STABILITY CONDITIONS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS BY LYAPUNOV'S DIRECT METHOD

BY

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DEDICATION

Dedicated to the memory of my Father and Mother, both of whom saw me begin this work but could not live long enough to see me finish it.

DECLARATION

I, Ibrahim Z. B. Tizhe, do hereby declare that this thesis has been written by me and it is a record of my own research work. It has not been presented in any previous application for a higher degree. All information derived from other sources have been acknowledged in both the text and references.

.....
Name and Signature

Nov. 1987
.....
Date

CERTIFICATION

This thesis entitled STABILITY CONDITIONS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS BY LYAPUNOV'S DIRECT METHOD by IBRAHIM Z.B. TIZHE meets the requirements of the regulations governing the award of the degree of MASTER OF SCIENCE IN MATHEMATICS of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation .

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ABSTRACT

We consider the general second order linear differential equation of the form

$$(p(x)y'(x))' + q(x)y(x) = 0 \quad a \leq x < \infty$$

where $p>0$ and locally absolutely continuous on $[a,\infty)$ and q is measurable and locally Lebesgue integrable on $[a,\infty)$.

Our aim is to use Lyapunov's direct method to study the stability properties associated with the equation above.

Knowles, by means of certain Lyapunov functions studied these properties by placing conditions on p and q which ensure that (i) all solutions of the equation are bounded or (ii) all solutions tend to zero at infinity or (iii) all solutions lie in $L^r[a,\infty)$ for some r, $0<r<\infty$.

Conditions opposite to (i), (ii) and (iii) are also studied but not in details.

We applied some of the main results to solve some problems.

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CHAPTER I

PRELIMINARIES

1.1 Introduction

In this work, we shall consider the general second order linear differential equation of the form

$$(p(x)y'(x))' + q(x)y(x) = 0 \qquad a \le x < \infty \qquad (1.1)$$

where $p>0$ and locally absolutely continuous on $[a,\infty)$, and q is measurable and locally Lebesgue integrable on $[a,\infty)$.

We intend to study the various stability properties associated with solutions of (1.1).

The Russian Mathematician, A.M. Lyapunov (1857-1918) was one of those who opened the way to the study of stability theory when, in 1892, he published his work on the stability of motions in Russian with the French translation appearing later in 1907. He treated stability by two methods. His first method involves the knowledge of an explicit solution of the equation while the second method which came to be known as the 'direct method' does not require an explicit solution.

It is a well known fact that it is not always easy to get explicit solutions to most of differential equations. This, therefore, accounts for why the second method of Lyapunov became a very useful tool for the study of stability problems associated with solutions of differential equations. Thus, all we need is to assure ourselves that a solution exists and from that, we can begin to make conclusions on the stability properties under certain conditions.

Basically and as pointed out by Boyce and Di Prima in [4], Lyapunov's second method is a generalisation of the physical principles that for a conservative system, (i) a rest position is stable if and only if the potential energy is a local minimum and (ii) the total energy is a constant during any motion.

In this work, we shall make use of the idea of the total energy of a conservative system to define functions known as energy or Lyapunov functions to estimate the solutions of (1.1). These functions are quite difficult to construct. For the purpose of this study, however, we shall make use of the Lyapunov function given by Knowles in [18] which is defined by

$$E(x) = \{\Theta(x)p(x)y'(x) + \gamma(x)y(x)\}^2 + \Phi^2(x)y(x) \qquad (1.2)$$

where $\Theta>0$, $\Phi>0$, and γ are arbitrary real-valued locally absolutely continuous functions defined on $[a,\infty)$ for any solution y of (1.1).

From (1.2) we obtain

$$E(a)\exp\left[\tfrac{1}{2}\int_a^x (A+C)\right]\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \le E(x) \le E(a)\exp\left[\tfrac{1}{2}\int_a^x (A+C)\right]\exp\left(\int_a^x F^{\frac{1}{2}}\right) \qquad \ldots.(1.3)$$

for upper and lower bounds for the solution of (1.1) where A,B,C and F are expressed in terms of Θ,Φ,γ, p and q.

From the values of A and C, (1.3) becomes

$$K\Theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \le E(x) < K\Theta(x)\Phi(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right) \qquad (1.4)$$

where $K = E(a)\{\Theta(a)\Phi(a)\}^{-1}$.

In this regard, a lot of attention has been given to the problem of finding sufficient conditions on p and q such that if y is a solution of (1.1), at least one of the following situat .. hold :

(i) y is bounded on $[a,\infty)$

(ii) $y(x) \to 0$ as $x \to \infty$

(iii) For some r, $0<r<\infty$, all solutions of (1.1) lie in $L^r[a,\infty)$.

In particular, if we set $p = 1$, (1.1) becomes

$$y''(x) + q(x)y(x) = 0 \qquad (1.5)$$

and the problem will be that of finding sufficient conditions on q such that if y is a solution of (1.5), at least one of (i) - (iii) will hold.

Conditions for (i) and (ii) have been given in Bellman [1,2,3], Cesari [6], Chiou[7], Fedosov and Lidskii[12], Hinton [14], Knowles [16,17,18], Lazer[20], Meir, et.al.[21,22] and Wintner [27] .

To study (iii), Hinton in [14] and Knowles in [18] derived both upper and lower bounds for the function

$$z(x) = \{u^2(x) + v^2(x)\}^{\frac{1}{2}} \qquad (1.6)$$

where u and v are two linearly independent solutions of (1.1). In particular, it has been shown by Knowles in [18] that by estimating

$$H(x) = \{\Theta(x)p(x)z'(x) + \gamma(x)z(x)\}^2 + \Phi^2(x)z^2(x)+p^2(x)\Theta^2(x)z^{-2}(x) \qquad .....(1.7)$$

where Θ,Φ,γ, p and q are as defined earlier, one can obtain upper and lower bounds for z which can be used to derive conditions for (iii). Conditions for which $r=2$ have been treated in Knowles [16,17].

Next, we shall seek conditions on p and q which guarantee that conditions opposite to (i),(ii) and (iii) hold. That is, if y is any solution of (1.1) ,

(iv) y is not bounded on $[a,\infty)$

(v) y does not tend to zero at infinity

(vi) For some r, $o<r<\infty$, no non-trivial solution of (1,1) lies in $L^r[a,\infty)$.

Conditions for (iv) and (v) have been given in Knowles [18].; To derive conditions for (v), in particular, the result obtained by Dolan and Klaasen in [9] shall be used.

We shall not go into details on conditions for (vi), but we shall make some mention of the special case where r=2 for which results have been obtained in Knowles [16,17].

1.2 Notations and Definitions.

Definition 1.2.1

By a differential equation, we shall mean the ordinary differential equation, and throughout this work, the convention that a dash (') denotes differentiation shall be used when the meaning is not clear.

Definition 1.2.2

Given the second order differential equation (1.1), by a solution of the equation in the interval $[a,\infty)$, we

mean a continuous function Ψ such that Ψ' and Ψ'' exist and satisfy

$$(p(x)\Psi'(x))' + q(x)\Psi(x) = 0 \qquad (1.8)$$

for every $x \in [a,\infty)$.

Definition 1.2.3

The differential equation (1.1) is said to be linear if the left hand side is expressed as a linear function of y, y' and y''.

Definition 1.2.4.

Let (X,β) be a real measurable space. Then the extended real-valued function f defined on X is said to be measurable (with respect to the σ-algebra β) if the set

$$\{ x: f(x) = \alpha \} \in \beta$$

for each constant α .

Definition 1.2.5

A function f is said to be Lebesgue integrable on $[a,\infty)$ if

$$\int_a^\infty f(x)dx < \infty .$$

Suppose Φ and Ψ are scalar functions of the variable x and the scalar parameter ε . The variable x ranges over some domain D and ε belongs to some interval I (which is $[a,\infty)$ in our case) then we shall define the order relations involving the order symbols O (large O) and o

(small o) as follows :-

Definition 1.2.6: Large O

Let x be fixed. We say that $\Phi=O(\Psi)$ in $[a,\infty)$ if there exists a function $K(x)$ such that $|\Phi| \leq K(x)|\Psi|$ for all ε in $[a,\infty)$. In other words, Φ/Ψ is bounded provided that $\Psi \neq 0$ in $[a,\infty)$.

Definition 1.2.7: small o

Again, with x fixed, we say $\Phi=o(\Psi)$ as $\varepsilon \to \varepsilon_o$, a limit point in $[a,\infty)$, if given any $\delta(x) > 0$, there exists a neighbourhood N_δ of ε_o such that $|\Phi| < \delta(x)|\Psi|$. In other words, $\Phi/\Psi \to 0$ if $\Psi \neq 0$ in $[a,\infty)$.

Definition 1.2.8

Equation (1.1) is said to be oscillatory if all its non-trivial solutions have an infinite number of zeros in $[a,\infty)$ and these nontrivial solutions are termed oscillatory solutions.

Definition 1.2.9

A real-valued function f defined on an interval I is said to be absolutely continuous on I if given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\sum_{i=1}^{n} | f(y_i) - f(x_i) | < \varepsilon$$

for every finite collection $\{ (x_i, y_i) \}$ of non-overlapping intervals with

$$\sum_{i=1}^{n} | y_i - x_i | < \delta$$

Definition 1.2.10

A measurable function f *defined on* $[a,\infty)$ *is said to be in* $L^r[a,\infty)$ *if*

$$\int_a^\infty |f|^r < \infty , \quad 0<r<\infty$$

Definition 1.2.11

A function $E(x)$ *is said to be positive definite if*

(a) $E(x)$ *is continuous together with its first partial derivatives in a certain open region* Ω *about the origin.*

(b) $E(o) = 0$

(c) $E(x) > 0$ *for* $x \neq 0$, $x \in \Omega$.

Definition 1.2.12.

E(x) is said to be a Lyapunov Function for the system

$$x' = X(x) \qquad X(o)=0 \tag{1.9}$$

if for some x *in the neighbourhood of the origin,* $E(x)$ *is positive definite and satisfies*

$$E'(x) = X.\text{grad } E(x) \leq 0$$

where gradient of $E(x)$ *is expressed as a row vector.*

1.3 Stability in the sense of Lyapunov

1.3.1 General

The concept of stability has to do with the following question ; suppose y *is a solution of (1.1) on* $[a,\infty)$. *Does a small change in* y *result in a small perturbation of* y *such that the resulting function is a*

new solution still valid on $[a,\infty)$ *and which remains as close as possible to* y *? If the answer is in the affirmative, then* y *is said to be stable. Otherwise, it is unstable.*

La Salle and Lefschetz in [19] *also gave a geometrical representation of the concept of stability of the solutions of the autonomous system*

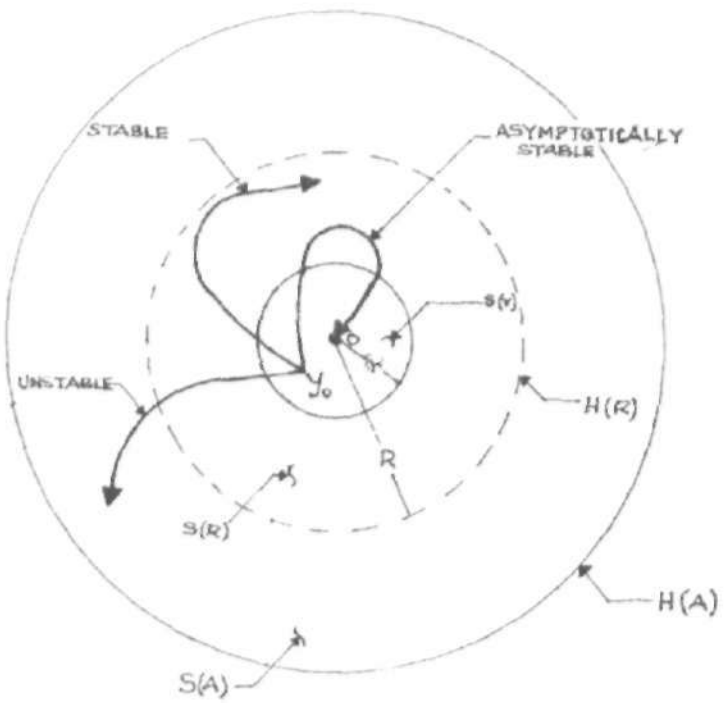
$$y' = Y(y) \ , \quad Y(o) = 0 \qquad (1.10)$$

by first making the following assumptions

$$S(R) = \{ y: \ ||y|| < R \}$$

$$H(R) = \{ y: \ ||y|| = R \}$$

$$S_r^R(R) = \{ y: r \leq ||y|| \leq R \}$$



Viewing from the diagram above, we shall say that the origin is :

(a) stable whenever for each $R<A$, there is an $r \leq R$ such that if a path g^+ initiates at point y_o of the spherical region $S(r)$, then it remains in the spherical region $S(R)$ ever after; i.e., a path starting in $S(r)$ never reaches the boundary sphere $H(R)$ of $S(R)$.

(b) asymptotically stable whenever it is stable and in addition every path g^+ starting inside some $S(R_o)$, $R_o>0$ tends to the origin as time increases infinitely.

(c) unstable whenever for some R and any r, no matter how small, there is always in the spherical region $S(r)$ a point y such that the path g^+ through y reaches the boundary sphere $H(R)$.

From the geometrical interpretation, it is clear why much attention has been given to the study of boundedness and asymptotic behaviour of solutions when considering stability properties of solutions of differential equations.

1.3.2 Lyapunov's Stability Theorems.

With respect to the Lyapunov function defined in 1.2.12, we now state without proof, Lyapunov's stability theorems associated with it.

Theorem 1.1

If there exists in some neighbourhood Ω of the origin a Lyapunov function $E(x)$, then the origin is said to be stable for the system (1.9).

Theorem 1.2.

If $E(x)$ is a Lyapunov function for (1.9) with the property that $-E'(x)$ is positive definite along the solution path of (1.9), then the origin is asymptotically stable.

Although these theorems are for the system (1.9) which is a system of first order differential equations, it is equally applicable to (1.1).

CHAPTER II

ESTIMATES OF SOLUTIONS

2.1 Bounds on Solutions

Let $\Theta>0$, $\Phi>0$ and γ be arbitrary real-valued locally absolutely continuous functions defined on $[a,\infty)$. For any real-valued solution y of (1.1), we define a Lyapunov function

$$E(x)=\{\Theta(x)p(x)y'(x) + \gamma(x)y(x)\}^2 + \Phi^2(x)y^2(x). \qquad (2.1)$$

We now derive estimates for the solutions of (1.1) by giving some bounds on $E(x)$.

Differentiating (2.1) with respect to x, we have

$$E'=2\{\Theta py'+\gamma y\}\{\Theta'py'+\Theta p'y'+\Theta py''+\gamma'y+\gamma y'\}+2\Phi y\{\Phi'y+\Phi y'\}$$

$$= 2\xi E^{\frac{1}{2}}(\Theta'py'-\Theta qy+\gamma'y+\gamma y')+2\eta E^{\frac{1}{2}}(\Phi'y+\Phi y') \qquad (2.2)$$

where $\xi = (\Theta py'+\gamma y)E^{-\frac{1}{2}}$, (2.3)

$$\eta = \Phi y E^{-\frac{1}{2}}, \qquad (2.4)$$

and $\xi^2 + \eta^2 = 1$. (2.5)

From (2.4), $y = \eta E^{\frac{1}{2}}/\Phi$ (2.6)

and from (2.3), $y' = (\xi E^{\frac{1}{2}}-\gamma y)/\Theta p$

$$= \left[\xi E^{\frac{1}{2}}-(\gamma\eta E^{\frac{1}{2}}/\Phi)\right]/\Theta p$$

$$\text{or} \quad y' = \frac{\xi E^{\frac{1}{2}}}{\Theta p} - \frac{\gamma\eta E^{\frac{1}{2}}}{\Phi\Theta p}. \qquad (2.7)$$

Putting the values of y and y' from (2.6) and (2.7) respectively into (2.2), we have

$$E'=2\xi E^{\frac{1}{2}}\left\{\Theta' p\left[\frac{\xi E^{\frac{1}{2}}}{\Theta p} - \frac{\gamma\eta E^{\frac{1}{2}}}{\Phi\Theta p}\right] - \frac{\Theta q\eta E^{\frac{1}{2}}}{\Phi} + \frac{\gamma'\eta E^{\frac{1}{2}}}{\Phi} + \gamma\left[\frac{\xi E^{\frac{1}{2}}}{\Theta p} - \frac{\gamma\eta E^{\frac{1}{2}}}{\Phi\Theta p}\right]\right\}$$

$$+ 2\eta E^{\frac{1}{2}} \left\{ \frac{\Phi' \eta E^{\frac{1}{2}}}{\Phi} + \Phi \left[\frac{\xi E^{\frac{1}{2}}}{\Theta p} - \frac{\gamma \eta E^{\frac{1}{2}}}{\Phi \Theta p} \right] \right\}$$

$$= 2 \left[\frac{\xi E \Theta'}{\Theta} - \frac{\Theta' \gamma \eta \xi E}{\Phi \Theta} - \frac{\Theta q \eta \xi E}{\Phi} + \frac{\gamma' \eta \xi E}{\Phi} + \frac{\gamma \xi^2 E}{\Theta p} - \frac{\gamma^2 \eta \xi E}{\Phi \Theta p} + \frac{\Phi' \eta^2 E}{\Theta p} - \frac{\gamma \eta^2 E}{\Theta p} \right]$$

$$= 2 \left[\frac{\Theta'}{\Theta} + \frac{\gamma}{\Theta p} \right] \xi^2 E + 2 \left[\frac{\Phi'}{\Phi} - \frac{\gamma}{\Theta p} \right] \eta^2 E + 2 \left[-\frac{q\Theta}{\Phi} - \frac{\Theta' \gamma}{\Theta \Phi} + \frac{\gamma'}{\Phi} - \frac{\gamma^2}{\Phi \Theta p} + \frac{\Phi}{\Theta p} \right] \eta \xi E$$

$$= 2 \left[\frac{\Theta'}{\Theta} + \frac{\gamma}{\Theta p} \right] \xi^2 E + 2 \frac{\Theta}{\Phi} \left[-q - \frac{\Theta' \gamma}{\Theta^2} + \frac{\gamma'}{\Theta} - \frac{\gamma^2}{\Theta^2 p} + \frac{\Phi^2}{\Theta^2 p} \right] \eta \xi E + 2 \left[\frac{\Phi'}{\Phi} - \frac{\gamma}{\Theta p} \right] \eta^2 E$$

$$= 2 \left[\frac{\Theta'}{\Theta} + \frac{\gamma}{\Theta p} \right] \xi^2 E + 2 \frac{\Theta}{\Phi} \left[-q + \left(\frac{\gamma}{\Theta} \right)' - \frac{1}{p} \left(\frac{\gamma}{\Theta} \right)^2 + \frac{1}{p} \left(\frac{\Phi}{\Theta} \right)^2 \right] \eta \xi E + 2 \left[\frac{\Phi'}{\Phi} - \frac{\gamma}{\Theta p} \right] \eta^2 E$$

$$\text{or} \quad \frac{E'}{E} = 2 \left[\frac{\Theta'}{\Theta} + \frac{\gamma}{\Theta p} \right] \xi^2 + 2 \frac{\Theta}{\Phi} \left[-q + \frac{1}{p} \left(\frac{\Phi}{\Theta} \right)^2 - \frac{1}{p} \left(\frac{\gamma}{\Theta} \right)^2 + \left(\frac{\gamma}{\Theta} \right)' \right] \eta \xi + 2 \left[\frac{\Phi'}{\Phi} - \frac{\gamma}{\Theta p} \right] \eta^2$$

$$= A\xi^2 + 2B\xi\eta + C\eta^2 .$$

$$\text{i.e.} \quad \frac{E'(x)}{E(x)} = A(x)\xi^2 + 2B(x)\xi\eta + C(x)\eta^2 . \qquad (2.8)$$

$$\text{where} \quad A = 2 \left[\frac{\Theta'}{\Theta} + \frac{\gamma}{\Theta p} \right] , \qquad (2.9)$$

$$B = \frac{\Theta}{\Phi} \left[-q + \frac{1}{p} \left(\frac{\Phi}{\Theta} \right)^2 - \frac{1}{p} \left(\frac{\gamma}{\Theta} \right)^2 + \left(\frac{\gamma}{\Theta} \right)' \right] , \qquad (2.10)$$

$$\text{and} \quad C = 2 \left[\frac{\Phi'}{\Phi} - \frac{\gamma}{\Theta p} \right] \qquad (2.11)$$

To find bounds on the quadratic expression on the right of (2.8), subject to (2.5), let

$$f(\xi,\eta) = A\xi^2 + 2B\xi\eta + C\eta^2$$

and we use the Lagrange multiplier rule.

