MATHEMATICAL MODELLING OF FLUID EXCHANGE THROUGH BLOOD CAPILLARIES

by

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DECLARATION

I hereby declare that this thesis has been composed by myself and that it is a record of my own research work. It has not been accepted in any previous application for a higher degree. All sources of information are specifically acknowledged by means of reference.

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NOVEMBER 1986

CERTIFICATION

This thesis titled " MATHEMATICAL MODELLING OF FLUID EXCHANGE THROUGH BLOOD CAPILLARIES" by Aishatu Saleh meets the regulation governing the award of the degree of Master of Science of Ahmadu Bello University, and is approved for its contribution to scientific knowledge and literary presentation.

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A B S T R A C T .

Title : Mathematical Modelling of Fluid-Exchange through Blood Capillaries .

Flow properties of blood play an increasing role in the diagnostic and therapy of many diseases as well as in different artificial organs. The vessels of human circulatory system range in internal diameter from approximately 2.5 cm to 3μ ($\mu=10^{-4}$ cm). The vessels with an internal dimension less than about 500μ are commonly reffered to as the microvessels. Microcirculatory vessels are small artries (70-500μ), arterioless (l0-70μ); capillaries (4-10μ), venules (10-100μ) and small viens (110-500μ).

The first interest in mathematical modelling of the microcirculation developed from experimental investigations to determine the arrangement of blood vessels in tissue. These experiments were being conducted as a result of interest in the mechanism by which oxygen was transported from blood to tissue and how this transfer could be controlled. The first theoretical study on the exchange of oxygen across the capillary walls and the tissue surrounding them was made by Krogh (1919) for which he won his Nobel prize. He observed in detail the distribu-tion of capillaries in muscle, and proposed a functional unit of capillary beds. Krogh's model in composed of a long straight cylindrical capillary and a concentric cylinder of homogeneous tissue surrounding it. The present thesis, which is concerned with the mathematical

modelling of fluid exchange through blood capillaries, comprises of four chapters.

The first chapter introduces the subject 'Biomechanics' and outlines the historical background of the subject and its applications to clinical problems in the cardiovascular system, quantitative physiology, orthopedics, artifical limb and etc. The second chapter describes the various aspects of the microcirculation. The factors, Starting hypothesis, capillary filtration coefficient, capillary and tissue hydrostatic pressures, plasma and tissue osmotic pressures are discussed. Finally this chapter discusses a variety of approaches and formulations for the mathematical modelling of the microcirculation. In the third chapter a mathematical model is developed to see the effects of the variable filtration coefficient, lymph flow and reflection coefficient on the fluid exchange between blood capillary and the tissue space. It is assumed that the exchange of fluid across the capillary wall obey Starling's law and that the filtration coefficient and the reflection coefficient of the capillary wall are not uniform along the length of capillary but is a linear function of the distance from the arterial end. It is further assumed that the flow of fluid through the tissue region is governed by a generalized Darey's equation. Under these assumptions the velocity and pressure distributions in the capillary and tissue regions are obtained. We also obtain the average

interstitial fluid pressure and the average effective pressure difference between the blood and interstitial fluid. While chapter three is concerned about the uniform capillary radius, chapter four is devoted to fluid exchange between blood capillary of slowly but arbitrary varying cross section and the tissue space. An analytical solution in the general case is obtained by perturbation technique.

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CHAPTER ONE

INTRODUCTION

1.1 Biomechanics:

Biomechanics is the application of the theory of mechanics to biology. The science of mechanics comprises the study of motion (or equilibrium) and the forces which cause it. Thus thermodynamics, heat and mass transfer, cybernatics, computing methods etc, are areas covered by mechanics. The term biology denotes the whole of the sciences which deal with living things. Since the biological world is part of the physical world around us, it is natural that it is an object of inquiry in mechanics.

Biomechanics aims at understanding the mechanics of living systems. It is an ancient subject that covers a very wide territory, but this work is concerned on the physiological applications. The motivation for this work comes from the realization that physiology can be more understood by biomechanics in the same way as an airplane cannot be understood without aerodynamics. For an organ, biomechanics helps us to understand its normal functions, predict changes due to alterations, and propose methods of artificial intervention. Thus diagnosis, surgery, and prothesis are closely associated with biomechanics.

1.2 Historical Background:

It is of more than passing interest to note that the involvement of the physical scientist in biology is not recent. Robert Aroke in 1665 first used the word

"cell" for the individual isolated spaces in cork and recognized the cell as a microscopical living matter. It required, however, 150 years for the biologists to recognize and state clearly that the unit of all living matter is the cell. The ancient philosopher Aristotle (384 - 322 B.C.) was one of those that tried to clearify the connection between "physics" - which for him was a general description of the universe, and the study of living things.

Some of the great names that contributed significantly to the development of biomechanics include William Harvey (1578 - 1658), who, in 1615 was credited with the discovery of blood circulation. The actual discovery of capillaries was made 45 years later by Marcello Malpighi (1628 - 1694). The constancy of the period of the pendulum was discovered by Galileo (1564 - 1642), who used the pendulum to measure the pulse rate of people. The physical discoveries of Galileo and the demonstrations of Santorio (1561 - 1686) and of Harvey led to the attempt to explain vital processes in term of mechanics. Galileo showed that mathematics was the key to science, without which nature could not be properly understood. This inspired Descartes (1596 - 1650) to work on physiology. He proposed a complicated model of animal structure, including the function of nerves, on theoretical grounds. Subsequent investigations failed to confirm many of his findings. These errors of fact

caused the loss of confidence in Descartes' approach - a lesson that should be kept in mind by all theoriticians.

Robert Hooke gave us Hooke's law in mechanics and the word "cell" in biology. Robert Boyle (1621 - 1691) studied the lung and discussed the function of air in water with respect to fish respiration. Thomas Young (1773 - 1829), a London physician, gave us the Young's modulus of elasticity. Poiseuille (1799 - 1869) invented the mercury manometer to measure the blood pressure in the aorta of a dog while he was a medical student, and discovered Poiseuille's law of viscous flow, upon graduation. Van Helmholtz (1821 - 1894) who was a professor of physiology, pathology, anatomy and finally professor of physics, wrote his paper on the "Law of conservation of energy" in barracks while he was in military service, fresh out of medical school. His contribution ranged over optics, acoustics, thermodynamics, electrodynamics, physiology and medicine. The physiologist Fick (1827 - 1901) was the author of Fick's law of mass transfer. The hydrodynamists Korteweg (1878) and Lamb (1898) wrote beautiful papers on wave propagation in blood vessels. Otto Frank (1865-1944) worked out a hydrodynamic theory of circulation.

The list perhaps suffices to show that there were, and of course are, people who would be equally happy to work on living subjects as well as inanimate objects. Most often, what a scientist picks up and works on depends a great deal on chance, and the biological world is so rich a field that

one should not permit the opportunities there to slip by unnoticed.

There were equally those physiologists who tried to clearify biomechanics. These include William Harvey who concieved the principle of blood circulation. Marcello Malpighi who discovered the capillary. Stephen Hales (1677 - 1761) measured the arterial blood pressure in the horse and correlated it to hemorrhage. He also measured the distensibility of the aorta and used it to explain how the intermittent pumping of the heart can be converted to a smooth flow in the blood vessels. He introduced the concept of peripheral resistance in blood flow and showed that the main site of this resistance was in the minute blood vessels in the tissue.

Otto Frank worked out a hydrodynamic theory of circulation. Starling (1866 - 1926) proposed the law of mass transfer across biological membranes and clarified the concept of water balance in the body. Krogh (1874 - 1949) won his Nobel prize on the mechanics of microcirculation, while Hill (1886 - 1977) won his Nobel prize on the mechanics of the muscle. Their contributions form the foundation on which biomechanics rests.

1.3 Applications:

Most biomechanics work aims at increasing basic knowledge about living systems and any artificial interventions. Any serious projects in bioengineering, except those which are well understood, need advanced basic

understanding. For example, prosthetic heart valves are now common - place, but a need exists for improving their service life, minimizing blood trauma, eliminating undesirable blood - artificial material interaction and simplifying anticoagulation managements. To achieve these improvements in a second generation of valves, one must understand the flow of blood in the heart, through the valves and in the aorta. To study this flow, one needs all the modern resources. It requires the same kind of attention engineers have given to the development of airplanes and rockets.

Some topics of concentration of current biomechanics are :

A. <u>Clinical Problems in the Cardiovascular system.</u>

(i) Prosthetic heart valve

(ii) Heart assist devices, such as the left ventricle assist pump, the aorta ballon pump, body acceleration synchronized with heart beat, peripheral cuffs and diastolic counterpulsation.

(iii) Extracorporeal circulation. Heart-lung machine and hemodialysis machine.

(iv) Heart replacement.

(v) Postoperative trauma, pulmonary edema and atelectasis.

(vi) Arterial pulse wave analysis.

(vii) Ultrasound applications. Phonoangiography,

turbulent noise analysis and pseudo-sound generation at artheroscelerotic constrictions in arteries.

B. Quantitive Physiology.

(i) Systems analysis of physiology.

(ii) Rheology of biological tissues such as blood, muscles, bones, connective tissues and artificial implantable materials.

(iii) Analysis of fluid transfer across biological membranes and blood vessels.

(iv) Diffusion analysis such as pulmonary function and indicator dilution method.

(v) Interfaces and surfactunts in the lung. Thrombogenic tendency of blood on artificial implantable materials. Recent work shows that platelet action and thrombosis on interfaces are shear stress dependent, thus projecting mechanics to the foreground of this important problem.

(vi) Microcirculation biomechanics has contributed to every aspect of microcirculation research .

C. Additional Applications.

(i) Surgery: Biomechanics of injury and the healing of surgical wounds are topics of great importance. New surgical procedures such as arterio-venous reversal and attachment of implantable mechanical devices are studied.

(ii) Implantable Artificial Materials: Materials for prosthesis require a detail study of their mechanical properties and biocompatibility.

(iii) Orthopedics, Orthosis and Orthodontics' mechanics of bone and cartilage: Reaction of bone to stress resulting to either in growth or reabsorption. Joint lubrication, joint replacement and orthopedic implants.

(iv) Artificial Limbs: Design of artificial limb is a classical objective of biomechanics.

(v) Artificial Internal Organs: Renal replacement, artificial kidney, artificial heart and other organs. Here an infinitely large vista is open in front of us.

(vi) Wheelchairs and Beds: Chairs to enable disabled patients to function as normally as possible. Use of back muscle, eye movement, or voice to steer and control such chairs. Design of beds to minimize trauma of bed-bound patients.

(vii) Occupational Safety and Health; Examples are; mechanics of pulmonary function testing, detection of black lung disease, mechanics of exercises and athletics.

(viii) Highway Safety; Head injury research. Seat belt research. Impact analysis. Safety of internal organs. Design of cars from the point

of view of protecting the passengers.

(ix) Flight Safety: Vibration and unpact of human bodies. Human tolerance to acceleration. Man's reaction to environment such as zero gravity, high heat, low heat, or low oxygen atmosphere.

(x) Flying and Swimming in Nature: Flagellar motion. Microbial locomotion swimming. Flight of insects and birds.

(xi) Design of Socially Acceptable Monitoring Instruments. Instruments to meter and record a variety of parameters such as blood pressure, heart rate, respiration rate, electrocardiograms, and electromyograms to be carried or worn by patients in a socially acceptable way over a long period of time.

Every one of these topics can make a significant contribution to the welfare of human beings and provide extremely rewarding careers for bioengineers.

CHAPTER TWO

MICROCIRCULATION.

2.1 Introduction:

The vessels of human circulatory system range in internal diameter from approximately 2.5 cm to 3μ ($\mu=10^{-4}$ cm). The vessels with an internal dimension less than about 500μ are commonly referred to as the **microvessels**. Microcirculatory vessels are; small arteries (70 - 500 μ), arterioles (10 - 70μ), capillaries (4 - 10μ), venules (10 -110μ) and small viens (110 - 500μ).

The first interest in mathematical **modelling** of the microcirculation developed from experimental investigations to determine the arrangement of blood vessels in tissue. These experiments were being conducted as a result of interest in the mechanism by which oxygen was transported from blood to tissue and how this transfer could be controlled. Ranvier (1874) and Spalteh**oltz** (1885), had determined the geometric arrangement of blood vessels in straited muscle, but the work of August Krogh (1919, 1929), is credited with motivating and laying the basis for the first conceptual models of microcirculation. Krogh theorized that the rate of oxygen transport was related to the number and distribution (arrangement) of capillaries in tissue, and to the permeability to oxygen of the capillary walls and surrounding tissue. His first experimental efforts were aimed at obtaining an adequate knowledge of the number, distribution and surface area of capillaries in muscle tissue. Krogh's investigations showed

that, in a cross - section of straited muscle open capillaries were distributed quite uniformly. Based on this observation, he concluded that each capillary could be regarded as running parallel to the muscle tissues and supplying a concentric region of tissue surrounding that capillary, and this region was independent of the other parallel capillaries and tissues. He determined the average radius of a hypothetical tissue cylinder by counting the capillaries in each cross - section and dividing the cross - section's area by the number of capillaries he found. Krogh recognized that such an ideal physical geometric arrangement shown in F.I.G. 1, was amenable to mathematical description. Though not a mathematician, Krogh persuaded a colleague, the Danish mathematician Erlang, to describe his conceptual model in terms of mathematics. The result was the now famous Krogh - Erlang equation given by

$$U(r,z) = U_o(z) - \frac{K}{2D_r}\left| r_t^2 \log\left(\frac{r}{r_c}\right) - \frac{1}{2}(r^2 - r_c^2)\right| \qquad (1)$$

on the tissue annulus r, $r_c \leq r \leq r_t$, and where $U(r,z)$ is the concentration (or partial pressure) of oxygen in volumes of gas per volume of tissue, the tissue radius is r_t, the capillary radius is r_c, the radial oxygen diffusion coefficient is D_r and the constant rate of oxygen consumption is K. Although a highly simplified model, Krogh's tissue cylinder model was a major step in the study of substrate supply to living tissue. He not only initiated the analytical study but set the course for its study from 1919 to the present day.

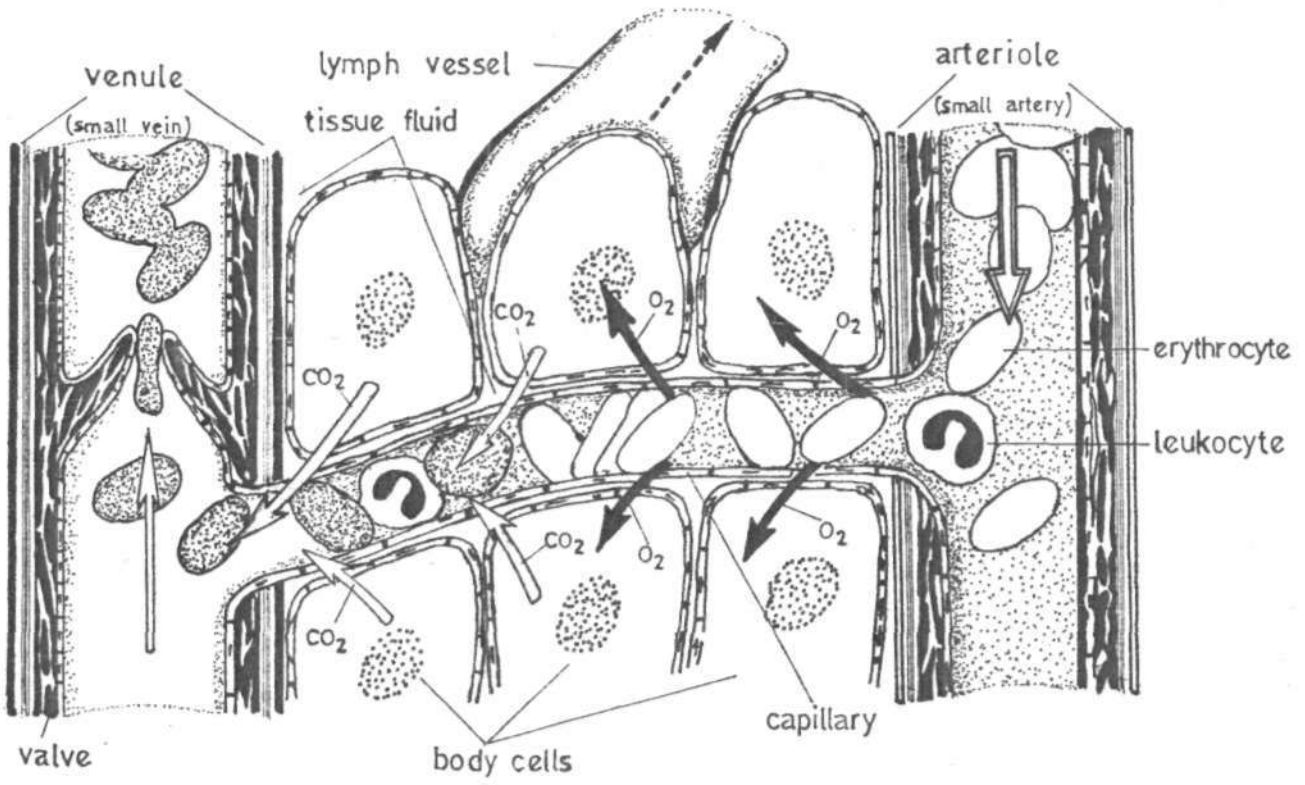


FIG. 1 : Diagram showing the connection between the small blood vessels through capillaries; Note the lymph capillary, a part of tissue drainage.

Krogh also was the first to measure (determine) rates at which oxygen diffuses through tissues, and to originate many other measurements essential to the modelling process. In spite of the passage of time and the development of new modern methods, many of his techniques and parameter values are still found to be accurate and valid for use in modelling studies of today.

Following Krogh's initial studies, several investigators considered the diffusion of oxygen, lactic acid and other metabolities through tissues. Hill (1928) derived equations which extended the steady state models of Krogh to the unsteady state. However, Hill did not apply his models to geometries or conditions representative of living tissues. Thus his results contributed mostly to the formulation of concepts and equations describing the diffusion process rather than applications to actual circulatory systems. Other general treatments of diffusion models were later published by Crank (1956) and by Carslaw and Jaeger (1959).

In the early 1950s, Roughton (1952) and Thews (1950) developed steady and unsteady state equations for a Krogh tissue cylinder. Thews' treatment obtained a number of solution for cases of physiological interest. He presented graphical displays of subtrate diffusing from a supplying capillary with an assumed oxygen profile into tissue spaces with different geometries. Roughton in his work, considered that oxygen combines with (is bound by) a substance in the tissue through a reversible reaction. This model is an idealization of oxygen diffusing through red muscle which

contains the binding protein myoglobin. Several special cases of his models were treated but solutions of the completely general equations were not obtained. Roughton's models treated only the tissue conditions and did not predict oxygen conditions in the capillary.

The first attempts to include intra-capillary conditions seen to have been Kety's (1870) work. Kety hypothesized that under certain conditions, the oxygen content of capillary blood should fall linearly from the arterial to the venous end. He combined this hypothesis with the nonlinear oxygen hemoglobin dissociation curve to give the first model including oxygen distribution both within the capillary and in the Krogh tissue cylinder. He generated monograms from which capillary and tissue oxygen could be determined.

Opitz and Schneider (1950) applied Krogh's concepts to a determination of oxygen supply to nerve cells of the brain. Since this application was in a tissue quite different than that of straited muscle, their analytical modelling work was necessarily accompanied by extensive experimental investigations which related to brain microcirculatory anatomy and oxygen metabolism. Their foundings led to the inclusion of additional terms in the equations describing the Krogh tissue cylinder, which accounted for axial diffusion as well as the previsouly considered radial components. Analytical solutions for these expanded equations were obtained by Becker and Thews and were later described by Thews (1960) in the summary article. Thews also considered oxygen gradients in the capillary as well as in the tissue cylinder.

Two additional features were added to the Krogh's cylinder by Blum (1960). He considered a finitely permeable or semipermeable capillary wall and considered a constant, linear and nonlinear (Michaelis-Meuten) metabolic r in the tissue space. Since he did not consider any bound substrate in the capillary blood, his results would not apply to the substrate oxygen or the by - product carbon-dioxide. Such results would describe the passive transport of substrate such as glucose, bilirubin, or any other dissolved, flow transported diffusable substrate.

With the development of numerical approximation techniques and the increasing availability of electronic digital computers by the 1960s, several investigators became interested in the mathematical description of substrate supply to tissue. This interest centered on a more complete description of the Krogh cylinder as well as other geometries. These extended systems described substrate movement from supplying arterioles to the capillaries by time dependent flow, with a nonlinear kinetic reaction occurring in the capillary blood, followed by diffusion within the capillary vessels, then across a finitely permeable capillary wall and finally movement into a tissue space were it diffused and was bound and/or consumed. The metabolic functional dependence on substrate level could be zero order (constant), first order (linear), or Michaelis-Meuten (nonlinear). Reneau (1966, 1967, 1969, 1970) considered the Krogh

cylinder model and some generalizations. This work was followed or paralleled by several new modelling status, including the steady state models considered by Thews (1968) and Gonzalez-Fernandez (1968), and later the time dependent effects studied by Fletcher (1972, 1973, 1975). Other approaches using multi capillary profusion were considered by Metzger (1967, 1969, 1973), and by Grunewald and Lübbers (1966, 1973a, 1973b).

2.2 MICROCIRCULATORY FLUID EXCHANGE.

The principal site of passage of small lipid- in soluble molecules is believed to be the capillaries. The most detailed information available on such substances is that obtained on water movement in single capillaries by the occlusion technique developed by Landis (1927) and extended by Intaglietta and Zweifach (1974).

The capillary barrier to lipid - insoluble substances appears to be increasingly porous as the molecular size decreases. In fact, the dimensions of the channels available for exchange are sufficiently large relative to the size of the water molecule, that in all probability, most water transport across the endothelial barrier is a hydrodynamic rather than a diffusional phenomenon. Since blood is propelled through the microcirculation by intraluminal pressure gradients, at an average pressure higher than the local tissue pressives, the fluid component of blood could ultimately leak into the tissue unless the concomitant transcapillary pressure gradients were opposed. The effect

is provided by the colloid osmotic pressure that arises from the difference in concentration in plasma proteins between the blood and tissue compartments, where the capillary wall and the materials in its immediate vicinity constitute a functional barrier to the passage of proteins.

2.2.1 Starling's Hypothesis.

Published at the turn of the century, Starling's views (Starling (1896)) on the forces that govern the movement of water across endothelial membranes have enjoyed a unique and preeminent place in the thinking of physiologists. The Starling hypothesis of fluid exchange is based on the concept that the balance of fluid between blood and tissue compartments is a consequence of the exchange that occurs at the level of blood capillaries, caused by interaction of hydroaulic and colloid osmotic pressures, and mediated by the permeability characteristics of the interface between blood and tissue. In the commonly accepted explanation of this process (Landis and Pappenheimer (1963)), blood enters the exchange vessels of the microcirculation at a comparatively high pressure, and owing to viscous losses, exits from the system at a lower pressure. The blood colloid osmotic pressure is assumed to be straddled by these input and output hydraulic pressures, in such a fashion that outward filtration in the arterial (or high - pressure) portion of the network is balanced by absorption in the venous (or

low - pressure) region. Historically, the analysis of fluid exchange dealt primarily with the interplay of intravascular pressure and plasma colloid osmotic pressure neglecting extravascular factors. When Guyton (1963) presented data in support of a negative interstitial tissure pressure, attention was focused to the osmotic and hydrostatic pressures in the interstitial ground substance and current theories attempt to includes these extravascular forces.

The general expression for Starling's hypothesis is

$$M = KA[(P_c - P_t) - \sigma(\pi_p - \pi_t)] \qquad (2)$$

where M is the rate of fluid movement across the capillary, P_c is the capillary hydrostatic pressure, P_t the tissue hydrostatic pressure, π_p the plasma colloid osmotic pressure, π_t the tissue colloid osmotic pressure, σ the reflection coefficient of the capillary membrane, K the capillary filtration coefficient and A the capillary surface area. In this instance, π_p and π_t represent the colloid osmotic pressures which would be developed with a perfectly semi-permeable membrance. The quantity $\sigma(\pi_p - \pi_t)$ is the effective osmotic pressure difference across the capillary membrane.

The classical formulation of this hypothesis at the capillary level assumes a cylindrical capillary of fixed radius, uniform hydraulic conductivity, a reflection coefficient of unity and a linear drop in hydrostatic pressure. Under normal conditions, the sum of the outward

forces exceeds that of the inward forces at the arterial end of the capillary. The opposite is true at the venous end of the capillary. On the average, the outward and the inward forces very neatly balance and the lymphatic system drains any excess of filtrate out of the tissue. The paracapillary flow that is filtered out of the arterial side and reabsorbed in the venous side is believed to be the order of 1% of blood flow. The lymph flow is estimated to about one - tenth of the paracapillary flow.

The Starling equation represents relationships among factors that are not constant. All of the factors in the equation may vary according to local conditions.

2.2.2 Capillary Filtration Coefficient.

Landis (1927) measured the rate of fluid exchange in single capillaries as a function of hydraulic pressure and from these data and capillary surface area, computed the capillary filtration coefficient. This technique was subsequently employed by Intagleitta and Zweifach (1974) to obtain coefficients in other vascular beds. The differences in filtration coefficient or permeability among organs presumably reflect variations in wall structure of the exchange vessel. In addition, there is evidence that within a given capillary, the filtration coefficient is not constant but may be substantially greater at the venous end (Wiederhielm (1967), Eriksson and Myrhage (1972)). This variation in filtration coefficient will tend to localize

the exchange process near the venous end of the capillary.

2.2.3 Capillary Hydrostatic Pressure.

The hydrostatic pressure in the capillary bed is a function of several hemodynamic factors. Changes in arterial resistance, venous pressure and venous resistance all have a direct influence on capillary pressure. Since these factors are readily varied under physiological conditions, capillary hydrostatic pressure is perhaps the most changeable of the Starling forces. Within a capillary bed, spatial and temporal variability is also evident and periodic fluctuations in flow are frequently observed. These changes are due to either sphincter or arteriolar vasometer activity. Associated with the flow changes are parallel variations in hydrostatic pressure in the capillary. During high flow, the capillary may be filtering fluid along the entire length but when flow falls drastically or stops, the capillary pressure decreases to near the venular value and the balance of forces may favour reabsorption.

2.2.4 Tissue Hydrostatic Pressure:

Hydrostatic pressure in the tissue has been measured by a variety of techniques and the findings have not been uniform. Many data indicate that this pressure is at or very near zero (atmosphere) under normal conditions. A more complete analysis of tissue hydrostatic pressure can be carried out by including the effects of the gradient of permeability along the capillary and the lymphatic drainage. The tissue pressure is assumed not equal to the

lymphatic pressure. This assumption is necessary in view of the probable role of lymphatic pumping in the extraction of excess tissue fluid. It appears that negative tissue pressures can arise from two factors :

(1) high systemic colloid osmotic pressures, or

(2) mechanical pumping in the lymphatics.

Since plasma colloid osmotic pressure is usually similar to the average capillary pressure, it becomes apparent that the determining factor is the terminal lymphatic pressure.

2.2.5 Plasma Colloid Osmotic Pressure:

The magnitude of the colloid osmotic pressure of the plasma is a function of the protein concentration and the reflection coefficient of the membrane for the protein. Under normal circumstances, plasma protein concentration can be considered as a constant but some increase would be expected along the capillary length if substantial filtration occurred. While plasma protein concentration is identical in all vascular beds, effective colloid osmotic pressure may differ, since the reflection coefficient may not be the same in all beds. The reflection coefficient is probably unity for skeletal and cardiac muscle capillaries, but it may be quite low in the liver sinusoids. In adipose tissue there is evidence that the reflection coefficient decreases during sympathetic nerve simulation. This may reflect a change either in size of pores or in the number of large pores.

2.2.6 Tissue Colloid Osmotic Pressure.

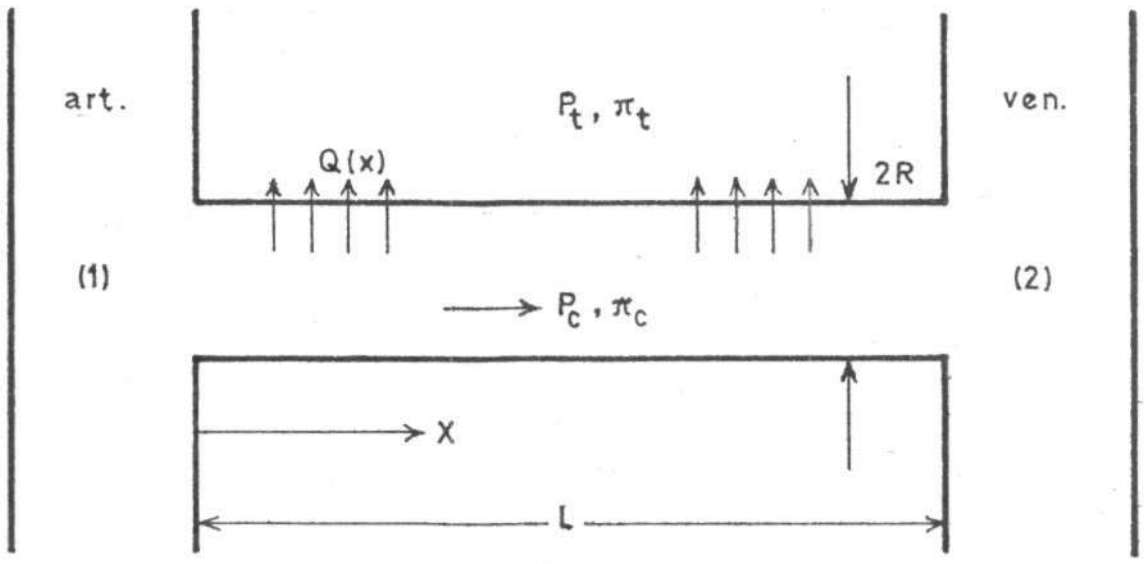
The colloid osmotic pressure of the interstitial fluid has not been measured directly but is usually considered to be near if not identical to lymph colloid osmotic pressure. Lymph protein levels and colloid osmotic pressures vary considerably among organs. In skin and skeletal muscle, lymph protein is about 20% that of plasma, while in intestine it is about 50% and in liver 80% to 70% that of plasma. These values may change drastically when capillary filtration increases. For example, elevation of venous pressure in the intestine causes an outward filtration and consequent dilution of extravascular fluid and drop in lymph protein concentration. As a result, tissue colloid osmotic pressure falls, and the forces favouring filtration are reduced. Eventually, the tissue proteins are diluted sufficiently that the Starling forces reach a new equilibrium and filtration ceases. Thus, it appears that adjustments in tissue colloid osmotic pressure can be an important factor in regulating tissue fluid volume. It is worth noting, that lymph protein concentrations may not reflect the full colloid osmotic pressure of the tissue fluids.

2.3 MATHEMATICAL MODELLING OF MICROCIRCULATORY FLUID EXCHANGE.

Transluminal filtration is the movement of fluid across the endothelial wall of a capillary from the blood to the extravascular space. The movement of fluid through-

out an organism has always been recognized as a critical element in the maintenance of basic transport processes necessary for life. This fluid movement which is based on the physical principle of convective flow brings water, dissolved low-molecular weight materials, and other substances from the intravascular space through the endothelial wall of the capillaries, through the extravascular tissue to the cells. The major driving forces for the fluid flux were first identified by Starling (1896), and his ideas have dominated the theoretical and experimental work in this area since that time. Two scales of this convective flow may be considered. At the microscopic level, the problem is to describe the motion of the fluid from the lumen of the capillary, through the capillary wall into the tissue. At the macroscopic level, one is concerned with the movement of fluid between different organic segments or tissue elements in order to establish an overall fluid movement within an organ. This review will focus on the microscopic phenomena involved in fluid transport in capillaries.

In order to formulate a quantitative description of fluid exchange in the microcirculation, we will focus our attention to a single capillary as a representation for the entire capillary network of the tissue. We assume that the capillary has a length L and a constant radius R; see FIG.2. According to Starling's law (Landis and Pappenheimer (1963)), the filtration or absorption of fluid across the capillary walls depends upon the net effect of four interdependent



FIG, 2. GEOMETRY OF CAPILLARY.

forces: (1) hydrostatic capillary blood pressure P_c, (2) hydrostatic interstitial fluid pressure P_i, (3) osmotic pressure of the plasma proteins Π_c, and (4) osmotic pressure π_i of the proteins in the interstitial fluid immediately outside the capillary walls. This law is then formulated as follow:

$$Q = k(P_c - \pi_c - P_i + \pi_i)$$

or it can be written as ;

$$Q = k(x)\{[p_c(x) - p_i] - [\pi_c(x) - \pi_i]\} \qquad (3)$$

The quantity Q represents the rate of flow per unit area of wall surface, with a plus sign to indicate filtration and a minus sign to indicate absorption. The proportionality k is a measure of the permeability of the capillary wall to fluid, it is a function of x and is called the filtration coefficient.

2.3.1 Formulation of the Problem of Transcapillary Fluid Exchange:

The purpose of investigating the details of transcapillary fluid movement is to determine the amount of fluid that is filtered or absorbed by the capillary under various physiological conditions. This means that, in addition to the local value given in Eq. (1) , knowledge of the integral value, i.e., the filtration rate Q_f, is of great physiological significance :

$$Q_f = A \int_0^1 k(x^*)\{(p_c(x^*) - p_i) - (\pi_c(x^*) - \pi_i)\}\, dx^* \qquad (4)$$

where $x^* = x/L$ and $A = 2\pi RL$ is the capillary surface area. The filtration rate is equal to the difference between the

blood volume flows at the arterial and venous ends of the capillary. We will now, present a brief outline of the models for transcapillary fluid exchange which are presently in use.

2.3.2 Models for Transcapillary Fluid Exchange.

When the filtration coefficient k is constant, Eq. (4) reduces to

$$Q_f = k.A. \int_0^1 \{ (p_c(x^*) - p_i) - (\pi_c(x^*) - \pi_i) \} dx^* \quad (5)$$

We will consider various models based on different assumptions.

Model 1.

The assumptions $p_c(x^*) = p = \text{constant}$

$\pi_c(x^*) = \pi = \text{constant}$

lead to

$$Q_f = k.A \{ (p - p_i) - (\pi - \pi_i) \} \quad (6)$$

These assumptions are very restrictive and do not comply with the conservation laws for momentum and mass. The major advantage of the model is its simplicity while its disadvantage is limited accuracy and inability to reflect physiological behaviour. This model has been used extensively by Wiederhelm (1968, 1979) who modified it slightly by splitting the capillary into an arterial and a venous part.

Model 2:

A more realistic approach is to permit the intraluminal hydrostatic pressure to vary linearly along the capillary axis. The assumptions

$$p_c(x^*) = p_i(1 - \omega x^*) \quad (\omega = \text{constant})$$

$$\pi_c(x^*) = \pi = \text{constant}$$

lead to

$$Q_f = k.A \{(p_1(1 - \frac{\omega}{2}) - p_i) - (\pi - \pi_i)\} \qquad (7)$$

This equation reflects the classical concept of trans-capillary fluid exchange as expressed in numerous physiological textbooks. In this case, the capillary filters fluid into the extracapillary space at the arterial end and reabsorbs it at the venous end.

Model 3:

In this model, the intraluminal hydrostatic pressure is determined from the momentum equation. The plasma osmotic pressure is assumed constant $\pi_c(x^*) = \pi = \text{constant}$. This leads to an analytical solution:

$$Q_f = K.A \quad \frac{\cosh 4\sqrt{\epsilon} - 1}{2\sqrt{\epsilon}, \sinh 4\sqrt{\epsilon}} \left(\frac{p_1 + p_2}{2} - p_i + \pi_i\right) - \pi \Big\} \qquad (8)$$

ϵ is a dimensionless parameter which is defined as

$$\epsilon = \frac{k\mu}{R} \left(\frac{L}{R}\right)^2 \qquad (9)$$

In the following, this parameter will be denoted as fluid-filtration parameter. In Eq. (9), μ is the blood viscosity. In the double limit $\epsilon \to 0$ and $p_E \to p_A$, Eq. (8) becomes identical to Eq.(6) of Model 1. In the limiting case $\epsilon \to 0$ and $p_2 = p_1 (1 - \omega)$, Eq. (8) becomes identical to Eq. (7) of Model 2. Model 3 has been used by Apelblat et al. (1974) and An and Salathe (1976).

Model 4:

The assumption $p_c(x^*) = p_1(1 - \omega x^*)$ (ω = constant).

and $\pi_c(x^*)$ resulting from a differential equation that reflects the conservation of the mass of proteins in the capillary lead to

$$Q_f = k.A \left\{ p_1 \left(1 - \frac{\omega}{2}\right) - p_i + \pi_i - \int_0^1 \pi_c(x^*)\, dx^* \right\}$$

$$= (1 - H_F) \cdot Q_1 \cdot (1 - C_1/C_2) \qquad (10)$$

In this equation, c_1 and c_2 are the plasma protein concentrations in the arterial and venous ends of the capillary respectively. This model has been used by Deen et al (1972) to study glomerular filtration.

Model 5:

This model does not employ any assumptions regarding $p_c(x^*)$ and $\pi_c(x^*)$. The hydrostatic pressure $p_c(x^*)$ is determined by the momentum equation, the osmotic pressure $\pi_c(x^*)$ by the conservation of proteins in the capillary. By using a mass balance, Starling's law can be written as

$$Q(x^*) = 2\pi R.k \left\{ (p_c(x^*) - p_i) - (\pi_c(x^*) - \pi_j) \right\} \qquad (11)$$

Let us denote

$$P_e(x^*) = p_c(x^*) - p_i + \pi_i \qquad (12)$$

that appears in Eq. (11) as 'effective pressure'. The local osmotic pressure of the blood can be calculated from the axially varying plasma protein concentration C according to the emperical equation

$$\pi_c = \alpha.c + \beta.c^2 + \delta.c^3 \qquad (13)$$

The coefficients α, β and δ have been reported by Landis and Pappenheimer (1963) to be

$$\alpha = 2.1 \frac{\text{mm Hg}}{\text{g/100 m}\ell}\ ; \quad \beta = 0.16 \frac{\text{mm Hg}}{(\text{g/100 m}\ell)^2}\ ; \quad \delta = 0.009 \frac{\text{mm hg}}{(\text{g/100 m}\ell)^3}$$

The blood volume flow $Q(x^*)$ in Eq. (11) is related to the axial pressure gradient through the generalized Poiseuille equation

$$\frac{dp_e}{dx^*} = - \frac{8\mu}{\pi R^4} Q(x^*) \tag{14}$$

where μ is the blood viscosity in the capillary. Conservation of plasma proteins in the blood lumen leads to the following equation for the plasma protein concentration

$$C(x^*) = C_1 . \frac{(1-H_F)Q_1}{Q(x^*)-H_F Q_1} \tag{15}$$

Equations (11) to (15) to (13) form a set of coupled nonlinear differential equations which are subject to the boundary conditions

$$\left.\begin{aligned} x^* = 0 : \quad p_e &= p_1 - p_i + \pi_i \\ Q &= Q_1 \\ C &= C_1 \end{aligned}\right\} \tag{16}$$

Rather than using the volume flow at the capillary entrance, one may prescribe the effective pressure p_{e_2} at the venous end of the capillary, i.e., $x^* = 1$, as a boundary condition. From the numerical solution we obtain the pressure, volume flow and plasma protein concentration as a function of the axial position x^*. The total filtration rate Q_F follows directly from Eq.(15)

$$Q_f = Q_1 - Q_2 = (1 - H_F).Q_1.(1 - C_{1/C_2}) \tag{17}$$

where the subscript 2 stands for the venous end of the capillary. This model has been used by Huss et al. (1975), and in a more sophisticated and extended version by Papenfuss and Gross (1978) and Papenfuss et al. (1979).

Model 6:

Oka and Murata (1970) studied the flow of blood in a capillary with permeable wall.

Basic assumptions:

(i) The capillary tube between an arteriole and a venule is a straight rigid tube of a uniform circular cross section as in FIG.3 where R is the radius and L is the length of the tube.

(ii) The fluid is transported across the tube wall in accordance with Starling's hypothesis.

(iii) The rate of flow per unit area of the wall surface is very small.

(iv) The hydrostatic interstitial fluid pressure, the osmotic pressure of the plasma and the osmotic pressure of the interstitial fluids are constant. The hydrostatic capillary pressure is a function of z only and the filtration coefficient is constant.

(v) The blood is a homogenous incompressible Newtonian fluid.

(vi) The motion of the fluid is laminar and steady.

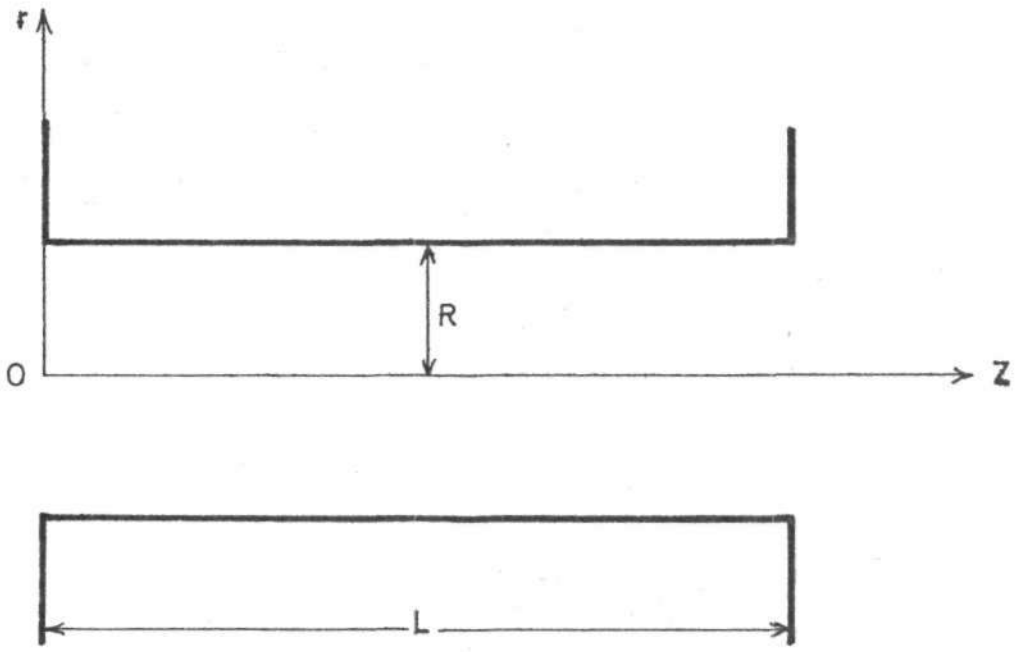
(vii) The motion has axial symmetry.

(viii) The motion is so slow that the inertia term can be neglected.

(ix) No body force acts on the fluid.

(x) There is no slip at the wall.

(xi) The end effect is negligible.



FIG,3, CAPILLARY TUBE

(xii) The non-dimensional factor $\varepsilon = \frac{kn}{R}$ is very small compared with unity.

Starling's law:

$$M = k(p_c - \pi_c - p_i + \pi_i) \qquad (18)$$

The quantity M represents the rate of flow per unit area of wall surface, and k is the filtration coefficient.

Equations of Motion:

From the assumptions (i) ~ (ix), the Navier-Stokes and continuity equations were reduced to

$$\frac{\partial p}{\partial r} = \eta\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2} + \frac{\partial^2 v}{\partial z^2} \right) , \qquad (19)$$

$$\frac{\partial p}{\partial z} = \eta \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} \right) , \qquad (20)$$

while the continuity equation was reduced to

$$\frac{1}{r}\frac{\partial}{\partial r}(rv) + \frac{\partial u}{\partial z} = 0 \qquad (21)$$

where u and v are the z - and r - component of the velocity, respectively, and η is the coefficient of viscosity of blood.

Boundary Conditions:

$$\frac{\partial u}{\partial r} = 0 \quad \text{at} \quad r = 0 , \qquad (22)$$

$$v = 0 \quad \text{at} \quad r = 0 , \qquad (23)$$

$$u = 0 \quad \text{at} \quad r = R , \qquad (24)$$

$$v = k(p - a) \quad \text{at} \quad r = R , \qquad (25)$$

$$\bar{p} = p_a \quad \text{at} \quad z = 0 , \qquad (26)$$

$$\bar{p} = p_v \quad \text{at} \quad z = L , \qquad (27)$$

where p_a and p_v are the arterial and venous pressures respectively and $\bar{p}$ is the average pressure over the cross

section of the tube. α is regarded as a constant being equal to $\pi_c + p_i - \pi_i$.

Results Obtained:

$$U = \frac{R^2 \Delta p}{4\eta L}(1 - \zeta^2)\{1 + \varepsilon f(\zeta, \xi)\} \quad , \tag{28}$$

$$V = \frac{\varepsilon R \Delta p}{\eta}\left(2\zeta - \zeta^3\right)\left(\frac{\Delta\alpha}{\Delta p} - \xi\right) \quad , \tag{29}$$

$$p = p_a - \xi \Delta p + \varepsilon g(\zeta, \xi)\Delta p \tag{30}$$

with

$$f(\zeta, \xi) = \frac{8}{\beta^2}\left(\xi^2 - 2\frac{\Delta\alpha}{\Delta p}\xi + \frac{\Delta\alpha}{\Delta p} - \frac{1}{3} + \frac{1}{2}\beta^2 - \frac{1}{4}\beta^2\zeta^2\right) \tag{31}$$

$$g(\zeta, \xi) = -\frac{8}{3\beta^2}\left(\xi^3 - 3\frac{\Delta\alpha}{\Delta p}\xi^2 - \left\{1 - 3\frac{\Delta\alpha}{\Delta p} - \frac{3}{4}\beta^2\left(1 - \zeta^2\right)\right\}\xi - \frac{3}{4}\beta^2\frac{\Delta\alpha}{\Delta p}\left(1 - \zeta^2\right)\right) \tag{32}$$

where $\Delta p = p_a - p_v$ (33)

$$\Delta\alpha = p_a - \alpha \tag{34}$$

$$\beta = \frac{R}{L} \tag{35}$$

$$\zeta = \frac{r}{R} \tag{36}$$

$$\xi = \frac{z}{L} \tag{37}$$

Model 7:

Dinnar (1972) extended Oka and Murata (1970) by considering the capillary as well as the surrounding tissue space.

Basic assumptions:

(i) The capillary as well as the surrounding tissue space was considered as region I (capillary) and region II (tissue space), see FIG. 4.

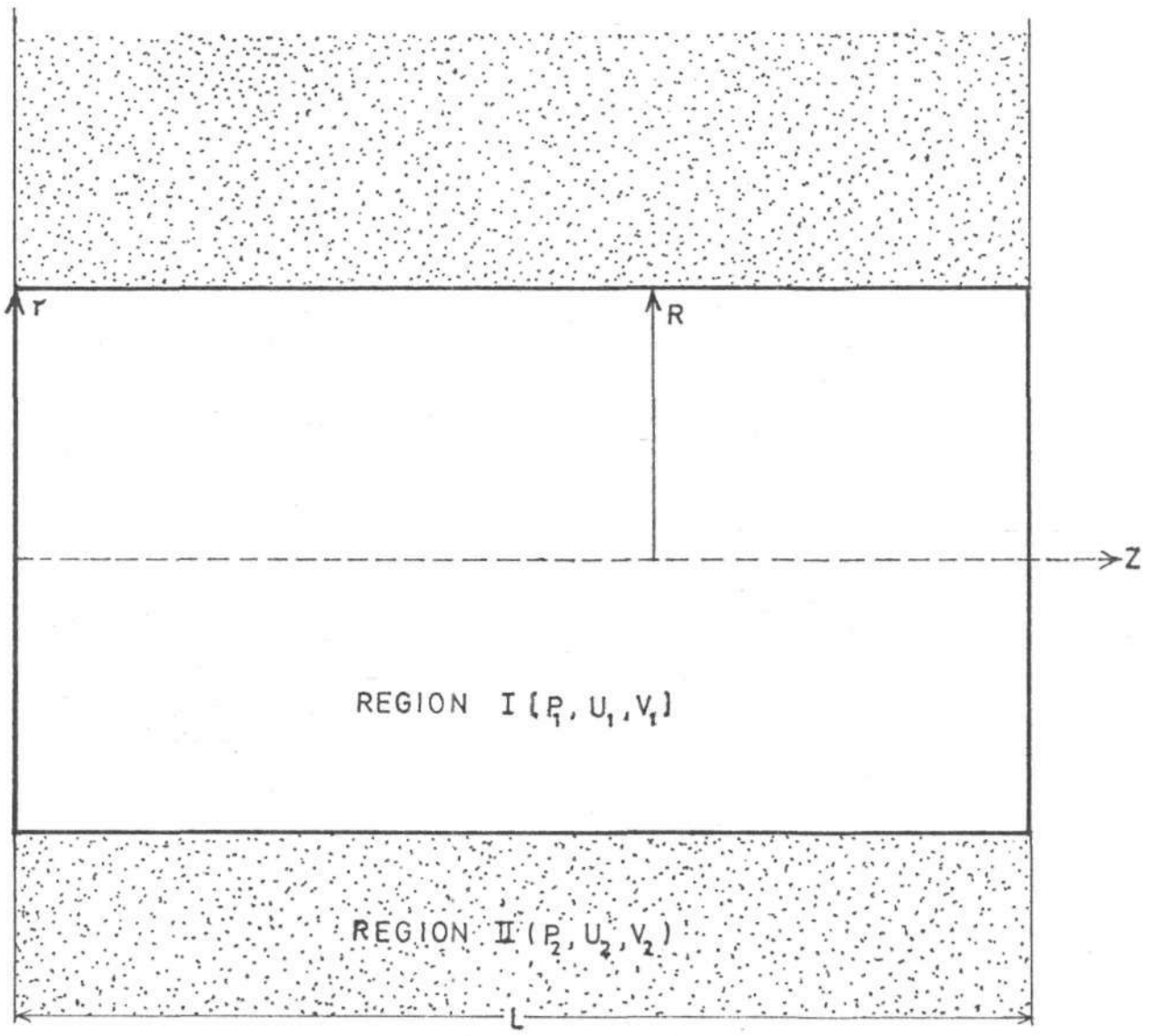


FIG. 4. GEOMETRY OF THE PROBLEM

(ii) The rate of flow through the capillary wall is much smaller than the rate of flow through the tube.

