INVESTIGATING THE EFFECTS OF AN OBLATE SATELLITE ON EQUILIBRIUMPOINTS FOR PHOTOGRAVITATIONAL ELLIPTIC RESTRICTED THREE-BODY PROBLEM (ER3BP)

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DECEMBER, 2019

DECLARATION

I declare that the work in this dissertation titled **Investigating Effects of an Oblate Satellite on Equilibrium Points for Photogravitational Elliptic Restricted Three-Body Problem** has been performed by me under the supervision of Dr. (Mrs.) A. Umar and Prof. B. Sani in the department of Mathematics, Ahmadu Bello University Zaria. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other institution.

SA'AD Maryam____

Name of student Signature Date

CERTIFICATION

This Thesis titled "**INVESTIGATING THE EFFECTS OF AN OBLATE SATELLITE ON EQUILIBRIUM POINTS FOR PHOTOGRAVITATIONAL ELLIPTIC RESTRICTED THREE-BODY PROBLEM (ER3BP).**"by SA'AD Maryam meets the regulations governing the award of the degree of Master of Science of the Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

Dr. (Mrs.) A. Umar ______________________________________
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Member, Supervisory Committee Signature Date

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External, Examiner Signature Date

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Prof. Hafiz A Abubakar______ ____________________ ___________________
Dean, School of Postgraduate studies Signature Date

DEDICATION

This research work is dedicated to Almighty ALLAH, the creator of the universe. Also to my late father, Mallam Sa'ad Yusha'u, may ALLAH forgive him (Ameen), and to all members of my family.

ACKNOWLEDGMENT

My greatest appreciation goes to Almighty ALLAH my creator who has been my greatest companion and guide. To Him I owe my life, my love and all my progress.

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ABSTRACT

We investigate in the elliptic framework of the restricted problem of three-bodies, the motion of an oblate infinitesimal particle in the vicinity of a luminous primary and an oblate secondary. The locations and stability of the equilibrium points are found to be affected by the eccentricity, oblatenessand radiation pressure parameters. We highlights the effects of the said parameters on the locations of the triangular points and stabilityusing CEN X-3 model. The triangular points are also found to be stable for $0 < \mu < \mu_C$; where μ is the mass ratio $\left(\mu \leq \frac{1}{2}\right)$. Further analysis indicates that collinear points remain unstable in spite of the introduction of these parameters.

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Figure 5.13: Effect of semi major axis on the collinear points $L_{i(1,2,3)}$ of CEN X-3

LIST OF SYMBOLS

μ	Mass ratio
μ_c	Critical value of the mass parameter
m_1	Mass of the bigger primary
m_2	Mass of the smaller primary
Ω	Potential force function
A	Oblateness of the third body
A_2	Oblateness of the smaller primary body
r_1	Distance from the third body to the bigger primary
r_2	Distance from the third body to the smaller primary
$a = 1 - \alpha$	Semi-major axis of the system
e	Eccentricity of the system
$q = 1 - \beta$	Radiation coefficient of the bigger primary.
n	Mean motion

CHAPTER ONE

INTRODUCTION

1.1 Historical Background

Classical mechanics may be defined as that branch of mechanics which deals with the deterministic motion of bodies. This is the oldest branch of mechanics. It dates back to the scientific evolution and is said to have originated in Galileo's verification of the heliocentric theory by means of telescope. It is also connected to Galileo’s study of falling bodies although some of the ideas are found in the early works of Oresume, who studied the motion of a uniformly accelerated body. The law of inertia was already known to Davinci while Kepler discovered several laws of planetary motion. (Amuda, 2014)

The origin of classical mechanics goes back to Koperniks theory of the solar system. This theory opened up the possibility that celestial and terrestrial matter might be of the same nature, hence, governed by the same law of mechanics using Hugyens results on centripetal acceleration. Hooke and Wren realized that this force diminishes as the inverse square of the distance. Newton identified this force as the same force, which makes objects fall near the surface of the earth and succeeded in computing the orbits of celestial bodies using the inverse square law. Based on these laws, it was possible to solve what appeared complicated, the two body problem which describes the motion of two bodies of finite masses moving under a mutual gravitational attraction. The motion of any of the planets around the sun constitutes a two –body problem. Example is the solar system, which consists of the Sun and its nine planets. (Laraba, 2012)

The most celebrated problem of space dynamics is the problem of three-body. The three-body problem is defined in terms of three bodies with arbitrary masses attracting one another according to the Newton law of gravitation which are free to move in space in any given manner. (Hussain, 2017)

A classical example of the 3BP is the Sun- Planet -Planet system. When they are considered as point masses, they form what is known as the main problem of the lunar theory. The degenerated case of the 3BP is the restricted three body problem (R3BP) which describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses called the primaries moving in circular or elliptic orbits around their common center of mass on account of their mutual gravitational attraction and the infinitesimal mass not influencing the motion of the primaries.

The R3BP is one of the most widely studied areas in space dynamics as well as celestial mechanics. Significant results have been produced by well-known mathematicians and scientists in an attempt to understand and predict the motion of natural bodies. The application of R3BP spans solar system dynamics, lunar theory, motion of space crafts and stellar dynamics. A typical example of the R3BP is seen in a system made up of the Sun and Jupiter as primaries and then a Trojan asteroid assuming the role of the infinitesimal mass in the Sun –Jupiter system. This problem began with Euler in 1767, in connection with his lunar theories which brought about his major accomplishment in the introduction of synodic (rotating) coordinate system. This led to the discovery of the Jacobian integral by Jacobi in 1836.(Amuda, 2014)

In recent times, many properties such as shape, surface area, light, perturbing forces are taken into consideration in describing the motion of satellites (both artificial and natural), meteorites, asteroid and their stability.

1.2 Statement of the Problem

Singh and Umar (2013b) investigated in the elliptic framework of the R3BP the application of double pulsars to the axisymmetric R3BP in which both pulsars are oblate. They found that the triangular and collinear points are unstable.Their study however did not consider one of the primaries to be luminous. In this study we shall consider the motion of an oblate third body (satellite, test particle, asteroids, comet or circumbinary planet) around a luminous primary and an oblate secondary. This is applicable to both the solar system (Sun-Planet-Moon-system) and stellar system(Cen X-3).

