LINEAR COMPLEMENTARITY PROBLEMS

(THEORY AND APPLICATIONS)

By

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DECLARATION

I hereby declare that this thesis is as a result of my own research work, and that it is not a published or an unpublished work of another author. It has not been accepted or submitted for a degree of any kind in any other University. Sources of information have been acknowledged both in the texts and references.



A.O. AJIBADE

1997

CERTIFICATION

This thesis entitled "Linear Complementarity Problems (Theory and application)" by A.O. Ajibade meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University and is approved for its contribution to Knowledge and literary presentation.

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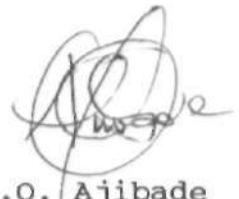
Wonderful mum

Late Maria Monisola Ajibade

and

my Grandmother

Madam Janet Egbekemi Ajibade



A.O. Ajibade
1997

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TABLE OF CONTENTS

Page

Title Page i

Declaration ii

Certification.....iii

Dedication iv

Acknowledgement.....v

Table of contents vi

Abstract viii

CHAPTER ONE.....1

Introduction.....1

1.1 Definition of Linear Complementarity problem..... .1

1.2 Linear programming problems and LCP.....3

1.3 Solving the linear complementarity problem.....6

1.4 Definition of Almost complementary basic feasible solution.....8

1.5 Definition of Adjacent almost complementary basic feasible solution 8

1.6 Lemke's complementary pivoting algorithm.....9

1.7 Termination of Lemke's algorithm.....13

1.8 Numerical examples using Lemke's algorithm.....15

1.9 Main convergence theorem.....21

CHAPTER TWO.....26

Introduction.....26

CHAPTER ONE

INTRODUCTION

As a form of introduction to this study, it is necessary to define what a linear complementarity problem is and the approach to the solution of such problems. The thesis is divided into five main chapters, the first which is the introductory chapter familiarizes the reader with the basic definitions of terms used in this thesis. Also, the complementary pivoting algorithm which solves the linear complementarity problem (LCP) is discussed in the chapter.

The second chapter gives an alternative approach to solving the LCP using the principal pivoting method. Furthermore, in the chapter, the condition that warrants the existence of a complementary solution to the LCP is also discussed.

Chapter three discussed the large, structured LCP. That is, linear complementarity problems of large order in which the matrix involved has a particular structure. An algorithm which executes these types of problem is also presented in the chapter.

Applications of the linear complementarity problem are discussed in chapter four. These consist of the relationship between LCP and other programming problems. The last chapter contains the conclusion of the whole work.

1.1 DEFINITION: (Linear Complementarity Problem)

Let M be a square matrix of order n and let q be a given column n-vector. The linear complementarity problem is to determine

vectors $W \in \mathbb{R}_n$ and $Z \in \mathbb{R}_n$ such that the following three conditions are satisfied.

$$W - MZ = q \qquad 1.1.1$$

$$W^T Z = 0 \qquad 1.1.2$$

$$W \geq 0,\ Z \geq 0 \qquad 1.1.3$$

for $W = (w_1, w_2, \ldots w_n)^T$ and $Z = (z_1, z_2, \ldots z_n)^T$.

From (1.1.2) and (1.1.3), it can be seen that $w_j \geq 0$ and $z_j \geq 0$ with $w_j z_j = 0$ $(j=1,2 \ldots n)$.

A solution to this problem, written as $(W,Z) = (w_1, w_2, \ldots w_n, z_1, z_2, \ldots z_n)$ is called complementary feasible solution. This solution is said to be feasible because of its non negativity assumption (1.1.3), and it is called complementary because $W^T Z = 0$ (1.1.2) which implies that $w_j z_j = 0$ $(j=1,2 \ldots n)$. Therefore, from (1.1.2) and (1.1.3) it can be observed that w_j and z_j cannot be non-zero simultaneously, which means if $w_j > 0$ then automatically $z_j = 0$ and we say that z_j is the complement of w_j.

Alternatively, given the square matrix M and the column n-vector q as above, the linear complementary problem can be defined as obtaining a vector $Z \in \mathbb{R}^n$ that satisfies the following conditions.

$$MZ + q \geq 0 \qquad 1.1.4$$

$$(MZ + q)^T Z = 0 \qquad 1.1.5$$

$$Z \geq 0 \qquad 1.1.6$$

Going by these definitions above, it could be seen that if $W = MZ + q$ then the two definitions are exactly the same. Here (w_j, z_j) is a pair of complementary variables and a solution (W,Z)

or $V - AX = -b$ 1.2.4

These constraint equations (1.2.3) and (1.2.4) could be combined and written in matrix form thus:

$$\begin{pmatrix} U \\ \cdots \\ V \end{pmatrix} - \begin{pmatrix} -A^T Y \\ \cdots \\ AX \end{pmatrix} = \begin{pmatrix} C^T \\ \cdots \\ -b \end{pmatrix}$$

This could be rewritten as

$$\begin{pmatrix} U \\ \cdots \\ V \end{pmatrix} - \begin{pmatrix} 0X - A^T Y \\ \cdots\cdots \\ AX - 0Y \end{pmatrix} = \begin{pmatrix} C^T \\ \cdots \\ -b \end{pmatrix}$$

$$\begin{pmatrix} U \\ \cdots \\ V \end{pmatrix} \geq 0, \quad \begin{pmatrix} X \\ \cdots \\ Y \end{pmatrix} \geq 0 \quad \text{and} \quad \begin{pmatrix} U \\ \cdots \\ V \end{pmatrix}^T \begin{pmatrix} X \\ \cdots \\ Y \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} U \\ \cdots \\ V \end{pmatrix} - \begin{pmatrix} 0 & \vdots & -A^T \\ \cdots & \cdots & \cdots \\ A & \vdots & 0 \end{pmatrix} \begin{pmatrix} X \\ \cdots \\ Y \end{pmatrix} = \begin{pmatrix} C^T \\ \cdots \\ -b \end{pmatrix} \qquad \text{1.2.5}$$

$$q = \begin{pmatrix} C^T \\ \cdots \\ -b \end{pmatrix}$$

then (1.2.5) is seen to be a LCP. Solving the linear program (1.2.1) can be achieved by solving the linear complementary problem (1.2.5).

The demonstration of this formulation with numerical example will be discussed in chapter four under the applications of linear complementarity problems.

1.3 SOLVING THE LINEAR COMPLEMENTARITY PROBLEM

In obtaining a solution to the linear complementarity problem, it is necessary to consider the column n-vector q and then determine the approach to solve the linear complementarity problem.

If the column n-vector q is non negative in (1.1.1) (ie. $q_i \geq 0$ i=1,2,... n), then we immediately have a solution satisfying (1.1.1), (1.1.2) and (1.1.3) by simply letting Z = 0 and W = q. In this case the complementary feasible solution (W,Z) will be given by (q,0) where 0 is an n-vector with all entries being zero.

Let I be an identity matrix, it is well known that the columns in I represent a set of linearly independent vectors which span the vector space $\mathbb{R}_n$. Therefore the columns of I form a basis for $\mathbb{R}_n$. Now if (1.1.1) is multiplied through by I, we have

$$IW - IMZ = Iq$$

This could be written as

$$IW - MZ = q \qquad 1.3.1$$

If we now consider the complementary feasible solution $(W,Z)=(q,0)$, it can be seen that the variables that are not associated with the basis I all equal to zero. A complementary feasible solution which associates the value zero to all variables that are non-basic is called complementary basic feasible solution.

On the other hand, if the column n-vector q is not non-negative, that is, there exists at least one q_i in q such that $q_i < 0$, then a new column vector "e" and an artificial variable z_o are introduced into the system resulting in the following system; where $e=(1,1...1)^T$, column n-vector of ones.

$$W - MZ - ez_o = q \qquad 1.3.2$$

$$W^TZ = 0 \qquad 1.3.3$$

$$W, Z \geq 0, \quad z_o \geq 0 \qquad 1.3.4$$

If we take $Z = 0$, then we have $W=q+ez_o$, where w_i $(i=1,2...n)$ are the basic variables. To obtain the starting solution to the above system, suppose $-q_j$ = maximum $\{-q_i : 1 \leq i \leq n\}$ then remove w_j from the set of basic variables and replace it with z_o and the initial complementary basic solution will be given by

$$(w_1, w_2, \ldots w_{j-1}, z_o, w_{j+1}, \ldots w_n, 0, 0, \ldots 0) \qquad 1.3.5.$$

The solution to the above system is obtained through a sequence of pivots performed on (1.3.5) that satisfy (1.3.2), (1.3.3) and (1.3.4). In solving the LCP completely, we attempt to

drive the artificial variable z_o to the zero level.

1.4 DEFINITION: (Almost Complementary basic feasible solution)

Considering the system defined above in (1.3.2), (1.3.3) and (1.3.4), a feasible solution (W,Z,z_o) to this system is called an almost complementary basic feasible solution if it satisfies the following conditions:

i (W,Z,z_o) is a basic feasible solution for (1.3.2) and (1.3.4)

ii Neither w_s nor z_s are basic for one $s \in (1,2, \ldots n)$

iii z_o is basic and exactly one variable from the complementary pair (w_j, z_j) is basic for j 1,2, ..., n such that $j \neq s$.

It is important at this point to give the definition of another type of solution which is necessary in solving the LCP.

1.5 DEFINITION: (Adjacent Almost Complementary basic feasible solution)

Given an almost complementary basic feasible solution (W,Z,z_o) where w_s and z_s are both nonbasic, another almost complementary basic feasible solution (W,Z,z_o) is obtained by the introduction of either w_s or z_s into the basis if pivoting drives a variable other than z_o out of the basis. This new solution is called Adjacent almost complementary basic feasible solution. It should be noted that an adjacent almost complementary basic feasible solution can only occur if after pivoting, z_o still remains in the basis.

It we take a critical look at definitions (1.4) and (1.5), we can deduce that each almost complementary basic feasible solution has at most two adjacent almost complementary basic feasible solutions. The first, if w_s is introduced into the basis and the other, if z_s is introduced into the basis.

1.6 LEMKE'S COMPLEMENTARY PIVOTING ALGORITHM

In solving the linear complementarity problem (M,q) where q is not a non-negative vector, an algorithm is used which was proposed by Lemke (1968). After the introduction of the artificial variable z_o associated with the column vector "e" into the system as it was done in (1.3.2). In detached coefficient tableau form (1.3.2) then becomes

W	Z	z_o	=
I	-M	-e	q

$W \geq 0$ $Z \geq 0$ $z_o \geq 0$ (1.6.1)

In each stage, the algorithm deals with a basis which is a square non-singular submatrix of order n of the coefficient matrix in (1.6.1). The column vectors in the basis are called the basic column vectors while the variables associated with these vectors are called the basic variables. Other variables that are not basic are called non-basic variables.

The solution of (1.6.1) corresponding to a given basis is obtained by setting all the non-basic variables equal to zero and then solving the remaining system of linear equations for values of

the basic variables.

The computational step used in the algorithm is the **PIVOT** step, which is also the main step in the simplex algorithm for solving linear programs. In each stage of the algorithm, the basis is changed by bringing into the basis exactly one nonbasic vector. The variable in (1.6.1) associated with this column vector is called the **ENTERING VARIABLE.**

Since a basis for (1.6.1) always consist of n column vectors, when a new vector is introduced into the basis, one of the existing column vectors has to be dropped from the basis, and the variable associated with it will be removed from the set of basic variables, this is called the **LEAVING VARIABLE.** The leaving variable has to be determined according to a rule known as minimum ratio test in linear programming terminology. This test guarantee that the new basis obtained after the pivot step will also be a feasible basis. That is, each basic variables will be non negative.

This algorithm moves among adjacent almost complementary basic feasible solutions until either a complementary basic feasible solution is obtained or a direction indicating unboundedness of the region, defined by (1.3.2) and (1.3.4) is found. Later in this study, it will be seen that if the matrix **M** satisfies certain conditions, then the algorithm terminates in a finite number of steps with a complementary basic feasible solution.

In all subsequent stages of the algorithm, there is a unique way to continue the algorithm, which is to pick as the entering variable, the complement of the variable that just dropped from the

basis. This is known as **complementary pivot rule.** It is this rule that will be employed together with the minimum ration test in the Lemke's complementary pivoting algorithm. The algorithm goes thus:

INITIALIZATION STEP:

If $q \geq 0$, stop

$(W,Z) = (q,0)$ is a complementary basic feasible solution. Otherwise, in the tableau (1.6.1) let $-q_s$ = maximum $(-q_i : 1 \leq i \leq n)$ and update the tableau by pivoting at row s and z_0 column, this will result in w_s leaving the basis and z_0 entering the basis. Thus the basic variables z_0 and w_j $j=1,2,\ldots n$ $j \neq s$ are non negative. By the complementary rule since w_s has left the basis, its complement z_s automatically become the entering variable in the next iteration.

Let $y_s = z_s$ and go to the main step.

MAIN STEP:

STEP I: Let $\mathbf{d}^*$ be the updated column in the current tableau under the variable y_s. If $\mathbf{d}^* \leq 0$ go to step IV. Otherwise, determine the index r by the following minimum ratio test where $\mathbf{q}^*$ is the updated right-hand side column denoting the values of the basic variables.

$$q_r^*/d_r^* = \underset{i \leq i \leq n}{Minimum}\{q_i^*/d_i^* \quad : \quad d_i^* > 0\}$$

If the variable at the r^{th} row is z_0, go to step III otherwise to to step II.

