EXTENSION OF BURR V DISTRIBUTION: ITS PROPERTIES AND APPLICATION TO REAL-LIFE DATA

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Declaration

I declare that the work in this dissertation titled "Extension of Burr V Distribution: Its Properties and Application to Real-life Data" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

_____ _____

Alfa Saidu Aliyu Date

Certification

This dissertation titled "EXTENSION OF BURR V DISTRIBUTION: ITS PROPERTIES AND APPLICATION TO REAL-LIFE DATA" by Saidu Aliyu ALFA meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University and is approved for its contribution to knowledge and literary presentation.

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Dedication

To Maimuna (Ummi) Abdullahi Alfa, may Allah have mercy on her, ameen.

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All praises and thanks are to the Giver of knowledge, Master of the world for giving me the opportunity to complete this work. My sincere appreciation goes to my parents Alhaji Aliyu Alfa Musa and his dear wives; Hajiyas- Maimuna, Aisha and Hauwa, for their guidance, prayers and moral support, may Almighty Allah grant you the best in your final abode, Amen. This work would not have been possible without the professional guidance, comments and suggestions of my supervisor Dr. H. G. Dikko who was able to go through my work despite his tight schedule, workload and exegesis of running and heading the department. I am grateful and may Almighty Allah continue to reward you abundantly, amen.

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Abstract

More than a decades ago, a new general family of continuous distribution known as beta-G Distribution was proposed. This new generator has been used to propose some distributions in recent years with simplified mathematical treatment. An extension of Burr V (BV) distribution was therefore proposed, using the logit of the beta random variable and some of its properties studied. The Probability Density Function (PDF) of the Beta Burr V (BBV) was defined and verified. Hazard Rate Function, Survival Function and Asymptotic Behaviour of the distribution were obtained, the distribution was also found to be bimodal. The method of maximum likelihood was proposed to estimate the parameters of the distribution. The performance of the new distribution was evaluated using a real-life data. The extended distribution was found to be superior and quite flexible when subjected to scrutiny with burr V, beta Dagum and Beta Power Exponential distributions by fitting it to two real empirical datasets, thus; it was found that it can serve as a good alternative distribution to model positive real data in many areas.

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CHAPTER ONE: INTRODUCTION

1.1 Background to the Study

Extended or generalized distributions have been extensively studied in recent years. Amoroso (1925) was the pioneer researcher to start generalizing continuous distributions, discussing the generalized gamma distribution to fit observed distribution of income rate. Since then, numerous authors have developed various classes of generalized distributions. Well-known distributions have been generated or extended in many ways. Some of the well-established generators are Marshal-Olkin generated family (MO-G) by Marshall and Olkin (1997), the beta-G by Eugene *et al*. (2002), Jones (2004), gamma-G (type 1) by Zografos and Balakrishnan (2009), Kumaraswamy-G (Kw-G for short) by Cordeiro and De Castro (2011), gamma-G (type 2) by Ristic and Balakrishnan (2012), gamma-G (type 3) by Torabi and Hedesh (2012), McDonald-G (Mc-G) by Alexander *et al. (*2012), log-gamma-G by Amini *et al*. (2014), exponentiated generalized-G by Cordeiro *et al*. (2013), Weibull-G by Bourguignion, *et al*. (2014) among others. Recent developments have been geared to define new families by introducing shape parameters to control skewness, kurtosis and tail weights thus providing great flexibility in modeling skewed data in practice (Jones, 2004; Cordeiro *et al*., 2013).

Evidently, one of the most used generalized distribution generator is the beta-G. The earliest of the beta extended distributions is the class of distributions generated from the logit of a beta random variable with cumulative distribution function that involve employing two parameters whose role is to introduce skewness and to vary tail weights (Eugene *et al.*, 2002). Jones (2004) discusses general beta family influenced by its order statistics and shows that it has beautiful

distributional properties and potential for interesting statistical applications. The generalization method used is the logit of beta distribution.

In this Research, we will define and study a four-parameter beta-Burr type V distribution. To attain this, we will let *F(x)* be the CDF of the Burr type V distribution as given in Equation (1.7) such that the emerging distribution is being referred to as the beta-Burr type V distribution (BBV) using appropriate transformations and employing the logit of beta. We will then prove that this is a genuine distribution, in such a way that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(x)dx = 1$, where $g(x)$ is the PDF of the proposed distribution. Then, we will also be define and discuss some properties of the extended distribution.

1.2 Aim and Objectives of the Study

The aim of this work is to extend Burr V (BV) distribution and study some of its properties and evaluate its performance using a real-life data. This will be achieved through the following objectives:

i. To define and express the pdf of the Beta Burr V (BBV) distribution as a mixture of beta- G density function using the logit of beta and derive some of its statistical properties.

ii. To estimate the parameters of the BBV distribution by Maximum Likelihood procedure.

iii. To evaluate the performance of the BBV compared to other traditional distributions.

1.3 Scope of the study

The study will focus only on building or proposing a new BBV distribution and then exploring the verification of its PDF. The expression of the CDF will be defined; the asymptotic behavior of the proposed distribution, hazard rate function, and order Statistics will be given too. Its

parameter will then be estimated using the method of Maximum Likelihood Estimation (MLE) procedure.

1.4 Statement of the problem

One of the interests of statisticians and researchers is to simplify and get better flexibility out of probability models or distributions. Many lifetime data used for statistical analysis follow a particular probability distribution and hence the technical know-how of the appropriate distribution that any circumstances follows greatly better the sensitivity, strength and efficacy of the statistical tests related to it. As it was found in the work by Jones (2004) generalization of continous distribution yield a better and more flexible distribution, though the baseline distributions can sometimes outperform the reulting hybridised distribution- this is mostly depend on the parameters in place.

Different distributions are used for modeling these real-life data. Some of these real-life data do not follow any of the existing well-known probability distributions or are ineptly described by the models. It is as a result of these that building or developing a new distribution that could better explain most of these situations or circumstances and thus provide greater elasticity in the modeling of real-life data.

1.5 Significance of the Study

At the end of the proposition, the distribution, its properties and the parameter estimates will increase the flexibility of the baseline distribution- Burr type V distribution and easily model various data sets that cannot be properly fitted by the existing distributions. The hazard and survival functions of the proposed distribution will allow for its applications in reliability

analysis. Afterwards, the study will compare the proposed model to some existing generalizations of the Beta distribution and other generalized distribution to identify the most fitted model using a real-life data set.

1.6 Motivation

Ever since the introduction of the Burr distribution, it has somewhat been neglected as an option in the analysis of lifetime data. Even though there are researches that indicate that this distribution possesses sufficient flexibility to make it a possible model for various types of data; for example, beta-Burr X was used to examine the strengths of 1.5 cm glass fibers (Merovci *et al*. 2016) and Paranaiba *et al.* (2011) introduced beta-Burr XII to discuss the cancer recurrence.

However, the distributions such as Burr type V and beta individually fail to address the variety of shapes the distribution should exhibit and to what direction it should take. That is, to combine both characteristic of shape and scale parameters, and lacking in-terms of flexibility to model real-life data set in statistics.

Hence, it has become imperative to have an extension of the Burr V distribution using the logit of beta. This we hope will be better than each of them individually in terms of the estimate of their characteristics in their parameters and application to real-life data.

CHAPTER TWO: LITERATURE REVIEW

There are numerous continuous univariate distributions found in statistics literature. Recent developments focus on new techniques for building meaningful distributions. Several methods of introducing one or more parameters to generate new distributions have been studied over the years. In addition, several families of distributions were proposed by compounding some useful lifetime and truncated discrete distributions. Beta distribution is a natural extension of the uniform distribution by adding two parameters (Amusan, 2010). The beta distribution model circumstances which are confined to occur within a range outlined by lowest and highest value. It is the most important distribution that has bounded support. One major benefit of beta distribution is its ability of fitting skewed data that cannot be properly fitted by existing distributions (Gomes *et al.*, 2013).

In recent years, there has been a great deal of interest among statisticians and applied researchers in constructing flexible distributions to facilitate better modeling of lifetime data. Several authors have introduced more flexible distributions to model monotone or unimodal failure rates-which are mostly referred to as generalized distributions. These classes of generalized distributions have been receiving considerable attention over the past two decades, in particular after the recent works of Eugene *et al*. (2002) and Jones (2004).

Eugene *et al*. (2002) introduced the beta-Normal distribution (BND) and its applications. In their work, they studied some properties of the beta-normal distributions.

One property discussed was the shape property. The beta-normal distribution was found to be symmetric about the location parameter, μ when $\alpha = \beta$ (α *and* β are shape parameters). It was also noted that the class of beta-normal distributions uses the normal

distribution N ($\mu;\sigma$) as the distribution generator. Another property noted was that the beta-normal distribution can model both unimodal and bimodal data. The bimodal region for the BN distribution was studied by Famoye *et al*. (2004). The values of α and β were noted to affect the modality of the distribution, since they are the shape parameters while μ is the location parameter and σ is the scale parameter.

The researchers further mentioned the shape properties of the unimodal beta-normal distribution. Included in these, is that the distribution is right skewed when $\alpha > \beta$ and that the increase in the degree of right skewness is proportional to the increase in α . Another shape property is that the distribution is left skewed when $\alpha < \beta$.

The relationships between the first two moments and the parameters of the beta-normal distribution were also noted. The first moment of the beta normal (BN) distribution was considered and it was concluded that the exact first moments cannot be evaluated. As a result, the application of the differential equations technique used by Bose and Gupta (1959) to derive some exact first moments was taken into consideration. The maximum likelihood method was used to estimate the four parameters of the model. The applications of the Beta Normal distribution were also considered and it was noted that both the gamma and Lagrange-gamma distributions provided poor fits for some of the data sets fitted with the BN Distribution.

