CONGRUENCES ON REGULAR w - SEMIGROUPS.

By

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DECLARATION

This thesis is not a published or unpublished work of another author. No part of this thesis has been submitted for a degree in any other university. Information derived from the published or unpublished work of others has been acknowledged in the text.

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CERTIFICATION.

This thesis titled "CONGRUENCES ON REGULAR ω-SEMIGROUPS" by Abdullahi Umar meets the regulation governing the award of the degree of Master of Science (M.Sc.) of Ahmadu Bello University, and is approved for its contribution to scientific knowledge and literary presentation.

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DEDICATED

TO

My parents, brothers and sisters.

ABDULLAHI UMAR.

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In the name of Allah, the beneficient, the most merciful.

thank Allah for giving me the health, guidance all I needed to complete this work.

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PREFACE

Munn [8] established a structure theorem for simple, and also general ω - semigroups which represents an extension of the structure theorem for bisimple ω-semigroups due to Reilly [I2]. Congruences on a bisimple ω-semigroup were studied by Munn and Reilly [9], in particular they constructed all idempotent separating congruences and the quotient relative to the least group congruence on such a semigroup. Group congruences were characterised by Ault [I]. Iyengar [7] established the homomorphism theorem between two simple ω- semigroups which gives an insight into the study of congruences on simple ω- semigroups. Petrich [11] determined all congruences on a simple ω - semigroup where he classified them as group and non-group congruences. Determination of congruences throughout [11] is based on their description in [I0].

The main purpose of this thesis is to characterize all non-group congruences on simple ω-semigroups using the approach of Munn and Reilly [9]. In chapter one some basic definitions and results that are needed in the main body of the thesis are given. In chapter two we look at the structure of a regular ω-semigroup in general. In chapter three we characterize all the non-group congruences ie. idempotent separating and pseudo-idempotent separating congruences on a simple ω - semigroup. In chapter four we find the maximum group homorphic image of a simple ω-semigroup, determine the nature of $S/\sigma v_H$ and also establish a homomorphism theorem of a simple ω- semigroup into a group G*.

Chapter One

Basic Definitions and Preliminaries.

Most of the results of this chapter are from Clifford and Preston [2] and J.M. Howie [6].

By a groupoid (S,μ) we shall mean a non-empty set S on which a binary operation $\mu: S\times S \longrightarrow S$. We shall say that (S,μ) is a semigroup if μ is associative, i.e. if

$$\forall\ x,y,z \in S\ ((x,y)\mu,z)\mu = (x,(y,z)\mu)\ \mu) \qquad (1.1)$$

(Here and elsewhere we shall write function symbols on the right). For convenience we shall also write $(x,y)\mu$ simply as xy and usually refer to the semigroup operation as multiplication. The formula (1.1) then becomes $(xy)z = x(yz)$, and the semigroup (S, μ) is then denoted by $(S,.)$.

If $(S,.)$ is a semigroup, then a non-empty subset T of S is called a subsemigroup of S if it is closed with respect to multiplication, i.e. if $(\forall x, y \in T)xy \in T$.

An element e of S is called an idempotent if it satisfies the equation $e^2 = e$.

By a semilattice we shall mean a commutative semigroup of idempotents.

A relation ρ on a set X is called a partial ordering of X if

(i) $a \rho a$ for every $a \in X$.

(ii) $a \rho b$ and $b \rho a$ implies $a = b$.

(iii) $a \rho b$ and $b \rho c$ implies $a \rho c$

Lemma I.2 [4,Lemma I.I] Let E be a semilattice, then the relation "$\leq$" defined by the rule that $e \leq f$ if and only if $ef = fe = e$ is a partial ordering on E, called the natural partial ordering.

Proof (i) $e.e = e^2 = e$, and hence $e \leq e$ (ii) If $e \leq f$ and $f \leq e$ then $ef = fe = e$ and $fe = ef = f$, whence $e = f$.
(iii) If $e \leq f$ and $f \leq g$ then $ef = fe = e$ and $fg = gf = f$, and $eg = (ef)g = e(fg) = ef = e$, $ge = g(fe) = (gf)e = fe = e$. Hence $e \leq g$.

Let S be any semigroup, and let 1 be a symbol not representing any element of S. If we extend the given binary operation in S to one in $S \cup \{1\}$ by defining $1.1 = 1$ and $1.a = a.1 = a$ for every $a \in S$. It is quickly verified that $S \cup \{1\}$ is a semigroup with identity element. Let us define $S^1 = \begin{cases} S & \text{if } S \text{ has an identity element} \\ S \cup \{1\} & \text{otherwise.} \end{cases}$

By a left (right) ideal of a semigroup S we mean a non-empty subset A of S such that $SA \subseteq A$ ($AS \subseteq A$). By two-sided ideal, or simply ideal, we mean a subset of S which is both a left and a right ideal of S

If A is a non-empty subset of a groupoid S, the intersection of all left ideals of S containing A (S being one such) is a left ideal of S containing A and contained in every other such left ideal of S. We call it the left ideal of S generated by A. If S is a semigroup,

the left ideal of S generated by A is simply $A \cup SA = S^1A$. With the analogous definition, we see that the right ideal of S generated by A is $A \cup AS = AS^1$, and that the (two-sided) ideal of A generated by A is $A \cup S\ A \cup AS \cup SAS = S^1AS^1$. If, in particular, A consists of a single element a, then we call S^1a, aS^1 and S^1aS^1 the <u>principal</u> left, right and two-sided ideal of A, respectively generated by a. Evidently every ideal (whether one or two-sided) is a subsemigroup but not every subsemigroup is an ideal.

If ϕ is a mapping from a semigroup $(S,.)$ into a semigroup $(T,.)$, we say that ϕ is a homomorphism if $(\forall x,y \in S)\ (xy)\phi = (x\phi)(y\phi)$.
If ϕ is both one-one and onto we shall call it an isomorphism. An isomorphism from S onto S will be called an automorphism of S. If there exists an isomorphism ϕ from S onto T we say that S and T are isomorhpic and write $S \simeq T$.

An element a of a semigroup S is regular if $a \in aSa$, that is, if $axa=a$ for some x in S. A semigroup is called regular if every element of S is regular.

We note that if $axa=a$ then $e=ax$ is an idempotent element of S such that $ea=a$ for $(ax)(ax) = (axa)x=ax=e$ and $ea=axa=a$. Similarly $f = ax$ is an idempotent such that $af=a$. We also note that if a is a regular element of S, then the principal ideal $aS^1 = a \cup aS$ generated by a is just aS, for $a=af$ implies $a \in aS$. Similarly $S^1a=Sa$.

<u>Lemma. I.3</u> [2.Lemma I.3] An element a of a semigroup S is regular if and only if the principal right (left) ideal of S generated by a has an idempotent generator e, i.e, $aS^1 = eS^1$ $(S^1a = S^1e)$.

Proof:

If a is regular, then axa = a for some x in S, and e = ax is an idempotent element of S such that ea = a. Clearly $aS^1 = eS^1$. Assume conversely that $aS^1=eS^1$ with $e^2=e$. Then a=ex with x in S^1, so $ea=e^2x = ex=a$, and e=ay with y in S^1, so that a=ea=aya. If y=1, then $a=a^2$, and a=aaa. Hence, in any case, ae aSa and a is regular.

Two elements a and b of a semigroup S are said to be inverse of each other if aba = a and bab = b.

Lemma I.4 [2. Lemma I.14] If a is a regular element of a semigroup S, say axa=a with x in S, then a has at least one inverse in S, in particular xax.

Proof Let b = xax

aba = a(xax)a=ax(axa) = axa =a

bab = (xax)a(xax)=x(axa)(xax)=xa(xax)

= x(axa)x=xax=b

Hence b is an inverse of a.

A semigroup S in which every element has a unique inverse is called an inverse semigroup.

Lemma I.5 [2,Lemma I.16] If e,f, fe and ef are all idempotent elements of a semigroup, then ef and fe are inverses of each other.

Proof: $(ef)(fe)(ef) = ef^2e^2f = ef\,ef = (ef)^2 = ef$ and by symmetry (fe)(ef)(fe) =fe.

Theorem I.6 [2,Theorem 1.7] The following three conditions on a semigroup S are equivalent:

(i) S is regular, and any two idempotent elements of S commute with each other ;

(ii) every principal right ideal and every principal left ideal has a unique idempotent generator;

(iii) S is an inverse semigroup (i.e., every element of S has a unique inverse in S).

Lemma 1.7 [6, Proposition I.4(d)&(f)]

If a and e are any elements of an inverse semigroup S with $e^2=e$, then aa^{-1} , $a^{-1}a$, $a^{-1}ea$ and aea^{-1} are idempotents.

An element e of a semigroup S is called a left (right) identity element of S if ea=(ae=a) for all a ∈ S. e is called a two-sided identity (or simply an identity) element of S if it is both a left and a right identity element of S.

We note that if S contains a left identity e and a right identity f, then e=f; for ef=f since e is a left identity and ef =e since f is a right identity.

If e is an idempotent element of a semigroup S, then eS consists of all elements a of S for which e is a left identity, that is ea=a. For if a=ex for some x in S, then $ea=e^2x$ = ex=a, and the converse is evident. Similarly Se consists of all elements of S for which e is a right identity, and eSe is the set of all elements of S for which e is a two-sided identity. We evidently have eSe = Se∩eS. eS and Se are subsemigroups of S and so is their intersection eSe. Moreover eSe has the identity element e.

Lemma 1. 8 [6, Lemma 1.11]

If e and f are idempotent elements of an inverse semigroup S, then Se∩Sf = Sef . Thus for any elements a,b∈S

$$Sa \cap Sb = Sa^{-1}a \cap Sb^{-1}b = Sa^{-1}ab^{-1}b$$

Proof: If a ∈ Se∩Sf, then ae = af = a, so that aef=af=a, and a ∈ Sef. Conversely if ae Sef, then aef=afe=a, and so ae=af=a, that is ae Se∩Sf.

A relation ρ on a Set S is said to be an <u>equivalence relation</u> if the following are satisfied:
(i) $a \rho a$ for every $a \in S$ (ii) $a \rho b$ implies $b \rho a$ and (iii) $a \rho b$ and $b \rho c$ implies $a \rho c$.

Two elements of a semigroup S are said to be $\mathcal{L}$ -($\mathcal{R}$-) equivalent if they generate the same principal left (right) ideal of S, and $\mathcal{J}$ - equivalent if they generate the same principal ideal of S. We write $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Then $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ and $\mathcal{J}$ called <u>Green's relations</u> are equivialence relations on S such the $\mathcal{H} \leq \mathcal{L} \leq \mathcal{D} \leq \mathcal{J}$ and $\mathcal{H} \leq \mathcal{R} \leq \mathcal{D} \leq \mathcal{J}$. We denote by H_a (L_a, R_a, D_a and J_a) the $\mathcal{H}$ - ($\mathcal{L}$-, $\mathcal{R}$ - , $\mathcal{D}$- & $\mathcal{J}$-) class of S containing the element a.

A semigroup S is said to be <u>bisimple</u> if it contains only one $\mathcal{D}$ - class and <u>simple</u> if it contains only one $\mathcal{J}$ - class.