1.3 Aim and Objectives

The aim of this research work is to investigate and examine the motion of an oblate third body in the frame work of ER3BP when the primary is radiating and the secondary is an oblate spheroid. In line with this idea, the objectives of the study are to:

1. Locate the collinear and triangular equilibrium points in the neighbourhood of binary System CEN X-3.
2. Examine the linear stability of the equilibrium points in the neighbourhood of the binary system CEN X-3.

1.4 Significance/Justification of the Study

The study of the restricted three-body problem over the last 200 years has produced significant results and is the backbone of space technology. Man-made satellites are modelled and built as test particles in orbits of celestial bodies. Results considering the shape of the Earth and radiation effect of the Sun, in the Sun-Earth-Satellite system are examples. The equilibrium points of the R3BP are of enormous importance to space application in the past, present and future. The equilibrium solutions of these systems are widely used in many branches of astronomy, both for constant and variable masses as in the case of the Roche model for binary star system (Lyapunov, 1956).

The collinear equilibrium points are good spots for space-based observatories given their location and accessibility and they provide easy access to orbits in the case of lunar and Earth orbits; though all of them are unstable. This means a spacecraft to be kept at or orbiting around them will require correction maneuvers typically to be performed at the expenses of a propellant mass. The triangular equilibrium points are also very interesting from the point of view of astronomical objects; possible location of interplanetary dust and asteroids, hence they have been suggested as convenient sites to locate future space colonies. As such scientists and astronomers have been devoted to set up a colony at one of the two triangular points of the Earth-Moon system.

This research exposes us to the behavior of an oblate test particle when perturbations such as radiation pressure force and oblateness of the primaries act on it.The participating bodies in the classical R3BP are assumed to be strictly spherical in shape, but in reality, it has been proved that several heavenly bodies are sufficiently oblate or triaxial, for instance,Earth, Jupiter and Saturn, are oblate while the moon of the Earth, Pluto and its moon Charon are triaxial. Also, the minor planets and meteoroids have irregular shapes. In these cases, on account of the small dimensions of the bodies in comparison with their distances from the primaries, they are considered to be point masses, but in many cases the dimension of the bodies are larger than the distances from their respective primaries. Thus, the results obtained are far from realistic. The lack of sphericity, (triaxiality, oblateness and so on) of the celestial bodies causes large perturbation from two- body orbit. The motions of artificial Earth satellites are examples of this.

Theseforesaid perturbations can actually cause a large deviation in the location of libration points and may change the entire pattern of their stability which is very vital in the lunch of artificial satellites and various outer space probes. The motions of artificial Earth satellites are examples of this and also several studies that have considered one or both primaries as oblate spheroids. These include among many others SubbaRao and Sharma (1975), Sharma (1987), Khanna and Bhatnagar (1999), Singh and Ishwar (1999) Douskos and Markellos (2006) Abdurraheem and Singh (2006) Kushvah (2008), Singh and Begha (2011), Singh and Leke (2012) and Singh and Umar (2015).

Hence in connection to the above studies, it is reasonable to study the case when the bigger primary is a source of radiation and the smaller one an oblate spheroid. This model will have enormous applications in various astronomical problems such as space mission design.

1.5 Research Methodology

The effects of the oblateness of an artificial satellite on the orbits around the triangular points of the Earth –Moon system was studied by Singh and Umar (2013a). Thus following Singh and Umar (2013a), we wish to modify the equations of motion of the infinitesimal particle with respect to the primaries, locate the equilibrium points and study their linear stability when the primary is radiating and the secondary is an oblate spheroid in the vicinity of the binary system CEN X-3.Lastly, we intend to use theMathematicasoftware 10.3 for numerical computations and plotting of graphs where necessary. We shall show both numerically and graphically the effects of the perturbing forces (radiation pressure of the primary, oblateness of the secondary body and third body (test particle) as well, and the eccentricity of the orbit of the primaries on the position and stability of the equilibrium points.

1.6 Theoretical Frame Work

1.6.1 Three Body Problem (3BP)

The study of the motion of three particles which are free to move in space under their mutual gravitational influence and initially move in any given manner is called the three-body problem (3BP).

1.6.2 RESTRICTED THREE BODY-PROBLEMS (R3BP)

The restricted three-body problem (R3BP) is a simplified form of the general 3BP, in which one of the bodies is of infinitesimal mass, and therefore does not influence the motion of the remaining two massive bodies called the primaries.

1.6.3 Circular Restricted Three Body-Problem (CR3BP)

At the Newtonian level, the CR3BP deals with the motion of the infinitesimal mass in the gravitational field of the primaries, which revolve in circular orbit around theircenter of mass.Consider an isolated dynamical system consisting of three gravitationally interacting point masses, m_1, m_2, and m_3. Suppose, however, that the third mass, m_3, is so much smaller than the other two that it has a negligible effect on their motion. Suppose, further, that the first two masses, m_1 and m_2, execute circular orbits about their common center of mass. Later, we shall investigate this simplified problem, which is generally known as the circular restricted three-body problem.

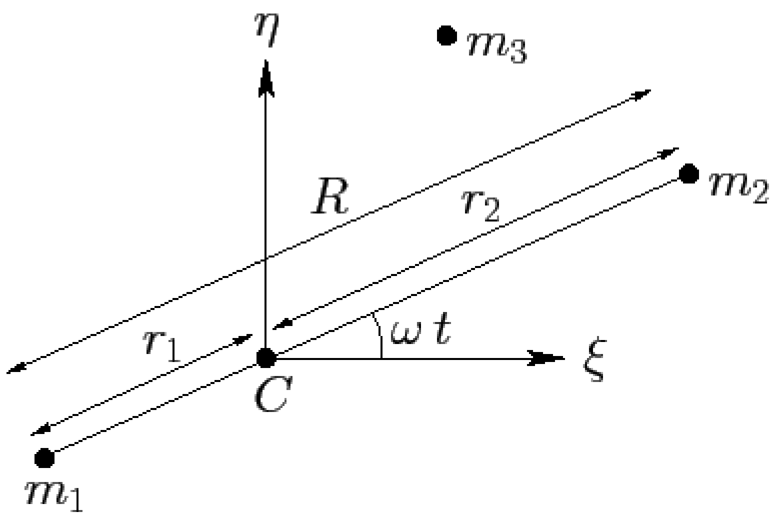

Figure 1.1:*The circular restricted three-body problem.*

1.6.4 Elliptical Restricted Three Body-Problem (ER3BP)

The ER3BP deals with the motion of the infinitesimal mass in the gravitational field of the primaries, which revolve in elliptic orbits around their center of mass.

