty$$

$$+ \left\{\int_{t_0}^{t} \left\|h'(s) - h_n'(s)\right\|_2^2 d\mu \int_{t_0}^{t} \left\|h_n''(s)\right\|_2^2 d\mu\right\}^{1/2} \|g\|_\infty.$$

Taking the supremum over $g \in L^\infty(\mathcal{C})$ *with* $\|g\|_\infty \leq 1$ *and letting* $n \to \infty$, *we conlude that*

$\int_{t_0}^{t} h_n'(s)^* h_n''(s) d\mu \to \int_{t_0}^{t} h'(s)^* h''(s) d\mu \quad L^1(\hat{\mathfrak{C}})$, *and so* $E_{t_0}\left(\int_{t_0}^{t} h_n'(s)^* h_n''(s) d\mu\right) \to$

$E_{t_0}\left(\int_{t_0}^{t} h'(s)^* h''(s) d\mu\right)$ *in* $L^1(\hat{\mathfrak{C}})$. *Hence*

$$E_{t_0}\left(I(h')^* I(h'')\right) = L^1 - \lim E_{t_0}\left(I(h_n')^* I(h_n'')\right)$$

$$= \pm L^1 - \lim \int_{t_0}^{t} E_{t_0}\left(h_n'(s)^* h_n''(s)\right) d\mu = \pm \int_{t_0}^{t} E_{t_0}\left(h'(s)^* h''(s)\right) d\mu. \blacksquare$$

3.20 Theorem

Let $h \in \mathfrak{H}[t_0, t]$. Then

$$E_{t_0}\left(\left|\int_{t_0}^{t} h d\psi\right|^2\right) = \int_{t_0}^{t} E_{t_0}\left(|\beta h(s)|^2\right) d\mu.$$

Proof

It is easy to see that $\beta h \in \mathfrak{H}[t_0, t]$ and so we can write $h = h_1 + h_2$ *with* $h_1, h_2 \in \mathfrak{H}[t_0, t]$, h_1 even and h_2 odd. (In fact $h_1 = \frac{1}{2}(h + \beta h)$ and $h_2 = \frac{1}{2}(h - \beta h)$)

By Theorem 3.19, we have

$E_{t_0}\left(I(h_i)^* I(h_j)\right) = (2\delta_{ij} - 1)\int_{t_0}^{t} E_{t_0}\left(h_i(s)^* h_j(s)\right)d\mu = \int_{t_0}^{t} E_{t_0}\left((\beta h_i(s))^* \beta h_j(s)\right)d\mu$. *Summing over $i, j = 1,2$ gives the required result.*

■

3.21 Theorem

Let $h \in \mathfrak{H}_{loc}(0,\infty)$. Then

$Z_t = \left|\int_{t_0}^{t} hd\psi_s\right|^2 - \int_{t_0}^{t} |\beta h(s)|^2 d\mu, t \in \Re_+$, *defines a centred L^1 – martingale adapted to the family to* $\{\mathcal{C}_t, t \in \Re_+\}$. *In particular,* $\left\{\left|\int_{t_0}^{t} hd\psi\right|^2 : t \in \Re_+\right\}$ can be written as the sum of a centred *L^1 – martingale and a positive increasing process*.

Proof

We have already established that $Z_t \in L^1(\mathcal{C}_t)$ for all $t \in \Re_+$. *For $0 \le t_0 \le t$, write*

$\left|\int_0^t hd\psi\right|^2 = \left(\int_0^{t_0} hd\psi + \int_{t_0}^{t} hd\psi\right)^* \left(\int_0^{t_0} hd\psi + \int_{t_0}^{t} hd\psi\right)$. By Theorem 2.14, we have

$$E_{t_0}\left(\left(\int_0^{t} hd\psi\right)^{*}\left(\int_{t_0}^{t} hd\psi\right)\right)=\left(\int_0^{t_0} hd\psi\right)^{*}E_{t_0}\left(\int_{t_0}^{t} hd\psi\right)=0,$$

And similarly

$$E_{t_0}\left(\left(\int_{t_0}^{t} hd\psi\right)^{*}\left(\int_0^{t_0} hd\psi\right)\right)=E_{t_0}\left(\int_{t_0}^{t} hd\psi\right)^{*}\int_0^{t_0} hd\psi=0.$$

Therefore

$$E_{t_0}\left(\left|\int_0^{t} hd\psi\right|^2\right)=\left|\int_0^{t_0} hd\psi\right|^2+E_{t_0}\left(\left|\int_{t_0}^{t} hd\psi\right|^2\right)=\left|\int_0^{t_0} hd\psi\right|^2+E_{t_0}\left(\int_{t_0}^{t}|\beta h(s)|^2 d\mu\right)$$

By Theorem 3.20,

$$=\left|\int_0^{t_0} hd\psi\right|^2+E_{t_0}\left(\int_0^{t}|\beta h(s)|^2 d\mu\right)-E_{t_0}\left(\int_0^{t_0}|\beta h(s)|^2 d\mu\right)=\left|\int_0^{t_0} hd\psi\right|^2+E_{t_0}\left(\int_0^{t}|\beta h(s)|^2 d\mu\right)-\int_0^{t_0}|\beta h(s)|^2 d\mu,$$

i.e., $M_{t_0}(Z_t)=Z_{t_0}$.

particular, $\tau(Z_t)=\tau(E_0(Z_t))=\tau(Z_0)=0.$ *Hence* $\{Z_t : t\in\Re_+\}$ *is a centred* L^1 *– martingale.*

■

CHAPTER FOUR

Stochastic Integrals in an arbitrary probability gauge space

4.0 Introduction

In this chapter we review the stochastic integrals in an arbitrary Probability gauge space.

