LIMIT SETS, PROLONGATIONS AND RELATED TOPICS IN THE THEORY OF GENERAL DYNAMICAL SEMI-SYSTEMS

By

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DECLARATION

This thesis is not published or an unpublished work of another author. No part of this thesis has been submitted for a degree in any other University. Where the work of others have been used, the source is acknowledged in the text.

CERTIFICATION

This Thesis entitled

LIMIT SETS, PROLONGATIONS AND RELATED TOPICS IN THEORY OF GENERAL DYNAMICAL SEMI-SYSTEMS by MICHAEL OLUWOLE ADEWALE meets the regulations governing the award of the degree of DOCTOR OF PHILOSOPHY of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literacy presentation.

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To my parents,

ABIGAIL OLAPERI and JOSEPH ADENIRAN ADEWALE,

who, owing to their Love and enlightenment invested on my education out of their meagre income,

my wife,

EBUNLOMO ADEYOOLA,

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CHAPTER 0

INTRODUCTION

§ 0.1.

The classical theory of dynamical systems is based on the so-called qualitative theory of ordinary differential equations. In this theory, usually, we are not interested in solving explicitly the differential equation, but rather, we are looking for Theorems that deal with the general properties of solutions of equations belonging to some classes of equations or about properties of families of equations, as well as about properties of some subsets of spaces in which we consider corresponding equations with respect to solutions of those equations.

In particular, to that theory belong such topics as: stability of trajectories and motions given by differential equations, asymptotic behaviour of solutions (including existence and non-existence of limit sets and the properties of those sets), invariance of some sort with respect to mappings induced by solutions; topological properties of families of curves representing trajectories of solutions and many others.

The theory of dynamical systems has as its background the qualitative theory of (autonomous) ordinary differential equations with uniqueness while the theory of semi-systems generalizes that part of the theory of (autonomous) ordinary differential equations without uniqueness to the left and the delay differential equations as well.

The theory of processes and pseudo-processes generalizes the theory of non-autonomous ordinary differential equations together with delay-differential equations as well.

A processes is a direct generalization of the concept of topological dynamical system. The transition from dynamical system to processes is, roughly speaking analogous to the transition from autonomous to non-autonomous systems of ordinary differential equations.

The classical theory of dynamical systems has been dealt with by many authors, some of them generalized it by introducing the so-called dynamical semi-systems, pseudo-dynamical systems semi-systems, generalized dynamical systems, local dynamical semi-systems and many others.

Also, some particular theories (or rather, sub-theories), for example the theory of stability, has been investigated and was generalized independently of the general ones. There appeared some natural trends with a view to making such generalizations which would give rise to some kind of uniformity and a good classification of various types of general theories of (generalized) dynamical systems, as well as the possibilities of reasonable uniform presentation of various versions of stabilities, asymptotic behaviour, limit sets, prolongational limit sets, prolongations and so on.

§ 0.2. CLASSICAL DYNAMICAL SYSTEMS

In order to introduce some fundamental notions in a natural way, we consider first of all, classical dynamical systems induced by the system of autonomous differential equations. Let $f: R^n \longrightarrow R^n$ be a given continuous function. Consider the following ordinary differential equatio

$$(0.2.1) \qquad x' = f(x), \qquad (\text{where } x' = \frac{dx}{dt})$$

(representing in fact the following system of n ordinary differential equations

$$(0.2.1_o) \qquad x_i' = f_i(x_1, x_2, \ldots, x_n); \quad (i=1,2,\ldots,n)).$$

Let us consider the following in general:

Assumption (A):

For every $x_o \in R^n$, there exists exactly one solution of the equation (0.2.1) passing through the point $(x_o, 0)$ belonging to $R^n \times R$ (which means that, the unique solution satisfies the initial condition

$$(0.2.2) \qquad x(0) = x_o$$

defined and being of class C^1 in R. We denote this solution by $x(x_o;\cdot)$.

Thus $x(x_o;\cdot): R \longrightarrow R^n$ is such a function of the class C^1, that is,

$$x'(x_o;t) = f(x(x_o;t)) \qquad \text{for } t \in R \quad \text{and}$$

$$x(x_o;0) = x_o.$$

We use here, the natural notation:

$$x'(x_o;t) = \left(\frac{\partial}{\partial t} x(x_o;t)\right)$$

if x is considered as a function of two variables t (interpreted usually as time) and x_o.

Remark 0.2.1

Assumption (A) above is equivalent to the following condition (A'): For every $(x_o,t) \in R^n \times R$ there exists exactly one solution x of the equation (0.2.1) satisfying the initial condition $x(t_o) = x_o$ defined and being of the class C^1 in R.

It is well known that for every $x_o \in R^n$ and every $t,s \in R$, we have

$$x(x(x_o,t),s) = x(x_o,t+s). \tag{0.2.3}$$

We now state the following:

Assumption (B):

For every fixed $t \in R$, the mapping $R^n \ni x_o \longmapsto x(x_o;t) \in R^n$ is continuous.

Remark 0.2.2

In the classical theory of ordinary differential equations, there are proved well known theorems about the continuous dependence of solutions on initial data. Such theorems permit us to replace the above assumption (B) with suitable conditions that appeared as assumptions in those theorems. This is however not essential here and we shall limit ourselves to assumption (B) only.

It is pertinent to notice that the mapping

$$R^n \times R \ni (x_o,t) \longmapsto x(x_o;t) \in R^n \tag{0.2.4}$$

induces a family of mappings of R^n into itself. This family is indexed by R: For each fixed $t \in R$ we consider the mapping

$$R^n \ni x_o \longmapsto x(x_o;t) \in R^n. \tag{0.2.5}$$

The condition (0.2.2) implies that for $t = 0$ we have the identity mapping $R^n \longrightarrow R^n$ while condition (0.2.3) gives the following connection between composition operation of family of our mappings and the addition: the composition of two mappings corresponding to t and s is the mapping corresponding to t+s.

In that sense, we approach the elementary and classical theory of (continuous) dynamical systems.

We now forget for the moment about how we arrived at the mapping (0.2.4), instead, we consider it together with the assumption that the conditions (0.2.2) and (0.2.3) are satisfied as a given mapping from $R^n \times R$ into R^n. Repeating now formally the observations given above, we can state that the mapping (0.2.4) induces the family $\{\pi_t\}_{t \in R}$ of mappings of R^n into itself, defined as follows:

For every fixed $t \in R$, the function π_t is given by the formula

$$(0.2.6) \qquad \pi_t(x_o) = \pi(x_o;t) \qquad \text{for } x_o \in R^n.$$

The conditions (0.2.2) and (0.2.3) give the formulas

$$(0.2.7) \qquad \pi_o = \text{identity mapping of } R^n$$

and

$$(0.2.8) \qquad \pi_t \circ \pi_s = \pi_{t+s} \qquad \text{for every } t,s \in R.$$

The mapping

$$(0.2.9) \qquad R^n \times R \ni (x,t) \longmapsto \pi_t(x) \in R^n$$

is continuous (since it is in fact the mapping (0.2.4)).

So, we obtain a family of mappings $\{\pi_t\}$ indexed by R, having a group structure with respect to the group operation being the composition, with the neutral element being the identity mapping. This group is abelian and it is isomorphic with the additive group (R,+). Of course, each of the mappings belonging to the family above is a bijection of R^n into itself; the inverse mapping for π_t is clearly π_{-t}.

All the mappings are continuous and moreover, the dependence

of the elements of the family on the parameter (index) t is not only continuous but also is such that the mapping (0.2.9) is continuous. Such a family forms the so-called dynamical system on R^n.

Remark 0.2.3

If the system (0.2.1) does not satisfy the uniqueness condition, to the left or to the right, then we can define dynamical semi-systems.

Roughly speaking, a dynamical semi-system is defined in the same manner as a dynamical system in one direction of time (either positive or negative) and because of lack of uniqueness or existence in the other direction it is not defined for $t \leq 0$ (or $t \geq 0$ respectively).

Remark 0.2.4

The evolution of the theory of dynamical systems has given rise to some class of problems which do not need the topological structures in R^n and R as well as the continuity of π, neither for the formulation of problems nor for their solutions. Because of this fact, there appeared the notion of pseudo-dynamical system $(R^n, R; \pi)$ in which R^n is a simple set, $(R,+)$, a group and the mapping π fulfils only the first two conditions, (i.e. the identity property and the group peoperty). Since every dynamical system is evidently a pseudo-dynamical system, we have some kind of generalization.

§ 0.3. NONAUTONOMOUS DIFFERENTIAL EQUATIONS

It is known that the system of non-autonomous differential equations of the form:

$$(0.3.1) \qquad x' = f(x,t); \quad \left(x' = \frac{dx}{dt}\right)$$

in n-space, where $f: \in R^n \times R \longrightarrow R^n$ is continuous and satisfies the uniqueness condition does not basically give rise to dynamical systems or semi-systems without further adjustments being made. In this case one can propose generalisations by using the theory of so-called processes.

The system (0.3.1) can be however reduced to an autonomous system of the form

$$(0.3.2) \qquad \frac{dx}{dt} = f(x,\xi); \quad \frac{d\xi}{dt} = 1$$

in the (n+1)-space. This is arrived at by treating the independent variable t as a new coordinate.

While the construction (0.3.2) above is valid, it has the effect of destroying some of the latent structure of the original equation. For example, the new equation will not have any bounded motions, nor any periodic motions, nor any almost periodic motions. Because of this fact, the theory of topological dynamics has not developed into a powerful technique in applications to non-autonomous differential equations.

Howbeit, Sell (1967a), using the suggestion given in Miller (1965a) developed the basic theory for viewing the solutions of non-autonomous differential equations as dynamical systems. He affirmed that such system like (0.3.1) that satisfies natural existence and uniqueness conditions for solutions induce a dynamical system in the space R^{n+1}.

Instead of treating $R^n \times R$ as the new phase space, one can consider the product $R^n \times \mathcal{F}$, where $\mathcal{F}$ is the function space consisting of the translates of f. That is, a typical element of $\mathcal{F}$ would be f_τ where $f_\tau(x,t) = f(x,\tau+t)$. By introducing an appropriate topology on $\mathcal{F}$, one can define „a local dynamical system" on $R^n \times \mathcal{F}$. If the solution of (0.3.1) is defined for all time t, this becomes a dynamical system on R^n

CONSTRUCTION OF THE DYNAMICAL SYSTEM.

Let $f \in C(R^n \times R, R)$ be an admissible function that satisfies the global existence property and let $\hat{X}$ be the product space $R^n \times \mathcal{F}$. A metric d on $\hat{X}$ is defined by

$$(0.3.3) \qquad d((x,f),(\hat{x},\hat{f})) = |x-\hat{x}| + \rho(f,\hat{f})$$

where ρ is any metric on $\mathcal{F}$.

We define the mapping $\pi : \hat{X} \times R \longrightarrow \hat{X}$ by

$$(0.3.4) \qquad \pi(x,f;\tau) = (\varphi(x,f,\tau), f_\tau)$$

where $\varphi(x,f;\tau)$ denotes the solution of (0.3.1) that satisfies the condition

$$(0.3.5) \qquad \varphi(x,f,0) = x.$$

We have that $\pi(x,f,0) = (\varphi(x,f,0), f_0) = (x,f)$. So, π satisfies the identity property. From the definition of π, it could be easily proved that the triple $(\hat{X},R;\pi)$ is a dynamical system.

§ 0.4. NON AUTONOMOUS DELAY DIFFERENTIAL EQUATIONS

Suppose that $r>0$ is a given real number, R^n is an n-dimensional space over reals, $C([a,b],R^n)$ is the space of continuous functions mapping the interval $[a,b]$ into R^n with the topology of uniform convergence. It $[a,b]=[-r,0]$, we let $C = C([-r,0],R^n)$. If $\sigma \in R$, $\alpha \geq 0$ and $x \in C([\sigma-r, \sigma+\alpha],R^n)$, then for any $t \in [\sigma,\sigma+\alpha]$ we let $x_t \in C$ be defined by

$$(0.4.1) \qquad x_t(\tau) = x(t+\tau), \quad -r \leq \tau \leq 0.$$

If D is a subset of $R \times C$, $f: D \longrightarrow R^n$ is a given function and "$\cdot$" represents the right-hand derivative, we say that the equation

(0.4.2) $\dot{x}(t) = f(t,x_t)$

is a nonautonomous delay differential equation on D and we will denote it by "DDE".

A function x is said to be a solution of (0.4.2) on $(\sigma-r,\sigma+\alpha)$ if there are $\sigma \in R$ and $\alpha>0$ such that $x \in C([\sigma-r,\sigma+\alpha],R^n)$; $(t,x_t) \in D$ and x(t) satisfies equation (0.4.2) for $t \in [\sigma,\sigma+\alpha]$.

For given $\sigma \in R$, $\varphi \in C$, we say that $x(\sigma,\varphi,f)$ is a solution of equation (0.4.2) with initial value φ at σ or simply a solution through the point (σ,φ) if there is $\alpha>0$ such that $x(\sigma,\varphi,f)$ is a solution of equation (0.4.2) on $[\sigma-r,\sigma+\alpha]$ and $x_\sigma(\sigma,\varphi,f) = \varphi$.

Equation (0.4.2) is a very general type of equation and it includes ordinary differential equation (r=0), $\dot{x}(t) = f(x(t))$. We say that equation (0.4.2) is autonomous if $f(t,\varphi) = g(\varphi)$, where $g(\varphi)$ does not depend on t.

§ 0.5. FUNDAMENTAL NOTATIONS

By R, we shall denote the space of real numbers, so, $R^n = R \times \ldots \times R$ (n times) is the n-dimensional real Euclidean space. This usual notation had already been used in sections 0.1 - 0.4.

The set R will be considered sometimes as a set only, and sometimes as a topological space (provided with the natural topology) sometimes as a group (R,+) or a topological group as well as an ordered group $(R,+,\leq)$. It will be clear in each particular context (otherwise it will be stated explicitly). The same is with respect to R^n.

We shall use the usual logic notation and the classical set theory terminology and notation. The symbol := stands for „equal by the definition", the symbol $\Longleftrightarrow$ stands for „if and only if" while the symbol $\overset{df}{\Longleftrightarrow}$ stands for „if and only if by the definition".

We put $R_+ := \{t \in R;\ t \geq 0\}$, $R_- := \{t \in R:\ t \leq 0\}$.

The set of all integers will be denoted by Z while N denotes the set of all positive integers and $N \cup \{0\}$ will be denoted by N_0.

§ 0.6. HISTORICAL AND TERMINOLOGICAL REMARKS

The name "Dynamical system" (in French "système Dynamique") and (in Polish "Układy dynamiczne") is a natural extension of the terminology taken from the theory of ordinary differential equations. In the classical Theory of ordinary differential equations, this name has been used for systems having the right-hand side independent of the time (in the usual mechanical interpretation of the independent variable as time).

Nowadays, the name "Autonomous system" is used most frequently for such systems of differential equations.

We know that such systems of differential equations induce the fundamental examples of dynamical systems in the general sense. Because of the topological assumptions required in the definition of the general dynamical systems, there is also used, the name "Topological Dynamics" in the place of "dynamical systems".

Recently, many authors proposed their terminologies for generalized systems, including local ones. There are for example, dynamical semi-systems (or semi-dynamical systems) and local dynamical semi-systems and so on. The definitions are not compatible with one another and because of this fact, we cannot keep to a particular terminology met in the mathematical literatures.

It is impossible to present here the full bibliography concerning the theory (or rather theories) of dynamical systems and their generalizations. Let us mention only some important books and papers. Generally, theory of global dynamical systems

in R^n and general metric spaces is treated in the book of Bhatia and Szego as well as in their earlier script (preprint) of then (see Bhatia and Szego (1967) and (1970)), while the local dynamical systems and semi-systems are investigated in the script of Bhatia and Hajek (1969), (where the term semi-dynamical systems is used) and in the book by Sell, G.R. (Sell (1966), (1967a) and (1967b)), in which the name "Topological dynamics" is employed, together with Sibirsky (1975).

A very good introduction for the generalized dynamical systems is given in the paper by Sell (1971), while some special cases of local pseudo-dynamical semi-systems are considered by Pelczar (see Pelczar (1978) and (1980)).

Generalized systems are discussed in particular by Bronshtein (1973), (1962), (1968a,b), Pelczar (1980) (part II) and (1973), Roxin (1971). For further refefences, the rich bibliography presented in the book of Bhatia and Szego (1967) and Sell (1971) will be of great help.

This thesis is concerned with the theories of Dynamical systems and semi-systems (and their generalizations - for example, pseudo-dynamical semi-systems) and the so called pseudo-processes as well, and families of such systems.

The theorems that answer the following questions are stated and proved.

- What kind of properties of dynamical systems (and their generalizations) are preserved passing to the limits of sequences of such systems?

- Which properties of the trajectories, inverse trajectories, limit sets, prolongational limit sets and prolongations in dynamical semi-systems and processes are preserved when passing to the limits of the sequences of such semi-systems and processes?

The main results are all the theorems in sections 3.1 and 3.2, theorem 3.3.2, theorems 4.3.1 - 4.3.4. Supplementary results that are also original and new are dealt with in chapter 5 with respect to cartesion products of dynamical systems and processes together with asymptotic hulls of processes as well.

There are some results which are extensions and generalizations of known ones. For instance, we mention in particular that from theorem 3.1.3 we obtain as corollany (see Remar 3.1.3) the known result on the upper

semicontinuity of the mapping

$$x \longmapsto \Lambda^{+}(x)$$

and

$$x \longmapsto \Lambda^{-}(x)$$

The thesis contains also some known results which are mentioned in order to make it self-contained and so, it presents the original results which form its main part and also a complete outline of that of the theory of dynamical systems and semi-systems which deals with limit sets, prolongational limit sets and prolongations.

CHAPTER I

GENERAL DEFINITIONS

We recall the definition of (continuous) dynamical system in the classical sense. In particular, we employ only the notations which will be used often in this study.

Along with the above, we give the definitions in the classical sense of the concepts like: periodicity and almost periodicity of motions, Stability of sets and of motions, Limit sets, prolongational limit sets and prolongations including their respective properties.

We state briefly the pointwise and uniform convergence in one variable and also in two variables of the sequences $\{(X,R;\pi_n)\}$ and $\{(X,R_+,\pi_n)\}$ of pseudo-dynamical systems and also pseudo-dynamical semi-systems respectively.

The fundamental definitions of systems and semi-systems in general topological spaces are discussed. However, our study deals with systems and semi-systems in metric spaces only and the group R or semi-groups R_+ and R_-. Some known theorems are stated without proofs.

§ 1.1.

Definition 1.1.1

Let X be a topological space, $(G;+)$ be a topological abelian group and let π be a mapping from $X \times G$ into X. The triple $(X,G;\pi)$ is said to be a dynamical (continuous) system if and only if

(1.1.1) $\quad \pi(x,0) = x \quad$ for every $x \in X$

(1.1.2) $\pi(\pi(x,s),t) = \pi(x,t+s)$ for every $x \in X$, for every $t,s \in G$

(1.1.3) π is continuous.

We should note that we put the "continuity" in brackets because in our definitions, the term "dynamical" will cover the continuity property of the considered systems, while in some books and research papers, the "continuity" is explicity underlined in the terminology.

It is clear that the conditions (1.1.1) and (1.1.2) above are equivalent to the following conditions: The mapping induces the family $\{\pi_t\}_{t \in G}$ of mappings of X into itself provided with the abelian group structure with respect to the composition operation with the neutral element being the identity in X, so that relations (0.2.7) and (0.2.8) are satisfied. Of course, the definition of π_t is such that $\pi_t(x) = \pi(x,t)$ (see (0.2.6)).

A special case and simultaneously a fundamental model (and an example showing classical applications) of such dynamical system is that obtained previously on the basis of differential equation theory; in this case, $X = R^n$, $G = R$, π = mapping (0.2.4). Repeat now that starting with equation (0.2.1) for which the existence and uniqueness of solutions are assumed only for $t \geq 0$, we can repeat our reasoning and we shall obtain as a direct generalization the so-called dynamical semi-systems in the sense more general than was discussed in section 0.2.

Definition 1.1.2

Let X be a topological space, $(G;+)$ be an abelian semi-group with the neutral element 0, and let finally π be the mapping of $X \times G$ into X. The triple $(X,G;\pi)$ is said to be a dynamical semi-system if and only if the conditions (1.1.1) - (1.1.3)

in the above definition 1.1.1 are satisfied.

It is clear that in this case the family $\{\pi_t\}_{t \in G}$ has a natural semi-group (abelian) structure with the neutral element being the identity in X.

If we assume the uniqueness and existence properties for local solutions only, we can obtain in a similar way, the theory of the so-called local dynamical systems (or semi-systems) (see Bhatia and Hajek (1969) and Pelczar (1980)). There are also more general theories (see for instance Pelczar (1980), (1978)) concerning such cases in which we omit the topological structures in X and G (and necessarily, the condition (1.1.3) in definition 1.1.1). For details, see the next section.

§ 1.2. LOCAL AND GLOBAL PSEUDO-DYNAMICAL AND DYNAMICAL SYSTEMS AND SEMI-SYSTEMS

Let X be a non-empty set and let (G;+) be an abelian semi-group with a neutral element 0 and let A be a subset of the cartesian product $X \times G$ such that

(1.2.1) $\quad X \times \{0\} \subset A.$

We should note that condition (1.2.1) implies that $A \neq \emptyset$. Let us put for $x \in X$

(1.2.2) $\quad I_x := \{t \in G : (x,t) \in A\}.$

Since $0 \in I_x$ for every $x \in X$, we have that $I_x \neq \phi$ for every $x \in X$. Let now π be the mapping from A into X.

Definition 1.2.1

The quadruple $(X,G,A;\pi)$ is said to be a local pseudo-dynamical semi-system (shortly written as loc.PDSS) if and only if

(1.2.3) $\pi(x,0) = x$ for every $x \in X$

(1.2.4) If $x \in X$, $t \in I_x$ and $s \in I_{\pi(x,t)}$, then $t+s \in I_x$

(1.2.5) For every $x \in X$, $t \in I_x$, $s \in I_{\pi(x,t)}$, we have $\pi(\pi(x,t),s) = \pi(x,s+t)$.

Definition 1.2.2

If $(X,G,A;\pi)$ is a loc.PDSS such that $A = X \times G$, then $(X,G,A;\pi)$ is said to be a global pseudo-dynamical semi-system or simply a pseudo-dynamical semi-system (shortly written as PDSS); in that case, we write $(X,G;\pi)$ instead of $(X,G,X \times G;\pi)$.

Definition 1.2.3

If $(X,G,A;\pi)$ (or $(X,G;\pi)$) is a loc.PDSS (respectively: PDSS) such that $(G;+)$ is a group and moreover: if $x \in X$ and $t \in I_x$, then $-t \in I_{\pi(x,t)}$ (and so necessarily we have: $\pi(\pi(x,t),-t) = x$ for every $x \in X$ and $t \in I_x$), then $(X,G,A;\pi)$, (respectively: $(X,G;\pi)$) is said to be a local pseudo-dynamical system (respectively: pseudo-dynamical system (shortly written respectively as: loc. PDS and PDS).

Remark 1.2.1

It is clear that the condition (1.4.1) above is automatically satisfied if $A = X \times G$, so in the case of global PDS, we do not need to assume separately this condition.

Definition 1.2.4

Let $(X,G,A;\pi)$ be a loc.PDSS (or PDSS or loc.PDS or PDS); in the case of PDSS or PDS we have of course $(X,G;\ )$. We say that $(X,G,A;\pi)$ is a local dynamical semi-system (or respectively - a dynamical semi-system; or a local dynamical system, or a dynamical system) if and only if

(1.2.6) X is a topological space

(1.2.7) $(G;+)$ is a topological semi-group (or in the case of loc.PDS or PDS - a topological group)

(1.2.8) A is open in the product topology in $X \times G$ (this condition is trivially satisfied in the global case: $A = X \times G$

(1.2.9) $\pi: A \longrightarrow X$ is continuous.

We like to point out that some authors use the terminology „semi-dynamical systems" in the place of „dynamical semi-systems".

Hence forth, we replace the set G with R, the set of real numbers. Instead of triple $(X,G;\pi)$ we have the triple $(X,R;\pi)$.

Given a dynamical system on X, the space X and the map are respectively called the phase space and the phase map (of the dynamical system). The sets $\pi(x) = \pi(x,R)$, $\pi_+(x) = \pi(x,R_+)$ and $\pi_-(x) = \pi(x,R_-)$ are respectively refered to as the trajectory, the positive semi-trajectory and the negative semi-trajectory through the point $x \in X$.

For every $x \in X$, the map π induces a continuous map

(1.2.10) $\pi^x: R \longrightarrow X$

defined by the formula:

(1.2.11) $\pi^x(t) = \pi(x,t)$.

This mapping π^x is called the motion through the point $x \in X$.

Moreover, for every $t \in R$, the map π induces a map

(1.2.12) $\pi_t: X \longrightarrow X$

defined by

(1.2.13) $\pi_t(x) = (x,t)$.

The map π_t is called the <u>transition</u> (or action)

Theorem 1.2.1

The map π_{-t} defined by $\pi_{-t}(x) = \pi(x,-t)$ is the inverse of the map π_t.

It is worth it to mention that in some standard text books on Dynamical systems, the symbol π is deleted. In those books, the image $\pi(x,t)$ of a pair $(x,t) \in X \times R$ is simply expressed as xt. The identity and group properties are as follows:

$$x0 = x \quad \text{for every } x \in X$$
$$(xt)s = x(t+s) \quad \text{for every } x \in X, \text{ and } t,s \in R.$$

In this study, we adopt throughout the notation used in the definition 1.1.1 above. (The above definitions are due to Pelczar [5]).

§ 1.3.

In this section, we give the definitions of periodicity and almost periodicity of motions and the stability of motions and sets.

Definition 1.3.1

The motion π^x through the point $x \in X$ is said to be periodic if there is a $T \neq 0$ such that $\pi(x,t) = \pi(x,t+T)$ for all $t \in R$. A number $T \in R$ will be called the period of the motion.

A set D of real numbers is called relatively dense if there is a $T > 0$ such that

$$D \cap (t-T, t+T) \neq \emptyset \quad \text{for all } t \in R.$$

Definition 1.3.2

The motion π^x through the point $x \in X$ is said to be almost periodic if for every $\varepsilon > 0$ there exists a relatively dense subset of numbers $\{\tau_n\}$, called displacement, such that $\rho(\pi(x,t), \pi(x,t+\tau_n)) < \varepsilon$ for all $t \in R$, and each τ_n.

Let (X,ρ) be a given metric space.

Definition 1.3.3

A set M is said to be positively Lyapunov stable (or positivel stable in the Lyapunov sense) if for every $\varepsilon > 0$ and $x \in M$ there is a $\delta = \delta(x,\varepsilon) > 0$ such that $\pi(S(x,\delta),R_+) \subset S(M,\varepsilon)$.

The stability of the trajectory $\pi(x)$ through the point $x \in X$ could be obtained by substituting $\pi(x)$ in the place of M in definition 1.3.3 above.

Definition 1.3.4

The motion π^x through $x \in X$ is said to be positively stable in the Lyapunov sense if given $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that $\rho(x,y) < \delta(\varepsilon)$ implies that $\rho(\pi(y,t),\pi(x,t)) < \varepsilon$ for $t \in R_+$.

A motion π^x through $x \in X$ is said to be negatively Lyapunov stable or stable in both directions if the condition in the above definition 1.3.4 is satisfied with $t \in R_+$ being replaced by $t \in R_-$ or with $t \in R$.

The definitions 1.3.1 - 1.3.4 are due to Bhatia and Szego (1970).

Definition 1.3.5

A set $M \subset X$ is said to be invariant in the dynamical system (semi-system) $(X,G;\pi)$ if and only if

$$\pi(M) = \bigcup \{\pi(x) : x \in M\} \subset M.$$

If $G = R$ then M is said to be positively (negatively) invarian if and only if

$$\pi_+(M) = \bigcup \{\pi_+(x) : x \in M\} \subset M$$

(respectively:

$$\pi_-(M) = \bigcup \{\pi_-(x) : x \in M\} \subset M$$)

§ 1.4. LIMIT SETS, PROLONGATIONAL LIMIT SETS AND PROLONGATIONS.

In this section, we give the definitions of limit sets, prolongational limit sets and prolongations and some of their properties.

Assume that (X,ρ) is a metric space and $(X,R;\pi)$ is a dynamical system.

Definition 1.4.1

Let $x \in X$ be given. Put

(1.4.1) $$\Lambda^+(x) := \{ y \in X: \text{there is a sequence } \{t_n\} \text{ in } R \text{ with } t_n \longrightarrow +\infty \text{ and } \pi(x,t_n) \longrightarrow y \}$$

(1.4.2) $$\Lambda^-(x) := \{ y \in X: \text{there is a sequence } \{t_n\} \text{ in } R \text{ with } t_n \longrightarrow -\infty \text{ and } \pi(x,t_n) \longrightarrow y \}$$

For $x \in X$, the set $\Lambda^+(x)$ is called its positive (or omega) limit set, and the set $\Lambda^-(x)$ is called its negative (or alpha) limit set. In cases (1.4.1) and (1.4.2) the point $y \in X$ is called the positive and negative limit point of the point $x \in X$ respectively.

The set of all positive limit points of all points $x \in B \subset X$ will be called the positive limit set of the set B and is denoted by $\Lambda^+(B)$. Thus

$$\Lambda^+(B) := \cup \{\Lambda^+(x): x \in B\}, \quad \text{similarly}$$

$$\Lambda^-(B) := \cup \{\Lambda^-(x): x \in B\}.$$

Remark 1.4.1

Let π be a continuous map, then it is possible to prove in our case that

(1.4.3) $\Lambda^+(x) := \bigcap \{\overline{\pi_t(\pi_+(x))} : t \in R_+\}$, and similarly

(1.4.4) $\Lambda^-(x) := \bigcap \{\pi_t(\pi_-(x) : t \in R_-\}$.

The maps $\Lambda^+ : X \ni x \longmapsto \Lambda^+(x)$ and $\Lambda^- : X \ni x \longmapsto \Lambda^-(x)$ play important roles in the theory of stability in pseudo-dynamical systems and dynamical systems.

We now state of the properties of limit sets which we often make use in this study. These properties are stated either as Lemmas, propositions or Theorems in standard text books on dynamical systems, hence we omit their proofs.

