A STUDY OF COMMON FIXED POINT APPROXIMATIONS FOR FINITE FAMILIES OF TOTAL ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP IN HYPERBOLIC SPACES

BY

ISMA'IL, DANBABA GARBA

DEPARTMENT OF MATHEMATICS

FACULTY OF PHYSICAL SCIENCES

AHMADU BELLO UNIVERSITY, ZARIA,

NIGERIA.

OCTOBER, 2017

A STUDY OF COMMON FIXED POINT APPROXIMATIONS FOR FINITE FAMILIES OF TOTAL ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP IN HYPERBOLIC SPACES

BY

Isma'il, Danbaba GARBA

B.Sc. (A.B.U, 2010)

M.Sc./Sci./20653/2012-2013

A DISSERTATION SUBMITTED TO THE SCHOOL OF POST-GRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTER DEGREE IN MATHEMATICS

DEPARTMENT OF MATHEMATICS

FACULTY OF PHYSICAL SCIENCES

AHMADU BELLO UNIVERSITY, ZARIA

NIGERIA.

OCTOBER, 2017

DECLARATION

I declare that the work in this dissertation titled "A Study of Common Fixed Point Approximations for Finite Families of Total Asymptotically Nonexpansive Semigroup in Hyperbolic Spaces" has been performed by me in the Department of Mathematics, Ahmadu Bello University, Zaria, under the supervision of Professor Bashir Ali and Professor Yohanna Tella. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other Institution.

____________________________ ____________
GARBA, Danbaba Isma'il Date
M.Sc./Sci./20653/2012-2013

CERTIFICATION

This dissertation titled "A Study of Common Fixed Point Approximations for Finite Families of Total Asymptotically Nonexpansive Semigroup in Hyperbolic Spaces" by GARBA, Danbaba Isma'il (M.Sc./Sci./20653/2012-2013), meets the regulations governing the award of Master degree of the Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

________________________ ____________
Prof. Bashir Ali Date
Chairman, Supervisory Committee

________________________ ____________
Prof. Yohanna Tella Date
Member, Supervisory Committee

________________________ ____________
Prof. E. J. D. Garba Date
External Examiner

________________________ ____________
Prof. A. A. Tijjani qquad Date
Head of Department

________________________ ____________
Prof. S. Z. Abubakar Date
Dean Postgraduate School

DEDICATION

To my Dearest Mother; Hajiya Khadijah A. Aliyu and Lovely Daughter; Fatima G. Danbaba.

ACKNOWLEDGEMENT

There are several reasons for the success of the research described in this dissertation. First, I start by expressing my gratitude to ALLAHU (SWT); for giving me the ability to complete this research work in good health, for this I say ALHAMDULILLAH. Then, I would like to mention the contribution of my main supervisor; Professor Bashir Ali, whose ideas, guidance, patience and support have been essential at all stages. I would like to thank him for giving me an insight into his perception of mathematics. Also, for hours he devoted of mathematical and non-mathematical conversations, as well as careful reading of the draft of this dissertation. My sincere thanks are due to my other supervisor; Professor Yohanna Tella, for his advice, corrections and encouragements. My thanks are also to Doctor A. T. Imam, for time taken to go through the draft of this dissertation and for suggesting many corrections.

I would like to thank Professors; A. A. Tijjani (Head of Department), G.U. Garba, D. Singh, B.K. Jha, J. Singh, Babangida Sani and A. O. Ajibade, Doctors; I. A. Fulatan, H. M. Jibril, Y. M. Baraya, A. J. Alkali, A. Mohammed, M. A. Ibrahim and Mrs. A. Umar. I also wish to thank Mrs. M. I. Yakubu. I appreciate all of the above for the mathematical and moral training I received from them in one way or another through the years.

I also like to thank Mal. Musa Balarabe for most enjoyable time I have had corresponding with him. I sincerely hope that this is not the end of our co-operation but merely a stage of it. My thanks also goes to Doctor Murtala H. Harbau (B.U.K) for providing me with a number of relevant materials and time taken for explaining one concept or another which also helped in the completion of this dissertation.

I would like to thank my friends and colleagues who have made positive effects in my mathematical and private life. I would especially like to mention; Muh'd Buhari Usman (Master), Lawan Suleiman, Yusuf Lawal, Yusuf Ibrahim, Abubakar Abdulkarim, Sirajo Yahaya, Mohammed Idris Shagari, Nurraddeen Mukhtar, Kamilu Uba Kamfa, Sabith Yunusa Ja'afar and many others too numerous to mention here, especially the members of the M.Sc. Mathematics class of 2012/2013.

I owe the following a duty, and to acknowledge my mother; Hajiya Khadijah A. Aliyu, my step father; Alhaji Ahmed Usman, my brothers; Abubakar (jnr), Muhammed Sani and Aliyu, my sisters; Hajiya Habiba, Hajiya Pharm. Samira and Khadijah (jnr). I have had an extraordinary fortune that these are also the people that I have loved and love most. I am grateful to them for standing by me even in the most difficult moments in my life. And to my father; Garba A. Danbaba, I say thank you.

I have to express that I feel incredibly grateful to my lovely ideal wife, Nafisa Mustapha and to my dearest daughter Fatima I. Danbaba; whose liberal attitude, never ending support, unconditional love and generosity leave me speechless. I am grateful to them for standing by me during all these years and for bearing with me when I am doing Mathematics. I am so blessed to have such an amazing wife. There is nothing I can give her in return, but I hope that this dissertation at least makes her proud.

Finally, I would like to express my gratefulness to my father in-law; Alhaji Mustapha Y. Maigoro, my mothers in-law; Hajiya Fatima F. Babajo and Engr. Fatima Mohammed, and all of my brothers and sisters in-law. They have inspired, helped and encouraged me to the success of this research work.

ABSTRACT

In this dissertation, a multi-step iterative scheme was used to establish the strong and $\Delta-$convergence theorems for finite families of uniformly asymptotic regular, total asymptotically nonexpansive semigroup in a uniformly convex hyperbolic space. We also used a different method of proof, to establish the polar and $\Delta-$convergence theorems for finite families of uniformly asymptotic regular, uniformly L-Lipschitzian and total asymptotically nonexpansive semigroup in a complete $CAT(0)$ space. We then studied the modified Mann iteration scheme for approximating common fixed point of a uniformly asymptotic regular family of total asymptotically nonexpansive semigroup in a complete $CAT(0)$ space. We proved that the sequences generated by the iterative schemes converges to a common fixed point of a finite family of uniformly asymptotic regular and total asymptotically nonexpansive semigroup in hyperbolic spaces.

TABLE OF CONTENTS

CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

The study of non-linear operators had its beginning about the start of the twentieth century, with the investigation into the existence properties of solutions to certain boundary valued problems arising in ordinary and partial differential equations. The earliest techniques, largely devised by Picard (1893), involved the iteration of an integral operator to obtain solutions to such problems. In 1922, these techniques of Picard were given precise abstract formulation by Cacciopoli (1931) and Banach (1932), a powerful tool in analysis for establishing existence and uniqueness of solution of problems of different kinds. The fact that, fixed point theorem is an important tool for solving equation of the form $Tx = x$; where T is a self-map defined on a subset of some suitable space, leads to the significance of this area. And it is useful in the theory of Newtonian and non-Newtonian calculus. The fixed point of a map plays the role of an equilibrium or stable point of the body of a system defined in terms of the operator. It is also known that the concept of equilibrium system is very crucial in other scientific areas that includes: biology, ecology, economics, physics, medicine, chemistry and also in engineering among others. Various authors have generalized Banach-Cacciopoli contraction mapping principle in different spaces which includes: Banach space, Hilbert space, Metric space, Hyperbolic space and even the $CAT(0)$ space.

Goebel and Kirk (1972) introduced the concept of asymptotically nonexpansive mappings as a generalization of nonexpansive mappings. Alber et al.

(2006), introduced the class of total asymptotically nonexpansive mappings, which generalizes several classes of maps, which are extensions of asymptotically nonexpansive mappings. The concept of fixed point theory in $CAT(k)$ space was first introduced by Kirk (2003, 2004). His work was followed by a series of new works by many authors, mainly focusing on the $CAT(0)$ space, which is a special case of the $CAT(k)$ space, all results in $CAT(0)$ space immediately apply to any $CAT(k)$ space with $k \leq 0$. As it is well known, the construction of common fixed points of nonexpansive semigroup and asymptotically nonexpansive semigroup is an important problem in the theory of nonexpansive mappings in nonlinear operator theory and applications. These has applications in: image recovery, signal processing problems and convex feasibility problems. (see; Yao and Shahzad (2011), Chidume and Chidume (2006), Marino and Xu (2006), Xu (2004), Shimoji and Takahashi (2001), among others). Takahashi (1969) proved the fixed point theorem for a noncommutative semigroup of nonexpansive mappings which generalises DeMarr's (1963) result.

The concept of approximation of common fixed point of asymptotically nonexpansive mappings in $CAT(0)$ space and convergence theorems for finite families of total asymptotically nonexpansive mappings in hyperbolic space, were discussed in Ugwunnadi and Ali (2016) and in Ali (2016). In this dissertation, using the definitions of the semigroup of the one parameter family of nonexpansive mappings and the uniformly asymptotic regularity of a map, their results were a bit modified and presented in the same complete $CAT(0)$ space.

1.2 Statement of the Problem

Ali (2016) proved the strong and $\Delta-$convergence for finite families of total asymptotically nonexpansive mappings in hyperbolic spaces, while Ugwunnadi and Ali (2016) studied the modified Mann iterative scheme and proved that the proposed sequence in the iterative scheme converges to a common fixed point of total asymptotically nonexpansive mappings in a complete $CAT(0)$ space. In this dissertation, using the concept of uniformly asymptotic regularity of self mappings and the semigroup of the one parameter family of nonexpansive mappings, we shall study and prove the strong, polar and $\Delta-$convergence theorems in a uniformly convex hyperbolic space and also in a complete $CAT(0)$ space.

1.3 Aim and Objectives

This dissertation is aimed at establishing new results in the field of non-linear operator theory. The primary objectives of this study are to:

1. establish the strong and $\Delta-$convergence theorems, using the uniformly asymptotic regularity of self mappings on the total asymptotically quasi-nonexpansive semigroup in a uniformly convex hyperbolic space.

2. prove (1) above, using a different method of proof to show polar and $\Delta-$convergence theorems, using the uniformly asymptotic regularity of self mappings on the total asymptotically quasi-nonexpansive semigroup in a complete $CAT(0)$ space.

3. establish the notion of convergence theorems in hyperbolic space and also in a complete $CAT(0)$ space, using the concept of the semigroup of the one parameter family of nonexpansive mappings.

1.4 Research Methodology

The method used in this dissertation is by consulting necessary and relevant books and articles, in literature; on the fixed point theorem, approximation of common fixed points, convergence for finite families, uniformly asymptotic regular family, total asymptotically nonexpansive semigroup, $CAT(0)$ space, hyperbolic space and so on. These articles were reviewed thoroughly to cover a major part of the work done on polar and $\Delta-$ convergence in hyperbolic and $CAT(0)$ spaces. The work done on the common fixed point approximations for finite families of total asymptotically nonexpansive semigroup in hyperbolic space are then taken in the settings of a complete $CAT(0)$ space.

In the theorems, we assumed that the finite families of mappings are uniformly asymptotic regular and are semigroup of nonexpansive mappings, if they satisfy some restricted and appropriate conditions, then the sequence either polar, strongly or $\Delta-$converge to a point in the set of all common fixed points in the space. And, in the proofs, we considered a sequence from the proposed iterative schemes and applied the concepts of: a metric space, the uniformly asymptotic regularity of a map, the total asymptotically quasi-nonexpansive semigroup of self mappings, the definitions of polar convergence, strong convergence and $\Delta-$convergence. Using these concepts, we successfully showed that the limit of the sequence exists in the set of all common fixed points in a complete

$CAT(0)$ space and also in a uniformly convex hyperbolic space.

1.5 Outline of the Dissertation

The dissertation contains four other chapters apart from the introductory chapter. The outline of the remaining chapters are as follows:

Chapter II: In this chapter, we present a survey of the necessary and relevant literature for fixed point theorem, $CAT(0)$ space and semigroup of nonexpansive mappings.

Chapter III: In this chapter, we establish the strong and $\Delta-$convergence theorems for finite families of total asymptotically nonexpansive semigroup of mappings in both the hyperbolic space and the $CAT(0)$ space.

Chapter IV: In this chapter, we introduce the approximation of common fixed point of a family of uniformly asymptotic regular semigroup of mappings in a complete $CAT(0)$ space, using the modified Mann iterative scheme.

Chapter V: In the final chapter, we present the summary and conclusion of the results obtained in this dissertation, along with some directions/recommendations for further research.

1.6 Preliminaries

In this section we give some basic and important definitions and concepts which are useful and related to the context of this dissertation.

1.6.1 Metric Space

Definition 1.6.1 *Let X be a nonempty set and $\mathbb{R}$ denote the set of real numbers. A metric d on X is a real-valued function $d : X \times X \to \mathbb{R}$ which satisfies the following conditions: For any $x, y, z \in X$,*

M1: $d(x, y) \geq 0$

M2: $d(x, y) = 0$ *if and only if* $x = y$,

M3: $d(x, y) = d(y, x)$, *and*

M4: $d(x, y) \leq d(x, z) + d(z, y)$

The pair (X, d) is called a metric space.

Example 1.6.1 *Let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $d(x, y) = |x - y|, \forall x, y \in \mathbb{R}$. Then $(\mathbb{R}, d)$ is a metric space, called the usual metric on $\mathbb{R}$.*

Example 1.6.2 *Let X be an arbitrary nonempty set. Define $d : X \times X \to \mathbb{R}$ by;*

$$d(x, y) = \begin{cases} 1, & x \neq y, \\ \\ 0, & x = y. \end{cases}$$

for all $x, y \in X$. Then (X, d) is a metric space called the trivial or discrete metric.

1.6.2 Fixed Point

Definition 1.6.2 *Let X be a nonempty set and $T : X \to X$ be self mappings. Then a point $x \in X$ is called a fixed point of T if: $Tx = x$.*
The set of all fixed points of T is represented by: $F(T) := \{x \in X : Tx = x\}$.
The set of all common fixed points of T is represented by: $\mathcal{F} := \bigcap_{i=1}^{n} F(T_i) \neq \emptyset$, $\forall i \in \mathbb{N}$. (see; Alber et al. (2006))

Let (X, d) be a metric space. A self mappings $T : X \to X$ is called nonexpansive if: $d(Tx, Ty) \leq d(x, y)$, for every $x, y \in X$. A map T is called quasi-nonexpansive if: $F(T) := \{x \in X : Tx = x\} \neq \emptyset$ and $d(Tx, p) \leq d(x, p)$, for every $x \in X$ and $p \in F(T)$. The class of quasi-nonexpansive mappings properly contained the class of nonexpansive mappings with fixed points.

The mappings T is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that: for every $n \in \mathbb{N}$, $d(T^n x, T^n y) \leq k_n d(x, y)$, for all $x, y \in X$.

If $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that: for every $n \in \mathbb{N}$, $d(T^n x, p) \leq k_n d(x, p)$, for all $x \in X$ and $p \in F(T)$, then T is called asymptotically quasi-nonexpansive.

The mappings T is called total asymptotically nonexpansive, if there exists infinitesimal real sequences $\{u_n\}$ and $\{v_n\}$ of nonnegetive numbers (i.e $u_n, v_n \to 0$ as $n \to \infty$) and a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that: $d(T^n x, T^n y) \leq d(x, y) + u_n \psi(d(x, y)) + v_n$, for every $x, y \in X$.

And T is called total asymptotically quasi-nonexpansive, if: $F(T) \neq \emptyset$ and there exists infinitesimal real sequences $\{u_n\}$ and $\{v_n\}$ of nonnegative numbers (i.e $u_n, v_n \to 0$ as $n \to \infty$) and a strictly increasing function $\psi : [0, \infty) \to [0, \infty)$ with $\psi(0) = 0$ such that: $d(T^n x, p) \leq d(x, p) + u_n \psi(d(x, p)) + v_n$, for every $x, \in X$ and $p \in F(T)$. (see: Ali (2016))

A self mappings $T : X \to X$ is called L-Lipschitz (or L-Lipschitzian), if there exists a constant $L > 0$ such that: $d(Tx, Ty) \leq Ld(x, y)$, for every $x, y \in X$. T is called a contraction (or strict contraction), if $L < 1$ and its nonexpansive if $L = 1$.

The mappings T is called uniformly L-Lipschitz (or uniformly L-Lipschitzian), if for every constant $L > 0$, there exists $n \in \mathbb{N}$ such that:
$d(T^n x, T^n y) \leq L^n d(x, y) \leq Ld(x, y)$, for every $x, y \in X$.

Example 1.6.3 *The map $T : \mathbb{R} \to \mathbb{R}$ defined by:*

1. *(i) $Tx = tanx$ (ii) $Tx = sinx$; are non-linear.*

2. *(i) $Tx = 5x + 10$ (ii) $Tx = ax$, for any constant a; are linear.*

3. *$Tx = x + 1$: T is nonexpansive and Lipschitz.*
   Since, $d(Tx, Ty) = |Tx - Ty| = |(x+1) - (y+1)| \leq L|x - y| = Ld(x, y)$. Clearly, T is Lipschitz with $L > 0$, its a contraction with $L < 1$ and its nonexpansive with $L = 1$.

1.6.3 $CAT(0)$ Space

Let (X, d) be a metric space, $\forall x, y \in X$. Then, we have the following definitions:

Geodesic Path

A geodesic path from x to y is an isometry. A map $T : X \to X$ is an isometry (distance preserving) if for any $x, y \in X$, $d(Tx, Ty) = d(x, y)$.

Geodesic Segment

The image of a geodesic path is called a geodesic segment. A geodesic segment joining two points x, y in a metric space X is represented by $[x, y]$, where $[x, y] := \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$. A subset K of a metric space X is convex if $\forall x, y \in K$, $[x, y] \subset K$.

Geodesic Space

A metric space (X, d), is called a geodesic space if any two distinct points of X are joined by the geodesic segment.

Geodesic Triangle

A geodesic triangle $(\triangle(x, y, z))$, consist of three distinct points $x, y, z \in X$ joined by three geodesic segments in a geodesic space.

Comparison Triangle

A comparison triangle $(\overline{\triangle}(x, y, z))$ or $(\triangle(\overline{x}, \overline{y}, \overline{z}))$ of a geodesic triangle $(\triangle(x, y, z))$ is a triangle in the Euclidean space $(\mathbb{R}^2)$ such that: $d(x, y) = d_{\mathbb{R}^2}(\overline{x}, \overline{y})$, $d(y, z) = d_{\mathbb{R}^2}(\overline{y}, \overline{z})$ and $d(z, x) = d_{\mathbb{R}^2}(\overline{z}, \overline{x})$.

A geodesic space is called a $CAT(0)$ space if for every geodesic triangle ($\triangle$) and its comparison triangle ($\overline{\triangle}$), the following inequality holds:

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}),$$

where; $x, y \in \triangle$ and $\overline{x}, \overline{y} \in \overline{\triangle}$.

Also,
A metric space (X, d) is a $CAT(0)$ space if it is geodesically connected and if every geodesic triangle ($\triangle$) in X, is at least as thin as its comparison triangle ($\overline{\triangle}$) in the Euclidean space ($\mathbb{R}^2$). (see; Kirk and Panyanak (2008))

A $CAT(0)$ space X, is said to be complete if every Cauchy sequence $\{x_n\}$ in X, converges to a point $x \in X$. A complete $CAT(0)$ space is often called the Hadamard space.

CAT means *Cartan Alexandrov Topogonov*. Examples of the $CAT(0)$ spaces includes; $\mathbb{R}-$tree, Hadamard space, Hilbert ball equipped with hyperbolic metric and so on. For details on these spaces, (see; Abramenco and Brown (2008), Dhompongsa and Panyanak (2008), Burago et al. (2001), Bridson and Haefliger (1999), Ballmann (1995) and Brown (1989)).