STEP II: The basic variable at row r is either w_k or z_k for some $k \neq s$. The variable y_s enters the basis and the tableau is updated by pivoting at the r^{th} row and the y_s column. If the variable that just left the basis is w_k, then its complement z_k becomes the next entering variable and we write $y_s = z_k$. If the variable that just left the basis is z_k then $y_s = w_k$. Go to step I.

STEP III: Here y_s enters the basis and z_0 leaves the basis. Pivot at y_s column and z_0 row producing a complementary basic feasible solution stop.

STEP IV: Stop, here the solution is unbounded.
Let $R = \{(W,Z,z_0) + \lambda d : \lambda \geq 0\}$ be defined such that every point in R satisfies the system (1.3.2), (1.3.3), (1.3.4). (W, Z, z_0) is the almost complementary basic feasible solution associated with the last tableau and entries in d are such that it has a 1 at the row corresponding to y_s, $-d_i^*$ at the row corresponding to other current basic variables and zero elsewhere.

A critical look ar R as it was defined in step IV suggests unboundedness in the solution set because for any $\lambda \geq 0$, there exist a solution which satisfies (1.3.2), (1.3.3), (1.3.4). In this case, we say the algorithm terminates with a ray.

1.7 TERMINATION OF LEMKE'S ALGORITHM:

When solving the linear complementarity problem (M,q) using the Lemke's complementary pivoting algorithm, there are exactly two ways possible in which the algorithm can terminate.

The first type of termination can occur if at some stage of the algorithm, the artificial variable z_0 leaves the basis, then a complementary feasible basis is obtained. The solution of the tableau (1.6.1) corresponding to this final basis is a solution to the linear complementarity problem (1.1.1), (1.1.2), (1.1.3). Hence, the first type of termination is when the algorithm stops with a complementary basic feasible solution.

On the other hand, if at some stage of the algorithm, the pivot column turns out to be non positive which implies that the choice of the leaving variable is practically impossible, then the algorithm terminates with a ray. This is called RAY TERMINATION. When this happens, then the algorithm is unable to solve the LCP.

It should be noted that if the Lemke's complementary pivoting algorithm fails to produce a solution, it does not mean that the problem has no solution. This will be demonstrated with numerical examples as the study advances.

The theorem that ascertain the finiteness of the number of iterations in the Lemke's complementary pivots algorithm is given by Bazaraa and Shetty [1]. This theorem has to do with the degeneracy of the almost complementary basic feasible solution.

THEOREM 1.1

Suppose that each almost complementary basic feasible solution to (1.3.2), (1.3.3), (1.3.4) is non-degenerate (ie each basic variable is positive). Then none of the points generated by the complementary pivoting algorithm is repeated and furthermore the algorithm must terminate in a finite number of steps.

PROOF: Let (W,Z,z_0) be an almost complementary basic feasible solution where both w_s and z_s are non-basic. Then (W,Z,z_0) has at most two adjacent almost complementary basic feasible solution; one obtained by introducing w_s into the basis and the other obtained by introducing z_s into the basis. Since (W,Z,z_0) is non degenerate, then these adjacent almost complementary basic feasible solutions are distinct from (W,Z,z_0).

To show that none of these points generated by the complementary pivoting algorithm are repeated. Let $(W,Z,z_0)_v$ be the point generated at a general iteration v. By contradiction, suppose $(W,Z,z_0)_{k+\alpha} = (W,Z,z_0)_k$ for some positive integers k and α where $k+\alpha$ is the smallest index where repetition is first observed. By non-degeneracy assumption, $\alpha > 1$. Also, using the complementary pivot rule $\alpha > 2$.

But since $(W,Z,z_0)_{k+\alpha-1}$ is adjacent to $(W,Z,z_0)_k$, if $k = 1$, and since $(W,Z,z_0)_k$ has exactly one adjacent almost complementary basic solution then we have $(W,Z,z_0)_{k+\alpha-1} = (W,Z,z_0)_{k+1}$ and hence, repetition occurs at iteration $k + \alpha - 1$ contradicting our assumption that the first repetition occurs at iteration $k+\alpha$. If $k \geq 2$ then $(W,Z,z_0)_{k+\alpha-1}$ is adjacent to $(W,Z,z_0)_k$ and hence must be equal to either

$(W,Z,z_o)_{k-1}$ or $(W,Z,z_o)_{k+1}$. In either case, repetition occurs at iteration $k+\alpha-1$ which contradicts our assumption. Thus, the points generated by the algorithm are distinct. Since there is only a finite number of almost complementary basic feasible solutions and since none of them is repeated, the algorithm stops in a finite number of steps with either a complementary basic feasible solution or with ray termination.

1.8 NUMERICAL EXAMPLES USING LEMKE'S ALGORITHM

In order to further facilitate the understanding of Lemke's complementary pivot algorithm, it is important to accompany the algorithm with numerical examples which will show the reader how the complementary pivot rules together with the minimum ratio test are being used in the algorithm.

A total of two examples will be treated here. The first one is when the linear complementarity problem is solvable by the Lemke's complementary pivot algorithm. By this, it implies a case where the algorithm terminates with a complementary basic feasible solution. The second case is when the algorithm fails to produce a solution, that is when the algorithm stops with a ray termination.

EXAMPLE 1.8.1

Find $Z \in \mathbb{R}_4$ and $W \in \mathbb{R}_4$ that satisfy

$$W - MZ = q$$

$$W^TZ = 0$$

$$W \geq 0,\ Z \geq 0$$

where

$$M = \begin{pmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -2 \\ 1 & -1 & 2 & -2 \\ 1 & 2 & -2 & 4 \end{pmatrix} \text{and} \quad q = \begin{pmatrix} 2 \\ 2 \\ -2 \\ -6 \end{pmatrix}$$

Introducing the artificial variable z_0 we have

$$W - MZ - ez_0 = q$$

Therefore, the initial tableau (1.6.1) will be formed thus;

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	q
w_1	1	0	0	0	0	0	1	1	-1	2
w_2	0	1	0	0	0	0	-1	2	-1	2
w_3	0	0	1	0	-1	1	-2	2	-1	-2
w_4	0	0	0	1	-1	-2	2	-4	-1	-6

Since the most negative q_i is $q_4 = -6$, we have to pivot in the row corresponding to q_4 and the column corresponding to z_0. Hence the pivot element is the one with asterisk. The aim is to bring the pivot column into the basis, and drop the column corresponding to w_4. The pivot element is made equal to 1 and all other entries in the pivot column are made equal to zero. Since the pivot operation drives w_4 out of the basis, by the complementary rule, its complement z_4 automatically becomes the entering variable in the next iteration. Go to iteration 1 with the entering variable $y_s = z_4$.

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	q	Ratio
w_1	1	0	0	-1	1	2	-1	5	0	8	8/5
w_2	0	1	0	-1	1	2	-3	6	0	8	4/3
w_3	0	0	1	-1	0	3	-4	6*	0	4	⅔
z_0	0	0	0	-1	1	2	-2	4	1	6	3/2

Now, $y_n = z_4$ enters the basis. To determine the leaving variable, consider the strictly positive entries d^*_1 in the column corresponding to the entering variable and calculate q^*_1/d^*_1. The variable in the row corresponding to the minimum of these ratios becomes the leaving variable. The entry at the intersection of the column of the entering variable and the row of the leaving variable becomes the pivot for the next iteration. From the above tableau, using the minimum ratio test, w_3 is the leaving variable with minimum ratio of 2/3 and 6* is the pivot. Divide the pivot row by the pivot to get 1 in the position of the pivot and make all other entries on that column equal to zero. Since the pivot operation has driven w_3 out of the basis, its complement z_3 becomes the entering variable for the next iteration. Hence go to the next iteration with $y_n = z_3$.

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	q	Ratio
w_1	1	0	-5/6	-1/6	1	-½	7/3*	0	0	14/3	2
w_2	0	1	-1	0	1	-1	1	0	0	4	4
z_4	0	0	1/6	-1/6	0	½	-⅔	1	0	⅔	
z_0	0	0	-⅔	-⅓	1	0	⅔	0	1	10/3	5

Now, $y_s = z_3$ enters the basis. By the minimum ratio test w_1 leaves the basis and hence by the complementary rule, its complement z_1 becomes the entering variable for the next iteration. We pivot through 7/3* and go to the next iteration.

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	q	Ratio
z_3	3/7	0	-5/14	-1/14	3/7	-3/14	1	0	0	2	14/3
w_2	-3/7	1	-9/14	**1/14**	4/7	-11/14	0	**0**	**0**	**2**	7/2
z_4	2/7	0	-1/14	-3/14	2/7	5/14	0	1	0	2	7
z_0	-2/7	0	-3/7	-2/7	5/7*	1/7	0	0	1	2	14/5

Now $y_s = z_1$ enters the basis. By the minimum ratio test, z_0 leaves the basis. Hence, pivot through 5/7* to get the next tableau.

	w_1	w_2	w_3	w_4	z_1	z_2	z_3	z_4	z_0	q
z_3	3/5	0	-1/10	1/10	0	-3/10	1	0	-3/5	4/5
w_2	-1/5	1	-3/10	3/10	**0**	-9/10	0	**0**	-4/5	2/5
z_4	2/5	0	1/10	-1/10	0	3/10	0	**1**	-2/5	6/5
z_1	-2/5	0	-3/5	-2/5	**1**	1/5	0	0	-7/5	14/5

Since the condition for termination is attained. That is, driving the artificial variable z_0 to the zero level, the algorithm terminates. The corresponding solution is given thus:

Non-basic variables $w_1 = w_3 = w_4 = z_2 = 0$

basic variables $z_1 = 14/5$, $z_3 = 4/5$, $z_4 = 6/5$, $w_2 = 2/5$.

Hence the complementary basic feasible solution is

$(W,Z) = (0, 2/5, 0, 0, 14/5, 0, 4/5, 6/5)$.

EXAMPLE 1.8.2

Find $Z \in \mathbb{R}_2$ and $W \in \mathbb{R}_2$ that satisfy

$$W - MZ = q$$

$$W^T Z = 0$$

$$W \geq 0,\ Z \geq 0$$

where

$$M = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \text{ and } q = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Introducing the artificial variable z_0, we have

$$W - MZ - ez_0 = q$$

which give rise to the initial tableau

	w_1	w_2	z_1	z_2	z_0	q
w_1	1	0	0	-3	-1*	-2
w_2	0	1	-2	1	-1	-1

Since max $\{-q_i : i = 1,2\}$ is $-q_1$, we pivot z_0 into the basis and w_1 leaves the basis. The complement of w_1 becomes the entering variable for the next iteration. Pivot through -1* and proceed to iteration 1 with $y_0 = z_1$.

	w_1	w_2	z_1	z_2	z_0	q
z_0	-1	0	0	3	1	2
w_2	-1	1	-2	4	0	1

Since the pivot column (z_1 column) turns out not to be strictly positive, the choice of leaving variable using the minimum

ratio test is practically impossible. Hence, the algorithm terminates with a ray.

$R = (W,Z,z_o) + \lambda(W, Z, z_o)$ where (W,Z,z_o) is the almost complementary basic feasible solution corresponding to the last tableau.

ie. $(W,Z,z_o) = (0, 1, 0, 0, 2)$

If d^*_1 and d^*_2 are the entries in the pivot column then $(\bar{W},\bar{Z},\bar{z}_o)=(0,-d^*_2,1,0,-d^*_1)$

where the value 1 is assigned to the entering variable, and $-d^*_1$ is assigned to each of the corresponding basic variables in the last tableau and zero elsewhere. Hence we have $(\bar{W},\bar{Z},\bar{z}_o) = (0,2,1,0,0)$. Therefore the algorithm terminates with ray

$R = (0,1,0,0,2) + \lambda(0,2,1,0,0)$ and hence the Lemke's algorithm cannot solve the linear complementarity problem.

1.9 MAIN CONVERGENCE THEOREM

In order to study the convergence of the Lemke's complementary pivoting algorithm, it is necessary to give the definition of certain types of matrices. Also, the statement of a Lemma will be required in the proof of the main convergence theorem.

DEFINITION (Copositive plus matrix)

Let M be a square matrix of order n, then the matrix is said to be:

A POSITIVE SEMI-DEFINITE MATRIX if

$$Z_T MZ \geq 0 \quad \forall \quad Z \in \mathbf{R}_n$$

A **COPOSITIVE MATRIX** if $Z^TMZ \geq 0 \quad \forall \quad Z \geq 0$

Furthermore, M is said to be

A COPOSITIVE PLUS MATRIX If it is copositive and if $Z^T(M+M^T)=0$, $Z \geq 0$ satisfies $Z^TMZ = 0$

The lemma that follows now gives certain implications of ray termination when each almost complementary basic feasible solution is non degenerate we state without proof the lemma as follows:

LEMMA 1.2

Suppose that each almost complementary basic feasible solution of the system defined by the equation (1.3.2), (1.3.3), (1.3.4) is non degenerate. Suppose also that the Lemke's complementary pivoting algorithm is used to solve this system and further, suppose that ray termination occurs. In particular, assume that at termination, we have the almost complementary basic feasible solution $(\bar{W},\bar{Z},\bar{z}_o)$ and the extreme direction $(\hat{W},\hat{Z},\hat{z}_o)$ giving the ray

$R = \{(\bar{W},\bar{Z},\bar{z}_o) + \lambda(\hat{W},\hat{Z},\hat{z}_o) : \lambda \geq 0\}$ Then:

i. $(\hat{W},\hat{Z},\hat{z}_o) \neq (0,0,0)$, $\hat{W} \geq 0$ $\hat{Z} \geq 0$, $\hat{z}_o \geq 0$

ii. $\hat{W} - M\hat{Z} - e\hat{z}_o = 0$

iii. $\bar{W}^T\bar{Z} = \bar{W}^T\hat{Z} = \hat{W}^T\bar{Z} = \hat{W}^T\hat{Z} = 0$

iv. $\hat{Z} \neq 0$

v. $\hat{Z}^TM\hat{Z} = -e^T\hat{Z}\hat{z}_o \leq 0$

The proof of this lemma was given by Bazaraa and Shetty [1].

