A STUDY OF PROPERTIES AND APPLICATIONS OF WEIBULL-BURR XII DISTRIBUTION

By

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DEPARTMENT OF STATISTICS,

AHMADU BELLO UNIVERSITY, ZARIA,

NIGERIA

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DECLARATION

I declare that the work in this thesis entitled "A Sudy of Properties and Applications of Weibull-Burr XII Distribution" has been performed by me in the Department of Statistics, Ahmadu Bello University, Zaria under the supervision of Dr. A. Yahaya and Prof. O.E. Asiribo. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

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Sa′ad MOHAMMED

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Date

CERTIFICATION

This thesis titled "A Sudy of Properties and Applications of Weibull-Burr XII Distribution" by Sa'ad MOHAMMED meets the regulations governing the award of the degree of Masters of Science in Statistics of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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Dr. A. Yahaya Date
Chairman, Supervisory Committee

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Member, Supervisory Committee

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Dean Postgraduate School

DEDICATION

This research is dedicated to my Father, Mal. Sa'ad Sani (Late)

ACKNOWLEDGEMENT

My sincere thanks are to the members of staff of Statistics Department, ABU Zaia. I am garteful to my supervisor Dr. A. Yahaya whose advice led to the achievement of this research. I am equally grateful to all my course mates for their contributions. Lastly, I exprees my gratitude to all my family members especially our elder brother Alh. Mohammed Sa'ad Toro for their support throughout this programe.

ABSTRACT

In recent times, lots of efforts have been made to define new probability distributions that cover different aspect of human endeavors with a view to providing alternatives in modeling real data. A five-parameter distribution, called Weibull-Burr XII (Wei-Burr XII) distribution is studied and investigated to serve as an alternative model for skewed data set in life and reliability studies. Some of its statistical properties are obtained, these include moments, moment generating function, characteristics function, quantile function and reliability (survival) functions. The distribution's parameters are estimated by the method of maximum likelihood. We evaluated the performance of the new distribution compared with other competing distributions based on application on real data and it was concluded that Weibull-Burr XII distribution perfom best using BIC, AIC and CAIC. It was also concluded that the distribution can be used to model highly skewed data (skewed to the right)

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CHAPTER ONE

INTRODUCTION

1.0.1 Background of the study

Probability distributions are recently receiving alot of attention with regards to introducing new generators for univariate continuous type of probability distributions by introducing additional parameter(s) to the base line distribution. This seemed necessary to reflect current realities that are not captured by the conventional probability distributions since it has been proven to be useful in exploring tail properties of the distribution under study (Tahir, et.al; 2016).

This idea of adding one or more parameter(s) to the baseline distribution has been in practice for a quite long time. Several distributions have been proposed in the literature to model lifetime data. Some of these distributions include: a two-parameter exponential-geometric distribution introduced by Adamidis and Loukas in 1998 which has a decreasing failure rate. Following the same idea of the exponential geometric distribution, the exponential-Poisson distribution was introduced by Kus (2007) with also a decreasing failure rate and discussed some of its properties. Marshall and Olkin (1997) presented a simpler technique for adding a parameter to a family of distributions with application to the exponential and Weibull families. Adamidis et al. (2005) suggested the extended exponential-geometric (EEG) distribution which generalizes the exponential geometric distribution and discussed some of its statistical properties along with its hazard rate and survival functions.

Some of the well-known class of generators include the following: Kumaraswamy-G (Kw-G) proposed by Cordeiro and de Castro (2011), McDonald-G (Mc- G) introduced by Alexander et al. (2012), gamma-G type 1 presented by Zografos and Balakrishanan (2009), exponentiated generalized (exp-G) which was derived by Cordeiro et al. (2013), others are weibull-power function by Tahir et. al. (2010), exponentiated T-X proposed by Alzaghal et al.(2013). Most recently, a New Weibull-G Family of Distributions by Tahir, (2016), The Weibull–G family of probability distributions by Bourguignon et al. (2014). This research is motivated by the work done by Bourguignon et al. (2014) - The Weibull–G family of probability distributions who introduced a generator based on the Weibull random variable called a Weibull-G family. In this research, we propose an extension of the Burr XII pdf called the Weibull-Burr XII distribution based on the Weibull-G class of distributions defined by Bourguignon et al (2014). i.e. we propose a new distribution with five parameters, referred to as the Weibull-Burr XII (Wei-BXII) distribution, which contains as special sub-models the Weibull and Burr XII distributions.

1.0.2 Statement of the problem

It has been anticipated that a generalized model is more flexible than a conventional or ordinary model and its applicability is preferred by many data analysts in analyzing statistical data. It is imperative to mention that through generalizations, the convetional logistic distribution with only two parameters (location and scale) has been propagated into type I, type II and type III generalized logistic distributions which has three parameters each as indicated in Balakrishnan and Leung (1988). So, there is a genuine desire to search for some generalizations or modifications of the Burr XII distribution that can provide more flexibility in lifetime modeling.

1.0.3 Purpose of the Study

Existing literature focus on generalizations or modifications of the Weibull distribution that can provide more flexibility in modeling lifetime data such as; Weibull-Log logistic distribution by Broderick (2016), Weibull-Lomax distribution by Tahir, (2015), etc. Less attention is given to generalization of Weibull and Burr XII distributions. Where the later distribution was discovered by Burr in1942 as a two parameter family. An additional scale parameter was introduced by Tadikamalla in 1980. It is a very popular distribution for modelling lifetime data.

The purpose of this research focuses mainly on generalization of a Burr XII distribution to a five-parameter distribution, called the Weibull-Burr XII (Wei-BurrXII) distribution for modelling skewed data set (skewed to the right).

1.0.4 Aim and Objectives of the Study

The aim of this research is to study Weibull-Burr XII probability distribution and investigate its properties and applications. This is expected to be achieved through the following objectives by:

1. establishing the Weibull-Burr XII distribution;
2. establishing some statistical properties of Weibull-Burr XII distribution such as; moments, moment generating function, quantile function, characteristics function, survival function and hazard rate function;
3. estimating the parameters of the proposed model by the method of maximum likelihood estimation;
4. evaluating how well the Weibull-Burr XII distribution perform when compared with other Weibull–G family of distributions based on application on real life data.

1.0.5 Significance of the study

Many models were introduced in the literature by extending some distributions with Burr XII distribution. e.g. the Beta- Burr XII (BBXII) distribution discussed by Paranaíba et al. (2011) where it was concluded that application of the Beta-BXII distribution indicated that it had provided a better fit than other statistical models used in lifetime data analysis, the Kumaraswamy -Burr XII distribution introduced by Paranaíba et. al. (2013). Therefore, the significance of this study is mainly to propose a new model (Wei-Burr XII distribution) that is much more flexible than the Burr XII distribution.

1.0.6 Limitations of the study

The limitation of this research is that, it did not consider estimating parameters of the Weibull-Burr XII distribution using other methods like Bayesian method. Some other properties of probability distribution are also not considered in this research work. e.g Rényi entropy, incomplete moments, e.t.c.

1.0.7 Definition of terms

Reliability is generally regarded as the likelihood that a product or service is functional during a certain period of time under a specified operation.

Survival function is the probability that a patient, device, or other object of interest will survive beyond a specified time. It is also known as the survivor function or reliability function.
S(x) = Pr(an object will survive beyond time x).

Hazard function (also known as the failure rate, hazard rate, or force of mortality) is the ratio of the probability density function to the survival function. Failure rate is the frequency with which an engineered system or component fails, expressed in failures per unit of time (Evans,et.al. 2000)

H(x)= Pr(an object will fail at time x+t given that it survive up to time x)

Akaike Information Criterion (AIC) is a measure of the relative quality of statistical models for a given set of data. Given a collection of models for the data, AIC estimates the quality of each model, relative to each of the other models. Hence, AIC provides a means for model selection. Given a set of candidate models for the data, the preferred model is the one with the minimum AIC value. Mathematically,

AIC $=2k\text{-}2ll$

Where ll is the log-likelihood function for the model and k is the number of estimated parameters in the model.

Bayesian Information Criterion (BIC) or Schwarz criterion is also a criterion for model selection among a finite set of models. The model with the lowest BIC is preferred. Computed by;

BIC = ln(n)k-2ll where n is the sample size and k is the number of estimated parameters in the model.

Consistent Akaike Information Criterion (CAIC) is mathematically defined by

CAIC = $-2ll+ 2kn/(n\text{-}k\text{-}1)$ where ll = log likelihood.

CHAPTER TWO

LITERATURE REVIEW

The Weibull distribution was firstly identified as a valid distribution in 1939 and was used to measure the weakening strength of materials and in 1951 for other variety of applications. The Weibull distribution is named after its discoverer, Wallodi Weibull, a Physicist from Swedish. The distribution receives many contributions since its origin. Some argue that Weibull may not be the first person to propose this distribution. The name Fréchet distribution is also sometimes used due to the fact that it was Fréchet who first identified this distribution to be an external distribution. According to Hallinan (1993), it was Weibull who proposed a scale parameter and a location parameter that added flavour to this well-known distribution.