1.6.5 Photo-gravitational Restricted Three Body-Problem

An R3BP is called photogravitational when one or both of the primaries is /are a source of light.

1.6.6 Radiation and Radiation Pressure

Radiation pressure implies an interaction between electromagnetic radiation and bodies of various types, including clouds of particles or gases. The interaction can be absorption, reflection, or both (the common case). Bodies also emit radiation and thereby experience resulting pressure. Radiation from a material or body is the transfer of heat energy in the form of electromagnetic wave which does not require an electric medium or the emitting of energy. Any hot object radiates heat energy in form of electromagnetic wave of all wavelengths. However, the heat emitted by a surface depends on the nature of the surface, temperature of the body and the surface area. Example of a radiating body is the Sun in the solar system.

The radiation pressure force F_p changes with distance by the same law as the gravitational force F_g and acts opposite to radiation pressure. It is possible that the force will lead to a reduction of the effective mass of the particle, and since this reduction depends on the properties of the particles, it is acceptable to speak about a reduced mass, thus, the resultant force on the particle is

$$F = (F_g - F_p) = F_g \left(1 - \frac{F_p}{F_g}\right) = q \,,$$

Where $q = F_g \left(1 - \frac{F_p}{F_g}\right)$ is the mass reduction factor and the force of the body is given by $F_p = (1-q)F_g$, such that $0 < (1-q) << 1$.

We note that;

If q=1, the radiation pressure has no effect.

If $0 < q < 1$, the gravitational force is greater than the radiation force.

If q = 0, the radiation force balances the gravitational force.

If $q < 0$, the radiation pressure overrides the gravitational attraction.

Following these discoveries, several studies such as Hamilton and Burns (1992), Singh and Ishwar (1999), Kunitsyn (2000), Singh (2009), Singh and Leke (2010), Singh and Umar (2012 a,b), Singh and Taura (2013) have produced significant results in the study of the restricted three- body problem under different assumptions, by taking into account the radiation pressure forces.

1.6.7 Ellipsoid

An ellipsoid is a closed quadric surface that is a three-dimensional analogue of an ellipse.

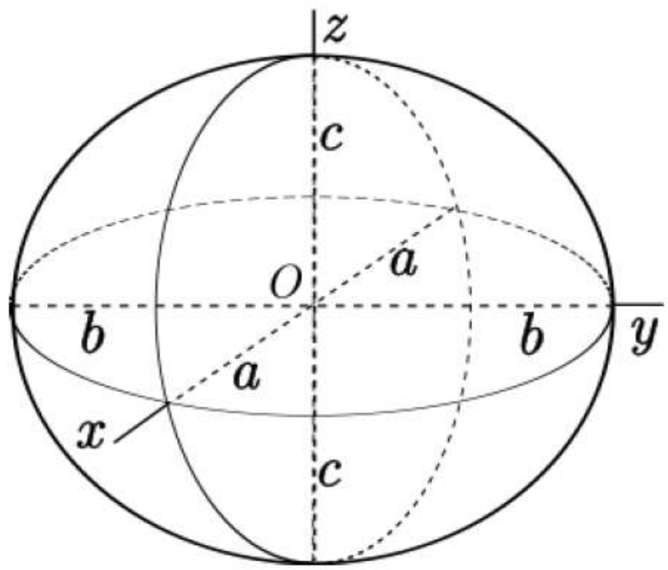

Figure 1.2: Shape of an ellipsoid.

1.6.8 Oblateness.

In the figure above, if $a = b > c$, we have an oblate spheroid. In other words, an ellipsoid having a polar axis shorter than the diameter of the equatorial circle whose plane bisects it is known as an oblate spheroid, i.e., its two out of three moments of inertia are equal.

We denote $A_i \, (i = 1,2)$ for the oblateness coefficients of the bigger and smaller primary respectively such that $0 < A_i << 1$(McCuskey, 1963). The values $A_i = \frac{AE_i^2 - AP_i^2}{5R^2} (i = 1,2)$, where AE_1 and AE_2 are the equatorial radii, AP_1 and AP_2, the polar radii of the bigger and smaller primaries respectively and R the distance between the primaries. Oblate spheroids are contracted along a line, whereas prolate spheroids are elongated. It can be formed by rotating an ellipse about its minor axis, forming an equator with the end points of the major axis. As with all ellipsoids, it can also be described by the lengths of three mutually perpendicular principal axes, which are in this case two arbitrary equatorial semi major axes and one semi –minor axis.

The R3BP assumes that masses concerned are spherically symmetrical in homogeneous layers, but it is found that celestial bodies, such as Saturn and Jupiter are sufficiently oblate (Beatty et al.1999). The minor planets (e.g. Ceres) and meteoroids have irregular shapes (Millis et al.1978; Norton & Chitwood 2008). The oblateness or triaxiality of a body can produce perturbation deviation from the two- body motion. The orbits of the fifth satellite of Jupiter Amalthea is one of the most striking examples of perturbations arising from oblateness in the solar system (Moulton, 1914).

Rotation in stars produces an equatorial bulge due to centrifugal force, and as a resultof the rapid rotation after formation of Neutron stars, white and black dwarfs, they may be considered oblate. A neutron star on formation can rotate at rate of nearly a thousand rotation per second (Du et al.2009).The millisecond pulsar PRSB1937+21, spinning about 642 times a second and the pulsar PSRJ1748-422ad, spinning 716 times a second are some of the swiftest spinning pulsar (Hessels et al. 2006).This has motivated several researchers such as Subbarao & Sharma (1975), Elipe &Ferrer (1985), Singh and Ishwar (1999), Abdurraheem & Singh (2006), Vishnu et al (2008),Singh (2011,2012), Singh & Umar (2012a,b), Singh & Leke (2013), Singh & Umar (2013a,b,c, 2014a,b,c,2015,) to include oblateness of one or both primaries in their studies. It is the approximate shape of many planets and celestial bodies, including Saturn, Jupiter and to a lesser extend the

Earth. It is therefore the most used geometric figure for defining reference ellipsoids, upon which cartographic and geodetic system are based.