<u>Theorem 1.9</u> [2, Theorem 2.11]
(i) If a $\mathcal{D}$-class D of a semigroup S contains a regular element, then every element of D is regular,
(ii) If D is regular, then every $\mathcal{L}$-class and $\mathcal{R}$-class contained in D contains an idempotent.

<u>Proof</u>: From lemma 1.3 an element of a semigroup S is regular if and only if R_a [L_a] contains an idempotent. It follows that an $\mathcal{R}$-class R ($\mathcal{L}$-class L) contains a regular element, then it contains an idempotent, and every element of R(L) is regular. Since every $\mathcal{R}$-class of S contained in D meets every $\mathcal{L}$-class of S contained in D, (i) is evident.

But then (ii) is also immediate from lemma 1.3.

The following lemas 1.10 & 1.11 are evident.

Lemma 1.10 [2,Lemma 2.12] If a and a^{-1} are inverse elements of a semigroup S, then $e=aa^{-1}$ and $f= a^{-1}a$ are idempotents such that $ea=af=a$ and $a^{-1}e = fa^{-1} a = a^{-1}$. Hence $e \in R_a \cap L_{a^{-1}}$ and $f \in R_{a^{-1}} \cap L_a$. The elements a, a^{-1}, e and f all belong to the same $\mathcal{D}$-class.

Lemma 1.11 [2,Lemma 2.13]

(i) If a is a regular element of a semigroup S, then $aS^1 = aS$ and $S^1a = Sa$.

(ii) If a and b are regular elements of S, then $a \mathcal{L} b$ ($a \mathcal{R} b$) and only if $Sa = Sb$ ($aS = bS$).

Lemma 1.12 [2,Lemma 2.14] Any idempotent element e of a semigroup S is a right identity element of L_e, a left identity element of R_e, and a two-sided identity element of H_e.

Proof: If $a \in L_e$ then $a \in Se$, and hence $ae = a$. If $a \in R_e$ then $a \in eS$, and $ea=a$. If $a \in H_e = R_e \cap L_e$, then $ea=ae=a$.

Lemma 1.13 [2,Lemma 2.15] No $\mathcal{H}$-class can contain more than one idempotent.

Proof: If e and f are idempotents such that $H_e = H_f$, then by lemma 1.12, each is a two-sided identity of the other, and hence $e = f$.

Theorem 1.14 [2, Theorem 2.18] Let a be a regular element of a semigroup S.

(i) Every inverse of a lies in D_a

(ii) An $\mathcal{H}$-class H_b contains an inverse of a if and only if both of the $\mathcal{H}$-classes $R_a \cap L_b$ and $R_b \cap L_a$ contain idempotents.

(ii) No H-class contains more than one inverse of a.

Corollary 1.15 [2, Corollary 2.19 (i)]

A semigroup S is an inverse semigroup if and only if each L-class and each R-class contains exactly one idempotent.

Corollary 1.16 [6, Corollary 1.3]

A semigroup S is simple if and only if $S a S = S$ for every $a \in S$ i.e. if and only if for every $a, b \in S$ there exist x,y in S such that $x a y = b$.

A two sided (left, right) ideal M of a semigroup S is called minimal if it does not properly contain any two-sided (left, right) ideal of S. Let M be a minimal ideal, if A is any other ideal of S of the same type as M, either $M \subseteq A$ or $M \cap A = \phi$.

Since two-sided ideals A and B of a semigroup S always contain there set product AB(an ideal), it follows that there can be at most one minimal two-sided ideal of S. If S has a minimal two-sided ideal K, then K is called the kernel of S. Since K is contained in every two sided ideal of S, it may be characterised as the intersection of all the two-sided ideals of S. If this intersection is empty, then S does not have a kernel; this is the case, for example, for an infinite cyclic semigroup. Every finite semigroup evidently has a kernel

Proposition 1.17 [6, Proposition 1.5]. If a semigroup S has a kernel K, then K is a simple semigroup.

A relation ρ on a groupoid S is said to be right (left) compatible if $a \rho b$ (a, b in S) implies $a c \rho b c$ ($c a \rho c b$) for every $c \in S$.

A right (left) compatible equivalence relation on S will be called a right (left) congruence on S. By a congruence on S we mean an equivalence relation on S which is both right and left compatible.

Let ρ be a congruence on a groupoid S. Let A and B be members of S/ρ, i.e., equivalence classes of S mod ρ. Let $a_1, a_2 \in A$ and $b_1, b_2 \in B$. From $a_1 \rho a_2$ we have $a_1 \cdot b_1 \rho a_2 \cdot b_1$, since ρ is right compatible. From $b_1 \rho b_2$ we have $a_2 \cdot b_1 \rho a_2 \cdot b_2$, since ρ is left compatible. From the transitivity of ρ, we conclude that $a_1 \cdot b_1 \rho a_2 \cdot b_2$. Hence the set product AB of A and B is contained in some equivalence class C. If we define $A.B = C$, then S/ρ becomes a groupoid which we called the factor (or quotient) groupoid of S modulo ρ

We denote by $a\rho$ ($a \in S$) the equivalence class mod ρ to which a belongs. The above definition of ($\circ$) is simply $a\rho \circ b\rho = (ab)\rho$, for all a,b in S. Denoting by ρ'' the natural mapping of S upon S/ρ, we have $a\rho'' = a\rho$ for all a in S, and so $a\rho'' \circ b\rho'' = (ab)\rho''$. Thus ρ'' is a homomorphism, and we call it the natural (or canonical) homomorphism of S upon S/ρ. Since ρ'' is a homomorphism, S/ρ is a semigroup if S is a semigroup.

Theorem 1.18 [Main homomorphism Theorem]. [2,Theorem 1.5]

Let θ be a homomorphism of a groupoid S upon a groupoid S^1. Let $\rho = \theta.\theta^{-1}$ i.e., $a \rho b$ (a,b in S) if and only if $a\theta = b\theta$. Then ρ is a congruence on S, and there exists an isomorphism Ψ of S/ρ upon S^1 such that $\rho''\Psi = \theta$ where ρ'' is the natural homomorphism of S upon S/ρ

We mostly consider these types of congruences

I. A congruence ρ on a semigroup S is called idempotent separating if no two distinct idempotents are ρ-equivalent.

2. A congruence ρ on a semigroup S is called an inverse semigroup congruence if S/ρ is an inverse semigroup

3. A congruence ρ on a semigroup S is called locally idempotent separating if no two distinct $\mathcal{D}$-equivalent idempotents are ρ - equivalent

4. A congruence ρ on a semigroup S is called pseudo idempotent separating if it is locally idempotent separating but not idempotent separating.

5. A congruence ρ on a semigroup S is called a group congruence if S/ρ is a group

Theorem I.I9 [4, Lemma 4.8]

Let S be a regular semigroup, then a congruence ρ on S is idempotent separating if and only if $\rho \subseteq H$.

Theorem I.20 [6, theorem 3.I]

If S is an inverse semigroup with semilattice of idempotents E, then the relation

$\sigma = \{(x,y) \in S \times S;\ (\exists e \in E)\ ex=ey\}$ is the minimum group congruence on S.

Theorem I.2I [6, theorem 3.2] If S is an inverse semigroup with semilattice of idempotents E, then the relation

$\mu = \{(a,b) \in S \times S:\ (\forall e \in E)\ a^{-1}ea = b^{-1}eb\}$ is the maximum idempotent seperating congruence on S.

Chapter Two

In this chapter we look at the structure of regular ω-semigroups in general as in [8, § 1].

Let S be a semigroup whose set E of idempotents is non-empty. We define a partial ordering $\geq$ on E by the rule that $e \geq f$ if and only . If $E = \{e_i : i \in N\}$, where N denotes the set of all non-negative integers, and if the elements of E form the chain $e_0 > e_1 > e_2 > \cdots$, then S is called an ω- semigroup.

Some examples of regular ω-semigroups. With a few minor exceptions we shall use the terminology of [2&3]. The set of all non-negative integers will be denoted by N.

1. The union of an ω-chain of groups.

Let $\{G_i : i \in N\}$ be a set of pairwise -disjoint groups and for each $i \in N$ let γ_i be a homomorphism of G_i into G_{i+1}. For each pair $(i,j) \in N \times N$ such that $i < j$ let $\alpha_{i,j} = \gamma_i \gamma_{i+1} \cdots \gamma_{j-1}$ and for each $i \in N$ let $\alpha_{i,i}$ denote the identity automorphism of G_i. Let $S = \bigcup_{i=0} G_i$ and define a multiplication on S by the rule that

$$a_i b_j = (a_i \alpha_{i,t})(b_j \alpha_{j,t}) \qquad (a_i \in G_i,\ b_j \in G_j),$$

where $t = \max\{i,j\}$. Then S is a regular ω-semigroup. In fact, if e_i denotes the identity of G_i for all $i \in N$, then $e_i \geq e_j$ if and only if $i \leq j$. Write $T_n = \bigcup_{i=n} G_i$ $(n \in N)$. Then it is clear from the law of multiplication that T_n is an ideal of S for all $n \in N$. Moreover, $\bigcap^{\infty} T_n = \Phi$.

Hence S has no kernel.

2. The bicyclic semigroup B.

Let $B = N \times N$ and define a multiplication in B by the rule that

$$(m,n)(p,q) = (m-n+t, q-p+t),$$

where $t = \max\{n,p\}$.

Then B is a bisimple ω-semigroup [8, e.g.1.2]. The set of idempotents of B is $\{(n,n): n \in N\}$ and $(m,m) \geq (n,n) \Leftrightarrow m \leq n$. We call B the bicylic semigroup

3. The semigroup $S(G,\alpha)$.

The bicyclic semigroup can be generalised as follows. Let G be any group and let α be an endomorphism of G. Let $S=N\times G\times N$ and define a multiplication in S by

$$(m,g,n)(p,h,q)=(m-n+t,\ g\alpha^{t-n}.\ h\alpha^{t-p},\ q-p+t),$$

where $t=\max\{n,p\}$ and α^0 denotes the identity automorphism of G. Then S is a bisimple ω-semigroup, which we denote by $S(G,\alpha)$; moreover, every bisimple ω- semigroup is, to within isomorphism of this type [12,Thrms.2.2&2.8]. Such a semigroup is necessarily regular [2,theorem 2.11]

4. The semigroup B_d.

Let d be any positive integer and let B_d be defined by $B_d = \{(m,n) \in B: m \equiv n \pmod d\}$,

where B is the bicyclic semigroup (e.g.2). Then B_d is a subsemigroup of B. Further more it can be shown that B_d is a simple regular ω-semigroup with exactly d $\mathcal{D}$-classes. The $\mathcal{D}$-classes are the subsets

$D_i = \{(m,n) \in B: m \equiv i \pmod d \text{ and } n \equiv i \pmod d \text{ o} \leq i \leq d\}$,

and each D_i is a subsemigroup of B_d isomorphic to B itself.