Let A be a von Neumann algebra acting on a Hilbert space H, and let τ be a faithful normal trace on A with $\tau(I)=1$. Let $\{ A_\alpha, \alpha \in \Re_+ \}$ be a family of von Neumann subalgebras of A such that

(i) $\left(\bigcup_{\alpha\in\Re_+} A_\alpha\right)'' = A$ (ii) $\bigcap_{\beta>\alpha} A_\beta = A_\alpha$,

(iii) $\left(\bigcup_{\beta<\alpha} A_\beta\right)'' = A_\alpha$, (iv) $A_\beta \subset A_\alpha$ if $\beta < \alpha$.

For $1 \leq p < \infty$ *and* $\alpha \in \Re_+$, let L^p(A $_\alpha$) be the completion of A $_\alpha$ with respect to the norm $\|x\|_p = \tau(|x|^p)^{1/p}$. Then L^p(A $_\alpha$), is the non-commutative analogue of the Lebesque space associated with A $_\alpha$ and τ, *with* L^∞(A $_\alpha$) = A $_\alpha$.

Let $\{E_\alpha : \alpha \in \Re_+\}$ *be a family of conditional expectations of* A *onto* A $_\alpha$ *whose extension to* L^p(A $_\alpha$) is still denoted by E_α.

A process is a family $(X_\alpha)\alpha \in \Re_+$ *with* $X_\alpha \in L^1$(A $_\alpha$). For each $\alpha \in \Re_+$ we denote by E_α the conditional expectation from L^p(A $_\infty$) onto L^p(A $_\alpha$). We do not distinguish between the expectations for differing values of $p \in [1, \infty)$ since each one is the appropriate extension of the expectation of A $_\infty$ onto A $_\alpha$. A process (X_α) is martingale (resp. supermartingale, submartingale) if

$E_\beta(X_\alpha) = X_\beta, \beta \leq \alpha (respectively; E_\beta(X_\alpha) \leq X_\beta, E_\beta(X_\alpha) \geq X_\beta)$ If (X_α) is a supermartingale with $X_\alpha \geq 0, \alpha \in \Re_+, and\ \tau(X_\alpha) \to 0\ as\ \alpha \to \infty\ then\ (X_\alpha)$ is called a potential.

4.1 Definition

Let $\alpha \to \infty \ (X_\alpha) \subseteq L^1$ (A $_\alpha$) be a process. We say that (X_α) is of class D if and only if

$S(X_\alpha) := \left\{ \sum_{i=1}^{n} E_{\alpha_{i-1}} \left(X_{\alpha_i} - X_{\alpha_{i-1}} \right) : 0 \le \alpha_0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n < \infty, n \in N \right\}$ is weakly relatively compact in L^1 (A).

4.2 Definition

Let (X_α) be a submartingale. We say that (X_α) has a *Doob–Meyer decomposition* if we can write $X_\alpha = U_\alpha + A_\alpha, \alpha \in \Re_+, where$ (U_α) is a martingale and (A_α) is a positive increasing process, $i.e., 0 \le A_{\alpha_1} \le A_{\alpha_2}$ *for* $0 \le \alpha_1 \le \alpha_2$.

4.3 Remark

Let (X_α) be an L^2*–martingale. If* $\left(X_\alpha^* X_\alpha\right)$ *(respectively* $\left(X_\alpha X_\alpha^*\right)$*) is of class D then it has a Doob–Meyer decomposition. We write* $\left(\langle X \rangle_\alpha\right)$ *for the positive inceasin g process in this decomposition.*

We recall that a positive increasing process, (A_α), which is continuous as a map $\Re_+ \to L^1(A\ \ )$ defines a vector measure on the Borel $\sigma - field\ of\ [0,t]\ for\ any\ t \in \Re_+$.

This measure is (the extension of) the set function dA given by

$$dA([s_1, s_2]) = A_{s_2} - A_{s_1} \qquad s_1 \le s_2$$

(With no change for $[s_1, s_2), (s_1, s_2], (s_1, s_2)$ because (A_α) is continuous). The measure has variation $d|A|$ equal to the semivariation with respect to A , given by $E \to \|dA(E)\|_1$.

4.4 Lemma

(i) If (X_α) *is an* L^2 *– martingale with* $(|X_\alpha|^2)$ *of class D then for* $0 \le s_1 \le s_2$

$$d|\langle X\rangle|([s_1, s_2]) = \tau\left(|X_{s_2}|^2 - |X_{s_1}|^2\right)$$

(ii) If (X_α) *is is an* L^2 *– martingale with* $(X_\alpha^* X_\alpha)$ *and* $(X_\alpha X_\alpha^*)$ *of class D then*

$$d|\langle X\rangle| = d|\langle X^*\rangle|$$

Proof

$$(i)\ d|\langle X\rangle|([s_1, s_2]) = \|d\langle X\rangle([s_1, s_2])\|_1 = \left\|\langle X\rangle_{s_2} - \langle X\rangle_{s_1}\right\|_1 = \tau\left(\langle X\rangle_{s_2} - \langle X\rangle_{s_1}\right)$$

If $|X_s|^2 = V_s + \langle X \rangle_s$, *with* (V_s) *a martingale, then as* $E_{s_1}(V_{s_2}) = V_{s_1}$ *we have*

$$\tau(\langle X \rangle_{s_2} - \langle X \rangle_{s_1}) = \tau(E_{s_1}(\langle X \rangle_{s_2} - \langle X \rangle_{s_1})) = \tau(E_{s_1}(V_{s_2} + \langle X \rangle_{s_2} - (V_{s_1} + \langle X \rangle_{s_1})))$$

$$= \tau\left(E_{s_1}\left(|X_{s_2}|^2 - |X_{s_1}|^2\right)\right) = \tau\left(|X_{s_2}|^2 - |X_{s_1}|^2\right)$$