1.6.9 Synodic or Rotating Co-ordinate System

If the system of coordinates is such that the $\xi - \eta$ plane rotates in the positive direction with an angular velocity equal to that of the common velocity of one primary with respect to the other keeping the origin fixed, then the coordinate system is known as a synodic co-ordinate system. The synodic co-ordinate system is accelerated, since it is rotating with a velocity.

1.6.10 Eccentricity

The orbital eccentricity of an astronomical body is the amount by which its orbit deviates from a perfect circle and is usually denoted by e. Where for $e < 1$, the orbit is elliptic; for e = 0, the orbit is circle; for $e = 1$ the orbit is parabolic and for e >1, the orbit is hyperbolic.

1.6.11 Equilibrium points

The equilibrium points are those points at which the velocity and the acceleration of an infinitesimal mass are zero. CR3BP admits five equilibrium points in the plane of motion of the primaries, three are the collinear points L_1, L_2, & L_3 lying on the line connecting the primaries, while the other two are the triangular points L_4& L_5 forming equilateral triangles with the primaries. The collinear equilibrium points were found by Euler in 1767 while the triangular equilibrium points were worked out by Lagrange in 1772.

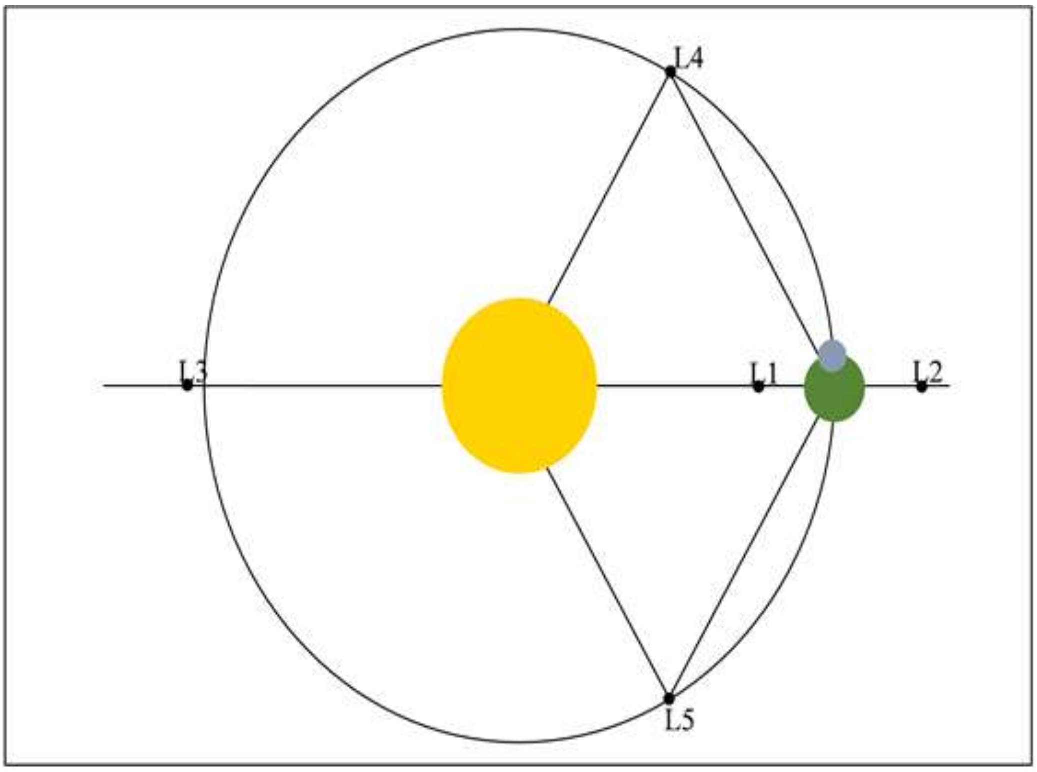

Figure 1.3: The five Equilibrium Points of the Sun-Earth-Moon System

Aside from the five equilibrium points of the classical restricted three-body problem, further studies have revealed other types of equilibrium points, examples are the out – of-plane equilibrium points, which are referred to as co-planer points. These equilibrium points have been found in the studies of the restricted three-body problem with variable masses (Singh and Leke 2010,2012, 2013). In the case of problem of constant masses, these points are found by expressing the equation of motion in three- dimensional form, when radiation or oblateness is involved (see Radzievsky 1950,1953; Douskos and Markellos 2006; Shankaran et al.2011;Singh and Umar 2013a,b,c and 2015).

1.6.12 Stability of Equilibrium Points

The motion of an infinitesimal particle near one of equilibrium points is said to be stable if for a given small displacement and small velocity of the particle oscillates for considerable time around that point and when the time elapses it returns to the same point, otherwise unstable.

1.6.13 Stability of a Linear System

In ordinary differential equations, the stability of linear systems is determined by the eigen values of the coefficient matrix. The locations of the infinitesimal body would be displaced a little from the equilibrium point due to perturbations. If the resultant motion of the infinitesimal body is a rapid departure from the vicinity of the point, we call such location of equilibrium point an "unstable one" otherwise it is stable. In order to examine the stability of the orbit in the vicinity of the equilibrium points, we apply this small displacement method by shifting the origin or coordinates of the infinitesimal mass and linearizing the equations of motion around the coordinates of the equilibrium. The variational equations of motion corresponding to the dynamical system are derived, which in turn through the trial solutions are transformed to a matrix form and a characteristic equation of the variational equations of the dynamical system is obtained. The stability of the solutions depends on the nature of the characteristic roots.

(i) For complex roots; the equilibrium point is asymptotically stable when all roots have negative real parts, and unstable when some or all roots have positive real part, while multiple complex roots can either be stable or unstable.

(ii) For pure imaginary roots; equilibrium point is stable, though not asymptotically stable. If there are multiple roots, the solution contains mixed terms (i.e., periodic and secular terms), the equilibrium point is unstable.

(iii) For real roots; the equilibrium point is stable if all the roots are both real and negative, but unstable if any of the roots is positive. This statement is also true for multiple roots.

1.6.14 Kepler's Laws of Planetary Motion

- Each planet moves, relative to the sun, in an elliptical orbit, the Sun being at one of the two foci of the ellipse.
- The rate of motion in the elliptical orbit is such that the vector pointing to the position of the planet relative to the sun spans equal areas of the orbital plane in equal times.
- The square of the orbital period T, is proportional to the cube of the semi-major axis of the orbital ellipse.