With the definition of copositive plus matrix and lemma 1.2 at hand, the theorem that ascertain the convergence of the Lemke's complementary pivoting algorithm can now be stated as follows.

THEOREM 1.3

Suppose that each almost complementary basic feasible solution to the system defined by (1.3.2), (1.3.3), (1.3.4) is non degenerate and suppose that the matrix M is copositive plus, then the complementary pivoting algorithm stops in a finite number of steps. If the system defined by (1.1.1) and (1.1.2) is consistent, then the algorithm stops with a complementary basic feasible solution to the system defined by (1.1.1), (1.1.2) and (1.1.3). On the other hand, if the system defined in (1.1.1) and (1.1.2) is inconsistent, then the algorithm stops with ray termination.

PROOF: By theorem 1.1, the complementary pivoting algorithm stops in a finite number of steps. Now suppose that the algorithm stops with ray termination. In particular, suppose that $(\bar{W},\bar{Z},\bar{z}_0)$ is the almost complementary basic feasible solution and $(\hat{W},\hat{Z},\hat{z}_0)$ is the extreme direction associated with the final tableau. By lemma (1.2),

$$\hat{Z} \geq 0,\ \hat{Z} \neq 0 \text{ and } \hat{Z}^T M\hat{Z} = e^T\hat{Z}\hat{z}_0 \leq 0 \qquad 1.9.1$$

But since M is copositive-plus, $\hat{Z}^T M\hat{Z} \geq 0$. From (1.9.1), it follows that $0 = \hat{Z}^T M\hat{Z} = -e^T Z z_0$. Since $\hat{Z} \neq 0$, $\hat{z}_0 = 0$. But since $(\hat{W},\hat{Z},\hat{z}_0)$ is a direction of the set defined by (1.3.2) and (1.3.3), $\hat{W} - M\hat{Z} - e\hat{z}_0 = 0$ and hence $\hat{W} = M\hat{Z}$ 1.9.2

Now we show that $q^T\hat{Z} < 0$. Since $\hat{Z}^T M\hat{Z} = 0$ and M is copositive-plus, $(M + M^T)\hat{Z} = 0$. This with lemma 1.2 (iii) and the fact that $\bar{W} = q + M\bar{Z} + e\bar{z}_0$ implies

$$0 = \bar{W}^T\bar{Z} = (q + M\bar{Z} + e\bar{z}_0)^T\hat{Z} = q^T\hat{Z} - \bar{Z}^T M\hat{Z} + \bar{z}_0 e^T\hat{Z} \qquad 1.9.3$$

From (1.9.2), $M\hat{Z} = \hat{W}$ and hence from lemma 1.2 (iii) it follows tht $\hat{Z}^T M\hat{Z} = 0$. Furthermore, $\hat{z}_o > 0$ and $e^T\hat{Z} > 0$ by (1.9.1). Substituting in (1.9.3), it follows that $q^T\hat{Z} < 0$.

To summarize, we showed that $M\hat{Z} = \hat{W} \geq 0$. Since $(M + M^T)\hat{Z} = 0$, then $M^T\hat{Z} = -M\hat{Z} \leq 0$, $-I\hat{Z} \leq 0$ and $q^T\hat{Z} < 0$. Thus, the system $M^TY \leq 0$, $-IY \leq 0$ and $q^TY < 0$ has a solution say $Y = \hat{Z}$. It was shown by Bazaraa and Shetty [1] that the system $W - MZ = q$, $W \geq 0$, $Z \geq 0$ has no solution.

Now if the system defined by (1.1.1) and (1.1.2) is consistent, then the algorithm must stop with a complementary basic feasible solution, because otherwise the algorithm would stop with ray termination, which as was shown above, is only possible if the system (1.1.1) and (1.1.2) is inconsistent. If the system defined by (1.1.1) and (1.1.2) is inconsistent, then the algorithm obviously could not stop with a complementary basic feasible solution and hence must stop with ray termination.

If another condition is imposed on the matrix M, then in theorem (1.3), the consistency of the system (1.1.1) and (1.1.2) can be guaranteed and hence the complementary pivoting algorithm will terminate with a complementary basic feasible solution. This condition was presented, by Bazaraa and Shetty [1] as a form of Corollary to theorem 1.3 that if the matrix M has non negative entries with positive diagonal elements, then the complementary pivoting algorithm stops in a finite number of steps with a complementary basic feasible solution.

By the assumption on M, the system $W - MZ = q$, $W, Z \geq 0$ has a

solution by choosing Z sufficient large so that $W = MZ+q \geq 0$. The result then follows from theorem (1.3) by noting that M is a copositive-plus matrix.

CHAPTER TWO

INTRODUCTION

The most widely used algorithm for linear complementarity problem is the Lemke's algorithm (Todd [12]). It would be recalled that Lemke's method introduces an artificial variable z_o into the system (1.1.1), (1.1.2) and (1.1.3) to obtain the system defined by (1.3.2), (1.3.3), (1.3.4). Solving this problem, pivot operations are applied to (1.3.2) maintaining (1.3.3) and (1.3.4) until either z_o leaves the basis which leads to a solution or a desired pivot column is non positive and a ray termination which signifies unboundedness is encountered.

In this chapter, another alternative way of dealing with unboundedness is investigated. It is to this effect that two algorithms were proposed for the solution of the linear complementarity problems.

2.1 ALGORITHM A

This algorithm is only applicable when $q < 0$, that is $q_i < 0$ $(i = 1,2, \ldots, n)$. Let d be a row vector of order n $[d \in \mathbb{R}^n]$. Now for this algorithm, we assume $q < 0$ and take $d > 0$ sufficiently large so that $qd + M$ has at least one negative row. A transformation is then performed by adjoining a row with associated free basic variable f to obtain:

$$\left.\begin{aligned} W &= MZ + q.1, \; W \geq 0, \; Z \geq 0 \\ f &= dZ + (-1).1 \quad f, \text{ free} \end{aligned}\right\} \qquad 2.1.1$$

The basic variable f need not be non negative that is why it

was said to be free to take any value.

We now pivot on -1 to make 1 basic and f non basic, leading to the following system.

$$\left.\begin{array}{ll} W = (M + qd)Z + (-q).f & W \geq 0,\ Z \geq 0 \\ 1 = dZ + (-1).f & f, \text{ free} \end{array}\right\} \quad 2.1.2$$

Now relabel W, Z, f and 1 by X, Y, 1 and g to obtain

$$\left.\begin{array}{ll} X = (qd + M)Y + (-q).1 & X \geq 0,\ Z \geq 0 \\ g = dY + (-1).1 & g, \text{ free} \end{array}\right\} \quad 2.1.3$$

Algorithm A produces a sequence of solutions to the system (2.1.3). These solutions correspond to solutions of the system (2.1.1), that is, $w_i = 0$ iff $x_i = 0$ and $z_i = 0$ iff $y_i = 0$.

Assuming that the system (2.1.3) is non-degenerate, then a basic solution to (2.1.3) has (n+1) basic variables; one is g and the rest are strictly positive. It should be recalled that a solution (X,Y) is complementary if $X^TY = 0$. This definition could be extended a bit by saying that the solution is said to be i-almost complementary if $X^TY = x_iy_i$.

Furthermore, G_i (i = 1,2,..., n) has for nodes all i-almost complementary basic solution of (2.1.3).

By searching G_i from (X,Y) to find (X',Y') it means, tracing the unique path in G_i from the complementary solution (X,Y) to the different complementary solution (X'Y'). This operation is performed by a sequence of pivots in (2.1.3).

Before proceeding to give the statement of Algorithm A, it would be necessary to give a brief explanation of the pivoting

techniques used in the algorithm. The pivoting operations make use of the elements on the principal diagonal of the matrix and is given the name principal pivoting technique.

2.2 PRINCIPAL PIVOTING TECHNIQUE

Searching through the graph G_1 from one complementary solution to another using the principal pivoting technique, the reciprocal of the i^{th} entry in the i^{th} row (column) of the matrix is used in the following way.

If the tableau under consideration is given as

	y_1	$y_2 \ldots\ldots y_i \ldots\ldots y_n$	1
x_1	m_{11}	$m_{12} \ldots\ldots m_{1i} \ldots\ldots m_{1n}$	$-q_1$
x_2	m_{21}	$m_{22} \ldots\ldots m_{2i} \ldots\ldots m_{2n}$	$-q_2$
,	,	, ,	,
,	,	, ,	,
,	,	, , ,	
x_i	m_{i1}	$m_{i2} \ldots\ldots m_{ii} \ldots\ldots m_{in}$	$-q_i$
,	,	, ,	,
,	,	, ,	,
,	,	, , ,	
x_n	m_{n1}	$m_{n2} \ldots\ldots m_{ni} \ldots\ldots m_{nn}$	$-q_n$
g	d_1	$d_2 \ldots\ldots d_i \ldots\ldots d_n$	-1

2.2.1

Search G_i means pivot $\langle x_i, y_i \rangle$ through m_{ii} using the following steps.

: Find the reciprocal m_{ii}^{-1} of m_{ii}

: Multiply every element in column i by m_{ii}^{-1}

: Multiply every element in row i by $-m_{ii}^{-1}$

: For all $j \neq i$ $k \neq i$ use the relation

$$m_{jk} = m_{jk} - m_{ji}\, m_{ii}^{-1}\, m_{ik}$$

$$d_j = d_j - m_{ij}\, m_{ii}^{-1}\, d_i$$

$$q_j = q_j - m_{ji}\, m_{ii}^{-1}\, q_i$$

Searching G_i in tableau (2.2.1) using the principal pivoting technique already described, we obtain the tableau.

	y_1	x_i	y_n	1
x_1	$m_{11}-m_{1i}m_{ii}^{-1}m_{i1}$	$\ldots\ m_{1i}m_{ii}^{-1}$	$\ldots m_{1n}-m_{1i}m_{ii}^{-1}m_{in}$	$-q_1-m_{1i}m_{ii}^{-1}q_i$
x_2	$m_{21}-m_{2i}m_{ii}^{-1}m_{i1}$	$\ldots\ m_{2i}m_{ii}^{-1}$	$\ldots m_{2n}-m_{2i}m_{ii}^{-1}m_{in}$	$-q_2-m_{2i}m_{ii}^{-1}q_i$
,	,	,	,	,
,	,	,	,	,
,	,	,	,	,
,	,	,	,	,
y_i	$-m_{ji}m_{ii}^{-1}$	$\ldots\ m_{ii}^{-1}$	$-m_{in}m_{ii}^{-1}$	$q_im_{ii}^{-1}$
,	,	,	,	,
,	,	,	,	,
,	,	,	,	,
,	,	,	,	,
x_n	$m_{n1}-m_{ni}m_{ii}^{-1}m_{i1}$	$\ldots\ m_{ni}m_{ii}^{-1}$	$\ldots m_{nn}-m_{ni}m_{ii}^{-1}m_{in}$	$-q_n-m_{ni}m_{ii}^{-1}q_i$
g	$d_1-m_{i1}m_{ii}^{-1}d_i$	$\ldots\ d_im_{ii}^{-1}$	$\ldots d_n-m_{in}m_{ii}^{-1}d_i$	$-1-d_im_{ii}^{-1}q_i$

It is to be remarked that the algorithm is independent of the choice of the row vector d. That is, different choices of d will still produce the same result as long as the choice of that vector

d satisfies the condition that qd + M must have at least one negative row. Secondly, the algorithm must terminate since we have a finite number of pair of basic solutions.

Having the principal pivoting technique at hand and formulating our tableau from system (2.1.3) we can proceed to give the statement of the algorithm as follows.

2.3 ALGORITHM A (OPERATION STEPS)

INITIALIZATION STEP:

Set (X,Y) = (-q,0) as the initial solution to (2.1.3). Set i=1.

STEP I

Search G_i from (X,Y) to find another solution (X',Y') with associated value g' for g. If g'>0 go to step IV otherwise replace (X,Y) by (X',Y'). If i < n, go to step III otherwise go to step II.

STEP II

If g' = -1, stop with an indication of failure otherwise set i = 0.

STEP III

Increase i by 1 and go to step I.

	z_1	z_2	1
w_1	0	3	-2
w_2	2	-1	-1
f	1	2	-1*

Pivoting in the lower right corner of the tableau so as to pivot 1 into the basis and f out of the basis. Use the approach that was given in the principal pivoting technique (2.2). That is, every entries in the pivot row should be multiplied by negative of the reciprocal of the pivot element. Also every entries in the pivot column are multiplied by the reciprocal of the pivot element. All other entries are obtained by the expression $m_{jk} = m_{jk} - m_{j1}m_{11}^{-1}m_{1k}$. To be particular about the above tableau,

$$m_{11}^{-1} = -1$$

Elements of the pivot column are -2x(-1), (-1)x(-1) which gives 2,1.

Elements of the pivot row are 1 x 1, 2 x 1 (ie to be multiplied by the negative of m_{11}^{-1}.