(iii) The osmotic pressures are constant.

(iv) The hydrostatic capillary pressure is a function of z only. The hydrostatic interstitial fluid pressure is also a function of z and is not influenced by the filtration coefficient which is constant.

(v) The pressure in the venule is determined by the flow properties.

(vi) All other assumptions are the same as in Oka and Murata (1970) except for (xii).

Starling's law:

$$V = k(P_1 - P_2 - \pi_1 + \pi_2) \qquad (38)$$

where 1 denotes inside of the capillary, 2 the outside of the capillary, and k is the filtration coefficient. The pressure outside the capillary is determined by the interstitial fluid volume which itself depneds upon the capillary pressure, the osmotic pressure, the permeability of the capillary and the lymph flow. Hence in general $P = P_1 - P_2$ is determined by P_1, π_1, π_2 and k.

Equations of Motion:

The governing equations used for the cylindrical coordinate system (r, Φ, z) were

Momentum equations:

$$\frac{\partial P_i}{\partial r} = \eta\left(\frac{\partial^2 v_i}{\partial r^2} + \frac{1}{r}\frac{\partial v_i}{\partial r} - \frac{v_i}{r^2} + \frac{\partial^2 v_i}{\partial z^2}\right) , \qquad (39)$$

$$\frac{\partial P_i}{\partial z} = \eta\left(\frac{\partial^2 u_i}{\partial r^2} + \frac{1}{r}\frac{\partial u_i}{\partial r} + \frac{\partial^2 u_i}{\partial z^2}\right) , \qquad (40)$$

Continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_i) + \frac{\partial u_i}{\partial z} = 0 \quad , \tag{41}$$

where i = 1 is for region I and i = 2 is for region II.

Boundary Conditions:

$$\frac{\partial u_1}{\partial r} = 0 \ , \quad v_1 = 0 \quad \text{at} \quad r = 0 \tag{42}$$

$$u_1 = 0, \quad u_2 = 0, \ v_1 = v_2 = k\{p_1 - p_2 - \alpha\} \ \text{at} \ r = R \tag{43}$$

where

$$\alpha = \pi_1 - \pi_2 \tag{44}$$

$$\bar{p}_1 = p_a, \ u_2 = 0 \quad \text{at} \quad z = 0 \tag{45}$$

$$\bar{p}_1 = p_v, \ u_2 = 0 \quad \text{at} \quad z = L \tag{46}$$

and $\bar{p}_1$ is the average pressure over the cross-section. p_a and p_v are the arterial and venous pressures, respectively.

Results Obtained:

$$U_1 = -\frac{1}{4\eta}\left(\frac{16k\eta}{R^3}\right)^{1/2}(Ee^{\mu z} - Fe^{-\mu z})\left\{(R^2 - r^2) - \frac{1}{8}\left(\frac{16k\eta}{R^3}\right)(R^4 - r^4)\right\} \tag{47}$$

$$V_1 = \frac{k}{R^3}[Ee^{\mu z} + Fe^{-\mu z}]\left\{(2rR^2 - r^3) - \frac{1}{12}\left(\frac{16\eta}{R^3}\right)(3rR^4 - \frac{10}{6}r^5)\right\} \tag{48}$$

$$P_1 = P_{2O} + \alpha + [Ee^{\mu z} + Fe^{-\mu z}]\left\{1 - \ - \frac{1}{4}\left(\frac{16k\eta}{R^3}\right)\left(r^2 + \tfrac{1}{2}R^2\right)\right\} \tag{49}$$

$$U_2 = -k\frac{Y_O(\mu R)J_O(\mu r) - Y_O(\mu r)J_O(\mu R)}{Y_O(\mu R)J_1(\mu R) - Y_1(\mu R)J_O(\mu R)}[Ee^{\mu z} + Fe^{-\mu z}] \tag{50}$$

$$V_2 = k\frac{Y_O(\mu R)J_1(\mu r) - Y_1(\mu r)J_O(\mu R)}{Y_O(\mu R)J_1(\mu R) - Y_1(\mu R)J_O(\mu R)}[Ee^{\mu z} + Fe^{-\mu z}] \tag{51}$$

$$P_2 = P_{2\,0} \tag{52}$$

where $E = E'(P_a - P_v)$ (53)

$$F = F'(P_a - P_v) \tag{54}$$

and $$E' = \frac{P_a}{P_a - P_v}\left(e^{+\delta} - \frac{(P_v/P_a)) - h(e^{\delta} - 1)}{e^{\delta} - e^{-\delta}}\right) \tag{55}$$

$$F' = \frac{P_a}{P_a - P_v}\left(e^{\delta} - (P_v/P_a)) - \frac{h(e^{-\delta} - 1}{e^{\delta} - e^{-\delta}}\right) \tag{56}$$

$$h = \frac{P_{20} + \alpha}{P_a} \tag{57}$$

$$\delta = \sqrt{\frac{16k\eta L^2}{R^3}} \tag{58}$$

Model 8:

Apelblat et al (1974) discussed the mathematical analysis of capillary tissue fluid exchange. The paper was an extension of Dinnar (1972). Two different models were discussed; (1) The hydrostatic pressures were not considered to be constant. This model was used for a 2 regions and 3 regions i.e., capillary, tissue region and a membrane between the capillary and tissue region;
(2) The hydrostatic capillary pressure was considered while the hydrostatic interstitial pressure was assumed constant. This model was used for a uniform and non-uniform tube. In both models, the osmotic pressures were assumed constant. We are only concerned with the case of 2 regions, thus we are considering the first model.

Basic assumptions.

(i) The basic functional unit; the KROGH model considered (see FIG. 5) includes a uniform cylindrical capillary of radius R with rigid walls. The cylindrical space surroun-

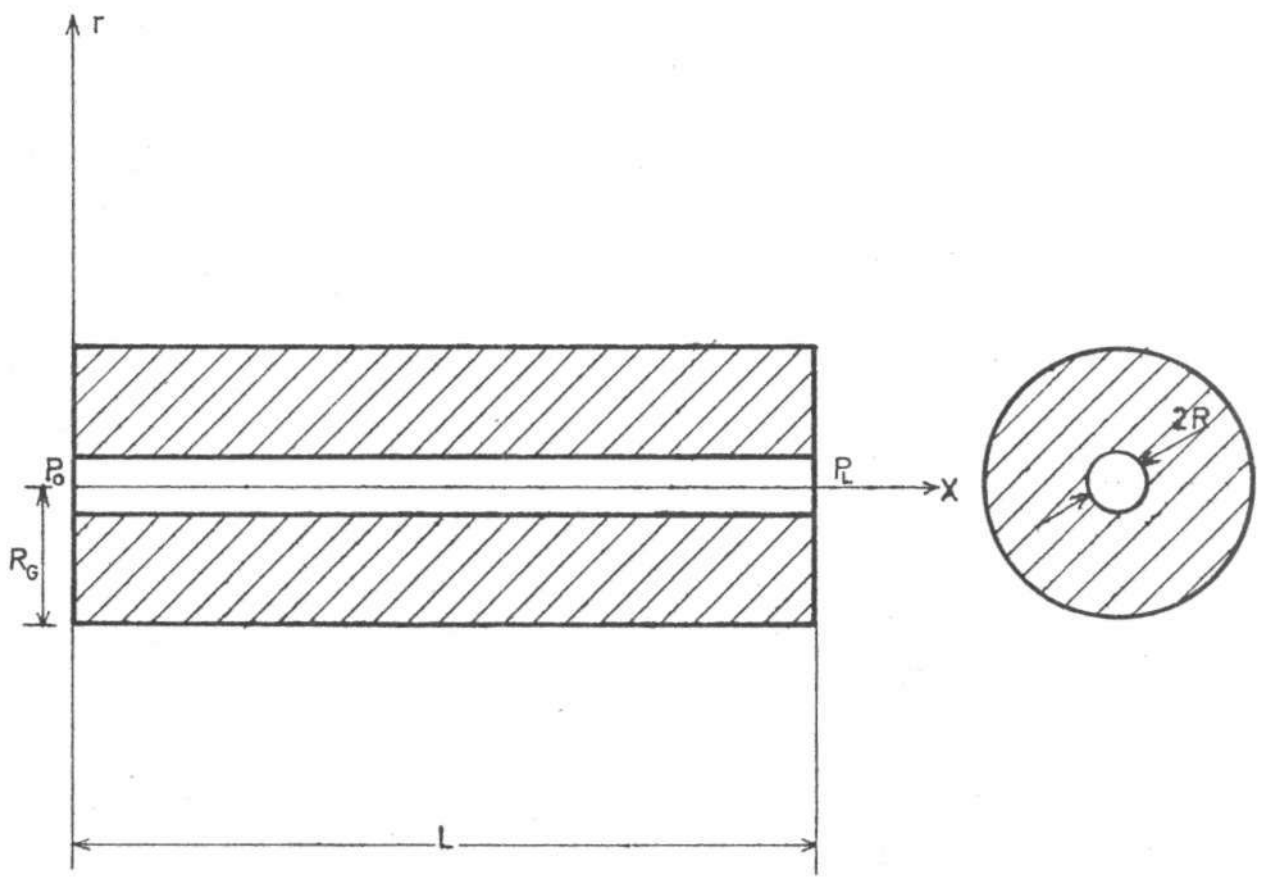


FIG. 5. BASIC FUNCTIONAL UNIT

ding the capillary of length L is composed of a tissue space up to a radius R_G. P_o and P_L are the inlet and outlet pressures respectively.

(ii) The osmotic pressures are constant. The hydrostatic capillary fluid is a function of x while the hydrostatic interstitial fluid is a function of x and r.

(iii) In the tissue region, the Darcy law is applied. The Darcy law states that there is proportionaity between the flow and the applied pressure gradient.

i.e., $U = -\frac{K}{\eta} \nabla P$ where K is the so- called Darcy constant.

(iv) All other assumptions are the same as in Models 6 and 7.

Equations of Motion:

The governing equations used for the fluid flow in the capillary were the Navier - Stokes equation and the equation of continuity.

$$\frac{\partial P}{\partial x} = \eta \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial 1} \right) \qquad 0 \leq x \leq L \qquad (59)$$

$$\frac{\partial P}{\partial r} = 0 \text{ or } P = P(x) \qquad (6\tfrac{1}{4})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0 \qquad (61)$$

In the above, u and v, are the axial and radial components of velocity respectively and η is the viscosity.

The equations of motion and the continuity equation for the tissue region were

$$\bar{U}(x,r) = -\frac{K}{\eta} \left(\frac{\partial \bar{p}}{\partial x}\right) \qquad (62)$$

$$\bar{V}(x,r) = -\frac{K}{\eta}\left(\frac{\partial \bar{P}}{\partial r}\right) \tag{63}$$

$$\frac{\partial^2 \bar{P}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{P}}{\partial r} + \frac{\partial^2 \bar{P}}{\partial x^2} = 0 \tag{64}$$

where the quantities in the tissue region are denoted by subscript, bar.

Boundary Conditions;

$$P(0,r) = P_o \tag{65}$$

$$P(L,r) = P_L \tag{66}$$

$$U(x,R) = 0 \tag{67}$$

$$P(x,R) = \bar{P}(x,R) \tag{68}$$

$$V(x,R) = \bar{V}(x,R) \tag{69}$$

$$\bar{V}(x,R_a) = 0 \tag{70}$$

$$\bar{U}(0,r) = 0 \tag{71}$$

$$\bar{U}(L,r) = 0 \tag{72}$$

Results Obtained:

$$U(x,r) = \frac{1}{4\eta}\frac{dP}{dx}(r^2-R^2) \tag{73}$$

$$V(x,r) = \frac{1}{16\eta}\frac{d^2P}{dx^2}(2R^2r - r^3) \tag{74}$$

$$P(x) = D + Gx - \frac{16K}{R^3}\sum_{n=1}^{\infty}\frac{C_n Z_n(R)}{\alpha_n}\cos \alpha_n x \tag{75}$$

$$\bar{U}(x,r) = \frac{K}{\eta}\sum_{n=1}^{\infty}\alpha_n C_n Y_n(r)\cos \alpha_n x \tag{76}$$

$$\bar{V}(x,r) = \frac{K}{\eta}\sum_{n=1}^{\infty}\alpha_n C_n Z_n(r)\cos \alpha_n x \tag{77}$$

$$\bar{P}(x.r) = A + \sum_{n=1}^{\infty} C_n Y_n(r)\cos \alpha_n x \tag{78}$$

where $$D = P_o + \frac{16KG}{R^3}\sum_{n=1}^{\infty}\frac{C_n Z_n(R)}{\alpha_n} \tag{79}$$

$$G = \frac{P_L - P_o}{L + \epsilon} \tag{80}$$

$$A = D + GL/2$$

$$C_n = \frac{2G[(-1)^n - 1]}{\alpha_n L\left[\alpha_n Y_n(R) + \frac{16K}{R^3} Z_n(R)\right]} = G\delta_n \quad (82)$$

$$\varepsilon = \frac{16K}{R^3} \sum_{n=1}^{\infty} \left[1 - (-1)^n\right] \frac{\delta_n Z_n(R)}{\alpha_n} \quad (83)$$

$$Z_n = \frac{dY_n(r)}{dr} \quad (84)$$

$$Y_n(r) = \left[K_1(\alpha_n R_G) I_0(\alpha_n r) + I_1(\alpha_n R_G) K_0(\alpha_n r)\right] / I_1(\alpha_n R_G) \quad (85)$$

$$\alpha_n = n\pi/L, \ldots\ldots, n = 1, 2, 3. \quad (86)$$

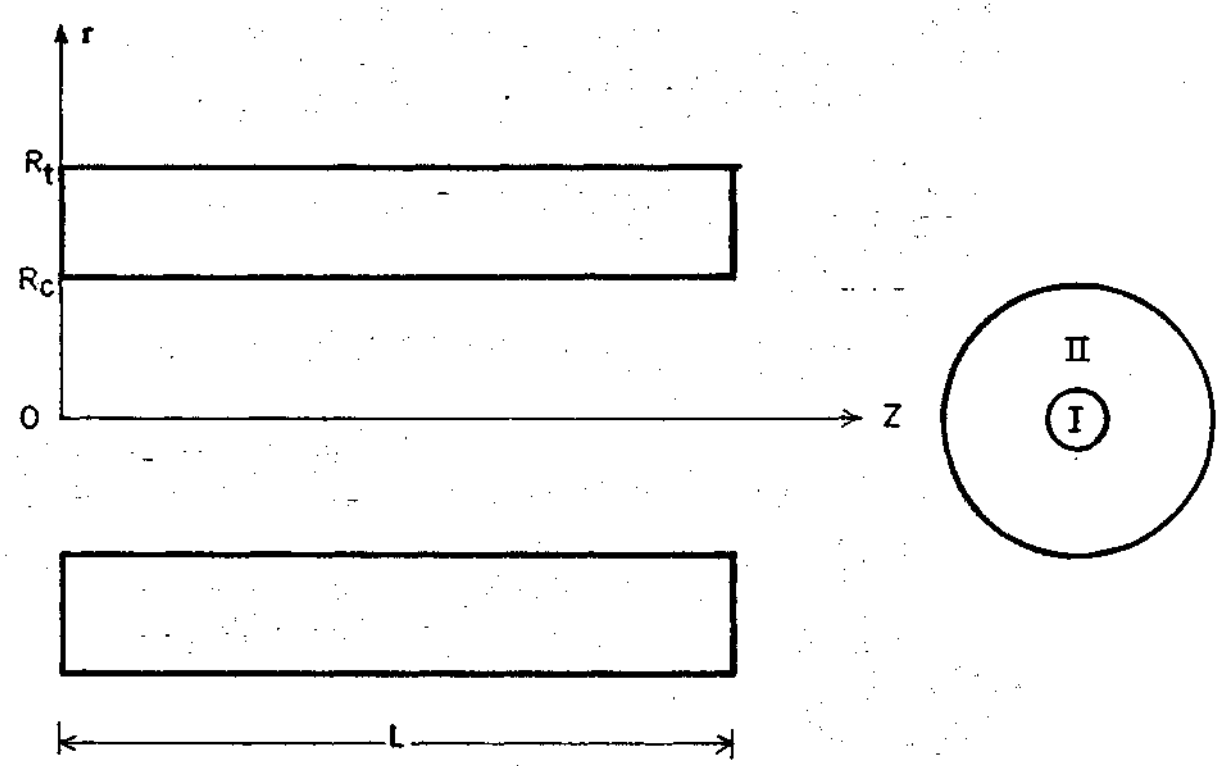
Model 9.

Murata (1978) extended Apelblat et al. (1974) by studying the effects of filtration coefficient and lymph flow on fluid exchange.

Basic assumptions;

(i) The Krogh cylinder arrangement is also used (see FIG.6). The z - axis coincides with the center line of the capillary and the origin of the coordinate system is located at the arterial end. Here, R_c and L are the radius and length of the capillary (I), respectively, and R_t is the outer radius of tissue region (II).

(ii) The tissue spaces are complex composite systems. Many tissue cells are distributed in the tissue spaces, and the interstitial space lying among tissue cells contains many elongated collagen fibers. Thus, the tissue region in the Krogh cylinder is assumed to be



FIG,6, KROGH CYLINDER AND CYLINDRICAL CO-ORDINATE SYSTEM

a homogeneous, isotropic porous medium.

(iii) The interstitial fluid is assumed to be the same homogenous incompressible Newtonian fluid as plasma and that the movements of interstitial fluid obey a generalized Darcy's equation, as derived theoretically by Tam (1969).

(iv) The osmotic pressures are constant. The hydrostatic capillary pressure is a function of z while the hydrostatic interstitial pressure is a function of z and r.

(v) The filtration coefficient is a linear function of the distance from the arterial end. We consider k as a linear function of z as follows:

$$k = k_a + \frac{k_v - k_a}{L} z \, , \quad k_v > k_a$$

where k_a and k_v are the values of the filtration coefficient at the arterial and venous ends of the capillary, respectively.

(vi) The lymphatic capillaries are assumed to be distributed uniformly on the outer surface of the tissue region of the Krogh cylinder. Fluid is also assumed to enter into the lymphatic capillary at a constant velocity from the outer surface of the tissue region of the Krogh cylinder and that the lymph flow is proportional to the filtration rate.

(vii) All other assumptions are the same as those of Model 6 except for (xii).

Starling's law:

$$Q = k(p_c - \Pi_c - p_i + \Pi_i) \tag{88}$$

where the quantity Q represents the rate of flow per unit area of wall surface, and k is the filtration rate.

Equations of Motion:

The equation of motion and the continuity equation used in the capillary (region I) were

$$\nabla P_1 = \eta \nabla^2 V_1 \tag{89}$$

$$\nabla . V_1 = 0 \tag{90}$$

where V_1 is the velocity vector, P_1 is the fluid pressure, and η is the coefficient of viscosity. Eq. (89) is the linearized Navier - Stokes equation. The fundamental equations in the tissue region (region II) were

$$\nabla P_2 = \eta(\nabla^2 V_2 - \sigma^2 V_2) \tag{91}$$

$$\nabla . V_2 = 0 \tag{92}$$

where Eq.(91) is a generalized Darcy's equation derived by Tam (1969). The quantity σ^2 represents the resistance to flow by spherical particles.

Boundary Conditions.

$$\frac{\partial U_1}{\partial r} = 0 \ , \quad V_1 = 0 \ , \qquad \text{at} \quad r = 0 \tag{93}$$

$$U_1 = 0 \ , \qquad , \qquad \text{at} \quad r = R_c \tag{94}$$

$$V_1 = k(P_1 - P_2 + \Pi_o) , \qquad \text{at} \quad r = R_c \tag{95}$$

$$P_1 = P_a \ , \qquad \text{at} \quad z = 0 \tag{96}$$

$$P_1 = P_v \ , \qquad \text{at} \quad z = L \tag{97}$$

$$V_2 = V_o \ , \qquad \text{at} \quad r = R_t \tag{98}$$

$$U_2 = 0 \ , \qquad \text{at} \quad r = R_c \tag{99}$$

$$V_2 = k(P_1 - P_2 + \Pi_e) , \qquad \text{at} \quad r = R_c \qquad (100)$$

$$U_2 = 0 , \qquad \text{at} \quad z = 0 \qquad (101)$$

$$U_2 = 0 , \qquad \text{at} \quad z = L \qquad (102)$$

where Π_e is the difference between the osmotic pressure of the plasma proteins Π_1 and the osmotic pressure of the proteins in the interstitial fluid Π_2, i.e., $\Pi_e = \Pi_2 - \Pi_1$. Π_e is a constant according to assumption (iv). P_a and P_v are the arterial and venous pressures, respectively. Both P_a and P_v are regarded as constant.

Results Obtained:

In region I.

$$U_1 = (1 - \zeta^2) \left\{ 1 + \frac{4}{3} \kappa\beta^2 \frac{d}{d\xi} \frac{\Psi}{\xi} \right\} \qquad (103)$$

$$V_1 = -\frac{1}{3}\kappa 3 (2\zeta - \zeta^3) \frac{d^2\Psi}{d\xi} \qquad (104)$$

$$P_1 = P_a - \xi - \frac{4}{3}\kappa B^2 \Psi(\xi) \qquad (105)$$

where

$$\Psi(\xi) = \frac{3+6\Theta+2\Theta^2+3\alpha(3+2\Theta)}{3(1+\Theta)} \xi - \frac{3+4\Theta+3\alpha}{1+\Theta} \xi^2 + 2\left(\frac{3 - 4\Theta^2 - 3\alpha\Theta}{3(1+\Theta)}\right) \xi^3 + 2\Theta\xi^4 \qquad (106)$$

$$\alpha = \gamma V_0 / 2\kappa\beta \qquad (107)$$

and $$\zeta = \frac{r}{R_c}, \quad \xi\xi = \frac{Z}{L}, \quad \beta = \frac{L}{R_c}, \quad \nabla P = P_a - P_v \qquad (108)$$

$$\gamma\gamma = \frac{R_t}{R_c}, \quad \kappa = \frac{nk_a}{R_c}, \quad \Theta = \frac{1}{2}\frac{k_v - k_a}{k_a}, \quad p_a = \frac{P}{\Delta P} \qquad (109)$$

In region II.

$$U_2 = -\frac{16\kappa}{\beta} \sum_{n=1}^{\infty} A_n(\Theta) f_n(\zeta) \sin \lambda_n \beta \xi \qquad (110)$$

$$V_2 = \frac{\gamma V_0}{\xi} + \frac{16\kappa\kappa}{\beta} \sum_{n=1}^{\infty} A_n(\Theta) g_n(\zeta) \cos \lambda_n \beta \xi \qquad (111)$$

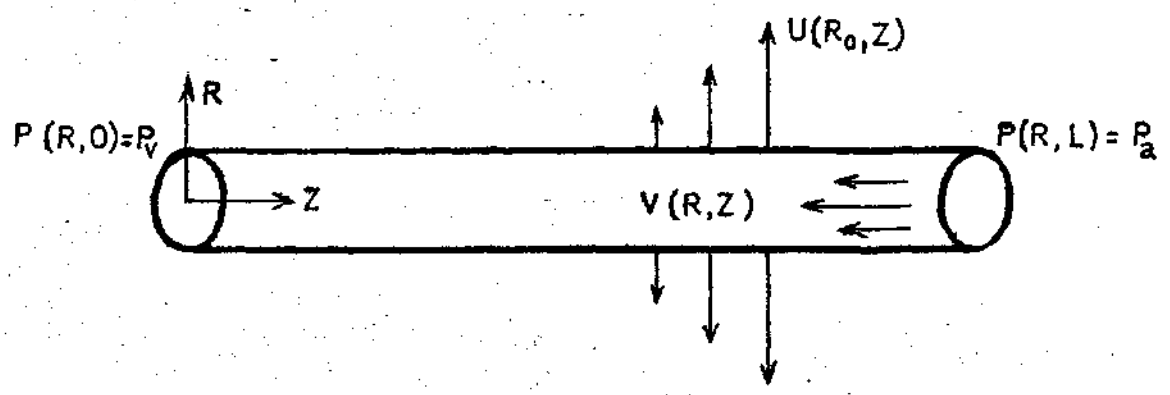
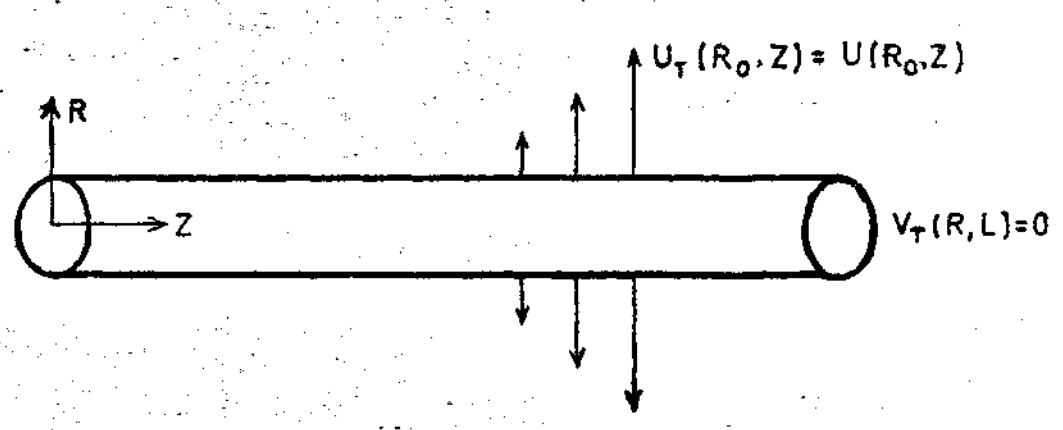


FIG. 7(a). MODEL FOR INTRALUMINAL FLOW



FIG, 7(b), MODEL FOR EXTRALUMINAL FLOW

(iii) Flow is transported across the capillary wall in accordance with Starling's hypothesis.

(iv) The filtration coefficient and tissue collodial osmotic pressure are considered as constants.

(v) The capillary hydrostatic pressure is determined by the velocity field in the capillary, while the tissue hydrostatic pressure is related to the velocity field in the tissue. The capillary collodial osmotic pressure is a function of the solute concentration of the capillary. Specifically, a functional form is assumed for Π, i.e.,

$$\Pi = \Pi_1 C + \Pi_2 C^2 \qquad (123)$$

where Π_1 and Π_2 are constants and C is the solute concentration.

(vi) Flow is steady and the plasma is assumed to be a Newtonian fluid wherein the colloids are a dilute solute.

(vii) Presence of erythrocytes is neglected.

(viii) The wall of the capillary is permeable to the fluid but impermeable to the solute. This wall is infinitesimally thin and separates the fluid in the capillary (intraluminal flow) from the fluid flux in the surrounding interstitial space (extraluminal flow).

(ix) The intraluminal flow is assumed to be that of an in compressible Navier-Stokes fluid, while the extraluminal flow is according to Darcy's law where the hydrodynamic conductivity of the tissue, K_t, is the ratio of the magnitude of local fluid velocity to that of the local pressure

gradient and is assumed constant.

(x) Inertia terms in the momentum equation can be neglected.

(xi) The ratio $\frac{R_o}{L} \equiv \varepsilon^2$ is very small compared with unity.

Starling's law :

$$\dot{M} = K \left\{ P - \Pi - (P_t - \Pi_t) \right\} \qquad (124)$$

where $\dot{M}$ is the local fluid volume flux per unit capillary surface area, P is the capillary hydrostatic pressure, P_t is the tissue hydrostatic pressure, Π is the capillary collodial osmotic pressure, Π_t is the tissue collodial osmotic pressure and K is the filtration coefficient.

Equations of Motion:

The nondimensionalized system of differential equations describing conservation of mass, momentum and solute mass concentration for the intraluminal flow were

$$\Gamma_2 \frac{1}{r} \frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial z} = 0 , \qquad (125)$$

$$- \frac{\partial p}{\partial z} + \Gamma_1 \left\{ \varepsilon^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial v}{\partial r} + O\left[\varepsilon^6\right] \right\} = 0, \qquad (126)$$

$$- \frac{\partial p}{\partial r} + \Gamma_1 \Gamma_2 O\left[\varepsilon^6\right] = 0 , \qquad (127)$$

$$v \frac{\partial c}{\partial z} + \Gamma_2 u \frac{\partial c}{\partial r} = \Gamma_2 \left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial c}{\partial r} + O\left[\varepsilon^4\right] \right\} \qquad (128)$$

where Γ_1 and Γ_2 are similarity parameters which are of order of unity and $\frac{R_o}{L} \equiv \varepsilon^2$, where the higher - order terms have been neglected for simplicity, and where u and v are, respectively the radial and axial components

of velocity, c is the mass concentration

of the solute and p is the hydrostatic pressure.

For the extraluminal flow, the corresponding nondimensional equation was

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p_t}{\partial r}\right) + \varepsilon^2\frac{\partial^2 p_t}{\partial z^2} = 0 \qquad (129)$$

where p_t is the tissue hydrostatic pressure and this differential equation follows from the combination of Darcy's law with the equation of conservation of mass.

Boundary Conditions:

For the intraluminal flow,

$$v(1, z) = 0 \ , \qquad (130)$$

$$\frac{\partial v}{\partial r}(0,z) = 0 \ , \qquad (131)$$

$$U(1,z) = \varepsilon^2\Gamma_0\{\varepsilon\,\Gamma_3+p(1,z)-1-(p_t(1,z)-p_{t\infty}) - [\Pi_1(c(1,z)-1) + \Pi_2(c^2(1,z)-1)]\}, \qquad (132)$$

$$U(0,z) = 0 \ , \qquad (133)$$

$$U(r,1) = 0 \ , \qquad (134)$$

$$\frac{\partial c}{\partial r}(1,z) = c(1, z)\ u\ (1, z) \ , \qquad (135)$$

$$\frac{\partial c}{\partial r}\ (0, z) = 0 \ , \qquad (136)$$

$$c(r, 1) = 1 \ , \qquad (137)$$

$$p(r, 1) = 1 \ , \qquad (138)$$

$$p(r, 0) = p_v = 1-\varepsilon^2 p^{(2)}(0)-\varepsilon^3 p^{(3)}(0)+O[\varepsilon^4] \qquad (139)$$

where $$\frac{P_a - [\Pi_1 C_a + \Pi_2 C_a^2] - (P_{t\infty} - \Pi_{t\infty})}{P_a} \equiv \varepsilon\,\Gamma_3 \qquad (140)$$

and P_a = pressure scale, arteriolar pressure

C_a = Concentration scale, arteriolar concentration

Γ_o is a similarity parameter of order of unity.

For the extraluminal flow,

$$-\frac{\partial p}{\partial r}t\ (1,z) = \varepsilon^2\Gamma_4\{\varepsilon\ \Gamma_3+p(1,z) - 1- \left[p_t(1,z)-p_{t\infty}\right] - \left[\ \Pi_1(c(1,z) - 1) + \Pi_2(c^2(1,z)-1)\right]\} \quad (141)$$

for $0 < z < 1$

$$-\frac{\partial p}{\partial z}t\ (r,\ 1) = 0\ , \quad (142)$$

$$-\frac{\partial p}{\partial z}t\ (r,\ 0) = 0\ , \quad (143)$$

for $0 < r < 1$

$$\lim_{r^2+z^2\to\infty} p_t(r,z) = p_{t\infty} \quad (144)$$

where $$\frac{R_o\ K}{K_t} = \Gamma_4\varepsilon^2 \quad (145)$$

Perturbation Expansion:

The velocity, pressure and solute concentration for the intraluminal flow have been assumed to have the form

$$\Phi(r,z;\varepsilon) = \phi^{(o)}(r,z) + \varepsilon\ \Phi^{(1)}(r,z) + O\left[\varepsilon^2\right] \quad (146)$$

where $\Phi = u,\ v,\ p,\ c$.

The extraluminal pressure field was expanded in an analogous manner :

$$p_t\ (r,\ z;\ \varepsilon) = p_{t\infty} + \varepsilon p_t^{(1)}(r,\ z) + O\left[\varepsilon^2\right] \quad (147)$$

Results Obtained:

The expansions for the intraluminal flow were

$$v(r,\ z;\ \varepsilon) = \frac{p^{(2)}(0)}{4\Gamma_1}\ (r^2 - 1) + O[\ \varepsilon\ ] \quad (148)$$

$$u(r,z;\ \varepsilon) = -\ \varepsilon^3 2\Gamma_o\Gamma_3 r\ \{\frac{r^2}{2} - 1\} + O[\varepsilon^4] \quad (149)$$

$$p(r,z;\ \varepsilon) = 1 + \varepsilon^2 p^{(2)}(0)\ (z-1) + O[\varepsilon^3] \quad (150)$$

$$c(r,\ z;\ \varepsilon) = 1 + O[\varepsilon^3] \quad (151)$$

For the extraluminal flow ;

$$p_t(r,z;\ \varepsilon) = p_{t\infty} - \varepsilon^3(\ln\varepsilon^2)\Gamma_2\Gamma_4 + \varepsilon^3\Gamma_3\Gamma_4\ \{-\ln r + \tfrac{1}{2}\ln 4z|z-1|\}$$

$$- \varepsilon^4(\ln\varepsilon^2)\ \Gamma_4 p^{(2)}(0)\ (z-1)$$

$$+ \varepsilon^4 p^{(2)}(0)\Gamma_4\ \{-\ (z-1)\ln\tfrac{1}{2}r + (z-\tfrac{1}{2})\ln z - z + \tfrac{1}{2}\}$$

$$+ \ldots\ldots \qquad (152)$$

$$u_t(r,\ z;\ \varepsilon) = \varepsilon^3\ \frac{\Gamma_3\Gamma_4}{r} + \varepsilon^4 p^{(2)}(0)\Gamma_4\ \frac{(z-1)}{r} + \ldots, \qquad (153)$$

$$v_t(r,z;\varepsilon) = -\varepsilon^3\Gamma_3\Gamma_4\ \frac{1-2z}{2z(1-z)} + \varepsilon^4(\ln\varepsilon^2)\Gamma_4 p^{(2)}(0)$$

$$- \varepsilon^4\Gamma_4 p^{(2)}(0)\ \{\ -\ln\frac{r}{2} - \frac{1}{2z} + \ln z\ \} \qquad (154)$$

$$+ \ldots\ldots$$

CHAPTER THREE

MATHEMATICAL MODELLING OF TRANSCAPILLARY FLUID EXCHANGE: UNIFORM CAPILLARY.

3.1. Introduction

The exchange of fluid across blood capillary walls, that is, filtration or absorption, is one of the most important physiological functions. Under normal conditions, the fluids are filtered from the blood capillaries into the tissue spaces at the arterial ends of the capillaries and the filtrates which pass into the tissue spaces are reabsorbed into the blood capillaries at the venous ends. However, a small portion of the filtrates enters the lymphatic capillaries and drains back through them into the blood. It is well known that the exchange of fluid between the blood and the tissue spaces as well as the transport of interstitial fluid into lymphatics plays an important role in the mechanism maintaining the partition of body fluids between the circulation and the interstitial space. Further, it may be expected that the flow of fluid across the capillary walls has effects upon the transport of dissolved substances in blood (Murata (1975)). Therefore, it is very interesting to consider theoretically this problem on the basis of some experimental results.

According to Starling's law (Landis and Pappenheimer (1963)), the filtration or absorption of fluid across the capillary walls depends upon the net effect of four independent forces: (1) hydrostatic capillary

blood pressure P_c, (2) hydrostatic interstitial fluid pressure P_i, (3) osmotic pressure of the plasma proteins π_c, and (4) osmotic pressure π_i of the proteins in the interstitial fluid immediately outside the capillary walls. This law is then formulated as follows:

$$Q = k(P_c - \pi_c - P_i + \pi_i)$$

The quantity Q represents the rate of flow per unit area of wall surface, with a plus sign to indicate filtration and a minus sign to indicate absorption. The proportionality factor k is a measure of the permeability of the capillary wall to fluid and is called the filtration coefficient. In the capillary of a frog's mesentery, Landis (1927) found that the rates of flow across the capillary walls were, on the average, proportional to the difference between the hydrostatic capillary pressure and the osmotic pressure of the plasma. Pappenheimer and Soto-Rivera (1948) obtained similar results for the capillaries of perfused hindlimbs of cats and dogs.

Wiederhielm (1968) pointed out some defects of this traditional model for fluid exchange. After careful examination of the validity of some assumptions implied in this traditional model, he derived the following conclusions: (1) the filtration coefficient may be 50 to 100% larger in the venous capillary than in the arterial capillary; (2) the surface area of the arterial capillaries may be only one-sixth to one-fourth

of the surface area of the venous capillaries; (3) the hydrostatic and osmotic pressures of interstitial fluid are not negligible compared with the corresponding blood values, and the average value of the osmotic pressure of interstitial fluid is 5 to 8 mmHg, when referred to the atmospheric pressure; (4) the hydrostatic interstitial fluid pressures range from 0 to 5 mmHg by direct measurements, whereas the pressures recorded in subcutaneous capsules are negative and average - 6 mmHg. On the basis of these conclusions, he has calculated the change in the interstitial space volume with an analog computer by considering the relationship among four fluid fluxes into and out of interstitial space.

To explain the effects of the exchange of fluid across the capillary wall on the flow of the blood within the capillary, Oka and Murata (1970) have used a simplified model for a capillary and blood. They assumed that the hydrostatic interstitial fluid pressure P_i, the osmotic pressure of the plasma π_c, and the osmotic pressure of the interstitial fluid π_i were constant. They further assumed that the capillary was a straight tube of a uniform circular cross section and that the filtration coefficient was constant along the capillary. On these assumptions they solved approximately the Navier-Stokes equation and obtained the expressions for the velocity and pressure distributions within the capillary. From their results it

has been shown that

$$\pi_c + p_i - \pi_i = (p_a + p_v)/2$$

where p_a and p_v are the arterial and venous pressures, respectively, provided that there is a balance between the outflow and the inflow across the capillary wall.

Apelblat et al. (1974) studied analytically the capillary-tissue fluid exchange from various points of view. To consider the characteristics of tissues, they used the single capillary model introduced by Krogh (1919) as the functional unit of a capillary-tissue system. Within the capillary they solved approximately the Navier-Stokes and continuity equation using lubrication-layer theory. On the other hand, they treated the tissue regions as isotropic porous media to which the Darcy law is applicable. Since the nonslip condition at the capillary wall in the tissue region was not satisfied, the condition for the tangential component of velocity at the capillary wall in the tissue region was left unspecified. In this model it was assumed that the filtration coefficient was uniform along the capillary and that there was no lymph flow. In the next place, to consider the removal of lymph from the tissue spaces by the lymphatics and the leakage of plasma through large pore systems at the venous end of the capillary, they discussed the improved Wiederhielm model. Instead of the integral description of the material balance obtained by Wiederhielm, they presented a local

description by solving the differential equations of the fluid exchange in the conical capillary. In this model, they considered the two-segment capillary with two different filtration coefficients. However, they assumed that the interstitial fluid pressure was uniform over the tissue region, so that the effect of the interstitial fluid pressure on the transcapillary fluid exchange was neglected.

An and Salathe (1976) showed that large gradients of interstitial fluid pressure normal to the capillary occur in the tissue spaces adjacent to the capillary wall. They indicated that these gradients have a significant effect on fluid exchange so that it is not possible to assume that the interstitial fluid pressure in the surrounding tissue is uniform. They considered a straight circular capillary with variable radius surrounded by a tissue space of infinite extent and have assumed that the fluid exchange between the capillary and the surrounding tissue is governed by Starling's law. The movements of fluid within the capillary and within the tissue space were assumed to be governed by Poiseuille's law and by Darcy's law, respectively. An and Salathe obtained, based on the assumption that the ratio of radius to length of the capillary is very small compared with unity and that the filtration coefficient is very small, an approximate solution of the pressure for arbitrary variation of capillary radius and permeability. Numerical results were

also obtained for a linear variation in radius and filtration coefficient. However, the tangential component of the velocity at the capillary wall in the tissue space did not satisfy the nonslip condition because the Darcy equation was used. Further, the effect of flow of fluid from the tissue space to the lymphatic capillaries on the pressure distributions was neglected in this model.

Murata (1978) theoretically analyzed the effects of the variable filtration coefficient and lymph flow on the fluid exchange between the blood capillary and the tissue space using a simplified model. The Krogh cylinder was used as the functional unit of a capillary-tissue system. It was assumed that the exchange of fluid across the capillary wall obeyed Starling's law and that the filtration coefficient of the capillary was not unifomm along the length of the capillary but was a linear function of the distance from the arterial end. It was further assumed that the flow of fluid through the tissue region was governed by a generalized Darcy's equation as proposed by Tam and that the motion of fluid in the capillary obeyed Stoke's equation. Under these assumptions, the velocity and pressure distributions in the capillary and tissue regions were obtained. He also obtained the average interstitial fluid pressure and the average effective pressure difference between the blood and interstitial fluid.

In this chapter, we shall analyze the effects of the variable filtration coefficient and lymph flow on the fluid exchange between the blood capillary and the tissue space. We shall also consider the fluid exchange at the interface between the tissue space and the lymphatic capillaries. The Krogh cylinder is used as a functional unit of a capillary-tissue system. The exchange of fluid across the capillary wall is assumed to obey a modified Starling's law and the filtration coefficient is not uniform along the length of the capillary but is a linear function of the distance from the arterial end. The transfer of fluid at the interface between the tissue and lymphatic capillary is assumed to be governed by a modified Starling's law with a reflection coefficient. It is further assumed that the flow of fluid through the tissue region is governed by a generalized Darcy's equation as proposed by Tam and that the motion of fluid in the capillary obeys the Navier-Stokes equations. Under these assumptions, the velocity and pressure distributions in the capillary and tissue regions are obtained.

3.2. ASSUMPTIONS:

In most tissue the blood capillaries are arranged in a complicated three-dimensional network and the lymphatic capillary networks are also distributed through the interstitial spaces along with the blood capillaries. The flow of fluid across the capillary wall and through the interstitial spaces is highly

complex. It is very difficult to analyze theoretically these fluid movements without using a simplified model. To simplify the calculations of the transcapillary fluid exchange we shall use the following assumptions:

(i) The Krogh cylinder arrangement is used as a geometrical representation of a capillary-tissue system. The Krogh cylinder is composed of a long, uniform, straight cylindrical capillary and a concentric cylinder of tissue surrounding it (Fig. 8). To obtain the distribution of oxygen concentration in a tissue space, Krogh (1919) has observed in detail the distribution of capillaries in muscle and has proposed this model as a functional unit of a capillary-tissue system. Since the work of Krogh, this model has been used by many workers.

(ii) Blood is a homogeneous incompressible Newtonian fluid. Since a capillary is comparable in diameter to the red cells, it is of course an oversimplification to treat the blood within a capillary as a homogeneous fluid. However, it is very difficult to consider theoretically the effects of the red cells on the flow of fluid across the capillary wall.

(iii) The fluid is transported across the blood capillary wall in accordance with a modified Starling's law. It seems likely that fluid can pass through the endothelial cells and the intercellular cements between endothelial cells. Therefore, we can assume that the

pathway of fluid is distributed continuously over the blood capillary wall. There is experimental evidence that the capillary permeability to water is not uniform along the length of the capillary but is larger in the venous capillary than in the arterial capillary. Therefore, we assume that the filtration coefficient is a linear function of the distance from the arterial end. Since the regional difference in the concentration of proteins is plasma and interstitial fluid may be neglected, we assume that these osmotic pressures are constant.

(iv) The transfer of fluid at the interface between the tissue space and lymphatic capillaries is governed by a modified Starling's law with a reflection coefficient which is assumed to be equal to one. The regional difference in the concentration of proteins in plasma and interstitial fluid may be neglected, thus the osmotic pressures are assumed constant.

(v) The tissue spaces are complex composite systems. Many tissue cells are distributed in the tissue spaces, and the interstitial space lying among tissue cells contain many elongated collagen fibers. The spaces between the collagen fibers are filled with the amorphous muco polysaccharide gel. Thus, in effect the entire interstitial fluid space has physical characteristics of a soft sponge (Guyton et al. (1957)). It seems that the tissue cells are more or less uniformly distributed around the blood capillary. Therefore, we

assume that the tissue region, the Krogh cylinder is a homogeneous,isotropic porous medium.

(vi) The interstitial fluid is composed of both free interstitial fluid and fluid in a gel state. To bring this problem into a mathematically solvable form, we assume that the interstitial fluid is the same homogeneous incompressible Newtonian fluid as plasma and that the movements of interstitial fluid obey a generalized Darcy's equation, as derived theoretically by Tam (1969).

(vii) Since the lymphatic capillary networks are distributed through the interstitial spaces along with the blood capillaries, the terminal vessels of the two systems must lie very near each other. Most commonly, the lymphatic capillaries are believed to end blindly in interstitial spaces at varying distances from the blood capillaries. To introduce the lymphatic capillary into the Krogh cylinder without destroying the axial symmetry, we assume that the lymphatic capillaries are distributed uniformly on the outer surface of the tissue region of the Krogh cylinder. The structural features of the lymphatic capillary allow fluid to enter the lymphatic capillary, whenever the pressure outside the lymphatic capillary is greater than the pressure inside, and to prevent backward movement of the fluid out of the lymphatic capillary into the tissue space, whenever the pressure inside the lymphatic capillary becomes greater than outside (Guyton et al. (1975)).

Therefore, we assume that the fluid enters into the lymphatic capillary at a certain velocity from the outer surface of the tissue region of the Krogh cylinder.

(viii) The pulsatory characteristics of the blood flow are largely damped out in the blood capillary and the flow is sufficiently slow for the viscous properties of the blood to be extremely important. Therefore, we can assume that the motion of the fluid is laminar and steady and that the inertia term in the equation of motion can be neglected.

(ix) We assume that the rate of flow per unit area of the blood capillary wall is very small, since the value of the filtration coefficient is very small.

(x) The end effects are negligible, because the flow is sufficiently slow and the ratio of the radius to length of the cylinder is very small.

(xi) There is no slip at the capillary wall. This assumption is valid unless the mean free path of the fluid molecules becomes large compared with the dimensions of the bounding surfaces.

(xii) No body force acts on the fluid.

(xiii) The motion has an axial symmetry.