Let (X,ρ) be a given metric space and $(X,R;\pi)$ a DS (so π is continuous). Then the statements of the following theorems hold true

Theorem 1.4.1

For any $x \in X$

(1.4.5) $\Lambda^+(x)$ and $\Lambda^-(x)$ are closed invariant sets

(1.4.6) $\overline{\pi_+(x)} = \pi_+(x) \cup \Lambda^+(x)$ and $\overline{\pi_-(x)} = \pi_-(x) \cup \Lambda^-(x)$

(1.4.7) $\overline{\pi(x)} = \pi(x) \cup \Lambda(x)$.

Theorem 1.4.2

For any $x \in X$, $\pi_+(x) \subseteq \Lambda^+(x)$ if and only if π^x is periodic.

Theorem 1.4.3

If for $x \in X$, π^x is a periodic motion, then $\Lambda^+(x) = \Lambda^-(x) = \pi(x) = \overline{\pi(x)}$.

Theorem 1.4.4

For any $x \in X$, $\overline{\pi_+(x)}$ is compact if and only if $\Lambda^+(x)$ is compact.

The concept of trajectory has been mentioned in the previous section. The form and the properties of $\pi(x)$ do not contain

usually any information about the properties of neighbouring trajectories. To overcome these limitations, the concept of prolongational limit sets and prolongations were developed.

Let $x \in X$ be given where (X,ρ) is a metric space. Put

(1.4.8) $J^+(x) := \{ y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R_+ such that $x_n \longrightarrow x$, $t_n \longrightarrow +\infty$ and $\pi(x_n,t_n) \longrightarrow y \}$

(1.4.9) $J^-(x) := \{ y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R_- such that $x_n \longrightarrow x$, $t_n \longrightarrow -\infty$ and $\pi(x_n,t_n) \longrightarrow y \}$

(1.4.10) $D^+(x) := \{ y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R_+ such that $x_n \longrightarrow x$, and $\pi(x_n,t_n) \longrightarrow y \}$

(1.4.11) $D^-(x) := \{ y \in X$: there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R_- such that $x_n \longrightarrow x$, and $\pi(x_n,t_n) \longrightarrow y \}$.

For any $x \in X$, the sets $J^+(x)$ and $J^-(x)$ are respectively called the <u>positive</u> and <u>negative prolongational limit sets</u> of the point $x \in X$ while the sets $D^+(x)$ and $D^-(x)$ are respectively called the <u>positive</u> and <u>negative prolongations</u> of the point $x \in X$.

We can consider mappings

$$J^+ : x \longmapsto J^+(x), \quad J^- : x \longmapsto J^-(x)$$

and

$$D^+ : x \longmapsto D^+(x), \quad D^- : x \longmapsto D^-(x).$$

These mappings play important roles in the theory of stability.

It is worthwhile to note that for any $x \in X$, $D^+(x) \supset \pi_+(x)$, $D^-(x) \supset \pi_-(x)$, $J^+(x) \supset \Lambda^+(x)$ and $J^-(x) \supset \Lambda^-(x)$.

These inclusions may be proper.

From the definitions and the properties of the sets $J^+(x)$ and $D^+(x)$ it follows (see for instance Bhatia and Szego (1967), (1970)). that

$$D^+(x) = \overline{\left\{\bigcup\left\{\pi_+(y): \rho(x,y)<\alpha\right\}: \alpha>0\right\}} \tag{1.4.12}$$

$$J^+(x) = \bigcap\left\{D^+(\pi(x,t)): t\in R\right\}. \tag{1.4.13}$$

These formulae are considered even as definitions in several books; here we limit ourselves to metric spaces and so we adopt (1.4.8)-(1.4.11).

$J^-(x)$ and $D^-(x)$ can be presented analogously.

We state some of basic properties of the above sets.

Theorem 1.4.5

For any x X

$D^+(x)$ is closed and positively invariant (1.4.14)

$J^+(x)$ is closed and invariant (1.4.15)

$D^+(x) = \pi_+(x)\cup J^+(x)$. (1.4.16)

Analogous results hold for both $J^-(x)$ and $D^-(x)$.

Theorem 1.4.6

Let the space X be locally compact. Then for any $x\in X$, $D^+(x)$ is connected whenever it is compact. If $D^+(x)$ is not connected, then, it has no compact component.

Theorem 1.4.7

Let X be locally compact. Then $\wedge^+(x) \neq \emptyset$ whenever $J^+(x)$ is non-empty and compact.

Theorem 1.4.8

Let X be locally compact. Then $J^+(x) \neq \emptyset$ and compact if and only if $D^+(x)$ is compact.

Theorem 1.4.9

Let $x,y\in X$. Then $y\in J^+(x)$ if and only if $x\in J^-(x)$.

The definitions and theorems stated in this section are taken from Bhatia and Szego (1967), (1970).

§ 1.5. SEQUENCES OF DYNAMICAL SYSTEMS

In order to exclude any misfits we state formally in this section definitions pertaining to sequences of pseudo-dynamical systems $(X,R;\pi_n)$ and pseudo-dynamical semi-systems $(X,R_+;\pi_n)$ and $(X,R_-;\pi_n)$.

Definition 1.5.1

The sequence $\{\pi_n\}$ is said to be convergent (pointwise) to π on $X \times R$ if and only if $\forall x \forall t \forall \varepsilon > 0, \exists n_o : \rho(\pi_n(x,t),\pi(x,t)) < \varepsilon$ $\forall n > n_o$.

Definition 1.5.2

The sequence $\{\pi_n\}$ is said to be convergent uniformly in $t \in R$ to π, written shortly as:

$$\pi_n \longrightarrow \pi \quad \text{uniformly in } t,$$

if and only if $\forall x \in X \; \forall \varepsilon > 0 \; \exists n_o$:

$$\rho(\pi_n(x,t),\pi(x,t)) < \varepsilon \quad \forall n > n_o; \quad \forall t \in R.$$

Definition 1.5.3

The sequence $\{\pi_n\}$ is said to be convergent uniformly to π if and only if $\forall \varepsilon > 0 \; \exists n_o$:

$$\rho(\pi_n(x,t),\pi(x,t)) < \varepsilon \quad \forall n > n_o, \; \forall (x,t) \in X \times R.$$

Analogous statements hold for pseudo-dynamical semi-systems with R_+ or R_- replacing R.

§ 1.6. PSEUDO-PROCESSES

Let $(R,+)$ be an abelian semi-group with neutral element 0, H be a sub-semi-group of R, such that $0 \in H$, and let X be a non-empty set (called in this study) a space.

Suppose that μ is a mapping from $R \times X \times H$ into X.

Definition 1.6.1

We say that the quadruple $(X,R,H;\mu)$ is a pseudo-process if and only if

(1.6.1) $\mu(g,x,0) = x$ for $g \in R$ and $x \in X$

and

(1.6.2) $\mu(g+h,\mu(g,x,h),k) = \mu(g,x,h+k)$ for $g \in R$, $x \in X$ and $k,h \in H$.

The notion of the pseudo-process (see Pelczar (1985a) and (1973)), is a direct generalization of the notion of processes introduced by Dafermos (1971).

In this study, we limit ourselves to the case of X being a metric space, R being additive group of real numbers and H equal to the semi-group R_+ of non-negative real numbers.

There are some fundamental connections between pseudo-processes and the classical dynamical systems. If μ does not depend on the first variable, then putting formally:

$$\pi(x,s) := \mu(t,x,s) \quad \text{for } (x,s) \in X \times H,\ t \in R$$

we obtain a dynamical semi-system in the sense of Pelczar (1983), (called in Pelczar (1978), pseudo-dynamical semi-system), that is the triple $(X,H;\pi)$ such that

$\pi : X \times H \longrightarrow X$ satisfies the conditions

$\pi(x,0) = x$ and $\pi(\pi(x,t),s) = \pi(x,t+s)$ for $x \in X$ and $t,s \in H$.

If X is a topological space, H a topological semi-group (group) and π is continuous, then our triple is a dynamical semi-system (dynamical system). On the other hand, if we have a pseudo-process $(X,R,H;\mu)$, then we can construct an associated semi-system $(\tilde{X},H;\tilde{\pi})$ by putting

$$\widetilde{X} := X \times R$$

and

(1.6.3) $\quad \widetilde{\pi}((x,t),s) := (\mu(t,x,s),t+s) \quad \text{for } ((x,t),s) \in \widetilde{X} \times H.$

Such a reduction of pseudo-processes to corresponding semi-systems is presented in Pelczar (1973) in order to investigate certain stability questions concerning pseudo-processes by replacing them with suitable stability problems, treated with respect to semi-systems or dynamical systems.

HOW PSEUDO-PROCESSES ARISE FROM THE SOLUTIONS OF DELAY DIFFERENTIAL EQUATIONS.

Let f be a continuous function from $R \times R^n$ into R^n such that for every $(t_0,y_0) \in R \times R^n$ there exists exactly one (saturated to the right) solution of the initial Cauchy problem

(1.6.4) $\quad y' = f(t,y), \quad y(t_0) = y_0$

defined on the interval $[t_0,\infty)$. We denote this unique solution by $\varphi(\cdot,y_0,t_0)$ and we put

(1.6.5) $\quad \widetilde{\mu}(t,x,s) := \varphi(t+s,x,t)$

for $t \in R$, $s \in R_+$, $x \in R^n$.

We obtain in that way a process $(R^n,R,R_+;\widetilde{\mu})$. Indeed, we have first of all:

$$\widetilde{\mu}(t,x,0) = \varphi(t;x,t) = x \quad \text{for every } t \in R.$$

Moreover, putting for fixed t,s

$$\psi(r) := \varphi(t+s+r;x,t)$$

and

$$\lambda(r) := \varphi(t+s+r;\ \varphi(t+s;x,t),t+s)$$

we get

$$\psi'(r) = f(t+s+r,\ \varphi(t+s+r;x,t) = f(t+s+r,\psi(r))$$

and similarly

$$\lambda'(r) = f(t+s+r,\varphi(t+s+r;\ \varphi(t+s;x,t),t+s)) = f(t+s+r,\lambda(r)).$$

We have also

$$\lambda(0) = \psi(0) = \varphi(t+s;x,t).$$

So we get two solutions ψ and λ of the initial Cauchy problem

$$(1.6.6) \qquad \begin{cases} y' = g(\xi,y) \\ y(0) = \varphi(t+s;x,t) \end{cases}$$

where

$$g(\xi,y) := f(t+s+\xi,y).$$

From the assumptions on the uniqueness of solutions of our original Cauchy problem (1.6.4) for every $(t^0,y^0) \in R \times R^n$, it follows that we have also the uniqueness of (1.6.6). This implies that $\psi(r) = \lambda(r)$ for every r and so

$$\varphi(t+s+r;x,t) = \varphi(t+s+r;\ \varphi(t+s;x,t),t+s)$$

which means that

$$\tilde{\mu}(t,x,s+r) = \tilde{\mu}(t+s,\tilde{\mu}(t,x,s),r)$$

for every $t \in R$, $s,r \in R_+$ and $x \in X$.
This proves our assertion: $(R^n,R,R_+,\tilde{\mu})$ is a pseudo-process.

As we have in the cases of dynamical semi-systems, dynamical systems and pseudo-dynamical systems and semi-systems notions of limit sets, prolongational limit sets, prolongations and stability and so on, so also we have parallel notions in pseudo-processes.

Definition 1.6.2

Suppose μ is a pseudo-process on X, the trajectory $\mu[\sigma,x]$ through the point $(\sigma,x) \in R \times X$ is the set in $R \times X$ defined by

$$\mu[\sigma,x] = \{(\sigma+t, \mu(\sigma,x,t)) : t \in R_+\} .$$

Definition 1.6.3

Suppose μ is a pseudo-process on X, the orbit $\mu(\sigma,x)$ through the point (σ,x) is the set X defined by

$$\mu(\sigma,x) = \{\mu(\sigma,x,t) : t \in R_+\}.$$

Definition 1.6.4

A pseudo-process μ is said to be T-periodic if there is a $T > 0$ such that $\mu(\sigma+T,x,t) = \mu(\sigma,x,t)$ for all $\sigma \in R$, $t \in R_+$ and $x \in X$.

Definition 1.6.5

A point $x \in X$ is said to be an equilibrum or critical point of a pseudo-process μ if there is $\sigma \in R$ such that $\mu(\sigma,x) = x$, that is $\mu(\sigma,x,t) = x$ for all $t \in R_+$.

BIBLIOGRAPHICAL NOTE.

The definition of limit sets is due to G.D.Birkhoff (1927) [Vol.1, pp.654-672]. This concept has been previously used by H.Poincare without a formal definition. Alternative equivalent definitions of limit sets were used by S.Lefschetz (1957) and by T.Ura (1964).

The formal definition of prolongations is due to T.Ura (1964). This concept has been previously used by H.Poicare [1, Vol.1, p. 44] in a special case without a formal definition. The concept of prolongational limit set is due to J.Auslander, N.P.Bhatia and P.Seibert.

A definition of stability (of motion) was given in a precise form by A.M.Lyapunov (1966), The concept of stability of equilibrum was previously discussed by Lagrange and by Dirichlet.

The first complete development of stability theory of dynamical systems is due to V.I.Zubov (1954) and S.Lefschetz (1957). In a sense, in the theory of stability of dynamical systems or semi-systems, one has achieved a synthesis between the point of view of Lyapunov and that of Poincare. In fact, Lyapunov was interested in local (stability) properties of n-dimensional systems, while Poincare was interested in a global property of planar systems.

There is no definite bibliography about the definition of processes. However, there appears definition of processes in Hale (1977). In the same year of the publication of Hale's book, Dafermos (1971) appeared and later Pelczar (1973)

(The references quoted in the first four paragraphs of this bibliography are as in Bhatia and Szego (1970), while those in the last paragraph are as in this study).

CHAPTER II.

TRAJECTORIES, SEMI-TRAJECTORIES, INVERSE TRAJECTORIES AND FAMILY OF TRAJECTORIES.

We give some properties of trajectories, semi-trajectories, inverse trajectories and the family of trajectories. They are discussed respectively in sections 1,2 and 3 of this chapter. Some additional remarks on inverse trajectories are given in the last section 4. We also give some general definitions that are pertinent to the results dealt with in this chapter generally.

§ 2.1.

Let us recall the classical definition of upper semi-continuity of multivalued mappings.

Definition 2.1.1

Suppose that (X,ρ) is a metric space.

Let H be a mapping from the space X into 2^X. The mapping H is said to be upper semi-continuous at the point $x \in X$ if and only if

$$\left.\begin{array}{ll} x_n \in X & y_n \in H(x_n) \\ x_n \longrightarrow x & y_n \longrightarrow y \end{array}\right\} \Longrightarrow y \in H(x)$$

Theorem 2.1.1

Let (X,ρ) be a given metric space and $(\Delta,R;\pi)$ be a dynamical system (or dynamical semi-system). Suppose that $x_0 \in X$ and $J^+(x_0) \neq \emptyset$ and $J^+(x_0) \subset \pi_+(x_0)$. Then the mapping

$$\pi_+ : X \ni x \longmapsto \pi_+(x)$$

is upper semi-continuous at the point $x_0 \in X$.

Proof:

Let $\{x_n\}$ and $\{y_n\}$ be sequences of elements of X such that

(2.1.1) $\quad x_n \longrightarrow x_o \quad$ as $n \longrightarrow \infty$

(2.1.2) $\quad y_n \longrightarrow y_o \quad$ as $n \longrightarrow \infty$

(2.1.3) $\quad y_n \in \Pi_+(x_n), \quad$ for $n=1,2,\ldots$.

We have to show that

(2.1.4) $\quad y_o \in \Pi_+(x_o)$.

Relation (2.1.3) implies that for some $t_n \geq 0$ we have

(2.1.5) $\quad y_n = \Pi(x_n, t_n)$.

We consider two cases here:

Case (1^o): $\{t_n\}$ is bounded

Case (2^o): $\{t_n\}$ is not bounded from above.

Case (1^o); It means that there exists a convergent subsequence $\{t_{k_n}\}$ of $\{t_n\}$ such that $t_{k_n} \longrightarrow t_o \geq 0$ as $n \longrightarrow \infty$. So we have that

(2.1.6) $\quad \underset{n \longrightarrow \infty}{\mathrm{Lt}}\, y_{k_n} = \underset{n \longrightarrow \infty}{\mathrm{Lt}}\, \Pi(x_{k_n}, t_{k_n}) = \Pi(x_o, t_o)$

Applying relation (2.1.2) we have that $y_o = \Pi(x_o, t_o) \in \Pi_+(x_o)$. Thus, we have proved (2.1.4) in the case (1^o).

Case (2^o); It means that there exists a subsequence $\{t_{p_n}\}$ of $\{t_n\}$ such that $t_{p_n} \longrightarrow \infty$ as $n \longrightarrow \infty$.

Thus we have that $y_{p_n} = \Pi(x_{p_n}, t_{p_n})$, $t_{p_n} \longrightarrow \infty$ as $n \longrightarrow \infty$. Using relation (2.1.2) we get

(2.1.7) $\quad y_o = \underset{n \longrightarrow \infty}{\mathrm{Lt}}\, \Pi(x_{p_n}, t_{p_n})$

and this implies that

(2.1.8) $y_o \in J^+(x_o)$ (see relation (2.1.1))

From the assumptions, we have that $J^+(x_o) \subset \pi_+(x_o)$ and using relation (2.1.8), we obtain again (2.1.4).

We have shown in both cases that $y_o \in \pi_+(x_o)$, so the mapping $X \ni x \longmapsto \pi_+(x)$ is upper semi-continuous at the point $x_o \in X$.

Corollary 2.1.1

Let (X,ρ) be a given metric space and $(X,R;\pi)$ be a DS (or DSS) with $\pi_+(x)$ being periodic and positively Lyapunov stable. Then the mapping

$$X \ni x \longmapsto \pi_+(x)$$

is upper semi-continuous at the point $x_o \in X$.

Proof:

From the assumptions we have that $\pi_+(x)$ is periodic. This means that

(2.1.9) $\pi_+(x) = \pi(x) = \Lambda^+(x)$.

Relation (2.1.9) implies that $J^+(x) \neq \emptyset$.

Moreover, from the assumptions, we have that $\pi_+(x)$ is positively Lyapunov stable. This implies that

(2.1.10) $D^+(\pi_+(x)) = \pi_+(x)$ (see Bhatia and Szego (1967)).

Relation (2.1.10) implies that

(2.1.11) $D^+(x) \subset \pi_+(x)$.

From the definition of $D^+(x)$ we get

(2.1.12) $J^+(x) \subset \pi_+(x)$.

Thus, by virtue of Theorem 2.1.1 we obtain the assertion.

The next theorem is the symmetric case of Theorem 2.1.1.

Theorem 2.1.2

Let (X,ρ) be a given metric space and $(X,R;\pi)$ be a DS or DSS. Let $J^-(x_0) \neq \emptyset$ and $J^-(x_0) \subset \pi_+(x_0)$ for any $x_0 \in X$. Then the mapping

$$X \ni x \longmapsto \pi_-(x)$$

is upper semi-continuous at the point $x_0 \in X$.

The proof of Theorem 2.1.2 is parallel to the proof of Theorem 2.1.1. We need only to replace J^+ by J^-, π_+ by π_- and $t_n \longrightarrow +\infty$ by $t_n \longrightarrow -\infty$ respectively.

Theorem 2.1.3

Let (X,ρ) be a given metric space and $(X,R;\pi)$ be dynamical semi-system (DSS) or dynamical system (DS). For fixed $x \in X$, let π^x be positively Lyapunov stable. Then $D^+(x) \subset \overline{\pi_+(x)}$.

Proof:

Let $p \in D^+(x)$. This means that there exist sequences $\{x_n\}$ in X and $\{t_n\}$ in R_+ such that

$$x_n \longrightarrow x \quad \text{as} \quad n \longrightarrow \infty \tag{2.1.13}$$

$$t_n \geq 0 \tag{2.1.14}$$

and

$$\pi(x_n,t_n) \longrightarrow p \quad \text{as} \quad n \longrightarrow \infty . \tag{2.1.15}$$

Case (1): $\{t_n\}$ is bounded.

The sequence $\{t_n\}$ is bounded, and so we can assume without loss of generality that $t_n \longrightarrow t \in R_+$.

Using relation (2.1.15) and continuity axiom we get

$$\pi(x_n,t_n) \longrightarrow \pi(x,t) \in \overline{\pi_+(x)}. \tag{2.1.16}$$

Applying relation (2.1.15), then relation (2.1.16) we get

$$p \in \overline{\pi_+(x)}. \tag{2.1.17}$$

Case (2): $\{t_n\}$ is not bounded.

This means that we can assume that $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$. From the assumptions, we have that π^x is positively Lyapunov stable. This means that

$$(2.1.18)\quad \begin{cases} \forall \varepsilon > 0 \ \exists \delta > 0 : \rho(x,y) < \delta \implies \\ \rho(\pi(x,t),\pi(y,t)) < \varepsilon \quad \forall t \geq 0 \end{cases}$$

Fix $\eta > 0$. For this η, we can find $\delta > 0$ such that the relation (2.1.18) holds true for $\varepsilon = \eta/2$. For this δ we take n_o such that

$$(2.1.19)\quad \rho(x,x_n) < \delta \quad \text{for } n > n_o$$

Such an n exists since $x_n \longrightarrow x$ as $n \longrightarrow \infty$.

For $n > n_o$, we then have from relation (2.1.18) that

$$(2.1.20)\quad \rho(\pi(x,t_n),\pi(x_n,t_n)) < \varepsilon = \eta/2 \quad \forall n > n_o.$$

We now take n_1 such that

$$(2.1.21)\quad \rho(\pi(x_n,t_n),p) < \varepsilon = \eta/2 \quad \text{for } n > n_1.$$

Such n_1 exists since relation (2.1.15) holds.

For $n_* = \max(n_o,n_1)$ we have

$$(2.1.22)\quad \rho(\pi(x,t_n),p) \leq \rho(\pi(x_n,t_n),p) + \rho(\pi(x_n,t_n),\pi(x,t_n))$$
$$\leq \varepsilon + \varepsilon = \eta/2 + \eta/2 = \eta \ ; \quad \text{for } n > n_*$$

In other words, we proved that

$$(2.1.23)\quad \forall \eta > 0 \quad \exists n_* : \rho(\pi(x,t_n),p) < \eta \quad \text{for } n > n_*.$$

Relation (2.1.23) implies that

$$(2.1.24)\quad p = \lim_{n \longrightarrow \infty} \pi(x,t_n).$$

Relation (2.1.24) combined with the fact that $t_n \longrightarrow \infty$ implies that

$$(2.1.25)\quad p \in \Lambda^+(x).$$

But $\overline{\pi_+(x)} = \pi_+(x) \cup \Lambda^+(x)$.

So, we have that $p \in \overline{\pi_+(x)}$. In both cases, we showed that $p \in \overline{\pi_+(x)}$. This completes the proof of the theorem.

Corollary 2.1.2

Under the assumptions of Theorem 2.1.3, we have that $\overline{\pi_+(x)} = D^+(x)$.

The next theorem is the symmetric case of Theorem 2.1.3.

Theorem 2.1.4

Let (X,ρ) be a given metric space and $(X,R;\pi)$ be a dynamical system or dynamical semi-system. For fixed $x \in X$, let π^x be negatively Lyapunov stable. Then $D^-(x) \subset \overline{\pi_-(x)}$.

We omit the proof of Theorem 2.1.4 since it is similar to that of Theorem 2.1.3.

Theorem 2.1.5

Let (X,ρ) be a given metric space and $(X,R;\pi)$ be a dynamical system. For $x_0 \in X$, let $J^+(x_0) \neq \emptyset$, $J^-(x_0) = \emptyset$, $J^+(x_0) \subset \pi_+(x_0)$ and $J^-(x_0) \subset \pi_-(x_0)$, then the mapping

$$x \longmapsto \pi(x)$$

is upper semi-continuous at the point $x_0 \in X$.

Proof:

It is sufficient to observe that if any two mappings

$$A : x \longmapsto A(x) \quad \text{and} \quad B : x \longmapsto B(x)$$

with $A(x), B(x) \subset X$ are upper semi-continuous at some point $x_0 \in X$, then the mapping

$$x \longmapsto A(x) \cup B(x)$$

is also upper semi-continuous at the point x_0. So, by applying Theorems 2.1.1 and 2.1.2, the assertion holds.

§ 2.2. INVERSE TRAJECTORIES.

We shall adopt the usual notation:

By $\mathcal{P}(X)$ we denote the family of all non-empty subsets of X.

If (X,ρ) is a metric space, then for any subset A of X ($A \neq \emptyset$) and $x \in X$ we put

$$\rho(x,A) := \inf\{\rho(x,y): y \in A\}$$

and for $\varepsilon > 0$

$$B(A,\varepsilon) := \{y: \rho(y,A) < \varepsilon\}.$$

Of course, $B(\{x\},\varepsilon) = B(x,\varepsilon)$ is the usual open ball centred at x with its radius being ε.

Definition 2.2.1 (see Adewale and Pelczar (1983)).

Let $(X,R_+;\pi)$ be a dynamical semi-system and $x \in X$ be fixed. We put

$$(2.2.1) \quad \pi^{-1}(x) := \{s \in X: x \in \pi(s)\}$$

and we call the set (2.2.1) the inverse trajectory through the point $x \in X$.

Definition 2.2.2 (see for instance Bajaj (1970)).

Let $(X,R_+;\pi)$ be a dynamical semi-system. A point $x \in X$ is said to be a start point if and only if the only point $y \in X$ such that $x \in \pi(y)$ is $y = x$. This is to say that for all $t > 0$, for all $y \in X$, $\pi(y,t) \neq x$.

Remark 2.2.1

If we have the dynamical system $(X,R;\pi)$, then formula (2.2.1) gives the negative semi-trajectory of the point $x \in X$.

Remark 2.2.2

If a mapping $G: x \longmapsto G(x) \in \mathcal{P}(R^n)$ is such that G(x) is compact for every $x \in X$, then G is upper semi-continuous at the point $x_* \in X$ if and only if $\forall \varepsilon > 0 \ \exists \delta > 0: \rho(x,x_*) < \delta \Longrightarrow G(x) \subset B(G(x_*),\varepsilon)$.

Theorem 2.2.1 (Adewale and Pelczar (1983)).

Let $(R^n, R_+; \pi)$ be a dynamical semi-system. Suppose

(2.2.2) for every $x \in R^n$ there is exactly one start point $s(x)$ such that $x \in \pi(s(x))$

(2.2.3) the set of all start points in R^n is closed

(2.2.4) $\pi^{-1}(x) = [s(x), x] := \{\pi(s(x), t): 0 \leq t \leq t_x\}$ where $t_x = \inf\{t \geq 0: x = \pi(s(x), t)\}$

(2.2.5) the mapping $R^n \ni x \longmapsto \pi^{-1}(x) \in \mathcal{P}(R^n)$ is upper semi-continuous.

Then the mapping

$$R^n \ni x \longmapsto s(x)$$

is continuous.

Proof:

We observe that the set $\pi^{-1}(x)$ is compact because the mapping π^x is continuous and $[0, t_x]$ is compact.

So, we can apply the above Remark 2.2.1 with respect to the mapping

$$x \longmapsto \pi^{-1}(x).$$

Thus, for every fixed $x^* \in R^n$ and every $\varepsilon > 0$ we can find $\delta > 0$ such that

(2.2.6) $$\rho(x^*, y) < \delta \Longrightarrow \pi^{-1}(y) \subset B(\pi^{-1}(x^*), \varepsilon).$$

Let now x^* be fixed. Let $\{x_n\}$ be a sequence of elements of R^n such that $x_n \longrightarrow x^*$.

For every $\varepsilon > 0$, there is n^* such that

(2.2.7) $$\pi^{-1}(x_n) \subset B(\pi^{-1}(x^*), \varepsilon) \quad \text{for } n > n^*.$$

Because of the compactness of $\pi^{-1}(x^*)$ we have the compactness of $B(\pi^{-1}(x^*), \varepsilon)$.

Consider the sequence $\{s(x_m)\}$. For m sufficiently large, $s(x_m)$ belongs to $B(\pi^{-1}(x^*),\varepsilon)$; (because $s(x_m) \in \pi^{-1}(x_m) \subset B(\pi^{-1}(x^*),\varepsilon)$).

Since the set $B(\pi^{-1}(x^*),\varepsilon)$ is compact, we can find a subsequence $\{s(x_{k_m})\}$ of $s(x_m)$ which is convergent to some element $\hat{y} \in R^n$.

So, we have that

(2.2.8) $$\hat{y} = \lim_{m \to \infty} s(x_{k_m})$$

On the other hand, $\hat{y} \in \pi^{-1}(x^*)$ (because from our assumptions, (see (2.2.4)) it follows directly that $\pi^{-1}(x^*)$ is closed). By the uniqueness of start points, we have that

(2.2.9) $\hat{y} = s(x^*)$ and so, the sequence $\{s(x_{k_m})\}$ is convergent to $s(x^*)$.

We apply now the classical reasoning which permits us to observe that if a sequence $\{a_m\}$ of elements of R^n has the following properties

a) there is at most one accumulation point of $\{a_m\}$

b) for every subsequence $\{a_{p_m}\}$ of $\{a_m\}$, there is a subsequence $\{a_{q_p}\}$ which is convergent to some a,

then the sequence $\{a_m\}$ is convergent to the unique accumulation point.

So, $\{s(x_m)\}$ is convergent to $s(x^*)$. This means that the mapping

$$R^n \ni x \longmapsto s(x)$$

is continuous.

Theorem 2.2.2 (Adewale and Pelczar [1]).

Let (R^n, R_+, π) be a dynamical semi-system. Suppose

(s) for every $x \in X$ there is exactly one $s(x)$ such that $x \in \pi(s(x))$, where $s(x)$ is a start point

(b) $\pi^{-1}(x) = [s(x),x]$

(c) the mapping $x \longmapsto s(x)$ is continuous

(d) for every $\varepsilon>0$ there is $\delta>0$ such that $\rho(z,x)<\delta \Longrightarrow |t_x-t_z| < \varepsilon$

then, the mapping $x \longmapsto \pi^{-1}(x)$ is upper semi-continuous.

Proof:

Let $\varepsilon>0$ be fixed and we take $c>0$ such that if $|t-t_x|<c$ then

(2.2.10) $\rho(\pi(s(x),t), \pi(s(x),t_x))< \frac{\varepsilon}{2}$.

We take $\delta_1>0$ such that if $\rho(z,x)<\delta_1$ then

(2.2.11) $|t_z-t_x|< c.$

We take $\eta>0$ such that if $y \in B(s(x),\eta)$ then

(2.2.12) $\rho(\pi(y,t),\pi(s(x),t))< \frac{\varepsilon}{2}$

for $t\in[0,t_x+c]$; (this is possible because of the uniform continuity of π on compact sets).