To this effect, let us define the auxiliary function

$$F(\xi,\eta,\lambda) = A\xi^2 + 2B\xi\eta + C\eta^2 - \lambda(\xi^2+\eta^2-1) \qquad (2.12)$$

where λ is the Lagrange multiplier.

Let $F_i = \frac{\partial F}{\partial i}$, $i = \xi, \eta, \lambda$.

At maximum and minimum values, $F_i = 0$ for each i.

$$\text{Thus } F_\xi = 2A\xi + 2B\eta - 2\lambda\xi = 0, \qquad (2.13)$$

$$F_\eta = 2B\xi + 2C\eta - 2\lambda\eta = 0 \qquad (2.14)$$

$$\text{and } F_\lambda = \xi^2 + \eta^2 - 1 . \qquad (2.15)$$

From (2.13) and (2.14) ,

$$\frac{B}{A-\lambda} = \frac{C-\lambda}{B}$$

$$\text{or } \lambda^2 - (A+C)\lambda + AC - B^2 = 0$$

$$\text{Thus } \lambda_{\substack{M\\m}} = (A+C) \pm \{(A+C)^2 - 4(AC-B^2)\}^{\frac{1}{2}} / 2$$

$$= \tfrac{1}{2}(A+C) \pm \tfrac{1}{2}\{A^2+2AC+C^2-4AC+4B^2\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2}(A+C) \pm \tfrac{1}{2}\{A^2-2AC+C^2+4B^2\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2}(A+C) \pm \tfrac{1}{2}\{(A+C)^2+4B^2\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2}(A+C) \pm \{\tfrac{1}{4}(A+C)^2 + B^2\}^{\frac{1}{2}} \qquad (2.16)$$

and these are the eigenvalues of the matrix $\begin{pmatrix} A & B \\ B & C \end{pmatrix}$.

$$\text{Also } F_\xi = 0 = F_\eta \iff \begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

The right of (2.8) can be expressed as

$$A\xi^2 + 2B\xi\eta + C\eta^2 = (\xi,\eta)\begin{pmatrix} A & B \\ B & C \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = (\xi,\eta)\ \lambda \begin{pmatrix} \xi \\ \eta \end{pmatrix} = \lambda . \qquad (2.17)$$

To know which of λ_M and λ_m is the minimum and maximum , we let $\xi = \cos\tau$ and $\eta = \sin\tau$. (2.18)

Then $f(\xi,\eta) = A\cos^2\tau + 2B\cos\sin\tau + C\sin^2\tau$

$$= \tfrac{1}{2}A\{1+\cos 2\tau\} + B\sin 2\tau + \tfrac{1}{2}C(1-\cos 2\tau)$$

$$= \tfrac{1}{2}\{(A+C) + (A-C)\cos 2\tau + 2B\sin 2\tau\}.$$

Let us put $A - C = D\cos t$ (2.19)

and $2B = D\sin t$ (2.20)

where $D = \{(A-C)^2 + 4B^2\}^{\frac{1}{2}}$ (2.21)

Then $f(\xi,\eta) = \tfrac{1}{2}\{(A+C) + D\cos(2\tau - t)\}$

$$= \tfrac{1}{2}\{(A+C) + D\cos(2\tau - t)\}\ . \qquad (2.22)$$

Since $-1 \leq \cos(2\tau - t) \leq 1$,

$$\tfrac{1}{2}(A+C) - \tfrac{1}{2}D \leq \tfrac{1}{2}\{(A+C) + D\cos(2\tau - t)\} < \tfrac{1}{2}(A+C) + \tfrac{1}{2}D$$

or $\tfrac{1}{2}(A+C) - \tfrac{1}{2}\{(A-C)^2+4B^2\}^{\frac{1}{2}} \leq f(\xi,\eta) \leq \tfrac{1}{2}(A+C) + \tfrac{1}{2}\{(A-C)^2+4B^2\}^{\frac{1}{2}}$

or $\tfrac{1}{2}(A+C) - \{\tfrac{1}{4}(A-C)^2+B^2\}^{\frac{1}{2}} \leq f(\xi,\eta) \leq \tfrac{1}{2}(A+C) + \{\tfrac{1}{4}(A-C)^2+B^2\}^{\frac{1}{2}}$

or $\lambda_m \leq A\xi^2 + 2B\xi\eta + C\eta^2 \leq \lambda_M$.

Thus for each $x \geq a$,

$$\lambda_m(x) \leq A(x)\xi^2 + 2B(x)\xi\eta + C(x)\eta^2 \leq \lambda_M(x).$$

i.e. the quadratic expression on the right of (2. 8) is bounded below and above by $\lambda_m(x)$ *and* $\lambda_M(x)$ *respectively.*

Hence $\lambda_m(x) \leq E'(x)/E(x) \leq \lambda_M(x)$. (2.23)

Considering the left hand side of (2.23) and integrating over $[a,x]$, we get

$$\tfrac{1}{2}\int_a^x (A+C) - \int_a^x \left\{ \frac{1}{4}\left[2\left(\frac{\Theta'}{\Theta} - \frac{\Phi'}{\Phi} + \frac{2\gamma}{\Theta P}\right)\right]^2 + \frac{\Theta^2}{\Phi^2}\left[-q + \frac{1}{p}\left(\frac{\Phi}{\Theta}\right)^2 - \frac{1}{p}\left(\frac{\gamma}{\Theta}\right)^2 + \left(\frac{\gamma}{\Theta}\right)'\right]^2\right\}^{\frac{1}{2}} \leq \int_a^x \frac{E'(x)}{E(x)}$$

or $\tfrac{1}{2}\int_a^x (A+C) - \int_a^x \left\{ \frac{1}{4}\left[-2\left(\frac{\Phi'}{\Phi} - \frac{\Theta'}{\Theta} - \frac{2\gamma}{\Theta P}\right)\right]^2 + \frac{\Theta^2}{\Phi^2}\left[-q + \frac{1}{p}\left(\frac{\Phi}{\Theta}\right)^2 - \frac{1}{p}\left(\frac{\gamma}{\Theta}\right)^2 + \left(\frac{\gamma}{\Theta}\right)'\right]^2\right\}^{\frac{1}{2}} \leq \ln\left(\frac{E(x)}{E(a)}\right)$

or $\tfrac{1}{2}\int_a^x (A+C) - \int_a^x \left\{ \left[\frac{\Phi'}{\Phi} - \frac{\Theta'}{\Theta} - \frac{2\gamma}{\Theta P}\right]^2 + \frac{\Theta^2}{\Phi^2}\left[-q + \frac{1}{p}\left(\frac{\Phi}{\Theta}\right)^2 - \frac{1}{p}\left(\frac{\gamma}{\Theta}\right)^2 + \left(\frac{\gamma}{\Theta}\right)'\right]^2\right\}^{\frac{1}{2}} < \ln\left(\frac{E(x)}{E(a)}\right)$

$$i.e.\quad \tfrac{1}{2}\int_a^x (A+C) + \int_a^x \left(-F^{\frac{1}{2}}\right) \leq \ln\left(\frac{E(x)}{E(a)}\right) \qquad (2.24)$$

$$where\quad F = \left[\frac{\Phi'}{\Phi} - \frac{\Theta'}{\Theta} - \frac{2\gamma}{\Theta P}\right]^2 + \frac{\Theta^2}{\Phi^2}\left[-q+\frac{1}{p}\left(\frac{\Phi}{\Theta}\right)^2 - \frac{1}{p}\left(\frac{\gamma}{\Theta}\right)^2 + \left(\frac{\gamma}{\Theta}\right)'\right]^2 . \qquad (2.25)$$

Thus from (2.24), we have

$$\exp\left[\tfrac{1}{2}\int_a^x (A+C) + \int_a^x \left(-F^{\frac{1}{2}}\right)\right] \leq \frac{E(x)}{E(a)}$$

$$or\quad E(a)\exp\left[\tfrac{1}{2}\int_a^x (A+C) + \int_a^x \left(-F^{\frac{1}{2}}\right)\right] \leq E(x)$$

$$or\quad E(a)\exp\left[\tfrac{1}{2}\int_a^x (A+C)\right]\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) . \qquad (2.26)$$

Similarly, by considering the right hand side of (2.23) and integrating over $[a,x]$, *we have*

$$E(x) \leq E(a)\exp\left(\tfrac{1}{2}\int_a^x (A+C)\right)\exp\left(\int_a^x F^{\frac{1}{2}}\right). \qquad (2.27)$$

Combining (2.26) and (2.27), we have

$$E(a)\exp\left(\tfrac{1}{2}\int_a^x (A+C)\right)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) \leq E(a)\exp\left(\tfrac{1}{2}\int_a^x (A+C)\right)\exp\left(\int_a^x F^{\frac{1}{2}}\right) . \qquad (2.28)$$

From left hand side of (2.28),

$$E(a)\exp\left(\tfrac{1}{2}\int_a^x (A+C)\right)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x)$$

$$\text{or } E(a)\exp\left(\tfrac{1}{2}\int_a^x 2\left(\frac{\Theta}{\Theta}' + \frac{\Theta}{\Phi}'\right)\right)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) \text{ by (2.9) and (2.11)}$$

$$\text{or } E(a)\exp\left(\int_a^x \frac{\Theta}{\Theta}' + \int_a^x \frac{\Phi}{\Phi}'\right)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x)$$

$$\text{or } E(a)\exp\left(ln\left(\frac{\Theta(x)}{\Theta(a)}\right) + ln\left(\frac{\Phi(x)}{\Phi(a)}\right)\right)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x)$$

$$\text{or } E(a)\exp\left(ln\left[\frac{\Theta(x)\Phi(x)}{\Theta(a)\Phi(a)}\right]\right)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x)$$

$$\text{or } E(a)\left(\Theta(a)\Phi(a)\right)^{-1}\Theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x)$$

$$\text{or } K\,\Theta(x)\,\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) \qquad (2.29)$$

where $K = E(a)\,(\,\Theta(a)\,\Phi(a))^{-1}$.

Similarly, by considering the right hand side of (2.28), we have

$$E(x) \leq K\,\Theta(x)\,\Phi(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right) . \qquad (2.30)$$

Combining (2.29) and (2.30), we have,

$$K\Theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) \leq K\Theta(x)\Phi(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right) \qquad (2.31)$$

and this is an estimate for $E(x)$.

2.1.1. Examples.

As an application of (2.31), we shall consider the following examples :

1. Let $P=1$, $q>0$, $\Theta^2 = q^{-\frac{1}{2}}$, $\Phi^2=q^{\frac{1}{2}}$, $\gamma\Theta^{-1}=q'(4q)^{-1}$,

then

$$K\Theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) = E(a)\{\theta(a'\Phi(a))^{-1}\Theta(x)\Phi(x)\left(-\int_a^x F^{\frac{1}{2}}\right)$$

$$= E(a)\exp\left(-\int_a^x\left\{\left[\frac{\Phi'}{\Phi}-\frac{\Theta'}{\Theta}-\frac{2\gamma}{\Theta P}\right]+\frac{\Theta}{\Phi}\left[-q+\frac{1}{p}\left(\frac{\Phi}{\Theta}\right)^2-\frac{1}{p}\left(\frac{\gamma}{\Theta}\right)+\left(\frac{\gamma}{\Theta}\right)'\right]\right\}\right)$$

$$= E(a)\left(-\int_a^x\left\{\left[\frac{1}{4}q'q^{-1}+\frac{1}{4}q'q^{-1}-\tfrac{1}{2}q'q^{-1}\right]\right.\right.$$

$$\left.\left.+\,q^{-\frac{1}{2}}\left[-q+q-\left(\frac{1}{4}q'q^{-1}\right)^2+\frac{1}{4}(q''q^{-1}-(q')^2q^{-2})\right]\right\}\right)$$

$$= E(a)\exp\left(-\int_a^x q^{-\frac{1}{2}}\left[-\frac{1}{16}(q')^2(q^{-1})^2+\frac{1}{4}q''q^{-1}-\frac{1}{4}(q')^2q^{-2}\right]\right)$$

$$= E(a)\exp\left(-\int_a^x q^{-\frac{1}{2}}\left[\frac{1}{4}q''q^{-1}-\frac{5}{16}(q')^2q^{-2}\right]\right)$$

$$= E(a)\exp\left(-\int_a^x\frac{1}{4}\left[q''q^{-3/2}-\frac{5}{4}(q')^2q^{-5/2}\right]\right)$$

$$= E(a)\exp\left(-\int_a^x R\right)$$

$$\text{where}\quad R=\frac{1}{4}\left|q''q^{-3/2}-\frac{5}{4}(q')^2q^{-5/2}\right|. \qquad (2.32)$$

$$\text{i.e.}\quad K\Theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) = E(a)\exp\left(-\int_a^x R\right). \qquad (2.33)$$

Similarly, it can be shown that

$$K\Theta(x)\Phi(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right) = E(a)\exp\left(\int_a^x R\right). \qquad (2.34)$$

$$E = (\Theta py' + \gamma y)^2 + \Phi^2 y^2 \quad \text{by} \quad (2.1)$$

$$= (\Theta y' + \gamma y)^2 + \Phi^2 y^2 \quad \text{since} \quad p = 1$$

$$= (q^{-1/4} y' + q^{-1/4} q' (4q)^{-1})^2 + q^{\frac{1}{2}} y^2$$

$$i.e\ E = q^{-\frac{1}{2}} (y' + q' (4q)^{-1})^2 + q^{\frac{1}{2}} y^2 . \qquad (2.35)$$

Putting (2.33), (2,34) and (2,35) into (2,31), we have

$$E(a)\exp\left(-\int_a^x R\right) \leq q^{-\frac{1}{2}} (y' + q' (4q)^{-1})^2 + q^{\frac{1}{2}} y^2 \leq E(a)\exp\left(\int_a^x R\right) . \quad (2.36)$$

If $q' q^{-3/2}$ is bounded, then

$$K_1 \left[q^{-\frac{1}{2}} (y')^2 + q^{\frac{1}{2}} y^2\right] \leq q^{-\frac{1}{2}} (y' + q' (4q)^{-1})^2 + q^{\frac{1}{2}} y^2 \leq K_2 \left[q^{-\frac{1}{2}} (y')^2 + q^{\frac{1}{2}} y^2\right] \quad (2.37)$$

for some positive constants K_1 and K_2. To obtain (2.37), we have used the fact that $(a+b)^2 \leq 2 (a^2 + b^2)$.

If in addition, we know that $R \in L[a,\infty)$,

i.e. $\int_a^\infty R < \infty$,

then $\int_a^x R < \infty$ for $x \geq a$.

Let $E(a)\exp \int_a^\infty R = C_2 < \infty$ for some positive constant C_2.

Then $E(a)\exp \int_a^x R \leq E(a)\exp \int_a^\infty R = C_2 < \infty$.

Since $\int_a^x R < \infty$, then $-\int_a^x R > -\infty$.

Let $E(a)\exp\left(-\int_a^\infty R\right) = C_1 > 0$ for some constant C_1 .

Then $E(a)\exp\left(-\int_a^x R\right) \geq E(a)\exp\left(-\int_a^\infty R\right) = C_1 > 0.$

Thus from (2.36),

$$0<C_1 \leq E(a)\exp\left(-\int_a^x R\right) \leq q^{-\frac{1}{2}}(y'+q'(4q)^{-1})^2+q^{\frac{1}{2}}y^2 \leq E(a)\exp\left(\int_a^x R\right) \leq C_2<\infty$$

$$\text{or } 0<C_1 \leq \{q(x)\}^{\frac{1}{2}}\{y(x)\}^2 + \{q(x)\}^{-\frac{1}{2}}\{y'(x)\}^2 \leq C_2 \qquad (2.38)$$

for all $x \geq a$.

2. Assume $q > o$, and locally absolutely continuous on $[a, \infty)$. Let $\Phi=1$, $\Theta^2 = (pq)^{-1}$ and $\gamma = 0$.

Since $\Theta^2 = (pq)^{-1}$

$$\Theta'\Theta^{-1} = -\tfrac{1}{2}(pq)'(pq)^{-1}. \qquad (2.39)$$

Thus $A = 2(\Phi'\Theta^{-1} + \gamma\Theta^{-1}p^{-1})$

$= 2\Theta'\Theta^{-1}$ since $\gamma = 0$

$= 2\left[-\tfrac{1}{2}(pq)'(pq)^{-1}\right]$ by (2.29)

or $A = -(pq)'(pq)^{-1}$. (2.40)

Also $C = (\Phi'\Phi^{-1} - \gamma\Theta^{-1}p^{-1}) = 0$. (2.41)

Since $\Theta\Phi^{-1} = (pq)^{-\frac{1}{2}}$,

then $(\Phi\Theta^{-1})^2 = \left[(pq)^{\frac{1}{2}}\right]^2 = pq$.

Thus $B = (pq)^{-\frac{1}{2}}\left(-q + \frac{pq}{p} - 0 + 0\right)$

$= (pq)^{-\frac{1}{2}}(-q + q - 0 + 0)$

i.e. $B = 0$ (2.42)

Now let $A_+(x) = \max(0, A(x))$

and $-A_-(x) = \min(0, A(x))$.

If we set $g = (Pq)'(pq)^{-1}$,

then $g_- = (-g)_+$

and $g_+ = (-g)_-$

From (2.1),

$$E = (\Theta py' + \gamma y)^2 + \Phi^2 y^2$$

$$= (\Theta py')^2 + y^2 \quad \text{since } \gamma=0 \text{ and } \Phi=1$$

$$= ((pq)^{-\frac{1}{2}} py')^2 + y^2$$

$$= ((pq^{-1})^{\frac{1}{2}} y')^2 + y^2$$

i.e. $$E = (pq^{-1})(y')^2 + y^2 \qquad (2.43)$$

Also under these conditions (2.25) becomes

$$F = (-\Theta'\Theta^{-1})^2$$

or $$F^{\frac{1}{2}} = -\Theta'\Theta^{-1}$$

$$= \tfrac{1}{2}(pq)'(pq)^{-1} \quad \text{by (2.39)}$$

i.e. $$F^{\frac{1}{2}} = -\frac{A}{2} \quad \text{by (2.40)}$$

Using the left hand side of (2.28),

$$E(a)\exp\left(\int_a^x \frac{A}{2}\right) \exp\left(-\int_a^x \left(-\frac{A}{2}\right)\right) \leq E(x), \text{ since } C = 0.$$

$$\text{or } E(a)\exp\int_a^x A \leq E(x)$$

$$\text{or } E(a)\exp\int_a^x (-g_+) \leq E(x)$$

$$\text{or } E(a)\exp\left(-\int_a^x g_+\right) \leq E(x) . \qquad (2.44)$$

Similarly, by considering the right hand side of (2.28), we can show that

$$E(x) \leq E(a)\exp\left(\int_a^x g_-\right) \tag{2.45}$$

Hence $$E(a)\exp\left(\int_a^x g_+\right) \leq E(x) \leq E(a)\exp\left(\int_a^x g_-\right) \tag{2.46}$$

2.2 Joint Bounds on Independent Solutions

Let u and v denote two linearly independent real-valued solutions of (1.1) on $[a,\infty)$ with

$$P(uv' - u'v) = 1. \tag{2.47}$$

Defining

$$Z(x) = \left[u^2(x) + v^2(x)\right]^{\frac{1}{2}}, \tag{2.48}$$

we shall now find estimates for Z adopting the method of section 2.1.