Weibull models are among the best models that define several types of experimental failures of materials and phenomena. They are extensively used in reliability and life analysis. In addition to the traditional two-parameter and three parameter Weibull distributions in the statistics literature, many other Weibull-related distributions are available.

Over the last three decades, various articles have been published on this great distribution. Hallinan (1993) gave a perceptive review by presenting a number of past facts on the many forms of this distribution as used by practitioners, and possible confusions and errors that arise due to this non-uniqueness. Johnson et al. (1994) presented a complete chapter for a systematic understanding of this distribution. More recently, Murthy et al. (2003) presented a graph that contains approximately every feature relating to the Weibull distribution and its extensions.

On the other hand, the base line distribution called Burr Type XII distribution or simply the Burr XII distribution is also a continuous probability distribution for a non-negative random variable. It is also known as the Singh–Maddala distribution, is also receiving greater applicability in reliability phenomena and one of a number of different distributions sometimes called the generalized log-logistic distribution.

Burr XII (BXII) distribution was firstly raised by Burr Irving W. (1942) as a two parameter family. An additional scale parameter was introduced by Tadikamalla (1980). The BXII distribution, having characterized by Logistic and Weibull as parental sub models, is a very common distribution for modelling survival data and for modelling phenomenon with monotone failure rates. The BXII distribution can fit a wide range of experimental data. Different values of its parameters cover a broad set of skewness and kurtosis. Hence, it is applied in various fields of study such as finance, hydrology, engineering and reliability to model a variety of data sets. Examples of data modelled by the BXII distribution are household income, crop prices, insurance risk, travel time, flood levels, and failure data (Mead, 2014).

Other applications of Burr XII distribution include quantal response, approximation of certain distribution and development of non-normal control charts. A number of conventional theoretical distributions are limiting procedures of Burr distributions. When modeling monotone hazard rates for example, the Weibull distribution may be an initial choice because of its ability to shape negatively and positively skewed density shapes. However, it does not provide a judicious parametric fit for modeling failures rate (phenomenon) that are non-monotonic such as the bathtub shaped and the unimodal failure rates that are very common in survival and engineering studies. Such bathtub hazard curves always have nearly flat middle portions and the corresponding densities have a non-zero anti-mode. Unimodal failure rates can be seen in course of a disease whose mortality rate reaches a top after some finite period and then declines gradually (Paranaiba, 2011).This distribution has algebraic tails which are very effective for modeling phenomenon that occur with smaller frequency compared with those models based on exponential tails. Hence, Burr XII appeared to be a good candidate for modeling failure time data (Zimmer et al., 1998). Shao (2004) examined maximum likelihood estimation for the three-parameter BXII distribution. Shao et al. (2004) considered models that can be applied using the extended three parameter BXII distribution with application to flood data analysis. According to Soliman (2005), this distribution shields the curve shape characteristics for a very large number of distributions. The versatility and flexibility of the BXII distribution turns it quite attractive as a tentative model for data whose underlying distribution is predictable. The current paper by Shao et al (2004) contains a newly introduced model for statistical analysis referred to as the "extended Burr XII distribution". Shao et al. (2004) also derived various statistical properties of the distribution including its moments, quantiles functions, and estimation of parameters procedure by the method of maximum likelihood. It was further tried by means of an application to real data from China.

Many models were introduced in the literature by extending some distributions with BurrXII distribution. For example, to add flavour to the Burr XII distribution, numerous generalizations of the distribution have been proposed, the beta Burr XII (BBXII) distribution discussed by Paranaíba et al. (2011), the Kumaraswamy Burr XII distribution introduced by Paranaíba et al. (2013), and recently, McDonald Burr XII presented by Antonio et al. (2014).

CHAPTER THREE

RESEARCH METHODOLOGY

3.0.8 INTRODUCTION

We introduce a new model called the Weibull-Burr XII distribution which extends the Burr XII distribution. Various structural properties of the new distribution are derived including explicit expressions for the hazard and reliability functions, moments, moment generating function, quantile function and characteristics function. Maximum Likelihood method is applied for parameter estimation of the proposed model

3.0.9 Burr XII distribution

The three-parameter Burr XII distribution is defined by the cumulative distribution and density function. The probability density function (pdf) of the Burr XII distribution is given by;

$$g\left(x;c,k,s\right) = cks^{-c}x^{c-1}(1+\left(x/s\right)^{c})^{-k-1}, \tag{3.0.1}$$

$$x \geq 0, \quad c, \ k, s > 0.$$

The cumulative distribution function (cdf) of the Burr XII distribution

$$G\left(x;c,k,s\right) = 1-\left(1+\left(x/s\right)^{c}\right)^{-k}, \tag{3.0.2}$$

$$x \geq 0, \quad c, \ k, s > 0.$$

The suevival and hazard rate functions of the Burr XII distribution are given by

$S_B\left(x\right) = \left(1+\left(x/s\right)^{c}\right)^{-k}$ and

$$h_B(x) = cks^{-c}x^{c-1}\left(1+(x/s)^c\right)^{-1}$$

respectively.

Note that k and c are shape parameters and s is a scale parameter.

3.0.10 Weibull distribution

The probability density function of Weibull distribution is given by;

$$g(x;\alpha,\beta) = \alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, \qquad \beta>0, \alpha>0,\ x>0 \tag{3.0.3}$$

Where, $\alpha > 0$ is the shape parameter and $\beta > 0$ is the scale parameter of the distribution. The Weibull distribution is related to a number of other probability distributions; in particular, it includes the exponential distribution ($\beta = 1$)

The cumulative distribution function of Weibull distribution is given by;

$$G(x;\alpha,\beta) = 1 - e^{-\alpha x^{\beta}} \tag{3.0.4}$$

3.0.11 The pdf of the generalized Weibull-G family

The pdf of the generalized Weibull-G family of distribution is;

$$g(x;\alpha,\beta,\boldsymbol{\xi}) = \alpha\beta g(x;\boldsymbol{\xi}) \times \frac{G(x;\boldsymbol{\xi})^{\beta-1}}{1-G(x;\boldsymbol{\xi})^{\beta+1}} exp\left\{-\alpha\left[\frac{G(x;\boldsymbol{\xi})}{1-G(x;\boldsymbol{\xi})}\right]^{\beta}\right\} \tag{3.0.5}$$

which depends on a parameter vector $\boldsymbol{\xi}$

$$g(x;\alpha,\beta,c,k,s) =$$

$$\alpha\beta g\left(x;c,k,s\right)\frac{G\left(x;c,k,s\right)^{\beta-1}}{1-G\left(x;c,k,s\right)^{\beta+1}}\times exp\left\{-\alpha\left[\frac{G\left(x;c,k,s\right)}{1-G\left(x;c,k,s\right)}\right]^{\beta}\right\} \tag{3.0.6}$$

3.0.12 The Cumulative Distribution Function of Weibull-G Family

The cumulative distribution function of the Weibull-G Family of Distribution is given by;

$$G\left(x;\alpha,\beta,\xi\right)=\int_{0}^{\frac{G(x;\xi)}{1-G(x;\xi)}}\alpha\beta t^{\beta-1}e^{-\alpha t^{\beta}}dt, \tag{3.0.7}$$

it implies that

$$G\left(x;\alpha,\beta,\xi\right)=-e^{-\alpha t^{\beta}}\Big|_{0}^{\frac{G(x;\xi)}{1-G(x;;\xi)}}$$

$$=-e^{-\alpha\left[\frac{G(x;\xi)}{1-G(x;;\xi)}\right]^{\beta}}+e^{-\alpha(0)^{\beta}}$$

$$=1-e^{-\alpha\left[\frac{G(x;\xi)}{1-G(x;;\xi)}\right]^{\beta}},\qquad x\subseteq R,\alpha,\beta>0$$

3.0.13 The pdf of the Weibull-Burr XII Ditribution based on the generalized Weibull-G pdf

The pdf of the generalized Weibull-G family of distribution is;

$$g\left(x;\alpha,\beta,\xi\right)=\alpha\beta g\left(x;\xi\right)\frac{G\left(x;\xi\right)^{\beta-1}}{1-G\left(x;\xi\right)^{\beta+1}}exp\left\{-\alpha\left[\frac{G\left(x;\xi\right)}{1-G\left(x;\xi\right)}\right]^{\beta}\right\} \tag{3.0.8}$$

which depends on a parameter vector ξ

$$g\left(x;\alpha,\beta,c,k,s\right) \tag{3.0.9}$$

$$= \alpha\beta g\left(x;c,k,s\right)\frac{G\left(x;c,k,s\right)^{\beta-1}}{1-G\left(x;c,k,s\right)^{\beta+1}} exp\left\{-\alpha\left[\frac{G\left(x;c,k,s\right)}{1-G\left(x;c,k,s\right)}\right]^{\beta}\right\}$$

