ALGEBRAIC STUDY OF SOFT LATTICE THEORY AND ITS APPLICATIONS TO DISTRIBUTED COMPUTING SYSTEM

By

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MARCH, 2016

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Ph.D/Sci./02945/2010-2011

A THESIS SUBMITTED TO THE
SCHOOL OF POST GRADUATE STUDIES,
AHMADU BELLO UNIVERSITY, ZARIA-NIGERIA,
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE AWARD OF DOCTOR OF PHILOSOPHY
IN MATHEMATICS

DEPARTMENT OF MATHEMATICS
AHMADU BELLO UNIVERSITY, ZARIA,
NIGERIA

MARCH, 2016

DECLARATION

I declare that the work in this thesis titled "ALGEBRAIC STUDY OF SOFT LATTICE THEORY AND ITS APPLICATIONS TO DISTRIBUTED COMPUTING SYSTEM" has been performed by me in the Department of Mathematics under the supervision of Dr. A. M. Ibrahim, Professor D. Singh and Professor B. Sani. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

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Date

CERTIFICATION

This thesis titled " ALGEBRAIC STUDY OF SOFT LATTICE THEORY AND ITS APPLICATIONS TO DISTRIBUTED COMPUTING SYSTEM" by Omeiza Ahmed YUSUF (Ph.D/Sci./02945/2010-2011) meet the regulations governing the award of the degree of Doctor of Philosophy of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This thesis is dedicated to my lovely family.

ACKNOWLEDGEMENT

In the name of Allah, the most beneficent the most merciful, peace be upon Prophet Muhammad (S.A.W). My heartfelt gratitude goes to my supervisor Dr. A. M. Ibrahim for his perseverance and tolerance throughout the period of this work, who despite his tight schedule found time to go through the work chapter by chapter and suggesting valuable corrections. My sincere gratitude goes to my other supervisors, Professor D. Sigh and Professor B. Sani, for their assistance towards the completion of this thesis. I would like to express my sincere gratitude to the Head of Department, Professor B. Sani, the Post-graduate Coordinator, Dr. A. Yahaya. At this juncture, it is imperative to acknowledge the support of my friends and colleagues,. Abdulsamad Imam Tanko, Musa Balarebe, Muhammad Jamilu Ibrahim, Junaidu Abba Muktar, Dr. Alhaji Jibril Alkali, Chinedu Peter, Abdulazeez Shehu, Balami Holly-heavy, Richard Tyokyaa, Haruna Lawal, Balogun Funmilola, Auwal Bichi, and Rosemary Dtzer. I would ever remain grateful to my late father, Mal. Yusuf Ahmed Onota, my late wife, Mrs. Surrayya Yusuf, my mother, Mrs. Barikisu Jato, and my brother Musa Jimoh Yusuf for giving me a solid foundation on which I have built up my dreams. Finally, I owe my most sincere gratitude to my wife, Mrs. Fatima Yusuf and my children; Aisha Yusuf, Saeed Yusuf and Bilkisu Yusra Yusuf for their support and patience towards the success of this programme.

ABSTRACT

In this thesis, we crisply present the fundamentals of soft set theory to emphasize that soft set has enough developed basic supporting tools through which various algebraic structures in theoretical point of view could be developed. The concepts of conjunction and disjunction are redefined as binary operations on soft sets and their properties are presented. A perception named soft Boolean algebra is introduced where some related results were established. It is shown that if SB is a collection of all soft sets under a common universe U, then $(SB, \wedge, \vee, \emptyset, \tilde{U})$ is a Boolean algebra. For any two soft sets (F, A), $(G, B) \in (SB)$, domination, idempotents, absorption and complement laws are satisfied, where $\emptyset$ and $\tilde{U}$ are unique. We define soft lattice in terms of the redefined conjunction and disjunction and present some examples. Upper bound and least upper bound, lower bound and greatest lower bound were defined in soft set context. Soft lattice is redefine in terms of supremum and infimum and it is shown that the two definitions are equivalent. Given any soft semilattice (Γ, E), where $\Gamma(e_1) \subseteq \Gamma(e_2)$ if and only if $\Gamma(e_1) \wedge \Gamma(e_2) = \Gamma(e_1)$, $\forall\ e_1, e_2 \in E$, we show that $((\Gamma, E), \subseteq)$ is an ordered soft set in which every pair of elements has greatest lower bound. The idea of soft lattice is extended to distributed soft lattice, modular soft lattice and isomorphic soft lattice and their properties are presented with some related results. We established that if (Γ, E) is an ordered soft set and $A, B \subseteq E$, such that $\theta : (F, A) \to (G, B)$ is defined by $\theta(F(e_1)) = \{F(e_2) \in (F, A) : F(e_1) \subseteq F(e_2), \forall F(e_1) \in (F, A)\}$, then (F, A) is isomorphic to the range of θ ordered by containment $\subseteq$. Finally, some applications of soft lattice theory to distributed computing system are presented where it is shown that, a predicate is linear if and only if it is meet-closed. If B is a linear predicate with the efficient advancement property, then there exists an efficient algorithm to determine the least consistent cut that satisfy B (if any). We presented an algorithm to detect a linear predicate of a consistent cut and showed that the slice of a distributed computing is uniquely

defined for all predicate.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

Most of our real life problems in engineering, social and medical science, economics, environment etc., involve imprecise data and their solutions involve the use of mathematical principles based on uncertainty and imprecision. To handle such uncertainties, a number of theories have been proposed. According to (Molodtsov, 1999) some of these are probability theory, fuzzy sets theory, intuitionistic fuzzy sets, interval mathematics and rough sets, etc. All these theories, however, are associated with an inherent limitation, which is the inadequacy of the parameterization tool associated with these theories.

Fuzzy set was developed by (Zadeh, 1965), in an attempt to deal with the problems of uncertainties. This theory has been found to be appropriate to some extent.

Let A be a subset of set X, μ_A called indicator function, is defined as

$$\mu_A = \begin{cases} 1, \ if \ x \in A, \\ 0, \ if \ x \notin A. \end{cases}$$

Obviously there is one-to-one correspondence between a set and its indicator function.

Let U be a universe. A fuzzy set X over U is a set defined by a function μ_X representing a mapping

$$\mu_X : U \longrightarrow [0, \ 1].$$

Here, μ_X is called the membership function of X, and the value $\mu_X(u)$ is called

the grade of membership of $u \in U$, and represents the degree of u belonging to the fuzzy set X. Thus, a fuzzy set X over U can be represented as follows:

$$X = \{(\mu_X(u)/u) : \ u \in U, \ \mu_X(u) \in [0,1]\}$$

One of the deficiencies of fuzzy set is how to set the membership function (Molodtsov, 1999).

Given the various observations on some of the existing tools for solving uncertainty problems, soft set theory emerged to soften these limitations.

Definition1.1.1: Soft set (Molodtsov, 1999)

Let U be a universe set and let E be a set of parameters (each parameter could be a word or a sentence). Let $P(U)$ denote the power set of U.

A pair $(F, \ E)$ is called a *soft set* over a given universe set U if and only if F is a mapping of a set of parameters E into the power set of U. That is, $F \ : \ E \longrightarrow \ P(U)$. Clearly, a soft set over U is a parameterized family of subsets of a given universe U. Also, for any $e \in E$, $F(e)$ is considered as the set of e–approximate elements of the soft set (F, E).

Soft sets could be regarded as neighbourhood systems, and they are special cases of context-dependent fuzzy sets (Molodtsov, 1999). In soft set theory the problem of setting the membership function, among other related problems, simply does not arise. This makes the theory very convenient and easy to apply in practice.

Example 1.1.1

(i) Let (X, τ) be a topological space, that is, X is a set and τ is a toplogy (a family of subsets of X called the open sets of X). Then, the family of neighborhoods $T(x)$ of point x, where $T(x) = \{V \in \tau \mid x \in V\}$, may be considered

as the soft set $(T(x),\ \tau)$.

(ii) Let A be a fuzzy set and μ_A be the membership function of the fuzzy set A, that is, μ_A is a mapping of U into $[0,\ 1]$, let $F(\alpha) = \{x \in U \mid \mu_A(x) \geq \alpha\},\ \alpha \in [0,\ 1]$ be a family of $\alpha-$ level sets for function μ_A. If the family F is known, $\mu_A(x)$ can be found by means of the definition:

$$\mu_A(x) = Sup_{\substack{\alpha \in [0,\ 1], \\ x \in F(\alpha)}} \alpha$$

. Hence every fuzzy set A may be considered as the soft set $(F,\ [0,\ 1])$.

(iii) Let $U = \{C_1,\ C_2,\ C_3,\ C_4,\ C_5,\ C_6,\ C_7,\ C_8,\ C_9,\ C_{10}\}$ be the set of Cars under consideration, E be a set of parameters defined as follows:

$E = \{e_1 =$ expensive, $e_2 =$ beautiful, $e_3 =$ manual gear, $e_4 =$ cheap, $e_5 =$ automatic gear, $e_6 =$ in good repair, $e_7 =$ in bad repair$\}$. Then the soft set (F, E) describes the attractiveness of the Cars under consideration.

Li (2010) used the concept of soft sets to defined soft lattice, gave some properties of soft lattice and discussed the relationship between soft lattice and fuzzy sets. The notion of soft lattice, soft sublattice, complete soft lattice, modular soft lattice, distributive soft lattice and soft chain were studied using the conventional conjunction and disjunction (Karaaslan et al., 2012).
However, in this thesis we redefine the concepts of conjunction and disjunction and thereby define soft lattice via the redefined notions and obtain results. The concept of soft Boolean algebra is proposed with some theorems. The application of soft lattice to distributed computing system is introduced.

1.2 Statement of Problem

The concept of soft sets proposed by (Molodtsov, 1999) received much attentions in research domain due to its resourcefulness in engineering, control system, computer science and real life situations. However, the idea of soft lattice theory in an algebraic perspective and its application to distributed computing system have not been studied.

1.3 Justification

Modern set theory formulated by George Cantor is fundamental for the whole of mathematics. The issues associated with the notion of set are the concept of vagueness and uncertainties. Mathematics requires that all mathematical notions including set must be exact. This vagueness and uncertainties or the representation of imperfect knowledge has been a serious problem for so long a time for logicians and mathematicians. As a result the concept of approximate solution is introduced. Such concepts include the notion of soft set theory, where an approximate solution is provided.

1.4 Aim and Objectives

The aim of this research is to develop algebraic theorems on soft lattice defined via conjunction and disjunction, and present its application to distributed computing system.

The research objectives are to:
(i) redefine the concepts of conjunction and disjunction as binary operations on soft sets and present their properties;

(ii) define soft lattice in terms of the redefined conjunction and disjunction with illustrations;

(iii) introduce soft Boolean algebra and obtain some results;

(iv) extend the concept of soft lattice to distributed, modular and isomorphic soft lattices;

(v) apply soft lattice theory to distributed computing system.

1.5 Methodology

We used the existing literature on development and fundamentals of soft set theory to obtain new results. Various researches on lattice theory including its applications were studied, as guiding principles in extending soft set theory to lattice theory, called soft lattice.

CHAPTER TWO

LITERATURE REVIEW

The origin of soft set theory could be traced to the work of Pawlak (1999) titled *Hard and Soft Set* in Proceeding of the International Workshop on rough sets and knowledge discovery at Banff. His notion of soft sets is a unified view of classical, rough and fuzzy sets. This motivated Molodtsov (1999) titled *soft set theory: first result*. Therein, the basic notions of the theory of soft sets and some of its possible applications were presented. For positive motivation, the work discussed some future problems with regards to the theory.

In order to solidify the theory of soft set, Maji et al. (2002), defined some basic terms of the theory such as equality of two soft sets, subset and super set of a soft set, complement of a soft set, null soft set, and absolute soft set with examples. Binary operations like AND, OR, union and intersection were also defined. De Morgan's laws and a number of results are verified in soft set theory context.

Chen et al. (2005) focused their discussion on the parameterization reduction of soft sets and its applications. First they pointed out that the results of soft set reductions offered in Maji et al. (2002) can be restructured. Therein they observed that the algorithms used to first compute the choice value to select the optimal objects for the decision problems can be redefined and they illustrated this with an example. Finally, they proposed parameterization reduction of soft sets and compared it with the concept of attributes reduction in rough sets theory. Mushrif et al. (2006) studied the texture classification via soft set theory.

Aktas and Cagman (2007) extended the basic properties of soft sets in Maji et al. (2002) and compared the soft sets to the related concept of fuzzy sets

and rough sets. They then gave a definition of soft groups, and derived their basic properties.

Herawan et al. (2010) gave an alternative approach for attribute reduction in multi-valued information system under soft set theory. The work emphasized that based on the notion of multi-soft sets and AND operation, attribute reduction can be defined. It is shown that the results obtained are equivalent with Pawlak's rough reduction. Also, Xu et al. (2010) introduced the notion of vague soft set which is an extension of soft set. Therein the basic properties of vague soft sets were presented and discussed.

Li (2010) defined the soft lattices, soft sub-lattices, discussed its properties, and studied the relation between the soft lattices and the fuzzy soft set. Also, Babitha and Sunil (2010), introduced the concept of soft set relations as a soft subset of the Cartesian product of the soft sets and many related concepts such as equivalent soft set relation, partition, composition and function are discussed. Nagarajan and Meenambigai (2011) extended the study of soft lattices by using soft set theory. The notions of soft lattices, soft distributive lattices, soft modular lattices, soft lattice ideals, soft lattice homomorphisms are introduced and several related properties are investigated.

Ozturk and Inan (2011) illustrated the interconnections between the various operations in soft set and defined the notion of restricted symmetric difference of soft sets and investigated its properties. However, Manemaran (2011) further discussed fuzzy soft sets algebraic structures and defined fuzzy soft group. Operations on fuzzy soft groups and some related results were proved. Furthermore, definitions of fuzzy soft functions and fuzzy soft homomorphism were also defined. Finally, the theorems on homomorphic image and homomorphic pre image were discussed in details.

Ghosh et al. (2011) also extended soft set to ring theory and fuzzy soft ideal

theory. Also, Atagun and Sezgin (2011) studied soft subrings and soft ideals of a ring. Moreover, the concept of soft subfields of a field and soft sub module of a left R-module were defined. Some related properties about soft substructures of rings, fields and modules are investigated and illustrated by some examples.

Sezgin and Atagun (2011) proved that certain De Morgan's law holds in soft set theory with respect to different operations on soft sets. Soft set is being currently extended to intuitionistic fuzzy soft sets and the concept of intuitionistic fuzzy soft sets to semi group theory. The notion of intuitionistic fuzzy soft ideals over a semi group is introduced with their basic properties investigated (Zhou et al., 2011). Also, some lattice structures of the set of all intuitionistic fuzzy soft ideals of a semi group were derived.

Yang and Guo (2011) studied the concept of anti-reflexive kernel, symmetric kernel, reflexive closure, and symmetric closure of a soft set relation. Finally, soft set relation mapping and inverse soft set relation mappings were proposed and some related properties were discussed. Also, in the same year, (Ge and Yan, 2011) further investigated the operational rules given by (Maji *et al.*, 2002 and Ali *et al.*, 2009) to obtain some necessary and sufficient conditions that made corresponding operational rules to hold.

Marudai and Rajendran (2011) studied (Molodtsov, 1999) notion of soft set and fuzzy soft set considering the fact that the parameters are mostly fuzzy hedges or fuzzy parameters. Also study the notion of fuzzy soft lattice on groups, homomorphic image, preimage of fuzzy soft lattice, arbitrary family of fuzzy soft lattice and fuzzy normal soft lattice using T-norms.

Dauda and Ibrahim (2013) developed partial ordering in soft set context. In the same year, Kurt (2013) introduced and investigated the basic concepts of soft set theory and soft group theory. Some examples for soft groups and soft subgroups are also studied.

Akram and Feng (2013) introduced the concept of soft Lie subalgebras (respectively, soft Lie ideals) and state some of their fundamental properties. Also the concept of soft intersection Lie subalgebras (respectively, soft Lie ideals) are introduced and some related properties are investigated. Soft set theory has potential applications in many different fields which include the smoothness of functions, game theory, operations research, Riemann integration, Perron integration, probability theory, and measurement theory (Molodtsov, 1999).

In Molodtsov (1999) applications of soft set to stability and regularization, game theory and operational research were discussed in details. Extension of soft set theory to real analysis and its applications were also presented.

Soft set approaches have been extended to the application of the concept of intuitionistic fuzzy soft sets over semigroup theory Maji et al. (2002). In Maji et al. (2002), the notion of intuitionistic fuzzy soft ideals over a semigroup were introduced and their basic properties were investigated. Some lattice structures of the set of all intuitionistic fuzzy soft ideals of a semigroup were also derived.

Maji et al. (2002) also, applied soft set theory to solve a decision making problem using rough set theory and an algorithm to select the optimal choice of an object was provided. This algorithm uses fewer parameters to select the optimal object for decision problems.

However, in decision making problem in Maji et al. (2002), there is a straightforward relationship between the decision values of objects and the conditional parameters. That is, the decision values are computed with respect to the conditional parameters. This is quite different in the case of rough sets. In rough set theory the decision attributes are not computed according to the conditional attributes.

Karadogan (2008) worked on the application of fuzzy soft set theory in the

selection of underground mining method. Hoever, In (Feng et al., 2008), soft set theory is used to initiate the study of semirings. The notions of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphism are introduced and several related properties are investigated.

Soft set theory based classification algorithm was proposed in Ali et al. (2009) which can be applied to texture classification. This proposed algorithm has very low computational complexity when compared with Bayes classification technique and yield very good classification accuracy .

Cagman et al. (2010) also extended soft set to fuzzy parameterized (fp) soft sets and proposed a decision making method based on FP-soft set theory. With examples, the method was successfully applied to the problems that contain uncertainties.

In Herawan et al. (2010), an alternative approach for attribute reduction in multi-valued information system under soft set theory is presented. Based on the notion of multi-soft sets and AND operation, attribute reduction is defined.

Chet and Das (2010) extended Sanchez's approach to medical diagnosis using interval-valued fuzzy soft sets and exhibit the technique with a hypothetical case study. Also, application of soft set theory has also been extended to the concept of bijective soft set and some of its operations such as the restricted AND and the relaxed AND operations on a bijective soft set, dependency between two bijective soft sets, bijective soft decision system, significance of bijective soft set with respect to bijective soft decision system and reduction of bijective soft decision system were extensively discussed in (Gong et al., 2010). With these notions and operations, an application of bijective soft decision soft set in decision-making problems is revealed.

Kharal and Ahmad (2010) defined mapping on soft classes and studied several properties of images and inverse images of soft sets supported by examples

and counterexamples. These notions were applied to the problem of medical diagnostics in medical expert systems.

Cagman et al. (2011), defined soft matrices and their operations to construct a soft max-min decision making method which can be successfully applied to the problems that contain uncertainties. And also, extended soft set to fuzzy parameterized fuzzy soft sets and explicate their applications in decision making method.

redefined some of the operations of soft sets. By using these new definitions they also construct a decision making method which selects a set of optimum elements from the alternatives. Also, Kalaichelvi and Malini (2011) applied fuzzy soft sets to investment decision making problems.

Babitha and John (2011) defined generalized intuitionistic fuzzy soft sets and set theoretical operations with illustrating examples. It also proposed a similarity measure for generalized intuitionistic fuzzy soft set and used it to find out the similarity between synthetic texture and natural textures. A congenial method for solving multi criteria decision making problem in generalized intuitionistic fuzzy soft set frame work is presented.

In Sut (2012), the concept of soft relations and fuzzy soft relations were discussed and applied in a decision making problem.

Said and Smarandache (2013) combined the concepts of intuitionistic neutrosophic set and soft set together by introducing a new concept called intuitionistic neutrosophic soft set, with its operations namely equal, subset, union, and intersection. And the applications of intuitionistic neutrosophic soft set in decision making problems are presented.

Lamport (1978) in seminar paper argued that the order of events that can be observed in a distributed computation is only partial. Lamport (1978) also

called this partial order the happened-before order and presented a mechanism called logical clocks that gave a timestamp in a totally ordered domain preserving the happened-before order.

Chandy and Lamport (1985) defined a consistent cut, also called a consistent global state. Let E be the set of events of a computation and $\rightarrow$ be the happened-before order on E. A subset G of E is a consistent cut if whenever it contains all element f then it contains an elements e that happened-before f, this concept is identical to the notion of order ideal in the lattice theory.

Mattern (1989) showed that the set of all consistent cuts of a distributed computation forms a lattice. This result is a special case of the theorem in lattice theory that the set of all ideals of a partial order form a distributive lattice.

Charron-Bost (1991) gave a lower bound on the dimension of vector clocks using dimension theory of partial order. Dimension theory of partial order was initiated by Dushnik and Miller (1941).

Chase and Garg (1995) defined linear predicates for efficient detection of global predicates. Chase and Garg (1995) also show that a predicate B is linear if and only if the set of consistent cuts satisfying the predicate B is closed under the meet operation of the lattice of consistent cuts.

Garg and Skawrananond (2001) defined a special types of partial order called string and showed that Fidge-Mattern vector clock corresponds to a string realizer of a poset. Garg and Skawrananond (2001) also applied Dilworth's theorem for the dimension of a finite distributive lattice to show that any vector clock mechanism that can timestamp a consistent cut of a distributed computation on N processes must have dimension at least N.

Sen and Garg (2002) developed algorithms to compute slices for temporal logic

formulas. These algorithms are useful in detecting temporal logic formulas in a distributed computation. Sen and Garg (2002) implemented a tool called Partial Order Trace Analyzer for evaluating temporal logic formulas on partial order traces.

Feng et al. (2008) extended the study of soft set to soft semirings. The notions of soft semirings, soft subsemirings, soft ideals, idealistic soft semirings and soft semiring homomorphisms were introduced, and several related properties were investigated.

The theory of BCK/BCI–algebras is now being studied via soft set theory Jun et al. (2008) applied the notion of soft sets to commutative ideals of BCK–algebras. Implicative soft ideals and commutative soft ideals were also investigated.

Soft set theoretic approach for dimensionality reduction and fuzzy parameterized fuzzy soft set theory and its applications were discussed in Cagman et al. (2010) where soft set theory is applied to commutative ideals in BCK-algebras.

Qin and Hon (2010) dealt with algebraic structure of soft sets. The lattice structures of soft sets were constructed. The concept of soft equality was introduced and some related properties were derived. It is proved that soft equality is a congruence relation with respect to some operations and the soft quotient algebra was introduced.

The notion of commutative soft ideals and commutative idealistic soft BCK-algebras are introduced in (Jun et al., 2011). Example to show that there are no relations between positive implicative idealistic soft BCK-algebras and commutative idealistic soft BCK-algebras is provided.

Jun et al. (2011) applied the notion of soft sets to the theory of BCK- algebras. The notions of positive implicative soft ideals and positive implicative idealistic

soft BCK- algebra were introduced and their basic properties were derived.

Ghosh *et a*l., (2011) defined fuzzy soft ring and study some of its algebraic properties. Thereafter, defined fuzzy soft ideal and discussed a few of its properties.

Liu and Xu (2011) applied the idea of soft set theory to lattice implication algebra, the novel concept of (implicative) filteristic soft lattice implication algebras which related to (implicative) filter for short (IF) F-soft lattice implication algebras are introduced. Basic properties of (IF) F-soft lattice implication algebras are derived. Some characterizations for a soft set to be a (IF) F-soft lattice implication algebras are provided.

Ibrahim *et al.*, (2012) reinvestigated relations and function in soft set context with emphasize on matrix representation of the composition of soft set relations and how it satisfied War shall algorithm.

Yilmaz and Kazanci (2013) inspired by the study of algebraic structures of soft sets, to initiate research on the connection between soft sets and lattice structures. Also applying the notion of soft sets to lattices. Some related notions, such as those of soft lattices (ideas, filter) are defined, and several basic properties are discussed by using the soft set theory.

Zheng et al. (2013) introduced the concepts of soft lattice implication subalgebras, endow a parameter set as lattice implication algebra, and further discussed its equivalent characterization. Then, new operations of soft lattice implication subalgebras are introduced, under which two soft lattice implication subalgebras is also a soft lattice implication subalgebra. Finally, the concepts of image and preimage of a soft lattice implication subalgebra and their properties are presented.

Abood and Kadhim (2014) dealt with the algebraic structure on some collec-

tion of soft lattice, defined the notions of Borel soft lattice, product Borel soft lattice, intersection (union) Borel soft lattice and measurable soft lattice.

Having gone through the existing literature on soft set theory, we observed that introduction of soft Boolean algebra, development of soft lattice theory in an algebraic context and its application to distributed computing system are paramount to the enrichment of soft set theory.

CHAPTER THREE

FUNDAMENTALS OF SOFT SET THEORY

In this chapter, fundamentals of soft set theory, operations and its various algebraic structures are presented with redefined concept of conjunction and disjunction as binary operations.