Defining

$$H(x) = \left[\Theta(x)p(x)z'(x) + \gamma(x)q(x)\right]^2 + \Phi^2(x)z^2(x) + \Theta^2(x)p^2(x)z^{-2}(x) \tag{2.49}$$

for all $x \geq a$, we have

$$\begin{aligned}
H' &= 2(\Theta pz' + \gamma z)(\Theta' pz' + \Theta p'z' + \Theta pz'' + \gamma' z + \gamma z') + 2\Phi z(\Phi' z + \Phi z') \\
&\quad + 2\Theta p(\Theta p)' z^{-2} - 2(\Theta p)^2 z^{-3} z' \\
&= 2(\Theta pz' + \gamma z)(\Theta' pz' + \Theta(p'z' + pz'') + \gamma' z + \gamma z') + 2\Phi'\Phi z^2 \\
&\quad + 2\Phi^2 z'z + 2\Theta p(\Theta p)' z^{-2} - 2(\Theta p)^2 z^{-3} z' \\
&= 2(\Theta pz' + \gamma z)(\Theta pz')(\Theta'\Theta^{-1} + \gamma(\Theta p)^{-1}) + 2(\Theta pz' + \gamma z)(\gamma z)(\Theta'\Theta^{-1} + \gamma(\Theta p)^{-1}) \\
&\quad - 2(\Theta pz' + \gamma z)(\gamma z)(\Theta'\Theta^{-1} + \gamma(\Theta p)^{-1}) + 2(\Theta pz' + \gamma z)(\gamma' z - \Theta qz)
\end{aligned}$$

$$+\ 2(\Theta pz'+\gamma z)(\Phi^2 z(\Theta p)^{-1}-2\gamma\phi^2(\Theta p)^{-1}-2\gamma\Theta^2(\Theta p)^{-1}z^2)$$

$$+\ 2\Phi'\Phi^{-1}(\Phi^2z^2)+2(\Theta p)'(\Theta p)^2z^{-2}+\gamma(p\Theta)^{-1}(p\Theta)^2z^{-2}$$

$$= 2(\Theta'\Theta^{-1}+\gamma(\Theta p)^{-1})(\Theta pz'+\gamma z)^2$$

$$+2(\gamma'\Phi^{-1}+\gamma\Theta'\Theta^{-1}\Phi^{-1}-\Theta q\Phi^{-1}+\Phi(\Theta p)^{-1}-\gamma^2\Phi^{-1}(\Theta p)^{-1})(\Theta pz'+\gamma z)(\Phi z)$$

$$+2(\Phi'\Phi^{-1}-\gamma(p\Theta)^{-1})\Phi^2z^2\ +2((p\Theta)'(p\Theta)^{-1}+\gamma(p\Theta)^{-1}(p\Theta)^2z^{-2}). \qquad (2.50)$$

or $H'H^{-1} = [2\ (\Theta'\Theta^{-1}+\gamma(\Theta p)^{-1})(\Theta pz'+\gamma z)^2$

$$+2(\gamma'\Phi^{-1}+\gamma\Theta'\Theta^{-1}\Phi^{-1}-\Theta q\Phi^{-1}+\Phi(\Theta p)^{-1}-\gamma^2\Phi^{-1}(\Theta p)^{-1})(\Theta pz'+\gamma z)(\Phi z)$$

$$+2(\Phi'\Phi^{-1}-\gamma(p\Theta)^{-1})\Phi^2z^2+2((p\Theta)'(p\Theta)^{-1}+\gamma(p\Theta)^{-1}(p\Theta)^2z^{-2})\}H^{-1}$$

Thus

$$H'H^{-1} = A\xi^2 + 2B\xi\eta + C\eta^2 + D\zeta^2 \qquad (2.51)$$

where

$$\xi = (\Theta pz' + \gamma z)H^{-\frac{1}{2}}\ , \qquad (2.52)$$

$$\eta = \Phi H^{-\frac{1}{2}}z\ , \qquad (2.53)$$

$$\zeta = p\Theta H^{-\frac{1}{2}}\ z^{-1} \qquad (2.54$$

$$D = 2((p\Theta)'(p\Theta)^{-1} + \gamma(p\Theta)^{-1})\ , \qquad (2.55)$$

and A,B,C are as given by (2.9), (2.10) and (2.11) respectively.

As in Section 2.1, we note here that for each $x \geq a$, the quadratic form on the right hand side of (2.51) is bounded above and below by eigenvalues of the matrix

$$\begin{pmatrix} A(X) & B(x) & O \\ B(x) & C(x) & O \\ O & O & D(x) \end{pmatrix}$$

i.e. the solution to

$$\det \begin{pmatrix} A(x)-\lambda & B(x) & 0 \\ B(x) & C(x)-\lambda & 0 \\ 0 & 0 & D(x)-\lambda \end{pmatrix} = 0$$

or $(A(x)-\lambda)(C(x)-\lambda)(D(x)-\lambda)-B^2(x)(D(x)-\lambda) = 0$

or $((A(x)-\lambda)(C(x)-\lambda)-B^2(x))(D(x)-\lambda) = 0.$

If $(A(x)-\lambda)(C(x)-\lambda)-B^2(x) = 0,$

then we get, as in (2.16),

$$\lambda_{\substack{M \\ m}}(x) = \tfrac{1}{2}\{A(x)+C(x)\} + \{\tfrac{1}{4}(A(x)-C(x))^2+B^2(x)\}^{\frac{1}{2}}$$

Also if $D(x)-\lambda = 0$

then $\lambda = D(x)$

∴ at each value of x, this matrix has eigenvalues $\lambda_M(x)$, $\lambda_m(x)$ *and* $D(x)$.

We will assume, in the sequel, that

$$D \leq \lambda_M. \tag{2.56}$$

Thus from (2.51)

$$H'H^{-1} = A\xi^2 + 2B\xi\eta + C\eta^2 + D\zeta^2$$

$$\leq \lambda_M$$

$$= \tfrac{1}{2}(A+C) + \{\tfrac{1}{4}(A-C)^2 + B^2\}^{\frac{1}{2}}$$

$$= \tfrac{1}{2}(A+C) + F^{\frac{1}{2}} \quad \text{by (2.25)}$$

$$\text{or} \quad \int_a^x H'H^{-1} \leq \tfrac{1}{2}\int_a^x (A+C) + \int_a^x F^{\frac{1}{2}}$$

$$\text{or} \quad \ln\left(\frac{H(x)}{H(a)}\right) \leq \tfrac{1}{2}\int_a^x (A+C) + \int_a^x F^{\frac{1}{2}}$$

$$or \quad H(x)\,H^{-1}(a) \leq \exp\left(\tfrac{1}{2}\int_a^x (A+C)\right) \exp\left(\int_a^x F^{\frac{1}{2}}\right)$$

$$i.e.\ H(x) \leq H(a)\exp\left(\tfrac{1}{2}\int_a^x (A+C)\right) \exp\left(\int_a^x F^{\frac{1}{2}}\right) . \qquad (2.57)$$

$$Now \quad \tfrac{1}{2}(A+C) = (\Theta'\Theta^{-1} + \Phi'\Phi^{-1})$$

$$or \quad \tfrac{1}{2}\int_a^x (A+C) = \int_a^x (\Theta'\Theta^{-1}+\Phi'\Phi^{-1})$$

$$= \left[\ ln\ (\Theta\Phi)\ \right]_a^x$$

$$= ln\left(\frac{\Theta(x)\Phi(x)}{\Theta(a)\Phi(a)}\right)$$

$$or \quad \exp\left(\tfrac{1}{2}\int_a^x (A+C)\right) = \frac{\Theta(x)\Phi(x)}{\Theta(a)\Phi(a)} .$$

Thus (2.57) becomes

$$H(x) \leq H(a)\ (\Theta(a)\Phi(a))^{-1}\ \Theta(x)\Phi(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right)$$

$$= K\Theta(x)\Phi(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right)$$

$$i.e. \quad H(x) \leq K\Theta(x)\Phi(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right) \quad for \quad x \geq a \qquad (2.58)$$

where $K = H(a)\ (\Theta(a)\Phi(a))^{-1}$.

From (2.49),

$$H = (\Theta pz' + \gamma z)^2 + \Phi^2 z^2 + \Theta^2 p^2 z^{-2}$$

$$or\ H \geq \Phi^2 z^2 + \Phi^2 p^2 z^{-2} \qquad since \quad (\Theta pz' + \gamma z)^2 > 0 \qquad (2.59)$$

$$or \quad H \geq \Phi^2 z^2 \qquad since \quad \Theta^2 p^2 z^{-2} > 0.$$

$$Thus \quad K\Phi\Theta \exp\left(\int_a^x F^{\frac{1}{2}}\right) \geq H \geq \Phi^2 z^2 \qquad by \quad (2.58)$$

$$or \quad K\Phi\Theta \exp\left(\int_a^x F^{\frac{1}{2}}\right) \geq \Phi^2 z^2$$

$$or \quad K\Phi^{-1}\Theta \exp\left(\int_a^x F^{\frac{1}{2}}\right) \geq z^2 \quad since \ \Phi^2 > 0. \qquad (2.60)$$

Combining (2.58) and (2.59), we have

$$K\Theta\Phi \exp\left(\int_a^x F^{\frac{1}{2}}\right) \geq H \geq \Phi^2 z^2 + \Theta^2 p^2 z^{-2}$$

$$or \quad K\Theta\Phi \exp\left(\int_a^x F^{\frac{1}{2}}\right) \geq \Theta^2 p^2 z^{-2} \quad since \ \Phi^2 z^2 \geq 0$$

$$or \quad K^{-1}\Theta^{-1}\Phi^{-1} \exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq \Theta^{-2} p^{-2} z^2$$

$$or \quad K^{-1}\Phi^{-1}\Theta p^2 \exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq z^2 . \qquad (2.61)$$

Combining (2.60) and (2.61), we have

$$K^{-1} p^2 \Theta\Phi^{-1} exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq z^2 \leq K\Theta\Phi^{-1} exp\left(\int_a^x F^{\frac{1}{2}}\right)$$

$$or \quad K^{-1} p^2(x)\Theta(x)\{\Theta(x)\}^{-1} exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq z^2(x) \leq K\Theta(x)\{\Phi(x)\}^{-1}$$

$$exp\left(\int_a^x F^{\frac{1}{2}}\right) . \qquad .... \ (2.62)$$

Thus u and v are jointly bounded, since $z^2 = u^2 + v^2$.

CHAPTER III

STABILITY RESULTS I

3.1 Introduction.

In this chapter, we shall use the results in chapter II to investigate stability problems of the following types :-

(i) All solutions of (1.1) are bounded.

(ii) All solutions of (1.1) tend to zero at ∞.

(iii) For some r, $0<r<\infty$, all solutions of (1.1) lie in $L^r[a,\infty)$.

We shall assume $p=1$ throughout this chapter.

3.2 Boundedness conditions for solutions:

In this section, we give conditions for which (i) holds. Wintner [27] gave the following theorem.

Theorem 3.2.1.

If there exists constants a and $w \neq 0$ such that either

$$\int_a^\infty |q - w^2| < \infty \qquad (3.1)$$

or

$$\int_a^\infty |q'| < \infty \text{ and } q(x) \to w^2 \text{ as } x \to \infty , \qquad (3.2)$$

then both y and y' are bounded on $[a,\infty)$ for any solution of (1.1).

Proof:

If (3.1) is assumed, set $\Theta = 1$, $\gamma = 0$ and $\Phi^2 = w^2$ (3.3)

Then $F = \left[\frac{\Phi'}{\Phi} - \frac{\Phi'}{\Theta} - \frac{2\gamma}{\Theta} \right]^2 + \frac{\Theta^2}{\Phi^2} \left[-q + \frac{\Phi^2}{\Theta^2} - \frac{\gamma^2}{\Theta^2} + \left(\frac{\gamma}{\Theta}\right)' \right]^2$ since $p=1$

$$= \frac{1}{w^2} (- q + w^2)^2$$

$$= \{ - \frac{1}{w} (q - w^2) \}^2$$

or $$|F| = \left| - \frac{1}{w} (q - w^2) \right|^2$$

$$F^{\frac{1}{2}} = \left|\frac{1}{w}\right| \left|q - w^2\right|$$

$$\int_a^x F^{\frac{1}{2}} = \left|\frac{1}{w}\right| \int_a^x \left| q - w^2 \right|$$

$$\leq \left|\frac{1}{w}\right| \int_a^\infty \left| q - w^2 \right| < \infty \quad \text{by} \quad (3.1).$$

Thus $$\int_a^x F^{\frac{1}{2}} < \infty. \tag{3.4}$$

Similarly, $$-\infty < -\int_a^\infty F^{\frac{1}{2}} \tag{3.5}$$

By (2.31),

$$k\Theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) \leq k\Theta(x)\Phi(x)\exp\int_a^x F^{\frac{1}{2}} .$$

$$\Longrightarrow \qquad 0 \leq E(x) < \infty \tag{3.6}$$

by (3.3), (3.4) and (3.5).

But $$E(x) = \{\Theta(x)p(x)y'(x) + \gamma(x)y(x)\}^2 + \Phi^2(x)y^2(x)$$

$$= (y'(x))^2 + w^2y^2(x) \tag{3.7}$$

Since $p = 1$ and by (3.3).

Putting (3.7) into (3.6), we get

$$0 \leq (y'(x))^2 + w^2y^2(x) < \infty .$$

$\Longrightarrow$ y' and y are bounded.

If on the other hand we assume (3.2), we use (2.46) which is

$$E(a)\exp\left(-\int_a^x g_+\right) \leq E(x) \leq E(a)\exp\left(\int_a^x g_-\right) .$$

But $g = (pq)'(pq)^{-1} = q'q^{-1}$, since $p = 1$.

Thus $$\int_a^x g = \int_a^x q'q^{-1}$$

$$\leq \int_a^{\infty} | q'q^{-1} |$$

$$\leq \int_a^{\infty} | q'q^{-1} |$$

$$= \frac{1}{q} \int_a^{\infty} | q' |$$

$$< \infty \quad , \text{ since } \int_a^x | q' | < \infty .$$

$$\therefore \int_a^x g < \infty \quad . \tag{3.8}$$

By definition, $g_+ = \max(0,g)$

$$= \max(0, q'q^{-1})$$

$$= \begin{cases} 0 & \text{if } q' \leq 0 \\ q'q^{-1} & \text{if } q' > 0 \end{cases} \quad \text{since } q > 0 .$$

Also $g_- = \max(0, -g)$

$$= \begin{cases} q'q^{-1} & \text{if } q' \leq 0 \\ 0 & \text{if } q' > 0. \end{cases}$$

Now, two cases arise:

Case (i) : $q' \leq 0$

$\Rightarrow$ $g_+ = 0$ and $g_- = q'q^{-1} = g$.

Thus (2.46) becomes

$$E(a) \leq E(x) < \infty. \tag{3.9}$$

Case (ii) : $q' > 0$

$\Rightarrow$ $g_+ = g$ and $g_- = 0$.

Then (2.46) becomes

$$E(a) \exp(-\infty) \leq E(x) \leq E(a) \exp(0)$$

$$0 \leq E(x) \leq E(a) \tag{3.10}$$

From (3.9) and (3.10), $E(x)$ is bounded.

i.e. $E(x) = (y'(x))^2 + w^2 y^2(x)$ is bounded.

Hence y' and y are bounded.

The next theorem is given by knowles in [18] .

Theorem 3.2.2.

Let q *be positive and differentiable on* $[a,\infty)$, *and assume that on* $[a,\infty)$,

$$\frac{q'}{q} + \frac{4W'}{W} \geq 0 , \tag{3.11}$$

where W *is positive and twice differentiable, and*

$$\int_a^\infty |W''|\, q^{-\frac{1}{2}}\, W^{-1} < \infty. \tag{3.12}$$

Then $|y| \leq W$ *on* $[a,\infty)$.

Proof:

We set $\Theta = 1$, $\Phi^2 = q$ *and* $\gamma = -W'W^{-1}$.

Since $p = 1$, *then from* (2.25),

$$F = \left[\frac{q'}{2q} + \frac{2W'}{W}\right]^2 + \frac{1}{q}\left[-q + \Phi^2 - \gamma^2 + \gamma'\right]^2$$

$$= \left[\frac{q'}{2q} + \frac{2W'}{W}\right]^2 + \left[\frac{1}{q^{\frac{1}{2}}}(-\gamma^2 + \gamma')\right]^2 \text{ since } \Phi^2 = q$$

$$= \left[\tfrac{1}{2}\left(\frac{q'}{q} + \frac{4W'}{W}\right)\right]^2 + \left[-q^{\frac{1}{2}}W''W^{-1}\right]$$

$$|F| \leq \left|\tfrac{1}{2}\left(\frac{q'}{q} + \frac{4W'}{W}\right)\right|^2 + \left|-W''q^{-\frac{1}{2}}W^{-1}\right|^2$$

$$\therefore\ F^{\frac{1}{2}} \leq \left[\tfrac{1}{2}\left(\frac{q'}{q} + \frac{4W'}{W}\right) + |W''|q^{-\frac{1}{2}}W^{-1}\right] \text{ by (3.11) and since}$$

$q > o$ *and* $w > o$.

$$\text{Thus } \int_a^x F^{\frac{1}{2}} \leq \tfrac{1}{2}\int_a^x \left(\frac{q'}{q} + \frac{4w'}{w}\right) + \int_a^x |w''|\, q^{-\frac{1}{2}}w^{-1}$$

$$= \tfrac{1}{2}\left[\ln\left(\frac{q(x)}{q(a)}\right) + 4\ln\left(\frac{w(x)}{w(a)}\right)\right] + \int_a^x |w''|q^{-\frac{1}{2}}w^{-1}$$

$$ln\left(\frac{q(x)}{q(a)}\right)^{\frac{1}{2}} + ln\left(\frac{w(x)}{w(a)}\right)^{2} + \int_a^x |w''|q^{-\frac{1}{2}}w^{-1}$$

$$= ln\{q(x)/q(a))^{\frac{1}{2}}(w(x)/w(a))^{2}\} + \int_a^x |w''|q^{-\frac{1}{2}}w^{-1}$$

$$exp\int_a^x F^{\frac{1}{2}} \leq \{(q(x)/q(a))^{\frac{1}{2}}(w(x)/w(a))^{2}\}exp\int_a^x |w''|q^{-\frac{1}{2}}w^{-1}$$

$$\leq (q^{-\frac{1}{2}}(a)w^{-2}(a)exp\int_a^x |w''|q^{-\frac{1}{2}}w^{-1})q^{\frac{1}{2}}(x)w^{2}(x). \qquad (3.13)$$

But $k\Theta(x)\Phi(x)exp\int_a^x F^{\frac{1}{2}} = Kq^{\frac{1}{2}}(x)exp\int_a^x F^{\frac{1}{2}}$ since $\Theta=1$, $\Phi^2=q$

$$\leq kq^{\frac{1}{2}}(x)\{q^{-\frac{1}{2}}(a)w^{-2}(a)exp\int_a^x |w''|q^{-\frac{1}{2}}w^{-1}\}q^{\frac{1}{2}}(x)w^{2}(x)$$

$$\leq K\{q^{-\frac{1}{2}}(a)w^{-2}(a)exp\int_a^x |w''|q^{-\frac{1}{2}}w^{-1}\}q(x)w^{2}(x)$$

$$\leq K\{q^{-\frac{1}{2}}(a)w^{-2}(a)exp\int_a^\infty |w''|q^{-\frac{1}{2}}w^{-1}\}q(x)w^{2}(x)$$

$$= cq(x)w^{2}(x).$$

where $c = k\{q^{-\frac{1}{2}}(a)w^{-2}(a)exp\int_a^\infty |w''|q^{-\frac{1}{2}}w^{-1}\} < \infty$ by (3.12).

$$i.e. \quad K\Theta(x)\Phi(x)exp\int_a^x F^{\frac{1}{2}} \leq cq(x)w^{2}(x). \qquad (3.14)$$

But $E(x)=[y'(x)+\gamma(x)y(x)]^{2} + \Phi^{2}(x)y^{2}(x)$, since $p=1,\Theta=1$

$$=[y'(x)-w'w^{-1}y(x)]^{2}+q(x)y^{2}(x), \quad since\ \gamma=-w'w^{-1}, \Phi^{2}=q.$$

$$i.e.\ E(x) = [y'(x) - w'w^{-1}y(x)]^{2} + q(x)y^{2}(x). \qquad (3.15)$$

From (2.31)

$$E(x) \leq K\Theta(x)\Phi(x)exp\int_a^x F^{\frac{1}{2}}.$$

Then $[y'(x)-w'w^{-1}y(x)]^{2}+q(x)y^{2}(x) \leq cq(x)w^{2}(x)$, by (3.14) and (3.15)

$$\Rightarrow \quad q(x)y^2(x) \leq cq(x)w^2(x) \quad \text{since } \left[y'(x)-w'w^{-1}y(x)\right]^2 \geq 0$$

$$\text{or} \qquad y^2(x) \leq w^2(x)$$

$$\text{or} \qquad |y(x)| \leq |w(x)|$$

$$\therefore \qquad |y| \leq w.$$

Hence the theorem.