1.6.15 Newton's Laws of Motion

- A body continues in its state of rest, or in a uniform motion in a straight line, unless it is compelled to change that state by a force impressed upon it.
- The acceleration of motion of a body is directly proportional to the force to which it is subjected, and inversely proportional to its mass, and takes place in the direction in which the force acts.
- To every action there is an equal and opposite reaction; or, the mutual actions of two bodies are always equal and oppositely directed.

1.6.16 Perturbation

In astronomy, perturbation is the deviation in the motion of a celestial object caused either by the gravitational force of a passing object or by a collision with it. For example, predicting the Earth's orbit around the Sun would be rather straightforward if not for the slight perturbations in its orbital motion caused by the gravitational influence of the other planets.

1.6.17 Stability in the sense of Lyapunov

A dynamic system $\dot{x} = f(x)$ is *Lyapunov stable* or *internally stable* about an equilibrium point x_{eq} if state trajectories are confined to a bounded region whenever the initial condition x_0 is chosen sufficiently close to x_{eq}.

Mathematically, given $R > 0$ there always exists $r > 0$ so that if $\|x_0 - x_{eq}\| < r$, then $\|x(t) - x_{eq}\| < R$ for all $t > 0$. As seen in figure 1.4, R defines a desired confinement region, while r defines the neighborhood of x_{eq} where x_0 must belong so that $x(t)$ does not exit in the confinement region.

An equilibrium point x_{eq} of $\dot{x} = f(x)$ ($i.e., f(x_{eq}) = 0$) is

i. Locally stable, if for every $R > 0$ there exist $r > 0$, such that $\|x(0) - x_{eq}\| < r$, implies $\|x(t) - x_{eq}\| < R$ for all $t \geq 0$.

ii. Locally asymptotically stable, if locally stable and $\|x(0) - x_{eq}\| < r \Rightarrow \lim_{t \to \infty} x(t) = x_{eq}$

iii. Globally asymptotically stable, if it is asymptotically stable for all $x(0) \in R^n$

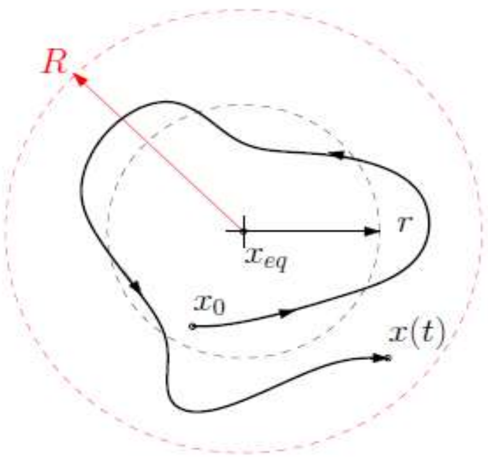

Figure 1.4: Lyapunov Stability Region

CHAPTER TWO

LITERATURE REVIEW

2.1Introduction

The R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of two massive bodies which move in either eliptic, hypabolic or parabolic orbits around their common center of mass. This is based on the condition that their mutual attraction and the infinitesimal mass does not influence the motion of the primaries. The approximate circular motion of the planets around the sun and the small masses of asteroids and the satellites of the planets compared to the planets masses originally suggested the formulation of the restricted three body. There is an enormous literature devoted to works that have been done in the field of the restricted three-body problem. This includes both analytic and numerical developments. Review of some published works as related to the study is given in this chapter. The R3BP is of fundamental importance in mechanics, with significant application to astrodynamics, but because no general solution in the CR3BP is available, particular solutions are sought so as to obtain insight into the problem. These particular solutions are referred to as the equilibrium or libration points. There are five such points for the classical restricted problem, two triangular and three collinear, (Szebehely 1967). The efforts of many famous Mathematicians have been devoted to this difficult problem, including Singh and Umar (2012a,b, 2013a,b 2014a,b,c 2015),Singh and Amuda(2014),Singh and Taura (2013) and Singh and Muhammad(2007). In 1772, Euler first introduced a synodic (rotating) coordinate system. Jacobi in 1866 subsequently discovered an integral of motion in this coordinate system which he independently discovered and which is now known as the Jacobi integral. While Hill in 1878 used this integral to show that the

Earth-Moon distance remains bounded from above for all time (assuming his model for the Sun-Earth-Moon system is valid), and Szebehely, V. (1967a)Gave the most precise lunar theory of his time.

Furthermore the condition for linear stability of the triangular points was established by Routh (1875) when the condition $0 < \mu < \mu_c$ is satisfied, all roots of the characteristics equation are pure imaginary, which lead to pure oscillatory solution. This solution was later referred to as "critical solution" by Lyapunov in 1956.The collinear points were found to be unstable in the linear and nonlinear sense, as Routh in 1875 original result shows when interpreted in the light of Lyapunov's. While Radzievskii (1950,1953) studied the photogravitational R3BP, and found that an allowance for direct solar radiation pressure result in a change in the positions of the equilibrium point. El-Shaboury (1990) had a possibility of nine libration points for small values of oblateness in the photogravitational R3BP when infinitesimal mass is an axisymmetric body and one of the finite masses is a spherical luminous body, while the other is assumed to be an axisymmetric non luminous body.

Subsequently Ragouse and Zagouras (1993) studied the existence of the out-of-plane equilibrium points in the photogravitational restricted 3BP. They noted that indeed these equilibrium points exist. The location and stability of five Lagrangian equilibrium points in the planer, circular restricted three-body was investigated by Murray (1994) when the third body is acted on by a variety of drag forces. The approximate locations of the displaced equilibrium points were calculated for small mass ratios and a simple criterion for their linear stability was derived.In addition a different kind of restricted three-body problem formulated by Robe (1977) was studied by Murray (1994) when linear drag forces are present. They discussed in particular, the stability of the models' equilibrium points discussed by Robe and when the body is assumed to be a Roche ellipsoid.