3.1 Soft Set Theory

In this section, some basic definitions of soft theory and some suitable examples are given. Let U be a universal set and E be the set of all possible parameters under consideration with respect to U. Let the power set of U (i.e., the set of all subsets of U) be denoted by $P(U)$ and A is a subset of the parameters, E (A$\subseteq E$). The parameters are attributes, characteristics or properties associated with the objects in U. Then we have the following: definitions.

Definition 3.1.1 A pair (F, E) is called a soft set over U if and only if F is a mapping of E into the set of all subsets of the set U. That is, a soft set is a parametrized family of subsets of the set U. For all $e \in E$, $F(e)$ is considered as the set of $e-$approximate elements of the soft set (F, E).

Definition 3.1.2

A soft set (F, E) over a universe U is said to be *null* or *empty* soft set denoted by $\widetilde{\emptyset}$, if $\forall e \in E$, $F(e) = \emptyset$.

Definition 3.1.3

A soft set (F, A) over a universe U is called *absolute* or *universal soft set* denoted by $\widetilde{(F, A)}$ or $\tilde{U}$, if $\forall e \in E$, $F(e) = U$.

Definition 3.1.4

Let $E = \{e_1, e_2, e_3, \quad . \ . \ ., \ e_n\}$ be a set of parameters. The *not-set* of E denoted by $\neg E$ is defined as $\neg E = \{\neg e_1, \neg e_2, \neg e_3, \quad . \ . \ ., \neg e_n)$.

Definition 3.1.5

The *complement* of a soft set (F, E), denoted by $(F, \ E)^c$, is defined as $(F, \ E)^c \ = (F^c, \neg E)$.

Where F^c: $\neg E \ \rightarrow P(U)$ is a mapping given by $F^c(\alpha) \ = U \ - \ F(\neg\alpha)$, $\alpha \in \neg E$. F^c is called the soft complement function of F. Consequently, $(F^c)^c = F$ and $((F, E)^c)^c = (F, E)$

Definition 3.1.6

Let $(F, \ A)$ and $(G, \ B)$ be any two soft sets over a common universe U, $(F, \ A)$ is called a soft subset of $(G, \ B)$, denoted by $(F, \ A) \ \tilde{\subset} \ (G, \ B)$ if ;

(i) $A \ \subset \ B$, and

(ii) $\forall e \in \ A, \ F(e) = G(e)$

$(F, \ A)$ is said to be a soft super set of $(G, \ B)$, if $(G, \ B)$ is a subset of $(F, \ A)$ and it is denoted by $(F, \ A) \ \tilde{\supset} \ (G, \ B)$.

Definition 3.1.7

Two soft sets $(F, \ A)$ and $(G, \ B)$ over a common universe U are said to be soft equal, denoted by $(F, \ A) = \ (G, \ B)$, if $(F, \ A)$ is a soft subset of $(G, \ B)$ and $(G, \ B)$ is a soft subset of (F, A).

Definition 3.1.8

If (F, A) and (G, B) are two soft sets then "$(F, A) \ AND \ (G, B)''$ denoted by $(F, A) \wedge (G, B)$ is defined as $(F, A) \wedge (G, B) = (H, \ A \times B)$, where

$H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, $\forall(\alpha, \beta) \in A \times B$.

Definition 3.1.9

If (F, A) and (G, B) are two soft sets then "(F, A) OR $(G, B)''$ denoted by $(F, A) \vee (G, B)$ is defined as $(F, A) \vee (G, B) = (P, A \times B)$, where, $P(\alpha, \beta) = F(\alpha) \cup G(\beta)$, $\forall$ $(\alpha, \beta) \in A \times B$.

3.2 Soft Set Operations

Definition 3.2.1

Let (F, A) and (G, B) be two soft sets over a common universe U. The union or extended union of (F, A) and (G, B), denoted by $(F, A) \widetilde{\cup} (G, B)$ or $(F, A) \widetilde{\cup}_E (G, B)$, is the soft set (H, C) satisfying the following conditions:

$$\text{(i) } C = A \cup B, \text{ (ii) } \forall\ e \in C,\ H(e) = \begin{cases} F(e) & \text{if } e \in A \backslash B \\ G(e) & \text{if } e \in B \backslash A \\ F(e) \cup G(e) & \text{if } e \in A \cap B \end{cases}$$

Definition 3.2.2

The intersection of two soft sets (F, A) and (G, B) over a common universe set U is the soft set (H, C), where $C = A \cap B$, and $\forall e \in C$, $H(e) = F(e)$ or $G(e)$, we write $(F, A) \widetilde{\cap} (G, B) = (H, C)$

Definition 3.2.3

The extended intersection of soft sets $(F, A)$and (G, B) over a common universe U, denoted by $(F, A) \widetilde{\cap}_E (G, B)$, is the soft set (H, C), where $C = A \cup B$ $\forall e \in C$

and

$$H(e) = \begin{cases} F(e) & \text{if } e \in A \backslash B \\ G(e) & \text{if } e \in B \backslash A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}$$

Definition 3.2.4

The restricted intersection of soft sets $(F, A)$and (G, B) over a common universe U, denoted by $(F, A) \widetilde{\cap}_R (G, B)$, is the soft set (H, C), where $C = A \cap B \neq \emptyset$ such that $H(e) = F(e) \cap G(e), \ \forall e \in C$.

Definition 3.2.5

Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The restricted union of (F, A) and (G, B), denoted by $(F, \ A) \ \widetilde{\cup}_R \ (G, B)$, is defined as $(F, A) \ \widetilde{\cup}_R \ (G, B) = (H, C)$, where $C = A \cap B$, and $\forall e \in C, \ H(e) = F(e) \cup G(e)$.

Definition 3.2.6

Let (F, A) and (G, B) be two soft sets over a common universe U such that $A \cap B \neq \emptyset$. The restricted difference of (F, A) and (G, B) denoted by $(F, A) \ \sim_R \ (G, B)$, is defined as $(F, A) \ \sim_R \ (G, B) = (H, C)$, where $C = A \cap B$, and $\forall e \in C, \ H(e) = F(e) \backslash G(e)$.

Definition 3.2.7

The restricted symmetric difference of two soft sets (F, A) and (G, B) over a common universe U is defined as

$$(F, A) \ \triangle (G, B) = ((F, A) \ \cup_R \ (G, B)) \sim_R ((F, A) \ \cap_R \ (G, B)) .$$

3.3 Properties of Soft Set Operations

In this section, we present various properties of soft set operations like: union or extended union, restricted union, intersection, extended intersection, restricted intersection, differences, symmetric differences, conjunction and disjunction.

Proposition 3.3.1 Qin and Hon (2010) Properties of the union ($\widetilde{\cup}$) operation.

Let (F,A), (G,B), $(H,\ C)$ be soft set over a common universe U. Then

i. $(F,A)\,\widetilde{\cup}\,\big((G,B)\,\widetilde{\cup}\,(H,C)\big) = \big((F,A)\,\widetilde{\cup}\,(G,B)\big)\,\widetilde{\cup}\,(H,C)$

ii. $(F,A)\,\widetilde{\cup}U_A = U_A$,

iii. $(F,A)\,\widetilde{\cup}\emptyset_A = (F,A)$,

iv. $(F,A)\,\widetilde{\cup}\,(G,B) = \emptyset_A$ if and only if $(F,A) = \emptyset_A$ and $(G,B) = \emptyset_A$

v. $(F,A)\,\widetilde{\cup}\,\big((G,B)\,\widetilde{\cap}\,(H,C)\big) = \big((F,A)\,\widetilde{\cup}\,(G,B)\big)\,\widetilde{\cap}\,\big((F,A)\,\widetilde{\cup}\,(H,C)\big)$,

vi. $\big((F,A)\,\widetilde{\cap}\,(G,B)\big)\,\widetilde{\cup}\,(H,C) = \big((F,A)\,\widetilde{\cup}\,(H,C)\big)\,\widetilde{\cap}\,\big((G,B)\,\widetilde{\cup}\,(H,C)\big)$.

Proposition 3.3.2 Maji *et al.*(2003) Properties of the restricted union ($\widetilde{\cup}_{\mathbf{R}}$) operation.

Let (F,A), (G,B), $(H,\ C)$ be soft set over a common universe U. Then

i. $(F,A)\,\widetilde{\cup}_R\,\big((G,B)\,\widetilde{\cup}_R\,(H,C)\big) = \big((F,A)\,\widetilde{\cup}_R\,(G,B)\big)\,\widetilde{\cup}_R\,(H,C)$

ii. $(F,A)\,\widetilde{\cup}_R U_A = U_A$,

iii. $(F,A)\,\widetilde{\cup}_R\emptyset_A = (F,A)$,

iv. $(F,A)\,\widetilde{\cup}_R\,(G,B) = \emptyset_A$ if and only if $(F,A) = \emptyset_A$ and $(G,B) = \emptyset_A$

v. $(F,A)\,\widetilde{\cup}_R\,\big((G,B)\,\widetilde{\cap}\,(H,C)\big) = \big((F,A)\,\widetilde{\cup}_R\,(G,B)\big)\,\widetilde{\cap}\,\big((F,A)\,\widetilde{\cup}_R\,(H,C)\big)$,

vi. $\big((F,A)\,\widetilde{\cap}\,(G,B)\big)\,\widetilde{\cup}_R\,(H,C) = \big((F,A)\,\widetilde{\cup}_R\,(H,C)\big)\,\widetilde{\cap}\,\big((G,B)\,\widetilde{\cup}_R\,(H,C)\big)$.

vii. $(F,A)\,\widetilde{\cup}_R\,\big((G,B)\,\widetilde{\cap}_E\,(H,C)\big) = \big((F,A)\,\widetilde{\cup}_R\,(G,B)\big)\,\widetilde{\cap}_E\,\big((F,A)\,\widetilde{\cup}_R\,(H,C)\big)$,

viii. $\big((F,A)\,\widetilde{\cap}_E\,(G,B)\big)\,\widetilde{\cup}_R\,(H,C) = \big((F,A)\,\widetilde{\cup}_R\,(H,C)\big)\,\widetilde{\cap}_E\,\big((G,B)\,\widetilde{\cup}_R\,(H,C)\big)$.

Proposition 3.3.3 Maji *et al.*(2003) Properties of the intersection ($\widetilde{\cap}$) operation.

Let $(F,A)\,,\ (G,B)\,,\ (H,\ C)$ be soft set over a common universe U. Then

i. $(F,A)\,\widetilde{\cap}\,\big((G,B)\,\widetilde{\cap}\,(H,C)\big) = \big((F,A)\,\widetilde{\cap}\,(G,B)\big)\,\widetilde{\cap}\,(H,C)$

ii. $(F,A)\,\widetilde{\cap}U_A = (F,A)$,

iii. $(F,A)\,\widetilde{\cap}\emptyset_A = \emptyset_A$,

iv. $(F,A)\,\widetilde{\cap}\,\big((G,B)\,\widetilde{\cup}_R\,(H,C)\big) = \big((F,A)\,\widetilde{\cap}\,(G,B)\big)\,\widetilde{\cup}_R\,\big((F,A)\,\widetilde{\cap}\,(H,C)\big)$,

v. $\big((F,A)\,\widetilde{\cup}\,(G,B)\big)\,\widetilde{\cap}\,(H,C) = \big((F,A)\,\widetilde{\cap}\,(H,C)\big)\,\widetilde{\cup}\,\big((G,B)\,\widetilde{\cap}\,(H,C)\big)$.

vi. $(F,A)\,\widetilde{\cap}\,\big((G,B)\,\widetilde{\cup}\,(H,C)\big) = \big((F,A)\,\widetilde{\cap}\,(G,B)\big)\,\widetilde{\cup}\,\big((F,A)\,\widetilde{\cap}\,(H,C)\big)$,

vii. $\big((F,A)\,\widetilde{\cup}\,(G,B)\big)\,\widetilde{\cap}\,(H,C) = \big((F,A)\,\widetilde{\cap}\,(H,C)\big)\,\widetilde{\cup}\,\big((G,B)\,\widetilde{\cap}\,(H,C)\big)$.

viii. $(F,A)\,\widetilde{\cap}\,((G,B)\sim_R(H,C)) = \big((F,A)\,\widetilde{\cap}\,(G,B)\big)\sim_R\big((F,A)\,\widetilde{\cap}\,(H,C)\big)$,

ix. $((F,A)\sim_R(G,B))\,\widetilde{\cap}\,(H,C) = \big((F,A)\,\widetilde{\cap}\,(H,C)\big)\sim_R\big((G,B)\,\widetilde{\cap}\,(H,C)\big)$.

Proposition 3.3.4 Maji *et al.*(2003) Properties of the extended intersection ($\widetilde{\cap}_{\mathbf{E}}$) Operation.

Let $(F,A)\,,\ (G,B),\ (H,\ C)$ be soft set over a common universe U. Then

i. $(F,A)\ \widetilde{\cap}_E\big((G,B)\ \widetilde{\cap}_E\,(H,C)\big) = \big((F,A)\ \widetilde{\cap}_E\,(G,B)\big)\ \widetilde{\cap}_E\,(H,C)$

ii. $(F,A)\ \widetilde{\cap}_E U_A = (F,A)$,

iii. $(F,A)\ \widetilde{\cap}_E \emptyset_A = \emptyset_A$,

iv. $(F,A)\ \widetilde{\cap}_E \big((G,B)\,\widetilde{\cup}_R\,(H,C)\big) = \big((F,A)\ \widetilde{\cap}_E\,(G,B)\big)\,\widetilde{\cup}_R\,\big((F,A)\ \widetilde{\cap}_E\,(H,C)\big)$,

v. $\big((F,A)\,\widetilde{\cup}_R\,(G,B)\big)\,\widetilde{\cap}_E\,(H,C) = \big((F,A)\,\widetilde{\cap}_E\,(H,C)\big)\,\widetilde{\cup}_R\,\big((G,B)\,\widetilde{\cap}_E\,(H,C)\big)$.

Proposition 3.3.5 Maji *et al.*(2003) Properties of the restricted difference ($\sim_{\mathbf{R}}$) operation.

Let $(F,A)\,,\ (G,B)\,,\ (H,\ C)$ be soft set over a common universe U. Then

i. $(F,A)\sim_R \emptyset_A = (F,A)$

ii. $(F,A)\sim_R (F,A) = \emptyset_A$

iii. $U_A \sim_R (F,A) = (F,A)^c$

iv. $(F,A)\sim_R \big((G,B)\ \widetilde{\cap}\,(H,C)\big) = ((F,A)\sim_R (G,B))\,\widetilde{\cup}_R\,((F,A)\sim_R (H,C))$

v. $(F,A)\sim_R \big((G,B)\ \widetilde{\cup}_R\,(H,C)\big) = ((F,A)\sim_R (G,B))\,\widetilde{\cap}\,((F,A)\sim_R (H,C))$

vi. $(F,A)\sim_R \big((G,B)\ \widetilde{\cap}_E\,(H,C)\big) = ((F,A)\sim_R (G,B))\,\widetilde{\cup}\,((F,A)\sim_R (H,C))$

vii. $(F,A)\sim_R \big((G,B)\ \widetilde{\cup}\,(H,C)\big) = ((F,A)\sim_R (G,B))\,\widetilde{\cap}_E\,((F,A)\sim_R (H,C))$

Proposition 3.3.6 Maji *et al.*(2003) Properties of Conjunction ($\wedge$) and disjunction ($\vee$)

Let $(F,A)\,,\ (G,B)\,,\ (H,\ C)$ be soft set over a common universe U. Then

1. *Idempotent properties*

(i) $(F,A)\vee(F,A) = (F,A)\ \forall\ (x,y)\in(A\times A)$, where $x=y$

(ii) $(F,A)\wedge(F,A) = (F,A)\ \forall\ (x,y)\in(A\times A)$, where $x=y$.

Proof

(i) Let $(F,A) \vee (F,A) = (H,\ A \times A)\,,\ \forall (x,x) \in A \times A,\ H\,(x,x) = F\,(x) \cup F\,(x) = F(x)$

i.e., $(F,A) \vee (F,A) = (F,A)$

Similarly, for a(ii)

$$(F,A) \wedge (F,A) = (F,A)$$

Example 3.3.1

Let $U = \{h_1, h_2, h_3, h_4\}$ be the set of common universe and let $A = \{e_1, e_2, e_3\}$,
$(F,A) = \{F\,(e_1) = \{h_1, h_2\}\,, F\,(e_2) = \{h_2, h_3\}\,,\ F\,(e_3) = \{h_3\}\ \}$

$$(F,A) \vee (F,A) = (H, A \times A\ ),$$

where $A \times A = \{(e_1,e_1)\,,(e_1,e_2)\,,(e_1,e_3)\,,(e_2,e_1)\,,(e_2,e_2)\,,(e_2,e_3)\,,(e_3,e_1)\,,(e_3,e_2)\,,(e_3,e_3)\}$

$$(H, A \times A\ ) = \left\{ \begin{array}{c} H\,(e_1,e_1) = \{h_1,h_2\}\,, H\,(e_1,e_2) = \{h_1,h_2,h_3\}\,, H\,(e_1,e_3) = \{h_1,h_2,h_3\}\,, \\ H\,(e_2,e_1) = \{h_1,h_2,h_3\}\,, H\,(e_2,e_2) = \{h_2,h_3\}\,, H\,(e_2,e_3) = \{h_2,h_3\} \\ H\,(e_3,e_1) = \{h_1,h_2,h_3\}\,, H\,(e_3,e_2) = \{h_2,h_3\}\,, H\,(e_3,e_3) = \{h_3\} \end{array} \right\}$$

Hence, $(F,A) \wedge (F,A) = (F,A)$ if $\forall\ (e_1,e_2) \in A \times A$, $e_1 = e_2$

2. *Commutative properties*

(i) $(F,A) \vee (G,B) = (G,B) \vee (F,A)\ \forall\ (x,y) \in (A \times B)$, where $x = y$

(ii) $(F,A) \wedge (G,B) = (G,B) \wedge (F,A)\ \forall\ (x,y) \in (A \times B)$, where $x = y$

Proof

b(i) Let $(F,A) \vee (G,B) = (H, A \times B)\,,\ \forall\,(x,y) \in A \times B,\ H\,(x,y) = F(x) \cup G(y)$

$$= G\,(x) \cup F\,(y)$$

$$= H\,(y,x)\,,\ \forall (y,x) \in B \times A$$

$$= (H, B \times A),$$

i.e., $(F, A) \vee (G, B) = (H, A \times B) = (H, B \times A) = (G, B) \vee (F, A)$

hence, $(F, A) \vee (G, B) = (G, B) \vee (F, A)$.

Similarly, for b(ii) $(F, A) \wedge (G, B) = (G, B) \wedge (F, A)$

Example 3.3.2

Let $U = \{h_1, h_2, \ldots, h_{10}\}$ be set of common universe and let $A = \{e_1, e_2, e_3\}$, $B = \{e_3, e_4, e_5\}$

$(F, A) = \{F(e_1) = \{h_2, h_4, h_7, h_8\}, F(e_2) = \{h_1, h_3, h_4\}, F(e_3) = \{h_6, h_9, h_{10}\}\}$

$(G, B) = \{G(e_3) = \{h_6, h_9, h_{10}\}, G(e_4) = \{h_2, h_3, h_7\}, G(e_5) = \{h_5, h_6, h_8\}\}$

$(F, A) \wedge (G, B) = (H, A \times B)$, where $A \times B = \left\{ \begin{array}{c} (e_1, e_3), (e_1, e_4), (e_1, e_5), (e_2, e_3), (e_2, e_4) \\ (e_2, e_5), (e_3, e_3), (e_3, e_4), (e_3, e_5) \end{array} \right\}$

$$(H, A \times B) = \left\{ \begin{array}{c} H(e_1, e_3) = \{\emptyset\}, H(e_1, e_4) = \{h_2, h_7\}, H(e_1, e_5) = \{\emptyset\}, H(e_2, e_3) = \{\emptyset\}, \\ H(e_2, e_4) = \{h_3\}, H(e_2, e_5) = \{\emptyset\}, H(e_3, e_3) = \{h_6, h_9, h_{10}\}, \\ H(e_3, e_4) = \{\emptyset\}, H(e_3, e_5) = \{h_6\} \end{array} \right\}$$

$(G, B) \wedge (F, A) = (H, B \times A)$, where $B \times A = \left\{ \begin{array}{c} (e_3, e_1), (e_3, e_2), (e_3, e_3), (e_4, e_1), (e_4, e_2) \\ (e_4, e_3), (e_5, e_1), (e_5, e_2), (e_5, e_3) \end{array} \right\}$

$$(H, B \times A) = \left\{ \begin{array}{c} K(e_3, e_1) = \{\emptyset\}, K(e_3, e_2) = \{\emptyset\}, K(e_3, e_3) = \{h_6, h_9, h_{10}\}, \\ K(e_4, e_1) = \{h_2, h_7\}, \\ K(e_4, e_2) = \{h_3\}, K(e_4, e_3) = \{\emptyset\}, K(e_5, e_1) = \{\emptyset\}, \\ K(e_5, e_2) = \{\emptyset\}, K(e_5, e_3) = \{h_6\} \end{array} \right\}$$

Hence, $(H, A \times B) = (H, B \times A)$ if $\forall\ (x, y) \in (A \times B)$, then $x = y$.

$\Rightarrow (F, A) \wedge (G, B) = (G, B) \wedge (F, A)$.

3. *Associative properties*

(i) $((F, A) \vee (G, B)) \vee (H, C) = (F, A) \vee ((F, A) \vee (G, B))$ $\forall\ (x, y, z) \in (A \times B)$, where $x = y = z$.

(ii) $((F, A) \wedge (G, B)) \wedge (H, C) = (F, A) \wedge ((F, A) \wedge (G, B))$ $\forall\ (x, y, z) \in (A \times B)$, where $x = y = z$.

Proof

From LHS (i):

$((F, A) \vee (G, B)) \vee (H, C) = (K, A \times B) \vee (\mathrm{H}, \mathrm{C})$

$$= (\mathrm{J},\ \mathrm{A} \times \mathrm{B} \times \mathrm{C}) \tag{3.3.1}$$

From RHS (ii):

$(F, A) \vee ((F, A) \vee (G, B)) = (F, A) \vee (\mathrm{L},\ \mathrm{B} \times \mathrm{C})$

$$= (\mathrm{M},\ \mathrm{A} \times \mathrm{B} \times \mathrm{C}) \tag{3.3.2}$$

Equating (5.5.1) and (5.5.3), we have ,

$$((F, A) \vee (G, B)) \vee (H, C) = (F, A) \vee ((F, A) \vee (G, B))$$

Similarly, $((F, A) \wedge (G, B)) \wedge (H, C) = (F, A) \wedge ((F, A) \wedge (G, B))$.

4. *Distributive properties*

(i) $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

(ii) $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

Proof

From LHS of (i):

$(F, A) \vee ((G, B) \wedge (H, C)) = (F, A) \vee (\mathrm{K},\ \mathrm{B} \times \mathrm{C})$

$$= (\mathrm{J},\ \mathrm{A} \times \mathrm{B} \times \mathrm{C}) \tag{3.3.3}$$

From RHS of (i)

$((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C)) = (L, A \times B) \wedge (\mathrm{M},\ \mathrm{A} \times \mathrm{C})$

$$= (O, A \times B \times A \times C) \tag{3.3.4}$$

Equating (5.5.1) and (5.5.3), we have,

$(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$, If $C = A \times C$.

Similarly,

$$(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C)) .$$

5. *Absorption properties*

(i) $(F, A) = (F, A) \vee ((F, A) \wedge (G, B)) \ \forall\ (x, y) \in (A \times B)$, where $x = y$.

(ii) $(F, A) = (F, A) \wedge ((F, A) \vee (G, B)) \ \forall\ (x, y) \in (A \times B)$, where $x = y$.

Proof

From LHS of (i):

$(F, A) = (F, A) \vee ((F, A) \wedge (G, B)) = ((F, A) \vee (F, A)) \wedge ((F, A) \vee (G, B))$

$= (F, A) \wedge ((F, A) \vee (G, B))$

$= (F, A)$ By definition (ii)

Hence,

$$(F, A) = (F, A) \vee ((F, A) \wedge (G, B))$$

Similarly, for (ii)

$$(F, A) = (F, A) \wedge ((F, A) \vee (G, B))$$

6. *Complement properties*

(i) $(F, A) \vee (F, A)^c = U$

(ii) $(F, A) \wedge (F, A)^c = \emptyset$

Proof

From LHS of (i):

$(F, A) \vee (F, A)^c = (F, A) \vee (F^c, \neg A)$

$= (H, A \times \neg A)\,,\ \forall x \in (A \times \neg A)$, then

$H(x) = F(x) \cup F^c(x) = U$ Hence, $(F, A) \vee (F, A)^c = U$.