Other entries in the next tableau include

$m_{11} = 0 - (-2)(-1)(1) = -2$

$m_{12} = 3 - (-2)(-1)(2) = -1$

$m_{21} = 2 - (-1)(-1)(1) = 1$

$m_{22} = -1 - (-1)-1)(2) = -3$

Hence, the next tableau can now be given thus:

	z_1	z_2	f
w_1	-2	-1	2
w_2	1	-3	1
1	1	2	-1

We now replace W,Z,1 and f by X,Y,g and 1 to get the next tableau

	y_1	y_2	1
x_1	-2*	-1	2
x_2	1	-3	1
g	1	2	-1

The next thing is to search G_1 (ie pivot $\langle x_1, y_1 \rangle$) using the principal pivoting technique.

	x_1	y_2	1
y_1	-½	-½	1
x_2	-½	-7/2*	2
g	-½	3/2	0

Obtaining a complementary solution. However, the value of is zero (not strictly positive). We now search G_2 (i.e pivc $\langle x_2, y_2 \rangle$).

	x_1	x_2	1
y_1	-3/7	1/7	5/7
y_2	-1/7	-2/7	4/7
g	-5/7	-3/7	6/7*

Obtaining a complementary solution with g positive. Replace X,Y,g and 1 with W,Z, 1 and f and pivot in the lower right corner, to obtain the complementary basic feasible solution.

	w_1	w_2	1
z_1	1/6	½	5/6
z_2	1/3	0	2/3
f	5/6	½	7/6

Hence (W,Z) = (0,0,5/6,2/3) is a complementary basic feasible solution to the linear complementarity problem.

2.4 ALGORITHM B [Extension of Lemke's Algorithm]

This second algorithm is a direct extension of Lemke's algorithm. It deals with a more general case where there is no restriction on the column vector q. That is $q \not< 0$.

Consider the system (1.3.2), (1.3.4). Exactly as in Lemke's algorithm, we first introduce z_o to replace w_i in the basis where

$$q_i = \min_{1 \le j \le n} q_j$$

It can be assumed without loss of generality that $q_i < 0$ otherwise W = q, Z = 0 solves the LCP. Take i=1 so that $q_1 < q_j$ $1 \le j \le n$, and from this solution we make a transformation ϕ-(d_o, d) choosing $d_o > 0$ so that $1 + q_1 d_o < 0$ and choose $d > 0$ so that $-M^j + q_1 d < 0$ where M^j denotes the j^{th} row of the matrix M. As in

algorithm A, the transformation is performed by introducing a row with associated free basic variable f to obtain.

$$\begin{bmatrix} z_o \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = e w_1 + \begin{bmatrix} -M^1 \\ M^2 - M^1 \\ \vdots \\ M^n - M^1 \end{bmatrix} Z + \begin{bmatrix} -q_1 \\ q_2 - q_1 \\ \vdots \\ q_n - q_1 \end{bmatrix} .1$$

$$\left.\begin{aligned} f \;&=\; d_o w_1 + dZ + (+1).1 \\ &\quad W \geq 0,\ Z \geq 0,\ z_o \geq 0 \text{ free} \end{aligned}\right\} \quad 2.4.1$$

Pivot on the +1 to make 1 basic and f non basic

$$\begin{bmatrix} z_o \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} 1+q_1 d_o \\ 1-(q_2-q_1)\,d_o \\ \vdots \\ 1-(q_n-q_1)\,d_o \end{bmatrix} w_1 + \begin{bmatrix} -M^1+q_1 d \\ M^2-M^1-(q_2-q_1)\,d \\ \vdots \\ M^n-M^1-(q_n-q_1)\,d \end{bmatrix} Z + \begin{bmatrix} -q_1 \\ q_2-q_1 \\ \vdots \\ q_n-q_1 \end{bmatrix} f$$

$$\left.\begin{aligned} 1 &= -d_o\, w_1 - dZ + (+1)f \\ W &\geq 0 \quad Z \geq 0 \; z_o \geq 0 \; f \text{ free} \end{aligned}\right\} \quad 2.4.2$$

Now re-label W, Z, z_o, f and 1 by X,Y,y_o, 1 and g to obtain

$$\begin{bmatrix} y_o \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 1+q_1 d_o \\ 1-(q_2-q_1)\, d_o \\ \vdots \\ 1-(q_n-q_1)\, d_o \end{bmatrix} x_1 + \begin{bmatrix} -M^1+q_1 d \\ M^2-M^1-(q_2-q_1)\, d \\ \vdots \\ M^n-M^1-(q_n-q_1)\, d \end{bmatrix} Y + \begin{bmatrix} -q_1 \\ q_2-q_1 \\ \vdots \\ q_n-q_1 \end{bmatrix} .1$$

$$g = -d_o\, x_1 - dY + (+1)\ .1$$

$$X \geq 0,\ Y \geq 0,\ y_o \geq 0,\ g \text{ free}$$

The second algorithm (Algorithm B) produces a sequence of solutions to the system (2.4.3). It is supposed that if the system (2.4.3) is bounded, then these exist a one to one correspondence between the basic solutions to (2.4.3) with g positive and the basic solutions to (2.4.1).

In this case $w_i = 0$ iff $x_i = 0$ and $z_i = 0$ iff $y_i = 0$. It is also assumed that (2.4.3) is non degenerate so that in any basic solution, the n basic variables other than g ae strictly positive values. Here, a solution to (2.4.3) is said to be complementary if $X^TY = 0$ and is said to be i-almost complementary if $y_o = 0$ and $X^TY = x_i y_i$.

By the same definition of G_i as in algorithm A, and using the principal pivoting technique described in (2.2), Todd [12] gave the statement of algorithm B as follows:

Hence a solution to the LCP.

It would be important to demonstrate the procedures in this algorithm with numerical example.

Algorithm B will be used to solve the problem in example (1.8.2) so that comparism can be made between results obtained by algorithm A and that of algorithm B.

EXAMPLE 2.5.1

Use algorithm B to solve the LCP (M,q) where

$$M = \begin{bmatrix} 0 & 3 \\ 2 & -1 \end{bmatrix} \quad and \quad q = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Introducing the artificial variable z_0, we obtain the tableau.

	z_0	z_1	z_2	1
w_1	1*	0	3	-2
w_2	1	2	-1	-1

Employing the principal pivoting technique, we pivot z_0 into the basis and w_1 out of the basis to get the next tableau. That is pivoting $<z_0, w_1>$.

	W_1	Z_1	Z_2	1
Z_o	1	0	-3	2
W_2	1	2	-4	1

In the selection of d_o and the vector d, the conditions for selections must be strictly adhered to. That is $1 + q_1 d_o < 0$, $d_o > 0$ and $-M^1 + q_1 d < 0$, $d > 0$.

It is to be noted that the choice of d_o and d is not unique, the only thing about it is that each d_o and d selected must satisfy the above conditions for instance to select d_o:

$$1 + q_1 d_o < 0 \qquad \text{where } q_1 = -2$$

$$\Rightarrow \quad 1 + (-2)d_o < 0$$

$$\Rightarrow \quad -2d_o < -1$$

$$\Rightarrow \quad d_o > \tfrac{1}{2}$$

This means that d_o can be any number greater than $\frac{1}{2}$. For this problem let $d_o = 1$.

To select the row vector d use the condition $-M^1 + q_1 d < 0$

Since $-M^1 = (0, -3)$ and $q_1 = -2$

we have

$$(0, -3) + (-2)(d_1, d_2) < 0$$

$$\Rightarrow \quad -2d_1 < 0$$

$$-3 \; -2d_2 < 0$$

$$\Rightarrow \quad d_1 > 0$$

$$d_2 > -3/2$$

If we suppose $d = (d_1, d_2) = (1,1)$ for this problem, then we have the next tableau.

	w_1	z_1	z_2	1
z_o	1	0	-3	2
w_2	1	2	-4	1
f	1	1	1	1*

Whose last row $(d_o, d, 1)$ satisfies the selection condition. To pivot 1 into and f out of the basis, we pivot in the lower right corner to give the next tableau.

	W_1	Z_1	Z_2	f
Z_o	-1	-2	-5	2
W_2	0	1	-5	1
1	-1	-1	-1	1

If we replace W,Z,1 and f by X,Y,g and 1 respectively we obtain the next tableau where z_o is replaced by y_o.

	X_1	Y_1	Y_2	1
Y_o	-1	-2.	-5	2
X_2	0	1	-5	1
g	-1	-1	-1	1

We now search G_o introducing y_1 into the basis. That is Pivot $\langle y_1, y_o \rangle$.

	X_1	Y_o	Y_2	1
Y_1	-½	-½	-5/2	1
X_2	-½	-½	-15/2	2
g	-½	½	3/2	0

Eliminate the column for y_o from further consideration.

	X_1	Y_2	1
Y_1	-½.	-5/2	1
X_2	-½	-15/2	2
g	-½	3/2	0

Obtaining a complementary solution with g = 0. Record this solution and search G_1. That is pivot $\langle x_1, y_1 \rangle$.

	y_1	y_2	1
x_1	-2	-5	2
x_2	1	-5*	1
g	1	4	-1

The above tableau produces a complementary solution with g negative. We now search G_2. That is pivot $\langle y_2, x_2 \rangle$.

	y_1	x_2	1
x_1	-3*	1	1
y_2	-1/5	-1/5	1/5
g	9/5	-4/5	-1/5

This tableau also produces a complementary solution with g negative. This is not a solution, hence we search G_1 again.

	x_1	x_2	1
y_1	-1/3	1/3	1/3
y_2	-1/15	-2/15	4/15
g	-3/5	-1/5	2/5*

This tableau now produces a complementary solution with g positive. Now pivot 1 into and g out of the basis to get the tableau.

	w_1	w_2	1
z_1	1/6	½	5/6
z_2	1/3	0	2/3
f	3/2	½	5/2

Where X,Y,g,1 have been replaced by W,Z,1 and f.

Hence $z_1 = 5/6$ $z_2 = 2/3$, $w_1 = w_2 = 0$ solve the LCP.

This same problem has again been solved with another choice of $d_o = 1$ and d = (1,0) and it was discovered that the choice of d_o or d does not have any effect on the complementary solution as long as the conditions for the selections are strictly adhered to.

2.6 DEFINITION OF P-MATRIX

Let M be a square matrix of order n, then the matrix is said to be a p-matrix if it has all its principal minors positive.

This definition is required to study the existence and uniqueness of solution to a given linear complementarity problem.

2.7 GENERALIZED COMPLEMENTARITY PROBLEMS

The issue of existence and uniqueness of solutions to generalized linear complementarity problem relies mostly on the type of matrix M by which the LCP is defined. It has been shown by Murty [9] that the linear complementarity problem has a unique complementary basic feasible solution for each choice of data vector q iff the coefficient matrix M is a p-matrix.

This section would be concerned with a characterization of p-matrices which has to do with all complementary points for a single

given q rather than a unique feasible complementary solution for all q. The result here relates well with the principal pivoting approach for solving the LCP.

If we label the orthants which partitioned $\mathbb{R}^n$ into 2^n subspaces by θ_k k = 1,2,..., 2^n. It should be noted that the orthants partitioned $\mathbb{R}_n$ as the quadrants partitioned $\mathbb{R}_2$. Studying the linear complementarity problem (M,q) associated with θ_k. The problem can be defined as follows.

Problem θ_k

Find a $Z \in \mathbb{R}^n$ satisfying

$$Z \in \theta_k \quad ,$$

$$W = W(Z) = MZ + q \qquad \in \theta_k. \qquad 2.7.1$$

$$W^TZ = 0$$

Since the linear complementarity problem must have solutions in $\mathbb{R}_{n+}$, we can transform the 2_n problems θ_k (2.7.1) into 2_n problem on $\mathbb{R}_{n+}$ by the conditions that: Corresponding to each orthant θ_k, there exist a unique diagonal matrix P_k (with plus or minus ones on the diagonal) and such that $Z \in \theta_k$ iff $P_k Z \geq 0$. Then the problems θ_k (2.7.1) can be written as usual complementarity problems thus:

Problem K

Given an n x n matrix M and a column n-vector q, find $Z \in \mathbb{R}^n_+$ such that

$$W = P_kMP_kZ + P_kq \in \mathbb{R}^n_+ \qquad 2.7.2$$

$$W^TZ = 0$$

A solution to problem θ_k is obtained from that of problem k

(2.7.2) by multiplying by P_k.

In order to characterize p-matrices as desired, it is necessary to define some important terms to be used in the characterization.

2.8 DEFINITIONS

Some of the basic definitions needed for the characterization of p-matrices include non degeneracy of the coefficient matrix M, non degeneracy of the vector q and pigeonholing of the set of complementary points for a given M and q.

2.8.1 Given the vector $q \in \mathbb{R}^n$ and the square matrix of order n, suppose $Z \in \mathbb{R}^n$ satisfies the complementary condition $(MZ + q)_i Z_i = 0$ $i = 1,2,\ldots,n$. Then, the complementary point S of (M,q) associated with Z is defined by

$$S = Z + (MZ + q)$$

or $$S = Z + W(Z)$$

where $W = MZ + q$

Problem θ_k (2.7.1) is equivalent to showing the existence of a complementary point in θ_k.

2.8.2 Let Σ denote the set of all complementary points for a given vector q and matrix M. The set Σ is said to be pigeonholed if the interior of each orthant in $\mathbb{R}^n$ contains one and only one complementary point.

2.8.3 A given square matrix of order n is said to be a non-degenerate matrix if all the principal minors of the matrix M are non-zero.

If we go by the definition of a p-matrix it would be observed tht every p-matrices are non-degenerate matrices but the converse needs not be true.

2.8.4 Given S, a complementary point of (M,q), S is said to be a non-degenerate complementary point if it has no zero components.

It should be noted that, pigeonholing implies tht M is non-degenerate, it also implies that all the complementary points are non degenerate and that the 2^n problem θ_k all have unique solution. Therefore, the notion of pigeonholing is much stronger than the notion of non-degeneracy used in complementary theory.