Nadarajah and Gupta (2004) later derived a more general expression for the nth moment of the BN which was introduced by Eugene *et al.* (2002). The proofs of the results were much simpler to follow, owing to fact that the calculations made were based on the use of the complementary error function and some of its properties.

Nadarajah and Kotz (2004) got motivated by the work of Eugene *et al*. (2002) to generalize the Gumbel distribution thereby introducing the beta-Gumbel distribution (BGD). It was generated from the logit of a beta random variable. A generalization of the Gumbel distribution was proposed with the hope of attracting broad application in engineering.

The Gumbel distribution is most widely used in statistical distribution for problems mainly in engineering. They found the generalizations to have allowed greater flexibility of its tail, because most problems studied using Gumbel distribution are concerned with the tail behavior of one or more variables.

They noted that the PDF of the beta Gumbel distribution exhibited a single mode and obtained the hazard rate function which is an important quantity that characterizes life phenomena. An expression for the nth moment of the distribution was also obtained after which the variance, skewness, and kurtosis measures were computed. It was noted that the beta Gumbel distribution was more flexible than the Gumbel distribution. The parameters of their proposed distribution (beta Gumbel) were estimated by method of Maximum Likelihood.

Nadarajah and Kotz (2004) introduced the beta Frechet (BFr) distribution by taking *G(x)* to be the Frechet distribution, then derived the analytical shapes of the density and hazard rate functions and calculated the asymptotic distribution of the extreme order statistics, Barreto-Souza *et al*. (2011) got another expression for the moments and present some additional mathematical properties for the BFr distribution.

Furthermore, Nadaraja and Kotz (2006) introduced the beta exponential distribution (BED) by generalization of the exponential distribution, again using the logit of a beta random variable just like the previous distributions studied. The distribution was extended with the goal that it will

attract wider applications in reliability analysis. In the paper, the authors provided a comprehensive treatment of statistical properties of the beta-exponential distribution. The paper also discussed and derived expressions for the moment generating function, characteristic function, the first four moments, variance, skewness, kurtosis, mean deviation about the mean, mean deviation about the median, Renyi entropy, and the Shannon entropy.

It was showed that the exponentiated exponential distribution was a particular case of BED (Gupta and Kundu, 2003). The exponential distribution was another particular case of BED showed. They established that the shapes of the PDF and the hazard rate function of the BED depend only on the parameter α (shape parameter). The PDF was noted to monotonically decrease if $\alpha < 1$. It was also noted if $\alpha > 1$ then the PDF had a unique mode at $x = x_0$ with the PDF increasing for $x > x_0$ and the PDF decreasing for $x > x_0$. It was noted that if $\alpha < 1$, then the hazard rate function monotonically decreases with x. while, if $\alpha > 1$, and then the hazard rate function monotonically increases with *x*.

The moment generating function and the characteristics function were derived for the BED. The nth moment of the BED was obtained from the moment generating function and from this; the first four moments were obtained. The first three central moments, skewness and the kurtosis of the distribution were also obtained. It was noted that the skewness and the kurtosis depend only on α and β (Nadaraja and Kotz, 2006).

The possibility of the sum of two random variables with a BED with common parameters α, β and λ having the gamma distribution, and their ratio has the F distribution was also exploded. It was noted that they were possible for the case when $\alpha = 1$. Aside its mathematical sophistication, the BED can be used as an improved model for failure time data. This was because it displays

both increasing and decreasing failure rates and the shape of the failure rate function depends only on the parameter α .

The beta-Weibull distribution (BWD) with four parameters was proposed by Famoye *et al.* (2005). Like the previously mentioned generalized distributions, it was also generated from the logit of a beta random variable. It was shown that the beta-Weibull distribution is unimodal and some results on the non-central moments were obtained. Lee *et al*. (2007) gave some properties of the hazard function, entropies and an application to censored data for the BWD. Additional mathematical properties of this distribution were derived by Carrasco *et al*. (2008) such as expressions for the moments. The parameters were estimated using the maximum likelihood technique and a likelihood ratio test was derived for the beta-Weibull distribution.

The Weibull distribution is known to have wide applications in many disciplines. Different generalizations of the Weibull distribution have also been studied. The Weibull-exponential distribution was proposed by Oguntunde *et al*. (2015). Mudholkar and Kollia (1994) introduced a generalized Weibull distribution with an additional shape parameter. There was later a paper on the application of the generalized Weibull distribution to model survival data by Mudholkar and Hutson (1996). They showed that the graph of the pdf of the distribution may be increasing, decreasing or bathtub depending on the values of the parameters. The hazard function was further shown to be unimodal. An application of the exponentiated Weibull distribution on extreme value data was also done. The exponentiated exponential distribution was also shown to be a special case of the exponentiated Weibull family.

It was also noted that the biases of the maximum log likelihood estimates from beta-Weibull distribution were smaller than those from exponentiated Weibull model with comparable standard errors when $\beta < 1$.

Akinsete *et al.* (2008) defined and studied a four-parameter distribution called beta-Pareto distribution (BPD), just like the previous distributions studied it, therefore, it was generated from the logit of a beta random variable. The Pareto distribution is mainly used in modeling the heavy-tailed distributions that include data on income distribution, city population size and size of firms. Some other quantities measured in the physical, biological, technological and social systems of various kinds.

Many generalizations of the Pareto distribution exist; one among these is the generalized Pareto distribution (GPD) by Pickands in 1975. The GPD was found to be useful in modeling extreme value data due to its long tail feature. Properties discussed in the paper, include: the unimodality of the distribution, the unimodal or decreasing hazard rate, the expressions for the mean, mean deviation, variance, skewness, kurtosis, Renyi and Shannon entropies, maximum likelihood estimates of the parameters and applications to real-life data.

Akinsete *et al*. (2008) noted that when $\alpha = \beta = 1$ (where α, β are shape parameters) the BPD reduces to the Pareto distribution with parameters k and θ as shape and scale parameters respectively. It was also shown that when $\alpha = 1$, the BPD reduces to the Pareto distribution with parameters $k\beta = c$ and θ. The hazard rate of the BPD was derived, and it was noted that this reduces to the hazard rate of the Pareto distribution when $\alpha = \beta = 1$. The BPD was noted to have unimodal failure rate when $\alpha > 1$ and a decreasing failure rate when $0 < \alpha \leq 1$. The moments of the BPD were obtained using the beta integral. Using this, the mean, variance, skewness and

kurtosis were also obtained. The mean deviation from the mean, the mean deviation from the median, the Renyi entropy and the Shannon entropy were obtained. The method of maximum likelihood was also used to estimate the parameters.

According to Akinsete and Lowe (2009), the problem of estimating the reliability of components is of utmost importance in many areas of research, for example, in medicine, engineering and control systems. If *X* represents a random strength capable of withstanding a random amount of stress *Y* in a component, the quantity *R=P(Y<X)* measures the reliability of the component. In the paper, the authors defined and studied the beta-Rayleigh distribution (BRD) and obtained a measure of reliability when both *X* and *Y* has the beta-Rayleigh distribution.

The BRD was observed to be a special case of the Weibull distribution with parameters α and β as , where α = 2. It is also a special case of the chi distribution with parameter $v = 2$. It was noted that, if *X* is a random variable with a BRD with parameters α, β and σ, then $Y = 1 - e^{\frac{-1}{2}(\frac{X}{\sigma})^2}$ has the beta density with parameters α and β. Also if you set $\alpha = \beta = 1$, BRD reduces to the Rayleigh distribution with parameter σ. When $\alpha = 1$, the BRD is noted to reduce to the Rayleigh distribution with parameter $k = \frac{\sigma}{\sqrt{\beta}}$. The variance, skewness and kurtosis were obtained. The parameters of the BRD were estimated by the method of maximum likelihood estimation.

Inspired by the work of Eugene *et al.* (2002), Kozubowski and Nadarajah (2008) introduced the beta-Laplace distribution (BLD) which they generated using the logit of a beta random variable. They discussed some of the hypothetical properties of the distribution, including, for instance;

modality and concavity of the density, moments and related parameters, and stochastic representations that aid in random variate generation from the model.

Shittu and Adepoju (2013) used existing two parameters Nakagami distribution to extend and introduce two extra shape parameters- thus defining the beta-Nakagami distribution which has a better shape, broader tails and a class of hazard rate functions depending on the parameters. In the paper, they studied the statistical properties of the proposed distribution which include moments and moment generating function among others. The parameters of the defined model were estimated by method of maximum likelihood, to enable the derivation of fisher information matrix. Application to real-life indicates that Beta-Nakagami distribution apart from being more flexible has better representation of data than Nakagami distribution.

Chukwu and Ogunde (2015) introduced and studied the beta Gompertz-Mekaham (BGM) distribution. In the paper, the mortality model was derived. Various properties of the distribution were discussed in the paper; these include the asymptotic behaviors, Cumulative density function and hazard functions among others. Also, they derived the estimation of parameters by the method of maximum likelihood. Real-life application indicates that new distribution apart from being more flexible has better representation of data than its sub-model i.e Gompertz-Mekaham distribution.

Paranaíba *et al*. (2011) defined and studied a five parameter lifetime distribution; the beta-Burr XII (BBXII) distribution, which extends several distributions widely used in the lifetime literature. The new model was much more flexible than the Burr XII (BXII), beta Weibull (BW) and log-logistic (LL) sub-models. It was found to be useful to model lifetime data with a unimodal shaped hazard rate function. They provided a mathematical treatment of the

distribution including the mean deviations, Bonferroni and Lorenz curves and expansions for the densities of the order statistics. They derived explicit closed form expressions for the moments and moment generating function which hold in generality for any parameter values. They gave linear functions to compute the moments of the order statistics. The estimation of parameters was approached by the method of maximum likelihood and the observed information matrix is derived. One application of the BBXII distribution showed that it could provide a better fit than other statistical models used in lifetime data analysis.