Furthermore, Singh and Ishwar (1999) generalized the R3BP by considering both primaries to be source of radiation and as well as oblate spheroids. They observed that the equations of motion and the locations of the equilibrium points are affected by the radiation pressure forces and oblateness of the primaries. They found out that the triangular points are stable for $0 < \mu < \mu_C$ and unstable for $\mu_c < \mu \leq \frac{1}{2}$, where μ and μ_C are the mass ratio and critical mass ratio, and further established that the range of stability depends upon radiating and oblateness coefficients. The same year Khanna and Bhatnagar (1999) investigated the R3BP by modeling the bigger primary as an oblate spheroid and the smaller primary as a triaxial rigid body. They maintained the unstable nature of the collinear points and the stable nature of the triangular points for some values of the mass parameter. They further observed that the triangular points have long or short periodic elliptical orbits in the same range of the mass parameter. In addition Sharma et al. (2001a) further studied the existence and stability of libration points in the R3BP, considering the case where both primaries are triaxial rigid bodies. They found three collinear points which are unstable and two triangular points which are stable for certain values of the mass parameter. The stability of equilibrium points in the Relativistic restricted three- body problem was carried out by Douskos and Perdios (2002). They observed that the stability of the triangular points so determined were contrary to other findings as the region of linear stability in the parameter space was obtained. The position of the collinear points were approximated by series expansion and their stability was similarly determined. Further, they found that these collinear points are always unstable. The linear stability of triangular equilibrium points in the generalized photogravitational restricted three- body problem with pointing –Robertson drag was studied by Ishwar and Kushvah (2006). Here, they considered the smaller primary to be an oblate spheroid while the bigger one is a radiating body. Obviously, the equations of motion depend on the radiation pressure force, oblateness and PR drag. They verified all classical results involving photogravitational and oblateness in restricted three body problem and concluded with the help of the roots of the characteristics equation, that triangular equilibrium points are unstable. Singh and Leke (2010) studied the stability of the photo gravitational restricted three body problem with variable masses in which the masses of the luminous primaries vary isotropically in accordance with the unified Meshcherskii law, and their motion takes place within the frame work of the Gylden-Meshcheskii problem. They showed that for the system with constant masses the collinear and coplanar points are unstable, while the triangular points are conditionally stable. They also showed that the stability of equilibrium points varying with time are unstable using the Lyaponov characteristics numbers(LCN).

Also Singh and Umar (2012a) looked at motion of an infinitesimal mass around seven equilibrium points in the framework of the elliptic restricted three body problem under the assumption that the primary of the system is a non –luminous, oblate spheroid and the secondary is luminous. Practical application of the problem is the study of the dynamical evolution of dust particles in orbits around a binary system with a dark degenerate primary and a secondary stellar companion. The condition of stability of the motion around the triangular points was shown to depend on the mass ratio and the critical mass ratio of the system. Singh and Umar (2012b) followed up immediately by studying these equilibrium points in the elliptic restricted three body problem with radiating and oblate primaries, and applying their study to Gamma Leporis and Altair. Several studies such as Singh and Amuda (2014) have all examined the photogravitational CR3BP with different perturbing forces. Also Singh and Umar (2013a) investigated the equilibrium points and collinear points in the axi-symmetric R3BP when both primaries and the third body are oblate. Subsequently Singh and Umar (2013b, 2014a) investigated the effects of the luminosity and oblateness of both primary bodies and the effect of the triaxiality of the bigger primary on the collinear libration points of the binary systems Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB and Xi Bootis, and binary pulsars, moving in elliptic orbits around their common centre of mass. They found that, the collinear points are affected by the eccentricity oblateness, radiation and triaxiality factors; the collinear points however remain unstable. Singh and Umar (2015) examined the effect of the oblateness of an artificial satellite on the orbits around the triangular points of the Earth-moon system in the axisymmetric ER3BP. They concluded that the frequencies of the long and short periodic orbits around the triangular points with their orientations, eccentricity, semi-major and semi-minor axes are influenced by the eccentricity of the orbits, oblateness and semi-major axis of the primaries and of the third body as well.

CHAPTER THREE

LOCATIONS AND LINEAR STABILITY OF THE TRIANGULAR EQUILIBRIUM POINTS

3.1 Introduction

In this chapter the equations of motion of an oblate test particle under the influence of a luminous primary and oblate secondary are presented in a dimensionless-pulsating coordinates. The locations and linear stability of the triangular equilibrium points are thereafter obtained.

3.2 Mathematical Model

As already indicated, the works of Singh and Umar (2013b) were carried out on the assumption that both the primaries are oblate. But in our own case we consider one of the primaries to be radiating. Thus, we write the equations of motion of an oblate test particle under the influence of a luminous primary and oblate secondary in a dimensionless-pulsating coordinate system (ξ, η, ζ) while adopting their own parameters. The equations in our case will therefore be as follows;

$$\begin{aligned} \xi'' - 2\eta' &= \Omega_\xi \\ \eta'' + 2\xi' &= \Omega_\eta \\ \zeta'' &= \Omega_\zeta \end{aligned} \tag{3.1}$$

where the force function is

$$\Omega = (1-e^2)^{-1/2}\left[\frac{\xi^2+\eta^2}{2} + \frac{1}{n^2}\left\{\frac{(1-\mu)q}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)A}{2r_1^3} + \frac{\mu A}{2r_2^3} + \frac{\mu A_2}{2r_2^3}\right\}\right] \tag{3.2}$$

and

$$r_1^2 = (\xi + \mu)^2 + \eta^2 + \zeta^2$$
$$r_2^2 = (\xi + \mu - 1)^2 + \eta^2 + \zeta^2 \qquad (3.3)$$

The mean motion is

$$n^2 = \frac{1}{a}\left(1 + \frac{3e^2}{2} + \frac{3A_2}{2}\right) \qquad (3.4)$$

where the prime represents differentiation w.r.t. the eccentric anomaly E and $r_i (i = 1,2)$ are the distances between the third body and the primaries, n, a, e, A and, A_2 are the mean motion, semi-major axis, eccentricities of the orbits, and oblateness respectively.