A regular ω-semigroup is necessarily an inverse semigroup [12,2.1]. It is convienient to distinguish between the case in which S has a kernel and that in which it has not. It has been shown by Munn [8,2.6] that S has no kernel if and only if it is the union of semilattice of groups, the semilattice in this case being an ω-chain. The structure of a regular ω-semigroup without a kernel is therefore determined by an infinite sequence of groups G_i and homomorphisms γ_i.

$$G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} G_2 \xrightarrow{\gamma_2} \cdots \xrightarrow{\gamma_{n-1}} G_n \xrightarrow{\gamma_n} \cdots,$$

in accordance with a theorem of clifford.

On the other hand, if S has a kernel K then K is a simple regular ω-semigroup; further if $K \neq S$ then the multiplication in S can be expressed in terms of that of K and of finitely many groups by means of certain connecting homomorphisms [8, Thm 2.7].

Munn [8] reduced the problem of determining the structure of regular ω-semigroups (with kernel) to that of simple regular ω-semigroups, and classified the simple regular ω-semigroup completely as follows:

Let d be a positive integer and let $\{G_i : i=0,1, \ldots d-1\}$ be a set of d pairwise-disjoint groups with identities f_i. Let γ_{d-1} be a homomorphism of G_{d-1} into G_0 and, if $d > 1$ let γ_i be a homomorphism of G_i into $G_{i+1} (i=0,1,\ldots d-2)$. Thus we have a sequence

$$G_0 \xrightarrow{\gamma_0} G_1 \xrightarrow{\gamma_1} \cdots G_{d-1} \xrightarrow{\gamma_{d-1}} G_0.$$

For all $n \in N$ let $\gamma_n = \gamma_{n(\mathrm{mod}\ d)}$. For $m,n \in N$ and $m < n$ write $\alpha_{m,n} = \gamma_m \gamma_{m+1} \cdots \gamma_{n-1}$.

And for all $n \in N$ let $\alpha_{n,n}$ denote the identity automorphism of $G_{n\ (\mathrm{mod}\ d)}$.

Let S be the set of all ordered triples

$$(m, a_i, n),$$

where $m,n \in N$, $0 \leq i \leq d-1$ and $a_i \in G_i$. Define a multiplication in S by the rule that.

$$(m,a_i,n)(p,b_j,q) = (m-n+t, (a_i \alpha_{u,\omega})(b_j \alpha_{v,w}), q-p+t)$$

where $t=\max'\{n,p\}$, $u=nd+i$, $v=pd+j$ and $\omega = \max'\{u,v\}$. Denote the groupoid so formed by $S(d,G_i, \gamma_i)$.

Theorem 2.1 [8, Theorems 3.3&4.11]

$S(d,G_i,\gamma_i)$ is simple regular ω-semigroup with exactly d $\mathcal{D}$-classes.

Conversely, every simple ω-semigroup has the above structure.

Corollary 2.2 [8, Theorem 3.3]

The set of idempotents of $S(d,G_i,\gamma_i)$ is

$$E_s = '\{(m,f_i,m) = e_{md+i}: f_i \in G_i, m \in N\}$$

Theorem 2.3 [8, Theorem 2.1]

Let S be a regular ω-semigroup. Then S is an inverse semigroup with an identity and $\mathcal{H}$- is a congruence on S.

Let S be a regular ω-semigroup and let $'\{e_n: n \in N\}$ be the set of idempotents of S, where $e_m \geq e_n$ if and only if $m \leq n$. Let the $\mathcal{R}$-($\mathcal{L}$-) class of S containing e_n be denoted by $R_n(L_n)$ for all $n \in N$.

With the above notation and the fact that the outer components of the elements of $S(d,G_i,\gamma_i)$ behave as in the bisimple ω-semigroup under multiplication, one can easily see that

$R_{md+i} = '\{(m,a_i,n) \in S: a_i \in G_i, n \in N\}$ $(m \in N; i=0,1,\ldots,d-1)$

$L_{nd+i} = '\{(m,a_i,n) \in S: a_i \in G_i, m \in N\}$ $(n \in N; i=0,1,\ldots,d-1)$

$D_i = \{(m,a_i,n) \in S: a_i \in G_i, m \in N, n \in N\}$ $(i=0,1,\ldots,d-1$

Corollary 2.4

Let $S=S(d,G_i,\gamma_i)$. Then (m,g_i,n) and (p,h_j,q) are

1. $\mathcal{R}$ - related if and only if $m=p$ & $i=j$
2. $\mathcal{L}$ - related if and only if $n=q$ & $i=j$
3. $\mathcal{D}$ - related if and only if $i=j$
4. $\mathcal{H}$ - related if and only if $m=p, n=q$ & $i=j$

Theorem 2.5 [8, Lemma 4.4]

(i) D_i is a bisimple ω-semigroup with identity e_i.

(ii) The $\mathcal{R}$-($\mathcal{L}$-) classes of D_i are the sets $R_{nd+i}(L_{nd+i})(n \in N$,

Theorem 2.6 Let $S(d, G_i, \gamma_i)$ be a simple regular ω-semigroup. Then $S(d,G_i, \gamma_i)/_{\mathcal{H}} \cong B_d$

Proof: Define a mapping $\theta: S(d,G_i,\gamma_i) \longrightarrow B_d$ by

$$(m,g_i,n)\,\theta = (md+i, nd+i).$$

Clearly θ is single - valued and onto. Next we show that θ is a homomorphism.

$$[(m,g_i,n)(p,h_j,q)]\,\theta = (m-n+t, g_i\alpha_{nd+i,td+u}\cdot h_j\alpha_{pd+j,td+u}, q-p+t)\theta$$

where $td+u=\max\{nd+i, pd+j\}$

$$=((m-n+t)d+u,\ (q-p+t)d+u)$$

$$=(md+i, nd+i)(pd+j, qd+j).$$

$$=(m,g_i,n)\theta\cdot(p,h_j,q)\theta$$

$\therefore$ θ is a homomorphism.

Now $(m,g_i,n)\theta = (p,h_j,q)\theta$

$\Leftrightarrow$ $(md+i, nd+i) = (pd+j, qd+j)$

$\Leftrightarrow$ $md+i = pd+j$ and $nd+i = qd+j$

$\Leftrightarrow$ $m=p$, $n=q$ and $i=j$ since $0 \leq i,j \leq d-1$

$\Leftrightarrow$ $(m,g_i,n)\ \mathcal{H}\ (p,h_j,q)$. Hence $\theta\cdot\theta^{-1} = \mathcal{H}$ and the result follows.

Lemma 2.7 For any pseudo-idempotent separating congruence ρ on a regular ω-semigroup S and $m,n \in N$

a) $(m,f_i,m)\rho(m,f_j,m) \iff (n,f_i,n)\rho(n,f_j,n)$

b) $(m,f_i,m)\rho(m+1,f_j,m+1) \iff (n,f_i,n)\ (n+1,f_j,n+1)$

Proof:

a) $(n,f_o,m)(m,f_i,m)(m,f_o,n) = (n,f_i,n)$

$(n,f_o,m)\ (m,f_j,m)\ (m,f_o,n) = (n,f_j,\ n)$

and $(m,f_o,n)(n,f_i,n)(n,f_o,m) = (m,f_i,m)$

$(m,f_o,n)(n,f_j,n)\ (n,f_o,m) = (m,f_j,m)$

b) $(n,f_o,m)(m,f_i,m)(m,f_o,n) = (n,f_i,n)$

$(n,f_o,m)(m+1,f_j,m+1)(m,f_o,n) = (n+1,f_j,n+1)$

and $(m,f_o,n)(n,f_i,n)\ (n,f_o,m) = (m,f_i,m)$

$(m,f_o,n)\ (n+1,f_j,n+1)(n,f_o,m) = (m+1,f_j,m+1)$

Hence the proof.

Lemma 2.8 For any pseudo-idempotent separating congruence on a regular ω-semigroup S

$(e_m, e_n) \in \rho \Rightarrow (e_m,e_k) \in \rho\ \forall\ k \ni m \leq k \leq n$ (or $n \leq k \leq m$)

Proof: Suppose $m \leq k \leq n$

and $(e_m,e_n) \in \rho \Rightarrow (e_m e_k, e_n e_k) \in \rho \Rightarrow (e_k,e_n) \in \rho$

by the transitive property $(e_m,e_k) \in \rho$. Hence the proof.

Theorem 2.9 A congruence on $S(d,G_i, \gamma_i)$ is either a group congruence or a locally idempotent separating congruence.

Proof: Let ρ be a congruence on $S(d,G_i, \gamma_i)$. Suppose ρ is not a group congruence, then we will show that no two distinct $\mathcal{D}$-equivalent idempotents are ρ equivalent.

Suppose $(e_{md+i}, e_{(m+k)d+i}) \in \rho$ for some $m,k,i \in N$ with $k > o$ and $o \leq i \leq d-1$. Without loss of generality we can take $k = 1$ i.e.

$(e_{md+i}, e_{(m+1)d+i}) \in \rho$. Now using lemmas 2.7&2.8 and the trainsitive property of the congruence we can easily see that

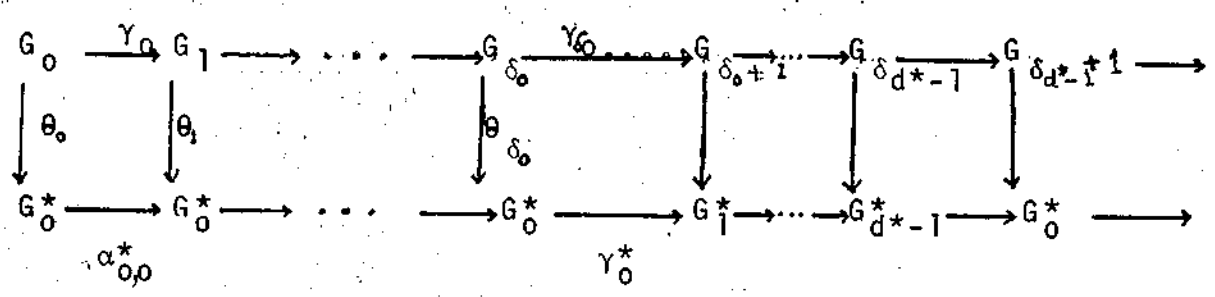
$(e_m, e_n) \in \rho \ \forall\ m,n \in N \Rightarrow \rho$ is a group congruence, contrary to assumption. Therefore no two distinct $\mathcal{D}$-equivalent idempotents are ρ-equivalent if ρ is not a group congruence, hence it must be a locally idempotent separating congruence.