3.3. FUNDAMENTAL EQUATIONS AND BOUNDARY CONDITIONS:

The Krogh cylinder and a cylindrical coordinate system (r, ϕ, z) are shown in Fig. 8. The z-axis coincides with the centre line of the capillary and the origin of this coordinate system is located at the

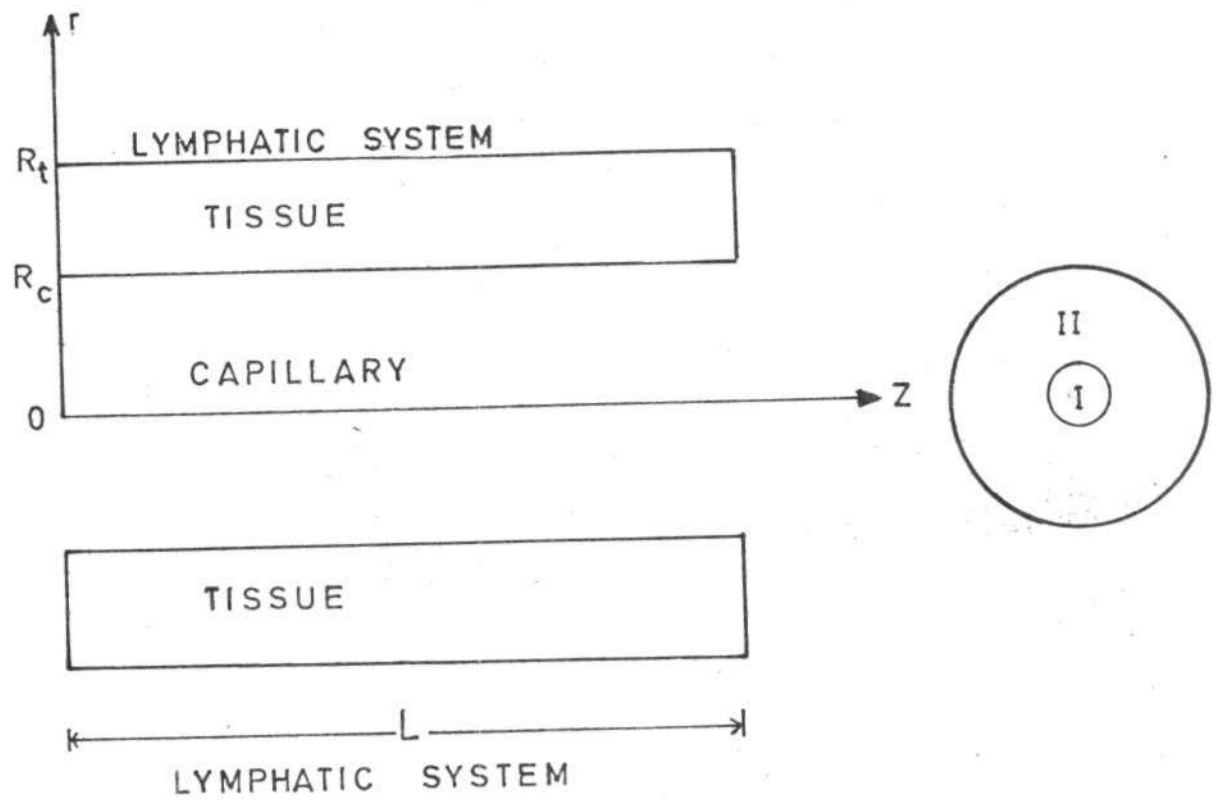


FIG. 8: KROGH CYLINDER AND CYLINDRICAL CO-ORDINATE SYSTEM

arterial end. Here, R_c and L are the radius and the length of the capillary (I), respectively, and R_t is the outer radius of tissue region (II).

Our fundamental equations are the equation of motion and the continuity equation. From assumptions (ii), (viii), and (xii), the equation of motion and the continuity equation in the capillary (region I) are

$$\nabla P_1 = \eta \nabla^2 V_1 \qquad (1)$$

$$\nabla . V_1 = 0 \qquad (2)$$

where V_1 is the velocity vector, P_1 is the fluid pressure, and η is the coefficient of viscosity. Equation (1) is the linearized Navier-Stokes equation.

From assumptions (v), (vi), (viii), and (xii), the fundamental equations in the tissue region (region II) are

$$\nabla P_2 = \eta(\nabla^2 V_2 - \sigma^2 V_2) \qquad (3)$$

$$\nabla . V_2 = 0 \qquad (4)$$

where the subscript (2) denotes the quantities in the tissue region. Equation (3) is a generalized Darcy's equation derived by Tam (1969). He has regarded a form of porous medium as a particle cloud and has considered slow viscous flow past a large collection of spherical particles. Using a point-force approximation, he derived a set of equations governing the viscous interaction between a large number of particles. On taking the ensemble average of the set of equations, he obtained averaged equations

describing the mean fluid flow with particles embedded in it, that is, a generalized Darcy's equation (3). The quantity σ^2 in equation (3) represents the resistance to flow by spherical particles, and its value depends upon the particle size and the volume concentration of particles. In the case where σ^2 is so large that the first term in parentheses on right-hand side of equation (3) can be neglected compared with the second term, equation (3) is reduced to Darcy's original equation.

The boundary conditions are:

$$\frac{\partial U_1}{\partial r} = 0 \qquad V_1 = 0 \text{ at } r = 0 \quad (1.1)$$

$$U_1 = 0 \qquad \text{at } r = R_c \quad (1.2)$$

$$V_1 = k[(P_1-P_2)+\sigma(z)\Pi_e] \qquad \text{at } r = R_c \quad (1.3)$$

$$P_1 = P_a \qquad \text{at } z = 0 \quad (1.4)$$

$$P_1 = P_v \qquad \text{at } z = L \quad (1.5)$$

$$V_2 = k_\ell[P_2 - \sigma_\ell\Pi_\ell] \qquad \text{at } r = R_t \quad (2.1)$$

$$U_2 = 0 \qquad \text{at } r = R_c \quad (2.2)$$

$$V_2 = k[(P_1-P_2)+\sigma(z)\Pi_e] \qquad \text{at } r = R_c \quad (2.3)$$

$$U_2 = 0 \qquad \text{at } z = 0 \quad (2.4)$$

$$U_2 = 0 \qquad \text{at } z = L \quad (2.5)$$

where U and V are the z- and r-components of the velocity, respectively. The quantities in region I and region II are denoted by subscripts 1 and 2, respectively.

The boundary condition (1.1) expresses the axial symmetry of fluid motion. The boundary conditions (1.2) and (2.2) express the nonslip condition at the

capillary wall. The boundary conditions (1.3) and (2.3) express the modified Starling's law where $\sigma(z)$ is the reflection coefficient. In these equations, Π_e is the difference between the osmotic pressure of the plasma proteins Π_1 and the osmotic pressure of the proteins in the interstitial fluid immediately outside the capillary wall Π_2, that is, $\Pi_e = \Pi_2 - \Pi_1$ Now we regard Π_e as a constant according to assumptions (iii) and (ix). The filtration coefficient k is not uniform along the length of the capillary. According to assumption (iii), we consider that k is a linear function of z as follows:

$$k = k_a + \frac{k_v - k_a}{L} z, \; k_v > k_a \qquad (5)$$

where k_a and k_v are the values of the filtration coefficient at the arterial and venous ends of the capillary, respectively. The boundary condition (2.1) expresses the flow of fluid from the tissue region into the lymphatic capillaries. According to assumption (iv), the velocity is $k_\ell[P_2 - \sigma_\ell \Pi_\ell]$ where $k_\ell = P_L - (\Pi_L - \Pi_t)$ and $\sigma_\ell = 1$. k_ℓ is the hydrodynamic conductivity of the interface of the lymphatic capillaries and the area. P_L and Π_L are the hydrostatic and osmotic pressure of the lymph, while Π_t is the osmotic pressure of the tissue space. σ_ℓ is the reflection coefficient.

The boundary conditions (1.4) and (1.5) express the fact that the fluid enters the capillary at the

arterial end and leaves it at the venous end. Here P_a and P_v are the arterial and venous pressures, respectively. Both P_a and P_v are regarded as constant. The boundary conditions (2.4) and (2.5) express the fact that the Krogh cylinder is an isolated functional unit.

3.4. SOLUTIONS OF THE EQUATIONS;

We shall introduce the following non dimensional variables:

$$\left.\begin{aligned} \zeta = \frac{r}{R_c},\ \xi = \frac{z}{L},\ u = \frac{U}{U_o},\ v = \frac{V}{U_o},\ p = \frac{P}{\Delta P} \\ K = \frac{\eta k_a}{R_c},\ K_L = \frac{\eta k_\ell}{R_c},\ \theta = \tfrac{1}{2}\,\frac{k_v - k_a}{k_a},\ \pi_e = \frac{\Pi_e}{\Delta P} \end{aligned}\right\} \quad (6)$$

where

$$\Delta P = P_a - P_v,\ U_o = \frac{R_c^2\ \Delta P}{4\eta L} \qquad (7)$$

From Oka and Murata (1970), if the value of k is very small, equations (1) and (2) can be written approximately as follows:

$$\frac{\partial P_1}{\partial r} = 0 \qquad (8)$$

$$\frac{\partial P_1}{\partial z} = \eta\left(\frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r}\,\frac{\partial U_1}{\partial r}\right) \qquad (9)$$

$$\frac{1}{r}\,\frac{\partial}{\partial r}(rV_1) + \frac{\partial U_1}{\partial z} = 0 \qquad (10)$$

To convert the above equations into non dimensionalized form we have;

from (8),

$$\frac{\Delta P}{R_c}\,\frac{\partial\ p_1}{\partial \zeta} = 0 \Rightarrow \frac{\partial p_1}{\partial \zeta} = 0$$

from (9),

$$\frac{\Delta P}{\eta L}\frac{\partial p_1}{\partial \xi} = \frac{U_o}{R_c^2}\frac{\partial^2 u_1}{\partial \zeta^2} + \frac{U_o}{R_c^2 \zeta}\frac{\partial u_1}{\partial \zeta}$$

$$\frac{4U_o}{R_c^2}\frac{\partial p_1}{\partial \xi} = \frac{U_o}{R_c^2}\frac{\partial^2 u_1}{\partial \zeta^2} + \frac{U_o}{R_c^2 \zeta}\frac{\partial u_1}{\partial \zeta}$$

multiplying by R_c^2/U_o we have

$$\frac{\partial p_1}{\partial \xi} = \frac{1}{4}\left[\frac{\partial^2 u_1}{\partial \zeta^2} + \frac{1}{\zeta}\frac{\partial u_1}{\partial \zeta}\right] \iff \frac{\partial p_1}{\partial \xi} = \frac{1}{4\zeta}\frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial u_1}{\partial \zeta}\right)$$

from (10)

$$\frac{1}{R_c^2 \zeta}\frac{\partial}{\partial \zeta}(R_c.\zeta.U_o v_1) + \frac{U_o}{L}\frac{\partial u_1}{\partial \xi} = 0$$

$$\frac{U_o}{R_c}\frac{1}{\zeta}\frac{\partial}{\partial \zeta}(\zeta.v_1) + \frac{U_o}{L}\frac{\partial u_1}{\partial \xi} = 0$$

multiplying by R_c/U_o we have

$$\frac{1}{\zeta}\frac{\partial}{\partial \zeta}(\zeta.v_1) + \frac{R_c}{L}\frac{\partial u_1}{\partial \xi} = 0 \iff \frac{1}{\zeta}\frac{\partial}{\partial \zeta}(\zeta.v_1) + \frac{1}{\beta}\frac{\partial u_1}{\partial \xi} = 0$$

Thus in non dimensionalized form, equations (8) to (10) can be written as follows:

$$\frac{\partial p_1}{\partial \zeta} = 0 \tag{11}$$

$$\frac{\partial p_1}{\partial \xi} = \frac{1}{4\zeta}\frac{\partial}{\partial \zeta}\left(\zeta \frac{\partial u_1}{\partial \zeta}\right) \tag{12}$$

$$\frac{1}{\zeta}\frac{\partial}{\partial \zeta}(\zeta.v_1) + \frac{1}{\beta}\frac{\partial u_1}{\partial \xi} = 0 \tag{13}$$

where

$$\beta = \frac{L}{R_c} \tag{14}$$

Taking equation (12), we can write

$$\frac{d}{d\zeta}(\zeta \frac{du_1}{d\zeta}) = 4\zeta \frac{dp_1}{d\xi}$$

integrating,

$$\frac{du_1}{d\zeta} = 2\zeta \frac{dp_1}{d\xi} + \frac{C_1}{\zeta}$$

using boundary condition (1.1), C = 0

integrating again,

$$u_1 = \zeta^2 \frac{dp_1}{d\xi} + D$$

Using boundary condition (1.2) $u_1 = 0$ at $\zeta = 1$,

$$D = - (\frac{dp_1}{d\xi})$$

thus

$$u_1 = \zeta^2 \frac{dp_1}{d\xi} - (\frac{dp_1}{d\xi})$$

$$<=> \quad u_1 = - (1 - \zeta^2) \frac{dp_1}{d\xi} \qquad (15)$$

Taking equation (13) and substituting for u_1 we have,

$$\frac{1}{\zeta} \frac{d}{d\zeta}(\zeta v_1) - \frac{1}{\beta}(1 - \zeta^2) \frac{d^2 p_1}{d\xi^2} = 0$$

$$\frac{d}{d\zeta}(\zeta v_1) = \frac{1}{\beta}(\zeta - \zeta^3) \frac{d^2 p_1}{d\xi^2}$$

Integrating,

$$\zeta v_1 = \frac{1}{\beta}(\frac{\zeta^2}{2} - \frac{\zeta^4}{4}) \frac{d^2 p_1}{d\xi^2} + C$$

$$v_1 = \frac{1}{\beta}(\frac{\zeta}{2} - \frac{\zeta^3}{4}) \frac{d^2 p_1}{d\xi^2} + C$$

using boundary condition (1.1), C = 0

thus,

$$v_1 = \frac{1}{\beta}(\frac{\zeta}{2} - \frac{\zeta^3}{4}) \frac{d^2 p_1}{d\xi^2}$$

$$<=> \quad v_1 = \frac{1}{4\beta}(2\zeta - \zeta^2) \frac{d^2 p_1}{d\xi^2} \tag{16}$$

Equations (3) and (4) may be written as

$$\frac{\partial P_2}{\partial r} = \eta(\frac{\partial^2 V_2}{\partial r^2} + \frac{\partial^2 V_2}{\partial z^2} + \frac{1}{r}\frac{\partial V}{\partial r} - \frac{V}{r^2} - \sigma^2 V) \tag{17}$$

$$\frac{\partial P_2}{\partial z} = \eta(\frac{\partial^2 U_2}{\partial r^2} + \frac{\partial^2 U_2}{\partial z^2} + \frac{1}{r}\frac{\partial U_2}{\partial r} - \sigma^2 U) \tag{18}$$

$$\frac{\partial V_2}{\partial r} + \frac{\partial U_2}{\partial z} + \frac{V_2}{r} = 0 \tag{19}$$

To convert the above equations into non dimensionalized form we have;

from (17),

$$\frac{\Delta P}{\eta R_c}\frac{\partial p_2}{\partial \zeta} = (\frac{U_o}{R_c^2}\frac{\partial^2 v_2}{\partial \zeta^2} + \frac{U_o}{L^2}\frac{\partial^2 v_2}{\partial \xi^2} + \frac{U_o}{R_c^2}\frac{1}{\zeta}\frac{\partial v_2}{\partial \zeta}$$

$$- \frac{U_o}{R_c^2}\frac{v_2}{\zeta} - \sigma^2 R_c^2 v_2)$$

$$\frac{4U_o L}{R_c^3}\frac{\partial p_2}{\partial \zeta} = \frac{U_o}{R_c^2}(\frac{\partial^2 v_2}{\partial \zeta^2} + \frac{R_c^2}{L^2}\frac{\partial^2 v_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial v_2}{\partial \zeta} - \frac{v_2}{\zeta^2} - \sigma^2 R_c^2 v_2)$$

multiplying by R_c^2/U_o we have,

$$\frac{4L}{R_c}\frac{\partial p_2}{\partial \zeta} = (\frac{\partial^2 v_2}{\partial \zeta^2} + \frac{R_c^2}{L^2}\frac{\partial^2 v_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial v_2}{\partial \zeta} - \frac{v_2}{\zeta^2} - \sigma^2 R_c^2 v_2)$$

$$<=> \quad \frac{\partial p_2}{\partial \zeta} = \frac{1}{4\beta}(\frac{\partial^2 v_2}{\partial \zeta^2} + \frac{1}{\beta^2}\frac{\partial^2 v_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial v_2}{\partial \zeta} - \frac{v_2}{\zeta^2} - s^2 v_2)$$

where $\beta = \frac{L}{R_c}$ and $s = \sigma R_c$

from (18),

$$\frac{\Delta P}{\eta L}\frac{\partial p_2}{\partial \xi} = \left(\frac{U_o}{R_c^2}\frac{\partial^2 u_2}{\partial \zeta^2} + \frac{U_o}{L^2}\frac{\partial^2 u_2}{\partial \xi^2} + \frac{U_o}{R_c^2}\frac{1}{\zeta}\frac{\partial u_2}{\partial \zeta} - \sigma^2 U_o u_2\right)$$

$$\frac{4U_o}{R_c^2}\frac{\partial p_2}{\partial \xi} = \frac{U_o}{R_c^2}\left(\frac{\partial^2 u_2}{\partial \zeta^2} + \frac{R_c^2}{L^2}\frac{\partial^2 u_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial u_2}{\partial \zeta} - \sigma^2 R_c^2 u_2\right)$$

multiplying by R_c^2/U_o we have

$$\frac{4\partial p_2}{\partial \xi} = \left(\frac{\partial^2 u_2}{\partial \zeta^2} + \frac{R_c^2}{L^2}\frac{\partial^2 u_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial u_2}{\partial \zeta} - \sigma^2 R_c^2 u_2\right)$$

$$\Leftrightarrow \quad \frac{\partial p_2}{\partial \xi} = \frac{1}{4}\left(\frac{\partial^2 u_2}{\partial \zeta^2} + \frac{1}{\beta^2}\frac{\partial^2 u_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial u_2}{\partial \zeta} - s^2 u_2\right)$$

from (19),

$$\frac{U_o}{R_c}\frac{\partial v_2}{\partial \zeta} + \frac{U_o}{L}\frac{\partial u_2}{\partial \xi} + \frac{U_o}{R_c}\frac{v_2}{\zeta} = 0$$

multiplying by R_c/U_o we have

$$\frac{\partial v_2}{\partial \zeta} + \frac{R_c}{L}\frac{\partial u_2}{\partial \xi} + \frac{v_2}{\zeta} = 0$$

$$\frac{\partial v_2}{\partial \zeta} + \frac{1}{\beta}\frac{\partial u_2}{\partial \xi} + \frac{v_2}{\zeta} = 0$$

$$\Leftrightarrow \quad \frac{1}{\zeta}\frac{\partial}{\partial \zeta}(\zeta v_2) + \frac{1}{\beta}\frac{\partial u_2}{\partial \xi} = 0$$

Thus in non dimensionalized form, equations (17) to (19) can be written as follows:

$$\frac{\partial p_2}{\partial \zeta} = \frac{1}{4\beta}\left(\frac{\partial^2 v_2}{\partial \zeta^2} + \frac{1}{\beta^2}\frac{\partial^2 v_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial v_2}{\partial \zeta} - \frac{v_2}{\zeta^2} - s^2 v_2\right) \quad (20)$$

$$\frac{\partial p_2}{\partial \xi} = \frac{1}{4}\cdot\left(\frac{\partial^2 u_2}{\partial \zeta^2} + \frac{1}{\beta^2}\frac{\partial^2 u_2}{\partial \xi^2} + \frac{1}{\zeta}\frac{\partial u_2}{\partial \zeta} - s^2 u_2\right) \quad (21)$$

$$\frac{1}{\zeta}\frac{\partial}{\partial\zeta}(\zeta v_2) + \frac{1}{\beta}\frac{\partial u_2}{\partial\xi} = 0 \qquad (22)$$

Now we will define a stream function as follows:

$$u_2 = -\frac{1}{\zeta}\frac{\partial\psi}{\partial\zeta}, \quad v_2 = \frac{1}{\beta\zeta}\frac{\partial\psi}{\partial\xi}$$

writing equations (20) and (21) in the stream function we have,

$$\frac{\partial p_2}{\partial\zeta} = \frac{1}{4\beta^2}\left(-\frac{1}{\zeta^2}\frac{\partial^2\psi}{\partial\xi\partial\zeta} + \frac{1}{\zeta}\frac{\partial^3\psi}{\partial\xi\partial\zeta^2} + \frac{1}{\beta^2\zeta}\frac{\partial^3\psi}{\partial\xi^3} - \frac{s^2}{\zeta}\frac{\partial\psi}{\partial\xi}\right) \qquad (23)$$

$$\frac{\partial p_2}{\partial\xi} = \frac{1}{4}\left(-\frac{1}{\zeta^3}\frac{\partial\psi}{\partial\zeta} + \frac{1}{\zeta^2}\frac{\partial^2\psi}{\partial\zeta^2} - \frac{1}{\zeta}\frac{\partial^3\psi}{\partial\zeta^3} - \frac{1}{\beta^2\zeta}\frac{\partial^3\psi}{\partial\zeta\partial\xi^2}\right.$$

$$\left. + \frac{s^2}{\zeta}\frac{\partial\psi}{\partial\zeta}\right) \qquad (24)$$

To eliminate pressure we differentiate (23) partially by $\frac{\partial}{\partial\xi}$ and (24) by $\frac{\partial}{\partial\zeta}$, thus we have,

$$\frac{\partial}{\partial\xi}\left(\frac{\partial p_2}{\partial\zeta}\right) = \frac{1}{4\beta^2}\left(-\frac{1}{\zeta^2}\frac{\partial^3\psi}{\partial\xi^2\partial\zeta} + \frac{1}{\zeta}\frac{\partial^4\psi}{\partial\xi^2\partial\zeta^2} + \frac{1}{\beta^2\zeta}\frac{\partial^4\psi}{\partial\xi^4}\right.$$

$$\left. - \frac{s^2}{\zeta}\frac{\partial^2\psi}{\partial\xi^2}\right) \qquad (25)$$

$$\frac{\partial}{\partial\zeta}\left(\frac{\partial p_2}{\partial\xi}\right) = \frac{1}{4}\left(\frac{3}{\zeta^4}\frac{\partial\psi}{\partial\zeta} - \frac{1}{\zeta^3}\frac{\partial^2\psi}{\partial\zeta^2} - \frac{2}{\zeta^3}\frac{\partial^2\psi}{\partial\zeta^2} + \frac{1}{\zeta^2}\frac{\partial^3\psi}{\partial\zeta^3}\right.$$

$$+ \frac{1}{\zeta^2}\frac{\partial^3\psi}{\partial\zeta^3} - \frac{1}{\zeta}\frac{\partial^4\psi}{\partial\zeta^4} + \frac{1}{\beta^2\zeta^2}\frac{\partial^3\psi}{\partial\xi^2\partial\zeta} - \frac{1}{\beta^2\zeta}\frac{\partial^4\psi}{\partial\zeta^2\partial\xi^2}$$

$$\left. - \frac{s^2}{\zeta^2}\frac{\partial\psi}{\partial\zeta} + \frac{s^2}{\zeta}\frac{\partial^2\psi}{\partial\zeta^2}\right) \qquad (26)$$

(25) - (26) gives,

$$0 = \frac{1}{2\beta^2\zeta}\frac{\partial^4\psi}{\partial\xi^2\partial\zeta^2} + \frac{1}{4\beta^4\zeta}\frac{\partial^4\psi}{\partial\xi^4} + \frac{1}{4\zeta}\frac{\partial^4\psi}{\partial\zeta^4} - \frac{1}{2\beta^2\zeta^2}\frac{\partial^3\psi}{\partial\xi^2\partial\zeta}$$

$$- \frac{1}{2\zeta^2}\frac{\partial^3\psi}{\partial\zeta^3} \quad \frac{1}{4\zeta^4}(3-\zeta^2 s^2)\frac{\partial\psi}{\partial\zeta} - \frac{s^2}{4\beta^2\zeta}\frac{\partial^2\psi}{\partial\xi^2} + \frac{3}{4\zeta^3}\frac{\partial^2\psi}{\partial\zeta^2} - \frac{s^2}{4\zeta}\frac{\partial^2\psi}{\partial\zeta^2}$$

$$<=> \quad (\frac{1}{\beta^2}\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2} - \frac{1}{\zeta}\frac{\partial}{\partial\zeta})(\frac{1}{\beta^2}\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\zeta^2} - \frac{1}{\zeta}\frac{\partial}{\partial\zeta}$$

$$- s^2)\ \psi = 0 \qquad (27)$$

Taking the first bracket, i.e.,

$$\frac{1}{\beta^2}\frac{\partial^2\psi}{\partial\xi^2} + \frac{\partial^2\psi}{\partial\zeta^2} - \frac{1}{\zeta}\frac{\partial\psi}{\partial\zeta} = 0$$

To solve the above equation, we shall apply variable separation method. Let us assume the solution in the form

$$\psi = Z(\zeta)X(\xi) \qquad (28)$$

where Z and X are functions of ζ and ξ, respectively.

Substituting (28) into the above equation we have,

$$\frac{1}{\beta^2}\frac{d^2X}{d\xi^2}Z + \frac{d^2Z}{d\zeta^2}X - \frac{1}{\zeta}\frac{dZ}{d\zeta}X = 0$$

dividing by XZ,

$$\frac{1}{X\beta^2}\frac{d^2X}{d\xi^2} + \frac{1}{Z}\frac{d^2Z}{d\zeta^2} - \frac{1}{Z\zeta}\frac{dZ}{d\zeta} = 0$$

$$<=> \quad \frac{1}{Z}\frac{d^2Z}{d\zeta^2} - \frac{1}{Z\zeta}\frac{dZ}{d\zeta} = -\frac{1}{X\beta^2}\frac{d^2X}{d\xi^2} = \lambda^2$$

where λ is a positive constant.

Thus we have two separate equations

$$\frac{Z''}{Z} - (\frac{1}{\zeta})\frac{Z'}{Z} = \lambda^2, \quad -\frac{X''}{X} = \beta^2\lambda^2$$

$$<=> \quad Z'' - (\frac{1}{\zeta})Z' - Z\lambda^2 = 0 \qquad (29)$$

and

$$X'' + \beta^2\lambda^2 X = 0 \qquad (30)$$

The general solution of (30) is given by

$$X(\xi) = C_1 \cos \frac{n\pi}{\beta} \beta\xi + C_2 \sin \frac{n\pi}{\beta} \beta\xi \qquad (31)$$

The boundary conditions in this region are:

$U_2 = 0$ at $z = 0$

$U_2 = 0$ at $z = L$

In non dimensionalized form,

$\psi = 0$ at $\xi = 0$

$\psi = 0$ at $\xi = 1$

Applying the boundary conditions to (31) we have,

$$X(\xi) = C_2 \sin \frac{n\pi}{\beta} \beta\xi$$

Let us write $\lambda_n = \frac{n\pi}{\beta}$, $n = 1,2,$-------

$$\therefore \quad X(\xi) = C_2 \sin \lambda_n \beta\xi \qquad (32)$$

For equation (29), we put $\phi = \frac{Z}{\zeta}$

Substituting in (29) we have,

$$\zeta \frac{d^2\phi}{d\zeta^2} + 2\frac{d\phi}{d\zeta} - \frac{1}{\zeta}(\phi + \zeta \frac{d\phi}{d\zeta}) - \zeta\phi\lambda^2 = 0$$

$$\zeta \frac{d^2\phi}{d\zeta^2} + \frac{d\phi}{d\zeta} - (\frac{1}{\zeta} + \zeta\lambda^2)\phi = 0$$

multiplying by ζ,

$$\zeta^2\phi'' + \zeta\phi' + i^2(1 + \zeta^2\lambda^2)\phi = 0$$

This is a Bessel's differential equation of order 1.

The general solution is given by

$$\phi(\zeta) = A_1 I_1(\lambda\zeta) + B_1 K_1(\lambda\zeta) \qquad (33)$$

or we can write

$$\phi(\zeta) = A_1 I_1(\lambda_n\zeta) + B_1 K_1(\lambda_n\zeta) \qquad (34)$$

But

$$\phi = \frac{Z}{\zeta} \quad \Rightarrow \quad Z(\zeta) = \phi\zeta$$

Thus

$$Z(\zeta) = \zeta[A_1 I_1(\lambda_n \zeta) + B_1 K_1(\lambda_n \zeta)] \tag{35}$$

Hence $\psi = Z(\zeta)X(\xi)$ is given by

$$\psi = C_2 \sin \lambda_n \beta\xi [\zeta A_1 I_1(\lambda_n \zeta) + \zeta B_1 K_1(\lambda_n \zeta)] \tag{36}$$

Taking the second bracket,

$$\frac{1}{\beta^2} \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \zeta^2} \quad \frac{1}{\zeta} \frac{\partial \psi}{\partial \zeta} - s^2 \psi = 0$$

After applying the variable separation method we obtain two separate equations

$$Z'' - (\frac{1}{\zeta}) Z' - (s^2 + \lambda^2) Z = 0 \tag{37}$$

and

$$X'' + \beta^2 \lambda^2 X = 0 \tag{38}$$

The general solution of (38) is given as

$$X(\xi) = D_1 \cos \frac{n\pi}{\beta} \beta\xi + D_2 \sin \frac{n\pi}{\beta} \beta\xi \tag{39}$$

After applying the same boundary conditions as before, we obtain,

$$X(\xi) = D_2 \sin \lambda_n \beta\xi \tag{40}$$

Similarly using $\phi = \frac{Z}{\zeta}$ for equation (37), we obtain a Bessel's differential equation as follows:

$$\zeta^2 \phi'' + \zeta^2 \phi' + i^2 (1 + \zeta^2 \lambda_{sn}^2) \phi = 0$$

where $\lambda_{sn} = (s^2 + \lambda_n^2)^{\frac{1}{2}}$, n = 1,2,----------

The general solution is given by

$$\phi(\zeta) = A_2 I_1(\lambda_{sn} \zeta) + B_2 K_1(\lambda_{sn} \zeta) \tag{41}$$

But

$$\phi = \frac{Z}{\zeta} \Rightarrow Z(\zeta) = \phi\zeta$$

Thus

$$Z(\zeta) = \zeta[A_2 I_1(\lambda_{sn}\zeta) + B_2 K_1(\lambda_{sn}\zeta)] \tag{42}$$

Hence

$\psi = Z(\zeta)X(\xi)$ is given by

$$\psi = D_2 \sin \lambda_n \beta\xi[\zeta A_2 I_1(\lambda_{sn}\zeta) + \zeta B_2 K_1(\lambda_{sn}\zeta)] \tag{43}$$

Thus the solution of (26) is given by

$$\psi = C_2 \sin \lambda_n \beta\xi[\zeta A_1 I_1(\lambda_n\zeta) + \zeta B_1 K_1(\lambda_n\zeta)]$$

$$+ D_2 \sin \lambda_n \beta\xi[\zeta A_2 I_1(\lambda_{sn}\zeta) + \zeta B_2 K_1(\lambda_{sn}\zeta)] \tag{44}$$

C_2 and D_2 are constants thus we may write just one constant, i.e.,

$$\psi = \zeta[A_1 K_1(\lambda_n\zeta) + B_1 K_1(\lambda_n\zeta) + A_2 I_1(\lambda_{sn}\zeta)$$

$$+ B_2 K_1(\lambda_{sn}\zeta)] C_2 \sin \lambda_n \beta\xi \tag{45}$$

We will now apply boundary conditions (2.1) and (2.2), i.e.,

$$V_2 = k_\ell[P_2 - \sigma_\ell \Pi_\ell] \quad \text{at } r = R_t$$

$$U_2 = 0 \quad \text{at } r = R_c$$

where $\sigma_\ell = 1$ and $\Pi_\ell = P_L - (\Pi_L - \Pi_t)$

In non dimensionalized form we have

$$v_2 = 4\beta_\ell[p_2 - \pi_\ell] \quad \text{at } \zeta = \gamma \tag{2.1}$$

$$u_2 = 0 \quad \text{at } \zeta = 1 \tag{2.2}$$

Equation (44) must satisfy the above boundary conditions. Using the stream function, i.e.,

$$v_2 = \frac{1}{\beta\zeta} \frac{\partial\psi}{\partial\xi}$$

we have

$$4\beta\kappa_\ell[p_2 - \pi_\ell] = \frac{1}{\beta\zeta}\cdot\zeta[A_1I_1(\lambda_n\zeta) + B_1K_1(\lambda_n\zeta)$$

$$+ A_2I_1(\lambda_{sn}\zeta) + B_2K_1(\lambda_{sn}\zeta)]\ C_2\lambda_n\beta\cos\lambda_n\beta\xi$$

$$\Leftrightarrow\quad 4\beta\kappa_\ell[p_2 - \pi_\ell] = \frac{1}{\beta}[A_1I_1(\lambda_n\zeta) + B_1K_1(\lambda_n\zeta)$$

$$+ A_2I_1(\lambda_{sn}\zeta) + B_2K_1(\lambda_{sn}\zeta)]C_2\lambda_n\beta\cos\lambda_n\beta\xi$$

In this form ψ will not satisfy boundary condition (2.1).

Now

$$v_2 = \frac{1}{\beta\zeta}\frac{\partial\psi}{\partial\xi} = -4\beta\kappa_\ell\pi_\ell + 4\beta K_\ell p_2(\gamma,\xi) \text{ at } \zeta = \gamma$$

thus integrating,

$$\frac{1}{\beta}\psi = -4\beta\ _\ell\pi_\ell\xi\zeta + 4\beta^2\kappa_\ell\zeta\int_{\zeta=\gamma} p_2(\gamma,\xi)d\xi + C \text{ at } \zeta = \gamma$$

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\zeta\xi + 4\beta^2\kappa_\ell\zeta\int_{\zeta=\gamma} p_2(\gamma,\xi)d\xi + C(\zeta) \text{ at } \zeta = \gamma$$

i.e.

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma\int_{\zeta=\gamma} p_2(\gamma,\xi)d\xi + C(\zeta)$$

$$\text{at } \zeta = \qquad (46)$$

thus,

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma\int_{\zeta=\gamma} p_2(\gamma,\xi)d\xi$$

$$+ \zeta[A_1I_1(\lambda_n\zeta) + B_1K_1(\lambda_n\zeta) + A_2I_1(\lambda_{sn}\zeta)$$

$$+ B_2K_1(\lambda_{sn}\zeta)]C_2\sin\lambda_n\beta\xi \qquad (47)$$

Therefore ψ will satisfy boundary condition (2.1) only if it is in this form.

Using

$$u_2 = -\frac{1}{\zeta}\frac{\partial\psi}{\partial\zeta},\quad v_2 = \frac{1}{\beta\zeta}\frac{\partial\psi}{\partial\xi}$$

we have

$$u_2 = -[\lambda_n A_1 I_o(\lambda_n\zeta) - \lambda_n B_1 K_o(\lambda_n\zeta)$$

$$+ \lambda_{sn} A_2 I_o(\lambda_{sn}\zeta) - \lambda_{sn} B_2 K_o(\lambda_{sn}\zeta)]C_2 \sin \lambda_n\beta\xi \quad (48)$$

$$v_2 = -4\beta\kappa_\ell \pi_\ell + 4\beta\kappa_\ell P_2(\gamma,\xi) + [A_1 I_1(\lambda_n\zeta) + B_1 K_1(\lambda_n\zeta)$$

$$+ A_2 I_1(\lambda_{sn}\zeta) + B_2 K_1(\lambda_{sn}\zeta)]C_2\lambda_n\beta \cos \lambda_n\beta\xi \quad (49)$$

where

$$\frac{d}{d\zeta}(\zeta I_1(\lambda_n\zeta)) = \zeta\lambda_n I_o(\lambda_n\zeta)$$

and

$$\frac{d}{d\zeta}(\zeta K_1(\lambda_n\zeta)) = -\zeta\lambda_n K_o(\lambda_n\zeta)$$

Now applying boundary conditions (2.1)(i.e. equation 46) and (2.2) in equation (47) we have,

$$0 = -[\lambda_n\{A_1 I_o(\lambda_n) - B_1 K_o(\lambda_n)\}$$

$$+ \lambda_{sn}\{A_2 I_o(\lambda_{sn}\zeta) - B_2 K_o(\lambda_{sn}\zeta)\}]C_2 \sin \lambda_n\beta\xi \quad (50)$$

$$0 = [A_1 I_1(\lambda_n\gamma) + B_1 K_1(\lambda_n\gamma) + A_2 I_1(\lambda_{sn}\gamma)$$

$$+ B_2 K_1(\lambda_{sn}\gamma)]C_2\lambda_n\beta \cos \lambda_n\beta\xi \quad (51)$$

from (50), $A_1 I_1(\lambda_n\gamma) + B_1 K_1(\lambda_n\gamma) = 0$

$$\therefore B_1 = \frac{-A_1 I_1(\lambda_n\gamma)}{K_1(\lambda_n\gamma)} \quad (52)$$

Also from (50), $A_2 I_1(\lambda_{sn}\gamma) + B_2 K_1(\lambda_{sn}\gamma) = 0$

$$\therefore B_2 = \frac{-A_2 I_1(\lambda_{sn}\gamma)}{K_1(\lambda_{sn}\gamma)} \quad (53)$$

Substituting in (49) we have,

$$0 = -[\lambda_n\{A_1 I_o(\lambda_n) + \frac{A_1 I_1(\lambda_n\gamma)}{K_1(\lambda_n\gamma)} \times K_o(\lambda_n)\}$$

$$+ \lambda_{sn}\{A_2 I_o(\lambda_{sn}) + \frac{A_2 I_1(\lambda_{sn}\gamma)}{K_1(\lambda_{sn}\gamma)} \times K_o(\lambda_{sn})\}]C_2 \sin \lambda_n\beta\xi$$

$$\Leftrightarrow \quad 0 = -[\lambda_n A_1\{\frac{I_o(\lambda_n)K_1(\lambda_n\gamma) + I_1(\lambda_n\gamma)K_o(\lambda_n)}{K_1(\lambda_n\gamma)}\}$$

$$+ \lambda_{sn} A_2\{\frac{I_o(\lambda_{sn})K_1(\lambda_{sn}\gamma) + I_1(\lambda_{sn}\gamma)K_o(\lambda_{sn})}{K_1(\lambda_{sn}\gamma)}\}]C_2 \sin \lambda_n\beta\xi$$

i.e.,

$$0 = -[\lambda_n A_1 \frac{F(\lambda_n, \lambda_n\gamma)}{K_1(\lambda_n\gamma)} + \lambda_{sn} A_2 \frac{F(\lambda_{sn}, \lambda_{sn}\gamma)}{K_1(\lambda_{sn}\gamma)}]C_2 \sin \lambda_n\beta\xi$$

where

$$F(\lambda_n, \lambda_n\gamma) = I_o(\lambda_n)K_1(\lambda_n\gamma) + K_o(\lambda_n)I_1(\lambda_n\gamma)$$

thus

$$A_2 = -\frac{\lambda_n}{\lambda_{sn}} A_1 \frac{F(\lambda_n, \lambda_n\gamma)}{F(\lambda_{sn}, \lambda_{sn}\gamma)} \times \frac{K_1(\lambda_{sn}\gamma)}{K_1(\lambda_n\gamma)}$$

Substituting (51), (52) and (53) in (46) we have

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma \int_{\zeta=\gamma} p_2(\gamma, \xi)d\xi$$

$$+ \zeta[A_1 I_1(\lambda_n\zeta) - \frac{A_1 I_1(\lambda_n\gamma)}{K_1(\lambda_n\gamma)} \times K_1(\lambda_n\zeta)$$

$$- \frac{\lambda_n}{\lambda_{sn}} A_1 \frac{F(\lambda_n, \lambda_n\gamma)}{F(\lambda_{sn}, \lambda_{sn}\gamma)} \times \frac{K_1(\lambda_{sn}\gamma)}{K_1(\lambda_n\gamma)} \times I_1(\lambda_{sn}\zeta)$$

$$+ \frac{\lambda_n}{\lambda_{sn}} A_1 \frac{F(\lambda_n, \lambda_n\gamma)}{F(\lambda_{sn}, \lambda_{sn}\gamma)} \times \frac{K_1(\lambda_{sn}\gamma)}{K_1(\lambda_n\gamma)} \times \frac{I_1(\lambda_{sn}\gamma)}{K_1(\lambda_{sn}\gamma)} \times K_1(\lambda_{sn}\zeta)]$$

$$\times C_2 \sin \lambda_n\beta\xi$$

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma\int_{\zeta=\gamma} P_2(\gamma,\xi)d\xi$$

$$+ \zeta A_1\left[\frac{I_1(\lambda_n\zeta)K_1(\lambda_n\gamma) - I_1(\lambda_n\gamma)K_1(\lambda_n\zeta)}{K_1(\lambda_n\gamma)}\right.$$

$$\left. - \frac{\lambda_n}{\lambda_{sn}}\frac{F(\lambda_n,\lambda_n\gamma)}{F(\lambda_{sn},\lambda_{sn}\gamma)} x \frac{1}{K_1(\lambda_n\gamma)}\{K_1(\lambda_{sn}\gamma)I_1(\lambda_{sn}\zeta)\right.$$

$$\left. - I_1(\lambda_{sn}\gamma)K_1(\lambda_{sn}\zeta)\}\right]C_2 \sin\lambda_n\beta\xi$$

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma\int_{\zeta\ \gamma} P_2(\gamma,\xi)d\xi$$

$$+ \zeta A_1\left[\frac{G(\lambda_n\zeta,\lambda_n\gamma)}{K_1(\lambda_n\gamma)} - \frac{\lambda_n}{\lambda_{sn}}\frac{F(\lambda_n,\lambda_n\gamma)}{F(\lambda_{sn},\lambda_{sn}\gamma)} x \frac{1}{K_1(\lambda_n\gamma)}\right.$$

$$\left. x\ G(\lambda_{sn}\gamma)\right]\ x\ C_2 \sin\lambda_n\beta\xi\ x\ G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)]x\ C_2 \sin\lambda_n\beta\xi$$

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma\int_{\zeta=\gamma} P_2(\gamma,\xi)d\xi$$

$$+ \zeta A_1\left[\frac{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma) - \lambda_n F(\lambda_n,\lambda_n\gamma)G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)}{\lambda_{sn}F(\lambda_{sn},\lambda_n\gamma)K_1(\lambda_n\gamma)}\right]$$

$$x\ C_2 \sin\lambda_n\beta\xi$$

Since A_1, C_2 and $\lambda_{sn}F(\lambda_{sn},\lambda_n\gamma)K_1(\lambda_n\gamma)$ are all constants,

Let

$$\frac{A_1\ x\ C_2}{\lambda_{sn}F(\lambda_{sn},\lambda_n\gamma)x\ K_1(\lambda_n\gamma)} = B_n/\lambda_n$$

$$\therefore\ \psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma\int_{\zeta=\gamma} P_2(\gamma,\xi)d\xi$$

$$+ \zeta\sum_{n=1}^{\infty}\frac{B_n}{\lambda_n}\{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma) - \lambda_n F(\lambda_n,\lambda_n\gamma)$$

$$G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\}\ x\ \sin\lambda_n\beta\xi$$

$$\psi = -4\beta^2\kappa_\ell\pi_\ell\gamma\xi + 4\beta^2\kappa_\ell\gamma\int_{\zeta=\gamma} p_2(\gamma,\xi)d\xi$$

$$+ \zeta \sum_{n=1}^{\infty} \frac{B_n}{\lambda_n} g_n(\zeta) \sin \lambda_n\beta\xi \qquad (55)$$

where

$$g_n(\zeta) = \lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma)$$

$$- \lambda_n F(\lambda_n,\lambda_n\gamma)G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)$$

Now,

$$u_2 = -\frac{1}{\zeta}\frac{\partial\psi}{\partial\zeta}$$

therefore using (55),

$$u_2 = -\frac{1}{\zeta} \times \zeta \sum_{n=1}^{\infty} \frac{B_n}{\lambda_n} \{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)\lambda_n F(\lambda_n\zeta,\lambda_n\gamma)$$

$$- \lambda_n F(\lambda_n,\lambda_n\gamma)\lambda_{sn}F(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\} \sin \lambda_n\beta\xi$$

$$u_2 = -\sum_{n=1}^{\infty} B_n\lambda_{sn}\{F(\lambda_{sn},\lambda_{sn}\gamma)F(\lambda_n\zeta,\lambda_n\gamma)$$

$$- F(\lambda_n,\lambda_n\gamma)F(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\} \sin \lambda_n\beta\xi$$

where

$$\frac{d}{d\zeta}(\zeta G(\lambda_n\zeta,\lambda_n\gamma)) = \zeta\lambda_n F(\lambda_n\zeta,\lambda_n\gamma)$$

$$\therefore \quad u_2 = -\sum_{n=1}^{\infty} B_n f_n(\zeta) \sin \lambda_n\beta\xi \qquad (56)$$

where

$$f_n(\zeta) = \lambda_{sn}\{F(\lambda_{sn},\lambda_{sn}\gamma)F(\lambda_n\zeta,\lambda_n\gamma)$$

$$- F(\lambda_n,\lambda_n\gamma)F(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\}$$

Also we have,

$$v_2 = -\frac{1}{\beta\zeta}\frac{\partial\psi}{\partial\xi}$$

Thus using (55)

$$v_2 = \frac{1}{\beta\zeta}[-4\beta^2\kappa_{\ell}\pi_{\ell}\gamma + 4\beta^2\kappa_{\ell}\gamma\ P_2(\gamma,\xi)$$

$$+ \zeta \sum_{n=1}^{\infty} \frac{B_n}{\lambda_n} g_n(\zeta)\lambda_n\beta \cos \lambda_n\beta\xi]$$

$$\therefore v_2 = -\frac{4\beta\kappa_{\ell}\pi_{\ell}\gamma}{\zeta} + \frac{4\beta\kappa_{\ell}\gamma}{\zeta} P_2(\gamma,\xi)$$

$$+ \sum_{n=1}^{\infty} B_n g_n(\zeta) \cos \lambda_n\beta\xi \qquad (57)$$

we now have

$$v_2 = \frac{-4\beta\kappa_{\ell}\pi_{\ell}\gamma}{\zeta} + \frac{4\beta\kappa_{\ell}\gamma}{\zeta} P_2(\gamma,\xi)$$

$$+ \sum_{n=1}^{\infty} B_n\{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma)$$

$$- \lambda_n F(\lambda_n,\lambda_n\gamma)G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\} \cos \lambda_n\beta\xi$$

multiply by ζ

$$\zeta.v_2 = -4\beta\kappa_{\ell}\pi_{\ell}\gamma + 4\beta\kappa_{\ell}\gamma\ P_2(\gamma,\xi)$$

$$+ \zeta\sum_{n=1}^{\infty} B_n\{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma)$$

$$- \lambda_n F(\lambda_n,\lambda_n\gamma)G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\} \cos \lambda_n\beta\xi$$

differentiating,

$$v_2 + \zeta \frac{\partial v_2}{\partial\zeta} = \zeta \sum_{n=1}^{\infty} B_n\{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)\lambda_n F(\lambda_n\zeta,\lambda_n\gamma)$$

$$- \lambda_n F(\lambda_n,\lambda_n\gamma)\lambda_{sn}F(\lambda_{sn}\zeta,\lambda_{sn}\gamma) \cos \lambda_n\beta\xi$$

dividing by ζ,

$$\frac{v_2}{\zeta} + \frac{\partial \Psi_2}{\partial \zeta} = \sum_{n=1}^{\infty} B_n \{\lambda_{sn} F(\lambda_{sn}, \lambda_{sn}\gamma) \lambda_n F(\lambda_n \zeta, \lambda_n \gamma)$$

$$- \lambda_n F(\lambda_n, \lambda_n \gamma) \lambda_{sn} F(\lambda_{sn} \zeta, \lambda_{sn} \gamma)\} \cos \lambda_n \beta \xi$$

differentiating again,

$$\frac{1}{\zeta} \frac{\partial v_2}{\partial \zeta} - \frac{v_2}{\zeta^2} + \frac{\partial^2 v_2}{\partial \zeta^2} = \sum_{n=1}^{\infty} B_n \{\lambda_{sn} F(\lambda_{sn}, \lambda_{sn}\gamma) \lambda_n^2 G(\lambda_n \zeta, \lambda_n \gamma)$$

$$- \lambda_n F(\lambda_n, \lambda_n \gamma) \lambda_{sn}^2 G(\lambda_{sn} \zeta, \lambda_{sn} \gamma)\} \cos \lambda_n \beta \xi \qquad (58)$$

where

$$\frac{d}{d\zeta} (F(\lambda_n \zeta, \lambda_n \gamma)) = \lambda_n G(\lambda_n \zeta, \lambda_n \gamma)$$