3.3 Variational Equation

Let (ξ_0, η_0) be the coordinates of an equilibrium point under consideration; and (α, β) be the small displacements in the coordinates of the equilibrium point. Then the displacement, velocity and acceleration of the infinitesimal particle are:

$$\xi = \xi_0 + \alpha, \eta = \eta_0 + \beta$$

$$\dot{\xi} = \dot{\alpha}, \dot{\eta} = \dot{\beta} \qquad (3.5)$$

$$\ddot{\xi} = \ddot{\alpha}, \ddot{\eta} = \ddot{\beta}$$

Making use of equation (3.5) in equation (3.1), we obtain

$$\Omega_\xi = \Omega_\xi(\xi, \eta) = \Omega_\xi(\xi_0 + \alpha, \eta_0 + \beta)$$
$$\Omega_\eta = \Omega_\xi(\xi, \eta) = \Omega_\eta(\xi_0 + \alpha, \eta_0 + \beta) \qquad (3.6)$$

Expanding (3.6) using Taylor's expansion, and considering only linear terms in α *and* β , we have:

$$\Omega_{\xi} = \Omega_{\xi}\left(\xi, \eta\right) = \Omega_{\xi}\left(\xi_0, \eta_0\right) + \alpha\,\Omega^0_{\xi\xi} + \beta\,\Omega^0_{\xi\eta}$$
$$\Omega_{\eta} = \Omega_{\eta}\left(\xi, \eta\right) = \Omega_{\eta}\left(\xi_0, \eta_0\right) + \alpha\,\Omega^0_{\eta\xi} + \beta\,\Omega^0_{\eta\eta} \qquad (3.7)$$

Where the super script denotes that the partial derivatives are to be evaluated at the equilibrium point $\left(\xi_0, \eta_0\right)$.

At equilibrium point, $\Omega_{\xi}\left(\xi_0, \eta_0\right) = \Omega_{\eta}\left(\xi_0, \eta_0\right) = 0$

Hence equations (3.7) becomes

$$\Omega_{\xi} = \alpha\,\Omega^0_{\xi\xi} + \beta\,\Omega^0_{\xi\eta}$$
$$\Omega_{\eta} = \alpha\,\Omega^0_{\eta\xi} + \beta\,\Omega^0_{\eta\eta} \qquad (3.8)$$

Substituting equations (3.5) and (3.8) in the equations of motion (3.1), we obtain the variational equations of motion as:

$$\ddot{\alpha} - 2\dot{\beta} = \alpha\,\Omega^0_{\xi\xi} + \beta\,\Omega^0_{\xi\eta}$$
$$\ddot{\beta} + 2\dot{\alpha} = \alpha\,\Omega^0_{\eta\xi} + \beta\,\Omega^0_{\eta\eta}$$

(3.9)3.4 Characteristic Equation

Let us take trial solutions for the variational equations (3.9) as:

$\alpha = Ae^{\lambda t}$ *and* $\beta = Be^{\lambda t}$ (3.10)

where A, B *and* λ are parameters to be determined

Now

$$\dot{\alpha} = A\lambda e^{\lambda t} \qquad \dot{\beta} = B\lambda e^{\lambda t}$$
$$\ddot{\alpha} = A\lambda^2 e^{\lambda t} \qquad \ddot{\beta} = B\lambda^2 e^{\lambda t} \qquad (3.11)$$

Substituting equations (3.10) and (3.11) in equations (3.9), we have

$$A\lambda^2 e^{\lambda t} - 2B\lambda e^{\lambda t} = Ae^{\lambda t}\Omega^0_{\xi\xi} + Be^{\lambda t}\Omega^0_{\xi\eta}$$
$$B\lambda^2 e^{\lambda t} + 2A\lambda e^{\lambda t} = Ae^{\lambda t}\Omega^0_{\eta\xi} + Be^{\lambda t}\Omega^0_{\xi\xi}$$

Dividing by $e^{\lambda t}$,we have

$$A\lambda^2 - 2B\lambda = A\Omega^0_{\xi\xi} + B\Omega^0_{\xi\eta}$$
$$B\lambda^2 + 2A\lambda = A\Omega^0_{\eta\xi} + B\Omega^0_{\xi\xi}$$

or

$$A\left(\lambda^2 - \Omega^0_{\xi\xi}\right) + B\left(-2\lambda - \Omega^0_{\xi\eta}\right) = 0$$
$$B\left(\lambda^2 - \Omega^0_{\eta\eta}\right) + A\left(2\lambda - \Omega^0_{\eta\xi}\right) = 0$$

The last equation will be a non-trivial solution for $A \text{ and } B$ if

$$\begin{vmatrix} \lambda^2 - \Omega^0_{\xi\xi} & -2\lambda - \Omega^0_{\xi\eta} \\ 2\lambda - \Omega^0_{\eta\xi} & \lambda^2 - \Omega^0_{\eta\eta} \end{vmatrix} = 0$$

Expanding the determinant yields

$$\lambda^4 - \left(\Omega^0_{\eta\eta} + \Omega^0_{\xi\xi} - 4\right)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 = 0 \quad (3.12)$$

Equation (3.12) constitutes the characteristic equation of the system.

3.5 Jacobian Constant

Multiplying equations (3.1) by $2\xi', 2\eta'$ and 2ζ respectively, we obtain

$$2\xi'\xi'' - 4\xi'\eta' = \frac{2\xi'\partial\Omega}{\partial\xi}$$

$$2\eta'\eta'' + 4\eta'\xi' = \frac{2\eta'\partial\Omega}{\partial\eta}$$

$$2\xi'\xi'' = \frac{2\xi'\partial\Omega}{\partial\xi}$$

Adding these equations together, we get

$$2\xi'\xi'' + 2\eta'\eta'' + 2\zeta'\zeta'' = 2\left(\xi'\frac{\partial\Omega}{\partial\xi} + \eta'\frac{\partial\Omega}{\partial\eta} + \zeta'\frac{\partial\Omega}{\partial\zeta}\right)$$

$$\frac{d}{dt}\left(\xi'^2 + \eta'^2 + \zeta'^2\right) = 2\frac{d\Omega}{dt}$$

Integrating we have

$\xi'^2 + \eta'^2 + \zeta'^2 = 2\Omega - C$ (3.13)where C in the constant of integration known as Jacobian constant and equation (3.13) is called Jacobian Integral