1.6.4 Hyperbolic Space

Definition 1.6.3 *A geodesic space (X, d) is called hyperbolic, if for any $x, y, z \in X$,*

$$d(\frac{1}{2}x \oplus \frac{1}{2}z, \frac{1}{2}z \oplus \frac{1}{2}y) \leq \frac{1}{2}d(x, y).$$

The class of hyperbolic spaces includes: normed spaces, $CAT(0)$ spaces amongst others. The following is an example of a hyperbolic space which is not a normed space. (see; Reich and Shafrir (1990))

Example 1.6.4 *Let $\mathbb{D}$ be a unit disc in a complex plane $\mathbb{C}$. Define $d : \mathbb{D}\times\mathbb{D} \to \mathbb{R}$ by:*

$$d(z,w) = \log(\frac{1+|\frac{z-w}{1-z\overline{w}}|}{1-|\frac{z-w}{1-z\overline{w}}|})$$

Then, $(\mathbb{D}, d)$ is a complete hyperbolic metric space.

From the example above, we have that the class of hyperbolic spaces are more general than the class of normed spaces.

1.6.5 Uniformly Convex Hyperbolic Space

Definition 1.6.4 *Let (X, d) be a hyperbolic space. Then X is called uniformly convex if for any $a \in X$, for every $r > 0$ and for each $\epsilon > 0$:*

$$\delta_a(r,\epsilon) = \inf\{1 - \frac{1}{r}d(\frac{1}{2}x \oplus \frac{1}{2}y, a) : d(x,a) \leq r, d(y,a) \leq r, d(x,y) \geq \epsilon\} > 0.$$

1.6.6 Semigroup of The One Parameter Family of Non-Expansive Mappings

Definition 1.6.5 *Let K be a nonempty subset of a metric space X. A one parameter family $\partial := \{T(t) : K \to K, t \in \mathbb{R}^+\}$, where $\mathbb{R}^+$ denotes the set of non-negative real numbers of maps is called a semigroup of self mappings from K into K satisfying:*

S1: $T(0)x = x$, *for all* $x \in K$;

S2: $T(s+t)x = T(s)T(t)x$, *for all* $s, t \in \mathbb{R}^+$;

S3: For each $x \in K$, *the mapping* $t \longmapsto T(t)x$ *is continuous, (*$\lim_{t \to 0} T(t)x = x$*)*

Let X be a nonempty set together with the binary operation $(+, .)$, then X is said to be a semigroup if it satisfies the closure and associative properties under the binary operators. A semigroup is also known as an associative magma.

Uniformly L-Lipschitzian Nonexpansive Semigroup

A one parameter family $\partial := \{T(t) : K \to K, t \in \mathbb{R}^+\}$, is said to be uniformly L-Lipschitzian nonexpansive semigroup if conditions $S1 - S3$ above are satisfied and in addition:

S4: for each $t > 0$, there exists a bounded function $L(t) : (0, \infty) \to [0, \infty)$ such that; $d(T^n(t)x, T^n(t)y) \leq L(t)d(x, y), \quad \forall \quad x, y \in K.$

A uniformly L-Lipschitzian semigroup of a one parameter family ∂ is called nonexpansive (or contraction) if: $L(t) = 1$ $(or L(t) < 1)$, for all $t > 0$.

Total Asymptotically Nonexpansive Semigroup

A one parameter family $\partial := \{T(t) : K \to K, t \geq 0\}$, is said to be total asymptotically nonexpansive semigroup, if conditions $S1 - S3$ above are satisfied and in addition:

S4: for each $t \geq 0$, there exists functions $u, v : [0.\infty) \to [0, \infty)$ and strictly in-

creasing and continuous functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that: $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) = 0$ and $\psi(0) = 0$, then;
$d(T(t)x, T(t)y) \leq d(x,y) + u(t)\psi(d(x,y)) + v(t), \quad \forall x, y \in K$.

Total Asymptotically Quasi-Nonexpansive Semigroup

A one parameter family $\partial := \{T(t) : K \to K, t \geq 0\}$, is said to be total asymptotically quasi-nonexpansive semigroup, if conditions $S1 - S3$ above are satisfied and in addition:

S4: for each $t \geq 0$, there exists functions $u, v : [0.\infty) \to [0,\infty)$ and strictly increasing and continuous functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that: $\lim_{t\to\infty} u(t) = \lim_{t\to\infty} v(t) = 0$ and $\psi(0) = 0$. If $F(T) := \{p \in X : Tp = p\} \neq 0$, then:
$d(T(t)x, p) \leq d(x,p) + u(t)\psi(d(x,p)) + v(t), \quad \forall x \in K$ and $p \in F(T)$
.

Asymptotically Regular and Uniformly Asymptotically Regular

A one parameter family $\partial := \{T(t) : K \to K, t \geq 0\}$, is said to be asymptotic regular if;

$$\lim_{t\to\infty} d(T(s+t)x, T(t)x) = 0, \quad \forall \quad t \in [0,\infty) \quad \textit{and} \quad x \in K$$

It is also said to be uniformly asymptotic regular if: for any $t \geq 0$ and for any bounded subset C of K,

$$\lim_{t\to\infty} \sup_{x\in C} d(T(s+t)x, T(t)x) = 0.$$

1.6.7 Polar and $\Delta-$Convergence

Definition 1.6.6 *(Ali (2016)) A sequence $\{x_n\}$ in a complete $CAT(0)$ space X is said to $\Delta-$converge to a point x, if x is a unique asymptotic centre of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. This is written as $\Delta - \lim_{n\to\infty} x_n = x$.*

A sequence $\{x_n\}$ in a complete $CAT(0)$ space X is said to polar converge to a point $x \in X$, if for every $y \in X$, $y \neq x$, there exists $N_y \in \mathbb{N}$ such that; $d(x_n, x) < d(x_n, y)$, $\forall n \geq N_y$.

The sequence $\{x_n\}$ converge $\Delta-$strongly to a point x, if the limit $\lim_{n\to\infty} d(x_n, x)$ exists and for any $y \neq x$, $\lim_{n\to\infty} d(x_n, x) \leq \liminf_{n\to\infty} d(x_n, y)$.

Let $\{x_n\}$ be a bounded sequence in a complete $CAT(0)$ space X. For $x \in X$, we set: $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$.
The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by :
$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$,
and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set:
$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}$.

It is well known that in a $CAT(0)$ space, $A(\{x_n\})$ consist of exactly one point, see (Proposition 7 of Dhompongsa et al. (2006))

1.6.8 Demiclosedness Property

Definition 1.6.7 *The mappings $T : X \to X$ is said to be demiclosed at a point, if for any sequence $\{x_n\}$ in X which converges weakly to a point $x \in X$, with $d(x_n, Tx_n) \to 0$, as $n \to \infty$, then $Tx = x$, for all $x \in X$.*
(see; Chang et al. (2012))

CHAPTER TWO

LITERATURE REVIEW

The earliest techniques of solving differential equations was largely devised by E. Picard, who published a very influential paper in 1893, formed the basis of fixed point theory, and throughout the last century, many papers have been published using this idea. This fact has helped in developing fixed point theory to be an independent research area of non-linear mathematical analysis. In this chapter, we present a review of some of these works which are related to the subject of this dissertation.

2.1 Fixed Point Theorems

In 1922, the techniques which was largely devised by Picard were given precise abstract formulation by Cacciopoli (1931) and Banach (1932), in what is now widely refer to as the 'Contraction Mapping Principle'. In 1910, Brouwer introduced an earlier fixed point theorem, called the Brouwer Fixed Point Theorem. This theorem concerns continuous mappings and has an advantage over the 'Contraction Mapping Principle', in the sense that it applies to a much larger class of mappings. For the iterative formula in the 'Contraction Mapping Principle', it was observed that if the constant condition on the operator is weakened, the operator may no longer have a fixed point. Krasnonelskii (1955) however proved with an example that, if the Picard iterative formula is replaced by another formula, the iterative sequence converges to the fixed point. In general, if E is a normed linear space and T is a nonexpansive map, a generalization of Krasnonelskii (1955) which has been proved successfully in the approximation of fixed points (when they exist) of T was given by Schaefer (1957). However, the most general iterative formula for approximation of fixed points of nonexpansive mappings was introduced by Mann (1953), called the Mann iteration formula. In 1967, Browder and Petryshyn (1967) proved that if a Banach space $E = H$, a Hilbert space, then the limit exist and is a fixed point of the map T. Reich (1980) extended this result to a uniformly smooth Banach space. Kirk (1981) obtained the same result but in an arbitrary Banach space, under the additional assumption that T has pre-compact range. Recently, Morales and Jung (2000) proved the same result for T, a continuous pseudocontraction map in a real reflexive Banach space with uniformly Gateaux differentiable norm.

Halpern (1967) introduced a recursion formula in the framework of a Hilbert space. Under appropriate conditions on the domain of T, and some restrictions on the parameter sequence, he proved strong convergence of the sequence to a fixed point of T. Under a different restriction on the parameter sequence used above, Lion (1977) improved the result of Halpern (1967), still in Hilbert space. Lion (1977) proved strong convergence of the sequence to a fixed point of T, where the real sequence satisfies some conditions. Reich (1980) proved that the result of Halpern remains true when E is uniformly smooth. It was observed that both Halpern's and Lion's conditions on the real sequence excluded the canonical choice of the parameter sequence. This was overcome by Wittmann (1992), who proved, still in Hilbert space, the strong convergence of the sequence to a fixed point of T, if the real sequence satisfies other conditions which include the canonical choice of the parameter sequence. Reich (1994) extended the result of Wittmann to Banach space, which are uniformly smooth and have weakly sequentially continuous duality maps (e.g l_p space,

$1 < p < \infty$). This space exclude the L_p space, $1 < p < \infty, p \neq 2$. Shioji and Takahashi (1997) extended Wittmann's result to real Banach space with uniformly Gateaux differentiable norm, in which each nonempty closed convex and bounded subset has the fixed point property for nonexpansive mappings (e.g L_p space).

Goebel and Kirk (1972) introduced the concept of asymptotically nonexpansive mappings as an important generalization of nonexpansive mappings. Alber et al. (2006), introduced the class of total asymptotically nonexpansive mappings, which generalizes several classes of maps, which are extensions of asymptotically nonexpansive mappings. These classes of maps are extensively studied by many authors (see Chang et al. (2012, 2013), Kohlenbach and Leustean (2010), Chidume and Ofeodu (2007, 2009), Chidume and Ali (2007), Goebel and Kirk (1972), to list but a few) by virtue of their importance as generalizations of nonexpansive mappings.

2.2 Semigroup of The One Parameter Family of Nonexpansive Mappings

Takahashi (1969) proved the fixed point theory for a noncommutative semigroup of nonexpansive mappings which generalizes DeMarr's (1963) result, he proved that any discrete left amenable semigroup has a common fixed point. Mitchell (1970) generalizes Takahashi's result by showing that any discrete left reversible semigroup has a common fixed point; see (Lau (1973)). Takahashi (1981) proved the nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in Hilbert space. Lau and Takahashi (1987) considered and proved the problem of weak convergence of a right reversible semitopological semigroup in a uniformly convex Banach space with Frechet differentiable norm. Later that same year, Kim and Kim (1987) proved the weak convergence theorems for semigroups of asymptotically nonexpansive type of a right reversible semitopological semigroup and strong convergence theorems for a commutative case. Takahashi and Zhang (1988, 1989) established the weak convergence of an almost-orbit of Lipschitzian semigroups of a noncommutative semitopological semigroup. After this, Lau in some of his results which includes; (Lau and Takahashi (1990, 1996, 2003) amongst others) proved the existence of common fixed points for nonexpansive mappings related to reversibility or amenability of a semigroup. Kakavandi and Amini (2011) proved a nonlinear ergodic theorem for a nonexpansive semigroup in $CAT(0)$ space as well as strong convergence theorem for a commutative semitopological semigroup. Anakkanmatee and Dhompongsa (2011) extended Rode (1982) result on common fixed points of semigroups of nonexpansive mappings in Hilbert space to $CAT(0)$ space settings.

Recently, Suantai and Phuengrattana (2014) proved in their result the existence, the weak convergence and the strong convergence theorems of some fixed point for semigroup of total asymptotically nonexpansive mappings in a uniformly convex Banach space. They proved both strong and weak convergence theorems to a fixed point in the family of semigroups of nonexpansive mappings in a uniformly convex Banach space.

2.3 Fixed Point Theorems and Semigroup of The One Parameter Family of Nonexpansive Mappings in Hyperbolic spaces ($CAT(0)$ Spaces)

Lim (1976) introduced the concept of $\Delta-$convergence in general metric space. After defining $\Delta-$convergence as given in chapter one above, he was able to prove it in a general metric space. Kirk (2003, 2004) introduced the concept of fixed point theory in $CAT(k)$ spaces which is a generalization of the $CAT(0)$ spaces, all results in $CAT(0)$ space immediately apply to any $CAT(k)$ space with $k \leq 0$. Being able to achieve these, Kirk and Panyanak (2008) studied the concept of $\Delta-$convergence and also introduced it in the hyperbolic and $CAT(0)$ space; where they introduced the concept of convergence in a geodesic space. Later that same year, Dhompongsa and Panyanak (2008) gave the $CAT(0)$ space analogs of results on weak convergence of the Picard, the Mann and the Ishikawa iterative schemes, which was earlier proved in a uniformly convex Banach space in 1976 by Z. Opial. Anakkanmatee and Dhompongsa (2011) extended Rode (1982) result on common fixed points of semigroups of nonexpansive mappings in Hilbert space to $CAT(0)$ space settings. Chang et al. (2012, 2013) were able to prove strong and $\Delta-$convergence theorems for mixed type total asymptotically nonexpansive mappings in $CAT(0)$ space. They were also able to establish the demiclosedness principle of those class of maps in the $CAT(0)$ space. Basarir and Sahin (2013) studied a multi-step iterative process for fixed point of generalized nonexpansive mappings in a $CAT(0)$ space, they were able to prove strong and $\Delta-$convergence of new multi-step and the S-iteration processes for fixed point of generalized nonexpansive mappings in $CAT(0)$ space. They also established the demiclosedness principle for those class of maps in the same space.

Chidume and Ali (2007) introduced an iterative scheme and studied the convergence of the sequence in the scheme to a common fixed point of finite family of non-self asymptotically nonexpansive mappings in a uniformly convex Banach space. Recently, Ali (2016) modified the scheme which was introduced by Chidume and Ali (2007), he defined a sequence and proved both strong and $\Delta-$convergence of the sequence to a point in the set of fixed points using a finite family of uniformly L-Lipschitzian total asymptotically nonexpansive mappings in a uniformly convex hyperbolic space and in a complete $CAT(0)$ space. More recently, Ugwunnadi and Ali (2016) studied the modified Mann iterative scheme for approximating common fixed point of total asymptotically nonexpansive mappings in a complete $CAT(0)$ space. They proved that using sequences of parameters satisfying some certain appropriate and restricted conditions, the sequences of parameters converges strongly to a point in the set of fixed points in the space. The concept used by Ugwunnadi and Ali (2016) was originally used in Hilbert space by Yao et al. (2009).

CHAPTER THREE

CONVERGENCE THEOREMS FOR FINITE FAMILIES OF TOTAL ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP IN HYPERBOLIC SPACES

In this chapter, we prove the necessary and sufficient conditions for the strong convergence of the scheme defined in lemma 3.2.1 below, to a common fixed point of a uniformly asymptotically regular family $\partial := \{T_i(t) : K \longrightarrow K, t \geq 0\}$ of total asymptotically quasi-nonexpansive semigroup in a uniformly convex hyperbolic space. We also prove both polar and $\Delta-$convergence theorems for a uniformly asymptotically regular family $\partial := \{T_i(t) : K \longrightarrow K, t \geq 0\}$ of uniformly L-Lipschitzian and total asymptotically quasi-nonexpansive semigroup in a complete $CAT(0)$ space. The results presented in this chapter are the semigroup version of the results in Ali (2016).

In Ali (2016), it was proved that if the finite family of mappings: $T_is, i = 1, 2, 3, ..., m$ are total asymptotically quasi-nonexpansive, under some certain appropriate and restricted conditions, the sequence $\{x_n\}$ either strongly, polar or $\Delta-$converges to a point in $\mathcal{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. In line with this, we now present his results in a different manner, (i.e) by letting the finite family of mappings to be uniformly asymptotic regular and then total asymptotically quasi-nonexpansive semigroup of the ground set.

Remark: Note that the difference between Ali (2016) and the results in this chapter is that: here, we used the assumption that the finite families of mappings are uniformly asymptotic regular and also a semigroup of nonexpansive mappings. We now have the following results.

3.1 Preliminaries

We shall make use of the following theorems and lemmas to prove our main results:

Theorem 3.1.1 *(Chang et al. (2012)); Let K be a closed and convex subset of a complete CAT(0) space X and* $T : K \longrightarrow X$ *be a uniformly L-Lipchitzian and total asymptotically nonexpansive mapping. Let* $\{x_n\}$ *be a bounded sequence in K such that* $x_n \rightharpoonup x$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$*. Then,* $Tx = x$*.*

Lemma 3.1.2 *(Dhompongsa and Panyanak (2008)); Let E be a complete CAT(0) space,* $\{x_n\}$ *be a bounded sequence in E with* $A(\{x_n\}) = \{p\}$ *and* $\{u_n\}$ *be a subsequence of* $\{x_n\}$*, with* $A(\{u_n\}) = \{u\}$*. If the sequence* $\{d(x_n, u)\}$ *converges, then* $p = u$*.*

Lemma 3.1.3 *(Dhompongsa and Panyanak (2008)); Let E be a complete CAT(0) space. Then,*
$d^2((1-\alpha)x \oplus \alpha y, a) \leq (1-\alpha)d^2(x, a) + \alpha d^2(y, a) - \alpha(1-\alpha)d^2(x, y)$,
$\forall \alpha \in [0, 1]$ *and* $x, y, a \in E$*.*

Lemma 3.1.4 *(Dhompongsa et al. (2007)); Let E be a complete CAT(0) space, K be a closed and convex subset of E. If* $\{x_n\}$ *is a bounded sequence in K, then the asymptotic centre of* $\{x_n\}$ *is in K.*

Lemma 3.1.5 *(Khamsi and Khan (2011)); Let (E,d) be a uniformly convex hyperbolic space,* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in E. For any* $\lambda \in (0, 1)$*, if there exist* $r \in [0, \infty)$ *such that;* $\limsup_{n\to\infty} d(x_n, a) \leq r$, $\limsup_{n\to\infty} d(y_n, a) \leq r$ *and* $\limsup_{n\to\infty} d(1-\lambda)x_n \oplus \lambda y_n, a) = r$*, then* $\lim_{n\to\infty} d(x_n, y_n) = 0$*.*

Lemma 3.1.6 *(Kirk and Panyanak (2008)); Every bounded sequence in a* *complete CAT(0) space has a $\Delta-$convergence subsequence.*

Lemma 3.1.7 *(Tan and Xu (1993)); Let $\{\lambda_n\}$ and $\{\sigma_n\}$ be non-negative sequences of real numbers such that: $\lambda_{n+1} \leq \lambda_n + \sigma_n$, $\forall n \geq 1$ and $\sum_{n=1}^{\infty} \sigma_n < \infty$, then $\lim_{n\to\infty} \lambda_n$ exists. Moreover, if there exists a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$, such that: $\lambda_{n_j} \to 0$ as $j \to \infty$, then $\lambda_n \to 0$ as $n \to \infty$.*

3.2 Main Results

We designate the set $\{1, 2, 3, ..., m\}$ by I and assume that $\mathcal{F} := \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Now, we state and prove the convergence theorems for a uniformly asymptotic regular finite family of total asymptotically quasi-nonexpansive semigroup in a uniformly convex hyperbolic space and in a complete $CAT(0)$ space.