(ii) *If* (X_α) *is an with* L^2 *– martingale* $(|X_\alpha|^2)$ *of class D then for* $0 \le s_1 \le s_2$

$d|\langle X \rangle|([s_1, s_2]) = \tau\left(|X_{s_2}|^2 - |X_{s_1}|^2\right)$ *and since* τ *is trace then*

$$\tau|X_\alpha^* X_\alpha| = \tau|X_\alpha X_\alpha^*|$$

$$\Rightarrow d|\langle X \rangle| = d|\langle X^* \rangle|$$

4.5 Definition

Let (X_α) *be an* L^2 *– martingale. Then the function* $\alpha \to \tau(|X_\alpha|^2)$ is increasing and continuous. We shall denote the Stieltjes measure defined by this function by $d|\langle X \rangle|$.

4.6 Definition

(i) *Let* $F=[r,t), r<t$, *and* $X\in L^1(A_r)$. *Then* $h(s)=X\chi_F(s)$ *is process. All processes of this form are called elementry.*

(ii) *Let* $h(s)$ *be a process. We say* $h(s)$ *is simple on* $[0,t]$ *if it is finite linear combination of elementry processes on* $[0,t)$. *We note that if* $t>0$ *then a simple proces on* $[0,t]$ *can always be written in the form* $\sum_{i=1}^{n} X_i\chi_{F_i}$, *where*

$F_i=[s_i,t_i)(1\le i\le n-1), F_n=\{t\}$ *and*

$0=s_1<t_1=S_2<t_2=s_3<\ldots.t_{n-1}=t.$

(iii) *Let* (X_α) *be an* L^2 *−martingale and* $h(s)$ *a simple process on* $[0,t]$, *say*

$h(s)=\sum_{i=}^{n} h(s)\chi_{[s_i,t_i)}(s)$ *on* $[0,t)$ *and* $[s_i,t_i)$ *form a disjoint partition of* $[0,t)$. *Then we define the right integral of h by*

$\int_0^t h(s)dX_s=\sum_{i=1}^{n} h(s_i)(X_t-X_{s_i})$, *and the left integral of h by*

$\int_0^t dX_s h(s)=\sum_{i=1}^{n}(X_{t_i}-X_{s_i})h(s_i)$.

We note immediately that

$$\left(\int_0^t h(s)dX_s\right)^* = \int_0^t dX_s^* h(s)^*.$$

4.7 Theorem (contraction property)

Let (X_α) *be an* L^2 *– martingale and* f *a simple process on* $[0,t]$

(i) *If* $(X_\alpha^* X_\alpha)$ *is class D then*

$$\left\|\int_0^t dX_s f(s)\right\|_2^2 = \tau\left(\int_0^t |f^*(s)|^2 d\langle X\rangle_s\right) \le \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s|$$

(ii) *If* $(X_\alpha X_\alpha^*)$ *is class D then*

$$\left\|\int_0^t f(s)dX_s\right\|_2^2 = \tau\left(\int_0^t |f(s)|^2 d\langle X^*\rangle_s\right) \le \int_0^t \|f(s)\|_\infty^2 d|\langle X\rangle_s|$$

Proof

The inequality follows from the positivity of τ,

if $|X_\alpha|^2 = U_s + \langle X\rangle_s$, *then*

$$\tau\left(\int_0^t |h^*(s)|^2 d\langle X\rangle_s\right) = \sum_i \tau\left(|h_{i-1}^*|^2\left(\langle X\rangle_i - \langle X\rangle_{i-1}\right)\right) = \sum_i \tau\left(|h_{i-1}^*|^2\left(|X_i|^2 - |X_{i-1}|^2\right)\right)$$

$$= \sum_i \tau\left(\Delta X_i |h_{i-1}^*|^2 \Delta X_i^*\right) \le \sum_i \left\| |h_{i-1}^*|^2 \right\|_\infty \tau(\Delta X_i \Delta X_i)$$

$$= \sum_i \|h_{i-1}\|_\infty^2 \tau\left(|X_i|^2 - |X_{i-1}|^2\right) = \int_0^t \|h(s)\|_\infty^2 d|\langle X\rangle_s|, \text{ where } \Delta X_i = X_i - X_{i-1}. \quad \blacksquare$$

4.8 Definition

Let $f : \Re_+ \to \mathbf{A}$ *be a process and* (X_α) *an* L^2*(bounded)martingale.Let* $d|\langle X\rangle|$ *denote the measure derived from* $s \to \tau(X_s^* X_s)$ *and* $\mathbf{P}\langle X\rangle$ *denote the class of all* L^2*(bounded)martingales. We write* $f \in \mathbf{P}\ \langle X\rangle$ *if and only if*

(i) f *is the* $d|\langle X\rangle|$ *almost everywhere* $\|\|_\infty$ lim*it of simple* $\mathbf{A}$ *valued processes* (f_n),

(ii) $\forall \varepsilon > 0, \exists \delta > 0$ *such that if* $F \subseteq \Re_+$ *is a Borel set and* $d|\langle X\rangle|(F) < \delta$ *then*

$\int_F \|h_n(s)\|_\infty^2 d|\langle X\rangle_s| < \varepsilon$ *for* $n = 1,2,3,\ldots.$

We say that the sequence (h_n) *defines h and that* (h_n) *is uniformly absolutely continuous* $(u.a.c)$ *with respect to* $d|\langle X\rangle|$.