2.8.5 The vector $q \in \mathbb{R}^n$ is non-degenerate with respect to M, if for each $Z \in \mathbb{R}^n$ at most n of the 2^n variables are equal to zero. That is to say that the vector q does not lie in any subspace generated by (n-1) or less column vectors in the set $\{I_{.1}, I_{.2}, \ldots I_{.n}, -M_{.1}, -M_{.2} \ldots -M_{.n}\}$. It should be noted that if q is non degenerate with respect to M, then the complementary points of (M,q) are non-degenerate but not conversely.

The characterization of the matrix M as a p-matrix in terms of pigeonholing of Σ for any one q, such that the LCP (M,q) has only non-degenerate complementary points characterizes M as a p-matrix.

THEOREM 2.1

Let M be a p-matrix and $q \in \mathbb{R}^n$. Then

i. Problem θ_k (2.7.1) has a unique solution for each K.

ii. If q is such that (M,q) has no degenerate complementary points then the complementary points are pigeonholed.

PROOF: From the established characterization of a p-matrix under the hypothesis, problem K (2.7.2) has a unique solution in $\mathbb{R}^n_+$ for each $q \in \mathbb{R}^n$. This is so because $P_k M P_k$ is also a p-matrix for each K whenever M is a p-matrix and hence problem θ_k (2.7.1) has a unique solution.

The second part of this theorem will be proved by contradiction. Suppose S is a complementary point which is degenerate. Let S have r zeros, then if we pick any non degenerate matrix M and a zero vector q = 0, we have all complementary points equal to zero and so Z = W = 0 is the unique solution for each K. This result holds for any non-degenerate matrix that need not be a p-matrix but in particular, it holds for M, a p-matrix because every p-matrix is non degenerate. The solution Z = W = 0 implies that the interior of each orthant cannot contain any complementary point. Hence Σ is not pigeonholed. This contradiction is the result of the assumption that the complementary points are degenerate. Therefore, whenever the complementary points are non degenerate, then the set Σ of complementary points is pigeonholed.

In order to present the main characterization result, it is important to give the following definition concerning the set of

all complementary points of the linear complementarity problem (M,q).

2.9 DEFINITION A_M and H_M

Given a square matrix M of order n. Let A_M and H_M be defined thus:

A_M ($q \in \mathbb{R}^n$: (M,q) has only non-degenerate complementary points)

H_M ($q \in \mathbb{R}^n$: Σ is pigeonholed)

From definitions (2.8.2) and (2.8.4), it could be noted that $H_M \subseteq A_M$. Now, as a corollary to theorem (2.1) we present the following result:

COROLLARY 2.2

If M is a p-matrix then $A_M = H_M$.

PROOF: From definitions (2.8.2) and (2.8.4), it was noted that $H_M \subseteq A_M$. But, considering the fact that M is a p-matrix, from theorem (2.1) (ii) we have $A_M \subseteq H_M$ and the result follows.

The next result which was proved by Habetler and Kostreva [7] is concerned with H_M whenever M is not a p-matrix.

THEOREM 2.3

If M is not a p-matrix then $H_M = \phi$.

Habetler and Kostreva [7] proved this theorem by contradiction that whenever $H_M \neq \phi$ then M must be a p-matrix.

Using the above results and definitions, the main result is now obtained and is given in form of a theorem.

THEOREM 2.4

For an n x n matrix M, the following are equivalent.

i. M has positive principal minors

ii. $H_M = A_M$

iii. $H_M \neq \phi$

PROOF: Assume (i) holds. That is M has positive principal minors, this implies by definition that M is a p-matrix and hence by corollary (2.2) $A_M = H_M$ and (ii) holds.

A_M is always non-empty since it is the complement of a union of finite number of subspaces of n-1 or less. Thus (ii) => (iii) since $A_M \neq \phi$ and $H_M = A_M$.

Finally (iii) => (i) follows from theorem (2.3).

This result gives the converse of theorem (2.1) and has given the characterization of p-matrices as it is so desired.

In conclusion, it can be deduced from the above result that the linear complementarity problem (M,q) has a unique solution whenever M is a p-matrix. Furthermore, for any p-matrix M, there exist a vector $q \in \mathbb{R}^n$ for which the LCP (M,q) has non degenerate complementary points.

CHAPTER THREE

INTRODUCTION

Most of the linear complementarity problems are solvable in a mathematical sense. However, most of the existing methods cannot be used in practice because many matrices of practical problems are of large order. This is why the main thrust of algorithmic development has been oriented towards processing the LCP (M,q) for various classes of matrices. In most cases, the types of matrices studied have had no structural endowments like special patterns or sparseness.

This chapter will study the underlying theory of an efficient method for solving a class of specially structured linear complementarity problems of very large size.

3.1 LARGE, STRUCTURED LINEAR COMPLEMENTARITY PROBLEMS

Two classes of such large, structured linear complementarity problem (M,q) will be studied together with methods for solving them. The first case which is the tridiagonal case will be discussed together with an algorithm for solving the LCP. The second case to be studied together with an algorithm for solving it, is the block partitioned case.

This section deals with large scale linear complementarity problem (M,q) in which the coefficient matrix M is a tridiagonal Minkowski matrix. This implies tht M satisfies the following three conditions.

$$m_{ij} = 0 \qquad \text{if } |i - j| > 1 \qquad 3.1.1$$

$m_{ij} \leq 0$ if $i \neq j$ 3.1.2

M has positive principal minors 3.1.3

3.2 THE TRIDIAGONAL CASE

The introduction of tridiagonality condition (3.1.1) in the linear complementarity problem (M,q) results in desirable sparseness, the more so as n increases. The kind of pivoting methods normally used to solve LCP would rapidly increase the density of M, particularly when it is irreducible; for then, conditions (3.1.2) and (3.1.3) imply M^{-1} is a positive matrix. In this study we exhibit a principal pivoting algorithm (2.2) for solving special large scale LCP by taking advantage of the three conditions satisfied by M, the tridiagonal Minkowski matrix.

It is necessary at this point to establish the following notations in order to enhance a smooth study henceforth.

T_n = class of all n x n tridiagonal matrices

Z_n = class of all n x n matrices with non positive off diagonal entries.

P_n = class of all n x n matrices with positive principal minors.

The matrix under consideration here belongs to the intersection of these three classes. There is considerable amount known about $Z_n \bigcap P_n$. Indeed Fielder and Ptak [6] proved that for a matrix $M \in Z_n$ and $M \in P_n$, M has positive leading principal minors and there exist a non negative inverse of the matrix M. Moreso if M were an irreducible Minkowski matrix, then M^{-1} would be strictly

positive.

To solve the LCP (M,q) by pivoting method when $M \in T_n \cap Z_n \cap P_n$ with n large one has to make good use of what linear complementarity theory has to say about matrices of this kind. One of such theory by Samelson, et al [11] states that the LCP (M,q) has a unique solution for every q Iff M has positive principal minors. That is, there exist a unique solution to the LCP (M,q) for every q iff M is a p-matrix.

A solution technique for the LCP with $M \in P_n$ is given as a principal pivoting method as described in (2.2). This method rests on some work of Tucker [13] showing that if $M \in P_n$ so also is the principal rearrangement P^TMP for any permutation matrix P as well as the pivotal transform of M given by

$$\overline{M} = \begin{pmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12} \\ M_{21}M_{11}^{-1} & M_{22}-M_{21}M_{11}^{-1}M_{12} \end{pmatrix}$$

where

$$M = \begin{pmatrix} M_{11} & M_{21} \\ M_{21} & M_{22} \end{pmatrix}$$

This fact will be used in the algorithm presented below. Chandrasekaran [2] made an observation regarding the class Z_n,

which states that; If $M \in Z_n$ and Z satisfies the condition $q + MZ \geq 0$, then $z_i > 0$ whenever $q_i < 0$. This point can best be appreciated by looking at the formulation in definition (1.1.1) of the LCP (M,q). Conditions (1.1.2) and (1.1.3) imply that $w_i z_i = 0$ i=1,2,...,n. Combining this fact with the above Chandrasekaran's observation, we find that $q_i < 0$ implies $w_i = 0$ in a solution of (M,q). The objective is to enforce this condition, and since $M \in Z_n$, w_i needs not become non-zero once it has been set equal to zero.

In dealing with (1.1.1), it is often convenient to use tableau representation.

	1	z_1	z_2	z_n
w_1	q_1	M_{11}	M_{12}	M_{1n}
w_2	q_2	M_{21}	M_{22}	M_{2n}
⋮	⋮	⋮	⋮	⋮
w_n	q_n	M_{n1}	M_{n2}	M_{nn}

In this case z_i's (i = 1,2,...,n) are regarded as non basic variables while w_i's (i = 1,2,...,n) are the basic variables. Pivoting procedure involves finite sequence of changes between basic and non basic variables. In particular, for all those q_i that initially are negative, an exchange of w_i and z_i can, and must be made. These pivot steps are mandatory. If performing the mandatory pivots generate additional mandatory pivots, these too are executed. This method is efficient due to the fact that the

original sparsity of the problem can be maintained.

Specifically, it should be noted that if a pivot causes a basic variable w_i to be non basic, the variable should be constrained to be zero thereafter. In this case, attention is confined to the computational refinement of a method suggested by Saigal [10] for the problem considered by Chandrasekaran.

The algorithm, which uses the principal pivoting technique [2.2] for its operation is hereby presented thus:

3.3 ALGORITHM FOR THE LCP (M,q) WITH $M \in Z_n$

INITIALIZATION STEP:

Let $M^{(0)} = M$

$q^{(0)} = q$

$I^{(0)} = \{i : q_i < 0\}$

starting with $Z^{(0)} = 0$, $W^{(0)} = q^{(0)}$

Let k = 0 and go to step I

STEP I: If $I^{(k)} = \phi$ then,

$Z^{(k)} = 0$, $W^{(k)} = q^{(k)}$ is the solution

If $I^{(k)} \neq \phi$

Choose $i \in I^{(k)}$, if $m_{ii}^{(k)} \leq 0$ stop, the problem has no solution.

If $M_{ii}^{(k)} > 0$ pivot on $m_{ii}^{(k)}$ making $z_i^{(k)}$ basic and $w_i^{(k)}$ non basic to obtain $q^{(k+1)}$ and $M^{(k+1)}$.

STEP II: Drop column i from further consideration.

Define $w_i^{(k+1)} = Z_i^{(k)}$

$Z_i^{(k+1)} = W_i^{(k)}$

$w_j^{(k+1)} = w_j^{(k)}$ $\quad j \neq i$

$$z_j^{(k+1)} = z_j^{(k)} \qquad j \neq i$$

$$I^{(k+1)} = \{i : q_i^{(k+1)} < 0\}$$

Return to step I with k replaced by k+1.

The algorithm presented above involves scanning of the vector q and its successive pivotal transforms for negative entries. The search within a cycle is done component by component. Whenever a negative component is found, a principal pivot is executed and a variable P, which counts the number of pivots since the cycle began is advanced by 1. The current cycle ends when the last q component has been tested and is followed by a principal pivot if it is negative. If $P \geq 1$ then P is reset to zero and a new cycle begins but if the cycle ends with $P = 0$, then the solution is at hand and the procedure stops.

A flowchart for the above algorithm, is given by R.W. Cottle et al. [3] and is presented in Appendix [A], while a computer program in Fortran was developed from this flowchart and is presented in Appendix [B].

The algorithm for Minkowski linear complementarity problems presented above will solve the linear complementarity problem (M,q) whenever M is a Minkowski matrix. However the tridiagonal property that was assumed for M makes it possible to prevent the number of stored non zero entries from growing. Indeed, the problem is set originally with at most 3n-2 non-zero entries. By analyzing what a simple principal pivot actually accomplishes under the present hypothesis, the number of arithmetic operations required to pass from one tableau to the next can be sharply decreased.

Furthermore, improvements can be made by assuming two additional properties. The first of these properties is the symmetry of M which would make it positive definite and hence a Stieltjes matrix. The second property is about the sign configuration of q: that is, all its negative entries come first ie $q_i < 0$ $i = 1, 2, \ldots l$ and $q_i \geq 0$ $i = l+1, \ldots, n$. Assumption of the second property makes it easier to scan through the vector q for negative entries which in turn makes it easier to solve the LCP (M,q).

The second case of large structured LCP to be studied is the block partitioned case and this will be examined in the following section.

3.4 THE BLOCK PARTITIONED CASE

In this case, the matrix to be considered is the type that is symmetric, positive definite and partitioned into m^2 blocks M_{ij} so that the diagonal blocks are square.

Moreover, some problems possesses diverse partitioned structure from the start. For example M might have block tridiagonal structure [$M_{ij} = 0$ if $|i-j| > 1$], its diagonal blocks M_{ii} might themselves be tridiagonal.

As a result of the large scale nature of the problem, extensions to both principal pivoting methods and various forms of factorization can be hindered by storage problems. Therefore, the algorithm for large scale block partitioned LCP presented here is a block ITERATIVE procedure that calls for solution of subproblems

each of which is also a LCP. When the diagonal blocks M_{ii} are tridiagonal and Minkowski, the corresponding subproblem can be solved with the algorithm (3.3) for Minkowski LCP discussed in the tridiagonal case.

The algorithms to be described generates a sequence of iterates Z^k, $k = 1, 2, \ldots$ converging to a solution. The value Z^k is determined by a specific transformation on Z^{k-1} where z° is the initial vector supplied by the user. The entire sequence is denoted by $\{Z^k\}$ and its dependence on z° is minimal.

3.5 ALGORITHM FOR THE BLOCK PARTITIONED LCP

The algorithm for the solution of block partitioned LCP as was given by Cottle et al [4] is based on the block successive over-relaxation algorithms for linear system.