Lemma 3.2.1 *; Let (X, d) be a hyperbolic space and K be a closed convex nonempty subset of X. Let $\partial := \{T_i(t) : K \to K, t \geq 0\}$ be a uniformly asymptotic regular family of total asymptotically quasi-nonexpansive semigroup of X, with sequences $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{v_{in}(t_n)\}_{n=1}^{\infty}$ and mappings $\psi_i : [0, \infty) \to [0, \infty)$ satisfying $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ and $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, for $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0, 1)$. Let $\{t_n\}$ be an increasing sequence in $[0, \infty)$. Assume that there exists constants $M_i, \overline{M_i}$ such that $\psi_i(r_i) \leq M_i r_i, \forall r_i \geq \overline{M_i}$, for $i \in I$. Let $\{x_n\}$ be a sequence defined iteratively by $x_1 \in K$, such that:*
$x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1(t_n)x_n$, for $n \geq 1, m = 1$, and

$$
\begin{cases}
x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1(t_n) y_{n+m-2} \\
y_{n+m-2} = (1-\alpha_n)x_n \oplus \alpha_n T_2(t_n) y_{n+m-3} \\
y_{n+m-3} = (1-\alpha_n)x_n \oplus \alpha_n T_3(t_n) y_{n+m-4} \\
\vdots \\
y_n = (1-\alpha_n)x_n \oplus \alpha_n T_m(t_n) x_n, \qquad n \geq 1, m \geq 2
\end{cases} \tag{3.2.1}
$$

Then $\{x_n\}$ *is bounded and the limits* $\lim_{n\to\infty} d(x_n, x^\star)$ *and* $\lim_{n\to\infty} d(x_n, \mathcal{F})$ *both exists.*

Proof. we start by first considering the case were $m \geq 2$. Since ψ_i is an increasing function for each $i \in I$, it follows that $\psi_i(r_i) \leq \psi(\overline{M_i})$, whenever $r_i \leq \overline{M_i}$ and by hypothesis, $\psi_i(r_i) \leq M_i r_i$ when $r_i \geq \overline{M_i}$. In either case $\psi_i(r_i) \leq \psi(\overline{M_i}) + M_i r_i, \quad i \in I$, for some constants $M_i, \overline{M_i} > 0$. Now, set $\omega_n := \Sigma_{i=1}^m u_{in}(t_n) M_i$ and let $x^\star \in \mathcal{F}$. Then we have;

$$
\begin{aligned}
d(x_{n+1}, x^\star) &= d((1-\alpha_n)x_n \oplus \alpha_n T_1(t_n) y_{n+m-2}, x^\star) \qquad \text{since } T_1(t_n)x^\star = x^\star \\
&\leq (1-\alpha_n) d(x_n, x^\star) + \alpha_n d(T_1(t_n) y_{n+m-2}, T_1(t_n) x^\star) \\
&\leq (1-\alpha_n) d(x_n, x^\star) + \alpha_n [d(y_{n+m-2}, x^\star) + u_{1n}(t_n)\psi_1(d(y_{n+m-2}, x^\star)) \\
&\quad + v_{1n}(t_n)] \\
&\leq (1-\alpha_n) d(x_n, x^\star) + \alpha_n (1 + u_{1n}(t_n) M_1) d(y_{n+m-2}, x^\star) + \alpha_n u_{1n}(t_n) \\
&\quad \psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n) \\
&= (1-\alpha_n) d(x_n, x^\star) + \alpha_n (1 + u_{1n}(t_n) M_1) d[(1-\alpha_n)x_n \oplus \alpha_n T_2(t_n) \\
&\quad y_{n+m-3}, x^\star] + \alpha_n u_{1n}(t_n) \psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n)
\end{aligned}
$$

$$\begin{aligned}
&\leq (1-\alpha_n)d(x_n,x^\star)+\alpha_n(1+u_{1n}(t_n)M_1)[(1-\alpha_n)d(x_n,x^\star)+\alpha_n \\
&\quad d(T_2(t_n)y_{n+m-3},T_2(t_n)x^\star)]+\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1})+\alpha_n v_{1n}(t_n) \\
&\leq (1-\alpha_n)d(x_n,x^\star)+\alpha_n(1+u_{1n}(t_n)M_1)[(1-\alpha_n)d(x_n,x^\star) \\
&\quad +\alpha_n[d(y_{n+m-3},x^\star)+u_{2n}(t_n)\psi_2(d(y_{n+m-3},x^\star))+v_{2n}(t_n)]] \\
&\quad +\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1})+\alpha_n v_{1n}(t_n) \\
&\leq (1-\alpha_n)d(x_n,x^\star)+\alpha_n(1+u_{1n}(t_n)M_1)[(1-\alpha_n)d(x_n,x^\star) \\
&\quad +\alpha_n[(1+u_{2n}(t_n)M_2)d(y_{n+m-3},x^\star)+\alpha_n u_{2n}(t_n)\psi_2(\overline{M_2}) \\
&\quad +\alpha_n v_{2n}(t_n)]]+\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1})+\alpha_n v_{1n}(t_n)
\end{aligned}$$

Since $\psi_i(r_i) \leq \psi(\overline{M_i})+M_i r_i$, for all $i \in I$, then we have; $\psi_1(d(y_{n+m-2},x^\star)) \leq \psi_1(\overline{M_1})+M_1 d(y_{n+m-2},x^\star)$

$$\begin{aligned}
&= (1-\alpha_n)d(x_n,x^\star)+\alpha_n(1-\alpha_n)(1+u_{1n}(t_n)M_1)d(x_n,x^\star)+\alpha_n^2(1+u_{1n}(t_n) \\
&\quad M_1)(1+u_{2n}(t_n)M_2)d(y_{n+m-3},x^\star)+\alpha_n^2(1+u_{1n}(t_n)M_1)u_{2n}(t_n)\psi_2(\overline{M_2}) \\
&\quad +\alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n)+\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1})+\alpha_n v_{1n}(t_n) \\
&= (1-\alpha_n)d(x_n,x^\star)+\alpha_n(1-\alpha_n)(1+u_{1n}(t_n)M_1)d(x_n,x^\star)+\alpha_n^2(1+u_{1n}(t_n) \\
&\quad M_1)(1+u_{2n}(t_n)M_2)d[(1-\alpha_n)x_n\oplus\alpha_n T_3(t_n)y_{n+m-4},x^\star]+\alpha_n^2(1+u_{1n}(t_n)M_1) \\
&\quad u_{2n}(t_n)\psi_2(\overline{M_2})+\alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n)+\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1})+\alpha_n v_{1n}(t_n) \\
&= (1-\alpha_n)d(x_n,x^\star)+\alpha_n(1-\alpha_n)(1+u_{1n}(t_n)M_1)d(x_n,x^\star)+\alpha_n^2(1+u_{1n}(t_n)M_1) \\
&\quad (1+u_{2n}(t_n)M_2)d[(1-\alpha_n)x_n+\alpha_n T_3(t_n)y_{n+m-4},T_3(t_n)x^\star]+\alpha_n^2(1+u_{1n}(t_n)M_1) \\
&\quad u_{2n}(t_n)\psi_2(\overline{M_2})+\alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n)+\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1})+\alpha_n v_{1n}(t_n) \\
&\leq (1-\alpha_n)d(x_n,x^\star)+\alpha_n(1-\alpha_n)(1+u_{1n}(t_n)M_1)d(x_n,x^\star)+\alpha_n^2(1+u_{1n}(t_n)M_1) \\
&\quad (1+u_{2n}(t_n)M_2[(1-\alpha_n)d(x_n,x^\star)+\alpha_n d(T_3(t_n)y_{n+m-4},T_3(t_n)x^\star)]+\alpha_n^2(1+u_{1n} \\
&\quad (t_n)M_1)u_{2n}(t_n)\psi_2(\overline{M_2})+\alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n)+\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) \\
&\quad +\alpha_n v_{1n}(t_n)
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n(1-\alpha_n)(1+u_{1n}(t_n)M_1)d(x_n, x^\star) + \alpha_n^2(1+u_{1n}(t_n)M_1) \\
&(1+u_{2n}(t_n)M_2)[(1-\alpha_n)d(x_n, x^\star) + \alpha_n[d(y_{n+m-4}, x^\star) + u_{3n}(t_n)\psi_3(d(y_{n+m-4}, x^\star)) \\
&+v_{3n}(t_n)]] + \alpha_n^2(1+u_{1n}(t_n)M_1)u_{2n}(t_n)\psi_2(\overline{M_2}) + \alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n) \\
&+\alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n) \\
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n(1-\alpha_n)(1+u_{1n}(t_n)M_1)d(x_n, x^\star) + \alpha_n^2(1+u_{1n}(t_n)M_1) \\
&(1+u_{2n}(t_n)M_2[(1-\alpha_n)d(x_n, x^\star) + \alpha_n[(1+u_{3n}(t_n)M_3)d(y_{n+m-4}, x^\star) + u_{3n}(t_n) \\
&\psi_3(\overline{M_3}) + v_{3n}(t_n)]] + \alpha_n^2(1+u_{1n}(t_n)M_1)u_{2n}(t_n)\psi_2(\overline{M_2}) + \alpha_n^2(1+u_{1n}(t_n)M_1) \\
&v_{2n}(t_n) + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n)
\end{aligned}
$$

Continuing in this fashion/method of deduction, we have;

$$
\begin{aligned}
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n(1-\alpha_n)(1+u_{1n}(t_n)M_1)d(x_n, x^\star) + \alpha_n^2(1-\alpha_n) \\
&(1+u_{1n}(t_n)M_1)(1+u_{2n}(t_n)M_2)d(x_n, x^\star) + \cdots + \alpha_n^{h-1}(1-\alpha_n)(1+u_{1n} \\
&(t_n)M_1)(1+u_{2n}(t_n)M_2)\cdots(1+u_{h-1n})(t_n)M_{h-1})d(x_n, x^\star) + \cdots + \alpha_n^m \\
&(1-\alpha_n)(1+u_{1n}(t_n)M_1)(1+u_{2n}(t_n)M_2)\cdots(1+u_{mn}(t_n)M_m)d(x_n, x^\star) \\
&+\alpha_n v_{1n}(t_n) + \alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n) + \alpha_n^3(1+u_{1n}(t_n)M_1)(1+u_{2n}(t_n) \\
&M_2)v_{3n}(t_n) + \alpha_n^4(1+u_{1n}(t_n)M_1)(1+u_{2n}(t_n)M_2)(1+u_{3n}(t_n)M_3)v_{4n}(t_n) \\
&+\cdots + \alpha_n^m(1+u_{1n}(t_n)M_1)(1+u_{2n}(t_n)M_2)(1+u_{3n}(t_n)M_3)(1+u_{4n}(t_n) \\
&M_4)\cdots(1+u_{m-1n}(t_n)M_{m-1})v_{mn}(t_n) + \alpha_n v_{1n}(t_n) + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) \\
&+\alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n) + \alpha_n^2(1+u_{1n}(t_n)M_1)u_{2n}(t_n)\psi_2(\overline{M_2}) + \cdots \\
&+\alpha_n^m\cdots(1+u_{m-1n}(t_n)M_{m-1})v_{mn}(t_n) + \alpha_n v_{1n}(t_n) + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) \\
&+\alpha_n^2(1+u_{1n}(t_n)M_1)v_{2n}(t_n) + \alpha_n^2(1+u_{1n}(t_n)M_1)u_{2n}(t_n)\psi_2(\overline{M_2}) + \cdots \\
&+\alpha_n^m(1+u_{1n}(t_n)M_1)(1+u_{2n}(t_n)M_2) + \cdots(1+u_{m-1n}(t_n)M_{m-1})v_{mn}(t_n) \\
&+\cdots + \alpha_n^m(1+u_{1n}(t_n)M_1)(1+u_{2n}(t_n)M_2) + \cdots(1+u_{m-1n}(t_n)M_{m-1}) \\
&u_{mn}(t_n)\psi_m(\overline{M_m})
\end{aligned}
$$

$$\leq d(x_n, x^\star)[1 + u_{1n}(t_n)M_1 + u_{2n}(t_n)M_2(1 + u_{1n}(t_n)M_1) + u_{3n}(t_n)M_3$$
$$(1 + u_{1n}(t_n)M_1)(1 + u_{2n}(t_n)M_2) + \cdots + u_{mn}(t_n)M_m(1 + u_{1n}(t_n)M_1)$$
$$(1 + u_{2n}(t_n)M_2)(1 + u_{3n}(t_n)M_3) \cdots (1 + u_{m-1n}(t_n)M_{m-1})] + \alpha_n v_{1n}(t_n)$$
$$+\Sigma_{j=1}^m \alpha_n^j [u_{jn}(t_n)\psi_j(\overline{M_j}) + v_{jn}(t_n)] \prod_{k=1}^{j} (1 + u_{kn}(t_n)M_k)$$

since $\omega_n := \Sigma_{i=1}^m u_{in}(t_n)M_i$, then we get;

$$\leq d(x_n, x^\star)[1 + \binom{m}{1}\omega_n + \binom{m}{2}\omega_n^2 + \binom{m}{3}\omega_n^3 + \binom{m}{4}\omega_n^4 + \cdots + \binom{m}{m}\omega_n^m] + \alpha_n v_{1n}(t_n)$$
$$+\Sigma_{j=1}^m \alpha_n^j [u_{jn}(t_n)\psi_j(\overline{M_j}) + v_{jn}(t_n)] \prod_{k=1}^{j} (1 + u_{kn}(t_n)M_k)$$

Let δ_m, be a positive real number defined by; $\delta_m := [\binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \binom{m}{4} + \cdots + \binom{m}{m}]$

$$\leq d(x_n, x^\star)[1 + \delta_m\omega_n] + \alpha_n v_{1n}(t_n) + \Sigma_{j=1}^m \alpha_n^j [u_{jn}(t_n)\psi_j(\overline{M_j}) + v_{jn}(t_n)]$$
$$\prod_{k=1}^{j} (1 + u_{kn}(t_n)M_k)$$

Using the Maclaurin's series expansion, on $\delta_m\omega_n$; $e^{\delta_m\omega_n} = 1 + \delta_m\omega_n + \frac{(\delta_m\omega_n)^2}{2!} + \cdots$, then we write:

$$\leq d(x_n, x^\star)e^{\delta_m\omega_n} + \alpha_n v_{1n}(t_n) + \Sigma_{j=1}^m \alpha_n^j [u_{jn}(t_n)\psi_j(\overline{M_j}) + v_{jn}(t_n)]$$
$$\prod_{k=1}^{j} (1 + u_{kn}(t_n)M_k)$$
$$\leq d(x_1, x^\star)e^{\delta_m\Sigma_{n=1}^\infty\omega_n} + \alpha_n v_{1n}(t_n) + \Sigma_{j=1}^m \alpha_n^j [u_{jn}(t_n)\psi_j(\overline{M_j}) + v_{jn}(t_n)]$$
$$\prod_{k=1}^{j} (1 + u_{kn}(t_n)M_k)$$
$$\leq d(x_1, x^\star)e^{\delta_m\Sigma_{n=1}^\infty\omega_n} + \alpha_n v_{1n}(t_n) + \Sigma_{j=1}^m \alpha_n^j [u_{jn}(t_n)\psi_j(\overline{M_j}) + v_{jn}(t_n)]$$
$$e^{\Sigma_{k=1}^j u_{kn}(t_n)M_k} < \infty.$$

By using Maclaurin's series expansion, on the last term, we get the above expression. This implies that $\{x_n\}$ is bounded, and by setting

$V_n := max_{i \leq j < m} v_{in}(t_n) + u_{jn}(t_n)\psi_i(\overline{M_i})$, there exist a positive integer $M > 0$ such that;

$$d(x_{n+1}, x^\star) \leq d(x_n, x^\star) + (\delta_m \omega_n + V_n)M$$

Since its true for each $x^\star$ in $\mathcal{F}$, we have;-

$$d(x_{n+1}, \mathcal{F}) \leq d(x_n, \mathcal{F}) + (\delta_m \omega_n + V_n)M$$

So therefore, by lemma 3.1.7, $\lim_{n\to\infty} d(x_n, x^\star)$ and $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exist. Now, for $m = 1$, we have;

$$\begin{aligned}
d(x_{n+1}, x^\star) &= d((1-\alpha_n)x_n \oplus \alpha_n T_1(t_n)x_n, x^\star) \qquad [T_1(t_n)x^\star = x^\star] \\
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n d(T_1(t_n)x_n, T_1(t_n)x^\star) \\
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n[d(x_n, x^\star) + u_{1n}(t_n)\psi_1(d(x_n, x^\star)) + v_{1n}(t_n)] \\
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n(1 + u_{1n}(t_n)M_1)d(x_n, x^\star) + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) \\
&\quad + \alpha_n v_{1n}(t_n)
\end{aligned}$$

Since $\psi_i(r_i) \leq \psi(\overline{M_i}) + M_i r_i$, for all $i \in I$, then we write; $\psi_1(d(x_n, x^\star)) \leq \psi_1(\overline{M_1}) + M_1 d(x_n, x^\star)$, expanding and reducing the expression above gives;

$$\begin{aligned}
&= (1 + \alpha_n u_{1n}(t_n)M_1)d(x_n, x^\star) + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n) \\
&\leq (1 + \omega_n)d(x_n, x^\star) + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n) \\
&\leq d(x_n, x^\star)e^{\omega_n} + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n) \\
&\leq d(x_1, x^\star)e^{\Sigma_{n=1}^{\infty}\omega_n} + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n) < \infty.
\end{aligned}$$

By using Maclaurin's series expansion on the first term, we get the expression above. Thus, $\{x_n\}$ is bounded. Using the deduction above, and lemma 3.1.7, the limits $\lim_{n\to\infty} d(x_n, x^\star)$ and $\lim_{n\to\infty} d(x_n, \mathcal{F})$ both exist.

This completes the proof. □

Theorem 3.2.2 *; Let K be a nonempty closed convex subset of a hyperbolic space E. Let $\partial := \{T_i(t) : K \to K, t \geq 0\}$ be a uniformly asymptotic regular family of total asymptotically quasi-nonexpansive semigroup of E, with sequences $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{v_{in}(t_n)\}_{n=1}^{\infty}$ and mappings $\psi_i : [0,\infty) \to [0,\infty)$ satisfying $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ and $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, for $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $\{x_n\}$ be a sequence defined iteratively as in lemma 3.2.1. Let $\{t_n\}$ be an increasing sequence in $[0,\infty)$. Then, $\{x_n\}$ converges to a common fixed point of the family of $\partial := \{T_i(t) : K \longrightarrow K, t \geq 0\}$ if and only if $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.*

Proof. For the necessity; we assume that $x_n \longrightarrow x^\star \in \mathcal{F}$, then we want to show that; $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. By definition of a normed space on the metric, $d(x_n, \mathcal{F}) = \inf_{x\in\mathcal{F}} \|x_n - x\|$. Since, $d(x_n, \mathcal{F}) = \inf_{x\in\mathcal{F}} \|x_n - x\| \leq \|x_n - x^\star\| \longrightarrow 0$, as $n \to \infty$, $\Longrightarrow 0 \leq d(x_n, \mathcal{F}) = \inf_{x\in\mathcal{F}} \|x_n - x\| \leq \|x_n - x^\star\| \longrightarrow 0$, as $n \to \infty$. Applying Sandwich theorem, $d(x_n, \mathcal{F}) \longrightarrow 0$, as $n \to \infty$, $\Longrightarrow \lim_{n\to\infty} d(x_n, \mathcal{F}) = \liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

To prove the sufficiency; we assume that $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F})$ exists, then $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0 = \lim_{n\to\infty} d(x_n, \mathcal{F})$. Thus, by definition, given $\epsilon > 0$, there exists a positive integer N_0 and $b^\star \in \mathcal{F}$, such that: $\forall n \geq N_0$, $d(x_n, b^\star) < \frac{\epsilon}{2}$. Then, for any $k \in \mathbb{N}$, we have for $n \geq N_0$,

$$\begin{aligned} d(x_{n+k}, x_n) &\leq d(x_{n+k}, b^\star) + d(b^\star, x_n) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{2\epsilon}{2} = \epsilon \end{aligned}$$

So therefore, $d(x_{n+k}, x_n) < \epsilon$ and so $\{x_n\}$ is Cauchy.