From RHS of (ii)

$(F, A) \wedge (F, A)^c = (F, A) \wedge (F^c, \neg A)$

$= (H, A \times \neg A)\,,\ \forall x \in (A \times \neg A)$, then

$H(x) = F(x) \cap F^c(x) = \emptyset$ Hence, $(F, A) \wedge (F, A)^c = \emptyset$.

7. *Identity properties*

(i) $(F, A) \vee \emptyset = (F, A)$

(ii) $(F, A) \wedge \emptyset \ = \emptyset$

(iii) $(F, A) \vee U{=}U$

(iv) $(F, A) \wedge U = (F, A)$

Proof

From LHS (i):

$$\begin{aligned}
(F, A) \vee \emptyset &= (F, A) \vee ((F, A) \wedge (F, A)^c) \\
&= ((F, A) \vee (F, A)) \wedge ((F, A) \vee (F, A)^c) \\
&= (F, A) \wedge ((F, A) \vee (F, A)^c) \\
&= (F, A) \wedge U \\
&= (F, A).
\end{aligned}$$

Hence, $(F, A) \vee \emptyset = (F, A)$

From LHS of (ii):

$$\begin{aligned}
(F, A) \wedge \emptyset &= (F, A) \wedge ((F, A) \wedge (F, A)^c) \\
&= ((F, A) \wedge (F, A)) \wedge ((F, A) \wedge (F, A)^c) \\
&= ((F, A) \wedge (F, A)) \wedge (F, A)^c \\
&= ((F, A) \wedge (F, A)^c) \wedge ((F, A) \wedge (F, A)^c) \\
&= \emptyset \wedge \emptyset \\
&= \emptyset.
\end{aligned}$$

Hence, $(F, A) \wedge \emptyset = \emptyset$.

From LHS (iii):

$$\begin{aligned}(F,A)\vee U &= (F,A)\vee((F,A)\vee(F,A)^c)\\ &= ((F,A)\vee(F,A))\vee((F,A)\vee(F,A)^c)\\ &= (F,A)\vee((F,A)\vee(F,A)^c)\\ &= (F,A)\vee U\\ &= U\end{aligned}$$

Hence, $(F,A)\vee\emptyset = U$.

From LHS of (iv):

$$\begin{aligned}(F,A)\wedge U &= (F,A)\wedge((F,A)\vee(F,A)^c)\\ &= ((F,A)\wedge(F,A))\vee((F,A)\wedge(F,A)^c)\\ &= (F,A)\vee((F,A)\wedge(F,A)^c)\\ &= (F,A)\vee\emptyset\\ &= (F,A)\end{aligned}$$

Hence, $(F,A)\wedge U = (F,A)$.

8. *De Morgan's properties*

(i) $((F,\ A)\vee(G,B))^c = (F,\ A)^c\wedge(G,\ B)^c$

(ii) $((F,\ A)\wedge(G,B))^c = (F,\ A)^c\vee(G,\ B)^c$

Proof

(i) Suppose that $(F,\ A)\vee(G,\ B) = (O,\ A\times B)$.

$\Rightarrow ((F,\ A)\vee(G,B))^c = (O,\ A\times B)^c = (O^c,\ \neg(A\times B))$. Now,

$$(F,\ A)^c\wedge(G,\ B)^c = (F^c,\ \neg A)\wedge(G^c,\ B)$$

$$= (J,\ \ \neg A \times \neg B)$$

$$= (J,\ \neg (A \times B))$$

Now, take $(\neg\alpha,\ \ \neg\beta) \in\ \neg (A \times B)$.

$\Rightarrow O^c (\neg\alpha,\ \neg\beta) = U - O (\alpha,\ \beta)$

$$= U - [F(\alpha) \cup G(\beta)]$$

$$= [U - F(\alpha)] \cap [U - G(\beta)]$$

$$= F^{\mathbf{c}} (\neg\alpha) \cap G^{\mathbf{c}} (\neg\beta)$$

$$= J (\neg\alpha,\ \neg\beta)$$

$\Rightarrow\ O^c = J$. Hence, proved.

(ii) Suppose that $(F,\ A) \wedge (G,\ B) = (H,\ A \times B)$.

$\Rightarrow ((F,\ \ A) \wedge (G, B))^c = (H,\ A \times B)^c = (H^c,\ \neg(A \times B))$. Now,

$$(F,\ A)^c \vee (G,\ B)^c = (F^c,\ \neg A) \vee (G^c,\ B)$$

$$= (K,\ \ \neg A \times \neg B)$$

$$= (K,\ \neg (A \times B))$$

Now, take $(\neg\alpha,\ \ \neg\beta) \in\ \neg (A \times B)$

$\Rightarrow H^c (\neg\alpha,\ \neg\beta) = U - H (\alpha,\ \beta)$

$$= U - [F(\alpha) \cap G(\beta)]$$

$$= [U - F(\alpha)] \cup [U - G(\beta)]$$

$$= F^{\mathbf{c}} (\neg\alpha) \cup G^{\mathbf{c}} (\neg\beta)$$

$$= J (\neg\alpha,\ \neg\beta)$$

$\Rightarrow H^c = K$. Hence, proved.

3.4 Algebraic Structures via Soft Sets

In this section we present various algebraic structures via soft sets with reference to the binary operations, union, restricted union, intersection, restricted intersection, extended intersection, restricted differnce, conjunction and disjunction on soft sets. The structures, are semigroup, commutative monoid, semiring and lattices.

1. Semigroup

i. $(S(F,E), \cup)$ is a semigroup, since for all (F,A), (G,B), $(H,C) \in S(F,E)$
$(F,A) \cup ((G,B) \cup (H,C)) = ((F,A) \cup (G,B)) \cup (H,C)$,

ii. $(S(F,E), \cap)$ is a semigroup, since for all (F,A), (G,B), $(H,C) \in S(F,E)$
$(F,A) \cap ((G,B) \cap (H,C)) = ((F,A) \cap (G,B)) \cap (H,C)$,

iii. $(S(F,E), \bigvee)$ is a semigroup, since for all (F,A), (G,B), $(H,C) \in S(F,E)$
$(F,A) \bigvee ((G,B) \bigvee (H,C)) = ((F,A) \bigvee (G,B)) \bigvee (H,C)$

iv. $(S(F,E), \bigwedge)$ is a semigroup, since for all (F,A), (G,B), $(H,C) \in S(F,E)$
$(F,A) \bigwedge ((G,B) \bigwedge (H,C)) = ((F,A) \bigwedge (G,B)) \bigwedge (H,C)$,

v. $(S(F,E), \bigcup_R)$ is a semigroup, since for all (F,A), (G,B), $(H,C) \in S(F,E)$ $(F,A) \bigcup_R ((G,B) \bigcup_R (H,C)) = ((F,A) \bigcup_R (G,B)) \bigcup_R (H,C)$

vi. $(S(F,\mathrm{E}), \bigcap_R)$ is a semigroup, since for all (F,A), (G,B), $(H,C) \in S(F,E)$ $(F,A) \bigcap_R ((G,B) \bigcap_R (H,C)) = ((F,A) \bigcap_R (G,B)) \bigcap_R (H,C)$,

vii. $(S(F,E), \bigcup_E)$ is a semigroup, since for all (F,A), (G,B), $(H,C) \in S(F,E)$ $(F,A) \bigcup_E ((G,B) \bigcup_E (H,C)) = ((F,A) \bigcup_E (G,B)) \bigcup_E (H,C)$,

viii. $(S(F,A), \bigcap_E)$ is a semigroup, since for all (F,A), $(G,B), (H,C) \in S(F,E)$ $(F,A) \bigcap_E ((G,B) \bigcap_E (H,C)) = ((F,A) \bigcap_E (G,B)) \bigcap_E (H,C)$.

2. Commutative monoids

i. $(S(F,E), \bigcup_R)$ is a monoid with $\emptyset_E$ as an identity element, $(S(F,A), \bigcup_R)$ is a subsemigroup of $(S(F,E), \bigcup_R)$,

ii. $(S(F,E), \bigcap_R)$ is a monoid with U_E as an identity element, $(S(F,A), \bigcap_R)$ is a subsemigroup of $(S(F,E), \bigcap_R)$,

iii. $(S(F,E), \bigcup_E)$ is a monoid with $\emptyset_\emptyset$ as an identity element, $(S(F,A)_A, \bigcup_E)$ is a subsemigroup of $(S(F,E), \bigcup_E)$,

iv. $(S(F,E), \bigcap_E)$ is a monoid with $U_\emptyset$ as an identity element, $(S(F,A)_A, \bigcap_E)$ is a subsemigroup of $(S(F,E), \bigcap_E)$,

3. Semirings

i. $(S(F,E), \bigcup_R, \bigcap_R)$ is a commutative, idempotent hemiring with identity U_E, since $(F,A) \bigcap_R U_E = (F,A)$, for every (F,A) in $S(F,E)$.

ii. $(S(F,E), \bigcup_R, \bigcup_E)$ is a commutative, idempotent hemiring with identity $\emptyset_\emptyset$, since $(F,A) \bigcup_E \emptyset_\emptyset = (F,A)$, for every (F,A) in $S(F,E)$,

iii. $(S(F,E), \bigcup_R, \bigcap_E)$ is a commutative, idempotent hemiring with identity $\emptyset_\emptyset$, since $(F,A) \bigcap_E \emptyset_\emptyset = (F,A)$, for every (F,A) in $S(F,E)$,

iv. $\left(S(F,E), \bigcap_R, \bigcup_R, \right)$ is a commutative, idempotent hemiring with identity $\emptyset_\emptyset$, since $(F,A) \bigcup_R \emptyset_E = (F,A)$, for every (F,A) in $S(F,E)$,

v. $\left(S(F,E), \bigcap_R, \bigcup_E\right)$ is a commutative, idempotent hemiring with identity $\emptyset_\emptyset$, since $(F,A) \bigcup_E \emptyset_\emptyset = (F,A)$, for every (F,A) in $S(F,E)$,

vi. $(S(F,E), \bigcap_R, \bigcap_E)$ is a commutative, idempotent hemiring with identity $\emptyset_\emptyset$, since $(F,A) \bigcap_E \emptyset_\emptyset = (F,A)$, for every (F,A) in $S(F,E)$,

vii. $(S(F,E), \bigcup_E, \bigcap_R)$ is a commutative, idempotent hemiring with identity U_E,

since $(F,A)\bigcap_R U_E=(F,A)$, for every (F,A) *in* $S(F,E)$,

viii. $(S(F,E),\bigcap_E,\bigcup_R)$ is a commutative, idempotent hemiring with identity $\emptyset_E$, since $(F,A)\bigcup_R\emptyset_E=(F,A)$, for every (F,A) *in* $S(F,E)$.

4. Lattices

i. $\left(S(F,E),\emptyset_\emptyset,U_E,\bigcup_R,\bigcap_E\right)$ and $\left(S(F,E),U_E,\emptyset_\emptyset,\bigcap_R,\bigcup_E\right)$ are lattices with $\left(S(F,A),\emptyset_{\mathrm{A}},U_A,\bigcup_R,\bigcap_E\right)$ and $(S(F,A),U_A,\emptyset_{\mathrm{A}},\bigcap_R,\bigcup_E)$ as their sublattices respectively.

ii. $\left(S(F,E),\emptyset_\emptyset,U_E,\bigcup_E,\bigcap_R\right)$ and $\left(S(F,E),U_E,\emptyset_\emptyset,\bigcap_E,\bigcup_R\right)$ are lattices with $\left(S(F,A),\emptyset_{\mathrm{A}},U_A,\bigcup_E,\bigcap_R\right)$ and $(S(F,A),U_A,\emptyset_{\mathrm{A}},\bigcap_E,\bigcup_R)$ as their sublattices respectively.

iii. $\left(S(F,A),\emptyset_A,U_A,\bigcup_R,\bigcap_R\right)$ and $\left(S(F,A)_A,U_A,\emptyset_\emptyset,\bigcap_R,\bigcup_R\right)$ are Boolean algebras, since for $(F,A)\in S(F,A)$, then $(F,A)\bigcup_R(F,A)^c=U_A$ and $(F,A)\bigcap_R(F,A)^c=\emptyset_A$. This implies that each element of the set $S(F,A)_A$ is complemented.

iv. $\left(S(F,A),\emptyset_A,U_A,\bigcup_R,\bigcap_R\right)$ and $\left(S(F,A)_A,U_A,\emptyset_\emptyset,\bigcap_R,\bigcup_R\right)$ are De Morgan's algebras, since for $(F,A),(G,B)\in S(F,A)_A$, then $((F,A)\bigcup_R(G,B))^c=(F,A)^c\bigcap_R(G,B)^c$.

This implies that De Morgan's law holds for both $\left(S(F,A),\emptyset_A,U_A,\bigcup_R,\bigcap_R\right)$ and $\left(S(F,A),U_A,\emptyset_\emptyset,\bigcap_R,\bigcup_R\right)$.

Remark 3.4.1

While proving some of the properties of conjunction ($\wedge$) and disjunction ($\vee$) operations, we encountered some difficulties due to the cross product between the parameters of soft sets. Based on this note, we redefined the concept of conjunction and disjunction as binary operations on soft sets. With these redefined conjunction and disjunction many algebraic structures were derived.

CHAPTER FOUR

SOFT BOOLEAN ALGEBRA

4.1 Redefined Concept of Conjunction $(\wedge)$ and Disjunction $(\vee)$ in Soft Set Theory

Based on the difficulties we encountered in the *Definition 3.1.8* and *3.1.9* we hereby redefine the Concept of Conjunction $(\wedge)$ and Disjunction $(\vee)$ as follows:

Definition 4.1.1

Given any two soft sets (F, A) and (G, B) over a common universe U, the disjunction of (F, A) and (G, B) denoted by $(F, A) \vee (G, B)$ is defined as $(F, A) \vee (G, B) = (H,\ A \cup B)$, where $H(\alpha) = F(\alpha) \cup G(\alpha)$, $\forall \alpha \in A \cup B$.

Definition 4.1.2

Given any two soft sets (F, A) and (G, B) over a common universe U, the conjunction of (F, A) and (G, B) denoted by $(F, A) \wedge (G, B)$ is defined as $(F,\ A) \wedge (G, B) = (P,\ A \cap B)$, where $P(\alpha) = F(\alpha) \cap G(\alpha)$, $\forall\ \alpha \in A \cap B$.

Example 4.1.1

Let $U = \{h_1, h_2, h_3, \ldots, h_8\}$ be an universe. Let $A = \{e_1, e_2, e_3\}$, $B = \{e_1, e_3, e_4\}$, $C = \{e_2, e_3, e_5\}$ be any given sets of parameters, such that

$$(F, A) = \{F(e_1) = \{h_1, h_2, h_3\},\ F(e_2) = \{h_2, h_3\},\ F(e_3) = \{h_4\}\}$$

$$(G, B) = \{G(e_1) = \{h_1, h_3, h_4\},\ G(e_3) = \{h_3, h_4\},\ G(e_4) = \{h_4\}\}$$

$$(H, C) = \{H(e_2) = \{h_1, h_4, h_5\},\ H(e_3) = \{h_3\},\ H(e_5) = \{h_1, h_2\}\}$$

$$(F, A) \vee (G, B) = (H,\ A \cup B),\ \text{where } A \cup B = \{e_1, e_2, e_3, e_4\}$$

$$(H, A \cup B) \;=\; \{H(e_1) = \{h_1, h_2, h_3, h_4\}, H(e_2) = \{h_2, h_3, h_4\}, H(e_3) = \{\ h_3, h_4\},$$
$$H(e_4) = \{h_4\}\}$$

$$(F, A) \wedge (G, B) = (H,\ A \cap B),\ \text{where } A \cap B = \{e_1, e_3\ \}$$

$$(H, A \cap B) = \{H(e_1) = \{h_1,\ h_3\ \}, H(e_3) = \{h_3, h_4\}\}$$

Example 4.1.2

Let $U = \{h_1, h_2, h_3, \ldots, h_8\}$ be an universe. Let $A = \{e_1, e_2, e_3\}$,
$B = \{e_1, e_3, e_4\}$, $C = \{e_2, e_3, e_5\}$ be any given sets of parameters, such that

$$(F, A) = \{F(e_1) = \{h_1, h_2, h_3\},\ \ F(e_2) = \{h_2, h_3\},\ \ F(e_3) = \{h_4\}\}$$

$$(G, B) = \{G(e_1) = \{h_1, h_3, h_4\},\ G(e_3) = \{h_3, h_4\},\ G(e_4) = \{h_4\}\}$$

$$(H, C) = \{H(e_2) = \{h_1, h_4, h_5\},\ \ H(e_3) = \{h_3\},\ \ H(e_5) = \{h_1, h_2\}\}$$

$(F, A) \vee ((G, B) \vee (H, C)) = (F, A) \vee (T, B \cup C)$, where $(B \cup C) = \{e_1, e_2, e_3, e_4, e_5\}$,

$$(T, B \cup C) = \left\{ \begin{array}{c} T(e_1) = \{h_1, h_3, h_4\}, T(e_2) = \{h_1, h_4, h_5\}, T(e_3) = \{h_3, h_4\}, \\ T(e_4) = \{h_4\}, T(e_5) = \{h_1, h_2\} \end{array} \right\}$$

$(F, A) \vee (T, B \cup C) = (L, A \cup (B \cup C))$, where $A \cup (B \cup C) = \{e_1, e_2, e_3, e_4, e_5\}$,

$$(F, A) \vee (T, B \cup C) = \left\{ \begin{array}{c} L(e_1) = \{h_1, h_2, h_3, h_4\},\ L(e_2) = \{h_1, h_2, h_3, h_4, h_5\}, \\ L(e_3) = \{h_3, h_4\}, L(e_4) = \{h_4\}, L(e_5) = \{h_1, h_2\} \end{array} \right\}$$

$$= (F, A) \vee ((G, B) \vee (H, C)).$$

Also, for $((F, A) \vee (G, B)) \vee (H, C)$,

$(F,A) \vee (G,B) = (P, A \cup B)$, where $(A \cup B) = \{e_1, e_2, e_3, e_4\}$

$$(F,A) \vee (G,B) = (P, A \cup B) = \left\{ \begin{array}{c} P(e_1) = \{h_1, h_2, h_3, h_4\}, P(e_2) = \{h_2, h_3\}, \\ P(e_3) = \{h_3, h_4\}, P(e_4) = \{h_4\} \end{array} \right\}$$

$(P, A \cup B) \vee (H,C) = (L, (A \cup B) \cup C)$, where $(A \cup B) \cup C = \{e_1, e_2, e_3, e_4, e_5\}$

$$(P, A \cup B) \vee (H,C) = \left\{ \begin{array}{c} L(e_1) = \{h_1, h_2, h_3, h_4\}, L(e_2) = \{h_1, h_2, h_3, h_4, h_5\}, \\ L(e_3) = \{h_3, h_4\}, L(e_4) = \{h_4\}, L(e_5) = \{h_1, h_2\} \end{array} \right\}$$

$$= ((F,A) \vee (G,B)) \vee (H,C)$$

Consequently, $(F,A) \vee ((G,B) \vee (H,C)) = ((F,A) \vee (G,B)) \vee (H,C)$.

4.2 Properties of the Redefined Conjunction ($\wedge$)and Disjunction ($\vee$)

In this section we present some related properties of the redefined Conjunction and disjunction.

Proposition 4.2.1

Given any soft sets (F,A), (G,B) and (H,C) over a common universe U, then the following properties are satisfied without any restriction:

1. *Idempotent laws*

(i) $(F,A) \vee (F,A) = (F,A)$

(ii) $(F,A) \wedge (F,A) = (F,A)$

2. *Commutativity laws*

(i) $(F,A) \vee (G,B) = (G,B) \vee (F,A)$

(ii) $(F, A) \wedge (G, B) = (G, B) \wedge (F, A)$

3. *Associativity laws*

(i) $((F, A) \vee (G, B)) \vee (H, C) = (F, A) \vee ((G, B) \vee (H, C))$

(ii) $((F, A) \wedge (G, B)) \wedge (H, C) = (F, A) \wedge ((G, B) \wedge (H, C))$

4. *Distributive laws*

(i) $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

(ii) $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

5. *Absorption laws*

(i) $(F, A) \vee ((F, A) \wedge (G, B)) = (F, A)$

(ii) $(F, A) \wedge ((F, A) \vee (G, B)) \ = (F, A)$

6. *Complement laws*

(i) $(F, A) \vee (F, A)^c = \tilde{U}$

(ii) $(F, A) \wedge (F, A)^c = \widetilde{\emptyset}$

7. *Identity laws*

(i) $(F, A) \vee \widetilde{\emptyset} = (F, A)$

(ii) $(F, A) \wedge \widetilde{\emptyset} = \widetilde{\emptyset}$

8. *Universe laws*

(i) $(F, A) \vee \tilde{U} = \tilde{U}$

(ii) $(F, A) \wedge \tilde{U} = (F, A)$

9. *De Morgan's laws*

(i) $((F,\ A) \vee (G,B))^c = (F,\ A)^c \wedge (G,\ B)^c$

(ii) $((F,\ A) \wedge (G,B))^c = (F,\ A)^c \vee (G,\ B)^c$

Proof of (3) and (9)

3. *Associativity*

Given any soft sets (F,A), (G,B) and (H,C) over a common universe U, then
(i) $((F,A) \vee (G,B)) \vee (H,C) = (F,A) \vee ((G,B) \vee (H,C))$

Proof

From RHS of (i)

Let $\alpha(x) \in (F,A) \vee ((G,B) \vee (H,C))$

$\Rightarrow\ x \in A$ or $\alpha(x) \in ((G,B) \vee (H,C))$

$\Rightarrow\ x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)\,,$

where $\alpha(x) = F(x) \cup (G(x) \cup H(x)),\ \forall x \in A \cup (B \cup C)$

$$= (F(x) \cup G(x)) \cup H(x)$$

$\Rightarrow x \in (A \cup B) \cup C$

$\Rightarrow \alpha(x) \in ((F,A) \vee (G,B)) \vee (H,C)$

Hence,

$$(F,A) \vee ((G,B) \vee (H,C))\ \tilde{\subset}\ ((F,A) \vee (G,B)) \vee (H,C) \qquad (4.2.1)$$

Conversely,

Let $\beta(x) \in ((F,A) \vee (G,B)) \vee (H,C)$

$\Rightarrow \beta(x) \in ((F,A) \vee (G,B))$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$,

where $\beta(x) = (F(x) \cup G(x)) \cup H(x)$, $\forall x \in (A \cup B) \cup C$

$$= F(x) \cup (G(x) \cup H(x))$$

$\Rightarrow \ x \in A$ or $\beta(x) \in (G(x) \cup H(x))$

$\Rightarrow \ x \in A$ or $x \in (B \cup C)$

i.e, $x \in A \cup (B \cup C)$,

$\Rightarrow \beta(x) \in (F,A) \vee ((G,B) \vee (H,C))$

Hence,

$$((F,A) \vee (G,B)) \vee (H,C) \ \tilde{\subseteq} \ (F,A) \vee ((G,B) \vee (H,C)) \tag{4.2.2}$$

From (4.2.1) and (4.2.2) we have

$$(F,A) \vee ((G,B) \vee (H,C)) = ((F,A) \vee (G,B)) \vee (H,C).$$

Similarly, $(F,A) \wedge ((G,B) \wedge (H,C)) = ((F,A) \wedge (G,B)) \wedge (H,C)$

9. *De Morgan's laws*

Proof

Given any soft sets (F,A), (G,B) over a common universe U, then the following De Morgan laws hold.

(i) $((F,A) \vee (G,B))^c = (F,A)^c \wedge (G,B)^c$

From LHS

Let $\alpha(x) \in ((F,A) \vee (G,B))^c$

$\Rightarrow \alpha(x) \notin (F,A) \vee (G,B)$

$\Rightarrow x \notin A \cup B$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A^c$ and $x \in B^c$,

where $\alpha(x) = F^c(x) \cap G^c(x) = \left(\tilde{U} - F(x)\right) \cap \left(\tilde{U} - G(x)\right)$

$\Rightarrow x \in A^c \cap B^c$

$\Rightarrow \alpha(x) \in (F,A)^c \wedge (G,B)^c$.

Hence,

$$((F,A) \vee (G,B))^c \tilde{\subset} (F,A)^c \wedge (G,B)^c \tag{4.2.3}$$

Conversely,

From RHS

Let $\beta(x) \in (F,A)^c \wedge (G,B)^c$

$\Rightarrow x \in A^c \cap B^c$,

where $\beta(x) = F^c(x) \cap G^c(x) = \left(\tilde{U} - F(x)\right) \cap \left(\tilde{U} - G(x)\right)$

$\Rightarrow x \in A^c$ and $x \in B^c$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \notin A \cup B$

$\Rightarrow \beta(x) \notin (F,A) \vee (G,B)$

$\Rightarrow \beta(x) \in ((F,A) \vee (G,B))^c$.