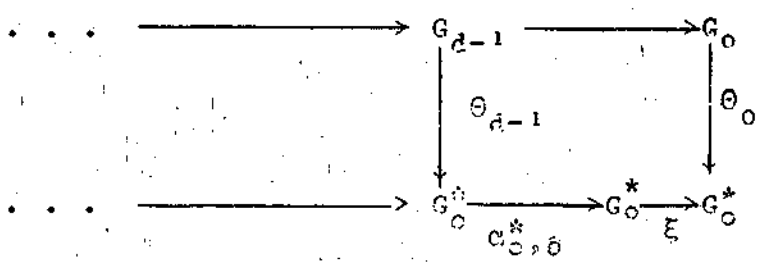
Theorem 2.10 [7, Theorem 2.1]

Let $S=S(d,G_i,\gamma_i)$ and $S^* =S(d^*,G^*_{i^*},\gamma^*_{i^*})$, $d^* \leq d$, be two simple regular ω-semigroups. Let $-1= \delta_{-1} < \delta_o < \ldots < \delta_{d^*-1} < \delta_{d^*}=d-1$, be a set of integers and let $i^*=j$ whenever $\delta_{j-1} \leq i \leq \delta_j$. Suppose $\theta_i : G_i \longrightarrow G^*_{i^*} (G^*_{d^*} = G^*_o)$, are homomorphisms of G_i into $G^*_{i^*}$, $o \leq i \leq d-1$ and $\xi : G^*_o \longrightarrow G^*_o$ an inner automorphism of G^*_o defined by $x\xi = zxz^{-1}$ for some $z \in G^*_o$, such that

(2.11) $\bigcup\{ G_i\theta_i \mid o < i < \delta_{d^*-1}\} = \bigcup\{ G^*_j \mid o \leq j \leq d^*-1\}$

and the following diagram (2.12) is commutative





Define $\Theta : S \longrightarrow S^*$ by

(2.13) $(m,a_i,n)\,\Theta = k^m\,(\lambda_i,a_i,\lambda_i)h^n$

where $\lambda_i = 0$ or 1 according as $i \leq$ or $> \delta_{d^*-1}$, $h=k^{-1} = (0,z,1)$ with $h^0 = k^0 = (0,\ f_0^*,\ 0)$. Then Θ is a homomorphism of S onto S^*. conversely every homomorphism of S onto S^* can be obtained in the above fashion.

Let $\rho = \Theta \circ \Theta^{-1}$, then $S/\rho \cong S(d^*,\ C^*_{i*},\ \gamma^*_{i*})$ (Theorem 1.18 ρ is not a group congruence since S/ρ is not a group, it's also abvious that ρ is not an idempotent separating congruence if $d^*<d$ (or $\{\delta\} \neq \phi$). Therefore from Theorem 2.9 ρ must be a pseudo-idempotent separating congruence.

For any pseudo-idempotent separating congruence ρ on $S(d,G_i,\gamma_i)$ $S/\rho \cong S(d^*,G^*_{i*},\ \gamma^*_{i*})$ for some $d^*< d$. Therefore ρ can be characterized by $\Theta_0\Theta^{-1}$ where Θ is of the above type.

Theorem 2.14: Let ρ be a pseudo-idempotent separating congruence on $S(d,G_i,\gamma_i)$. Then $\exists$ a non-empty sequence of integers $k_0,k_1,\ k_2,\ldots,\ k_t$ satisfying $-1 = k_{-1}<k_0<k_1<\ldots<k_t<k_{t+1}=d-1$

and $(m,g_i,n)\rho(p,h_j,q) \Rightarrow$
- $m=p,\ n=q$ for $k_{v-1}<i,j\leq k_v, 0\leq v \leq t+1$
- or $m=p+1,\ n=q+1$ for $i\leq k_0$ and $j > k_t$
- or $m+1=p,\ n+1=q$ for $j \leq k_0$ and $i > k_t$

Proof

Suppose ρ is a pseudo-idempotent separating congruence on $S(d,\ G_i,\ \gamma_i)$. Then $(m,g_i,n)\rho(p,h_j,q) \Rightarrow (m,g_i,n)^{-1}\rho(p,h_j,q)^{-1}$ because ρ is a

congruence on an inverse semigroup.

Now $(m,g_i,n)(m,g_i,n)^{-1}\rho(p,h_j,q)(p,h_j,q)^{-1}$

and $(m,g_i,n)^{-1}(m,g_i,n)\rho(p,h_j,q)^{-1}(p,h_j,q)$

$\Longrightarrow$ $(m,f_i,m)\rho(p,f_j,p)$ and $(n,f_i,n)\rho(q,f_j,q)$

i.e $(e_{md+i}, e_{pd+j})\varepsilon\rho$ and $(e_{nd+i}, e_{qd+j})\varepsilon\rho$.

If $md+i \geq (p+1)d+j \Rightarrow (e_{pd+j}, e_{(p+1)d+j})\varepsilon\rho$

so $md+i < (p+1)d+j$

$i < j$, $m \leq p + 1$

$i > j$, $m \leq p$

likewise ,

$j < i$, $p \leq m + 1$

$j > i$, $p \leq m$

Thus for $i < j$, $m \leq p + 1 \leq m + 1$ i.e $m = p$ or $p+1$

$i > j$, $m \leq p \leq m + 1$ i.e $p = m$ or $m + 1$

Reversing the roles of m & n and p & q we have

for $i < j$, $n = q$ or $q + 1$

$i > j$, $q = n$ or $n + 1$

By lemma 2.7 we find

$i < j \Rightarrow (e_i, e_j)\varepsilon\rho$ or $(e_{d+i}, e_j)\varepsilon\rho$

$i > j \Rightarrow (e_i, e_j)\varepsilon\rho$ or $(e_i, e_{d+j})\varepsilon\rho$

Let $k_o, k_1, \ldots k_t$ be the δ's of theorem 2.10 satisfying $-1 = k_{-1} < k_o < \ldots < k_t < k_{t+1} = d-1$ and $(e_j, e_{k_j})\varepsilon\rho$. It follows therefore $(e_i, e_j)\varepsilon\rho$ whenever $k_{v-1} < i, j \leq k_v$, $0 \leq v \leq t+1$; $(e_{d+i}, e_j)\varepsilon\rho$ whenever $i \leq k_o, j > k_t$; and $(e_i, e_{d+j})\varepsilon\rho$ whenever $j \leq k_o$, $i > k_t$.

$\therefore$ $m = p$, $n = q$ for $k_{v-1} < i, j \leq k_v$, $0 \leq v \leq t + 1$

or $m = p+1$, $n = q + 1$ for $i \leq k_o$, $j > k_t$

or $m + 1 = p$, $n+1 = q$ for $j \leq k_o$, $i > k_t$.

Chapter Three

In this chapter we characterize all the idempotent separating and pseudo-idempotent separating congruences on the simple regular ω-semigroup $S(d,G_i,\gamma_i)$.

Let $S=S(d,G_i,\gamma_i)$ and λ be any congruence on S.

Define $A_\lambda \triangleq \{g_i \in G_i / ((o,g_i,o),e_i) \in \lambda, i=o,1, \ldots,d-1\}$, set $A_\lambda \cap G_i = A_\lambda^i$

$U_{n,i} \triangleq \{(n,g_i,n)/g_i \in G_i, n \in N\}$, $\quad i=o,1 \ldots,d-1$

Clearly $U_{n,i}$ is a subgroup of S $\forall n \in N$ and $i=o,1,\ldots,d-1$.

Note also $A_{\lambda \cap H}=A_\lambda$ since the H-class containing e_{nd+i} is $U_{n,i}$.

Definition: A subset A of $G=\bigcup_i G_i$ is '$\{\gamma_i\}$ - admissible $\Leftrightarrow$ $a_i \in A \Rightarrow a_i \alpha_{i,nd+j} \in A$ for some $n \in N$ and for all $j, o \leq j \leq d-1$ and hence for all $n,j \in N$ (set of non-negative integers).

Lemma 3.1 For any congruence λ on S, A_λ is '$\{\gamma_i\}$' admissible and $A_\lambda^i \triangleq G_i$, $o \leq i \leq d-1$.

Proof. Let $\lambda_{o,i}=\lambda \cap (U_{o,i} \times U_{o,i})$. Then $\lambda_{o,i}$ is a congruence on $U_{o,i}$ and so $e_i \lambda_{o,i}$ is a normal subgroup of $U_{o,i}$. Also $e_i \lambda_{o,i}$ is the image of A_λ^i under the isomorphism $a_i \longrightarrow (o,a_i\ o)$ from G_i onto $U_{o,i}$. Thus A_λ^i is a normal subgroup of G_i.

Let $a_i \in A_\lambda$. Write $x = (o,a_i,o)$, $z=(o,f_i,n)$. Note that for $\alpha_{i,nd+j}$ to have a meaning $nd+j \geq i$.

Now since $a_i \in A_\lambda \Rightarrow (x,e_i) \in \lambda$ for some $o \leq i \leq d-1$

$\Rightarrow (zxz^{-1},\ ze_iz^{-1}) \in \lambda$

But $zxz^{-1}=(o,f_i,n)(o,a_i,o)(n,f_j,o)=(o,a_i\alpha_{i,nd+j},o)$

$ze_iz^{-1}=(o,f_j,n)(o,f_i,o)(n,f_j,o)=(o,f_j,o)$

$\therefore ((o,a_i\alpha_{i,nd+j},o),\ (o,f_j,o)) \in \lambda \Rightarrow a_i\alpha_{i,nd+j} \in A_\lambda$.

Lemma 3.2

$A_{\sigma \cap H} = A_\sigma = \bigcup A_\sigma^i$ and $A_\sigma^i = \bigcup_{k=1}^{\infty} \operatorname{Ker} \alpha_{i, kd+i}$. Where σ is the minimum group congruence on S, H is the Green's relation and $\operatorname{Ker} \alpha_{i, kd+i} = \{a_i \in G_i \mid a_i \alpha_{i, kd+i} = f_i (\text{the identity of } G_i)\}$.

$0 < i < d-1$ $k \in N$.

Proof: Let $a_i \in A_\sigma^i$ then $((0, a_i, 0), (0, f_i, 0)) \in \sigma$, $0 \leq i \leq d-1$, so $\exists$ $m \in N$ such that $e_m (0, a_i, 0) = e_m e_i$ and we may choose $m = kd+i$, $0 \leq i \leq d-1$, $k \in N$

$\therefore (k, f_i, k)(0, a_i, 0) = (k, f_i, k)(0, f_i, 0)$

$= (k, a_i \alpha_{i, kd+i}, k) = (k, f_i, k)$

$\therefore a_i \in \bigcup_{k=1}^{\infty} \operatorname{Ker} \alpha_{i, kd+i}$.

Conversely, let $a_i \in \bigcup_{k=1}^{\infty} \operatorname{Ker} \alpha_{i, kd+i}$. Then $a_i \alpha_{i\, kd+i} = f_i$ for some $n \in N$, then $(n, f_i, n)(0, a_i, 0) = (n, a_i \alpha_{i, nd+i}, n) = (n, f_i, n)$ $= (n, f_i, n)(0, f_i, 0)$.

$\therefore a_i \in A_\sigma^i$.

We now consider idempotent separating congruence on the simple regular ω-semigroup $S(d, G_i, \gamma_i)$

Lemma 3.3 Let λ be an idempotent separating congruence on a regular semigroup S. Then λ is $\mathcal{D}$ - separating. $\mathcal{D}$ is the Green's relation.

Proof: $\lambda \subseteq H \subseteq \mathcal{D}$. (since S is regular)

Lemma 3.4

(i) Let λ be an idempotent separating congruence on a simple regular ω-semigroup $S(d, G_i, \gamma_i)$.