Example:

Let us consider the Bessel differential equation

$$x^2y'' + xy' + (x^2-n^2)y = 0 \tag{3.16}$$

of order $n \geq 0$,

Suppose $n = 0$, then (3.16) becomes

$$y'' + \frac{y'}{x} + y = 0 \tag{3.17}$$

The substitution

$$y = z \exp\left(-\tfrac{1}{2}\int_a^x \frac{du}{u}\right) \tag{3.18}$$

transforms (3.17) into

$$z'' + \left(1 + \frac{1}{4x^2}\right)z = 0 \tag{3.19}$$

which is of the form (1.5).

Let $q(x) = 1 + \frac{1}{4x^2}$,

then

$$\int_a^\infty |q(x)-1|dx = \frac{1}{4}\int_a^\infty \left|\frac{dx}{x^2}\right|$$

$$= \left[\frac{1}{4}\left|-\frac{1}{x}\right|\right]_a^\infty$$

$$= \frac{1}{4a}$$

$$< \infty$$

$$\text{i.e.} \quad \int_a^\infty |q(x)-1|dx < \infty. \tag{3.20}$$

This establishes (3.1) with $w^2 = 1$.

Also $$\int_a^\infty |q'(x)|\, dx = \left| \frac{1}{4x^2} \right|_a^\infty$$

$$= \frac{1}{4a^2}$$

$$< \infty .$$

i.e. $$\int_a^\infty |q'(x)|\, dx < \infty , \qquad (3.21)$$

and $$\lim_{x\to\infty} q(x) = \lim_{x\to\infty} \left(1 + \frac{1}{4x^2}\right) = 1$$

i.e. $q(x) \to 1$ as $x \to \infty$. (3.22)

(3.21) and (3.22) establish (3.2) with $w^2=1$. We can therefore appeal to the conclusion of Theorem 3.2.1 to conclude that z and z' are bounded on $[a,\infty)$ for any solution z of (3.19). Going back to (3.18), it implies that y and y' are also bounded on $[a,\infty)$ for any solution y of (1.1) provided that $\int_a^\infty \frac{du}{u} < \infty$.

3.3. Asymptotic Behaviour of Solutions.

In this section, we shall give conditions for which (ii) holds.

We start with the following theorem given by Fedosov and Lidskii in [12] .

Theorem 3.3.1

Let q be positive and differentiable on $[a,\infty)$ and assume that $q' \geq \alpha q$ for some positive non-increasing absolutely continuous function α with

$$\int_a^\infty \alpha(x)\, dx = \infty . \qquad (3.23)$$

Then $q^{-\frac{1}{2}}y'(x) \to 0$ *and* $y(x) \to 0$ *for all solutions* y *of (1.1).*

Proof:

Let ε *be such that* $0 < \varepsilon < \frac{1}{2}$ *be given. Without loss of generality, we assume that* $0<\alpha(x)<\varepsilon$, $\forall\ x \geq a$ .....(3.24)

Since $q' \geq \alpha q$,

then $q'q^{-1} \geq \alpha$. (3.25)

Integrating (3.25), we have

$$\int_a^x q'q^{-1} \geq \int_a^x \alpha$$

$$\log q(x)\Big|_a^x \geq \int_a^x \alpha$$

$$\log\left(\frac{q(x)}{q(a)}\right) \geq \int_a^x \alpha \to \infty \text{ as } x \to \infty \text{ by (3.23)}$$

$$\Rightarrow \quad \frac{q(x)}{q(a)} \to \infty \text{ as } x \to \infty .$$

Hence $q(x) \to \infty$ *as* $x \to \infty$.

In particular, we can assume that

$$q^{-\frac{1}{2}}(x) \leq 16, \quad \forall\ x \geq a. \tag{3.26}$$

By (2.25),

$$F = \left[\frac{\Phi'}{\Phi} - \frac{\Theta^1}{\Theta} - \frac{2\gamma}{\Theta}\right]^2 + \frac{\Theta^2}{\Phi^2}\left[-q + \left(\frac{\Phi}{\Theta}\right)^2 - \left(\frac{\gamma}{\Theta}\right)^2 + \left(\frac{\gamma}{\Theta}\right)'\right]^2 \quad \text{since } p=1.$$

If we set $\Theta = 1$, $\Phi^2 = q$ *and* $\gamma = \frac{\alpha}{4}$, (3.27)

$$F = \left[\frac{\Phi'}{\Phi} - 2\gamma\right]^2 + \frac{1}{\Phi^2}\left[-\gamma^2 + \gamma'\right]^2$$

$$F^{\frac{1}{2}} = \sqrt{\left(\frac{\Phi'}{\Phi} - 2\gamma\right)^2 + \left(\frac{1}{\Phi}(-\gamma^2+\gamma')\right)^2}$$

$$\leq \left|\frac{\Phi'}{\Phi} - 2\gamma\right| + \left|\frac{1}{\Phi}(\gamma' - \gamma^2)\right|$$

since $\sqrt{a^2+b^2} \leq |a| + |b|$ (3.28)

$$= \left|\frac{q'q^{-1}}{2} - \frac{\alpha}{2}\right| + \left|\frac{1}{q^{\frac{1}{2}}}\left(\frac{\alpha'}{4} - \frac{\alpha^2}{16}\right)\right| .$$

Thus $|F^{\frac{1}{2}}| \leq \frac{1}{2}|q'q^{-1} - \alpha| + (16q^{\frac{1}{2}})^{-1}|\alpha^2 - 4\alpha'|$

or $$\int_a^x F^{\frac{1}{2}} \leq \frac{1}{2}\int_a^x |q'q^{-1} - \alpha| + \int_a^x (16q^{\frac{1}{2}})^{-1}|\alpha^2 - 4\alpha'|$$

$$\leq \frac{1}{2}\int_a^x |q'q^{-1} - \alpha| + \int_a^x |\alpha^2 - 4\alpha'| \quad \text{by (3.26)}$$

$$\leq \frac{1}{2}\int_a^x q'q^{-1} - \frac{1}{2}\int_a^x \alpha + \int_a^x \alpha^2 - 4\int_a^x \alpha' \quad \text{by (3.25) and } \alpha' \leq 0$$

$$\leq \frac{1}{2}\log\left(\frac{q(x)}{q(a)}\right) + \int_a^x (\alpha^2 - \tfrac{1}{2}\alpha) - 4(\alpha(x) - \alpha(a)) .$$

Since $\alpha \leq \varepsilon$ by (3.24),

$$\alpha - \tfrac{1}{2} \leq \varepsilon - \tfrac{1}{2}$$

$$\alpha(\alpha - \tfrac{1}{2}) \leq \alpha(\varepsilon - \tfrac{1}{2}) , \quad \text{since } \alpha > 0$$

$$= -\alpha(\tfrac{1}{2} - \varepsilon) .$$

Thus $$\int_a^x F^{\frac{1}{2}} \leq -4(\alpha(x) - \alpha(a)) + \log\left(\frac{q(x)}{q(a)}\right)^{\frac{1}{2}} - (\tfrac{1}{2} - \varepsilon)\int_a^x \alpha$$

$$\leq O(1) + \log\left(\frac{q(x)}{q(a)}\right)^{\frac{1}{2}} - (\tfrac{1}{2} - \varepsilon)\int_a^x \alpha \qquad (3.29)$$

From (2.21),

$$E(x) \leq K\Theta(x)\Phi(x)\exp\int_a^x F^{\frac{1}{2}}$$

$$= kq^{\frac{1}{2}}(x)\exp\left[O(1)+1\bullet g\left(\frac{q(x)}{q(a)}\right)^{\frac{1}{2}} - (\tfrac{1}{2}-\varepsilon)\int_a^x \alpha\right] \text{ by (3.27) and (3.29)}$$

$$= kq^{\frac{1}{2}}(x)\left(\frac{q(x)}{q(a)}\right)^{\frac{1}{2}} \exp\{-(\tfrac{1}{2}-\varepsilon)\int_a^x \alpha\}$$

$$= \frac{k}{q^{\frac{1}{2}}(a)}\, q(x)\exp\{-(\tfrac{1}{2}-\varepsilon)\int_a^x \alpha\}.$$

i.e. $$E(x) < C_1\, q(x)\exp\{-(\tfrac{1}{2}-\varepsilon)\int_a^x \alpha\}, \qquad (3.30)$$

where $C_1 = \dfrac{k}{q^{\frac{1}{2}}(a)}$.

From (2.1),

$$E(x) = \{\Theta(x)y'(x)+\gamma(x)y(x)\}^2 + \Phi^2(x)y^2(x), \text{ since } p=1$$

$$\text{or } E(x) = \{y'(x) + \frac{\alpha(x)}{4}y(x)\}^2 + q(x)y^2(x) \qquad (3.31)$$

by (3.27).

Putting (3.31) into (3.30), we get

$$\{y'(x) + \frac{\alpha(x)}{4}\, y(x)\}^2 + q(x)y^2(x) \le c_1 q(x)\exp\{-(\tfrac{1}{2}-\varepsilon)\int_a^x \alpha\}. \qquad (3.32)$$

Since $q^{-\frac{1}{2}} \le 16 = 4^2$,

$$\Rightarrow \quad q > (\tfrac{1}{4})^2$$

$$\therefore \quad q > (\tfrac{\alpha}{4})^2 \text{ since } o < \alpha(x) \le \varepsilon \text{ and } o<\varepsilon<\tfrac{1}{2}.$$

Hence, if a is chosen large enough,

$$\{q^{-\frac{1}{2}}(x)y'(x)+y(x)\}^2 + y^2(x) \le c_1\exp\{-(\tfrac{1}{2}-\varepsilon)\int_a^x \alpha\}$$

$$\{q(x)\}^{-1}\{y'(x)\}^2 + y^2(x) \le c_2\exp\{-(\tfrac{1}{2}-\varepsilon)\int_a^x \alpha\} \qquad (3.33)$$

for some constant c_2 .

By (3.22), $\int_a^\infty \alpha(x)\,dx = \infty$.

Thus $-(\tfrac{1}{2}-\varepsilon)\int_a^\infty \alpha(x)\,dx \to -\infty$

and $\exp\{-(\tfrac{1}{2}-\varepsilon)\int_a^\infty \alpha(x)\,dx\} \to 0$.

From (3.33),

$$\{q(x)\}^{-1}\ \{y(x)\}^2 + y^2(x) \to 0 \text{ as } x \to \infty$$

$$\text{or } \{q(x)\}^{-1}\ \{y(x)\}^2 \to 0 \text{ and } y^2(x) \to 0 \text{ as } x \to \infty$$

$$\text{or } q^{-\frac{1}{2}}(x)y(x) \to 0 \text{ and } y(x) \to 0 \text{ as } x \to \infty.$$

Hence the theorem.

The next result which is similar to that of Meir et.al. [22] was given by Knowles in [18].

Theorem 3.3.2.

Assume that q is positive, absolutely continuous and non-decreasing on $[a,\infty)$. If there exists a positive function w such that w' is non-negative and absolutely continuous on $[a,\infty)$, and

$$\text{(i)}\quad \limsup \{w(x)\}^{-1} \int_a^x q^{-\frac{1}{2}}|w''| = \lambda < \infty, \tag{3.34}$$

$$\text{(ii)}\quad \limsup \{w(x)\}^{-1} \int_a^x (w' - \frac{q'}{q})_+ = \mu < \infty, \tag{3.35}$$

then for any $\varepsilon > 0$,

$$\{q(x)\}^{-1}\{y'(x)\}^2 + \left[1+\{w'(x)\}^2\{q(x)\}^{-1}\right] y^2(x)$$

$$\leq c\ \{1+w^2(x)\}^{\frac{1}{2}} \exp\{(\mu+\tfrac{1}{2}\lambda+\varepsilon-\tfrac{1}{2})w(x)\}, \tag{3.36}$$

for some constant c and all $x \geq a$. In particular,

if $w(x) \to \infty$ as $x \to \infty$ and $2\mu + \lambda < 1$,

then $q^{-\frac{1}{2}}(x)y'(x) \to 0$ and $y(x) \to 0$ as $x \to \infty$

for all solutions y of (1.1).

Proof:

To prove (3.36), let $\varepsilon > 0$ be given and choose a so that

$$\int_a^x q^{-\frac{1}{2}}|w''| \leq (\lambda+\varepsilon)w(x) \tag{3.37}$$

and $\int_a^x (w' - \frac{q'}{q})_+ \leq (\mu + \frac{1}{2}\varepsilon) w(x)$ (3.38)

For all $x \geq a$.

Set $\theta^2 = e^w q^{-1}$, $\phi^2 = e^w\{1 + (w'/4q^{\frac{1}{2}})^2\}$, $\gamma\theta^{-1} = \frac{w'}{4}$. (3.39)

We note that $(\phi^2)' = 2\phi'\phi$

$$\text{or } (\phi^2)' \ \phi^{-2} = 2\,\phi'\phi^{-1}$$

$$\text{Thus } \phi'\phi^{-1} = \tfrac{1}{2}(\phi^2)' \ \phi^{-2} \quad (3.40)$$

Also $2\gamma\theta^{-1} = \frac{2w'}{4} = \frac{1}{2}w'$ (3.41)

Thus $\phi'\phi^{-1} - 2\gamma\theta^{-1} = \frac{1}{2}(\phi^2)'\phi^{-2} - \frac{1}{2}w'$. (3.42)

Since $\phi^2 = e^w\{1 + (w'/4q^{\frac{1}{2}})^2\}$,

$$2\log\phi = \log e^w + \log\{1 + (w'/4q^{\frac{1}{2}})^2\}$$

$$= w + \log\{1 + (w'/4q^{\frac{1}{2}})^2\}$$

$$2\phi'\phi^{-1} = w' + \{1+(w'/4q^{\frac{1}{2}})^2\}^{-1}\{2(w'/4q^{\frac{1}{2}})\}\{(w'/4q^{\frac{1}{2}})\}'$$

$$(\phi^2)'\phi^{-2} = w' + 2(w'/4q^{\frac{1}{2}})\{1+(w'/4q^{\frac{1}{2}})^2\}^{-1}\{(w'/4q^{\frac{1}{2}})\}' \quad (3.43)$$

Substituting (3.43) into (3.42), we get

$$\phi'\phi^{-1} - 2\gamma\theta^{-1} = \tfrac{1}{2}w' + \tfrac{1}{2}.2\left(\frac{w'}{4q^{\frac{1}{2}}}\right)\left[1+\left(\frac{w'}{4q^{\frac{1}{2}}}\right)^2\right]^{-1}\left[\left(\frac{w'}{4q^{\frac{1}{2}}}\right)\right]' - \tfrac{1}{2}w'$$

$$= \tfrac{1}{2}.2\left(\frac{w'}{4q^{\frac{1}{2}}}\right)\left[1+\left(\frac{w'}{4q^{\frac{1}{2}}}\right)^2\right]^{-1}\left[\tfrac{1}{4}w''q^{-\frac{1}{2}} - \tfrac{1}{8}w'q'q^{-3/2}\right]$$

$$= 2\left(\frac{w'}{4q^{\frac{1}{2}}}\right)\left[1+\left(\frac{w'}{4q^{\frac{1}{2}}}\right)^2\right]^{-1}\left[\tfrac{1}{8}w''q^{-\frac{1}{2}} - \tfrac{1}{16}w'q'q^{-3/2}\right].$$

$$\text{Thus } \left|\phi'\phi^{-1} - 2\gamma\theta^{-1}\right| = \left|2\left(\frac{w'}{4q^{\frac{1}{2}}}\right)\left[1+\left(\frac{w'}{4q^{\frac{1}{2}}}\right)^2\right]^{-1}\right| \cdot \left|\tfrac{1}{8}w''q^{-\frac{1}{2}} - \tfrac{1}{16}w'q'q^{-3/2}\right|$$

$$\leq \left|\tfrac{1}{8}w''q^{-\frac{1}{2}} - \tfrac{1}{16}w'q'q^{-3/2}\right|, \text{since } \left|2\left(\frac{w'}{4q^{\frac{1}{2}}}\right)\left[1+\left(\frac{w'}{4q^{\frac{1}{2}}}\right)^2\right]^{-1}\right| < 1$$

$$\leq \left|\tfrac{1}{8}w''q^{-\frac{1}{2}}\right| + \left|\tfrac{1}{16}w'q'q^{-3/2}\right|$$

$$= \frac{1}{8}\left|w''\right|q^{-\frac{1}{2}} + \frac{1}{16}\, w'q'q^{-3/2}, \quad \text{since } w'>0, q>0.$$

$$\text{i.e. } \left|\phi'\phi^{-1} - 2\gamma\theta\right| \le \frac{1}{8}\left|w''\right|q^{-\frac{1}{2}} + \frac{1}{16}\, w'q'q^{-3/2}. \qquad (3.44)$$

$$\text{Now } \left(\frac{w'}{q^{\frac{1}{2}}}\right)' = w''q^{-\frac{1}{2}} - \tfrac{1}{2}\, w'q'q^{-3/2}$$

$$\therefore \int_a^x \left(\frac{w'}{q^{\frac{1}{2}}}\right)' = \int_a^x w''q^{-\frac{1}{2}} - \tfrac{1}{2}\int_a^x w'q'q^{-3/2}$$

$$w'q^{-\frac{1}{2}}\Big|_a^x = \int_a^x w''q^{-\frac{1}{2}} - \tfrac{1}{2}\int_a^x w'q'q^{-3/2} \qquad (3.45)$$

$$\text{Thus } w'(x)q^{-\frac{1}{2}}(x) - w'(a)q^{-\frac{1}{2}}(a) = \int_a^x w''q^{-\frac{1}{2}} - \int_a^x w'q'q^{-3/2}$$

$$\text{or } w'(x)q^{-\frac{1}{2}}(x) = w'(a)q^{-\frac{1}{2}}(a) + \int_a^x w''q^{-\frac{1}{2}} - \int_a^x w'q'q^{-3/2}.$$

Hence for $x \ge a$,

$$0 \le w'(x)q^{-\frac{1}{2}}(x) \le w'(a)q^{-\frac{1}{2}}(a) + \int_a^x w''q^{-\frac{1}{2}}, \quad \text{since} \quad \int_a^x w'q'q^{-3/2} \ge 0.$$

$$\text{or } 0 \le w'(x)q^{-\frac{1}{2}}(x) \le O(1) + \int_a^x w''q^{-\frac{1}{2}}$$

$$0 \le w'(x)q^{-\frac{1}{2}}(x) \le O(1) + (\lambda+\varepsilon)w(x) \qquad (3.46)$$

by (3.37).

$$\text{Also } \tfrac{1}{2}\int_a^x w'q'q^{-3/2} = -\left[w'q^{-\frac{1}{2}}\right]_a^x + \int_a^x w''q^{-\frac{1}{2}}$$

$$\le w'q^{-\frac{1}{2}}\Big|_a^x + \int_a^x \left|w''\right| q^{-\frac{1}{2}}$$

$$= w'(x)q^{-\frac{1}{2}}(x) - w'(a)q^{-\frac{1}{2}}(a) + \int_a^x |w''|q^{-\frac{1}{2}}$$

$$\le w'(x)q^{-\frac{1}{2}}(x) + \int_a^x |w''|q^{-\frac{1}{2}}, \text{since } w'(a)q^{-\frac{1}{2}}(a) \ge 0$$

$$= O(1) + \int_a^x |w''| q^{-\frac{1}{2}}$$

$$\text{i.e.} \quad \int_a^x w' q' q^{-3/2} \leq O(1) + 2 \int_a^x |w''| q^{-\frac{1}{2}} \qquad (3.47)$$

By (2.28),

$$E(a) \exp\left[\tfrac{1}{2}\int_a^x (A+C)\right] \exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) \leq E(a) \exp\left[\tfrac{1}{2}\int_a^x (A+C)\right] \exp\left(\int_a^x F^{\frac{1}{2}}\right).$$