Let $\lim_{n\to\infty} x_n = b$, we want to show that $b \in F$. Let $T_i(t_n) = \{T_1(t_n), T_2(t_n), T_3(t_n), \cdots, T_m(t_n)\}$. Since $\lim_{n\to\infty} d(x_n, \mathcal{F}) = 0$, there exists a positive number N large enough and $b^\star \in \mathcal{F}$, such that $n \geq N$ implies that $d(b^\star, x_n) < \dfrac{\epsilon}{8(1+w_1)}$, $d(b, x_n) < \dfrac{\epsilon}{8(1+w_1)}$ and $(v_{in}(t_n) + u_{in}(t_n)\psi_i(\overline{M_i})) < \dfrac{\epsilon}{4}$ then, we have;

$$\begin{aligned} d(b^\star, b) &\leq d(b^\star, x_n) + d(b, x_n) < \frac{\epsilon}{8(1+w_1)} + \frac{\epsilon}{8(1+w_1)} \\ &= \frac{2\epsilon}{8(1+w_1)} = \frac{\epsilon}{4(1+w_1)} \end{aligned}$$

Thus, we have the following estimates, for $n \geq N$ and arbitrary $T_i(t_n), i = 1, 2, 3, \cdots, m$.

$$\begin{aligned} d(b, T_i(t_n)b) &\leq d(b, x_n) + d(x_n, b^\star) + d(b^\star, T_i(t_n)b) \\ &= d(b, x_n) + d(x_n, b^\star) + d(T_i(t_n)b^\star, T_i(t_n)b), \qquad [T_i(t_n)b^\star = b^\star] \\ &\leq d(b, x_n) + d(x_n, b^\star) + (1+w_1)d(b^\star, b) + v_{in}(t_n) + u_{in}(t_n)\psi_i(\overline{M_i}) \\ &< \frac{\epsilon}{8(1+w_1)} + \frac{\epsilon}{8(1+w_1)} + (1+w_1)\frac{\epsilon}{4(1+w_1)} + \frac{\epsilon}{4} \\ &= \frac{\epsilon}{8(1+w_1)} + \frac{\epsilon}{8(1+w_1)} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{4(1+w_1)} + \frac{\epsilon}{4} + \frac{\epsilon}{4} \\ &= \frac{\epsilon + 2\epsilon(1+w_1)}{4(1+w_1)} \leq \epsilon \end{aligned}$$

This implies that $b \in Fix(T_i(t_n)), \forall i = 1, 2, 3, \cdots, m$ and thus, $b \in \mathcal{F}$.

This completes the proof. □

Corollary 3.2.3 *; Let K be a nonempty closed convex subset of a complete hyperbolic space X. Let $\partial := \{T_i(t) : K \to K, t \geq 0\}$ be a uniformly asymptotic regular family of total asymptotically quasi-nonexpansive semigroup of X, with $\mathcal{F} \neq \emptyset$, sequences $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{v_{in}(t_n)\}_{n=1}^{\infty}$ and mappings $\psi_i : [0,\infty) \to [0,\infty)$ satisfying $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ and $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, for $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be* *a sequence in* $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. *Let* $\{x_n\}$ *be a sequence defined iteratively as in lemma 3.2.1. Let* $\{t_n\}$ *be an increasing sequence in* $[0,\infty)$. *Then,* $\{x_n\}$ *converges to a common fixed point of the family of* $\partial := \{T_i(t) : K \to K, t \geq 0\}$ *if and only if* $\liminf_{n\to\infty} d(x_n, \mathcal{F}) = 0$.

Now, to be able to prove our next theorem, we start by proving the following lemma which will be used in this chapter.

Lemma 3.2.4 *; Let* X *be a uniformly convex hyperbolic space and* K *be a nonempty closed convex subset of* X. *Let* $\partial := \{T_i(t) : K \to K, t \geq 0\}$ *be a uniformly asymptotic regular family of total asymptotically quasi-nonexpansive semigroup of* X, *with sequences* $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{V_{in}(t_n)\}_{n=1}^{\infty}$ *and mappings* $\psi_i : [0,\infty) \to [0,\infty)$ *satisfying* $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ *and* $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, *for* $i \in I$. *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence in* $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. *Let* $\{x_n\}$ *be a sequence defined iteratively as in lemma 3.2.1. Let* $\{t_n\}$ *be an increasing sequence in* $[0,\infty)$. *Then,* $\lim_{n\to\infty} d(x_n, T_1(t_n)x_n) = \lim_{n\to\infty} d(x_n, T_2(t_n)x_n) = \lim_{n\to\infty} d(x_n, T_3(t_n)x_n) = \cdots = \lim_{n\to\infty} d(x_n, T_m(t_n)x_n) = 0$.

Proof. Since for some $x^\star \in \mathcal{F}$, the limit $\lim_{n\to\infty} d(x_n, x^\star)$ exist by lemma 3.2.1. Let $\lim_{n\to\infty} d(x_n, x^\star) = l$, so that from the prove of lemma 3.2.1, we can have the following relations by taking limit superior through the inequalities;

$$\begin{aligned}
d(x_{n+1}, x^\star) &= d((1-\alpha_n)x_n \oplus \alpha_n T_1(t_n)y_{n+m-2}, x^\star), \qquad [T_1(t_n)x^\star = x^\star] \\
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n d(T_1(t_n)y_{n+m-2}, T_1(t_n)x^\star) \\
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n[d(y_{n+m-2}, x^\star) + u_{1n}(t_n)\psi_1(d(y_{n+m-2}, x^\star)) \\
&\quad +v_{1n}(t_n)] \\
&\leq (1-\alpha_n)d(x_n, x^\star) + \alpha_n(1 + u_{1n}(t_n)M_1)d(y_{n+m-2}, x^\star) + \alpha_n u_{1n}(t_n) \\
&\quad \psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n)
\end{aligned}$$

So that, we can write that;

$$d(x_{n+1}, x^\star) - (1-\alpha_n)d(x_n, x^\star) \leq \alpha_n(1 + u_{1n}(t_n)M_1)d(y_{n+m-2}, x^\star) + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n)$$

Now attaching the $\limsup_{n\to\infty}$ on both sides of the inequality we get;

$$\begin{aligned}\limsup_{n\to\infty}[d(x_{n+1}, x^\star) - (1-\alpha_n)d(x_n, x^\star)] &\leq \limsup_{n\to\infty}[\alpha_n(1 + u_{1n}(t_n)M_1)d(y_{n+m-2}, x^\star) \\ &\quad + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n)] \\ \limsup_{n\to\infty}[\alpha_n d(x_n, x^\star)] &\leq \limsup_{n\to\infty}[\alpha_n(1 + u_{1n}(t_n)M_1)d(y_{n+m-2}, x^\star) \\ &\quad + \alpha_n u_{1n}(t_n)\psi_1(\overline{M_1}) + \alpha_n v_{1n}(t_n)]\end{aligned}$$

Pushing the $\limsup_{n\to\infty}$ through, we have;

$$\alpha l \leq \alpha \limsup_{n\to\infty} d(y_{n+m-2}, x^\star)$$

And as such, we write that;

$$\begin{aligned} l = \limsup_{n\to\infty} d(x_{n+1}, x^\star) &\leq \limsup_{n\to\infty} d(y_{n+m-2}, x^\star) \\ &\leq \limsup_{n\to\infty} d(y_{n+m-3}, x^\star) \\ &\leq \limsup_{n\to\infty} d(y_{n+m-4}, x^\star) \\ &\vdots \\ &\leq \limsup_{n\to\infty} d(y_{n+1}, x^\star) \\ &\leq \limsup_{n\to\infty} d(x_n, x^\star) = l \end{aligned}$$

Now, for $2 \leq h \leq m$, the $\limsup_{n\to\infty} d((1-\alpha_n)x_n \oplus \alpha_n T_h(t_n)y_{n+m-h-1}, x^\star) \leq l$. From this and $\lim_{n\to\infty} d(x_n, x^\star) = l$, using lemma 3.1.5, we have;

$\lim_{n\to\infty} d(x_n, T_h(t_n)y_{n+m-h-1}) = 0$, for $2 \le h \le m$. Observe that;

$$\begin{aligned} d(x_n, T_h(t_n)y_{n+m-h-1}) &= d((1-\alpha_n)x_n \oplus \alpha_n T_h(t_n)y_{n+m-h-1}, x_n) \\ &\le (1-\alpha_n)d(x_n,x_n) + \alpha_n d(T_h(t_n)y_{n+m-h-1}, x_n) \\ &= \alpha_n d(T_h(t_n)y_{n+m-h-1}, x_n) \longrightarrow 0, \quad as \quad n \to \infty \end{aligned}$$

Thus,

$$\begin{aligned} d(x_n, T_h(t_n)x_n) &\le d(x_n, T_h(t_n)y_{n+m-h-1}) + d(T_h(t_n)y_{n+m-h-1}, T_h(t_n)x_n) \\ &\le d(x_n, T_h(t_n)y_{n+m-h-1}) + (1+u_{hn}(t_n)M_h)d(y_{n+m-h-1}, x_n) \\ &\quad +u_{hn}(t_n)\psi_h(\overline{M_h}) + v_{hn}(t_n) \longrightarrow 0, \quad as \quad n \to \infty \end{aligned}$$

And

$$\begin{aligned} d(x_{n+1}, x_n) &= d((1-\alpha_n)x_n \oplus \alpha_n T_h(t_n)y_{n+m-2}, x_n) \\ &\le (1-\alpha_n)d(x_n,x_n) + \alpha_n d(T_1(t_n)y_{n+m-2}, x_n) \\ &= \alpha_n d(T_1(t_n)y_{n+m-2}, x_n) \longrightarrow 0, \quad as \quad n \to \infty \end{aligned}$$

Now,

$$\begin{aligned} d(x_n, T_h(t)x_n) &\le d(x_n, T_h(t_n)x_n) + d(T_h(t_n)x_n, T_h(t_n)y_{n+m-h-1}) \\ &\quad +d(T_h(t_n)y_{n+m-h-1}, T_h(t)x_n) \\ &\le d(x_n, T_h(t_n)x_n) + d(T_h(t_n)y_{n+m-h-1}, T_h(t)x_n) \\ &\quad +(1+u_{hn}(t_n)M_h)d(y_{n+m-h-1}, x_n) + u_{hn}(t_n)\psi_h(\overline{M_h}) \\ &\quad +v_{hn}(t_n) \end{aligned}$$

Consider the following; for any $t \geq 0$, $\{T_i(t) : t \geq 0\}$ is a uniformly asymptotic regular, then we have; $\lim_{n\to\infty} d(T_h(t)T_h(t_n)y_{n+m-h-1}, T_h(t_n)x_n)$
$\leq \lim_{n\to\infty} \sup_{z\in K} d(T(t)T(t_n)z, T(t_n)z) = 0$. where K is any bounded subset of X containing both $\{x_n\}$ and $\{y_{n+m-h-1}\}$. Since $T_i(t)$ is continuous, we get;

$$\begin{aligned}
d(x_n, T_h(t)x_n) &\leq d(x_n, T_h(t_n)x_n) + d(T_h(t_n)x_n, T_h(t)T_h(t_n)y_{n+m-h-1}) \\
&\quad + d(T_h(t)T_h(t_n)y_{n+m-h-1}, T_h(t)x_n) \\
&\leq d(x_n, T_h(t_n)x_n) + d(T_h(t_n)x_n, T_h(t+t_n)y_{n+m-h-1}) \\
&\quad + (1+u_{hn}(t)M_h)d(T_h(t_n)y_{n+m-h-1}, x_n) + u_{hn}(t)\psi_h(\overline{M_h}) \\
&\quad + v_{hn}(t) \\
&\leq d(x_n, T_h(t_n)x_n) + \sup_{z\in K, s\in\mathbb{R}^+} d(T_h(t_n)z, T_h(s+t_n)z) \\
&\quad + (1+u_{hn}(t)M_h)d(T_h(t_n)y_{n+m-h-1}, x_n) + u_{hn}(t)\psi_h(\overline{M_h}) \\
&\quad + v_{hn}(t)
\end{aligned}$$

Using this and the uniform asymptotic regularity of ∂, we have;

$$\begin{aligned}
\lim_{n\to\infty} d(x_n, T_h(t)x_n) &\leq \lim_{n\to\infty} [d(x_n, T_h(t_n)x_n) + \sup_{z\in K, s\in\mathbb{R}^+} d(T_h(t_n)z, T_h(s+t_n)z) \\
&\quad + (1+u_{hn}(t)M_h)d(T_h(t_n)y_{n+m-h-1}, x_n) + u_{hn}(t)\psi_h(\overline{M_h}) \\
&\quad + v_{hn}(t)]
\end{aligned}$$

Thus, we have;

$$\lim_{n\to\infty} d(x_n, T_h(t)x_n) = 0, \quad \forall \quad t \geq 0.$$

This completes the proof. □

Theorem 3.2.5 *; Let X be a uniformly convex hyperbolic space and K be a* *closed convex nonempty subset of X. Let $\partial := \{T_i(t) : K \to K, t \geq 0\}$ be a uniformly asymptotic regular family of uniformly L-Lipschitzian total asymptotically quasi-nonexpansive semigroup of X, with sequences $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{v_{in}(t_n)\}_{n=1}^{\infty}$ and mappings $\psi_i : [0,\infty) \to [0,\infty)$ satisfying $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ and $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, for $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$ and assume that each $T_i(t)$ is demiclosed at zero. Let $\{x_n\}$ be a sequence defined iteratively as in lemma 3.2.1. Let $\{t_n\}$ be an increasing sequence in $[0,\infty)$. Then, $\{x_n\}$ $\Delta-$converges to an element of $\mathcal{F}$.*

Proof. Let $W_\Delta(\{x_n\}) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\})$. We now show that $W_\Delta(\{x_n\}) \subset \mathcal{F}$ and also that $W_\Delta(\{x_n\})$ consists only of a single point. Let $u \in W_\Delta(\{x_n\})$, then by the definition of $\Delta-$convergence, there exist a subsequence say $\{u_n\} \subset \{x_n\}$ such that $\{u\} = A(\{u_n\})$. By lemma 3.1.6, there exists a convergence subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$, for some $v \in K$. But $lim_{n\to\infty} d(v_n, T_i(t)v_n) = 0$, for each $i \in I$, and $t \geq 0$. By the demiclosedness property of each $T_i(t)$, we have $v \in \mathcal{F}$. As the limit $lim_{n\to\infty} d(v_n, v)$, exists, so $u = v \in \mathcal{F}$, when limit exists, its unique and as such, since $u = v$ and $\{u\} = A(\{u_n\})$, then $\{u\} \in W_\Delta(\{x_n\}) \subset \mathcal{F}$.

Next, we show that $W_\Delta(\{x_n\})$ has a single element. Let $\{u_n\}$ be a subsequence of $\{x_n\}$, such that $\{u\} = A(\{u_n\})$ and let $\{x\} = A(\{x_n\})$. Since $\{u\} \in W_\Delta(\{x_n\}) \subset \mathcal{F}$, the limit $lim_{n\to\infty} d(x_n, u)$ exists and by lemma 3.1.2, $x = u$ and so $W_\Delta(\{x_n\})$ has a single element which implies the conclusion that $\{x_n\}$ $\Delta-$converges to an element of $\mathcal{F}$. This completes the proof. □

Next, we present Δ and polar convergence theorems for finite families of total asymptotically nonexpansive semigroup in the framework of a complete $CAT(0)$ space. This next result is a corollary to the previous lemma 3.2.4, but we shall present them using a different method of proof.

Corollary 3.2.6 *; Let X be a complete $CAT(0)$ space and K be a nonempty closed convex subset of X. Let $\partial := \{T_i(t) : K \to K, t \geq 0\}$ be a uniformly asymptotic regular family of a uniformly continuous, total asymptotically quasi-nonexpansive semigroup of X, with sequences $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{v_{in}(t_n)\}_{n=1}^{\infty}$ and mappings $\psi_i : [0,\infty) \to [0,\infty)$ satisfying $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ and $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, for $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $\{x_n\}$ be a sequence defined iteratively as in lemma 3.2.1. Let $\{t_n\}$ be an increasing sequence in $[0,\infty)$. Then, $\lim_{n\to\infty} d(x_n, T_1(t_n)x_n) = \lim_{n\to\infty} d(x_n, T_2(t_n)x_n) = \lim_{n\to\infty} d(x_n, T_3(t_n)x_n) = \cdots = \lim_{n\to\infty} d(x_n, T_m(t_n)x_n) = 0$.*

Proof. Since $\{x_n\}$ is bounded, for some $x^\star \in F$, there exists positive real numbers γ and M with $d^2(x_n, x^\star) \leq \gamma$, for all $n \geq 1$ and by using lemma 3.1.3 and the recursion formula in lemma 3.2.1, we have;