Hence,

$$(F,A)^c \wedge (G,B)^c \,\tilde{\subset}\, ((F,A) \vee (G,B))^c \tag{4.2.4}$$

From (4.2.3) and (4.2.4),

$$((F,A) \vee (G,B))^c = (F,A)^c \wedge (G,B)^c$$

Similarly, the proof of the remaining properties of the redefined conjunction and disjunction follow.

4.3 Concept of Soft Boolean Algebra

In this section, we establish the concept of soft Boolean algebra and present the proof with respect to the redefined concept of conjunction and disjunction.

Theorem 4.3.1

Let SB be a collection of all soft sets under a common universe U. Let $\emptyset \in SB$ be an empty soft set and $\tilde{U}$ be the universal soft set. Then SB equipped with the two binary operations $\wedge$ and $\vee$ is a Boolean algebra. That is $\left(SB,\ \wedge,\ \vee, \emptyset,\ \tilde{U}\right)$ is a Boolean algebra.

Proof

Let (F,A), (G,B) and $(H,C) \in \left(SB,\ \wedge,\ \vee, \emptyset,\ \tilde{U}\ \right)$ be any soft sets over a common universe U, $\emptyset \in SB$ an empty soft set and $\tilde{U}$ the universal soft set. To show that $\left(SB,\ \wedge,\ \vee, \emptyset,\ \tilde{U}\ \right)$ is a Boolean algebra, it is enough to verify that commutative, associative, distributive, identity and complement laws are satisfied:

1. *Commutative*

Let $(F,A)\,,\quad (G,B)\in\Big(SB,\ \ \wedge,\ \ \vee,\emptyset,\ \tilde{U}\Big)$, then

(i) $(F,A)\vee(G,B)=(G,B)\vee(F,A)$

(ii) $(F,A)\wedge(G,B)=(G,B)\wedge(F,A)$

From LHS of (i)

Let $\alpha(x)\in(F,A)\vee(G,B)$

$\Rightarrow x\in A\cup B$,

where $\alpha(x)=F\,(x)\cup G\,(x)=G(x)\cup F(x)$

$\Rightarrow x\in B\cup A$

$\Rightarrow \alpha(x)\in(G,B)\vee(F,A)\,.$

Hence,

$$(F,A)\vee(G,B)\ \tilde{\subset}\ (G,B)\vee(F,A) \tag{4.3.1}$$

Conversely,

Let $\beta(x)\in(G,B)\vee(F,A)\Rightarrow x\in B\cup A$,

where $\beta(x)=G(x)\cup F(x)=F\,(x)\cup G\,(x)$

$\Rightarrow x\in A\cup B$

$\Rightarrow \beta(x)\in(F,A)\vee(G,B)\,.$

Hence,

$$(G,B)\vee(F,A)\ \tilde{\subset}\ (F,A)\vee(G,B) \tag{4.3.2}$$

From (4.2.5) and (4.2.6),

$$(F,A)\vee(G,B)=\ \ (G,B)\vee(F,A)\,.$$

Similarly, a(ii) holds.

2. *Associativity*

Given any soft sets (F, A), (G, B) and (H, C) over a common universe U, then

(i) $((F, A) \vee (G, B)) \vee (H, C) = (F, A) \vee ((G, B) \vee (H, C))$

(ii) $((F, A) \wedge (G, B)) \wedge (H, C) = (F, A) \wedge ((G, B) \wedge (H, C))$

LHS of (i)

Let $\alpha(x) \in (F, A) \vee ((G, B) \vee (H, C))$

$\Rightarrow$ $x \in A$ or $\alpha(x) \in ((G, B) \vee (H, C))$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

$\Rightarrow x \in A \cup (B \cup C)$,

where $\alpha(x) = F(x) \cup (G(x) \cup H(x)), \forall x \in A \cup (B \cup C) = (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow x \in (A \cup B) \cup C$

$\Rightarrow \alpha(x) \in ((F, A) \vee (G, B)) \vee (H, C)$

Hence,

$$(F, A) \vee ((G, B) \vee (H, C)) \ \tilde{\subset} \ ((F, A) \vee (G, B)) \vee (H, C) \tag{4.3.3}$$

Conversely,

Let $\beta(x) \in ((F, A) \vee (G, B)) \vee (H, C)$

$\Rightarrow \beta(x) \in ((F, A) \vee (G, B))$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cup C$,

where $\beta(x) = (F(x) \cup G(x)) \cup H(x)$, $\forall x \in (A \cup B) \cup C$

$$= F(x) \cup (G(x) \cup H(x))$$

$\Rightarrow$ $x \in A$ or $\beta(x) \in (G(x) \cup H(x))$

$\Rightarrow$ $x \in A$ or $x \in (B \cup C)$

i.e, $x \in A \cup (B \cup C)$,

$\Rightarrow \beta(x) \in (F, A) \vee ((G, B) \vee (H, C))$

Hence,

$$((F, A) \vee (G, B)) \vee (H, C) \ \tilde{\subset} \ (F, A) \vee ((G, B) \vee (H, C)) \tag{4.3.4}$$

From (4.2.7) and (4.2.8) we have

$$(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C).$$

Similarly, $(F, A) \wedge ((G, B) \wedge (H, C)) = ((F, A) \wedge (G, B)) \wedge (H, C)$

3. *Distributive*

(i) $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

(ii) $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

From LHS of (i)

Let $\alpha(x) \in ((F, A) \vee ((G, B) \wedge (H, C)))$

$\Rightarrow x \in A$ or $\alpha(x) \in ((G, B) \wedge (H, C))$

i.e $x \in A$ or $x \in B$ and $x \in C$

$\Rightarrow$ $x \in A \cup (B \cap C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$,

where $\alpha(x) = ((F(x) \cup G(x)) \cap (F(x) \cup H(x)))$

$\Rightarrow \ \alpha(x) \in (((F,\ A) \vee (G,\ B)) \wedge ((F,\ A) \vee (H, C))$.

Hence,

$$((F, A) \vee ((G, B) \wedge (H, C))) \ \tilde{\subset} \ (((F,\ A) \vee (G,\ B)) \wedge ((F,\ A) \vee (H, C)). \tag{4.3.5}$$

Conversely,

Let $\beta(x) \in (((F,\ A) \vee (G, B)) \wedge ((F,\ A) \vee (H, C))$

$\Rightarrow \beta(x) \in ((F,\ A) \vee (G, B))$ and $\beta(x) \in \ ((F,\ A) \vee (H, C)$

$\Rightarrow \ x \in A \cup B$ and $x \in A \cup C$

where $\beta(x) = ((F(x) \cup G(x)) \cap ((F(x) \cup H(x))$

$$= F(x) \cup ((G(x) \cap H(x))$$

$\Rightarrow \ x \in A \cup (B \cap C)$

$\Rightarrow \ \beta(x) \in (F, A) \vee ((G, B) \wedge (H, C))$.

Hence,

$$(((F,\ A) \vee (G, B)) \wedge ((F,\ A) \vee (H, C)) \ \tilde{\subset} \ ((F, A) \vee ((G, B) \wedge (H, C))) \tag{4.3.6}$$

From (4.2.9) and (4.2.10),

$$((F, A) \vee ((G, B) \wedge (H, C))) = (((F,\ A) \vee (G, B)) \wedge ((F,\ A) \vee (H, C)).$$

Similarly, *3* (ii) holds.

4. *Existence of Identity*

(i) $(F, A) \vee \widetilde{\emptyset} = (F, A)$

(ii) $(F, A) \wedge \tilde{U} = (F, A)$

5. *Complement*

(i) $(F, A) \vee (F, A)^c = \tilde{U}$

(ii) $(F, A) \wedge (F, A)^c = \widetilde{\emptyset}$

4. and *5.* are trivial and easy to proof.

Hence, $\left(SB, \ \wedge, \ \vee, \emptyset, \ \tilde{U}\right)$ is a Boolean algebra named soft Boolean algebra.

4.4 Some Results on Soft Boolean Algebra

Here, we present some related results on soft Boolean algebra.

Theorem 4.4.1

Let $\left(SB, \ \wedge, \ \vee, \emptyset, \ \tilde{U}\right)$ be a soft Boolean algebra. Then for any (F, A), $(G, B) \in SB$,

1. *Domination law*

(i) $(F, A) \vee \tilde{U} = \tilde{U}$

(ii) $(F, A) \wedge \widetilde{\emptyset} = \widetilde{\emptyset}$

2. *Idempotent law*

(i) $(F, A) \vee (F, A) = (F, A)$

(ii) $(F, A) \wedge (F, A) = (F, A)$

3. *Absorption law*

(i) $(F, A) = (F, A) \vee ((F,\ A) \wedge (G, B))$

(ii) $(F, A) = (F, A) \wedge ((F,\ A) \vee (G, B))$

4. *Complement law*

(i) $(F, A) \vee (F, A)^c = \tilde{U}$

(ii) $(F, A) \wedge (F, A)^c = \widetilde{\emptyset}$

5. *Uniqueness of* U *and* $\emptyset$

(i) $(F, A) \vee \widetilde{\emptyset} = (F, A)$

(ii) $(F, A) \wedge \tilde{U} = (F, A)$

Proof

1. *(i)*

From LHS of *(i),*

Let $\alpha(x) \in (F, A) \vee \tilde{U}$

$\Rightarrow x \in A \cup U,$

where $\alpha(x) = F(x) \cup \tilde{U} = \tilde{U}$

$\Rightarrow \alpha(x) \in \tilde{U}$

Hence,

$$(F, A) \vee \tilde{U} \tilde{\subset} \tilde{U} \tag{4.4.1}$$

Conversely,

Let $\beta(x) \in \tilde{U}$

where $\beta(x) = F(x) \cup \tilde{U}$

$\Rightarrow x \in A \cup U$

$\Rightarrow \ \beta(x) \in (F, A) \vee \tilde{U}$

Hence,

$$\widetilde{U} \ \tilde{\subset} \ (F, A) \vee \tilde{U} \tag{4.4.2}$$

From (4.4.1) and (4.4.2) we have

$(F, A) \vee \tilde{U} = \tilde{U}$

Similarly, 1 (ii) follows.

2. (i)

From LHS

Let $\alpha(x) \in (F, A) \vee (F, A)$

$\Rightarrow x \in A \cup A = A$

$\Rightarrow x \in A$

$\Rightarrow \alpha(x) \in (F, A)$.

Hence,

$$(F, A) \vee (F, A) \ \tilde{\subset} \ (F, A) \tag{4.4.3}$$

Conversely,

Let $\beta(x) \in (F, A)$

$\Rightarrow x \in A$

$\Rightarrow x \in A \cup A$

$\Rightarrow \beta(x) \in (F,A) \vee (F,A)$

Hence,

$$(F,A) \ \tilde{\subset} \ (F,A) \vee (F,A) \tag{4.4.4}$$

From (4.4.3) and (4.4.4),

$$(F,A) \vee (F,A) = (F,A)\,.$$

Similarly, 2 (ii) follows.

Finally, the remaining results follow similarly.

Theorem 4.4.2

Let $\left(SB, \ \wedge, \ \vee, \emptyset, \ \tilde{U}\right)$ be a Boolean algebra. For any soft sets $(F,A)\,,\ (G,\ B)$ and $(H,\ C) \in SB$, the following hold.

1. (i) $(F,A)\vee((F,A) \vee ((G,B) \vee (H,C))) = (F,A)\vee(((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c\vee((F,A) \vee ((G,B) \vee (H,C))) = (F,A)^c \vee(((F,A) \vee (G,B)) \vee (H,C))$

2. (i) $(F,A)\wedge((F,A) \vee ((G,B) \vee (H,C))) = (F,A)\wedge(((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c\wedge((F,A) \vee ((G,B) \vee (H,C))) = (F,A)^c \wedge(((F,A) \vee (G,B)) \vee (H,C))$

3. (i) $(F,A)\vee((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A)\wedge(((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c\vee((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A)^c \wedge(((F,A) \vee (G,B)) \vee (H,C))$

4. (i) $(F,A)\wedge((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A)\vee(((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c\wedge((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A)^c\vee(((F,A) \vee (G,B)) \vee (H,C))$

Proof

1 (i)

Let $(J,O) = (F,A) \vee ((G,B) \vee (H,C))$ and

$$(P,K) = ((F,A) \vee (G,B)) \vee (H,C)$$

We show that

$$(F,A) \vee (J,O) = (F,A) \vee (P,K) \tag{4.4.5}$$

$$(F,A)^c \vee (J,O) = (F,A)^c \vee (P,K) \tag{4.4.6}$$

From LHS of (4.4.5)

Let $\alpha(x) \in (F,A) \vee (J,O)$

$\Rightarrow \alpha(\ x) \in (F,A) \vee ((F,A) \vee ((G,B) \vee (H,C)))$ *By definition*

$\Rightarrow\ x \in A$ or $\alpha(\ x) \in ((F,A) \vee ((G,B) \vee (H,C)))$

$\Rightarrow\ x \in A$ or $\alpha(\ x) \in ((F,A) \vee (G,B))$ or $\alpha(\ x) \in (H,C)$

$\Rightarrow\ x \in A$ or $\ x \in (A \cup B)$ or $x \in C$

$\Rightarrow\ x \in A \cup (A \cup B) \cup C$

Where $\alpha(\ x) =\ F\,(x) \cup (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow\ \alpha(\ x) \in (F,A) \vee (((F,A) \vee (G,B)) \vee (H,C))$

$\Rightarrow\ \alpha(\ x) \in (F,A) \vee (P,K)$ *By definition*

Hence,

$$(F,A) \vee (J,O)\, \tilde{\subset}\, (F,A) \vee (P,K) \tag{4.4.7}$$

Also, from RHS of (4.4.5)

Let $\beta(\ x) \in (F, A) \vee (P, K)$

$\Rightarrow\ \beta(\ x) \in (F, A) \vee (((F, A) \vee (G, B)) \vee (H, C))$, *By definition*

where $\beta(\ x) = F\,(x) \cup ((F(x) \cup G(x)) \cup H(x))$

$$\beta(\ x) = F\,(x) \cup (F\,(x) \cup (G(x) \cup H(x)))$$

$\Rightarrow\ x \in A$ or $(\ x \in A$ or $\ x\ \in B \cup C)$

$\Rightarrow\ x \in A \cup (\ A \cup (B \cup C))$

$\Rightarrow\ \beta(\ x) \in (F, A) \vee ((F, A) \vee ((G, B) \vee (H, C)))$

$\Rightarrow \beta(\ x) \in (F, A) \vee (J, O)$ *By definition*

Hence,

$$(F, A) \vee (P, K) \tilde{\subset} (F, A) \vee (J, O) \tag{4.4.8}$$

From (4.4.7) and (4.4.8),

$$(F, A) \vee (J, O) = (F, A) \vee (P, K)\,.$$

From LHS of (4.4.6)

Let $\alpha(x) \in (F, A)^c \vee (J, O)$

$\Rightarrow \alpha(x) \in (F, A)^c \vee ((F, A) \vee ((G, B) \vee (H, C)))$ *By definition*

$\Rightarrow\ x \in A^c$ or $\alpha(x) \in ((F, A) \vee ((G, B) \vee (H, C)))$

$\Rightarrow\ x \in A^c$ or $\alpha(x) \in (((F, A) \vee (G, B)) \vee (H, C))$

$\Rightarrow\ x \in A^c$ or $x \in (A \cup B)$ or $x \in C$,

where $\alpha(x) = F^c\,(x) \cup (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow\ \alpha(x) \in (F, A)^c \vee (((F, A) \vee (G, B)) \vee (H, C))$

$\Rightarrow\ \alpha(x) \in (F,A)^c \vee (P,K)$ *By definition*

Hence,

$$(F,A)^c \vee (J,O)\,\tilde{\subset}(F,A)^c \vee (P,K) \tag{4.4.9}$$

Also, form RHS of (4.4.6)

Let $\beta(x) \in (F,A)^c \vee (P,K)$

$\Rightarrow\ \beta(x) \in (F,A)^c \vee (((F,A) \vee (G,B)) \vee (H,C))$, *By definition*

where $\beta(x) = F^c(x) \cup ((F(x) \cup G(x)) \cup H(x))$

$$= F^c(x) \cup (F(x) \cup (G(x) \cup H(x)))$$

$\Rightarrow\ x \in A^c$ or ($x \in A$ or $x \in B \cup C) \Rightarrow\ x \in A^c \cup (A \cup (B \cup C))$

$\Rightarrow\ \beta(x) \in (F,A)^c \vee ((F,A) \vee ((G,B) \vee (H,C)))$

$\Rightarrow \beta(x) \in (F,A)^c \vee (J,O)$ *By definition*

Hence,

$$(F,A)^c \vee (P,K)\,\tilde{\subset}(F,A)^c \vee (J,O) \tag{4.4.10}$$

From (4.4.9) and (4.4.10),

$$(F,A)^c \vee (J,O) = (F,A)^c \vee (P,K).$$

Similarly 2 (i) and (ii) follows.

3 (i)

Let $(J,O) = (F,A) \vee ((G,B) \vee (H,C))$ and

$$(P,K) = ((F,A) \vee (G,B)) \vee (H,C)$$

We show that

$$(F, A) \vee (J, O) \neq (F, A) \wedge (P, K) \tag{4.4.11}$$

$$(F, A)^c \vee (J, O) \neq (F, A)^c \wedge (P, K) \tag{4.4.12}$$

From LHS of (4.4.11)

Let $\alpha(x) \in (F, A) \vee (J, O)$

$\Rightarrow \alpha(\ x) \in (F, A) \vee ((F, A) \vee ((G, B) \vee (H, C)))$ *By definition*

$\Rightarrow\ x \in A$ or $\alpha(\ x) \in ((F, A) \vee ((G, B) \vee (H, C)))$

$\Rightarrow\ x \in A$ or $\alpha(\ x) \in ((F, A) \vee (G, B))$ or $\alpha(\ x) \in (H, C)$

$\Rightarrow\ x \in A$ or $\ x \in (A \cup B)$ or $x \in C$

$\Rightarrow\ x \in A \cup (A \cup B) \cup C$

Where $\alpha(\ x) =\ F\,(x) \cup ((F(x) \cup G(x)) \cup H(x))$

$\Rightarrow\ \alpha(\ x) \in (F, A) \vee (((F, A) \vee (G, B)) \vee (H, C))$

$\Rightarrow\ \alpha(\ x) \in (F, A) \vee (P, K) \neq (F, A) \wedge (P, K)$ *By definition*

Hence,

$$(F, A) \vee (J, O)\, \widetilde{\not\subset}\, (F, A) \wedge (P, K) \tag{4.4.13}$$

Also, from RHS of (4.4.11)

Let $\beta(\ x) \in (F, A) \wedge (P, K)$

$\Rightarrow\ \beta(\ x) \in (F, A) \wedge (((F, A) \vee (G, B)) \vee (H, C))$, *By definition*

where $\beta(\ x) = F\,(x) \cap ((F(x) \cup G(x)) \cup H(x))$

$$= F\,(x) \cap (F\,(x) \cup (G(x) \cup H(x)))$$

$\Rightarrow\ x \in A$ and ($x \in A$ or $x\ \in B \cup C$)

$\Rightarrow\ x \in A \cap (\ A \cup (B \cup C))$

$\Rightarrow\ \beta(\ x) \in (F, A) \wedge ((F, A) \vee ((G, B) \vee (H, C)))$

$\Rightarrow \beta(\ x) \in (F, A) \wedge (J, O) \neq (F, A) \vee (J, O)$ *By definition*

Hence,

$$(F, A) \wedge (P, K) \,\widetilde{\not\subset}\, (F, A) \vee (J, O) \tag{4.4.14}$$

From (4.4.13) and (4.4.14),

$$(F, A) \vee (J, O) \neq (F, A) \wedge (P, K)\,.$$

From LHS of (4.4.12)

Let $\alpha(x) \in (F, A)^c \vee (J, O)$

$\Rightarrow \alpha(x) \in (F, A)^c \vee ((F, A) \vee ((G, B) \vee (H, C)))$ *By definition*

$\Rightarrow\ x \in A^c$ or $\alpha(x) \in ((F, A) \vee ((G, B) \vee (H, C)))$

$\Rightarrow x \in A^c$ or $\alpha(x) \in (((F, A) \vee (G, B)) \vee (H, C))$

$\Rightarrow\ x \in A^c$ or $x \in (A \cup B)$ or $x \in C$,

where $\alpha(x) = F^c\,(x) \cup (F(x) \cup G(x)) \cup H(x)$

$\Rightarrow\ \alpha(x) \in (F, A)^c \vee (((F, A) \vee (G, B)) \vee (H, C))$

$\Rightarrow\ \alpha(x) \in (F, A)^c \vee (P, K) \neq (F, A)^c \wedge (P, K)$ *By definition*

Hence,

$$(F, A)^c \vee (J, O)\, \widetilde{\not\subset} (F, A)^c \wedge (P, K) \tag{4.4.15}$$

Also, form RHS of (4.4.12) Let $\beta(x) \in (F, A)^c \wedge (P, K)$

$\Rightarrow\ \beta(x) \in (F,A)^c \wedge (((F,A) \vee (G,B)) \vee (H,C))$, *By definition*

where $\beta(x) = F^c(x) \cap ((F(x) \cup G(x)) \cup H(x))$

$$= F^c(x) \cap (F(x) \cup (G(x) \cup H(x)))$$

$\Rightarrow\ x \in A^c$ and ($x \in A$ or $x \in B \cup C$)

$\Rightarrow\ x \in A^c \cap (A \cup (B \cup C))$

$\Rightarrow\ \beta(x) \in (F,A)^c \wedge ((F,A) \vee ((G,B) \vee (H,C)))$

$\Rightarrow \beta(x) \in (F,A)^c \wedge (J,O) \neq (F,A)^c \vee (J,O)$ *By definition*

Hence,

$$(F,A)^c \wedge (P,K) \,\widetilde{\not\subset} (F,A)^c \vee (J,O) \tag{4.4.16}$$

From (4.4.15) and (4.4.16),

$$(F,A)^c \wedge (J,O) \neq (F,A)^c \vee (P,K).$$

Similarly 4 (i) and (ii) follows.

Corollary 4.4.1

In the theorem 4.3.1, the associativity can be neglected.

Proof

From **Theorem 4.4.2**

(i) $(F,A) \vee (J,O) = (F,A) \vee (P,K)$ (4.2.5)

(ii) $(F,A)^c \vee (J,O) = (F,A)^c \vee (P,K)$ (4.2.6)

We now show that $(J,O) = (P,K)$

i.e $(F,A) \vee ((G,B) \vee (H,C)) = ((F,A) \vee (G,B)) \vee (H,C)$ (4.4.17)

Now, from LHS of (4.4.17)

$(F,A) \vee ((G,B) \vee (H,C)) = ((F,A) \vee ((G,B) \vee (H,C))) \wedge \tilde{U}$ *Identity law*

$= ((F,A) \vee ((G,B) \vee (H,C))) \wedge ((F,A) \vee (F,A)^c)$ *Complement law*

$= (((F,A) \vee ((G,B) \vee (H,C))) \wedge (F,A)) \vee (((F,A) \vee ((G,B) \vee (H,C))) \wedge (F,A)^c)$

Distributive law

$= ((((F,A) \vee (G,B)) \vee (H,C)) \wedge (F,A)) \vee ((((F,A) \vee (G,B)) \vee (H,C)) \wedge (F,A)^c)$

Commutative law

$= (((F,A) \vee (G,B)) \vee (H,C)) \wedge ((F,A) \vee (F,A)^c)$ *Distributive law*

$= (((F,A) \vee (G,B)) \vee (H,C)) \wedge \tilde{U}$ *Identity law*

$$= ((F,A) \vee (G,B)) \vee (H,C)$$

Hence,

$$(F,A) \vee ((G,B) \vee (H,C)) = ((F,A) \vee (G,B)) \vee (H,C)$$

Remark 4.4.1

The concept of soft Boolean algebra has been studied with respect to the redefined concept of conjunction and disjunction as binary operations on soft set. We note that this notion could be extended to the idea of soft lattice theory and its application to distributed computing system in the following chapter.

CHAPTER FIVE

SOFT LATTICE THEORY AND APPLICATION

5.1 Soft Lattice Theory

There are two standard ways of defining lattice in classical setting: viz., based on algebraic structure or based on the notion of order. In this chapter, we present soft lattice theory based on algebraic structure and on the notion of order with its related application to distributed system.