If we replace $A = 2(\theta'\theta^{-1} + \gamma\theta^{-1})$ by $A_1 = 2(\theta'\theta^{-1})_+ + 2\gamma\theta^{-1}$,

we get

$$A_1 + C = 2(\theta'\theta^{-1})_+ + 2\gamma\theta^{-1} + 2\phi'\phi^{-1} - 2\gamma\theta^{-1}$$

$$= 2(\theta'\theta^{-1})_+ + 2\phi'\phi^{-1}.$$

Thus $\frac{1}{2}\int_a^x (A_1 + C) = \int_a^x \{(\theta'\theta^{-1})_+ + \phi'\phi^{-1}\}$

$$\exp\{\tfrac{1}{2}\int_a^x (A_1 + C)\} = \exp \int_a^x \{(\theta'\theta^{-1})_+ + \phi'\phi^{-1}\}$$

$$= \exp \int_a^x (\theta'\theta^{-1})_+ \exp \int \phi'\phi^{-1}$$

$$\text{i.e.} \quad \exp\{\tfrac{1}{2}\int_a^x (A_1 + C)\} = \frac{\phi(x)}{\phi(a)} \exp \int_a^x (\theta'\theta^{-1})_+ . \qquad (3.48)$$

By (2.25),

$$F = \left[\frac{\phi'}{\phi} - \left(\frac{\theta}{\theta}\right)'_+ - \frac{2\gamma}{\theta}\right]^2 + \frac{\theta^2}{\phi^2}\left[-q + \left(\frac{\phi}{\theta}\right)^2 - \left(\frac{\gamma}{\theta}\right) + \left(\frac{\gamma}{\theta}\right)'\right]' \text{ since } p=1$$

$$= \{\phi'\phi^{-1} - (\theta'\theta^{-1})_+ - 2\gamma\theta^{-1}\}^2 + \left[\frac{1}{q+(w'/4)^2}\right]\{-q+q+\frac{w'^2}{4} - \frac{w'^2}{4} + \frac{w''}{4}\}$$

$$= \{\phi'\phi^{-1} - (\theta'\theta^{-1})_+ - 2\gamma\theta^{-1}\}^2 + \left[\frac{1}{q+(w'/4)^2}\right](w''/4)$$

$$\leq \{\phi'\phi^{-1} - (\theta'\theta^{-1})_+ - 2\gamma\theta^{-1}\}^2 + \frac{1}{q}(w''/4), \text{ since } \frac{1}{q} > \frac{1}{q+(w'/4)^2}$$

$$= \{\Phi'\Phi^{-1} - 2\gamma\Theta^{-1} - (\Theta'\Theta^{-1})_+\}^2 + \{w''/4q^{\frac{1}{2}}\}^2$$

$$F^{\frac{1}{2}} \leq \left|\Phi'\Phi^{-1} - 2\gamma\Theta^{-1} - (\Theta'\Theta^{-1})_+\right| + \left|w''/4q^{\frac{1}{2}}\right|$$

$$\exp \int_a^x F^{\frac{1}{2}} \leq \exp \int_a^x (|\Phi'\Phi^{-1} - 2\gamma\Theta^{-1} - (\Theta'\Theta^{-1})_+| + |w''/4q^{\frac{1}{2}}|). \quad (3.49)$$

$$E(x) \leq E(a) \exp \{\tfrac{1}{2} \int_a^x (A+C)\} \exp \left(\int_a^x F^{\frac{1}{2}} \right) \quad \text{by} \quad (2.28)$$

$$\leq E(a) \frac{\Phi(x)}{\Phi(a)} \exp \int_a^x (\Theta'\Theta^{-1})_+ \exp \int_a^x (|\Phi'\Phi^{-1} - 2\gamma\Theta^{-1} - (\Theta'\Theta^{-1})_+| + |w''/4q^{\frac{1}{2}}|).$$

by (3.48) and (3.49),

$$\text{or } E(x) \leq K\Phi(x) \exp \int_a^x (\Theta'\Theta^{-1})_+ \exp \int_a^x (|\Phi'\Phi^{-1} - 2\gamma\Theta^{-1} - (\Theta'\Theta^{-1})_+| + |w''/4q^{\frac{1}{2}}|) \quad (3.50)$$

where $K = \frac{E(a)}{\Phi(a)}$.

Now $(\Theta^2)' = 2\Theta\Theta'$

$$\therefore \Theta'\Theta^{-1} = \tfrac{1}{2} (\Theta^2)' \Theta^{-2}$$

$$\text{Thus} \int_a^x (\Theta'\Theta^{-1})_+ = \tfrac{1}{2} \int_a^x \{(\Theta^2)' \Theta^{-2}\}_+$$

$$= \tfrac{1}{2} \int_a^x \left\{ \frac{e^w w' q - e^w q' q^{-2}}{e^w q^{-1}} \right\}_+$$

$$= \tfrac{1}{2} \int_a^x \left\{ w' - \frac{q'}{q} \right\}_+ .$$

$$\text{i.e.} \quad \int_a^x (\Theta'\Theta^{-1})_+ \leq \tfrac{1}{2} (\mu + \tfrac{1}{2} \varepsilon) w(x) \quad (3.51)$$

by (3.38).

By (3.50),

$$H(x) \leq K\Phi(x) \exp \int_a^x (\Theta'\Theta^{-1})_+ \exp \int_a^x (|\Phi'\Phi^{-1} - 2\gamma\Theta^{-1} - (\Theta'\Theta^{-1})_+| + |w''/4q^{\frac{1}{2}}|)$$

$$\leq k\Phi(x)\exp\Big\{\int_a^x(\theta'\theta^{-1})_+\Big\}\exp\Big\{\int_a^x(\theta'\theta^{-1})_+\Big\}\exp\Big\{\int_a^x\big(|\Phi'\Phi^{-1}-2\gamma\theta^{-1}|+\tfrac{1}{4}|w''/q^{\frac{1}{2}}|\big)\Big\}$$

$$\leq k\Phi(x)\exp\Big\{2\int_a^x(\theta'\theta^{-1})_+\Big\}\exp\Big\{\int_a^x\big(\tfrac{1}{8}|w''|q^{-\frac{1}{2}}+\tfrac{1}{16}w'q'q^{-3/2}\big)+\tfrac{1}{4}\int_a^x|w''|q^{-\frac{1}{2}}\Big\}$$

by (3.44).

$$=k\Phi(x)\exp\Big\{2\int_a^x(\theta'\theta^{-1})_+\Big\}\exp\Big\{\tfrac{3}{8}\int_a^x|w''|q^{-\frac{1}{2}}+\tfrac{1}{16}\int_a^x w'q'q^{-3/2}\Big\}$$

$$=k\Phi(x)\exp\Big\{2\int_a^x(\theta'\theta^{-1})_+\Big\}\exp\Big[\tfrac{3}{8}\int_a^x|w''|q^{-\frac{1}{2}}+\tfrac{1}{16}\,|\,O(1)+2\int_a^x|w''|q^{-\frac{1}{2}}\Big]$$

by (3.47).

$$\leq k\Phi(x)\exp\Big\{2\int_a^x(\theta'\theta^{-1})_+\Big\}\exp\Big\{\tfrac{3}{8}\int_a^x|w''|q^{-\frac{1}{2}}+\tfrac{1}{8}\int_a^x|w''|q^{-\frac{1}{2}}\Big\}$$

$$=k\Phi(x)\exp\Big\{2\int_a^x(\theta'\theta^{-1})_+\Big\}\exp\Big\{\tfrac{1}{2}\int_a^x|w''|q^{-\frac{1}{2}}\Big\}$$

$$\leq k\Phi(x)\exp\Big\{2\Big|\tfrac{1}{2}(\mu+\tfrac{1}{2}\varepsilon)w(x)\Big|\Big\}\exp\Big\{\tfrac{1}{2}(\lambda+\varepsilon)w(x)\Big\}\quad\text{by (3.37) and (3.51)}$$

$$=k\Phi(x)\exp\Big\{(\mu+\tfrac{1}{2}\varepsilon)w(x)+\tfrac{1}{2}(\lambda+\varepsilon)w(x)\Big\}$$

$$=k\Phi(x)\exp\Big\{(\mu+\tfrac{1}{2}\lambda+\varepsilon)w(x)\Big\}$$

$$=k\{\exp(w/2)\}\{1+(w'/4q^{\frac{1}{2}})\}^{\frac{1}{2}}\exp\{(\mu+\tfrac{1}{2}\lambda+\varepsilon)w(x)\}\text{by (3.39).}$$

i.e. $$E(x)\leq k\{1+(w'/4q^{\frac{1}{2}})\}^{\frac{1}{2}}\exp\{(\mu+\tfrac{1}{2}\lambda+\varepsilon+\tfrac{1}{2})w(x)\}. \qquad (3.52).$$

From (2.1),

$$E(x)=\{\theta(x)y'(x)+\gamma(x)y(x)\}^2+\Phi^2(x)y^2(x)\qquad\text{since } p=1$$

$$=\{e^{w/2}q^{-\frac{1}{2}}(x)y'(x)+e^{w/2}q^{-\frac{1}{2}}(x)\frac{w'(x)}{4}y(x)\}^2$$

$$+e^{w}\{1+\frac{1}{16}\{w'(x)\}^2\{q(x)\}^{-1}\}y^2(x).\ \text{by (3.39)}$$

$$= e^{W}\{q^{-\frac{1}{2}}(x)y'(x) + \tfrac{1}{4}w'(x)q^{-\frac{1}{2}}(x)y(x)\}^2 + e^{W}\{1 + \frac{1}{16}(w'(x)^2(q(x))^{-1}\}y^2(x). \qquad (3.53)$$

Putting (3.53) into (3.52), we get

$$e^{W}\left[\{q^{-\frac{1}{2}}(x)y'(x) + \tfrac{1}{4}w'(x)q^{-\frac{1}{2}}(x)y(x)\}^2 + \{1 + \frac{1}{16}(w'(x)^2(q(x))^{-1}\}y^2(x)\right]$$
$$\leq k\{1 + (w'/4q^{\frac{1}{2}})^2\}^{\frac{1}{2}} \exp\{(\mu + \tfrac{1}{2}\lambda + \varepsilon + \tfrac{1}{2})w(x)\}$$

or

$$\{q^{-\frac{1}{2}}(x)y'(x) + \tfrac{1}{4}w'(x)q^{-\frac{1}{2}}(x)y(x)\}^2 + \{1 + \frac{1}{16}(w'(x))^2(q(x))^{-1}\}y^2(x)$$
$$\leq k\{1 + \frac{1}{16}(w'(x))^2(q(x))^{-1}\}^{\frac{1}{2}} \exp\{(\mu + \tfrac{1}{2}\lambda + \varepsilon - \tfrac{1}{2})w(x)\}. \qquad (3.54)$$

By (3.45),

$$w'(x)q^{-\frac{1}{2}}(x) \leq O(1) + (\lambda + \varepsilon)w(x)$$

$$|w'(x)q^{-\frac{1}{2}}(x)| \leq |\lambda + \varepsilon|\, w(x)$$

i.e. $|w'|\, q^{-\frac{1}{2}}(x) = O(w)$. (3.55)

Applying this to (3.54), we get

$$\{q^{-\frac{1}{2}}(x)y'(x)\}^2 + \{1 + (w'(x)^2(q(x))^{-1}\}y^2(x)$$
$$\leq C\{1 + (w'(x)^2\}^{\frac{1}{2}} \exp\{(\mu + \tfrac{1}{2}\lambda + \varepsilon - \tfrac{1}{2})w(x)\} \text{ for some constant}$$

C, since $\frac{1}{16}(w'(x))^2(q(x))^{-1} \leq (w'(x))^2(q(x))^{-1}$.

This proves (3.36).

Next, if $2\mu + \lambda < 1$,

$$\mu + \tfrac{1}{2}\lambda < \tfrac{1}{2}$$

$$\mu + \tfrac{1}{2}\lambda - \tfrac{1}{2} < 0.$$

Thus $(\mu + \frac{1}{2}\lambda - \frac{1}{2})w(x) < 0$, since $w > 0$.

$$\Rightarrow \quad \exp(\mu + \tfrac{1}{2} - \tfrac{1}{2})w(x) \to 0 \quad \text{as} \quad x \to \infty \qquad (3.56)$$

since $w(x) \to \infty$ as $x \to \infty$.

$\therefore$ (3.36) can be written as

$$(q(x))^{-1}(y'(x))^2 + \{1 + (w'(x))^2(q(x))^{-1}\}\, y^2(x)$$

$$\leq C\{1 + (w'(x))^2\}^{\frac{1}{2}} \exp\{(\mu + \tfrac{1}{2}\lambda - \tfrac{1}{2})w(x)\}.\ \exp \epsilon$$

$\to 0$ as $x \to \infty$ by (3.56).

$\Rightarrow (q(x))^{-1}(y'(x))^2 \to 0$ and $y^2(x) \to 0$ as $x \to \infty$

$\Rightarrow q^{-\frac{1}{2}}(x)y'(x) \to 0$ and $y(x) \to 0$ as $x \to \infty$,

for all solutions y of (1.1).

This proves the theorem.

Lastly, we consider the following Corollary.

Corollary 3.3.3.(Meir et. al [22])

If there exists a positive function v, differentiable on $[a,\infty)$, and such that

$$\int_a^{\infty} v^{-1}(x)\,dx = \infty, \tag{3.57}$$

$$\liminf_{x \to \infty} \frac{v'(x)}{v(x)q^{\frac{1}{2}}(x)} \geq 0, \tag{3.58}$$

and

$$\liminf_{x \to \infty} \frac{q'(x)v(x)}{q(x)} > 0, \tag{3.59}$$

then $q^{-\frac{1}{2}}(x)y'(x) \to 0$ and $y(x) \to 0$ as $x \to \infty$ for all solutions y of (1.1).

Proof:

Let $w(x) = 1 + \int_a^x v^{-1}(t)\,dt > 0$, since $v > 0$.

$w(x) \to \infty$ as $x \to \infty$ by (3.57).

Also $w'(x) = v^{-1}(x) > 0$, since $v > 0$.

Thus $w(x) > 0$, $w'(x) > 0$ and $w(x) \to \infty$ as $x \to \infty$. (3.60)

To verify (3.34),

Let $|v'(x)| = v'(x) + 2(-v'(x))_+$.

Since $w''(x) = -v^{-2}(x)v'(x)$,

then $|w''(x)| = v^{-2}(x)|v'(x)|$, since $v > 0$.

$$= v^{-2}(x)\{v'(x) + 2(-v'(x))_+\}$$

$$= v^{-2}(x)v'(x) + 2v^{-2}(x)(-v'(x))_+$$

Thus $q^{-\frac{1}{2}}(x)|w''(x)| = v^{-2}(x)v'(x)q^{-\frac{1}{2}}(x) + 2v^{-2}(x)(-v'(x))_+ q^{-\frac{1}{2}}(x)$,

or $$\int_a^x q^{-\frac{1}{2}}(t)|w''(t)|dt = \int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)v'(t)dt + 2\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+ dt. \quad \ldots\ldots(3.61)$$

Now $$\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)v'(t)dt$$

$$= v^{-2}(t)q^{-\frac{1}{2}}(t)v(t)\Big|_a^x + \tfrac{1}{2}\int_a^x v^{-2}(t)v(t)q^{-3/2}(t)q'(t)dt + 2\int_a^x q^{-\frac{1}{2}}(t)v(t)v^{-3}(t)v'(t)dt$$

$$= v^{-1}(t)q^{-\frac{1}{2}}(t)\Big|_a^x + \tfrac{1}{2}\int_a^x v^{-1}(t)q^{-3/2}(t)q'(t)dt + 2\int_a^x q^{-\frac{1}{2}}(t)v^{-2}(t)v'(t)dt$$

or $$-\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)v'(t)dt = v^{-1}(t)q^{-\frac{1}{2}}(t)\Big|_a^x + \tfrac{1}{2}\int_a^x v^{-1}(t)q^{-3/2}(t)q'(t)dt$$

$$\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)v'(t)dt = -v^{-1}(t)q^{-\frac{1}{2}}(t)\Big|_a^x - \tfrac{1}{2}\int_a^x v^{-1}(t)q^{-3/2}(t)q^{-1}(t)dt$$

$$= -v^{-1}(x)q^{-\frac{1}{2}}(x) + v^{-1}(a)q^{-\frac{1}{2}}(a) - \tfrac{1}{2}\int_a^x v^{-1}(t)q^{-3/2}(t)q^{-1}(t)dt$$

$$\leq v^{-1}(a)q^{-\frac{1}{2}}(a), \quad \text{since } v>0,\ q>0,\ q'\geq 0.$$

i.e. $$\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)v'(t)dt \leq v^{-1}(a)q^{-\frac{1}{2}}(a). \quad (3.62)$$

Putting (3.62) into (3.61), we get

$$\int_a^x q^{-\frac{1}{2}}(t)\,|w''(t)|\,dt \le v^{-1}(a)q^{-\frac{1}{2}}(a) + 2\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+\,dt. \quad (3.63)$$

From (3.63), two cases arise.

First, if $\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+\,dt < \infty$ for all x, (3.64)

then $$\lim_{x\to\infty} \frac{\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+\,dt}{w(x)} = 0 \quad (3.65)$$

since $w(x) \to \infty$ as $x \to \infty$.

If, on the other hand $\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+\,dt$ is not bounded or it diverges, in which case (3.64) does not hold, the limit is infinity.

i.e. $$\lim_{x\to\infty} \frac{\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+\,dt}{w(x)} = \lim_{x\to\infty} \frac{v^{-2}(x)q^{-\frac{1}{2}}(x)(-v'(x))_+}{w'(x)}$$

by L'Hospital's rule.

$$\text{or} \quad \lim_{x\to\infty} \frac{\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+\,dt}{w(x)} = \lim_{x\to\infty} \frac{v^{-2}(x)q^{-\frac{1}{2}}(x)(-v'(x))_+}{v^{-1}(x)}$$

$$\text{or} \quad \lim_{x\to\infty} \frac{\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+\,dt}{w(x)} = \lim_{x\to\infty} \frac{(-v'(x))_+}{v(x)q^{\frac{1}{2}}(x)} \quad (3.66)$$

Now $(-v'(x))_+ = \max(0, -v'(x))$

$$= \begin{cases} 0 & \text{if } v'(x) > 0 \\ -v'(x) & \text{if } v'(x) < 0 \end{cases} \quad (3.67)$$

If $v'(x) > 0$,

$$\text{then } \lim_{x\to\infty} \frac{\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+ dt}{w(x)} = 0 \qquad (3.68)$$

But if $v'(x) < 0$,

$$\lim_{x\to\infty} \frac{\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+ dt}{w(x)} = \lim_{x\to\infty} \frac{-v'(x)}{v(x)q^{\frac{1}{2}}(x)} \qquad (3.69)$$

We note that $\lim\sup_{x\to\infty} \frac{-v'(x)}{v(x)q^{\frac{1}{2}}(x)} \leq 0$ by (3.58).

Also $\lim\sup_{x\to\infty} \frac{-v'(x)}{v(x)q^{\frac{1}{2}}(x)} \geq 0$

$$\therefore \lim\sup_{x\to\infty} \frac{-v'(x)}{v(x)q^{\frac{1}{2}}(x)} = 0. \qquad (3.70)$$

By (3.58),

$$\lim\inf_{x\to\infty} \frac{v'(x)}{v(x)q^{-\frac{1}{2}}(x)} \geq 0$$

Also $\lim\inf_{x\to\infty} \frac{v'(x)}{v(x)q^{-\frac{1}{2}}(x)} \leq 0$ since $v'(x)<0$ here.

$$\Rightarrow \lim\inf_{x\to\infty} \frac{v'(x)}{v(x)q^{\frac{1}{2}}(x)} = 0$$

$$\therefore \lim\inf_{x\to\infty} \frac{-v'(x)}{v(x)q^{\frac{1}{2}}(x)} = 0. \qquad (3.71)$$

Combining (3.70) and (3.71), we have

$$\lim_{x\to\infty} \frac{-v'(x)}{v(x)q^{\frac{1}{2}}(x)} = 0. \qquad (3.72)$$

Putting (3.72) into (3.66), we have

$$\lim_{x\to\infty} \frac{\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+dt}{w(x)} = 0. \qquad (3.73)$$

Combining (3.65), (3.68) and (3.73), we have

$$\int_a^x v^{-2}(t)q^{-\frac{1}{2}}(t)(-v'(t))_+dt = O(w(x)).$$

(3.63) can now be written as

$$\int_a^x q^{-\frac{1}{2}}(t)|w''(t)|dt = v^{-1}(a)q^{-\frac{1}{2}}(a) + O(w(x)).$$

Thus $\lim_{x\to\infty} \sup \frac{1}{w(x)} \int_a^x q^{-\frac{1}{2}}(t)|w''(t)|dt = \lim_{x\to\infty} \frac{1}{w(x)} v^{-1}(a)q^{-\frac{1}{2}}(a)$

$= 0$ since $w(x)\to\infty$ as $x\to\infty$

$< \infty$

i.e. $\lim_{x\to\infty} \sup \frac{1}{w(x)} \int_a^x q^{-\frac{1}{2}}(t)|w''(t)|dt = 0 < \infty$

$\therefore$ (3.34) is satisfied with $\lambda = 0$.