Most recently, in their paper Merovci *et al*. (2016) proposed a distribution which generalizes the Burr type X distribution. They named it beta-Burr type X (BBX) and have a special sub models. They derived the CDF, PDF, hazard function and limit of PDF. Additionally, some of the mathematical and statistical properties like quantile function, skewness, kurtosis, r^{th} moment and order statistic were also provided in the paper. The model parameters were estimated using maximum likelihood estimation and they derived the observed information matrix. They compared all the distribution models i.e sub-models from BBX by considering criteria like; log likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian Information Criterion (BIC) for the data set. The better distribution which was BBX Corresponded to smaller LL, AIC, AICC and BIC values to Burr type X, Rayleigh and G Exponential. The distribution of the data is skewed to the left (skewness = –0.95 and kurtosis = 1.01). The BBX distribution was studied and shown to be very good in modeling left skewed data. Simulation study was carried out under varying sample size to assess the performance of the model. Finally, application of a real data set by using the goodness-of-fit was illustrated. It was shown that the new distribution provides a better fit than its sub-models and it was very good model for left skewed data.

Navid and Muhammad (2013) proved that small samples sizes with higher censoring rates are closely related to the limiting figures of the variance covariance matrix. They concluded that, the approximate variance covariance matrix can be used for analysis of the unknown parameters of the Burr type V distribution. They further showed that, the maximum likelihood point and interval estimates can effectively be applied to the real-life situations using moderate sample sizes. However, they could not obtain the explicit expression for the limiting Fisher information matrix analytically for Burr type V distribution under left censored samples.

The beta-hyperbolic secant (BHS) distribution was introduced by Fischer and Vaughan (2010). They gave some properties of the distribution, like the moments, special and limiting cases and compared it with other distributions using a real data set.

The beta gamma distribution (BG) was introduced by Kong *et al*. (2007). In their paper, they derived some properties of the limit of the density function and of the hazard function, gave an expression for the moments when the shape parameter α is an integer and made an application on real data set to further examine the flexibility of the distribution.

Condino and Domma (2013) proposed a distribution called beta-Dagum Distribution (BDaD) which includes some important distributions as special case, such as Dagum, Beta-Burr III and Beta-Fisk distributions. They showed that the probability density function could be expressed as a mixture of densities of Dagum random variables. Some of the mathematical properties of the new distribution were obtained. The usefulness and better flexibility of the new distribution was illustrated through real data set of Bank of Italy on income and consumption.

Adepoju and Chukwu (2014) used existing two parameter power exponential distribution to extend it by two extra shape parameters to give rise to the beta-Power Exponential Distribution

(BPED) which has a better shape, broader tails and a class of hazard rate functions depending on the parameters. They studied in detail some of the statistical properties of the proposed distribution which include moments and moment generating function among others. The parameters of the model were estimated by method of maximum likelihood which actually paves way for the derivation of Fisher information matrix. Real-life application indicates that Beta-Power Exponential Distribution apart from its flexibility has better representation of data than exponentiated exponential distribution.

Mead (2014) proposed a new model which he called the beta exponentiated Burr XII distribution. It has as special sub-models some well-known and new distributions; such as the beta-Burr XII, beta exponentiated log-logistic, beta exponentiated Lomax, beta log-logistic, beta Lomax, exponentiated Burr XII, exponentiated log-logistic, exponentiated Lomax, Burr XII, log-logistic, and Lomax, among several others. He derived structural properties of the new distribution; including explicit expressions for the moments, incomplete moments, mean deviations, quantile function, and Rényi entropy. The method of maximum likelihood was used to estimate the model parameters. He obtained the observed information matrix and used two real data sets to illustrate the importance and flexibility of the proposed model.

Daniyal and Aleem (2014) proposed the generalization of Burr XII and Weibull Distributions by combining the two distinct models linearly. They studied some of its properties, these include; Cumulative distribution function, hazard rate, failure rate, inverse hazard function, odd function and the cumulative hazard function, rth moment, moment generating function, characteristic function, moments, mean and variance, Renyi and Beta entropies, mean deviation from mean and mean time between failures (MTBF). They approached the parameter estimation by method of maximum likelihood estimation and constructed the information matrix.

CHAPTER THREE: METHODOLOGY

3.1 The Beta-Burr V (BBV) Distribution

In this chapter, the procedure of defining and extending the Burr V distribution to a new class of distribution, called beta-Burr type V (BBV) distribution is established and looking at the properties and some useful features of the extended distribution. Before that, we are going to explore bit by bit beta and burr V distribution and conclusively the method of generalization.

3.1.1 Beta distribution

Beta distribution is a continuous probability distribution with two positive shape parameters and it is the conjugate prior of the binomial distribution.

The probability density function (PDF) of the beta distribution is given by equation (3.1):

$$f(x) = \frac{(1-x)^{\beta-1} x^{\alpha-1}}{B(\alpha,\beta)}, \quad 0 < x < 1, \alpha > 0, \beta > 0 \tag{3.1}$$

where $B(\alpha,\beta)$ is the beta function defined by (3.2) and (3.3):

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \tag{3.2}$$

and in terms of gamma function it is defined by:

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{3.3}$$

where $\Gamma(.)$ is the gamma function which is defined by:

$$\Gamma\alpha = \int_0^\infty x^{\alpha-1} e^{-\alpha x} dx = (\alpha-1)\Gamma(\alpha-1) = (\alpha-1)! \tag{3.4}$$

The integral that defines the beta function can be generalized by changing the interval of integration from *(0,1)* to *(0,x)* where $x \in (0,1)$

The *incomplete beta function* is defined as follows

$$B(x;\alpha,\beta)=\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt \tag{3.5}$$

where

$x\in(0,1)$

From (Gomes *et al*., 2013) Of course, the ordinary (complete) beta function is $B(\alpha,\beta)=B(1;\alpha,\beta)$

The beta function is simply the normalizing constant, so it is clear that *f* (beta density function), is a valid probability density function. If $\alpha \geq 1$, *f* is defined at 0, and if $\beta \geq 1$, *f* is defined at 1. In these cases, it is customary to extend the support set to these endpoints.

The beta distribution has just two parameters and yet a rich variety of shapes. Qualitatively, the first order properties of *f* depend on whether each parameter is less than, equal to, or greater than 1. Beta distribution is very flexible and can be used to analyze different types of data sets. Many of the baseline distributions encountered in practice can easily be transmuted into the standard beta distribution. In econometrics, quite often the data are analyzed by using baseline distributions (Dila, 2015).

The cumulative distribution function (CDF) of the beta distribution is given by

$$F(x)=\frac{1}{B(\alpha,\beta)}\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt \tag{3.6}$$

where $0<\alpha,\beta<\infty$

The distribution function *F* is sometimes known as the *regularized incomplete beta function.*

This can be expressed sometimes as

$$F(x) = \frac{B(x;\alpha,\beta)}{B(\alpha,\beta)}, \quad 0 < x < 1 \tag{3.7}$$

In some cases, the distribution function *F(x)* and its inverse, the quantile function $F^{-1}(x)$, can be computed in closed form, without resorting to special functions.

3.1.2 Burr Distribution

Burr (1942) introduced a family of twelve cumulative distribution functions for modeling lifetime data. Burr distribution (XII) is a continuous probability distribution for a random variable and a unimodal family of distributions with a wide variety of shapes. Burr distribution is also known as the Singh–Maddala distribution and is one of a number of different distributions sometimes called the "generalized log-logistic distribution". This distribution is used to model a wide variety of phenomena including crop prices, household income, option market price distributions, health, risk (insurance) and travel time.

Of the dozen distributions introduced, Burr III, IV, V, IX, and XII families have a variety of density shapes (Rodriguez, 1977). Some of the forms of Burr distributions are related by simple transformation. For example, the Burr type III distribution can be obtained from Burr type II distribution by replacing *x* with *ln(x)* (Johnson *et al.*, 1995). Similarly, Burr type XII distribution can be derived from Burr type III distribution by replacing *x* with *(1/x)* (Burr and Cislak, 1968). The twelve distributions of Burr are found in Burr, (1942) and Johnson *et al*., (1995) and are

given below. The range of x over which the given expression is to be used is written to the right when it is not $(-\infty,\infty)$. Constant k, r and c are positive numbers.

1. $F(x)=x, \qquad (0,1),$ (3.8)

2. $F(x)=\left(e^{-x}+1\right)^{-r},$ (3.9)

3. $F(x)=\left(x^{-k}+1\right)^{-r}, \qquad (0,\infty),$ (3.10)

4. $F(x)=\left[\left(\frac{c-x}{x}\right)^{1/c}+1\right]^{-r}, \qquad (0,c),$ (3.11)

5. $F(x)=\left(ke^{-\tan x}+1\right)^{-r}, \qquad \left(-\frac{\pi}{2},\frac{\pi}{2}\right),$ (3.12)

6. $F(x)=\left(ke^{-c\sinh x}+1\right)^{-r},$ (3.13)

7. $F(x)=2^{-r}\left(1+\tanh x\right)^{r},$ (3.14)

8. $F(x)=\left(\frac{2}{\pi}\arctan e^{x}\right)^{r},$ (3.15)

9. $F(x)=1-\frac{2}{k\left[\left(1+e^{x}\right)^{r}-1\right]+2},$ (3.16)

10. $F(x)=\left(1-e^{-x^{2}}\right)^{r}, \qquad (0,\infty),$ (3.17)

11. $F(x)=\left(x-\frac{1}{2\pi}\sin 2\pi x\right)^{r}, \qquad (0,1)$ (3.18)

12. $F(x)=1-\left(1+x^{c}\right)^{-k}, \qquad (0,\infty)$ (3.19)

Burr type V distribution

A continuous random variable X is said to follow a Burr type V distribution if its probability density function is given by equation (1.20):

$$f(x)=\begin{cases}\alpha\beta e^{-\tan x}\sec^2 x\left(1+\beta e^{-\tan x}\right)^{-\alpha-1} & \dfrac{-\pi}{2}<x<\dfrac{\pi}{2}\\ 0 & \textit{elsewhere}\end{cases} \tag{3.20}$$

and its cumulative distribution function is given by equation (3.20):

$$F(x)=(1+\beta e^{-\tan x})^{-\alpha} \tag{3.21}$$

where α and β are the location parameters of the distribution.

and $\alpha, \beta > 0$

The location parameter determines the position of the data distribution along the x-axis. The location is either estimated from the data or specified based on historical process knowledge. The location parameter defines the shift of the data. A positive location value shifts your distribution to the right, while a negative location value shifts your data distribution to the left.