Definition 5.1.1

Let (Γ, E) be a soft set. Let $A,\ B,\ C \subseteq E$ such that $(F, A), (G, B)$ and (H, C) are all defined. Then (Γ, E) together with the binary operations $\vee$ and $\wedge$ is called soft lattice if the following axioms are satisfied:

$L1$: (i) $(F, A) \vee (G, B) = (G, B) \vee (F, A)$

(ii) $(F, A) \wedge (G, B) = (G, B) \wedge (F, A)$ (Commutative laws)

$L2$: (i) $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$

(ii) $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$

(Associative laws)

$L3$: (i) $(F, A) \vee (F, A) = (F, A)$

(ii) $(F, A) \wedge (F, A) = (F, A)$ (Idempotent laws)

$L4$: (i) $(F, A) = (F, A) \vee ((F, A) \wedge (G, B))$

(ii) $(F, A) = (F, A) \wedge ((F, A) \vee (G, B))$, (Absorption laws)

We denote the soft lattice (Γ, E) by $L(\Gamma, E)$ or simply L. Where $\vee$ and $\wedge$ are

as defined in Definition 4.1.1 and Definition 4.1.2.

Example 5.1.1

Let $U = \{h_1,\ h_2,\ h_3,\ \ldots, h_8\}$ be the set of universe under consideration. Let $E = \{e_1,\ e_2,\ e_3,\ e_4, e_5\}$ be the set of parameters on U. Let Γ: $E \to P(U)$, then (Γ, E) is a soft set. Let $A = \{e_1,\ e_2,\ e_3\}$, $B = \{e_1,\ e_2, e_4,\}$ and $C = \{e_1, e_3, e_4,\ e_5\}$ where A, B and C are subset of E such that (F, A), (G, B), (H, C) are all defined.

Let $(F, A) = \{\ F(e_1) = \{h_1,\ h_2,\ h_4\},\ F(e_2) = \{h_1,\ h_2,\ h_4, h_5\}, F(e_3) = \{h_1,\ h_2\}\}$

$(G, B) = \{\ G(e_1) = \{h_1,\ h_3,\ h_4\},\ G(e_2) = \{h_1,\ h_4\},\ G(e_4) = \{h_2,\ h_3,\ h_4\}\}$

$(H, C) = \{\ H(e_1) = \{h_2,\ h_4\},\ H(e_3) = \{h_1,\ h_2,\ h_5\},\ H(e_4) = \{h_3,\ h_4,\ h_5\},$ $H(e_5) = \{h_1,\ h_2,\ h_4,\ h_5\}\}$. Then the soft set (Γ, E) is a soft lattice.

To see this, we verify the axioms L_1, L_2, L_3 and L_4.

L_1 (i) Commutativity

$(F, A)\vee(G, B) = (P, A \cup B) = \{P(e_1) = \{h_1,\ h_2,\ h_3, h_4\},\ P(e_2) = \{h_1,\ h_2,\ h_4, h_5\},$

$P(e_3) = \{h_1,\ h_2\},\ \ P(e_4) = \{h_2,\ h_3,\ h_4\}\}$ and
$(G, B)\vee(F, A) = (T, B \cup A) = \{T(e_1) = \{h_1,\ h_2,\ h_3, h_4\},\ T(e_2) = \{h_1,\ h_2,\ h_4, h_5\},$
$T(e_3) = \{h_1,\ h_2\}, T(e_4) = \{h_2,\ h_3,\ h_4\}\}$
$\Rightarrow (P, A \cup B) = (T, B \cup A)$
Hence, $(F, A) \vee (G, B) = (G, B) \vee (F, A)$

L_2 (ii) Associativity

$(F, A)\vee((G, B) \vee (H, C)) = (F, A)\vee(P, B \cup C)$ where $B\cup C = \{e_1,\ e_2,\ e_3,\ e_4, e_5\}$,

$(P, B \cup C) = \{P(e_1) = \{h_1,\ h_2,\ h_3, h_4\},\ P(e_2) = \{h_1,\ h_2,\ h_4, h_5\},$
$P(e_3) = \{h_1,\ h_2,\ h_5\}, P(e_4) = \{h_2,\ h_3,\ h_4,\ h_5\},\ P(e_2) = \{h_1,\ h_2,\ h_4, h_5\}\ ,$
$P(e_5) = \{h_1,\ h_2,\ h_4, h_5\}\}$ then

$(F, A) \vee (P, B \cup C) = (T, A \cup (B \cup C))$ where $A \cup (B \cup C) = \{e_1,\ e_2,\ e_3,\ e_4, e_5\}$
$(T, A \cup (B \cup C)) = \{T(e_1) = \{h_1,\ h_2,\ h_3, h_4\},\ T(e_2) = \{h_1,\ h_2,\ h_4, h_5\},$
$T(e_3) = \{h_1,\ h_2, h_5\}, T(e_4) = \{h_2,\ h_3,\ h_4,\ h_5\},\ T(e_5) = \{h_1,\ h_2,\ h_4, h_5\}\ \}$

Also we have,

$((F, A) \vee (G, B)) \vee (H, C) = (J, A \cup B) \vee (H, C)$ where $A \cup B = \{e_1,\ e_2,\ e_3, e_4\}$,

$(J, A \cup B) = \{J(e_1) = \{h_1,\ h_2,\ h_3, h_4\},\ J(e_2) = \{h_1,\ h_2,\ h_4, h_5\},$
$J(e_3) = \{h_1,\ h_2\}, J(e_4) = \{h_2,\ h_3,\ h_4\}\}$ then

$(J, A \cup B) \vee (H, C) = (K, (A \cup B) \cup C)$ where $(A \cup B) \cup C = \{e_1,\ e_2,\ e_3, e_4, e_5\}$,

$$(K, (A \cup B) \cup C) = \{K(e_1) = \{h_1,\ h_2,\ h_3, h_4\},\ K(e_2) = \{h_1,\ h_2,\ h_4, h_5\},$$

$$K(e_3) = \{h_1,\ h_2, h_5\},\ K(e_4) = \{h_2,\ h_3,\ h_4,\ h_5\},$$

$$K(e_5) = \{h_1,\ h_2,\ h_4, h_5\}\ \}.$$

$\Rightarrow (T, A \cup (B \cup C)) = (K, (A \cup B) \cup C)$
Hence, $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee\ (G, B)) \vee (H, C)$

L_3 (iii) Idempotent

$(F, A) = \{\ F(e_1) = \{h_1,\ h_2,\ h_4\},\ F(e_2) = \{h_1,\ h_2,\ h_4, h_5\},\ F(e_3) = \{h_1,\ h_2\}\},$

$$(F, A) \vee (F, A) = (H, A \cup A) = (H, A)$$

$(H, A) = \{\ H(e_1) = \{h_1,\ h_2,\ h_4\}\ H(e_2) = \{h_1,\ h_2,\ h_4, h_5\}, H(e_3) = \{h_1,\ h_2\}\}.$

$\Rightarrow (H, A) = (F, A)$

Hence, $(F,A) \vee (F,A) = (F,A)$

L_4(iv) Absorption

$$(F,A) = (F,A) \vee ((F,A) \wedge (G,B))$$

$(F,A) = \{\, F(e_1) = \{h_1,\ h_2,\ h_4\},\ F(e_2) = \{h_1,\ h_2,\ h_4, h_5\}, F(e_3) = \{h_1,\ h_2\}\}$ and

$(F,A) \vee ((F,A) \wedge (G,B)) = (F,A) \vee (P, A \cap B)$, where $A \cap B = \{e_1,\ e_2\}$,

$(P, A \cap B) = \{\ P(e_1) = \{h_1,\ h_4\},\ P(e_2) = \{h_1,\ h_4\},$

$(F,A) \vee (P, A \cap B) = (K, A \cup (A \cap B))$

$= \{\ K(e_1) = \{h_1,\ h_2,\ h_4\},\ K(e_2) = \{h_1,\ h_2,\ h_4, h_5\},$ $K(e_3) = \{h_1,\ h_2\}\}$.

$(F,A) = (F,A) \vee ((F,A) \wedge (G,B))$

Hence, the soft set (Γ, E) defined on U is a soft lattice.

Definition 5.1.2

A soft set (Γ, E) is called an ordered soft set if the parameter E is ordered.

Example 5.1.2

Let U be a universal set, let $P(U)$ be the power set of U. Let $E = \{e_1,\ e_2,\ e_3,\ e_4,\ e_5\}$ where e_1 stands for fair, e_2= good, e_3=better, e_4= best, e_5= excellent, and $e_1 \subseteq\ e_2 \subseteq\ \ e_3 \subseteq\ e_4 \subseteq e_5$. Let Γ: $E \to P(U)$, then (Γ, E) is an ordered soft set.

Remark 5.1.1

If (Γ, E) is order soft set then for $A,\ B,\ C \subseteq E$, $(F,A), (G,B), (H,C)$ are all

order soft sets.

Definition 5.1.3

A binary relation $\subseteq$ defined on the set of parameters E is a partial order on E if for every $A, B, C \subseteq E, (F, A), (G, B), (H, C)$ are defined such that the following axioms are satisfied:

(i) $(F, A) \widetilde{\subseteq} (F, A)$ (Reflexivity)

(ii) $(F, A) \widetilde{\subseteq} (G, B)$ and $(G, B) \widetilde{\subseteq} (F, A) \Rightarrow (F, A) = (G, B)$ (Antisymmetry)

(iii) $(F, A) \widetilde{\subseteq} (G, B)$ and $(G, B) \widetilde{\subseteq} (H, C) \Rightarrow (F, A) \widetilde{\subseteq} (H, C)$ (Transitivity)

If, in addition, for every $A, B \subseteq E$,

(iv) $(F, A) \widetilde{\subseteq} (G, B)$ or $(G, B) \widetilde{\subseteq} (F, A)$, then we say $\widetilde{\subseteq}$ is a total order on E.

A non- empty soft set (Γ, E) with a partial order on it is called *partially ordered soft set* denoted as $\left((\Gamma, E), \widetilde{\subseteq}\right)$. If the relation is total order then we say that $\left((\Gamma, E), \widetilde{\subseteq}\right)$ is called *totally ordered soft set.*

Example 5.1.3

Let U be a universe set, let $P(U)$ be the power set of U.

Let $E = \{e_1 = fair,\ e_2 = good,\ e_3 = better,\ e_4 = best,\ e_5 = excellent\}$ be the set of ordered parameters defined on U. Let $\Gamma : E \to P(U)$, then (Γ, E) is an ordered soft set

Let $A, B, C \subseteq E$ such that $A = \{e_1.e_2,\ e_3\}$, $B = \{e_1.e_2,\ e_3\}$,$C = \{e_1.e_2,\ e_3,\ e_4\}$, and $(F, A), (G, B),\ (H, C)$ are all defined. Then we show that (Γ, E) is a partial ordered soft set.

Let $(F,A) = \{\, F(e_1) = \{h_1,\ h_2,\ h_4\},\ F(e_2) = \{h_1,\ h_4\},\ F(e_3) = \{h_2,\ h_3, h_4\}$

$(G,B) = \{\, G(e_1) = \{h_1,\ h_2,\ h_4\},\ G(e_2) = \{h_1,\ h_4\},\ G(e_3) = \{h_2,\ h_3,\ h_4\}$

$(H,C) = \{\, H(e_1) = \{h_1, h_2,\ h_3, h_4\},\ H(e_2) = \{h_1,\ h_4,\ h_5\},\ H(e_3) = \{h_2,\ h_3,\ h_4\},$

$$H(e_4) = \{h_2,\ h_3,\ h_4,\ h_5\}\}$$

(i) Since, $F(e_1) \subseteq F(e_1),\ \ F(e_2) \subseteq F(e_2),\ F(e_3) \subseteq F(e_3)$,

$\Rightarrow (F,A)\,\widetilde{\subseteq}\,(F,A)$ (Reflexivity)

(ii) $F(e_1) \subseteq G(e_1),\ \ F(e_2) \subseteq G(e_2),\ F(e_3) \subseteq G(e_3)$,

$\Rightarrow (F,A)\,\widetilde{\subseteq}\,(G,B)$

Also, $G(e_1) \subseteq F(e_1),\ \ G(e_2) \subseteq F(e_2),\ G(e_3) \subseteq F(e_3)$,

$\Rightarrow (G,B)\,\widetilde{\subseteq}\,(F,A)$

$i.e\ \ (G,B) = (F,A)$ (Antisymmetry)

(iii) $F(e_1) \subseteq G(e_1),\ \ F(e_2) \subseteq G(e_2),\ F(e_3) \subseteq G(e_3)$,

$\Rightarrow (F,A)\,\widetilde{\subseteq}\,(G,B)$ and

$$G(e_1) \subseteq H(e_1),\ \ G(e_2) \subseteq H(e_2),\ G(e_3) \subseteq H(e_3),$$

$\Rightarrow (G,B)\,\widetilde{\subseteq}\,(H,C)$

$i.e\ \ (F,A)\,\widetilde{\subseteq}\,(H,C)$ (Transitivity)

Hence, we say (Γ, E) is a partial ordered soft set defined on U.

Definition 5.1.4 Upper bound

Let (Γ, E) be a partially ordered soft set. Let $A \subseteq E$ such that (F, A) is also a partially ordered soft set. Then a set $\Gamma(e)$ in (Γ, E) is called an *upper bound* for (F, A) if $F(a) \subseteq \Gamma(e), e \in E \; \forall F(a) \in (F, A)$, and $a \in A$

Definition 5.1.5 Least upper bound

Let (Γ, E) be a partially ordered soft set. Let $A \subseteq E$ such that (F, A) is also a partial ordered soft set. Then a set $\Gamma(e)$ in $(\Gamma, E), e \in E$ is called the *least upper bound* (lub), or *supremum* of (F, A) $(sup(F, A))$ if $\Gamma(e)$ is an *upper bound* of (F, A), and if $\forall F(a) \in (F, A), \; \forall a \in A, \; \exists e_1 \in E$ such that $F(a) \subseteq \Gamma(e_1), \forall F(a) \in (F, A) \Rightarrow \Gamma(e) \subseteq \Gamma(e_1)$ {i.e., $\Gamma(e)$ is the smallest among the *upper bound* of (F, A)}.

Example 5.1.4

Let (Γ, E) be a partially ordered soft set defined on a universal set $U = \{h_1, h_2, \; \ldots, \; h_6\}$. Let $E = \{e_1, e_2, \; e_3, \; e_4, \; e_5\}$ be the set of parameters. Let $A \subseteq E$ where $A = \{e_1, e_2, \; e_3\}$. Then (F, A) is a partial order soft subset of (Γ, E).

Let $(\Gamma, E) = \{\Gamma(e_1) = \{h_1, h_2, h_3, h_4\}, \Gamma(e_2) = \{h_1, h_2, h_3, h_4, h_5\},$
$\Gamma(e_3) = \{h_1, h_2, h_4\}, \; \Gamma(e_4) = \{h_2, h_3, h_4, \; h_5, h_6\}, \Gamma(e_5) = \{h_1, h_2, h_3, \; h_4, \; h_5\}\}$
and

$(F, A) = \{F(e_1) = \{h_1, h_2\}, \; F(e_2) = \{h_1, h_2, \; h_3\}, \; F(e_4) = \{h_4\}\}.$

We say that $\Gamma(e_1) \in (\Gamma, E)$ is an *upper bound* of (F, A), since for every $F(a)$ in (F, A), $F(a) \subseteq \Gamma(e_1)$, $\Gamma(e_2) \in (\Gamma, E)$ is an *upper bound* of (F, A), since for every $F(a)$ in (F, A), $F(a) \subseteq \Gamma(e_2)$ and also $\Gamma(e_5) \in (\Gamma, E)$ is an *upper bound* of (F, A), since for every $F(a)$ in (F, A), $F(a) \subseteq \Gamma(e_5)$. $\Gamma(e_1)$ is the *least upper bound* or s*upremum* of (F, A), since there exists $\Gamma(e_2)$ and $\Gamma(e_5)$ in (Γ, E) such that $\Gamma(e_1)$ is an *upper bound* of (F, A) and $F(a) \subseteq \Gamma(e_2) \subseteq \Gamma(e_5)$ for every $F(a)$ in $(F, A) \Rightarrow \Gamma(e_1) \subseteq \Gamma(e_2) \subseteq \Gamma(e_2)$. Hence the *supremum* of

(F, A) is $\Gamma(e_1)$, written $sup\{(F, A)\} = \Gamma(e_1)$.

Definition 5.1.6 Lower bound

Let (Γ, E) be a partially ordered soft set. Let $A \subseteq E$ such that (F, A) is also a partially ordered soft set. Then a set $\Gamma(e)$ in (Γ, E) is called a *lower bound* for (F, A) if $\Gamma(e) \subseteq F(a), e \in E$, $\forall F(a) \in (F, A)$ and $\forall a \in A$

Definition 5.1.7 Greatest lower bound

Let (Γ, E) be a partially ordered soft set. Let $A \subseteq E$ such that (F, A) is also a partial ordered soft set. Then a set $\Gamma(e)$ in $(\Gamma, E), e \in E$ is called the *greatest lower bound* (g. l. b), or *infimum* of (F, A) $(imf(F, A))$ if $\Gamma(e)$ is a *lower bound* of (F, A), and if $\forall F(a) \in (F, A)$, $\forall a \in A$, $\exists e_1 \in E$ such that $\Gamma(e_1) \subseteq F(a), \forall F(a) \in (F, A), \forall a \in A \Rightarrow \Gamma(e_1) \subseteq \Gamma(e)$ {i.e., $\Gamma(e)$ is the greatest among the lower bound of (F, A)}.

Example 5.1.5

Let (Γ, E) be a partially ordered soft set defined on a universal set $U = \{h_1, h_2, \ldots, h_7\}$. Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the set of parameters. Let $A \subseteq E$ where $A = \{e_1, e_2, e_3\}$. Then (F, A) is a partially order soft subset of (Γ, E).

Let $(\Gamma, E) = \{\Gamma(e_1) = \{h_1, h_2, h_3, h_4\}, \Gamma(e_2) = \{h_1, h_2, h_3\},$
$\Gamma(e_3) = \{h_1, h_2, h_4\}, \Gamma(e_4) = \{h_2, h_3, h_4, h_5, h_6\}, \Gamma(e_5) = \{h_1, h_2\}\}$ and
$(F, A) = \{F(e_1) = \{h_1, h_2, h_3, h_4, h_5\}, F(e_2) = \{h_1, h_2, h_3, h_4, h_5, h_6\}, F(e_3) = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}\}$

We say that $\Gamma(e_1) \in (\Gamma, E)$ is a *lower bound* of (F, A), since for every $F(a)$ in (F, A), $F(a) \supseteq \Gamma(e_1)$, $\Gamma(e_2) \in (\Gamma, E)$ is a *lower bound* of (F, A), since for every $F(a)$ in (F, A), $F(a) \supseteq \Gamma(e_2)$ and also $\Gamma(e_5) \in (\Gamma, E)$ is a *lower bound* of (F, A), since for every $F(a)$ in (F, A), $F(a) \supseteq \Gamma(e_5)$. $\Gamma(e_1)$ is the *greatest*

lower bound or *infimum* of (F, A), since there exists $\Gamma(e_2)$ and $\Gamma(e_5)$ in (Γ, E) such that $\Gamma(e_1)$ is an *lower bound* of (F, A) and $F(a) \supseteq \Gamma(e_2) \supseteq \Gamma(e_5)$ for every $F(a)$ in $(F, A) \Rightarrow \Gamma(e_1) \supseteq \Gamma(e_2) \supseteq \Gamma(e_2)$. Hence the *infimum* of (F, A) is $\Gamma(e_1)$, written $inf\{(F, A)\} = \Gamma(e_1)$.

Definition 5.1.8 Soft lattice interms of supremum and infimum

A partially ordered soft set (Γ, E) is a soft lattice denoted by $L(\Gamma, E)$ if and only if for every partial ordered soft subset (F, A) of (Γ, E), the supremum of (F, A) and the infimum of (F, A) exists in (Γ, E).

Definition 5.1.9

Let (Γ, E) be a soft set. Let $A,\ B,\ C \subseteq E$ such that $(F, A), (G, B)$ and (H, C) are all defined. Then (Γ, E) together with the binary operations $\vee$ and $\wedge$ is called soft semilattice if the following axioms are satisfied:

L1: (i) $(F, A) \vee (G, B) = (G, B) \vee (F, A)$

(ii) $(F, A) \wedge (G, B) = (G, B) \wedge (F, A)$ (Commutative laws)

L2: (i) $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$

(ii) $(F, A) \vee ((G, B) \vee (H, C)) = ((F, A) \vee (G, B)) \vee (H, C)$ (Associative laws)

L3: (i) $(F, A) \vee (F, A) = (F, A)$

(ii) $(F, A) \wedge (F, A) = (F, A)$ (Idempotent laws)

5.2 Some Theorems on Soft Lattice Theory

In this section, we present some results of soft lattice theory.

Theorem 5.2.1

A finite partially ordered soft set with a finite ordered parameter sets defined on it is a soft lattice.

Proof

Suppose that (Γ, E) is a partially ordered soft, and E is a set of ordered parameters defined on the soft set (F, E). Since the soft set (Γ, E) is finite then supremum and infimum of (Γ, E) exist. The result follows from thoerem 5.1.8.

Theorem 5.2.2

Definition 5.1.1 and Definition 5.1.8 are equivalent.

Proof

To prove that the two definitions are equivalent, we need to show the following constructions:

(i) If L is a lattice by Definition 5.1.1, then define $\subseteq$ on L by $\Gamma(a) \subseteq F(b)$ if and only if $\Gamma(a) = \Gamma(a) \wedge \Gamma(b)$, for $\Gamma(a), \Gamma(b) \in L$

(ii) If L is a lattice by the Definition 5.1.8, then define the operation $\wedge$ and $\vee$ by $\Gamma(a) \vee \Gamma(b) = sup\{\Gamma(a), \Gamma(b)\}$, and $\Gamma(a) \wedge \Gamma(b) = inf\{\Gamma(a), \Gamma(b)\}$.

Suppose that L is a lattice by Definition 5.1.1, and $\subseteq$ is defined as in (i), and $F(a), F(b)$ and $F(c) \in L$.

From $\Gamma(a) \wedge \Gamma(a) = \Gamma(a)$, it follows that

$\Gamma(a) \subseteq \Gamma(a)$. (Reflexivity)

If $\Gamma(a) \subseteq \Gamma(b)$ and $\Gamma(b) \subseteq \Gamma(a)$,

then, $\Gamma(a) = \Gamma(a) \wedge \Gamma(b)$ and $\Gamma(b) = \Gamma(b) \wedge \Gamma(a)$;

Hence $\Gamma(a) = \Gamma(b)$. (Antisymmetry)

Also, if $\Gamma(a) \subseteq \Gamma(b)$ and $\Gamma(b) \subseteq \Gamma(c)$

Then, $\Gamma(a) = \Gamma(a) \wedge \Gamma(b)$ and $\Gamma(b) = \Gamma(b) \wedge \Gamma(c)$,

so, $\Gamma(a) = \Gamma(a) \wedge (\Gamma(b) \wedge \Gamma(c))$;

$$= (\Gamma(a) \wedge \Gamma(b)) \wedge \Gamma(c)$$

$$= \Gamma(a) \wedge \Gamma(c).$$

Hence, $\Gamma(a) \subseteq \Gamma(c)$. (Transitivity)

This shows that $\subseteq$ is a partially ordered on L.

Also, from $\Gamma(a) = \Gamma(a) \wedge (\Gamma(a) \vee \Gamma(b))$ and $\Gamma(b) = \Gamma(b) \wedge (\Gamma(a) \vee \Gamma(b))$

We have, $\Gamma(a) \subseteq \Gamma(a) \vee \Gamma(b)$ and $\Gamma(b\ ) \subseteq \Gamma(a) \vee \Gamma(b)$,

$\Rightarrow \Gamma(a) \vee \Gamma(b)$ is an upper bound of both $\Gamma(a)$ and $\Gamma(b)$.

Now, if $\Gamma(a) \subseteq \Gamma(u)$ and $\Gamma(b) \subseteq \Gamma(u)$, where $\Gamma(u) = (\Gamma(a) \vee \Gamma(b))$,

then $\Gamma(a) \vee \Gamma(u) = (\Gamma(a) \vee \Gamma(u)) \vee \Gamma(u) = \Gamma(u)$, and

$\Gamma(b) \vee \Gamma(u) = (\Gamma(b) \vee \Gamma(u)) \vee \Gamma(u) = \Gamma(u)$, so

$$(\Gamma(a) \vee \Gamma(u)) \vee (\Gamma(b) \vee \Gamma(u)) = \Gamma(u) \vee \Gamma(u) = \Gamma(u);$$

Hence, $(\Gamma(a) \vee \Gamma(b)) \vee \Gamma(u) = \Gamma(u)$,

Now, $(\Gamma(a) \vee \Gamma(b)) \wedge \Gamma(u) = (\Gamma(a) \vee \Gamma(b)) \wedge [(\Gamma(a) \vee \Gamma(b)) \vee \Gamma(u)]$

$$= \Gamma(a) \vee \Gamma(b) \text{ (by the absorption law),}$$

$\Rightarrow \Gamma(a) \vee \Gamma(b) \subseteq \Gamma(u)$. Thus, $\Gamma(a) \vee \Gamma(b) = sup\{\Gamma(a), \Gamma(b)\}$.