4.9 Lemma

Let (X_α) *be an* L^2 *– bounded martingale and* $g, h \in \mathrm{P}\langle X\rangle$, $\lambda \in \Re$ *then*

(i) $\lambda g + h \in \mathrm{P}\langle X\rangle$.

(ii) *For each* $t \geq 0$ *the function* $s \to \|h(s)\|_\infty^2$ *is* $d|\langle X\rangle|$ – *integrable on* $[0,t]$ *and*

$$\int_0^t \|h(s)\|_\infty^2 \, d|\langle X\rangle_s| = \lim_{n\to\infty} \int_0^t \|h_n(s)\|_\infty^2 \, d|\langle X\rangle_s|.$$

(iii) $\mathrm{P}\langle X\rangle = \mathrm{P}\langle X^*\rangle$ *and* $h \in \mathrm{P}\langle X\rangle \Rightarrow h^* \in \mathrm{P}\langle X\rangle$.

Proof

Since $g, h \in \mathrm{P}\langle X\rangle$, *then* $\lambda h \in \mathrm{P}\langle X\rangle$ *by definition*

$\Rightarrow \lambda g + h \in \mathrm{P}\langle X\rangle$ *by definition*

(ii) Since (h_n) converges point wise $d|\langle X\rangle|$ almost everywhere $\|\cdot\|_\infty$ to h and $(\Re_+, d|\langle\ \rangle|)$ is a totally finite measure space then $s \to \|h(s)\|_\infty$ is measurable. So therefore is

$s \to \|h(s)\|_\infty^2$ *and, moreover,* $\|h_n(s)\|_\infty^2 \to \|h(s)\|_\infty^2$ $n \to \infty$ *for* $d|\langle X\rangle|$ *almost every* $s \in \Re_+$.

(iii) Since P $\langle X^*\rangle$ *is the predual of all* L^2*(bounded)martingales* P $\langle X\rangle$. ■

4.10 Theorem

Let (X_α) *be an* L^2 *– bounded martingale.Let* (f_n) *a sequence of simple process that define* $f \in$ P $\langle X\rangle$. *Then for* $t \geq 0$ *the sequences*

$\left(\int_0^t dX_s f_n(s)\right)$, $\left(\int_0^t f_n(s)dX_s\right)$, *are Cauchy in* L^2(A). *There* lim*its which we denote by*

$\int_0^t dX_s f(s)$ *and* $\int_0^t f(s)dX_s$ *respectively (the left ,respectively right,* int*egral of* f *with respect to* (X_α)) *are independent of the sequence* (f_n) *defining* f.

Proof

Consider the right integral. From Theorem 4.7

$$\left\| \int_0^t (h_n(s) - h_m(s)) dX_s \right\|_2^2 \leq \int_0^t \| h_n(s) - h_m(s) \|_\infty^2 \, d|\langle X \rangle_s|$$

Now let $\varepsilon > 0$ *and choose* $\delta > 0$ *so that* $d|\langle X \rangle|(F) < \delta$ *gives*

$$\int_F \| h_n(s) \|_\infty^2 \, d|\langle X \rangle_s| < \frac{\varepsilon}{6} \quad for \ n = 1, 2, \ldots$$

Now on $[0,t]$ (f_n) converges $d|\langle X \rangle|$ *almost uniformly to* f hence there is a Borel set

$C \subseteq [0,t]$ *with* $d|\langle X \rangle|([0,t]/C) < \delta$ *and* $h_n \to h$ *uniformly on* C. *Set* $d|\langle X \rangle|([0,t]) = U$

and $C' = [0,t]/C$.

By choosing $N(\varepsilon)$ large enough we can ensure that

$$\| h(s) - h_n(s) \|_\infty < \frac{1}{2} \sqrt{\frac{\varepsilon}{2U}} \quad for \ s \in C \ \ and \ n \geq N(\varepsilon)$$

It follows for $m, n \geq N(\varepsilon), s \in C$

$$\|h_n(s)-h_m(s)\|_\infty{}^2 \le \left(\frac{1}{2}\sqrt{\frac{\varepsilon}{6U}}+\frac{1}{2}\sqrt{\frac{1}{6U}}\right)^2 = \frac{\varepsilon}{6U}.$$

Hence

$$\|h_n(s)-h_m(s)\|_\infty{}^2 d|\langle X\rangle_s| < \frac{\varepsilon}{6U} d|\langle X\rangle|(C) \le \frac{\varepsilon}{6}.$$

Moreover,

$$I_{C'} = \int_{C'} \|h_m(s)-h_n(s)\|_\infty{}^2 d|\langle X\rangle_s| \le 2\int_{C'} \|h_m(s)\|_\infty{}^2 d|\langle x\rangle_s| + 2\int_{C'} \|h_n(s)\|_\infty{}^2 d|\langle X\rangle_s|$$

$$but\ d|\langle X\rangle|(C') < \delta, so\ I_{C'} < \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon. Thus \int_0^t \|h_n(s)-h_m(s)\|_\infty{}^2 d|\langle X\rangle_s| < \frac{5\varepsilon}{6} < \varepsilon.$$

Suppose now that (h_n) and (g_n) define h Then the compound sequence

$$q_n = \begin{cases} h_n\ if\ n = 2r & r \in N, \\ g_n\ if\ n = 2r+1 & r \in N, \end{cases}$$

also define h and so $\left(\int_0^t q_n(s) dX_s\right)$ *is Cauchy in* L^2*.Clearly*

$\left(\int_0^t h_n(s)dX_s\right)$ and $\left(\int_0^t g_n(s)dX_s\right)$ converges to the same limit as $\left(\int_0^t q_n(s)dX_s\right)$, showing that the limit is independent of (h_n). ■