Then $(m, g_i, n)\, \lambda\, (p, h_j, q) \Leftrightarrow i=j, m=p, n=q$ and $g_i h_i^{-1} \in A_\lambda$

(ii) For any set $A \subseteq G$ such that A is $\{\gamma_i\}$- admissible and $A \cap G_i$ is a normal subgroup of G_i, i=o,1, . . . d-1, there exists an idempotent separating congruence λ of $S(d,G_i,\gamma_i)$ such that $A=A_\lambda$.

Proof:

(i) Suppose λ is idempotent separating then $\lambda \subseteq H$ (Theorem 1.19)

$\therefore$ i=j, m=p and n=q (corollary 2.4)

Now $(m,g_i\ n)\ \lambda(m,h_i,n)$

$= (o,e_i,m)(m,g_i,n)(n,h_i^{-1},o)\ \lambda(o,e_i,m)(m,h_i,n)(n,h_i^{-1},o)$

$= (o,g_ih_i^{-1},o)\ \lambda\ (o,e_i,o)$

$= g_ih_i^{-1} \in A_\lambda$.

(ii) Define λ by $(m,g_i,n)\lambda(p,h_j,q) \Leftrightarrow$ i=j, m=p, n=q and $g_ih_i^{-1} \in A$.

Since $A \cap G_i \triangleq G_i$, we have λ an equivalence relation.

Suppose $(m,g_i,n)\lambda(m,h_i,n)$ then $g_ih_i^{-1} \in A$.

Let (p,h_j,q) be an arbitrary element of $S(d,G_i,\gamma_i)$

$(m,g_i,n)(p,h_j,q) = (m-n+t, g_i\alpha_{nd+i,td+u} \cdot h_j\alpha_{pd+j,td+u}, q-p+t)$

where $td+u = \max\{nd+i, pd+j\}$

and $(m,h_i,n)(p,h_j,q) = (m-n+t, h_i\alpha_{nd+i,td+u} \cdot h_j\alpha_{pd+j,td+u}, q-p+t)$

and $(g_i\alpha_{nd+i,td+u}\ h_j\alpha_{pd+j,td+u})(h_i\alpha_{pd+j,td+u})^{-1}$

$= (g_ih_i^{-1})\alpha_{nd+i,td+u} \in A$ since A is $\{\gamma_i\}$- admissible.

$\therefore$ λ is a right congruence.

Similarly λ is a left congruence

$\therefore$ λ is a congruence relation and is idempotent separating

Also $a_i \in A_\lambda \Leftrightarrow (o,a_i,o)\ \lambda\ (o,f_i,o)$

$\Leftrightarrow a_if_i^{-1} = a_i \in A$

$\therefore$ $A = A_\lambda$.

Define a semigroup $S(d, {}^{G}i/_{A_\lambda}i, \gamma_i^*)$

where $\gamma_i^* : {}^{G}i/_{A_\lambda^i} \longrightarrow {}^{G}{}_{i+1}/_{A_\lambda^{i+1}}$ $\quad 0 \le i \le d-1$ is

defined by $(A_\lambda^i a_i)\ \gamma_i^* = A_\lambda^{i+1}(a_i \gamma_i)$ ——— (3.5)

γ_i^* is single valued since A_λ is $\{\gamma_i\}$ - admissible over $\{G_i\}$, also γ_i^* is a homomorphism of ${}^{G}i/_{A_\lambda^i}$ into ${}^{G}{}_{i+1}/_{A_\lambda^{i+1}}$

Note that $G_n = G_{n(\text{mod } d)}$ and $A_\lambda^n = A_\lambda^{n(\text{mod } d)}$

Multiplication in $S(d, {}^{G}i/_{A_\lambda}i, \gamma_i^*)$ is thus defined by

$$(m, A_\lambda^i a_i, n)(p, A_\lambda^j a_j, q) = (m-n+t, (A_\lambda^i a_i)\ \alpha^*_{u,w} (A_\lambda^j a_j) \alpha^*_{v,w}, q-p+t)$$

——— (3.6)

where $t = \max\{n,p\}, u = nd+i, v = pd+j, w = \max\{u,v\}$

and $\alpha^*_{i,j} = \gamma_i^* \gamma_{i+}^* \ldots\ldots \gamma_{j-1}^*$. Clearly from (3.5) and (3.6)

$$(A_\lambda^i a_i)\ \alpha^*_{nd+i,td+u} = A_\lambda^u (a_i \alpha_{nd+i,td+u})$$

Theorem 3.7 Let λ be an idempotent separating congruence on $S = S(d, G_i, \gamma_i)$ then

$$S/_\lambda \simeq S(d, {}^{G}i/_{A^i}{}_\lambda, \gamma_i^*) = S^* \text{ defined above.}$$

Proof: Define $\theta : S \longrightarrow S(d, {}^{G}i/_{A_\lambda^i}, \gamma_i^*)$ by

$(m, g_i, n)\ \theta = (m, A_\lambda^i g_i, n)$

clearly θ is single valued and onto.

$$(m, g_i, n)\theta . (p, h_j, q)\theta = (m, A_\lambda^i\ g_i, n)(p, A_\lambda^j h_j, q)$$

$$= (m-n+t, (A_\lambda^i g_i)\alpha^*_{u,w} (A_\lambda^i h_j)\alpha^*_{v,w}, q-p+t) \quad \text{using 3.6}$$

$$= (m-n+t, A_\lambda^{u_1}(g_i \alpha_{u,w}) . A_\lambda^{u_1}(h_j \alpha_{v,w}), q-p+t) \quad \text{writing } w = td+u$$

$= (m-n+t,\ A^{u}_{\lambda}(g_i\alpha_{u,w})(h_j\ \alpha_{v,w},\ q-p+t)$

$= (m-n+t,\ g_i\alpha_{u,w}h_j\alpha_{v,w},\ q-p+t)\theta$

$= \left[(m,g_i,n)(p,h_j,q)\right]\theta$

$\therefore$ θ is a homomorphism of S onto S^*

Now $(m,a_i,n)\ \theta = (p,\ b_j,\ q)\theta$

$<=>\ (m,A^{i}_{\lambda}a_j,n) = (p,\ A^{j}_{\lambda}b_j,q)$

$<=>\ i = j,\ m = p,\ n = q,\ A^{i}_{\lambda}a_i\ =\ A^{j}_{\lambda}b_j$

$<=>\ (m,a_j,\ n)\lambda(p,b_j,q)$. Hence $\theta\cdot\theta^{-1} = \lambda$ and the result follows.

Next we consider pseudo-idempotent separating congruences on on the simple regular ω-semigroup $S(d,G_i,\ \gamma_i)$.

Theorem 3.3 Let $k_o,k_1,k_2\ .\ .\ .\ .,\ k_t$ be a non-empty sequence of integers satisfying $-1 = k_{-1}<k_o<k_1<\ldots<k_t<k_{t+1} = d-1$.

Define $\rho = \rho(k_o,k_1,\ldots,k_t)$ on $S(d,G_i,\gamma_i)$ by

$$(m,g_i,n)\rho(p,h_j,q) \iff \begin{cases} m=p,n=q;\ g_i\alpha_{i,wd+\ell}\cdot h_j^{-1}\alpha_{j,wd+\ell}\in A_\rho \\ \text{for some } w \geq 0,\ \ \ell = \max\{i,j\} \\ k_{v-1}<i,j\leq k_v,\ 0\leq v\leq t+1 \\ \text{or } m = p+1,\ n=q+1; g_i(h_j^{-1}\alpha_{j,d+i})\in A_\rho \\ \qquad\qquad \text{for } i\leq k_o\ \&\ j > k_t \\ \text{or } m+1=p,\ n+1 = q;\ (g_i\alpha_{i,d+j})h_j^{-1}\in A_\rho \\ \qquad\qquad \text{for } \ i\leq k_o\ \&\ i > k_t. \end{cases}$$

Then ρ is a pseudo-idempotent separating congruence. Conversely, every pseudo-idempotent separating congruence on $S(d,\ G_i,\ \gamma_i)$ can be characterized in a similar fashion.

Proof

Let $\rho = \rho(k_o,k_1,\ldots,k_t)$ be as defined above. Since $A_\rho \cap G_i$

is a normal subgroup of G_i then we have ρ an equivalence relation. To show that ρ is a congruence we have to show that ρ is both left and right compatible.

Now for $k_{v-1} < i, j \leq k_v, 0 \leq v \leq t+1$

$(m, g_i, n)\ \rho\ (p, h_j, g) \iff (m, g_i, n)\ \rho\ (m, h_j, n)$ and

$g_i \alpha_{i,wd+\ell} \cdot h_j^{-1} \alpha_{j,wd+\ell} \in A_\rho$ for some $w \geq 0$, $\ell = \max\{i,j\}$

Let (r, c_k, s) be an arbitrary element of $S(d, G_i, \gamma_i)$

$(m, g_i, n)(r, c_k, s) = (m-n+t,\ g_i \alpha_{nd+i,td+u}\ c_k \alpha_{rd+k,td+u},\ s-r+t)$

$(m, h_i, n)(r, c_k, s) = (m-n+t,\ h_i \alpha_{nd+j,t_1 d+u_1}\ c_k \alpha_{rd+k,td+u},\ s-r+t)$

$td+u = \max\{nd+i,\ rd+k\}$

$t_1 d+u_1 = \max\{nd+j,\ rd+k\}$

Case 1.

$rd+k \leq nd+i,\ nd+j \implies t = n = t_1$, $u = i$, $u_1 = j$

$(m, g_i, n)(r, c_k, s) = (m,\ g_i c_k \alpha_{rd+k,nd+i},\ s-r+n)$

$(m, h_j, n)(r, c_k, s) = (m,\ h_j\ c_k \alpha_{rd+k,nd+j},\ s-r+n)$

and $(g_i c_k \alpha_{rd+k,nd+i}) \alpha_{i,wd+\ell} (h_j c_k \alpha_{rd+k,nd+j})^{-1} \alpha_{j,wd+\ell}$

$= g_i \alpha_{i,wd+\ell} \cdot h_j^{-1} \alpha_{j,wd+\ell} \in A_\rho$ for some $w \geq 0$

Case 2. $rd+k \geq nd+i,\ nd+j \implies t = r = t_1$, $u = k = u_1$

$(m, g_i, n)(r, c_k, s) = (m-n+r,\ g_i \alpha_{nd+i,rd+k}\ c_k,\ q)$

$(m, h_j, n)(r, c_k, s) = (m-n+r,\ h_j \alpha_{nd+j,rd+k}\ c_k,\ q)$

and $(g_i \alpha_{nd+i,rd+k}\ c_k) \alpha_{k,wd+k} \cdot (c_k^{-1} h_j^{-1} \alpha_{nd+j,rd+k}\ \alpha_{k,wd+k}$

$= (g_i \alpha_{i,wd+\ell} \cdot h_j^{-1} \alpha_{j,wd+\ell}) \alpha_{\ell,(r-n)d+k} \in A_\rho$ since A_ρ is $\{\gamma_i\}$-admissible for some $w \geq 0$.