$$\begin{aligned}
d^2(y_n, x^\star) &= d^2((1-\alpha_n)x_n \oplus \alpha_n T_m(t_n)x_n, x^\star), \qquad [T_m(t_n)x^\star = x^\star] \\
&\leq (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n d^2(T_m(t_n)x_n, x^\star) - \alpha_n(1-\alpha_n) \\
&\quad d^2(x_n, T_m(t_n)x_n) \\
&= (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n d^2(T_m(t_n)x_n, T_m(t_n)x^\star) - \alpha_n \\
&\quad (1-\alpha_n)d^2(x_n, T_m(t_n)x_n) \\
&\leq (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n[(1+u_{mn}(t_n)M_m)d(x_n, x^\star) + u_{mn}(t_n) \\
&\quad \psi_m(\overline{M_m}) + v_{mn}(t_n)]^2 - \alpha_n(1-\alpha_n)d^2(x_n, T_m(t_n)x_n) \\
&= (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n[(1+u_{mn}(t_n)M_m)^2 d^2(x_n, x^\star) + [u_{mu}(t_n) \\
&\quad \psi_m(\overline{M_m}) + v_{mn}(t_n)]^2 + 2(u_{mn}(t_n)\psi_m(\overline{M_m}) + v_{mn}(t_n)) \\
&\quad (1+u_{mn}(t_n)M_m)d(x_n, x^\star)] - \alpha_n(1-\alpha_n)d^2(x_n, T_m(t_n)x_n)
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)d^2(x_n,x^\star)+\alpha_n[d^2(x_n,x^\star)+(2u_{mn}(t_n)M_m+u_{mn}^2(t_n)M_m^2) \\
&\quad d^2(x_n,x^\star)+[u_{mn}(t_n)\psi_m(\overline{M_m})+v_{mn}(t_n)]^2+2(u_{mn}(t_n)\psi_m(\overline{M_m}) \\
&\quad +v_{mn}(t_n))(1+u_{mn}(t_n)M_m)d(x_n,x^\star)]-\alpha_n(1-\alpha_n)d^2(x_n,T_m(t_n)x_n) \\
&= d^2(x_n,x^\star)+\alpha_n[2u_{mn}(t_n)M_md^2(x_n,x^\star)+u_{mn}^2(t_n)M_m^2d^2(x_n,x^\star) \\
&\quad +2(u_{mn}(t_n)\psi_m(\overline{M_m})+v_{mn}(t_n))(1+u_{mn}(t_n)M_m)d(x_n,x^\star) \\
&\quad +v_{mn}^2(t_n)+(u_{mn}(t_n)\psi_m(\overline{M_m}))^2+2v_{mn}(t_n)(u_{mn}(t_n)\psi_m(\overline{M_m}))] \\
&\quad -\alpha_n(1-\alpha_n)d^2(x_n,T_m(t_n)x_n) \\
&\leq d^2(x_n,x^\star)+\alpha[2u_{mn}(t_n)M_m\gamma+u_{mn}^2(t_n)M_m^2\gamma+2(u_{mn}(t_n)\psi_m(\overline{M_m}) \\
&\quad +v_{mn}(t_n))(1+u_{mn}(t_n)M_m)\gamma+v_{mn}^2(t_n)+(u_{mn}(t_n)\psi_m(\overline{M_m}))^2 \\
&\quad +2v_{mn}(t_n)(u_{mn}(t_n)\psi_m(\overline{M_m}))]-\alpha_n(1-\alpha_n)d^2(x_n,T_m(t_n)x_n) \\
&= d^2(x_n,x^\star)+2\alpha_nu_{mn}(t_n)M_m\gamma+\alpha_nu_{mn}^2(t_n)M_m^2\gamma+2\alpha_nu_{mn}(t_n)M_m \\
&\quad v_{mn}(t_n)\gamma+2\alpha_nv_{mn}(t_n)\gamma+2\alpha_nu_{mn}(t_n)\psi_m(\overline{M_m})\gamma+2\alpha_nu_{mn}(t_n) \\
&\quad \psi_m(\overline{M_m})u_{mn}(t_n)M_m\gamma+\alpha_nv_{mn}^2(t_n)+\alpha_n(u_{mn}(t_n)\psi_m(\overline{M_m}))^2 \\
&\quad +2\alpha_nv_{mn}(t_n)u_{mn}(t_n)\psi_m(\overline{M_m})-\alpha_n(1-\alpha_n)d^2(x_n,T_m(t_n)x_n) \\
&\leq d^2(x_n,x^\star)+\alpha_n[(2u_{mn}(t_n)M_m+u_{mn}^2(t_n)M_m^2+2(v_{mn}(t_n)+u_{mn}(t_n)M_m \\
&\quad v_{mn}(t_n)+u_{mn}(t_n)\psi_m(\overline{M_m})+u_{mn}(t_n)\psi_m(\overline{M_m})u_{mn}(t_n)M_m))\gamma+v_{mn}^2(t_n) \\
&\quad +(u_{mn}(t_n)\psi_m(\overline{M_m}))^2+2v_{mn}(t_n)u_{mn}(t_n)\psi_m(\overline{M_m})]-\alpha_n(1-\alpha_n)d^2(x_n, \\
&\quad T_m(t_n)x_n) \qquad\qquad [\omega_n=\Sigma_{i=1}^m u_{in}(t_n)M_i] \\
&= d^2(x_n,x^\star)+\alpha_n[(2\omega_n+\omega_n^2+2(v_{mn}(t_n)+\omega_nv_{mn}(t_n)+u_{mn}(t_n)\psi_m(\overline{M_m}) \\
&\quad +u_{mn}(t_n)\psi_m(\overline{M_m})\omega_n))\gamma+v_{mn}^2(t_n)+(u_{mn}(t_n)\psi_m(\overline{M_m}))^2+2v_{mn}(t_n) \\
&\quad u_{mn}(t_n)\psi_m(\overline{M_m})]-\alpha_n(1-\alpha_n)d^2(x_n,T_m(t_n)x_n)
\end{aligned}
$$

Now, putting ω_n together and where we have M, its power increases to 2, that is $M \leq M^2 \; \forall \; M \geq 1$, then the above becomes;

$$\leq \quad d^2(x_n, x^\star) + 7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2(x_n, T_m(t_n)x_n)$$

Thus, we write that;

$$d^2(y_n, x^\star) \quad \leq \quad d^2(x_n, x^\star) + 7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2(x_n, T_m(t_n)x_n)$$

Also, we have that;

$$\begin{aligned}
d^2(y_{n+1}, x^\star) &= d^2((1-\alpha_n)x_n \oplus \alpha_n T_{m-1}(t_n)y_n, x^\star), \qquad [T_{m-1}(t_n)x^\star = x^\star] \\
&\leq (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n d^2(T_{m-1}(t_n)y_n, x^\star) - \alpha_n(1-\alpha_n)d^2(x_n, \\
&\quad T_{m-1}(t_n)y_n) \\
&= (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n d^2(T_{m-1}(t_n)y_n, T_{m-1}(t_n)x^\star) - \alpha_n(1-\alpha_n) \\
&\quad d^2(x_n, T_{m-1}(t_n)y_n) \\
&\leq (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n[(1+u_{m-1n}(t_n)M_{m-1})d(y_n, x^\star) + u_{m-1n}(t_n) \\
&\quad \psi_{m-1}(\overline{M_{m-1}}) + v_{m-1n}(t_n)]^2 - \alpha_n(1-\alpha_n)d^2(x_n, T_{m-1}(t_n)y_n) \\
&= (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n[(1+u_{m-1n}(t_n)M_m)^2 d^2(y_n, x^\star) + [u_{m-1n}(t_n) \\
&\quad \psi_{m-1}(\overline{M_{m-1}}) + v_{m-1n}(t_n)]^2 + 2(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}) + v_{m-1n}(t_n)) \\
&\quad (1+u_{m-1n}(t_n)M_{m-1})d(y_n, x^\star)] - \alpha_n(1-\alpha_n)d^2(x_n, T_{m-1}(t_n)y_n) \\
&= (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n[d^2(y_n, x^\star) + (2u_{m-1n}(t_n)M_{m-1} + u_{m-1n}^2(t_n) \\
&\quad M_{m-1}^2)d^2(y_n, x^\star) + [u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}) + v_{m-1n}(t_n)]^2 + 2(u_{m-1n} \\
&\quad (t_n)\psi_{m-1}(\overline{M_{m-1}}) + v_{m-1n}(t_n))(1+u_{m-1n}(t_n)M_{m-1})d(y_n, x^\star)] - \alpha_n \\
&\quad (1-\alpha_n)d^2(x_n, T_{m-1}(t_n)y_n)
\end{aligned}$$

$$
\begin{aligned}
&= (1-\alpha_n)d^2(x_n,x^\star)+\alpha_n d^2(y_n,x^\star)+\alpha^n[2u_{m-1n}(t_n)M_{m-1}d^2(y_n,x^\star) \\
&\quad +u^2_{m-1n}(t_n)M^2_{m-1}d^2(y_n,x^\star)+2(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})+v_{m-1n}(t_n)) \\
&\quad (1+u_{m-1n}(t_n)M_{m-1})d(y_n,x^\star)+v^2_{m-1n}(t_n)+(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}))^2 \\
&\quad +2v_{m-1n}(t_n)(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}))]-\alpha_n(1-\alpha_n)d^2(x_n,T_{m-1}(t_n)y_n), \\
&\qquad [d^2(y_n,x^\star)\le\gamma] \\
&\le (1-\alpha_n)d^2(x_n,x^\star)+\alpha_n d^2(y_n,x^\star)+\alpha[2u_{m-1n}(t_n)M_{m-1}\gamma+u^2_{m-1n}(t_n) \\
&\quad M^2_{m-1}\gamma+2(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})+v_{m-1n}(t_n))(1+u_{m-1n}(t_n)M_{m-1}) \\
&\quad \gamma+v^2_{m-1n}(t_n)+(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}))^2+2v_{m-1n}(t_n)(u_{m-1n}(t_n) \\
&\quad \psi_{m-1}(\overline{M_{m-1}}))]-\alpha_n(1-\alpha_n)d^2(x_n,T_{m-1}(t_n)y_n) \\
&= (1-\alpha_n)d^2(x_n,x^\star)+\alpha_n d^2(y_n,x^\star)+2\alpha_n u_{m-1n}(t_n)M_{m-1}\gamma+\alpha_n u^2_{m-n}(t_n) \\
&\quad M^2_{m-1}\gamma+2\alpha_n u_{m-1n}(t_n)M_{m-1}v_{m-1n}(t_n)\gamma+2\alpha_n v_{m-1n}(t_n)\gamma+2\alpha_n u_{m-1n}(t_n) \\
&\quad \psi_{m-1}(\overline{M_{m-1}})\gamma+2\alpha_n u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})u_{m-1n}(t_n)M_{m-1}\gamma+\alpha_n v^2_{m-1n} \\
&\quad (t_n)+\alpha_n(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}))^2+2\alpha_n v_{m-1n}(t_n)u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}) \\
&\quad -\alpha_n(1-\alpha_n)d^2(x_n,T_{m-1}(t_n)y_n \\
&\le (1-\alpha_n)d^2(x_n,x^\star)+\alpha_n d^2(y_n,x^\star)+\alpha_n[(2u_{m-1n}(t_n)M_{m-1}+u^2_{m-1n}(t_n)M^2_{m-1} \\
&\quad +2(v_{m-1n}(t_n)+u_{m-1n}(t_n)M_{m-1}v_{m-1n}(t_n)+u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}) \\
&\quad +u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})u_{m-1n}(t_n)M_{m-1}))\gamma+v^2_{m-1n}(t_n)+(u_{m-1n}(t_n) \\
&\quad \psi_{m-1}(\overline{M_{m-1}}))^2+2v_{m-1n}(t_n)u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})]-\alpha_n(1-\alpha_n) \\
&\quad d^2(x_n,T_{m-1}(t_n)y_n), \qquad\qquad [\omega_n=\Sigma_{i=1}^{m-1}u_{in}(t_n)M_i] \\
&= (1-\alpha_n)d^2(x_n,x^\star)+\alpha_n d^2(y_n,x^\star)+\alpha_n[(2\omega_n+\omega_n^2+2(v_{m-1n}(t_n)+\omega_n v_{m-1n} \\
&\quad (t_n)+u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})+u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})\omega_n))\gamma+v^2_{m-1n}(t_n) \\
&\quad +(u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}}))^2+2v_{m-1n}(t_n)u_{m-1n}(t_n)\psi_{m-1}(\overline{M_{m-1}})] \\
&\quad -\alpha_n(1-\alpha_n)d^2(x_n,T_{m-1}(t_n)y_n)
\end{aligned}
$$

Putting ω_n together and where we have M, its power increases to 2, that is $M \leq M^2$, $\forall\ M \geq 1$, and replacing, $d^2(y_n, x^\star) \leq d^2(x_n, x^\star) + 7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2(x_n, T_m(t_n)x_n)$ above, we get;

$$\begin{aligned}
d^2(y_{n+1}, x^\star) &\leq (1-\alpha_n)d^2(x_n, x^\star) + \alpha_n d^2(y_n, x^\star) - \alpha_n(1-\alpha_n)d^2(x_n, T_{m-1}(t_n)y_n) \\
&\quad +7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 \\
&\leq d^2(x_n, x^\star) + \alpha_n[7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2(x_n, T_m(t_n)x_n)] \\
&\quad +7\alpha_n\omega_n M^2\gamma + \alpha_n\omega_n^2 - \alpha_n(1-\alpha_n)d^2(x_n, T_{m-1}(t_n)y_n) \\
&= d^2(x_n, x^\star) + \alpha_n[7\omega_n M^2\gamma + \omega_n^2](1+\alpha_n) - (\alpha_n)^m(1-\alpha_n)[d^2(x_n, \\
&\quad T_{m-1}(t_n)y_n) + d^2(x_n, T_m(t_n)x_n)]
\end{aligned}$$

Thus, continuing in this fashion and using; $x_{n+1} = (1-\alpha_n)x_n \oplus \alpha_n T_1(t_n)y_{n+m-2}$,

$$\begin{aligned}
d^2(x_{n+1}, x^\star) &\leq d^2(x_n, x^\star) + \alpha_n[7\omega_n^2 M^2\gamma + \omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j - \alpha_n^m(1-\alpha_n) \\
&\quad [\Sigma_{j=1}^{m-1} d^2(x_n, T_{m-j}(t_n)y_{n+j-1}) + d^2(x_n, T_m(t_n)x_n)]
\end{aligned}$$

so that we can have;

$$\begin{aligned}
&\alpha_n^m(1-\alpha_n)[\Sigma_{j=1}^{m-1} d^2(x_n, T_{m-j}(t_n)y_{n+j-1}) + d^2(x_n, T_m(t_n)x_n)] \\
&\quad \leq d^2(x_n, x^\star) - d^2(x_{n+1}, x^\star) + \alpha_n[7\omega_n^2 M^2\gamma + \omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j
\end{aligned}$$

Now, using the comparison test of a series, which states that; "if $0 \leq a_n \leq Kb_n$, for $K \geq 1$ then, Σb_n converges implies that Σa_n converges". The necessary condition for Σb_n to converges, is that the $\lim_{n\to\infty} b_n = 0$ and that the convergence depends on the tail end of the series. If its finite, then it converges. that is for $M \geq 0$, $\lim_{n\to\infty} b_n < M$. When a series is finite, we use the oscillating/oscillatory method as follows; so that for Σb_n to converges, let $m \in N$, then we write;

$$\begin{aligned}
&\Sigma_{n=1}^{m}[d^2(x_n,x^\star)-d^2(x_{n+1},x^\star)+\alpha_n[7\omega_n^2M^2\gamma+\omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j]\\
&=\Sigma_{n=1}^{m}[d^2(x_n,x^\star)-d^2(x_{n+1},x^\star)]+\Sigma_{n=1}^{m}[\alpha_n[7\omega_n^2M^2\gamma+\omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j]]\\
&\le d^2(x_1,x^\star)-d^2(x_{m+1},x^\star)+\Sigma_{n=1}^{m}[\alpha_n[7\omega_n^2M^2\gamma+\omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j]]\\
&\le d^2(x_1,x^\star)+\Sigma_{n=1}^{\infty}[\alpha_n[7\omega_n^2M^2\gamma+\omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j]]<\infty
\end{aligned}$$

Thus, Σb_n is finite and as such, it converges, then we can also write that;

$$\begin{aligned}
0 &\le \Sigma_{n=1}^{\infty}(\alpha_n^m(1-\alpha_n)[\Sigma_{j=1}^{m-1}d^2(x_n,T_{m-j}(t_n)y_{n+j-1})+d^2(x_n,T_m(t_n)x_n)])\\
&\le \Sigma_{n=1}^{\infty}(d^2(x_n,x^\star)-d^2(x_{n+1},x^\star)+\alpha_n[7\omega_n^2M^2\gamma+\omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j)<\infty
\end{aligned}$$

Now, since Σb_n is finite and it converges, so does Σa_n, by the comparison test of a series. since; $\Sigma_{n=1}^{\infty}(d^2(x_n,x^\star)-d^2(x_{n+1},x^\star)+\alpha_n[7\omega_n^2M^2\gamma+\omega_n^2]\Sigma_{j=0}^{m-1}\alpha_n^j)<\infty$, then this also implies that;
$\Sigma_{n=1}^{\infty}(\alpha_n^m(1-\alpha_n)[\Sigma_{j=1}^{m-1}d^2(x_n,T_{m-j}(t_n)y_{n+j-1})+d^2(x_n,T_m(t_n)x_n)])<\infty$
and by our choice of $\{\alpha_n\}_{n=1}^{\infty}$, we have;
$\alpha_n^m(1-\alpha_n)[\Sigma_{j=1}^{m-1}d^2(x_n,T_{m-j}(t_n)y_{n+j-1})+d^2(x_n,T_m(t_n)x_n)]$
$=(\alpha_n^m-\alpha_n^{m+1})[\Sigma_{j=1}^{m-1}d^2(x_n,T_{m-j}(t_n)y_{n+j-1})+d^2(x_n,T_m(t_n)x_n)]\longrightarrow 0$ as $n\to\infty$. Thus, we have;

$$\begin{aligned}
\lim_{n\to\infty}d(x_n,T_m(t_n)x_n) &= \lim_{n\to\infty}d(x_n,T_{m-1}(t_n)y_n)\\
&\vdots\\
&= \lim_{n\to\infty}d(x_n,T_h(t_n)y_{n+m-h-1})\\
&\vdots\\
&= \lim_{n\to\infty}d(x_n,T_h(t_n)y_{n+m-2})=0. \qquad 2\le h\le m
\end{aligned}$$

Hence, the remaining of the proof follows as in lemma 3.2.4 above. $\square$

Theorem 3.2.7 *; Let E be a complete $CAT(0)$ space and K be a nonempty closed convex subset of E. Let $\partial := \{T_i(t) : K \to K, t \geq 0\}$ be a uniformly asymptotic regular family of uniformly L-Lipschitzian total asymptotically quasi-nonexpansive semigroup of E, with sequences $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{v_{in}(t_n)\}_{n=1}^{\infty}$ and mappings $\psi_i : [0,\infty) \to [0,\infty)$ satisfying $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ and $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, for $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $\{x_n\}$ be a sequence defined iteratively as in lemma 3.2.1. Let $\{t_n\}$ be an increasing sequence in $[0,\infty)$. Then, $\{x_n\}$ $\Delta-$converges to an element of $\mathcal{F}$.*

Proof. Let $W_\Delta(\{x_n\}) := \bigcup_{\{u_n\}\subset\{x_n\}} A(\{u_n\})$. We now show that $W_\Delta(\{x_n\}) \subset \mathcal{F}$ and also that $W_\Delta(\{x_n\})$ consists only of a single point. Let $u \in W_\Delta(\{x_n\})$, then by the definition of $\Delta-$convergence there exists a subsequence say $\{u_n\}$ of $\{x_n\}$ such that $\{u\} = A(\{u_n\})$. By lemma 3.1.6, there exists a convergence subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v$, for some $v \in K$. But $lim_{n\to\infty} d(v_n, T_i(t_n)v_n) = 0$, for each $i \in I$, and $t \geq 0$. By the demiclosedness property of each $T_i(t)$, we have $v \in \mathcal{F}$. As the limit $lim_{n\to\infty} d(v_n, v)$ exists, so $u = v \in \mathcal{F}$, when limit exists, its unique and as such, since $u = v$ and $\{u\} = A(\{u_n\})$, then $\{u\} \in W_\Delta(\{x_n\}) \subset \mathcal{F}$.

Next, we show that $W_\Delta(\{x_n\})$ has a single element. Let $\{u_n\}$ be a subsequence of $\{x_n\}$, such that $\{u\} = A(\{u_n\})$ and let $\{x\} = A(\{x_n\})$. Since $\{u\} \in W_\Delta(\{x_n\}) \subset \mathcal{F}$, the limit $lim_{n\to\infty} d(x_n, u)$ exists and by lemma 3.1.2, $x = u$ and so $W_\Delta(\{x_n\})$ has a single element which implies the conclusion that $\{x_n\}$ $\Delta-$converges to an element of $\mathcal{F}$. This completes the proof. $\square$

The $CAT(0)$ spaces are also the rotund metric spaces (staple rotund see; (Staples (1976))). The polar and $\Delta-$convergence of the $CAT(0)$ space, coincides in a complete rotund metric space. As a consequence of this statement and theorem 3.2.7, we have the following theorem.

Theorem 3.2.8 *; Let E be a complete $CAT(0)$ space and K be a closed convex nonempty subset of E. Let $\partial := \{T_i(t) : K \to K, t \geq 0\}$ be a uniformly asymptotic regular family of uniformly L-Lipschitzian total asymptotically quasi-nonexpansive semigroup of E, with sequences $\{u_{in}(t_n)\}_{n=1}^{\infty}$, $\{v_{in}(t_n)\}_{n=1}^{\infty}$ and mappings $\psi_i : [0, \infty) \to [0, \infty)$ satisfying $\Sigma_{n=1}^{\infty} u_{in}(t_n) < \infty$ and $\Sigma_{n=1}^{\infty} v_{in}(t_n) < \infty$, for $i \in I$. Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0, 1)$, Let $\{x_n\}$ be a sequence defined iteratively as in lemma 3.2,1 above and let $\{t_n\}$ be an increasing sequence in $[0, \infty)$. Then, $\{x_n\}$ polar converges to an element of $\mathcal{F}$.*

CHAPTER FOUR

COMMON FIXED POINT APPROXIMATION OF TOTAL ASYMPTOTICALLY NONEXPANSIVE SEMIGROUP IN $CAT(0)$ SPACES

In this chapter, the modified Mann iteration scheme is used to approximate common fixed point of a family of uniformly asymptotic regular $\partial := \{T_{i(n)}(t) : K \longrightarrow K, t \geq 0\}$ $L_{i(n)}(t)-$Lipchitzian and total asymptotically quasi-nonexpansive semigroup in a complete $CAT(0)$ space. We were able to prove that the iterative scheme proposed in theorem 4.2.1, converges strongly to a common fixed point of a total asymptotically quasi-nonexpansive semigroup in a complete $CAT(0)$ space. The result presented in this chapter is the semigroup version of the result in Ugwunnadi and Ali (2016).