4.11 Theorem

Let (X_s) be an L^2 – bounded martingale. Then $\left(\int_0^t h_n(s)dX_s\right)$ and $\left(\int_0^t dX_s h_n(s)\right)$ are martingales. Moreover the relations of Theorem 4.7 extend to $h \in_P \langle X \rangle$, i.e.,

$$\left\|\int_0^t dX_s h(s)\right\|_2^2 \le \int_0^t \|h(s)\|_\infty^2 d\left|\langle X\rangle_s\right| \text{ and } \left\|\int_0^t h(s)dX_s\right\|_2^2 \le \int_0^t \|h(s)\|_\infty^2 d\left|\langle X\rangle_s\right|$$

Proof

Let $h \in_P \langle X \rangle$ be defined by (h_n). We verified that for each $n \in N$

$\left(\int_0^t h_n(s)dX_s\right)$ and $\left(\int_0^t dX_s h_n(s)\right)$ *are martingales. For $t \in \Re_+$, the conditional* exp*ectation operator E_t is $\| \|_2$ continuous so for $t_1 \le t_2$,*

$$E_{t_1}\left(\int_0^{t_2} h(s)dX_s\right) = E_{t_1}\left(\| \|_2 - \lim_n \int_0^{t_2} h_n(s)dX_s\right)$$

$$= \|\cdot\|_2 - \lim_n E_{t_1}\left(\int_0^{t_2} h_n(s)dX_s\right) = \|\cdot\|_2 - \lim_n \int_0^{t_1} h_n(s)dX_s = \int_0^{t_1} h(s)dX_s.$$

Similarly for the left integral.

The inequalities required follow from lemma 4.9 (ii), Theorem 4.7 and Theorem 4.10

4.12 Theorem

Let $g, h \in P\langle X\rangle$ *be defined by* (h_n) *and* (g_n) *respectively and let* $Y_\alpha = \int_0^\alpha h(s)dX_s$. *Suppose that*

(i) $g_n, h_n, m, n \in N$ *are u.a.c.with respect to* $d|\langle X\rangle|$ *and*

(ii) $g_n, n \in N$ *are u.a.c.with respect to* $d|\langle Y\rangle|$.*Then*

$$gh \in P\langle X\rangle,\ g \in P\langle Y\rangle \text{ and } \int_0^\alpha g(s)dY_s = \int_0^\alpha g(s)h(s)dX_s.$$

Proof

By ignoring the union of two $d|\langle X\rangle|$ *null sets we have* $g_n h_n \to gh$ *point wise,*

$g_n h_n$ *simple, and* $g_n h_n, n \in N$, *u.a.c. with respect to* $d|\langle X\rangle|$ *and so* $gh \in$ P $\langle X\rangle$. *Since for* $s \leq t$,

$$\tau\left(|Y_t|^2 - |Y_s|^2\right) = \tau\left(|Y_t - Y_s|^2\right) = \left\|\int_s^t h dX\right\|_2^2 \leq \int_0^t \|h\|_\infty^2 d|\langle X\rangle|.$$

Then any $d|\langle X\rangle|$ *null set is* $d|\langle Y\rangle|$ *null. So* $g_n \to g - d|\langle Y\rangle|$ *a.e. and this with (ii) shows*

$g \in$ P $\langle X\rangle$ *and that* (g_n) *define* g *in* P $\langle Y\rangle$. *So* $\int_0^\alpha g_n dY \to \int_0^\alpha g dY$ *in* $\|\cdot\|_2$. *However, if* (q_n) *is elementary*, $q(s) = q\chi_{[r,u)}(s)$ *with* $h \in$ A $_r$ $0 \leq r \leq u \leq \alpha$, *and* (h_n) *are simple processes defining* h

We have

$$\int_0^\alpha q(s) dY_s = q(Y_u - Y_r) = q\int_r^u h(s) dX_s = q \lim_n \int_r^u h_n(s) dX_s.$$

But multiplication by q *is* $\|\cdot\|_2$ *continuous*. Bearing in mind that $\int_r^u h_n(s) dX_s$ is just a finite sum we have

$$q.\lim_n \int_r^u h_n(s)dX_s = \lim_n \int_r^u qh_n(s)dX_s = \lim_n \int_0^a q(s)f_n(s)dX_s.$$

Now $q_n(s)h_n(s)$ is a sequence of simple processes converging $d|\langle X\rangle|$ almost everywhere to $q(s)h(s)$ *and* $q(s)h_n(s)$ *are u.a.c.with respect to* $d|\langle X\rangle|$. *So*

$gh \in P\ \langle X\rangle$ *and* $\lim_n \int_0^a q(s)h_n(s)dX_s = \int_0^a q(s)h(s)dX_s$, *by theorem* 4.13.