Case 3. $nd+i \leq rd+k \leq nd+j \implies n = p$ $i \leq k \leq j$

$(m, g_i, n)(r, c_k, s) = (m,\ g_i \alpha_{i,k}\ c_k,\ q)$

$(m, h_j, n)(r, c_k, s) = (m, h_j\ c_k\ \alpha_{k,j}, q)$

and $(g_i \alpha_{i,k}\ c_k) \alpha_{k,wd+j} (c_k^{-1} \alpha_{k,j} \cdot h_j^{-1}) \alpha_{j,wd+j}$

$= g_i \alpha_{i,wd+j} h_j^{-1}\ \alpha_{j,wd+j} \in A_\rho$

Case 4. $nd+j \leq rd+k \leq nd+i$

This is similar to case 3.

Next we consider the case $i \leq k_o$ & $j > k_t$

$(m, g_i, n)\rho(p, h_j, q) \Longleftrightarrow (p+1, g_i, q+1)\rho(p, h_j, q)$ and $g_i h_j^{-1} \alpha_{j,d+i} \in A_\rho$

$(p+1, g_i, q+1)(r, c_k, s) = (p-q+t_1,\ g_i \alpha_{(q+1)d+i, t_1 d+u_i}\ c_k \alpha_{rd+k, t_1 d+u_1},\ s-r+t_1)$

$(p, h_j, q)(r, c_k, s) = (p-q+t,\ h_j \alpha_{qd+j, td+u}\ c_k \alpha_{rd+k, td+u},\ s-r+t)$

where $td+u = \max\{qd+j, rd+k\}$, $t_1 d + u_1 = \max\{(q+1)d+i, rd+k\}$

Case 1. $rd+k \leq qd+j \Longrightarrow t = q$, $t_1 = q+1$, $u = j$, $u_1 = i$

$(p+1, g_i, q+1)(r, c_k, s) = (p+1,\ g_i c_k \alpha_{rd+k, (q+1)d+i},\ s-r+q+1)$

$(p, h_j, q)(r, c_k, s) = (p,\ h_j c_k \alpha_{rd+k, qd+j},\ s-r+q)$

and $(g_i c_k \alpha_{rd+k, (q+1)d+i})(c_k^{-1} \alpha_{rd+k, qd+j} \cdot h_j^{-1}) \alpha_{j,d+i}$

$= g_i h_j^{-1} \alpha_{j,d+i} \in A_\rho$

Case 2. $(q+1)d+i \leq rd+k \Longrightarrow t = r = t_1$, $u = k = u_1$

$(p+1, g_i, q+1)(r, c_k, s) = (p-q+r,\ g_i \alpha_{(q+1)d+i, rd+k} \cdot c_k,\ s)$

$(p, h_j, q)(r, c_k, s) = (p-q+r,\ h_j \alpha_{qd+j, rd+k} \cdot c_k,\ s)$

and $(g_i \alpha_{(q+1)d+i, rd+k}\ c_k) \alpha_{k, wd+k} (c_k^{-1} h_j^{-1} \alpha_{qd+j, rd+k}) \alpha_{k, wd+k}$

$= (g_i\ h_j^{-1} \alpha_{j,d+i}) \alpha_{(q+1)d+i, (w+r)d+k} \in A_\rho$ is $\{\gamma_i\}$ - admissible

Case 3. $qd+j \leq rd+k \leq (q+1)d+i$

$\Longrightarrow t = r$, $u = k$, $t_1 = q+1$, $u_1 = i$.

$(p+1, g_i, q+1)(r, c_k, s) = (p+1, g_i c_k \alpha_{rd+k, (q+1)d+i},\ s-r+q+1)$

$(p, h_j, q)(r, c_k, s) = (p-q+r,\ h_j \alpha_{qd+j, rd+k} \cdot c_k,\ s)$

(i) $r = q \Longrightarrow j \leq k$ so $k > k_t$

The above products are

$(p+1,\ g_i c_k\ \alpha_{k,d+i},\ s+1)$ and $(p,\ h_j \alpha_{j,k} \cdot c_k,\ s)$,

and $(g_i c_k\ \alpha_{k,d+i})\ (c_k^{-1} h_j^{-1} \alpha_{j,k}) \alpha_{k,d+i}$

$= g_i\ h_j^{-1} \alpha_{j,d+i} \in A_\rho$

(ii) $r = q + 1 \Rightarrow k \leq i \leq k_o$

The above products are

$(p+1,\ g_i\ c_k\ \alpha_{k,i},\ s)$ and $(p+1,\ h_j \alpha_{j,d+k} \cdot c_k,\ s)$,

and $(g_i \cdot c_k \alpha_{k,i})\ (c_k^{-1} \cdot h_j^{-1} \alpha_{j,d+k}) \alpha_{k,i}$

$= g_i\ h_j^{-1} \alpha_{j,d+i} \in A_\rho$

The case $j \leq k_o$ & $i > k_t$ is handed similarly. Therefore ρ is right compatible. We can also show that ρ is left compatible similarly, hence ρ is a congruence. It is also abvious that ρ is pseudo-idempotent separating congruence on $S(d,\ G_i,\ \gamma_i)$.

Conversely, suppose ρ is a pseudo-idempotent separating congruence on $S(d, G_i, \gamma_i)$. Then by theorem 2.14 $\exists$ a non-empty sequence of integers $k_o, k_1, k_2, \ldots, k_t$ satisfying

$-1 = k_{-1} < k_o < k_1 < \ldots < k_t < k_{t+1} = d-1$

$$(m, g_i, n)\rho(p, h_j, q) \Longrightarrow \begin{cases} m=p,\ n=q \text{ for } k_{v-1} < i,j \leq k_v, 0 < v \leq t+1 \\ \text{or } m=p+1,\ n=q+1 \text{ for } i \leq k_o \ \&\ j > k_t \\ \text{or } m+1=p, n+1=q \text{ for } i \leq k_o \ \&\ i > k_t. \end{cases}$$

Now, for $k_{v-1} < i,j \leq k_v, 0 \leq v \leq t+1$

$(m,\ g_i,\ n)\rho(m,\ h_j, n)$

$\Rightarrow (0, e_o, m)(m, g_i, n)(n, h_j^{-1}, 0)\rho(0, e_o, m)(m, h_j, n)\ (n, h_j^{-1}, 0)$

$\Leftrightarrow (0, g_i \alpha_{nd+i, nd+\ell} \cdot h_j^{-1} \alpha_{nd+j, nd+\ell}, 0)\rho(0, e_j, 0), \quad \ell = \max\{i,j\}$

$g_i \alpha_{i,\ell} \cdot h_j^{-1} \alpha_{j,\ell} \in A_\rho$

For $i \leq k_o$, $j > k_t$

$(m+1, g_i, n+1)\ \rho\ (m, h_j, n)$

$\Rightarrow$ $(0, e_o, m+1), (m+1, g_i\ m+1)\ (n+1, h_j^{-1}\alpha_{j,d+i}, 0)$

$\rho(0, e_o, m+1)\ (m, h_j, n)(n+1, h_j^{-1}\alpha_{j,d+i}, 0).$

$\Rightarrow$ $(0, g_i, h_j^{-1}\alpha_{j,d+i}, 0)\ \rho\ (0, h_j\alpha_{j,d+i}h_j^{-1}\alpha_{j,d+i}, 0)$

$\Rightarrow$ $(0, g_i h_j^{-1}\alpha_{j,d+i}, 0)\ \rho\ (0, e_i, 0).$

$\Rightarrow$ $g_i\ h_j^{-1}\ \alpha_{j,d+i} \in A_\rho$

The case $j \leq k_o$ and $i > k_t$ is similar.

Chapter Four

In this chapter we find the maximum group homomorphic image of the simple regular ω-semigroup $S = S(d, G_i, \gamma_i)$, $i = 0, \ldots, d-1$ and also establish a homomorphism theorem of $S(d, G_i, \gamma_i)$ into a group G^*.

Let $G = \cup G_i$ be the union of an ω-chain of groups (as in chapter 2 example 1.). Also let N denote the set of non-negative integers. Define a relation ρ on $\cup G_i \times N$ by the rule that

$((a_i, m)(b_j, n)) \in \rho \Leftrightarrow a_i \alpha_{md+i, td+u} = b_j \alpha_{nd+j, td+u}$ for some t, u, $td+u \geq md+i, nd+j$.

Lemma 4.1 ρ is an equivalence relation on $\cup G_i \times N$.

Further if we define

$(a_i, m)\rho \cdot (b_j, n)\rho = (a_i \alpha_{md+i, td+u} \cdot b_j \alpha_{nd+j, td+u}, t)\rho$

where $td+u = \max\{md+i, nd+j\}$. Then $(\cup G_i \times N)/\rho$ under the above binary operation is a group.

Proof: ρ is clearly an eqivalence relation.

$$\{(a_i, m_1)\rho (b_j, m_2)\rho\}(c_k, m_3)\rho = (a_i \alpha_{m_1 d+i, t_1 d+u_1} \cdot b_j \alpha_{m_2 d+j, t_1 d+u_1}, t_1)\rho \, (c_k, m_3)\rho$$

$$= ((a_i \alpha_{m_1 d+i, t_1 d+u_1} \cdot b_j \alpha_{m_2 d+j, t_1 d+u_1}) \alpha_{t_1 d+u_1, t_2 d+u_2} \cdot c_k \alpha_{m_3 d+k, t_2 d+u_2}, t_2)\rho$$

$$= (a_i \alpha_{m_1 d+i, t_2 d+u_2} \cdot b_j \alpha_{m_2 d+j, t_2 d+u_2} \cdot c_k \alpha_{m_3 d+k, t_2 d+u_2}, t_2)\rho$$

where $t_1 d+u_1 = \max\{m_1 d+i, m_2 d+j\}$.

$t_2 d+u_2 = \max\{t_1 d+u_1, m_3 d+k\}$

$= \max\{m_1 d+i, m_2 d+j, m_3 d+k\}$

By symmetry we see that the binary operation is associative. Note that $(f_i,o)\rho=(f_j,o)\rho=(f_i,m)\rho=(f_j,n)\rho$. The identity of $H=(\cup G_i X N)/\rho$ is $(f_i,n)\rho, o\leq i \leq d-1, n\in N$.

For $(a_i,m)\rho.(f_i,m)\rho=(f_i,m)\rho(a_i,m)\rho=(a_i,m)\rho$.

Inverse of $(a_i,m)\rho$ is $(a_i^{-1},m)\rho$, where a_i^{-1} is the inverse of a_i in G_i. This completes the proof.

Lemma 4.2 Let $H=(\cup G_i X N)/\rho$ be the group defined in lemma 4.1. Define a multiplication on the set ZxH by the rule that

$(m,(a_i,n)\rho).(p,(b_j,q)\rho)$

$=(m+p,(a_i\alpha_{nd+i,td+u}.b_j\alpha_{(p+q)d+j,td+u},t-p)\rho)$

where $td+u=\max\{nd+i,(p+q)d+j\}$ $o\leq u\leq d-1$, and Z is the set of integers. With respect to this multiplaction ZxH is a group.