We take $\delta_2>0$ such that if $z\in B(x,\delta_2)$, then

(2.2.13) $s(z)\in B(s(x),\eta)$

(This is possible because of assumption (c)).

We take $\delta = \min(\delta_1,\delta_2)$.

Now, if $z\in B(x,\delta)$, then $s(z)\in B(s(x),\eta)$ and so

(2.2.14) $\rho(\pi(s(x),t), \pi(s(z),t))< \frac{\varepsilon}{2}$

for $t\in[0,t_x+c]$.

So, in particular

(2.2.15) $\rho(\pi(s(x),t), \pi(s(z),t))< \frac{\varepsilon}{2}$ for $t\in[0,t_x]$.

If $t_z>t_x$, then we have

(2.2.16) $[t_x,t_z]\subset[t_x,t_x+c]$.

Now, if $t \in [t_x, t_z]$, then

$$(2.2.17) \qquad \rho(\pi(s(z),t), \pi(s(x),t_x)) \leq \rho(\pi(s(z),t),\pi(s(x),t)) + \rho(\pi(s(x),t), \pi(s(x),t_x)) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

So, we have proved that

$$(2.2.18) \qquad \rho(\pi(s(z),t),\pi(s(x),t_x)) = \rho(\pi(s(z),t),x)$$

for $t \in [t_x, t_z]$.

Moreover we have also

$$(2.2.19) \qquad \rho(\pi(s(z),t),\pi(s(x),t)) < \frac{\varepsilon}{2}$$

for $t \in [0, t_x]$ and so

$$(2.2.20) \qquad \pi^{-1}(z) \subset B(\pi^{-1}(x), \varepsilon) .$$

Thus, we have shown that if element $z \in B(x,\delta)$ then $\pi^{-1}(z) \subset B(\pi^{-1}(x),\varepsilon)$ Thus the mapping $R^n \ni x \longmapsto \pi^{-1}(x)$ is upper semi-continuous at the point z, with z being an arbitrary element of R^n

§ 2.3.

Suppose (X,ρ) is a metric space and $(X,R_+;\pi)$ is a PDSS. For $t \leq 0$, and $x \in X$ we put

$$(2.3.1) \qquad \pi(x,t) := \{ w \in X : \pi(w,-t) = x . \}$$

Definition 2.3.1

Let $x \in X$ be fixed. We say that the point x is negatively quasi-stable if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in B(x,\delta)$, $t \leq 0$ and $z \in \pi(y,t)$, then there exists $w \in \pi(x,t)$ such that $\rho(z,w) \leq \varepsilon$.

Theorem 2.3.1

Let (X,ρ) be a given metric space and $(X,R_+;\pi)$ be a PDSS. Let $x_0 \in X$ be negatively quasi-stable. Then the mapping $x \longmapsto \pi^{-1}(x)$ is upper semi-continuous at the point x_0.

Proof:

We know (see Definition 2.2.1) that

$$\pi^{-1}(x) = \{ y \in X: x \in \pi(y) \} \quad (= \{ y \in X: \exists\, t \leq 0, \pi(y,t) = x \}.$$

So,

$$\pi^{-1}(x) = \bigcup \{ \pi(x,t): t \leq 0 \}.$$

Let $\{x_n\}$ and $\{y_n\}$ be sequences of elements of X such that

(2.3.2) $\quad x_n \longrightarrow x_o \quad$ as $\quad n \longrightarrow \infty$

(2.3.3) $\quad y_n \longrightarrow y_o \quad$ as $\quad n \longrightarrow \infty$

(2.3.4) $\quad y_n \in \overline{\pi^{-1}(x_n)} \quad$ for $n=1,2,\ldots$.

We have to show that

(2.3.5) $\quad y_o \in \overline{\pi^{-1}(x_o)}$.

For every fixed n, there is a sequence $\{y_n^m\}$ of elements of $\pi^{-1}(x_n)$ such that

(2.3.6) $\quad y_n^m \longrightarrow y_n \quad$ as $m \longrightarrow \infty$, $\quad$ for every fixed n.

Since $y_n \longrightarrow y_o$ as $n \longrightarrow \infty$, we can apply the usual method and find a sequence $\{m_n\}$ of positive integers such that

(2.3.7) $\quad m_n \longrightarrow \infty \quad$ as $n \longrightarrow \infty$

and

(2.3.8) $\quad y_n^{m_n} \in \pi^{-1}(x_n), \quad y_n^{m_n} \longrightarrow y_o \quad$ as $n \longrightarrow \infty$.

So, putting

(2.3.9) $\quad w_n := y_n^{m_n}$

we obtain $w_n \longrightarrow y_o$ as $n \longrightarrow \infty$ and $w_n \in \pi^{-1}(x_n)$, $n=1,2,\ldots$.

Let now $\varepsilon > 0$ be arbitrarily fixed. We find $\delta > 0$ such that for $\xi \in B(x_o, \delta)$ we have such that for $t \leq 0$ and $z \in \pi(\xi,t)$ there exists $w \in \pi(x_o,t)$ such that

(2.3.10) $\rho(z,w) \leq \frac{\varepsilon}{2}$.

We choose n_o so large that

(2.3.11) $\rho(x_n,x_o) < \delta$ for $n \geq n_o$

and

(2.3.12) $\rho(w_n,y_o) \leq \frac{\varepsilon}{2}$ for $n \geq n_o$.

Relations $w_n \in \pi^{-1}(x_n)$ $(n=1,2,\ldots)$ mean that for every n there exists $t_n \leq 0$ such that

(2.3.13) $\pi(w_n,-t_n) = x_n$ (or $w_n \in \pi(x_n,t_n)$).

For $n \geq n_o$ we can find $z_n \in \pi(x_o,t_n)$ such that

(2.3.14) $\rho(z_n,w_n) \leq \frac{\varepsilon}{2}$.

From triangle inequality, we obtain

(2.3.15) $\rho(y_n,y_o) < \varepsilon$ for $n \geq n_o$.

Since $z_n \in \pi(x_o,t_n)$, (with $t_n \leq 0$) we have $z_n \in \pi^{-1}(x_o)$ and so, we have proved that for every fixed $\varepsilon > 0$, there exists a point $z \in \pi^{-1}(x_o)$ (being of the form $z = z_n$ for a suitable n, for instance $n = n_o$), such that

(2.3.16) $\rho(z,y_o) < \varepsilon$.

This means that relation (2.3.5) holds true.

This completes the proof of the Theorem.

(The Author is indebted to Professor A.Pelczar for suggestions owing to which the definition 2.3.1 was stated which enhances the proof of the Theorem 2.3.1).

§ 2.4.

In this section, we give some properties that are preserved passing to the limits of the families of trajectories and inverse trajectories.

(2.4.8) $\rho(\pi(x,0),x) = 0$

So

(2.4.9) $\pi(x,0) = x.$

Continuity property is automatically satisfied, since π is the uniform limit of the sequence $\{\pi_n\}$.

We shall finish the proof by giving the group property:

Since π_n is a PDS, n=1,2,..., we have

(2.4.10) $\pi_n(\pi_n(x,t),s) = \pi_n(x,t+s)$, $\forall x \in X$, $t,s \in R$.

Moreover, we have

(2.4.11) $\lim_{n \to \infty} \pi_n(\pi_n(x,t),s) = \lim_{n \to \infty} \pi_n(x,t+s)$ (see relation (2.4.10))

From the fact that $\{\pi_n\}$ is convergent uniformly to π we have

(2.4.12) $\lim_{n \to \infty} \pi_n(x,t+s) = \pi(x,t+s)$

From triangle inequality we get

$$(2.4.13)\quad \begin{cases} \rho(\pi(\pi(x,t),s),\pi_n(\pi_n(x,t),s)) \leq \\ \qquad \rho(\pi(\pi(x,t),s),\pi(\pi_n(x,t),s)) + \\ \qquad + \rho(\pi(\pi_n(x,t),s),\pi_n(\pi_n(x,t),s)). \end{cases}$$

The uniform convergence of $\{\pi_n\}$ to π (together with the continuity of π and π_n) implies that the right hand side of relation (2.4.13) tends to zero as $n \longrightarrow \infty$.

So,

(2.4.14) $\lim_{n \to \infty} \pi_n(\pi_n(x,t),s) = \pi(\pi(x,t),s)$

and then (see relations (2.4.11) and (2.4.12)) we get the equation

Theorem 2.4.1

Let (X,ρ) be a given metric space and $(X,R;\pi_n)$ be (for every n), a DS, such that the sequence $\{\pi_n\}$ converges uniformly to π where π is the mapping from $X \times R$ into X. Then the triple $(X,R;\pi)$ is a DS.

Proof:

For the triple $(X,R;\pi)$ to be a PDS, the map π should satisfy the properties of a PDS viz:

(2.4.1) Initial value or identity property, i.e. for every $\xi \in X$, $\pi(\xi,0) = \xi$.

(2.4.2) Continuity.

(2.4.3) Group property:

$$\pi(\pi(\xi,t),s) = \pi(\xi,t+s) \quad \forall\, \xi \in X,\ t,s \in R.$$

From the assumptions, $\{\pi_n\}$ is convergent uniformly to π . This implies that

$$(2.4.4) \quad \begin{cases} \forall\, \varepsilon>0 \quad \exists\, n_o: \\ \rho(\pi_n(x,t),\pi(x,t)) < \varepsilon \quad \forall n > n_o,\ \forall (x,t) \in X \times R \end{cases}$$

So, in particular we have that

$$\forall\, \varepsilon>0 \quad \exists n_o \in N:$$

$$(2.4.5) \qquad (\pi_n(x,0),\pi(x,0)) < \varepsilon \quad \forall n > n_o,\ \forall (x,0) \in X \times R.$$

From the assumptions, we have that π_n is a PDS for every n. This implies that

$$(2.4.6) \qquad \pi_n(x,0) = x, \quad n=1,2,\ldots .$$

Combining relations (2.4.5) and (2.4.6) we get

(2.4.7) for every $\varepsilon>0$ $\rho(\pi(x,0),x) < \varepsilon$

and this implies that

(2.4.8) $\rho(\pi(x,0),x) = 0$

So

(2.4.9) $\pi(x,0) = x.$

Continuity property is automatically satisfied, since π is the uniform limit of the sequence $\{\pi_n\}$.

We shall finish the proof by giving the group property:

Since π_n is a PDS, n=1,2,..., we have

(2.4.10) $\pi_n(\pi_n(x,t),s) = \pi_n(x,t+s), \forall x \in X, t,s \in R.$

Moreover, we have

(2.4.11) $\lim_{n \to \infty} \pi_n(\pi_n(x,t),s) = \lim_{n \to \infty} \pi_n(x,t+s)$ (see relation (2.4.10))

From the fact that $\{\pi_n\}$ is convergent uniformly to π we have

(2.4.12) $\lim_{n \to \infty} \pi_n(x,t+s) = \pi(x,t+s)$

From triangle inequality we get

$$(2.4.13)\quad \begin{cases} \rho(\pi(\pi(x,t),s),\pi_n(\pi_n(x,t),s)) \leq \\ \qquad \rho(\pi(\pi(x,t),s),\pi(\pi_n(x,t),s)) + \\ \qquad + \rho(\pi(\pi_n(x,t),s),\pi_n(\pi_n(x,t),s)). \end{cases}$$

The uniform convergence of $\{\pi_n\}$ to π (together with the continuity of π and π_n) implies that the right hand side of relation (2.4.13) tends to zero as $n \longrightarrow \infty$.

So,

(2.4.14) $\lim_{n \to \infty} \pi_n(\pi_n(x,t),s) = \pi(\pi(x,t),s)$

and then (see relations (2.4.11) and (2.4.12)) we get the equation

(2.4.15) $\quad \pi(\pi(x,t),s) = \pi(x,t+s) \quad \forall x \in X,\ t,s \in R.$

The proof is completed.

Theorem 2.4.2

Let $(X,R_+;\pi)$ be a PDSS in the given metric space (X,ρ) and $(X,R_+;\pi_n)$ be a PDSS for every n in the same metric space such that the sequence $\{\pi_n\}$ converges uniformly to π. If for every n, the motion π_n^x is positively Lyapunov stable, then π^x is positively Lyapunov stable.

Proof:

The motion π_n^x $(n=1,2,\ldots,)$ is positively Lyapunov stable. This means that

$$(2.4.16)\quad \begin{cases} \forall n \in N\ \forall \varepsilon > 0\ \exists \delta > 0 : \rho(x,y) < \delta \Longrightarrow \\ \rho(\pi_n(x,t),\pi_n(y,t)) < \varepsilon \qquad \text{for } t \geq 0 \end{cases}$$

We fix $\varepsilon > 0$ and we find $n_o \in N$ such that

$$(2.4.17)\quad \rho(\pi_n(x,t),\pi(x,t)) < \frac{\varepsilon}{3} \quad \forall n > n_o\ \forall (x,t) \in X \times R_+.$$

For the fixed $\varepsilon > 0$, we find $n_1 \in N$ such that

$$(2.4.18)\quad \rho(\pi_n(y,t),\pi(y,t)) < \frac{\varepsilon}{3} \quad \forall n > n_1 \geq n_o; \forall (y,t) \in X \times R_+.$$

(The choice of n_1 is possible because of the uniform convergence of π_n to π).

We take $n > n_1$ (for instance, we can put $n=n_1+1$) in such a way that $\rho(x,y) < \delta$ implies

$$(2.4.19)\quad \rho(\pi_n(x,t),\pi_n(y,t)) < \frac{\varepsilon}{3} \qquad \text{for } t \geq 0.$$

Thus, for $t \in R_+$ we have chosen y such that $\rho(y,x) < \delta$.

From triangle inequality we get

$$(2.4.20)\quad \begin{cases} \rho(\pi(y,t),\pi(x,t)) \leq \rho(\pi_n(x,t),\pi(x,t)) + \\ \qquad + \rho(\pi_n(x,t),\pi_n(y,t)) + \rho(\pi_n(y,t),\pi(y,t)) \\ \qquad \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{cases}$$

So, we have proved that

$$(2.4.21)\quad \begin{cases} \forall \varepsilon>0 \ \exists \delta>0: \rho(x,y)<\delta \implies \\ \rho(\pi(x,t), \pi(y,t))<\varepsilon \quad \text{for } t\geq 0 \end{cases}$$

which means that π^x is positively Lyapunov stable. This completes the proof of the Theorem.

Remark 2.4.1

If we assume that the convergence of $\{\pi_n\}$ to π is pointwise, then the assertion of Theorem 2.4.2 need not be satisfied. (An example is given by A.Pwlczar -[not published yet]).

Theorem 2.4.3

Let (X,ρ) be a given metric space and $(X,R_+;\pi_n)$ for every n as well as $(X,R_+;\pi)$ be pseudo-dynamical semi-systems such that the sequence $\{\pi_n\}$ converges to π uniformly in $t\in R_+$; then $\rho_H(\pi_n(x),\pi(x)) \longrightarrow 0$ as $n \longrightarrow \infty$ where ρ_H is the Hausdorff metric.

Proof:

We recall that

$$(2.4.22)\quad \begin{aligned} \rho_H(A,B) &:= \max\{\rho^*(A,B), \rho^*(B,A)\} = \\ &= \max\{\sup\{\rho(a,B): a\in A\}, \sup\{\rho(b,A): b\in B\}\} \end{aligned}$$

From the assumptions we have that

$$(2.4.23)\quad \begin{cases} \forall \varepsilon>0 \ \exists n_o \ \forall t\in R_+ \ \forall n>n_o \\ \rho(\pi_n(x,t), \pi(x,t))<\varepsilon \end{cases}$$

In our case, we have for every $x\in X$:

$$\begin{aligned} \rho_H(\pi_n(x),\pi(x)) &= \max\{\sup\{\rho(y,\pi_n(x)): y\in\pi(x)\}, \sup\{\rho(z,\pi(x)): z\in\pi_n(x)\} \\ &= \max\{\sup\{\inf\{\rho(y,z): z\in\pi_n(x)\}: y\in\pi(x)\}, \\ &\qquad \sup\{\inf\{\rho(y,z): y\in\pi(x)\}: z\in\pi_n(x)\}\} \\ &= \max\{\sup\{\inf\{\rho(\pi(x,t),\pi_n(x,s): s\in R_+\}: t\in R_+\}, \\ &\qquad \sup\{\inf\{\rho(\pi_n(x,s),\pi(x,t)): t\in R_+\}: s\in R_+\}\}. \end{aligned}$$

Let us fix $x \in X$ and $\varepsilon > 0$. We the choose n_o according to relation (2.4.23). For every fixed $t \in R_+$ we have:

$$(2.4.24) \quad \inf\{\rho(\pi(x,t), \pi_n(x,s)) : s \in R_+\} \leq \rho(\pi(x,t), \pi_n(x,t)) \quad \forall n > n_o.$$

So, we have that

$$(2.4.25) \quad \sup\{\inf\{\rho(\pi(x,t), \pi_n(x,s)) : s \in R_+\} : t \in R_+\} \leq \varepsilon \quad \forall n > n_o.$$

Similarly we can show that there is n_1 such that:

$$(2.4.26) \quad \sup\{\inf\{\rho(\pi_n(x,s), \pi(x,t)) : t \in R_+\} : s \in R_+\} \leq \varepsilon \quad \forall n > n_1.$$

So, for $n_* = \max(n_o, n_1)$ we have

$$\rho_H(\pi_n(x), \pi(x)) \leq \varepsilon \quad \forall n > n_*.$$

Thus we have proved that for every x, $\rho_H(\pi_n(x), \pi(x)) \longrightarrow 0$ as $n \longrightarrow \infty$.

Remark 2.4.2

Let the statement of Theorem 2.4.2 hold true. Then for every $\varepsilon > 0$, $\rho_H(\overline{\pi_n(x)}, \overline{\pi(x)}) \leq \varepsilon$ for n sufficiently large.

Proof:

From Theorem 2.4.3 we have that

$$(2.4.27) \quad \rho_H(\pi_n(x), \pi(x)) \leq \varepsilon \quad \text{for n sufficiently large.}$$

Relation (2.4.27) means that for n sufficiently large:

$$(2.4.28) \quad \rho^*(\pi_n(x), \pi(x)) \leq \varepsilon$$

$$(2.4.29) \quad \rho^*(\pi(x), \pi_n(x)) \quad .$$

Observe that:

$$\rho^*(A,B) = \sup\{\rho(a,B) : a \in A\} = \sup\{\rho(a,\bar{B}) : a \in \bar{A}\} = \rho^*(\bar{A},\bar{B}).$$

So, we have that for large n

$$(2.4.30) \quad \rho^*(\pi_n(x), \pi(x)) = \rho^*(\overline{\pi_n(x)}, \overline{\pi(x)})$$

and

$$(2.4.31)\qquad \rho^*(\pi(x),\pi_n(x)) = \rho^*(\overline{\pi(x)},\overline{\pi_n(x)}).$$

Since $\overline{\pi_n(x)}$ and $\overline{\pi(x)}$ are closed sets, we may assume without loss of generality that there are elements $y \in \overline{\pi_n(x)}$ and $z \in \overline{\pi(x)}$ such that

$$(2.4.32)\qquad \rho^*(\overline{\pi_n(x)},\overline{\pi(x)}) \leq \rho(y,z).$$

From triangle inequality we get

$$(2.4.33)\qquad \rho(y,z) \leq \rho(y,\pi_n(x)) + \rho(z,\pi(x)) + \rho^*(\pi_n(x),\pi(x))$$
$$0 + 0 + \varepsilon = \varepsilon.$$

Combining relations (2.4.32) and (2.4.33) we get

$$(2.4.34)\qquad \rho^*(\overline{\pi_n(x)},\overline{\pi(x)}) \leq \rho(y,z) \leq \varepsilon \text{ for } n \text{ sufficiently large.}$$

Using similar reasoning, we can show that

$$(2.4.35)\qquad \rho^*(\overline{\pi(x)},\overline{\pi_n(x)}) \leq \varepsilon \qquad \text{for } n \text{ sufficiently large.}$$

So, we have that

$$(2.4.36)\qquad \rho_H(\overline{\pi_n(x)},\overline{\pi(x)}) = \max(\rho^*(\overline{\pi_n(x)},\overline{\pi(x)}),\ \rho^*(\overline{\pi(x)},\overline{\pi_n(x)})) \leq \varepsilon$$
for n sufficiently large.

Corollary 2.4.1

If the assumptions of Theorem 2.4.3 are satisfied, then $\rho_H(\overline{\pi_n(x)},\overline{\pi(x)}) \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 2.4.1

Suppose that $(X,R;\pi_n)$ for $n=1,2,\ldots$ and $(X,R;\pi)$ are PDS with (X,ρ) being a given metric space. Let $x \in X$ be fixed. Let the set $\pi_n(x)$ for every n be bounded and the sequence $\{\pi_n\}$ converge uniformly to π in $t \in R$, then the set $\pi(x)$ is bounded.

Proof:

Suppose the contrary; thus $\pi(x)$ is not bounded.

So, in particular, there exists a sequence $\{t_m\}$ of real numbers, such that the sequence $\{\rho(\pi(x,t_m),x)\}$ is not bounded.

Without loss of generality, we can assume that

(2.4.37) $\rho(\pi(x,t_m),x) \longrightarrow \infty$ as $m \longrightarrow \infty$.

We take $\varepsilon_o > 0$ fixed arbitrarily and we choose $n_o \in N$ in such a way that

(2.4.38) $\rho(\pi_n(x,t),\pi(x,t)) < \varepsilon_o$ for $n > n_o$ $\forall t \in R$.

This is possible because of the uniform convergence of π_n to π in $t \in R$.

We consider now the set $\pi_{n_o}(x)$. Since it is bounded, there is a positive real number r_{n_o} such that

(2.4.39) $\rho(\pi_{n_o}(x,t),x) < r_{n_o}$ $\forall t \in R$.

From triangle inequality we get

(2.4.40) $$\rho(\pi(x,t_m),x) \leq \rho(\pi(x,t_m), \pi_{n_o}(x,t_m)) + \rho(\pi_{n_o}(x,t_m),x) \leq \varepsilon_o + r_{n_o}$$

and so, the sequence $\{\rho(\pi(x,t_m),x)\}$ is bounded.

This gives a contradiction, and proves our Lemma.

Corollary 2.4.2

If $X = R^k$ and $\overline{\pi_n(x)}$ is for every n compact and the sequence $\{\pi_n\}$ converges uniformly to π in $t \in R$, then the trajectory $\pi(x)$ has its closure being also compact.

Theorem 2.4.4

Let $(R^k,R;\pi_n)$, n=1,2,... and $(R^k,R;\pi)$ be DSS and let $x \in R^k$ be fixed. Suppose that the sequence $\{\pi_n\}$ converges to π uniformly in $t \in R$ and $\overline{\pi_n(x)}$ is compact for every n, then $\rho_H(\overline{\pi_n(x)},\overline{\pi(x)}) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof:

We recall that

$$\rho_H(A,B) = \max\{\sup\{\rho(a,B): a \in A\}, \sup\{\rho(b,A): b \in B\}\}$$

with

$$\rho(a,B) = \inf\{\rho(a,y): y \in B\}$$
$$\rho(b,A) = \inf\{\rho(b,x): x \in A\}.$$

Let $\varepsilon > 0$ be fixed. We take n_o such that

(2.4.41) $\quad \rho(\pi_n(x,t),\pi(x,t)) < \varepsilon \quad \forall n > n_o,\ t \in R.$

We fix $n > n_o$ in the sequel. Let $z \in \pi_n(x)$ be given. This element z is of the form $z = \pi_n(x,s)$ for some $s \in R$. Since $\rho(\pi(x,s),\pi_n(x,s)) < \varepsilon$ for $n > n_o$, we have

(2.4.42) $\quad \rho(z,\pi(x)) = \rho(\pi_n(x,s),\pi(x)) < \varepsilon\ .$

This is true for every element $z \in \overline{\pi_n(x)}$ and this implies that

(2.4.43) $\quad \rho(z,\overline{\pi(x)}) \leq \varepsilon$ for every $z \in \overline{\pi_n(x)}$.

So, we have that

(2.4.44) $\quad \sup\{\rho(z,\overline{\pi(x)}): z \in \overline{\pi_n(x)}\} \leq \varepsilon$

Similarly, for every $w \in \pi(x)$, we have $w = \pi(x,s)$ for some $s \in R$ and

(2.4.45) $\quad \rho(w,\pi_n(x)) = \rho(\pi(x,s),\pi_n(x,s)) < \varepsilon\ .$

Thus

(2.4.46) $\quad \rho(w,\pi_n(x)) = \rho(\pi(x,s),\pi_n(x)) < \varepsilon\ .$

This is true for every $w \in \overline{\pi(x)}$ and this implies that

(2.4.47) $\quad \rho(w,\overline{\pi_n(x)}) \leq \varepsilon$ for every $w \in \overline{\pi(x)}$

and so,

(2.4.48) $\quad \rho_H(\overline{\pi_n(x)},\overline{\pi(x)}) = \max\{\sup\{\rho(w,\overline{\pi_n(x)}): w \in \overline{\pi(x)}\ ,$
$\sup\{\rho(z,\overline{\pi(x)}): z \in \overline{\pi_n(x)}\}\} \leq \varepsilon.$

This is true for every fixed $n > n_o$. Thus we have proved that

(2.4.49) $\quad \rho_H(\overline{\pi_n(x)},\overline{\pi(x)}) \longrightarrow 0$ as $n \longrightarrow \infty$.

CHAPTER III

LIMIT SETS, PROLONGATIONAL LIMIT SETS AND PROLONGATIONS.

In this chapter, we state some properties of limit sets, prolongational limit sets and prolongations together with these properties that are preserved passing to the limits of the families of pseudo-dynamical systems.

Section 1 of the chapter is devoted to limit sets and prolongational limit sets while the properties of prolongations are treated in section 2. Some known results from other papers are quoted in section 0.

§ 3.0.

Theorem 3.0.1

Let $(X,R;\pi)$ be a PDS in the given metric space (X,ρ). For $x \in X$, let the motion π^x be positively Lyapunov stable and $\Lambda^+(x) \neq \emptyset$, then the mapping

$$\Lambda^+ : X \ni \xi \longmapsto \Lambda^+(\xi) \in 2^X$$

is upper semi-continuous at the point $x \in X$.

Proof:

Let sequences $\{x_m\}$ and $\{y_m\}$ of elements of X be such that

(3.0.1) $\quad x_m \longrightarrow x \quad$ as $\quad m \longrightarrow \infty$

(3.0.2) $\quad y_m \longrightarrow y \quad$ as $\quad m \longrightarrow \infty$

(3.0.3) $\quad y_m \in \Lambda^+(x_m) \quad$ for $m=1,2,\ldots$.

We have to show that

(3.0.4) $\quad y \in \Lambda^+(x)$.

Let

(3.0.13) $\tau_m := s_m(\frac{1}{m})$.

Without loss of generality, we may assume that $\tau_m \longrightarrow \infty$ as $m \longrightarrow \infty$. If not, we can find a subsequence. Such a subsequence exists because $\{\tau_m\}$ cannot be bounded. So, we have proved that, there exists a sequence $\{\tau_m\}$ of real numbers tending to $+\infty$ such that

(3.0.14) $\rho(\pi(x_m,\tau_m),y_m) \leq \frac{1}{m}$ for $m=1,2,\ldots$

From triangle inequality we get

(3.0.15) $\rho(\pi(x,\tau_m),y) \leq \rho(y_m,y) + \rho(\pi(x_m,\tau_m),y_m) + \rho(\pi(x_m,\tau_m),\pi(x,\tau_m))$.

Let now $\varepsilon>0$ be given. We put $\eta = \frac{1}{3}\varepsilon$. There is $n_o \in N$ such that

(3.0.16) $\rho(y_m,y)<\eta \quad \forall\, m>n_o$ (see relation (3.0.2)).

Since $\rho(\pi(x_m,\tau_m),y_m) < \frac{1}{m}$, we can find $n_1 \in N$ such that

(3.0.17) $\rho(\pi(x_m,\tau_m),y_m)<\eta \quad \forall\, m>n_1$.

From the assumptions π^x is positively Lyapunov stable. So, there is a $\delta>0$ such that $\rho(x,x_m)<\delta$ implies that

(3.0.18) $\rho(\pi(x_m,t),\pi(x,t))<\eta$ for $t \geq 0$.

Thus $\rho(\pi(x_m,\tau_m),\pi(x,\tau_m))<\eta$ if $\rho(x_m,x)<\delta$.

But $\rho(x,x_m)<\delta$ holds true if m is sufficiently large, say if $m>n_2$.

So, if $n_* = \max(n_o,n_1,n_2)$, then for $m>n_*$ we have from relation (3.0.15) that

(3.0.19) $\rho(\pi(x,\tau_m),y) \leq \eta+\eta+\eta = \varepsilon$.

Thus, we have proved that $\pi(x,\tau_m) \longrightarrow y$ as $m \longrightarrow \infty$ and from the fact that $\tau_m \longrightarrow \infty$ as $m \longrightarrow \infty$, we have shown that $y \in \Lambda^+(x)$.

This completes the proof of the assertion. (Theorem 3.0.1 with its proof is due to Pelczar [1]. We present it here because we want to make this study reasonably self-contained).

Theorem 3.0.2

Let $(X,R;\pi)$ be a PDS in the given metric space (X,ρ). For $x \in X$, let the motion π^x be negatively Lyapunov stable and $\Lambda^-(x) \neq \emptyset$, then the mapping

$$\Lambda^- : x \longmapsto \Lambda^-(x)$$

is upper semi-continuous at the point $x_o \in X$.
(Theorem 3.0.2 is the symmetric case of Theorem 3.0.1, hence we omit its proof).

§ 3.1.

Let $(X,R;\pi_n)$, $(n=1,2,\ldots)$ and $(X,R;\pi)$ be PDS. For a given point $x \in X$ we denote by $\Lambda_n^+(x)$, $\Lambda_n^-(x)$, $\Lambda^+(x)$, $\Lambda^-(x)$: the positive and negative limit sets in the n-th PDS and positive and negative limit sets of x in $(X,R;\pi)$ respectively.

Analogous notations: $J_n^+(x)$, $J_n^-(x)$, $J^+(x)$, $J^-(x)$ and $D_n^+(x)$, $D_n^-(x)$, $D^+(x)$, $D^-(x)$ will be used for corresponding prolongational limit sets and prolongations respectively.