The result for a simple q follows by linearity. If g is not simple and (g_n) are the simple processes (defining g) of the hypothesis then $g_n h \in P\ \langle X\rangle$ *and* $g_n h \to ghd|\langle X\rangle|a.e.$

But $g_n h = \lim_n g_n h_m - d|\langle X\rangle|$ *a.e.and* $(g_n h_m)_{m-1}^{\infty}$ *are u.a.c.with respect to* $d|\langle X\rangle|$. *It follows that*

$$\int \|g_n h\|_\infty^2 d|\langle X\rangle| = \lim_n \int \|g_n h_m\|_\infty^2 d|\langle X\rangle|.$$

Thus $(g_n h)$ *are u.a.c.with respect to* $d|\langle X\rangle|$. *Now*

$$\left\| \int_0^a g_n h dX - \int_0^a gh dX \right\|_2^2 \le \int_0^a \|g_n h - gh\|_\infty^2 d|\langle X\rangle|.$$

But

$\|g_n h - gh\|_\infty^2 \to 0 \quad d|\langle X\rangle|$ *a.e.and*

$\int_0^\alpha \|g_n h - gh\|_\infty^2 d|\langle X\rangle| \le 2\int_0^\alpha \|g_n h\| d|\langle X\rangle| + 2\int_0^\alpha \|gh\|_\infty^2 d|\langle X\rangle|$. *It follows that*

$\int_0^\alpha \|g_n h - gh\|_\infty^2 d|\langle X\rangle| \to 0$ *as* $n \to \infty$. *Then*

$$\int_0^\alpha g(s) dY_s = \lim_n \int_0^\alpha g_n(s) dY_s = \lim_n \int_0^\alpha g_n(s) h(s) dX_s = \int_0^\alpha g(s) h(s) dX_s.$$

CHAPTER FIVE

STOCHASTIC DIFFERENTIAL EQUATIONS

5.1 Introduction

We considered the case of $H = L^2(\Re_+)$ with complex conjugation, for $t \geq 0$, $\Im_t$ denotes the W*-subalgebra generated by $\{\psi(z): ess\sup z \subseteq [0,t]\}$, and $L^p(\Im_t)$ is the completion of $\Im_t$ with respect to $\|\|_p$. Thus, for $1 \leq p < \infty$, $L^p(\Im_t)$ may be considered as a subspace of $L^p(\Im)$.

5.2 Definition

A map $X : \Re_+ \to L^2(\Im)$ is said to be adapted if $X_t \in L^2(\Im_t)$ for each $t \in \Re_+$. A map $F : L^2(\Im) \times \Re_+ \to L^2(\Im)$ is said to be adapted if for any $t \in \Re_+$ and $u \in L^2(\Im_t)$ we have $F(u,t) \in L^2(\Im_t)$.

In particular, if $X : \Re_+ \to L^2(\Im)$ and $F : L^2(\Im) \times \Re_+ \to L^2(\Im)$ are both adapted , then so is the map $t \mapsto F(X,t)$.

For $u_1,\ldots\ldots,u_n \in L_{loc}^2(\Re_+)$, the Wick-ordered monomial $:\psi(u_1\chi_{[o,t]})\ldots\psi(u_n\chi_{[o,t]}):$ is denoted by W_t. It was shown in [1] that (W_t) is an L^∞ – *martingale* adapted to the family $\{\Im_t : t \in \Re_+\}$. We considered stochastic differential equations driven by W_t,

$dX_t = F(X_t,t)dW_t + dW_tG(X_t,t) + H(X_t,t)dt$ and seek a solution X_t *in* $L^2(\Im_t)$, for $t \geq 0$. As usual, the stochastic differential equation actually means the equivalent integral equation,

$X_t = z + \int_{t_0}^{t} F(X_s,s)dW_s + \int_{t_0}^{t} dW_sG(X_s,s) + \int_{t_0}^{t} H(X_s,s)ds$, where $z \in L^2(\Im_{t_0})$ is the initial value $X_{t_0} = z$. part of the existence problem is to show that these integrals are well defined. The first and second integrals are to be $It\hat{o} - Clifford$ integrals and the third an $L^2(\Im)$ Bochner-integral. For these to be well defined, it is sufficient for the maps $s \mapsto F(X_s,s)$ and $s \mapsto G(X_s,s)$ to be adapted and $L^2 - continuous$ and the map $s \mapsto H(X_s s)$ to be $L^2 - continuous$. This is the case if $s \mapsto X_s$ *and* F,G *are continuous*. To ensure that $s \mapsto X_s$ is adapted we then require that H be adapted. Thus, we shall suppose that F, G, H are continuous adapted maps and seek an $L^2 - continuous$ adapted solution X_t, for $t \geq 0$.

Our method of proof is just the usual picard-iteration scheme and so we must impose a Lipschitz condition on F, G and H.

5.3 Definition

A map F: $L^2(\Im)\times \Re_+ \to L^2(\Im)$ is said to satisfy a locally uniform Lipschitz condition if for each $T \geq 0$ There is a constant K>0 such that $\|F(X,s)-F(Y,s)\|_2 \leq K\|X-Y\|_2$ for all $0 \leq s \leq T$ and all X,Y in $L^2(\Im)$.

Fix $u_1,....u_n \in L^2{}_{loc}(\Re_+)$ and let $W_t =: \psi(u_1\chi_{[0,t]}).....\psi(u_n\chi_{[0,t]})$: for $t \in \Re_+$. Then $W_s W_s^* = a_s 1$,

where $a_s = \int_0^s .....\int_0^s |u(s_1,.....s_n)|^2 ds_1.....ds_n$ with

$u(s_1,.....,s_n) = (n!)^{1/2}\lambda(u_n \otimes .....\otimes u_1)(s_1,....s_n)$ *in chapter four*. Let v denote the measure on $\Re_+$ given by the positive increasing function $s \mapsto a_s$, that, $v([\alpha,\beta]) = a_\beta - a_\alpha$ for $\alpha \leq \beta$ in $\Re_+$, and let $f:\Re_+ \to L^2(\Im)$ be measurable and adapted with $\int_0^t \|f(s)\|^2{}_2 dv(s) < \infty$. It was shown in [1] that the *Itô – Clifford* stochastic integral satisfies the *following isometry property*; $\left\|\int_0^t f(s)dW_s\right\|_2^2 = \int_0^t \|f(s)\|_2^2 dv(s)$.