Proof: Suppose $(m,(a_i,n)\rho)=(m,(a_k,n')\rho)$

$(p,(b_j,q)\rho)=(p,(b_\ell,q')\rho)$

then we have $a_i\alpha_{nd+i,t_1d+u_1}=a_k\alpha_{n'd+k,t_1d+u_1}$

$b_j\alpha_{qd+j,t_2d+u_2}=b_\ell\alpha_{q'd+\ell,t_2d+u_2}$

for some t_1, u_1, t_2, u_2 $t_1d+u_1\geq nd+i, n'd+k$

$t_2d+u_2\geq qd+j, q'd+\ell$

let $td+u=\max\{nd+i,(p+q)d+j\}$

$t'd+u'=\max\{n'd+k,(p+q')d+\ell\}$

$t''d+u''=\max\{t_1d+u_1,(t_2+p)d+u_2,td+u,t'd+u'\}$

$(m,(a_i,n)\rho)(p,(b_j,q)\rho)=(m+p,(a_i\alpha_{nd+i,td+u}.b_j\alpha_{(p+q)d+j,td+u},t-p)\rho)$

$(m,(a_k,n')\rho)(p,(b_\ell,q')\rho)=(m+p,(a_i\alpha_{n'd+k,t'd+u'}.b_\ell\alpha_{(p+q)d+\ell,t'd+u'},t'-p)\rho)$

Now $(a_i\alpha_{nd+i,td+u}.b_j\alpha_{(p+q)d+j,td+u},t-p)\rho)$

$$= ((a_i \alpha_{nd+i,\, td+u} \cdot b_j \alpha_{(p+q)d+j,\, td+u}) \alpha_{td+u,\, t''d+u''}\, t''-p)\rho$$

$$= (a_i \alpha_{nd+i,\, t''d+u''} \cdot b_j \alpha_{(p+q)d+j,\, t''d+u''}, t''-p)\rho$$

$$= (a_i \alpha_{nd+i,\, t_1 d+u_1} \alpha_{t_1 d+u_1,\, t''d+u''} \cdot b_j \alpha_{(p+q)d+j,\, (t_2+p)d+u_2} \alpha_{(t_2+p,\, u_2,\, td+u}\, t''-p)\rho$$

$$= (a_k \alpha_{n'd+k,\, t''d+u''} \cdot b_\ell \alpha_{(p+q')d+\ell,\, t''d+u''}, t''-p)\rho$$

$$= ((a_k \alpha_{n'd+k,\, t'd+u'} b_\ell \alpha_{(p+q')d+\ell,\, t'd+u'}) \alpha_{t'd+u',\, t''d+u''}, t''-p)\rho$$

$$= (a_k \alpha_{n'd+k,\, t'd+u'} \cdot b_\ell \alpha_{(p+q')d+t\ell,\, t'd+u'}, t'-p)\rho$$

Hence the multiplication is well-defined.

$$= (m_1 \cdot m_2 \cdot m_3^{-1} \cdot \eta_3,\ (a \cdot^{\alpha} \eta_d + 1, t_3 d + u_3 \cdot {}^{b}\!\int^{\alpha} \eta_d + \int,\ t_2 d + u_2 \cdot {}^{C} k^{\alpha} m_d + k,\ t_4 d + u_4)^{\alpha}\ (t + m_3^{-1} \eta_1,\ (t + n_3^{-1} m_2^{-1} \eta_2)\ d + u_1 \cdot {}^{b}\!\int^{\alpha} \eta_d + \int,\ t_2 d + u_2 \cdot {}^{C} k^{\alpha} m_d + k,\ t_4 d + u_4)\ \rho)$$

$$= (m_1 \cdot m_2^{-1} \cdot \eta_2,\ (a \cdot^{\alpha} \eta_d + 1, t d + u \cdot {}^{b}\!\int^{\alpha} m_d + \int,\ t d + u \cdot {}^{C} k^{\alpha} m_d + k,\ t_2 d + u_2)\ \rho)\ (m_3^{-1} \eta_3,\ (C\ k, \eta_3)\ \rho)$$

$$= (m_1 \cdot m_2^{-1} \cdot m_3^{-1} \cdot \eta_3,\ ((a \cdot^{\alpha} \eta_d + 1, t d + u \cdot {}^{b}\!\int^{\alpha} m_d + \int,\ t d + u)^{\alpha}\ (t + \eta_2 - m_2)\ d + u,\ t_1 d + u_1 \cdot {}^{C} k^{\alpha} m_d + k,\ t_1 d + u_1,\ t_2 + \eta_3 - m_3)\ \rho)$$

where $t d + u = \max \{\eta_d + 1, m_d + \int\},\ t_d + u_1 = \max \{(t + \eta_2 - m_2)\ d + u, m_d + k\}$

$$= (m_1 \cdot m_2^{-1} \cdot m_3^{-1} \cdot \eta_3,\ (a \cdot^{\alpha} \eta_d + 1, (t_1 + m_2^{-1} \cdot \eta_1)\ d + u_1 \cdot {}^{b}\!\int^{\alpha} m_d + \int,\ (t_1 + m_2^{-1} \cdot \eta_2)\ d + u_2 \cdot {}^{C} k^{\alpha} m_d + k,\ t_4 d + u_4)^{\alpha}\ (t + \eta_2 \cdot m_2^{-1} \cdot m_3^{-1} \cdot \eta_3)\ d + u_3,\ t_4 d + u_4 \qquad 4.3$$

and $(m, (a \cdot {}^{\gamma} m)\ \rho)' \{(m_2^{-1} \eta_2\ (b \cdot {}^{\gamma}\!\eta_2)\ \rho)(m_3^{-1} \eta_3\ (C\ k, \eta_3)\ \rho)$

$$= (m_1 \cdot (a \cdot {}^{\gamma}\!\eta_1)\ \rho)\ (m_2^{-1} m_3\ \eta_2^{-1} \eta_3 \cdot {}^{b}\!\int^{\alpha} \eta_d + k, t_d + u_2 \cdot {}^{C} k^{\alpha} m_d + k, t_d + u_2 \cdot t_3^{-1} \eta_3^{-1} m_3)\ \rho\ )$$

It follows from the definition of ρ that expression (4.3) is equal to expression (4.4)

Thus the binary operation is associative.

N a x t we check for associativity

32.

The identity of $\mathbb{Z}xH$ is $(o,(f_i,o)\ \rho), o \leq i \leq d-1$. To prove this we first note that

$(m,(a_i,n)\ \rho)(o,(f_i,o)\rho\ ) = (m,(a_i,n)\ \rho\ )$.

If $n+m \geq o, (o,(f_i,o)\rho\ )\ (m,(a_i,n)\rho) = (m,(a_i,n)\ \rho)$

If $n+m < o, (o,(f_i,o)\ \rho)\ (m,(a_i,n)\rho\ ) = (m,(a_i\alpha_{(n+m)d+i,i},-m)\rho$

$= (m,(a_i\alpha_{i,-(n+m)d+i},-m)\rho\ )$

$= (m,(a_i,-m+n+m)\ \rho) = (m,(a_i,n)\rho\ )$.

Note that $(a_i,n)\rho = (a_i\alpha_{i,md+j},n+m)$ for $n,m \geq o, o \leq i,j \leq d-1$

Inverse of $(m,(a_i,n)\ \rho)$ is $(-m,(a_i^{-1}\ \alpha_{i,kd+j},^{md+j \geq i}_{n+m+k})\ \rho)$

where $n+m+k \geq o$, $k \geq o$.

for $(m,(a_i,n)\rho\ )\ (-m,(a_i^{-1}\alpha_{i,kd+i},m+n+k)\rho)$

$= (m,(a_i\alpha_{i,kd+i},n+k)\ \rho)(-m,\ (a_i^{-1}\alpha_{i,kd+i},m+n+k)\rho)$

$= (o,(a_i\alpha_{i,kd+i}\cdot a_i^{-1}\alpha_{i,kd+i},n+k+m)\rho)$

$= (o,(f_i,n+k+m)\ \rho)\ = (o,(f_i,o)\rho)$

and

$(-m,(a_i^{-1}\ \alpha_{i,kd+i},n+m+k)\ \rho)\ \ (m,(a_i,n)\rho\ )$

$= (-m,(a_i^{-1}\alpha_{i,kd+i},n+m+k)\rho\ )(m,(a_i\alpha_{i,kd+i},n+k)\ \rho)$

$= (o,(f_i,n+k)\rho) = (o,(f_i,o)\rho)$.

This completes the proof.

Theorem 4.5 . Let $S=S(d,G_i,\gamma_i)$ be a simple regular ω-semigroup. Then

$$S/\sigma \simeq \mathbb{Z}xH$$

where $\mathbb{Z}xH$ has been introduced in Lemma 4.2 and σ is the minimum group congruence on S. (theorem 1.20)

Proof: Define $\theta : S \longrightarrow \mathbb{Z}xH$ by

$$(m,a_i,n)\ \theta = (m-n,(a_i,n)\rho)$$

First we show that θ is onto.

Let $k\in\mathbb{Z}$ and $(a_i,n)\rho$ be any element of $H(a_i\in G_i, n\in N)$

If $k \geq o$, then $(k,(a_i,n)\rho) = (k+n,a_i,n)\theta$.

On the other hand if $k < o$ then

$$(k,(a_i,n)\ \rho\ ) = (k,\ (a_i\alpha_{i,-kd+i},n-k)\rho)$$

$$=(n,(a_i\alpha_{i,-kd+i}),n-k)\ \theta$$

Now let (m,a_i,n) and (p,b_j,q) be any two elements of S.

Then $(m,a_i,n)\theta.\ (p,b_j,q)\theta$

$$=(m-n+p-q,(a_i\alpha_{nd+i,td+u},b_j\alpha_{pd+j,td+u},t-p+q)\rho)$$

where $td+u=\max\{nd+i,pd+j\}$

$$=(m-n+t,a_i\alpha_{nd+i,td+u}\cdot b_j\alpha_{pd+j,td+u},q-p+t)\theta \text{ if } m-n+p-q \geq o$$

$$=[(m,a_i,n)(p,b_j,q)]\ \theta.$$

On the other hand if $m-n+p-q<o$, then

$$(m-n+p-q,(a_i\alpha_{nd+i,td+u}\cdot b_j\alpha_{pd+j,td+u},t-p+q)\rho\ )$$

$= \{(m-n+p-q, (a_i\alpha_{nd+i,td+u} \cdot b_j\alpha_{pd+j,td+u}, t-p+q)\rho)^{-1}\}^{-1}$

$= \{(n-m+q-p, c_u, t+m-n)\rho\}^{-1}$

where $\varepsilon_u = (a_i\alpha_{nd+i,td+u} \cdot b_j\alpha_{pd+j,td+u})^{-1} = b_j^{-1}\alpha_{pd+j,td+u} \cdot a_i^{-1}\alpha_{nd+i,td+u}$

$= \{(q-p+t, c_u, m-n+t)\theta\}^{-1}$

$= \{[(q, b_j^{-1}, p)(n, a_i^{-1}, m)]\theta\}^{-1}$

$= \{[(q, b_j^{-1}, p)(n, a_i^{-1}, m)]^{-1}\}\theta$

$= [(m, a_i, n)(p, b_j, q)]\theta.$

Hence θ is a homomorphism,

Since $S\theta$ is a group and σ is the minimum group congruence on S,

it follows $\sigma \subseteq \theta o \theta^{-1}$

Now $(m, a_i, n)\theta = (p, b_j, q)\theta \Leftrightarrow (n, a_i^{-1}, m)\theta = (q, b_j^{-1}, p)\theta$

from [5; §v, 1.7]

$(n-m, (a_j^{-1}, m)\rho) = (q-p, (b_j^{-1}, p)\rho)$

$n-m = q-p$ and $(a_i^{-1}, m)\rho = (b_j^{-1}, p)\rho$

$\therefore a_i^{-1}\alpha_{md+i,td+u} = b_j^{-1}\alpha_{pd+j,td+u} \Leftrightarrow a_i\alpha_{md+i,td+u} = b_j\alpha_{pd+j,td+u}$

for some t, u $td+u \geq md+i, pd+j$.