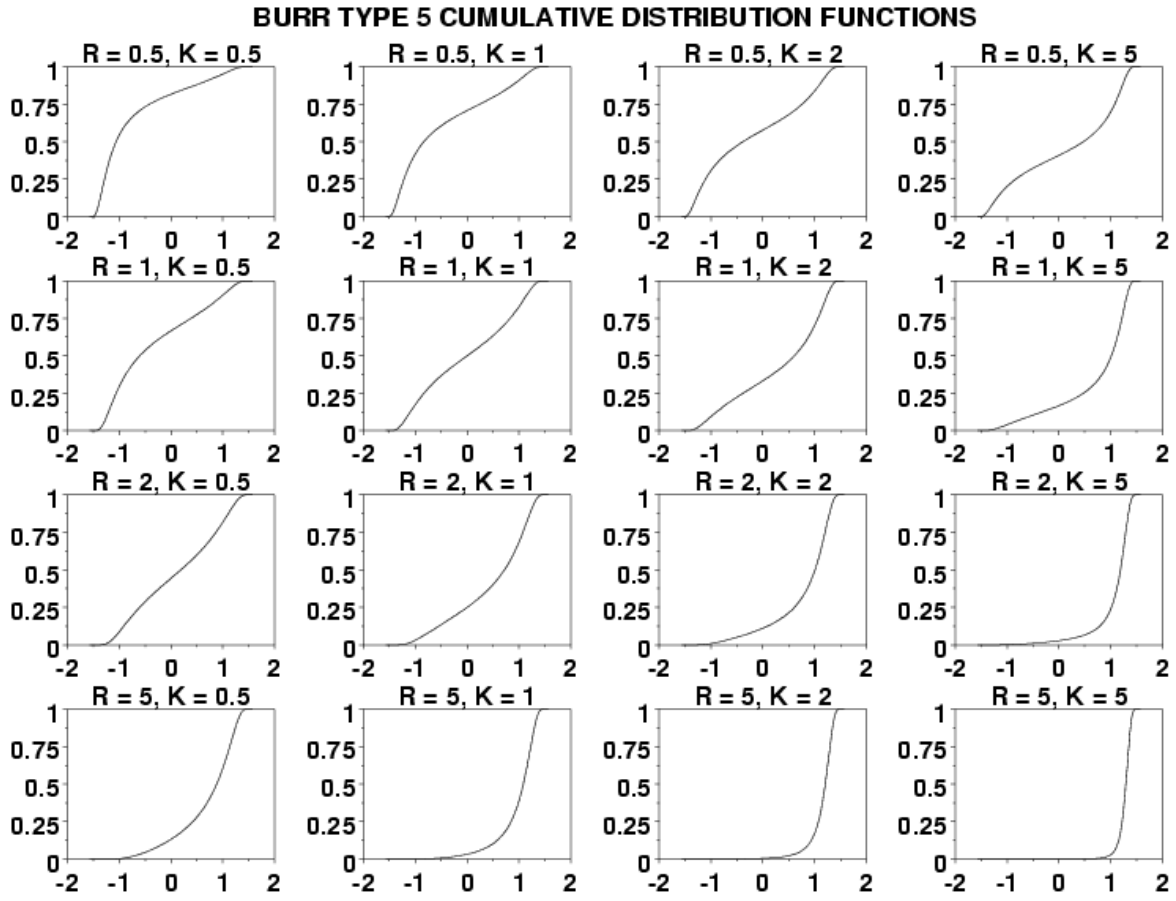


Figure 3.1: The Effect of the Different Values of the Location Parameter on Burr V Distribution

Burr distributions are the heavy tailed distribution which means that these distributions have more heavy tails as compared to the exponential distribution (Daniyal and Aleem, 2014).

3.1.3 Generalization of the beta distribution

Generalized beta distributions have been widely studied in statistics and a lot of researchers have come up with various classes of these distributions, The main use of the beta distribution as stated earlier has led to it being used in the generalization of different distributions, which has

been introduced and studied by different authors in literature such as: Nadarajah and Gupta, (2004) defined the beta-Frechet distribution; Famoye *et al*. (2005) defined the beta-Weibull; Nadaraja and Kotz (2006) defined the beta-exponential distribution; Kong *et al*. (2007) proposed the beta-gamma distribution. Fischer and Vaughan (2010) introduced the beta-hyperbolic secant distribution, beta-Gumbel (BGU) distribution was introduced by Nadarajah and Kotz (2004); the beta-Pareto distribution was defined and studied by Akinsete *et al*. (2008); the beta-Rayleigh distribution was proposed by Akinsete and Lowe (2009); beta-Burr XII by Paranaiba *et al*. (2011); and very recently, Merovci *et al*. (2016) developed the beta-Burr type X distribution.

The generalization of beta distribution was made prominent by Jones (2004), where the author talked about the families of distributions arising from distributions of order statistics. The author proposed an alternative derivation of this family of distributions as a result of applying the inverse probability integral transformation to the beta distribution. He further studied some properties which include: the location and scale parameters, the distribution function, the moments, modality, the tail weight, and the limiting distributions.

Suppose *F(x)* is the cumulative distribution function of a random variable *X* that follows a beta distribution. Merovci *et al*. (2016) stated that "For any continuous baseline, the cumulative distribution function for the beta-G distribution

G(x) is given as:

$$G(x,\alpha,\beta)=\frac{1}{B(\alpha,\beta)}\int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}\,dt \tag{3.22}$$

where $0<\alpha,\beta<\infty$

Its probability density function is $g(x)=G'(x)$ which is given by (3.23)

$$g(x,\alpha,\beta)=\frac{1}{B(\alpha,\beta)}F(x)^{\alpha-1}\left(1-F(x)\right)^{\beta-1}F'(x) \tag{3.23}$$

The probability density function *f (x) = F′ (x)* has been studied by many authors assuming various type of CDF of *F*(*x*)."

The above work by Jones (2004) introduced a general family that triggered other authors to consider generalized distributions with some statistical applications. Thereafter, few authors defined and studied various forms of beta-compounded distributions. For example, Eugene *et al*. (2002) introduced the beta-normal distribution and studied some of its properties. Nadarajah and Gupta (2004) later derived a more general expression for the n^{th} moment of the beta-normal distribution. The beta-exponential and the beta-Gumbel were both defined and studied by Nadaraja and Kotz (2006) and Nadarajah and Kotz (2004) respectively. Most recently, the beta-Burr X proposed by Merovci *et al.* (2016) where they studied some properties of the distribution and test its flexibility with its sub-model and other known distributions. We will now proceed to extend the burr V distribution as follows:

3.1.4 Definition

Suppose that *F(x)* denotes the probability density function (PDF) of a random variable *X*. The PDF of a generalized class of distribution for the random variable *X* according to Jones (2004) is defined by:

$$g(x)=\frac{1}{B(a,b)}[F(x)]^{a-1}[1-F(x)]^{b-1}f(x) \tag{3.24}$$

Where

F(x) is the cumulative density of the baseline distribution

f(x) = F'(x) is the probability density of the baseline distribution.

Equation (3.24) is the probability density function (PDF) corresponding to the cumulative distribution function (CDF) *G(x)* given by:

$$G(x)=\frac{1}{B(\alpha,\beta)}\int_0^{F(x)} t^{\alpha-1}(1-t)^{\beta-1}\,dt \tag{3.25}$$

Equation (3.24) is referred the probability density function *f (x) = F′ (x)* has been studied by many authors assuming various type of CDF of *F*(*x*), it has been used to define and study various forms of beta-compounded distributions such as Akinsete and Lowe (2009), Famoye *et al*. (2005), Nadaraja and Kotz (2006), Merovci *et al*. (2016) Paranaíba *et al*. (2011), and so on, where they take on different types of cumulative distribution functions *F(x)* to build new distributions.