In the same way we see that $\Gamma(a) \wedge \Gamma(b) = inf\{\Gamma(a), \Gamma(b)\}$..

Similarly, from definition 5.1.5 and 5.1.7, the $sup\{\Gamma(a), \Gamma(b)\} = \Gamma(a) \vee \Gamma(b)$ and the $inf\{\Gamma(a), \Gamma(b)\} = \Gamma(a) \wedge \Gamma(b)$, the result follows.

Theorem 5.2.3

Given a soft semilattice (Γ, E), such that $\Gamma(e_1) \subseteq \Gamma(e_2)$ if and only if $\Gamma(e_1) \wedge \Gamma(e_2) = \Gamma(e_1)$. Then $((\Gamma, E), \subseteq)$ is an ordered soft set in which every pair of elements has greatest lower bound.

Proof

Let $((\Gamma, E), \subseteq)$ be a soft semilattice, and define $\subseteq$ as above. First we show that $\subseteq$ is a partial order.

Let $\Gamma(e_1)$, $\Gamma(e_1)$ and $\Gamma(e_1) \in (\Gamma, E)$.

(i) $\Gamma(e_1) \wedge \Gamma(e_1) = \Gamma(e_1) \implies \Gamma(e_1) \subseteq \Gamma(e_1)$ (Reflexive)

(ii) Antisymmetry

If $\Gamma(e_1) \subseteq \Gamma(e_2)$ and $\Gamma(e_2) \subseteq \Gamma(e_1)$, then

$$\Gamma(e_1) = \Gamma(e_1) \wedge \Gamma(e_2)$$

$$= \Gamma(e_2) \wedge \Gamma(e_1)$$

$$= \Gamma(e_2)$$

(iii) Transitivity

If $\Gamma(e_1) \subseteq \Gamma(e_2)$ and $(e_2) \subseteq \Gamma(e_3)$, then
$\Gamma(e_1) = \Gamma(e_1) \wedge \Gamma(e_2)$ and $\Gamma(e_2) = \Gamma(e_1) \wedge \Gamma(e_3)$
$\Rightarrow \Gamma(e_1) = \Gamma(e_1) \wedge (\Gamma(e_2) \wedge \Gamma(e_3))$

$$= (\Gamma(e_1) \wedge \Gamma(e_2)) \wedge \Gamma(e_3)$$

$$= \Gamma(e_1) \wedge \Gamma(e_3)$$

$\Rightarrow \Gamma(e_1) = \Gamma(e_1) \wedge \Gamma(e_3)$. Hence,
$\Gamma(e_1) \subseteq \Gamma(e_3)$.

$$\Gamma(e_1) \wedge \Gamma(e_3) = (\Gamma(e_1) \wedge \Gamma(e_2)) \wedge \Gamma(e_3)$$

$$= \Gamma(e_1) \wedge ((e_2) \wedge \Gamma(e_3)) = \Gamma(e_1) \wedge \Gamma(e_2) = \Gamma(e_1),$$

so $F(e_1) \subseteq F(e_3)$.

Since, $(\Gamma(e_1) \wedge \Gamma(e_2)) \wedge F(e_1) = \Gamma(e_1) \wedge (\Gamma(e_1) \wedge \Gamma(e_2))$

$$= (\Gamma(e_1) \wedge \Gamma(e_1)) \wedge \Gamma(e_2)$$

$$= \Gamma(e_1) \wedge \Gamma(e_2)$$

We have, $\Gamma(e_1) \wedge \Gamma(e_2) \subseteq F(e_1)$

Similarly, $\Gamma(e_1) \wedge \Gamma(e_2) \subseteq \Gamma(e_2)$.

Thus, $\Gamma(e_1) \wedge \Gamma(e_2)$ is a lower bound for $\{\Gamma(e_1), \ \Gamma(e_2)\}$. To see that it is the greatest lower bound, suppose $\Gamma(e_3) \subseteq \Gamma(e_1)$ and $\Gamma(e_3) \subseteq \Gamma(e_2)$. Then

$$\Gamma(e_3) \wedge (\Gamma(e_1) \wedge \Gamma(e_2)) = (\Gamma(e_3) \wedge \Gamma(e_1)) \wedge \Gamma(e_2)$$

$$= \Gamma(e_3) \wedge \Gamma(e_2) = \Gamma(e_3)$$

So, $\Gamma(e_3) \subseteq \Gamma(e_1) \wedge \Gamma(e_2)$, as desired.

5.3 Hasse Diagram for Soft Lattice

Hasse diagram named after Helmut Hasse (1897-1979), was devised as a technique for making drawings of partially ordered sets by hands. Hasse diagram

can also be created automatically using graph drawing technique (Aeschlimann and Schmid 1992).

Soft set can be represented on Hasse Diagram. Let (Γ, E) be a partial ordered soft set, let E be the set of parameters such that Γ: $E \to P(U)$. Let $\Gamma(a), \Gamma(b), \Gamma(c) \in (\Gamma, E)$, we say that $\Gamma(b)$ covers $\Gamma(a)$ or $\Gamma(a)$ is covered by $\Gamma(b)$, if $\Gamma(a) \subseteq \Gamma(b)$, and whenever $\Gamma(a) \subseteq \Gamma(c) \subseteq \Gamma(b) \Rightarrow \Gamma(a) = \Gamma(c)$ or $\Gamma(c) = \Gamma(b)$. We use the notation $\Gamma(a) \prec \Gamma(b)$ to denote $\Gamma(a)$ is covered by $\Gamma(b)$.

The closed interval $[\Gamma(a), \Gamma(b)]$ is defined to be the set of $\Gamma(c)$ in (Γ, E) such that $\Gamma(a) \subseteq \Gamma(c) \subseteq \Gamma(b)$, and the open interval $(\Gamma(a), \Gamma(b))$ is the set of $\Gamma(c)$ in (Γ, E) such that $\Gamma(a) \subset \Gamma(c) \subset \Gamma(b)$. Now, we represent each element of (Γ, E) by a small circle "o". If $\Gamma(a) \subseteq \Gamma(b)$ then we draw the circle for $\Gamma(b)$ above the circle for $\Gamma(a)$, joining the two circles with a line segment. From this diagram we can recaptured the relation $\subseteq$ by noting that $\Gamma(a) \subseteq \Gamma(c)$ holds if and only if for some finite $\Gamma(c_1), \Gamma(c_1), \ldots, \Gamma(c_n)$ from (Γ, E) we have $\Gamma(a) = \Gamma(c_1) \prec \Gamma(c_1) \ldots \Gamma(c_{n-1}) \prec \Gamma(c_n) = \Gamma(b)$

5.3.1 Some examples of Hasse diagram of soft lattice.

Let (Γ, E) be a partially ordered soft set, let E be the set of parameters such that Γ: $E \to P(U)$. Let $E = \{a, b, c, d, e, f, g, h\}$.

$$(\Gamma, E) = \left\{ \begin{array}{lll} \Gamma(a) = \{h_1, h_2\}, & \Gamma(b) = \{h_1, h_2, h_3\}, & \Gamma(c) = \{h_1, h_2\}, \\ \Gamma(d) = \{h_1, h_2\}, & \Gamma(e) = \{h_1, h_2\}, & \Gamma(f) = \{h_2, h_3\}, \\ \Gamma(g) = \{h_1\}, & \Gamma(h) = \{h_1, h_2, h_3, h_4\}, & \end{array} \right\}$$

Let $\Gamma(a), \Gamma(b), \Gamma(c), \Gamma(d), \Gamma(e), \Gamma(f), \Gamma(g), \Gamma(h) \in (\Gamma, E)$, we use the Hasse Diagram to represent the following:

(i) $\Gamma(a) \prec \Gamma(b)$, $\Gamma(a) \prec \Gamma(c)$ and $\Gamma(d) \prec \Gamma(c)$

(ii) $\Gamma(d) \prec \Gamma(a) \prec \Gamma(h)$,$\Gamma(f) \prec \Gamma(b)$, $\Gamma(g) \prec \Gamma(e) \prec \Gamma(a)$ and $\Gamma(e) \prec$

$\Gamma(c) \prec \Gamma(h)$

(iii) $\Gamma(c) \prec \Gamma(a)$, $\Gamma(d) \prec \Gamma(a)$,$\Gamma(c) \prec \Gamma(b)$ and $\Gamma(d) \prec \Gamma(b)$

(iv) $\Gamma(b) \prec \Gamma(a)$ or $\Gamma(a) \prec \Gamma(b)$

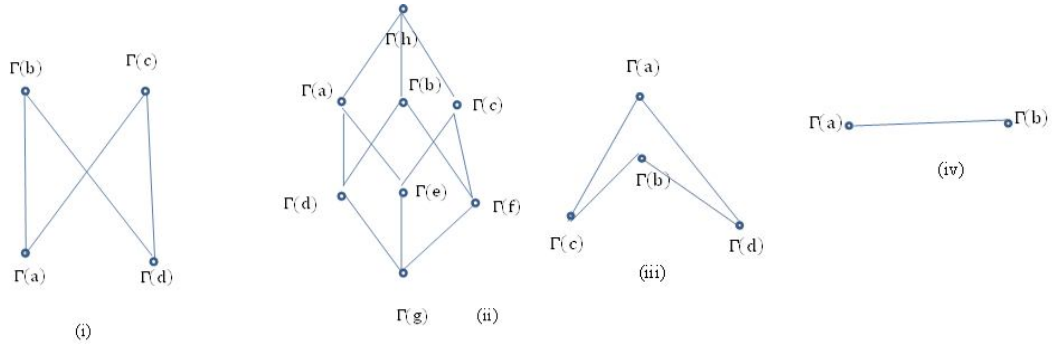


Figure 5.3.1: Representation of elements of (Γ, E) in Hasse Diagrams (i) - (iv)

5.4 Isomorphic Soft Lattices and Soft Sublattices

The word isomorphism is used to signify that two structures are the same except for the nature of their elements. So many researchers have studied isomorphic lattices and sublattices in standard or classical setting. In this section we present isomorphic soft lattices and soft sublattices where some related results are derived.

Definition 5.4.1

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B),and (H, C) are all defined. Then two soft lattices (F, A) and (G, B) are isomorphic if there is a bijection $\alpha : (F, A) \to (G, B)$ and for $\forall F(e_1)$, $F(e_2) \in (F, A)$,

(i). $\alpha(F(e_1) \vee F(e_2)) = \alpha(F(e_1)) \vee \alpha(F(e_2))$ or
(ii). $\alpha(F(e_1) \wedge F(e_2)) = \alpha(F(e_1)) \wedge \alpha(F(e_2))$. Such an α is called an isomorphism.

Example 5.4.1

Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, $A = \{e_1, e_2, e_3, e_4\}$, $B = \{e_1, e_2, e_3, e_4\}$.

Let $(F, A) = \{F(e_1) = \{h_1, h_2, h_3\}, F(e_2) = \{h_2, h_3\}, F(e_3) = \{h_3, h_4, h_5\}, F(e_4) = \{h_4\}\}$

$(G, B) = \{G(e_1) = \{h_4, h_5, h_6\}, G(e_2) = \{h_4, h_5\}, G(e_3) = \{h_2, h_4, h_6\}, G(e_4) = \{h_2\}\}$

From the above information, we see that

$\alpha : (F, A) \to (G, B)$ defined by $\alpha(|F(e_i)|) = |G(e_i)|$, $\forall F(e_i) \in (F, A)$, and $G(e_i) \in (G, B)$ is a bijection, since every element in (F, A) has a unique images in (G, B).
Also α is an isomorphism since $\alpha(|F(e_1) \vee F(e_2)|) = \alpha(|F(e_1)|) \vee \alpha(|F(e_2)|)$, $\forall F(e_1), F(e_2) \in (F, A)$ and $e_1, e_2 \in A$

Remark 5.4.1

It is useful to note if $\alpha : (F, A) \to (G, B)$ is an isomorphism then
$\alpha^{-1} : (G, B) \to (F, A)$ is an isomorphism, and if $\beta : (G, B) \to (H, C)$ is an isomorphism then $\beta o \alpha : (F, A) \to (H, C)$ is an isomorphism.

Definition 5.4.2

If (F, A) and (G, B) are two partially ordered soft sets (possets) and $\alpha : (F, A) \rightarrow (G, B)$ is a map, then we say α is order-preserving if $\alpha(F(e_i)) \subseteq (G(e_i))$ holds in (G, B) whenever $F(e_i) \subseteq G(e_i)$ holds in (F, A), $\forall e_i \in A$.

Example 5.4.2

Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7, h_8\}$, $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, $A = \{e_1, e_2, e_3\}$, $B = \{e_1, e_2, e_3, e_4\}$, $(F, A) = \{F(e_1) = \{h_2, h_3\}, F(e_2) = \{h_1, h_2, h_3\}, F(e_3) = \{h_1, h_2, h_3, h_4\}\}$ and $(G, B) = \{G(e_1) = \{h_2, h_3\}, G(e_2) = \{h_1, h_2, h_3\}, G(e_3) = \{h_1, h_2, h_3, h_4\}, G(e_4) = \{h_1, h_2, h_3, h_4, h_5\}\}$ The mapping $\alpha : (F, A) \rightarrow (G, B)$ be defined by $\alpha(|F(e_i)|) = |G(e_j)|$ if and only if $e_i = e_j$ and $(|F(e_i)|) = (|G(e_j)|)$ is order-preserving.

Theorem 5.4.1

Two soft lattices (F, A) and (G, B) are isomorphic if and only if there is a bijection $\alpha : (F, A) \rightarrow (G, B)$ such that both α and α^{-1} are order-preserving.

Proof

If $\alpha : (F, A) \rightarrow (G, B)$ is an isomorphism, and $F(e_1) \subseteq F(e_2)$ holds in (F, A), where $i, j, k = 1, 2, 3, ...$, then $F(e_1) = F(e_i) \wedge F(e_j)$ so,

$$\alpha(F(e_i)) = \alpha(F(e_i) \wedge F(e_j))$$

$$= \alpha(F(e_i)) \wedge \alpha(F(e_j))$$

Hence, $\alpha(F(e_i)) \subseteq \alpha(F(e_j))$ and thus, α is order-preserving. As α^{-1} is an isomorphism, it is also order-preserving.

Conversely, let $\alpha : (F, A) \rightarrow (G, B)$ be a bijection such that both α and α^{-1} are order-preserving. For $F(e_i), F(e_j) \in (F, A)$, we have

$F(e_i) \subseteq F(e_i) \vee F(e_j)$ and $F(e_j) \subseteq F(e_i) \vee F(e_j)$ so,

$\alpha(F(e_i)) \subseteq \alpha(F(e_i) \vee F(e_j))$ and $\alpha(F(e_j)) \subseteq \alpha(F(e_i) \vee F(e_j))$, hence,

$\alpha(F(e_i)) \vee \alpha(F(e_j)) \subseteq \alpha(F(e_i) \vee F(e_j))$

Furthermore, if $\alpha(F(e_i)) \vee \alpha(F(e_j)) \subseteq F(e_k)$, where $F(e_k)$ is arbitrary set in (F, A), then $\alpha(F(e_i)) \subseteq F(e_k)$ and $\alpha(F(e_j)) \subseteq F(e_k)$ hence,

$F(e_i) \subseteq \alpha^{-1}(F(e_k))$ and $F(e_j) \subseteq \alpha^{-1}(F(e_k))$ so,

$F(e_i) \vee F(e_j) \subseteq \alpha^{-1}(F(e_k))$, and thus $\alpha(F(e_i) \vee F(e_j)) \subseteq F(e_k)$

$\Rightarrow \alpha(F(e_i)) \vee \alpha(F(e_j)) = \alpha(F(e_i) \vee F(e_j))$

Similarly, let $\alpha : (F, A) \to (G, B)$ be a bijection, such that both α and α^{-1} are order-preserving. For $F(e_i), F(e_j) \in (F, A)$, we have

$F(e_i) \subseteq F(e_i) \wedge F(e_j)$ and $F(e_j) \subseteq F(e_i) \wedge F(e_j)$ so,

$\alpha(F(e_i)) \subseteq \alpha(F(e_i) \wedge F(e_j))$ and $\alpha(F(e_j)) \subseteq \alpha(F(e_i) \wedge F(e_j))$, hence,

$\alpha(F(e_i)) \wedge \alpha(F(e_j)) \subseteq \alpha(F(e_i) \wedge F(e_j))$

Furthermore, if $\alpha(F(e_i)) \wedge \alpha(F(e_j)) \subseteq F(e_k)$, where $F(e_k)$ is arbitrary set in (F, A), then $\alpha(F(e_i)) \subseteq F(e_k)$ and $\alpha(F(e_j)) \subseteq F(e_k)$ hence,

$F(e_i) \subseteq \alpha^{-1}(F(e_k))$ and $F(e_j) \subseteq \alpha^{-1}(F(e_k))$ so,

$F(e_i) \wedge F(e_j) \subseteq \alpha^{-1}(F(e_k))$, and thus $\alpha(F(e_i) \wedge F(e_j)) \subseteq F(e_k)$

$\Rightarrow \alpha(F(e_i)) \wedge \alpha(F(e_i)) = \alpha(F(e_i) \wedge F(e_j))$

Theorem 5.4.2

Let (Γ, E) be an ordered soft set, and $A, B \subseteq E$, such that (F, A) and (G, B) are defined. Let $\emptyset : (F, A) \to (G, B)$ be defined as $\emptyset(F(e_i)) =$

$\{F(e_j) \in (F,A) : F(e_j) \subseteq F(e_i),\ \forall F(e_i) \in (F,A)\}$.

Then (F,A) is isomorphic to the range of $\emptyset$ ordered by the containment $\subseteq$.

Proof

If $F(e_i) \subseteq F(e_j)$, then $F(e_k) \subseteq F(e_i)$

$\Rightarrow F(e_k) \subseteq F(e_j)$, (By transitivity)

and hence, $\emptyset(F(e_i)) \subseteq \emptyset(F(e_j))$,

Since $F(e_i) \subseteq \emptyset(F(e_i))$ (By reflexivity)

$\Rightarrow \emptyset(F(e_i)) \subseteq \emptyset(F(e_j))$

$\Rightarrow F(e_i) \subseteq F(e_j)$

Thus, $F(e_i) \subseteq F(e_j)$ if and only if $\emptyset(F(e_i)) \subseteq \emptyset(F(e_j))$

That $\emptyset$ is one - to - one then follows by antisymmetry.

Theorem 5.4.3

Let (Γ, E) be a soft lattice. Let $A, B \subseteq E$, such that (F,A) and (G,B) are all defined. If $\alpha :\ (F,A) \to (G,B)$ is a mapping then the following holds:

(i) $(F,A) \subseteq (G,B) \Rightarrow \alpha((F,A)) \subseteq (G,B)$

(ii) $\alpha((F,A) \wedge (G,B)) \subseteq (G,B)$.

Proof

(i) $(F,A) \subseteq (G,B) \Rightarrow \alpha((F,A)) \subseteq (G,B)$

Suppose $(F,A) \subseteq (G,B)$

$\Rightarrow (F,A) = (F,A) \wedge (G,B)$

$\Rightarrow \alpha((F,A)) = \alpha((F,A)) \wedge (G,B)$

$= \alpha((F,A)) \wedge (G,B)$

$\Rightarrow \alpha((F,A)) = \alpha((F,A)) \wedge (G,B)$

$\Rightarrow \alpha((F,A)) \subseteq (G,B).$

(ii) $\alpha((F,A) \wedge (G,B)) \subseteq (G,B).$

Let $G(b) \in \alpha((F,A) \wedge (G,B))$

$\Rightarrow G(b) \subseteq \alpha((F,A))$ and $G(b) \subseteq (G,B)$

$\Rightarrow G(b) \in \alpha((F,A)) \wedge (G,B)$

$\Rightarrow G(b) \in (G,B) \wedge (G,B)$

$\Rightarrow G(b) \in (G,B)$

Hence, $\alpha((F,A) \wedge (G,B)) \subseteq (G,B)$

Theorem 5.4.4

If $\alpha : (F,A) \to (G,B)$ is a bijective mapping, then $\alpha^{-1} : (G,B) \to (F,A)$ is also a bijective mapping.

Proof

Let $G(b_i) \neq G(b_j)$ for $G(b_i), G(b_j) \in (G,B)$

$\Rightarrow \alpha^{-1}(G(b_i)) = F(a_i)$ for $F(a_i) \in (F,A)$ and $\alpha^{-1}(G(b_j)) = F(a_j)$ for $F(a_j) \in (F,A)$

$\Rightarrow G(b_i) = \alpha(F(a_i))$ and $G(b_j) = \alpha(F(a_j))$

$\Rightarrow \alpha(F(a_i)) \neq \alpha(F(a_j))$ since α is one-one

$\Rightarrow \alpha^{-1}(G(b_i)) \neq \alpha^{-1}(G(b_j))$. Hence, α^{-1} is one - one.

Let $F(a) \in (F, A)$. Since α is one-one, there exists a unique element $G(b)$ in (G, B) such that $\alpha(F(a)) = G(b)$

$\Rightarrow F(a) = \alpha^{-1}(G(b))$

$\Rightarrow \alpha^{-1}$ is one-one

$\Rightarrow \alpha^{-1}$ is bijective.

Theorem 5.4.5

Let $\alpha : (F, A) \to (G, B)$ and $\beta : (G, B) \to (H, C)$ be two bijective mappings of soft lattices. Then $\beta o \alpha : (F, A) \to (H, C)$ is also a bijective mapping.

Proof

Suppose $F(a_i), F(a_j) \in (F, A)$. Then $F(a_i) \neq F(a_j)$

$\Rightarrow \alpha(F(a_i)) \neq \alpha(F(a_j))$, since α is injective

$\Rightarrow \beta(F(a_i)) \neq \beta(F(a_j))$, since β is injective

$\Rightarrow \beta o \alpha(F(a_i)) \neq \beta o \alpha(F(a_j))$

Hence, $\beta o \alpha$ is one-one.

Let $H(c) \in (H, C)$ there exists $G(b)$ in (G, B) such that $\beta(G(b)) = H(c)$ as β is onto.

Since, α is onto there exists $F(a)$ in (F, A) such that $\alpha(F(a)) = G(b)$.

Then $\beta(\alpha(F(a))) = H(c)$ for all $H(c)$ in (H, C)

$\Rightarrow \beta o \alpha(F(a)) = H(c)$

$\Rightarrow \beta o \alpha$ is onto. Hence $\beta o \alpha$ is bijective.

Definition 5.4.3

Let (Γ, E) be a soft lattice and $(F, A) \neq \emptyset$ is a soft subset of (Γ, E) such that $\forall\ F(e_i), F(e_j) \in (F, A)$ both $F(e_i) \vee F(e_j)$ and $F(e_i) \wedge F(e_j)$ are in (F, A), where $\vee$ and $\wedge$ are the soft lattice operations of (Γ, E). Then we say that (F, A) with the same operations (restricted to (F, A)) is a soft sublattice of (Γ, E).

Remark 5.4.2

If (F, A) is a soft sublattice of (Γ, E) then for $F(e_i), F(e_j) \in (F, A)$ we will of course have $F(e_i) \subseteq F(e_j) \in (F, A)$ if and only if $F(e_i) \subseteq F(e_j) \in (\Gamma, E)$. It is interesting to note that given a soft lattice (Γ, E), one can often find soft subsets (F, A) which as partially ordered soft sets (using the same order relation) are soft lattices, but which do not qualify as soft sublattice as the operation $\vee$ and $\wedge$ do not agree with those of the original soft lattice (Γ, E).

Example 5.4.1

Let $U = \{h_1, h_2, h_3, h_4, h_5, h_6, h_7\}$ be the universal set, let $P(U)$ be the power set of U.

Let $E = \{e_1, e_2, e_3, e_4, e_5\}$ be the sets of parameters, and $e_1 \subseteq e_2 \subseteq e_3 \subseteq e_4 \subseteq e_5$. Let $A = \{e_1, e_2, e_3, e_5\} \subseteq E$. Let (Γ, E) be a soft lattice and (F, A) be soft subset of (Γ, E). Let

$$(\Gamma, E) = \{\Gamma(e_1) = \{h_1, h_2, h_4\}, \Gamma(e_2) = \{h_1, h_2, h_4\}, \Gamma(e_3) = \{h_1, h_2, h_3, h_4\},$$

$$\Gamma(e_4) = \{h_1, h_2, h_3, h_4\}, \Gamma(e_5) = \{h_1, h_2, h_3,\ h_4,\ h_5\}\}$$

$$(F, A) = \{F(e_1) = \{h_1, h_2, h_4\},\ F(e_2) = \{h_1, h_2, h_4\},\ F(e_3) = \{h_1, h_2, h_3, h_4\},$$
$$F(e_5) = \{h_1, h_2, h_3,\ h_4,\ h_5\}\}$$

From **Figure 5.4.1**, using the informations in (Example 5.4.1) we note that

(F, A) as partially ordered soft set is a soft lattice, but (F, A) is not a soft sublattice of soft lattice (Γ, E).