Now

$$\frac{\partial v_2}{\partial \xi} = \frac{4\beta \kappa_\ell \gamma}{\zeta} p_2'(\gamma, \xi) - \sum_{n=1}^{\infty} B_n \{\lambda_{sn} F(\lambda_{sn}, \lambda_{sn}\gamma) G(\lambda_n \zeta, \lambda_n \gamma)$$

$$- \lambda_n F(\lambda_n, \lambda_n \gamma) G(\lambda_{sn} \zeta, \lambda_{sn} \gamma)\} \lambda_n \beta \sin \lambda_n \beta \xi$$

$$\frac{\partial^2 v_2}{\partial \xi^2} = \frac{4\beta \kappa_\ell \gamma}{\zeta} p_2''(\gamma, \xi)$$

$$- \sum_{n=1}^{\infty} B_n \{\lambda_{sn} F(\lambda_{sn}, \lambda_{sn}\gamma) G(\lambda_n \zeta, \lambda_n \gamma)$$

$$- \lambda_n F(\lambda_n, \lambda_n \gamma) G(\lambda_{sn} \zeta, \lambda_{sn} \gamma)\} \lambda_n^2 \beta^2 \cos \lambda_n \beta \xi$$

$$\frac{1}{\beta^2} \frac{\partial^2 v_2}{\partial \xi^2} = \frac{4\kappa_\ell \gamma}{\beta \zeta} p''(\gamma, \xi)$$

$$- \sum_{n=1}^{\infty} B_n \{\lambda_{sn} F(\lambda_{sn}, \lambda_{sn}\gamma) G(\lambda_n \zeta, \lambda_n \gamma)$$

$$- \lambda_n F(\lambda_n, \lambda_n \gamma) G(\lambda_{sn} \zeta, \lambda_{sn} \gamma)\} \lambda_n^2 \cos \lambda_n \beta \xi \qquad (59)$$

$$s^2 v_2 = \frac{-4\beta \kappa_\ell \pi_\ell \gamma s^2}{\zeta} + \frac{4\beta \kappa_\ell \gamma s^2}{\zeta} p_2(\gamma, \xi)$$

$$+ s^2 \sum_{n=1}^{\infty} B_n \{\lambda_{sn} F(\lambda_{sn}, \lambda_{sn}\gamma) G(\lambda_n \zeta, \lambda_n \gamma)$$

$$- \lambda_n F(\lambda_n, \lambda_n \gamma) G(\lambda_{sn} \zeta, \lambda_{sn} \gamma)\} \cos \lambda_n \beta \xi \qquad (60)$$

thus (57)+(58)-(59) gives,

$$\frac{1}{\zeta}\frac{\partial v_2}{\partial \zeta^2} - \frac{v_2}{\zeta^2} + \frac{\partial^2 v_2}{\partial \zeta^2} + \frac{1}{\beta^2}\frac{\partial^2 v_2}{\partial \xi^2} - s^2 v_2$$

$$= \sum_{n=1}^{\infty} B_n\{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)\lambda_n^2 G(\lambda_n\zeta,\lambda_n\gamma)$$

$$- \lambda_n F(\lambda_n,\lambda_n\gamma)\lambda_{sn}^2 G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\}\cos\lambda_n\beta\xi$$

$$+ \frac{4\kappa_\ell\gamma}{\beta\zeta}p_2^n(\gamma,\xi) - \sum_{n=1}^{\infty} B_n\{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma)$$

$$- \lambda_n F(\lambda_n,\lambda_n\gamma)G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\}\lambda_n^2\cos\lambda_n\beta\xi$$

$$+ \frac{4\beta\kappa_\ell\pi_\ell\gamma s^2}{\zeta} - \frac{4\beta\kappa_\ell\gamma s^2}{\zeta}p_2(\gamma,\xi) - s^2\sum_{n=1}^{\infty}B_n\{\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)$$

$$\times G(\lambda_n\zeta,\lambda_n\gamma) - \lambda_n F(\lambda_n,\lambda_n\gamma)G(\lambda_{sn}\zeta,\lambda_{sn}\gamma)\}\cos\lambda_n\beta\xi$$

from (20) we have,

$$\frac{4\beta\partial p_2}{\partial\zeta} = \frac{4\beta\kappa_\ell\pi_\ell\gamma s^2}{\zeta} - \frac{4\beta\kappa_\ell\pi_\ell\gamma s^2}{\zeta}p_2(\gamma,\xi) + \frac{4\kappa_\ell\gamma}{\beta\zeta}p_2''(\gamma,\xi)$$

$$- s^2\sum_{n=1}^{\infty} B_n\lambda_{sn}F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma)\cos\lambda_n\beta\xi$$

integrating,

$$p_2 = \kappa_\ell\pi_\ell\gamma s^2\ln\zeta - \kappa_\ell\gamma s^2\ln\zeta\, p_2(\gamma,\xi)$$

$$+ \frac{\kappa_\ell\gamma}{\beta^2}p_2''(\gamma,\xi)\ln\zeta$$

$$- \frac{s^2}{4\beta}\sum_{n=1}^{\infty} B_n\frac{\lambda_{sn}}{\lambda_n}F(\lambda_{sn},\lambda_{sn}\gamma)F(\lambda_n\zeta,\lambda_n\gamma)\cos\lambda_n\beta\xi + C$$

where

$$\int I_1(\lambda_n\zeta)d\zeta = \frac{1}{\lambda_n}I_0(\lambda_n\zeta),\quad \int K_1(\lambda_n\zeta)d\zeta = -\frac{1}{\lambda_n}K_0(\lambda_n\zeta)$$

Let $C = p_o$, thus

$$p_2(\zeta,\xi) = p_o + \gamma\kappa_\ell s^2[\pi_\ell - p_2(\gamma,\xi) + \frac{1}{\beta^2 s^2} p_2''(\gamma,\xi)]\ln\zeta$$

$$- \frac{s^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(\zeta) \cos \lambda_n \beta\xi \qquad (61)$$

where

$$k_n(\zeta) = (\lambda_{sn}/\lambda_n) F(\lambda_{sn}, \lambda_{sn}\gamma) F(\lambda_n\zeta, \lambda_n\gamma)$$

In (60), the order of $\frac{1}{\beta^2 s^2} = 0\ (\frac{1}{10^4} \cdot \frac{1}{10^6}) = 0(\frac{1}{10^{10}})$ which is very small compared with π_ℓ which has an order of 1. The orders of β and s are 10^2 and 10^3 respectively. We can therefore neglect $\frac{1}{\beta^2 s^2} p_2''(\gamma,\xi)$ in comparison to π_ℓ. Then we have,

$$p_2(\zeta,\xi) = p_o + \gamma\kappa_\ell s^2[\pi_\ell - p_2(\gamma,\xi)]\ln\zeta$$

$$- \frac{s^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(\zeta) \cos \lambda_n \beta\xi \qquad (62)$$

$$p_2(\gamma,\xi) = p_o + \gamma\kappa_\ell s^2[\pi_\ell - p_2(\gamma,\xi)]\ln\gamma$$

$$- \frac{s^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta\xi$$

Let $a = \gamma\kappa_\ell \ln\gamma$

thus

$$p_2(\gamma,\xi)(1+as^2) = p_o + as^2\pi_\ell - \frac{s^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta\xi$$

$$\therefore\ p_2(\gamma,\xi) = \frac{p_o + as^2\pi_\ell}{1 + as^2} - \frac{s^2}{4\beta(1+as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta\xi \qquad (63)$$

substituting in (62), we have

$$p_2(\zeta,\xi) = p_o + \gamma\kappa_\ell s^2[\pi_\ell - \frac{p_o + as^2\pi_\ell}{1 + as^2}$$

$$+ \frac{s^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta\xi]\ln\zeta$$

$$- \frac{s^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(\zeta) \cos \lambda_n \beta\xi \qquad (64)$$

subrtituting (62) in (57), we have

$$v_2 = \frac{-4\beta\kappa_\ell \pi_\ell \gamma}{\zeta} + \frac{4\beta\kappa_\ell \gamma}{\zeta} [\frac{p_o + as^2\pi_\ell}{1 + as^2} - \frac{s^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n$$

$$+ \sum_{n=1}^{\infty} B_n g_n(\zeta) \cos \lambda_n \beta\xi \qquad (65)$$

$$p_o = \frac{1}{[1 + \theta + \omega]} x[C_o(1 + \theta) + \sigma_o \pi_e \{(1 + \theta) - \delta^*(\frac{3+4\theta}{6})\}$$

$$+ C_1(\frac{3 + 4\theta}{6}) - 4\beta^2 \kappa\omega \frac{2 + 3\theta}{3} \pi_\ell + \pi_\ell \omega], \text{ ------(82)}$$

where

$$\frac{\gamma V_o}{\kappa\beta} = \omega = \frac{\gamma \kappa_\ell}{\kappa(1 + as^2)}$$

To obtain B_n, we have

$$B_n g_n(1) = 2 \int_o^1 \{4\kappa\beta[C_o - p_o + \sigma_o(1 - \delta^*\xi)\pi_e$$

$$+ C_1\xi + 8\beta\gamma V_o\{p_o - \pi_\ell\}\xi^2 + 2\theta C_o\xi - 2\theta p_o\xi$$

$$+ 2\theta\sigma_o(1 - \delta^*\xi)\pi_e\xi + 2\theta C_1\xi^2 + 16\beta\gamma V_o\theta\{p_o - \pi_\ell\}\xi^3]$$

$$- 4\gamma V_o\{p_o - \pi_\ell\}] \cos \lambda_n\beta\xi d\xi$$

integrating,

$$B_n g_n(1) = 2\{4\kappa\beta C_o \sin \frac{\lambda_n\beta}{\lambda_n\beta} \xi \Big|_o^1 - 4\kappa\beta p_o \sin \frac{\lambda_n\beta}{\lambda_n\beta} \xi \Big|_o^1$$

$$+ 4\kappa\beta\sigma_o\pi_e \sin \frac{\lambda_n\beta}{\lambda_n\beta} \xi \Big|_o^1 - 4\kappa\beta\sigma_o\delta^*\pi_e\{\xi \sin \frac{\lambda_n\beta}{\lambda_n\beta} \xi \Big|_o^1$$

$$+ \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2} \xi \Big|_o^1\} + 4\kappa\beta C_1\{\xi \sin \frac{\lambda_n\beta}{\lambda_n\beta} \xi \Big|_o^1 + \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2} \xi \Big|_o^1\}$$

$$+ 32\kappa\beta^2\gamma V_o\{p_o - \pi_\ell\}\{\xi^2 \sin \frac{\lambda_n\beta}{\lambda_n\beta} \xi \Big|_o^1 + 2\xi \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2} \xi \Big|_o^1$$

Thus the solutions of equations (3) and (4) which satisfy the boundary conditions (2.1), (2.2), (2.4) and (2.5), are given as follows:

$$u_2 = - \sum_{n=1}^{\infty} B_n f_n(\zeta) \sin \lambda_n \beta \xi \qquad \text{----------(66)}$$

$$v_2 = - \frac{4\beta\kappa_\ell \pi_\ell \gamma}{\zeta} + \frac{4\beta\kappa_\ell \gamma}{\zeta} \left[\frac{P_o + as^2 \pi_\ell}{1 + as^2}\right.$$

$$\left. - \frac{S^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta \xi\right]$$

$$+ \sum_{n=1}^{\infty} B_n g_n(\zeta) \cos \lambda_n \beta \xi \qquad \text{----------(67)}$$

$$p_2 = p_o + \gamma\kappa_\ell s^2 \left[\frac{\pi_\ell - p_o}{1 + as^2}\right.$$

$$\left. + \frac{S^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta \xi\right] \ln \zeta$$

$$- \frac{S^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(\zeta) \cos \lambda_n \beta \xi \qquad \text{----------(68)}$$

with

$$f_n(\zeta) = \lambda_{sn}\{F(\lambda_{sn}, \lambda sn\gamma) F(\lambda_n \zeta, \lambda_n \gamma)$$

$$- F(\lambda_n, \lambda_n \gamma) (\lambda_{sn} \zeta, \lambda_{sn} \gamma)\}, \qquad \text{----------(69)}$$

$$g_n(\zeta) = \lambda_{sn} F(\lambda_{sn}, \lambda_{sn}\gamma) G(\lambda_n \zeta, \lambda_n \gamma)$$

$$- \lambda_n F(\lambda_n, \lambda_n \gamma) G(\lambda_{sn} \zeta, \lambda_{sn} \gamma), \qquad \text{----------(70)}$$

$$k_n(\zeta) = (\lambda_{sn}/\lambda_n) F(\lambda_{sn}, \lambda_{sn}\gamma) F(\lambda_n \zeta, \lambda_n \gamma), \qquad \text{----(71)}$$

$$\lambda_n = n\pi/\beta \quad (n = 1,2,3,\text{----}), \qquad \text{---------(72)}$$

$$\lambda_{sn} = (\lambda_n^2 + s^2)^{\frac{1}{2}}, \qquad \text{----------(73)}$$

$$F(x,y) = I_o(x)K_1(y) + K_o(x)I_1(y), \text{-------(74)}$$

$$G(x,y) = I_1(x)K_1(y) - K_1(x)I_1(y), \text{-------(75)}$$

Where $I_i(x)$ $K_i(x)$ are the modified Bessel functions of order i, and p_o and the B_n's are arbitrarily set constants. The nondimensional parameters used in equations (65), (66) and (67) are defined as follows:

$$\gamma = \frac{R_t}{R_c}, \quad s = \sigma R_c, \text{ ----------(76)}$$

To determine the constants p_o and B_n and the function $p_1(\xi)$, we use boundary conditions (1.3) and (2.3) with equations (5), (16), (67) and (68). Boundary conditions (1.3) and (2.3) are

$$V_1 = k(P_1 - P_2 - \sigma(z)\pi_e) \text{ at } r = R_c \quad (1.3)$$

$$V_2 = k(P_1 - P_2 - \sigma(z)\pi_e) \text{ at } r = R_c \quad (2.3)$$

Now we shall use $V_1 = V_2$ at $r = R_c$.

In nondimensionalized form we have

$$v_1 = v_2 \text{ at } \zeta = 1$$

i.e.

$$\frac{1}{4\beta}(2\zeta - \zeta^3)\frac{d^2p_1}{d\xi^2} = \frac{-4\beta\kappa_\ell\pi_\ell\gamma}{\zeta} + \frac{4\beta\kappa_\ell\gamma}{\zeta}\left[\frac{p_o + a\,\pi_\ell s^2}{1 + as^2}\right.$$

$$\left. - \frac{s^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma)\cos\lambda_n\beta\xi\right]$$

$$+ \sum_{n=1}^{\infty} B_n g_n(\zeta)\cos\lambda_n\beta\xi \quad \text{at } \zeta = 1$$

Thus,

$$\frac{1}{4\beta}\frac{d^2p_1}{d\xi^2} = -4\beta\kappa_\ell\pi_\ell\gamma + 4\beta\kappa_\ell\gamma\left[\frac{p_o + a\pi_\ell s^2}{1 + as^2}\right]$$

$$- \frac{\kappa_\ell s^2\gamma}{(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma)\cos \lambda n\beta\xi + \sum_{n=1}^{\infty} B_n g_n(1)\cos \lambda_n\beta\xi$$

$$\frac{d^2p_1}{d\xi^2} = 4\beta\left[\frac{4\beta\kappa_\ell\gamma}{1 + as^2}\{p_o - \pi_\ell\} + \sum_{n=1}^{\infty} B_n\{g_n(1)\right.$$

$$\left. - \frac{\kappa_\ell s^2\gamma}{1 + as^2} k_n(\gamma)\} \cos \lambda_n\beta\xi\right], \text{ ----------(77)}$$

In nondimensionalized form, boundary condition (2.3) is given by

$$V_2 = 4\kappa\beta(1 + 2\theta\xi)[p_1 - p_2 + \sigma_o(1 - \delta^*\xi)\pi_e] \text{ at } \zeta=1.$$

$$p_2(\zeta = 1,\xi) = p_o - \frac{s^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(1)\cos \lambda_n\beta\xi$$

thus we have,

$$\frac{4\beta\kappa_\ell\gamma}{1 + as^2}\{p_o - \pi_\ell\} - \frac{\kappa_\ell\gamma s^2}{1+as^2} \sum_{n=1}^{\infty} B_n k_n(\gamma)\cos \lambda_n\beta\xi$$

$$+ \sum_{n=1}^{\infty} B_n g_n(1)\cos \lambda_n\beta\xi$$

$$= 4\kappa\beta(1 + 2\theta\xi)[p_1 - p_o + \frac{s^2}{4\beta} \sum_{n=1}^{\infty} B_n k_n(1)\cos \lambda_n\beta\xi$$

$$+ \sigma_o\{1 - \delta^*\xi)\pi_e]$$

rearranging, we obtain

$$4\kappa\beta(1 + 2\theta\xi)[p_1 - p_o + \sigma_o(1 - \delta^*\xi)\pi_e]$$

$$- \frac{4\beta\kappa_\ell\gamma}{1 + as^2}\{p_o - \pi_\ell\} = \sum_{n=1}^{\infty} B_n\{g_n(1) - \kappa s^2(1+2\theta\xi)k_n(1)$$

$$- \frac{\kappa_\ell\gamma s^2}{1 + as^2} k_n(\gamma)\} \cos \lambda_n\beta\xi, \quad \text{----------(78)}$$

integrating (77),

$$\frac{dp_1}{d\xi} = 4\beta[\frac{4\beta\kappa_\ell\gamma}{1 + as^2}\{p_o - \pi_\ell\}\xi + \sum_{n=1}^{\infty} B \frac{1}{\lambda_n\beta}$$

$$\{g_n(1) - \frac{\kappa_\ell s^2\gamma}{1 + as^2} k_n(\gamma)\}x \sin \lambda_n\beta\xi] + C_1$$

integrating again,

$$p_1 = 4\beta[\frac{4\beta\kappa_\ell\gamma}{1 + as^2}\{p_o - \pi_\ell\}\frac{\xi^2}{2}$$

$$- \sum_{n=1}^{\infty} B_n \frac{1}{\lambda_n^2\beta^2}\{g_n(1) - \frac{\kappa_\ell s^2\gamma}{1 + as^2} k_n(\gamma)\}$$

$$x \cos \lambda_n\beta\xi] + C_1 \xi + C_o$$

Thus we have

$$p_1 = C_o + C_1 \xi + \frac{8\kappa_\ell\beta^2\gamma}{1 + as^2}\{p_o - \pi_\ell\}\xi^2$$

$$- \frac{4}{\beta} \sum_{n=1}^{\infty} B_n \frac{1}{\lambda_n^2}\{g_n(1) - \frac{\kappa_\ell s^2\gamma}{1 + as^2} k_n(\gamma)\}\cos \lambda_n\beta\xi, \quad \text{--(79)}$$

where C_o and C_1 are arbitrarily set constants.

Substituting (79) into (78),

$$4\kappa\beta(1 + 2\theta\xi)[C_o + C_1\xi + \frac{8\kappa_\ell\beta^2\gamma}{1 + as^2}\{p_o - \pi_\ell\}\xi^2$$

$$- \frac{4}{\beta}\sum_{n=1}^{\infty} B_n \frac{1}{\lambda_n^2}\{g_n(1) - \frac{\kappa_\ell s^2\gamma}{1 + as^2} k_n(\gamma)\}\cos \lambda_n\beta\xi - p_o$$

$$+ \sigma_o(1 - \delta^*\xi)\pi_e] - \frac{4\beta\kappa_\ell\gamma}{1 + as^2}\{p_o - \pi_\ell\}$$

$$= \sum_{n=1}^{\infty} B_n\{g_n(1) - \kappa s^2(1 - 2\theta\xi)k_n(1) - \frac{\kappa_\ell s^2}{1+as^2} k_n(\gamma)\}\cos \lambda_n\beta\xi$$

$$\Leftrightarrow 4\kappa\beta(1 + 2\theta\xi)[C_o - p_o + \sigma_o(1 - \delta^*\xi)\pi_e$$

$$+ C_1\xi + \frac{8\kappa_\ell\beta^2\gamma}{1 + as^2}\{p_o - \pi_\ell\}\xi^2] - \frac{4\kappa_\ell\beta\gamma}{1 + as^2}\{p_o - \pi_\ell\}$$

$$= \sum_{n=1}^{\infty} B_n\{g_n(1)[1 + \frac{16\kappa}{\lambda_n^2}(1 + 2\theta\xi)] - \kappa s^2(1+2\theta\xi)k_n(1)$$

$$\pm - \frac{\kappa_\ell\gamma s^2}{1 + as^2} k_n(\gamma)[1 + \frac{16\kappa}{\lambda_n^2}(1 + 2\theta\xi)]\}$$

$$\times \cos \lambda_n\beta\xi, \text{ ----------(80)}$$

The orders of B_n and κ_ℓ may be estimated as κ, thus neglecting powers of order of κ^2, we obtain

$$\sum_{n=1}^{\infty} B_n g_n(1) \cos \lambda_n\beta\xi = 4\kappa\beta(1 + 2\theta\xi)$$

$$[C_o - p_o + \sigma_o(1 - \delta^*\xi)\pi_e + C_1\xi + \frac{8\kappa_\ell\beta^2\gamma}{1 + as^2}$$

$$\{p_o - \pi_\ell\}\xi^2] - \frac{4\kappa_\ell\beta\gamma}{1 + as^2}\{p_o - \pi_\ell\}, \text{-----(81)}$$

Expanding the right-hand side of equation (81) in the Fourier cosine series, we have

$$f(x) = a_o + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_o = \frac{1}{L} \int_o^L f(x)dx$$

$$a_n = \frac{2}{L} \int_o^L f(x)\cos nxdx$$

In our case, let $\frac{\kappa_\ell \beta}{1 + as^2} = V_o$ then,

$$a_o = \int_o^1 \{4\kappa\beta(1 + 2\theta\xi)[C_o - p_o + \sigma_o(1 - \delta^*\xi)\pi_e + C_1\xi$$

$$+ 8\beta\gamma V_o\{p_o - \pi_\ell\}\xi^2] - 4\gamma V_o\{p_o - \pi_\ell\}\}d\xi$$

$$= \int_o^1 \{4\kappa\beta[C_o - p_o + \sigma_o(1 - \delta^*\xi)\pi_e + C_1\xi$$

$$+ 8\beta\gamma V_o\{p_o - \pi_\ell\}\xi^2 + 2\theta C_o\xi - 2\theta p_o\xi + 2\theta\sigma_o(1 - \delta^*\xi)\xi\pi_e + 2\theta C_1\xi^2$$

$$+ 16\beta\gamma V_o\theta\{p_o - \pi_\ell\}\xi^3] - 4\gamma V_o\{p_o - \pi_\ell\}\}d\xi$$

integrating,

$$a_o = 4\kappa\beta C_o\xi \Big|_o^1 - 4\kappa\beta p_o \Big|_o^1 + 4\kappa\beta\sigma_o(\xi - \delta^*\frac{\xi^2}{2})\pi_e \Big|_o^1$$

$$+ 4\kappa\beta C_1 \frac{\xi^2}{2} \Big|_o^1 + 32\kappa\beta^2\gamma V_o\{p_o - \pi_\ell\} \frac{\xi^3}{3} \Big|_o^1 + 8\kappa\beta\theta C_o \frac{\xi^2}{2} \Big|_o^1$$

$$- 8\kappa\beta\theta p_o \frac{\xi^2}{2} \Big|_o^1 + 8\kappa\beta\theta\sigma_o(\frac{\xi^2}{2} - \delta^*\frac{\xi^3}{3})\pi_e \Big|_o^1 + 8\kappa\beta\theta C_1 \frac{\xi^3}{3} \Big|_o^1$$

$$+ 64\kappa\beta^2\gamma V_o\theta\{p_o - \pi_\ell\} \frac{\xi^4}{4} \Big|_o^1 - 4\gamma V_o\{p_o - \pi_\ell\}\xi \Big|_o^1$$

$$a_o = 4\kappa\beta C_o - 4\kappa\beta p_o + 4\kappa\beta\sigma_o(1 - \frac{\delta^*}{2})\pi_e + 2\kappa\beta C_1$$

$$+ \frac{32}{3}\kappa\beta^2\gamma V_o\{p_o - \pi_\ell\} + 4\kappa\beta\theta C_o - 4\kappa\beta\theta p_o + 8\kappa\beta\sigma_o(\frac{1}{2} - \frac{\delta^*}{3})$$

$$+ \frac{8}{3}\kappa\beta^2\theta C_1 + 16\kappa\beta^2\theta\gamma V_o\{p_o - \pi_\ell\} - 4\gamma V_o\{p_o - \pi_\ell\}$$

divided by $4\kappa\beta$

$$\therefore a_o = C_o(1+\theta) - p_o(1+\theta) + \sigma_o\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\}$$

$$+ C_1\frac{(3+4\theta)}{6} + 8\beta\gamma V_o\frac{2+3\theta}{6}\{p_o - \pi_\ell\} - \frac{\gamma V_o}{\kappa\beta}\{p_o - \pi_\ell\}$$

$$= C_o(1+\theta) - p_o[(1+\theta) - 8\beta\gamma V_o(\frac{2+3\theta}{6}) + \frac{\gamma V_o}{\kappa\beta}] + C_1(\frac{3+4\theta}{6})$$

$$+ \sigma_o[(1+\theta) - \delta^*(\frac{3+4\theta}{6})]\pi_e - \pi_\ell[8\beta\gamma V_o(\frac{2+3\theta}{6}) - \frac{\gamma V_o}{\kappa\beta}]$$

but $\frac{1}{2}\, a_o = 0$ thus,

$$p_o[(1+\theta) - 8\beta\gamma V_o(\frac{2+3\theta}{6}) + \frac{\gamma V_o}{\kappa\beta}] = C_o(1+\theta)$$

$$+ \sigma_o\pi_e\{(1+\theta) - \delta^*(\frac{3+4\theta}{6})\} + C_1(\frac{3+4\theta}{6})$$

$$- \pi_\ell[8\beta\gamma V_o(\frac{2+3\theta}{6}) - \frac{\gamma V_o}{\kappa\beta}]$$

thus,

$$p_o = \frac{1}{[1+\theta - 8\beta\gamma V_o(\frac{2+3\theta}{6}) + \frac{\gamma V_o}{\kappa\beta}}$$

$$\times [C_o(1+\theta) + \sigma_o\pi_e\{(1+\theta) - \delta^*(\frac{3+4\theta}{6})\}$$

$$+ C_1(\frac{3+4\theta}{6}) - \pi_\ell\{8\beta\gamma V_o(\frac{2+3\theta}{6}) - \frac{\gamma V_o}{\kappa\beta}\}]$$

The order of $\beta\gamma V_o$ is $0(10^{-2})$ which can be neglected in comparison to $\gamma V_o/\kappa\beta$ which is of order $0(10)$ and is also neglected in comparison to 1.

Then we can write,

$$- 2 \sin \frac{\lambda_n\beta}{\lambda_n^3\beta^3}\xi \Big|_0^1 + 8\kappa\beta\theta C_o\{\xi \sin \frac{\lambda_n\beta}{\lambda_n\beta}\xi \Big|_0^1 + \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2}\xi \Big|_0^1\}$$

$$- 8\kappa\beta\theta p_o\{\xi \sin \frac{\lambda_n\beta}{\lambda_n\beta}\xi \Big|_0^1 + \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2}\xi \Big|_0^1\}$$

$$+ 8\kappa\beta\theta\sigma_o\pi_e\{\xi \sin \frac{\lambda_n\beta}{\lambda_n\beta}\xi \Big|_0^1 + \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2}\xi \Big|_0^1\}$$

$$- 8\kappa\beta\theta\sigma_o\delta^*\pi_e\{\xi^2 \sin \frac{\lambda_n\beta}{\lambda_n\beta}\xi \Big|_0^1 + 2\xi \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2}\xi \Big|_0^1$$

$$- 2 \sin \frac{\lambda_n\beta}{\lambda_n^3\beta^3}\xi \Big|_0^1\} + 8\kappa\beta\theta C_1\{\xi^2 \sin \frac{\lambda_n\beta}{\lambda_n\beta}\xi \Big|_0^1$$

$$+ 2\xi \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2}\xi \Big|_0^1 - 2 \sin \frac{\lambda_n\beta}{\lambda_n^3\beta^3}\xi \Big|_0^1\} - 4\gamma V_o\{p_o - \pi_\ell\}$$

$$\sin \frac{\lambda_n\beta}{\lambda_n\beta}\xi \Big|_0^1 + 64\kappa\beta^2\gamma V_o\theta\{p_o - \pi_\ell\}\{\xi^3 \sin \frac{\lambda_n\beta}{\lambda_n\beta}\xi \Big|_0^1$$

$$+ 3\xi^2 \cos \frac{\lambda_n\beta}{\lambda_n^2\beta^2}\xi \Big|_0^1 - 6\xi \sin \frac{\lambda_n\beta}{\lambda_n^3\beta^3}\xi \Big|_0^1 - 6 \cos \frac{\lambda_n\beta}{\lambda_n^4\beta^4}\xi \Big|_0^1]$$

$$B_n g_n(1) = 2[- 4\kappa\beta\sigma_o\delta^*\pi_e \frac{(-1)^n - 1}{\lambda_n^2\beta^2} + 4\kappa\beta C_1 \frac{(-1)^n - 1}{\lambda_n^2\beta^2}$$

$$+ 64\kappa\beta^2\gamma V_o \frac{(-1)^n}{\lambda_n^2\beta^2}\{p_o - \pi_\ell\} + 8\kappa\beta\theta C_o \frac{(-1)^n - 1}{\lambda_n^2\beta^2}$$

$$- 8\kappa\beta\theta p_o \frac{(-1)^n - 1}{\lambda_n^2\beta^2} + 8\kappa\beta\theta\sigma_o\pi_e \frac{(-1)^n - 1}{\lambda_n^2\beta^2}$$

$$- 16\kappa\beta\theta\sigma_o\pi_e\delta^* \frac{(-1)^n}{\lambda_n^2\beta^2} + 16\kappa\beta\theta C_1 \frac{(-1)^n}{\lambda_n^2\beta^2}$$

$$+ 192\kappa\beta^2\gamma V_o\theta\{p_o - \pi_\ell\}\frac{(-1)^n}{\lambda_n^2\beta^2} - 64 \times 6\kappa\beta^2\gamma V_o\theta\{p_o-\pi_\ell\}\frac{(-1)^n}{\lambda_n^4}$$

$$= 2[8\kappa\beta\sigma_o\pi_e\{[\theta - \frac{\delta^*}{2}]\frac{(-1)^n - 1}{\lambda_n^2\beta^2} - 2\theta\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2}\}$$

$$+ 4\kappa\beta C_1\frac{(-1)^n - 1}{\lambda_n^2\beta^2} + 16\kappa\beta\theta C_1\frac{(-1)^n}{\lambda_n^2\beta^2} + 8\kappa\beta\theta C_o\frac{(-1)^n - 1}{\lambda_n^2\beta^2}$$

$$- 8\kappa\beta\theta p_o\frac{(-1)^n - 1}{\lambda_n^2\beta^2} + 64\kappa\beta^2\gamma V_o\frac{(-1)^n}{\lambda_n^2\beta^2}\{p_o - \pi_\ell\}$$

$$+ 192\kappa\beta^2\gamma V_o\theta\{p_o - \pi_\ell\}\frac{(-1)^n}{\lambda_n^2\beta^2}$$

$$- 64 \times 6\kappa\beta^2\gamma V_o\theta\{p_o - \pi_\ell\}\frac{(-1)^n - 1}{\lambda_n^4\beta^4}]$$

Substituting for p_o

$$B_n\mathcal{G}_n(1) = 2[8\kappa\beta\sigma_o\pi_e\{[\theta - \frac{\delta^*}{2}]\frac{(-1)^n - 1}{\lambda_n^2\beta^2} - 2\theta\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2}\}$$

$$+ 4\kappa\beta C_1\frac{(-1)^n - 1}{\lambda_n^2\beta^2} + 16\kappa\beta\theta C_1\frac{(-1)^n}{\lambda_n^2\beta^2} + 8\kappa\beta\theta C_o\frac{(-1)^n - 1}{\lambda_n^2\beta^2}$$

$$- 8\kappa\beta\theta\{C_o\frac{(1 + \theta)}{1 + \theta + \omega} + \frac{\sigma_o\pi_e}{1 + \theta + \omega}\{(1 + \theta) - \delta^*\frac{(3 + 4\theta)}{6}\}$$

$$+ C_1\frac{(3 + 4\theta)}{6(1 + \theta + \omega)} - 4\kappa\beta^2\omega\frac{(2 + 3\theta)\pi_\ell}{3(1 + \theta + \omega)}$$

$$+ \frac{\pi_\ell\omega}{1 + \theta + \omega}\}\frac{(-1)^n - 1}{\lambda_n^2\beta^2}$$

$$- 64\kappa\beta^2\gamma V_o\frac{(-1)^n\pi_\ell}{\lambda_n^2\beta^2} + 64\kappa\beta^2\gamma V_o\{C_o\frac{(1 +\theta)}{1 + \theta + \omega}$$

$$+ \frac{\sigma_o\pi_e}{1 + \theta + \omega}\{(1 - \theta) - \delta^*\frac{(3 - 4\theta)}{6}\} + C_1\frac{(3 + 4\theta)}{6(1 + \theta + \omega)}$$

$$- 4\kappa\beta^2\omega\frac{(2+3\theta)}{3(1+\theta+\omega)}\pi_\ell + \frac{\pi_\ell\omega}{1+\theta+\omega}\}\frac{(-1)^n}{\lambda_n^2\beta^2}$$

$$+ 192\kappa\beta^2\theta\{C_o\frac{(1+\theta)}{1+\theta+\omega} + \frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta)-\delta^*\frac{(3+4\theta)}{6}\}$$

$$+ C_1\frac{(3+4\theta)}{6(1+\theta+\omega)} - 4\kappa\beta^2\omega\frac{(2+3\theta)}{3(1+\theta+\omega)}\pi_\ell$$

$$+ \frac{\pi_\ell\omega}{1+\theta+\omega}\}\frac{(-1)^n}{\lambda_n^2\beta^2} - 192\kappa\beta^2\gamma V_o\theta\frac{(-1)^n}{\lambda_n^2\beta^2}\pi_\ell$$

$$+ 64 \times 6\kappa\beta^2\gamma V_o\theta\frac{(-1)^n-1}{\lambda_n^4\beta^4}\pi_\ell - 64 \times 6\kappa\beta^2\gamma V_o\theta\{C_o\frac{(1+\theta)}{1+\theta+\omega}$$

$$+ \sigma_o\pi_e\{(1+\theta)-\delta^*\frac{(3+4\theta)}{6}\} + C_1\frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$- 4\kappa\beta^2\omega\frac{(2+3\theta)}{3(1+\theta+\omega)}\pi_\ell + \frac{\pi_\ell\omega}{1+\theta+\omega}\}\frac{(-1)^n-1}{\lambda_n^4\beta^4}]$$

Neglecting terms of κ^2, i.e., terms with $\kappa\beta^2\gamma V_o$, we have

$$B_n g_n(1) = 2[8\kappa\beta\sigma_o\pi_e(\{\theta-\frac{\delta^*}{2}\}\frac{(-1)^n-1}{\lambda_n^2\beta^2} - 2\theta\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2})$$

$$- 8\kappa\beta\frac{\sigma_o\pi_e\theta}{1+\theta+\omega}\{(1+\theta)-\delta^*\frac{(3+4\theta)}{6}\}\frac{(-1)^n-1}{\lambda_n^2\beta^2}$$

$$+ 8\kappa\beta C_o\frac{\theta\omega}{1+\theta+\omega}\frac{(-1)^n-1}{\lambda_n^2\beta^2} - 8\kappa\beta\pi_\ell\frac{\theta\omega}{1+\theta+\omega}\frac{(-1)^n-1}{\lambda_n^2\beta^2}$$

$$+ 8\kappa\beta\{\frac{3-4\theta^2+3\omega}{3(1+\theta+\omega)}\frac{(-1)^n-1}{2} + 2\theta(-1)^n\}\frac{C_1}{\lambda_n^2\beta^2}]$$

Hence,

$$B_n = 16\kappa\beta\{\frac{3-4\theta^2+3\omega}{3(1+\theta+\omega)}\frac{(-1)^n-1}{2} + 2\theta(-1)^n\}\frac{C_1}{\lambda_n^2\beta^2 g_n(1)}$$

$$+ 16\kappa\beta \frac{\theta\omega}{1 + \theta + \omega}\{C_o - \pi_\ell\}\frac{(-1)^n - 1}{\lambda_n^2\beta^2 g_n(1)}$$

$$+ 16\kappa\beta\sigma_o\pi_e[\{\frac{\theta\omega}{1 + \theta + \omega} - \delta^*\frac{(3 - 4\theta^2 + 3\omega)}{6(1 + \theta + \omega)}\}\frac{(-1)^n - 1}{\lambda_n^2\beta^2 g_n(1)}$$

$$- 20\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2 g_n(1)}], \qquad \text{----------(83)}$$

From equation (79),

$$p_1 = C_o + C_1\xi + \frac{8\kappa_\ell\beta^2\gamma}{1 + as^2}\{p_o - \pi_\ell\}\xi^2 - \frac{4}{\beta}\sum_{n=1}^{\infty} B_n \frac{1}{\lambda^2}\{g_n$$

$$- \frac{\kappa_\ell s^2\gamma}{1 + as^2} k_n(\gamma)\} \cos \lambda_n\beta\xi$$

B_n and κ_ℓ have the same power of κ, thus neglecting powers of κ^2, and let $\kappa_\ell\beta/1 + as^2 = V_o$, we obtain

$$p_1 = C_o + C_1\xi + 8\beta\gamma V_o\{p_o - \pi_\ell\}\xi^2$$

$$- \frac{4}{\beta}\sum_{n=1}^{\infty} B_n \frac{g_n(1)}{\lambda_n^2} \cos \lambda_n\beta\xi, \qquad \text{----------(84)}$$

We shall now subtaitute equations (82) and (83) into boundary conditions (1.4) and (1.5)

$p_1 = p_a$ at $z = 0$ (1.4)

$p_1 = p_v$ at $z = L$ (1.5)

In nondimensionalized form, we have

$p_1 = p_a$ at $\xi = 0$ (1.4)

$p_1 = (p_a - 1)$ at $\xi = 1$ (1.5)

Substituting the equations into the boundry conditions we obtain,

$$p_a = C_o - \frac{4}{\beta} \sum_{n=1}^{\infty} B_n \frac{g_n(1)}{\lambda_n^2}, \qquad \text{---------- (85)}$$

$$(p_a - 1) = C_o + C_1 + 8\beta\gamma V_o\{p_o - \pi_\ell\}$$

$$- \frac{4}{\beta} \sum_{n=1}^{\infty} B_n \frac{g_n(1)}{\lambda_n^2} (-1)^n, \qquad \text{---------- (86)}$$

We shall use the following Cosine Fourier Series expansion

$$\xi = \frac{1}{2} + 2 \sum \frac{1}{n^2\pi^2}[(-1)^n - 1] \cos n\pi\xi$$

$$\xi^2 = \frac{1}{3} + 4 \sum \frac{1}{n^2\pi^2} (-1)^n \cos n\pi\xi$$

$$\xi^3 = \frac{1}{4} + 6 \sum [\frac{1}{n^2\pi^2} (-1)^n - 2\frac{(-1)^n - 1}{n^4\pi^4}] \cos n\pi\xi$$

$$\xi^4 = \frac{1}{5} + 8 \sum [\frac{1}{n^2\pi^2} (-1)^n - \frac{6}{n^4\pi^4} (-1)^n] \cos n\pi\xi$$

which gives

$$\sum \frac{1}{n^2\pi^2}[(-1)^n - 1] \cos n\pi\xi = \frac{1}{2}(\xi - \frac{1}{2})$$

$$\sum \frac{1}{n^2\pi^2} (-1)^n \cos n\pi\xi = \frac{1}{4}(\xi^2 - \frac{1}{3})$$

$$\sum \frac{1}{n^2\pi^2} \cos n\pi\xi = \frac{1}{4}(\xi^2 - 2\xi + \frac{2}{3})$$

for $\xi = 0$, it reduces to $\sum \frac{1}{n^2\pi^2} = \frac{1}{6}$, $\sum \frac{1}{n^2\pi^2} (-1)^n = -\frac{1}{12}$

we obtain

$$\sum \frac{1}{n^4\pi^4}[(-1)^n - 1]\cos n\pi\xi = \frac{1}{48}(-4\xi^3 + 6\xi^2 - 1)$$

$$\sum \frac{1}{n^4\pi^4} (-1)^n \cos n\pi\xi = \frac{1}{48}(-\xi^4 + 2\xi^2 - \frac{7}{15})$$

$$\sum \frac{1}{n^4\pi^4} \cos n\pi\xi = \frac{1}{48}(-\xi^4 + 4\xi^3 - 4\xi^2 + \frac{8}{15})$$

for $\xi = 0$, it reduces to $\sum \frac{1}{n^4\pi^4} = \frac{1}{90}$, $\sum \frac{1}{n^4\pi^4} (-1)^n$

$$= -\frac{7}{8 \times 90}$$

$$\sum (\frac{(-1)^n - 1}{n^4\pi^4}) = -\frac{1}{48}$$

From (85) we have,

$$p_a = C_o - \frac{4}{\beta} \sum_{n=1}^{\infty} [16\kappa\beta\{\frac{3 - 40^2 + 3\omega}{3(1 + 0 + \omega)} \frac{(-1)^n - 1}{2}$$

$$+ 20(-1)^n\} \frac{C_1}{\lambda_n^2 \beta^2 g_n(1)} + 16\kappa\beta \frac{\theta\omega}{1 + 0 + \omega}$$

$$x \{C_o - \pi_\ell\} \frac{(-1)^n - 1}{\lambda_n^2 \beta^2 g_n(1)} + 16\kappa\beta\sigma_o\pi_e\{[\frac{0\omega}{1 + 0 + \omega}$$

$$- \delta^*\{\frac{3 - 40^2 + 3\omega}{6(1 + 0 + \omega)}] \frac{(-1)^n - 1}{\lambda_n^2 \beta^2 g_n(1)} - 20\delta^* \frac{(-1)^n}{\lambda_n^2 \beta^2 g_n(1)}\}.$$

$$= C_o - 64\kappa C_1 \sum_{n=1}^{\infty} \{\frac{3 - 40^2 + 3\omega}{3(1 + 0 + \omega)} \frac{(-1)^n - 1}{2}$$

$$+ 20(-1)^n\} \frac{\beta^2}{\lambda_n^4 \beta^4} - 64\kappa\{C_o - \pi_\ell\} \sum_{n=1}^{\infty} \frac{\theta\omega}{1 + 0 + \omega} \frac{(-1)^n - 1}{\lambda_n^4 \beta^4}$$

$$- 64\kappa\sigma_o\pi_e \sum_{n=1}^{\infty} [\{\frac{0\omega}{1 + 0 + \omega} - \delta^*\frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\} \frac{(-1)^n - 1}{\lambda_n^4 \beta^4}$$

$$- 20\delta^* \frac{(-1)^n - 1}{\lambda_n^4 \beta^4} \beta^2$$

$$p_a = C_o - \frac{64\kappa\beta^2}{2} C_1 \sum_{n=1}^{\infty} \frac{3 + 40^2 + 3\omega)}{3(1 + 0 + \omega)} \frac{(-1)^n - 1}{\lambda_n^4 \beta^4}$$

$$- 64\kappa\beta^2 C_1 \sum_{n=1}^{\infty} 20 \frac{(-1)^n}{\lambda_n^4 \beta^4} - 64\kappa\beta^2\{C_o - \pi_\ell\}$$

$$x \sum_{n=1}^{\infty} \frac{0\omega}{1 + 0 + \omega} \frac{(-1)^n - 1}{\lambda_n^4 \beta^4} - 64\kappa\beta^2\sigma_o\pi_e \sum_{n=1}^{\infty} \frac{0\omega}{1 + 0 + \omega}$$

$$- \delta^*\{\frac{(3 + 40^2 + 3\omega)}{6(1 + 0 + \omega^1)}\} \frac{(-1)^n - 1}{\lambda_n^4 \beta^4} + 64\kappa\beta^2\sigma_o\pi_e\delta^* \sum_{n=1}^{\infty} 20 \frac{(-1)^n}{\lambda^4 \beta^4}$$

Using the series expansion and recalling that

$\lambda_n = \frac{n\pi}{\beta}$; we have

$$p_a = C_o - 32\kappa\beta^2 C_1 \frac{(3 - 40^2 + 3\omega)}{3(1 + 0 + \omega)} \times (-\frac{1}{48})$$

$$- 64\kappa\beta^2 C_1 \, 20 \times (\frac{-7}{8 \times 90}) - 64\kappa\beta^2 \{C_o - \pi_\ell\} \frac{\theta\omega}{1 + 0 + \omega}$$

$$\times (-\frac{1}{48}) - 64\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1 + 0 + \omega} - \delta^* \frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\}$$

$$\times (-\frac{1}{48}) + 64\kappa\beta^2 \sigma_o \pi_e \delta^* \, 20 \times (-\frac{7}{8 \times 90})$$

$$= C_o + \frac{2}{3}\kappa\beta^2 C_1 \frac{(3 - 40^2 + 3\omega)}{3(1 + 0 + \omega)} + \frac{16 \times 7}{90}\kappa\beta^2 C_1 0$$

$$+ \frac{4}{3}\kappa\beta^2 \{C_o - \pi_\ell\} \frac{\theta\omega}{1 + 0 + \omega} + \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1 + 0 + \omega}$$

$$- \delta^* \frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\} - \frac{16 \times 7}{90}\kappa\beta^2 \sigma_o \pi_e 0\delta^*$$

thus

$$p_a = C_o\{1 + \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1 + 0 + \omega}\}$$

$$+ \frac{2}{3}\kappa\beta^2 \{\frac{30(1 + \omega) + 160^2 + 560(1 + \omega)}{30(1 + 0 + \omega)}\} C_1$$

$$- \frac{4}{3}\kappa\beta^2 \pi_\ell \frac{\theta\omega}{1 + 0 + \omega} + \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1 + 0 + \omega}$$

$$- \delta^* \frac{[15(1 + \omega) + 80^2 + 280(1 + \omega)]}{30(1 + 0 + \omega)}\}, \text{-------(87)}$$

From (86),

$$(p_a - 1) = C_o + C_1 + 8\beta\gamma V_o \{\frac{C_o(1 + 0)}{1 + 0 + \omega}$$

$$+ \frac{\sigma_o \pi_e}{1+0+\omega}\{(1+0) - \delta^*\frac{(3+40)}{6}\} + C_1\frac{(3+40)}{6(1+0+\omega)}$$

$$- 4\kappa\beta^2\omega\ \frac{(2+30)}{3(1+0+\omega)}\pi_\ell + \frac{\pi_\ell\omega}{1+0+\omega} - \pi_\ell\}$$

$$- \frac{4}{\beta}\sum_{n=1}^{\infty}[16\kappa\beta\{\frac{3-40^2+3\omega}{3(1+0+\omega)}\ \frac{(-1)^n-1}{2} + 20(-1)^n\}\frac{C_1}{\lambda_n^2\beta^2 g_n}$$

$$+ 16\kappa\beta\ \frac{0\omega}{1+0+\omega}\{C_o - \pi_\ell\}\frac{(-1)^n-1}{\lambda_n^2\beta^2 g_n(1)}$$

$$+ 16\kappa\beta\sigma_o\pi_e\{[\frac{0\omega}{1+0+\omega} - \delta^*\frac{(3-40^2+3\omega)}{6(1+0+\omega)}]\frac{(-1)^n-1}{\lambda_n^2\beta^2 g_n(1)}$$

$$- 20\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2 g_n(1)}]\ \frac{g_n(1)}{\lambda_n^2}\ (-1)^n$$

$$= C_o + C_1 + 8\beta\gamma V_o\{\frac{C_o(1+0)}{1+0+\omega} + \frac{\sigma_o\pi_e}{1+0+\omega}\{(1+0)$$

$$- \delta^*\frac{(3+4)}{6}\} + \frac{C_1(3+40)}{6(1+0+\omega)}$$

$$- 4\kappa\beta^2\omega\ \frac{(2+30)}{3(1+0+\omega)}\pi_\ell + \frac{\pi_\ell\omega}{1+0+\omega} - \pi_\ell\}$$

$$- \frac{64}{2}\kappa\beta^2 C_1\sum_{n=1}^{\infty}\frac{3-40^2+3\omega}{3(1+0+\omega)}\ \frac{((-1)^n-1)(-1)^n}{\lambda_n^4\beta^4}$$

$$- 64\kappa\beta^2 C_1\sum_{n=1}^{\infty}20\ \frac{(-1)^n\times(-1)^n}{\lambda_n^4\beta^4}$$

$$- 64\kappa\beta^2\{C_o - \pi_\ell\}\sum_{n=1}^{\infty}\frac{\theta\omega}{1+\theta+\omega}\ \frac{((-1)^n-1)(-1)^n}{\lambda_n^4\beta^4}$$

$$- 64\kappa\beta^2\sigma_o\pi_e\sum_{n=1}^{\infty}\{\frac{\theta\omega}{1+\theta+\omega} - \delta^*\frac{(3-40^2+3\omega)}{6(1+\theta+\omega)}\}\frac{((-1)^n-1)(-1)^n}{\lambda_n^4\beta^4}$$