In certain situation, it is natural to consider simultaneously changing the values of all variables associated with a coordinate line of grid points in an approximation scheme. Such methods are known as Line or Block - iterative techniques. The word "Block" refers to the fact that the variables which are simultaneously changed correspond to a diagonal block of the matrix M of the system $MZ + q = 0$. In the corresponding linear system $MZ + q = 0$, we conformably partition the vectors Z and q and the matrix M. We

view Z as the direct sum of vector $Z_i \in \mathbb{R}^{n_i}$ where $\sum_{i=1}^{r} n_i = n$. Thus

we have,

$$Z = (z_1, z_2 \ldots, z_n)$$

$$= (z_{11}, z_{12}, \ldots, z_{1,n1}, z_{21}, z_{22}, \ldots z_{2n2}, \ldots z_{m1}, z_{m2} \ldots z_{mnm})$$

where z_{ij} corresponds to the grid point at row i and column j.

Cryer [5] gave a description for the modified point successive over relation (S.O.R.) algorithm for th linear complementarity problem (M,q) where M is symmetric and positive definite.

By a slight change of notion, we pass from Point-iterative to Block-iterative technique. In particular, M is partitioned into submatrices M_{ij} [i,j = 1,2,...,m] where the diagonal block M_{ii} is of order n_i, the vectors Z and q are partitioned conformably with the partition of M. We then obtain the modified block S.O.R. algorithm for the LCP (M,q) where M is positive definite and M_{ii} are symmetric for each i. The algorithm to be presented gives an approximate solutions to the LCP unlike other algorithms that produce exact solution. To obtain an exact solution to large LCP is possible by the use of Lemke's pivoting algorithm or the principal pivoting technique. However, this is not an easy task because of the computing and storage of the non zero entries in the problem. The proposed algorithm is given thus:

3.6 MODIFIED BLOCK S.O.R. ALGORITHM:

INITIALIZATION STEP:

Let $Z^{(0)} = (Z_1^{(0)}\ Z_2^{(0)} \ \ldots\ Z_n^{(0)})$ be an arbitrary non negative vector and $\alpha \in (0,2)$.

Set k = 0, i = 1 and choose $\epsilon > 0$ small go to step I.

STEP I: Let $Z_i^{(k+1)}$ solve the linear complementarity problem

$$\left(\sum_{j<i} M_{ij} Z_j^{(k+1)} + \sum_{j>i} M_{ij} Z_j^{(k)} + q_i \ , \ M_{ii} \right)$$

STEP II:

$$\text{Let} \qquad \alpha_i^{(k+1)} = Max \left\{ \bar{\alpha} : \bar{\alpha} \leq \alpha \ , \ Z_i^{(k)} + \bar{\alpha} \ (\bar{Z}_i^{(k+1)} - Z_i^{(k+1)} \right\} \geq 0$$

Let $Z_i^{(k+1)} = Z_i^{(k)} + \alpha_i^{(k)} \ (Z_i^{(k+1)} - Z_i^{(k)})$

STEP III:

If i = m go to step IV otherwise return to step I with i replaced by i + 1.

STEP IV:

$$Define \quad J = \left\{ (i,j) : (Z_i^{(k+1)})_j > 0 \ \ or \ \left(\sum_{r=1}^{m} M_{ir} Z_r^{(k+1)} + q_i \right)_j < 0 \right\}$$

An approximate solution is at hand. Otherwise return to step I with k replaced by k+1 and i=1.

$$If \quad \max_{(i,j)\in J} \left| \left(\sum_{r=1}^{m} M_{ir} Z_r^{(k+1)} + q_i \right)_j \right| < \epsilon \qquad stop.$$

In general, if M_{ii} is an arbitrary positive definite matrix, then the standard method for solving (M_{ii}, q_i) may be used. That is, Lemke's pivoting algorithm or the principal pivoting technique. However whenever the modified block S.O.R. algorithm is used on problems whose diagonal blocks M_{ii} are tridiagonal Minkowski matrices, then the subproblems (M_{ii} q_i) can be solved by the algorithm proposed in (3.3) for Minkowski matrices.

CHAPTER FOUR

4.1 APPLICATIONS OF THE LINEAR COMPLEMENTARITY PROBLEM

Linear complementarity problems can be applied in many aspects of optimization. For instance, a linear programming problem can be formulated into the linear complementarity problems and hence solved by a solution algorithm for the LCP. Also, quadratic programming problems are formulated into a LCP and hence solved by the LCP solution algorithms. The linear complementarity problems could also be used to solve non-zero sum (or a bi-matrix) games.

Furthermore, LCP has been used to study problems in the plastic analysis of structures, in the flexural behaviour of reinforced concrete beams, in the free boundary problems for journal bearings, in the study of finance models and in many other areas.

This study will concentrate on some of these applications, discussing their formulation into a linear complementarity problem. Four of these several applications will be considered in detail and they include Linear Programming, Quadratic Programming, Two person bi-matrix games and the Journal bearing problems.

4.2 L.C.P APPLICATION IN LINEAR PROGRAMMING PROBLEMS

Consider the symmetric form of the linear programming problem

$$\text{Minimize } CX$$

$$\text{subject to } AX \geq b$$

$$X \geq 0$$

Recall from section (1.2) that this problem can be formulated

into a linear complementarity problem W - MZ = q where W,M,Z and q are as defined in the formulation (1.2.6).

We now demonstrate the formulation with a numerical example which will further aid the understanding of the subject matter.

EXAMPLE 4.2.1

Solve the following linear programming problem.

$$\begin{aligned} \text{Minimize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \geq 4 \\ & 3x_1 + 2x_2 \geq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

This is a linear programming problem with

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$$

$$b = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$$

$$c = (2, 1)$$

Therefore, by the formulation (1.2.6) we have

$$M = \begin{pmatrix} 0 & -A^T \\ A & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -2 & -3 \\ 0 & 0 & -1 & -2 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix}$$

$$q = \begin{pmatrix} C^T \\ -b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -4 \\ -6 \end{pmatrix}$$

Let

$$W = \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} U_1 \\ U_2 \\ V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \quad , \quad Z = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix}$$

Therefore $x_1 = z_1$ and $x_2 = z_2$ will be an optimal solution to the linear programming problem.

W - MZ = q gives

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -2 & -3 \\ 0 & 0 & -1 & -2 \\ 2 & 1 & 0 & 0 \\ 3 & 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -4 \\ -6 \end{pmatrix}$$

Solving this problem using the Lemke's complementary pivoting algorithm as in section (1.8), we obtain the complementary basic feasible solution.

$(W,Z) = (w_1, w_2, w_3, w_4, z_1, z_2, z_3, z_4)$

$= (0, 0, 0, 0, 2, 0, 0, 0,)$

Hence $X = (x_1, x_2) = (z_1, z_2) = (2,0)$ is an optimal solution to the original linear programming problem.

This same problem, when solved by the simplex algorithm for linear program yields exactly the same solution $x_1 = 2$, $x_2 = 0$ which must be the case.

4.3 QUADRATIC PROGRAMMING PROBLEM AND LCP

Linear complementarity problem can also be used in the area of non-linear programming to aid in the solution of quadratic programming problem. Formulation of a quadratic programming problem into a LCP will be examined together with the relationship between a Karush-Kuhn-Tucker point for the quadratic programming

problem and the complementary feasible solution to the resulting LCP.

Consider the quadratic programming problem (QPP)

$$\text{Minimize } CX + \tfrac{1}{2} X^T DX$$

$$\text{subject to } AX \geq b$$

$$X \geq 0 \qquad 4.3.1$$

where A is a matrix of order m x N and D is a symmetric matrix of order N. The dual of this problem is given by

$$\text{Maximize } b^T Y$$

$$\text{subject to } A^T Y \leq [C + X^T D]^T \qquad 4.3.2$$

$$Y \geq 0$$

Introducing the slack variable $U \in \mathbb{R}^N$, the constraint equation in (4.3.2) becomes

$$A^T Y + U = [C + X^T D]^T$$

$$\text{or} \quad U + A^T Y = C^T + DX \qquad (D \text{ is symmetric})$$

$$\text{or} \quad U - DX + A^T Y = C^T \qquad 4.3.3$$

Similarly, introducing the surplus variable $V \in \mathbb{R}^n$, the constraint equation in (4.3.1) becomes

$$AX - V = b$$

$$\text{or} \quad V - AX = -b \qquad 4.3.4$$

Equations (4.3.3) and (4.3.4) combined together in matrix notation gives the following matrix equation.

$$\begin{pmatrix} U \\ V \end{pmatrix} - \begin{pmatrix} DX - A^T Y \\ AX + OY \end{pmatrix} = \begin{pmatrix} C^T \\ -b \end{pmatrix}$$

A point $X \in \mathbb{R}^N$ is said to be a Karush-Kuhn-Tucker point for the quadratic programming problem (4.3.1) if there exist a dual vector $Y \in \mathbb{R}_m$, slack vector $U \in \mathbb{R}_n$ and surplus vector $V \in \mathbb{R}_n$ such that X, Y, U, V together satisfy.

$$\begin{pmatrix} U \\ V \end{pmatrix} - \begin{pmatrix} D & -A^T \\ A & O \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} C^T \\ -b \end{pmatrix}$$

$$\begin{pmatrix} U \\ V \end{pmatrix} \geq 0 \begin{pmatrix} X \\ X \end{pmatrix} \geq 0 \text{ and } \begin{pmatrix} U \\ V \end{pmatrix}^T \begin{pmatrix} X \\ Y \end{pmatrix} = 0$$

If we take

$$W = \begin{pmatrix} U \\ V \end{pmatrix}$$

$$Z = \begin{pmatrix} X \\ Y \end{pmatrix}$$

$$M = \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix}$$

$$q = \begin{pmatrix} C^T \\ -b \end{pmatrix}$$

then (4.3.5) is seen to be a linear complementarity problem. Solving this LCP (4.3.5) provides a Karush-Kuhn-Tucker point for the QPP (4.3.1). If D is a positive semi definite matrix, (that is $X^TDX \geq 0, X \in \mathbb{R}^n$) this Karush-Kuhn-Tucker point is an optimal solution for (4.3.1).

We illustrate the concept with a numerical example:

EXAMPLE 4.3.6

Solve the quadratic programming problem using linear complementarity pivoting algorithm.

Maximize $2x_1 + 3x_2 - 2x_1^2$

subject to

$$x_1 + 4x_2 \leq 4$$

$$x_1 + x_2 \leq 2$$

$$x_1, \quad x_2 \geq 0$$

Converting this problem to minimization problem we have

$$\text{Minimize} \quad -2x_1 + -3x_2 + 2x_1^2$$

subject to

$$-x_1 + -4x_2 \geq -4$$

$$-x_1 + -x_2 \geq -2$$

$$x_1, \quad x_2 \geq 0$$

If we compare this minimization problem with the QPP (4.3.1)

We have $C = (-2, -3)$

$$D = \begin{pmatrix} 4 & 0 \\ 0 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} -4 \\ -2 \end{pmatrix}$$

Now, going by the formulation (4.3.5) of the LCP, we have the following matrices.

$$M = \begin{pmatrix} D & -A^T \\ A & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 1 & 1 \\ 0 & 0 & 4 & 1 \\ -1 & -4 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}$$

Solving the LCP (M,q) will result to solving the quadratic programming problem. That is, the values of z_1 and z_2 obtained from the LCP (m,q) is the solution to the QPP.

$$q = \begin{pmatrix} C^T \\ -b \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 4 \\ 2 \end{pmatrix}$$

Using Lemke's complementary pivoting algorithm the LCP (M,q) formulated above yields the complementary basic feasible solution $(W,Z) = (w_1,w_2,w_3,w_4,z_1,z_2,z_3,z_4) = (0,0,0,49/64.5/16,59/64,3/4,0)$.

Therefore $x_1 = z_1 = 5/16$ and $x_2 = z_2 = 59/64$ solve the original quadratic program.

This same problem, when solved using the modified Wolfe Simplex Algorithm for QPP yields exactly the same solution x_1=5/16 and x_2 = 59/64 which must be the case.

This solution results in 3.19 as the value of the objective function.

The next application of the LCP to be examined is in the aspect of two person non-zero-sum games. The study will focus on its formulation into linear complementarity problem.

4.4 BIMATRIX GAMES AND LCP

This is a kind of game that consist of two players. In each play of the game, player I picks one out of a possible set of his m choices and independently, player II picks one out of a possible set of his N choices. If it happens that in a play, player I picks his choice i and player II picks his choice j, then player I loses an amount a'_{ij} say, and player II loses an amount b'_{ij} say, where $A' = (a'_{ij})$ and $B' = (b'_{ij})$ are given loss matrices.

If $a'_{ij} + b'_{ij} = 0$ for all i and j, we call the game zero sum game. In this case it is possible to develop the concept of optimum strategy for playing the game.

In Bimatrix games, it is difficult to define an optimum strategy. However, in this case an equilibrium pair of strategies can be defined and the problem of computing an equilibrium pair of strategies can be formulated into a linear complementarity problem.

Suppose player I picks his choice i with a probability of x_i, the column vector $X = (x_i) \in \mathbb{R}^m$ completely defines player I's strategy. Similarly, let the probability vector $Y = (y_j) \in \mathbb{R}^n$ be player II's strategy. If player I adopts strategy X and player II adopts strategy Y, the expected loss of player I is $X^TA'Y$ and that of player II is $X^TB'Y$.