Theorem 3.1.1

Let (X,ρ) be a metric space; $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ be dynamical systems. Suppose that $x \in X$ is such that $\pi_+(x)$ is relatively compact and that $\{\pi_n\}$ converges to π uniformly in t. Then for every $\varepsilon > 0$ there exists $n_o \in N$ such that

(3.1.1) $\quad \Lambda_n^+(x) \subset B(\Lambda^+(x),\varepsilon) \qquad$ for $n > n_o$.

Proof:

Assume the contrary. So, there exists $\varepsilon^{o} > 0$ such that for every n^{o} there exists $n \geq n^{o}$ for which

(3.1.2) $\quad \Lambda_n^+(x) \not\subset B(\Lambda^+(x), \varepsilon^{o})$.

This means that there exists a sequence $\{y_n\}$ of points of X such that

(3.1.3) $\quad y_n \in \Lambda_n^+(x) \qquad n=1,2,\ldots$

but

(3.1.4) $\quad y_n \notin B(\Lambda^+(x), \varepsilon^{o})$.

The relations (3.1.3) mean that for every $n \in N$ there exists a sequence $\{t_m^n\}_{m=1,2,\ldots}$ such that

(3.1.5) $\quad t_m^n \longrightarrow \infty$ as $m \longrightarrow \infty$

and for every fixed n:

(3.1.6) $\quad \rho(\pi_n(x,t_m^n), y_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

From (3.1.6) it follows that for every n there exists m_n such that

(3.1.7) $\quad \rho(\pi_n(x,t_{m_n}^n), y_n) \leq \frac{1}{n} \qquad n=1,2,\ldots$.

Without loss of generality we may assume that

(3.1.8) $\quad t_{m_n}^n \longrightarrow \infty \qquad$ as $n \longrightarrow \infty$.

Let us put

(3.1.9) $\quad s_n := t_{m_n}^n$.

Since $\overline{\pi_+(x)}$ is compact we may assume without loss of generality that the sequence $\{\pi(x,s_n)\}$ is convergent to some y belonging to $\overline{\pi_+(x)}$.

From (3.1.8) it follows however that $y \in \Lambda^+(x)$.

The relation (3.1.7) implies that

$$\rho(\pi_n(x,s_n),y_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

and so

(3.1.10) $\quad y_n \longrightarrow y \qquad \text{as } n \longrightarrow \infty .$

Since $y \in \Lambda^+(x)$ we get from (3.1.10) in particular that

(3.1.11) $\quad \rho(y_n, \Lambda^+(x)) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$

which contradicts (3.1.4) since (3.1.4) implies necessarily that

(3.1.12) $\quad \rho(y_n, \Lambda^+(x)) \geq \varepsilon^o .$

This completes the proof of our Theorem.

Remark 3.1.1

If the compactness of the set $\overline{\pi_+(x)}$ is not assumed then the above Theorem 3.1.1 does not holds true.

Theorem 3.1.2

For the given metric space (X,ρ) let $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ be PDS. Suppose the sequence $\{\pi_n\}$ converges to uniformly in $t \in R_+$ and the sequence $\{z_n\}$ of elements of X is such that $\{z_n\}$ converges to $z \in X$ and $z_n \in \Lambda_n^+(y)$ $(n=1,2,\ldots)$, then $z \in \Lambda^+(y)$.

Proof:

We have $z_n \in \Lambda_n^+(y)$, $n=1,2,\ldots$.

So,

$$\forall\, n \in N \quad \exists \{t_n^k\}_{k=1,2,\ldots},\ t_n^k \longrightarrow \infty \text{ as } k \longrightarrow \infty$$

and

(3.1.13) $\quad \pi_n(y,t_n^k) \longrightarrow z_n \qquad \text{as } k \longrightarrow \infty .$

Let $\varepsilon > 0$ be fixed. Take n_o so large that $\frac{1}{n} < \frac{\varepsilon}{3}$ for $n > n_o$.

$\forall n \ \exists k_n$:

$$(3.1.14) \qquad \rho(\pi_n(y,t_n^{k_n}),z_n) \leq \frac{1}{n} .$$

We may assume without loss of generality that $t_n^{k_n} \geq n$ for every fixed n. This means that $t_n^{k_n} \longrightarrow \infty$ as $n \longrightarrow \infty$.

From the assumptions, we have that $\{\pi_n\}$ converges to uniformly in $t \in R_+$. This means in particular that there exists n_1 such that

$$(3.1.15) \qquad \rho(\pi_n(y,t_n^{k_n}),\pi(y,t_n^{k_n})) < \frac{\varepsilon}{3} \qquad \forall n > n_1 .$$

Moreover, since $\rho(z_n,z) \longrightarrow 0$ as $n \longrightarrow \infty$, we have that there exists $n_2 \in N$ such that

$$(3.1.16) \qquad \rho(z,z_n) < \frac{\varepsilon}{3} \qquad \forall n > n_2 .$$

From triangle inequality we obtain

$$(3.1.17) \qquad \rho(\pi(y,t_n^{k_n}),z) \leq \rho(\pi(y,t_n^{k_n}),\pi_n(y,t_n^{k_n})) + \rho(\pi_n(y,t_n^{k_n}),z_n) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for $n > \max(n_o,n_1,n_2)$.

Thus, we have proved that

$$(3.1.18) \qquad \pi(y,t_n^{k_n}) \longrightarrow z \qquad \text{as } n \longrightarrow \infty .$$

This means that $z \in \Lambda^+ y)$.

Theorem 3.1.3

Let (X,ρ) be a given metric space; and $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ be PDS such that the sequence $\{\pi_n\}$ converges to π uniformly in $t \in R_+$. Suppose that $x \in X$ is such that

$$(3.1.19) \qquad \begin{cases} \text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ for which} \\ \tilde{x} \in B(x,\delta) \Longrightarrow \rho(\pi_n(x,t),\pi_n(\tilde{x},t)) < \varepsilon \\ \text{for every } t \geq 0 \text{ and every } n. \end{cases}$$

If $\{z_n\}$ and $\{x_n\}$ are such that $z_n \longrightarrow z$, $x_n \longrightarrow x$, $z_n \in \Lambda_n^+(x_n)$ for $n=1,2,\ldots$, then $z \in \Lambda^+(x)$.

Remark 3.1.2

From the assumptions, it follows in particular that every motion π_n^x is positively Lyapunov stable, but our condition is stronger: (δ is chosen for ε only, uniformly for every n). The stability of every π_n^x means of course that:

$$\begin{cases} \forall\, \varepsilon>0 \ \forall n\in N \ \exists \delta = \delta_{n,\varepsilon} \text{ such that} \\ \tilde{x} \in B(x,\delta) \longrightarrow \rho(\pi_n(x,t),\pi_n(\tilde{x},t)) < \varepsilon \qquad \text{for } t \geq 0. \end{cases}$$

Proof of the Theorem

The sequence $\{\pi_n\}$ is assumed to be convergent to π uniformly in $t\in R_+$. This means that,

$$(3.1.20) \qquad \begin{cases} \forall\, y\in X \quad \forall\, \varepsilon>0 \quad \exists n_o : \\ \rho(\pi_n(y,t),\pi(y,t)) < \varepsilon \qquad \text{for } n>n_o, \quad t\in R_+ \end{cases}$$

The relations $z_n \in \Lambda_n^+(x_n)$ $(n=1,2,\ldots)$ mean that there exists a sequence $\{t_n^k\}_{k=1,2,\ldots}$ of positive numbers such that $t_n^k \longrightarrow +\infty$ as $k \longrightarrow \infty$, and

$$(3.1.21) \qquad \pi_n(x_n,t_n^k) \longrightarrow z_n \quad \text{as } k \longrightarrow \infty.$$

From the assumptions, we have that $z_n \longrightarrow z$ as $n \longrightarrow \infty$. This means that,

$$(3.1.22) \qquad \rho(z_n,z) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty.$$

Relation (3.1.21) is equivalent to the condition

$$(3.1.23) \qquad \rho(\pi_n(x_n,t_n^k),z_n) \longrightarrow 0 \quad \text{as } k \longrightarrow \infty.$$

Since $t_n^k \longrightarrow +\infty$ as $k \longrightarrow \infty$, we can find a sequence $\{k_n\}$ of natural numbers, $k_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that $t_n^{k_n} \geq n$, $n\in N$.

So, we may assume that

$$t_n^{k_n} \longrightarrow +\infty \text{ as } n \longrightarrow \infty . \tag{3.1.24}$$

Let $\varepsilon > 0$ now be fixed. Let us put $\eta = \frac{\varepsilon}{4}$.

Applying (3.1.20) we find n_1 such that

$$\rho(\pi_n(x,t_n^{k_n}),\pi(x,t_n^{k_n})) < \eta \quad \text{for } n > n_1, \forall t_n^{k_n}. \tag{3.1.25}$$

Using (3.1.22), we find n_2 such that

$$\rho(z,z_n) < \eta \qquad \text{for } n > n_2. \tag{3.1.26}$$

Relations (3.1.23) implies that there exists n_3 such that

$$\rho(\pi_n(x_n,t_n^{k_n}),z_n) < \eta \qquad \text{for } n > n_3 \tag{3.1.27}$$

Using the assumption (3.1.19) we find $\delta > 0$ such that

$$\tilde{x} \in B(x,\delta) \Longrightarrow \rho(\pi_n(x,t),\pi_n(\tilde{x},t)) < \eta \quad \text{for } t \geq 0,\ n \in N.$$

We find n_4 so large that

$$\rho(x,x_n) < \delta \qquad \text{for } n > n_4.$$

Now, putting $n_* = \max(n_1,n_2,n_3,n_4)$ we obtain for $n > n_*$:

$$\begin{cases} \rho(\pi_n(x_n,t_n^{k_n}),z_n) < \eta \ , \ \rho(z,z_n) < \eta \ , \\ \rho(\pi_n(x_n,t_n^{k_n}),\pi_n(x,t_n^{k_n})) < \eta \text{ and } \rho(\pi_n(x,t_n^{k_n}),\pi(x,t_n^{k_n})) < \eta \end{cases} \tag{3.1.28}$$

The triangle inequality gives

$$\begin{cases} \rho(\pi(x,t_n^{k_n}),z) \leq \rho(\pi(x,t_n^{k_n}),\pi_n(x,t_n^{k_n})) + \\ \qquad + \rho(\pi_n(x,t_n^{k_n}),\pi_n(x_n,t_n^{k_n})) + \\ \qquad + \rho(\pi_n(x_n,t_n^{k_n}),z_n) + \rho(z_n,z) \end{cases} \tag{3.1.29}$$

and then by using (3.1.28) we get

(3.1.30) $\rho(\pi(x,t_n^{k_n}),z) < \eta+\eta+\eta+\eta = \varepsilon$ for $n > n_*$.

Thus, we have proved that

(3.1.31) $\pi(x,t_n^{k_n}) \longrightarrow z$ as $n \longrightarrow \infty$.

This means (see (3.1.24)) that $z \in \Lambda^+(x)$.

The proof is completed.

Theorem 3.1.4

Let (X,ρ) be a given metric space with $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ being PDS. Suppose the sequence $\{\pi_n\}$ converges uniformly to π and π^x is positively Lyapunov stable. Let the sequences $\{x_n\}$ and $\{y_n\}$ of elements of X be such that $x_n \longrightarrow x$, $y_n \longrightarrow y$ and $y_n \in \Lambda_n^+(x_n)$ for $n=1,2,\ldots$, then $y \in \Lambda^+(x)$.

Proof:

Let $y_n \in \Lambda_n^+(x_n)$ for every $n \in N$. This means that, for every fixed n, there exists a sequence $\{t_k^n\}_{k=1,2,\ldots}$ of positive numbers such that

(3.1.32) $t_k^n \longrightarrow \infty$ as $k \longrightarrow \infty$

and

(3.1.33) $\pi_n(x_n,t_k^n) \longrightarrow y_n$ as $k \longrightarrow \infty$.

Relation (3.1.32) means that

(3.1.34) $\rho(\pi_n(x_n,t_k^n),y_n) \longrightarrow 0$ as $k \longrightarrow \infty$

Moreover we have

(3.1.35) $\rho(y,y_n) \longrightarrow 0$ as $n \longrightarrow \infty$

By virtue of relation (3.1.32), we can find a strictly increasing sequence of positive integers $\{k_n\}$ such that $t_{k_n}^n \geq n$ and so

(3.1.36) $t_{k_n}^n \longrightarrow \infty$ as $n \longrightarrow \infty$

Applying relations (3.1.34) and (3.1.36) we observe that, for every fixed n, there is $r(n) \in N$ such that

$$(3.1.37) \qquad \rho(\pi_n(x_n, t^n_{k_{r(n)}}), y_n) \leq \frac{1}{n} \qquad \text{for } n=1,2,\dots .$$

Without loss of generality, we may assume that $r(n+1) > r(n) > n$, $n=1,2,\dots$. In particular, we have that $r(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Putting

$$(3.1.38) \qquad \sigma_n := t^n_{k_{r(n)}}, \qquad n=1,2,\dots$$

we have

$$(3.1.39) \qquad \sigma_n \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty$$

and

$$(3.1.40) \qquad \rho(\pi_n(x_n, \sigma_n), y_n) \leq \frac{1}{n}, \quad n=1,2,\dots .$$

Let $\varepsilon > 0$ be given. We put $\eta = \frac{\varepsilon}{4}$. We can find n_1 such that (see (3.1.35))

$$(3.1.41) \qquad \rho(y, y_n) < \eta \qquad \text{for } n > n_1$$

and n_2 such that

$$(3.1.42) \qquad \rho(\pi_n(x_n, \sigma_n), y_n) < \eta \qquad \text{for } n > n_2$$

(see (3.1.40)).

Because of the Lyapunov stability of π^x we can find $\delta > 0$ such that

$$(3.1.43) \qquad \rho(x, \tilde{x}) < \delta \implies \rho(\pi(x,t), \pi(\tilde{x},t)) < \varepsilon \qquad \text{for } t \geq 0$$

Let n_3 be so large that

$$(3.1.44) \qquad \rho(x, x_n) < \delta \qquad \text{for } n > n_3.$$

Combining (3.1.43) and (3.1.44) we observe that

$$(3.1.45) \qquad \rho(\pi(x, \sigma_n), \pi(x_n, \sigma_n)) < \eta \qquad \text{for } n > n_3.$$

Finally, using the uniform convergence of $\{\pi_n\}$ to π, we can find n_4 so large that

(3.1.46) $\rho(\pi_n(x_n,\sigma_n),\pi(x_n,\sigma_n)) < \eta$ for $n > n_4$.

From the triangle inequality we obtain

(3.1.47)
$$\begin{cases} \rho(\pi(x,\sigma_n),y) \le \rho(y,y_n) + \rho(\pi_n(x_n,\sigma_n),y_n)) + \\ \qquad + \rho(\pi_n(x_n,\sigma_n),\pi(x_n,\sigma_n)) + \\ \qquad + \rho(\pi(x_n,\sigma_n),\pi(x,\sigma_n)) \end{cases}$$

which gives (see (3.1.41), (3.1.42), (3.1.45), (3.1.46))

(3.1.48) $\rho(\pi(x,\sigma_n),y) < \eta+\eta+\eta+\eta = \varepsilon$ for $n > \max(n_1,n_2,n_3,n_4)$.

Thus we have proved that

(3.1.49) $\pi(x,\sigma_n) \longrightarrow y$ as $n \longrightarrow \infty$

and by virtue of(3.1.39), we obtain $y \in \Lambda^+(x)$ which finishes the proof.

Corollary 3.1.1 (of Theorem 3.1.2)

In the given metric space (X,ρ), let $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ be PDS such that the sequence $\{\pi_n\}$ converges uniformly to π . Suppose that the sequence $\{z_n\}$ of elements of X is such that $z_n \longrightarrow z \in X$ and $z_n \in \Lambda_n^+(x)$ for every n and for fixed $x \in X$, then $z \in \Lambda^+(x)$.

Corollary 3.1.2

In the given metric space (X,ρ), let $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ be PDS. Let the sequence $\{\pi_n\}$ converge uniformly to π . For fixed $x \in X$, let $z \in \Lambda_n^+(x)$ for every n; then $z \in \Lambda^+(x)$.

Remark 3.1.3

If in Theorem 3.1.3 above, we assume that $(X,R;\pi_n) = (X,R;\pi)$ for every $n \in N$; which means that $\pi_n = \pi$ for every n and so, $\Lambda_n^+(x_n) = \Lambda^+(x_n)$ for every n, then we have the same result as in Theorem 3.0.1 above, which means that the mapping

$$X \ni x \longmapsto \Lambda^{+}(x)$$

is upper semi-continuous at the point $x \in X$.

Remark 3.1.4

Let (X,ρ) be a given metric space. If $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ are DS, the statements of Theorems 3.1.2 - 3.1.4; and of corollaries 3.1.1 and 3.1.2 hold true for $\Lambda_n^{-}(x)$ and $\Lambda^{-}(x)$.

Theorem 3.1.5

Let the metric space (X,ρ) be given with $(X,R;\pi)$ being a DS. Then, the mapping

$$X \ni x \longmapsto J^{+}(x)$$

is upper semi-continuous at the point $x_o \in X$.

Proof:

We have to show that if sequences $\{x_n\}$ and $\{y_n\}$ of elements of X are such that

(3.1.50) $\quad x_n \longrightarrow x_o \qquad$ as $n \longrightarrow \infty$

(3.1.51) $\quad y_n \longrightarrow y_o \qquad$ as $n \longrightarrow \infty$

(3.1.52) $\quad y_n \in J^{+}(x) \quad$ for $n=1,2,\ldots$,

then

(3.1.53) $\quad y_o \in J^{+}(x_o)$.

Relations (3.1.52) means that for every n there exists a sequence $\{t_n^s\}$ of real non-negative numbers and a sequence $\{x_n^s\}$ of elements of X such that for every n:

(3.1.54) $\quad x_n^s \longrightarrow x_n \qquad$ as $s \longrightarrow \infty$

(3.1.55) $\quad t_n^s \longrightarrow \infty \qquad$ as $s \longrightarrow \infty$

and

(3.1.56) $\quad \pi(x_n^s, t_n^s) \longrightarrow y_n \qquad$ as $s \longrightarrow \infty$.

Relations (3.1.56) mean that for every n

$$(3.1.57)\qquad \rho(\pi(x_n^{s},t_n^{s}),y_n) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

while (3.1.50) and (3.1.51) mean that

$$(3.1.58)\qquad \rho(x_n,x_o) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

and respectively

$$(3.1.59)\qquad \rho(y_n,y_o) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty .$$

From (3.1.54) - (3.1.56), it follows that we can find a sequence of positive integers $\{s_n\}$ such that

$$(3.1.60)\qquad s_n \longrightarrow \infty$$

$$(3.1.61)\qquad t_n^{s_n} \geq n \qquad n=1,2,\ldots$$

$$(3.1.62)\qquad \rho(x_n^{s_n},x_n) \leq \frac{1}{n} \qquad n=1,2,\ldots$$

$$(3.1.63)\qquad \rho(\pi(x_n^{s_n},t_n^{s_n}),y_n) \leq \frac{1}{n} \qquad n=1,2,\ldots .$$

So, putting

$$(3.1.64)\qquad \sigma_n := t_n^{s_n}, \qquad \hat{x}_n := x_n^{s_n}$$

we get

$$(3.1.65)\qquad \sigma_n \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty$$

$$(3.1.66)\qquad \rho(\hat{x}_n,x_n) \leq \frac{1}{n} \qquad n=1,2,\ldots$$

$$(3.1.67)\qquad \rho(\pi(\hat{x}_n,\sigma_n),y_n) \leq \frac{1}{n} , \qquad n=1,2,\ldots .$$

We have

$$\rho(\hat{x}_n,x_o) \leq \rho(x_n,\hat{x}_n) + \rho(x_n,x_o)$$

and so, (see (3.1.66)

$$(3.1.68)\qquad \rho(\hat{x}_n,x_o) \longrightarrow 0 \qquad \text{as } n \longrightarrow \infty$$

The triangle inequality gives also

(3.1.69) $$\rho(\pi(\hat{x}_n,\sigma_n),y_o)\le\rho(\pi(\hat{x}_n,\sigma_n),y_n)+\rho(y_n,y_o)$$

which implies (see (3.1.67) and (3.1.59))

(3.1.70) $$\rho(\pi(\hat{x}_n,\sigma_n),y_o)\longrightarrow 0$$

and so

(3.1.71) $$\pi(\hat{x}_n,\sigma_n)\longrightarrow y_o \qquad \text{as } n\longrightarrow\infty .$$

This means (see (3.1.65)), that

$$y_o\in J^+(x_o).$$

This completes the proof of the Theorem.

Theorem 3.1.7

Let (X,ρ) be a given metric space. Suppose that $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ are PDS. Assume that $x\in X$ and there are positive real numbers λ and s such that the sequence $\{\pi_n\}$ converges uniformly in the set $[s,\infty)\times B(x,\lambda)$ to π. Assume that the sequences $\{x_n\}$ and $\{z_n\}$ of elements of X are such that

(3.1.72) $$x_n\longrightarrow x \qquad \text{as } n\longrightarrow\infty$$

(3.1.73) $$z_n\longrightarrow z \qquad \text{as } n\longrightarrow\infty$$

(3.1.74) $$z_n\in J_n^+(x_n) \qquad \text{for } n=1,2,\ldots$$

then

(3.1.75) $$z\in J^+(x).$$

Proof:

The relations (3.1.74) mean that there exist sequences $\{x_n^m\}$ of elements of X and $\{t_n^m\}$ of real numbers such that for every fixed n:

(3.1.76) $$x_n^m\longrightarrow x_n \qquad \text{as } m\longrightarrow\infty$$

(3.1.77) $t_n^m \longrightarrow \infty$ as $m \longrightarrow \infty$

and

(3.1.78) $\pi_n(x_n^m, t_n^m) \longrightarrow z_n$ as $m \longrightarrow \infty$.

From (3.1.72), it follows that for every n there exists $m_n^o \in N$ such that

(3.1.79) $\rho(x_n^m, x_n) \leq \frac{1}{n}$ for $m \geq m_n^o$.

For every fixed n we can find also (see (3.1.73)) a positive integer m_n^1 such that

(3.1.80) $\rho(\pi_n(x_n^m, t_n^m), z_n) \leq \frac{1}{n}$ for $m \geq m_n^1$

and moreover, there exists m_n^2 such that

(3.1.81) $t_n^m \geq n$ for $m \geq m_n^2$.

We put now

(3.1.82) $m_n := \max(m_n^o, m_n^1, m_n^2)$

and we take n_* so large that

(3.1.83) $n > s$ and $\frac{1}{n} < \frac{\lambda}{2}$ for $n > n_*$.

Without loss of generality we may assume that n_* is so large that

(3.1.84) $\rho(x, x_n) < \frac{\lambda}{2}$ for $n > n_*$

(see assumption (3.1.72)).

Thus, by virtue of the inequality

(3.1.85) $\rho(x_n^{m_n}, x) \leq \rho(x_n^{m_n}, x_n) + \rho(x_n, x)$

we obtain

$$\rho(x_n^{m_n}, x) < \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda \quad \text{for } n > n_*$$

(see (3.1.79), (3.1.83) and (3.1.84)), which means that

$$(3.1.86) \qquad x_n^{m_n} \in B(x,\lambda) \qquad \text{for } n > n_*.$$

Simultaneously we have (see (3.1.81) and (3.1.83))

$$(3.1.87) \qquad t_n^{m_n} > s \qquad \text{for } n > n_* .$$

Thus for $n > n_*$, the points $(x_n^{m_n}, t_n^{m_n})$ belong to the set $[s,\infty) \times B(x,\lambda)$ in which the sequence $\{\pi_n\}$ converges to π uniformly.

From the triangle inequality we obtain for every n:

$$(3.1.88) \qquad \left\{ \begin{aligned} \rho(\pi(x_n^{m_n}, t_n^{m_n}), z) \le \rho(\pi(x_n^{m_n}, t_n^{m_n}), \pi_n(x_n^{m_n}, t_n^{m_n})) + \\ + \rho(\pi_n(x_n^{m_n}, t_n^{m_n}), \pi(x_n^{m_n}, t_n^{m_n})) + \rho(z_n, z) \end{aligned} \right.$$

Let now $\varepsilon > 0$ be arbitrarily fixed. Find $n_o > n_*$ so large that

$$(3.1.89) \qquad \rho(\pi(x_n^{m_n}, t_n^{m_n}), \pi_n(x_n^{m_n}, t_n^{m_n})) < \frac{\varepsilon}{3} \qquad \text{for } n > n_o,$$

this is possible because of the uniform convergence of $\{\pi_n\}$ to π.

Because of (3.1.80) we may find $n_1 > n_*$ such that

$$(3.1.90) \qquad \rho(\pi_n(x_n^{m_n}, t_n^{m_n}), z_n) < \frac{\varepsilon}{3} \qquad \text{for } n > n_1.$$

Finally, we find $n_2 > n_*$ such that

$$(3.1.91) \qquad \rho(z, z_n) < \frac{\varepsilon}{3} \qquad \text{for } n > n_2$$

(see (3.1.73)). We take now $\hat{n} := \max(n_o, n_1, n_2)$.

By using (3.1.89), (3.1.90) and (3.1.91) we get from (3.1.88) the inequality

$$\rho(\pi(x_n^{m_n}, t_n^{m_n}) z) < \varepsilon \qquad \text{for } n > \hat{n}.$$

Thus we have proved that

$$(3.1.92) \qquad \pi(x_n^{m_n}, t_n^{m_n}) \longrightarrow z \qquad \text{as } n \longrightarrow \infty .$$

We know that

$$\rho(x_n^{m_n}, x) \leq \rho(x_n^{m_n}, x_n) + \rho(x_n, x)$$

and then

$$(3.1.93) \qquad x_n^{m_n} \longrightarrow x \qquad \text{as } n \longrightarrow \infty$$

(see (3.1.79)) and (3.1.72)).

We know also that

$$(3.1.94) \qquad t_n^{m_n} \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty$$

(see (3.1.81)). The relations (3.1.92), (3.1.93) and (3.1.94) give (3.1.75). The proof is completed.

Corollary 3.1.3

Let (X,ρ) be a given metric space and let $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ be PDS. Suppose that the sequence $\{\pi_n\}$ converges to π uniformly. If $z_n \in J_n^+(y)$, $z_n \longrightarrow z$, then $z \in J^+(y)$.

Corollary 3.1.4

Assume that $(X,R;\pi_n)$ and $(X,R;\pi)$ are as above (see Corollary 3.1.3). If $z \in J_n^+(y)$ for every n, then $z \in J^+(y)$.

Remark 3.1.4

Putting in the assumptions of Theorem 3.1.7, $\pi_n = \pi$, we obtain in that way the upper semi-continuity of the mapping

$$y \longmapsto J^+(y)$$

(see Theorem 3.1.5).

Remark 3.1.5

It is obvious that similar (symmetric) results hold true for J_n^- and J^- considered in the place of J_n^+ and J^+ respectively.

§ 3.2.

In this section, we give some properties of prolongations and also the properties that are preserved under suitable assumptions, passing to the limits of pseudo-dynamical systems.

Theorem 3.2.1

Let the metric space (X,ρ) be given and $(X,R;\pi)$ be a PDS. The mapping $X \ni x \longmapsto D^+(x)$ is upper semi-continuous at every point $x_o \in X$.

Proof:

Let the sequences $\{x_m\}$ and $\{y_m\}$ of elements of X be such that

(3.2.1) $x_m \longrightarrow x_o$ as $m \longrightarrow \infty$

(3.2.2) $y_m \longrightarrow y_o$ as $m \longrightarrow \infty$

(3.2.3) $y_m \in D^+(x_m)$ for $m=1,2,\ldots$.

We have to show that

(3.2.4) $y_o \in D^+(x_o)$.

Relations (3.2.3) mean that for every fixed $m \in N$ there are sequences $\{x_m^k\}$ in X and $\{t_m^k\}_{k=1,2,\ldots}$ in R, $t_m^k \geq 0$ such that

(3.2.5) $x_m^k \longrightarrow x_m$ as $k \longrightarrow \infty$

and

(3.2.6) $\pi(x_m^k, t_m^k) \longrightarrow y_m$ as $k \longrightarrow \infty$

Relation (3.2.5) implies that for every fixed m, we can find a positive number k_m such that

$$(3.2.7) \qquad \rho(x_m, x_m^k) \leq \frac{1}{m} \qquad \text{for } k > k_m.$$

Similarly, relation (3.2.6) implies that for every fixed m, there exists a positive integer r_m such that

$$(3.2.8) \qquad \rho(\pi(x_m^k, t_m^k), y_m) \leq \frac{1}{m} \qquad \text{for } k > r_m.$$

Thus, in particular, if $p_m := \max(k_m, r_m)$, then

$$(3.2.9) \qquad \rho(x_m, x_m^k) \leq \frac{1}{m} \qquad \text{for } k > p_m$$

and simultaneously

$$(3.2.10) \qquad \rho(\pi(x_m^k, t_m^k), y_m) \leq \frac{1}{m} \qquad \text{for } k > p_m.$$

In particular, for $k = p_m$, we have

$$(3.2.11) \qquad \rho(x, x_m^{p_m}) \leq \frac{1}{m} \qquad m = 1, 2, \ldots$$

and

$$(3.2.12) \qquad \rho(\pi(x_m^{p_m}, t_m^{p_m}), y_m) \leq \frac{1}{m} \qquad \text{for } m = 1, 2, \ldots$$

We put

$$(3.2.13) \qquad s_m := t_m^{p_m}, \quad s_m \geq 0, \quad v_m := x_m^{p_m}.$$

From triangle inequality we obtain

$$(3.2.14) \qquad \rho(v_m, x_o) \leq \rho(v_m, x_m) + \rho(x_m, x_o) \leq \frac{1}{m} + \rho(x_m, x_o)$$

and by relation (3.2.12) we obtain

$$(3.2.15) \qquad \rho(\pi(v_m, s_m), y_o) \leq \rho(\pi(v_m, s_m), y_m) + \rho(y_m, y_o) \leq \rho(y_m, y_o) + \frac{1}{m}$$

Relation (3.2.14) implies that

(3.2.16) $\quad v_m \longrightarrow x_o \qquad\qquad \text{as } m \longrightarrow \infty$

while relation (3.2.15) gives

(3.2.17) $\quad \pi(v_m, s_m) \longrightarrow y_o, \qquad s_m \geq 0.$

These two relations combined means that $y_o \in D^+(x_o)$. This completes the proof of the assertion.

Corollary 3.2.1

Let $(X,R;\pi)$ be a PDSS in the given metric space (X,ρ), then the mapping

$$X \ni x \longmapsto D(x) = D^+(x)$$

is upper semi-continuous at every point $x_o \in X$.