In Ugwunnadi and Ali (2016), it was proved that if the finite family of the mappings; $T_{i(n)}, i(n) = 1, 2, 3, ..., N$ are uniformly L-Lipschitzian and total asymptotically quasi-nonexpansive, under appropriate and restricted conditions, the sequence $\{x_n\}$ strongly converges to a point in; $\mathcal{F} := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. In line with this, we were able to present their result in a different manner, (i.e) by letting the finite family of mappings to be uniformly asymptotic regular and then total asymptotically quasi-nonexpansive semigroup of the ground set. We were able to obtain the following result:

4.1 Preliminaries

In this section, to prove the main result, we need the following lemmas:

Lemma 4.1.1 *(Dehghan and Rooin (2014)); Let X be a CAT(0) space. Then,* $d^2((1-\alpha)x \oplus \alpha y, z) \leq (1-\alpha)^2 d^2(x,z) + \alpha^2 d^2(y,z) + 2\alpha(1-\alpha)d^2\langle\overrightarrow{xz}, \overrightarrow{yz}\rangle$ $\forall \alpha \in [0,1]$ *and* $x, y, z \in X$.

Lemma 4.1.2 *(Kakavandi (2012)); Let X be a complete CAT(0) space,* $\{x_n\}$ *be a sequence in* X *and* $x \in X$. *Then,* $\{x_n\}$ $\Delta-$*converges to* x *if and only if* $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$, *forall* $y \in X$.

Lemma 4.1.3 *(Xu (2004)); Let* $\{a_n\}$ *be a sequence of nonnegative real number satisfying the following relation;* $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n$, $\forall n \geq 0$. *Where, (i)* $\{\alpha_n\} \subset [0,1], \sum \alpha_n = \infty$; *(ii)* $\limsup \sigma_n \leq 0$; *(iii)* $\gamma_n \geq 0; (n \geq 0), \sum \gamma_n < \infty$. *Then,* $a_n \to 0$ *as* $n \to \infty$.

Theorem 4.1.4 *(Chidume and Chidume (2006)); Suppose that* $\{x_n\}$ *is a sequence of positive terms and that the limit:* $L = \lim_{n\to\infty} \dfrac{x_{n+1}}{x_n}$ *exists.*
If $L < 1$, *then* $\lim_{n\to\infty} x_n = 0$.

Lemma 4.1.5 *; Let* $\partial := \{T_{i(n)}(t) : K \longrightarrow K, t \geq 0\}$ *be a uniformly asymptotic regular finite family of total asymptotically nonexpansive semigroup, with* $\{\mu^{i(n)}_{p(n)}(t_{i(n)})\}$, $\{\nu^{i(n)}_{p(n)}(t_{i(n)})\}$ *and mappings* $\phi_{i(n)} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ *satisfying* $\Sigma^{\infty}_{p(n)=1}$ $\mu^{i(n)}_{p(n)}(t_{i(n)}) < \infty$ *and* $\Sigma^{\infty}_{p(n)=1}\nu^{i(n)}_{p(n)}(t_{i(n)}) < \infty$. *Then, there exists sequences* $u_n(t_n)$ *and* $v_n(t_n) \subset [0,\infty)$ *such that;*
$d(T_{i(n)}(t_n)x, T_{i(n)}(t_n)y) \leq d(x,y) + u_n(t_n)\phi_{i(n)}(d(x,y)) + v_n(t_n)$, $\forall n \geq 1$ *and* $x, y \in K$. *Where;* $p(n) = j+1$, *if* $jN < n \leq (j+1)N, j = 0, 1, 2, \cdots$ *and* $n = jN + i(n); i(n) \in \{1, 2, 3, ..., N\}$.

4.2 Main Result

In our discussion here, we shall assume that $p(n) = j+1$, if $jN < n \leq (j+1)N$, $j = 0, 1, 2, \cdots$ and $n = jN + i(n)$; $i(n) \in \{1, 2, 3, ..., N\}$ and $u_n(t_n) := max_{1 \leq i(n) \leq N} \mu_{p(n)}^{i(n)}(t_{i(n)})$ and $v_n(t_n) := max_{1 \leq i(n) \leq N} \nu_{p(n)}^{i(n)}(t_{i(n)})$ and for each $n \geq 1$, $n = (p(n)-1)N + i(n)$. In this section, we state and prove the convergence theorem on approximation of common fixed point of uniformly asymptotic regular family of total asymptotically nonexpansive semigroup in a complete $CAT(0)$ space.

Theorem 4.2.1 *; Let X be a complete $CAT(0)$ space and K be a closed convex nonempty subset of X. Let $\partial := \{T_{i(n)}(t) : K \longrightarrow K, t \geq 0\}$ be a uniformly asymptotic regular family of uniformly $L_{i(n)}(t)-$Lipchitzian and total asymptotically nonexpansive semigroup of X, with $\{\mu_{p(n)}^{i(n)}(t_{i(n)})\}$, $\{\nu_{p(n)}^{i(n)}(t_{i(n)})\}$ and mappings $\phi_{i(n)} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying: $\Sigma_{p(n)=1}^{\infty} \mu_{p(n)}^{i(n)}(t_{i(n)}) < \infty$ and $\Sigma_{p(n)=1}^{\infty}$ $\nu_{p(n)}^{i(n)}(t_{i(n)}) < \infty$. Let $u_n(t_n) := max_{1 \leq i(n) \leq N} \mu_{p(n)}^{i(n)}(t_{i(n)})$ and $v_n(t_n) := max_{1 \leq i(n) \leq N}$ $\nu_{p(n)}^{i(n)}(t_{i(n)})$ such that: $\mathcal{F} = \bigcap_{i(n)=1}^{N} F(T_{i(n)}) \neq \emptyset$, where: $\mathcal{F} = F(T_N(t)T_{N-1}(t)T_{N-2}(t) \cdots T_3(t)T_2(t)T_1(t)) = F(T_1(t)T_N(t)T_{N-1}(t)T_{N-2}(t) \cdots T_3(t)T_2(t)) = \cdots = F(T_{N-1}(t)T_{N-2}(t) \cdots T_3(t)T_2(t)T_1(t)T_N(t)) \neq \emptyset$. Let $\{t_{i(n)}\}$ be an increasing sequence in $[0, \infty)$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence generated by $x_1 = x \in K$.*

$$\begin{cases} y_n = P_K((1-\alpha_n)x_n) \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n \end{cases} \quad (4.2.2)$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1), \{\beta_n\}_{n=1}^{\infty} \subset [c,d] \subset (0,1)$ satisfying $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \frac{u_n(t_n)}{\alpha_n} = 0, \lim_{n\to\infty} \frac{v_n(t_n)}{\alpha_n} = 0$. Assuming that there exists a constant $M := max_{1 \leq i(n) \leq N} M_{i(n)}$, such that $\phi_{i(n)}(r) \leq Mr, \forall r \geq 0$. Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $\mathcal{F}$.

Proof. We start the prove by letting $\delta_n := (1 + \beta_n u_{n+1}(t_{n+1})M)\alpha_n$. Since there exists $N_0 \geq 0$ such that $\frac{u_{n+1}(t_{n+1})}{\alpha_n} \leq \frac{\epsilon(1+\beta_n u_{n+1}(t_{n+1})M)}{\beta_n M}$ and $\frac{v_{n+1}(t_{n+1})}{\alpha_n} \leq \frac{(1+\beta_n u_{n+1}(t_{n+1})M)}{\beta_n}$, for all $n \geq N_0$ and for some $\epsilon > 0$ satisfying $0 \leq (1-\epsilon)\delta_n \leq 1$. For some $p \in \mathcal{F}$ and $n \geq N_0$, then we have from (4.2.2) above that:

$$\begin{aligned} d(y_n, p) &= d(P_K((1-\alpha_n)x_n), p) \\ &= d(P_K(\alpha_n(0) \oplus (1-\alpha_n)x_n), p) \\ &\leq d(\alpha_n(0) \oplus (1-\alpha_n)x_n, p) \\ &\leq \alpha_n d(0, p) + (1-\alpha_n)d(x_n, p) \end{aligned}$$

And

$$\begin{aligned} d(x_{n+1}, p) &= d((1-\beta_n)y_n \oplus \beta_n T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n, p) \\ &\leq (1-\beta_n)d(y_n, p) + \beta_n d(T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n, p), \quad [p = T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})p] \\ &= (1-\beta_n)d(y_n, p) + \beta_n d(T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n, T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})p) \\ &\leq (1-\beta_n)d(y_n, p) + \beta_n[d(y_n, p) + u_{n+1}(t_{n+1})\phi_{i(n+1)}(d(y_n, p)) \\ &\quad + v_{n+1}(t_{n+1})] \\ &= d(y_n, p) - \beta_n d(y_n, p) + \beta_n d(y_n, p) + \beta_n u_{n+1}(t_{n+1})\phi_{i(n+1)} \\ &\quad (d(y_n, p)) + \beta_n v_{n+1}(t_{n+1}) \\ &= d(y_n, p) + \beta_n u_{n+1}(t_{n+1})\phi_{i(n+1)}(d(y_n, p)) + \beta_n v_{n+1}(t_{n+1}) \end{aligned}$$

But using $\phi_{i(n)}(r) \leq Mr, \forall r \geq 0$, we then have; $\phi_{i(n+1)}(d(y_n, p)) \leq Md(y_n, p)$

$$\begin{aligned} d(x_{n+1}, p) &\leq d(y_n, p) + \beta_n u_{n+1}(t_{n+1})Md(y_n, p) + \beta_n v_{n+1}(t_{n+1}) \\ &= [1 + \beta_n u_{n+1}(t_{n+1})M]d(y_n, p) + \beta_n v_{n+1}(t_{n+1}) \end{aligned}$$

$$
\begin{aligned}
&\leq [1+\beta_n u_{n+1}(t_{n+1})M](\alpha_n d(0,p)+(1-\alpha_n)d(x_n,p))+ \\
&\quad \beta_n v_{n+1}(t_{n+1}) \\
&= [1+\beta_n u_{n+1}(t_{n+1})M]\alpha_n d(0,p)+[1+\beta_n u_{n+1}(t_{n+1})M] \\
&\quad (1-\alpha_n)d(x_n,p)+\beta_n v_{n+1}(t_{n+1})
\end{aligned}
$$

From our assumption above, we have that: $\delta_n := (1+\beta_n u_{n+1}(t_{n+1})M)\alpha_n$, $\beta_n u_{n+1}(t_{n+1})M \leq \epsilon(1+\beta_n u_{n+1}(t_{n+1})M)\alpha_n = \epsilon\delta_n$ and also $\beta_n v_{n+1}(t_{n+1}) \leq (1+\beta_n u_{n+1}(t_{n+1})M)\alpha_n = \delta_n$. Substituting this above and simplifying, we then write;

$$
\begin{aligned}
d(x_{n+1},p) &\leq \delta_n d(0,p)+d(x_n,p)+\beta_n u_{n+1}(t_{n+1})Md(x_n,p)-\delta_n d(x_n,p)+\delta_n \\
&= \delta_n d(0,p)+d(x_n,p)+\epsilon\delta_n d(x_n,p)-\delta_n d(x_n,p)+\delta_n \\
&= d(x_n,p)+\epsilon\delta_n d(x_n,p)-\delta_n d(x_n,p)+\delta_n d(0,p)+\delta_n \\
&= [1-\delta_n+\epsilon\delta_n]d(x_n,p)+\delta_n[d(0,p)+1] \\
&= [1-(1-\epsilon)\delta_n]d(x_n,p)+\delta_n[d(0,p)+1] \\
&= [1-(1-\epsilon)\delta_n]d(x_n,p)+\frac{(1-\epsilon)}{(1-\epsilon)}\delta_n[d(0,p)+1] \\
&= [1-(1-\epsilon)\delta_n]d(x_n,p)+\frac{(1-\epsilon)\delta_n[d(0,p)+1]}{(1-\epsilon)} \\
&\leq max\{d(x_n,p),\frac{[d(0,p)+1]}{(1-\epsilon)}\}
\end{aligned}
$$

Thus, we have;

$$
d(x_{n+1},p) \leq max\{d(x_n,p),\frac{[d(0,p)+1]}{(1-\epsilon)}\}, \quad \forall \quad n+1 \geq n
$$

And by induction, since it holds for all $n+1 \geq n$, then it also hold for all $n \geq N_0$. Then, we can write that;

$$d(x_n,p) \;\leq\; max\{d(x_{N_0},p), \frac{[d(0,p)+1]}{(1-\epsilon)}\}, \quad \forall \quad n \geq N_0. \qquad (4.2.3)$$

It then follows that $\{x_n\}$ is bounded, and as such $\{y_n\}$, $\{T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n\}$, and $\{T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})x_n\}$ are all bounded. Also,

$$\begin{aligned} d(y_n,x_n) &= d(P_K((1-\alpha_n)x_n),x_n) \\ &= d(P_K(\alpha_n(0)\oplus(1-\alpha_n)x_n),x_n) \\ &\leq d(\alpha_n(0)\oplus(1-\alpha_n)x_n,x_n) \\ &\leq \alpha_n d(0,x_n)+(1-\alpha_n)d(x_n,x_n) \\ &= \alpha_n d(0,x_n) \longrightarrow 0 \quad as \quad n\to\infty. \end{aligned}$$

Using lemma 3.1.3 and by letting $\overline{u_n(t_n)} = 2Mu_n(t_n) + u_n^2(t_n)M^2$, we have;

$$\begin{aligned} d^2(x_{n+1},p) &= d^2((1-\beta_n)y_n \oplus \beta_n T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n,p) \\ &\leq (1-\beta_n)d^2(y_n,p)+\beta_n d^2(T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n,p)-\beta_n(1-\beta_n) \\ &\quad d^2(y_n,T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) \\ &= (1-\beta_n)d^2(y_n,p)+\beta_n d^2(T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n,T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})p) \\ &\quad -\beta_n(1-\beta_n)d^2(y_n,T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n), \quad [p=T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})p] \\ &\leq (1-\beta_n)d^2(y_n,p)+\beta_n[d(y_n,p)+u_{n+1}(t_{n+1})\phi_{i(n+1)}(d(y_n,p)) \\ &\quad +v_{n+1}(t_{n+1})]^2-\beta_n(1-\beta_n)d^2(y_n,T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) \end{aligned}$$

Using $\phi_{i(n)}(r) \leq Mr, \forall r \geq 0$, we then have; $\phi_{i(n+1)}(d(y_n,p)) \leq Md(y_n,p)$

$$\begin{aligned} &\leq (1-\beta_n)d^2(y_n,p)+\beta_n[d(y_n,p)+u_{n+1}(t_{n+1})Md(y_n,p)+v_{n+1}(t_{n+1})]^2 \\ &\quad -\beta_n(1-\beta_n)d^2(y_n,T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) \end{aligned}$$

$$
\begin{aligned}
&= (1-\beta_n)d^2(y_n,p)+\beta_n[(1+u_{n+1}(t_{n+1})M)d(y_n,p)+v_{n+1}(t_{n+1})]^2-\beta_n \\
&\quad (1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \\
&= (1-\beta_n)d^2(y_n,p)+\beta_n[(1+u_{n+1}(t_{n+1})M)^2d^2(y_n,p)+2[(1+u_{n+1}(t_{n+1})M) \\
&\quad d(y_n,p)][v_{n+1}(t_{n+1})]+(v_{n+1}(t_{n+1}))^2]-\beta_n(1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \\
&= d^2(y_n,p)+2\beta_n u_{n+1}(t_{n+1})Md^2(y_n,p)+\beta_n u_{n+1}^2(t_{n+1})M^2d^2(y_n,p)+2\beta_n v_{n+1} \\
&\quad (t_{n+1})d(y_n,p)+2\beta_n v_{n+1}(t_{n+1})u_{n+1}(t_{n+1})Md(y_n,p)+\beta_n v_{n+1}^2(t_{n+1})-\beta_n \\
&\quad (1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \\
&= [1+\beta_n(2u_{n+1}(t_{n+1})M+u_{n+1}^2(t_{n+1})M^2)]d^2(y_n,p)+\beta_n v_{n+1}(t_{n+1})[2d(y_n,p) \\
&\quad +2u_{n+1}(t_{n+1})Md(y_n,p)+v_{n+1}(t_{n+1})]-\beta_n(1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \\
&= [1+\beta_n\overline{u_{n+1}(t_{n+1})}]d^2(y_n,p)+\beta_n v_{n+1}(t_{n+1})[2(1+u_{n+1}(t_{n+1})M)d(y_n,p) \\
&\quad +v_{n+1}(t_{n+1})]-\beta_n(1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \qquad (4.2.4)
\end{aligned}
$$

Since $\{y_n\}$ is bounded, then there exists $D>0$ such that: $d(y_n,p)\leq D \quad \forall n\geq 1$, so that we have;

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq d^2(y_n,p)+\beta_n\overline{u_{n+1}(t_{n+1})}D^2+\beta_n v_{n+1}(t_{n+1})[2(1+u_{n+1}(t_{n+1})M) \\
&\quad D+v_{n+1}(t_{n+1})]-\beta_n(1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \qquad (4.2.5)
\end{aligned}
$$

Also from (4.2.2) and lemma 3.1.3, we obtain;

$$
\begin{aligned}
d^2(y_n,p) &= d^2(P_K((1-\alpha_n)x_n),p) \\
&= d^2(P_K(\alpha_n(0)\oplus(1-\alpha_n)x_n),p) \\
&\leq d^2(\alpha_n(0)\oplus(1-\alpha_n)x_n,p) \\
&\leq \alpha_n^2d^2(0,p)+(1-\alpha_n)^2d^2(x_n,p)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \qquad (4.2.6)
\end{aligned}
$$

Now, substituting (4.2.6) into (4.2.5), we obtain;

$$
\begin{aligned}
d^2(x_{n+1},p) &\leq \alpha_n^2 d^2(0,p)+(1-\alpha_n)^2 d^2(x_n,p)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \\
&\quad +\beta_n\overline{u_{n+1}(t_{n+1})}D^2+\beta_n v_{n+1}(t_{n+1})[2(1+u_{n+1}(t_{n+1})M)D \\
&\quad +v_{n+1}(t_{n+1})]-\beta_n(1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \qquad (4.2.7)\\
&\leq \alpha_n^2 d^2(0,p)+(1-\alpha_n)^2 d^2(x_n,p)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \\
&\quad +\beta_n\overline{u_{n+1}(t_{n+1})}D^2+\beta_n v_{n+1}(t_{n+1})[2(1+u_{n+1}(t_{n+1})M)D \\
&\quad +v_{n+1}(t_{n+1})] \qquad (4.2.8)
\end{aligned}
$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded, then there exists $D_1>0$ such that;

$$
\alpha_n d^2(0,p)+2(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \leq D_1, \qquad \forall n\geq 1 \qquad (4.2.9)
$$