Next we verify (3.35).

For some $\alpha > 0$, $\left(\frac{q'}{q} - \frac{\alpha}{v}\right) \geq 0$.

Without loss of generality, let $\alpha=1$ (otherwise we can rescale v), then

$$q'q^{-1} - v^{-1} \geq 0$$

$$q'q^{-1} - w' \geq 0, \text{ since } w' = v^{-1}$$

$$\text{or } w' - q'q^{-1} \leq 0. \qquad (3.74)$$

Since $(w' - q'q^{-1})_+ = \max(0,\ w'-q'q^{-1})$

$= 0$ from (3.74),

then $\lim_{x\to\infty} \sup \{w(x)\}^{-1} \int_a^x (w'-q'q^{-1})_+ = 0 < \infty$.

$\therefore$ (3.35) holds for $\mu = 0$.

Consequently from (3.36),

$$\{q(x)\}^{-1}\{y'(x)\}^2 + (1 + \{w'(x)\}^2\{q(x)\}^{-1}\ y^2(x)$$

$$\leq C\{1+(w'(x)^2\}^{\frac{1}{2}} \exp\{(\mu+\tfrac{1}{2}\lambda+\varepsilon-\tfrac{1}{2})w(x)\}$$

$$\Rightarrow \{q(x)\}^{-1}\{y'(x)\}^2+(1+\{w'(x)\}^2\ \{q(x)\}^{-1}\ y^2(x)$$

$$\leq C\ \{\ 1 + (v^{-1}(x))^2\}^{\frac{1}{2}} \exp\ \{\ (\ \varepsilon-\tfrac{1}{2}\ )w(x)\}$$

since $\lambda = \mu = 0$ and $w' = v^{-1}$

$$= C\{1+(v^{-1}(x))^2\}^{\frac{1}{2}} \exp\ \{(\varepsilon-\tfrac{1}{2})(1+ \int_a^x v^{-1}(t)\,dt\ )\}$$

for any $\varepsilon > 0$ and $x \geq a$.

$= 0$, since $\varepsilon - \frac{1}{2} < 0$.

$$\Rightarrow \{q(x)\}^{-1}\{y'(x)\}^2+(1+\{w'(x)\}^2\{q(x)\}^{-1}\ y^2(x)\to 0 \text{ as } x\to\infty$$

$$\Rightarrow \{q(x)\}^{-1}\{y'(x)\}^2 \to 0 \text{ and } y^2(x) \to 0 \text{ as } x\to\infty$$

$$\therefore\ q^{-\frac{1}{2}}(x)y'(x) \to 0 \text{ and } y(x) \to 0 \text{ as } x \to \infty.$$

Hence the corollary.

Example.

Let us consider the equation

$$y'' + x^2y = 0 . \qquad (3.75).$$

To get the solutions of (3.75), we apply the following theorem which we shall state without proof.

Theorem 3.3.4 (Bellman [3]).

If $q^{\frac{1}{2}}(x) > 0$, for $x \geq 0$ and

$$\int_a^\infty \frac{(q^{\frac{1}{2}}(x))'}{q^{3/2}(x)}\,dx < \infty, \tag{3.76}$$

$$\int_a^\infty \left| \frac{d}{dx} \left(\frac{(q^{\frac{1}{2}}(x))'}{q(x)} \right) \right| dx < \infty, \tag{3.77}$$

then all solutions of

$$y'' + q(x)y = 0 \tag{3.78}$$

have the form

$$y = \frac{C_1 \cos\left(\int_0^x q^{\frac{1}{2}}(t)\,dt\right) + C_2 \sin\left(\int_0^x q^{\frac{1}{2}}(t)\,dt\right)}{q^{1/4}(x)} + O(q^{-1/4}(x)) \tag{3.79}$$

as $x \to \infty$, for some constants C_1 and C_2.

From (3.75) and (3.78),

$$q(x) = x^2$$

$$\therefore q^{\frac{1}{2}}(x) = x$$

Let $q^{\frac{1}{2}}(x) = x > 0$ for $x \geq a > 0$.

Then $$\int_a^\infty \frac{(q(x))'}{q^{3/2}(x)}\,dx = \int_a^\infty \frac{dx}{x^3} = \frac{1}{2a} < \infty \tag{3.80}$$

and $$\int_a^\infty \left| \frac{(q^{\frac{1}{2}}(x))'}{q(x)} \right| dx = \int_a^\infty \left| \frac{1}{x^2} \right| dx = \frac{1}{a} < \infty \tag{3.81}$$

(3.80) and (3.81) satisfy (3.76) and (3.77).

∴ all solutions of (3.75) have the form

$$y = \frac{C_1 \cos\left(\frac{x^2}{2}\right) + C_2 \sin\left(\frac{x^2}{2}\right)}{x^{\frac{1}{2}}} + O(x^{-\frac{1}{2}}) \qquad (3.82)$$

as $x \to \infty$.

$$y' = \frac{\{-C_1 x \sin(\frac{x^2}{2}) + C_2 x \cos(\frac{x^2}{2})\}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\{C_1\cos(\frac{x^2}{2}) + C_2\sin(\frac{x^2}{2})\}}{x}$$

$$+ \{O(x^{-\frac{1}{2}})\}' \qquad (3.83)$$

Thus

$$q^{-\frac{1}{2}}(x)y'(x) = \frac{\{-C_1 x\sin(\frac{x^2}{2}) + C_2 x\cos(\frac{x^2}{2})\}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}\{C_1\cos(\frac{x^2}{2}) + C_2\sin(\frac{x^2}{2})\}}{x^2}$$

$$+ \frac{\{O(x^{-\frac{1}{2}})\}'}{x} \qquad (3.84)$$

From (3.82), $y(x) \to 0$ as $x \to \infty$ (3.85)

and from (3.84), $q^{-\frac{1}{2}}(x)y'(x) \to 0$ as $x \to \infty$. (3.86)

Thus (3.85) and (3.86) establish the conclusion of Theorem 3.3.1 for $\alpha(x) = 2x^{-1}$ and $a=4$ in particular.

3.4 Conditions for existence of $L^r[a,\infty)$ - solutions.

In this section, we shall give conditions for which (iii) holds.

Theorem 3.4.1.

Let Ψ be positive and twice differentiable on $[a,\infty)$, and assume that

$$\int_a^\infty \psi^2 |\psi^{-4} - q - \psi''\psi^{-1}| < \infty \qquad (3.87)$$

Then for $x \geq a$,

$$C^{-1}\,\psi^2(x) \leq z^2(x) \leq C\,\psi^2(x) \qquad (3.88)$$

for some constant C. Also all solutions of (1.1) are in $L^r[a,\infty)$, $r > 0$, if and only if $\psi \in L^r[a,\infty)$.

Proof:

We set $\theta = \psi^2$, $\Phi = 1$ and $\gamma\theta^{-1} = -\psi'\psi^{-1}$. (3.89)

Since $\theta = \psi^2$,

then $\log\theta = 2\log\psi$

$$\Rightarrow \theta'\theta^{-1} = 2\psi'\psi^{-1} , \qquad (3.90)$$

Since $\Phi = 1$, then $\Phi' = 0$,

From (2.55), $D = 2(\theta'\theta^{-1} + \gamma\theta^{-1})$, since $p = 1$

$= 2(2\psi'\psi^{-1} - \psi'\psi^{-1})$ by (3.89) and (3.90)

$$\therefore D = 2\,\psi'\psi^{-1} , \qquad (3.91)$$

Now $\frac{1}{2}(A+C) = \frac{1}{2} \cdot 2\,(\theta'\theta^{-1} + \Phi'\Phi^{-1})$ by (2.9) and (2.11)

$= \theta'\theta^{-1}$, since $\Phi' = 0$

$= 2\psi'\psi^{-1}$ by (3.90)

$= D$ by (3.91)

$$\therefore \tfrac{1}{2}(A+C) = D . \qquad (3.92)$$

We note that $\frac{1}{2}(A+C) \leq \frac{1}{2}(A+C) + \{\frac{1}{4}(A-C) + B^2\}^{\frac{1}{2}}$

$= \lambda_M$ by (2.16)

i.e. $\frac{1}{2}(A+C) \leq \lambda_M$

or $\quad D \leq \lambda_M \qquad (3.93)$

By (2.25),

$$F = \{\phi'\phi^{-1} - \phi'\phi^{-1} - 2\gamma\theta^{-1}\}^2 + (\theta\phi^{-1})^2\{-q+(\phi\theta^{-1})^2 - (\gamma\theta^{-1})^2+(\gamma\theta^{-1})'\}^2, \text{since } p=1$$

$$= \{-2\psi'\psi^{-1} + 2\psi'\psi^{-1}\}^2 + \psi^4\{-q+\psi^{-4} - (\psi')^2\psi^{-2}+(-\psi'\psi^{-1})'\}^2$$

$$= \psi^4\{-q+\psi^{-4} - (\psi')^2\psi^{-2} - \psi''\psi^{-1} + (\psi')^2\psi^{-2}\}^2$$

$$= \psi^4\{\psi^{-4} - q - \psi''\psi^{-1}\}^2$$

or $F^{\frac{1}{2}} = \psi^2|\psi^{-4} - q - \psi''\psi^{-1}|.$

$$\Rightarrow \int_a^x F^{\frac{1}{2}} = \int_a^x \psi^2|\psi^{-4} - q - \psi''\psi^{-1}|.$$

$$\leq \int_a^\infty \psi^2|\psi^{-4} - q - \psi''\psi^{-1}|$$

$< \infty$ by (3.87).

i.e. $\int_a^x F^{\frac{1}{2}} < \infty$. $\qquad (3.94)$

Since $p = 1$, $\Phi = 1$ and $\Theta = \psi^2$

$$K^{-1}\psi^2(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq z^2(x) \leq K\psi^2(x)\exp\left(\int_a^x F^{\frac{1}{2}}\right) \text{ by (2.62)}$$

or $\qquad C^{-1}\psi^2(x) \leq z^2(x) \leq C\psi^2(x)$

where $\quad C = K\exp\left(\int_a^x F^{\frac{1}{2}}\right).$

This proves (3.88).

Now, from the right hand inequality of (3.88),

$$z^2 < C\psi^2$$

or $z^r \leq C^{r/2} \psi^r$ for some $r > 0$

or $(u^2+v^2)^{r/2} \leq M\psi^r \leq M|\psi|^r$ where $M = C^{r/2}$

$$\int_a^\infty (u+v)^{r/2} \leq M \int_a^\infty |\psi| < \infty \text{ if } \psi \in L^r[a,\infty).$$

But for $r > 0$, and two constants k_1 and k_2,

$$k_1(|u|^r + |v|^r) \leq (u^2+v^2)^{r/2} \leq k_2(|u|^r + |v|^r). \qquad (3.95).$$

Thus, considering the right hand inequality of (3.95) and integrating on $[a,\infty)$, we have

$$\int_a^\infty (u^2+v^2)^{r/2} \leq k_2 \int_a^\infty (|u|^r + |v|^r) < \infty \text{ if } \psi \in L^r[a,\infty).$$

i.e. $\int_a^\infty |u|^r + \int_a^\infty |v|^r < \infty$ if $\psi \in L^r[a,\infty)$.

$\Rightarrow \int_a^\infty |u|^r < \infty$ and $\int_a^\infty |v|^r < \infty$ if $\psi \in L^r[a,\infty)$.

i.e. $u \in L^r[a,\infty)$ and $v \in L^r[a,\infty)$ if $\psi \in L^r[a,\infty)$.

Thus $u, v \in L^r[a,\infty)$ if $\psi \in L^r[a,\infty)$.

If we consider the left hand inequalities of (3.88) and (3.95), we can prove the converse that

$\psi \in L^r[a,\infty)$ if $u, v \in L^r[a,\infty)$, $r > 0$.

Hence $u, v \in L^r[a,\infty)$ if and only if $\psi \in L^r[a,\infty)$.

This proves the theorem.

CHAPTER IV

STABILITY RESULTS II

4.1 Introduction

In this chapter we shall seek for conditions on p and q which guarantee that conditions opposite to (i), (ii) and (iii) in chapter III hold. That is

(iv) All non-trivial solutions of (1.1) are unbounded

(v) No non-trivial solution of (1.1) tends to zero at ∞

and (vi) For some r, $0<r<\infty$, no non-trivial solution of (1.1) lies in $L^r[a,\infty)$.

Before we discuss these problems, it is worth noting from Knowles [18] that by the left hand inequality of (2.31),

$$k\theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x)$$

$$\text{or } k\theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq \{\theta(x)p(x)y'(x)+\gamma(x)y(x)\}^2 + \Phi^2(x)y^2(x),$$

we see that (1.1) can have no non-trivial solutions that are bounded on $[a,\infty)$ if there exist functions Φ,θ and γ such that each of the following conditions are satisfied:

(A) Φ and γ are bounded on $[a,\infty)$,

(B) $p\theta y'$ is bounded on $[a,\infty)$ whenever y is a bounded real-valued solution of (1.1),

(C) $\theta\Phi\exp\left(-\int_a^x F^{\frac{1}{2}}\right)$ is unbounded on $[a,\infty)$, where F is given by (2.25).

Thus we need criteria giving sufficient conditions on p and q for (B) to hold. In this respect, the following lemma is of interest.

Lemma 4.1.1

(B) holds if there exists a function θ on $[a, \infty)$ satisfying at least one of the following conditions:

(i) $q \in L[a,\infty)$ and θ is bounded on $[a, \infty)$

(ii) $p\theta'$ is bounded, locally absolutely continuous and of bounded variation on $[a,\infty)$ and $\theta q \in L[a, \infty)$;

(iii) θ is locally absolutely continuous on $[a, \infty)$ $\theta' \leq 0$, $q \geq 0$, and $[(\theta^2 pq)']_+ \in L[a, \infty)$.

Proof:

(i) From (1.1), $(py')' = -qy$

$$\int_a^x (py')' = -\int_a^x qy$$

$$py' \Big|_a^x = -\int_a^x qy$$

$$p(x)y'(x) = p(a)y'(a) - \int_a^x qy$$

Thus $\theta(x)p(x)y'(x) = \theta(x)p(a)y'(a) - \theta(x) \int_a^x qy$

$$\leq \theta(x)p(a)y'(a) - \theta(x) \int_a^\infty qy$$

$< \infty$ whenever y is bounded, since θ is bounded and $q \in L[a,\infty)$.

Hence (B) holds.

(ii) From (1.1) $\theta (py')' = - \theta qy$

$$\int_a^x \theta(py')' = - \int_a^x \theta qy$$

$$\theta py' \Big|_a^x - \int_a^x (py')\theta' = - \int_a^x \theta qy$$

$$\theta py' \Big|_a^x - \int_a^x (p\theta')y' = - \int_a^x \theta qy$$

$$\theta py' \Big|_a^x - \left[p\theta' y \Big|_a^x - \int_a^x y(p\theta')' \right] = - \int_a^x \theta qy$$

$$\theta py' \Big|_a^x - p\theta' y \Big|_a^x + \int_a^x y(p\theta')' = - \int_a^x \theta qy$$

$$\theta(x)p(x)y'(x) - \theta(a)p(a)y'(a) - p(x)\theta'(x)y(x) + p(a)\theta'(a)y(a) + \int_a^x y(p\theta')' = - \int_a^x \theta qy$$

$$\theta(x)p(x)y'(x) = \theta(a)p(a)y'(a) - p(a)\theta'(a)y(a) + p(x)\theta'(x)y(x) - \int_a^x y(p\theta')' - \int_a^x \theta qy$$

$$\leq \theta(a)p(a)y'(a) - p(a)\theta'(a)y(a) + p(x)\theta'(x)y(x) - \int_a^\infty y(p\theta')' - \int_a^\infty \theta qy$$

$$= O(1) + p(x)\theta'(x)y(x) - \int_a^\infty y(p\theta')' - \int_a^\infty \theta qy$$

$$= O(1) - \int_a^\infty \theta qy \quad \text{since } p\theta' \text{ is bounded}$$

$< \infty$ whenever y is bounded, since $\theta q \in L[a,\infty)$.

Hence (B) holds.

(iii). Multiplying (1.1) by $p\theta^2 y'$, we get

$$p\theta^2 y'(py')' = p\theta^2 y'(-qy)$$

or $$\theta^2(py')'.py' = -(\theta^2 pq)(yy')$$

Thus $$\int_a^x \theta^2(py')'.py' = -\int_a^x (\theta^2 pq)(yy') \qquad (4.1)$$

We note that

$$\int_a^x \theta^2(py')'.py' = \theta^2(py')^2 \Big|_a^x - \int_a^x \theta^2(py')(py')' - \int_a^x (py')^2(\theta^2)'$$

$$2\int_a^x \theta^2(py')'.py' = \theta^2(py')^2 \Big|_a^x - \int_a^x (py')^2(\theta^2)'$$

$$\int_a^x \theta^2(py')'.py' = \tfrac{1}{2}\left[\theta^2(py')^2 \Big|_a^x - \int_a^x (py')^2(\theta^2)'\right] \qquad (4.2)$$

Combining (4.1) and (4.2), we have

$$\tfrac{1}{2}\left[\theta^2(py')^2\Big|_a^x - \int_a^x (py')^2(\theta^2)'\right] = -\frac{(\theta^2 pq)}{2}.y^2 \Big|_a^x - \int_a^x (\theta^2 pq)'\frac{y}{2}$$

$$= -\tfrac{1}{2}\left[(\theta^2 pq)y^2 \Big|_a^x - \int_a^x (\theta^2 pq)'y\right]$$

$$\Rightarrow \left[\theta^2(py')^2\Big|_a^x - \int_a^x (py')^2(\theta^2)'\right] = -\left[(\theta^2 pq)y^2\Big|_a^x - \int_a^x (\theta^2 pq)'y\right]$$

$$\left[\theta(x)p(x)y'(x)\right]^2 - \left[\theta(a)p(a)y'(a)\right]^2 - \int_a^x (py')^2(\theta^2)'$$

$$= -\theta^2(x)p(x)q(x)y^2(x) + \theta^2(a)p(a)q(a)y^2(a) + \int_a^x (\theta^2 pq)'y$$

$$\left[\theta(x)p(x)y'(x)\right]^2 = \left[\theta(a)p(a)y'(a)\right]^2 + \theta^2(a)p(a)q(a)y^2(a)$$

$$- \theta^2(x)p(x)q(x)y^2(x) + \int_a^x (py')^2(\theta^2)' + \int_a^x (\theta^2 pq)' y^2$$

$$= O(1) - \theta^2(x)p(x)q(x)y^2(x) + \int_a^x (py')^2(\theta^2)' + \int_a^x (\theta^2 pq)' \cdot y^2$$

$$\leq O(1) + \int_a^x (\theta^2 pq)' y^2 \quad \text{since } p>0,\ q\geq 0,\ \theta' \leq 0$$

$$\leq O(1) + \int_a^x \left[(\theta^2 pq)'\right]_+ \cdot y^2 \quad \text{since } (\theta^2 pq)' \leq \left[(\theta^2 pq)'\right]_+$$

$$\leq O(1) + \int_a^\infty \left[(\theta^2 pq)'\right]_+ \cdot y^2$$

$$< \infty \quad \text{since } \left[(\theta^2 pq)'\right]_+ \in L[a,\infty) \text{ whenever } y \text{ is bounded.}$$

Hence (B) holds.