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$$+ 64\kappa\beta^2\sigma_o\pi_e\delta^* \sum_{n=1}^{\infty} 20 \frac{(-1)^n \times (-1)^n}{\lambda_n^4\beta^4}$$

Note that $((-1)^n - 1)(-1)^n = 1 - (-1)^n$ and $(-1)^n \times (-1)^n = 1$, therefore using the series expansion,

$$P_a - 1 = C_o + C_1 + 8\beta\gamma V_o\{\frac{C_o(1+\theta)}{1+\theta+\omega} + \frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\} + C_1\frac{(3+4\theta)}{6(1+\theta+\omega)} - 4\kappa\beta^2\omega\frac{(2+3\theta)}{3(1+\theta+\omega)}\}$$

$$\pi_\ell - \pi_\ell\frac{(1+\theta)}{1+\theta+\omega}\} - 32\kappa\beta^2 C_1\{\frac{3-4\theta^2+3\omega}{3(1+\theta+\omega)} \times \frac{1}{48}$$

$$- 64\kappa\beta^2 C_1\frac{20}{90} - 64\kappa\beta^2\{C_o - \pi_\ell\}\frac{0\omega}{1+\theta+\omega} \times \frac{1}{48}$$

$$+ 64\kappa\beta^2\sigma_o\pi_e\delta^*\frac{20}{90}$$

$$- 64\kappa\beta^2\sigma_o\pi_e\{\frac{0\omega}{1+\theta+\omega} - \delta^*\frac{(3-4\theta^2+3\omega)}{6(1+\theta+\omega)}\}\frac{1}{48}$$

neglecting powers of κ^2,

$$P_a - 1 = C_o + C_1 + 8\beta\gamma V_o\frac{(1+\theta)}{1+\theta+\omega}C_o + 8\beta\gamma V_o\frac{\sigma_o\pi_e}{1+\theta+\omega}$$

$$\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\} + 8\beta\gamma V_o\frac{(3+4\theta)}{6(1+\theta+\omega)}C_1$$

$$- 8\beta\gamma V_o\frac{(1+\theta)}{1+\theta+\omega}\pi_\ell - \frac{2}{3}\kappa\beta^2 C_1\frac{(3-4\theta^2+3\omega)}{3(1+\theta+\omega)}$$

$$- \frac{64 \times 2}{90}\kappa\beta^2\theta C_1 - \frac{4}{3}\kappa\beta^2\{C_o - \pi_\ell\}\frac{\theta\omega}{1+\theta+\omega}$$

$$- \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{\theta\omega}{1+\theta+\omega} - \delta^*\frac{(3-4\theta^2+3\omega)}{6(1+\theta+\omega)}\}$$

$$+ 64\kappa\beta^2\sigma_o\pi_e\delta^*\frac{2\theta}{90}$$

$$= C_o + C_1 + 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} C_o + 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1+\theta+\omega}\{(1+\theta$$

$$- \delta^* \frac{(3+4\theta)}{5}\} + 8\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)} C_1$$

$$- 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} \pi_\ell - \frac{2}{3}\kappa\beta^2 \frac{30(1+\theta)+24\theta^2+64\theta(1+\omega)}{30(1+\theta+\omega)} C_1$$

$$- \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega} C_1 + \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega} \pi_\ell$$

$$- \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1+\theta+\omega} - \delta^* \frac{[15(1+\theta)+12\theta^2+32\theta(1+\omega)]}{30(1+\theta+\omega)}\}$$

Thus,

$$p_a - 1 = C_o[1 + 8\beta\gamma V_o \frac{(1+0)}{1+\theta+\omega} - \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}]$$

$$+ C_1[1 + 8\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$- \frac{2}{3}\kappa\beta^2 \frac{[30(1+\theta)+24\theta^2+64\theta(1+\omega)]}{30(1+\theta+\omega)}]$$

$$+ 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^* \frac{(3+4\theta)}{6}\}$$

$$- 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} \pi_\ell + \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega} \pi_\ell$$

$$- \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1+\theta+\omega} - \delta^* \frac{[15(1+\theta)+12\theta^2+32\theta(1+\omega)]}{30(1+\theta+\omega)}\} \quad -(8$$

Now (87) + (88) gives

$$2p_a - 1 = [2 + 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega}]C_o$$

$$+ [1 - \frac{2}{3}\kappa\beta^2 \{\frac{8\theta^2+8\theta(1+\omega)}{30(1+\theta+\omega)}\} + 8\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)}]C_1$$

$$- 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} \pi_\ell + 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1+\theta+\omega}\{(1+\theta)$$

$$- \delta^*\left(\frac{3 + 4\theta}{6}\right)\} + \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\delta^*\left\{\frac{4\theta^2 + 4\theta(1+\omega)}{30(1 + \theta + \omega)}\right\}, \quad ----(89)$$

from (89),

$$C_1\left[1 - \frac{2}{3}\kappa\beta^2\left\{\frac{8\theta^2 + 8\theta(1+\omega)}{30(1 + \theta + \omega)}\right\} + 8\beta\gamma V_o\frac{3 + 4\theta}{6(1 + \theta + \omega)}\right]$$

$$= 2p_a - 1 - \left[2 + 8\beta\gamma V_o\frac{(1 + \theta)}{1 + \theta + \omega}\right]C_o + 8\beta\gamma V_o\frac{(1 + \theta)}{1 + \theta + \omega}\pi_\ell$$

$$- 8\beta\gamma V_o\frac{\sigma_o\pi_e}{1 + \theta + \omega}\left\{(1 + \theta) - \delta^*\frac{(3 + 4\theta)}{6}\right\}$$

$$- \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\delta^*\left\{\frac{4\theta^2 + 4\theta(1 + \omega)}{30(1 + \theta + \omega)}\right\}$$

$$C_1 = \left[2p_a - 1 - \left\{2 + 8\beta\gamma V_o\frac{(1 + \theta)}{1 + \theta + \omega}\right\}C_o\right.$$

$$+ 8\beta\gamma V_o\frac{(1 + \theta)}{1 + \theta + \omega}\pi_\ell$$

$$- 8\beta\gamma V_o\frac{\sigma_o\pi_e}{1 + \theta + \omega}\left\{1 + \theta - \delta^*\frac{(3 + 4\theta)}{6}\right\}$$

$$\left. - \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\delta^*\left\{\frac{4\theta^2 + 4\theta(1+\omega)}{30(1 + \theta + \omega)}\right\}\right]\times\left[1 - \left\{8\beta\gamma V_o\frac{3 + 4\theta}{6(1 + \theta + \omega}\right.\right.$$

$$\left.\left. - \frac{2}{3}\kappa\beta^2\frac{[8\theta^2 + 8\theta(1 + \omega)]}{30(1 + \theta + \omega)}\right\}\right]$$

neglecting powers of κ^2,

$$= 2p_a - 1 - \left\{2 + 8\beta\gamma V_o\frac{(1 + \theta)}{1 + \theta + \omega}\right\}C_o$$

$$+ 8\beta\gamma V_o\frac{(1 + \theta)}{1 + \theta + \omega}\pi_\ell$$

$$- 8\beta\gamma V_o\frac{\sigma_o\pi_e}{1 + \theta + \omega}\left\{(1 + \theta) - \delta^*\frac{(3 + 4\theta)}{6}\right\}$$

$$- \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\delta^*\left\{\frac{4\theta^2 + 4\theta(1 + \omega)}{30(1 + \theta + \omega)}\right\}$$

$$- 2p_a\{8\beta\gamma V_o \frac{(3 + 4\theta)}{6(1 + \theta + \omega)} - \frac{2}{3}\kappa\beta^2 \frac{8\theta^2 + 8\theta(1+\omega)}{30(1 + \theta + \omega)}\}$$

$$+ 8\beta\gamma V_o \frac{(3 + 4\theta)}{6(1 + \theta + \omega)} - \frac{2}{3}\kappa\beta^2 \frac{8\theta^2 + 8\theta(1 + \omega)}{30(1 + \theta + \omega)}$$

$$+ 16\beta\gamma V_o \frac{(3 + 4\theta)}{6(1 + \theta + \omega)} C_o - \frac{4}{3}\kappa\beta^2 \frac{8\theta^2 + 8\theta(1 + \omega)}{30(1 + \theta + \omega)} C_o$$

$$\therefore C_1 = [2 - 16\beta\gamma V_o \frac{(3 + 4\theta)}{6(1 + \theta + \omega)}$$

$$+ \frac{4}{3}\kappa\beta^2 \{\frac{8\theta^2 + 8\theta(1 + \omega)}{30(1 + \theta + \omega)}\}]p_a$$

$$- 1 - [2 - 16\beta\gamma V_o \frac{\theta}{6(1 + \theta + \omega)} + \frac{4}{3}\kappa\beta^2 \frac{8\theta^2 + 8\theta(1+\omega)}{30(1 + \theta + \omega)}]C_o$$

$$+ 8\beta\gamma V_o \frac{(3 + 4\theta)}{6(1 + \theta + \omega)} - \frac{2}{3}\kappa\beta^2 \frac{8\theta^2 + 8\theta(1 + \omega)}{30(1 + \theta + \omega)}$$

$$+ 8\beta\gamma V_o \frac{(1 + \theta)}{1 + \theta + \omega} \pi_\ell - 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1 + \theta + \omega}\{(1+\theta) - \delta^*(\frac{3+4\theta}{6}$$

$$- \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \delta^* \{\frac{4\theta^2 + 4\theta(1 + \omega)}{30(1 + \theta + \omega)}\}$$

Substitute for C_1 in (87)

$$p_a = C_o + \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1 + \theta + \omega} C_o$$

$$+ \frac{2}{3}\kappa\beta^2 \{\frac{30(1 + \omega) + 16\theta^2 + 56\theta(1 + \omega)}{30(1 + \theta + \omega)}\}$$

$$x [2p_a - 16\beta\gamma V_o \frac{(3 + 4\theta)}{6(1 + \theta + \omega)} p_a - \frac{4}{3}\kappa\beta^2 \frac{8\theta^2 + 8\theta(1 + \omega)}{30(1 + \theta + \omega)} p_a$$

$$- 2C_o + 16\beta\gamma V_o \frac{\theta}{6(1 + \theta + \omega)} C_o - \frac{4}{3}\kappa\beta^2 \frac{8\theta^2 + 8\theta(1+\omega)}{30(1 + \theta + \omega)} C_o$$

$$+ 8\beta\gamma V_o \frac{(3 + 4\theta)}{6(1 + \theta + \omega)} - \frac{2}{3}\kappa\beta^2 \frac{8\theta^2 + 80(1 + \omega)}{30(1 + \theta + \omega)}$$

$$+ 8\beta\gamma V_o \frac{(1 + \theta)}{1 + \theta + \omega} \pi_\ell - 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1 + \theta + \omega}\{(1+\theta) - \delta^* \frac{(3+4\theta)}{6}\}$$

$$- \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \delta^* \{\frac{4\theta^2 + 40(1 + \omega)}{30(1 + \theta + \omega)}\}] - \frac{4}{3}\kappa\beta^2 \pi_\ell \frac{\theta\omega}{1 + \theta + \omega}$$

$$+ \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1 + \theta + \omega} - \delta^* \frac{[15(1 + \omega) + 8\theta^2 * 28\theta(1+\omega)]}{30(1 + \theta + \omega)}$$

neglecting powers of κ^2,

$$p_a = C_o + \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1 + \theta + \omega} C_o$$

$$+ \frac{4}{3}\kappa\beta^2 \{\frac{30(1 +\omega) + 16\theta^2 + 56\theta(1 + \omega)}{30(1 + \theta + \omega)}\} p_a$$

$$- \frac{2}{3}\kappa\beta^2 \{\frac{30(1 + \omega) + 16\theta^2 + 56\theta(1 + \omega)}{30(1 + \theta + \omega)}\}$$

$$- \frac{4}{3}\kappa\beta^2 \{\frac{30(1 + \omega) + 16\theta^2 + 56\theta(1 + \omega)}{30(1 + \theta + \omega)}\} C_o - \frac{4}{3}\kappa\beta^2 \pi_\ell \frac{\theta\omega}{1+\theta+\omega}$$

$$+ \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1 + \theta + \omega} - \delta^* \frac{[15(1+\omega) + 8\theta^2 + 280(1+\omega)]}{30(1 + \theta + \omega)}\}$$

thus

$$C_o = [\{1 - \frac{4}{3}\kappa\beta^2 \frac{[30(1+\omega) + 16\theta^2 + 56\theta(1+\omega)]}{(30(1 + \theta + \omega)}\} p_a$$

$$+ \frac{2}{3}\kappa\beta^2 \frac{[30(1+\omega) + 16\theta^2 + 56\theta(1+\omega)]}{30(1 + \theta + \omega)} + \frac{4}{3}\kappa\beta^2 \pi_\ell \frac{\theta\omega}{1 + \theta + \omega}$$

$$- \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1 + \theta + \omega} - \delta^* \frac{[15(1+\omega) + 8\theta^2 + 280(1+\omega)]}{30(1 + \theta + \omega)}\}]$$

$$\times [1 - \frac{4}{3}\kappa\beta^2 \{\frac{\theta\omega}{1 + \theta + \omega} - \frac{[30(1 +\omega) + 16\theta^2 + 56\theta(1+\omega)]}{30(1 + \theta + \omega)}\}]$$

neglecting powers of κ^2,

$$C_o = \{1 - \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}\}p_a + 8\beta\gamma V_o \frac{(3+2\theta)}{3(1+\theta+\omega)} \pi_\ell$$

$$+ \frac{2}{3}\kappa\beta^2 [\frac{15(1+\omega)+8\theta^2+28\theta(1+\omega)}{30(1+\theta+\omega)}] + \frac{4}{3}\kappa\beta^2 \pi_\ell \frac{\theta\omega}{1+\theta+\omega}$$

$$- \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1+\theta+\omega} - \delta^* \frac{[15(1+\omega)+80^2+280(1+\omega)]}{30(1+\theta+\omega)}\} \quad (90)$$

Substituting for C_1,

$$C_1 = [2 - 16\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)} + \frac{4}{3}\kappa\beta^2 \frac{8\theta^2+8\theta(1+\omega)}{30(1+\theta+\omega)}$$

$$- [2 - 16\beta\gamma V_o \frac{\theta}{6(1+\theta+\omega)} + \frac{4}{3}\kappa\beta^2 \frac{8\theta^2+8\theta(1+\omega)}{30(1+\theta+\omega)}$$

$$\times [p_a - \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega} p_a + \frac{2}{3}\kappa\beta^2 [\frac{15(1+\omega)+80^2+280(1+\omega)}{15(1+\theta+\omega)}$$

$$+ \frac{4}{3}\kappa\beta^2 \pi_\ell \frac{\theta\omega}{1+\theta+\omega} - \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1+\theta+\omega}$$

$$- \delta^* \frac{[15(1+\omega)+80^2+280(1+\omega)]}{30(1+\theta+\omega)}]$$

$$+ 8\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)} - \frac{2}{3}\kappa\beta^2 \frac{80^2+80(1+\omega)}{30(1+\theta+\omega)}$$

$$+ 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} \pi_\ell - 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1+\theta+\omega} \{(1+\theta)$$

$$- \delta^* \frac{(3+4\theta)}{6}\} - \frac{4}{3}\kappa\beta^2 \sigma_o \pi_e \delta^* \{\frac{40^2+40(1+\omega)}{30(1+\theta+\omega)}\}$$

neglecting powers of κ^2,

$$C_1 = -1 - [16\beta\gamma V_o \frac{(1+\theta)}{2(1+\theta+\omega)} - \frac{8}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}] p_a$$

$$- \frac{4}{3}\kappa\beta^2\left[\frac{3(1+\omega)+20^2+60(1+\omega)}{30(1+0+\omega)}\right] + 8\beta\gamma V_o \frac{3+40}{6(1+0+\omega)}$$

$$+ 8\beta\gamma V_o \frac{(3+20)}{3(1+0+\omega)} - 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1+0+\omega}\{(1+0)$$

$$- \delta^*\frac{(3+40)}{6}\} + \frac{8}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{0\omega}{1+0+\omega}$$

$$- \delta^*\frac{[3(1+\omega)+20^2+60(1+\omega)]}{6(1+0+\omega)}\}, \quad \text{----------(91)}$$

where

$$p_a = P_a/\Delta P$$

Substituting equations (82), (83) (90) and (91) into equations (84), (66), (67), and (68), we obtain the expressions for the velocity and pressure distributio in regions I and II.

For region I,

$$p_1 = \{1 - \frac{4}{3}\kappa\beta^2 \frac{0\omega}{1+0+\omega}\}p_a$$

$$+ \frac{2}{3}\kappa\beta^2\frac{[15(1+\omega)+80^2+280(1+\omega)]}{15(1+0+\omega)}$$

$$+ \frac{4}{3}\kappa\beta^2\pi_\ell \frac{0\omega}{1+0+\omega} - \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{0\omega}{1+0+\omega}$$

$$- \delta^*\frac{[15(1+\omega)+80^2+280(1+\omega)]}{30(1+0+\omega)}\} - \xi$$

$$- [16\beta\gamma V_o \frac{(1+0)}{2(1+0+\omega)} - \frac{8}{3}\kappa\beta^2 \frac{0\omega}{1+0+\omega}]p_a\xi$$

$$- \frac{4}{3}\kappa\beta^2\frac{[3(1+\omega)+20^2+60(1+\omega)]}{3(1+0+\omega)}\xi + 8\beta\gamma V_o \frac{(3+40)}{6(1+0+\omega)}\xi$$

$$- [\frac{8}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega} - 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega}]\pi_\ell\xi$$

$$+ \frac{8}{3}\kappa\beta^2 \frac{\sigma_o\pi_e}{1+\theta+\omega}\{\frac{\theta\omega}{1+\theta+\omega}$$

$$- \delta^*[\frac{3(1+\omega)+2\theta^2+6\theta(1+\omega)}{6(1+\theta+\omega)}]\} + 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} p_a$$

$$+ 8\beta\gamma V_o \frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\}$$

$$- 8\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)} - 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega}\pi_\ell]\xi^2$$

$$- \frac{4}{\beta}\sum_{n=1}^{\infty}[-16\kappa\beta\{\frac{3-4\theta^2+3\omega}{3(1+\theta+\omega)}(\frac{(-1)^n-1}{2} + 2\theta(-1)^n\} \frac{1}{\lambda_n^2\beta^2 g_n}$$

$$+ 16\kappa\beta \frac{\theta\omega}{1+\theta+\omega}\{p_a - \pi_\ell\}\frac{(-1)^n-1}{\lambda_n^2\beta^2 g_n(1)} + 16\kappa\beta\sigma_o\pi_e$$

$$\{[\frac{\theta\omega}{1+\theta+\omega} - \delta^*\frac{(3-4\theta^2+3\omega)}{6(1+\theta+\omega)}]\frac{(-1)^n-1}{\lambda_n^2\beta^2 g_n(1)}$$

$$- 2\theta\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2 g_n(1)}\} \times \frac{g_n(1)}{\lambda_n^2}\cos\lambda_n\beta\xi$$

Using the series expansion, we have

$$p_1 = \{1 - \frac{8}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}\}p_a + \frac{2}{3}\kappa\beta^2[\frac{15(1+\omega)+8\theta^2+28\theta(1+\omega)}{15(1+\theta+\omega)}$$

$$+ \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega} - \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{\theta\omega}{1+\theta+\omega}$$

$$- \delta^*[\frac{15(1+\omega)+8\theta^2+28\theta(1+\omega)}{30(1+\theta+\omega)}]\} - \xi$$

$$- [16\beta\gamma V_o \frac{(1+2\theta)}{2(1+\theta+\omega)} - \frac{8}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}] p_a\xi$$

$$- \frac{4}{3}\kappa\beta^2 \left[\frac{3(1+\omega)+2\theta^2+6\theta(1+\omega)}{3(1+\theta+\omega)}\right]\xi + 8\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)}\xi$$

$$- \left[\frac{8}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega} - 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega}\right]\pi_\ell \xi$$

$$- 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^* \frac{(3+4\theta)}{6}\}\xi$$

$$+ \frac{8}{3}\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1+\theta+\omega} - \delta^* \frac{[3(1+\omega)+2\theta^2+6\theta(1+\omega)]}{6(1+\theta+\omega)}\}\xi$$

$$+ [8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} p_a + 8\beta\gamma V_o \frac{\sigma_o \pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^* \frac{(3+4\theta)}{6}\}$$

$$- 8\beta\gamma V_o \frac{(3+4\theta)}{6(1+\theta+\omega)} - 8\beta\gamma V_o \frac{(1+\theta)}{1+\theta+\omega} \pi_\ell]\xi^2$$

$$+ 32\kappa\beta^2 \frac{(3-4\theta^2+3\omega)}{3(1+\theta+\omega)} \times \frac{1}{48}(-4\xi^3 + 6\xi^2 - 1)$$

$$+ 64\kappa\beta^2 2\theta \times \frac{1}{48}(-\xi^4 + 2\xi^2 - \frac{7}{15})$$

$$- 64\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}\{p_a - \pi_\ell\} \times \frac{1}{48}(-4\xi^3 + 6\xi^2 - 1)$$

$$- 64\kappa\beta^2 \sigma_o \pi_e \{\frac{\theta\omega}{1+\theta+\omega} - \delta^* \frac{(3-4\theta^2+3\omega)}{6(1+\theta+\omega)}\}$$

$$\times \frac{1}{48}(-4\xi^3 + 6\xi^2 - 1)$$

$$+ 64\kappa\beta^2 \sigma_o \pi_e \times 2\theta\delta^* \times \frac{1}{48}(-\xi^4 + 2\xi^2 - \frac{7}{15})$$

Note that $\gamma V_o = \omega\kappa\beta$

$$= \frac{\gamma\kappa_\ell\beta}{1+as^2}$$

thus,

$$p_1 = \{1 - \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}\}p_a$$

$$+ \frac{2}{3}\kappa\beta^2 \left[\frac{15(1+\omega)+8\theta^2+28\theta(1+\omega)}{15(1+\theta+\omega)}\right] + \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}\pi_\ell$$

$$- \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\left\{\frac{\theta\omega}{1+\theta+\omega} - \delta^*\left[\frac{15(1+\omega)+3\theta^2+28\theta(1+\omega)}{30(1+\theta+\omega)}\right]\right\} - \xi$$

$$- \left[16\kappa\beta^2\omega\frac{(1+\theta)}{2(1+\theta+\omega)} - \frac{8}{3}\kappa\beta^2\frac{\theta\omega}{1+\theta+\omega}\right]p_a\xi$$

$$- \frac{4}{3}\kappa\beta^2\left[\frac{3(1+\omega)+2\theta^2+6\theta(1+\omega)}{3(1+\theta+\omega)}\right]\xi + 8\kappa\beta^2\omega\frac{(3+4\theta)}{6(1+\theta+\omega)}\xi$$

$$- \left[\frac{8}{3}\kappa\beta^2\frac{\theta\omega}{1+\theta+\omega} - 8\kappa\beta^2\omega\frac{(1+\theta)}{1+\theta+\omega}\right]\pi_\ell\xi$$

$$- 8\kappa\beta^2\omega\frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\}\xi$$

$$+ \frac{8}{3}\kappa\beta^2\sigma_o\pi_e\left\{\frac{\theta\omega}{1+\theta+\omega} - \delta^*\left[\frac{3(1+\omega)+2\theta^2+6\theta(1+\omega)}{6(1+\theta+\omega)}\right]\right\}\xi$$

$$+ \left[8\kappa\beta^2\omega\frac{(1+\theta)}{1+\theta+\omega}p_a + 8\kappa\beta^2\omega\frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\}\right.$$

$$\left. - 8\kappa\beta^2\omega\frac{(3+4\theta)}{6(1+\theta+\omega)} - 8\kappa\beta^2\omega\frac{(1+\theta)}{1+\theta+\omega}\pi_\ell\right]\xi^2$$

$$- \frac{8}{3}\kappa\beta^2\frac{(3-4\theta^2+3\omega)}{3(1+\theta+\omega)}\xi^3 + 4\kappa\beta^2\frac{(3-4\theta^2+3\omega)}{3(1+\theta+\omega)}\xi^2$$

$$- \frac{2}{3}\kappa\beta^2\frac{(3-4\theta^2+3\omega)}{3(1+\theta+\omega)} - \frac{8}{3}\kappa\beta^2\theta\xi^4 + \frac{16}{3}\kappa\beta^2\theta\xi^2$$

$$- \frac{56}{45}\kappa\beta^2\theta + \frac{16}{3}\kappa\beta^2\frac{\theta\omega}{1+\theta+\omega}\{p_a - \pi_\ell\}\xi^3$$

$$- 8\kappa\beta^2 \frac{0\omega}{1 + 0 + \omega}\{p_a - \pi_\ell\}\xi^2 + \frac{4}{3}\kappa\beta^2 \frac{0\omega}{1 + 0 + \omega}\{p_a - \pi_\ell\}$$

$$+ \frac{16}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{0\omega}{1 + 0 + \omega} - \delta^*\frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\}\xi^3$$

$$- 8\kappa\beta^2\sigma_o\pi_e\{\frac{0\omega}{1 + 0 + \omega} - \delta^*\frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\}\xi^2$$

$$+ \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{0\omega}{1 + 0 + \omega} - \delta^*\frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\}$$

$$- \frac{8}{3}\kappa\beta^2\sigma_o\pi_e 0\delta^*\xi^4 + \frac{16}{3}\kappa\beta^2\sigma_o\pi_e 0\delta^*\xi^2 - \frac{56}{45}\kappa\beta^2\sigma_o\pi_e 0\delta^*$$

$$p_1 = p_a - \xi - \frac{4}{3}\kappa\beta^2[2\omega\frac{(3 + 20)}{1 + 0 + \omega}p_a$$

$$+ \frac{[3 + 60 + 20^2 - 6\omega(1 + 0)]}{3(1 + 0 + \omega)} - 2\omega\frac{(3 + 20)}{1 + 0 + \omega}\pi_\ell$$

$$+ 2\sigma_o\pi_e\{\frac{\omega(3 + 20)}{1 + 0 + \omega} + \delta^*\frac{[3 + 60 + 20^2 - 6\omega(1 + 0)]}{6(1 + 0 + \omega)}\}]\xi$$

$$+ \frac{4}{3}\kappa\beta^2[\frac{6\omega}{1 + 0 + \omega}p_a + \frac{(3 + 40)}{1 + 0 + \omega} - \frac{6\omega}{1 + 0 + \omega}\pi_\ell$$

$$+ 6\sigma_o\pi_e\{\frac{\omega}{1 + 0 + \omega} + \delta^*\frac{(3 + 40)}{6(1 + 0 + \omega)}\}]\xi^2$$

$$- \frac{4}{3}\kappa\beta^2[2\frac{(3 - 40^2 + 3\omega)}{3(1 + 0 + \omega)} - \frac{40\omega}{1 + 0 + \omega}p_a + \frac{40\omega}{1 + 0 + \omega}\pi_\ell$$

$$- 4\sigma_o\pi_e\{\frac{0\omega}{1 + 0 + \omega} - \delta^*\frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\}]\xi^3$$

$$- \frac{4}{3}\kappa\beta^2 20(1 + \sigma_o\pi_e\delta^*)\xi^4$$

Thus we can write p_1 as

$$p_1 = p_a - \xi - \frac{4}{3}\kappa\beta^2\Psi(\xi)$$

$$u_1 = -(1-\zeta^2)\frac{dp_1}{d\xi}$$

$$= -(1-\zeta^2)[-1 - \frac{4}{3}\kappa\beta^2\frac{d\Psi}{d\xi}]$$

$$\therefore u_1 = (1-\zeta^2)[1 + \frac{4}{3}\kappa\beta^2\frac{d\Psi}{d\xi}]$$

$$v_1 = \frac{1}{4\beta}(2\zeta - \zeta^3)\frac{d^2p_1}{d\xi^2}$$

$$= -\frac{1}{4\beta}(2\zeta - \zeta^3)\frac{4}{3}\kappa\beta^2\frac{d^2\Psi}{d\xi^2}$$

$$\therefore v_1 = -\frac{1}{3}\kappa\beta(2\zeta - \zeta^3)\frac{d^2\Psi}{d\xi^2}$$

In region I, the velocity and pressure distributions are given by

$$u_1 = (1-\zeta^2)[1 + \frac{4}{3}\kappa\beta^2\frac{d\Psi}{d\xi}], \quad \text{----------(92)}$$

$$v_1 = -\frac{1}{4\beta}(2\zeta - \zeta^3)\frac{d^2\Psi}{d\xi^2}, \quad \text{----------(93)}$$

$$p_1 = p_a - \xi - \frac{4}{3}\kappa\beta^2\Psi(\xi), \quad \text{----------(94)}$$

where

$$\Psi(\xi) = \xi[2\omega\frac{(3+2\theta)}{1+\theta+\omega}p_a + \frac{(3+6\theta+2\theta^2-6\omega(1+\theta)}{3(1+\theta+\omega)}$$

$$- 2\omega\frac{(3+2\theta)}{1+\theta+\omega}\pi_\lambda + 2\sigma_o\pi_e\{\omega\frac{(3+2\theta)}{1+\theta+\omega}$$

$$+ \delta^*\frac{[3+6\theta+2\theta^2-6\omega(1+\theta)]}{6(1+\theta+\omega)}\}] - \xi^2[\frac{6\omega}{1+\theta+\omega}p_a$$

$$+ \frac{3 + 40}{1 + 0 + \omega} - \frac{6\omega}{1 + 0 + \omega}\pi_\ell + 6\sigma_o\pi_e\{\frac{\omega}{1 + 0 + \omega}$$

$$+ \delta^*\frac{(3 + 40)}{6(1 + 0 + \omega)}\}] + \xi^3[2\frac{(3 - 40^2 + 3\omega)}{3(1 + 0 + \omega)} - \frac{40\omega}{1 + 0 + \omega}P_a$$

$$+ \frac{40\omega}{1 + 0 + \omega}\pi_\ell - 4\sigma_o\pi_e\{\frac{0\omega}{1 + 0 + \omega} - \delta^*\frac{(3 - 40^2 + 3\omega)}{6(1 + 0 + \omega)}\}]$$

$$+ \xi^4[20(1 + \sigma_o\pi_e\delta^*)], \text{----------(95)}$$

$$\Psi(\xi) = \xi_\circ\frac{1}{3(1 + 0 + \omega)}[3 + 60 + 20^2 - 6\omega\pi_\ell(3 + 20)$$

$$- 6\omega(1 + 0) + 6\omega(3 + 20)P_a + \sigma_o\pi_e\{6\omega(3 + 20)$$

$$+ \delta^*[3 + 60 + 20^2 - 6\omega(1 + 0)]\}]$$

$$- \xi^2\circ\frac{1}{1 + 0 + \omega}[3 + 40 - 6\omega\pi_\ell + 6\omega P_a + \sigma_o\pi_e\{6\omega_L$$

$$+ \delta^*(3 + 40)\}] + \xi^3\circ\frac{1}{3(1 + 0 + \omega)}[2(3 - 40^2 + 6\omega 0\pi_\ell$$

$$- 6\omega 0P_a + 3\omega) - 2\sigma_o\pi_e\{60\omega - \delta^*(3 - 40^2 + 3\omega)\}]$$

$$+ \xi^4[20(1 + \sigma_o\pi_e\delta^*)], \text{----------(96)}$$

where

$$\omega = \frac{\gamma V_o}{\kappa\beta} = \frac{\gamma\kappa_\ell}{\kappa(1 + as^2)}, \text{----------(97)}$$

For region II, p_2 is given as

$$p_2 = p_o + \frac{\gamma\kappa_\ell s^2}{1 + as^2}[\pi_\ell - p_o + \frac{s^2}{4\beta}\sum_{n=1}^{\infty} B_n k_n(\gamma)\cos\lambda_n\beta\xi]\ln\xi$$

$$-\frac{s^2}{4\beta}\sum_{n=1}^{\infty} B_n k_n(\zeta)\cos\lambda_n\beta\xi = \{1-\frac{4}{3}\kappa\beta^2\frac{\theta\omega}{1+\theta+\omega}\}p_a\frac{(1+\theta)}{1+\theta+\omega}$$

$$+\frac{2}{3}\kappa\beta^2\left[\frac{15(1+\omega)+8\theta^2+28\theta(1+\omega)}{15(1+\theta+\omega)}\right]\times\frac{(1+\theta)}{(1+\theta+\omega)}$$

$$+\frac{4}{3}\kappa\beta^2\frac{\theta\omega(1+\theta)}{(1+\theta+\omega)^2}\pi_\ell-\frac{4}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{\theta\omega}{1+\theta+\omega}$$

$$-\delta^*\frac{[15(1+\omega)+8\theta^2+28\theta(1+\omega)]}{30(1+\theta+\omega)}\}\frac{(1+\theta)}{(1+\theta+\omega)}$$

$$+\frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta)-\delta^*\frac{(3+4\theta)}{6}\}-\frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$-[16\beta\gamma V_o\frac{(1+\theta)}{2(1+\theta+\omega)}-\frac{8}{3}\kappa\beta^2\frac{\theta\omega}{1+\theta+\omega}]p_a\frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$-\frac{4}{3}\kappa\beta^2\left[\frac{3(1+\omega)+2\theta^2+6\theta(1+\omega)}{3(1+\theta+\omega)}\right]\frac{(3+\theta)}{6(1+\theta+\omega)}$$

$$+8\beta\gamma V_o\frac{(3+4\theta)^2}{36(1+\theta+\omega)^2}-[\frac{8}{3}\kappa\beta^2\frac{\theta\omega}{1+\theta+\omega}$$

$$-3\beta\gamma V_o\frac{(1+\theta)}{1+\theta+\omega}]\pi_\ell\frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$-8\beta\gamma V_o\frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta)-\delta^*\frac{(3+4\theta)}{6}\}\frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$+\frac{8}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{\theta\omega}{1+\theta+\omega}$$

$$-\delta^*\frac{[3(1+\omega)+2\theta^2+6\theta(1+\omega)]}{6(1+\theta+\omega)}\}\frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$-\frac{1}{4}\kappa\beta^2\omega\frac{(2+3\theta)}{3(1+\theta+\omega)}\pi_\ell+\frac{\pi_\ell\omega}{1+\theta+\omega}$$

$$+ \frac{\gamma\kappa_\ell s^2}{1 + as^2}\{\pi_\ell \frac{(1 + \theta)}{1 + \theta + \omega} - \{1 - \frac{4}{3}\kappa\beta^2 \frac{\theta\omega}{1+\theta+\omega}\}p_a \frac{(1 + \theta)}{1+\theta+\omega}$$

$$- \frac{2}{3}\kappa\beta \frac{[15(1 + \omega) + 8\theta^2 + 28\theta(1 + \omega)]}{15(1 + \theta + \omega)} \times \frac{1 + \theta}{1 + \theta + \omega}$$

$$- \frac{4}{3}\kappa\beta^2 \frac{\theta\omega(1 + \theta)}{(1 + \theta + \omega)^2} \pi_\ell + \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\{\frac{\theta\omega}{1 + \theta + \omega}$$

$$- \delta^* \frac{[15(1 + \omega) + 8\theta^2 + 28\theta(1 + \omega)]}{30(1 + \theta + \omega)}\} \frac{1 + \theta}{1 + \theta + \omega}$$

$$- \frac{\sigma_o\pi_e}{1 + \theta + \omega}\{(1 + \theta) - \delta^* \frac{(3 + 4\theta)}{6}\} + \frac{(3 + 4\theta)}{6(1 + \theta + \omega)}$$

$$+ [16\beta\gamma V_o \frac{(1 + \theta)}{2(1 + \theta + \omega)} - \frac{8}{3}\kappa\beta^2 \frac{\theta\omega}{1 + \theta + \omega}]p_a \frac{3 + 4\theta}{6(1 + \theta + \omega)}$$

$$+ \frac{4}{3}\kappa\beta^2 \frac{[3(1 + \omega) + 2\theta^2 + 6\theta(1 + \omega)]}{3(1 + \theta + \omega)} \frac{(3 + 4\theta)}{6(1 + \theta + \omega)}$$

$$- 8\beta\gamma V_o \frac{(3 + 4\theta)^2}{36(1 + \theta + \omega)^2} + [\frac{8}{3}\kappa\beta \frac{\theta\omega}{1 + \theta + \omega}$$

$$- 8\beta\gamma V_o \frac{(1 + \theta)}{1 + \theta + \omega}]\pi_\ell \frac{(3 + 4\theta)}{6(1 + \theta + \omega)}$$

$$+ 8\beta\gamma V_o \frac{\sigma_o\pi_e}{1 + \theta + \omega}\{(1 + \theta) - \delta^* \frac{(3 + 4\theta)}{6}\} \frac{(3 + 4\theta)}{6(1 + \theta + \omega)}$$

$$- \frac{8}{3}\kappa\beta \sigma_o\pi_e\{\frac{\theta\omega}{1 + \theta + \omega} - \delta^* \frac{[3(1+\omega) + 2\theta^2 + 6\theta(1+\omega)]}{6(1+\theta+\omega)}\} \frac{(3 + 4\theta)}{6(1+\theta+\omega)}$$

$$+ 4\kappa\beta^2\omega \frac{(2 + 3\theta)}{3(1 + \theta + \omega)}\pi_\ell$$

$$+ \frac{s^2}{4\beta} \sum_{n=1}^{\infty}[- 16\kappa\beta\{\frac{3 - 4\theta^2 + 3\omega}{3(1 + \theta + \omega)} \frac{(-1)^n - 1}{2} + 2\theta(-1)^n\} \frac{1}{\lambda_n^2\beta^2 q_n}$$

$$+ 16\kappa\beta \frac{0\omega}{1 + 0 + \omega}\{p_a - \pi_\ell\} \frac{(-1)^n - 1}{\lambda_n^2\beta^2 g_n(1)}$$

$$+ 16\kappa\beta \sigma_o\pi_e\{[\frac{0\omega}{1 + 0 + \omega} - \delta^*\frac{(3-40^2+3\omega)}{6(1+0+\omega)}] \frac{(-1)^n - 1}{\lambda_n^2\beta^2 g_n(1)}$$

$$- 20\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2 g_n(1)}\}]k_n(\gamma)\cos \lambda_n\beta\xi\} \ln \zeta$$

$$- \frac{s^2}{4\beta} \sum_{n=1}^{\infty}[- 16\kappa\beta\{\frac{3 - 40^2 + 3\omega}{3(1 + 0 + \omega)} \frac{(-1)^n-1}{2} + 20(-1)^n\}\frac{1}{\lambda_n^2\beta^2 g_n}$$

$$+ 16\kappa\beta \frac{\theta\omega}{1 + 0 + \omega}\{p_a - \pi_\ell\} \frac{(-1)^n - 1}{\lambda_n^2\beta^2 g_n(1)}$$

$$+ 16\kappa\beta\sigma_o\pi_e\{[\frac{\theta\omega}{1 + \theta + \omega} - \delta^*\frac{(3 - 4\theta^2 + 3\omega)}{6(1 + \theta + \omega)}] \frac{(-1)^n - 1}{\lambda_n^2\beta^2 g_n(1)}$$

$$- 2\theta\delta^*\frac{(-1)^n}{\lambda_n^2\beta^2 g_n(1)}\}]k_n(\gamma)\cos \lambda_n\beta\xi$$

$$p_2 = \{\frac{1 + \theta}{1 + \theta + \omega} - \frac{4}{3}\kappa\beta^2\frac{(9 + 21\theta + 11\theta^2)}{3(1 + \theta + \omega)}\}p_a - \frac{(3 + 4\theta)}{6(1+ \theta + \omega}$$

$$- \frac{4}{3}\kappa\beta^2\{\frac{(21 + 42\theta + 16\theta^2)\theta}{90(1 + \theta + \omega)^2} - \omega\frac{(45 + 113\theta + 68\theta^2)}{30(1 + \theta + \omega)^2}\}$$

$$+ \frac{\sigma_o\pi_e}{1 + \theta + \omega}\{(1 + \theta) - \delta^*\frac{(3 + 4\theta)}{6}\} + \frac{\pi_\ell\omega}{1 + \theta + \omega}$$

$$+ \frac{4}{3}\kappa\beta^2\omega\{\frac{3 + 6\theta + 2\theta^2 - \omega(6 + 9\theta)}{3(1 + \theta + \omega)^2}\}\pi_\ell$$

$$- \frac{4}{3}\kappa\beta^2\sigma_o\pi_e\{\omega\frac{(9 + 21\theta + 11\theta^2)}{3(1 + \theta + \omega)^2}$$

$$+ \delta^* \left[\frac{6\theta^2 + 16\theta^3 + 21\theta(1+\omega) + 36\theta^2(1+\omega) - 15\omega(3+4\theta)^2}{90(1+\theta+\omega)^2} \right] \}$$

$$+ \kappa\omega s^2 \left[\pi_\ell \left(\frac{(1+\theta)}{1+\theta+\omega} - \left\{ \frac{1+\theta}{1+\theta+\omega} \right. \right. \right.$$

$$\left. - \frac{4}{3}\kappa\beta^2\omega \frac{(9+21\theta+11\theta^2)}{3(1+\theta+\omega)^2} \right\} p_a + \frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$+ \frac{4}{3}\kappa\beta^2 \left\{ \frac{(21+42\theta+16\theta^2)\theta}{90(1+\theta+\omega)^2} - \omega \frac{(45+113\theta+68\theta^2)}{30(1+\theta+\omega)^2} \right\} \}$$

$$- \frac{\sigma_o \pi_e}{1+\theta+\omega} \left\{ (1+\theta) - \delta^* \frac{(3+4\theta)}{6} \right\}$$

$$- \frac{4}{3}\kappa\beta^2\omega \left\{ \frac{3+6\theta+2\theta^2-\omega(6+9\theta)}{3(1+\theta+\omega)^2} \right\}$$

$$+ \frac{4}{3}\kappa\beta^2\sigma_o\pi_e \left\{ \omega \frac{(9+21\theta+11\theta^2)}{3(1+\theta+\omega)^2} \right.$$

$$+ \delta^* \left[\frac{6\theta^2 + 16\theta^3 + 21\theta(1+\omega) + 36\theta^2(1+\omega) - 15\omega(3+4\theta)^2}{90(1+\theta+\omega)^2} \right]$$

$$+ \frac{s^2}{4\beta^3} \sum_{n=1}^{\infty} \left[16\kappa\beta \left\{ \frac{3-4\theta^2+3\omega}{3(1+\theta+\omega)} \frac{1-(-1)^n}{2} - 2\theta(-1)^n \right\} \frac{1}{\lambda_n^2 g_n(1)} \right.$$

$$- 16\kappa\beta \frac{\theta\omega}{1+\theta+\omega} \{p_a - \pi_\ell\} \frac{1-(-1)^n}{\lambda_n^2 g_n(1)}$$

$$- 16\kappa\beta\sigma_o\pi_e \left\{ \left[\frac{\theta\omega}{1+\theta+\omega} - \delta^* \frac{(3-4\theta^2+3\omega)}{6(1+\theta+\omega)} \right] \frac{1-(-1)^n}{\lambda_n^2 g_n(1)} \right.$$

$$\left. \left. + 2\theta\delta^* \frac{(-1)^n}{\lambda_n^2 g_n(1)} \right\} \right] k_n(\gamma) \cos \lambda_n \beta\xi \ln\zeta$$

$$- \frac{s^2}{4\beta^3} \sum_{n=1}^{\infty} \left[16\kappa\beta \frac{3-4\theta^2+3\omega}{3(1+\theta+\omega)} \frac{1-(-1)^n}{2} - 2\theta(-1)^n \right\} \frac{1}{\lambda_n^2 g_n(1)}$$

$$- 16\kappa\beta \frac{\theta\omega}{1+\theta+\omega} \{p_a - \pi_\ell\} \frac{1-(-1)^n}{\lambda_n^2 g_n(1)}$$

$$- 16\kappa\beta\sigma_o\pi_e\{[\frac{\theta\omega}{1+\theta+\omega} - \delta^*\frac{(3-4\theta^2+3\omega)}{6(1+\theta+\omega)}]\frac{1-(-1)^n}{\lambda_n^2 g_n(1)}$$

$$+ 2\theta\delta^*\frac{(-1)^n}{\lambda_n^2 g_n(1)}]]k_n(\zeta)\cos\lambda_n\beta\xi$$

For the terms in the bracket with $\ln\zeta$, terms with $\kappa^2 s^2$ can be neglected in comparison with terms of the order of $\kappa^2 s^4$. Thus we obtain p_2 as

$$p_2 = \{\frac{1}{2}(p_a + p_v)\}\frac{1+\theta}{1+\theta+\omega} - \frac{(\theta - 6\omega\pi_\ell)}{6(1+\theta+\omega)}$$

$$- \frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\}$$

$$- \frac{4}{3}\kappa\beta^2[\omega\frac{(9+21\theta+11\theta^2)}{3(1+\theta+\omega)^2}p_a$$

$$+ \{\frac{(21+42\theta+16\theta^2)}{90(1+\theta+\omega)^2} - \omega\frac{(45+113\theta+68\theta^2)}{30(1+\theta+\omega)^2}\}$$

$$- \omega\frac{[3+6\theta+2\theta^2-\omega(6+9\theta)]}{3(1+\theta+\omega)^2}\pi_\ell + \sigma_o\pi_e\{\omega\frac{(9+21\theta+11\theta^2)}{3(1+\theta+\omega)^2}$$

$$+ \delta^*\frac{[6\theta^2+16\theta^3+21\theta(1+\omega)+36\theta^2(1+\omega)-15\omega(3+4\theta)^2]}{90(1+\theta+\omega)^2}\{]$$

$$+ \frac{4}{3}\kappa\beta^2 \times \omega s^2[\frac{3}{4\beta^2}\{\frac{\theta+6\pi_\ell(1+\theta)}{6(1+\theta+\omega)} - \{\frac{1}{2}(p_a+p_v)\}\frac{1+\theta}{1+\theta+\omega}$$

$$- \frac{\sigma_o\pi_e}{\theta\quad\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\}\}$$

$$+ \frac{3s^2}{\beta^4}\sum_{n=1}^{\infty} A_n(\theta)k_n(\gamma)\cos\lambda_n\beta\xi]\ln\zeta$$

$$- \frac{4}{3}\kappa\beta^2 \times \frac{3s^2}{\beta^4}\sum_{n=1}^{\infty} A_n(\theta)k_n(\zeta)\cos\lambda_n\beta\xi$$

thus p_2 may be written as

$$p_2 = \{\frac{1}{2}(p_a + p_v)\} \frac{1 + \theta}{1 + \theta + \omega} - \frac{(\theta - 6\omega\pi_\ell)}{5(1 + \theta + \omega)}$$

$$+ \frac{\sigma_o \pi_e}{1 + \theta + \omega} \{(1 + \theta) - \delta^* \frac{(3 + 4\theta)}{6}\} - \frac{4}{3}\kappa\beta^2 \Phi(\zeta, \xi)$$

$$u_2 = - \sum_{n=1}^{\infty} B_n f_n(\zeta) \sin \lambda_n \beta\xi$$

$$= - \frac{16\kappa\beta}{\beta^2} \sum_{n=1}^{\infty} A_n(\theta) f_n(\zeta) \sin \lambda_n \beta\xi$$

$$\therefore u_2 = -\frac{16\kappa}{\beta} \sum_{n=1}^{\infty} A_n(\theta) f_n(\zeta) \sin \lambda_n \beta\xi$$

$$v_2 = - \frac{4\beta\kappa_\ell \pi_\ell \gamma}{\zeta} + \frac{4\beta\kappa_\ell \gamma}{\zeta} [\frac{p_o + as^2 \pi_\ell}{1 + as^2}$$

$$- \frac{s^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta\xi]$$

$$+ \sum_{n=1}^{\infty} B_n g_n(\zeta) \cos \lambda_n \beta\xi$$

$$= - \frac{4\beta\kappa_\ell \gamma}{(1 + as^2)\zeta} [\pi_\ell - p_a \frac{(1 + \theta)}{1 + \theta + \omega}$$

$$- \frac{\sigma_o \pi_e}{1 + \theta + \omega} \{(1 + \theta) - \delta^* \frac{(3 + 4\theta)}{6}\}$$

$$+ \frac{(3 + 4\theta)}{6(1 + \theta + \omega)} - \frac{\pi_\ell \omega}{1 + \theta + \omega}$$

$$+ \frac{s^2}{4\beta} \sum_{n=1}^{\infty} \frac{16\kappa\beta}{\beta^2} A_n(\theta) k_n(\gamma) \cos \lambda_n \beta\xi.]$$

$$+ \frac{16\kappa\beta}{\beta^2} \sum_{n=1}^{\infty} A_n(\theta) g_n(\zeta) \cos \lambda_n \beta\xi$$

$$\therefore v_2 = -\frac{4\kappa\beta\omega}{\zeta}\left[\pi_\ell\frac{(1+\theta)}{1+\theta+\omega} - p_a\frac{(1+\theta)}{1+\theta+\omega}\right.$$

$$-\frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\} + \frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$\left.+\frac{4\kappa s^2}{\beta^2}\sum_{n=1}^{\infty} A_n(0)k_n(\gamma)\cos\lambda_n\beta\xi\right]$$

$$+\frac{16\kappa}{\beta}\sum_{n=1}^{\infty} A_n(0)g_n(\zeta)\cos\lambda_n\beta\xi$$