So now, suppose that *F(x)* is the cumulative distribution function of Burr type V distribution as given by equation (3.20) and *f(x)* is the probability density function defined in (3.21). The proposed distribution beta-Burr type V can be obtained as follows:

$$g(x)=\frac{1}{B(a,b)}[F(x)]^{a-1}[1-F(x)]^{b-1}f(x)$$

where $F(x)=\left[1+\beta e^{-\tan x}\right]^{-\alpha}$

$$f(x)=\alpha\beta e^{-\tan x}\sec x\left(1+\beta e^{-\tan x}\right)^{-\alpha-1}$$

Beta function is $B(a,b)=\dfrac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

$$g(x)=\frac{1}{B(a,b)}[1+\beta e^{-\tan x}]^{-\alpha(a-1)}[1-(1+\beta e^{-\tan x})^{-\alpha}]^{b-1}\alpha\beta e^{-\tan x}\sec^2 x(1+\beta e^{-\tan x})^{-\alpha-1}$$

Hence the proposed distribution is given by equation (3.26)

$$g(x)=\frac{\alpha\beta e^{-\tan x}\sec^2 x}{B(a,b)}[1+\beta e^{-\tan x}]^{-\alpha a-1}[1-(1+\beta e^{-\tan x})^{-\alpha}]^{b-1} \qquad \alpha,\beta,a,b>0 \tag{3.26}$$

3.1.5 Investigation of proper probability density function

Using appropriate transformation, we will verify that the proposed density function is a probability density function. This will be achieved by integrating *g(x)* with respect to x and see whether the integral is equal to unity or otherwise. If it is equal to unity, the density function is a proper probability density function otherwise it is not.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(x)dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\alpha\beta}{B(a,b)}e^{-\tan x}\sec^2 x[1+\beta e^{-\tan x}]^{-\alpha a-1}[1-(1+\beta e^{-\tan x})^{-\alpha}]^{b-1}dx$$

let $\quad p=\beta e^{-\tan x}$

then, $dp=-p\sec^2 xdx \Rightarrow dx=\dfrac{dp}{-p\sec^2 x}$

as $x\to\dfrac{\pi}{2}$, $\quad p\to 0$ and

as $x\to-\dfrac{\pi}{2}$, $\quad p\to\infty$

so, we have,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(x)dx=\int_0^{\infty}\frac{\alpha}{B(a,b)}p\sec^2 x[1+p]^{-\alpha a-1}[1-(1+p)^{-\alpha}]^{b-1}\frac{dp}{p\sec^2 x}$$

$$\int_0^{\infty}\frac{\alpha}{B(a,b)}(1+p)^{-\alpha a-1}[1-(1+p)^{-\alpha}]^{b-1}dp$$

let $R=(1+p)^{-\alpha} \quad \Rightarrow dR=-\alpha(1+p)^{-\alpha-1}dp$

$$R^{-\frac{1}{\alpha}} = 1 + p$$

Hence, $dR = -\alpha R^{-\frac{1}{\alpha}(-\alpha-1)}$ $\qquad dR = -\alpha R^{1+\frac{1}{\alpha}} dp$

$$dp = \frac{dR}{-\alpha R^{1+\frac{1}{\alpha}}}$$

as $p \to \infty,\ R \to 0$ and

as $p \to 0,\ R \to 1$

hence, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(x)\,dx = \frac{\alpha}{B(a,b)} \int_0^1 R^{a+\frac{1}{\alpha}} [1-R]^{b-1} \frac{dR}{\alpha R^{1+\frac{1}{\alpha}}}$

$$= \frac{1}{B(a,b)} \int_0^1 R^{a+\frac{1}{\alpha}-1-\frac{1}{\alpha}} (1-R)^{b-1}\, dR$$

$$\frac{1}{B(a,b)} \int_0^1 R^{a-1} (1-R)^{b-1}\, dR = \frac{B(a,b)}{B(a,b)} = 1$$

therefore, $g(x)$ is indeed a proper probability density function of a continuous distribution.

3.1.6 Expansion of the Probability Density Function

In this subsection, we will expand the pdf of the BBV distribution.

The probability density function of the proposed distribution can be expanded using the binomial series expansion. Recall that the binomial series expansion of

$$(1-x)^n = 1 + {}^nC_1(-x) + {}^nC_2(-x)^2 + {}^nC_3(-x)^3 + \ldots$$

$$= 1 - {}^{n}C_1 x + {}^{n}C_2 x^2 - {}^{n}C_3 x^3 + \ldots$$

$$= \sum_{r=0}^{\infty} (-1)^r \, {}^{n}C_r x^r \tag{3.27}$$

Hence, using this expression, the term $[1-(1+\beta e^{-\tan x})^{-\alpha}]^{b-1}$ in equation (3.26) can be written as:

$$[1-(1+\beta e^{-\tan x})^{-\alpha}]^{b-1} = \sum_{r=0}^{\infty} (-1)^r \binom{b-1}{r} (1+\beta e^{-\tan x_i})^{-\alpha r} \tag{3.27}$$

Substituting (3.27) in (3.26), we have

$$f(x) = \frac{\alpha\beta}{B(a,b)} e^{-\tan x} \sec^2 x [1+\beta e^{-\tan x}]^{-\alpha a - 1} \sum_{r=0}^{\infty} (-1)^r \binom{b-1}{r} (1+\beta e^{-\tan x_i})^{-\alpha r}$$

$$= \frac{\alpha\beta e^{-\tan x}}{B(a,b)} \sec^2 x \sum_{r=0}^{\infty} (-1)^r \binom{b-1}{r} [1+\beta e^{-\tan x}]^{-(\alpha(r+a)+1)} \tag{3.28}$$

If b is an integer, the summation in (3.28) stops at $b-1$

Recall that $(1+x)^{-n}$ can be written as:

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!} x^2 - \frac{n(n+1)(n+2)}{3!} x^3 + \ldots$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{n+j-1}{j} x^j \tag{3.29}$$

Hence, taking $[1+\beta e^{-\tan x}]^{-[\alpha(r+a)+1]}$ from (3.28) and using the information in (3.29), we write

$$[1+\beta e^{-\tan x}]^{-[\alpha(r+a)+1]} = \sum_{j=0}^{\infty} (-1)^j \binom{\alpha(r+a)+j}{j} (\beta e^{-\tan x})^j$$

$$= \sum_{j=0}^{\infty} (-1)^j \binom{\alpha(r+a)+j}{j} \beta^j e^{-j\tan x} \tag{3.30}$$

Substituting (3.30) in (3.27) yields

$$f\left(x\right)=\frac{\alpha\beta}{B\left(a,b\right)}e^{-\tan x}\sec^2 x\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{r+j}\binom{b-1}{r}\binom{\alpha\left(r+a\right)+j}{j}\beta^j e^{-j\tan x}$$

therefore,

$$f\left(x\right)=\frac{\alpha}{B\left(a,b\right)}\sum_{r=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{r+j}\beta^{j+1}\binom{b-1}{r}\binom{\alpha\left(r+a\right)+j}{j}\sec^2 xe^{-\left(j+1\right)\tan x} \tag{3.31}$$

3.1.7 Asymptotic Behavior

In order to investigate the behavior of the proposed distribution, we proceed thus,

Theorem 1:

The limit of beta-Burr V density function as $x\to\frac{\pi}{2}$ is 0 and the limit as $x\to-\frac{\pi}{2}$ is 0.

Proof:

These can be shown by getting the limit of the beta-Burr density in equation (3.26) when

$x\to-\frac{\pi}{2}$ and when $x\to\frac{\pi}{2}$ respectively.

For $x\to-\frac{\pi}{2}$:

$$\lim_{x\to-\frac{\pi}{2}}g\left(x\right)=\lim_{x\to-\frac{\pi}{2}}\frac{\alpha\beta e^{-\tan x}\sec^2 x}{B\left(a,b\right)}\left(1+\beta e^{-\tan x}\right)^{-\alpha a-1}\left[1-\left(1+\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}$$

$$=\frac{\alpha\beta}{B\left(a,b\right)}\lim_{x\to-\frac{\pi}{2}}e^{-\tan x}\sec^2 x\lim_{x\to-\frac{\pi}{2}}\left(1+\beta e^{-\tan x}\right)^{-\alpha a-1}\lim_{x\to-\frac{\pi}{2}}\left[1-\left(1+\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}$$

$$= \frac{\alpha\beta}{B(a,b)} \lim_{x\to -\frac{\pi}{2}} e^{-\tan x} \sec^2 x \lim_{x\to -\frac{\pi}{2}} \frac{1}{\left(1+\beta e^{-\tan x}\right)^{\alpha a+1}} \lim_{x\to -\frac{\pi}{2}} \left[1-\left(1+\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1} = 0$$

Since $\lim_{x\to -\frac{\pi}{2}} \frac{1}{\left(1+\beta e^{-\tan x}\right)^{\alpha a+1}} = 0$

Similarly, as $x \to \frac{\pi}{2}$

$$\lim_{x\to \frac{\pi}{2}} g(x) = \lim_{x\to \frac{\pi}{2}} \frac{\alpha\beta}{B(a,b)} e^{-\tan x} \sec^2 x \left(1+\beta e^{-\tan x}\right)^{-\alpha a-1} \left[1-\left(\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}$$

$$= \frac{\alpha\beta}{B(a,b)} \lim_{x\to \frac{\pi}{2}} e^{-\tan x} \sec^2 x \lim_{x\to \frac{\pi}{2}} \left(1+\beta e^{-\tan x}\right)^{-\alpha a-1} \lim_{x\to \frac{\pi}{2}} \left[1-\left(\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}$$

$$\frac{\alpha\beta}{B(a,b)} \lim_{x\to \frac{\pi}{2}} 0 \lim_{x\to \frac{\pi}{2}} \sec^2 x \lim_{x\to \frac{\pi}{2}} \left(1+\beta e^{-\tan x}\right)^{-\alpha a-1} \lim_{x\to \frac{\pi}{2}} \left[1-\left(\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1} = 0$$

Since, $\lim_{x\to \frac{\pi}{2}} e^{-\tan x} = 0$

Since the limits when $x = -\frac{\pi}{2}$ and $x = \frac{\pi}{2}$ are both equal zero, this is an indication that the propose distribution has at least a mode.