Furthermore, $\int_0^t dW_s f(s) = \left(\int_0^t f(s)^* dW_s^*\right)^*$ *and* $W_s^* W_s = a_s 1$; *therefore we also have*

$\left\|\int_0^t dW_s f(s)\right\|_2^2 = \int_0^t \|f(s)\|_2^2 dv(s)$.

5.4 Theorem

Let $F, G, H : L^2(\Im) \times [t_0, \infty] \rightarrow L^2(\Im)$ be adapted and continuous, and

satisfy a locally uniform Lipschitz condition on $[t_0, \infty)$. Then, for any $Z \in L^2(\Im_t)$,

there is a unique continuous adapted $L^2 - process$ $\{X_t\}_{t \geq 0}$ satisfying the stochastic

differential equation

$dX_t = F(X_t, t)dW_t + dW_t G(X_t, t) + H(X_t, t)dt$ *on* $[t_0, \infty)$ *with* $X_{t_0} = z$.

Proof

Fix $T > 0$ and $t_0 \leq t \leq T$ set $X_t^{(0)} = z$ and, for $n \geq 0$, defined, inductively,

$$X_t^{(n+1)} = z + \int_{t_0}^{t} F\left(X_\tau^{(n)}, \tau\right) dW_\tau + \int_{t_0}^{t} dW_\tau G\left(X_\tau^{(n)}, \tau\right) + \int_{t_0}^{t} H\left(X_\tau^{(n)}, \tau\right) d\tau.$$

We claim that each $X_t^{(n)}, n \geq 1$, defines an adapted $L^2-continuous$ process on $[t_0, T]$.
By hypothesis, $F(z,\tau), G(z,\tau)$, and $H(z,\tau)$ are L^2-continuous in τ and belong to $L^2(\Im_\tau)$ for $t_0 \leq \tau \leq T$. Hence $X_t^{(1)}$ is well defined for $t_0 \leq t \leq T$. Furthermore, continuity implies boundedness on compact sets, and so, using the isometry property, we can easily see that $t \mapsto X_t^{(1)}$ is continuous: $[t_0, T] \to L^2(\Im)$.

Now, if $X_t^{(n)}$ is assumed to be adapted and continuous, then as above, each $F(X_\tau^{(n)},\tau), G(X_\tau^{(n)},\tau) and, H(X_\tau^{(n)},\tau)$ is adapted, $L^2-continuous$ on $[t_0, T]$, and hence bounded. Thus we see that $X_t^{(n+1)}$ is adapted, and on use of the isometry property, it follows that $t \mapsto X_t^{(n+1)}$ *is* $L^2-continuous$ *on* $[t_0, T]$. Hence, by induction, we have proved our claim.

Let us consider the convergence of the iteration. We have

$$\left\|X_t^{(n+1)} - X_t^{(n)}\right\|_2 \leq \left\|\int_{t_0}^{t} \left\{F\left(X_\tau^{(n)},\tau\right) - F\left(X_\tau^{(n-1)},\tau\right)\right\} dW\tau\right\|_2$$
$$+ \left\|\int_{t_0}^{t} dW_\tau \left\{G\left(X_\tau^{(n)},\tau\right) - G\left(X_\tau^{(n-1)},\tau\right)\right\}\right\|_2 + \left\|\int_{t_0}^{t} \left\{H\left(X_\tau^{(n)},\tau\right) - H\left(X_\tau^{(n-1)},\tau\right)\right\} d\tau\right\|_2.$$

Now, by the isometry property of the *Itô – Clifford* stochastic integral, the boundedness of the 'wave-function' $u_1,\ldots,u_n$ over $[t_0,T]$, and the Lipschitz condition, we have

$$\left\|\int_{t_0}^{t}\left\{F\left(X_\tau^{(n)},\tau\right)-F\left(X_\tau^{(n-1)},\tau\right)\right\}dW_\tau\right\|_2=\left\{\int_{t_0}^{t}\left\|F\left(X_\tau^{(n)},\tau\right)-F\left(X_\tau^{(n-1)},\tau\right)\right\|_2^2 d\nu(\tau)\right\}^{1/2}$$

$$\leq K_1\left\{\int_{t_0}^{t}\left\|F\left(X_\tau^{(n)},\tau\right)-F\left(X_\tau^{(n-1)},\tau\right)\right\|_2^2 d\tau\right\}^{1/2}$$

$$\leq K_2\left\{\int_{t_0}^{t}\left\|X_\tau^{(n)}-X_\tau^{(n-1)}\right\|_2^2 d\tau\right\}^{1/2}$$ Where K_1 ,K_2 are positive constants (depending on T).

A similar estimate holds for the integral involving G. Using the Lipschitz condition and Schwartz inequality, we also have

$$\left\|\int_{t_0}^{t}\left\{H\left(X_\tau^{(n)},\tau\right)-H\left(X_\tau^{(n-1)},\tau\right)\right\}d\tau\right\|_2\leq\int_{t_0}^{t}\left\|H\left(X_\tau^{(n)},\tau\right)-H\left(X_\tau^{(n-1)},\tau\right)\right\|_2 d\tau\leq K_3\int_{t_0}^{t}\left\|X_\tau^{(n)}-X_\tau^{(n-1)}\right\|_2 d\tau$$

$$\leq K_4\left\{\int_{t_0}^{t}\left\|X_\tau^{(n)}-X_\tau^{(n-1)}\right\|_2^2 d\tau\right\}^{1/2}$$ where K_3, K_4 are positive constants(depending on T)

Finally, using the inquality $(a+b+c)^2\leq 3\left(a^2+b^2+c^2\right)$ we obtain

$\left\|X_t^{(n-1)} - X_t^{(n)}\right\|_2^2 \le K_5 \int_{t_0}^{t} \left\|X_\tau^{(n)} - X_\tau^{(n-1)}\right\|_2^2 d\tau$ *where* K_5 *a positive constant* (depending on T). by iteration, $\left\|X_t^{(n+1)} - X_t^{(n)}\right\|_2^2 \le K_5^n \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 \ldots ds_{n-1} \left\|X_{s_{n-1}}^{(1)} - X^{(0)}_{s_{n-1}}\right\|_2^2$

But both $s \mapsto X_s^{(1)}$ and $s \mapsto X_s^{(0)} = z$ are $L^2-continuous$ and hence bounded on $[t_0, T]$, and therefore we obtain the estimate. For some positive constant K_6(depending on T).