In region II, the velocity and pressure distributions are given by

$$u_2 = -\frac{16\kappa}{\beta}\sum_{n=1}^{\infty} A_n(0)f_n(\zeta)\sin\lambda_n\beta\xi, \text{ ----------(98)}$$

$$v_2 = -\frac{4\kappa\beta\omega}{\zeta}\left[\pi_\ell\frac{(1+\theta)}{1+\theta+\omega} - p_a\frac{(1+\theta)}{1+\theta+\omega}\right.$$

$$-\frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\} + \frac{(3+4\theta)}{6(1+\theta+\omega)}$$

$$\left.+\frac{4\kappa s^2}{\beta^2}\sum_{n=1}^{\infty} A_n(0)k_n(\gamma)\cos\lambda_n\beta\xi\right]$$

$$+\frac{16\kappa}{\beta}\sum_{n=1}^{\infty} A_n(0)g_n(\zeta)\cos\lambda_n\beta\xi, \text{ ---------(99)}$$

$$p_2 = \{\frac{1}{2}(p_a + p_v)\}\frac{1+\theta}{1+\theta+\omega} - \frac{(\theta - 6\omega\pi_\ell}{6(1+\theta+\omega)}$$

$$+\frac{\sigma_o\pi_e}{1+\theta+\omega}\{(1+\theta) - \delta^*\frac{(3+4\theta)}{6}\} - \frac{4}{3}\kappa\beta^2\Phi(\zeta,\xi), \text{ --(100)}$$

where

$$A_n(0) = \frac{1}{\lambda_n^2 g_n(1)}\left[\{\frac{3 - 4\theta^2 + 3\omega}{3(1+\theta+\omega)}\frac{1-(-1)^n}{2} - 2\theta(-1)^n\right.$$

$$\left.-\frac{\theta\omega}{1+\theta+\omega}\{p_a - \pi_\ell\}(1-(-1)^n)\right.$$

$$- \sigma_o \pi_e \{ [\frac{0\omega}{1 + 0 + \omega} - \delta^* \frac{(3 - 40^2 + 3\omega}{6(1 + 0 + \omega)}] (1 - (-1)^n)$$

$$+ 20\delta^* (-1)^n \}], \text{ ---------- (101)}$$

$$\Phi(\zeta,\xi) = \omega \frac{(9 + 210 + 110^2)}{3(1 + 0 + \omega)^2} p_a - \omega \frac{[3+60+20^2-\omega(6+90)]}{3(1 + 0 + \omega)^2} \pi_\ell$$

$$+ \{ \frac{(21 + 42\theta + 16\theta^2)\theta}{90(1 + \theta + \omega)^2} - \omega \frac{(45 + 113\theta + 68\theta^2)}{30(1 + \theta + \omega)^2} \}$$

$$+ \sigma_o \pi_e \{ \omega \frac{(9 + 21\theta + 11\theta^2)}{3(1 + \theta + \omega)^2}$$

$$+ \delta^* \frac{[6\theta^2 + 16\theta^3 + 21\theta(1 + \omega) + 36\theta^2(1+\omega) - 15\omega(3+4\theta)^2]}{90(1 + \theta + \omega)^2} \}$$

$$- \omega s^2 [\frac{3}{4\beta^2} \{ \frac{\theta + 6\pi_\ell (1 + \theta)}{6(1 + \theta + \omega)} - \{ \frac{1}{2}(p_a + p_v) \} \frac{1 + \theta}{1 + \theta + \omega}$$

$$- \frac{\sigma_o \pi_e}{1 + \theta + \omega} \{ (1 + \theta) - \delta^* \frac{(3 + 4\theta)}{6} \} \}$$

$$+ \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(\theta) k_n(\gamma) \cos \lambda_n \beta\xi] \ln\zeta$$

$$+ \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(\theta) k_n(\zeta) \cos \lambda_n \beta\xi, \text{ ---------- (102)}$$

4.5. Volume Of Flow Per Unit Time:

The volume of the fluid flowing per unit time across the cross section of the capillary at a point z(or ξ) is given by

$$Q_c(z) = \int_0^{R_c} U_1(r,z) 2r dr, \text{ ---------- (103)}$$

$$Q_c(\xi) = 2\pi R_c^2 U_o \int_0^1 u_1(\zeta,\xi) \zeta d\zeta, \text{ ---------- (104)}$$

Using equations (92) and (94),

$$\therefore Q_c(\xi) = 2\pi R_c^2 U_o \int_o^1 (1 - \zeta^2)\{1 + \tfrac{4}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}\}\zeta d\zeta$$

$$= 2\pi R_c^2 U_o \int_o^1 [\zeta + \tfrac{4}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}\zeta - \zeta^3 - \tfrac{4}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}\zeta^3]d\zeta$$

$$= 2\pi R_c^2 U_o [\frac{\zeta^2}{2} + \tfrac{4}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}\frac{\zeta^2}{2} - \frac{\zeta^4}{4} - \tfrac{4}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}\frac{\zeta^4}{4}]_o^1$$

$$= 2\pi R_c^2 U_o [\frac{1}{4} + \frac{1}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}]$$

$$Q_c(\xi) = \frac{1}{2}\pi R_c^2 U_o [1 + \tfrac{4}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}]$$

$$Q_c(\xi)/Q_o = 1 + \tfrac{4}{3}\kappa\beta^2 \frac{d\Psi}{d\xi}, \quad \text{---------+ --------(105)}$$

$$= -\frac{dp_1}{d\xi}, \quad \text{---(106)}$$

where $Q_o = \frac{1}{2}\pi R_c^2 U_o$, ---(107)

The total amount of fluid entering the capillary per unit time, the total flow rate Q_{ct}, is given by

$$Q_{ct} = Q_c(\xi = 0), \quad \text{---(108)}$$

The value of the fluid flowing per unit time across the capillary wall z = 0 up to z = z is given by

$$Q_t(z) = \int_o^z V_1(r = R_c, z)2\pi R_c \, dz$$

$$Q_t = 2\pi R_c \, LU_o \int_o^\xi V_1(\zeta = 1, \xi)d\xi, \quad \text{---(109)}$$

Using equations (93) and (94)

$$\therefore Q_t(\xi) = 2\pi R_c LU_o \int_o^{\xi} - \frac{1}{3}\kappa\beta^2 \frac{d^2\Psi}{d\xi^2} d\xi$$

$$= - \frac{2}{3}\pi R_c LU_o \kappa\beta [\frac{d\Psi}{d\xi}]_o^{\xi} = -\frac{2}{3}\pi R_c LU_o \kappa\beta [(\frac{d\Psi}{d\xi})_{\xi=\xi} - (\frac{d\Psi}{d\xi})_{\xi=0}]$$

$$= \frac{2}{3}\pi R_c U_o \kappa\beta [(\frac{d\Psi}{d\xi})_{\xi=0} - (\frac{d\Psi}{d\xi})_{\xi=\xi}]$$

$$Q_t(\xi)/Q_o = \frac{4}{3}\kappa\beta^2 [(\frac{d\Psi}{d\xi})_{\xi=0} - (\frac{d\Psi}{d\xi})_{\xi=\xi}$$

$$= [- 1 - (\frac{dp_1}{d\xi})_{\xi=0} + (\frac{dp_1}{d\xi})_{\xi=\xi} + 1]$$

$$\therefore Q_t(\xi)/Q_o = (\frac{dp_1}{d\xi})_{\xi=\xi} - (\frac{dp_1}{d\xi})_{\xi=0}, \quad \text{----------(110)}$$

The net volume of fluid transferred to the tissue from the capillary per unit time, the total filtration rate Q_{tf}, is given by

$$Q_{tf} = Q_t(\xi = 1), \quad \text{----------(111)}$$

The volume of fluid flowing into the lymphatic capillaries from the tissue region per unit time, which is called by us the lymph flow Q_L, is given by

$$Q_L = \int_{z=o}^{z=L} V_2(r = R_t, z) 2\pi R_t dz, \quad \text{----------(112)}$$

$$= 2\pi R_t LU_o \int_{\xi=o}^{\xi=1} v_2(\zeta = \gamma, \xi) d\xi$$

$$= 2\pi R_t LU_o \int_o^1 [- 4\beta\pi_\ell\kappa_\ell + 4\beta\kappa_\ell \{\frac{p_o + a\pi_\ell s^2}{1 + as^2}$$

$$- \frac{s^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma) \cos \lambda_n \beta\xi\}$$

$$+ \sum_{n=1}^{\infty} B_n g_n(\gamma)\cos \lambda_n \beta\xi]d\xi$$

$$= 2\pi R_t L U_o[- 4\beta\pi_\ell\kappa_\ell \; + 4\beta\kappa_\ell \frac{p_o + a\pi_\ell s^2}{1 + as^2} \xi$$

$$- 4\beta\kappa_\ell \frac{s^2}{4\beta(1 + as^2)} \sum_{n=1}^{\infty} B_n k_n(\gamma)\sin \frac{\lambda_n \beta}{\lambda_n \beta} \xi$$

$$+ \sum_{n=1}^{\infty} B_n g_n(\gamma)\sin \frac{\lambda_n \beta}{\lambda_n \beta} \xi]_o^1$$

$$= 2\pi R_t L U_o[- 4\beta\kappa_\ell\pi_\ell + 4\beta\kappa_\ell \frac{(p_o + a\pi_\ell s^2)}{1 + as^2}]$$

$$= 2\pi R_t L U_o \cdot 4\beta\kappa_\ell \frac{(p_o - \pi_\ell)}{1 + as^2} = 8\pi\gamma R_c \beta R_c U_o \beta\kappa_\ell \frac{(p_o - \pi_\ell)}{1 + as^2}$$

$$= 8\pi R_c^2 U_o \gamma\beta^2\kappa_\ell \frac{(p_o - \pi_\ell)}{1 + as^2}$$

but $Q_o = \frac{1}{2}\pi R_c^2 U_o$

$$\therefore Q_L = 16 Q_o \gamma\beta^2\kappa_\ell \frac{(p_o - \pi_\ell)}{1 + as^2}$$

but $\omega = \frac{\gamma V_o}{\kappa\beta} \equiv \frac{\gamma\beta\kappa_\ell}{\kappa\beta(1 + as^2)} = \frac{\gamma\kappa_\ell}{\kappa(1 + as^2)}$

$$\therefore Q_L = 16\kappa\beta^2\omega(p_o - \pi_\ell) Q_o, \longrightarrow -(113)$$

The volume of fluid transferred to the tissue region per unit time over that portion of the surface of the capillary where the radial component of velocity $V_1(r = R_c, z)$ is positive, which is called by us the filtration rate Q_f, is a physiologically important quantity. The filtration rate Q_f is the maximum of (

From equations (110) and (96) we have

$$Q_t(\xi)/Q_o = \frac{4}{3}\kappa\beta^2\left[\left(\frac{d\Psi}{d\xi}\right)_{\xi=o} - \left(\frac{d\Psi}{d\xi}\right)_{\xi=\xi}\right]$$

and

$$\left(\frac{d\Psi}{d\xi}\right)_{\xi=o} - \left(\frac{d\Psi}{d\xi}\right)_{\xi=\xi} = 2\xi\,\frac{1}{1+\theta+\omega}\left[3+4\theta-6\omega\pi_\ell+6\omega p_a\right.$$

$$\left. + \sigma_o\pi_e\{6\omega + \delta^*(3+4\theta)\}\right] - 3\xi^2\,\frac{1}{3(1+\theta+\omega)}\left[6-8\theta^2+12\omega\theta\pi_\ell\right.$$

$$\left. - 12\theta\omega p_a + 6\omega - 2\sigma_o\pi_e\{6\theta\omega - \delta^*(3-4\theta^2+3\omega)\}\right.$$

$$\left. - \xi^3[8\theta(1+\sigma_o\pi_e\delta^*)\right.$$

$$\therefore\ Q_t(\xi)/Q_o = \frac{4}{3}\kappa\beta^2[2B\xi - 3C\xi^2 - 4D\xi^3]$$

when $\dfrac{dQ_t(\xi)}{d\xi} = 0$

we have $2B - 6C\xi - 12D\xi^2 = 0$

or $-B + 3C\xi + 6D\xi^2 = 0$

where

$$B = \frac{3+4\theta-6\omega\pi_\ell+6\omega p_a+\sigma_o\pi_e\{6\omega+\delta^*(3+4\theta)\}}{1+\theta+\omega}$$

$$C = \frac{2(3-4\theta^2+6\omega\theta\pi_\ell-6\omega\theta p_a+3\omega)-2\sigma_o\pi_e\{6\theta\omega-\delta^*(3-4\theta^2+3\omega)\}}{3(1+\theta+\omega)}$$

$$D = 2\theta(1+\sigma_o\pi_e\delta^*)$$

Thus we have

$$12\theta(1+\sigma_o\pi_e\delta^*)\xi^2 +$$

$$+\frac{[2(3-4\theta^2+6\omega\theta\pi_\ell-6\omega\theta p_a+3\omega)-2\sigma_o\pi_e\{6\theta\omega-\delta^*(3-4\theta^2+3\omega)\}]\xi^2}{1+\theta+\omega}$$

$$-\frac{[3+4\theta-6\omega\pi_\ell+6\omega p_a+\sigma_o\pi_e\{6\omega+\delta^*(3+4\theta)\}]}{1+\theta+\omega} = 0$$

Factorizing we have

$$[\xi(1+\sigma_o\pi_e\delta^*) - \frac{[3+40-6\omega\pi_\ell+6\omega p_a+\sigma_o\pi_e\{6\omega+\delta^*(3+40)\}]}{6(1+0+\omega)}]$$

$$x\ [120\xi + 6] = 0$$

Hence, either $120\xi = -6 \Rightarrow \xi = -\frac{1}{20}$

or

$$\xi(1+\sigma_o\pi_e\delta^*) = \frac{3+40-6\omega\pi_\ell+6\omega p_a+\sigma_o\pi_e\{6\omega+\delta^*(3+40)\}}{6(1+0+\omega)}$$

$$\xi = \frac{3+40-6\omega\pi_\ell+6\omega p_a+\sigma_o\pi_e\{6\omega+\delta^*(3+40)\}}{6(1+0+\omega)(1+\sigma_o\pi_e\delta^*)}$$

Thus z_o is the point at which $Q_t(z)$ has a maximum value and is given by

$$\xi_o = z_o/L = \frac{3+4\theta-6\omega\pi_\ell+6\omega p_a+\sigma_o\pi_e\{6+\delta^*(3+4\theta)\}}{6(1+\theta+\omega)(1+\sigma_o\pi_e\delta^*)};\text{---(114)}$$

we have

$$Q_t(\xi)/Q_o = \frac{4}{3}\kappa\beta^2[2B\xi - 3C\xi^2 - 4D\xi^3]$$

at $\xi = \xi_o$

$$= \frac{4}{3}\kappa\beta^2[\frac{2B}{\xi_o} - 3C - 4D\xi_o^2]\xi_o^2$$

but $6(1+\sigma_o\pi_e\delta^*)\xi_o = B$

Thus

$$Q_t(\xi)/Q_o = 16\kappa\beta^2(1+\sigma_o\pi_e\delta^*)[1 - \frac{C}{4(1+\sigma_o\pi_e\delta^*)}$$

$$- \frac{D}{3(1+\sigma_o\pi_e\delta^*)}\ \xi_o]\xi_o^2$$

after substituting for C and D we finally obtain

$$Q_t(\xi)/Q_o = 8\kappa\beta^2(1 + \sigma_o\pi_e\delta^*)[(1 - \sigma_o\pi_e\delta^*) + \frac{2\theta}{3}\xi_o(1 + 3\sigma_o\pi_e\delta^*)]\xi_o^2$$

Thus the filtration rate Q_f is given by

$$Q_f = Q_t(z = 0)$$

$$Q_f/Q_o = 8\kappa\beta^2(1 + \sigma_o\pi_e\delta^*)[(1 - \sigma_o\pi_e\delta^*) + \frac{2\theta}{3}\xi_o(1 + 3\sigma_o\pi_e\delta^*)]\xi_o^2, \quad \text{---(115)}$$

Let us put

$$Q_L = \delta Q_f \quad (0 \leq \delta \leq 1), \quad \text{---(116)}$$

then we have

$$16\kappa\beta^2\omega(p_o - \pi_\ell)Q_o = \delta 8\kappa\beta^2(1 + \sigma_o\pi_e\delta^*)[(1 - \sigma_o\pi_e\delta^*) + \frac{2\theta}{3}\xi_o(1 + 3\sigma_o\pi_e\delta^*)]\xi_o^2 Q_o$$

or

$$\omega(p_o - \pi_\ell) = \delta\frac{(1 + \sigma_o\pi_e\delta^*)}{2}[(1 - \sigma_o\pi_e\delta^*) + \frac{2\theta}{3}\xi_o(1 + 3\sigma_o\pi_e\delta^*)]\xi_o^2, \quad \text{---(117)}$$

Equation (116) corresponds to assumption (vii). Substituting equations (113 and (115) with (114) into equation (116) and solving this resultant equation, we get

for $\delta = 0$, $\omega = 0$

for $\delta \neq 0$, $\theta = 0$

$$4\omega^2[2(1 + \sigma_o\pi_e\delta^*)\{(p_a - \pi_\ell) + \sigma_o\pi_e\} - (1 + \sigma_o\pi_e\delta^*)^2 - \delta(p_a - \pi_\ell + \sigma_o\pi_e\delta^*)^2(1 - \sigma_o\pi_e\delta^*)]$$

$$+ 4\omega[2(1 + \sigma_o\pi_e\delta^*)\{p_a - \pi_\ell) + \sigma_o\pi_e\} - (1 + \sigma_o\pi_e\delta^*)^2$$

$$- \delta(1 - \sigma_o\pi_e\delta^*)(1 + \sigma_o\pi_e\delta^*)\{p_a - \pi_\ell + \sigma_o\pi_e\}]$$

$$- \delta^*(1 + \sigma_o\pi_e\delta^*)^2(1 - \sigma_o\pi_e\delta^*) = 0, \text{ ----------(118)}$$

For the special case if $\delta^* = 0$, $\sigma_o = 1$, we get

$$4\omega^2[\delta(p_a - \pi_\ell)^2 - 2(p_a - \pi_\ell + \pi_e) + 1]$$

$$+ 4\omega[\delta(p_a - \pi_\ell + \pi_e) - 2(p_a - \pi_\ell + \pi_e) + 1] + \delta = 0, \text{--(119)}$$

for $\delta \neq 0$, $\theta \neq 0$, we have

$$\xi_o^2 = \frac{1}{36(1 + \theta + \omega)^2(1 + \sigma_o\pi_e\delta^*)^2}[(3+4\theta)^2(1+\sigma_o\pi_e\delta^*)^2$$

$$+ \omega\{12(3 + 4\theta)(1 + \sigma_o\pi_e\delta^*)(p_a - \pi_\ell + \sigma_o\pi_e)\}$$

$$+ 36\omega^2(p_a - \pi_\ell + \sigma_o\pi_e)^2], \text{ ---+---+-----(120)}$$

Substituting for ξ_o^2 in equation (117), we get a cubic equation of the form

$$a_o\omega^3 + a_1\omega^2 + a_2\omega + a_3 = 0, \text{ ----------(121)}$$

where the coefficients a_o, a_1, a_2 and a_3 are given by

$$a_o = 9(1 - (\sigma_o\pi_e\delta^*)^2) \times 36(p_a - \pi_\ell + \sigma_o\pi_e)^2$$

$$+ 2160(1 + 3\sigma_o\pi_e\delta^*)(p_a - \pi_\ell + \sigma_o\pi_e)^3$$

$$- \frac{648}{\delta}(1 + \sigma_o\pi_e\delta^*)^2\{(1 + 0)(p_a - \pi_\ell + \sigma_o\pi_e)$$

$$- \frac{(3 + 4\theta)}{6}(1 + \sigma_o\pi_e\delta^*)\}, \text{ ----------+----(122)}$$

$$a_1 = 9(1 - (\sigma_o\pi_e\delta^*)^2)\{36(1 + \theta)(p_a - \pi_\ell + \sigma_o\pi_e)^2$$

$$+ 12(3 + 4\theta)(p_a - \pi_\ell + \sigma_o\pi_e)(1 + \sigma_o\pi_e\delta^*)\}$$

$$+ 108\theta(1 + 3\sigma_o\pi_e\delta^*)(3 + 4\theta)(1 + \sigma_o\pi_e\delta^*)x(p_a-\pi_\ell+\sigma_o\pi_e)^2$$

$$- \frac{648}{\delta}(1 + \sigma_o\pi_e\delta^*)^2\{2(1 + \theta)^2(p_a - \pi_\ell + \sigma_o\pi_e)$$

$$- 2\frac{(3 + 4\theta)}{6}(1 + \theta)(1 + \sigma_o\pi_e\delta^*)\}, \text{ ---------(123)}$$

$$a_2 = 9(1 -(\sigma_o\pi_e\delta^*)^2)\{12(3 + 4\theta)(1+\theta)(p_a-\pi_\ell+\sigma_o\pi_e)$$

$$x\ (1 + \sigma_o\pi_e\delta^*) + (3 + 4\theta)^2(1 + \sigma_o\pi_e\delta^*)^2\}$$

$$+ 180(1 + 3\sigma_o\pi_e\delta^*)(3 + 4\theta)^2(p_a- \pi_\ell+\sigma_o\pi_e)(1+\sigma_o\pi_e\delta^*)^2$$

$$- \frac{648}{\delta}(1 + \sigma_o\pi_e\delta^*)^2\{1 + 4\theta)^3(p_a - \pi_\ell + \sigma_o\pi_e)$$

$$- \frac{(3 + 4\theta)}{6}(1 + \theta)^2(1 + \sigma_o\pi_e\delta^*)\}, \text{ ----------(124)}$$

$$a_3 = 9(1 -(\sigma_o\pi_e\delta^*)^2)(3 + 4\theta)^2(1 + \theta)(1 + \sigma_o\pi_e\delta^*)^2$$

$$+ \theta(1 + 3\sigma_o\pi_e\delta^*)(3 + 4\theta)^3(1 + \sigma_o\pi_e\delta^*)^3, \text{ -----(125)}$$

In order to solve the cubic equation (121), we make the substitution

$$\omega = y - \frac{a_1}{3a_o}$$

Substituting, we obtain

$$a_o(y - \frac{a_1}{3a_o})^3 + a_1(y - \frac{a_1}{3a_o})^2 + a_2(y - \frac{a_1}{3a_o}) + a_3 = 0$$

i.e.

$$a_o(y^3 - 3y^2\frac{a_1}{3a_o} + 3y\frac{a_1^2}{9a_o^2} - \frac{a_1^3}{27a_o^3})$$

$$+ a_1(y^2 - 2y\frac{a_1}{3a_o} + \frac{a_1^2}{9a_o^2}) + a_2y - \frac{a_1a_2}{3a_o} + a_3 = 0$$

which gives

$$a_oy^3 + y(a_2 - \frac{a_1^2}{3a_o}) + (a_3 - \frac{a_1a_2}{3a_o} + \frac{2a_1^3}{27a_o^2}) = 0$$

or

$$y^3 + y(\frac{a_2}{a_o} - \frac{a_1^2}{3a_o^2}) + (\frac{a_3}{a_o} - \frac{a_1a_2}{3a_o^2} + \frac{2a_1^3}{27a_o^2}) = 0, \text{----(126)}$$

which is in the form

$$y^3 + Ay + B = 0$$

where

$$A = \frac{3a_oa_2 - a_1^2}{3a_o^2}, \quad B = \frac{27a_o^2a_3 - 9a_oa_1a_2 + 2a_1^3}{27a_o^3}$$

Now assume a solution in the form

$$y = \lambda \cos \phi$$

then we have

$$\lambda^3 \cos^3\phi + A\lambda \cos \phi + B = 0$$

which can be compared to

$$4 \cos^3\phi - 3 \cos \phi = \cos 3\phi$$

comparing coefficients we have

$$\frac{\lambda^3}{4} = \frac{A\lambda}{-3} = -\frac{B}{\cos \phi}$$

from $\frac{\lambda^3}{4} = -\frac{A\lambda}{3} \Rightarrow \lambda = 0$ or $\lambda = \pm\sqrt{\frac{4A}{-3}} = \pm 2\sqrt{\frac{-A}{3}}$

Similarly, $\cos 3\phi = \frac{3B}{A\lambda} = \cos \Psi$

Thus

$$\cos \Psi = \frac{3B}{2A} \sqrt{\frac{-3}{A}}$$

$$\phi = \frac{\cos^{-1}(\frac{3B}{2A} \sqrt{\frac{-3}{A}}) \pm 2\pi k}{3}, \quad n = 3, \; k = 0,1,2$$

Thus

$$y = \pm 2\sqrt{\frac{A}{-3}} \cos[\frac{2\pi k \pm \cos^{-1}(\frac{3B}{2A} \sqrt{\frac{-3}{A}})}{3}], \quad k = 0,1,2$$

finally we obtain the solution of ω as

$$\omega = y - \frac{a_1}{3a_o}, \qquad (127)$$

For calculations we take k = 0 and positive values of y because the other value of k will give negative values of y in which we are not interested. Thus for different values of k = 0,1,2 and also taking positive and negative values we will see that there are three roots of y, one root is positive for k = 0 and two roots are negative for k = 1,2.

For example, if

$$\cos^{-1}(\frac{3B}{2A} \sqrt{\frac{-3}{A}}) = 60° = \frac{\pi}{3}$$

$$y = \pm \lambda \cos[\frac{2\pi k}{3} \pm \frac{\pi}{9}]$$

$$\frac{2\pi k}{3} \pm \frac{\pi}{9} = \pm \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}, \frac{11\pi}{9}, \frac{13\pi}{9}, \text{ for } k = 0,1,2$$

Thus for positive λ, three different roots are $y_1 = \lambda \cos \pi/9$, $y_2 = \lambda \cos 5\pi/9$, $y_3 = \lambda \cos 7\pi/9$; for negative λ, we will get same roots. Only y_1 is

positive, y_2 and y_3 are negative. Thus we take y_1 only i.e.

$$y = 2\sqrt{\frac{-A}{3}}\cos\left[\frac{\cos^{-1}\left(\frac{3B}{2A}\sqrt{\frac{-3}{A}}\right)}{3}\right]$$

$$\omega = y - \frac{a_1}{3a_o}$$

3.6. Average Pressure:

The average value of the pressure over the volume of region I, which is denoted by $\langle P_1\rangle$, is defined by

$$\langle P_1\rangle = \frac{1}{L}\int_o^L P_1(z)dz, \text{ ----- (128)}$$

but

$$\Delta P\langle p_1\rangle = \frac{1}{L}\int_o^1 \Delta P\, p_1(\xi)L d\xi$$

$$\therefore \langle p_1\rangle = \int_o^1 p_1(\xi)d\xi = \int_o^1\{p_a - \xi - \frac{4}{3}\kappa\beta^2\Psi(\xi)\}d\xi$$

$$= \int_o^1[p_a - \xi - \frac{4}{3}\kappa\beta^2\{\xi\frac{1}{3(1+0+\omega)}(3+60+20^2$$

$$- 6\omega\pi_\ell(3+20) - 6\omega(1+0) + 6\omega(3+20)p_a$$

$$+ \sigma_o\pi_e\{6\omega(3+20) + \delta^*[3+60+20^2 - 6\omega(1+0)]\})$$

$$- \xi^2\cdot\frac{1}{1+0+\omega}(3+40 - 6\omega\pi_\ell + 6\omega p_a$$

$$+ \sigma_o\pi_e\{6\omega + \delta^*(3+40)\})$$

$$+ \xi^3\cdot\frac{1}{3(1+0+\omega)}(2[3 - 40^2 + 6\omega 0\pi_\ell - 6\omega 0 p_a + 3\omega]$$

$$- 2\sigma_o\pi_e\{60\omega - \delta^*(3 - 40^2 + 3\omega)\})$$

$$+ \xi^4[2\ (1 + \sigma_o\pi_e\delta^*)]\}]d\xi$$

$$= p_a - \frac{1}{2} - \frac{4}{3}\kappa\beta^2[\frac{1}{6(1 + 0 + \omega)}[3 + 60 + 20^2$$

$$- 6\omega\pi_\ell(3 + 20) - 6\omega(1 + 0) + 6\omega(3 + 20)p_a$$

$$+ \sigma_o\pi_e\{6\omega(3 + 20) + \delta^*[3 + 60 + 20^2 - 6\omega(1 + o)]\}]$$

$$- \frac{1}{3(1+0+\omega)}[3+40-6\omega\pi_\ell+6\omega p_a+\sigma_o\pi_e\{6\omega+\delta^*(3+40)\}]$$

$$+ \frac{1}{12(1 + 0 + \omega)}[2\{3 + 40^2 + 6\omega 0\pi_\ell - 6\omega 0p_a + 3\omega)$$

$$- 2\sigma_o\pi_e\{60\omega - \delta^*(3 - 40^2 + 3\omega)\}] + \frac{1}{5}\{20(1+\sigma_o\pi_e\delta^*)\}]$$

Thus we obtain

$$\langle p_1\rangle = \frac{1}{2}(p_a + p_v) \cdot \frac{2\kappa\beta^2}{45(1+0+\omega)}[20(1+0)+30\omega(1+0)(p_a-\pi_\ell)$$

$$- 3\omega(5 + 60) + \sigma_o\pi_e\{30\omega(1 + 0)$$

$$+ \delta^*[20(1 + 0) - 3\omega(5 + 60)]\}],\text{----------(129)}$$

Similarly, the average pressure over the volume of region II is defined by

$$\langle p_2\rangle = \frac{1}{\pi(R_t^2 - R_c^2)L}\int_0^L\int_{R_c}^{R_t} p_2(r,z)2\pi r dr dz,\text{---(130)}$$

$$\Delta P\langle p_2\rangle = \frac{1}{\pi R_c^2(\frac{R_t^2}{R_c^2} - 1)L}\int_0^1\int_{\zeta=1}^{\gamma}\Delta P p_2(\zeta,\xi)2\pi R_c.\zeta.R_c d\zeta.Ld\xi$$

$$\therefore \langle p_2\rangle = \frac{2}{\gamma^2 - 1}\int_0^1\int_{\zeta=1}^{\gamma} p_2(\zeta,\xi)\zeta.d\zeta.d\xi$$

$$= \int_0^1 [\frac{2}{\gamma^2 - 1} \int_1^\gamma p_2(\zeta,\xi)\zeta.d\zeta]d\xi$$

$$= \int_0^1 \frac{2}{\gamma^2 - 1}[\{\frac{1}{2}(p_a+p_v)\} \frac{1 + 0}{1 + 0 + \omega} \frac{\zeta^2}{2} - \frac{(0-6\omega\pi_\ell)}{6(1+0+\omega)} \frac{\zeta^2}{2}$$

$$+ \frac{\sigma_o\pi_e}{1(+ 0 + \omega}\{(1 + 0) - \delta^*\frac{(3 + 40)}{6}\} \frac{\zeta^2}{2}$$

$$- \frac{4}{3}\kappa\beta^2\{\omega\frac{(9 + 210 + 110^2)}{3(1 + 0 + \omega)^2} p_a \frac{\zeta^2}{2}$$

$$- \omega\frac{[3 + 60 + 20^2 - \omega(6 + 90)]}{3(1 + 0 + \omega)^2} \pi_\ell \frac{\zeta^2}{2}$$

$$+ \{\frac{(21 + 420 + 160^2)}{90(1 + 0 + \omega)^2} - \omega\frac{(45 + 1130 + 680^2)}{30(1 + 0 + \omega)^2}\} \frac{\zeta^2}{2}$$

$$+ \sigma_o\pi_e\{\omega \frac{(9 + 21 + 11^2)}{3(1 + 0 + \omega)^2}$$

$$+\delta^*\frac{[60^2 + 160^3 + 210(1 + \omega) + 360^2(1+\omega) - 15\omega(3+40)^2]}{90(1 + 0 + \omega)^2}\} \frac{\zeta^2}{2}$$

$$- \omega s^2[\frac{3}{4\beta^2}\{\frac{0 + 6\pi_\ell(1 + 0)}{6(1 + 0 + \omega)} - \{\frac{1}{2}(p_a + p_v) \frac{1 + 0}{1 + 0 + \omega}$$

$$- \frac{\sigma_o\pi_e}{1 + 0 + \omega}\{(1 + 0) - \delta^*\frac{(3 + 40)}{6}\}\}$$

$$+ \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(0)k_n(\gamma)\cos\lambda_n\beta\xi]\{\frac{\zeta^2}{2}\ln\zeta - \frac{\zeta^2}{4}\}$$

$$+ \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(0)\frac{\lambda_{sn}}{\lambda_n^2} F(\lambda_{sn},\lambda_{sn}\gamma)G(\lambda_n\zeta,\lambda_n\gamma)\cos\lambda_n\beta\xi\}]_1^\gamma d\xi$$

$$= \int_0^1 \{\frac{1}{2}(p_a + p_v) \frac{1 + 0}{1 + 0 + \omega} - \frac{(0 - 6\omega\pi_\ell)}{6(1 + 0 + \omega)}$$

$$+ \frac{\sigma_o\pi_e}{1 + 0 + \omega}\{(1+0) - \delta^*\frac{(3+40)}{6}\} - \frac{4}{3}\kappa\beta^2\{\omega\frac{(9+210+110^2)}{3(1+ 0 +\omega)^2} p_a$$

$$- \omega \frac{[3 + 6\theta + 2\theta^2 - \omega(6 + 9\theta)]}{3(1 + \theta + \omega)^2} \pi_\ell$$

$$+ \{ \frac{(21 + 42\theta + 16\theta^2)\theta}{90(1 + \theta + \omega)^2} - \omega \frac{(45 + 113\theta + 68\theta^2)}{30(1 + \theta + \omega)^2} \}$$

$$+ \sigma_o \pi_e \{ \omega \frac{(9 + 21\theta + 11\theta^2)}{3(1 + \theta + \omega)^2}$$

$$- \delta^* \frac{[6\theta^2 + 16\theta^3 + 21\theta(1+\omega) + 36\theta^2(1+\omega) - 15\omega(3+4\theta)^2]}{90(1 + \theta + \omega)^2} \}$$

$$- \omega s^2 \left[\frac{3}{4\beta^2} \{ \frac{\theta + 6\pi_\ell(1 + \theta)}{6(1 + \theta + \omega)} - \{ \frac{1}{2}(p_a + p_v) \} \frac{1 + \theta}{1 + \theta + \omega} \right.$$

$$\left. - \frac{\sigma_o \pi_e}{1 + \theta + \omega} \{ (1 + \theta) - \delta^* \frac{(3 + 4\theta)}{6} \} \right\}$$

$$+ \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(\theta) k_n(\gamma) \cos \lambda_n \beta \xi] \frac{2\gamma^2 \ln\gamma - \gamma^2 + 1}{2(\gamma^2 - 1)}$$

$$- \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(\theta) \frac{\lambda_{sn}}{\lambda_n^2} F(\lambda_{sn}, \lambda_{sn}\gamma) G(\lambda_n, \lambda_n\gamma) \cos \lambda_n \beta \xi \}] d\xi$$

$$= [\{ \frac{1}{2}(p_a + p_v) \} \frac{1 + \theta}{1 + \theta + \omega} \xi - \frac{(\theta - 6\omega\pi_\ell)}{6(1 + \theta + \omega)} \xi$$

$$+ \frac{\sigma_o \pi_e}{1 + \theta + \omega} \{ (1 + \theta) - \delta^* \frac{(3 + 4\theta)}{6} \} \xi$$

$$- \frac{4}{3} \kappa \beta^2 \{ \omega \frac{(9 + 21\theta + 11\theta^2)}{3(1 + \theta + \omega)^2} p_a \xi$$

$$- \omega \frac{[3 + 6\theta + + 2\theta^2 - \omega(6 + 9\theta)]}{3(1 + \theta + \omega)^2} \pi_\ell \xi$$

$$+ \{ \frac{(21 + 42\theta + 16\theta^2)\theta}{90(1 + \theta + \omega)^2} - \omega \frac{(45 + 113\theta + 68\theta^2)}{30(1 + \theta + \omega)^2} \} \xi$$

$$+ \sigma_o \pi_e \{ \omega \frac{(9 + 21\theta + 11\theta^2)}{3(1 + \theta + \omega)^2}$$

$$- \delta^* \left[\frac{60^2 + 160^3 + 210(1+\omega) + 360^2(1+\omega) - 15\omega(3+40)^2}{90(1+0+\omega)^2}\right]\}\xi$$

$$- \omega s^2 \left[\frac{3}{4\beta^2}\left\{\frac{0 + 6\pi_\ell(1+0)}{6(1+0+\omega)}\right.\right. \xi$$

$$- \left\{\frac{1}{2}(p_a + p_v)\right\} \frac{1+0}{1+0+\omega} \xi$$

$$- \frac{\sigma_o \pi_e}{1+0+\omega}\left\{(1+0) - \delta^*\frac{(3+40)}{6}\right\}\xi\}$$

$$- \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(0) k_n(\gamma) \sin\frac{\lambda_n \beta}{\lambda_n \beta} \xi] \frac{2\gamma^2 \ln\gamma - \gamma^2 + 1}{2(\gamma^2 - 1)}$$

$$+ \frac{3s^2}{\beta^4} \sum_{n=1}^{\infty} A_n(0) \frac{\lambda_{sn}}{\lambda_n^2} \{F(\lambda_{sn}, \lambda_{sn}\gamma) G(\lambda_n, \lambda_n\gamma) \sin\frac{\lambda_n \beta}{\lambda_n \beta} \xi\}]_0^1$$

hence

$$\langle p_2 \rangle = \frac{1}{2}(p_a + p_v) + \sigma_o \pi_e (1 - \frac{\delta^*}{2})$$

$$- \frac{1}{6(1+0+\omega)}[(0 - 6\omega\pi_\ell) + \frac{1}{2}(p_a + p_v) 6\omega$$

$$+ \sigma_o \pi_e \{6\omega - \delta^*(3\omega - 0)\}] - \frac{4}{3}\kappa\beta^2 \langle\phi\rangle, \quad \text{---(131)}$$

where

$$\langle\phi\rangle = \frac{(21 + 420 + 160^2)0}{90(1+0+\omega)^2} - 2\omega\frac{(3 + 60 + 20^2)}{6(1+0+\omega)^2} \pi_\ell$$

$$+ \omega^2 \pi_\ell \frac{(6 + 90)}{3(1+0+\omega)^2} + \omega\frac{(9 + 210 + 110^2)}{3(1+0+\omega)^2} p_a$$

$$- \omega\frac{(45 + 1130 + 680^2)}{30(1+0+\omega)^2} + \sigma_o \pi_e \{\omega\frac{(9 + 210 + 110^2)}{3(1+0+\omega)^2}$$

$$- \delta^{*}\left[\frac{60^{2} + 160^{3} + 210(1+\omega) + 360^{2}(1+0) - 15\omega(3+40)^{2}}{90(1 + 0 + \omega)^{2}}\right]\}$$

$$- \frac{3\omega s^{2}}{4\beta^{2}}\left[\frac{0 + 6\pi_{\ell}(1 + 0)}{6(1 + 0 + \omega)} - \frac{1}{2}(p_{a} + p_{v})\frac{(1 + 0)}{1 + 0 + \omega}\right.$$

$$- \frac{\sigma_{o}\pi_{e}}{1 + 0 + \omega}\{(1 + 0) - \delta^{*}\frac{(3 + 40)}{6}\}]\ \frac{2\gamma^{2}\ln\gamma - \gamma^{2} + 1}{2(\gamma^{2} - 1)}, \quad (132)$$

Since under normal conditions the value of κ is very small, the last terms on the right-hand side of Equations (129) and (132) are very small compared with other terms. Neglecting the last terms on the right-hand side of Equations (129) and (132), we get

$$\langle p_{1}\rangle = \frac{1}{2}(p_{a} + p_{v}), \quad (133)$$

$$\langle p_{2}\rangle = \frac{1}{2}(p_{a} + p_{v}) + \sigma_{o}\pi_{e}(1 - \frac{\delta^{*}}{2})$$

$$- \frac{1}{6(1 + 0 + \omega)}[(0 - 6\omega\pi_{\ell}) + \frac{1}{2}(p_{a} + p_{v})6\omega$$

$$+ \sigma_{o}\pi_{e}\{6\omega - \delta^{*}(3\omega - 0)\}], \quad (134)$$

The filtration rate Q_{f} and the lymph flow Q_{L} (or Q_{tf}) are determined by effective pressure difference between the inside and outside of the capillary. Let us define the effective pressure difference, which is defined by ΔP_{ef}, as follows:

$$\Delta P_{ef} = [\langle P_{1}\rangle - \langle\sigma(z)\rangle\Pi_{1}]$$

$$- [\langle P_{2}\rangle - \langle\sigma(z)\rangle\Pi_{2}], \quad (135)$$

$$\Delta p_{ef} = \frac{\Delta P_{ef}}{\Delta P} = \langle p_{1}\rangle - \langle p_{2}\rangle + (1 - \frac{\delta^{*}}{2})\sigma_{o}\pi_{e}$$

$$= \frac{1}{6(1 + 0 + \omega)}[(0 - 6\omega\pi_{\ell}) + 3\omega(p_{a} + p_{v})$$

$$+ \sigma_o \pi_e \{6\omega - \delta^*(3\omega - 0)\}$$

$$+ \frac{4}{3}\kappa\beta^2 [\langle\phi\rangle - \frac{1}{30(1 + 0 + \omega)}\{20(1 + 0)$$

$$+ 30\omega(1 + 0)(p_a - \pi_\ell) - 3\omega(5 + 60)$$

$$+ \sigma_o \pi_e \{30\omega(1 + 0) + \delta^*[20(1 + 0)$$

$$- 3\omega(5 + 60)]\}\}], \quad \text{---(136)}$$

In the case where the value of κ is very small, we get

$$\Delta p_{ef} = \frac{1}{6(1 + 0 + \omega)} [(0 - 6\omega\pi_\ell) + 3\omega(p_a + p_v)$$

$$+ \sigma_o \pi_e \{6\omega - \delta^*(3\omega - 0)\}], \quad \text{---(137)}$$

which gives

$$\omega = \frac{0 - 6(1 + 0)\Delta p_{ef} + \sigma_o \pi_e \delta^* 0}{6[\Delta p_{ef} - \frac{p_a + p_v}{2} + \pi_\ell - \sigma_o \pi_e (1 - \frac{\delta^*}{2})]} \quad \text{---(138)}$$

$$1 + 0 + \omega = \frac{\{3+40\} - 6(1+0)(p_a - \pi_\ell) - \sigma_o \pi_e \{6(1+0) - \delta^*(3+40)\}}{6[\Delta p_{ef} - (p_a - \pi_\ell) + \frac{1}{2} - \sigma_o \pi_e (1 - \delta^*/2)]}$$

Using Equation (137), Equations (113), (115) and (114) are rewritten as follows:

$$Q_L/Q_o = 16\kappa\beta^2\omega(p_o - \pi_\ell)$$

$$= \frac{16\kappa\beta^2\omega}{6(1 + 0 + \omega)} [6(p_a - \pi_\ell)(1 + 0) + \sigma_o \pi_e \{6(1 + 0)$$

$$- \delta^*(3 + 40)\} - \{3 + 40)]$$

$$Q_L/Q_o = \frac{16\kappa\beta^2}{6}\{6(1+\theta)\Delta p_{ef} - \theta - \sigma_o\pi_e\delta^*\}$$

$$= 8\kappa\beta^2\{2(1+\theta)\Delta p_{ef} - \frac{1}{3}(\theta + \sigma_o\pi_e\delta^*)\}, \text{--------(139)}$$

$$Q_f/Q_o = 8\kappa\beta^2(1+\sigma_o\pi_e\delta^*)[1-\sigma_o\pi_e\delta^*)$$

$$+ \frac{20}{3}\xi_o(1+3\sigma_o\pi_e\delta^*)]\xi_o^2$$

$$= 8\kappa\beta^2\frac{20}{3}(1+\sigma_o\pi_e\delta^*)[(1-\sigma_o\pi_e\delta^*)\frac{3}{2\theta}$$

$$+ \xi_o(1+3\sigma_o\pi_e\delta^*)]\xi_o^2$$

$$Q_f/Q_o = 8\kappa\beta^2\frac{20}{3}(1+\sigma_o\pi_e\delta^*)\{\frac{\Delta p_{ef}(1+3\sigma_o\pi_e\delta^*)}{(1+\sigma_o\pi_e\delta^*)}$$

$$+ \frac{(3+\theta)-3\sigma_o\pi_e\delta^*(1-\theta)}{20}\}\{\frac{\Delta p_{ef}}{(1+\sigma_o\pi_e\delta^*)}+\frac{1}{2}\}^2, \text{--(140)}$$

$$\xi_o = \frac{\Delta p_{ef}}{(1+\sigma_o\pi_e\delta^*)} + \frac{1}{2}, \text{----------(141)}$$

3.7 Numerical Results and Discussion:

On the basis of many assumptions fluid exchange is considered theoretically and the effects of variable filtration coefficient, reflection coefficient and lymph flow on the fluid exchange are investigated. As seen in the previous section, the expressions for the velocity and pressure distributions in region II are very complex. To exhibit our results graphically, we must make some numerical calculations and estimate the values of some parameters.

The data of Krogh (1919) suggest that the outer radius of the tissue cyliner R_t will range 2.8×10^{-3} cm for guinea pig muscles. Since the work of Krogh, some workers (Roughton 1952; Blum, 1960; Apelblat et al., 1974) have used values of R_t ranging from 1×10^{-3} to 20×10^{-3} cm. As the typical values of the geometrical parameters, we choose the capillary radius $R_c = 5 \times 10^{-4}$cm, the ratio of the capillary length to radius $\beta = 10^2$, and the ratio of the outer radius of the tissue cylinder to the capillary radius $\gamma = 10$.

The average values of filtration coefficient for various capillaries have been measured by many workers. The filtration coefficient varies very widely for capillaries. According to their data, the ratio k_r to k_a is about 1.8. Wiederhielm (1968) has suggested that capillary permeability to water may be 50 to about 100% larger in the venous capillary than in the arterial

capillary. Accordingly, the nondimensional parameter κ defined by Eq.(6) is very small compared with unity and it takes on values of 2×10^{-9} to about 2×10^{-7} if we estimate the radius of the capillary and the fluid viscosity to be 5×10^{-4} cm and 4×10^{-2} p, respectively. Therefore we choose $\kappa = 10^{-7}$ as the typical value. To consider the effects of the ratio k_v to k_a on fluid exchange, we use values of Θ ranging from 0 to 0.75. The values of P_a, P_v, π_1 and π_2 have been measured for various capillaries. We choose $\Delta P(= P_a - P_v) = 17_{mm}\ H_g$, $P_a = 1.9$, $P_v = 0.68$, and $\pi_e = -1.24$. The values chosen for the lymph osmotic pressure π_L, lymph hydrostatic pressure P_L, osmotic pressure of interstitial fluid π_2 are 13.5 mm Hg, $\pm$.79mm Hg and 4mm Hg respectively. The values of the reflection coefficient σ_0 and the reflection coefficient δ^* are 1 and 0, .2 respectively.