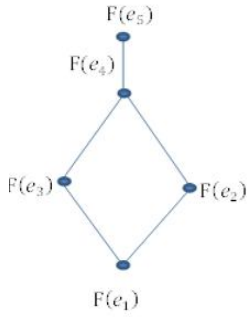


Figure 5.4.1: Soft lattice structure

Definition 5.4.4

A soft lattice (F, A) can be embedded into a soft lattice (Γ, E) if there is a soft sublattice of (Γ, E) isomorphic to (F, A); in this case we also say (Γ, E) contains a copy of (F, A) as a soft sublattice.

Example 5.4.2

Let (F, A) and (Γ, E) be two soft lattices, and (H, C) a soft sublattice of (Γ, E). If there exists an isomorphism $\alpha : (H, C) \to (F, A)$, such that for $\forall H(e) \in (H, C)$, we have $\alpha(H(e)) \in (F, A)$, then we say that (F, A) is embedded in (Γ, E) or (Γ, E) contain a copy of (F, A) as a soft sublattice.

5.5 Distributive and Modular Soft Lattices

There are several studied on distributive lattices and modular lattices in standard or classical setting. Here we present the concept of distributive soft lattices and modular soft lattices with some related results.

Definition 5.5.1

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B),and (H, C) are all defined. Then (Γ, E) together with the binary operations $\vee$ and $\wedge$ is called distributive soft lattice if the following axioms are satisfied:

D1: $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

D2: $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

Theorem 5.5.1

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B),and (H, C) are all defined. Then (Γ, E) satisfies D1 if and only if it satisfies D2.

Proof

Suppose that (Γ, E) satisfies D1. Then

$(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee ((F, A) \wedge (H, C))) \vee ((G, B) \wedge (H, C))$ (by L4 (i))

$= (F, A) \vee (((F, A) \wedge (H, C)) \vee ((G, B) \wedge (H, C)))$ (by L2 (i))

$= (F, A) \vee (((H, C) \wedge (F, A)) \vee ((H, C) \wedge (G, B)))$ (by L1 (ii))

$= (F, A) \vee ((H, C) \wedge ((F, A) \vee (G, B)))$ (by D1)

$= (F, A) \vee (((F, A) \vee (G, B)) \wedge (H, C))$ (by L1 (ii))

$= ((F, A) \wedge ((F, A) \vee (G, B))) \vee (((F, A) \vee (G, B)) \wedge (H, C))$ (by L4 (ii))

$= (((F, A) \vee (G, B)) \wedge (F, A)) \vee (((F, A) \vee (G, B)) \wedge (H, C))$ (by L1 (ii))

$= ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$ (by D1)

Similarly, suppose that (Γ, E) satisfies D2. Then

$(F,A)\wedge((G,B)\vee(H,C)) = ((F,A)\wedge((F,A)\vee(H,C)))\wedge((G,B)\vee(H,C)$ (by L4 (ii))

$= (F,A)\wedge(((F,A)\vee(H,C))\wedge((G,B)\vee(H,C)))$ (by L2 (ii))

$= (F,A)\wedge(((H,C)\vee(F,A))\wedge((H,C)\vee(G,B)))$ (by L1 (i))

$= (F,A)\wedge((H,C)\vee((F,A)\wedge(G,B)))$ (by D1)

$= (F,A)\wedge(((F,A)\wedge(G,B))\vee(H,C))$ (by L1 (i))

$= ((F,A)\vee((F,A)\wedge(G,B)))\wedge(((F,A)\wedge(G,B))\vee(H,C))$ (by L4 (i))

$= (((F,A)\wedge(G,B))\vee(F,A))\wedge(((F,A)\wedge(G,B))\vee(H,C))$ (by L1 (i))

$= ((F,A)\wedge(G,B))\vee((F,A)\wedge(H,C))$ (by D1)

Theorem 5.5.2

Let (Γ,E) be a soft lattice. Let $A,B,C\subseteq E$ such that (F,A), (G,B),and (H,C) are all defined. Then (Γ,E) together with the binary operations $\vee$ and $\wedge$ satisfies both of the following inequalities:

$((F,A)\wedge(G,B))\vee((F,A)\wedge(H,C))\subseteq(F,A)\wedge((G,B)\vee(H,C))$ and

$(F,A)\vee((G,B)\wedge(H,C))\subseteq((F,A)\vee(G,B))\wedge((F,A)\vee(H,C))$

Proof

To see this, we note that

$(F,A)\wedge(G,B)\subseteq(F,A)$ and

$(F,A)\wedge(G,B)\subseteq(F,A)\vee(H,C)$; hence,

$(F,A)\wedge(G,B)\subseteq(F,A)\wedge((G,B)\vee(H,C))$.

Thus to verify the distributive laws in a soft lattice (Γ,E), it suffices to check

either of the following inequalities;

$$(F, A) \wedge ((G, B) \vee (H, C)) \subseteq ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C)) \quad (5.5.1)$$

$$((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C)) \subseteq (F, A) \vee ((G, B) \wedge (H, C)) . \quad (5.5.2)$$

From LHS of (5.5.1):

$\alpha(x) \in ((F, A) \wedge ((G, B) \vee (H, C)))$

$\Rightarrow \ \alpha(x) \in (F, A)$ and $\alpha(x) \in \ ((G, B) \vee (H, C))$

$\Rightarrow \ \alpha(x) \in (F, A)$ and $\alpha(x) \in \ (G, B)$ or $\alpha(x) \in \ (H, C)$

Where $\alpha\,(x) = F(x) \cap (G(x) \cup H(x))$

$\Rightarrow x \in$A and $x \in B$ or $x \in C$

$\Rightarrow x \in A \cap (B \cup C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$ (By distribution)

$\alpha(x) \in ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

i.e., $(F, A) \wedge ((G, B) \vee (H, C)) \subseteq ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

Similarly, from LHS of (5.5.3):

$\beta\,(x) \in (((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C)))$

$\Rightarrow \beta\,(x) \in ((F, A) \vee (G, B))$ and $\beta\,(x) \in ((F, A) \vee (H, C))$

$\Rightarrow \beta\,(x) \in (F, A)$ or $\beta\,(x) \in (G, B)$ and $\beta\,(x) \in (F, A)$ or $\beta\,(x) \in (H, C)$

Where $\beta\,(x) = (F(x) \cup G(x)) \cap (F(x) \cup H(x))$

$\Rightarrow x \in A$ or $x \in B$ and $x \in A$ or $x \in C$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\Rightarrow x \in A \cup (B \cap C)$ (By distribution)

$\Rightarrow \beta(x) \in (F, A) \vee ((G, B) \wedge (H, C))$

i.e., $((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C)) \subseteq (F, A) \vee ((G, B) \wedge (H, C))$.

Hence, the soft lattice (Γ, E) satisfies the distributive laws.

Definition 5.5.2

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B),and (H, C) are all defined. Then (Γ, E) together with the binary operations $\vee$ and $\wedge$ is called modular soft lattice if the following axiom is satisfied:

M: $(F, A) \subseteq (G, B) \rightarrow (F, A) \vee ((G, B) \wedge (H, C)) = (G, B) \wedge ((F, A) \vee (H, C))$

Theorem 5.5.3

The modular law is obviously equivalent (for lattices) to the identity

$$((F, A) \wedge (G, B)) \vee ((G, B) \wedge (H, C)) = (G, B) \wedge (((F, A) \wedge (G, B)) \vee (H, C))$$

Proof

The proof of this theorem is straightforward, since

$(F, A) \subseteq (G, B)$ holds if and only if $(F, A) = (F, A) \wedge (G, B)$

Remark 5.5.1

We can see that every soft lattice satisfies

$$(F, A) \subseteq (G, B) \rightarrow (F, A) \vee ((G, B) \wedge (H, C)) \subseteq (G, B) \wedge ((F, A) \vee (H, C)).$$

So to verify the modular law it suffices to check the implication

$$(F,A) \subseteq (G,B) \rightarrow (G,B) \wedge ((F,A) \vee (H,C)) \subseteq (F,A) \vee ((G,B) \wedge (H,C))$$

Suppose, $(F,A) \subseteq (G,B)$,

$$\Rightarrow (F,A) \vee ((G,B) \wedge (H,C)) = ((F,A) \wedge (G,B)) \vee ((G,B) \wedge (H,C))$$

$= ((G,B) \wedge (F,A)) \vee ((G,B) \wedge (H,C))$ L1(ii)

$= (G,B) \wedge ((F,A) \vee (H,C))$ L4(i)

Hence, $(F,A) \vee ((G,B) \wedge (H,C)) = (G,B) \wedge ((F,A) \vee (H,C))$

Theorem 5.5.4

Every distributive soft lattice is a modular soft lattice.

Proof

Given that

$(F,A) \vee ((G,B) \wedge (H,C)) = ((F,A) \vee (G,B)) \wedge ((F,A) \vee (H,C))$. Since,

$(F,A) \vee (G,B) = (G,B)$, whenever $(F,A) \subseteq (G,B)$

$$\Rightarrow (F,A) \vee ((G,B) \wedge (H,C)) = (F,A) \vee ((F,A) \vee (G,B)) \wedge ((F,A) \vee (H,C))$$

$$= ((F,A) \vee (F,A)) \vee (G,B) \wedge ((F,A) \vee (H,C))$$

$$= (F,A) \vee (G,B) \wedge ((F,A) \vee (H,C))$$

$$= (G,B) \wedge ((F,A) \vee (H,C)).$$

Hence, the distributive soft lattice is a modular soft lattice.

Remark 5.5.2

The following result will give wonderful properties of modular and distributive soft lattices in terms of the *Diamond* soft lattices called M_3 and the *Pentagon* soft lattice called N_5 shown in Figure 2. In neither case is

$(F,A) \vee ((G,B) \wedge (H,C)) = ((F,A) \vee (G,B)) \wedge ((F,A) \vee (H,C))$, So, neither M_3 nor N_5 is a distributive soft lattice. For N_5 we see that

$(F,A) \subseteq (G\ ,B)$ but,

$(F,A) \vee ((G,B) \wedge (H,C)) \neq (G,B) \wedge ((F,A) \vee (H,C))$.

So, N_5 is not modular soft lattice.

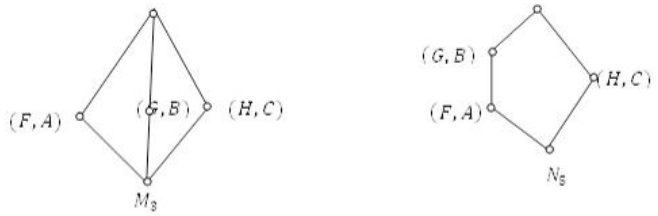


Figure 5.5.1: Soft lattice structure of M_3 "Diamond" and N_5 "Pentagon"

Corollary 5.5.1

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F,A), (G,B), and (H,C) are all defined. Then (Γ, E) is a modular soft lattice if and only if it does not contain the pentagon as a soft sublattice.

Theorem 5.5.5

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F,A), (G,B), and (H,C) are all defined. Then (Γ, E) is a nonmodular soft lattice if and only if

N_5 can be embedded into (Γ, E) .

Proof

From the **Remark 5,5.2** above it is clear that if N_5 can be embedded into (Γ, E), then (Γ, E) does not satisfy the modular law. Conversely, suppose that (Γ, E) does not satisfy the modular law. Then for some (F, A), (G, B), and $(H, C))$ in (Γ, E) we have

$(F, A) \subseteq (G, B)$ but

$(F, A) \vee ((G, B) \wedge (H, C)) \subset (F, A) \wedge ((G, B) \vee)(H, C))$

Let define $(F_1, A) = (F, A) \vee ((G, B) \wedge (H, C))$ and

$(G_1, B) = (G, B) \wedge ((F, A) \vee (H, C))$. Then

$(H, C) \wedge (G_1, B) = (H, C) \wedge [(G, B) \wedge ((F, A) \vee (H, C))]$

$= (H, C) \wedge [((F, A) \vee (H, C)) \wedge (G, B)]$ (by L1(b))

$= (H, C) \wedge [((H, C) \vee (F, A)) \wedge (G, B)]$ (by L1(a))

$= [(H, C) \wedge ((H, C) \vee (F, A))] \wedge (G, B)$ (by L2 (a))

$= (H, C) \wedge (G, B)$, (L4 (b))

and $(H, C) \vee (F_1, A) = (H, C) \vee [(F, A) \vee ((G, B) \wedge (H, C))]$

$= (H, C) \vee [((G, B) \wedge (H, C)) \vee (F, A)]$ (by L1(a))

$= (H, C) \vee [((H, C) \wedge (G, B)) \vee (F, A)]$ (by L1(b))

$= [(H, C) \vee ((H, C) \wedge (G, B))] \vee (F, A)$ (L2 (a))

$= (H, C) \vee (F, A)$ (L4 (a))

Now as $(H, C) \wedge (G, B) \subseteq (F_1, A) \subseteq (G_1, B)$ we have

$$(H,C) \wedge (G,B) \subseteq (H,C) \wedge (F_1,A) \subseteq (H,C) \wedge (G_1,B) = (H,C) \wedge (G,B)\,,$$

hence $(H,C) \wedge (F_1,A) = (H,C) \wedge (G_1,B) = (H,C) \wedge (G,B)$. Likewise

$$(H,C) \vee (G_1,B) = (H,C) \vee (F_1,A) = (H,C) \vee (F,A)\,.$$

Now it is straightforward to see that the diagram in Figure 3 give the desired copy of N_5 in (Γ, E).

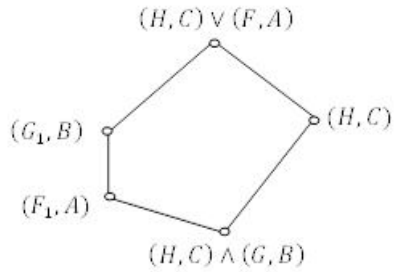


Figure 5.5.2: Copy of N_5 in (Γ, E)

Corollary 5.5.2

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B) and (H, C) are all defined. Then (Γ, E) is a distributive soft lattice if and only if it has neither M_3 nor N_5 as a soft sublattice.

Theorem 5.5.6

Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B) and (H, C) are all defined. Then (Γ, E) is a nondistributive soft lattice if and only if M_3 or N_5 can be embedded into (Γ, E).

Proof

If either M_3 or N_5 can be embedded into (Γ, E), then it is clear from earlier **Remark 5.5.2** that (Γ, E) cannot be a distributive lattice. Conversely, let us suppose that (Γ, E) is a nondistributive lattice and that (Γ, E) does not contain a copy of N_5 as a soft sublattice. Thus (Γ, E) is modular by Theorem

5.5.5. Since the distributive laws do not holds in (Γ, E), there must be soft sets $(F, A), (G, B), (H, C) \subseteq (\Gamma, E)$ such that

$((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C)) \subset (F, A) \wedge ((G, B) \vee (H, C))$.

We define

$(J, D) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C)) \vee ((G, B) \wedge (H, C))$

$(K, E) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C)) \wedge ((G, B) \vee (H, C)$

$(F_1, A) = ((F, A) \wedge (J, D)) \vee \ \ (K, E)$

$(G_1, B) = ((G, B) \wedge (J, D)) \vee \ \ (K, E)$

$(H_1, C) = ((H, C) \wedge (J, D)) \vee \ \ (K, E)$. Then it is easily seen that

$(K, E) \subseteq (F_1, A), (G_1, B), (H_1, C) \subseteq (J, D)$.

Now from $(F, A) \wedge (J, D) = (F, A) \wedge ((G, B) \vee (H, C))$, (by L4 (b))

and $(F, A) \wedge (K, E) = (F, A) \wedge (((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C)) \vee ((G, B) \wedge \ (H, C)))$
$= (((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))) \vee ((F, A) \wedge ((G, B) \wedge (H, C)))$ (by M)

$= ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$. It follows that

$$(K, E) \subset (J, D).$$

We now wish to show that the diagram in Figure 4 is a copy of M_3 in (Γ, E). To do this it suffices to shows that

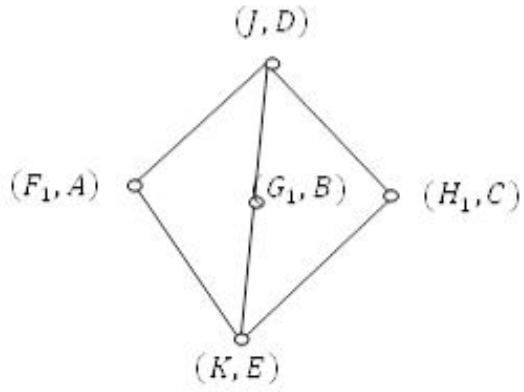


Figure 5.5.3: Copy of M_3 in (Γ, E)

$(F_1, A) \wedge (G_1, B) = (F_1, A) \wedge (H_1, C) = (G_1, B) \wedge (H_1, C) = (K, E)$ and (5.5.1)

$$(F_1, A) \vee (G_1, B) = (F_1, A) \vee (H_1, C) = (G_1, B) \vee (H_1, C) = (J, D)\,. \quad (5.5.3)$$

To verify case (5.5.1)

$$(F_1, A) \wedge (G_1, B) = (((F, A) \wedge (J, D)) \vee (K, E)) \wedge (((G, B) \wedge (J, D)) \vee (K, E))$$

$$= (((F, A) \wedge (J, D)) \wedge (((G, B) \wedge (J, D)) \vee (K, E))) \vee (K, E) \text{ (by M)}$$

$$= (((F, A) \wedge (J, D)) \wedge (((G, B) \vee (K, E)) \wedge (J, D))) \vee (K, E) \text{ (by M)}$$

$$= (((F, A) \wedge (J, D)) \wedge (J, D) \wedge ((G, B) \vee (K, E))) \vee (K, E) \text{ (L1(b))}$$

$$= (((F, A) \wedge (J, D)) \wedge ((G, B) \vee (K, E))) \vee (K, E) \text{ (by L3 (b))}$$

$$= ((F, A) \wedge ((G, B) \vee (H, C)) \wedge ((G, B) \vee ((F, A) \wedge (H, C)))) \vee (K, E)$$

(by L4)

$$= ((F, A) \wedge ((G, B) \vee (((G, B) \vee (H, C)) \wedge ((F, A) \wedge (H, C))))) \vee (K, E)$$

$$= ((F, A) \wedge ((G, B) \vee ((F, A) \wedge (H, C)))) \vee (K, E)$$

$$((F, A) \wedge (H, C) \subseteq (G, B) \vee (H, C))$$

$$= ((F, A) \wedge (H, C)) \vee ((G, B) \wedge (F, A)) \vee (K, E) \text{ (by M)}$$

$$= (K, E)\,.$$

Similarly, we verify case (5.5.3) by interchanging the operations $\vee$ for $\wedge$ in case *(1)*

5.6 Soft Lattice Distributed Computing

The application of soft lattice theory in distributed computing system becomes natural and imperative because when the underline set has attributes, the classical set theory will be unable to tackle such problems.
The following distributed computing problems can be solved by soft lattice theory:
(i) Detecting a predicate in a computation, that is determining whether there exists a *consistent cu*t of the computation satisfying the given predicate.

(ii) Computing the slice of a computation with respect to a predicate. A slice is a concise representation of all those global states of the computation that satisfy the given predicate.

(iii) Analyzing a partial order trace of a distributed program to determine whether it satisfies the given temporal logic.

(iv) Timestamping events and global states of a computation to capture the order relationship.
Below are solutions to some of the itemized problems above.

5.6.1 Detecting global predicate

Definition 5.6.1.1

Let (Γ, E) be the soft set of events of a computation and $\rightarrow$ be the happened-before order on (Γ, E). A soft subset (G, D) of (Γ, E) is a consistent cut if whenever it contains an element $G(f)$ then it contains all elements $G(e)$ that happened before $G(f)$.

This concept is identical to the notion of order ideal in the soft lattice theory.

Definition 5.6.1.2

A predicate is simply a soft Boolean function from the soft set of all *consistent cuts* to $\{0, 1\}$. Equivalently, a predicate specifies a soft subset of consistent cuts in which the soft Boolean function evaluate to 1.

We now define various classes of predicates.

Definition 5.6.1.3 Meet-closed predicates

A predicate B is meet-closed if for all consistent cuts (G, D), (H, C):

$$B\left((G, D)\right) \wedge B\left((H, C)\right) \Rightarrow B\left((G, D) \wedge (H, C)\right).$$

The classes of meet-closed predicate are useful because they allow us to compute the least consistent cut that satisfies a given predicate.

Definition 5.6.1.4 Crucial element

For a consistent cut $(G, D) \subsetneq (\Gamma, E)$ and a predicate B, we define
$G\left(d\right) \in (\Gamma, E) \sim (G, D)$ to be crucial for (G, D) as:

$$\text{Crucial}((G, D), G\left(d\right), B) = \forall\left(H, C\right) \supseteq (G, D) : B\left(G\left(d\right) \in (H, C)\right) \vee \neg B\left((H, C)\right).$$

Definition 5.6.1.5 Linear predicate

A predicate B is linear if for all consistent cuts $(G, D) \subsetneq (\Gamma, E)$,

$\neg B((G, D)) \rightarrow \exists G(d) \in (\Gamma, E) \sim (G, D) : \text{Crucial}((G, D), G(d), B)$.

Theorem 5.6.1.1

A predicate B is linear if and only if it is meet-closed.

Proof

First, assume that B is not closed under meet. We show that B is not linear. Since B is not closed under meets, there exist two consistent cuts (H, C), (K, A) such that $B((H, C))$ and $B((K, A))$ but not $B((H, C) \cap (K, A))$. Define (G, D) to be $(H, C) \cap (K, A)$ is a strict soft subset of $(H, C) \subseteq (\Gamma, E)$ because $B((H, C))$ but not $B((G, D))$. Therefore, (G, D) cannot be equal to (Γ, E). We show that B is not linear by showing that there does not exist any crucial element $G(d)$ for (G, D). A crucial element $G(d)$, if it exists, cannot be in $(\Gamma, E) \sim (G, D)$ because (K, A) does not contain $G(d)$ and still $B((K, A))$ holds.

Similarly, it cannot be in $(K, A) \sim (G, D)$ because (H, C) does not contain $G(d)$ and still $B((H, C))$ holds. It also cannot be in $(\Gamma, E) \sim ((H, C) \cup (K, A))$ because of the same reason. We conclude that there does not exist any crucial event for (G, D).

Now assume that B is not linear. This implies that there exists $(G, D) \subsetneq (\Gamma, E)$ such that $\neg B((G, D))$ and none of the elements in $(\Gamma, E) \sim (G, D)$ is crucial. We first claim that $(\Gamma, E) \sim (G, D)$ cannot be a singleton. Assume if possible $(\Gamma, E) \sim (G, D)$ contains only one element $G(d)$. Then, any consistent cut (H, C) that contain (G, D) and does not contain $G(d)$ must be equal to (G, D) itself. This implies that $\neg B((H, C))$ because we assumed $\neg B((G, D))$. There-

fore, $G(d)$ is crucial contradicting our assumption that none of the elements in $(\Gamma, E) \sim (G, D)$ is crucial.

Let $(W, T) = (\Gamma, E) \sim (G, D)$. For each $G(d) \in (W, T)$, we define $(H, C)_{G(d)}$ as the consistent cut that contains (G, D), does not contain $G(d)$ and still satisfies B. It is easy to see that (G, D) is the meet of all $(H, C)_{G(d)}$. Therefore, B is not meet-closed because all $(H, C)_{G(d)}$ satisfy B, but not their meets.

Example 5.6.1.1

Consider a distributed computation on two processes P_1 and P_2 and the predicate B to be true on a consistent cut if both the processes are in the critical section. Given any consistent cut (G, D) for which B does not hold, either P_1 is not in the critical section, or P_2 is not in the critical section. In the former case, the next event of P_1 after (G, D), entering the critical section is crucial and in the latter case the event of P_2 entering the critical section is crucial. This example can be easily generalized to any global Soft Boolean predicate that can be expressed as a conjunction of local predicates.

Theorem 5.6.1.2

If B is a linear predicate with the efficient advancement property, then there exists an efficient algorithm to determine the least consistent cut that satisfies B (if any).

Proof

An efficient algorithm to find the least consistent cut in which B is true is given in **Figure 5.6.1.1**. We search for the least consistent cut starting from the empty consistent cut. If the predicate is false in the consistent cut, then we find the crucial element using the efficient advancement property and then repeat the procedure. If this is the last state on the process, then we return false else we advance along the process that has the crucial event.

Algorithm to detect a linear predicate.