The strategy pair (X,Y) is said to be in equilibrium if no player benefit by unilaterally changing his own strategy while the other player keeps his strategy in the pair (X,Y) unchanged. That is

$$XA'Y \leq X\,A'Y \quad [\text{For all probability vector } X \in \mathbb{R}^m]$$

$$XB'Y \leq X\,B'Y \quad [\text{For all probability vector } Y \in \mathbb{R}^n]$$

Let α, β be arbitrary positive numbers such that

$$a_{ij} = a'_{ij} + \alpha > 0$$

$$b_{ij} = b'_{ij} + \beta > 0 \qquad \text{for all } i,j$$

Let $A = (a_{ij})$, $B = (b_{ij})$ and consider the linear complementarity problem

$$\begin{pmatrix} U \\ V \end{pmatrix} - \begin{pmatrix} 0 & A \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \pi \end{pmatrix} = \begin{pmatrix} -e_m \\ -e_N \end{pmatrix}$$

where e_m and e_N denote column m-vectors and column N-vectors of ones.

If $(U,V, \lambda \pi)$ solves the LCP (4.4.1) then:

$X = \lambda/\Sigma\lambda_i$ and

$Y = \pi/\Sigma\pi_i$

are equilibrium pair of strategies for the original game.

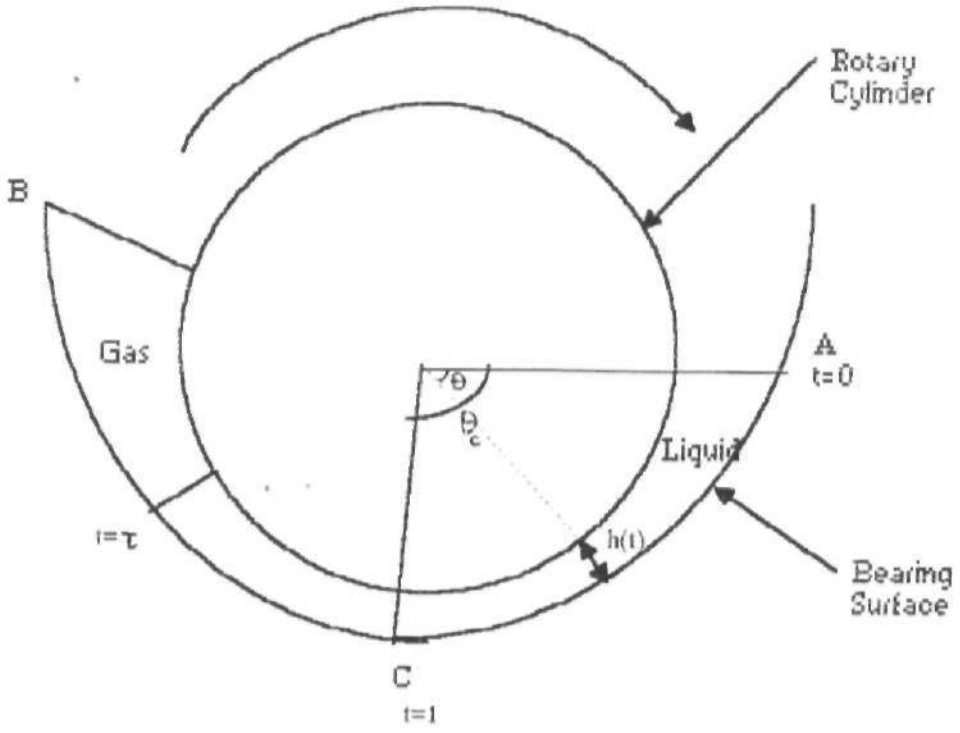
The last of the four areas of application of LCP to be examined is its application to the study of Lubrication of Journal Bearings.

4.5 JOURNAL BEARINGS AND LCP

A Journal Bearing consist of a rotary cylinder which is separated from a Bearing surface by a film of lubricating fluid (see figure I). The problem is to determine the pressure in the Lubricant.

FIGURE I

CROSS SECTION OF A JOURNAL BEARING



The depth of the film is minimum at C and we set $t = \theta/\theta_c$ where θ is the angle depicted at any point on the surface of the bearing as in figure I with θ_c as the angle depicted at the point of minimum dept. It could be observed that at the point C, $\theta = \theta_c$ and hence $t = \theta/\theta_c = 1$. In the case of partial bearing, the lubricant flows out at B and is replenished at A, while in the case of full bearing, the vapour formed at $t = \tau$ is assumed to condense at A into its previous liquid sate. The depth of the lubricating film is denoted h(t) and it satisfies the following condition.

$$h(t) > 0 \qquad t \in [0, T]$$

$$\frac{dh(t)}{dt} < 0 \qquad t \in [0, 1]$$

$$\frac{dh(t)}{dt} > 0 \qquad t \in [1, T]$$

The pressure on the film may be expected to increase between t=0 and t=1 and to decrease between t=1 and t=T. This is so because the depth of the lubricating film decreases with increasing $t \in [0, 1]$ and it increases with increasing $t \in [1, T]$. It is assumed that when $t = \tau$ the pressure becomes so low that the lubricant vapourizes.

The interface between the two boundaries of the lubricating fluid is called the free boundary. The mathematical formulation of the problem is as follows:

4.6 FREE BOUNDARY PROBLEM

The formulation of the free boundary problems and its conversion into a linear complementarity problem shall be the subject matter of this section. Also, going by the conversion of the problem into a LCP, the types of matrices involved and their implication toward the solution the LCP will be examined.

PROBLEM 4.6.1

Find a constant $\tau \in [0, T]$ and a function $p(t) \in c[0, T] \cap c^2(0\ T)$ which satisfies the following conditions.

$$(L_p) = \frac{d}{dt}\left(h^3(t)\ \frac{dp}{dt}\right) - \frac{dh}{dt} = 0; t \in (0, T) \qquad 4.6.2$$

$$P(0) = 0$$

$$P(t) = 0 \qquad t \in [\tau, T)$$

$$\frac{dp(\tau)}{dt} = 0$$

Equation (4.6.2) is known as Reynold's equation for the pressure in a lubricating film. To obtain a solution to this equation, Sommerfeld in Cryer [5] subdivide the interval [0, T] into N subintervals each of length δ so that $N = T/\delta$.

By using central differences, that is the forward difference operator Δ and the backward difference operator ∇, the Reynold's equation (4.6.2) can be approximated by finite difference equation.

$$(L_\delta)_i = \frac{1}{\delta^2}\Delta\left(h^3_{i-\frac{1}{2}}\nabla z_i\right) - \frac{1}{\delta}\Delta h_{i-\frac{1}{2}} = 0 \qquad [0 < i < N]$$

where $h_i = h(i\delta)$

$h_{i-\frac{1}{2}} = h[(1-\frac{1}{2})\delta]$

$\delta^2(L_\delta)_i$ can be written as

$$\delta^2(L_\delta)_i = \Delta\,[h^3_{i-\frac{1}{2}}(z_{i-1} - z_i)] - \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}) = 0$$

$$= [h^3_{i+\frac{1}{2}}(z_i - z_{i+1})] - [h^3_{i-\frac{1}{2}}(z_{i-1} - z_i)] - \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}) = 0$$

$$= h^3_{i+\frac{1}{2}}z_i - h^3_{i+\frac{1}{2}}z_{i+1} - h^3_{i-\frac{1}{2}}z_{i-1} + h^3_{i-\frac{1}{2}}z_i - \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}) = 0$$

$$= -h^3_{i-\frac{1}{2}}z_{i-1} + (h^3_{i+\frac{1}{2}} + h^3_{i-\frac{1}{2}})z_i - h^3_{i+\frac{1}{2}}z_{i+1} - \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}) = 0$$

Cryer [5] showed that this equation forms a discrete analogue of a problem which is equivalent to problem which is equivalent to problem (4.6.1). This discrete approximation is a linear complementarity problem of the following form. Let n = N-1 and define matrix $M \in \mathbb{R}_{n \times n}$ and vector $q \in \mathbb{R}_n$ such that

$$M_{ij} = \begin{cases} -h^3_{i+\frac{1}{2}} & \text{If } j = i+1 \\ h^3_{i+\frac{1}{2}} + h^3_{i-\frac{1}{2}} & \text{If } j = i \\ -h^3_{i-\frac{1}{2}} & \text{If } j = i-1 \\ 0 & \text{elsewhere} \end{cases} \qquad 4.6.3$$

for $i \leq n$, $j \leq n$

$$q_i = \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}) \qquad 1 \leq i \leq n$$

Going by the formulation 4.6.3, it will be observed that the matrix $M = (m_{ij})$ is a tridiagonal Minkowski matrix, hence the algorithm (3.3) proposed in chapter three is applicable to the LCP (M,q). Secondly, there is an integer k for which $q_i < 0$ if $i \leq k$ and $q_i \geq 0$ if $i > k$. Thus, in the journal bearing application, the negative entries in the constant column vector q all come at the beginning of the vector. Consequently the first k pivots of the algorithm's first cycle cause z_i $(1 \leq i \leq k)$ to become basic. Due to the tridiagonal nature of M, the only potentially negative entry in the constant column vector after the execution of the kth pivot is the (k+1)th entry. If it is non negative, then the algorithm terminates. Otherwise $\langle w_{k+1}, z_{k+1} \rangle$ is the next pivot of the cycle.

We continue in this manner until either the next entry in the constant column vector is non-negative or until n pivots have been accomplished.

In either case, the entire constant column vector will then be non negative and the solution will have been obtained in only one cycle of the algorithm.

CONCLUSION

This study has examined the linear complementarity problem by carefully defining the problem and examining some of the underlying theories that aid the study of linear complementarity problem. Also the relationship between LCP and other programming problems was examined as part of the applications of linear complementarity problem.

It was discovered that if the LCP (M,q) is consistent, there will exist a complementary basic feasible solution provided each almost complementary basic feasible solution is non-degenerate and the matrix M is copositive plus. By this assumption, the Lemke's complementary pivoting algorithm must terminate in a finite number of steps. Also the sufficient condition that guarantee the existence of a unique solution was examined. That is, the LCP (M,q) has a unique complementary basic feasible solution if the matrix M is a p-matrix. The p-matrix M was characterized in terms of all the complementary points for a single column vector q. In this case, Habetler and Kostreva [7] observed tht if M is a p-matrix then the set H_M of all $q \epsilon \mathbb{R}^n$ for which the set Σ (of complementary points) is pigeonholed is non-empty. That is, if M is a p-matrix, then there exist $q \epsilon \mathbb{R}^n$ for which the set Σ is pigeonholed.

Furthermore, other solution algorithms used to solve the linear complementarity problem were given, and so, there is a choice of approach for the solution of (M,q). All other algorithms proposed, apart from the Lemke's complementary pivoting algorithm

use the principal pivoting technique. These algorithms, which are based on principal pivoting are particularly useful when M is positive semidefinte. Some of the algorithms are stronger than the Lemke's complementary pivoting algorithm as could be seen in algorithm (2.4) which is a direct extension of Lemke's algorithm. For instance, in example (2.5.1), Lemke's algorithm fails to produce a solution while algorithm (2.1) and (2.4) produce a complementary feasible solution to the linear complementarity problem.

Considering the applications of linear complementarity problem, it was observed that, applying LCP approach in solving other programming problems like linear programming and quadratic programming gives a wider perspective for studying the existence of a solution. For instance, after the formulation of those problems into a LCP, if the matrix obtained is a p-matrix, then the probability of obtaining a complementary solution is one.

Kojima et al [14] proposed the path following algorithm for the linear complementarity problem and hence with a LCP (M,q) where M is a positive semidefinite matrix, if a strictly positive feasible solution is available, then ellipsoids each of which contains all the solutions of the LCP were proposed. Such an ellipsoid was used to compute a lower bound and an upper bound for each coordinate of the solutions of the LCP. If the computed lower bound is positive then we know that the variable is strictly positive over the solution set of the LCP. Hence by the complementarity condition, its complement is zero. If the LCP has

a unique non degenerate solution then the lower bound, and the upper bound for the solution, computed at each iteration both converge to the solution of the LCP.

Many interior point algorithms have been developed for the positive semidefinite LCP. They work as a primal-dual interior point algorithm when they are applied to a linear program and a quadratic program. Kojima et al [15] showed theoretical convergence of primal-dual interior point algorithm with the rule which depend on the lengths of steps from the current iterates in the primal-dual spaces to the respective boundaries of the primal dual feasible regions.

With regard to these two algorithms proposed by Kojima et al [15], the study of the LCP (M,q) enjoys a free hand in the choice of approach when the matrix involved is positive semidefinite. Consequently, linear programming and quadratic programming problems enjoy the same free hand as in the linearly complementarity problem.