The pdf and cdf of Bur XII distribution is respectively given by;

$$g\left(x;c,k,s\right) = cks^{-c}x^{c-1}(1+(x/s)^{c})^{-k-1}, \quad c,k,s>0. \tag{3.0.10}$$

and

$$G\left(x;c,k,s\right) = 1-(1+(x/s)^{c})^{-k} \tag{3.0.11}$$

$$1-G\left(x;c,k,s\right) = 1-\left[1-(1+(x/s)^{c})^{-k}\right] = (1+(x/s)^{c})^{-k} \tag{3.0.12}$$

Substituting equations 3.0.8, 3.0.9 and 3.0.10 into equation 3.0.7 , we have

$$g\left(x;\alpha,\beta,c,k,s\right) = \ \alpha\beta\ cks^{-c}x^{c-1}(1+(x/s)^{c})^{-k-1}$$

$$\times\left[\frac{\left(1-(1+(x/s)^{c})^{-k}\right)^{\beta-1}}{\left((1+(x/s)^{c})^{-k}\right)^{\beta+1}}\right] exp\left\{-\alpha\left[\frac{1-(1+(x/s)^{c})^{-k}}{(1+(x/s)^{c})^{-k}}\right]^{\beta}\right\}$$

$$g\left(x;\alpha,\beta,c,k,s\right) = \alpha\beta\ cks^{-c}x^{c-1}\left(\ 1+(x/s)^{c}\ \right)^{-k-1}$$

$$\times\left[\frac{\left(1-\frac{1}{(1+(x/s)^{c})^{k}}\right)^{\beta-1}}{\left((1+(x/s)^{c})^{-k}\right)^{\beta+1}}\right] exp\left\{-\alpha\left[\frac{\left(1-\frac{1}{(1+(x/s)^{c})^{k}}\right)}{(1+(x/s)^{c})^{-k}}\right]^{\beta}\right\}$$

$$= \alpha\beta\ cks^{-c}x^{c-1}\frac{(1+(x/s)^{c})^{-k}}{(1+(x/s)^{c})}$$

$$\times\left[\frac{\left((1+(x/s)^{c})^{k}-1\right)^{\beta-1} \div \left(\left((1+(x/s)^{c})^{k}\right)\right)^{\beta} \times \left((1+(x/s)^{c})^{-k}\right)}{\frac{1}{\left((1+(x/s)^{c})^{k}\right)^{\beta}\left((1+(x/s)^{c})^{k}\right)}}\right]$$

$$\times exp\left\{-\alpha\left[\frac{\left((1+(x/s)^{c})^{k}-1\right)\left((1+(x/s)^{c})^{k}\right)}{(1+(x/s)^{c})^{k}}\right]^{\beta}\right\}$$

$$= \alpha\beta\ cks^{-c}x^{c-1}\frac{(1+(x/s)^c)^{-k}}{(1+(x/s)^c)}\times$$

$$\left[\frac{\Big((1+(x/s)^c)^k-1\Big)^{\beta-1}\times\Big(\Big((1+(x/s)^c)^k\Big)\Big)^{-\beta+\beta}\times\Big((1+(x/s)^c)^k\Big)}{(1+(x/s)^c)^{-k}}\right]$$

$$\times exp\left\{-\alpha\Big[\Big((1+(x/s)^c)^k-1\Big)\Big]^\beta\right\}$$

$\therefore$ The proposed Weibull-Burr XII distribution is given by;

$$g\left(x;\alpha,\beta,c,k,s\right)=\alpha\beta\ cks^{-c}x^{c-1}\times\ \left(1+(x/s)^c\right)^{k-1}$$

$$\times\Big((1+(x/s)^c)^k-1\Big)^{\beta-1}exp\left\{-\alpha\Big[\Big((1+(x/s)^c)^k-1\Big)\Big]^\beta\right\} \tag{3.0.13}$$

$$\alpha,\beta,c,k,s>0,\ x\geq 0$$

3.0.14 Validity of the pdf of Weibull-Burr XII distribution and the Cumulative Distribution Function (cdf)

We know that when $\int_{-\infty}^{\infty} g\left(x\right)dx=1$, where $g\left(x\right)$ is a pdf and $-\infty<x<\infty$ of any continuous probability distribution depending on the interval then that pdf is valid. To test whether the Weibull-Burr XII distribution is valid, we show that

$$\int_0^\infty g\left(x;\alpha,\beta,c,k,s\right)dx=1$$

$$\int_0^\infty g\left(x;\alpha,\beta,c,k,s\right)dx=$$

$$\int_0^\infty\left[\begin{array}{c}\alpha\beta\ cks^{-c}x^{c-1}\ \left(1+(x/s)^c\right)^{k-1}\Big((1+(x/s)^c)^k-1\Big)^{\beta-1}\\ exp\left\{-\alpha\Big((1+(x/s)^c)^k-1\Big)^\beta\right\}\end{array}\right]dx \tag{3.0.14}$$

We apply integration by substitution, let

$$u=1+(x/s)^c$$

$$\frac{du}{dx} = cx^{c-1}s^{-c}$$

$$dx = \frac{du}{cx^{c-1}s^{-c}}$$

This implies that

$$\int_0^\infty g\left(x;\alpha,\beta,c,k,s\right)dx = \int_0^\infty \left[\alpha\beta\ cks^{-c}x^{c-1}u^{k-1}\left(u^k-1\right)^{\beta-1}\exp[-\alpha\left(u^k-1\right)^{\beta}].\tfrac{1}{cx^{c-1}s^{-c}}\right]du$$

$$= \int_0^\infty \left[\alpha\beta\ ku^{k-1}\left(u^k-1\right)^{\beta-1}exp[-\alpha\left(u^k-1\right)^{\beta}]\right]du \qquad (3.0.15)$$

Again, let $y = u^k - 1$

$$\frac{dy}{du} = ku^{k-1}$$

$$du = \frac{dy}{ku^{k-1}}$$

Substituting du into equation 3.0.15, we have

$$\int_0^\infty g\left(x;\alpha,\beta,c,k,s\right)dx = \int_0^\infty \alpha\beta\ ku^{k-1}(y)^{\beta-1}exp[-\alpha\left(y\right)^{\beta}].\frac{dy}{ku^{k-1}}$$

$$= \int_0^\infty \alpha\beta\ (y)^{\beta-1}\exp\left(-\alpha y^{\beta}\right)dy \qquad (3.0.16)$$

Similarly, let $z = \alpha y^{\beta}$

$$\frac{dz}{dy} = \alpha\beta y^{\beta-1}$$

$$dy = \frac{dz}{\alpha\beta y^{\beta-1}}$$

Substituting dy into 3.0.16, we have

$$\int_0^\infty g\left(x;\alpha,\beta,c,k,s\right)dx = \int_0^\infty \alpha\beta\ (y)^{\beta-1}\exp\left(-z\right).\frac{dz}{\alpha\beta y^{\beta-1}}$$

$$= \int_0^\infty \exp\left(-z\right)dz$$

$$= -exp\left(-z\right)\Big|_0^\infty = -exp\left(-\alpha y^{\beta}\right)\Big|_0^\infty$$

$$= -exp\left(-\alpha\left(u^k - 1\right)^\beta\right)\Big|\begin{matrix}\infty\\0\end{matrix}$$

$$= -exp\left(-\alpha\left((1 + (x/s)^c)^k - 1\right)^\beta\right)\Big|\begin{matrix}\infty\\0\end{matrix} \tag{3.0.17}$$

$$= -exp\left(-\alpha\left((1 + (\infty/s)^c)^k - 1\right)^\beta\right) - \left(-exp\left(-\alpha\left((1 + (0/s)^c)^k - 1\right)^\beta\right)\right)$$

$$\therefore \qquad \int_0^\infty g\left(x;\alpha,\beta,c,k,s\right)dx = 1$$

Hence,

$$g\left(x;\alpha,\beta,c,k,s\right) = \alpha\beta\ cks^{-c}x^{c-1} \times\ \left(1 + (x/s)^c\right)^{k-1}$$

$$\times\left((1 + (x/s)^c)^k - 1\right)^{\beta-1} exp\left\{-\alpha\left[\left((1 + (x/s)^c)^k - 1\right)\right]^\beta\right\},\quad \alpha,\beta,c,k,s > 0,\ x \geq 0,$$

is a valid probability density function.