Apelblat et al. (1974) considered the value of δ, which is defined by Eq (116), was 0.1 or 0.2. To estimate the effects of the lymph flow on the fluid exchange, we consider that the value of δ changes from 0 to 1. It is difficult to determine precisely the value of s because of a lack of knowledge of tissue structure. It is expected that the value of s is very large compared with unity. Then we choose $s = 10^3$, which is very large but satisfies the condition $\kappa s << 1$.

The values of our parameters are listed in Table 1. At the capillary wall there appear outflow and inflow in the region $\zeta < \zeta_0$ and $\zeta > \zeta_0$ respectively. The effects of the parameters δ and Θ on ζ_0 are shown in Fig. 9 and Fig. 10 respectively. As is evident from the figures, ζ_0 increases in δ or Θ, and varies greatly with Θ within the range of small values of δ

The average value of the pressure $< p_1 >$ over the volume of region I as a function of δ and for different values of Θ is presented in Fig. 11. In this figure, $< p_1 >$ is expressed in values relative to $(p_a + p_v)/2$, which is the value of $< P_1 > - < P_1 >\Theta = \omega = 0$. From this figure it can be seen that under the condition when the pressure difference ΔP is kept at a constant value the average pressure $< P_1 >$ decreases with increase in Θ or δ. The average pressure of the tissue fluid over the volume of region II as a function of δ with Θ as the parameter is presented in Fig. 12. In this figure $< P_2 >$ is expressed in values relative to $(P_a+P_v)/2 + \sigma_0 \pi_e (1 - \delta^*/2)$ which is the value of $< P_2>-<P_2>_{\Theta=\omega=0}$. As is seen from Fig. 12, under the condition that ΔP is constant, the average pressure $< P_2 >$ decreases with the increase in Θ or δ.

In Fig. 13, the volume of fluid flowing per unit time across the cross section of the capillary $Q_c(\zeta)$ as a function of ζ with Θ as the parameter is presented.

As is seen from Fig. 13, $Q_c(\zeta)/Q_c(\zeta = 0)$ has a minimum value at $\zeta = \zeta_0$ and its minimum value decreases with the increase in Θ. The lymph flow Q_L and the filtration rate Q_f as a function of δ with Θ as the parameter are presented in Figs 14 and 15 respectively. It can be seen that Q_L and Q_f increase in Θ or δ. In Fig. 16 the effective pressure difference ΔP_{ef} is shown as a function of δ with Θ as the parameter. It can be seen that under the condition where ΔP is kept at a constant value ΔP_{ef} increases with the increase in the value of Θ or δ. It can also be seen that when the value of δ is small the value of ΔP_{ef} is affected remarkably by the value of Θ. In Fig. 17 the lymph flow Q_L or the total filtration rate Θ as parameter.

Although several assumptions are involved in our calculations, our results will express fairly well the effects of the variable filtration coefficient, reflection coefficient and the lymph flow on fluid exchange.

TABLE 1

TYPICAL VALUES FOR PARAMETERS USED IN ANALYSIS

Parameter	Value
Capillary radius (R_c)	5×10^{-4} cm
Fluid viscosity (η)	4×10^{-2} p
Arterial capillary pressure (P_a)	32 mmHg
Venous capillary pressure (P_v)	15 mmHg
Plasma osmotic pressure (Π_1)	25 mmHg
Lymph osmotic pressure (Π_L)	13.5 mmHg
Lymph hydrostatic pressure (P_L)	$\frac{1}{2}$ .74 mmHg
Osmotic pressure of interstitial fluid Π_2	4 mmHg
π_e	-1.24
Reflection coefficient σ_o (at z=o)	1
Reflection coefficient parameter δ^*	0,.2
Ratio of the length to radius of capillary: $\beta = L/R_c$	10^2
Ratio of the outer radius of tissue region (R_t) to R_c : $\gamma = R_t/R_c$	10
Ratio of filtration coefficient at venous end (k_v) to that at arterial end (k_a): k_v/k_a	$1 \sim 2.5$
$\theta = (k_v/k_a - 1)/2$	$0 \sim 0.75$
Reduced filtration coefficient: $\kappa = \eta k_a/R_c$	10^{-7}
Ratio of lymph flow (Q_l to filtration rate (Q_f): $\delta = Q_l/Q_f$	$0 \sim 1$
Reduced flow resistance in tissue region: $s=\sigma R_c$	10^3

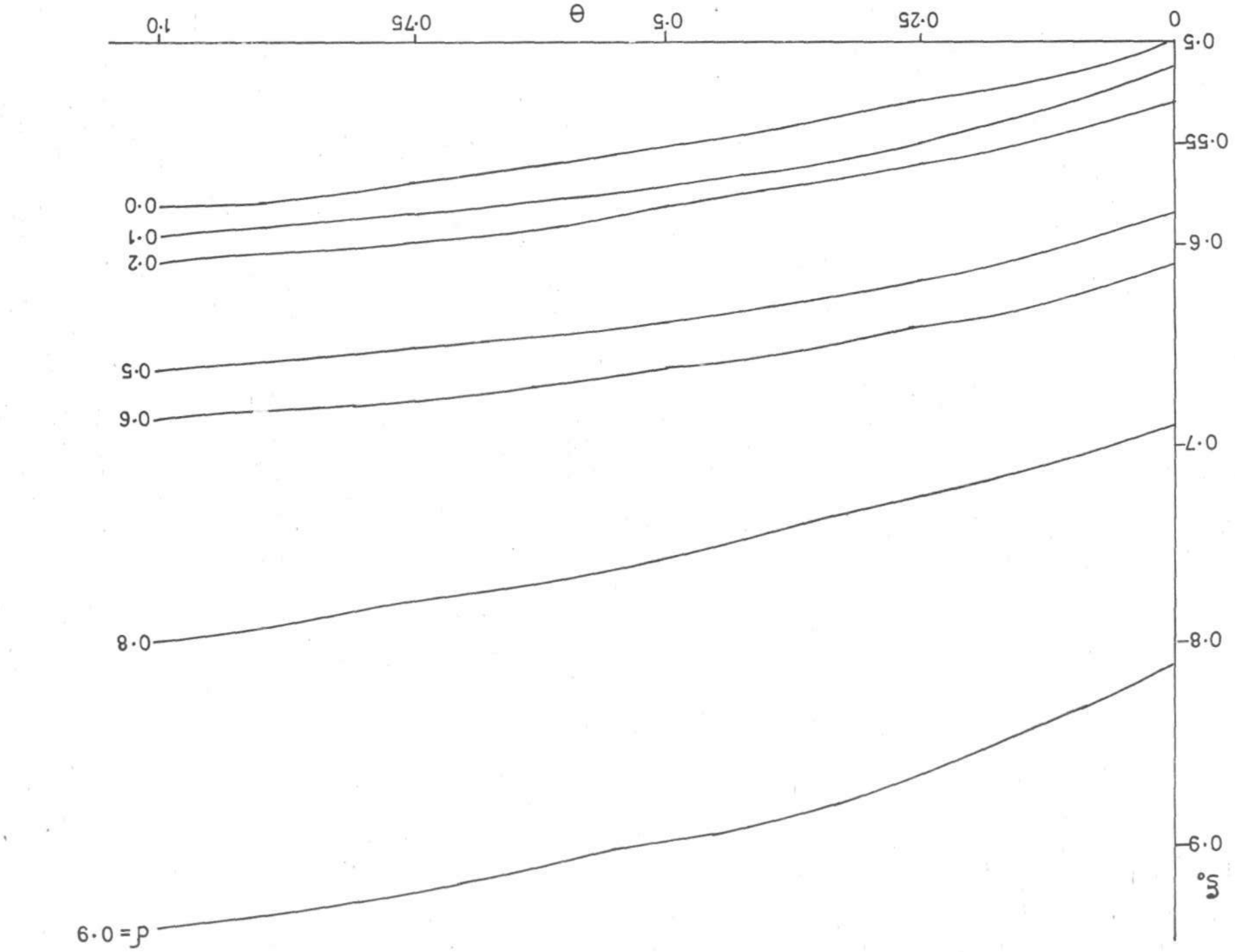


FIG.9. Variation of ξ_0 at which velocity V_1 vanishes with δ.

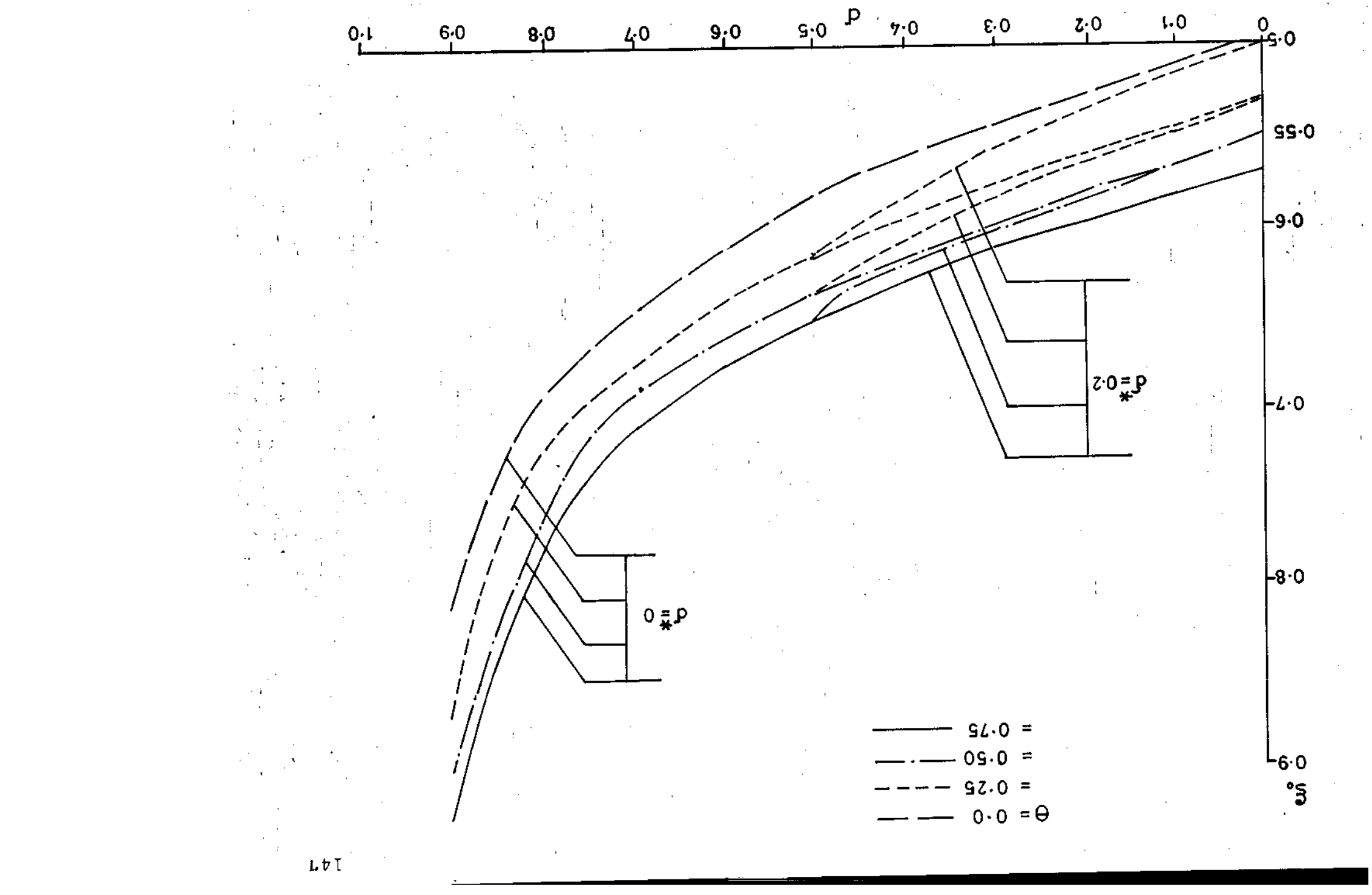


FIG.10. Variation of ξ_o at which velocity v_1 vanishes with θ

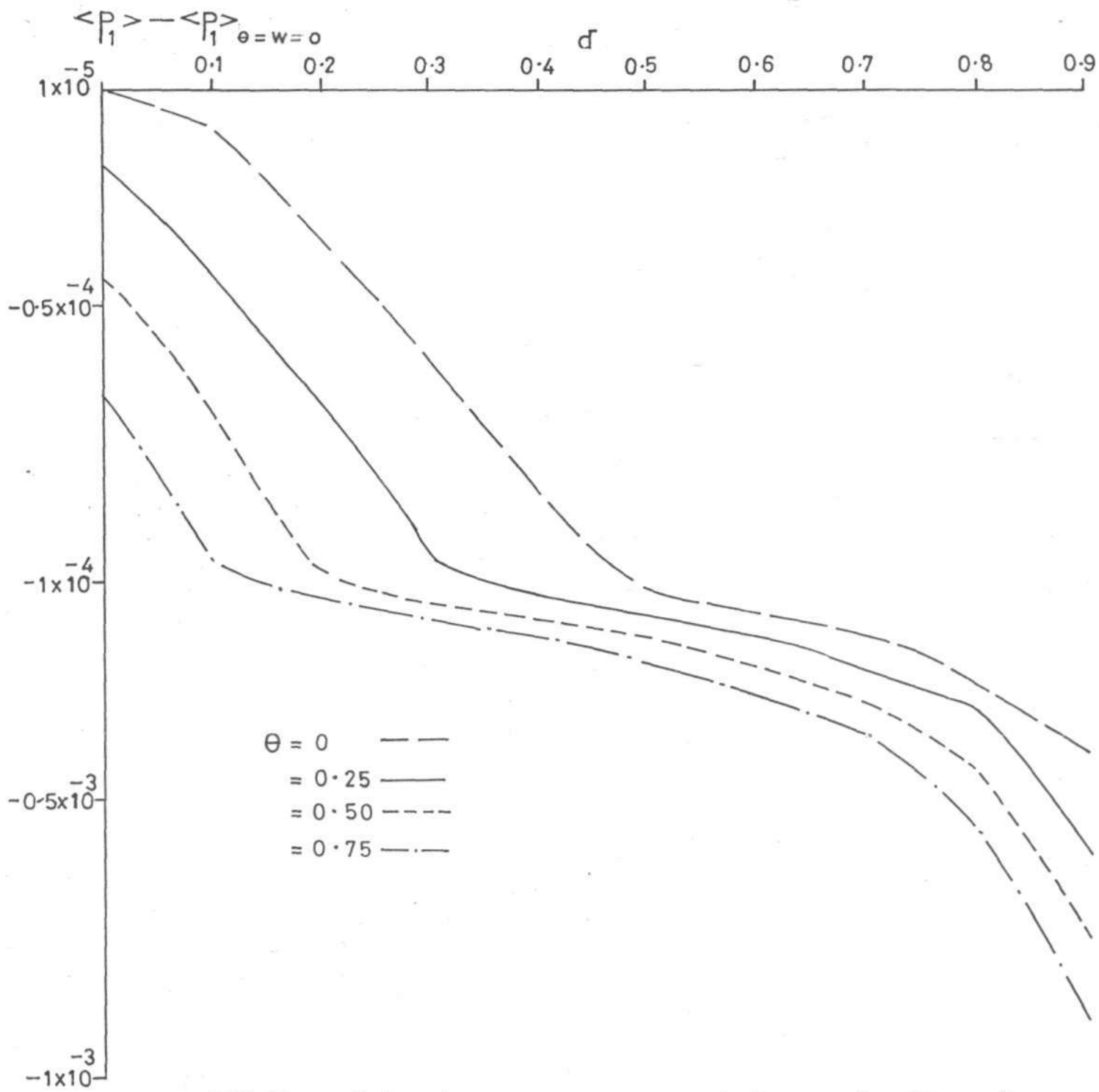


FIG.11. Reduced average pressure $\langle p_1 \rangle$ as a function of δ with θ as the parameter.

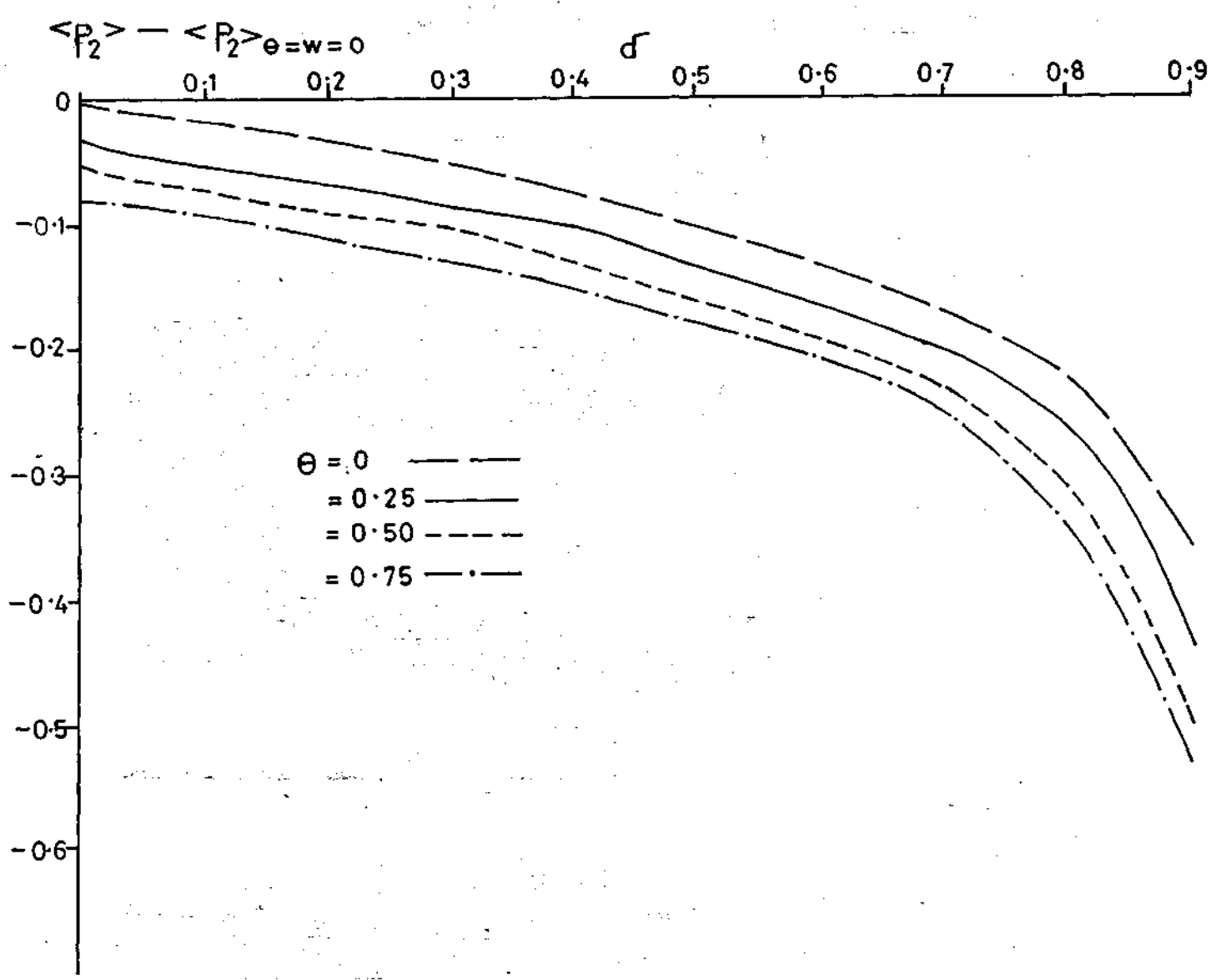


FIG.12. Red ed average pressure $\langle p_2 \rangle$ as a function of δ with θ as the parameter.

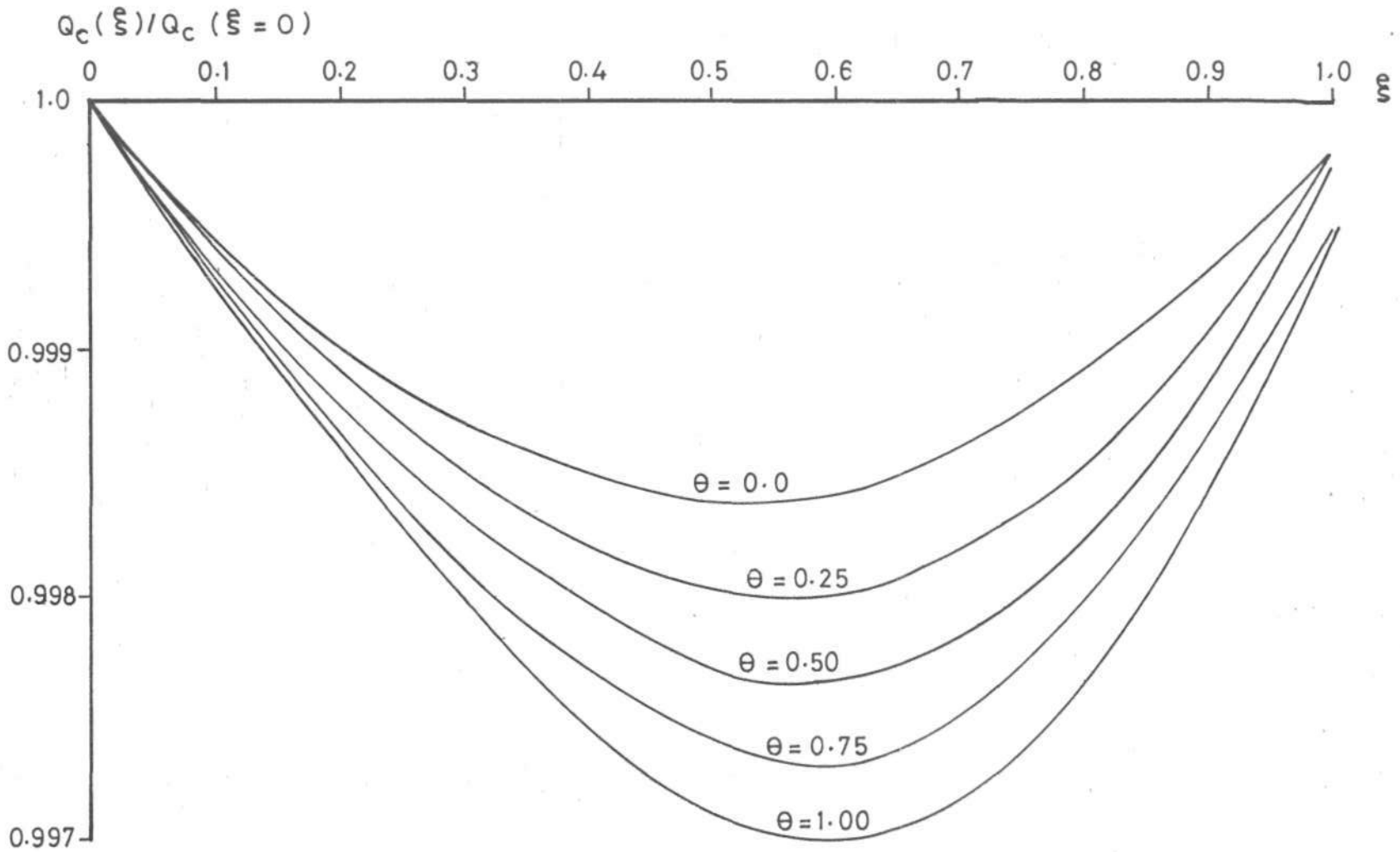


FIG.13. Reduced flux in the capillary $Q_c(\xi) / Q_c(\xi=0)$ as a function of ξ with θ as the parameter.

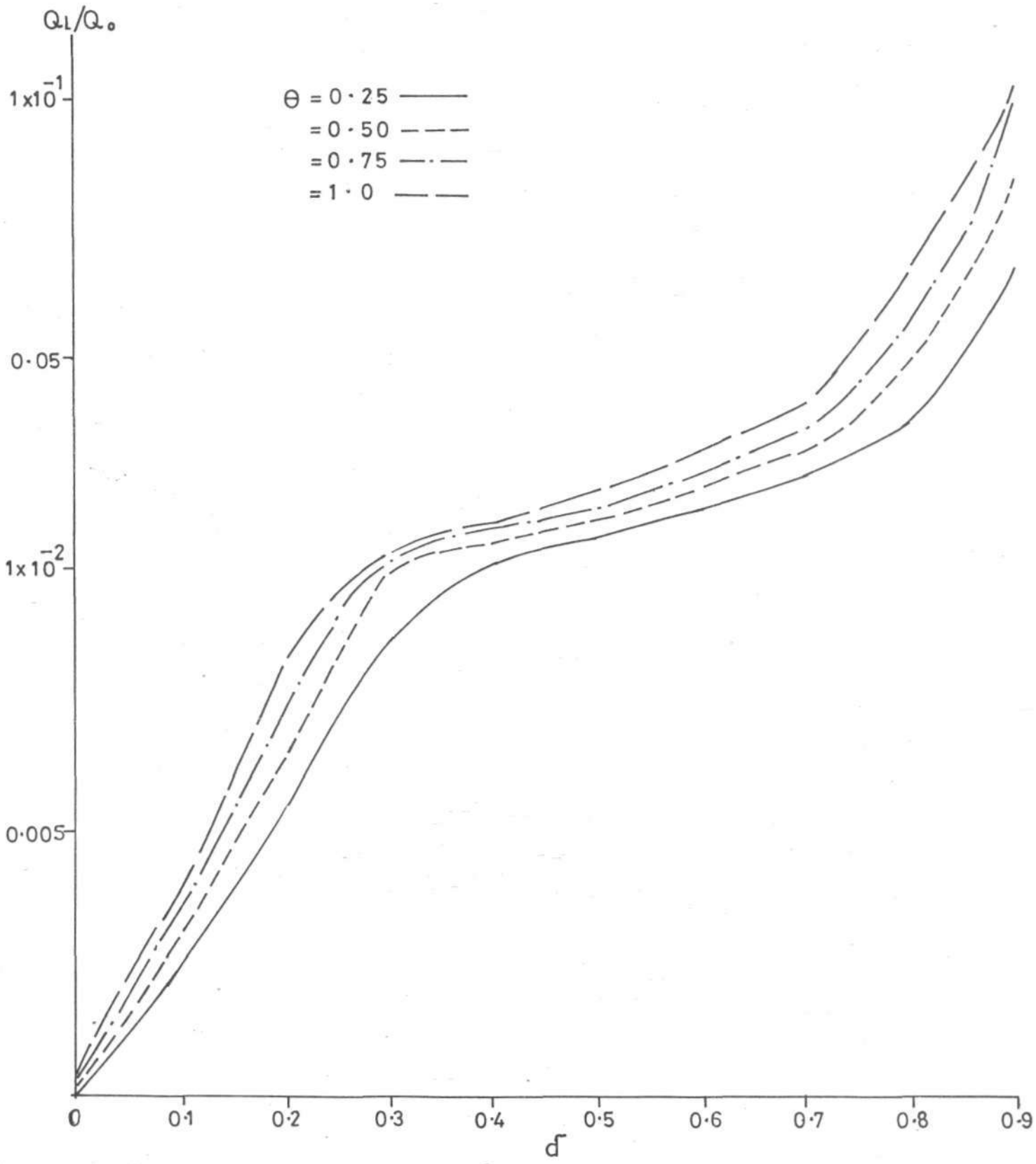


FIG.14. Reduced lymph flow Q_1/Q_o as a function of δ with Θ as the parameter.

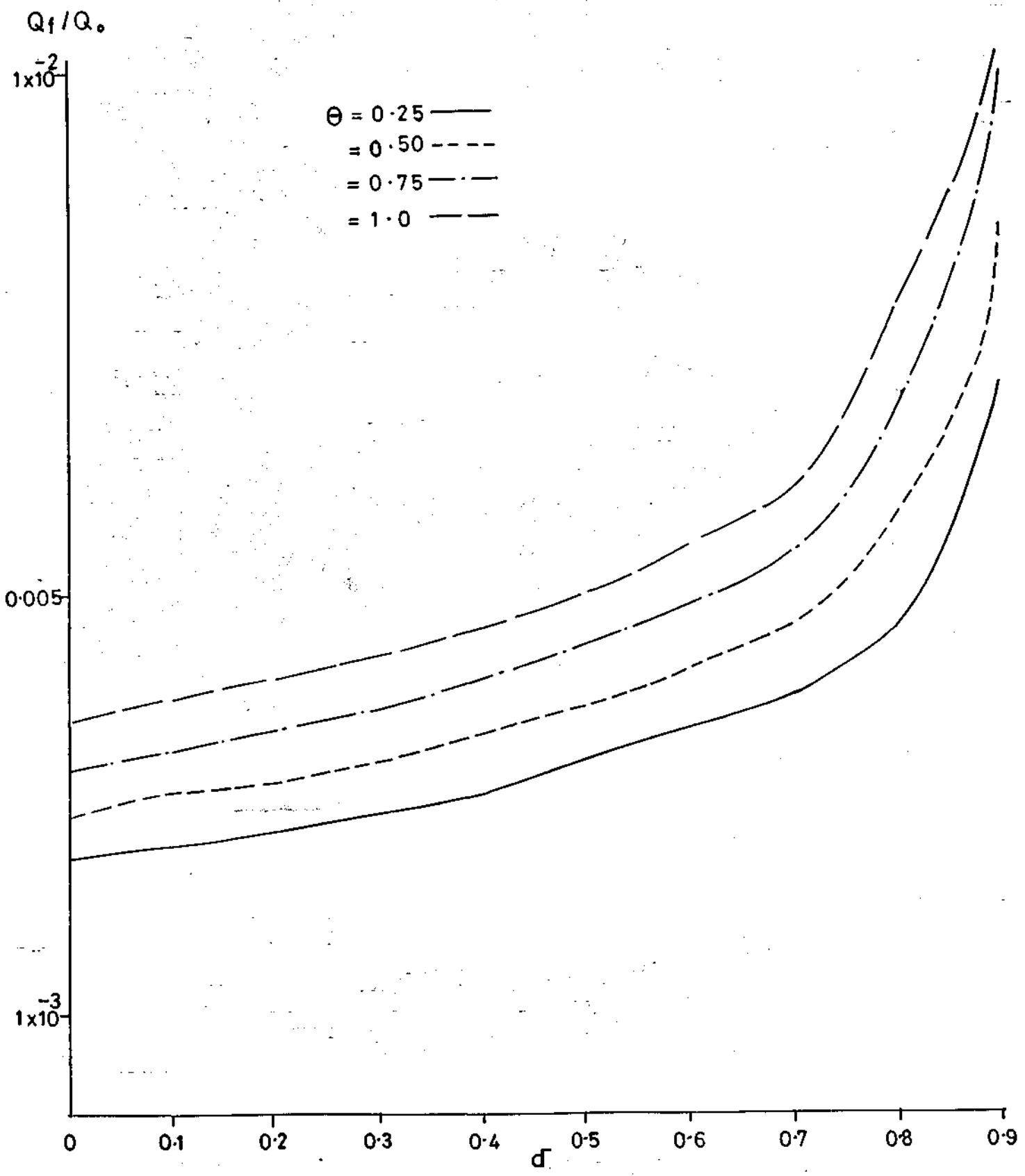


FIG.15. Reduced filtration rate Q_f/Q_o as a function of δ with Θ as the parameter.

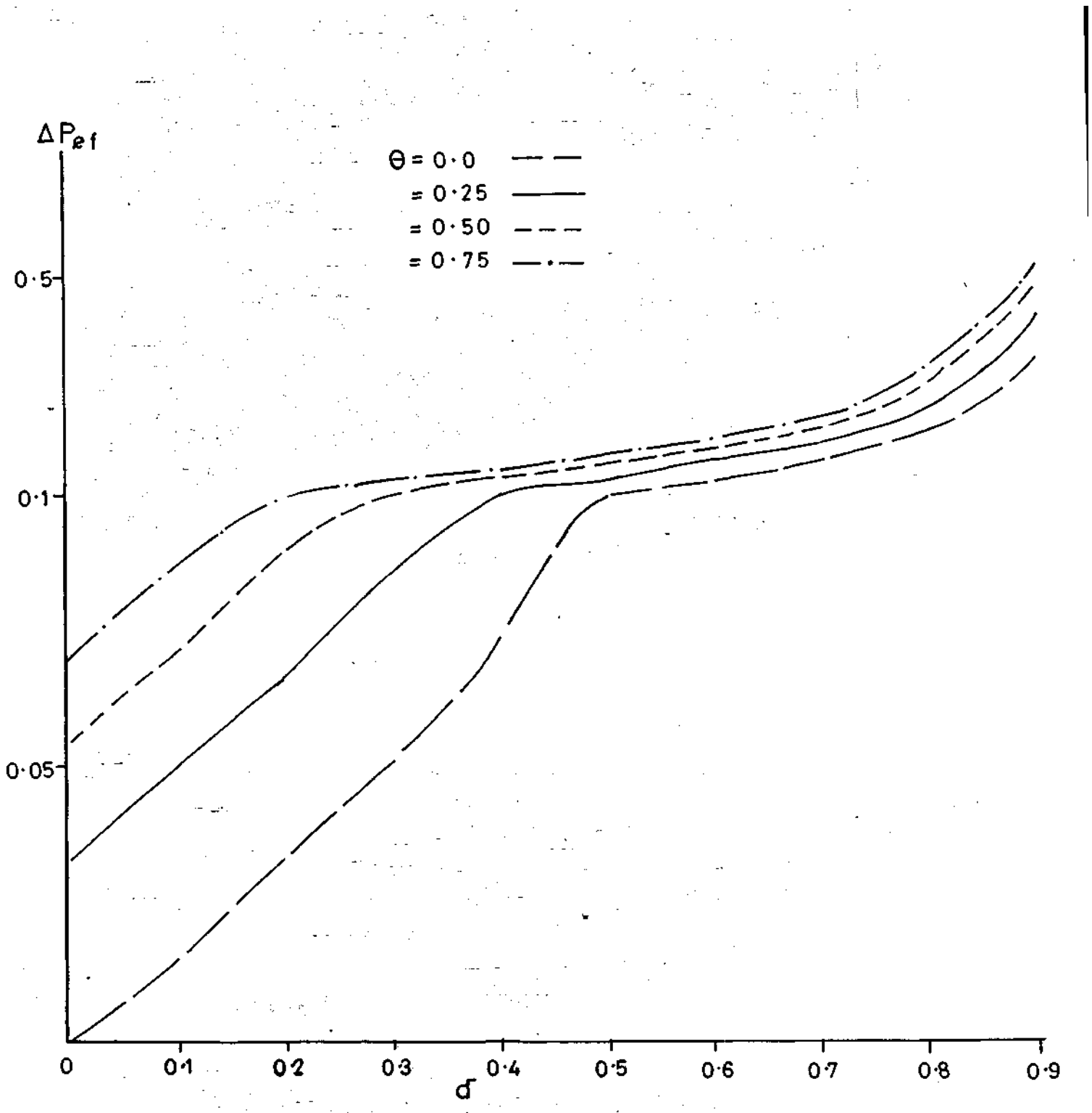


FIG.16. Reduced effective pressure difference Δp_{ef} as a function of δ with θ as the parameter.

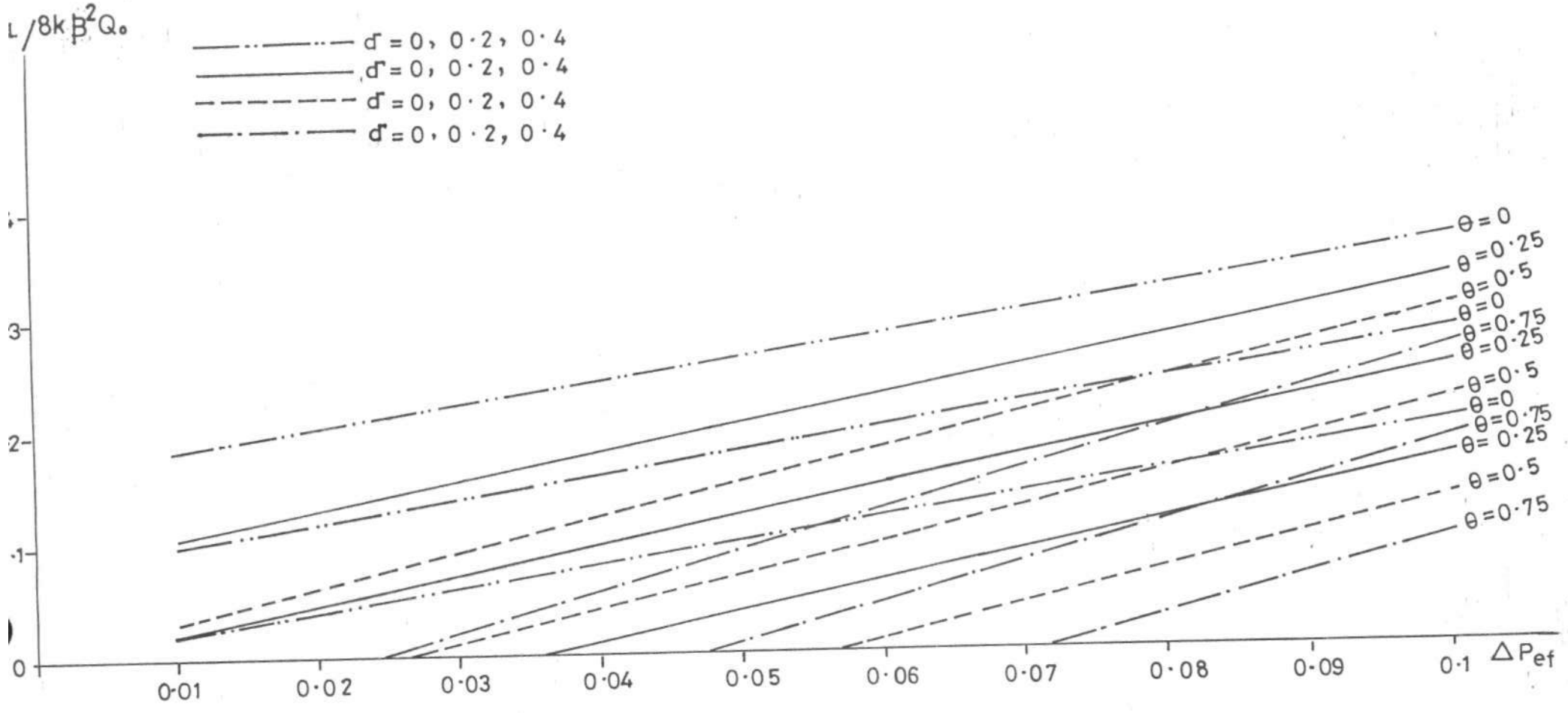


FIG. 17. Reduced lymph flow as a function of reduced effective pressure difference Δp_{ef} with θ as the parameter.

CHAPTER FOUR

MATHEMATICAL MODELLING OF TRANSCAPILLARY FLUID EXCHANGE: NON-UNIFORM CAPILLARY.

4.1. Introduction

The problem being discussed in this chapter concerns the study of the effects of permeability of the wall of a capillary by idealizing the tissue space as a porous medium bounding the capillary, and the capillary as a tube of slowly but arbitrary varying cross-section. The interstitial space surrounding the capillary contains tissue fluid lymph and lymphatic vessels. In our idealization, both the tissue fluid and lymph are taken to be the same fluid (i.e. of same viscosity), referred to as 'tissue fluid' Manton (1971) made a theoretical investigation of such flow for an arbitrary variation of a wall cross-section by the method of perturbation. In our case, an analytical solution is obtained by perturbation technique.

4.2. Assumptions:

(i) The capillary is a non-uniform tube whose radius varies in the axial direction (See Fig.18).

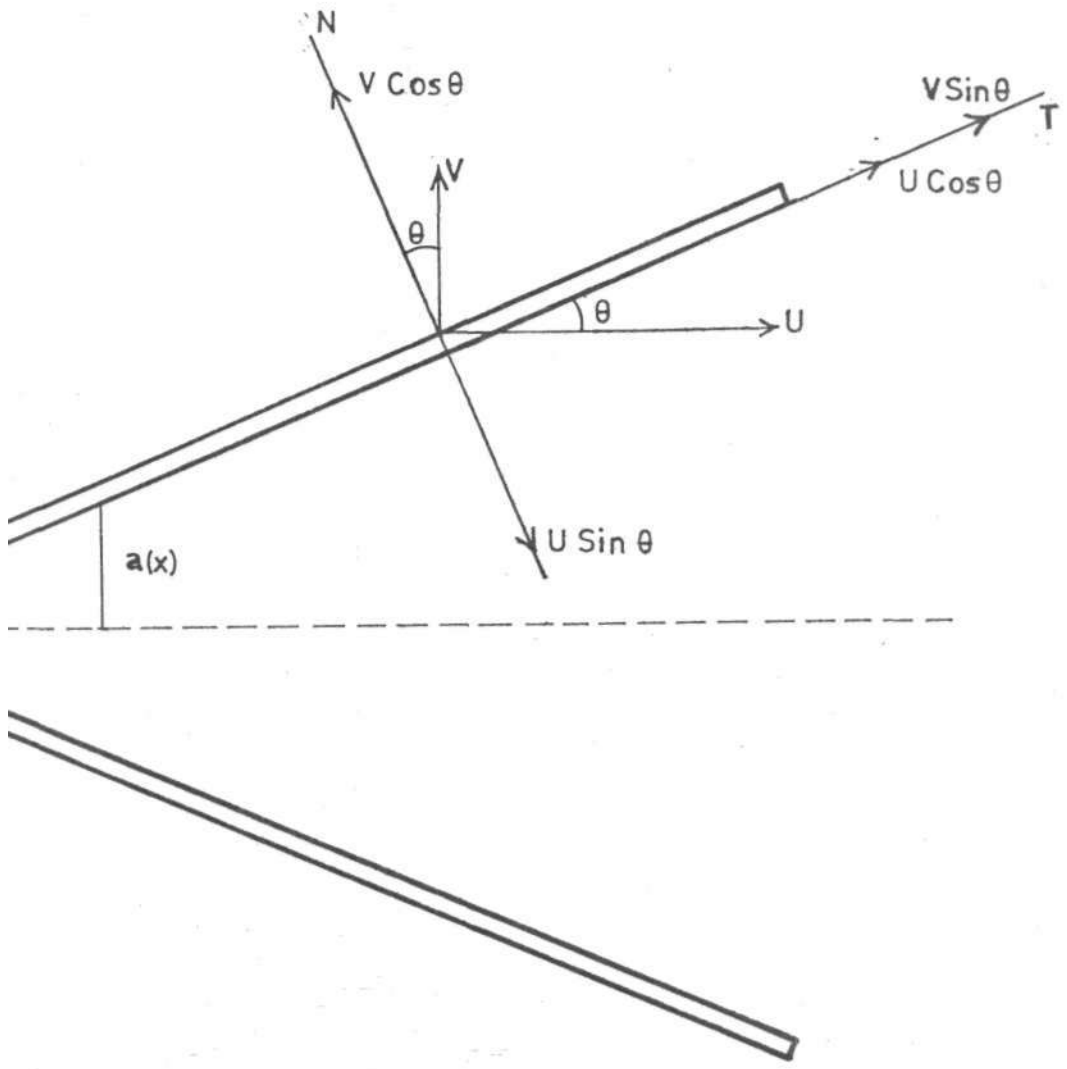
(ii) The blood is a homogeneous incompressible Newtonian fluid.

(iii) The flow is steady and axisymmetric.

(iv) The density as well as viscosity of blood and the tissue fluid are the same.

(v) The flow is governed by Navier - Stokes equations in the capillary, and by Darcy's law in the interstitial tissue space.

(vi) The transfer of fluid and fluid particles across the endothelial wall of the capillary is taken to be governned by Starling's hypothesis.



FIG,18, GEOMETRY OF THE PROBLEM

(vii) The permeability constant of the wall and the reflection coefficient of the wall are both functions of the axis of the capillary and both vary linearly along the axis of the capillary.

(viii) The Reynolds number of the blood flow is very small, hence the convective terms of the Navier - Stokes equations for the capillary are here neglected.

Hence, in this chapter, we consider the steady viscous flow of a Newtonian fluid through a non-uniform capillary with a permeable wall bounded by a tissue space. Due to permeability of the wall, we use a slip condition proposed by Jones (1973) which is an extension to curved surfaces of the slip condition of Beavers and Joseph (1967).

4.3. Formulation and Solution of the Problem:

We are considering the problem of a non-uniform tube (as shown in Fig. 18). With the cylindrical coordinates (x,r,θ), we take the capillary flow region to be given by

$$-\infty < r < +\infty,\ 0 < r < a(x) \text{ and } 0 < \theta < 2\pi;$$

so that $r = 0$ is axis of symmetry for the capillary and $r = a(x)$ is the permeable wall of the capillary. The external boundary of the tissue space is taken to be $r = b$(constant).

The normal and tangential components of the velocity are given by;

$$\text{Tan}\theta = \frac{da}{dx}$$

$$\text{Sin}\theta = \frac{\frac{da}{dx}}{\sqrt{1+(\frac{da}{dx})^2}}$$

$$\text{Cos}\Theta = \frac{1}{\sqrt{1+(\frac{da}{dx})^2}} \quad \text{thus,}$$

normal component of velocity = $v\cos\Theta - u\sin\Theta$ (1)

tangential component of velocity = $r\sin\Theta + u\cos\Theta$ (2)

4.4. Equations of Motion

For the flow in the tissue region, we have

$$u_1 = \frac{k}{\mu}\frac{\partial p_1}{\partial x}, \quad v_1 = -\frac{k}{\mu}\frac{\partial p_1}{\partial r} \tag{3}$$

and equation of continuity

$$\frac{\partial u_1}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv_1) = 0 , \tag{4}$$

where (u_1, v_1) are the components of Darcy velocity in (x,r) directions, respectively, p_1 is the pressure in the tissue region and k is the specific permeability of the region.

For the flow in the capillary, we have

$$\frac{1}{\rho}\frac{\partial p_2}{\partial x} = \nu\left(\frac{\partial^2 u_2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_2}{\partial r})\right) \tag{5}$$

$$\frac{1}{\rho}\frac{\partial p_2}{\partial r} = \nu\left(\frac{\partial^2 v_2}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial v_2}{\partial r}) - \frac{v_2}{r^2}\right) \tag{6}$$

and equation of continuity

$$\frac{\partial u_2}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv_2) = 0 \tag{7}$$

where (u_2, v_2) are the free flow velocity components in the (x,r) directions, respectively, p_2 is the pressure, ρ is the density and $\nu = \mu/\rho$ is the kinematic viscosity.

4.5. Boundary Conditions

Axial symmetry;

$$v_2 = 0, \frac{\partial u_2}{\partial r} = 0 \text{ at } r = 0, \tag{8}$$

Jones' slip condition at the capillary wall which is better for curved surfaces;

$$\frac{1}{2}\left(\frac{\partial u_2}{\partial r} + \frac{\partial v_2}{\partial x}\right) = \frac{\alpha}{\sqrt{k}}(q_{t_2} - q_{t_1}) \text{ at } r = a(x), \tag{9}$$

where α is a dimensionless constant,

Starling's law;

$$(q_n)_1 = (q_n)_2 = q_n = K(x)\left[(p_2 - p_1) - \sigma(x)(\pi_2 - \pi_1)\right] \text{ at } r = a(x), \tag{10}$$

and

$$v_1 = v_o \text{ at } r = b \tag{11}$$

where suffixes t and n denote the tangential and normal components of the velocity, respectively, and

$$(q_t)_m = \frac{u_m + v_m \frac{da}{dx}}{\sqrt{1 + \left(\frac{da}{dx}\right)^2}}, \quad (q_n)_m = \frac{v_m - u_m \frac{da}{dx}}{\sqrt{1 + \left(\frac{da}{dx}\right)^2}}, \tag{12}$$

where m = 1 or 2 according to whether we consider the tissue space or the capillary, $K(x)$ is the permeability constant of the wall; π_1 and π_2 are the colloidal osmotic pressures in the tissue space and in the capillary respectively, and $\sigma(x)$ is the reflection coefficient of the wall.