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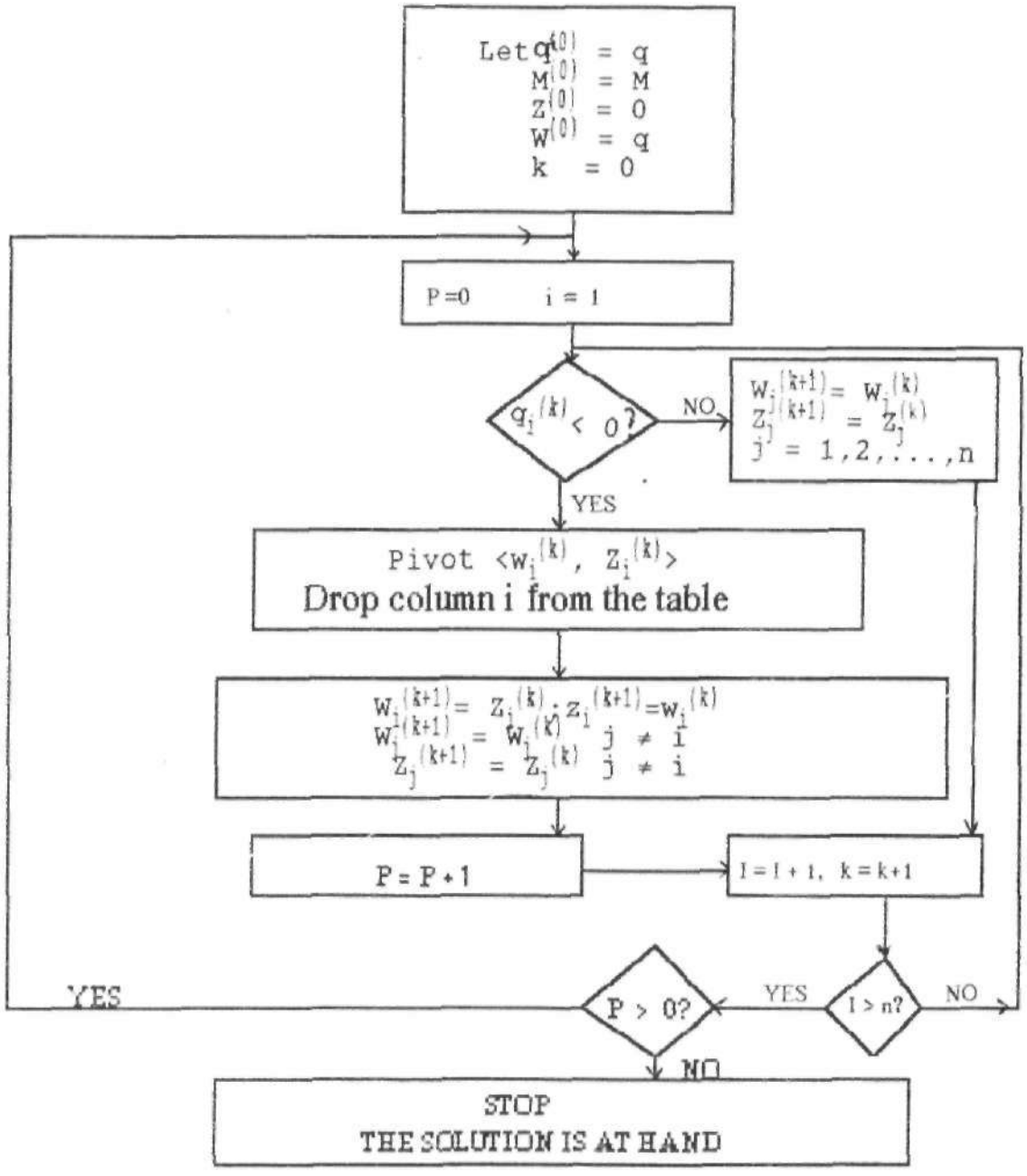
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Appendix A

Flowchart for the proposed algorithm (3.1.1) For Minkowski Linear Complementarity problems.



APPENDIX B

A Computer Program in FORTRAN to execute Minkowski and other Linear Complementarity Problems.

```
C       PROGRAM PIVOT
        COMMON M(10,10)
        DIMENSION T(10,10),Q(10),W(10),Z(10)
        READ(1,*)N
        DO 10 I = 1,N
        READ(1,*)Q(I)
 10     CONTINUE
        DO 57 I=1,N
        DO 15 J = 1,N
        READ(1,*)M(I,J)
 15     CONTINUE
 57     CONTINUE
        DO 31 I=1,N
        DO 32 J=1,N
        T(I,J)=M(I,J)
 32     CONTINUE
 31     CONTINUE
        NI = N+1
        DO 30 K = 1,N
        T(K,NI) = Q(K)
 30     CONTINUE
        DO 35 J = 1,N
        W(J) = T(J,NI)
```

```
      Z(J) = 0
35    CONTINUE
      KT=0
9     WRITE(2,71)KT
71    FORMAT(///,'AT THE END OF ITERATION ',I2,' ,WE HAVE:',//)
      WRITE(2,5)((T(I,J),J=1,NI),I=1,N)
5     FORMAT(5(3X,F7.3))
      WRITE(2,67)
67    FORMAT(/,'WHICH GIVES THE FOLLOWING SOLUTION',/)
      DO 77 J=1,N
      WRITE(2,87)J,W(J),J,Z(J)
87    FORMAT('W',I1,'=',F7.3,3X,'Z',I1,'=',F7.3)
77    CONTINUE
      I=1
36    IF(T(I,NI).LT.0) GO TO 40
      I=I+1
      IF(I.LE.N) GO TO 36
      GO TO 38
40    WRITE(2,56)
56    FORMAT(/,'THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT
     1 ITERATION.')
      DO 16 J=I,N
      IF(T(J,J).NE.0)GO TO 13
16    CONTINUE
      DO 17 J=1,N
      IF(T(J,J).NE.0)GO TO 13
```

```
17      CONTINUE
        GO TO 24
13      I=J
        CALL ITERATE(T,N,NI,I,Z,W)
        KT=KT+1
        GO TO 9
24      WRITE(2,14)
14      FORMAT('NO SOLUTION')
        GO TO 115
38      WRITE(2,85)
85      FORMAT(///,'THE COMPLEMENTARY BASIC FEASIBLE SOLUTION
     2 IS',///)
        DO 110 J = 1,N
        WRITE(2,105)J,W(J),J,Z(J)
105     FORMAT(1X,'W',I1,' = ',F7.3,3X,'Z',I1,' = ',F7.3,/)
110     CONTINUE
115     STOP
        END
C
        SUBROUTINE ITERATE(T,N,NI,I,Z,W)
        DIMENSION TI(10,10)
        DIMENSION T(10,10),W(10),Z(10)
        TI(I,I) = 1/T(I,I)
        DO 2 J = 1,N
        IF(J.EQ.I) GO TO 2
        TI(J,I) = T(J,I)*TI(I,I)
```

```
      TI(I,J)=T(I,J)*(-TI(I,I))
      DO 1 K = 1,NI
      IF(K.EQ.I) GO TO 1
      TI(J,K) = T(J,K)-T(J,I)*TI(I,I)*T(I,K)
1     CONTINUE
2     CONTINUE
      TI(I,NI)=T(I,NI)*(-TI(I,I))
      DO 4 L =1,N
      IF(L.NE.I) GO TO 7
      IF(W(L).EQ.T(L,NI)) GO TO 6
8     Z(L) = 0
      W(L) = TI(L,NI)
      GO TO 4
6     W(L) = 0
      Z(L) = TI(L,NI)
      GO TO 4
7     IF(W(L).EQ.T(L,NI)) GO TO 8
      GO TO 6
4     CONTINUE
      DO 12 J = 1,N
      DO 11 K = 1,NI
      T(J,K) = TI(J,K)
11    CONTINUE
12    CONTINUE
      RETURN
      END
```

GIVEN THE FOLLOWING MATRICES

$$M = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 2 & -2 \\ 1 & 2 & -2 & 4 \end{bmatrix} \qquad Q = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -6 \end{bmatrix}$$

AT THE END OF ITERATION 0 ,WE HAVE:

.000	.000	-1.000	-1.000	2.000
.000	.000	1.000	-2.000	2.000
1.000	-1.000	2.000	-2.000	-2.000
1.000	2.000	-2.000	4.000	-6.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= 2.000 Z1= .000

W2= 2.000 Z2= .000

W3= -2.000 Z3= .000

W4= -6.000 Z4= .000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 1 ,WE HAVE:

.500	-.500	-.500	-2.000	1.000
-.500	.500	.500	-1.000	3.000
-.500	.500	.500	1.000	1.000
2.000	1.000	-1.000	2.000	-8.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= 1.000 Z1= .000

W2= 3.000 Z2= .000

W3= .000 Z3= 1.000

W4= -8.000 Z4= .000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 2 ,WE HAVE:

2.500	.500	-1.500	-1.000	-7.000
.500	1.000	.000	-.500	-1.000
-1.500	.000	1.000	.500	5.000
-1.000	-.500	.500	.500	4.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= -7.000 Z1= .000

W2= -1.000 Z2= .000

W3= .000 Z3= 5.000

W4= .000 Z4= 4.000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 3 ,WE HAVE:

.400	-.200	.600	.400	2.800
.200	.900	.300	-.300	.400
-.600	.300	.100	-.100	.800
-.400	-.300	-.100	.100	1.200

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= 2.800

W2= .400 Z2= .000

W3= .000 Z3= .800

W4= .000 Z4= 1.200

THE COMPLEMENTARY BASIC FEASIBLE SOLUTION IS

W1 = .000 Z1 = 2.800

W2 = .400 Z2 = .000

W3 = .000 Z3 = .800

W4 = .000 Z4 = 1.200

GIVEN THE FOLLOWING MATRICES

$$M = \begin{matrix} 0 & 3 \\ 2 & -1 \end{matrix} \qquad Q = \begin{matrix} -2 \\ -1 \end{matrix}$$

AT THE END OF ITERATION 0 ,WE HAVE:

.000	3.000	-2.000
2.000	-1.000	-1.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= -2.000 Z1= .000

W2= -1.000 Z2= .000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 1 ,WE HAVE:

6.000	-3.000	-5.000
2.000	-1.000	-1.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= -5.000 Z1= .000

W2= .000 Z2= -1.000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 2 ,WE HAVE:

.167	.500	.833
.333	.000	.667

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= .833

W2= .000 Z2= .667

THE COMPLEMENTARY BASIC FEASIBLE SOLUTION IS

W1 = .000 Z1 = .833

W2 = .000 Z2 = .667

GIVEN THE FOLLOWING MATRICES

$$M = \begin{bmatrix} 1 & 1 & 2 & 0 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & 0 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} -9 \\ 5 \\ 3 \\ -5 \end{bmatrix}$$

AT THE END OF ITERATION 0 ,WE HAVE:

1.000	1.000	2.000	.000	-9.000
-1.000	1.000	-1.000	-1.000	5.000
1.000	-1.000	-1.000	-1.000	3.000
1.000	1.000	.000	2.000	-5.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= -9.000 Z1= .000

W2= 5.000 Z2= .000

W3= 3.000 Z3= .000

W4= -5.000 Z4= .000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 1 ,WE HAVE:

1.000	-1.000	-2.000	.000	9.000
-1.000	2.000	1.000	-1.000	-4.000
1.000	-2.000	-3.000	-1.000	12.000
1.000	.000	-2.000	2.000	4.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= 9.000

W2= -4.000 Z2= .000

W3= 12.000 Z3= .000

W4= 4.000 Z4= .000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 2 ,WE HAVE:

.500	-.500	-1.500	-.500	7.000
.500	.500	-.500	.500	2.000
.000	-1.000	-2.000	-2.000	8.000
1.000	.000	-2.000	2.000	4.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= 7.000

W2= .000 Z2= 2.000

W3= 8.000 Z3= .000

W4= 4.000 Z4= .000

THE COMPLEMENTARY BASIC FEASIBLE SOLUTION IS

W1 =	.000	Z1 =	7.000
W2 =	.000	Z2 =	2.000
W3 =	8.000	Z3 =	.000
W4 =	4.000	Z4 =	.000

GIVEN THE FOLLOWING MATRICES

$$M = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 2 \\ -1 & 2 & 5 \end{bmatrix} \qquad Q = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$$

AT THE END OF ITERATION 0 ,WE HAVE:

1.000	.000	3.000	-3.000
2.000	1.000	2.000	-2.000
-1.000	2.000	5.000	-1.000

WHICH GIVES THE FOLLOWING SOLUTION

W1=	-3.000	Z1=	.000
W2=	-2.000	Z2=	.000
W3=	-1.000	Z3=	.000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 1 ,WE HAVE:

1.000	.000	-3.000	3.000
2.000	1.000	-4.000	4.000
-1.000	2.000	8.000	-4.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= 3.000

W2= 4.000 Z2= .000

W3= -4.000 Z3= .000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 2 ,WE HAVE:

.625	.750	-.375	1.500
1.500	2.000	-.500	2.000
.125	-.250	.125	.500

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= 1.500

W2= 2.000 Z2= .000

W3= .000 Z3= .500

GIVEN THE FOLLOWING MATRICES

THE COMPLEMENTARY BASIC FEASIBLE SOLUTION IS

W1 =	.000	Z1 =	1.500
W2 =	2.000	Z2 =	.000
W3 =	.000	Z3 =	.500

GIVEN THE FOLLOWING MATRICES

$$M = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 5 \\ 2 & 1 & 2 \end{bmatrix} \qquad Q = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix}$$

AT THE END OF ITERATION 0 ,WE HAVE:

1.000	.000	3.000	-3.000
-1.000	2.000	5.000	-2.000
2.000	1.000	2.000	-1.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= -3.000		Z1=	.000
W2= -2.000		Z2=	.000
W3= -1.000		Z3=	.000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 1 ,WE HAVE:

1.000	.000	-3.000	3.000
-1.000	2.000	8.000	-5.000
2.000	1.000	-4.000	5.000

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= 3.000

W2= -5.000 Z2= .000

W3= 5.000 Z3= .000

THIS SOLUTION IS NOT FEASIBLE;GO TO NEXT ITERATION.

AT THE END OF ITERATION 2 ,WE HAVE:

1.000	.000	-3.000	3.000
.500	.500	-4.000	2.500
2.500	.500	-8.000	7.500

WHICH GIVES THE FOLLOWING SOLUTION

W1= .000 Z1= 3.000

W2= .000 Z2= 2.500

W3= 7.500 Z3= .000

THE COMPLEMENTARY BASIC FEASIBLE SOLUTION IS

W1 =	.000	Z1 =	3.000
W2 =	.000	Z2 =	2.500
W3 =	7.500	Z3 =	.000