3.0.15 Cumulative Distribution Function (cdf) of Weibull-Burr XII distribution

The cdf is obtained by the formula

$$G\left(x;\alpha,\beta,c,k,s\right) = \int_0^x g\left(t;\alpha,\beta,c,k,s\right)dt$$

it implies that

$$\int_0^x g\left(t;\alpha,\beta,c,k,s\right)dt = -exp\left(-\alpha\left((1 + (t/s)^c)^k - 1\right)^\beta\right)\Big|\begin{matrix}x\\0\end{matrix}$$

$$= -exp\left(-\alpha\left((1 + (t/s)^c)^k - 1\right)^\beta\right) - \left(-exp\left(-\alpha\left((1 + (0/s)^c)^k - 1\right)^\beta\right)\right)$$

$$\therefore \qquad G\left(x;\alpha,\beta,c,k,s\right) = 1 - exp\left(-\alpha\left((1 + (x/s)^c)^k - 1\right)^\beta\right),\quad x \geq 0 \tag{3.0.18}$$

3.0.16 Survival function of Weibull-Burr XII distribution

Another lifetime distribution representative is the survival function or reliability function, defined as;

$$S(x) = \Pr(x > 0) = 1 - F(x)$$

indicating the probability of surviving an age of x or becoming older than x where $F(x)$ is the cumulative distribution function.Thus Survival function of Weibull-Burr XII distribution is

$$S(x) = 1 - G(x; \alpha, \beta, c, k, s) = 1 - [1 - exp\left(-\alpha\left((1 + (x/s)^c)^k - 1\right)^\beta\right)], \quad x > 0$$

$$S(x) = exp\left(-\alpha\left((1 + (x/s)^c)^k - 1\right)^\beta\right), \quad x \geq 0$$

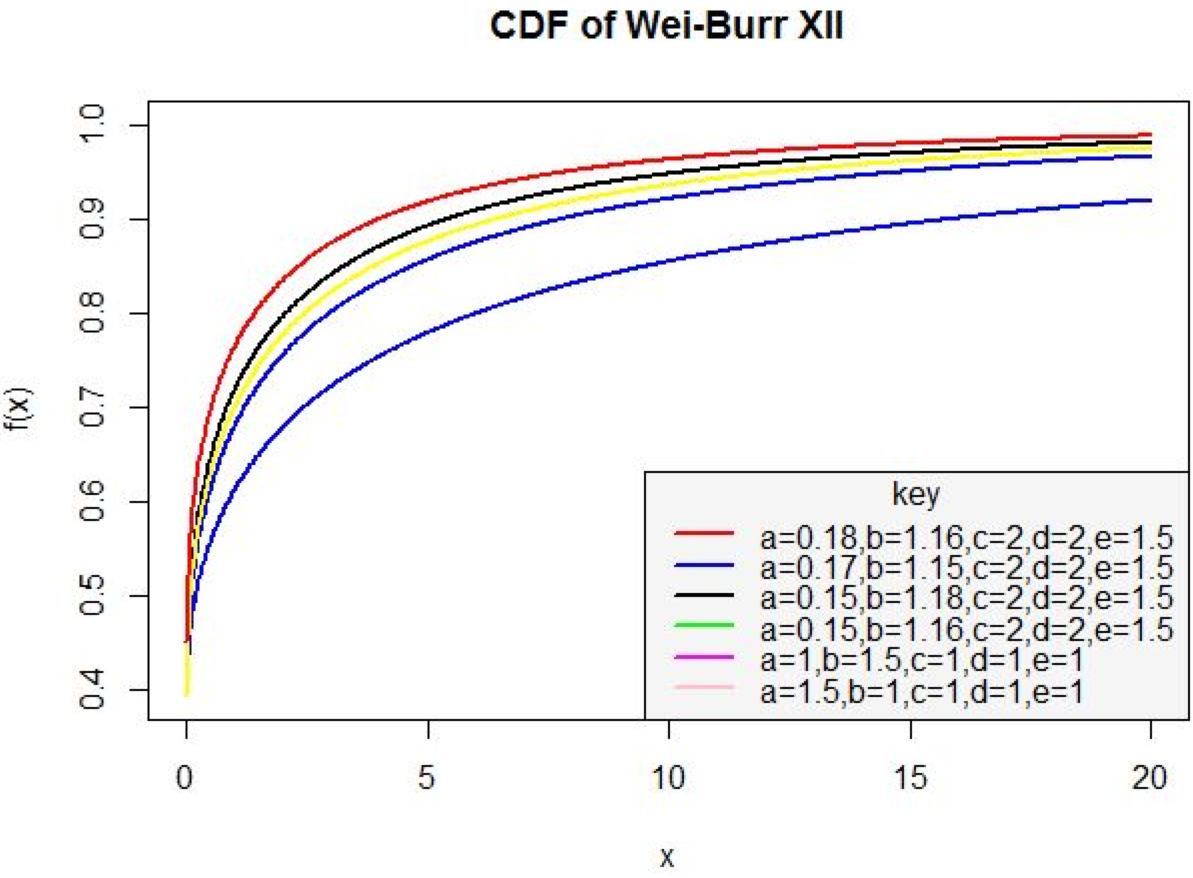

Figure 3.0.1: Graph of the CDF of the Weibull-Burr XII distribution

3.0.17 Hazard Rate Function of Weibull-Burr XII distribution

The reliability (survival) function examines the chance that breakdowns of organisms, of technical units etc. occur beyond a given point in time. To monitor the lifetime of a unit across the support of its lifetime distribution, the hazard rate $h(x)$ is used. In fact, the hazard rate usually is more informative about the underlying mechanism of failure than the other representatives of a lifetime distribution (Rinne, 2014). The Hazard Rate Function $h(x)$ is defined as;