From (4.2.7) and (4.2.9) above, we get;

$$
\begin{aligned}
\beta_n(1-\beta_n)d^2(y_n,T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) &\leq d^2(x_n,p)-d^2(x_{n+1},p)+\alpha_n^2 d^2(x_n,p) \\
&\quad +\alpha_n^2 d^2(0,p)+2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \\
&\quad +\beta_n\overline{u_{n+1}(t_{n+1})}D^2+\beta_n v_{n+1}(t_{n+1}) \\
&\quad [2(1+u_{n+1}(t_{n+1})M)D+v_{n+1}(t_{n+1})] \\
&= d^2(x_n,p)-d^2(x_{n+1},p)+\alpha_n[\alpha_n d^2(0,p) \\
&\quad +2(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle+\beta_n\frac{\overline{u_{n+1}(t_{n+1})}}{\alpha_n}D^2 \\
&\quad +\beta_n\frac{v_{n+1}(t_{n+1})}{\alpha_n}[2(1+u_{n+1}(t_{n+1})M)D \\
&\quad +v_{n+1}(t_{n+1})]+\alpha_n d^2(x_n,p)] \\
&\leq d^2(x_n,p)-d^2(x_{n+1},p)+\alpha_n[D_1+\beta_n \\
&\quad \frac{\overline{u_{n+1}(t_{n+1})}}{\alpha_n}D^2+\beta_n\frac{v_{n+1}(t_{n+1})}{\alpha_n}[2(1+u_{n+1} \\
&\quad (t_{n+1})M)D+v_{n+1}(t_{n+1})]+\alpha_n d^2(x_n,p)]
\end{aligned}
$$

Now, to complete the prove, we consider the following two cases;

CASE 1; Suppose that there exists $n_0 \in N$ such that $\{d(x_n, p)\}$ is non-decreasing, then in this case, $\{d(x_n, p)\}$ is convergent (by its monotonicity and boundedness). Then, from (4.2.10) above, we obtain;

$$\beta_n(1-\beta_n)d^2(y_n, T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \longrightarrow 0 \quad as \quad n \to \infty. \tag{4.2.11}$$

Which implies that by our choice of β_n,

$$d(y_n, T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \longrightarrow 0 \quad as \quad n \to \infty. \tag{4.2.12}$$

And

$$\begin{aligned} d(x_n, T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) &\leq d(x_n, y_n) + d(y_n, T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n) \\ &\longrightarrow 0 \quad as \quad n \to \infty \end{aligned}$$

Also, from (4.2.2) above, we have that;

$$\begin{aligned} d(x_{n+1}, x_n) &= d((1-\beta_n)y_n \oplus \beta_n T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n, x_n) \\ &\leq (1-\beta_n)d(y_n, x_n) + \beta_n d(T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n, x_n) \end{aligned}$$

Which implies that from (4.2.4) and (4.2.12) above, we write;

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0, \tag{4.2.13}$$

Furthermore, we obtain that;

$$
\begin{aligned}
d(x_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})x_n) &\leq d(x_n, y_n) + d(y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) + d(T^{p(n+1)}_{i(n+1)} \\
&\quad (t_{i(n+1)})y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n)})T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) + \\
&\quad d(T^{p(n+1)}_{i(n+1)}(t_{i(n)})T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})x_n) \\
&= d(x_n, y_n) + d(y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) + d(T^{p(n+1)}_{i(n+1)} \\
&\quad (t_{i(n+1)})y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)} + t_{i(n)})y_n) + d(T^{p(n+1)}_{i(n+1)} \\
&\quad T^{p(n+1)}_{i(n+1)}(t_{i(n+1)} + t_{i(n)})y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})x_n) \\
&\leq d(x_n, y_n) + d(y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) + \sup_{s\in\Re^+, z\in\{y_n\}} \\
&\quad d(T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})z, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)} + s)z) + [1 + u_{n+1} \\
&\quad (t_{i(n+1)})M]d(T^{p(n+1)}_{i(n+1)}(t_{i(n)})y_n, x_n) + v_{n+1}(t_{i(n+1)})
\end{aligned}
$$

Using the uniform asymptotic regularity of ∂, (4.2.12) and our assumption above, we get;

$$
\begin{aligned}
d(x_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})x_n) &\leq d(x_n, y_n) + d(y_n, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})y_n) + \sup_{s\in\Re^+, z\in\{y_n\}} \\
&\quad d(T^{p(n+1)}_{i(n+1)}(t_{i(n+1)})z, T^{p(n+1)}_{i(n+1)}(t_{i(n+1)} + s)z) + [1 + \\
&\quad u_{n+1}(t_{i(n+1)})M]d(T^{p(n+1)}_{i(n+1)}(t_{i(n)})y_n, x_n) + v_{n+1} \\
&\quad (t_{i(n+1)}) \longrightarrow 0 \quad as \quad n \to \infty
\end{aligned}
\tag{4.2.14}
$$

Hence, from (4.2.13),

$$
\begin{aligned}
d(x_{n+N}, x_n) &\leq d(x_{n+N}, x_{n+N-1}) + d(x_{n+N-1}, x_{n+N-2}) + d(x_{n+N-2}, x_{n+N-3}) \\
&\quad + \cdots + d(x_{n+2}, x_{n+1}) + d(x_{n+1}, x_n) \longrightarrow 0 \quad as \quad n \to \infty
\end{aligned}
\tag{4.2.15}
$$

Next, we show that $\forall t \geq 0$;

$$\lim_{n\to\infty} d(x_n, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)})T_{i(n+N-2)}(t_{i(n)})\cdots$$
$$T_{i(n+2)}(t_{i(n)})T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})x_n) \quad = \quad 0,$$

Using (4.2.15), it suffices to show that $\forall t \geq 0$;

$$\lim_{n\to\infty} d(x_{n+N}, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)})T_{i(n+N-2)}(t_{i(n)})\cdots$$
$$T_{i(n+2)}(t_{i(n)})T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})x_n) \quad = \quad 0,$$

But,

$$\begin{aligned}
d(x_{n+N-1}, T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1}) \quad &\leq \quad d(x_{n+N-1}, x_{n+N}) + d(x_{n+N}, T^{p(n+N)}_{i(n+N)} \\
&\quad (t_{i(n+N)})x_{n+N-1}) \\
&= \quad d(x_{n+N-1}, x_{n+N}) + d((1-\beta_{n+N-1}) \\
&\quad y_{n+N-1} \oplus \beta_{n+N-1}T^{p(n+N)}_{i(n+N)}(t_{i(n+N)}) \\
&\quad y_{n+N-1}), T^{p(n+N)}_{i(n+N)}(t_{i(n+N)}), x_{n+N-1}) \\
&\leq \quad d(x_{n+N-1}, x_{n+N}) + (1-\beta_{n+N-1}) \\
&\quad d(y_{n+N-1}, T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1}) \\
&\quad +\beta_{n+N-1}d(T^{p(n+N)}_{i(n+N)}(t_{i(n+N)}) \\
&\quad y_{n+N-1}), T^{p(n+N)}_{i(n+N)}(t_{i(n+N)}), x_{n+N-1}) \\
&\leq \quad d(x_{n+N-1}, x_{n+N}) + (1-\beta_{n+N-1}) \\
&\quad [\alpha_{n+N-1}d(0, T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1}) \\
&\quad +(1-\alpha_{n+N-1})d(x_{n+N-1}, T^{p(n+N)}_{i(n+N)}(t_{i(n+N)}) \\
&\quad x_{n+N-1})] + \beta_{n+N-1}(1+u_{n+N}(t_{n+N})M) \\
&\quad d(y_{n+N-1}, x_{n+N-1}) + \beta_{n+N-1}v_{n+N}(t_{n+N})
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\alpha_{n+N-1}+\beta_{n+N-1}(1-\alpha_{n+N-1})}d(x_{n+N-1},x_{n+N})+\\
&\frac{(1-\beta_{n+N-1})\alpha_{n+N-1}}{\alpha_{n+N-1}+\beta_{n+N-1}(1-\alpha_{n+N-1})}d(0,T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})\\
&x_{n+N-1})+\frac{\beta_{n+N-1}}{\alpha_{n+N-1}+\beta_{n+N-1}(1-\alpha_{n+N-1})}d(x_{n+N-1}\\
&[(1+u_{n+N}(t_{n+N})M)d(y_{n+N-1},x_{n+N-1})+v_{n+N}(t_{n+N}]\\
&\longrightarrow 0 \quad as \quad n\to\infty. \qquad (4.2.16)
\end{aligned}
$$

By (4.2.13), (4.2.14) and (4.2.3) we have the above expression. Now, since $\{T_{i(n)}(t_{i(n)})\}$ is Lipschitz with constant $L_{i(n)}(t_{i(n)})$ for each $i(n)\in\{1,2,3,...,N\}$ and for $L(t):=max_{1\leq i(n)\leq N}\{L_{i(n)}(t_{i(n)})\}$ for any positive number $n\geq 1$ and $t\geq 0$, $n=(p(n)-1)N+i(n)$, we have;

$$
\begin{aligned}
d(x_{n+N-1},T_{i(n+N)}(t_{i(n+N)})x_{n+N-1}) &\leq d(x_{n+N-1},T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1})+d\\
&(T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1},T^{p(n+N)}_{i(n+N)}(t_{i(n)})\\
&T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1})+d(T^{p(n+N)}_{i(n+N)}\\
&(t_{i(n)})T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1},T_{i(n+N)}\\
&(t_{i(n+N)})x_{n+N-1})\\
&= d(x_{n+N-1},T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1})\\
&+d(T^{p(n+N)}_{i(n+N)}(t_{i(n+N)})x_{n+N-1},T^{p(n+N)}_{i(n+N)}\\
&(t_{i(n+N)}+t_{i(n)})x_{n+N-1})+d(T^{p(n+N)}_{i(n+N)}\\
&(t_{i(n+N)}+t_{i(n)})x_{n+N-1},T_{i(n+N)}(t_{i(n+N)})\\
&x_{n+N-1})
\end{aligned}
$$

Since, for any $t\geq 0$ and $\{T_{i(n)}(t):t\geq 0\}$ is a uniformly asymptotic regular family, then we have that;

$$\lim_{n\to\infty} d(T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})x_{n+N-1}, T_{i(n+N)}^{p(n+N)}(t_{i(n+N)}+t)x_{n+N-1})$$
$$\leq \lim_{n\to\infty}\sup_{z\in K} d(T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})z, T_{i(n+N)}^{p(n+N)}(t_{i(n+N)}+t)z) = 0.$$

where K is a closed, convex, nonempty subset of X containing $\{x_n\}$ and $\{x_{n+N-1}\}$ (bounded sequences). Since $\{T_{i(n)}(t)\}$ is continuous, we get that;

$$\begin{aligned}
&\leq d(x_{n+N-1}, T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})x_{n+N-1}) + \sup_{z\in K, s\in\Re^+} d(T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})z,\\
&\quad T_{i(n+N)}^{p(n+N)}(t_{i(n+N)}+s)z) + L(t)d(T_{i(n+N)}^{p(n+N)}(t_{i(n)})T_{i(n+N)}^{p(n+N)-1}(t_{i(n+N)})\\
&\quad x_{n+N-1}, x_{n+N-1})\\
&\leq d(x_{n+N-1}, T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})x_{n+N-1}) + \sup_{z\in K, s\in\Re^+} d(T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})z,\\
&\quad T_{i(n+N)}^{p(n+N)}(t_{i(n+N)}+s)z) + L(t)[d(T_{i(n+N)}^{p(n+N)}(t_{i(n)})T_{i(n+N)}^{p(n+N)-1}(t_{i(n+N)}\\
&\quad)x_{n+N-1}, T_{i(n+N)}^{p(n+N)-1}(t_{i(n)})x_{n-1}) + d(T_{i(n+N)}^{p(n+N)-1}(t_{i(n)})x_{n-1}, x_{n-1})\\
&\quad +d(x_{n-1}, x_{n+N-1})]
\end{aligned} \tag{4.2.17}$$

Thus, since for each $n \geq N$, $n+N = n(modN)$, and also $n = (p(n)-1)N + i(n)$, so that we have; $n+N = (p(n)-1+1)N + i(n) = (p(n+N)-1) = p(n)$ and $i(n) = i(n+N)$, assume that $p(n+N) = p(n)$ also. Hence,
$d(T_{i(n+N)}^{p(n+N)}(t_{i(n)})T_{i(n+N)}^{p(n+N)-1}(t_{i(n+N)})x_{n+N-1}, T_{i(n)}^{p(n+N)-1}(t_{i(n)})x_{n-1})$

$$\begin{aligned}
&= d(T_{i(n)}^{p(n)}(t_{i(n)})T_{i(n)}^{p(n)}(t_{i(n)})x_{n+N-1}, T_{i(n)}^{p(n)}(t_{i(n)})x_{n-1})\\
&= d(T_{i(n)}^{p(n)}(2t_{i(n)})x_{n+N-1}, T_{i(n)}^{p(n)}(t_{i(n)})x_{n-1})\\
&= d(T_{i(n)}^{p(n)}(t_{i(n)})x_{n+N-1}, T_{i(n)}^{p(n)}(t_{i(n)})x_{n-1})\\
&\leq L(t)d(x_{n+N-1}, x_{n-1})
\end{aligned} \tag{4.2.18}$$

And also,

$$d(T_{i(n)}^{p(n+N)-1}(t_{i(n)})x_{n-1}, x_{n-1}) \quad = \quad d(T_{i(n)}^{p(n)}(t_{i(n)})x_{n-1}, x_{n-1}) \qquad (4.2.19)$$

Now, substituting (4.2.19), (4.2.18) into (4.2.17), we have;

$$\begin{aligned} d(x_{n+N-1}, T_{i(n+N)}(t_{i(n+N)})x_{n+N-1}) &\leq d(x_{n+N-1}, T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})x_{n+N-1}) + \\ &\sup_{z\in K, s\in\Re^+} d(T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})z, T_{i(n+N)}^{p(n+N)} \\ &(t_{i(n+N)} + s)z) + L(t)[L(t)d(x_{n+N-1}, \\ &x_{n-1}) + d(T_{i(n)}^{p(n)}(t_{i(n)})x_{n-1}, x_{n-1}) \\ &+d(x_{n-1}, x_{n+N-1})] \\ &= d(x_{n+N-1}, T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})x_{n+N-1}) \\ &+ \sup_{z\in K, s\in\Re^+} d(T_{i(n+N)}^{p(n+N)}(t_{i(n+N)})z, T_{i(n+N)}^{p(n+N)} \\ &(t_{i(n+N)} + s)z) + L^2(t)d(x_{n+N-1}, x_{n-1}) \\ &+L(t)d(T_{i(n)}^{p(n)}(t_{i(n)})x_{n-1}, x_{n-1}) + L(t) \\ &d(x_{n-1}, x_{n+N-1}) \end{aligned}$$

Using the definition of ∂, which is uniformly asymptotic regular, (4.2.15) and (4.2.16), we get that;

$$\lim_{n\to\infty} d(x_{n+N-1}, T_{i(n+N)}(t_{i(n+N)})x_{n+N-1}) \quad = \quad 0 \qquad (4.2.20)$$

Thus,

$$\begin{aligned} d(x_{n+N}, T_{i(n+N)}(t_{i(n+N)})x_{n+N-1}) &\leq d(x_{n+N}, x_{n+N-1}) + d(x_{n+N-1}, T_{i(n+N)} \\ &(t_{i(n+N)})x_{n+N-1}) \longrightarrow 0 \quad as \quad n\to\infty \qquad (4.2.21) \end{aligned}$$

Indeed, since $\{T_{i(n)}(t_{i(n)})\}$ is Lipchitzian, by using (4.2.21) above, we get;

$$d(x_{n+N}, T_{i(n+N)}(t_{i(n)})x_{n+N-1}) \longrightarrow 0 \quad as \quad n \to \infty$$

$$d(T_{i(n+N)}(t_{i(n)})x_{n+N-1}, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)})x_{n+N-2}) \longrightarrow 0 \quad as \quad n \to \infty$$

$$\vdots$$

$$d(T_{i(n+N)}(t_{i(n)}) \cdots T_{i(n+2)}(t_{i(n)})x_{n+1}, T_{i(n+N)}(t_{i(n)}) \cdots T_{i(n+1)}(t_{i(n)})x_n)$$

$$\longrightarrow 0 \quad as \quad n \to \infty$$

Adding them all together, we have;

$$d(x_{n+N}, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)})T_{i(n+N-2)}(t_{i(n)}) \cdots$$
$$T_{i(n+2)}(t_{i(n)})T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})x_n) \longrightarrow 0 \quad as \quad n \to \infty.$$

Which implies from (4.2.15) that;

$$d(x_n, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)})T_{i(n+N-2)}(t_{i(n)}) \cdots$$
$$T_{i(n+2)}(t_{i(n)})T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})x_n) \longrightarrow 0 \quad as \quad n \to \infty.$$

Also, since $\{T_{i(n)}(t_{i(n)})\}$ is Lipschitz with constant $L_{i(n)}(t_{i(n)})$ for each $i(n) \in \{1, 2, 3, ..., N\}$ and for $L(t) := max_{1 \le i(n) \le N}\{L_{i(n)}(t_{i(n)})\}$, we have;
$d(y_n, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)}) \cdots T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})y_n)$

$$\begin{aligned}
\le \;& d(y_n, x_n) + d(x_n, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)}) \cdots T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})x_n) \\
& + d(T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)}) \cdots T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})x_n, T_{i(n+N)}(t_{i(n)}) \\
& T_{i(n+N-1)}(t_{i(n)})T_{i(n+N-2)}(t_{i(n)}) \cdots T_{i(n+2)}(t_{i(n)})T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})y_n)) \\
\le \;& d(y_n, x_n) + L^n(t)d(y_n, x_n) + d(x_n, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)}) \cdots T_{i(n+1)} \\
& (t_{i(n)})T_{i(n)}(t_{i(n)})x_n)
\end{aligned}$$

$$\leq [1+L^n(t)]d(y_n,x_n) + d(x_n, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)})\cdots T_{i(n+1)}(t_{i(n)}) T_{i(n)}(t_{i(n)})x_n) \longrightarrow 0 \quad as \quad n\to\infty.$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$, which converges to a point $p \in \mathcal{F}$. By lemma 3.1.2, we may also assume that $\{x_{n_j}\}$ $\Delta-$converges to a point $p \in K$. Now, since the family of semigroup is finite, passing to a further subsequence if needed be, we may assume that for some $i(n) \in \{1,2,3,...,N\}$, it follows from lemma 3.1.6 that;

$$d(x_{n_j}, T_{i(n+N)}(t_{i(n)})T_{i(n+N-1)}(t_{i(n)})T_{i(n+N-2)}(t_{i(n)})\cdots T_{i(n+2)}(t_{i(n)})T_{i(n+1)}(t_{i(n)})T_{i(n)}(t_{i(n)})x_{n_j}) \longrightarrow 0 \quad as \quad j\to\infty.$$

Finally, we show that $x_n \longrightarrow p$, since $d(y_n,p) \leq D_1$ and since $D_1 > 0$, using (4.2.8) above, we obtain;

$$\begin{aligned} d^2(x_{n+1},p) &\leq (1-\alpha_n)^2 d^2(x_n,p) + \alpha_n^2 d^2(0,p) + \beta_n \overline{u_{n+1}(t_{n+1})}D^2 \\ &+\beta_n v_{n+1}(t_{n+1})[2(1+u_{n+1}(t_{n+1})M)D + v_{n+1}(t_{n+1})] \\ &+2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \end{aligned} \tag{4.2.22}$$

Thus,

$$d^2(x_{n+1},p) \leq (1-\alpha_n)^2 d^2(x_n,p) + \alpha_n\gamma_n + \sigma_n, \quad \forall n \geq 1$$

where;

$$\begin{aligned} \gamma_n &:= \alpha_n d^2(0,p) + 2(1-\alpha_n)\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \\ \sigma_n &:= \beta_n v_{n+1}(t_{n+1})[2(1+u_{n+1}(t_{n+1})M)D + v_{n+1}(t_{n+1})] + \beta_n\overline{u_{n+1}(t_{n+1})}D^2 \end{aligned}$$

With $\Sigma_{n=1}^{\infty}\sigma_n < \infty$. Since $\{x_{n_j}\}$ $\Delta-$converges to a point $p \in K$, then by lemma 3.1.7, we have $\limsup_{n\to\infty}\langle\overrightarrow{0p},\overrightarrow{x_np}\rangle \leq 0$, then it also follows from lemma 4.1.1 that $\{x_n\}$ converges strongly to a point $p \in \mathcal{F}$. Consequently, $\{y_n\}$ also converges strongly to a point $p \in \mathcal{F}$.