Theorem 3.2.2

Let (X,ρ) be a given metric space and $(X,R;\pi)$ be a PDS, then the mapping

$$X \ni x \longmapsto D^-(x)$$

is upper semi-continuous at every point $x_o \in X$.

Theorem 3.2.2 is the symmetric case of Theorem 3.2.1 hence we omit its proof.

Theorem 3.2.3

Let (X,ρ) be a given metric space and $(X,R;\pi_n)$ $(n=1,2,\ldots)$, $(X,R;\pi)$ are PDS; where the sequence $\{\pi_n\}$ converges uniformly to π in $R_+ \times B(x,\lambda)$ for some $\lambda > 0$. Suppose the sequences $\{x_n\}$ and $\{z_n\}$ of elements of X are such that

$$\begin{aligned} x_n &\longrightarrow x \in X && \text{as } n \longrightarrow \infty \\ z_n &\longrightarrow z \in X && \text{as } n \longrightarrow \infty \\ z_n &\in D_n^+(x_n) && \text{for } n=1,2,\ldots \end{aligned}$$

then $z \in D^+(x)$.

Proof:

The relations $z_n \in D_n^+(\varphi_n)$ $(n=1,2,\ldots)$ mean that for every fixed n, there exist sequences $\{x_n^k\}_{k=1,2,\ldots}$ in X and $\{t_n^k\}_{k=1,2,\ldots}$ of real numbers such that

(3.2.18) $\quad x_n^k \longrightarrow x_n \qquad$ as $k \longrightarrow \infty$

(3.2.19) $\quad t_n^k \geq 0$

and

(3.2.20) $\quad \pi_n(x_n^k, t_n^k) \longrightarrow z_n \qquad$ as $k \longrightarrow \infty$.

Relations $x_n \longrightarrow x$ as $n \longrightarrow \infty$ and $z_n \longrightarrow z$ as $n \longrightarrow \infty$ imply respectively that

(3.2.21) $\quad \rho(x_n, x) \longrightarrow 0 \qquad$ as $n \longrightarrow \infty$

and

(3.2.22) $\quad \rho(z_n, z) \longrightarrow 0 \qquad$ as $n \longrightarrow \infty$.

Relations (3.2.18) and (3.2.20) respectively imply that for every $n \in N$ there exists k_n^o such that

(3.2.23) $\quad \rho(x_n^k, x_n) \leq \frac{1}{n} \qquad$ for $k > k_n^o$

and also there exists k_n^1 such that

(3.2.24) $\quad \rho(\pi_n(x_n^k, t_n^k), z_n) \leq \frac{1}{n} \qquad$ for $k > k_n^1$.

We take $k_n := \max(k_n^o, k_n^1)$ and we take n_* so large that $n_* > k_n$ and $\frac{1}{n} < \frac{\lambda}{2}$ for $n > n_*$.

Thus in particular (see (3.2.23)) we get

(3.2.25) $\quad \rho(x_n^{k_n}, x_n) < \frac{\lambda}{2} \qquad$ for $n > n_*$.

Because of (3.2.21) we may assume also that n_* is so large that

(3.2.26) $\rho(x_n,x) < \frac{\lambda}{2}$ for $n > n_*$

From (3.2.25) and (3.2.26) we get by using the triangle inequality

$$\rho(x_n^{k_n},x) \le \rho(x_n^{k_n},x_n) + \rho(x_n,x) \le \frac{\lambda}{2} + \frac{\lambda}{2} = \lambda$$

for $n > n_*$, so

(3.2.27) $x_n^{k_n} \in B(x,\lambda)$ for $n > n_*$.

Let $\varepsilon > 0$ be arbitrarily fixed. Let us put $\beta = \frac{\varepsilon}{3}$. We find $n_o > n_*$ such that

(3.2.28) $\rho(\pi_n(x_n^{k_n},t_n^{k_n}), \pi(x_n^{k_n},t_n^{k_n})) < \beta$ for $n > n_o$

this is possible, since $\{\pi_n\}$ converges uniformly to π in the set $R_+ \times B(x,\lambda)$ and as we have proved, the condition (3.2.27) is fulfilled.

We take now $n_1 > n_*$ so large that $\frac{1}{n} < \beta$ for $n > n_1$; this will give (see (3.2.24))

(3.2.29) $\rho(\pi_n(x_n^{k_n},t_n^{k_n}),z_n) < \beta$ for $n > n_1$.

Furthermore there exists $n_2 > n_o$ such that

(3.2.30) $\rho(z_n,z) < \beta = \frac{\varepsilon}{3}$ for $n > n_2$.

Put $\hat{n} := \max(n_o,n_1,n_2)$.

Then from the triangle inequality we get for $n > \hat{n}$ (see (3.2.28), (3.2.29) and (3.2.30)):

(3.2.31) $$\rho(\pi(x_n^{k_n},t_n^{k_n}),z) \le \rho(z_n,z) + \rho(\pi_n(x_n^{k_n},t_n^{k_n}),z_n) + \rho(\pi_n(x_n^{k_n},t_n^{k_n}),\pi(x_n^{k_n},t_n^{k_n}))$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Thus we have proved that

$$(3.2.32)\qquad \pi(x_n^{k_n}, t_n^{k_n}) \longrightarrow z \qquad \text{as } n \longrightarrow \infty .$$

Using once again the triangle inequality which has been used in order to prove (3.2.27) we observe easily that

$$\rho(x_n^{k_n}, x) \leq \frac{1}{n} + \rho(x_n, x) \qquad \text{for } n > n_*$$

and so

$$(3.2.33)\qquad x_n^{k_n} \longrightarrow x . \qquad \text{as } n \longrightarrow \infty$$

The relations (3.2.19), (3.2.32) and (3.2.33) mean that

$$z \in D^+(x).$$

This completes the proof of the Theorem.

§ 3.3.

In this section we state some Theorems with regard to Poisson Stability, and wandering points.

Theorem 3.3.1

Let (X,ρ) be a metric space. Let $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ be PDS, where the sequence $\{\pi_n\}$ converges to π uniformly in $t \in R$.

Suppose that $x \in X$ is such that

$$(3.3.1)\qquad \begin{cases} \text{for every } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ for which} \\ \tilde{x} \in B(x,\delta) \Longrightarrow \rho(\pi_n(x,t), \pi_n(\tilde{x},t)) < \varepsilon \\ \text{for every } t \geq 0 \ (t \leq 0) \text{ for every } n \end{cases}$$

Let $x \in X$ be Poisson stable in the (PD) system $(X,R;\pi_n)$ $(n=1,2,\ldots)$, then $x \in X$ is Poisson stable in the (PD) system $(X,R;\pi)$.

Proof:

It is assumed that the point $x \in X$ is Poisson stable in the triple $(X,R;\pi_n)$ for every fixed n.

This means that

$$(3.3.2) \qquad x \in \Lambda_n(x) := \Lambda_n^+(x) \cup \Lambda_n^-(x) \quad \text{for every fixed } n \in N$$

Relations (3.3.2) mean that for every n we have

$$(3.3.3) \qquad x \in \Lambda_n^+(x)$$

or

$$(3.3.4) \qquad x \in \Lambda_n^-(x).$$

If the relation (3.3.3) is satisfied for infinitely many n, then we can apply Theorem 3.1.3 obtaining the relation $x \in \Lambda^+(x)$.

If (3.3.4) holds true for infinitely many n, then (see Remark 3.1.4) we get

$$x \in \Lambda^-(x).$$

So $x \in \Lambda(x) = \Lambda^+(x) \cup \Lambda^-(x)$ and then the proof is finished.

Theorem 3.3.2

Let (X,ρ) be a metric space with $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ being PDS. Let the sequence $\{\pi_n\}$ converge to π uniformly and π^x be Lyapunov stable. Suppose that the point $x \in X$ is Poisson stable in every PDS $(X,R;\pi_n)$ $(n=1,2,\ldots)$, then the point $x \in X$ is Poisson stable in the triple $(X,R;\pi)$.

Proof:

The proof is similar to that of Theorem 3.3.1, with only one change:

instead of Theorem 3.1.3, we have to apply here Theorem 3.1.4 (together with Remark 3.1.4).

We shall establish now a Theorem which is a simple corollary of Theorem 3.1.7. It will be about non-wandering points

Let us recall that a point $x \in X$ is said to be non-wandering in the PDS $(X,R;\pi)$ if $x \in J^+(x)$.

Theorem 3.3.3

Let (X,ρ) be a metric space. Suppose $(X,R;\pi_n)$ $(n=1,2,\ldots)$ and $(X,R;\pi)$ are PDS. Assume that for $x \in X$ there are positive real numbers λ and s such that the sequence $\{\pi_n\}$ converges to π uniformly in the set $[s,\infty) \times B(x,\lambda)$.

Assume finally that $\{x_n\}$ is a sequence of elements of X such that $x_n \longrightarrow x$, x_n is non-wandering in $(X,R;\pi_n)$ $(n=1,2,\ldots)$, then x is non-wandering in $(X,R;\pi)$.

Proof:

It is enough to apply Theorem 3.1.7, by putting $z_n = x_n$ $(n=1,2,\ldots)$ and $z = x$ in (3.1.73) and (3.1.74) respectively.

CHAPTER IV

LIMIT SETS, PROLONGATIONAL LIMIT SETS AND PROLONGATIONS IN PSEUDO-PROCESSES.

In this chapter, we state some properties of limit sets, prolongational limit sets and prolongations with regard to Pseudo-processes.

Section 1 of the chapter is devoted to the general definitions that are pertinent to the study while in section 2 we state some known results from other research papers. At the same time, we give some results that are obtained in this study.

In section 3, we state some properties that are preserved passing to the limits of the families of Pseudo-processes and in sections 4 and 5 are devoted to some other related results that we arrived at in this study.

§ 4.1.

In this section, we state some general definitions that form the basis of the results obtained in this study.

Let (X,ρ) be a metric space. Let $(R,X,R_+;\pi)$ be a psuedo-process (see Def. 1.6.1).

Definition 4.1.1

Let point $(t,x) \in R \times X$ be given. The limit set of the point (t,x) is the set $\Lambda(t,x)$ defined by the formula

$$\Lambda(t,x) := \{ y \in X : \text{there exists a sequence } \{s_n\} \text{ of non-negative numbers such that } s_n \longrightarrow \infty \text{ and } \pi(t,x,s_n) \longrightarrow y \text{ as } n \longrightarrow \infty \} . \tag{4.1.1}$$

Remark 4.1.1

The definition 4.1.1 above is a direct extension of that of the (positive) limit set in a dynamical system or semi-system.

So, it is obvious that $\Lambda(t,x)$ will have properties that run parallel with those of limit sets in dynamical systems or semi-systems.

Definition 4.1.2

We defined the (1)-prolongation of the point $(t,x) \in R \times X$ by the formula

$$(4.1.2) \quad D_{(1)}(t,x) := \{ y \in X: \text{there exist sequences } \{x_m\} \text{ of elements of X and } \{s_m\} \text{ of non-negative real numbers such that } x_m \longrightarrow x \in X \text{ and } \pi(t,x_m,s_m) \longrightarrow y \text{ as } m \longrightarrow \infty \}.$$

We define the (2)-prolongation of the point $(t,x) \in R \times X$ by the formula

$$(4.1.3) \quad D_{(2)}(t,x) := \{ y \in X: \text{there exist sequences } \{t_m\} \text{ and } \{s_m\} \text{ of real numbers such that } s_m \geq 0, \ t_m \longrightarrow t \text{ and } \pi(t_m,x,s_m) \longrightarrow y \text{ as } m \longrightarrow \infty \}.$$

We define the prolongation of the point $(t,x) \in R \times X$ by the formula

$$(4.1.4) \quad D(t,x) := \{ y \in X: \text{there exist sequences } \{x_m\} \text{ of elements of X, } \{t_m\} \text{ and } \{s_m\} \text{ of real numbers such that } s_m \geq 0, \ t_m \longrightarrow t, \ x_m \longrightarrow x \text{ and } \pi(t_m,x_m,s_m) \longrightarrow y \text{ as } m \longrightarrow \infty \}.$$

Remark 4.1.2

We see that the definition of $D(t,x)$ is an extension of that of $D^+(x)$ in dynamical systems or semi-systems. However, we should note that $D_{(1)}(t,x)$ and $D_{(2)}(t,x)$ are not to be construed in the same sense as $D^+(x)$ in dynamical systems.

(4.1.8) $\rho(\mu(t^0,x^0,s),\mu(t^0,x,s)) < \eta$

for $s > r$, $x \in X$ such that $\rho(x,x^0) < \sigma$.

Definition 4.1.3

Let (t,x) be a given point of $R \times X$. We define the (1)-prolongational limit set of the point (t,x) by the formula

(4.1.5) $J_{(1)}(t,x) := \{ y \in X$: there exist sequences $\{x_m\}$ of elements of X and $\{s_m\}$ of non-negative real numbers such that $x_m \longrightarrow x$, $s_m \longrightarrow \infty$ and $\mu(t,x_m,s_m) \longrightarrow y$ as $m \longrightarrow \infty \}$.

We define the (2)-prolongational limit set of the point (t,x) by the formula

(4.1.6) $J_{(2)}(t,x) := \{ y \in X$: there exist sequences $\{t_m\}$ and $\{s_m\}$ of real numbers such that $t_m \longrightarrow t$, $s_m \longrightarrow \infty$ and $\mu(t_m,x,s_m) \longrightarrow y$ as $m \longrightarrow \infty \}$.

We define the prolongational limit set of the point (t,x) by the formula

(4.1.7) $J(t,x) := \{ y \in X$: there exist sequences $\{x_m\}$ of elements of X, $\{t_m\}$ and $\{s_m\}$ of real numbers such that $x_m \longrightarrow x$, $t_m \longrightarrow t$, $s_m \longrightarrow \infty$ and $\mu(t_m,x_m,s_m) \longrightarrow y$ as $m \longrightarrow \infty \}$.

Remark 4.1.3

We see that the definition of $J(t,x)$ is an extension of that

[illegible] systems.

that a point $(t^o,x^o)\in R\times X$ is such that $\Lambda(t^o,x^o)$ is non-empty and compact. Then $\Lambda(t^o,x^o)$ is connected.

Theorem 4.2.1

If the point (t^o,x^o) satisfies the condition $(S)_{(1)}$, then the mapping

$$(4.2.2)\qquad X\ni x \longmapsto \Lambda(t^o,x)$$

is upper semi-continuous at the point $x^o\in X$.

Theorem 4.2.2

If the point (t^o,x^o) satisfies the condition $(S)_{(2)}$ then the mapping

$$(4.2.3)\qquad R\ni t \longmapsto \Lambda(t,x^o)$$

is upper semi-continuous at the point $t^o\in R$.

Theorem 4.2.3

If the point (t^o,x^o) satisfies the condition (S), then the mapping

$$(4.2.4)\qquad (t,x) \longmapsto \Lambda(t,x)$$

is upper semi-continuous at the point (t^o,x^o)

Proposition 4.2.4

Each of the sets $D_{(1)}(t,x)$, $D_{(2)}(t,x)$ and $D(t,x)$ is closed.

Theorem 4.2.4

For every fixed point (t^o,x^o), the mapping

$$(4.2.5)\qquad (t,x) \longmapsto D(t,x)$$

is upper semi-continuous at the point (t^o,x^o).

Corollary 4.2.2

For every fixed point (t^o,x^o) the mapping

$$X \ni x \longmapsto D_{(1)}(t^o,x) \tag{4.2.6}$$

is upper semi-continuous at x^o.

Corollary 4.2.3

For every fixed point (t^o,x^o), the mapping

$$R \ni t \longmapsto D_{(2)}(t,x^o) \tag{4.2.7}$$

is upper semi-continuous at t^o.

Proposition 4.2.5

For every $(t,x) \in R \times X$

$$\wedge[t,x] \subset D_{(1)}(t,x) \cap D_{(2)}(t,x) \cap D(t,x). \tag{4.2.8}$$

Theorem 4.2.5

Let $(t^o,x^o) \in R \times X$ be given such that the mapping

$$\mu(t^o,x^o,\cdot): R_+ \ni s \longmapsto \mu(t^o,x^o,s) \in X \tag{4.2.9}$$

is continuous. If (t^o,x^o) is stable, then $D(t^o,x^o) \subset \overline{\wedge[t^o,x^o]}$.

Theorem 4.2.6

Let (X,ρ) be locally compact, $(R,X,R_+;\mu)$ be a pseudo-process and let (t^o,x^o) be an element of $R \times X$ such that

$$\mu(t^o,x^o,s) = x^o \quad \text{for every } s \geq 0. \tag{4.2.10}$$

Assume that the mapping (4.2.9) is continuous and that there exists $\lambda > 0$ such that for every fixed $s \geq 0$ the mapping

$$(t^o-\lambda, t^o+\lambda) \ni t \longmapsto \mu(t,x^o,s) \in X \tag{4.2.11}$$

is continuous, then (t^0,x^0) satisfies the condition $(S)_{(2)}$ if and only if $D(t^0,x^0) = \overline{\pi[t^0,x^0]}$.

The above propositions 4.2.1 - 4.2.5, Theorems 4.2.1 - 4.2.6 and Corollaries 4.2.1 - 4.2.3 are due to Pelczar (1985a)

Theorem 4.2.7

Let (X,ρ) be a metric space and $(R,X,R_+;\pi)$ be a pseudo-process, then the set $J(t,x)$ is closed for any point $(t,x)\in R\times X$.

Proof:

Let $\{w_p\}$ be a sequence of points of $J(t,x)$ with

$$(4.2.12)\qquad w_p \longrightarrow w\in X \qquad \text{as } p\longrightarrow\infty .$$

For every fixed p, there exist sequences $\{x_n^p\}$ of elements of X and $\{t_n^p\}_{n=1,2,\ldots}$ in R and $\{s_n^p\}_{n=1,2,\ldots}$ in R_+ such that

$$(4.2.13)\qquad t_n^p \longrightarrow t \qquad \text{as } n\longrightarrow\infty$$

$$(4.2.14)\qquad x_n^p \longrightarrow x \qquad \text{as } n\longrightarrow\infty$$

$$(4.2.15)\qquad s_n^p \longrightarrow \infty$$

and

$$(4.2.16)\qquad \pi(t_n^p,x_n^p,s_n^p) \longrightarrow w_p \qquad \text{as } n\longrightarrow\infty .$$

For every p, we may find n_p such that

$$(4.2.17)\qquad s_{n_p}^p \geq p, \qquad p=1,2,\ldots$$

$$(4.2.18)\qquad \rho(x,x_{n_p}^p) \leq \frac{1}{p}, \qquad p=1,2,\ldots$$

$$(4.2.19)\qquad |t-t_{n_p}^p| \leq \frac{1}{p}, \qquad p=1,2,\ldots$$

and

(4.2.20) $$\rho(\mu(t^p_{n_p}, x^p_{n_p}, s^p_{n_p}), w_p) \leq \frac{1}{p}, \qquad p=1,2,\ldots .$$

Now we consider sequences $\{\sigma_p\}$, $\{h_p\}$, $\{\tau_p\}$ defined by

(4.2.21) $$\sigma_p := t^p_{n_p}, \quad h_p := x^p_{n_p}, \quad \tau_p := s^p_{n_p}$$

and then we get: $\tau_p \longrightarrow \infty$, $h_p \longrightarrow x$, $\sigma_p \longrightarrow t$ as $p \longrightarrow \infty$ and

(4.2.22) $$\mu(\sigma_p, h_p, \tau_p) \longrightarrow w \qquad \text{as } p \longrightarrow \infty .$$

To see this, from triangle inequality we get

(4.2.23) $$\rho(\mu(\sigma_p, h_p, \tau_p), w) \leq \rho(\mu(\sigma_p, h_p, \tau_p), w_p) + \rho(w_p, w)$$
$$< \frac{1}{p} + \rho(w_p, w).$$

Applying relations (4.2.12) we observe that the right hand side of (4.2.23) tends to zero as $p \longrightarrow \infty$.

Thus, we have

(4.2.24) $$w \in J(t,x) \qquad \text{(see (4.2.21))}.$$

So, the set $J(t,x)$ is closed.

Theorem 4.2.8

Each of the sets $J_{(1)}(t,x)$ and $J_{(2)}(t,x)$ is closed for any given metric space and a pseudo-process $(R,X,R_+;\mu)$.

We omit the proof of Theorem 4.2.8, because the method of proving it is analogous to that of Theorem 4.2.7.

Theorem 4.2.9

Let (X,ρ) be a metric space and $(R,X,R_+;\mu)$ be a pseudo-process. Suppose that μ is continuous on every point $(t^0,x^0,s^0) \in R \times X \times R_+$, then for every point $(t,x) \in R \times X$ we have:

(4.2.25) $\quad D(t,x) = \mu[t,x] \cup J(t,x).$

Proof:

Let $y \in D(t,x)$ for $(t,x) \in R_+ \times X$. This means that there exist sequences $\{x_m\}$ of points of X and $\{t_m\}, \{s_m\}$ of real numbers such that

(4.2.26) $\quad x_m \longrightarrow x \qquad$ as $m \longrightarrow \infty$

(4.2.27) $\quad t_m \longrightarrow t \qquad$ as $m \longrightarrow \infty$

(4.2.28) $\quad s_m \geq 0$

and

(4.2.29) $\quad \mu(t_m, x_m, s_m) \longrightarrow y \qquad$ as $m \longrightarrow \infty$.

Case (1): $\{s_m\}$ is bounded.

This implies that there exists a convergent subsequence $\{s_{n_m}\}$ of $\{s_m\}$ such that

(4.2.30) $\quad s_{n_m} \longrightarrow s \in R_+ \qquad$ as $m \longrightarrow \infty$.

By continuity axiom, we get

(4.2.31) $\quad \mu(t_m, x_m, s_{n_m}) \longrightarrow \mu(t,x,s) = y \in \mu[t,x].$

Case (2): $\{s_m\}$ is not bounded.

This implies that there exists a subsequence $\{s_{k_m}\}$ of $\{s_m\}$ such that

(4.2.32) $\quad s_{k_m} \longrightarrow \infty \qquad$ as $m \longrightarrow \infty$

and

(4.2.33) $\quad \mu(t_m, x_m, s_{k_m}) \longrightarrow y \qquad$ as $m \longrightarrow \infty$.

Relations (4.2.32) and (4.2.33) imply that

(4.2.34) $\quad y \in J(t,x)$.

Combining (4.2.31) and (4.2.34) we get

(4.2.35) $\quad D(t,x) \subset \wedge[t,x] \cup J(t,x)$.

The inverse inclusion is trivial since directly from the definition:

(4.2.36) $\quad \wedge[t,x] \subset D(t,x)$

as well as

(4.2.37) $\quad J(t,x) \subset D(t,x)$.

So

(4.2.38) $\quad \wedge[t,x] \cup J(t,x) \subset D(t,x)$.

So we have that

(4.2.39) $\quad D(t,x) = \wedge[t,x] \cup J(t,x)$.

Theorem 4.2.10.

Let (X,ρ) be a metric space and $(R,X,R_+;\wedge)$ be a pseudo-process. Suppose that $\wedge$ is continuous on every point $(t^0,x^0,s^0) \in R \times X \times R_+$ then

(4.2.40) $\quad D_{(1)}(t,x) = \wedge[t,x] \cup J_{(1)}(t,x)$

(4.2.41) $\quad D_{(2)}(t,x) = \wedge[t,x] \cup J_{(2)}(t,x)$.

The method of proving Theorem 4.2.10 follows analogously from that of Theorem 4.2.9.

Theorem 4.2.11

Let (X,ρ) be a metric space and $(R,X,R_+;\wedge)$ be a pseudo-process, then the mapping

(4.2.42) $\quad R \times X \to (t,x) \longmapsto J(t,x)$

is upper semi-continuous at the point $(t_o, x_o) \in R \times X$.

Proof:

If $\{z_m\}$, $\{x_m\}$ are sequences of elements of X and $\{t_m\}$ is a sequence of real numbers such that

(4.2.43) $\quad x_m \longrightarrow x_o \qquad$ as $m \longrightarrow \infty$

(4.2.44) $\quad t_m \longrightarrow t_o \qquad$ as $m \longrightarrow \infty$

(4.2.45) $\quad z_m \longrightarrow z_o \qquad$ as $m \longrightarrow \infty$

(4.2.46) $\quad z_m \in J(t_m, x_m) \qquad$ for $m=1,2,\ldots$

then, we have to show that

(4.2.47) $\quad z_o \in J(t_o, x_o)$.

Relations (4.2.46) mean that for every fixed m, there exist sequences $\{x_p^m\}$ in X, $\{t_p^m\}$ in R and $\{s_p^m\}$ in R_+ such that

(4.2.48) $\quad x_p^m \longrightarrow x_m \qquad$ as $p \longrightarrow \infty$

(4.2.49) $\quad t_p^m \longrightarrow t_m \qquad$ as $p \longrightarrow \infty$

(4.2.50) $\quad s_p^m \longrightarrow \infty \qquad$ as $p \longrightarrow \infty$

and

(4.2.51) $\quad f(t_p^m, x_p^m, s_p^m) \longrightarrow z_m \qquad$ as $p \longrightarrow \infty$.

For every fixed m, there exists a positive number p_m such that

(4.2.52) $\quad \rho(f(t_p^m, x_p^m, s_p^m), z_m) \leq \frac{1}{m} \qquad$ for $p \geq p_m$.

Without loss of generality, we may assume that the sequence $\{p_m\}$ is strictly increasing and we have that $p_m \longrightarrow \infty$ as $m \longrightarrow \infty$.

For every m, we can find a positive integer k_m such that

(4.2.53) $|t_m - t_p^m| \leq \frac{1}{m}$, $m=1,2,\ldots$ for $p \geq k_m$.

Moreover, for every m, we can find a number $l_m > 0$ such that

(4.2.54) $\rho(x_m, x_p^m) \leq \frac{1}{m}$, $m=1,2,\ldots$ for $p \geq l_m$.

Without loss of generality we can assume that $l_m \geq k_m$ and $l_m \geq p_m$.

Then putting

(4.2.55) $\theta_m := t_{l_m}^m$, $v_m := x_{l_m}^m$, $\lambda_m := s_{l_m}^m$, $\lambda_m \longrightarrow \infty$

as $m \longrightarrow \infty$ we get

(4.2.56) $\rho(\mu(\theta_m, v_m, \lambda_m), z_m) \leq \frac{1}{m}$; $m=1,2,\ldots$

and

(4.2.57) $|\theta_m - t_m| \leq \frac{1}{m}$, $m=1,2,\ldots$

also

(4.2.58) $\rho(x_m, v_m) \leq \frac{1}{m}$, $m=1,2,\ldots$.

Thus we get

(4.2.59) $\rho(x_o, v_m) \leq \rho(v_m, x_m) + \rho(x_m, x_o) < \frac{1}{m} + \rho(x_m, x_o)$.

Relation (4.2.59) implies that

(4.2.60) $v_m \longrightarrow x_o$ as $m \longrightarrow \infty$ (see (4.2.43))

and

(4.2.61) $\theta_m \longrightarrow t_o$.

From triangle inequality we get

(4.2.62) $\rho(\mu(\theta_m, v_m, \lambda_m), z_o) \leq \rho(\mu(\theta_m, v_m, \lambda_m), z_m) + \rho(z_m, z_o)$

$< \frac{1}{m} + \rho(z_m, z_o)$.

Using (4.2.45) on the right hand side of (4.2.62) we get

(4.2.63) $\mu(\vartheta_m, v_m, \lambda_m) \longrightarrow z_o$ as $m \longrightarrow \infty$.

Combining (4.2.55) and (4.2.63) we get

(4.2.64) $z_o \in J(t_o, x_o)$.

This completes the proof of the Theorem.

Corollary 4.2.4.

Let (X,ρ) be a metric space and $(R,X,R_+;\mu)$ be a pseudo-process, then for every fixed point (t_o,x_o) the mapping

(4.2.65) $x \longmapsto J_{(1)}(t_o,x)$

is upper semi-continuous at x_o.

Corollary 4.2.5

Let (X,ρ) be a metric space and (R,X,R_+,μ) be a FD, then for every fixed point (t_o,x_o) the mapping

(4.2.66) $t \longmapsto J_{(2)}(t,x_o)$

is upper semi-continuous at t_o.

Let $(R,X,R_+;\mu)$ be a pseudo-process with (X,ρ) being a metric space.

Definition 4.2.1

For every $(t,x) \in R \times X$ we put

(4.2.67) $\mu^{-1}[t,x] := \{ y \in X$: there is $s \leq t$ such that $\mu(t,x,t-s) = x \}$.

Definition 4.2.2

A point $(t,x) \in R \times X$ is said to be negatively quasi-stable if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

(4.2.68) $y \in B(x,\delta); |t-s| < \delta;\ z \in \mu^{-1}[s,y] \longrightarrow B(z,\varepsilon) \cap \mu^{-1}[t,x] \neq \emptyset$.

Remark 4.2.1

The condition $B(z,\varepsilon) \cap \mu^{-1}[t,x] \neq \emptyset$ is equivalent to saying that

(4.2.69) there exists $w \in \mu^{-1}[t,x]$ such that $\rho(z,w) < \varepsilon$.

Theorem 4.2.12

Let (X,ρ) be a metric space and $(R,X,R_+;\mu)$ be a pseudo-process. Assume that $(t^o,x^o) \in R \times X$ is negatively quasi-stable. Then the mapping

(4.2.70) $\quad (t,x) \longmapsto \overline{\mu^{-1}[t,x]}$

is upper semi-continuous at the point (t^o,x^o).

Proof:

Let $\{(t_n,x_n)\}$ and $\{y_n\}$ be such sequences of elements of $R \times X$ and X respectively that

(4.2.71) $\quad (t_n,x_n) \longrightarrow (t^o,x^o) \qquad$ as $n \longrightarrow \infty$

(4.2.72) $\quad y_n \longrightarrow y^o \qquad$ as $n \longrightarrow \infty$

and

(4.2.73) $\quad y_n \in \overline{\mu^{-1}\, t_n,x_n} \qquad$ for $n = 1,2,\ldots$.

We have to show that

(4.2.74) $\quad y^o \in \overline{\mu^{-1}[t^o,x^o]}$.

For every fixed n, there is a sequence $\{y_n^m\}$ of elements of $\mu^{-1}[t_n,x_n]$ such that

(4.2.75) $\quad y_n^m \longrightarrow y_n \qquad$ as $m \longrightarrow \infty$.