In order to solve the above equations, we introduce the stream functions ψ_2 and ψ_1 for the capillary and porous region, respectively, such that

$$u_m = \frac{1}{r}\frac{\partial \psi_m}{\partial r}, \quad v_m = -\frac{1}{r}\frac{\partial \psi_m}{\partial x} \tag{13}$$

We seek a solution of the problem when the radius a(x) of the tube varies slowly in the axial direction. In particular, the function a(x) is assumed to depend upon a slope parameter ε such that

$$a(x,\varepsilon) = a_o S(\frac{\varepsilon x}{a_o}) \ , \ (0 < \varepsilon \leq 1) \tag{14}$$

where a_o is a constant characterizing the radius of the tube.

We now introduce the non-dimensional quantities

$$\bar{K}_o = \frac{K_o \mu}{a_o} \ , \ x^* = \frac{\varepsilon x}{a_o} \ , \ z = \frac{r}{a_o} \ , \tag{15}$$

$$\rho_m(z,x^*;\varepsilon) = \frac{\psi_m}{\psi_c} \ , \ p_m(z,x^*;\varepsilon) = \frac{a_o^3}{\rho\nu\psi_o} p_m, \tag{16}$$

$$\pi_m(z,x^*;\varepsilon) = \frac{a_o^3}{\rho\nu\psi_o}\pi_m, \ N = \frac{a_o}{\sqrt{k}} \ , \tag{17}$$

$$N^2 = \frac{a_o^2}{k} \ , \ \bar{V}_o = \frac{V_o}{\psi_o / a_o^2} \ , \tag{18}$$

where ψ_o is a constant of the dimension of stream function and $z = r/a_o$ and $x^* = \varepsilon x/a_o$ are, respectively, the normalized radial and axial coordinates. Thus we can write,

$$a(x,\varepsilon) = a_o S(x^*) \tag{19}$$

$$K(x,\varepsilon) = K_o K(x^*) = K_o \left[1 + \delta x^* \right] \tag{20}$$

$$\sigma(x,\varepsilon) = \sigma_o \sigma(x^*) = \sigma_o \left[1 - \delta_1 x^* \right] \tag{21}$$

where $S(x^*)$ is the wall shape function, δ, δ_1 and σ are non-dimensional quantities. We will also assume the expensions

$$\Phi_m = \sum_{n=o}^{\infty} e^{2n} \Phi_m^{(2n)} (z, x^*) \tag{22}$$

$$P_m = \frac{x^*}{\varepsilon} F_m + \sum_{n=o}^{\infty} \varepsilon^{2n+i} \; p_m^{2n+1} \; (z,\; x^*) \qquad (23)$$

and

$$(q_t)_m = \frac{\psi_o}{a_o^2} \frac{\left[\frac{1}{z} \frac{\partial \phi_m}{\partial z} - \varepsilon^2 . \frac{1}{z} \frac{\partial \phi_m}{\partial x^*} . \frac{dS}{dx^*} \right]}{\sqrt{1+\varepsilon^2 \left(\frac{dS}{dx^*} \right)^2}} \qquad (24)$$

$$= \frac{\psi_o}{a_o^2} \left[\frac{1}{z} \frac{\partial \phi_m}{\partial z} - \varepsilon^2 . \frac{1}{z} \frac{\partial \phi_m}{\partial x^*} . \frac{dS}{dx^*} \right] \left[1 - \frac{1}{2} \varepsilon^2 \left(\frac{dS}{dx^*} \right)^2 \right]$$

thus

$$(q_t)_m = \frac{\psi_o}{a_o^2} \left[\phi_{mz} \Big|_z - \varepsilon^2 \frac{dS}{dx^*} \left\{ \phi_{mx^*} \Big|_z + \frac{1}{2} \phi_{mz} \Big|_z . \frac{dS}{dx^*} \right\} \right] + O(\varepsilon^4) \qquad (25)$$

and

$$(q_n)_m = - \frac{\psi_o}{a_o^2} . \varepsilon \left[\frac{1}{z} \left\{ \phi_{mx^*} + \frac{dS}{dx^*} \phi_{mz} \right\} \right] + O(\varepsilon^3) \qquad (26)$$

To convert the equations of motion and boundary conditions to non- dimensionalized forms, we first have to convert them in terms of the stream functions. Thus from equations (3), we obtain

$$\frac{\partial P_1}{\partial x} = - \frac{u}{Kr} \frac{\partial \Psi_1}{\partial r} \qquad (27)$$

$$\frac{\partial P_1}{\partial r} = \frac{\mu}{Kr} \frac{\partial \Psi_1}{\partial x} \qquad (28)$$

eliminating pressure, we obtain

$$\left[\frac{\partial^2 \Psi_1}{\partial r^2} - \frac{1}{r} \frac{\partial \Psi_1}{\partial r} + \frac{\partial^2 \Psi_1}{\partial x^2} \right] = 0 \qquad (29)$$

$$\left[\frac{\partial^4 \Psi_2}{\partial r^4} + \frac{\partial^4 \Psi_2}{\partial x^4} + 2\frac{\partial^4 \Psi_2}{\partial r^2 \partial x^2} - \frac{2}{r}\frac{\partial^3 \Psi_2}{\partial r \partial x^2} - \frac{2}{r}\frac{\partial^3 \Psi_2}{\partial r^3} + \frac{3}{r^2}\frac{\partial^2 \Psi_2}{\partial r^2} - \frac{3}{r^3}\frac{\partial \Psi_2}{\partial r} \right] = 0 \qquad (30)$$

Converting equations (29) and (30) into non -dimensional form, we have,

$$\left(\epsilon^2 \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right) \phi_1 = 0 \qquad (31)$$

$$\left(\epsilon^2 \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\left(\epsilon^2 \frac{\partial^2}{\partial x^{*2}} + \frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\phi_2 = 0 \qquad (32)$$

Zeroth order and Second order differential equations are obtained from equations (31) and (32) and can be written as

$$z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\phi_1^{(0)} = 0 \qquad (33)$$

$$\left[z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right]\left[z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right]\phi_2^{(0)} = 0 \qquad (34)$$

and

$$\frac{\partial^2 \phi_1^{(0)}}{\partial x^{*2}} + \left(\frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\phi_1^{(2)} = 0 \qquad (35)$$

$$\frac{\partial^2 \phi_2^{(0)}}{\partial x^{*2}} + \left(\frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\left(\frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\phi_2^{(2)} = 0 \qquad (36)$$

The boundary conditions in non-dimensionalized form are given as ; from equation (8), we have

$$\frac{\partial \phi_2}{\partial x^*} = 0, \quad \frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial \phi_2}{\partial z}\right) = 0 \quad \text{at} \quad z = 0 \qquad (37)$$

from equation (9),

$$\frac{\alpha}{\sqrt{K}}\frac{\Psi_2}{a_o^2}\Big[(\phi_{2z}|z-\phi_{1z}|z)-\varepsilon^2\frac{dS}{dx^*}\{\phi_{2x^*}|z+\frac{1}{2}\phi_{2z}|z\cdot\frac{dS}{dx^*}-\phi_{1x^*}|z-\frac{1}{2}\phi_{1z}|z\cdot\frac{dS}{dx^*}\}\Big]$$

$$=\tfrac{1}{2}\frac{\Psi_o}{a_o^3}\Big[(\phi_{2z}|z)_z-\varepsilon^2(\phi_{2x^*}|z)_{x^*}\Big]\quad\text{at } z=S(x^*) \tag{38}$$

Zeroth and Second order conditions can be written as

$$\frac{1}{\alpha N}\Big[(\phi^0_{2z}|z\Big]_z=2\Big[\phi^0_{2z}|z-\phi^0_{1z}|z\Big]\quad\text{at } z=S(x^*) \tag{39}$$

$$\frac{1}{\alpha N}\Big[(\phi^2_{2z}|z)_z-(\phi^0_{2x^*}|z)_{x^*}\Big]=2\Big|\phi^2_{2z}\Big[z-\phi^2_{1z}|z-\frac{dS}{dx^*}\{\phi^0_{2x^*}|z+\frac{1}{2}\phi^0_{2z}|z\cdot\frac{dS}{dx^*}-\phi^0_{1x^*}|z-\frac{1}{2}\phi^0_{1z}|z\cdot\frac{dS}{dx^*}\}\Big]\text{ at } z=S(x^*) \tag{40}$$

Normal velocity is given as

$$(q_n)z=S=-\frac{1}{2\pi S}\frac{\partial\Psi}{\partial x^*}$$

thus boundary condition (10), i.e. $(q_n)_1=(q_n)_2$ becomes,

$$-\frac{1}{2\pi S}\frac{\partial\Psi}{\partial x^*}=-\frac{1}{2\pi S}\frac{\partial\Psi_1}{\partial x^*}$$

or, $$\frac{\partial\Psi_2}{\partial x^*}-\frac{\partial\Psi_1}{\partial x^*}=0$$

$$\frac{\partial}{\partial x^*}\Big[\Psi_2-\Psi_1\Big]=0$$

$$\Psi_2-\Psi_1=\Psi_o\quad\text{(constant)}$$

thus, $\frac{\Psi_2}{\Psi_o} - \frac{\Psi_1}{\Psi_o} = 1$

or, $\phi_2 - \phi_1 = 1$ (42)

Zeroth and Second order conditions are given as

$$\phi_2^0 - \phi_1^0 = 1 \tag{43}$$

$$\phi_2^2 - \phi_1^2 = 1 \tag{44}$$

again from (10), $(q_n)_1 = K(x)\left[(P_2-P_1)-\sigma(x)(\pi_2-\pi_1)\right]_{r=a(x)}$

we have,

$$-\frac{\Psi_o}{a_o^2}.\varepsilon\left[\frac{1}{z}\{\phi_{1x^*} + \frac{dS}{dx^*}\phi_{1z}\}\right] + O(\varepsilon^3)$$

$$= K(x)\,\rho\frac{\nu\Psi_o}{a_o^3}\left[(P_2 - P_1) - \sigma(x)(\pi_2 - \pi_1)\right] \tag{45}$$

$$-\varepsilon\left[\frac{1}{z}\{\phi_{1x^*} + \frac{dS}{dx^*}\phi_{1z}\}\right] + O(\varepsilon^3)$$

$$= \frac{K(x)\mu}{a_o}\left[(P_2 - P_1) - \sigma(x)(\pi_2 - \pi_1)\right]$$

Let $\pi_2 - \pi_1 = \pi$ (constant),

thus,

$$-\varepsilon\left[\frac{1}{z}\{\phi_{1x^*} + \frac{dS}{dx^*}\phi_{1z}\}\right] + O(\varepsilon^3)$$

$$= \bar{K}_o(1 + \delta x^*)\left[P_2 - P_1 - \sigma_o(1-\delta_1(x^*))\pi\right] \tag{46}$$

at $z = S(x^*)$

or,

$$-\varepsilon\left[\frac{1}{z}\{\phi^0_{1x^*} + \frac{dS}{dx^*}\phi^0_{1z}\}\right] + O(\varepsilon^3)$$

$$= \bar{K}_o(1+\delta x^*)\left[\frac{x^*}{\varepsilon}(F_2-F_1) + \varepsilon(P_2^2 - P_1^2) + \varepsilon^3(P_2^3 - P_1^3)\right.$$

$$\left. + \ldots O(\varepsilon^5) - \sigma_o(1-\delta_1(x^*)\pi\right] \tag{47}$$

comparing like powers of ε, we obtain

$$F_2 - F_1 = 0 \tag{48}$$

$$-\frac{1}{z}\{\phi^0_{1x^*} + \frac{dS}{dx^*}\,\phi^0_{1z}\} = (P_2' - P_1')\,\bar{K}_o(1 + \delta x^*)$$

or,
$$-\left[\frac{\frac{1}{z}\{\phi^0_{1x^*} + \frac{dS}{dx^*}\,\phi^0_{1z}\}}{\bar{K}_o(1 + \delta x^*)}\right]_{x^*} = (P_2^{\sim} - P_1')_{x^*} \text{ at } z = S(x^*) \tag{49}$$

also, $\sigma_o(1 - \delta_1(x^*))\pi = 0$ (50)

From condition (π), V_o is assumed to be of order ε, thus,

$$-\frac{1}{r}\frac{\partial \Psi}{\partial x} = \varepsilon V_o \tag{51}$$

$$\frac{\partial \Psi}{\partial x} = -\varepsilon V_o r$$

$$\Psi = -\varepsilon V_o r x + \Psi_o \tag{52}$$

at $x = 0$, flux is constant (say Ψ_o) as $V_o = 0$, which means that there is no flow across the wall. Hence,

$$\frac{\Psi_1}{\Psi_o} = -\frac{\varepsilon V_o r x}{\Psi_o} + 1$$

$$= -\frac{\varepsilon V_o a_o z\, x^* a_o}{\Psi_o \varepsilon}$$

$$\frac{\Psi_1}{\Psi_o} = -\frac{V_o z}{\Psi_o / a_o^2}\, x^* + 1$$

thus,

$$\phi_1 = -\bar{V}_o \eta\, x^* + 1 \qquad \text{at} \qquad z = \eta \tag{53}$$

Zeroth and Second order conditions are given as ,

$$\phi_1^0 = -\bar{V}_o \eta\, x^* + 1 \qquad \text{at} \qquad z = \eta \tag{54}$$

$$\phi_1^2 = 0 \tag{55}$$

Finally the differential equations and boundary conditions can be summarized as ;

<u>For Zeroth order</u>:

$$z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\phi_1^0 = 0 \tag{56}$$

$$\left[z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right]\left[z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right]\phi_2^0 = 0 \tag{57}$$

$$\frac{1}{\alpha \dot{N}}\left[\phi_{2z}^0|z\right]_z = 2\left[\phi_{2z}^0|z - \phi_{1z}^0|z\right], \text{ at } z = S(x^*) \tag{58}$$

$$\phi_2^0 - \phi_1^0 = 1 \quad , \qquad \text{at} \quad z = S(x^*) \tag{59}$$

$$F_2 - F_1 = 0 \quad , \qquad \text{at} \quad z = S(x^*) \tag{60}$$

$$\phi_1^0 = -\bar{V}_o\, \eta\, x^* + 1 \ , \qquad \text{at} \quad z = \eta \tag{61}$$

<u>For Second order:</u>

$$\frac{\partial^2 \phi_1^{(0)}}{\partial x^{*2}} + \left(\frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\phi_1^{(2)} = 0 \tag{62}$$

$$\frac{\partial^2 \phi_2^{(0)}}{\partial x^{*2}} + \left(\frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\left(\frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\phi_2^{(2)} = 0 \tag{63}$$

$$\frac{1}{\alpha N}\left[(\phi_{2z}^2|z)_z - (\phi_{2x^*}|z)_{x^*}\right]$$

$$= 2\left[\phi_{2z}^2|z - \phi_{1z}^2|z - \frac{dS}{dx^*}\left\{\phi_{2x^*}^0|z + \frac{1}{2}\phi_{2z}^0|z \cdot \frac{dS}{dx^*} - \phi_{1x^*}^0|z - \frac{1}{2}\phi_{1z}^0|z \cdot \frac{dS}{dx^*}\right\}\right], \qquad \text{at} \quad z = S(x^*) \tag{64}$$

$$\phi_2^2 - \phi_1^2 = 0 \quad , \qquad \text{at} \quad z = S(x^*) \tag{65}$$

$$-\frac{1}{z}\left\{\phi_{1x^*}^0 + \frac{dS}{dx^*}\cdot\phi_{1z}^0\right\} = (P_2' - P_1')\,\bar{K}_o\,(1 + \delta x^*).$$

$$\text{or, } -\frac{1}{z}\left\{ \phi^0_{1x^*} + \left[\frac{\frac{dS}{dx^*}\,\phi^0_{1z}}{\bar{K}_o(1+\delta x^*)} \right]_{x^*} \right\} = (P_2' - P_1')_{x^*}, \text{ at } z = S(x^*) \tag{66}$$

$$\phi^2_1 = 0, \qquad \text{at} \qquad z = \eta \tag{67}$$

4.6 Solutions of Equations

Next, we will obtain solutions of the differential equations (56), (57), (62) and (63) subject to their boundary conditions (58) - (61), and (64) - (67).

from (56) $\quad z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\phi^0_1 = 0$

or,

$$\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\phi^0_1 = 0$$

integrating,

$$\frac{1}{z}\frac{\partial \phi^0_1}{\partial z} = 2c_{10}$$

$$\frac{\partial \phi_1}{\partial z} = 2c_{10}z$$

integrate again,

$$\phi^0_1 = c_{10}z2 + d_{10}$$

also from (57) we have,

$$\left[z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right]\left[z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\right]\phi^0_2 = 0$$

$$\text{let} \qquad z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right) = u \tag{68}$$

$$\therefore \qquad z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)u = 0$$

$$\rightarrow \quad u = A_1 z^2 + A_2 \tag{69}$$

$$\text{hence,} \quad z\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\phi^0_2 = A_1 z^2 + A_2 \tag{70}$$

$$\frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\right)\phi_2^0 = A_1 z + \frac{A_2}{z}$$

integrating,

$$\frac{1}{z}\frac{\partial\phi_2^0}{\partial z} = \frac{A_1 z^2}{2} + A_2 \log z + A_3$$

or,

$$\frac{\partial\phi_2^0}{\partial z} = A_1\frac{z^3}{2} + A_2 z \log z + A_3 z$$

integrating again,

$$\phi_2^0 = A_1 z^4 + A_2\left\{\frac{z^2}{\partial}\log z - \frac{z^2}{4}\right\} + A_3\frac{z^2}{2} + A_4 \quad (71)$$

from boundary condition (37), $A_2 = A_4 = 0$

thus we have

$$\phi_2^0 = c_{20}z^4 + d_{20}z^2 \quad (72)$$

$$\frac{\partial^2\phi_1^0}{\partial x^{*2}} + \left(\frac{\partial^2}{\partial z^2} - \frac{1}{z}\frac{\partial}{\partial z}\right)\phi_1^2 = 0$$

$$\frac{d^2}{dx^{*2}}(c_{10}z^2 + d_{10}) + z\frac{d}{dz}\left(\frac{1}{z}\frac{d}{dz}\right)\phi_1^{(2)} = 0 \quad (73)$$

or, $$z\frac{d}{dz}\left(\frac{1}{z}\frac{d}{dz}\right)\phi_1^{(2)} = -\frac{d^2c_{10}}{dx^{*2}}z^2 - \frac{d^2d_{10}}{dx^{*2}}$$

$$\frac{d}{dz}\left(\frac{1}{z}\frac{d}{dz}\right)\phi_1^{(2)} = -\frac{d^2c_{10}}{dx^{*2}}z - \frac{d^2d_{10}}{dx^{*2}}\frac{1}{z}$$

integrating,

$$\frac{1}{z}\frac{d\phi_1^{(2)}}{dz} = -\frac{d^2c_{10}}{dx^{*2}}\frac{z^2}{z} - \frac{d^2d_{10}}{dx^{*2}}\log z + c_{12}$$

$$\frac{d\phi_1^{(2)}}{dz} = -\frac{d^2c_{10}}{dx^{*2}}\frac{z^3}{2} - \frac{d^2d_{10}}{dx^{*}}z\log z + c_{12}z$$

integrate again,

$$\phi_1^2 = -\frac{d^2c_{10}}{dx^{*2}}\frac{z}{8} - \frac{d^2d_{10}}{dx^{*2}}\left\{\frac{1}{2}z^2\log z - \frac{z^2}{4}\right\} + c_{12}\frac{z^2}{2} + d_{12}$$

$$\phi_1^{(2)} = d_{12} + c_{12} z^2 - \frac{1}{8} \frac{d^2 c_{10}}{dx^{*2}} z^4 + \frac{1}{2} \frac{d^2 d_{10}}{dx^{*2}} z^2 (\frac{1}{2} - \log z) \tag{74}$$

using the same method as above for equation (63), we obtain,

$$\phi_2^{(2)} = d_{22} z^2 + c_{22} z^4 - \frac{1}{12} \frac{d^2 c_{10}}{dx^{*2}} z^6 \tag{75}$$

4.7 Pressure Distribution.

Pressure distributions may be obtained as,

For tissue region:

$$U_1 = - \frac{K}{\mu} \frac{\partial P_1}{\partial x} , \qquad V_1 = - \frac{K}{\mu} \frac{\partial P_1}{\partial r}$$

in non-dimensional form, we have

$$- \frac{K}{a_o^2} \varepsilon \frac{\partial P_1}{\partial x^*} = \frac{1}{z} \frac{\partial \phi_1}{\partial z} , \qquad \frac{K}{a_o^2} \frac{\partial P_1}{\partial z} = \varepsilon \frac{1}{z} \frac{\partial \phi_1}{\partial x^*}$$

or,

$$- \frac{1}{N^2} \varepsilon \frac{\partial P_1}{\partial x^*} = \frac{1}{z} \frac{\partial \phi_1}{\partial z} \quad (76), \qquad \frac{1}{N^2} \frac{\partial P_1}{\partial z} = \varepsilon \frac{1}{z} \frac{\partial \phi_1}{\partial x^*} \tag{77}$$

from (23),

$$P_1 = \frac{x^*}{\varepsilon} F_1 + \sum_{n=o}^{\infty} \varepsilon^{(2n+1)} (z, x^*, \varepsilon) \tag{78}$$

from (76) and (77), we have

$$- \frac{1}{N^2} \varepsilon \left[\frac{x^*}{\varepsilon} F_1 + \varepsilon P_1' + \varepsilon^3 P_1^3\right]_{x^*} = \frac{1}{z} \frac{\partial}{\partial z} \left[\phi_1^0 + \varepsilon^2 \phi_1^2 + \varepsilon^4 \phi_1^4 + \ldots\right] \tag{79}$$

$$\frac{1}{N^2} \left[\frac{x^*}{\varepsilon} F_1 + \varepsilon P_1' + \varepsilon^3 P_1^3\right]_z = \varepsilon \frac{1}{z} \frac{\partial}{\partial x^*} \left[\phi_1^0 + \varepsilon^2 \phi_1^2 + \varepsilon^4 \phi_1^4 + \ldots\right] \tag{80}$$

comparing like powers of ε, we get from (79),

$$(x^* F_1)_{x^*} = - N^2 . \frac{1}{z} \frac{\partial \phi_1^0}{\partial z}$$

from (67) we obtain,

$$(x^* F_1)_{x^*} = - 2N^2 c_{10} \tag{81}$$

also from (79),

$$(P_1')_{x*} = - N^2 \frac{1}{z} \frac{\partial \phi_1^2}{\partial z} \tag{82}$$

For capillary region;

$$\frac{\partial P_2}{\partial r} = \mu \left[\frac{1}{r^2} \frac{\partial^2 \phi_2}{\partial r^2 \partial x} - \frac{1}{r} \frac{\partial^3 \Psi_2}{\partial x^3} \right]$$

$$\frac{\partial P_2}{\partial x} = \mu \left[\frac{1}{r^3} \frac{\partial \Psi_2}{\partial r} - \frac{1}{r^2} \frac{\partial^2 \Psi_2}{\partial r^2} + \frac{1}{r} \frac{\partial^3 \Psi_2}{\partial r^3} + \frac{1}{r} \frac{\partial^3 \Psi_2}{\partial x^2 \partial r} \right]$$

in non-dimensional form, we have

$$\varepsilon \frac{\partial P_2}{\partial x^*} = \left[\frac{1}{z^3} \frac{\partial \phi_2}{\partial z} - \frac{1}{z^2} \frac{\partial^2 \phi_2}{\partial z^2} + \frac{1}{z} \frac{\partial^3 \phi_2}{\partial z^3} + \varepsilon^2 . \frac{1}{z} \frac{\partial^3 \phi_2}{\partial z \partial x^{*2}} \right] \tag{83}$$

$$\frac{\partial P_2}{\partial z} = \varepsilon \left[\frac{1}{z^2} \frac{\partial^2 \phi_2}{\partial z \partial x^*} - \frac{1}{z} \frac{\partial^3 \phi_2}{\partial z^2 \partial x^*} - \varepsilon^2 . \frac{1}{z} \frac{\partial^3 \phi_2}{\partial x^{*3}} \right] \tag{84}$$

from (83) and (84), we have

$$\varepsilon \left[\frac{x^*}{\varepsilon} F_2 + \varepsilon P_2' + \varepsilon^3 P_2^3 \right]_{x^*} = \left[\frac{1}{z^3} \frac{\partial \phi_2^0}{\partial z} - \frac{1}{z^2} \frac{\partial^2 \phi_2^0}{\partial z^2} + \frac{1}{z} \frac{\partial^3 \phi_2^0}{\partial z^3} \right]$$

$$+ \varepsilon^2 \left[\frac{1}{z^3} \frac{\partial \phi_2^2}{\partial z} - \frac{1}{z^2} \frac{\partial^2 \phi_2^2}{\partial z^2} + \frac{1}{z} \frac{\partial^3 \phi_2^2}{\partial z^3} \right.$$

$$\left. + \frac{1}{z} \frac{\partial^3 \phi_2^0}{\partial z \partial x^{*2}} \right] + O(\varepsilon^4) + \ldots \tag{85}$$

$$\left[\frac{x^*}{\varepsilon} F_2 + \varepsilon P_2' + \varepsilon^3 P_2^3 \right]_z = \varepsilon \left[\frac{1}{z^2} \frac{\partial^2 \phi_2^0}{\partial z \partial x^*} - \frac{1}{z} \frac{\partial^3 \phi_2^0}{\partial z^2 \partial x^*} \right]$$

$$+ \varepsilon^3 \left[\frac{1}{z} \frac{\partial^2 \phi_2^2}{\partial z \partial x^*} - \frac{1}{z} \frac{\partial^3 \phi_2^2}{\partial z^2 \partial x^*} - \frac{1}{z} \frac{\partial^3 \phi_2^0}{\partial x^{*3}} \right] + O(\varepsilon^5) + \ldots \tag{86}$$

comparing like powers of ε, we get from (85),

$$(x^* F_2)_{x^*} = \frac{1}{z^3} \frac{\partial \phi_2^0}{\partial z} - \frac{1}{z^2} \frac{\partial^2 \phi_2^0}{\partial z^2} + \frac{1}{z} \frac{\partial^3 \phi_2^0}{\partial z^3}$$

from (72) we obtain

$$(x^*F_2)_{x^*} = 16\, c_{20} \tag{87}$$

also from (85),

$$(P_2')_{x^*} = \frac{1}{z^3}\frac{\partial \phi_2^2}{\partial z^2} - \frac{1}{z^2}\frac{\partial^2 \phi_2^2}{\partial z^2} + \frac{1}{z}\frac{\partial^3 \phi_2^2}{\partial z \partial x^{*2}} \tag{88}$$

from (86),

$$(P_2')_z = \frac{1}{z^2}\frac{\partial^2 \phi_2^0}{\partial z \partial x^*} - \frac{1}{z}\frac{\partial^3 \phi_2^0}{\partial z^2 \partial x^*} \tag{89}$$

4.8 Determination of constants:

Finally, constants c_{10}, c_{20}, d_{10}, d_{20}, c_{12}, c_{22} d_{12} and d_{22} will be obtained using boundary conditions (58) - (61), and (64) - (67) as follows ; using bounday condition (60) i.e.,

$F_2 = F_1$, at $z = S(x^*)$

of, x^*F_2 x^*F_1

$$(x^*F_2)_{x^*} = (x^*F_1)_{x^*} \tag{90}$$

from (81) and (87), we get

$$16c_{20} = -2N^2 c_{10}$$

thus, $c_{10} = -\frac{8}{N^2} c_{20}$ (91)

using boundary condition (58) i.e.,

$$\frac{1}{\alpha N}\left[\phi^0_{2z}|z\right]_z = 2\left[\phi^0_{2z}|z - \phi^0_{1z}|z\right], \quad \text{at} \quad z = S(x^*)$$

$$d_{20} - C_{10} = 2\left(\frac{S}{\alpha N} - S^2\right)c_{20}$$

$$d_{20} = c_{10} + 2\left(\frac{S}{\alpha N} - S^2\right)c_{20}$$

$$= -\frac{8}{N^2}c_{20} + 2\left(\frac{S}{\alpha N} - S^2\right)c_{20}$$

thus,

$$d_{20} = 2c_{20}\left[\frac{S}{\alpha N} - S^2 - \frac{4}{N^2}\right] \qquad (92)$$

using boundary condition (61) i.e.,

$$\phi_1^0 = -\bar{V}_o \eta x^* + 1, \quad \text{at} \quad z = \eta$$

$$d_{10} + c_{10}\eta^2 = -\bar{V}_o \eta x^* + 1$$

$$d_{10} = -c_{10}\eta^2 - \bar{V}_o \eta x^* + 1$$

thus,

$$d_{10} = \frac{8\eta^2}{N^2} c_{20} - \bar{V}_o \eta x^* + 1 \qquad (93)$$

using boundary condition (59) i.e.,

$$\phi_2^{(0)} - \phi_1^{(0)} = 1, \qquad \text{at} \qquad z = S(x^*)$$

$$-c_{10}S^2 - d_{10} + c_{20}S^4 + d_{20}S^2 = 1$$

thus,

$$c_{20} = \left[2 - \bar{V}_o \eta x^*\right]\left[\frac{2S^3}{\alpha N} - \frac{8\eta^2}{N^2} - S^4\right]^{-1} \qquad (94)$$

using boundary condition (66);

now, from (82),

$$(P_1')_{x^*} = -N^2 \frac{1}{z} \frac{\partial \phi_1^{(2)}}{}$$

$$= -N^2\left[2c_{12} - \frac{1}{2} z^2 \frac{d^2 c_{10}}{dx^{*2}} - \log z \frac{d^2 d_{10}}{dx^{*2}}\right]$$

thus

$$(P_1')_{x^*} = -N^2\left[2c_{12} + \left\{\frac{4z^2}{N^2} - \log z \frac{8\eta^2}{N^2}\right\}\frac{d^2 c_{20}}{dx^*}\right] \qquad (95)$$

from (88),

$$(P_2')_{x^*} = \left[\frac{1}{z^3}\frac{\partial \phi_2^{(2)}}{\partial z} - \frac{1}{z^2}\frac{\partial^2 \phi_2^{(2)}}{\partial z^2} + \frac{1}{z}\frac{\partial^3 \phi_2^{(2)}}{\partial z^3} + \frac{1}{z}\frac{\partial^3 \phi_2^{0}}{\partial z \partial x^{*2}}\right]$$

$$= 16c_{22} - 4z^2\frac{d^2 c_{20}}{dx^{*2}} + 2\frac{d^2 d_{20}}{dx^{*2}}$$

thus, $(P_2')_{x^*} = 16c_{22} + \frac{d^2 c_{20}}{dx^{*2}}\left[\frac{4S}{\alpha N} - 4S^2 - 4z^2 - \frac{16}{N^2}\right]$

$$+ 4\left[\left(\frac{1}{\alpha N} - 2S\right)\frac{d^2 S}{dx^{*2}} - 2\left(\frac{dS}{dx^*}\right)^2\right]c_{20}$$

$$+ 8\left[\frac{1}{\alpha N}\frac{dS}{dx^*} - 2S\frac{dS}{dx^*}\right]\frac{dc_{20}}{dx^*} \qquad (96)$$

hence,

$$\left|(P_2')_{x^*} - (P_1')_{x^*}\right|_{at\ z=S} = 16c_{22} + 2N^2c_{12} + \frac{d^2c_{10}}{dx^{*2}}\left|-4S^2 + \frac{4S}{\alpha N}\right.$$

$$\left. - \frac{16}{N^2} - \log S.8\eta^2\right] + 8\frac{dc_{20}}{dx^*}\left[\frac{1}{\alpha N}.\frac{dS}{dx^*} - 2S\frac{dS}{dx^*}\right]$$

$$+ 4\left[\left(\frac{1}{\alpha N} - 2S\right)\frac{d^2S}{dx^{*2}} - 2\left(\frac{dS}{dx^*}\right)^2\right]c_{20} \qquad (97)$$

now from (66),

$$\left[-\frac{\frac{1}{z}\left\{\phi^0_{1x^*} + \frac{dS}{dx^*}\phi^0_{1z}\right\}}{\bar{K}_o(1+\delta x^*)}\right]_{x^*} = (P_2' - P_1')_{x^*},\ at\ z=S(x^*)$$

$$-\frac{1}{z}\left\{\phi^0_{1x^*x^*} + \frac{d^2S}{dx^{*2}}\phi^0_{1z} + \frac{dS}{dx^*}.\phi^0_{1zx^*}\right\}\bar{K}_o(1+\delta x^*) + \frac{1}{z}\left\{\phi^0_{1x} + \right.$$

$$\left. + \frac{DS}{dx^*}\phi^0_{1z}\right\}\bar{K}_o.\delta = \bar{K}_o^2\,(1+\delta x^*)^2\left[P_2' - P_1'\right]_{x^*}$$

$$- \frac{1}{z}\left\{\frac{d^2c_{10}}{dx^{*2}}z^2 + \frac{d^2d_{10}}{dx^{*2}} + \frac{d^2S}{dx^{*2}}(2zc_{10}) + \frac{dS}{dx^*}.2z\frac{dc_{10}}{dx^*}\right\}$$

$$(1+\delta x^*) + \frac{1}{z}\left\{\frac{dc_{10}}{dx^*}z^2 + \frac{dd_{10}}{dx^*} + \frac{dS_{10}}{dx^*}.2zc_{10}\right\}\delta$$

$$= \bar{K}_o(1+\delta x^*)^2\left[P_2' - P_1'\right]_{x^*}$$

thus,

$$\left[P_2' - P_1'\right]_{x^*} = -\frac{1}{z}\left[-\frac{8}{N^2}.\frac{d^2c_{20}}{dx^{*2}}.z^2 + \frac{8\eta^2}{N^2}.\frac{d^2S}{dx^{*2}}(2zc_{20}) - \right.$$

$$\left. - \frac{8}{N^2}\frac{dS}{dx^*}.2z\frac{dc_{20}}{dx^*}\right]\frac{1}{\bar{K}_o(1+\delta x^*)}$$

$$+ \frac{1}{z}\left[-\frac{8}{N^2}\frac{dc_{10}}{dx^*}z^2 + \frac{8\eta^2}{N^2}.\frac{dc_{20}}{dx^*}(2z\,c_{20})\right]\frac{\delta}{\bar{K}_o(1+\delta x^*)^2}$$

equating (97) and (98), we obtain,

$$C_{12} = -\frac{8}{N^2}c_{22} + \frac{1}{N^2}\frac{d^2c_{20}}{dx^{*2}}\left[4\eta^2\log S - 2\left(\frac{S}{\alpha N} - S^2 - \frac{4}{N^2}\right)\right.$$

$$\left. + \frac{4}{N^2}.\frac{(S^2-\eta^2)}{\bar{K}_o(1+\delta x^*)}.\frac{1}{S}\right] + \frac{8}{N^2}\frac{dc_{20}}{dx^*}.\frac{dS}{dx^*}\left[S + \frac{1}{N^2}\frac{1}{\bar{K}_o(1+\delta x^*)2\alpha N} - \frac{1}{}\right]$$

$$- \frac{4}{N^4} \frac{\delta}{\bar{K}_o(1+\delta x^*)^2} \frac{dc_{20}}{dx^*} \frac{(S^2-\eta^2)}{S} + \frac{4}{N^2}\left[(S + \frac{2}{N^2} \frac{1}{\bar{K}_o(1+\delta x^*)}\right.$$

$$\left. - \frac{1}{2\alpha N})\frac{d^2S}{dx^{*2}} + \left(\frac{dS}{dx^*}\right)^2 - \frac{2}{N^2} \cdot \frac{8}{\bar{K}_o(1+\delta x^*)^2} \cdot \frac{dS}{dx^*}\right] c_{20}$$

$$- \frac{\bar{v}_o \eta}{2N^2} \frac{\delta}{\bar{K}_o(1+\delta x^*)^2} \cdot \frac{1}{S}$$

Using boundary condition (67), i.e.,

$$\phi_1^{(2)} = 0 \quad , \quad \text{at} \quad z = \eta$$

$$d_{12} = - c_{12}\eta^2 + \frac{1}{8} \frac{d^2c_{10}}{dx^{*2}} \eta^4 - \frac{1}{2} \frac{d^2d_{10}}{dx^{*2}} \eta^2(\frac{1}{2}-\log \eta)$$

$$d_{12} = - c_{12}\eta^2 - \frac{1}{N^2} \eta^4 \cdot \frac{d^2c_{20}}{dx^{*2}} - \frac{4\eta^4}{N^2} \frac{d^2c_{20}}{dx^{*2}}(\frac{1}{2}-\log \eta) \qquad (100)$$

substituting the expression of c_{12} in (100), we obtain,

$$d_{12} = \frac{8\eta^2}{N^2} c_{22} - \frac{\eta^2}{N^2} \frac{d^2c_{20}}{dx^{*2}} \left[4\eta^2 \log S - 2(\frac{S}{\alpha N} - S^2 - \frac{4}{N^2}) + \right.$$

$$\left. + 3\eta^2 - 4\eta^2 \log \eta + \frac{4}{N^2} \cdot \frac{(S^2 - \eta^2)}{\bar{K}_o(1+\delta x^*)} \cdot \frac{1}{S}\right]$$

$$- \frac{8\eta^2}{N^2} \frac{dc_{21}}{dx^*} \cdot \frac{DS}{dx^*}\left[S + \frac{1}{N^2} \frac{1}{\bar{K}_o(1+\delta x^*)} - \frac{1}{2\alpha N}\right] + \frac{4\eta^2}{N^2} \frac{\delta}{\bar{K}_o(1+\delta x^*)^2}$$

$$\frac{dc_{10}}{dx^*} (\frac{S^2-\eta^2}{S}) - \frac{4\eta^2}{N^2} \left[(S+\frac{2}{N^2} \frac{1}{\bar{K}_o(1+\delta x^*)}\right.$$

$$\left. - \frac{1}{2\alpha N}) \frac{d^2S}{dx^{*2}} + (\frac{dS}{dx^*})^2 + (\frac{dS}{dx^*})^2 - \frac{2}{N^2} \frac{\delta}{\bar{K}_o(1+\delta x^*)^2} \cdot \frac{1}{8}\right.$$

Using boundary condition (64) i.e.,

$$\frac{1}{\alpha N}\left[(\phi^2_{2z}|z)_z - (\phi^0_{2x^*}|z)_{x^*}\right] = 2\left[\phi^2_{2z}|z - \phi^2_{1z}|z - \frac{dS}{dx^*}\{\phi^0_{2x^*}|z\right.$$

$$+ \frac{1}{2}\phi^0_{2z}|z.\frac{dS}{dx^*} - \phi^0_1 x^*|z - \frac{1}{2}\phi^0_{1z}|z.\frac{dS}{dx^*}\}\Big], \quad \text{at } z = S(x^*)$$

we have,

$$\frac{1}{\alpha N}\Big[8c_{22}\, z - 2\frac{d^2c_{10}}{dx^{*2}}.z^3 - \frac{d^2c_{20}}{dx^{*2}}.z^3 - \frac{d^2d_{20}}{dx^{*2}}.z\Big] =$$

$$= 2\Big[4c_{22}\, z^2 + 2d_{22} - \frac{1}{12}\frac{d^2c_{20}}{dx^{*2}}\,6z^4 + \{2c_{12} + \frac{1}{8}\frac{d^2c_{10}}{dx^{*2}}.4z^2$$

$$- \frac{1}{2}\frac{d^2d_{10}}{dx^*}(-2\log z)\} - \frac{dS}{dx^*}\{\frac{dc_{20}}{dx^*}z^3 + \frac{dd_{20}}{dx^*}\, z + \frac{1}{2}\frac{dS}{dx^*}$$

$$(c_{20}.4z^2 + 2d_{20})\} + \frac{dS}{dx^*}\{\frac{dc_{10}}{dx^*}.z + \frac{dd_{10}}{dx^*}.\frac{1}{z} + \frac{1}{2}.\frac{dS}{dx^*}(2c_{10})\}\Big]$$

substituting the expression of c_{12}, we obtain

$$d_{22} = 2(\frac{S}{\alpha N} - S^2 - \frac{4}{N^2})c_{22} - \frac{S^2}{4}\frac{d^2c_{20}}{dx^*}\Big[\frac{S}{\alpha N} + \frac{2}{(\alpha N)^2} - S^2 - \frac{8}{\alpha N^3} - \frac{1}{S}$$

$$- \frac{8}{N^2} - \frac{32\eta^2}{N^2}.\frac{1}{S^2}\log S + \frac{8}{S^2N^2}(\frac{S}{\alpha N} - S^2 - \frac{4}{N^2})$$

$$- \frac{16}{S^2N^4}.\frac{1}{\bar{K}_o(1+\delta x^*)}.\Big[\frac{(S^2 - \eta^2}{S}\Big] + \frac{dc_{20}}{dx^*}.\frac{dS}{dx^*}\Big[\frac{3S}{\alpha N} - \frac{S}{(\alpha N)^2} - \frac{1}{2}S^3$$

$$- \frac{4\eta^2}{N^2}.\frac{1}{S} + \frac{3S}{N^2} + \frac{8}{N^4}\,\frac{1}{\bar{K}_o(1+\delta x^*)} - \frac{4}{\alpha N^3}\Big] + c_{20}\Big[(\frac{S}{2(\alpha N)^2} + \frac{S^2}{\alpha N}$$

$$+ \frac{4S}{N^2} + \frac{8}{N^4}.\frac{1}{\bar{K}_o(1+\delta x^*)} - \frac{2}{\alpha N^3})\frac{d^2S}{dx^{*2}} + (\frac{3S}{\alpha N} - 2S^2 + \frac{4}{N^2})(\frac{dS}{dx^*})^2$$

$$- \frac{8}{N^4}\,\frac{\delta}{\bar{K}_o(1+\delta x^*)^2}\,\frac{dS}{dx^*}\Big] - \frac{dc_{20}}{dx^*}.\frac{4}{N^4}.\frac{\delta}{K_o(1+\delta x^*)^2}\,\frac{(S^2-\eta^2)}{S} +$$

$$+ \frac{dS}{dx^*}\,\frac{\bar{V}_o}{2}.\frac{1}{S}.\eta - \frac{\bar{V}_o\eta}{2N^2}\,\frac{\delta}{\bar{K}_o(1+\delta x^*)^2}.\frac{1}{S}$$

now using boundary condition (65) i.e.,

$$\phi_2(^2) = \phi_1(_2) \quad \text{on} \quad z = S(x^*)$$

$$d_{22}S^2 + c_{22}S^4 - \frac{1}{=2}\frac{d^2c_{20}}{dx^{*2}}\,S^6 = d_{12} + c_{12}S^2 - \frac{1}{8}\frac{d^2c_{10}}{dx^*} + \frac{1}{2}\frac{d^2d}{dx^{*2}}.S^2$$

$(\frac{1}{2} - \log S)$

we obtain

$$c_{22}\left[\frac{2S^3}{\alpha N} - \frac{8\eta^2}{N^2} - S^4\right] = \frac{d^2c_{20}}{dx^{*2}}\left[\frac{4\eta^4}{N^2}\log\left(\frac{\eta}{S}\right) - \frac{4\eta^2}{S}\frac{1}{\bar{K}_o(1+\delta x^*)}\right.$$

$$(S^2 - \eta^2) - \frac{2S^3}{\alpha N^3} + \frac{2S\eta^2}{\alpha N^3} - \frac{12\eta^2}{N^4} - \frac{3\eta^4}{N^2} + \frac{S^3}{4\alpha N}$$

$$+ \frac{2S^4}{4(\alpha N)^2} - \frac{16S^2\eta^2}{N^2}\log S - \frac{2S^4}{N^2} + \frac{S^6}{12}\Big] + \frac{dc_{20}}{dx^*}\cdot\frac{dS}{dx^*}\left[\frac{8\eta^2}{\alpha N^3} - \frac{8\eta^2 S}{N^2}\right.$$

$$- \frac{8\eta^2}{N^4}\frac{1}{\bar{K}_o(1+\delta x^*)}\frac{3S^4}{\alpha N} + \frac{S^3}{\alpha N^2} + \frac{4\eta^2 S^3}{N^2} + \frac{S^5}{2}\Big] + c_{20}\left[\left\{\frac{4S}{N^2}(S^2-\eta^2)\right.\right.$$

$$+ \frac{8}{N^4}\frac{1}{\bar{K}_o(1+\delta x^*)}(S^2 - \eta^2) - \frac{2}{\alpha N^3}(\eta^2-S^2) - \frac{4S^3}{N^2} - \frac{8S^2}{N^4}\frac{1}{\bar{K}_o(1+\delta x^*)}$$

$$+\frac{2S^2}{\alpha N^3} + \frac{S^4}{2(\alpha N)^2} - \frac{S^3}{(\alpha N)^2}\Big\}\frac{d^2S}{dx^{*2}} + \frac{dS}{dx^*}\left\{\frac{8}{N^4}\frac{\delta}{\bar{K}_o(1+\delta x^*)}(\eta^2-S^2)\right\}$$

$$\left.+ \left\{\frac{3S}{\alpha N} - 2S^2 + \frac{4}{N^2} + \frac{4}{N^2}(S^2 - \eta^2)\right\}\left(\frac{dS}{dx^*}\right)^2\right]$$

$$+ \left\{\frac{4\eta^2}{N^2}\frac{\delta}{\bar{K}_o(1+\delta x^*)^2}\frac{(S^2-\eta^2)}{S} + \frac{S^4}{N^2}\right\}\frac{dc_{20}}{dx^*} - \frac{\bar{V}_o}{2}\eta S\frac{dS}{dx^*}$$

or,

$$c_{22} = F\left[\frac{2S^3}{\alpha N} - \frac{8\eta^2}{N^2} - S^4\right]^{-1} = F.\left[(2 - \bar{V}_o\eta x^*)^{-1}\, c_{20}\right]$$

thus we can finally write,

$$c_{22} = \left[(2 - \bar{V}_o\eta x^*)^{-1}c_{20}\right]\left[\frac{d^2c_{20}}{dx^{*2}}\left\{\frac{4\eta^4}{N^2}\log\left(\frac{\eta}{S}\right) - \frac{4\eta^2}{S}\right.\right.$$

$$\frac{1}{\bar{K}_o(1+\delta x^*)}(S^2 - \eta^2) - \frac{2S^3}{\alpha N^2} + \frac{2S\eta^2}{\alpha N^3} - \frac{12\eta^2}{N^4} - \frac{3\eta^4}{N^2}$$

$$\left.+ \frac{S}{4\alpha N} + \frac{2S^4}{4(\alpha N)^2} - \frac{16S^2\eta^2}{N^2}\log S - \frac{2S^4}{N^2} + \frac{S^6}{12}\right\} + \frac{dc_{20}}{dx^*}\cdot\frac{dS}{dx^*}$$

$$\{\frac{8\eta^2}{\alpha N^3} - \frac{8\eta^2 S}{N^2} - \frac{8\eta^2}{N^4}\frac{1}{\bar{K}_o(1+\delta x^*)} - \frac{3S^4}{\alpha N} + \frac{S}{\alpha N^2} + \frac{4\eta^2 S^3}{N^2} + \frac{S^5}{2}\}$$

$$+c_o\Big[\Big(\frac{4S}{N^2}(S^2-\eta^2) + \frac{8}{N^4}\frac{1}{\bar{K}_o(1+\delta x^*)}(S^2-\eta^2) - \frac{2}{\alpha N^3}$$

$$(\eta^2-S^2) - \frac{4S}{N^2} - \frac{8S^2}{N^4}\frac{1}{\bar{K}_o(1+\delta x^*)} + \frac{2S^2}{\alpha N^3} - \frac{S^3}{(\alpha N)^2} + \frac{S^4}{2(\alpha N)^2}\Big)\frac{d^2S}{dx^{*2}}$$

$$+\frac{dS}{dx^*}\Big(\frac{8}{N^4}\frac{\delta}{\bar{K}_o(1+\delta x^*)}(\eta^2-S^2)\Big) + \Big(\frac{3S}{\alpha N} - 2S^2 + \frac{4}{N^2} + \frac{4}{N^2}(S^2-\eta^2)\Big)$$

$$\Big(\frac{dS}{dx^*}\Big)^2\Big\} + \Big(\frac{4\eta^2}{N^2}\frac{\delta}{\bar{K}_o(1+\delta x^*)^2}\frac{(S^2-\eta^2)}{S} + \frac{S^4}{N^2}\Big)\frac{dc_{20}}{dx^*} - \frac{\bar{V}_o}{2}\eta.S\frac{dS}{dx^*}\Big] \qquad (103)$$

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