CASE 2: suppose that $\{d(x_n, p)\}_{n=1}^{\infty}$ is not a monotone decreasing sequence, then lets set $\Upsilon_n := d^2(x_n, p)$ and let

$$\tau_n := max\{k \in N : k \leq n, \Upsilon_k \leq \Upsilon_{k+1}\}.$$

Then, τ is a non-decreasing sequence such that $\tau_n \longrightarrow \infty$ as $n \to \infty$ and $\Upsilon_k \leq \Upsilon_{k+1}$, for $n \geq N_0$ Now, following the same argument as in the first case above, we obtain that;

$$d(x_{\tau(n)}, T_{i(\tau(n+N))}(t_{i(\tau(n))})T_{i(\tau(n+N-1))}(t_{i(\tau(n))})T_{i(\tau(n+N-2))}(t_{i(\tau(n))})\cdots$$
$$T_{i(\tau(n+2))}(t_{i(\tau(n))})T_{i(\tau(n+1))}(t_{i(\tau(n))})T_{i(\tau(n))}(t_{i(\tau(n))})x_{\tau(n)}) \longrightarrow 0 \quad as \quad n \to \infty.$$

And also,

$$d(y_{\tau(n)}, T_{i(\tau(n+N))}(t_{i(\tau(n))})T_{i(\tau(n+N-1))}(t_{i(\tau(n))})T_{i(\tau(n+N-2))}(t_{i(\tau(n))})\cdots$$
$$T_{i(\tau(n+2))}(t_{i(\tau(n))})T_{i(\tau(n+1))}(t_{i(\tau(n))})T_{i(\tau(n))}(t_{i(\tau(n))})y_{\tau(n)}) \longrightarrow 0 \quad as \quad n \to \infty.$$

As in the first case, we also obtain that $\{x_{\tau(n)}\}$ $\Delta-$converges to some point $p \in \mathcal{F}$. Furthermore, for all $n \geq N_0$, we also obtain from (4.2.22) above that;

$$\begin{aligned} 0 &\leq d^2(x_{\tau(n+1)}, p) - d^2(x_{\tau(n)}, p) \\ &\leq \alpha_{\tau(n)}[\alpha_{\tau(n)})^2 d^2(0, p) + 2(1-\alpha_{\tau(n)})\langle\overrightarrow{0p},\overrightarrow{x_{\tau(n)}p}\rangle \\ &\quad +\beta_{\tau(n)}\frac{v_{\tau(n)+1}(t_{\tau(n)+1})}{\alpha_{\tau(n)}}[2(1+u_{\tau(n)+1}(t_{\tau(n)+1})M)D \end{aligned}$$

$$+v_{\tau(n)+1}(t_{\tau(n)+1})] + D^2\beta_{\tau(n)}\frac{\overline{u_{\tau(n+1)}(t_{\tau(n)+1})}}{\alpha_{\tau(n)}} - d^2(x_{\tau(n)}, p)] \tag{4.2.23}$$

It then follows from the above that;

$$\begin{aligned} d^2(x_{\tau(n)}, p) &\leq \alpha_{\tau(n)}[\alpha_{\tau(n)})^2 d^2(0, p) + 2(1 - \alpha_{\tau(n)})\langle\overrightarrow{0p}, \overrightarrow{x_{\tau(n)}p}\rangle \\ &+\beta_{\tau(n)}\frac{v_{\tau(n)+1}(t_{\tau(n)+1})}{\alpha_{\tau(n)}}[2(1 + u_{\tau(n)+1}(t_{\tau(n)+1})M)D \\ &+v_{\tau(n)+1}(t_{\tau(n)+1})] + D^2\beta_{\tau(n)}\frac{\overline{u_{\tau(n+1)}(t_{\tau(n)+1})}}{\alpha_{\tau(n)}}] \end{aligned} \tag{4.2.24}$$

Thus,

$$\lim_{n\to\infty} \Upsilon_{\tau(n)} = \lim_{n\to\infty} \Upsilon_{\tau(n)+1}$$

Furthermore, for all $n \geq N_0$, we have that; $\Upsilon_k \leq \Upsilon_{k+1}$, if $n \neq \tau(n)$, (i.e $\tau(n) < n$) because, $\Upsilon_j > \Upsilon_{j+1}$, for $\tau(n) \leq j \leq n$. it follows that $\forall n \geq N_0$, we have;

$$\begin{aligned} 0 &\leq \Upsilon_n \\ &\leq max\{\Upsilon_{\tau(n)}, \Upsilon_{\tau(n)+1}\} \\ &= \Upsilon_{\tau(n)+1} \end{aligned}$$

This implies that $\lim_{n\to\infty} \Upsilon_n = 0$ and hence, $\{x_n\}$ converges strongly to a point $p \in \mathcal{F}$. This completes the proof. □

Example 4.2.1 *Here, we discuss the direct application of theorem 4.1 on a typical example on the real line space. Consider the following: Let* $X = \mathbb{R}$ *and* $K = [0, \frac{1}{2}]$, *where* $K \subset X$. *Choose* $T(t)x = \frac{9}{10}x$, $\alpha_n = \frac{1}{2n}$ *and* $\beta_n = \dfrac{n+1}{4n-1}$. *Then, the modified Mann iterative scheme (4.2.2) above can be simplified as* *follows:*

$$\begin{aligned} y_n &= P_K((1-\alpha_n)x_n) \\ &= P_K((1-\frac{1}{2n})x_n) \\ &= (\frac{2n-1}{2n})x_n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

And also,

$$\begin{aligned} x_{n+1} &= (1-\beta_n)y_n \oplus \beta_n T_{i(n+1)}^{p(n+1)}(t_{i(n+1)})y_n \\ &= (1-\frac{n+1}{4n-1})(\frac{2n-1}{2n})x_n + (\frac{n+1}{4n-1})\frac{9}{10}(\frac{2n-1}{2n})x_n \\ &= (\frac{6n^2-7n+2}{8n^2-2n})x_n + (\frac{9(2n^2+n-1)}{10(8n^2-2n)})x_n, \quad \forall n \in \mathbb{N}. \end{aligned}$$

Take the initial point $x_1 = \frac{1}{10}$*, the numerical example using* $MATLAB$ *converges to zero(0). Hence, the iteration process of the sequence* $\{x_n\}$ *converges to zero(0), by using theorem 4.1.4 also.*

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

In this chapter, we present a summary and conclusion of the results presented in this dissertation. We also present a list of recommendations for future research in the line of problems addressed.

5.1 Summary

In conducting and compiling this dissertation, we were able to discuss the following results:

In the first chapter, we introduced the concept of fixed point theory, hyperbolic space, metric space, $CAT(0)$ space and semigroup of nonexpansive mappings, which we were able to do with the help of some important definitions and examples where necessary. The methodology and concepts used helped in achieving the desired results in the third and fourth chapters respectively.

In the second chapter, we gave some literature review on the concept studied in this dissertation, which includes; fixed point theorems and semigroup of the one parameter family of nonexpansive mappings.

In the third chapter, using an iterative scheme, we were able to establish the polar and $\Delta-$ convergence theorems for uniformly asymptotic regular finite family of total asymptotically quasi-nonexpansive semigroup in a uniformly convex hyperbolic space and also in a complete $CAT(0)$ space.

In the fourth chapter, we studied and used the modified Mann iterative scheme for approximating common fixed point of a uniformly asymptotic regular family of uniformly L-Lipschitzian and total asymptotically quasi-nonexpansive semigroup in a complete $CAT(0)$ space.

In the last chapter, we summarized and concluded the dissertation and provided some directions for future findings and research.

5.2 Conclusion

In this study, the iterative schemes considered in the third and fourth chapters were used to show three (3) important convergence of the semigroup of the one parameter family of nonexpansive mappings. These convergence includes: the strong convergence, the polar convergence and the $\Delta-$convergence. And also, the approximation of common fixed point theorems of such mappings have been established. Hence, we conclude that convergence can be obtained in a uniformly convex hyperbolic space and also in a complete $CAT(0)$ space, using both the concept of the semigroup of the one parameter family of nonexpansive mappings and that of uniformly asymptotic regularity of finite families of a map.

5.3 Recommendations

In this study, we were able to achieve the desired objectives by showing: strong, polar and $\Delta-$convergence in $CAT(0)$ spaces. With this, we recommend that this dissertation could be used to improve the study of the hyperbolic ($CAT(0)$) spaces by future researchers. The points listed below, will help in improving the study of the semigroup of the one parameter family of nonexpansive mappings in $CAT(0)$ spaces.

1 The convergence in the $CAT(0)$ space should be extended to common fixed point of a finite family of non-self mappings of total asymptotically nonexpansive semigroup.

2 Results involving variational inequality problem (V.I.P) and equilibrium problem (E.P) should be studied in the settings of the $CAT(0)$ spaces.

These are but a few directions we have for future research/investigations.

REFERENCES

Abramenco, P. and Brown, K. S. (2008). *Buildings: Theory and Applications*, volume 248. Springer New York.

Alber, Y. I., Chidume, C. E., and Zegeye, H. (2006). Approximating fixed points of total asymptotically nonexpansive mappings. *Fixed Point Theory and Application*, 44.

Ali, B. (2016). Convergence theorem for finite families of total asymptotically nonexpansive mappings in hyperbolic spaces. *Fixed points Theory and Applications.*

Anakkanmatee, W. and Dhompongsa, S. (2011). Rode's theorem on common fixed points of semigroup of nonexpansive mappings in cat(0) spaces. *Fixed Point Theory and Applications*, 34.

Ballmann, W. (1995). *Lectures on spaces of Nonlinear Curvature.* Birkhauser, Basel.

Banach, S. (1932). Theorie des operations lineires. *Warsaw.*

Basarir, M. and Sahin, A. (2013). On the strong and δ-convergence of new multistep and s-iteration processes in a cat(0)space. *Journal of Inequalities and Applications*, page 2013:482.

Branciari, A. (2000). A fixed point theorem of banach-cacciopoli type on a class of generalized metric spaces. *Publ. math.Debrecen*, 57:31–37.

Bridson, M. R. and Haefliger, A. (1999). *Metric spaces of non-positive curvature.* Springer, Berlin,Germany,.

Browder, F. E. and Petryshyn, W. V. (1967). Construction of fixed points of nonlinear mappings in hilbert spaces. *J. Math. Anal. Appl.*, 20:197–228.

Brown, K. S. (1989). *Buildings.* Springer,New York, NY, USA.

Burago, D., Burago, Y., and Ivanov, S. (2001). A course in metric geometry: Graduate studies in math. *Amer. Math. Soc., Providence, RI*, 33.

Caccioppoli, R. (1931). Un teorem generale sull'essistenza di el-ementiliaiti in una transformazione funzionale. *Rend. Accad. Naz. Lincei*, 13:498–502.

Chang, S. S., Wang, L., Lee, H. W. J., Chan, C. K., and Yang, L. (2012). Demiclosed principle and $\delta-$convergence theorems for total asymptotically nonexpansive mappings in cat(0) spaces. *Appl. Math. Comput.*, 219:2611–2617.

Chang, S. S., Wang, L., Lee, H. W. J., Chan, C. K., and Yang, L. (2013). Strong and $\delta-$convergence for mixed type total asymptotically nonexpansive mappings in cat(0) spaces. *Fixed Point Theory and Appl.*, 122.

Chidume, C. E. and Ali, B. (2007). Application of common fixed points of finite families of non-self asymptotically non-expansive mappings in banach spaces. *J. Math. Anal. Appl.*, 326:960–973.

Chidume, C. E. and Chidume, C. O. (2006). Iterative approximatins of fixed points of nonexpansive mappings. *J. Math. Anal. Appl.*, 318:288–295.

Chidume, C. E. and Ofeodu, E. U. (2007). Approximations of common fixed points for finite families of total asymptotically nonexpansive mappings. *J. Math. Anal. Appl.*, 333:128–141.

Chidume, C. E. and Ofeodu, E. U. (2009). A new iteration processes for approximations of common fixed points for finite families of total asymptotically nonexpansive mappings. *Inter. J. Math. and Math. Sci.*, 2009:17 pages.

Dehghan, H. and Rooin, J. (2014). Metric projections and convergence theorems for nonexpansive mappings in hadamard spaces. *[math.FA]*, 1.

DeMarr's, R. (1963). Common fixed points for commuting contraction mappings. *Pacific J. Math.*, 13:1139–1141.

Dhompongsa, S., Kirk, W. A., and Panyanak, B. (2007). Nonexpansive set-valued mappings in metric and banach spaces. *J. Nonlinear Convex Anal.*, 8:35–45.

Dhompongsa, S., Kirk, W. A., and Sims, B. (2006). Fixed points of uniformly lipschitzian mappings. *Nonlinear Anal.*, 65:762–772.

Dhompongsa, S. and Panyanak, B. (2008). On $\delta-$convergence theorems in cat(0) spaces. *Comput. Math. Appl.*, 56:2572–2579.

Goebel, K. and Kirk, W. A. (1972). A fixed point theorem for asymptotically nonexpansive mappings. *Proc. Amer. Math. Soc.*, 35:171–174.

Halpern, B. (1967). Fixed points of nonexpansive maps. *Bull. Amer. Math. Soc.*, 3:957–961.

Kakavandi, B. A. (2012). weak topologies in complete cat(0) spaces. *Proc. Amer. Math. Soc.*, s 0002-9939:117–435.

Kakavandi, B. A. and Amini, M. (2011). Non-linear ergodic theorem in complete non-positive curvature metric space. *Bull. Iran. Math. soc.*, 37:11–20.

Khamsi, M. A. and Khan, A. R. (2011). Inequalities in metric spaces with applications. *Nonlinear Anal.*, 74:4036–4045.

Kim, H. S. and Kim, T. H. (1987). Asymptotic behavior of semigroups of asymptotically nonexpansive type on banach spaces. *J. Korean Math. Soc.*, 24:169–178.

Kirk, W. A. (1981). *Locally nonexpansive mappings in Banach spaces*, volume 886. Springer-Verlag, Berlin.

Kirk, W. A. (2003). Geodesic geometry and fixed point theory in seminar of analysis (malaga/seville, 2002/2003). *Colecc. Abierta*, 64:195–225.

Kirk, W. A. (2004). Fixed point theorems in cat(0) spaces and r-tree. *Fixed Point Theory and Application*, 4:309–316.

Kirk, W. A. and Panyanak, B. (2008). A concept of convergence in geodesic spaces. *Nonlinear Anal.*, 68:3689–3696.

Kohlenbach, U. and Leustean, L. (2010). Asymptotically nonexpansive mapping in uniformly convex hyperbolic spaces. *J. Eur. Math. Soc.*, 12:71–92.

Krasnonelskii, M. A. (1955). Two observations about the methods of successive approximations. *Uspehi Math. Nauk*, 10:123–127.

Lau, A. and Takahashi, W. (1990). Invariant means and semigroup of nonexpansive mappings on uniformly banach spaces. *J. Math. Anal. Appl.*, 153:497–505.

Lau, A. and Takahashi, W. (1996). Invariant submeans and semigroup of nonexpansive mappings on uniformly banach spaces with normal structure. *J. Funct. Anal.*, 143:70–88.

Lau, A. and Takahashi, W. (2003). Nonlinear submeans on semigroups. *Topo. Methods Nonlinear Anal.*, 22:345–353.

Lau, A. T. (1973). Invariant means on almost periodic functions and fixed point properties. *Rocky Mountain J. Math*, 3:69–76.

Lau, A. T. and Takahashi, W. (1987). Weak convergence and nonlinear ergodic theorems for reversible semigroups of nonexpansive mappings. *Pacific J. Math*, 126:177–194.

Lim, T. C. (1976). Remarks on some fixed point theorems. *Proc. Amer. Math. Soc.*, 60.

Lion, P. L. (1977). Approximations de points fixed de contractions. *C. R. Acad. Sci. Paris Ser. A*, 284:1357–1359.

Mann, W. R. (1953). Mean valued methods in iteration. *Proc. Amer. Math. Soc.*, 4:506–510.

Marino, G. and Xu, H. K. (2006). A general iterative method for nonexpansive mappings in hilbert space. *J. Math. Anal. Appl.*, 318:43–52.

Mitchell, T. (1970). Fixed points of reversible semigroup of nonexpansive mappings. *Kodai Math. Semi. Rep.*, 22:322–323.

Morales, C. H. and Jung, J. S. (2000). Convergence of paths for pseudo-contractive mappings in banach spaces. *J. Math. Anal. Appl.*, 128:3411–3419.

Picard, E. (1893). Sur i'application des methodes d'approximations successive a l'etude de certaines equations differentielles ordinaires. *Journal of Mathematics*, 9:217–271.

Reich, S. (1980). Strong convergent theorems for resolvents of accretive operators in banach spaces. *J. Math. Anal. Appl.*, 75:287–292.

Reich, S. (1994). Approximating fixed points of nonexpansive mappings. *Panamer Math. J.*, 4:23–28.

Reich, S. and Shafrir, I. (1990). Nonexpansive iterations in hyperbolic spaces. *Nonlinear Anal.*, 15:537–558.

Rode, G. (1982). An ergodic theorem for semigroups of nonexpansive mappings in hilbert space. *J. Math. Anal. Appl.*, 85:172–178.

Schaefer, H. H. (1957). Uber die methode sukzessiver approximationen. *Jber. Deutch Math. Verein*, 59:383–389.

Shimoji, K. and Takahashi, W. (2001). Strong convergence to common fixed points of infinite nonexpansive mappings and applications. *Taiwanese J. Math.*, 5:387–404.

Shioji, S. and Takahashi, W. (1997). Strong convergence of approximated sequences for nonexpansive mappings in banach spaces. *Proc. Amer. Math. Soc.*, 125:3641–3645.

Staples, J. (1976). Fixed point theory in a uniformly rotund metric spaces, 14 (1976). *Bull. Austral. Math. Soc.*, 14:181–192.

Suantai, S. and Phuengrattana, W. (2014). Fixed point theorem for a semigroup of total asymptotically nonexpansive mappings in uniformly banach space. *Opuscula Mathimatica*, 34(1):183–197.

Takahashi, W. (1969). Fixed point theorem for amendable semigroups of nonexpansive mappings. *Kodai Math. Sem. Rep.*, 21:383–386.

Takahashi, W. (1981). A nonlinear ergodic theorem for an amendable semigroup of nonexpansive mappings in a hilbert space. *Proc. Amer. Maths. Soc.*, 81:253–256.

Takahashi, W. and Zhang, P. J. (1988). Asymptotic behavior of almost-orbits of semigroups of lipschitzian mappings in banach spaces. *Kodai Math. J.*, 11:129–140.

Takahashi, W. and Zhang, P. J. (1989). Asymptotic behavior of almost-orbits of semigroups of lipschitzian mappings. *J. Math. Anal. Appl.*, 142:242–249.

Tan, K. K. and Xu, H. K. (1993). Approximating fixed points of nonexpansive mappings by ishikawa iteration process. *J. Math. Anal. Appl.*, 178:301–308.

Ugwunnadi, G. C. and Ali, B. (2016). Approximation of common fixed points of total asymptotically nonexpansive mapping in cat(0) spaces. *Inter, Publ. USA*, 19:36–47.

Wittmann, R. (1992). Approximation of fixed points of nonexpansive mappings. *Arch. Math.(Basel)*, 58:486–491.

Xu, H. K. (2004). Viscosity approximation methods for nonexpansive mappings. *J. Math. Anal. Appl.*, 298:279–291.

Yao, Y. and Shahzad, N. (2011). New methods with perturbations for nonexpansive mappings in hilbert spaces. *Fixed Point Theory and Appl.*, 79.

Yao, Y., Zhou, H., and Liou, Y. C. (2009). Strong convergence of a modified krasnonelski-mann iteration algorithm for nonexpansive mappings. *J. Appl. Math.*, 29:383–389.