Since $y_n \longrightarrow y^o$, we can apply the usual method in order to find such a sequence $\{m_n\}$ of positive integers for which

$$m_n \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty$$

$$y_n^{m_n} \longrightarrow y^o \qquad \text{as } n \longrightarrow \infty$$

and

$$y_n^{m_n} \in \Lambda^{-1}[t_n, x_n] \qquad \text{for } n=1,2,\ldots .$$

So, putting

$$(4.2.76) \qquad w_n := y_n^{m_n} \qquad n=1,2,\ldots$$

we obtain

$$(4.2.77) \qquad w_n \in \Lambda^{-1}[t_n, x_n] \qquad n=1,2,\ldots .$$

Let now $\varepsilon > 0$ be arbitrarily fixed. We may (because of the negative quasi-stability) find $\delta > 0$ such that for

$$\xi \in B(x^o, \delta) \quad \text{and} \quad |t-t^o| < \delta$$

we have

$$(4.2.78) \qquad z \in \Lambda^{-1}[t, \xi] \Longrightarrow \text{there is } \nu \in \Lambda^{-1}[t^o, x^o] \cap B(z, \varepsilon/2)$$

choose n so large that

$$(4.2.79) \qquad \varrho(x^o, x_n) < \delta \quad \text{and} \quad |t^o - t_n| < \delta \qquad \text{for } n > n_o$$

and

$$(4.2.80) \qquad \varrho(w_n, y^o) < \frac{\varepsilon}{2} \qquad \text{for } n > n_o .$$

The relations (4.2.77) mean that for every n there exists s_n such that

$$s_n \leq t_n$$

and

$$(4.2.81) \qquad \mu(s_n, w_n, t_n - s_n) = x_n .$$

For $n \geq n_o$ we can find (see (4.2.79))

$$(4.2.82) \qquad z_n \in \Lambda^{-1}[t^o, x^o]$$

$$m_n \longrightarrow \infty \qquad \text{as } n \longrightarrow \infty$$

$$y_n^{m_n} \longrightarrow y^o \qquad \text{as } n \longrightarrow \infty$$

and

$$y_n^{m_n} \in \Lambda^{-1}[t_n, x_n] \qquad \text{for } n=1,2,\dots .$$

So, putting

(4.2.76) $$w_n := y_n^{m_n} \qquad n=1,2,\dots$$

we obtain

(4.2.77) $$w_n \in \Lambda^{-1}[t_n, x_n] \qquad n=1,2,\dots .$$

Let now $\varepsilon > 0$ be arbitrarily fixed. We may (because of the negative quasi-stability) find $\delta > 0$ such that for

$$\xi \in B(x^o, \delta) \quad \text{and} \quad |t-t^o| < \delta$$

we have

(4.2.78) $$z \in \Lambda^{-1}[t, \xi] \Longrightarrow \text{there is } w \in \Lambda^{-1}[t^o, x^o] \cap B(z, \varepsilon/2)$$

choose n so large that

(4.2.79) $$\rho(x^o, x_n) < \delta \quad \text{and} \quad |t^o - t_n| < \delta \qquad \text{for } n > n_o$$

and

(4.2.80) $$\rho(w_n, y^o) < \frac{\varepsilon}{2} \qquad \text{for } n > n_o .$$

The relations (4.2.77) mean that for every n there exists s_n such that

$$s_n \leq t_n$$

and

(4.2.81) $$\mu(s_n, w_n, t_n - s_n) = x_n .$$

For $n \geq n_o$ we can find (see (4.2.79))

(4.2.82) $$z_n \in \Lambda^{-1}[t^o, x^o]$$

such that

(4.2.83) $\rho(z_n, w_n) < \frac{\varepsilon}{2}$.

By virtue of (4.2.80) and (4.2.82) we get

(4.2.84) $\rho(z_n, y^o) < \varepsilon$ for $n \geq n_o$.

So, we have proved, that for every fixed $\varepsilon > 0$ and every $n \geq n_o$ there exists a point $z \in \pi^{-1}[t^o, x^o]$ (being of the form $z = z_n$ for a suitable n, for instance $n = n_o$) such that

(4.2.85) $\rho(z, y^o) < \varepsilon$.

So, in particular, for every $\varepsilon > 0$ there is z (being of the form $z = z_{n_o}$ (see (4.2.82)) belonging to

$$B(y^o, \varepsilon) \cap \overline{\pi^{-1}[t^o, x^o]}.$$

This means that (4.2.74) holds true. The proof is completed.

§ 4.3.

In this section we state some properties that are preserved passing to the limits of pseudo-processes.

Theorem 4.3.1

Let (X, ρ) be a metric space. Assume that $(R, X, R_+; \pi_n)$ $(n=1,2,\ldots$ is a pseudo-process such that for every fixed $t \in R$ and $s \in R_+$ the mapping

(4.3.1) $x \longmapsto \pi_n(t, x, s)$

is continuous.

Suppose that the sequence $\{\pi_n\}$ is convergent to a mapping

$$\pi : R \times X \times R_+ \longrightarrow X$$

uniformly. Then $(R, X, R_+; \pi)$ is a pseudo-process and the mapping

(4.3.2) $x \longmapsto \pi(t, x, s)$

is for every fixed $t \in R$, $s \in R_+$ continuous.

Proof:

Observe first of all that the continuity of (4.3.2) follows directly from the classical Theorem on the continuity of the limit of uniformly convergent sequences of continuous mappings.

For every n we have

$$(4.3.3) \qquad \mu_n(t,x,0) = x \qquad \text{for } x \in X,\ t \in R.$$

Passing to the limit as $n \longrightarrow \infty$ we get immediately

$$(4.3.4) \qquad \mu(t,x,0) = x \qquad \text{for } x \in X,\ t \in R.$$

For every n, we have also

$$(4.3.5) \qquad \mu_n(t+r, \mu_n(t,x,r), s) = \mu_n(t,x,s+r)$$

for $x \in X$, $t \in R$, $s,r \in R_+$.

We have for every fixed t,x,s and n the inequality

$$(4.3.6) \quad \begin{cases} \rho(\mu_n(t+r,\mu_n(t,x,r),s), \mu(t+r,\mu(t,x,r),s)) \leq \\ \rho(\mu_n(t+r,\mu_n(t,x,r),s), \mu(t+r,\mu_n(t,x,r),s)) \\ + \rho(\mu(t+r,\mu_n(t,x,r),s), \mu(t+r,\mu(t,x,r),s)). \end{cases}$$

Let $\varepsilon > 0$ be fixed. Find n^0 so large that

$$\rho(\mu_n(\tau,\xi,\sigma), \mu(\tau,\xi,\sigma)) < \frac{\varepsilon}{2} \qquad \text{for } n \geq n^0$$

for every τ, ξ, σ.

Thus the first member of the right hand of (4.3.6) is estimated by $\frac{\varepsilon}{2}$ for $n \geq n^0$.

Inorder to estimate the second member find $\delta > 0$ so small that

$$(4.3.7) \qquad \rho(\mu(t+r,y,s), \mu(t+r,\mu(t,x,r),s)) < \frac{\varepsilon}{2}$$

for $y \in B(\mu(t,x,r),\delta)$. Such a δ exists because of the continuity of the mapping (4.3.2) (in particular at the point $\mu(t,x,r)$).

We can find now n_1 so large that

(4.3.8) $\rho(\mu_n(t,x,r),\mu(t,x,r)) < \delta$ for $n \geq n_1$.

The conditions (4.3.7) and (4.3.8) give

$$\rho(\mu(t+r,\mu_n(t,x,r),s),\mu(t+r,\mu(t,x,r),s)) < \frac{\varepsilon}{2} \quad \text{for } n \geq n_1.$$

Thus

$$\rho(\mu_n(t+r,\mu_n(t,x,r),s),\mu(t+r,\mu(t,x,r),s)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for $n \geq \max(n^0, n_1)$.

So, we have proved that

$$\mu_n(t+r,\mu_n(t,x,r),s) \longrightarrow \mu(t+r,\mu(t,x,r),s)$$

as $n \longrightarrow \infty$. Since we have also clearly

$$\mu_n(t,x,s+r) \longrightarrow \mu(t,x,s+r) \qquad \text{as } n \longrightarrow \infty,$$

we get (see (4.3.5)) the equality

(4.3.9) $\mu(t+r,\mu(t,x,r),s) = \mu(t,x,s+r)$ for $x \in X$, $t \in R$; $s,r \in R_+$

We have proved that $(R,X,R_+;\mu)$ is a pseudo-process

Remark 4.3.1

If μ_n $(n=1,2,\ldots)$ are continuous, then μ is obviously continuous.

Theorem 4.3.2

Let (X,ρ) be a metric space and let $(R,X,R_+;\mu_n)$ $(n=1,2,\ldots)$ be pseudo-processes. Assume that $\{\mu_n\}$ converges to μ uniformly. If a point $(t^0,x^0) \in R \times X$ satisfies in every pseudo-process $(R,X,R_+;\mu_n)$ the condition $(S)_{(i)}$ or (S) or $S_{(i)}$ or S $(i=1,2)$, then this point satisfies the same condition $(S)_{(i)}$ or (S) or $S_{(i)}$ $(i=1,2)$ or - respectively - S in the pseudo-process $(R,X,R_+;\mu)$.

Proof:

Suppose that (t^0,x^0) satisfies the condition $(S)_{(1)}$ in every pseudo-process $(R,X,R_+;\mu_n)$ $(n=1,2,\ldots)$.

This means that

$$(4.3.10)\quad \begin{cases} \forall \eta > 0 \ \exists \sigma_n > 0 \ \exists r_n \geq 0: \\ (s \geq r_n, \rho(x,x^0) < \sigma_n) \Longrightarrow \rho(\mu_n(t^0,x^0,s), \mu_n(t^0,x,s)) < \eta \end{cases}$$

Let $\theta > 0$ be fixed and let us take $\eta = \frac{\theta}{3}$. Find n^0 so large that

$$(4.3.11)\quad \rho(\mu_n(t,x,s), \mu(t,x,s)) < \eta \quad \text{for } n \geq n^0 \text{ and for every } t,x,s.$$

Consider now $n = n^0$ and find σ_{n^0} and r_{n^0} according to (4.3.10). So, we have

$$(4.3.12)\quad \rho(\mu_{n^0}(t^0,x^0,s), \mu_{n^0}(t^0,x,s)) < \eta$$

for $x \in B(x^0, \sigma_{n^0})$ and $s \geq r_{n^0}$.

From the triangle inequality we get

$$(4.3.13)\quad \begin{cases} \rho(\mu(t^0,x^0,s), \mu(t^0,x,s)) \leq \rho(\mu(t^0,x^0,s), \mu_{n^0}(t^0,x^0,s)) + \\ \quad + \rho(\mu_{n^0}(t^0,x^0,s), \mu_{n^0}(t^0,x,s)) + \\ \quad + \rho(\mu_{n^0}(t^0,x,s), \mu(t^0,x,s)) \end{cases}$$

and so (see (4.3.12) and (4.3.12)) we obtain

$$(4.3.14)\quad \rho(\mu(t^0,x^0,s), \mu(t^0,x,s)) < 3\eta = \theta$$

for $s \geq r_{n^0}$ and x such that $\rho(x,x^0) < \sigma_{n^0}$.

So, for every $\theta > 0$ we can find $r\ (= r_{n^0})$ and $\sigma\ (= \sigma_{n^0})$ such that

$$(4.3.15)\quad \rho(\mu(t^0,x^0,s), \mu(t^0,x,s)) < \theta$$

if $s \geq r$ and $\rho(x,x^0) < \sigma$.

This means that (t^0,x^0) satisfies the condition $(S)_{(1)}$ in $(R,X,R_+;\mu)$.

The same method is used in the proof of the other implications:

$(S)_{(2)}$ in $(R,X,R_+;\mu_n) \Longrightarrow (S)_{(2)}$ in $(R,X,R_+;\mu)$

(S) in $(R,X,R_+;\mu_n) \Longrightarrow (S)$ in $(R,X,R_+;\mu)$

$S_{(1)}$ in $(R,X,R_+;\mu_n) \Longrightarrow S_{(1)}$ in $(R,X,R_+;\mu)$

$S_{(2)}$ in $(R,X,R_+;\mu_n) \Longrightarrow S_{(2)}$ in $(R,X,R_+;\mu)$

and

S in $(R,X,R_+;\mu_n) \Longrightarrow S$ in $(R,X,R_+;\mu)$.

So, we omit details.

We say that a pseudo-process $(R,X,R_+;\mu)$ is periodic if there exists $T>0$ such that

(4.3.16) $\mu(t+T,x,s) = \mu(t,x,s)$ for $t\in R$, $x\in X$, $s\geq 0$.

Every $T>0$ fulfilling the condition (4.3.16) is called a period of μ.

Theorem 4.3.3

Let (X,ρ) be a metric space. Suppose that $(R,X,R_+;\mu_m)$ $(m=1,2,\ldots)$ and $(R,X,R_+;\mu)$ are pseudo-processes. Assume that every pseudo-process $(R,X,R_+;\mu_m)$ is periodic and that there exist $\tilde{T}$, $\tilde{\tilde{T}}>0$, $\tilde{\tilde{T}}>\tilde{T}$ such that for every m there is a period T_m of μ_m such that

(4.3.17) $\tilde{T}\leq T_m\leq \tilde{\tilde{T}}$.

Assume moreover that for every fixed $(x,s)\in X\times R_+$ the mapping

(4.3.18) $t \longmapsto \mu(t,x,s)$

is continuous at every point $t\in R$. If the sequence $\{\mu_m\}$ is convergent to μ pointwise in (x,s) and uniformly in t in every bounded interval of the form $[t^*,t^*+a]$, (which means that $\forall \varepsilon>0\ \forall (x,s)\in X\times R_+\ \forall t\in R\ \forall a>0\ \exists m_0$ such that $m\geq m_0$, $t\in[t^*,t^*+a] \Longrightarrow \rho(\mu_m(t,x,s),\mu(t,x,s))<\varepsilon$) then the pseudo-process $(R,X,R_+;\mu)$ is periodic and has a period $T\in[\tilde{T},\tilde{\tilde{T}}]$.

Proof:

Consider periods T_m of μ_m for $m=1,2,\dots$. Because of the condition (4.3.17) we may assume without loss of generality, that the sequence $\{T_m\}$ is convergent to some $T \in [\tilde{T}, \tilde{\tilde{T}}]$.

We have: for a fixed $(x,s) \in X \times R_+$ and $t \in R$,

$$(4.3.19)\quad \begin{cases} \rho(\mu(t+T,x,s);\mu(t,x,s) \le \rho(\mu(t+T,x,s),\mu(t+T_m,x,s)) + \\ \qquad \rho(\mu(t+T_m,x,s),\mu_m(t+T_m,x,s)) + \\ \qquad \rho(\mu_m(t+T_m,x,s),\mu(t,x,s)) \end{cases}$$

Let $\eta > 0$ be arbitrarily fixed. We can find $\delta > 0$ such that

$$(4.3.20)\quad \rho(\mu(t+T,x,s),\mu(\sigma,x,s)) < \eta/3$$

for $|\sigma-(t+T)| < \delta$.

This is possible because of the continuity of the mapping (4.3.18).

Now, we can find m_o such that

$$(4.3.21)\quad |t+T-(t+T_m)| < \delta \qquad \text{for } m \ge m_o .$$

Thus the first member of the right hand side of (4.3.19) is estimated by $\eta/3$ if $m \ge m_o$.

The second member of the right hand side of (4.3.19) is estimated by $\eta/3$ for m sufficiently large, say for $m \ge m_1$ (because of the uniform convergence of $\{\mu_m\}$ to μ in t in bounded intervals).

Finally we observe that

$$\mu_m(t+T_m,x,s) = \mu_m(t,x,s)$$

and so, the third member of the right hand side of of (4.3.19) is estimated by $\eta/3$ for m sufficiently large, say for $m \ge m_2$.

Putting

$$(4.3.22)\quad m^* = \max(m_o, m_1, m_2)$$

we obtain

$$(4.3.23)\quad \rho(\mu(t+T,x,s),\mu(t,x,s)) < \eta .$$

Since $\eta > 0$ is arbitrarily fixed, we have

(4.3.24) $\rho(\mu(t+T,x,s),\mu(t,x,s)) = 0$

which means that

(4.3.25) $\mu(t+T,x,s) = \mu(t,x,s)$.

The proof is completed.

Theorem 4.3.4

Let (X,ρ) be a metric space and $(R,X,R_+;\mu_m)$, $(m=1,2,\ldots)$, $(R,X,R_+;\mu)$ be pseudo-processes such that the sequence $\{\mu_m\}$ converges uniformly to μ. Let the point (t_o,x_o) satisfy the condition S in the pseudo-processes $(R,X,R_+;\mu)$. Suppose the sequences $\{w_m\}$, $\{x_m\}$ in X and $\{t_m\}$ in R are such that $x_m \longrightarrow x_o \in X$, as $m \longrightarrow \infty$, $w_m \longrightarrow w \in X$ as $m \longrightarrow \infty$ $t_m \longrightarrow t_o$ and $w_m \in \Lambda_m(t_m,x_m)$ for $m=1,2,\ldots$, then $w \in \Lambda(t,x)$ (where $\Lambda_m(t,x)$ denotes the limit set of (t,x) in $(R,X,R_+;\mu_m)$).

Proof:

The relation $w_m \in \Lambda_m(t_m,x_m)$ for every fixed m, means that for every fixed m, there exists a sequence $\{\lambda_m^n\}$ of real numbers such that $\lambda_m^n \longrightarrow \infty$ as $n \longrightarrow \infty$ and

(4.3.26) $\mu_m(t_m,x_m,\lambda_m^n) \longrightarrow w_m$ as $n \longrightarrow \infty$.

For every fixed m, there exists a positive number n_m such that

(4.3.27) $\rho(\mu_m(t_m,x_m,\lambda_m^n),w_m) \leq \frac{1}{m}$ for $n \geq n_m$.

For every m, we can find a positive number k_m such that

(4.3.28) $\rho(\mu_m(t_m,x_m,\lambda_m^{k_m}),w_m) \leq \frac{1}{m}$, $m=1,2,\ldots$

and moreover we require that $k_m \geq n_m$.

It is assumed that $\{\mu_m\}$ is convergent to μ uniformly. This means that

$$\forall \varepsilon > 0 \quad \exists m^o :$$

$$(4.3.29)\qquad \rho(\mu_m(t_m,x_m,\lambda_m^{k_m}),\mu(t_m,x_m,\lambda_m^{k_m}))<\frac{\varepsilon}{4}\quad \text{for } m\geq m^0$$

$$\text{for } (t_m,x_m\lambda_m^{k_m})\in R\times X\times R_+.$$

We choose σ and δ in such a way that

$$(4.3.30)\qquad \rho(\mu(t,x,\lambda_m^{k_m}),\mu(t_0,x_0,\lambda_m^{k_m}))<\frac{\varepsilon}{4}$$

for $t\in(t_0-\delta,t_0+\delta)$ and x such that $\rho(x,x_0)<\sigma$. Such a choice is possible because the condition S is assumed for the point (t_0,x_0) with respect to PP $(R,X,R_+;\mu)$.

We can assume that the sequence $\{k_m\}$ is increasing as $m\longrightarrow\infty$ and then we choose m_1 such that

$$(4.3.31)\qquad |t-t_m|<\delta$$

$$(4.3.32)\qquad \rho(w,w_m)<\frac{\varepsilon}{4}$$

$$(4.3.33)\qquad \rho(x_0,x_m)<\sigma$$

$$(4.3.34)\qquad \lambda_m^{k_m}>m$$

and

$$(4.3.35)\qquad \frac{1}{m}<\frac{\varepsilon}{4}\quad \text{for}\quad m\geq m_1$$

from triangle inequality we get

$$(4.3.36)\qquad \rho(\mu(t_0,x_0,\lambda_m^{k_m}),w)\leq \rho(\mu_m(t_m,x_m,\lambda_m^{k_m}),w_m)+$$

$$+\rho(\mu_m(t_m,x_m,\lambda_m^{k_m}),\mu(t_m,x_m,\lambda_m^{k_m}))+$$

$$+\rho(\mu(t_m,x_m,\lambda_m^{k_m}),\mu(t_0,x_0,\lambda_m^{k_m}))+\rho(w,w_m).$$

Applying relations (4.3.28)-(4.3.30) and (4.3.32) we observe that the left hand side of (4.3.36) is estimated by

$$\frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \rho(w_n, w) = \frac{3}{4} + \rho(w_n, w) \leq \varepsilon \qquad \text{for } m \geq m_1$$

So, we get

(4.3.37) $\mu(t_o, x_o, \lambda_m^{k_m}) \longrightarrow w$ as $m \longrightarrow \infty$

(since $\rho(\mu_m(t_m, x_m, \lambda_m^{k_m}), w_m) \longrightarrow 0$).

Combining (4.3.34) with (4.3.37) we get

(4.3.38) $w \in \Lambda(t_o, x_o)$.

This completes the proof.

Corollary 4.3.1

Let (X, ρ) be a metric space and $(R, X, R_+; \mu_m)$ $(m=1,2,\ldots)$, $(R, X, R_+; \mu)$ be PD. Suppose the sequence $\{\mu_m\}$ converges to μ uniformly and the sequence $\{b_m\}$ in X is such that $b_m \longrightarrow b \in X$ as $m \longrightarrow \infty$ and $b_m \in \Lambda_m(t, x)$ for $m=1,2,\ldots$, then $b \in \Lambda(t, x)$.

Corollary 4.3.2

Let (X, ρ) be a metric space and $(R, X, R_+; \mu_m)$ $(m=1,\ldots)$, $(R, X, R_+; \mu)$ be PD such that the sequence $\{\mu_m\}$ converges to μ uniformly. Suppose the point (t_o, x_o) satisfies the condition $(S)_{(2)}$ and the sequences $\{z_m\}$ in X and $\{t_m\}$ in R are such that $z_m \longrightarrow z_o \in X$, $t_m \longrightarrow t_o$, $z_m \in \Lambda_m(t_m, x_o)$ for $m=1,2,\ldots$; then $z_o \in \Lambda(t_o, x_o)$.

Corollary 4.3.3

Let (X, ρ) be a metric space with $(R, X, R_+; \mu_m)$ $(m=1,2,\ldots)$ and $(R, X, R_+; \mu)$ being PP such that the sequence $\{\mu_m\}$ converges to μ uniformly. Assume that the point (t_o, x_o) satisfies the condition $(S)_{(1)}$ and the sequence $\{x_m\}$ and $\{y_m\}$ in X are such that $x_m \longrightarrow x_o \in X$, $y_m \longrightarrow y_o \in X$, $y_m \in \Lambda_m(t_o, x_m)$ for $m=1,2,\ldots$, then $y_o \in \Lambda(t_o, x_o)$.

Remark 4.3.1

If in Theorem 4.3.4 and in corollaries 4.3.2 and 4.3.3 we assume that $(R, X, R_+; \mu_m) = (R, X, R_+; \mu)$ for every $m \in N$ which means

that $\mu_m = \mu$ for every m, then we have the same results as in Theorems 4.2.3, 4.2.2 and 4.2.1 respectively which means that the mappings

(4.3.39) $\quad (t,x) \longmapsto \Lambda(t,x)$

(4.3.40) $\quad t \longmapsto \Lambda(t,x_o)$

and

(4.3.41) $\quad x \longmapsto \Lambda(t_o,x)$

are upper semi-continuous at the point (t_o,x_o) and at $t \in R$, $x \in R$ respectively.

Theorem 4.3.5

Let (X,ρ) be a metric space with $(R,X,R_+;\mu_p)$ $(p=1,2,\ldots)$ and $(R,X,R_+;\mu)$ being PD. Assume that the sequence $\{\mu_p\}$ converges uniformly to μ. Suppose that the sequences $\{y_p\},\{z_p\}$ in X and $\{t_p\}$ in R are such that $z_p \longrightarrow z \in X$, $y_p \longrightarrow y \in X$, $t_p \longrightarrow t \in R$ and $z_p \in D^p(t_p,y_p)$ for $p=1,2,\ldots$; then $z \in D(t,y)$ (where $D^p(t,y)$ denotes the prolongation of (t,y) in $(R,X,R_+;\mu_p)$).

Proof:

The relation $z_p \in D^p(t_p,y_p)$ for every $p \in N$ means that for every fixed p there exist sequences $\{y_p^r\}$ in X, $\{t_p^r\}$ in R and $\{s_p^r\}$ in R_+ such that

(4.3.42) $\quad y_p^r \longrightarrow y_p \qquad$ as $r \longrightarrow \infty$

(4.3.43) $\quad t_p^r \longrightarrow t_p \qquad$ as $r \longrightarrow \infty$

(4.3.44) $\quad s_p^r \geq 0$

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and

(4.3.45) $\quad \mu_p(t_p^r,y_p^r,s_p^r) \longrightarrow z_p \qquad$ as $r \longrightarrow \infty$.

For every fixed p, there exists a positive number r_p such that

(4.3.46) $\quad \rho(\mu_p(t_p^r,y_p^r,s_p^r),z_p) \leq \frac{1}{p} \qquad$ for $r \geq r_p$.

So, in particular

$$(4.3.47)\qquad \rho(\mu_p(t_p^{r_p}, y_p^{r_p}, s_p^{r_p}), z_p) \le \frac{1}{p}\,, \qquad p=1,2,\ldots$$

We can assume that $r_p \longrightarrow \infty$ and that $t_p^{r_p} \longrightarrow t$, $y_p^{r_p} \longrightarrow y$ as $p \longrightarrow \infty$.

So, putting

$$(4.3.48)\qquad \alpha_p := t_p^{r_p}, \quad w_p := y_p^{r_p}; \quad \lambda_p := s_p^{r_p}$$

we have

$$(4.3.49)\qquad \lambda_p \ge 0 \text{ and } \alpha_p \longrightarrow t, \quad w_p \longrightarrow y \text{ as } p \longrightarrow \infty.$$

Then relation (4.3.46) becomes

$$(4.3.50)\qquad \rho(\mu_p(\alpha_p, w_p, \lambda_p), z_p) \le \frac{1}{p}, \quad p=1,2,\ldots\,.$$

The sequence $\{\mu_p\}$ is assumed to be convergent to μ uniformly. So, if we fix $\gamma > 0$, then we can find $p_1 \in N$ such that

$$(4.3.51)\qquad \rho(\mu_p(s,y,\lambda), \mu(s,y,\lambda)) \le \frac{\gamma}{3} \qquad \text{for } p \ge p_1$$

It is assumed that $z_p \longrightarrow z$ as $p \longrightarrow \infty$. Thus we can find a positive number $p_2 \in N$ such that

$$(4.3.52)\qquad \rho(z, z_p) < \frac{\gamma}{3} \qquad \text{for } p \ge p_2.$$

Let now $\tilde{p}$ be such that $\frac{1}{\tilde{p}} \le \frac{\gamma}{3}$ and $\tilde{p} \ge \max(p_1, p_2)$.

From triangle inequality we get

$$(4.3.53)\qquad \rho(\mu(\alpha_p, w_p, \lambda_p), z) \le \rho(\mu(\alpha_p, w_p, \lambda_p), z_p) + \rho(\mu_p(\alpha_p, w_p, \lambda_p), \mu(\alpha_p, w_p, \lambda_p)) + \rho(z, z_p).$$

On using (4.3.50) - (4.3.52) on the right hand side of (4.3.53) we get for $p \ge \tilde{p}$.

$$(4.3.54)\qquad \rho(\mu(\alpha_p, w_p, \lambda_p), z) \le \gamma$$

Thus we have proved that $\mu(\alpha_p, w_p, \lambda_p) \longrightarrow z$ as $p \longrightarrow \infty$. So. $z \in D(t,y)$.

Remark 4.3.2

If in the above Theorem 4.3.5, we assume that $(R,X,R_+;\mu_p) = (R,X,R_+;\mu)$ for every $p\in N$ which means that $\mu_p = \mu$ for every p and so $D^p(t_p,y_p) = D(t_p,y_p)$ for every p, then we have the same result as in Theorem 4.2 which means that the mapping

$$(4.3.55) \qquad (t,y) \longmapsto D(t,y)$$

is upper semi-continuous at the point $(t_o,x_o)\in R\times X$.

Theorem 4.3.6

Let (X,ρ) be a metric space, $(R,X,R_+;\mu_p)$ $(p=1,2,\ldots)$ and $(R,X,R_+;\mu)$ be pseudo-process such that $\{\mu_p\}$ tends to μ uniformly. Denote by $B^p_{(1)}(t,y)$ (respectively $D_{(1)}(t,y)$) the corresponding prolongations (see Definition 4.1.2) of (t,y) in the p^{th} PP $(R,X,R_+;\mu_p)$ and $(R,X,R_+;\mu)$ respectively. Let $(t,y)\in R\times X$ be given.

Assume that μ is continuous at t uniformly with respect to the second and third variables:

$$(4.3.56) \qquad \forall\varepsilon>0 \ \exists\delta>0: |t-s|<\delta \Longrightarrow \rho(\mu(t,v,r),\mu(s,v,r))\le\varepsilon \quad \text{for every } (v,r)\in X\times R_+.$$

Suppose that sequences $\{y_p\}$ and $\{z_p\}$ of elements of X are convergent to y and z respectively and that $\{t_p\}$ is a sequence of real numbers convergent to t. If $z_p\in D^p_{(1)}(t_p,y_p)$, $p=1,2,\ldots$; then $z\in D_{(1)}(t,y)$.

Proof:

The relations $z_p\in D^p_{(1)}(t_p,y_p)$ for every p means that for every p there are sequences $\{y^k_p\}$ in X and $\{s^k_p\}$ in R_+ such that

$$(4.3.57) \qquad y^k_p \longrightarrow y_p \qquad \text{as } k\longrightarrow\infty$$

and

$$(4.3.58) \qquad \mu_p(t_p,y^k_p,s^k_p) \longrightarrow z_p \qquad \text{as } k\longrightarrow\infty.$$

The relations (3) permit us to say that for every η and every p there is a k_p such that

$$\rho(\mu_p(t_p,y_p^k,s_p^k),z_p)<\eta \qquad \text{for } k\geq k_p.$$

So in particular for every p there is m_p such that

$$(4.3.59) \qquad \rho(\mu_p(t_p,y_p^{m_p},s_p^{m_p}),z_p)<\frac{1}{p} \qquad (p=1,2,\ldots).$$

Without loss of generality we may assume that

$$(4.3.60) \qquad m_p \longrightarrow \infty \qquad \text{as } p \longrightarrow \infty$$

By using the standard method, we can prove that there is a subsequence of the sequence $\{y_p^{m_p}\}$ that converges to y.