$$h(x) = \frac{g(x)}{1 - G(x)}$$

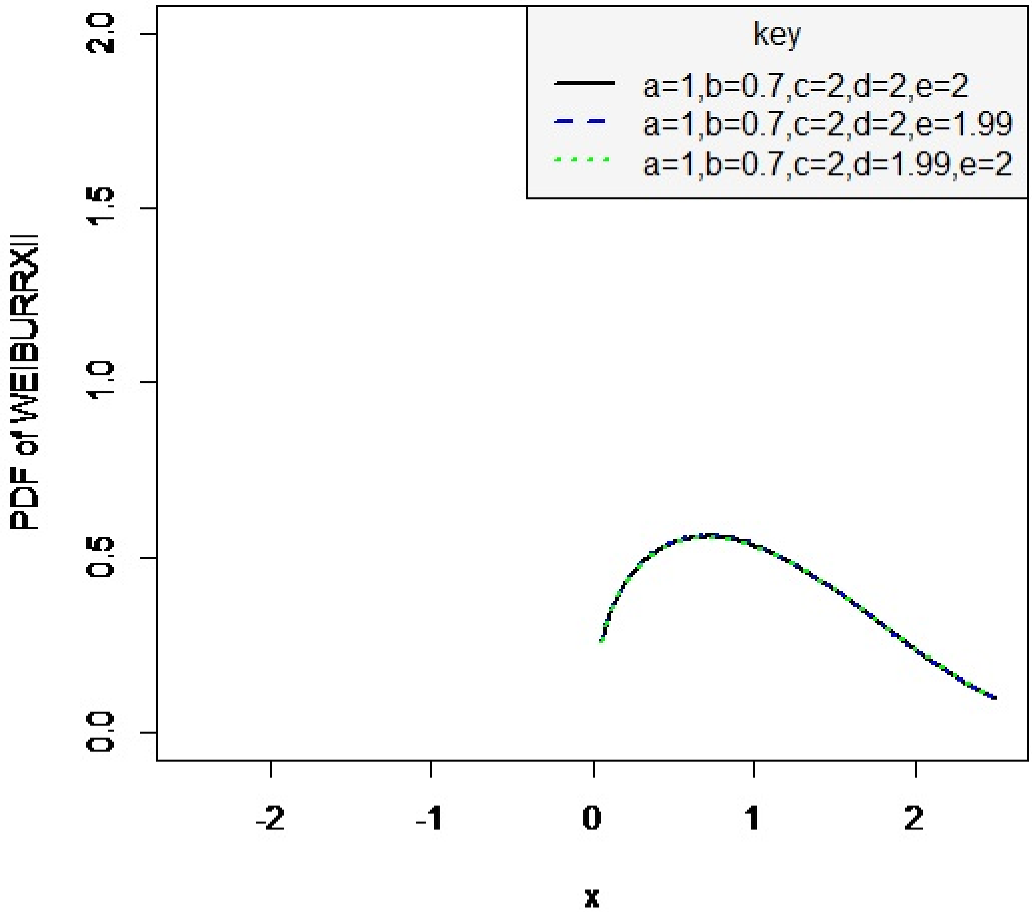

Figure 3.0.2: Graph of the pdf of the Weibull-Burr XII distribution

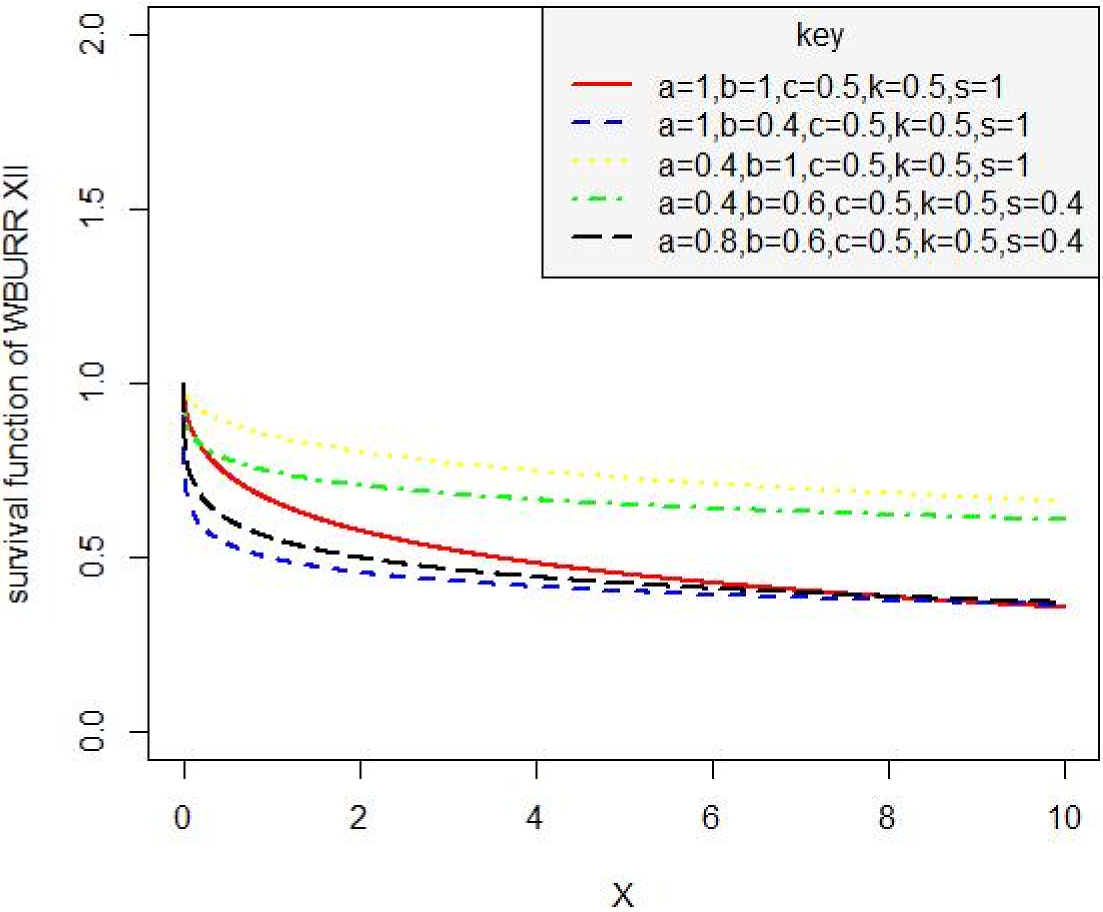

Figure 3.0.3: Graph of the Survival function of the Weibull-Burr XII distribution

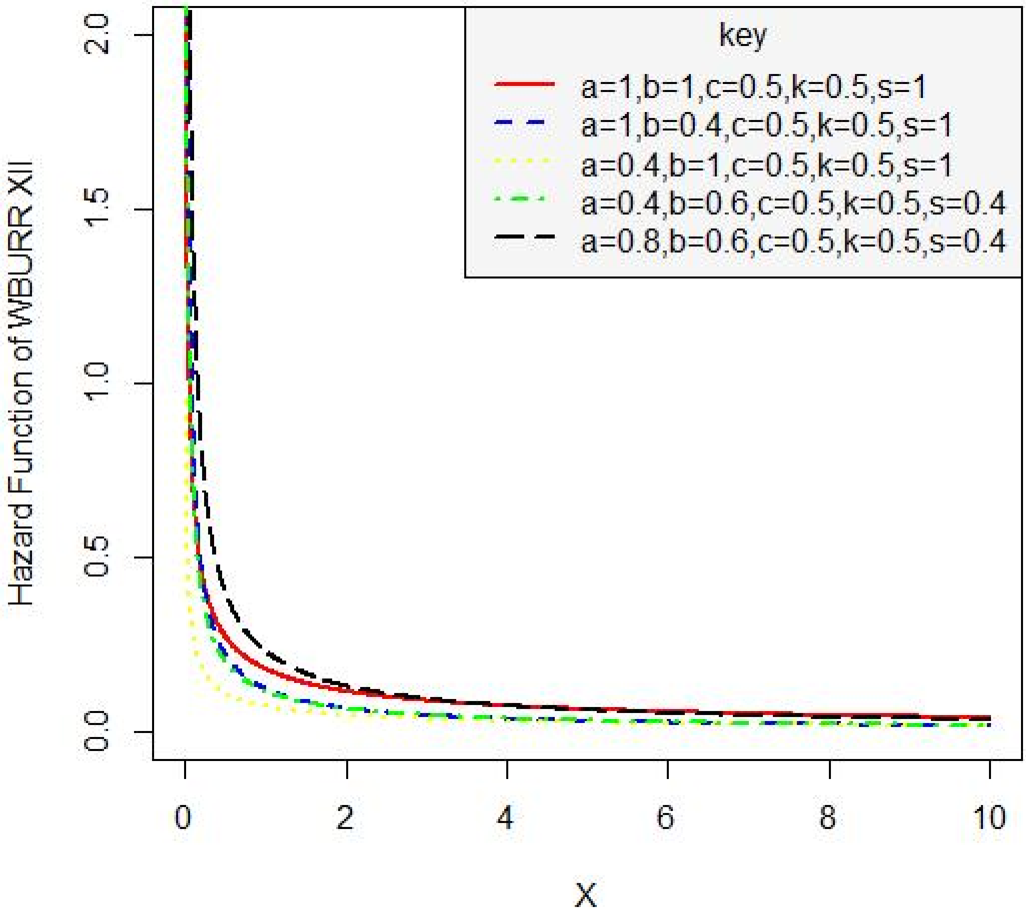

Figure 3.0.4: Graph of the Hazard function of the Weibull-Burr XII distribution

where $g(x)$ and $G(x)$ is the probability density function and cumulative distribution function of a distribution respectively. Therefore, the Hazard Rate Function of Weibull-Burr XII distribution is;

$$h\left(x;\alpha,\beta,c,k,s\right)=$$

$$\frac{\left\{\alpha\beta cks^{-c}x^{c-1}\ \left(1+\left(x/s\right)^{c}\right)^{k-1}\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta-1}exp\left\{-\alpha\left[\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)\right]^{\beta}\right\}\right\}}{1-\left[1-exp\left(-\alpha\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta}\right)\right]}$$

$$h\left(x;\alpha,\beta,c,k,s\right)=\alpha\beta cks^{-c}x^{c-1}\ \left(1+\left(x/s\right)^{c}\right)^{k-1}\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta-1},\quad x\geq 0$$

3.0.18 Quantile function of Weibull-Burr XII Distribution

Let $G\left(x;\alpha,\beta,c,k,s\right)=1-exp\left(-\alpha\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta}\right)=u$

Quantile function is defined as

$$Q\left(u\right)=G^{-1}\left(u\right)$$

now, $exp\left(-\alpha\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta}\right)=1-u$

$$\frac{1}{exp\left(\alpha\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta}\right)}=\frac{1-u}{1}$$

$$exp\left(\alpha\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta}\right)=\frac{1}{1-u}$$

$$\left(\alpha\left(\left(1+\left(x/s\right)^{c}\right)^{k}-1\right)^{\beta}\right)=In\left(\frac{1}{1-u}\right)$$

$$\left((1+(x/s)^c)^k-1\right)^\beta=\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)$$

$$(1+(x/s)^c)^k-1=\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}$$

$$(1+(x/s)^c)^k=1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}$$

$$(1+(x/s)^c)=\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}\right\}^{1/k}$$

$$(x/s)^c=\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}\right\}^{1/k}-1$$

$$\frac{x^c}{s^c}=\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}\right\}^{1/k}-1$$

$$x^c=s^c\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}\right\}^{1/k}-1$$

Therefore, $Q\left(u\right)=x=\sqrt[c]{\left[s^c\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}\right\}^{1/k}-1\right]}$

$$Q\left(u\right)=s\left[\left\{1+\left(\frac{1}{\alpha}In\left(\frac{1}{1-u}\right)\right)^{1/\beta}\right\}^{1/k}-1\right]^{\frac{1}{c}}$$

3.0.19 Expansion of the pdf of Weibull-Burr XII by using power series expansion

Recall that the pdf of Weibull-Burr XII distribution is given in equation (3.0.13) by
$g\left(x;\alpha,\beta,c,k,s\right)=\alpha\beta\ cks^{-c}x^{c-1}\times\ (1+(x/s)^c)^{k-1}$

$$\times\left((1+(x/s)^c)^k-1\right)^{\beta-1}exp\left\{-\alpha\left[\left((1+(x/s)^c)^k-1\right)\right]^\beta\right\}$$