Therefore for any n > K,

$$\left\|X_t^{(n+1)} - X_t^{(k+1)}\right\|_2 = \left\|\sum_{m=k+1}^{n}\left(X_t^{(m+1)} - X_t^{(m)}\right)\right\|_2 \le \sum_{m=k+1}^{n}\left\|X_t^{(m+1)} - X_t^{(m)}\right\|_2 \le \sum_{m=m+1}^{n}\left\{K_6^{\,n-1}T^{n-1}/(n-1)\right\}^{1/2}$$

It follows that there is $X_t \in L^2(\Im)$ such that $X_t^{(n)} \to X_t$ in $L^2(\Im)$ as $n\to\infty$, uniformly on $[t_0,T]$. Since each $X_t^{(n)}$ is adapted and $L^2-continuous$, the same is true of X_t .

We must now show that $\{X_t\} t \le t_0$ satisfies the stochastic differential equation. Clearly it obeys the initial condition $X_{t_0} = z$. Also

$$\left\|\int_{t_0}^{t} F\left(X_s^{(n)}, s\right) dW_s - \int_{t_0}^{t} F(X_s, s) dW_s\right\|_2^2 = \int_{t_0}^{t}\left\|F\left(X_s^{(n)}, s\right) - F(X_s, s)\right\|_2^2 dv(s) \le K_7^2 \int_{t_0}^{t}\left\|X_s^{(n)} - X_s\right\|_2^2 dv(s),$$

for some $K_7 > 0$, $\to 0$ as $n \to \infty$.

Since $X_s^{(n)} \to X_s$ in $L^2(\Im)$ uniformly for s in $[t_0, T]$. Similarly,

$$\int_{t_0}^{t} dW_s G(X_s^{(n)}, s) \to \int_{t_0}^{t} dW_s G(X_s, s)$$

and $\int_{t_0}^{t} H(X_s^{(n)}, s)ds \to \int_{t_0}^{t} H(X_s, s)ds$ in $L^2(\Im)$. Thus

$$X_t = \lim_n X_t^{(n+1)} = \lim_n \left(Z + \int_{t_0}^{t} F(X_s^{(n)}, s)dW_s + \int_{t_0}^{t} dW_s G(X_s^{(n)}, s) + \int_{t_0}^{t} H(X^{(n)}, s)ds \right)$$

$= Z + \int_{t_0}^{t} F(X_s, s)dW_s + \int_{t_0}^{t} dW_s G(X_s, s) + \int_{t_0}^{t} H(X_s, s)ds$, that is, $\{X_t\} t \geq t_0$ is a solution for $t \in [t_0, T]$.

However, since $T > t_0$ is arbitrary, we obtain a solution X_t for $t \geq t_0$.

To establish the uniqueness of the solution, suppose that $\{Y_t\} t \geq t_0$ is an adapted continuous solution with $Y_t = Z$. Then using the isometry property, Lipschitz condition, and the boundedness of $u_1, \ldots, u_n$ *on* $[t_0, t]$ *we obtain, as before,*

$$\|X_t - Y_t\|_2 \leq \left\| \int_{t_0}^{t} (F(X_s, s) - F(Y_s, s)) dW_s \right\|_2 + \left\| \int_{t_0}^{t} dW_s (G(X_s, s) - G(Y_s, s)) \right\|_2 +$$

$$\left\| \int_{t_0}^{t} (H(X_s, s) - H(Y_s, s)) ds \right\|_2 \leq K_8 \left\{ \int_{t_0}^{t} \|X_s - Y_s\|^2{}_2 ds \right\}^{1/2}$$

for some $K_8 > 0$ (which may depend on t). Now, $\|X_s - Y_s\|_2$ is continuous and therefore bounded on $[t_0, T]$, and therefore, by iteration, we get for any $n > 0$,

$\|X_t - Y_t\|^2{}_2 \leq K_8{}^n K_9 \int_{t_0}^{t} ds_1 \int_{t_0}^{s_1} ds_2 ...... \int_{t_0}^{s_{n-1}} ds_n = K_8{}^n K_9 (t - t_0)^n / n!$, where $K_9 > 0$ depends only on t. It follows that $X_t = Y_t$ for each $t \geq t_0$ that is, the solution is unique.

Summary and Conclusion

In this thesis we have seen that the notion of conditional expectation developed by Umegaki which was shown to be a faithful normal projection of norm one by Tomiyama has been reviewed. We also construct an integral of anti-commuting elements analogous to the Ito-integral for Brownian motion.

The notion of stochastic integral in an arbitrary gauge space was also discussed.

We show, in this thesis, that the solution to the stochastic differential equations

$$X_t = z + \int_{t_0}^{t} F(X_s, s)dW_s + \int_{t_0}^{t} dW_s G(X_s, s) + \int_{t_0}^{t} H(X_s, s)ds,$$

has a unique solution in the L^2-space of the Clifford algebra.

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