3.2 Cumulative Density Function of Beta-Burr Type V Distribution

The Cumulative distribution function of the beta-Burr type V distribution is derived as follows:

$$G(x) = \int_{-\frac{\pi}{2}}^{x} \frac{\alpha\beta}{B(a,b)} e^{-\tan y} \sec^2 y [1+\beta e^{-\tan y}]^{-\alpha a-1} [1-(1+\beta e^{-\tan y})^{-\alpha}]^{b-1} dy$$

let $p = \beta e^{-\tan y} \quad \Rightarrow \quad dp = -\beta \sec^2 y e^{-\tan y} dy$

$$dp = -p \sec^2 y dy$$

$$dy = \frac{dp}{-p \sec^2 y}$$

as $y \to -\frac{\pi}{2}$, $p \to \infty$ and

as $y \to x$, $p \to \beta e^{-\tan x}$

therefore,

$$G(x) = \int_{\beta e^{-\tan x}}^{\infty} \frac{\alpha}{B(a,b)} p \sec^2 y \left[1+p\right]^{-\alpha a-1} \left[1-\left(1+p\right)^{-\alpha}\right]^{b-1} \frac{dp}{p \sec^2 y}$$

$$= \int_{\beta e^{-\tan x}}^{\infty} \frac{\alpha}{B(a,b)} [1-(1+p)^{-\alpha}]^{b-1} dp$$

let $R = (1+p)^{-\alpha}$ $\qquad dR = -\alpha(1+p)^{-\alpha-1} dp$

$R^{-\frac{1}{2}} = 1+p$, $\qquad \therefore dR = -\alpha R^{\frac{\alpha+1}{\alpha}} dp \Rightarrow dp = \frac{dR}{-\alpha R^{\frac{\alpha+1}{\alpha}}}$

as $p \to \beta e^{-\tan x}$, $\qquad R \to (1+\beta e^{-\tan x})^{-\alpha}$ and

as $p \to \infty$, $\quad R \to 0$

$$G(x) = \int_0^{\left(1+\beta e^{-\tan x}\right)^{-\alpha}} R^{\frac{\alpha a+1}{\alpha}} \left[1-R\right]^{b-1} \frac{dR}{\alpha R^{\frac{\alpha+1}{\alpha}}}$$

$$G(x) = \int_0^{\left(1+\beta e^{-\tan x}\right)^{-\alpha}} \frac{\alpha}{B(a,b)} R^{\frac{\alpha a+1-\alpha-1}{\alpha}} \left(1-R\right)^{b-1} \frac{dR}{\alpha}$$

$$= \frac{1}{B(a,b)} \int_0^{(1+\beta e^{-\tan x})^{-\alpha}} R^{a-1}(1-R)^{b-1} dR$$

therefore

$$G(x) = \frac{B\left[\left\{1+\beta e^{-\tan x}\right\}^{-\alpha}, a, b\right]}{B(a,b)} \tag{3.32}$$

Where

$B\left(\left\{1+\beta e^{-\tan x}\right\}^{-\alpha}, a, b\right)$ is an incomplete beta function

3.3 Hazard Rate Function

The hazard rate function of a random variable *X* with the probability density function *g(x)* and a cumulative distribution function *G(x)* denoted by $h(x)$ is defined as the ratio of the density function *g(x)* to its survival function *S(x)=1- G(x)*. So, the hazard rate function *h(x)* of the four-parameter BBV distribution is given by:

$$h(x) = \frac{g(x)}{1-G(x)} \tag{3.33}$$

For the beta-Burr V distribution, the *g(x)* and *G(x)* are given respectively by equations (3.3) and (3.10). Using these expressions, the hazard rate function may be expressed as

$$h(x) = \frac{\dfrac{\alpha\beta e^{-\tan x}\sec^2 x\left[1+\beta e^{-\tan x}\right]^{-\alpha a-1}\left[1-\left(1+\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}}{B(a,b)}}{1-\dfrac{B\left[\left\{1+\beta e^{-\tan x}\right\}^{-\alpha}, a, b\right]}{B(a,b)}}$$

$$= \frac{\dfrac{\alpha\beta e^{-\tan x}\sec^2 x\left[1+\beta e^{-\tan x}\right]^{-\alpha a-1}\left[1-\left(1+\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}}{B(a,b)}}{\dfrac{B(a,b)-B(a,b)B\left[\left\{1+\beta e^{-\tan x}\right\}^{-\alpha},a,b\right]}{B(a,b)}}$$

Hence, the hazard rate function of the BBV is given by:

$$h(x) = \frac{\alpha\beta e^{-\tan x}\sec^2 x\left[1+\beta e^{-\tan x}\right]^{-\alpha a-1}\left[1-\left(1+\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}}{B(a,b)-B(a,b)B\left[\left\{1+\beta e^{-\tan x}\right\}^{-\alpha},a,b\right]} \tag{3.34}$$

It is also known as failure rate or force of mortality. The hazard function is not a density or a probability. However, it is thought of as the probability of failure in an infinitesimally small time period between $y+\partial y$ given that the subject has survived up till time *y*. In this sense, the hazard is a measure of risk: the greater the hazard between times y_1 and y_2, the greater the risk of failure in this time interval.

3.4 Survival Function

The survival function is the probability that a patient, device, or other object of interest will survive beyond a specified time. It is also known as the survivor function or reliability function.

Let *T* be a continuous random variable with cumulative distribution function *F*(*t*) on the interval [0,∞). The *survival function* or *reliability function* of T is defined by:

$$S(t) = P(\{T > t\}) = \int_t^{\infty} f(u)\,du = 1 - F(t) \tag{3.35}$$

In other words, the survivor function is the probability of survival beyond time *t*. One use of the survivor function is to predict quantiles of the survival time. The term reliability function is common in engineering while the term survival function is used in a broader range of applications, including human mortality.

So, the *survival function* of BBV is given as

$$S(x)=1-G(x)$$

$$=1-\frac{B\left[\left\{1+\beta e^{-\tan x}\right\}^{-\alpha},a,b\right]}{B(a,b)} \tag{3.36}$$

3.5 Maximum Likelihood Estimation of the Parameters of the Proposed Model and Fisher's Information Matrix

In this section, we are going to employ one of the most widely used methods for the estimation of parameters of distribution- the maximum likelihood estimation method (MLE). We will then later, obtain the elements for the Fisher's Information Matrix.

3.5.1 Maximum Likelihood Estimation

Here we derive the maximum likelihood estimates (MLEs) of the parameters of the Beta-Burr type V Distribution. To achieve such goal, we let $X_1, X_2, ...X_n$ be a random sample from a population *X* with probability density function $f(x:\theta)$ where θ is a vector of parameters and is unknown. The likelihood function $L(\theta)$ is defined as the product of the joint density of the random variables $X_1, X_2, ...X_n$. That is

$$L(\theta) = \prod_{i=1}^{n} f(x;\theta)$$

The value of θ that maximize the likelihood function $L(\theta)$ is called the maximum likelihood estimators of θ which is denoted by $\hat{\theta}$.

Let $X_1, X_2, ...X_n$ be a random sample from the beta-Burr type V distribution with parameters (a,b,α,β) defined in equation (3.3), then the joint distribution of $X_1, X_2, ...X_n$ is given by equation (3.37):

$$L(x:a,b,\alpha,\beta) = \prod_{i=1}^{n} g(x:a,b,\alpha,\beta)$$

$$= \prod_{i=1}^{n} \frac{\alpha\beta e^{-\tan x_i} \sec^2 x_i}{B(a,b)} (1+\beta e^{-\tan x_i})^{-\alpha a-1}[1-(1+\beta e^{-\tan x_i})^{-\alpha}]^{b-1} \quad (3.37)$$

$$= \frac{\alpha^n \beta^n e^{-\sum_{i=1}^{n} \tan x_i}}{B((a,b))^n} \prod_{i=1}^{n} \sec^2 x_i \left(1+\beta e^{-\tan x_i}\right)^{-\alpha a-1} \left[1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\right]^{b-1} \quad (3.38)$$

Taking log of (3.38), we have the log-likelihood given by equation (3.39):

$$L(x;a,b,\alpha,\beta) = n\log\alpha + n\log\beta - \sum_{i=1}^{n} \tan x_i - n\log\left(B(a,b)\right) + \sum_{i=1}^{n} \log\left(\sec^2 x_i\right) - \quad (3.39)$$
$$(\alpha a+1)\sum_{i=1}^{n} \log\left(1+\beta e^{-\tan x_i}\right) + (b-1)\sum_{i=1}^{n} \log\left[1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\right]$$

If we simplify this by recalling that $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

Then, we will have:

$$L(x;a,b,\alpha,\beta) = n\log\alpha + \log\beta - \sum_{i=1}^{n}\tan x_i - n\log\Gamma a - n\log\Gamma b + n\log(\Gamma a + b) + \sum_{i=1}^{n}\log\left(\sec^2 x_i\right) - (\alpha a + 1)\sum_{i=1}^{n}\log\left(1+\beta e^{-\tan x_i}\right) + (b-1)\sum_{i=1}^{n}\log\left[1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\right] \quad (3.40)$$

Differentiating (3.40) partially with respect to a, b, α and β, and equating to zero, yields equations (3.41), (3.42), (3.43) and (3.44) respectively:

$$\frac{\partial l(x;a,b,\alpha,\beta)}{\partial a} = \frac{n}{\Gamma a}\Gamma(a') + \frac{n}{\Gamma a + b}\Gamma(a+b)' - \alpha\sum_{i=1}^{n}\log\left(1+\beta e^{-\tan x_i}\right) = 0$$

$$n\psi(a) + n\psi(a+b) - \alpha\sum_{i=1}^{n}\log\left(1+\beta e^{-\tan x_i}\right) = 0 \quad (3.41)$$

$$\frac{\partial l(x;a,b,\alpha,\beta)}{\partial b} = -n\psi(b) + n\psi(a+b) + \sum_{i=1}^{n}\log\left[1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\right] = 0$$

$$-n\psi(b) + n\psi(a+b) + \sum_{i=1}^{n}\log\left[1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\right] = 0 \quad (3.42)$$

$$\frac{\partial l(x;a,b,\alpha,\beta)}{\partial \alpha} = \frac{n}{\alpha} - a\sum_{i=1}^{n}\log\left(1+\beta e^{-\tan x_i}\right) + (b-1)\sum_{i=1}^{n}\frac{\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\log\left(1+\beta e^{-\tan x_i}\right)}{1-\left(1+\beta e^{-\tan x_i}\right)} = 0$$

$$\frac{n}{\alpha} - a\sum_{i=1}^{n}\log\left(1+\beta e^{-\tan x_i}\right) + (b-1)\sum_{i=1}^{n}\frac{\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\log\left(1+\beta e^{-\tan x_i}\right)}{1-\left(1+\beta e^{-\tan x_i}\right)} = 0 \quad (3.43)$$

$$\frac{\partial l(x;a,b,\alpha,\beta)}{\partial \beta} = \frac{n}{\beta} - (\alpha a - 1)\sum_{i=1}^{n}\frac{e^{-\tan x_i}}{1+\beta e^{-\tan x_i}} + (b-1)\sum_{i=1}^{n}\frac{\alpha e^{-\tan x_i}\left(1+\beta e^{-\tan x_i}\right)^{-\alpha-1}}{1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}} = 0$$

$$\frac{n}{\beta}-(\alpha a-1)\sum_{i=1}^{n}\frac{e^{-\tan x_i}}{1+\beta e^{-\tan x_i}}+(b-1)\sum_{i=1}^{n}\frac{\alpha e^{-\tan x_i}\left(1+\beta e^{-\tan x_i}\right)^{-\alpha-1}}{1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}}=0 \qquad (3.44)$$