Also remember the power series expansion of

$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and

$$e^{-cx} = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!} x^k.$$

Expanding the pdf by using power series expansion in the exponential term above, we have;

$$exp\left\{-\alpha\left[\left((1+(x/s)^c)^k - 1\right)\right]^{\beta}\right\}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} \left((1+(x/s)^c)^k - 1\right)^{k\beta}$$

Substituting the above expression into equation (3.0.3) above, we get;

$$g(x; \alpha, \beta, c, k, s) = \alpha\beta\ cks^{-c}$$

$$\times\left((1+(x/s)^c)^k - 1\right)^{\beta-1} \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} x^{c-1} (1+(x/s)^c)^{k-1}$$

$$\times\left((1+(x/s)^c)^k - 1\right)^{k\beta}$$

$$= \alpha\beta\ cks^{-c} \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} x^{c-1} (1+(x/s)^c)^{k-1} \left((1+(x/s)^c)^k - 1\right)^{k\beta+\beta-1}$$

$$g(x; \alpha, \beta, c, k, s) =$$

$$\alpha\beta\ ck \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} \frac{x^{c-1}}{s^{-c}} (1+(x/s)^c)^{k-1} \left((1+(x/s)^c)^k - 1\right)^{k\beta+\beta-1}$$

$$= \alpha\beta\ ck \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^k}{k!} s^{-1} \left(\frac{x}{s}\right)^{c-1} (1+(x/s)^c)^{k-1} \left((1+(x/s)^c)^k - 1\right)^{k\beta+\beta-1}$$

Note that:

$$\left((1+(x/s)^c)^k - 1\right)^{k\beta+\beta-1} = \left[\frac{\left((1+(x/s)^c)^k - 1\right)}{(1+(x/s)^c)^k \times (1+(x/s)^c)^{-k}}\right]^{k\beta+\beta-1}$$

$$= \left[\frac{\left((1+(x/s)^c)^k - 1\right)}{(1+(x/s)^c)^k}\right]^{k\beta+\beta-1} \left((1+(x/s)^c)^k\right)^{k\beta+\beta-1}$$

$$= \left[1-(1+(x/s)^c)^{-k}\right]^{k\beta+\beta-1}\left((1+(x/s)^c)^k\right)^{k\beta+\beta-1}$$

Therefore,

$$g(x;\alpha,\beta,c,k,s)=$$

$$\alpha\beta\ cks^{-1}\sum_{k=0}^{\infty}\frac{(-1)^k\alpha^k}{k!}\left(\frac{x}{s}\right)^{c-1}(1+(x/s)^c)^{k-1}\left((1+(x/s)^c)^k\right)^{k\beta+\beta-1}$$

$$\times\left[1-(1+(x/s)^c)^{-k}\right]^{k\beta+\beta-1}$$

Similarly, recall that;

$$(1-z)^{b-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)}z^j$$

where $\Gamma(b)$ and $\Gamma(b-j)$ are gamma functions

Hence

$$\left[1-(1+(x/s)^c)^{-k}\right]^{k\beta+\beta-1}=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(k\beta+\beta)}{\Gamma(k\beta+\beta-j)\,j!}(1+(x/s)^c)^{-kj}$$

and

$$g(x;\alpha,\beta,c,k,s)=$$

$$\alpha\beta\ cks^{-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\alpha^k\Gamma(k\beta+\beta)}{j!k!\Gamma(k\beta+\beta-j)}\left(\frac{x}{s}\right)^{c-1}(1+(x/s)^c)^{k-1}$$

$$\times\left((1+(x/s)^c)^k\right)^{k\beta+\beta-1}(1+(x/s)^c)^{-kj}$$

$$=\alpha\beta\ cks^{-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\alpha^k\Gamma(k\beta+\beta)}{j!k!\Gamma(k\beta+\beta-j)}\left(\frac{x}{s}\right)^{c-1}$$

$$\times(1+(x/s)^c)^{k-1+k(k\beta+\beta-1)-kj}$$

$$g(x;\alpha,\beta,c,k,s)=\alpha\beta\ cks^{-1}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{j+k}\alpha^k\Gamma(k\beta+\beta)}{j!k!\Gamma(k\beta+\beta-j)}\left(\frac{x}{s}\right)^{c-1}$$

$$\times(1+(x/s)^c)^{k(k\beta+\beta-j)-1}$$

Let

$$W_{j,k} = \alpha\beta\ cks^{-c} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{j+k}\alpha^k \Gamma(k\beta+\beta)}{j!k!\Gamma(k\beta+\beta-j)}$$

so that

$$g(x;\alpha,\beta,c,k,s) = W_{j,k} x^{c-1} \left(1+(x/s)^c\right)^{k(k\beta+\beta-j)-1} \tag{3.0.19}$$

3.0.20 Moments

The nth moment of a random variable X having a pdf $g(x)$ is given by

$$E(x^n) = \int_0^{\infty} x^n g(x) dx \tag{3.0.20}$$

Subtituting equation 3.0.19 into 3.0.20, we obtain

$$E(x^n) = W_{j,k} \int_0^{\infty} x^{n+c-1} \left(1+(x/s)^c\right)^{k(k\beta+\beta-j)-1} dx \tag{3.0.21}$$

Now, let $y = 1 + \frac{x^c}{s^c} \implies \frac{dy}{dx} = \frac{cx^c}{s^c}$ and $dx = \frac{s^c dy}{cx^{-c}}$

Also, let $y = 1 + \frac{x^c}{s^c} \Longrightarrow x = (-1)^{\frac{1}{c}} s(1-y)^{\frac{1}{c}}$

Substituting for y and dx into 3.0.21, we obtain

$$E(x^n) = W_{j,k} \int_0^{\infty} \frac{s^c}{c} y^{k(k\beta+\beta-j)-1} \frac{s^c}{cx^{c-1}} dy$$

$$E(x^n) = W_{j,k} \int_0^{\infty} \frac{s^c}{c} x^{n+c-1-(c-1)} y^{k(k\beta+\beta-j)-1} dy$$

$$E(x^n) = W_{j,k} \frac{s^c}{c} \int_0^{\infty} x^n\ y^{k(k\beta+\beta-j)-1} dy$$

Recall that, $x = (-1)^{\frac{1}{c}} s(1-y)^{\frac{1}{c}}$

Substituting for x also gives;

$$E(x^n) = W_{j,k} \frac{s^c}{c} \int_0^{\infty} \left[(-1)^{\frac{1}{c}} s(1-y)^{\frac{1}{c}}\right]^n\ y^{k(k\beta+\beta-j)-1} dy$$

$$= W_{j,k} \frac{s^{c+n} \int_0^{\infty} (-1)^{\frac{n}{c}}}{c} y^{k(k\beta+\beta-j)-1} (1-y)^{\frac{n}{c}} dy$$

$$= W_{j,k} \frac{s^{c+n}(-1)^{\frac{n}{c}}}{c} y^{k(k\beta+\beta-j)-1}(1-y)^{\frac{n}{c}+1-1} dy$$

Recall that the beta function is expressed as

$$\beta(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1} dx$$

Hence,

$$E(x^n) = W_{j,k} \frac{s^{c+n}(-1)^{\frac{n}{c}}}{c} \beta\left(k(k\beta+\beta-j),\ \frac{n}{c}+1\right)$$

3.0.21 Moment Generating Function of Weibull-Burr XII Distribution

The moment generating function (mgf) of a random variable X following a pdf $g(x)$ is given by;

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} g(x) dx \tag{3.0.22}$$

Using Maclaurin's power series expansion,

$$e^{tx} = \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n$$

It implies that

$$M_X(t) = \int_0^\infty \sum_{n=0}^{\infty} \frac{t^n}{n!} x^n g(x;\alpha,\beta,c,k,s) dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^\infty x^n g(x;\alpha,\beta,c,k,s)\, dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \int_0^\infty x^n g(x;\alpha,\beta,c,k,s)\, dx$$

Threfore,

$$M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\ E(x^n)$$

3.0.22 Characteristic Function of Weibull-Burr XII Distribution

The characteristic function of a random variable x having a pdf g(x) is defined by;

$$\phi_X(t) = E\left(e^{itx}\right) g(x)\, dx \tag{3.0.23}$$

Recall that,

$$e^{itx} = cos\,(tx) - isin(tx)$$

Using power series expansion, it is known that

$$\cos(tx) \;= \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} x^{2n}$$

and

$$\sin(tx) \;= \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} x^{2n+1}$$