Thus, in all cases, (B) holds.

This proves the lemma.

4.2. Unboundedness of non-trivial solutions

In this section, we give conditions for which (iv) holds.

Theorem 4.2.1

Let $p=1$ and let q be positive and twice differentiable on

$[a,\infty)$. Assume also that q is bounded and q^{-1} is unbounded on $[a,\infty)$,

$$\left|\left(q^{-\frac{1}{2}}\right)'\right| = O(1) \quad \text{as} \quad x \to \infty, \tag{4.3}$$

$$q^{-\frac{1}{4}}\left(q^{-\frac{1}{4}}\right)'' \in L[a,\infty), \tag{4.4}$$

and for some β, $\frac{1}{2} \leq \beta < \frac{3}{4}$

$$q^{\beta} \in L[a,\infty). \tag{4.5}$$

Then all non-trivial solutions of (1.1) are unbounded on $[a,\infty)$.

Proof:

By (2.38), $q^{-\frac{1}{2}}(y')^2 + q^{\frac{1}{2}}y^2 \geq C > 0$ on $[a,\infty)$ (4.6)

for some constant C.

Now $q^{2\beta-(3/2)} > 0$, since $q > 0$. (4.7)

Multiplying both sides of (4.6) by (4.7), we have

$$q^{2\beta-(3/2)}q^{-\frac{1}{2}}(y')^2 + q^{2\beta-(3/2)}q^{\frac{1}{2}}y^2 \geq C\,q^{2\beta-(3/2)}$$

or $q^{2\beta-2}(y')^2 + q^{2\beta-1}y^2 \geq C\,q^{(4\beta-3)/2}$

or $q^{2(\beta-1)}(y')^2 + q^{2\beta-1}y^2 \geq C\,q^{-(3-4\beta)/2}$

or $q^{2\alpha}(y')^2 + q^{2\beta-1}y^2 \geq C\,q^{-\varepsilon}$

where $\alpha = \beta-1$ and $\varepsilon = (3-4\beta)/2$.

i.e. $(q^{\alpha})^2(y')^2 + q^{2\beta-1}y^2 \geq Cq^{-\varepsilon}$ (4.8)

Since $\alpha = \beta - 1$ and $\frac{1}{2} \leq \beta < \frac{3}{4}$,

then $\alpha + \frac{1}{2} \geq 0$ (4.9)

Also $q(x) = O(1)$, since q is bounded. (4.10)

Now let $\theta = q^{\alpha}$

$$\theta' = \alpha q^{\alpha-1} q'$$

$$= -2\alpha(q^{-\frac{1}{2}})' q^{\alpha+\frac{1}{2}}$$

$$|\theta'| = |2\alpha(q^{-\frac{1}{2}})' q^{\alpha+\frac{1}{2}}|$$

$$= |2\alpha||(q^{-\frac{1}{2}})'| q^{\alpha+\frac{1}{2}}$$

$$= O(1) \quad \text{as} \quad x \to \infty \qquad (4.11)$$

by (4.3) and (4.10).

Since $\theta' = \alpha q' q^{\alpha-1}$

$$\theta'' = \alpha\left[q'' q^{\alpha-1} + (\alpha - 1)(q')^2 q^{\alpha-2}\right]$$

$$= \alpha q'' q^{\alpha-1} + \alpha(\alpha-1)(q')^2 q^{\alpha-2}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}-3/2} + \alpha^2 (q')^2 q^{\alpha+1-3} - \alpha (q')^2 q^{\alpha+1-3}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \alpha^2 (q')^2 q^{\beta-3} - \alpha (q')^2 q^{\beta-3} \quad \text{, since } \alpha = \beta - 1$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \alpha^2 (q')^2 q^{\beta-3} - \alpha (q')^2 q^{\beta-1} + \frac{5\alpha}{4} (q')^2 q^{\beta-1}$$

$$- \frac{5\alpha}{4} (q')^2 q^{\beta-3}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \frac{5\alpha}{4} (q')^2 q^{\beta-3} - \frac{9\alpha}{4} (q')^2 q^{\beta-3} + \alpha^2 (q')^2 q^{\beta-3}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \frac{5\alpha}{4} (q')^2 q^{\alpha+\frac{1}{2}-5/2} - \frac{9\alpha}{4} (q')^2 q^{\beta-3} + \alpha^2 (q')^2 q^{\beta-3}$$

$$= \alpha q^{\alpha+\frac{1}{2}} (q'' q^{-3/2} + \frac{5}{4} (q')^2 q^{-5/2} - \frac{9\alpha}{4} - \alpha^2) (q')^2 q^{\beta-3}$$

$$|\theta''| = |\alpha q^{\alpha+\frac{1}{2}}(q''q^{-3/2} + \frac{5}{4}(q')^2 q^{-5/2}) - (\frac{9\alpha}{4} - \alpha^2)(q')^2 q^{\beta-3}|$$

$$\leq |\alpha| q^{\alpha+\frac{1}{2}} \left| q''q^{-3/2} + \frac{5}{4}(q')^2 q^{-5/2} \right| + \left| \frac{9\alpha}{4} - \alpha^2 \right| (q')^2 q^{\beta-3} ,$$

since $q > 0$.

Thus $$\int_a^x |\theta''| \leq \int_a^x |\alpha| q^{\alpha+\frac{1}{2}} \left| q''q^{-3/2} + \frac{5}{4}(q')^2 q^{-5/2} \right| + \int_a^x \left| \frac{9\alpha}{4} - \alpha^2 \right| (q')^2 q^{\beta-3}$$

..... (4.12)

We note that $q^{-\frac{1}{4}}(q^{-\frac{1}{4}})'' = -\frac{1}{4}\left[q''q^{-3/2} - \frac{5}{4}(q')^2 q^{-5/2} \right]$

$$\therefore \quad \left| q^{-\frac{1}{4}}(q^{-\frac{1}{4}})'' \right| \leq \left| q''q^{-3/2} \right| + \left| \frac{5}{4}(q')^2 q^{-5/2} \right| \qquad (4.13)$$

$$\in L\,[a, \infty), \quad \text{by (4.4).}$$

Also $$(q')^2 q^{-3} q^{\beta} = (-2q'q^{-3/2}q^{\beta})(-\tfrac{1}{2}q'q^{-3/2})$$

$$= (-2q'q^{-3/2}q^{\beta})(q^{-\frac{1}{2}})'$$

$$= \left| -2q'q^{-3/2}q^{\beta} \right| \left| (q^{-\frac{1}{2}})' \right| \text{ since } q > 0$$

$$= O(1) \text{ as } x \to \infty \qquad (4.14)$$

by (4.3).

Now applying (4.13) and (4.14) to (4.12), we get

$$\int_a^x |\theta''| \leq O(1) \qquad (4.15)$$

Next, from (4.5), $q^{\beta} \in L\,[a, \infty)$

$q^{\alpha}q \in L\,[a, \infty)$ since $\alpha = \beta - 1$

$\theta q \in L\,[a, \infty)$ since $\theta = q^{\alpha}$ (4.16)

Thus by (4.11), (4.15) and (4.16), Lemma 4.1.1[part (ii)] is satisfied. Consequently, (B) holds.

Since $\frac{1}{2} \leq \beta < \frac{3}{4}$,

$\Rightarrow \quad \varepsilon > 0 \quad$ and $\quad 2\beta - 1 \geq 0$.

Then by (4.8), all non-trivial solutions of (I.1) are unbounded on $[a,\infty)$.

This proves the theorem.

We give one more theorem on conditions that will guarantee that (iv) holds.

Theorem 4.2.2.

Set p=1 and assume that q is positive and three times differentiable on $[a,\infty)$. If q is bounded and q^{-1} is unbounded on $[a,\infty)$,

$$\phi^2 = \tfrac{1}{2}(q^{-\frac{1}{2}})'' + q^{\frac{1}{2}} - \tfrac{1}{4}\left[(q^{-\frac{1}{2}})'\right]^2 q^{\frac{1}{2}} > 0 \qquad (4.17)$$

$$\left|(q^{-\frac{1}{2}})'\right| = O(1) \quad \text{as} \quad x \to \infty \qquad (4.18)$$

$$\left|(q^{-\frac{1}{2}})''\right| = O(q^{\frac{1}{2}}) \quad \text{as} \quad x \to \infty \qquad (4.19)$$

$$\phi^{-2}\left[(q^{-\frac{1}{2}})''\right]_- \in L[a,\infty) \qquad (4.20)$$

and for some β, $\frac{1}{2} \leq \beta < \frac{3}{4}$, (4.21)

$$q^{\beta} \in L[a,\infty) \qquad (4.22)$$

then (1.1) has no non-trivial solutions that are bounded on $[a,\infty)$.

Proof:

By left hand inequality of (2.28), we have

$$E(a)\exp\left[\tfrac{1}{2}\int_a^x (A+C)\right]\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x). \qquad (4.23)$$

Now let $\theta^2 = q^{-\frac{1}{2}}$ (4.24)

$$\gamma = -\tfrac{1}{2}(q^{-\frac{1}{2}})'q^{\frac{1}{4}} \qquad (4.25)$$

and Φ^2 be as in (4.17).

We note that

$$2\log\theta = -\tfrac{1}{2}\log q$$

$$\text{or } 2\theta'\theta^{-1} = -\tfrac{1}{2}q'q^{-1}$$

$$\text{or } \theta'\theta^{-1} = -\tfrac{1}{4}q'q^{-1}. \qquad (4.26)$$

$$\text{Also } \gamma\theta^{-1} = -\tfrac{1}{2}(q^{-\frac{1}{2}})'q^{\frac{1}{4}}q^{\frac{1}{4}}$$

$$= -\tfrac{1}{2}\left[-\tfrac{1}{2}q'q^{-3/2}\right]q^{\frac{1}{2}}$$

$$= \tfrac{1}{4}q'q^{-1}. \qquad (4.27)$$

$$\text{Thus } \theta'\theta^{-1} + \gamma\theta^{-1} = -\tfrac{1}{4}q'q^{-1} + \tfrac{1}{4}q'q^{-1}$$

$$\text{i.e. } \theta'\theta^{-1} + \gamma\theta^{-1} = 0. \qquad (4.28)$$

We also note that

$$-q + \left(\frac{\Phi}{\theta}\right)^2 - \left(\frac{\gamma}{\theta}\right)^2 + \left(\frac{\gamma}{\theta}\right)'$$

$$= -q + \left\{\frac{\frac{1}{2}(q^{-\frac{1}{2}})'' + q^{\frac{1}{2}} - \frac{1}{4}\left[(q^{-\frac{1}{2}})'\right]^2 q^{\frac{1}{2}}}{q^{-\frac{1}{2}}}\right\} - \tfrac{1}{4}q'q^{-1} + \left(\tfrac{1}{4}q'q^{-1}\right)'$$

$$= -q + \left\{\tfrac{1}{2}(q^{-\frac{1}{2}})''q^{\frac{1}{2}} + q - \tfrac{1}{4}\left[(q^{-\frac{1}{2}})'\right]^2 q\right\} - \frac{1}{16}(q')^2q^{-2} + \tfrac{1}{4}\left\{q''q^{-1} - (q')^2q^{-2}\right\}$$

$$= -q + \left\{ -\tfrac{1}{4}\left[q'' q^{-1} - \frac{3}{2}(q')^2 q^{-2} \right] + q - \tfrac{1}{4}\left[-\tfrac{1}{2} q' q^{-3/2} \right]^2 q \right. - \frac{1}{16}(q')^2 q^{-2}$$

$$\left. + \tfrac{1}{4} q'' q^{-1} - \tfrac{1}{4}(q')^2 q^{-2} \right.$$

$$= -q - \tfrac{1}{4} q'' q^{-1} + \frac{3}{8}(q')^2 q^{-2} - \frac{3}{8}(q')^2 q^{-2} - \tfrac{1}{4} q'' q^{-1} + \tfrac{1}{4} q'' q^{-1}$$

$$= 0$$

$$\text{i.e.} \quad \left[-q + \left(\frac{\Phi}{\theta}\right)^2 - \left(\frac{\gamma}{\theta}\right)^2 + \left(\frac{\gamma}{\theta}\right)' \right] = 0 \qquad (4.29)$$

From (4.17),

$$\Phi^2 = \tfrac{1}{2}(q^{-\frac{1}{2}})'' + q^{\frac{1}{2}} - \tfrac{1}{4}\left[(q^{-\frac{1}{2}})' \right]^2 q^{\frac{1}{2}},$$

$$2\Phi'\Phi = \tfrac{1}{2}(q^{-\frac{1}{2}})''' + \tfrac{1}{2} q' q^{-\frac{1}{2}} - \tfrac{1}{4}\left\{ 2\left[(q^{-\frac{1}{2}})' \right](q^{-\frac{1}{2}})'' q^{\frac{1}{2}} + \left[(q^{-\frac{1}{2}})' \right]^2 \cdot \tfrac{1}{2} q' q^{-\frac{1}{2}} \right\}$$

$$\text{or} \quad \Phi'\Phi^{-1} = \left\{ \tfrac{1}{4}(q^{-\frac{1}{2}})''' + \tfrac{1}{4} q' q^{-\frac{1}{2}} - \tfrac{1}{4}(q^{-\frac{1}{2}})'' q^{\frac{1}{2}} - \frac{1}{8}\left[(q^{-\frac{1}{2}})' \right]^2 q' q^{-\frac{1}{2}} \Phi^{-2} \right\}$$

$$= \tfrac{1}{4}(q^{-\frac{1}{2}})''' \Phi^{-2} + \tfrac{1}{4} q' q^{-1} \left\{ q^{\frac{1}{2}} + \tfrac{1}{2}(q^{-\frac{1}{2}})'' - \tfrac{1}{4} \left| (q^{-\frac{1}{2}})' \right|^2 q^{\frac{1}{2}} \right\} \Phi^{-2}$$

$$= \tfrac{1}{4}(q^{-\frac{1}{2}})''' \Phi^{-2} + \tfrac{1}{4} q' q^{-1} \cdot \Phi^2 \Phi^{-2}$$

$$\therefore \quad \Phi'\Phi^{-1} = \tfrac{1}{4}(q^{-\frac{1}{2}})'' \Phi^{-2} + \tfrac{1}{4} q' q^{-1} \qquad (4.30)$$

Thus $\Phi'\Phi^{-1} - \gamma\theta^{-1} = \tfrac{1}{4}(q^{-\frac{1}{2}})''' \Phi^{-2} + \tfrac{1}{4} q' q^{-1} - \tfrac{1}{4} q' q^{-1}$ by (4.27) and (4.30)

$$= \tfrac{1}{4}(q^{-\frac{1}{2}})''' \Phi^{-2}$$

$$\text{i.e.} \quad \Phi'\Phi^{-1} - \gamma\theta^{-1} \geq -\tfrac{1}{4} \Phi^{-2} \left[(q^{-\frac{1}{2}})''' \right]_- \qquad (4.31)$$

By (2.11), $A = 2(\Phi'\Phi^{-1} + \gamma\theta^{-1})$, since $p = 1$

$$= 0 \quad \text{by (4.28).}$$

Thus $\tfrac{1}{2}(A+C) = \tfrac{1}{2}C$ since $A = 0$

$$= \tfrac{1}{2} \cdot 2(\Phi'\Phi^{-1} - \gamma\theta^{-1}) \text{ from (2.11) and since } p = 1$$

$$= \Phi'\Phi^{-1} - \gamma\Theta^{-1}$$

$$\geq -\tfrac{1}{4}\,\Phi^{-2}\left[(q^{-\frac{1}{2}})'''\right]_- \quad \text{by (4.31).}$$

$$\Rightarrow \tfrac{1}{2}\int_a^x (A+C) \geq -\tfrac{1}{4}\int_a^x \Phi^{-2}\left[(q^{-\frac{1}{2}})'''\right]_-$$

$$\geq -\tfrac{1}{4}\int_a^\infty \Phi^{-2}\left[(q^{-\frac{1}{2}})'''\right]_-$$

$$> -\infty \quad \text{by (4.20).}$$

Thus $\exp\left[\tfrac{1}{2}\int_a^x (A+C)\right] > \exp(-\infty) = 0.$

$$\text{i.e.} \quad \exp\left[\tfrac{1}{2}\int_a^x (A+C)\right] > 0. \qquad (4.32)$$

By (2.25), $F = \left[\Phi'\Phi^{-1} - \Theta'\Theta^{-1} - 2\gamma\Theta^{-1}\right]^2 + \frac{\Theta^2}{\Phi^2}\left[-q + \left(\frac{\Phi}{\Theta}\right)^2 - \left(\frac{\gamma}{\Theta}\right)^2 + \left(\frac{\gamma}{\Theta}\right)'\right]$ since $p=1$

$$= \left[\Phi'\Phi^{-1} - \gamma\Theta^{-1} - (\Theta'\Theta^{-1} + \gamma\Theta^{-1})\right]^2 + \frac{\Theta^2}{\Phi^2}\left[-q + \left(\frac{\Phi}{\Theta}\right)^2 - \left(\frac{\gamma}{\Theta}\right)^2 + \left(\frac{\gamma}{\Theta}\right)'\right]$$

$$= \left[\Phi'\Phi^{-1} - \gamma\Theta^{-1}\right]^2 \quad \text{by (4.28) and (4.29).}$$

$$F^{\frac{1}{2}} = \Phi'\Phi^{-1} - \gamma\Theta^{-1}$$

$$\geq -\tfrac{1}{4}\,\Phi^{-2}\left[(q^{-\frac{1}{2}})'''\right]_- \quad \text{by (4.31).}$$

$$-F^{\frac{1}{2}} \leq \tfrac{1}{4}\,\Phi^{-2}\left[(q^{-\frac{1}{2}})'''\right]_-$$

$$-\int_a^x F^{\frac{1}{2}} \leq \tfrac{1}{4}\int_a^x \Phi^{-2}\left[(q^{-\frac{1}{2}})'''\right]_- \leq \int_a^\infty \Phi^{-2}\left[(q^{-\frac{1}{2}})'''\right]_- < \infty \quad \text{by (4.20).}$$

$$\text{i.e.} \quad -\int_a^x F^{\frac{1}{2}} < \infty \qquad (4.33)$$

Applying (4.32) and (4.33) to (4.23), we get

$$0 \leq E(x) = \{\theta(x)y'(x) + \gamma(x)y(x)\}^2 + \phi^2(x)y^2(x)$$

since p=1 and by (2.1)

$$\Rightarrow \{\theta y' + \gamma y\}^2 + \phi^2 y^2 \geq C > 0 \qquad (4.34)$$

for some constant C

By (4.25), $\gamma = -\frac{1}{2}(q^{-\frac{1}{2}})' q^{\frac{1}{4}}$

$$|\gamma| = \tfrac{1}{2}\left|(q^{-\frac{1}{2}})'\right| q^{\frac{1}{4}}$$

$$\Rightarrow |\gamma| = O(q^{\frac{1}{4}}) \text{ as } x \to \infty \text{ by (4.18).} \qquad (4.35)$$

Also, by (4.17)

$$|\phi^2| = \tfrac{1}{2}\left|(q^{-\frac{1}{2}})''\right| + q^{\frac{1}{2}} + \tfrac{1}{4}\left|(q^{-\frac{1}{2}})'\right|^2 q^{\frac{1}{2}}$$

$$= O(q^{\frac{1}{2}}) \text{ as } x \to \infty \text{ by (4.18) and (4.19).}$$

i.e. $\phi^2 = O(q^{\frac{1}{2}})$ as $x \to \infty$. (4.36)

From (4.34), $(\theta y' + \gamma y)^2 + \phi^2 y^2 \geq C > 0$

$$\theta^2(y')^2 + 2\theta\gamma y'y + \gamma^2 y^2 + \phi^2 y^2 \geq C > 0$$

$$q^{-\frac{1}{2}}(y')^2 + 2q^{-\frac{1}{4}}(-\tfrac{1}{2}(q^{-\frac{1}{2}})')q^{\frac{1}{4}}y'y + (\tfrac{1}{4}\left[(q^{-\frac{1}{2}})'\right]^2 q^{\frac{1}{2}} + \tfrac{1}{2}(q^{-\frac{1}{2}})'' + q^{\frac{1}{2}} - \tfrac{1}{4}\left[(q^{-\frac{1}{2}})'\right]^2 q^{\frac{1}{2}})y^2$$

$$\geq C > 0$$

$$q^{-\frac{1}{2}}(y')^2 - (q^{-\frac{1}{2}})'y'y + \tfrac{1}{2}(q^{-\frac{1}{2}})''y^2 + q^{\frac{1}{2}}y^2 \geq C > 0$$

$$q^{-\frac{1}{2}}(y')^2 + \left|(q^{-\frac{1}{2}})'\right| y'y + \tfrac{1}{2}\left|(q^{-\frac{1}{2}})''\right| y^2 + q^{\frac{1}{2}}y^2 \geq C > 0$$

$$\Rightarrow q^{-\frac{1}{2}}(y')^2 + q^{\frac{1}{2}}y^2 \geq C_1 > 0 \qquad (4.37)$$

For some constant C_1.