Where $\psi(.)$ is the digamma function, which is the derivative of $\log\Gamma(.)$ in equations (3.41) and (3.42)

The maximum likelihood estimates (MLEs) obtained in Equations 3.41–3.44 cannot be solved algebraically. We can make use of software to get the MLE's numerically through maximization using the following algorithms such as: Nelder-Mead (NM), Newton-Raphson (NR), Simulated Annealing (SANN), and Berndt-Hall- Hall-Hausmann (BHHH) can be used in obtaining the MLEs of $\hat{\alpha},\hat{\beta},\hat{a}$ *and* $\hat{b}$. However, in this work we have used Nelder-Mead (NM)to obtain the estimates.

3.5.2 Fisher's Information Matrix

If X is a random variable with probability function $f(x:\theta)$ where $\theta=(\theta_1,\theta_2,...,\theta_n)$ is an unknown parameter vector, then, the fisher information $I(\theta)$ is an $n\times n$ matrix. The fisher (fisher, 1921) information matrix is needed in interval estimation and hypothesis testing. For a four parameters model, the information matrix is given by:

$$I(\theta)=I(\alpha,\beta,a,b)=\begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}$$

Where $I_{ii}=-\frac{1}{n}E\frac{\left(\partial^2\ln f(x:\theta_1,\theta_2,\theta_3,\theta_4)\right)}{\partial\theta_i\partial\theta_j}$

For *i=1,2,3,4* and *j=1,2,3,4*

Let $X_1, X_2, ..., X_n$ be a random sample from the BBV distribution, then the elements of the fisher Information Matrix can be obtained by re-differentiating (3.41), (3.42), (3.43), and (3.44) with respect to the parameters *a, b, α and β* respectively, these yields equations (3.45) to (3.54) as follows:

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial a^2} = n\left[\psi'(a)+\psi'(a+b)\right] \tag{3.45}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial b^2} = n\left(\psi'(a+b)-\psi'(b)\right) \tag{3.46}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial \alpha^2} = -\frac{n}{\alpha^2}-(b-1)\sum_{i=1}^{n}\frac{\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\left(\ln\left(1+\beta e^{-\tan x_i}\right)\right)^2}{1-\left(1+\beta e^{-\tan x_i}\right)} \tag{3.47}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial \beta^2} = -\frac{n}{\beta^2}+(\alpha a-1)\sum_{i=1}^{n}\frac{e^{-2\tan x_i}}{\left(1+\beta e^{-\tan x_i}\right)^2}-\alpha(b-1)$$
$$\sum_{i=1}^{n}e^{-2\tan x_i}\left[\frac{(\alpha+1)\left(1+\beta e^{-\tan x_i}\right)^{-\alpha-2}}{1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}}+\frac{\alpha\left(1+\beta e^{-\tan x_i}\right)^{-2(\alpha+1)}}{\left[1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\right]^2}\right] \tag{3.48}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial a\partial b} = n\psi'(a+b) \tag{3.49}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial a\partial \alpha} = -\sum_{i=1}^{n}\ln\left(1+\beta e^{-\tan x_i}\right) \tag{3.50}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial a\partial \beta} = -\alpha\sum_{i=1}^{n}\frac{e^{-\tan x_i}}{1+\beta e^{-\tan x_i}} \tag{3.51}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial b\partial\alpha}=\sum_{i=1}^{n}\frac{\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\ln\left(1+\beta e^{-\tan x_i}\right)}{1-\left(1+\beta e^{-\tan x_i}\right)} \tag{3.52}$$

$$\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial b\partial\beta}=\alpha\sum_{i=1}^{n}\frac{e^{-\tan x_i}\left(1+\beta e^{-\tan x_i}\right)^{-\alpha-1}}{1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}} \tag{3.53}$$

$$\begin{aligned}\frac{\partial^2 L(x:a,b,\alpha,\beta)}{\partial\alpha\partial\beta}=-a\sum_{i=1}^{n}\frac{e^{-\tan x_i}}{1+\beta e^{-\tan x_i}}+(b-1)\sum_{i=1}^{n}\left[\frac{e^{-\tan x_i}\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}}{1+\left(1+\beta e^{-\tan x_i}\right)\left[1-\left(1+\beta e^{-\tan x_i}\right)^{\alpha}\right]}\right]-\\ \frac{\alpha\left(1+\beta e^{-\tan x_i}\right)^{-\alpha-1}e^{-\tan x_i}\ln\left(1+\beta e^{-\tan x_i}\right)}{1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}}+\\ \frac{\alpha e^{-\tan x_i}\left(1+\beta e^{-\tan x_i}\right)^{-2\alpha-1}\ln\left(1+\beta e^{-\tan x_i}\right)}{\left(1-\left(1+\beta e^{-\tan x_i}\right)^{-\alpha}\right)^{-2}}\end{aligned} \tag{3.54}$$

Where $\psi'(.)$ is the tri-gamma functions which is the derivative of $\psi(.)$.

Equations 3.45 – 3.54 above are therefore the elements of the observed information matrix for the parameters (a,b,α,β).

The Fisher's Information Matrix is very useful tool for calculating the interval estimates, asymptotic variances, covariance and tests of hypothesis. In a nutshell, Fisher Information can be used to approximate the sampling distribution of the MLE when *n* is large enough, knowledge of the Fisher Information can be used to choose *n* such that the MLE produces an estimate close to the true value, construct a null hypothesis test, and compute confidence intervals.

3.6 Order Statistics

The order statistics of a random sample $X_1,...,X_n$ are the sample values placed in ascending order. They are denoted by $X_{(1)},...,X_{(n)}$. The order statistics are random variables that satisfy

$X_{(1)} \le X_{(2)} \le ... \le X_{(n)}$. The following are some statistics that are easily defined in terms of the order statistics.

Let $X_1, X_{2,}..., X_n$ be independent random sample from a probability distribution function $f(x)$ with an associated cumulative density function $F(x)$. The pdf of the i^{th} order statistics $x_{(i)}$ is defined by:

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f(x)\left(F(x)\right)^{i-1}\left[1-F(x)\right]^{n-i} \tag{3.55}$$

Recall that the binomial series expansion of

$$(1-x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + ...x^n$$

$$= \sum_{k=0}^{n} {}^nC_k x^k \tag{3.56}$$

Using this expansion, the term in equation (3.33) $\left[1-F(x)\right]^{n-i}$ can be written as

$$\left[1-F(x)\right]^{n-i} = \sum_{k=0}^{n-i} {}^{n-i}C_k \left[F(x)\right]^k$$

Therefore,

$$f_{i:n}(x) = \sum_{k=0}^{n-i} {}^{n-i}C_k f(x)\left[F(x)\right]^{k+i-1} \tag{3.57}$$

Hence, the i^{th} order statistic for the BBV distribution is given by:

$$f_{i:n}(x)=\sum_{k=0}^{n-i}{}^{n-i}C_k\,\frac{\alpha\beta e^{-\tan x}\sec^2 x}{B(a,b)}\left[1+\beta e^{-\tan x}\right]^{-\alpha a-1}$$

$$\left[1-\left(1+\beta e^{-\tan x}\right)^{-\alpha}\right]^{b-1}\left[\frac{B\left[\left\{1+\beta e^{-\tan x}\right\}^{-\alpha},a,b\right]}{B(a,b)}\right]^{k-i+1} \tag{3.57}$$

CHAPTER FOUR: RESULTS AND DISCUSSION

In this chapter, a real data set will be used to show the superiority and potentiality of the new distribution as compared to other Beta-G distributions. We first consider the widely used data from Lemonte and Cordeiro (2011).

A survival data set is applied to demonstrate that the proposed distribution is flexible and better to fit lifetime data. The data used represent the remission times (in months) of a random sample of 128 bladder cancer patients. The data below will be used to fit our proposed distribution (BBV) and some other distributions.

0.08, 2.09, 3.48, 4.87, 6.94 , 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46 , 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69

We also fit a five-parameter beta-Dagum (BDa) distribution introduced by Condino and Domma (2013) and three- parameter beta-Power Exponential (BPE) distribution introduced by Adepoju and Chukwu (2014) to make comparison with the BBV distribution. The BDa density function is given by equation (4.1)

$$BDa: \quad f_{BDa}(a,b,\alpha,\beta,\lambda,) = \sum_{j=0}^{\infty} \frac{\Gamma(a+b)(-1)^j}{\Gamma(a)\Gamma(b-j)!(a+j)} \left(1+\lambda x^{\alpha}\right)^{-\beta(a+j)} \tag{4.1}$$

$$BPE: \quad f_{BPE}(a,b,\lambda) = \frac{\lambda}{\beta(a,b+1)} \left(1-e^{-\lambda x}\right)^{a-1} \left(e^{-\lambda x}\right)^{b+1} \tag{4.2}$$

where $x > 0, \quad a,b,\alpha,\beta,\lambda > 0$

The goodness-of-fit test is the common and best criteria used to ascertain which of the model is best fit to the dataset used; it has been used by several authors in the literature. Below are the expressions of the criteria employed to choose the best model amongst the various distributions that are been considered. The criteria used are Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Consistent Akaike Information Criterion (CAIC).