Therefore,

$$\phi_X(t) = \int_0^{\infty} (cos\,(tx) - isin\,(tx))g(x)dx$$

$$\phi_X(t) = \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{t^n}{(2n)!} x^{2n} + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} x^{2n+1} \right) g(x)dx$$

$$= \int_0^{\infty} \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} x^{2n} g(x)\, dx + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} x^{2n+1} g(x)dx$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} \int_0^{\infty} x^{2n} g(x)\, dx + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} \int_0^{\infty} x^{2n+1} g(x)dx$$

Hence,

$$\phi_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{(2n)!} E\left(x^{2n}\right) + i \sum_{n=0}^{\infty} \frac{t^n}{(2n+1)!} E(x^{2n+1})$$

3.0.23 Estimation of Parameters of Weibull-Burr XII Distribution

At this point, we study the estimation of the unknown parameters of the Weibull-Burr XII distribution by the method of maximum likelihood. Let $x_1, \; x_2, \; \ldots, \; x_n$ be a sample of size n following the Weibull-Burr XII distribution then the likelihood and log-likelihood functions for the parameters $\Theta = (\alpha, \beta, c, k, s)^T$ can be expressed as

$$l\,(X/\alpha, \beta, c, k, s) = \left(\alpha\beta cks^{-c}\right)^n$$

$$\times \sum_{i=1}^{n}\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1}\sum_{i=1}^{n}{x_i}^{c-1}\sum_{i=1}^{n}\left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right]^{\beta-1}$$

$$\times exp\left\{-\alpha\sum_{i=1}^{n}\left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right]^{\beta}\right\}$$

and

$$l\,(\Theta) = n\text{log}\alpha + \text{ nlog}\beta + n\text{ log}c + \text{ log}k - cn\text{ log}s$$

$$+(k-1)\sum_{i=1}^{n}\log\left(1+\left(\frac{x_i}{s}\right)^c\right)+(c-1)\sum_{i=1}^{n}\log\left(x_i\right)$$

$$+(\beta-1)\sum_{i=1}^{n}\log\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right)-\alpha\sum_{i=1}^{n}\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right)^{\beta}$$

respectively.

The components of the vector $U\,(\Theta)$ are given by

$$U_\alpha = \frac{\partial l}{\partial\alpha} = \frac{n}{\alpha}-\sum_{i=1}^{n}\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right)^{\beta} = 0$$

$$U_\beta = \frac{\partial l}{\partial\beta} = \frac{n}{\beta}+\sum_{i=1}^{n}\log\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right)$$

$$-\sum_{i=1}^{n}\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right)^{\beta}\log\left(\left(1+\left(\frac{x_i}{s}\right)^c\right)^k-1\right) = 0$$

$$U_c = \frac{\partial l}{\partial c} =$$

$$\frac{n}{c} - n\log(s) \; + \sum_{i=1}^{n} \log(x_i) \; + (k-1) \sum_{i=0}^{n} \left\{ \frac{\left(\frac{x_i}{s}\right)^c ln\left(\frac{x_i}{s}\right)}{1+\left(\frac{x_i}{s}\right)^c} \right\}$$

$$+k(\beta-1) \sum_{i=0}^{n} \left\{ \frac{\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1} \left(\frac{x_i}{s}\right)^c ln\left(\frac{x_i}{s}\right)}{\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1} \right\}$$

$$-k\alpha\beta \sum_{i=0}^{n} \left\{ \left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right]^{\beta-1} \left(1+\left(\frac{x_i}{s}\right)^c\right)^k \left(\frac{x_i}{s}\right)^c ln\left(\frac{x_i}{s}\right) \right\} = 0$$

$$U_k = \frac{\partial l}{\partial k} =$$

$$\frac{n}{k} + \sum_{i=0}^{n} \log\left(1+\left(\frac{x_i}{s}\right)^c\right) + k(\beta-1) \sum_{i=0}^{n} \left\{ \frac{\left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1}}{\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1} \right\}$$

$$-k\alpha\beta \sum_{i=0}^{n} \left\{ \left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right]^{\beta-1} \left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1} \right\} = 0$$

$$U_s = \frac{\partial l}{\partial s} = \frac{cn}{k} - c(k-1) \sum_{i=0}^{n} \left\{ \frac{{x_i}^c}{\left(1+\left(\frac{x_i}{s}\right)^c\right) s^{c+1}} \right\}$$

$$+\frac{ck\alpha\beta}{s^{c+1}} \sum_{i=0}^{n} \left\{ x^c \left[\left(1+\left(\frac{x_i}{s}\right)^c\right)^k - 1\right]^{\beta-1} \left(1+\left(\frac{x_i}{s}\right)^c\right)^{k-1} \right\} = 0$$

CHAPTER FOUR

RESULTS AND DISCUSSION

The first real data set represents the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938 (Lee 1992). The data examined by Tahir et al (2015) are:

0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0

For the first data set, we fit the Weibull-Burr XII distribution defined in (13). Its fit is also matched with the commonly known Burr XII, Beta-Burr XII (Paranaíba et al. 2011) and Weibull define in (3.0.3) distributions.

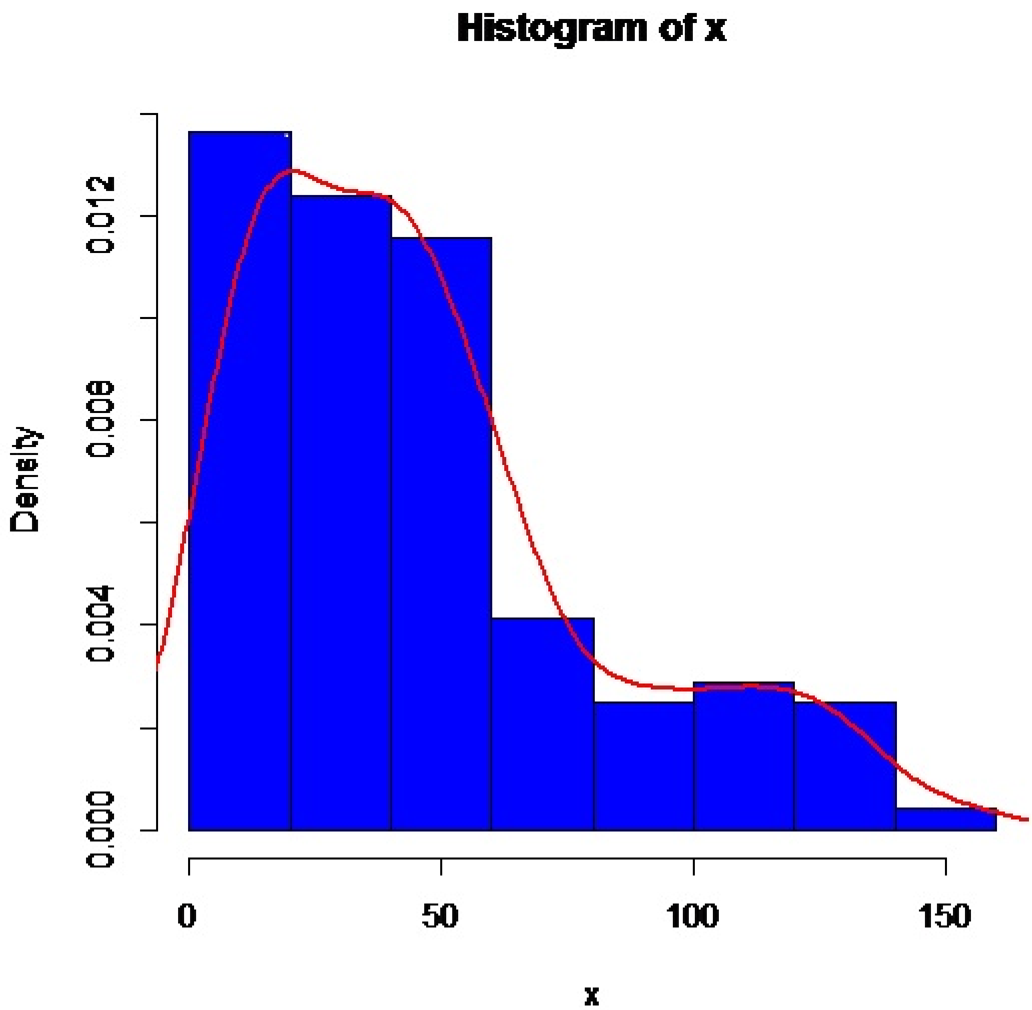

Figure 4.0.1: Histogram representing the survival times of 121 patients with breast cancer obtained from a large hospital in a period from 1929 to 1938