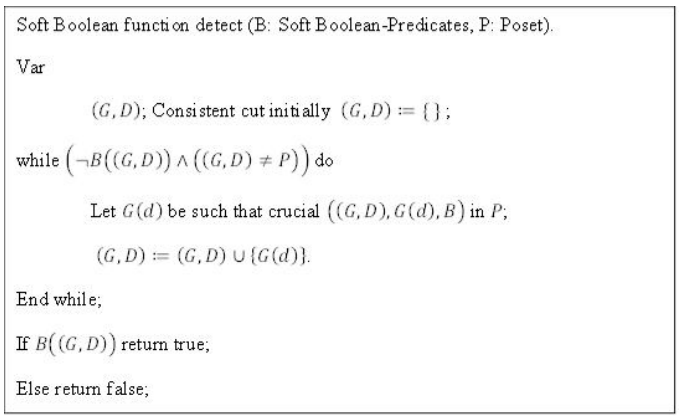


Figure 5.6.1: An efficient algorithm to detect a linear predicate

Assuming that crucial $((G, D), G(d), B)$ can be evaluate efficiently for a given poset, we can determine the least consistent cut that satisfies B efficiently even though the number of consistent cuts may be exponentially larger than the size of the poset. If the predicate B is join-closed, then one can search for the largest consistent cut that satisfies B in a fashion analogous to finding the least consistent cut when it is meet-closed.

Definition 5.6.1.6 (Regular predicate)

A predicate is regular if the soft set of consistent cuts that satisfy the predicate form a soft sublattice of the soft lattice of consistent cuts, Equivalently, a predicate B is regular with respect to P if it is closed under $\cap$ and $\cup$, i.e., for all consistent cuts (G, D), (H, C) of the poset P:

$$B((G, D)) \wedge B((H, C)) \Rightarrow B((G, D) \cup (H, C)) \wedge B((G, D) \cap (H, C)).$$

The set of consistent cuts that satisfy a regular predicates forms a soft sublattice of the soft lattice of all consistent cuts.

Example 5.6.1.2

Consider the predicate B as "there is no outstanding message in the channel". We show that this predicate is regular. Observe that B holds on a consistent cut (G, D) if and only if for all send events in (G, D) the corresponding receive events are also in (G, D). It is easy to see that if $B((G, D))$ and $B((H, C))$, then $B((G, D) \cup (H, C))$. To see that it holds for $(G, D) \cap (H, C)$, let $G(d)$ be any send event in $(G, D) \cap (H, C)$, let $H(d)$ be the receive event corresponding to $G(d)$.

From $B((G, D))$, we get that $H(d) \in (G, D)$ and from $B((H, C))$, we get that $H(d) \in (H, C)$. Thus, $H(d) \in (G, D) \cap (H, C)$. Hence, $B((G, D) \cap (H, C))$.

5.6.2 Slicing distributed computing

Suppose we are only interested in a soft subset of consistent cuts of a computation (but not all consistent cuts of a computation), namely those that satisfy some property of interest to us expressed as a predicate mapping is a consistent cut to a Boolean value.

Suppose the set of consistent cuts for which the predicate evaluates to true forms a soft sublattice of the soft lattice of consistent cuts. A soft sublattice of a distributive soft lattice is also a distributive soft lattice. Therefore, the soft sublattice generated by the consistent cuts satisfying the predicate is completely characterized by the join-irreducible elements of the soft sublattice.

Example 5.6.2.1

The distributed computation shown in Figure **5.6.2.2** consists of two processes P_1 and P_2. Process P_1 executes events **a** and **b**, whereas process P_2 executes events **c** and **d**. On executing **b**, P_1 sends a message to P_2, which is received by P_2 at **d.** The set of consistent cuts of the computation are shown in Figure

5.6.2.2.

Suppose we are interested in only those consistent cuts for which no messages are in transit, also known as *strongly consistent cuts*. They have been shaded in Figure **5.6.2.3** and are shown separately in Figure **5.6.2.4.** The set of strongly consistent cuts forms a soft sublattice and its join-irreducible element has been drawn with thick boundaries. The poset induced on the set of join-irreducible elements of the soft sublattice is shown in Figure **5.6.2.5**

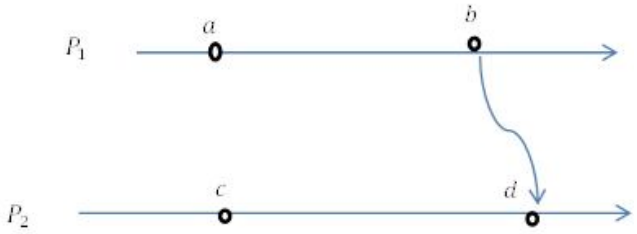


Figure 5.6.2: A distributed computation

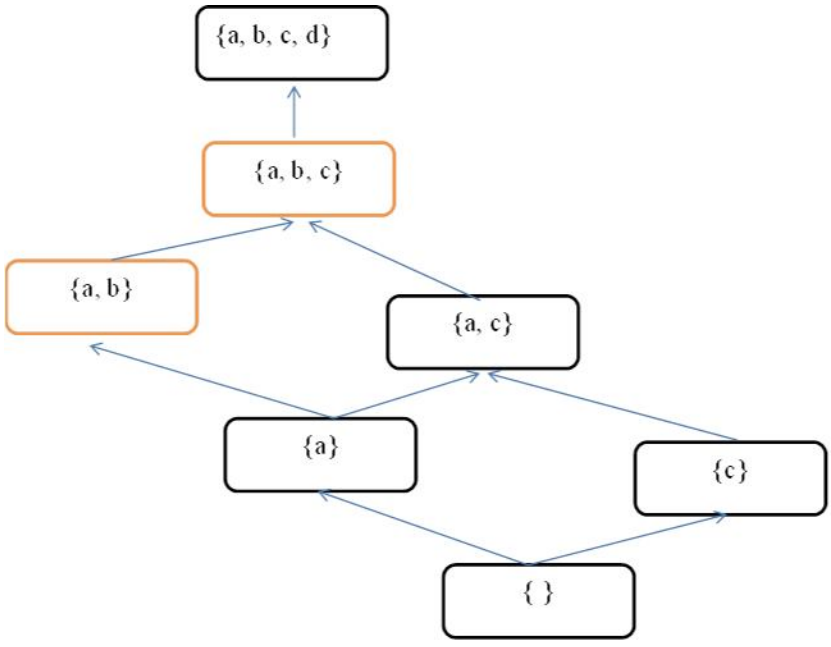


Figure 5.6.3: The distributive soft lattice generated by its consistent cuts

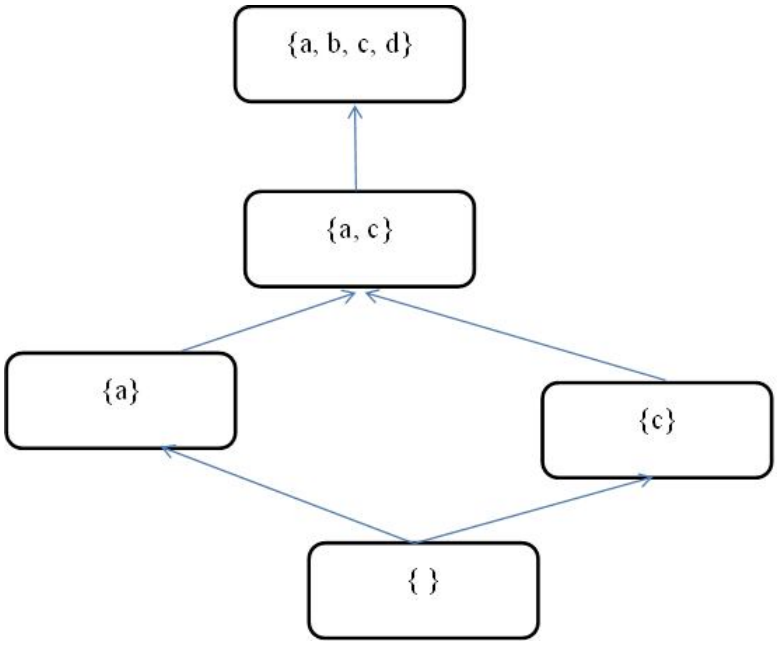


Figure 5.6.4: The soft sublattice containing all consistent cuts for which no messages are in transit

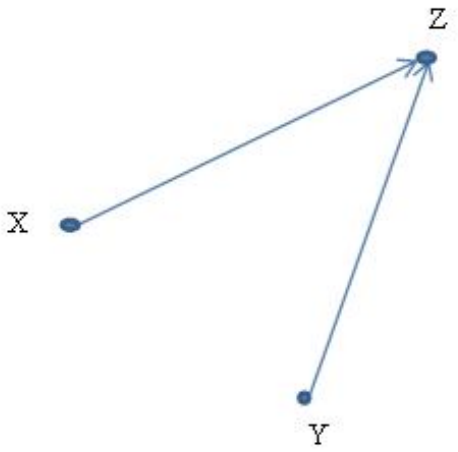


Figure 5.6.5: The poset induced on the set of join-irreducible elements of the soft sublattice

In case the set of consistent cuts that satisfy the predicate does not form a soft sublattice, we add one or more other consistent cuts, which do not satisfy the predicate, to complete the soft sublattice. The consistent cuts are added in such a way so as to minimize the total number of consistent cuts in the resulting soft sublattice. The soft sublattice is then represented using the set of its join-irreducible elements. This succinct representation of a possible large set of consistent cuts satisfying some property is referred to as a slice.

Theorem 5.6.2.1 The slice of a distributed computation is uniquely defined for all predicates.

Proof

Let D denote the set of all consistent cuts that satisfy the predicate. We show that the soft sublattice with the least number of consistent cuts that satisfy D is uniquely defined. Assume the contrary. Let X and Y be two distinct soft sublattices with the least number of consistent cuts such that:

(1) $Cardinality\,(X) = Cardinality\,(Y)$, and

(2) both X and Y contain D.

Consider $Z = X \cap Y$. Clearly, Z also contain D. Also, since $X \neq Y$, $Cardinality\,(Z) < Cardinality\,(X)$ and $Cardinality\,(Z) < Cardinality\,(Y)$. It can be proved that intersection of two soft sublattices is also a soft sublattice. This implies that Z is a soft sublattice that contain D and has fewer number of consistent cuts than either X or Y, a contradiction.

Remark 5.6.2.1

The slice for a predicate may contain consistent cuts that do not satisfy the predicate, namely that are added to complete the soft sublattice. A slice is *lean* if it contains only those consistent cuts that satisfy the predicate. Clearly,

the slice of a computation for a predicate is *lean* if and only if the predicate is regular.

Another way of looking at slice is that it specifies which events should be executed in an atomic fashion and the order in which they should be executed. For example, the slice shown in Figure **5.6.2.4** and redrawn in Figure **5.6.2.5** specifies that events **b** and **d** should be executed atomically after events **a** and **c** have been executed. This is expected because any consistent cut which includes the send event of a message but not its receive will have at least one message in transit.

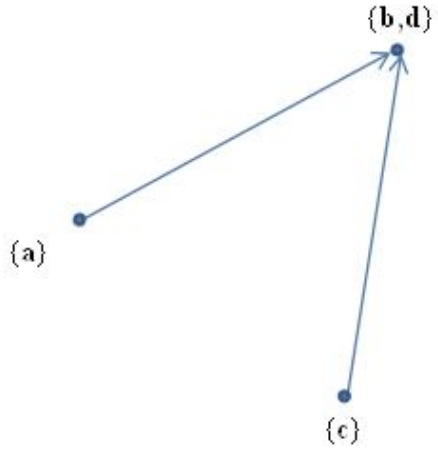


Figure 5.6.6: A slice that depict the events that are to execute atomically

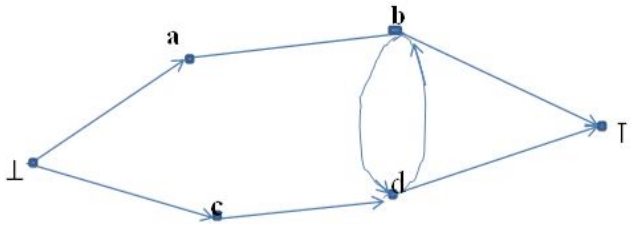


Figure 5.6.7: The graph representation of the slice in Figure 5.6.2.6

For algorithmic purposes, it is more convenient to represent a slice using a directed graph on events possibly containing cycles: all events that are to be executed atomically form a strongly connected component. The notion of consistent cut has to be extended appropriately.

We define a consistent cut (global state) on directed graph as a subset of vertices such that if the subset contains a vertex then it contains all its incoming neigbours. Observe that the empty $\emptyset$ and the set of all vertices are trivial consistent cuts.

We introduce a fictitious *global initial* and *global final* event, denoted by $\perp$ and $\top$**,** respectively.

The global initial event occurs before any other event on the processes and initializes the state of the processes. The global final event occurs after all other events on the processes. Any non-trivial consistent cut will contain the global initial event and not the global final event.

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Summary

This thesis presents a critical study of soft set theory, its algebraic structure and application in soft lattice distributed computing. It is made up of six chapters as shown summarily below.

Chapter one covers the general introduction of the thesis, which includes the historical background of the study, motivation and justification, and the aim and objectives of the study.

A critical literature survey of the fundamentals of soft set theory, in respect to its origin and general concept, application point of view and the algebraic point of view are presented in chapter two.

Chapter three presents a critical study of the fundamentals of soft set theory, its operation and various algebraic structures. We also redefined the concept of conjunction and disjunction as an algebraic operators on soft set and used in chapter four to propose soft Boolean algebra.

In chapter four, a perception named soft Boolean algebra is introduced and some related results are established.

Chapter five discusses soft lattice and some of its algebraic properties are introduced. Some important classes of lattices using the concept of soft sets theory are presented and some results are established. It also presents some applications of soft lattice theory in distributed computing system.

6.2 Conclusion

In this thesis, fundamentals of soft set theory, algebraic structures of soft set, soft Boolean algebras and soft lattice theory are presented. We introduced some important classes of lattices using the concept of soft sets theory and presented their properties; and some new results were obtained. We discussed the various applications of soft lattice theory to solve problems in distributed computing system.

The following new results are the main contributions of this thesis:

1 Redefined Concept of Conjunction (AND) and Disjunction (OR)

1.1 **Definition 4.1.1** Given any two soft sets (F, A) and (G, B) over a common universe U, the disjunction of (F, A) and (G, B) denoted by $(F, A) \vee (G, B)$ is defined as $(F, A) \vee (G, B) = (H,\ A \cap B)$, where $H(\alpha) = F(\alpha) \cup G(\alpha)$, $\forall \alpha \in A \cup B$.

1.2 **Definition 4.1.2** Given any two soft sets (F, A) and (G, B) over a common universe U, the conjunction of (F, A) and (G, B) denoted by $(F, A) \wedge (G, B)$ is defined as $(F,\ A) \wedge (G, B) =\ (P,\ A \cap B)$, where $P(\alpha) = F(\alpha) \cap G(\alpha)$, $\forall\ \alpha \in A \cap B$.

2 **Properties of the Redefined Conjunction ($\wedge$) and Disjunction ($\vee$)**

2.1 **Proposition 4.2.1** Given any soft sets (F, A), (G, B) and (H, C) over a common universe U, then the following properties are satisfied:

2.1.1 *Idempotent laws*

(i) $(F, A) \vee (F, A) = (F, A)$

(ii) $(F, A) \wedge (F, A) = (F, A)$

2.1.2 *Commutative laws*

(i) $(F, A) \vee (G, B) = (G, B) \vee (F, A)$

(ii) $(F, A) \wedge (G, B) = (G, B) \wedge (F, A)$

2.1.3 *Associative laws*

(i) $((F, A) \vee (G, B)) \vee (H, C) = (F, A) \vee ((G, B) \vee (H, C))$

(ii) $((F, A) \wedge (G, B)) \wedge (H, C) = (F, A) \wedge ((G, B) \wedge (H, C))$

2.1.4 *Distributive laws*

(i) $(F, A) \vee ((G, B) \wedge (H, C)) = ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

(ii) $(F, A) \wedge ((G, B) \vee (H, C)) = ((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C))$

2.1.5 *Absorption laws*

(i) $(F, A) \vee ((F, A) \wedge (G, B)) = (F, A)$

(ii) $(F, A) \wedge ((F, A) \vee (G, B)) \ = (F, A)$

2.1.6 *Complement laws*

(i) $(F, A) \vee (F, A)^c = \tilde{U}$

(ii) $(F, A) \wedge (F, A)^c = \widetilde{\emptyset}$

2.1.7 *Identity laws*

(i) $(F, A) \vee \widetilde{\emptyset} = (F, A)$

(ii) $(F, A) \wedge \widetilde{\emptyset} = \widetilde{\emptyset}$

2.1.8 *Universe laws*

(i) $(F, A) \vee \tilde{U} = \tilde{U}$

(ii) $(F, A) \wedge \tilde{U} = (F, A)$

2.1.9 *De Morgan's laws*

(i) $((F,\ A) \vee (G, B))^c = (F,\ A)^c \wedge (G,\ B)^c$

(ii) $((F,\ A) \wedge (G, B))^c = (F,\ A)^c \vee (G,\ B)^c$

[Proofs on page 38]

2.2 **Theorem 4.3.1** Let SB be a collection of all soft sets under a common universe U. Let $\emptyset \in SB$ be an empty soft set and $\tilde{U}$ be the universal soft set. Then SB equipped with the two binary operations $\wedge$ and $\vee$ is a Boolean algebra. That is $\left(SB,\ \wedge,\ \vee, \emptyset,\ \tilde{U}\right)$ is a Boolean algebra.

[Proofs on page 41]

2.3 **Theorem 4.4.1** Let $\left(SB,\ \wedge,\ \vee, \emptyset,\ \tilde{U}\right)$ be a Boolean algebra. Then for any $(F, A) \in SB$,

1. *Domination law*

(i) $(F, A) \vee \tilde{U} = \tilde{U}$

(ii) $(F, A) \wedge \widetilde{\emptyset} = \widetilde{\emptyset}$

2. *Idempotent law*

(i) $(F, A) \vee (F, A) = (F, A)$

(ii) $(F, A) \wedge (F, A) = (F, A)$

3. *Absorption law*

(i) $(F, A) = (F, A) \vee ((F,\ A) \wedge (G, B))$

(ii) $(F, A) = (F, A) \wedge ((F,\ A) \vee (G, B))$

4. *Complement law*

(i) $(F,A) \vee (F,A)^c = \tilde{U}$

(ii) $(F,A) \wedge (F,A)^c = \widetilde{\emptyset}$

5. *Uniqueness of* U *and* $\emptyset$

(i) $(F,A) \vee \widetilde{\emptyset} = (F,A)$

(ii) $(F,A) \wedge \tilde{U} = (F,A)$

[Proofs on page 47]

2.4 **Theorem 4.4.2** Let $\left(SB,\ \wedge,\ \vee, \emptyset,\ \tilde{U}\right)$ be a Boolean algebra. For any soft sets $(F,A)\,,\ (G,\ B)$ and $(H,\ C) \in SB$, the following hold.

2.4 .1 (i) $(F,A) \vee ((F,A) \vee ((G,B) \vee (H,C))) = (F,A) \vee (((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c \vee ((F,A) \vee ((G,B) \vee (H,C))) = (F,A)^c \vee (((F,A) \vee (G,B)) \vee (H,C))$

2.4 .2 (i) $(F,A) \wedge ((F,A) \vee ((G,B) \vee (H,C))) = (F,A) \wedge (((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c \wedge ((F,A) \vee ((G,B) \vee (H,C))) = (F,A)^c \wedge (((F,A) \vee (G,B)) \vee (H,C))$

2.4 .3 (i) $(F,A) \vee ((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A) \wedge (((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c \vee ((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A)^c \wedge (((F,A) \vee (G,B)) \vee (H,C))$

2.4 .4 (i) $(F,A) \wedge ((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A) \vee (((F,A) \vee (G,B)) \vee (H,C))$

(ii) $(F,A)^c \wedge ((F,A) \vee ((G,B) \vee (H,C))) \neq (F,A)^c \vee (((F,A) \vee (G,B)) \vee (H,C))$

[Proofs on page 50]

2.4 **Corollary 4.4.1** In the theorem 4.3.1, the associativity laws can be neglected.

[Proofs on page 55]

3 **Results of Soft Lattice Theory**

3.1 **Theorem 5.2.1** A finite partial ordered soft set with a finite ordered parameter sets defined on it is a soft lattice.

[Proofs on page 66]

3.2 **Theorem 5.2.2 Definition 5.1.1** and **Definition 5.1.8** are equivalent.

[Proofs on page 66]

3.3 **Theorem 5.2.3** Given a soft semilattice(Γ, E), define $\Gamma(\mathrm{e}_1) \subseteq \Gamma(\mathrm{e}_2)$ if and only if $\Gamma(\mathrm{e}_1) \wedge \Gamma(\mathrm{e}_2) = \Gamma(\mathrm{e}_1)$. Then $((\Gamma, E), \subseteq)$ is an ordered soft set in which every pair of elements has greatest lower bound.

[Proof on page 68]

4 **Theorem 5.4.1** Two soft lattices (F, A) and (G, B) are isomorphic if and only if there is a bijection $\alpha : (F, A) \to (G, B)$ such that both α and α^{-1} are order-preserving.

[Proof on page 73]

5. **Theorem 5.4.2** Let (Γ, E) be an ordered soft set, and $A, B \subseteq E$, such that (F, A) and (G, B) are defined. Let $\emptyset : (F, A) \to (G, B)$ be define as $\emptyset(F(e_1)) = \{F(e_2) \in (F, A) : F(e_2) \subseteq F(e_1), \forall F(e_1) \in (F, A)\}$. Then (F, A) is isomorphic to the range of $\emptyset$ ordered by $\subseteq$.

[Proof on page 75]

6. **Theorem 5.4.3** Let (Γ, E) be a soft lattice. Let $A, B \subseteq E$, such that (F, A) and (G, B) are all defined. If $\alpha : (F, A) \to (G, B)$ is a mapping then the following holds:

(i) $(F, A) \subseteq (G, B) \Rightarrow \alpha((F, A)) \subseteq \alpha((G, B))$

(ii) $\alpha((F, A) \vee (G, B)) = \alpha((F, A)) \vee \alpha((G, B))$

(iii) $\alpha((F, A) \wedge (G, B)) \subseteq \alpha((F, A)) \wedge \alpha((G, B))$, equality hold if α is one-one.

[Proofs on page 76]

7. **Theorem 5.4.4** If $\alpha : (F, A) \to (G, B)$ is a bijective mapping, then $\alpha^{-1} : (F, A) \to (G, B)$ is also a bijective mapping.

[Proof on page 78]

8. **Theorem 5.5.1** Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B),and (H, C) are all defined. Then (Γ, E) satisfies D1 if and only if it satisfies D2.

[Proof on page 81]

9. **Theorem 5.5.2** Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B),and (H, C) are all defined. Then (Γ, E) together with the binary operations $\vee$ and $\wedge$ satisfies both of the following inequalities:

$((F, A) \wedge (G, B)) \vee ((F, A) \wedge (H, C)) \subseteq (F, A) \wedge ((G, B) \vee (H, C))$ and

$(F, A) \vee ((G, B) \wedge (H, C)) \subseteq ((F, A) \vee (G, B)) \wedge ((F, A) \vee (H, C))$

[Proof on page 83]

10. **Theorem 5.5.3** The modular law is obviously equivalent (for lattices) to

the identity

$$((F, A) \wedge (G, B)) \vee ((G, B) \wedge (H, C)) = (G, B) \wedge (((F, A) \wedge (G, B)) \vee (H, C))$$

[Proof on page 85]

11. **Theorem 5.5.4** Every distributive soft lattice is a modular soft lattice.

[Proof on page 85]

12. **Theorem 5.5.5** Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A), (G, B),and (H, C) are all defined. Then (Γ, E) is a nonmodular soft lattice if and only if N_5 can be embedded into (Γ, E).

[Proof on page 87]

13. **Theorem 5.5.6** Let (Γ, E) be a soft lattice. Let $A, B, C \subseteq E$ such that (F, A),(G, B), and (H, C) are all defined. Then (Γ, E) is a nondistributive soft lattice if and only if M_3 or N_5 can be embedded into (Γ, E).

[Proof on page 89]

14. **Theorem 5.6.1** A predicate B is linear if and only if it is meet-closed.

[Proof on page 93]

15. **Theorem 5.6.2** If B is a linear predicate with the efficient advancement property, then there exists an efficient algorithm to determine the least consistent cut that satisfies B (if any).

[Proof on page 94]

16. **Theorem 5.6.2.1** The slice of a distributed computation is uniquely defined for all predicates. [Proof on page 99]

6.3 Recommendations

Soft set theory is a relatively new area in mathematics and very applicable in real life decision making problems that involve uncertainties. The notions of soft lattice implication algebras and soft lattice implication sub-algebras could be interesting for further studies. We recommend that the application of soft lattice be extended to graph decomposition, cryptography and analysis of partial order traces for temporal logic predicate.

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