If (as in Theorem 4.2.1) we let $\theta = q^{\alpha}$,

then $\theta'' = (q^{\alpha})''$

$$= \alpha\left[q'' q^{\alpha-1} + (\alpha-1)(q')^2 q^{\alpha-2}\right]$$

$$= \alpha q'' q^{\alpha-1} + \alpha(\alpha-1)(q')^2 q^{\alpha-2}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}-3/2} + \alpha^2 (q')^2 q^{\alpha+1-3} - \alpha (q')^2 q^{\alpha+1-3}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \alpha^2 (q')^2 q^{\beta-3} - \alpha (q')^2 q^{\beta-3}, \quad \text{since } \alpha = \beta - 1$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \alpha^2 (q')^2 q^{\beta-3} - \alpha (q')^2 q^{\beta-3} + \frac{3\alpha}{2}(q')^2 q^{\beta-3} - \frac{3\alpha}{2}(q')^2 q^{\beta-3}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \frac{3\alpha}{2}(q')^2 q^{\beta-3} - \frac{5\alpha}{2}(q')^2 q^{\beta-3} + \alpha^2 (q')^2 q^{\beta-3}$$

$$= \alpha q'' q^{\alpha+\frac{1}{2}} q^{-3/2} + \frac{3\alpha}{2}(q')^2 q^{\alpha+\frac{1}{2}-5/2} - \frac{5\alpha}{2}(q')^2 q^{\beta-3} + \alpha^2 (q')^2 q^{\beta-3}$$

$$= \alpha q^{\alpha+\frac{1}{2}} (q'' q^{-3/2} + \frac{3}{2}(q')^2 q^{-5/2}) - \frac{5\alpha}{2} - \alpha^2 (q')^2 q^{\beta-3}$$

$$\therefore \left| (q^\alpha)'' \right| \leq \left| \alpha \left| q^{\alpha+\frac{1}{2}} \left| q'' q^{-3/2} + \frac{3}{2}(q')^2 q^{-5/2} \right| \right| + \left| \frac{5\alpha}{2} - \alpha^2 \right| (q')^2 q^{\beta-3},$$

since $q > 0$.

Thus $$\int_a^X \left| (q^\alpha)'' \right| \leq \int_a^X |\alpha| q^{\alpha+1} \left| q'' q^{-3/2} + \frac{3}{2}(q')^2 q^{-5/2} \right| + \int_a^X \left| \frac{5\alpha}{2} - \alpha^2 \right| (q')^2 q^{\beta-3}$$

$$\leq \int_a^\infty |\alpha| q^{\alpha+\frac{1}{2}} \left| q'' q^{-3/2} + \frac{3}{2}(q')^2 q^{-5/2} \right| + \int_a^\infty \left| \frac{5\alpha}{2} - \alpha^2 \right| (q')^2 q^{-3} . q^\beta$$

$$= O(1) \quad \text{by (4.18), (4.19) and (4.22).}$$

i.e. $$\int_a^X \left| (q^\alpha)'' \right| \leq O(1) < \infty \qquad (4.38)$$

$$\Rightarrow (q^\alpha)'' \in L[a, \infty) \qquad (4.39)$$

By (4.11), (4.16) and (4.38), Lemma 4.1.1 (part ii) is satisfied. Then (B) holds.

(A) is guaranteed by (4.35) and (4.36)

(C) is guaranteed by the fact that $q^{-1} (= \theta^2)$ is unbounded.

Since (A), (B) and (C) are satisfied, the conclusion of the theorem follows.

4.3. Non-asymptotic behaviour of non-trivial solutions.

In this section, we shall give conditions for which (v) holds.

Before we state the theorem, we need the following result which was proved in [9] by Dolan and Klaasen :

If

$$\frac{\mu}{p} - \frac{q}{\mu} \in L\,[a, \infty) \qquad (4.40)$$

where $\mu>0$ and $\mu'\leq 0$ on $[a, \infty)$, then there are two possibilities. Either (1.1) is oscillatory, in which case (1.1) has no non-trivial solutions tending to zero at ∞, or (1.1) is non-oscillatory and the same conclusion holds provided that

$$\frac{\mu}{p} \notin L\,[a, \infty). \qquad (4.41)$$

Theorem 4.3.1

$$\text{If } \frac{\gamma\mu}{p},\ \left|\frac{\mu}{p} - \frac{q}{\mu} - \frac{\gamma^2\mu}{p} + \frac{(\gamma\mu)'}{\mu}\right| \in L\,[a, \infty) \qquad (4.42)$$

where $\mu>0$, $\mu'\leq 0$ and γ is bounded, then

(i) no non-trivial solution of (1.1) tends to zero at ∞ if (1.1) is oscillatory,

and (ii) no non-trivial solution of (1.1) tends to zero at ∞ if (1.1) is non-oscillatory and $\frac{\mu}{p} \notin L\,[b, \infty)$ for some $b \geq a$.

Proof:

By (2.10),

$$k\theta(x)\Phi(x)\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq E(x) = \{\theta(x)p(x)y'(x) + \gamma(x)y(x)\}^2 + \Phi^2(x)y^2(x)$$

i.e. $k\Theta\Phi \exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq (\Theta p y' + \gamma y)^2 + \Phi^2 y^2$.

With $\Phi = 1$ and $\Theta = \mu^{-1}$

$$k\mu^{-1}\exp\left(-\int_a^x F^{\frac{1}{2}}\right) \leq (p\mu^{-1}y' + \gamma y)^2 + y^2 \qquad (4.43)$$

From (2.25),

$$F = \left[0 + \mu'\mu^{-1} - \frac{2\gamma\mu}{p}\right]^2 + \mu^{-2}\left[-q + \frac{1}{p}\frac{1}{\mu^{-2}} - \frac{1}{p}(\gamma\mu)^2 + (\gamma\mu)'\right]^2$$

$$= \left[\mu'\mu^{-1} - \frac{2\gamma\mu}{p}\right]^2 + \left[-\frac{q}{\mu} + \frac{\mu}{p} - \frac{\gamma^2\mu}{p} + \frac{(\gamma\mu)'}{\mu}\right]^2$$

$$= \left[\mu'\mu^{-1} - \frac{2\gamma\mu}{p}\right]^2 + \left[\frac{\mu}{p} - \frac{q}{\mu} - \frac{\gamma^2\mu}{p} + \frac{(\gamma\mu)'}{\mu}\right]^2$$

$$F^{\frac{1}{2}} \leq \left|\mu'\mu^{-1} - \frac{2\gamma\mu}{p}\right| + \left|\frac{\mu}{p} - \frac{q}{\mu} - \frac{\gamma^2\mu}{p} + \frac{(\gamma\mu)'}{\mu}\right|$$

$$\int_a^x F^{\frac{1}{2}} \leq \int_a^x \left|\mu'\mu^{-1} - \frac{2\gamma\mu}{p}\right| + \int_a^x \left|\frac{\mu}{p} - \frac{q}{\mu} - \frac{\gamma^2\mu}{p} + \frac{(\gamma\mu)'}{\mu}\right|$$

$$\leq \int_a^x \left|\mu'\mu^{-1}\right| + 2\int_a^x \frac{\gamma\mu}{p} + \int_a^x \left|\frac{\mu}{p} - \frac{q}{\mu} - \frac{\gamma^2\mu}{p} + \frac{(\gamma\mu)'}{\mu}\right|$$

$$\leq 2\int_a^\infty \frac{\gamma\mu}{p} + \int_a^\infty \left|\frac{\mu}{p} - \frac{q}{\mu} - \frac{\gamma^2\mu}{p} + \frac{(\gamma\mu)'}{\mu}\right| + \int_a^\infty \left|\mu'\mu^{-1}\right|$$

$$= O(1) + \int_a^\infty \left|\mu\,\mu^{-1}\right| \quad \text{by} \quad (4.42)$$

$$= O(1) - \log\mu \quad , \text{ since} \quad \mu' \leq 0.$$

$$\text{i.e.} \quad \int_a^x F^{\frac{1}{2}} \leq O(1) - \log\mu \, . \qquad (4.44)$$

From (4.43) and (4.44),

$$k\,\mu^{-1}\exp(\log\mu) \leq (p\mu^{-1}y' + \gamma y)^2 + y^2$$

$$C \leq (p\mu^{-1}y' + \gamma y)^2 + y^2 \quad \text{for some constant } C$$

$$\text{or } (p\mu^{-1}y' + \gamma y)^2 + y^2 \geq C > 0 \qquad (4.45)$$

Since γ is bounded,

$$(p\mu^{-1}y')^2 + y^2 \geq C_1 > 0 \quad \text{for some constant } C_1. \qquad (4.46)$$

From (4.46) and on the strength of the result given by Dolan and Klaasen above, (i) and (ii) of the theorem hold. This proves the theorem.

4.4 Conditions for non existence of $L^r[a, \infty)$ - solutions

In this section, we shall give conditions for which (vi) holds. We shall depend heavily on the results obtained by Knowles in [16] and [17] for $r = 2$. Only sketches of the preliminary results will be mentioned and proofs of the theorems will be omitted.

By considering

$$\tau y(t) = -(p(t)y')' + q(t)y = 0 \qquad (4.47)$$

which is similar to (1.1) for some differential operator τ and by letting Ψ denote an arbitrary function for which Ψ' is locally absolutely continuous on $[a, \infty)$, Knowles in [16] notes that under the change of variables $(t \to s,\ y \to Y)$ defined by

$$S(t) = \int_a^t \Psi^2(r)\,dr \qquad (4.48)$$

$$Y(s) = y(t)(\Psi(t))^{-1} \qquad (4.49)$$

(4.47) becomes

$$\tau_1(s) = -(P(s)Y'(s))' + Q(s)Y(s) = 0, \quad 0 < s < s(\infty) \qquad (4.50)$$

for some other differential operator τ_1 ,

where $$P(s) = p(t)\Psi^{4}(t) \qquad (4.51)$$

$$Q(s) = q(t) - (p(t)\Psi'(t))'(\Psi(t))^{-1} \qquad (4.52)$$

We shall, therefore, work with the transformed equation (4.50) and results applying to (4.47) will be deduced via (4.48) and (4.49).

Let Ψ and Θ be arbitrary positive functions which are locally absolutely continuous on $[a,\infty)$ and for any real-valued solution Y of (4.50) define a Lypunov function

$$E(s) = \Theta^2(t)\left[P(s)Y'(s)\right]^2 + Y^2(s). \qquad (4.53)$$

Applying the techniques in Chapter II (Section 2.1), one can verify that

$$\frac{E'(s)}{E(s)} = A(s)\xi^2 + 2B(s)\xi\eta + C(s)\eta^2$$

where $\xi = \Theta P Y' E^{-\frac{1}{2}}$, $\eta = Y E^{-\frac{1}{2}}$ and $\xi^2 + \eta^2 = 1$

and for all $s \geq o$,

$$A(s) = 2\Theta'(t)\{\Theta(t)\Psi^2(t)\}^{-1}$$

$$B(s) = \{\Theta(t)P(s)\}^{-1} + \Theta(t)Q(s)$$

$$C(s) = 0$$

We also note that $$\lambda_o(s) \leq \frac{E'(s)}{E(s)} \leq \lambda_1(s) \qquad (4.54)$$

where $\lambda_1(s), \lambda_o(s) = \left[A(s) \pm \{A^2(s) + 4B^2(s)\}^{\frac{1}{2}}\right]/2$ for each $s \geq o$

are the eigenvalues of the matrix $\begin{bmatrix} A(s) & B(s) \\ B(s) & 0 \end{bmatrix}$

Integrating (4.54) with respect to s over $[o, s(t)]$ and using (4.48), (4.49), (4.51) and (4.52), we get

$$k\theta(t)\exp\left\{-\int_a^t F(r)\,dr\right\} \le E(s) \le k\theta(t)\exp\left\{\int_a^t F(r)\,dr\right\} \quad (4.55)$$

where $k = \dfrac{E(o)}{\theta(a)}$

and $$F = \left[\{\theta'\theta^{-1}\}^2 + \{(P\theta\psi^2)^{-1} + \theta\psi^2(q-(p\psi')'\psi^{-1})^2\right]^{\frac{1}{2}} \quad (4.56)$$

To simplify the notation, we assume without loss of generality that $k = 1$ and we replace θ by $w(t) = \theta(t)\psi^2(t)$. With these changes and integrating (4.55) with respect to s over $[o, s(\infty))$ we finally obtain

$$\int_a^\infty w(t)\exp\left[-\int_a^t F(r)\,dr\right] \le \int_o^{s(\infty)} E(s) \le \int_a^\infty w(t)\exp\left[\int_a^\infty F(r)\,dr\right]dt \quad (4.57)$$

where $$F = \left[\left|\frac{w'}{w} - \frac{2\psi'}{\psi}\right|^2 + \left|p^{-1}w^{-1} + w\{q-\psi^{-1}(p\psi')'\}\right|^2\right]^{\frac{1}{2}} \quad (4.58)$$

Now, we shall turn our attention to the left hand inequality of (4.57) and let us define

$$W(s) = P^{\frac{1}{2}}(s)\theta(t) = p^{\frac{1}{2}}(t)w(t). \quad (4.59)$$

We observe that if

(D) $$\int_a^\infty w(t)\exp\left[-\int_a^t F(r)\,dr\right]dt = \infty$$

and (E) $p^{\frac{1}{4}}WY' \in L^2[o, s(\infty))$ whenever $\tau_1 Y = o$ and $Y \in L^2[o, s(\infty))$,

then (4.50) has no solution in $L^2[o, s(\infty))$. By the change of variables

$y \to Y$, we can infer that (4.47) has no solutions in $L^2[a,\infty)$. Thus we need conditions on P and Q in (4.50) which guarantee that (E) holds.

We state these conditions in the following theorems omitting, as we have earlier mentioned, the proofs.

Theorem 4.4.1 ([16]).

Assume that $s(\infty) = \infty$. If there exists a non-negative function W which is locally absolutely continuous in $[o,s(\infty))$ and a decomposition $Q=Q_1+Q_2$ of Q such that for all $s \geq o$

(i) $P^{\frac{1}{2}}(s)\,|\,W'(s)\,| \leq K_1$

(ii) $-Q(s)W^2(s) \leq K_2$

(iii) $P^{-\frac{1}{2}}(s)W(s)\,\left|\,\int_o^s Q_2(r)\,dr\,\right| \leq K_3$

for some constants K_1, K_2 and K_3,

then (E) holds.

Theorem 4.4.2 ([16])

If there exist positive functions w and Ψ with w, Ψ' locally absolutely continuous in $[a, \infty)$, and a decomposition $q = q_1+q_2$ of q such that

(i) $p^{\frac{1}{2}}\left|\,(p^{\frac{1}{2}}w)'\,\right| \leq K_1$

(ii) $-\{q_1-(p\Psi')'\Psi^{-1}\}(p^{\frac{1}{2}}w)^2 \leq K_2$

(iii) $w\Psi^{-2}\left|\,\int_a^t q_2\Psi^2\,\right| \leq K_3$

(iv) $\int_a^\infty \Psi^2 = \infty$

(v) $\int_a^\infty w(t)\exp\left[-\int_a^t F(r)\,dr\right]dt = 0$

where F is given by (4.56), then (4.47) has no solution in $L^2[a, \infty)$.

Next, by defining another Lyapunov function

$$E(s) = \theta^2(t)\left[P(s)\,Y'(s)\right]^2 + \Phi^2(t)Y^2(s) \qquad (4.60)$$

from where he got, by the methods of [16],

$$K\int_a^\infty \theta(t)\Phi(t)\Psi^2(t)\exp\left[-\int_a^t F(r)\,dr\right]dt \le \int_0^{s(\infty)} E(s)\,ds \qquad (4.61)$$

where $K = \dfrac{E(a)}{\theta(a)\Phi(a)}$

and $F = \left[\left\{(C - A)/2\right\}^2 + B^2\right]^{\frac{1}{2}}$ (4.62)

where $A = 2\theta'\theta^{-1}$,

$B = \Phi(P\theta\Psi^2)^{-1} + \theta\Psi^2\Phi^{-1}\{q - (p\Psi')'\Psi^{-1}\}$, (4.63)

$C = 2\Phi'\Phi^{-1}$,

Knowles in [17] deduced that it follows from (4.61) that if

(F) $\int_a^\infty \theta(t)\Phi(t)\Psi^2(t)\exp\left[-\int_a^t F(r)\,dr\right]dt = \infty$,

(G) $P^{\frac{1}{2}}WY' \in L^2[0, s(\infty)$ whenever $\tau_1 Y = 0$ and $Y \in L^2[0, s(\infty)$,

(H) Φ is bounded,

where $W(s) = P^{\frac{1}{2}}(s)\theta(t) = p^{\frac{1}{2}}(t)\Psi^2(t)\theta(t)$ (4.64)

then (4.50) has no solution in $L^2[0, s(\infty)]$ and hence (4.47) has no solution in $L^2[a, \infty)$.

With these, we now state (without proof) the stronger version of Theorem 4.4.2.

Theorem 4.4.3. ([17])

Suppose that there exist positive functions θ, Φ and Ψ

with Φ bounded and θ, Φ and ψ' locally absolutely continuous on $[a,\infty)$, and a decomposition $q=q_1+q_2$ of q such that on $[a,\infty)$,

(i) $p^{\frac{1}{2}}\left|(p^{\frac{1}{2}}\theta\psi^2)'\right| \leq K_1$

(ii) $-\{q_1-(p\psi')'\psi^{-1}\}\, p\theta^2\psi^4 \leq K_3$

(iii) $\theta(t)\left|\int_a^t q_2\psi^2\right| \leq K_2$

(iv) $\int_a^\infty \psi^2 = \infty$

(v) $\int_a^\infty \theta(t)\Phi(t)\psi^2(t)\exp\left[-\int_a^t F(r)\,dr\right]dt = \infty,$

where K_1, K_2 and K_3 are constants and F is given by (4.62). Then (4.47) has no solutions in $L^2[a,\infty)$. Thus (vi) of section 4.1 is satisfied for $r=2$ by the provisions of Theorems 4.4.1, 4.4.2 and 4.4.3.

CONCLUSION

We have set out to study the stability properties associated with the solutions of the general second order linear differential equation (1.1).

With the fact that not all differential equations can be easily solved explicitly, we used Lyapunov's direct method which does not require explicit solutions of the equations before studying the properties of these solutions. To this effect, we defined a Lyapunov function $E(x)$ given by (1.2). Lower and upper bounds were derived for this function and that gave an estimate for it.

To study the stability properties of the solutions of (1.1), we dwelt mainly on the boundedness conditions for solutions, their asymptotic behaviour and the conditions for the existence of $L^2[a,\infty)$ solutions. These conditions were stated and proved in theorems and some examples involving Bessel differential equations of order $n \geq 0$ were used to illustrate the boundedness and asymptotic behaviour of solutions.

Lastly, conditions opposite to those mentioned above were considered. That is, conditions for which solutions of (1.1) are unbounded, conditions for which they do not tend to zero at infinity and conditions for the nonexistence of $L^2[a, \infty)$ solutions. As we were not set out to study

these aspects in details, most of the theorems in which they have been stated were without proofs.

Through the various theorems and examples, therefore, the work has illustrated the powerfulness of Lyapunov's direct method in studying stability properties associated with the solutions of second order linear differential equations.

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