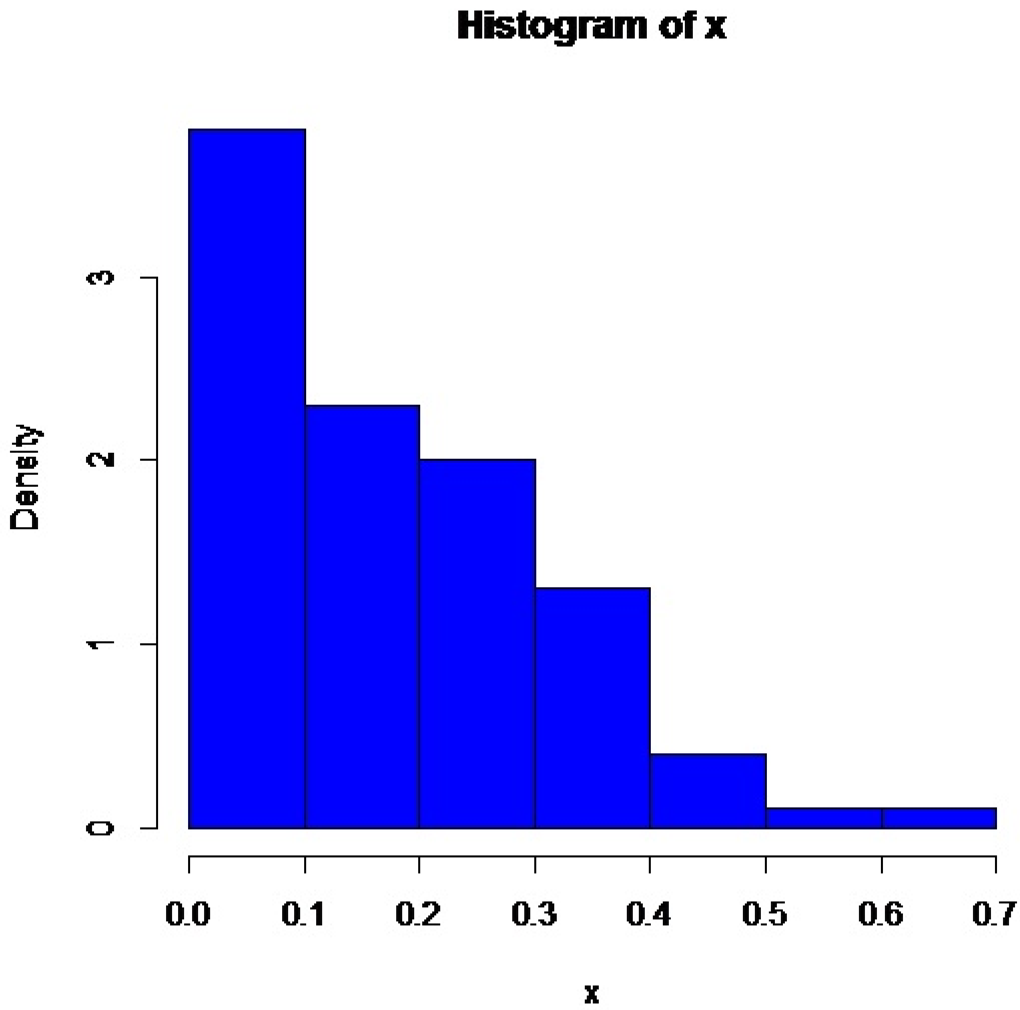

Figure 4.0.2: Graph of a Histogram showing the Survival Times (in days) for the Patients in Arm A of the Head-and-Neck-Cancer Trial

The second data set consist of Survival Times (in days) for the Patients in Arm A of the Head-and-Neck-Cancer Trial. Data is from Efron (1988);

7, 34, 42, 63, 64, 74, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140, 140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185, 218, 225, 241, 248, 273, 277, 279, 297, 319, 405, 417, 420,700,874.993, 440, 523, 523, 583, 594,789,897 1101, 1116, 1146, 1226, 1349, 1412, 1417

Table 2 Performance comparison of some distributions using the first data set

Distributions	AIC	CAIC	BIC	Ranks
Weibull-Burr XII *WBuD*	290.8001	291.0501	298.615	1
Weibull *WB*	1173.7555	1173.857	1179.347	3
Burr XII *BuD*	1365.175	1365.38	1373.562	4
Beta-Burr XII *BBuD*	332.299	333.352	331.296	2

Table 3 Performance comparison of some distributions using the second data set

Distributions	AIC	CAIC	BIC	Ranks
Weibull-Burr XII *WBuD*	-141.7826	-141.5326	-133.9071	1
Weibull *WB*	-75.98243	-75.85872	-70.77209	3
Burr XII *BuD*	816.4994	817.07	822.2948	4
Weibull-Uniform *WU*	-95.982	-97.034	-95.673	2

Considering the ranking in the last columns above, we say that Weibull-Burr XII distribution perfoms best based on these data set because it has the smallest value of AIC, BIC and CAIC compared to other distributions.

CHAPTER FIVE

Summary, Conclusion and Recommendations

5.0.24 Summary

We state and study a five-parameter continuous distribution called Weibull-Burr XII distribution. Furthermore, the importance of the new distribution is that it includes the parameters of the Weibull and Burr XII distributions. We study some of the mathematical and statistical properties of the proposed distribution. Parameter estimation is obtained by maximum likelihood technique. Two data sets are considered and performance comparison was made with other related distributions using BIC, AIC and CAIC. R-package is used for the analysis.

5.0.25 Conclusion

In the analysis carried out, we considered three measures of comparisons, i.e. BIC, AIC and CAIC (Bayesian Information Criterion, Akaike Information Criterion and Consistent Information Criterion respectively) and the model with the lowest value is always considered best. Therefore, based on the analysis carried out, it can be concluded that the Weibull-Burr XII distribution performs best compared to Burr XII, Beta-Burr XII, Weibull-Uniform and Weibull distributions.

5.0.26 Recommendations

Based on the research carried out, the following items were recommended:

i. Other mathematical properties like Renyi entropy, Incomplete moments, e.t.c. can be obatined

ii. Other methods of parameter estimation techniques should also be tested such as

Bayesian approach.

References

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Appendix

R-codes used in carying out the analysis

$library(AdequacyModel)$

$survival = c(7, 34, 42, 63, 64, 74, 83, 84, 91, 108, 112, 129, 133, 133, 139, 140,$

$140, 146, 149, 154, 157, 160, 160, 165, 173, 176, 185, 218, 225, 241, 248, 273, 277, 279, 297,$

$319, 405, 417, 420, 440, 523, 523, 583, 594, 1101, 1116, 1146, 1226, 1349, 1412, 1417)$

$WEBurrX11pdf_Burr = function(par, x)c = par[1]d = par[2]e = par[3]c * d * e^(- c) * x^(c - 1) *$

$function(par, x)c = par[1]d = par[2]e = par[3]1 - (1 + (x/e)^c)^(- d)$ set.seed(0) $result_1 =$

$goodness.fit(pdf = pdf_Burr, cdf = cdf_Burr, starts =$

$c(1, 1, 1), data = survival, method = "PSO", domain = c(0, Inf), prop = 0.1, lim_inf =$

$c(0, 0, 0), lim_sup = c(2, 2, 2), N = 121)result_1.library(AdequacyModel)survival =$

$c(0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5,$

$8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3$

$, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0,$

$27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0,$

$40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0,$

$49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0,$

$65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0,$

$111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0)$

$WEBurrX11$

$pdf_Burr = function(par, x)c = par[1]d = par[2]e = par[3]c * d * e^(- c) * x^(c - 1) * (1 + (x/e)^c)^(-$

$function(par, x)c = par[1]d = par[2]e = par[3]1 - (1 + (x/e)^c)^(- d)$

set.seed(0)

$result_1 = goodness.fit(pdf = pdf_Burr, cdf =$

$cdf_Burr, starts = c(1, 1, 1), data = survival, method =$

$"PSO", domain = c(0, Inf), prop = 0.1, lim_inf = c(0, 0, 0),$

$lim_sup = c(2, 2, 2), N = 121)$

$result_1$

$library(AdequacyModel)$

$survival = c(0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8,$

$7.4, 7.5, 8.4, 8.4, 10.3, 11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7,$

$16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5, 17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6,$

$24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0, 31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0,$

$39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0, 41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0,$

$47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0, 54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0,$

$65.0, 65.0, 67.0, 67.0, 68.0, 69.0, 78.0, 80.0, 83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0,$

$109.0, 111.0, 115.0, 117.0, 125.0, 126.0, 127.0, 129.0, 129.0, 139.0, 154.0)$

$WEBurrX11$

$pdf_W EBurrX11 = function(par, x)a = par[1]b = par[2]c = par[3]d = par[4]e = par[5]a * b * c * d$

$function(par, x)a = par[1]b = par[2]c = par[3]d = par[4]e = par[5]1 - exp(-a((1 + (x/e)^c)^d - 1)^b$

set.seed(0)

$result_1 = goodness.fit(pdf = pdf_o geil, cdf = cdf_o geil, starts = c(1, 1, 1, 1), data =$

$x, method =$

$"PSO", domain = c(0, Inf), prop = 0.1, lim_i nf = c(0, 0, 0, 0, 0),$

$lim_s up = c(2, 2, 2, 2, 2), N = 121)$

$result_1$