$\therefore$ $(t,f_u,t)(m,a_i,n)=(t,\ \alpha_i\alpha_{md+i,td+u}\ n-m+t)$
$(t,f_u,t)(p,b_j,q)=(t,\ b_j\alpha_{pd+j,td+u},\ q-p+t)$
but these are equal since $n-m=q-p$ and $(a_i,m)\rho=(b_j,p)\rho$

$\therefore$ $((m,a_i,n),(p,b_j q))\in\sigma$ and $\theta\circ\theta^{-1}\subseteq\sigma$.
Hence $\theta\circ\theta^{-1}=\sigma$ and the result follows.

Lemma 4.6 Let $S=S(d,G_i,\gamma_i)$. Then

$((m,a_i,n),(p,b_j,q))\in\sigma\ \forall H \Leftrightarrow m-n=p-q$

where σ is the minimum group congruence and H is the Green's relation on S.

Proof: First suppose $(m,a_i,n)\ \sigma\vee H(p,b_j,q)$

put $x=(m,a_i,n)$, $y=(p,b_j,q)$ since $\sigma\vee H=\sigma\circ H\circ\sigma$ (5, Theorem 3.7)

$\exists\, a,b\in S$ such that $(x,a)\in\sigma$, $(a,b)\in H$, $(b,y)\in\sigma$

Hence $a=(m',a_r,n')$, $b=(m',b_r,n')$, $0\leq r\leq d-1$.

$(x,a)\in\sigma\Rightarrow\exists\, k\in N$; $e_k\chi=e_k\alpha$ and we may suppose that $k=\ell d$
where $\ell <> m,m'$
Thus $(\ell,f_0,\ell)(m',a_r,n')=(\ell,f_0,\ell)(m,a_i,n)$
ie. $(\ell,\ a_r\alpha_{m'd+r,\ \ell d},n'-m'+\ell)=(\ell,\ a_i\alpha_{md+i,\ \ell d},n-m+\ell)$

$\Rightarrow n'-m'=n-m$

Similarly $(m',b_r,n')\ \sigma\ (p,b_j,q)\Rightarrow m'-n'=p-q$

$\therefore$ $m-n=p-q$.

Conversely let $m-n=p-q$. Assume $m<p$
Then $(p,f_j,p)(m,a_i,n)=(p,a_i\alpha_{md+i,pd+j}\ n-m+p)$
$=(p,a_i\alpha_{md+i,pd+j},q)$

$\therefore$ $x\ \sigma e_{pd+j}\chi$ and $e_{pd+j}\chi\ Hy$
$\Rightarrow$ $x\ \sigma\circ Hy$ $\Rightarrow x\sigma\vee Hy$.

Theorem 4.7 Let $S=S(d,G_i,\gamma_i)$. Then $S/(\sigma \vee H) \simeq \mathbb{Z}$

Proof: Define $\theta : S \longrightarrow \mathbb{Z}$ by

$$(m,a_i,n)\theta = m-n$$

Clearly θ is single - valued and onto.

$$[(m,a_i,n)(p,b_j,q)]\theta = (m-n+t,\ a_i\alpha_{nd+i,td+u}\cdot b_j\alpha_{pd+j,td+u},\ q-p+t)$$

$$=(m-n)+(p-q)$$

$$=(m,a_i,n)\theta + (p,b_j,q)\theta$$

By lemma 4.6 $\theta o\theta^{-1} = \sigma \vee H$ $\therefore S/(\sigma \vee H) \simeq \mathbb{Z}$

Theorem 4.8 Let $S=S(d,G_i,\gamma_i)$ and σ be the minimum group congruence on S. Then $H \subseteq \sigma \Leftrightarrow A_\sigma = \bigcup G_i$

$\Leftrightarrow \bigcup_{k=1}^{\infty} \ker \alpha_{i,kd+i} = G_i \ \forall i, 0 \leq i \leq d-1$

Proof: First let $H \subseteq \sigma$ and $a_i \in G_i$ then since $((0,a_i,0),(0,f_i,0)) \in H$ we have that $a_i \in A_\sigma$. If we write $\bigcup_{k=1}^{\infty} \ker \alpha_{i,kd+i} = K_i$ then $A_\sigma^i = K_i$ (lemma 3.3) and therefore $G_i \subseteq K_i \Rightarrow G_i = K_i$.

Conversely let $G_i = K_i \ \forall i,\ 0 \leq i \leq d-1$

Let $x\ H\ y$, then $x=(m,a_i,n), y=(m,b_i,n)$ for some i, $0 \leq i \leq d-1$ and $m,n \in N$. By hypothesis $a_ib_i^{-1} \in K_i$ $\therefore$ there exist k such that $(a_ib_i^{-1})\alpha_{i,kd+i} = f_i$. Thus $a_i\alpha_{i,kd+i} = b_i\alpha_{i,kd+i}$

Then $(m+k,f_i,m+k)(m,a_i,n) = (m+k,a_i\alpha_{md+i,(m+k)d+i},n+k)$

$$=(m+k,\ b_i\alpha_{md+i,(m+k)d+i},n+k)$$

$$=(m+k,\ f_i,m+k)(m,b_i,n)$$

$\therefore$ $H \subseteq \sigma$.

Theorem 4.9 Let $S=S(d,G_i,\gamma_i)$ and let G^* be a group. Let θ_i be a homormorphism of G_i into G^* such that

$\theta_i = \gamma_i\theta_{i+1}$, $0 \leq i < d-1$

$\theta_{d-1}\, C_z = \gamma_{d-1}\,\theta_0$ where $x\, C_z = zxz^{-1}$ $x \in G^*$ and z fixed in G^*.

Then $\theta: S \longrightarrow G^*$ defined by

$(m,a_i,n)\,\theta = z^{-m}(a_i\theta_i)z^{n}$ is a homomorphism of S into G^* and conversely every such homomorphism is obtained in this fashion.

Proof; $\theta : S \longrightarrow G^*$

$(m,a_i,n)\,\theta = z^{-m}\,(a_i\theta_i)\,z^{n}$

θ is clearly single-valued.

Suppose (m,a_i,n), $(p,b_j,q) \in S$

case (i) $p > n$

$$\begin{aligned}
&(m,a_i,n)\,\theta \cdot (p,b_j,q)\,\theta \\
&= z^{-m}\,(a_i\theta_i)\,z^{n}.z^{-p}\,(b_j\theta_j)\,z^{q} \\
&= z^{-m}(a_i\theta_i)\,z^{-(p-n)}\,(b_j\theta_j)\,z^{q} \\
&= z^{-m}\,z^{-(p-n)}(a_i\theta_i C_{z^{p-n}})(b_j\theta_j)\,z^{q}
\end{aligned}$$

from the given conditions we have $a_i\theta_i C_z = a_i\alpha_{i,d+j}\theta_j$

$a_i\theta\, C_{z^2} = (a_i\alpha_{i,d+j}\theta_j)\, C_z = a_i\alpha_{i,2d+j}\theta_j$ and by induction

$a_i\theta_i C_{z^{p-n}} = a_i\alpha_{i,(p-n)d+j}\,\theta_j$

$\therefore\quad (m,a_i,n)\theta(p,b_j,q)\theta = z^{-(m-n+p)}\, a_i\alpha_{i,(p-n)d+j}\theta_j b_j\theta_j z^{q}$

$= z^{-(m-n+p)}(a_i\alpha_{nd+i,pd+j}\cdot b_j)\theta_j z^{q}$

$= (m-n+p, a_i\alpha_{nd+i,pd+j}b_j, q)\theta$

$= [(m,a_i,n)(p,b_j,q)]\theta$

Case (ii) $n>p$ similar to case (i)

Case (iii) $n = p$ $(m,a_i,n)\theta(n,b_j,q)\theta$

$$= z^{-m} a_i \theta_i z^n . z^{-n} b_j \theta_j z^q$$

$$= z^{-m} a_i \theta_i b_j \theta_j z^q$$

$$= z^{-m} a_i \alpha_{i,t} \theta_t b_j \alpha_{j,t} \theta_t z^q \qquad t = \max\{i,j\}$$

$$= (m, a_i \alpha_{i,t} b_j \alpha_{j,t}, q)\theta$$

$$= \left[(m, a_i, n)(n, b_j, q)\right] \theta$$

Conversely suppose θ: $S \to G^*$ is a homomorphism of S into G*.

Then θ induces a homomorphism θ_i: $G_i \longrightarrow G^*$ defined by

$(o, a_i, \rho)\ \theta = a_i \theta_i$

Let $(o, f_o, 1)\ \theta = z \in G^*$

$(o, a_{d-1}\ \gamma_{d-1}, 1)\theta = \left[(o, a_{d-1}\gamma_{d-1}, 0)(o, f_o, 1)\right] \theta$

$= a_{d-1}\gamma_{d-1}\theta_o z \quad , \quad \forall \quad a_{d-1} \in G_{d-1}$

However, $(o, a_{d-1}\ \gamma_{d-1}, 1)\theta = \left[(o, f_o, 1)(o, a_{d-1}, o)\right]\theta$

$= z a_{d-1} \theta_{d-1}$

$\therefore \quad z a_{d-1}\ \theta_{d-1} z^{-1} = a_{d-1}\ \gamma_{d-1}\ \theta_o \quad \forall\ a_{d-1} \in G_{d-1}$

$\therefore \quad \theta_{d-1} C_z = \gamma_{d-1}\ \theta_o$

If $d > o$, for $o \leq i < d-1$

$(o, a_i \gamma_i, o)\theta = a_i \gamma_i \theta_{i+1} \qquad \forall\ a_i \in G_i$

Also $(o, a_i \gamma_i, o)\theta = \left[(o, a_i, \rho)(o, f_{i+1}, o)\right] \theta$

$= a_i \theta_i \qquad \forall\ a_i \in G_i$

$\therefore \quad \theta_i = \gamma_i \theta_{i+1}$

Moreover,

$$(m, a_i, n)\ \theta = \left[(1, f_o, o)^m (o, a_i, o)\ (o, f_o, 1)^n\right]\ \theta$$

$$= z^{-m}(a_i \theta_i)\ z^n .$$

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