So, without loss of generality we may require that putting

$w_p := y_p^{m_p}$, $r_p := s_p^{m_p}$ we get

$$(4.3.61) \qquad w_p \longrightarrow y \qquad \text{as } p \longrightarrow \infty$$

$$(4.3.62) \qquad r_p \geq 0 \qquad \text{for every p}$$

and (see (4.3.59))

$$(4.3.63) \qquad \rho(\mu_p(t_p,w_p,r_p),z_p)<\frac{1}{p}.$$

We have now

$$(4.3.64) \quad \begin{cases} \rho(\mu(t,w_p,r_p),z) \leq \rho(\mu(t,w_p,r_p),\mu(t_p,w_p,r_p)) + \\ \qquad + \rho(\mu(t_p,w_p,r_p),\mu_p(t_p,w_p,r_p)) + \\ \qquad + \rho(\mu_p(t_p,w_p,r_p),z_p) + \rho(z,z_p). \end{cases}$$

Let $\varepsilon>0$ be given. We find δ such that

$$(4.3.65) \qquad \rho(\mu(t,v,r),\mu(s,v,r))<\frac{\varepsilon}{4}$$

for $|t-s|<\delta$ and then we find p_o such that

(4.3.66) $|t-t_p| < \delta$ for $p \geq p_0$.

Because of the uniform convergence of $\{\mu_p\}$ to μ we can find p_1 such that

(4.3.67) $\rho(\mu(t_p,w_p,r_p),\mu_p(t_p,w_p,r_p)) < \frac{\varepsilon}{4}$ for $p \geq p_1$.

From (4.3.63) it follows that we can find p_2 such that

(4.3.68) $\rho(\mu_p(t_p,w_p,r_p),z_p) < \frac{\varepsilon}{4}$ for $p \geq p_2$.

(It is enough to find p_2 such that $\frac{1}{p_2} < \frac{\varepsilon}{4}$).

Finally, we can find p_3 such that

(4.3.69) $\rho(z,z_p) < \frac{\varepsilon}{4}$ for $p \geq p_3$,

(since $z_p \longrightarrow z$ as $p \longrightarrow \infty$).

Taking now

$$p^* = \max(p_0,p_1,p_2,p_3)$$

we get by (4.3.64) to (4.3.69)

(4.3.70) $\rho(\mu(t,w_p,r_p),z) < \varepsilon$ for $p \geq p^*$.

We have proved that

$$\mu(t,w_p,r_p) \longrightarrow z \qquad \text{as } p \longrightarrow \infty$$

and so (see (4.3.61) and (4.3.62)) we obtain

$$z \in D_{(1)}(t,y).$$

The proof is completed.

Theorem 4.3.7

Suppose that all assumptions of Theorem 4.3.6 are satisfied with only one change: in the place of the condition (4.3.56) we assume that

(4.3.71) $\forall \varepsilon>0\ \exists \delta>0: \rho(x,z)<\delta \Longrightarrow \rho(\mu(s,x,r),\mu(s,z,r))<\varepsilon$ for every $(s,r)\in R\times R_+$

and with the natural extension of the notation: $D^p_{(2)}(t,y)$ and $D_{(2)}(t,y)$ denote the appropriate prolongations in $(R,X,R_+;\mu_p)$ and $(R,X,R_+;\mu)$ respectively.

If

(4.3.72) $z_p\in D^p_{(2)}(t_p,y_p)$,

then

(4.3.73) $z\in D_{(2)}(t,y)$.

Proof:

The proof is similar to that of Theorem 4.3.6 with only one change: we use the condition (4.3.71) instead of (4.3.56) in order to obtain from a suitable inequality (analogous to (4.3.64)) the required condition saying that (4.3.73) holds true.

Concerning the discussion of prolongational limit sets (see Definition 4.1.3) we shall adopt the same natural convention: for a given (t,x) we shall denote by $J^p_{(1)}(t,x)$, $J^p_{(2)}(t,x)$ and $J^p(t,x)$ ($J_{(1)}(t,x)$, $J_{(2)}(t,x)$, $J(t,x)$) the corresponding prolongational limit set in the p^{th} pseudo-process $(R,X,R_+;\mu_p)$ (in $(R,X,R_+;\mu)$ respectively).

Theorem 4.3.8

Let (X,ρ) be a metric space and let $(R,X,R_+;\mu_p)$ $(p=1,2,\dots)$ and $(R,X,R_+;\mu)$ be pseudo-processes such that the sequence $\{\mu_p\}$ tends to μ uniformly. Let $(t,y)\in R\times X$ be fixed. Suppose sequences $\{y_p\},\{z_p\}$ in X be such that $y_p\longrightarrow y$ and $z_p\longrightarrow z$ as $p\longrightarrow\infty$. If $z_p\in J^p_{(1)}(t,y_p)$ for $p=1,2,\dots$, then $z\in J_{(1)}(t,y)$.

Proof:

The relation $z_p\in J^p_{(1)}(t,y_p)$ for every $p\in N$ means that for every p, there exist sequences $\{y^k_p\}$ in X and $\{\sigma^k_p\}$ in R_+ such that

(4.3.74) $$y_p^k \longrightarrow y_p \qquad \text{as } k \longrightarrow \infty$$

(4.3.75) $$\sigma_y^k \longrightarrow \infty \qquad \text{as } k \longrightarrow \infty$$

and

(4.3.76) $$\mu_p(t, y_p^k, \sigma_p^k) \longrightarrow z_p \qquad \text{as } k \longrightarrow \infty .$$

So, for every p there is q_p such that

(4.3.77) $$\rho(\mu_p(t, y_p^{q_p}, \sigma_p^{q_p}), z_p) < \frac{1}{p} .$$

Without loss of generality we may assume that $q_p \longrightarrow \infty$ as $p \longrightarrow \infty$ and (by using standard reasoning applied already several times above) there are subsequences of $\{y_p^{q_p}\}$ and $\{\sigma_p^{q_p}\}$ tending to y and to ∞ respectively. So, we may assume without loss of generality that putting

(4.3.78) $$v_p := y_p^{q_p} \quad \text{and} \quad \tau_p := \sigma_p^{q_p}$$

we get

(4.3.79) $$v_p \longrightarrow y, \qquad \tau_p \longrightarrow \infty$$

and

(4.3.80) $$\rho(\mu_p(t, v_p, \tau_p), z_p) < \frac{1}{p} \qquad \text{for every } p.$$

Let now $\varepsilon > 0$ be fixed. We can find p_o such that

(4.3.81) $$\rho(\mu_p(t,g,s), \mu(t,g,s)) < \frac{\varepsilon}{3} \quad \text{for } p \geq p_o$$

(This is possible because of the uniform convergence of $\{\mu_p\}$ to μ).

We now find p_1 such that

(4.3.82) $$\rho(z_p, z) < \frac{\varepsilon}{3} \quad \text{and} \quad \frac{1}{p} < \frac{\varepsilon}{3} \quad \text{for } p \geq p_1$$

and take

$$\hat{p} := \max(p_o, p_1).$$

We have by triangle inequality

$$\rho(\mu(t,v_p,\tau_p),z) \le \rho(\mu(t,v_p,\tau_p),\mu_p(t,v_p,\tau_p)) + \rho(\mu_p(t,v_p,\tau_p),z_p) + \rho(z,z_p) \tag{4.3.83}$$

which gives (see (4.3.81) and (4.3.82)) for $p \ge \hat{p}$ the estimation

$$\rho(\mu(t,v_p,\tau_p),z) \le \varepsilon . \tag{4.3.84}$$

We have proved that

$$\mu(t,v_p,\tau_p) \longrightarrow z$$

and so by virtue of (4.3.79) we get

$$z \in J_{(1)}(t,y). \tag{4.3.85}$$

This completes the proof.

Remark 4.3.3

In the case $\mu_p = \mu$ for every p, we get $J^p_{(1)}(t_p,y_p) = J_{(1)}(t_p,y_p)$ and then our Theorem 4.3.8 gives the upper semi-continuity of the mapping

$$z \longmapsto J_{(1)}(t,z)$$

at the point y.

Theorem 4.3.9

Let (X,ρ) be a metric space, $(R,X,R_+;\mu_p)$ $(p=1,2,\ldots)$ and $(R,X,R_+;\mu)$ be pseudo-processes such that the sequence $\{\mu_p\}$ converges to μ uniformly. Suppose that $(t_o,x_o) \in R \times X$ satisfies the condition (S) (see Definition 4.1.6) in $(R,X,R_+;\mu)$.

Assume that sequences $\{x_p\}$ and $\{z_p\}$ in X and $\{t_p\}$ in R are such that

$$x_p \longrightarrow x_o,\ z_p \longrightarrow z_o \text{ as } p \longrightarrow \infty \text{ and } t_p \longrightarrow t_o .$$

If $z_p \in J^p_{(1)}(t_p,x_p)$ for $p=1,2,\ldots$, then $z \in J_{(1)}(t_o,x_o)$.

Proof:

The relations $z_p \in J^p_{(1)}(t_p,x_p)$ for $p=1,2,\ldots$ mean that for every p there exist sequences $\{x_p^m\}$ in X, $\{s_p^m\}$ in R_+ such that

(4.3.86) $\qquad x_p^m \longrightarrow x_p \qquad$ as $m \longrightarrow \infty$

(4.3.87) $\qquad s_p^m \longrightarrow \infty \qquad$ as $m \longrightarrow \infty$

and

(4.3.88) $\qquad \mu_p(t_p,x_p^m,s_p^m) \longrightarrow z_p \quad$ as $m \longrightarrow \infty$.

From (4.3.88) it follows that (by using the same reasoning as before) in the proofs of the preceeding theorems, there is a sequence $\{m_p\}$ of positive integers such that

$$m_p \longrightarrow \infty \qquad \text{as } p \longrightarrow \infty$$

and

(4.3.89) $\qquad \rho(\mu_p(t_p,x_p^{m_p},s_p^{m_p}),z_p) \leq \frac{1}{p} \qquad (p=1,2,\ldots).$

Moreover, we may assume without loss of generality that putting

(4.3.90) $\qquad w_p := x_p^{m_p}, \quad r_p := s_p^{m_p}$

we get

(4.3.91) $\qquad w_p \longrightarrow x_o \quad$ and $\quad r_p \longrightarrow \infty \quad$ as $p \longrightarrow \infty$.

We have, by the triangle inequality

(4.3.92)
$$\left\{\begin{aligned} \rho(\mu(t_o,w_p,r_p),z) \leq\ & \rho(\mu(t_o,w_p,r_p),\mu(t_o,x_o,r_p)) + \\ & + \rho(\mu(t_o,x_o,r_p),\mu(t_p,w_p,r_p)) + \\ & + \rho(\mu(t_p,w_p,r_p),\mu_p(t_p,w_p,r_p)) + \\ & + \rho(\mu_p(t_p,w_p,r_p),z_p) + \rho(z_p,z) \end{aligned}\right.$$

Let $\varepsilon > 0$ be fixed.

Because of the condition (S) assumed for the point (t_o, x_o) we can find $r > 0$ and $\delta > 0$ such that if

$$(4.3.93) \qquad |t-t_o| < \delta, \quad \rho(x, x_o) < \delta \quad \text{and } s > r$$

then

$$(4.3.94) \qquad \rho(\mu(t_o, x_o, s), \mu(t, x, s)) < \frac{\varepsilon}{5}.$$

Because of the convergence (see (4.3.91)) we can find p_o such that

$$(4.3.95) \qquad \rho(w_p, x_o) < \delta \quad \text{and} \quad |t_p - t_o| < \delta \quad \text{for } p \geq p_o.$$

We can find p_1 such that

$$(4.3.96) \qquad r_p > r \qquad \text{for } p \geq p_1.$$

Combining (4.3.94)-(4.3.96) we observe that for $p \geq p_o$ we have

$$(4.3.97) \qquad \rho(\mu(t_o, w_p, r_p), \mu(t_o, x_o, r_p)) < \frac{\varepsilon}{5}$$

and

$$(4.3.98) \qquad \rho(\mu(t_o, x_o, r_p), \mu(t_p, w_p, r_p)) < \frac{\varepsilon}{5}$$

for $p \geq \max(p_o, p_1)$.

The uniform convergence of $\{\mu_p\}$ to μ implies that we can find p_2 such that

$$(4.3.99) \qquad \rho(\mu(t_p, w_p, r_p), \mu_p(t_p, w_p, r_p)) < \frac{\varepsilon}{5} \quad \text{for } p \geq p_2.$$

From (4.3.89) (see (4.3.90)) we get

$$(4.3.100) \qquad \rho(\mu_p(t_p, w_p, r_p), z_p) < \frac{\varepsilon}{5}$$

for p sufficiently large, say for $p \geq p_3$ (it is enough to take p_3 such that $\frac{1}{p_3} < \frac{\varepsilon}{5}$).

Finally, we can find p_4 so large that

$$(4.3.101) \qquad \rho(z,z_p) < \frac{\varepsilon}{5} \qquad \text{for } p \geq p_4.$$

Thus for $p \geq \hat{p} := \max(p_i : i=1,2,3,4)$ we get from (4.3.92) (see (4.3.96)-(4.3.100))

$$\rho(\mu(t_o,w_p,r_p),z) \leq \varepsilon$$

and so, we have proved that

$$(4.3.102) \qquad \mu(t_o,w_p,r_p) \longrightarrow z \qquad \text{as } p \longrightarrow \infty .$$

Relation (4.3.101) with (4.3.91) implies that

$$z \in J_{(1)}(t_o,x_o).$$

The proof is completed.

Theorem 4.3.10

Let (X,ρ) be a metric space, $(R,X,R_+;\mu_n)$ $(n=1,2,\ldots)$ and $(R,X,R_+;\mu)$ be pseudo-processes such that $\{\mu_n\}$ converges to μ uniformly. Suppose that $(t_o,x_o) \in R \times X$ satisfies the condition (S) (see Definition 4.1.6) in $(R,X,R_+;\mu)$. Assume that sequences $\{x_n\}$ and $\{y_n\}$ in X and $\{t_n\}$ in R are such that

$$x_n \longrightarrow x_o, \quad y_n \longrightarrow y_o, \quad t_n \longrightarrow t_o \text{ and } y_n \in J^n_{(2)}(t_n,x_n)$$

for $n=1,2,\ldots$, then $y_o \in J_{(2)}(t_o,x_o)$.

Proof:

The relation $y_n \in J^n_{(2)}(t_n,x_n)$ for every n means that for every n, there exist sequences $\{t^m_n\}$ in R, $\{s^m_n\}$ in R_+, $s^m_n \longrightarrow \infty$ as $m \longrightarrow \infty$, $t^m_n \longrightarrow t_n$ and

$$(4.3.103) \qquad \mu_n(t^m_n,x_n,s^m_n) \longrightarrow y_n \qquad \text{as } m \longrightarrow \infty .$$

For every fixed n, there exists a positive number m_n such that

(4.3.104) $\rho(\mu_n(t_n^m, x_n, s_n^m), y_n) \leq \frac{1}{n}$ for $m \geq m_n$.

So for every n, we can find a positive number b_n such that

(4.3.105) $\rho(\mu_n(t_n^{b_n}, x_n, s_n^{b_n}), y_n) \leq \frac{1}{n}$ $n=1,2,\ldots$

and moreover we may require that

(4.3.106) $b_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

It is assumed that $\{\mu_n\}$ is convergent to μ uniformly. This means that

(4.3.107)
$$\begin{cases} \sigma > 0 \quad \exists n_o \in N: \\ \\ \rho(\mu_n(h,v,\tau), \mu(h,v,\tau)) < \frac{\sigma}{4} \quad \text{for } n \geq n_o \end{cases}$$

for $(h,v,\tau) \in R \times X \times R_+$

We choose $\lambda > 0$, $\delta > 0$ and $r > 0$ such that

(4.3.108) $\rho(\mu(t^o, x^o, s_n^{b_n}), \mu(t,x,s_n^{b_n})) < \frac{\sigma}{4}$

for $s_n^{b_n} > r$ $(n=1,2,\ldots)$, $t \in (t^o-\delta, t^o+\delta)$ and $x \in X$ such that $\rho(x,x^o) < \lambda$.

Such a choice of λ, δ and r is possible because the condition (S) is assumed for the point (t_o, x_o) in $(R,X,R_+;\mu)$.

We can assume that $\{b_n\}$ is increasing and we find n_1 such that

(4.3.109) $|t - t_n^{b_n}| < \delta$

(4.3.110) $\rho(x, x_n) < \lambda$

(4.3.111) $\rho(y_n, y) < \frac{\sigma}{4}$

(4.3.112) $s_n^{b_n} > n$

and

(4.3.113) $\frac{1}{n} \leq \frac{\sigma}{4}$ for $n \geq n_1$.

Putting

$$t_n^* := t_n^{b_n}, \quad s_n^* := s_n^{b_n}$$

we obtain:

(4.3.114) $$t_n^* \longrightarrow t \text{ and } s_n^* \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

We get from triangle inequality

(4.3.115)
$$\begin{cases} \rho(\mu(t_n^*, x_n, s_n^*), y_o) \leq \rho(\mu_n(t_n^*, x_n, s_n^*), y_n) + \\ \quad + \rho(\mu(t_n^*, x_n, s_n^*), \mu(t, x, s_n^*)) + \\ \quad + \rho(y_n, y_o) + \rho(\mu_n(t_n^*, x_n, s_n^*), \mu(t_n^*, x_n, s_n^*)) \end{cases}$$

Using relations (4.3m105), (4.3.107), (4.3.108) and (4.3.111) we see that $\rho(\mu(t_n^*, x_n, s_n^*), y_o)$ is estimated by

$\frac{\sigma}{4} + \frac{\sigma}{4} + \frac{\sigma}{4} + \rho(\mu_n(t_n^*, x_n, s_n^*), y_n) \leq \sigma$.

Thus we get

(4.3.116) $$\rho(\mu(t_n^*, x_n, s_n^*), y_o) \leq \sigma$$

which means that

$$\mu(t_n^*, x_n, s_n^*) \longrightarrow y_o \quad \text{as } n \longrightarrow \infty$$

and we get (see (4.3.114))

$$y_o \in J_{(2)}(t_o, x_o).$$

Thus is the end of the proof.

Theorem 4.3.11

Let (X, ρ) be a metric space, $(R, X, R_+; \mu_n)$ $(n=1,2,\ldots)$ and $(R, X, R_+; \mu)$ be PP such that the sequence $\{\mu_n\}$ converges to μ uniformly. Suppose that sequences $\{h_n\}$ in X and $\{t_n\}$ in R are such that $h_n \longrightarrow h \in X$ as $n \longrightarrow \infty$ and $t_n \longrightarrow t_o$ and $h_n \in J_{(2)}^n(t_n, x_o)$ for $n=1,2,\ldots$; then $h \in J_{(2)}(t_o, x_o)$.

Proof:

The relations $h_n \in J^n_{(2)}(t_n, x_o)$ for every fixed n means that for every fixed n there exist sequences $\{t_n^k\}$ in R and $\{\lambda_n^k\}$ in R_+ such that

$$(4.3.117) \qquad t_n^k \longrightarrow t_n \qquad \text{as } k \longrightarrow \infty$$

$$(4.3.118) \qquad \lambda_n^k \longrightarrow \infty \qquad \text{as } k \longrightarrow \infty$$

and

$$(4.3.119) \qquad \mu_n(t_n^k, x_o, \lambda_n^k) \longrightarrow h_n \qquad \text{as } k \longrightarrow \infty .$$

For every fixed n, there exists a positive number k_n such that

$$(4.3.120) \qquad \rho(\mu_n(t_n^k, x_o, \lambda_n^k), h_n) \leq \frac{1}{n} \qquad \text{for } k \geq k_n .$$

So, for every n, we can find a positive number p_n such that

$$(4.3.121) \qquad \rho(\mu_n(t_n^{p_n}, x_o, \lambda_n^{p_n}), h_n) \leq \frac{1}{n} , \quad n=1,2,\ldots$$

and moreover we may require that

$$(4.3.122) \qquad p_n \longrightarrow \infty .$$

It is assumed that $\{\mu_n\}$ is convergent to μ uniformly. Thus means that

$$(4.3.123) \qquad \begin{cases} \forall \varepsilon > 0 \ \exists n_o \in N: \\ \\ \rho(\mu(\tau,w,\theta), \mu_n(\tau,w,\theta)) < \frac{\varepsilon}{3} \qquad \text{for } n \geq n_o \\ \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (\tau,w,\theta) \in R \times X \times R_+ . \end{cases}$$

It is assumed that $h_n \longrightarrow h$ as $n \longrightarrow \infty$. Thus we find now $n_1 \in N$ such that

$$(4.3.124) \qquad \rho(h_n, h) < \frac{\varepsilon}{3} \quad \text{and} \quad \frac{1}{n} < \frac{\varepsilon}{3} \quad \text{for } n \geq n_1$$

and putting

(4.3.125) $\quad v_n := t_n^{p_n}, \quad \beta_n := \lambda_n^{p_n}, \quad \beta_n \longrightarrow \infty$ as $n \longrightarrow \infty$

and using standard reasoning we can show that

(4.3.126) $\quad v_n \longrightarrow t_o \quad$ as $n \longrightarrow \infty$.

We have by the triangle inequality

$$(4.3.127)\quad \begin{cases} \rho(\mu(v_n, x_o, \ _n), h) \leq \rho(\mu_n(v_n, x_o, \beta_n), h_n) + \\ \qquad + \rho(h_n, h) + \rho(\mu_n(v_n, x_o, \beta_n), \mu(v_n, x_o, \beta_n)) \end{cases}$$

Using relations (4.3.121), (4.3.123) and (4.3.124) we get

(4.3.128) $\quad \rho(\mu(v_n, x_o, \beta_n), h) < \varepsilon \quad$ for n sufficiently large

which means that

$$\mu(v_n, x_o, \beta_n) \longrightarrow h \qquad \text{as } n \longrightarrow \infty$$

and we get (see (4.3.125))

$$h \in J_{(2)}(t_o, x_o).$$

This completes the proof.

Theorem 4.3.12

Let (X, ρ) be a metric space, $(R, X, R_+; \mu_n)$ $(n=1,2,\ldots)$ and $(R, X, R_+; \mu)$ be PD such that $\{\mu_n\}$ converges to μ uniformly. Assume that the sequences $\{w_n\}$ and $\{q_n\}$ in X and $\{t_n\}$ in R are such that

$$q_n \longrightarrow q \in X, \quad w_n \longrightarrow w \in X, \quad t_n \longrightarrow t \in R$$

and

$$q_n \in J^n(t_n, w_n) \quad \text{for } n=1,2,\ldots; \text{ then } q \in J(t,w).$$

Proof:

The relations $q_n \in J^n(t_n, w_n)$ for every n means that for every n there exist sequences $\{w_r^n\}$ in X, $\{t_r^n\}$ in R and $\{\lambda_r^n\}$ in R_+ such that

(4.3.129) $w_r^n \longrightarrow w_n$ as $r \longrightarrow \infty$

(4.3.130) $\lambda_r^n \longrightarrow \infty$ as $r \longrightarrow \infty$

(4.3.131) $t_r^n \longrightarrow t_n$ as $r \longrightarrow \infty$

and

(4.3.132) $\mu_n(t_r^n, w_r^n, \lambda_r^n) \longrightarrow q_n$ as $r \longrightarrow \infty$

(4.3.133)
$$\begin{cases} \forall n \in N \ \exists r_n \in N: \\ \rho(\mu_n(t_r^n, w_r^n, \lambda_r^n), q_n) \leq \frac{1}{n} \ \text{ for } r \geq r_n \end{cases}$$

So, there exists a sequence $\{k_n\}$ of positive integers such that

(4.3.134) $\rho(\mu_n(t_{k_n}^n, w_{k_n}^n, \lambda_{k_n}^n), q_n) \leq \frac{1}{n}$, $n=1,2,\ldots$

and moreover we require that

(4.3.135) $k_n \longrightarrow \infty$ as $n \longrightarrow \infty$

and putting

(4.3.136) $\tau_n := t_{k_n}^n$, $v_n := w_{k_n}^n$, $s_n := \lambda_{k_n}^n$

we get

(4.3.137) $\tau_n \longrightarrow t$, $v_n \longrightarrow w$

and also trivially

(4.3.138) $s_n \longrightarrow \infty$ as $n \longrightarrow \infty$.

We have obviously (because of (4.3.134))

(4.3.139) $\rho(\mu_n(\tau_n, v_n, s_n), q_n) \leq \frac{1}{n}$ for $n=1,2,\ldots$.

It is assumed that $\{\mu_n\}$ is convergent to μ uniformly. This means that

$$(4.3.140)\quad \begin{cases} \forall \gamma > 0 \quad \exists n_o \in \mathbb{N}: \\ \rho(\mu_n(f,g,h), \mu(f,g,h)) < \frac{\gamma}{3} \qquad \text{for } n \geq n_o \\ \qquad\qquad\qquad\qquad \text{for } (f,g,h) \in R \times X \times R_+ . \end{cases}$$

It is assumed that $\rho(q,q_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and we find now $n_1 \in \mathbb{N}$ such that

$$(4.3.141)\quad \rho(q,q_n) < \frac{\gamma}{3} \quad \text{and} \quad \frac{1}{n} < \frac{\gamma}{3} \quad \text{for } n \geq n_1$$

and by the triangle inequality we get

$$(4.3.142)\quad \begin{cases} \rho(\mu(\tau_n,v_n,s_n),q) \leq \rho(\mu_n(\tau_n,v_n,s_n),q_n) + \\ \qquad + \rho(q_n,q) + \rho(\mu_n(\tau_n,v_n,s_n),\mu(\tau_n,w_n,s_n)). \end{cases}$$

On applying (4.3.139)-(4.3.141) we get

$$(4.3.143)\quad \rho(\mu(\tau_n,v_n,s_n),q) \leq \gamma \qquad \text{for } n \geq \max(n_o,n_1)$$

which means that

$$\mu(\tau_n,v_n,s_n) \longrightarrow q \qquad \text{as } n \longrightarrow \infty$$

and we get (see (4.3.138))

$$q \in J(t,w).$$

Remark 4.3.4

In the case $\mu_n = \mu$ for every n in Theorems 4.3.11 and 4.3.12, we get respectively $J^n_{(2)}(t_n,x_o) = J_{(2)}(t_n,x_o)$ and $(J^n(t_n,w_n) = J(t_n,w_n))$, then we have the same results respectively as in corollary 4.3.5 and Theorem 4.2.11 above which means that the mappings

$$(4.3.144)\quad R \ni t \longmapsto J_{(2)_n}(t,x_o)$$

and

$$(4.3.145)\quad R \times X \ni (t,w) \longmapsto J(t,w)$$

are upper semi-continuous at $t \in R$ and at point $(t,w) \in R \times X$.

§ 4.4.

Theorem 4.4.1

Let (X,ρ) be a metric space and $(R,X,R_+;\mu)$ be a pseudo-process. For every $(\sigma,x) \in R \times X$ we have

$$(4.4.1) \qquad \Lambda(\sigma,x) = \bigcap_{t \geq 0} \overline{\{\mu(\sigma,x,\tau) : \tau \geq t\}}$$

Proof:

Assume that $y \in \Lambda(\sigma,x)$.

So, there is a sequence $\{s_m\}$ of real numbers such that

$$(4.4.2) \qquad y = \operatorname{Lt} \mu(\sigma,x,s_m) \qquad \text{as } m \longrightarrow \infty$$

and

$$(4.4.3) \qquad s_m \longrightarrow \infty \qquad \text{as } m \longrightarrow \infty .$$

Let $t \geq 0$ be fixed. For m sufficiently large say $m \geq m_0$ we have (see (4.4.3))

$$s_m \geq t.$$

This means that

$$(4.4.4) \qquad \mu(\sigma,x,s_m) \in \{\mu(\sigma,x,\tau) : \tau \geq t\} \quad \text{for } m \geq m_0.$$

So, y being the limit of the sequence $\{\mu(\sigma,x,s_m)\}$, $m = m_0, m_0+1, \ldots$ of elements of the set $\{\mu(\sigma,x,\tau) : \tau \geq t\}$ belongs to the closure of this set.

Thus

$$(4.4.5) \qquad y \in \overline{\{\mu(\sigma,x,\tau) : \tau \geq t\}}.$$

This is true for every $t \geq 0$. So y belongs to the intersection of these sets for $t \geq 0$:

$$(4.4.6) \qquad y \in \bigcap_{t \geq 0} \overline{\{\mu(\sigma,x,\tau) : \tau \geq t\}}$$

Thus we have proved the inclusion

$$(4.4.7) \qquad \Lambda(\sigma,x) \subset \bigcap_{t \geq 0} \overline{\{\mu(\sigma,x,\tau) : \tau \geq t\}}.$$

Let now y be an element of the set which is on the right hand side of inclusion (4.4.7). Let us consider t=1,2,...; of course

$$(4.4.8) \qquad y \in \overline{\{\mu(\sigma,x,\tau): \tau \geq n\}}.$$

Thus for every n there is a sequence $\{\tau_m^n\}$ such that $\tau_m^n \geq n$ and

$$(4.4.9) \qquad y = \operatorname{Lt} \mu(\sigma,x,\tau_m^n) \qquad \text{as } m \longrightarrow \infty .$$

So, for every fixed n there exists m_n such that

$$\rho(y,\mu(\sigma,x,\tau_m^n)) < \frac{1}{n} \qquad \text{for } m \geq m_n$$

and in particular for $s_n := \tau_{m_n}^n$ we have

$$(4.4.10) \qquad s_n \geq n$$

and

$$(4.4.11) \qquad \rho(y,\mu(\sigma,x,s_n)) < \frac{1}{n} .$$

From (4.4.10) we get

$$(4.4.12) \qquad s_n \longrightarrow \infty$$

while (4.4.11) gives

$$\mu(\sigma,x,s_n) \longrightarrow y \quad \text{as } n \longrightarrow \infty .$$

This means that

$$y \in \Lambda(\ ,x).$$

We have proved that

$$\bigcap_{t \geq 0} \overline{\{\mu(\sigma,x,\tau): \tau \geq t\}} \subset \Lambda(\sigma,x).$$

Thus we have that

$$(4.4.13) \qquad \Lambda(\sigma,x) = \bigcap_{t \geq 0} \overline{\{\mu(\sigma,x,\tau): \tau \geq t\}}.$$

This completes the proof.