The most popular information criterion is Akaike's (1973) information criterion (AIC) which compensates for the number of parameters in the model to encourage parsimony:

$$AIC = 2p - 2\ln(L)$$

Where:

p is the number of parameters in the model.

ln (L) is the log-likelihood function for the statistical model (deviance statistic for a given model).

When comparing two competing models, the model with the lowest AIC value would be selected as the model demonstrating better fit than the model it is compared with.

Consequently, additional information criteria, which have extended the AIC to give room for both model complexity and sample size is given below.

$$BIC = p\ln n - 2\ln(L)$$

Where:

p is the number of parameters in the model.

ln (L) is the log-likelihood function for the statistical model

n is the sample size.

$$CAIC = p\left[(\ln n)+1\right] - 2\ln(L)$$

Where:

p is the number of parameters in the model.

ln (L) is the log-likelihood function for the statistical model.

n is the sample size.

Given any two or more estimated models, the model with the lower value of AIC, BIC and CAIC is preferred; a lower AIC, BIC and CAIC implies either fewer explanatory variables, better fit, or both.

In order to compare the models above with the proposed BBV, we shall apply formal goodness-of -fit test to verify which distribution is the best. We considered the MLEs of the parameters (the standard errors which will be given in parentheses); we will also consider criteria like log likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian information criterion (BIC) for the data set. The better distribution is the one that corresponds to smaller LL, AIC, CAIC and BIC values of these criteria.

Below lists the MLEs, their standard errors (SE) in parentheses for the bladder cancer patients' data in table 4.1. The table indicates that the BBV distribution has the lowest values for the AIC, BIC and CAIC statistics among the fitted models for this data, and therefore it could be chosen as the best model. Moreover, the standard errors are much smaller compared with their estimates for the BBV distribution. Additionally, it is evident that the worst fit model was BDa distribution according to the data. The required numerical evaluations are implemented using the R software. The plot for beta Burr V distribution in Fig. 4.1 shows a bathtub-shaped hazard rate function and, therefore, indicates the appropriateness of the BBV distribution to fit these data, it also indicate that the distribution has at least a mode.

Table 4. 1: MLEs of the model parameters for the bladder cancer patients data, the corresponding SE (given in parentheses) and the measures AIC, BIC and CAIC

Model	MLE	Loglikelihood	AIC	BIC	CAIC
Beta-Power Exponential	a = 1.174473 (0.131994) b = 5.974162 (1.347064) α = 0.017775 (0.004285)	-413.3579	832.7158	841.2719	832.9093
Beta-Dagum	a = 15.83986 (1.26161) b = 19.14655 (2.74046) α = 0.56186 (0.08727) β = 6.67282 (1.82561) λ = 0.42873 (0.03835)	-412.4005	834.801	849.0612	835.2928
Beta-Burr V	a = 6.7091 (2.2773) b = 10.1948 (2.8099) α = 10.3665 (1.4322) β = 0.1126 (0.0289)	-25.98339	59.96678	71.3749	60.29198
Burr V	α = 1.029e+03 (8.424e+00) β = 1.131e-03 (1.006e-04)	-124.3364	252.6728	258.3769	252.7688

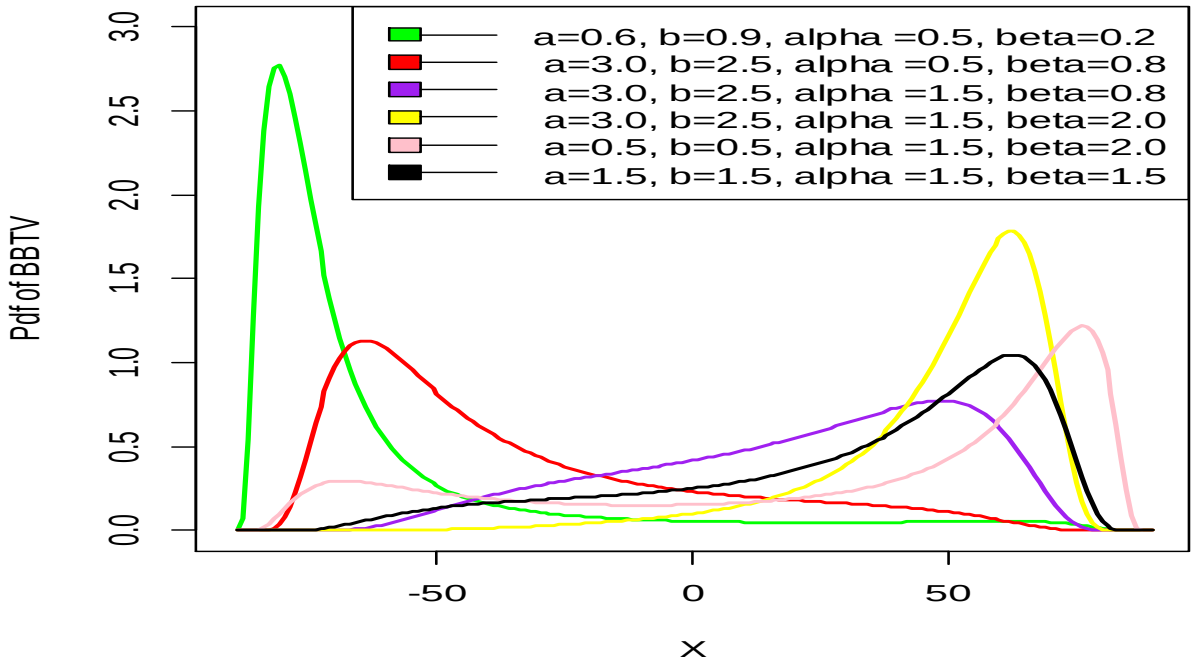


Figure 4. 1: Graph of Bata-Burr type V distribution for various parameter values

Data set two

The second dataset we are looking at to test the practicability of BBV distribution is a data set on lifetimes of 50 components, The data set is used by Aarset (1987) and its given below: (0.10,0.20,1,1,1,1,1,2,3,6,7,11,12,18,18,18,18,18,21,32,36,40,45,46,47,50,55,60,63,63,67,67,67, 67,72,75,79,82,82,83, 84,84,84,85,85,85,85,85,86,86)

The MLEs with the standard deviations of the parameters, -2log(L), AIC, BIC and CAIC are listed in Table 4.2. From these values, we note that the BBV model is still better than the BD, BP and Burr V distributions in terms of fitting to the data set as it has smaller AIC, BIC and CAIC hence, it was found to be better.

Table 4.2: MLEs of the model parameters for the Aarset data, the corresponding SE (given in parentheses) and the measures AIC, BIC and CAIC

Model	MLE	loglikelihood	AIC	BIC	CAIC
Beta-Power Exponential	a = 0.798190 (0.137661) b = 5.938166 (2.662827) α = 0.002555 (0.001155)	-240.186	486.372	482.4513	485.4513
Beta-Dagum	a = 1.2365 (0.4975) b = 0.5588 (0.1624) α = 1.2673 (0.5887) β = 2.5521 (1.2163) λ = 0.7993 (0.1318)	-262.1331	534.2662	532.3132	537.3132
Beta-Burr V	a = 0.17528 (0.05161) b = 0.24662 (0.04868) α = 4.41204 (1.29482) β = 4.35017 (0.09559)	-44.46686	96.93372	94.47892	98.47892
Burr V	α =44.53248 (17.18821) β = 0.05574 (0.02333)	-92.5929	189.1858	186.572	188.572

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The most important goal of this dissertation as stated in the objectives was to propose a continuous distribution which we called the beta-Burr type V distribution; it generalizes the Burr V by employing the logit of beta distribution introduced by Jones (2004). The CDF, PDF, hazard function and limit of PDF were derived. Subsequently, some of the mathematical and statistical properties were also provided. The model parameters were estimated by using maximum likelihood estimation. R statistical software was used in the execution of our results.

Finally, we subjected the distribution to real-life data set to assess the performance of the model by comparing it to three other distributions. We considered criteria like log likelihood (LL), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC) and Bayesian information criterion (BIC) in determining the best model.

5.2 Conclusion

In this work, we studied the beta-Burr type V distribution as a generalization of the Burr V distribution. We provided a better foundation for some mathematical properties for this distribution, including the derivation of the hazard rate function and order statistics. The model parameters were estimated by maximum likelihood and the observed information matrix was derived. An application of the BBV distribution to a real data set indicates that this distribution outperforms both the Burr V and other generalized distributions. We hope that the proposed extended model may attract wider applications in statistics.

5.3 Recommendation

Based on the model constructed, we recommend that;

1. When it is required to use a distribution that will fits better with both shape and location parameters and proved to be parsimonious, the BBV will be good alternative.

2. The proposed distribution can be used to model a continuous data set.

5.4 Contribution to Knowledge

In this research, we were able to contribute to the body of knowledge in so many ways, which include the following:

i. We proposed some formulae for the extended Burr V after conjoining it with the beta distribution to build beta-Burr type V.
ii. We gave the estimates of the parameters of the proposed model in this research.
iii. We proved that the estimators of the BBV are more flexible and fit in better than other distributions.

5.5 Areas of Further Research

i. There are other forms of Burr family of distribution yet to be explored; these include type II, IV VI, VII, VIII, IX and XI.
ii. Further research can be undertaken to determine the moments of the proposed distribution.

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