Theorem 4.4.2

Let (X,ρ) be a metric space and $(R,X,R_+;\mu)$ be a pseudo-process. For every $(\sigma,x)\in R\times X$ we have

$$(4.4.14)\qquad D(\sigma,x)=\bigcap\left\{\overline{\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s):\ \alpha>0,\beta>0}\right\}.$$

Proof:

Assume that

$$(4.4.15)\qquad y\notin D(\sigma,x).$$

So, there are sequences $\{x_m\}$ in X, $\{t_m\}$ in R and $\{s_m\}$ in R_+ such that

$$(4.4.16)\qquad x_m\longrightarrow x \quad \text{as } m\longrightarrow\infty$$

$$(4.4.17)\qquad t_m\longrightarrow\sigma \quad \text{as } m\longrightarrow\infty$$

$$(4.4.18)\qquad s_m\geq 0$$

and

$$(4.4.19)\qquad y=\operatorname{Lt}\mu(t_m,x_m,s_m) \quad \text{as } m\longrightarrow\infty.$$

Take $\alpha>0$, $\beta>0$ fixed. For m sufficiently large we have

$$(4.4.20)\qquad t_m\in B(\sigma,\alpha)$$

and

$$(4.4.21)\qquad x_m\in B(x,\beta)$$

and so

$$(4.4.22)\qquad \mu(t_m,x_m,s_m)\in\mu(B(\sigma,\alpha),B(x,\beta),s_m)$$

and then

$$(4.4.23)\qquad \mu(t_m,x_m,s_m)\in\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s).$$

Thus y being the limit of $\{\mu(t_m,x_m,s_m)\}$ belongs to

$$\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s)$$

This is true for every $\alpha>0$, $\beta>0$ arbitrarily fixed. So,

$$(4.4.24)\qquad y\in\bigcap\left\{\overline{\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s):\ \alpha>0,\ \beta>0}\right\}.$$

Let

$$(4.4.25)\qquad y\in\bigcap_{\substack{\alpha>0\\ \beta>0}}\Big\{\overline{\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s)}\Big\}$$
$$=\bigcap\Big\{\overline{\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s)}:\ \alpha>0,\ \beta>0\Big\}.$$

Then for every $\alpha>0$, $\beta>0$

$$(4.4.26)\qquad y\in\overline{\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s)}.$$

Thus for every α and β there is a sequence $\{y^n_{\alpha,\beta}\}$ such that

$$(4.4.27)\qquad y^n_{\alpha,\beta}\in\bigcup_{s\geq 0}\mu(B(\sigma,\alpha),B(x,\beta),s)$$

and

$$(4.4.28)\qquad y^n_{\alpha,\beta}\longrightarrow y\qquad \text{as } n\longrightarrow\infty.$$

Relations (4.4.28) imply that there is $s\geq 0$ such that

$$(4.4.29)\qquad y^n_{\alpha,\beta}\in\mu(B(\sigma,\alpha),B(x,\beta),s).$$

Then, finally, for every α and β we can find sequences $\{\sigma^n_{\alpha,\beta}\}$, $\{x^n_{\alpha,\beta}\}$ and $\{s^n_{\alpha,\beta}\}$ such that

$$(4.4.30)\qquad |\sigma-\sigma^n_{\alpha,\beta}|<\alpha,\quad \rho(x,x^n_{\alpha,\beta})<\beta,\quad s^n_{\alpha,\beta}\geq 0$$

and

$$(4.4.31)\qquad \mu(\sigma^n_{\alpha,\beta},x^n_{\alpha,\beta},s^n_{\alpha,\beta})\longrightarrow y\qquad \text{as } n\longrightarrow\infty.$$

Take

$$\alpha=1,\ \frac{1}{2},\ \ldots$$

$$\beta=1,\ \frac{1}{2},\ \ldots$$

and put

$$t^n_m:=\sigma^n_{\frac{1}{m},\frac{1}{m}}$$

$$\xi^n_m:=x^n_{\frac{1}{m},\frac{1}{m}}$$

$$r^n_m:=s^n_{\frac{1}{m},\frac{1}{m}}$$

For every fixed m we can find n_m so large that

$$(4.4.32) \qquad |\sigma - t_m^n| < \frac{1}{m} \qquad \text{for } n \geq n_m$$

$$(4.4.33) \qquad \rho(x, \xi_m^n) < \frac{1}{m} \qquad \text{for } n \geq n_m$$

and

$$(4.4.34) \qquad \rho(\mu(t_m^n, \xi_m^n, r_m^n), y) < \frac{1}{m} \qquad \text{for } n \geq n_m .$$

Without loss of generality we may assume that

$$n_m \longrightarrow \infty \quad \text{as} \quad m \longrightarrow \infty$$

and by standard arguments, we can show that there are suitable subsequences of $\{t_m^{n_m}\}$ and $\{\xi_m^{n_m}\}$ tending to σ and x respectively.

So, we may assume without loss of generality that these sequences themselves are convergent to σ and x respectively.

So, we have

$$(4.4.35) \qquad \rho(\mu(t_m^{n_m}, \xi_m^{n_m}, r_m^{n_m}), y) \longrightarrow 0 \quad \text{as } m \longrightarrow \infty$$

and

$$t_m^{n_m} \longrightarrow \sigma , \quad \xi_m^{n_m} \longrightarrow x \qquad \text{as } m \longrightarrow \infty .$$

Thus

$$y \in D(\sigma, x) .$$

The proof is completed.

CHAPTER V

CARTESIAN PRODUCTS OF DYNAMICAL SEMI-SYSTEMS AND PSEUDO-PROCESSES. SUPPLEMENTARY RESULTS.

In this chapter, we give results concerning the cartesian products of dynamical semi-systems and pseudo-processes. We also mention some supplementary results on processes. They are treated respectively in sections 1,2 and 3.

§ 5.1.

In this section, we give some results concerning cartesian products of dynamical semi-systems and their limit sets.

Let $(X,G;\pi)$ and $(\tilde{X},\tilde{G};\tilde{\pi})$ be two PDSS and put

$$\hat{G} = G \times \tilde{G} \tag{5.1.1}$$

and

$$\hat{X} = X \times \tilde{X}. \tag{5.1.2}$$

We introduce in $\hat{G}$ the usual algebraic operation. If "$+$" and "$\tilde{+}$" are additions in the semi-groups G and $\tilde{G}$ respectively, Then "$\hat{+}$" in $\hat{G}$ is defined as follows:

For (t,t) and (s,s) belonging to $G \times \tilde{G}$, we put

$$(t,\tilde{t}) \hat{+} (s,\tilde{s}) := (t+s,\ \tilde{t}\tilde{+}\tilde{s}). \tag{5.1.3}$$

It is known that $(\hat{G},\hat{+})$ is an abelian semi-group with the neutral element $(o,\tilde{o}) \in G \times \tilde{G}$ (o is the neutral element in G and $\tilde{o}$ is the neutral element in $\tilde{G}$).

We define now

$$\hat{\pi} := \pi \times \tilde{\pi} : \hat{X} \times \hat{G} \longrightarrow \hat{X} \tag{5.1.4}$$

by

$$\hat{\pi}((x,\tilde{x}),(t,\tilde{t})) \overset{\text{df}}{\Longleftrightarrow} (\pi(x,t),\tilde{\pi}(\tilde{x},\tilde{t})). \tag{5.1.5}$$

Theorem 5.1.1

$(\hat{X},\hat{G};\hat{\pi})$ is a PDSS.

Proof:

For the triple $(\hat{X},\hat{G};\hat{\pi})$ to be a PDSS, the map $\hat{\pi}$ should satisfy the following axioms:

(5.1.6) Identity Axiom

and

(5.1.7) Group Axiom

We have for every $(x,\tilde{x}) \in X \times \tilde{X}$:

$$(5.1.8)\qquad \hat{\pi}((x,\tilde{x}),(o,\tilde{o})) = (\pi(x,o),\tilde{\pi}(\tilde{x},\tilde{o})) = (x,\tilde{x}).$$

Thus

$$\hat{\pi}((x,\tilde{x}),(o,\tilde{o})) = (x,\tilde{x}) \quad \forall (x,\tilde{x}) \in \hat{X}.$$

We then have that $\hat{\pi}$ satisfies the Identity Axiom.

Let $(x,\tilde{x}) \in \hat{X}$ and $(t,\tilde{t}), (s,\tilde{s})$ belong to $\hat{G}$

$$\hat{\pi}(\hat{\pi}(x,\tilde{x}),(t,\tilde{t})),(s,\tilde{s})) = \hat{\pi}(\pi(x,t),\tilde{\pi}(\tilde{x},\tilde{t})),(s,\tilde{s})) =$$

$$(5.1.9)\qquad = ((\pi(\pi(x,t),s),\tilde{\pi}(\tilde{\pi}(\tilde{x},\tilde{t}),\tilde{s})) =$$

$$= (\pi(x,t+s),\tilde{\pi}(\tilde{x},\tilde{t}+\tilde{s})) = \hat{\pi}((x,\tilde{x}),(t+s,\tilde{t}+\tilde{s}))$$

Relation (5.1.9) shows that the map $\hat{\pi}$ satisfies the group axiom.

Remark 5.1.1

If G and $\tilde{G}$ are groups (and so $(X,G;\pi)$ and $(\tilde{X},\tilde{G};\tilde{\pi})$ are PDS) then clearly $(\hat{X},\hat{G};\hat{\pi})$ is also a PDS since $\hat{G}$ is a group.

Let (Y,R^2,π) be a PDS. Let z be a point of Y. We put

$$(5.1.10)\qquad \Lambda^+(z) := \{ y \in Y: \text{there is a sequence } \{(t_n,s_n)\} \text{ of elements of } R^2 \text{ such that } t_n \longrightarrow \infty,\ s_n \longrightarrow \infty \text{ and } \pi(z,(t_n,s_n)) \longrightarrow y \text{ as } n \longrightarrow \infty \}.$$

Similarly we define $\Lambda^{-}(z)$ (by replacing the conditions $t_n \longrightarrow \infty$, $s_n \longrightarrow \infty$ by $t_n \longrightarrow -\infty$ and $s_n \longrightarrow -\infty$ respectively).

This definition is a special case of the general definition taken from Pelczar (1965b).

We shall use definition 5.1.10 above with respect to $Y = \hat{X}$ defined previously as: $\hat{X} = X \times \tilde{X}$.

Theorem 5.1.2

Let $(X,R;\pi)$ and $(\tilde{X},R;\tilde{\pi})$ be PDS where (X,ρ) and $(\tilde{X},\tilde{\rho})$ are metric spaces. Let $\hat{X} = X \times \tilde{X}$ and $\hat{\pi} = \pi \times \tilde{\pi}$ be defined as previously. Then $\hat{\Lambda}^{+}((x,\tilde{x})) = \Lambda^{+}(x) \times \tilde{\Lambda}^{+}(\tilde{x}) \quad \forall x \in X, \tilde{x} \in \tilde{X}$.

Proof:

Let

(5.1.11) $\hat{m} = (m,\tilde{m})$ be a point of the set $\hat{\Lambda}^{+}((x,\tilde{x}))$.

There exists a sequence $\{(t_n,\tilde{t}_n)\}$ in R^2 such that

(5.1.12) $\hat{\pi}((x,\tilde{x}),(t_n,\tilde{t}_n)) \longrightarrow \hat{m}$ as $n \longrightarrow \infty$.

But

$$\hat{\pi}((x,\tilde{x}),(t_n,\tilde{t}_n)) = (\pi(x,t_n),\tilde{\pi}(\tilde{x},\tilde{t}_n)).$$

Thus (5.1.12) becomes

(5.1.13) $(\pi(x,t_n),\tilde{\pi}(\tilde{x},\tilde{t}_n)) \longrightarrow \hat{m} = (m,\tilde{m})$ as $n \longrightarrow \infty$.

Relations (5.1.13) imply that

(5.1.14) $\pi(x,t_n) \longrightarrow m$ as $n \longrightarrow \infty$

and

(5.1.15) $\tilde{\pi}(\tilde{x},\tilde{t}_n) \longrightarrow \tilde{m}$ as $n \longrightarrow \infty$.

So we get

(5.1.16) $m \in \Lambda^{+}(x)$

and

(5.1.17) $\tilde{m} \in \tilde{\Lambda}^{+}(\tilde{x})$.

From relations (5.1.16) and (5.1.17) we get

(5.1.18) $\quad (m,\tilde{m}) \in \Lambda^+(x) \times \tilde{\Lambda}^+(\tilde{x})$

which implies that

(5.1.19) $\quad \hat{m} \in \Lambda^+(x) \times \tilde{\Lambda}^+(\tilde{x})$.

So we have proved that

(5.1.20) $\quad \hat{\Lambda}^+((x,x)) \subset \Lambda^+(x) \times \tilde{\Lambda}^+(\tilde{x})$.

To prove the reverse inclusion, let us consider a point $\hat{y} = (y,\tilde{y})$ of the set $\Lambda^+(x) \times \tilde{\Lambda}^+(\tilde{x})$.

There exists a sequence $\{(s_n,\tilde{s}_n)\}$ in R^2 such that $s_n \longrightarrow \infty$, $\tilde{s}_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and

(5.1.21) $\quad (\pi((x,s_n),\tilde{\pi}(\tilde{x},\tilde{s}_n)) \longrightarrow \hat{y} = (y,\tilde{y}) \quad$ as $n \longrightarrow \infty$.

Relations (5.1.21) (see (5.1.5)) imply that

(5.1.22) $\quad \hat{\pi}((x,\tilde{x}),(s_n,\tilde{s}_n)) \longrightarrow \hat{y} = (y,\tilde{y}) \quad$ as $n \longrightarrow \infty$.

Thus we get (see (5.1.10))

(5.1.23) $\quad \hat{y} \in \hat{\Lambda}^+((x,\tilde{x}))$

which means that

(5.1.24) $\quad \Lambda^+(x) \times \tilde{\Lambda}^+(\tilde{x}) \subset \hat{\Lambda}^+((x,\tilde{x}))$.

Combining (5.1.20) and (5.1.24) we get

(5.1.25) $\quad \hat{\Lambda}^+((x,x)) = \Lambda^+(x) \times \tilde{\Lambda}^+(\tilde{x})$.

Thus proves the assertion

Corollary 5.1.1

Let (X,R,π) and $(\tilde{X},R,\tilde{\pi})$ be PDS where (X,ρ) and $(\tilde{X},\tilde{\rho})$ are metric spaces. Let $\hat{X} = X \times \tilde{X}$ and $\hat{\pi} = \pi \times \tilde{\pi}$ be defined as previously. Then $\hat{\Lambda}^-((x,\tilde{x})) = \Lambda^-(x) \times \tilde{\Lambda}^-(\tilde{x}))$ for $x \in X$, $\tilde{x} \in \tilde{X}$.

§ 5.2.

In this section, we give results concerning the cartesian products of pseudo-processes and their limit sets.

Let $(G,X,H;\mu)$ and $(\tilde{G},\tilde{X},\tilde{H};\tilde{\mu})$ be two PD and put

(5.2.1) $\quad \hat{H} = H \times \tilde{H}$

and

(5.2.2) $\quad \hat{G} = G \times \tilde{G}.$

For $(t,\tilde{t})$ and $(s,\tilde{s})$ belonging to $G \times \tilde{G}$ we define their addition as in (5.1.3) above. Then $H \times \tilde{H}$ is a sub-semi-group of $G \times \tilde{G}$.

We define now

(5.2.3) $\quad \hat{\mu} := \mu \times \tilde{\mu} : \hat{G} \times \hat{X} \times \hat{H} \longrightarrow \hat{X}$

by

(5.2.4) $\quad \hat{\mu}((t,\tilde{t}),(x,\tilde{x}),(s,\tilde{s})) := (\mu(t,x,s),\tilde{\mu}(\tilde{t},\tilde{x},\tilde{s})).$

Theorem 5.2.1

$(\hat{G},\hat{X},\hat{H};\hat{\mu})$ is a pseudo-process.

Proof:

For the quadruple $(\hat{G},\hat{X},\hat{H};\hat{\mu})$ to be a pseudo-process, the map $\hat{\mu}$ should satisfy both the Identity and group axioms:

(5.2.5) $\quad \hat{\mu}((t,\tilde{t}),(x,\tilde{x}),(o,\tilde{o})) = (\mu(t,x,o),\tilde{\mu}(\tilde{t},\tilde{x},\tilde{o})) =$

$$= (x,\tilde{x}) \quad \forall (x,\tilde{x}) \in \hat{X}.$$

Thus we get

(5.2.6) $\quad \hat{\mu}((t,\tilde{t}),(x,\tilde{x}),(o,\tilde{o})) = (x,\tilde{x}) \quad$ for $(x,\tilde{x}) \in \hat{X}.$

Let now $(x,\tilde{x}) \in \hat{X}$ and $(\sigma,\tilde{\sigma})$, $(s,\tilde{s})$ belong to $\hat{G}$.

$$(5.2.7)\quad \begin{cases} \hat{\mu}(((t,\tilde{t})\hat{+}(\sigma,\tilde{\sigma})),\hat{\mu}((t,\tilde{t}),(x,\tilde{x}),(\sigma,\tilde{\sigma})),(s,\tilde{s})) = \\ = \hat{\mu}(((t,\tilde{t})\hat{+}(\sigma,\tilde{\sigma}),(\mu(t,x,\sigma),\tilde{\mu}(\tilde{t},\tilde{x},\tilde{\sigma})),(s,\tilde{s}))= \\ = \hat{\mu}((t+\sigma,\tilde{t}\tilde{+}\tilde{\sigma}),(\mu(t,x,\sigma),\tilde{\mu}(\tilde{t},\tilde{x},\tilde{\sigma})),(s,\tilde{s})) = \\ = (\mu(t+\sigma,\mu(t,x,\sigma),s),\tilde{\mu}(\tilde{t}\tilde{+}\tilde{\sigma},\tilde{\mu}(\tilde{t},\tilde{x},\tilde{\sigma}),\tilde{s})) = \\ = (\mu(t,x,\sigma+s),\tilde{\mu}(\tilde{t},\tilde{x},\tilde{\sigma}\tilde{+}\tilde{s})) = \\ = \hat{\mu}((t,\tilde{t}),(x,\tilde{x}),(\sigma+s,\tilde{\sigma}\tilde{+}\tilde{s})) \end{cases}$$

Applying (5.2.6) and (5.2.7) we have proved that $(\hat{G},\hat{X},\hat{H};\hat{\mu})$ is a pseudo-process.

Let $(R,M,R^2;\mu)$ be a pseudo-process. Let (σ,w) be a point of $R \times M$. We put

$$(5.2.8)\quad \Lambda^+(\sigma,w) := \{ q \in M: \text{there is a sequence } \{(t_n,s_n)\} \text{ of elements of } R^2 \text{ such that } t_n \longrightarrow \infty,\ s_n \longrightarrow \infty \text{ and } \mu(\sigma,w,(t_n,s_n)) \longrightarrow q \text{ as } n \longrightarrow \infty \}.$$

Theorem 5.2.2

Let $(R,X,R_+;\mu)$ and $(R,\tilde{X},R_+;\tilde{\mu})$ be pseudo-processes where (X,ρ) and $(\tilde{X},\rho)$ are metric spaces. Let $\hat{X} = X \times \tilde{X}$ and $\hat{\mu} = \mu \times \tilde{\mu}$ be defined as above, then $\hat{\Lambda}^+((t,\tilde{t}),(x,\tilde{x})) = \Lambda^+(t,x) \times \tilde{\Lambda}^+(\tilde{t},\tilde{x})$.

Proof:

Let

(5.2.9) $\hat{p} = (p,\tilde{p})$ be a point of the set $\hat{\Lambda}^+((t,\tilde{t}),(x,\tilde{x}))$.

There exists a sequence $\{(r_m,\tilde{r}_m)\}$ in R_+^2 such that

(5.2.10) $\hat{\mu}((t,\tilde{t}),(x,\tilde{x}),(r_m,\tilde{r}_m)) \longrightarrow \hat{p} = (p,\tilde{p})$ as $m \longrightarrow \infty$.

This implies (see (5.2.4)) that

(5.2.11) $(\mu(t,x,r_m),\tilde{\mu}(\tilde{t},\tilde{x},\tilde{r}_m)) \longrightarrow \hat{p} = (p,\tilde{p})$ as $m \longrightarrow \infty$.

Relations (5.2.11) imply that

(5.2.12) $\mu(t,x,r_m) \longrightarrow p$ as $m \longrightarrow \infty$

and

$$\tilde{\mu}(\tilde{t},\tilde{x},\tilde{v}_m) \longrightarrow \tilde{p} \qquad \text{as } m \longrightarrow \infty . \tag{5.2.13}$$

So, we get

$$p \in \Lambda^+(t,x) \tag{5.2.14}$$

and

$$\tilde{p} \in \tilde{\Lambda}^+(\tilde{t},\tilde{x}). \tag{5.2.15}$$

Thus we get

$$\hat{p} = (p,\tilde{p}) \in \Lambda^+(t,x) \times \tilde{\Lambda}^+(\tilde{t},\tilde{x}) \tag{5.2.16}$$

which implies that

$$\hat{\Lambda}^+((t,\tilde{t}),(x,\tilde{x})) \subset \Lambda^+(t,x) \times \tilde{\Lambda}^+(\tilde{t},\tilde{x}). \tag{5.2.17}$$

To prove the reverse inclusion, let us consider a point $\hat{h} = (h,\tilde{h})$ of the set $\Lambda^+(t,x) \times \tilde{\Lambda}^+(\tilde{t},\tilde{x})$.

There exists a sequence $\{(v_m,\tilde{v}_m)\}$ such that

$$v_m \longrightarrow \infty \quad \text{and} \quad \tilde{v}_m \longrightarrow \infty \quad \text{as } n \longrightarrow \infty$$

and

$$(\mu(t,x,v_m),\tilde{\mu}(\tilde{t},\tilde{x},\tilde{v}_m)) \longrightarrow \hat{h} = (h,\tilde{h}) \quad \text{as } m \longrightarrow \infty \tag{5.2.18}$$

which implies (see (5.2.4)) that

$$\hat{\mu}((t,\tilde{t}),(x,\tilde{x}),(v_m,\tilde{v}_m)) \longrightarrow \hat{h} \qquad \text{as } m \longrightarrow \infty . \tag{5.2.19}$$

Thus (see (5.2.8)) we get

$$h \in \hat{\Lambda}^+((t,\tilde{t}),(x,\tilde{x})) \tag{5.2.20}$$

which means that

$$\Lambda^+(t,x) \times \tilde{\Lambda}^+(\tilde{t},\tilde{x}) \subset \hat{\Lambda}^+((t,\tilde{t}),(x,\tilde{x})). \tag{5.2.21}$$

Thus we (see (5.2.19)) get

$$\hat{\Lambda}^+((t,\tilde{t}),(x,\tilde{x})) = \Lambda^+(t,x) \times \tilde{\Lambda}^+(\tilde{t},\tilde{x}). \tag{5.2.22}$$

This completes the proof.

§ 5.3.

Let $(R,X,R_+;\mu)$ be a pseudo-process. By $H[\mu]$ we denote the family of mappings

(5.3.1) $\qquad R \times X \times R_+ \longrightarrow X$

defined as follows:

(5.3.2) $\begin{cases} v \in H[\mu] \text{ if and only if there exists a sequence } \{T_n\} \text{ of real numbers, such that } T_n \longrightarrow \infty \text{ and for every } t,x,s \in R \times X \times R_+, \\ \mu(t+T_n,x,s) \longrightarrow v(t,x,s) \quad \text{as } n \longrightarrow \infty . \end{cases}$

It is called the asymptotic hull of μ (see Dafermos (1971)).

If μ satisfies some additional conditions, then every $v \in H[\mu]$ is a pseudo-process; several results concerning that matter are obtained by Dafermos (1971, 1972a,b). For our purpose we shall simply assume that certain mapping $v \in H[\mu]$ is a pseudo-process. We shall consider below prolongations of the point $(t,x) \in R \times X$ with respect to distinct pseudo-processes.

We use the notations $D_\mu(t,x)$ and $D_{\tilde{\mu}}(t,x)$ for prolongations for μ and $\tilde{\mu}$ respectively and also $J_\mu(t,x)$, $J_{\tilde{\mu}}(t,x)$ denote the prolongational limit sets for μ and $\tilde{\mu}$ respectively.

Theorem 5.3.1

Let $(R,X,R_+;\mu)$ be a pseudo-process such that for every $(x,s) \in X \times R_+$ the mapping

(5.3.3) $\qquad t \longmapsto \mu(t,x,s)$

is continuous and there exists $\alpha > 0$ such that

(5.3.4) $\qquad \mu(t+\alpha,x,s) = \mu(t,x,s)$ for every t,x,s.

Assume that $\tilde{\mu}$ is a pseudo-process belonging to $H[\mu]$. Then for every $(t,x) \in R \times X$

$$D_{\tilde{\mu}}(t,x) \subset \bigcup \{ D_\mu(t+r,x) : r \in [0,\alpha] \} .$$

Proof:

Let z be a point of $D_{\tilde{\mu}}(t,x)$. So there are sequences $\{t_m\}$, $\{x_m\}$, $\{s_m\}$ such that

$$t_m \longrightarrow t, \quad x_m \longrightarrow x, \quad s_m \geq 0$$

and

(5.3.5) $\qquad z = \mathrm{Lt}\,\tilde{\mu}(t_m,x_m,s_m) \qquad$ as $m \longrightarrow \infty$.

On the other hand, there is a sequence $\{T_n\}$ of real numbers, such that $T_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and for every m

(5.3.6) $\quad \mu(t_m+T_n,x_m,s_m) \longrightarrow \tilde{\mu}(t_m,x_m,s_m)$ as $n \longrightarrow \infty$.

Thus we can find a sequence $\{n_m\}$ of positive integers such that $n_m \longrightarrow \infty$ as $m \longrightarrow \infty$ and

(5.3.7) $\qquad \rho(\mu(t_m+T_{n_m},x_m,s_m),\tilde{\mu}(t_m,x_m,s_m)) < \frac{1}{m}$.

We have

(5.3.8)
$$\begin{cases} \rho(z,\mu(t_m+T_{n_m},x_m,s_m)) \leq \rho(z,\tilde{\mu}(t_m,x_m,s_m)) + \\ \qquad + \rho(\tilde{\mu}(t_m,x_m,s_m),\mu(t_m+T_{n_m},x_m,s_m)) \\ \qquad \leq \rho(z,\tilde{\mu}(t_m,x_m,s_m)) + \frac{1}{m} \end{cases}$$

and so

(5.3.9) $\qquad \mu(t_m+T_{n_m},x_m,s_m) \longrightarrow z \qquad$ as $m \longrightarrow \infty$.

We can find sequences $\{k_m\}$ of integers and $\{r_m\}$ of real numbers such that

(5.3.10) $\qquad t_m+T_{n_m} = t_m+k_m\alpha+r_m, \qquad r_m \in [0,\alpha]$

This gives by (5.3.4):

(5.3.11) $\qquad \mu(t_m+T_{n_m},x_m,s_m) = \mu(t_m+r_m,x_m,s_m)$.

We may assume without loss of generality that $\{r_m\}$ is convergent to some number $r \in [0, \alpha]$.

So, we have

(5.3.12) $\quad t_m + r_m \longrightarrow t + r$

and then

(5.3.13) $\quad z \in D_\mu(t+r, x)$.

Thus we obtain

(5.3.14) $\quad z \in \bigcup \{ D_\mu(t+r, x) : r \in [0, \alpha] \}$.

The proof is completed.

Theorem 5.3.2

Let $(R, X, R_+; \mu)$ be a pseudo-process such that for every $(x, s) \in X \times R_+$ the mapping

(5.3.15) $\quad \tau \longmapsto \mu(\tau, x, s)$

is continuous and that there exists $\lambda > 0$ such that

(5.3.16) $\quad \mu(\tau + \lambda, x, s) = \mu(\tau, x, s)$ for every τ, x, s.

Assume that $\tilde{\mu}$ is a pseudo-process belonging to $H[\mu]$. Then, for every $(\tau, x) \in R \times X$

$$J_{\tilde{\mu}}(\tau, x) \subset \bigcup \{ J_\mu(\tau + r, x) : r \in [0, \lambda] \}.$$

Proof:

Let y be a point of $J_{\tilde{\mu}}(t, x)$. So, there are sequences $\{t_m\}$, $\{x_m\}$, $\{s_m\}$ such that

$$\tau_m \longrightarrow \tau, \quad x_m \longrightarrow x \text{ and } s_m \longrightarrow \infty \quad \text{as } m \longrightarrow \infty$$

and

(5.3.17) $\quad y = \mathrm{Lt}\, \tilde{\mu}(\tau_m, x_m, s_m) \quad$ as $m \longrightarrow \infty$.

On the other hand, there is a sequence $\{\sigma_k\}$ of real numbers such that $\sigma_k \longrightarrow \infty$ as $k \longrightarrow \infty$ and for every m

(5.3.18) $\mu(\tau_m + \sigma_k, x_m, s_m) \longrightarrow \tilde{\mu}(\tau_m, x_m, s_m)$ as $k \longrightarrow \infty$.

Thus, we can find a sequence $\{k_m\}$ of positive integers such that $k_m \longrightarrow \infty$ as $m \longrightarrow \infty$ and

(5.3.19) $\rho(\mu(\tau_m + k_m, x_m, s_m), \tilde{\mu}(\tau_m, x_m, s_m)) < \frac{1}{m}$.

From triangle inequality we get

$$(5.3.20)\quad \begin{cases} \rho(y, \mu(\tau_m + \sigma_{k_m}, x_m, s_m)) \leq \rho(y, \tilde{\mu}(\tau_m, x_m, s_m)) + \\ \qquad + \rho(\tilde{\mu}(\tau_m, x_m, s_m), \mu(\tau_m + \sigma_{k_m}, x_m, s_m)) \\ \qquad \leq \rho(y, \tilde{\mu}(\tau_m, x_m, s_m)) + \frac{1}{m} \end{cases}$$

and so, (see (5.3.17))

(5.3.21) $\mu(\tau_m + k_m, x_m, s_m) \longrightarrow y$ as $m \longrightarrow \infty$.

We can find a sequence $\{p_m\}$ of positive integers and $\{r_m\}$ of real numbers such that

(5.3.22) $\tau_m + \sigma_{k_m} = \tau_m + p_m\lambda + r_m, \quad r_m \in [0, \lambda]$.

This gives by (5.3.16)

(5.3.23) $\mu(\tau_m + \sigma_{k_m}, x_m, s_m) = \mu(\tau_m + r_m, x_m, s_m)$.

We may assume without loss of generality that $\{r_m\}$ is convergent to some number $r \in [0, \lambda]$.

So, we have

(5.3.24) $\tau_m + r_m \longrightarrow \tau + r$

and then

(5.3.25) $y \in J\mu(\tau + r, x)$.

Thus we get

(5.3.26) $y \in \bigcup \{J\mu(\tau + r, x) : r \in [0, \lambda]\}$.

This is the end of the proof.

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