3.6 Locations of the Equilibrium Points

3.6.1 Locations of triangular equilibrium points

The stationary solutions of the R3BP can be found by setting

$\dot{\xi} = \dot{\eta} = \dot{\zeta} = \ddot{\xi} = \ddot{\eta} = \ddot{\zeta} = 0$ in the equations of motion (3.1). They are the solutions of the equations, $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$ that is multiply equations(3.1) by $2\xi', 2\eta'$ and $2\zeta'$ respectively, we get

$$2\xi'\xi'' - 4\xi'\eta' = 2\xi'\Omega_\xi$$

$$2\eta'\eta'' - 4\eta'\xi' = 2\eta'\Omega_\eta$$

$$2\zeta'\zeta'' = 2\zeta'\Omega_\zeta$$

Adding all together,

$$2\xi'\xi'' + 2\eta'\eta'' + 2\zeta'\zeta'' = 2\left(\xi'\Omega_\xi + \eta'\Omega_\eta + \zeta'\Omega_\zeta\right) = 2\Omega_\varepsilon$$

Integrating w.r.t.ξ, we obtain

$$\xi'^2 + \eta'^2 + \zeta'^2 = 2\Omega - C$$

where (C=Jacobian constant)

Now, differentiating $w.r.t.\xi, \eta, \zeta$

$$\Omega_\xi = \left(1-e^2\right)^{-1/2}\left[\xi - \frac{1}{n^2}\left\langle \frac{(1-\mu)(\xi+\eta)q}{r_1^3} + \frac{\mu(\xi+\mu-1)}{r_2^3} + \frac{3(1-\mu)(\xi+\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu(\xi+\mu-1)\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2(\xi+\mu-1)}{2r_2^5}\right\rangle\right] \tag{3.14}$$

$$\Omega_\eta = \left(1-e^2\right)^{-1/2}\left[\eta\left(1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3(1-\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2}{2r_2^5}\right\}\right)\right] \tag{3.15}$$

$$\Omega_\zeta = \left(1-e^2\right)^{-1/2}\left[\zeta\left(\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3(1-\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2}{2r_2^5}\right)\right]$$

(3.16)

$\Omega_\varepsilon = \Omega_\eta = \Omega_\zeta = 0$ Singular points occur when

, Soequation (3.15) will befor the triangular points, $\eta \neq 0$

i.e.

$$\eta\left(1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3(1-\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2}{2r_2^5}\right\}\right)$$ Since $\eta \neq 0$, it follows that

$$1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3(1-\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2}{2r_2^5}\right\} = 0 \quad (3.17)$$

$$\xi - \xi_2 - \frac{1}{n^2}\left\{\begin{array}{l}\frac{(1-\mu)q\,(\xi-\xi_2)}{r_1^3} + \frac{\mu(\xi-\xi_2)}{r_2^3} + \frac{3(1-\mu)A\,(\xi-\xi_2)}{2r_1^5} + \frac{3\mu A\,(\xi-\xi_2)}{2r_2^5} + \\ \frac{3\mu A_2\,(\xi-\xi_2)}{2r_2^5}\end{array}\right\} = 0$$

(3.18)

From equations (3.14) and (3.17) we have

$$-\xi_1 - \frac{\mu}{n^2 r_2^3}(\xi - \xi_1 - \xi + \xi_2) - \frac{3\mu\mathrm{A}}{2n^2 r_2^5}(\xi - \xi_1 - \xi + \xi_2) - \frac{3\mu\mathrm{A}_2}{2n^2 r_2^5}(\xi - \xi_1 - \xi + \xi_2) = 0$$

(3.19)

$$\xi_1 + \frac{\mu(\xi_2 - \xi_1)}{n^2 r_2^3} + \frac{3\mu\mathrm{A}(\xi_2 - \xi_1)}{2n^2 r_2^5} + \frac{3\mu\mathrm{A}_2(\xi_2 - \xi_1)}{2n^2 r_2^5} = 0 \quad (3.20)$$

Subtracting (3.14) from (3.19)

$$\xi - \xi_2 - \frac{(1-\mu)q(\xi-\xi_2)}{n^2 r_1^3} - \frac{\mu(\xi-\xi_2)}{n^2 r_2^3} - \frac{3(1-\mu)\mathrm{A}(\xi-\xi_2)}{2n^2 r_1^5} - \frac{3\mu\mathrm{A}(\xi-\xi_2)}{2r_2^5 n^2} - \frac{3\mu\mathrm{A}_2(\xi-\xi_2)}{2n^2 r_2^5}$$

$$-\xi + \frac{(1-\mu)q(\xi-\xi_1)}{n^2 r_1^3} + \frac{\mu(\xi-\xi_2)}{n^2 r_2^3} + \frac{3(1-\mu)\mathrm{A}(\xi-\xi_1)}{2n^2 r_1^5} + \frac{3\mu\mathrm{A}(\xi-\xi_2)}{2n^2 r_2^5} + \frac{3\mu\mathrm{A}_2(\xi-\xi_2)}{2n^2 r_2^5} = 0$$

$$-\xi_2 - \frac{(1-\mu)q}{n^2 r_1^3}(\xi - \xi_2$$

$$\xi_2 + \frac{(1-\mu)q(\xi_1-\xi_2)}{n^2 r_1^3} + \frac{3(1-\mu)\mathrm{A}}{2n^2 r_1^5}(\xi_1-\xi_2) = \xi_2 - \frac{(1-\mu)q(\xi_2-\xi_1)}{n^2 r_1^3} - \frac{3(1-\mu)\mathrm{A}(\xi_2-\xi_1)}{2n^2 r_1^5} = 0 \quad (3.21)$$

Now, the distance between the primaries $\xi_2 - \xi_1 = 1 \Rightarrow \xi_1 = -\mu$ and $\xi_2 = 1-\mu$ (3.22)

Substituting (3.22) in (3.20) gives

$$\xi_1 + \frac{\mu(\xi_2-\xi_1)}{n^2 r_2^3} + \frac{3\mu\mathrm{A}(\xi_2-\xi_1)}{2n^2 r_2^5} + \frac{3\mu\mathrm{A}_2(\xi_2-\xi_1)}{2n^2 r_2^5} = -\mu + \frac{\mu}{n^2 r_2^3} + \frac{3\mu\mathrm{A}}{2n^2 r_2^5} + \frac{3\mu\mathrm{A}_2}{2n^2 r_2^5} = 0$$

$$-\mu\left[1 - \frac{1}{n^2 r_2^3} - \frac{3\mathrm{A}}{2n^2 r_2^5} - \frac{3\mu\mathrm{A}_2}{2n^2 r_2^5}\right] = 0 \quad -\mu \neq 0$$

Since $-\mu \neq 0$ then

$$1 - \frac{1}{n^2 r_2^3} - \frac{3A}{2n^2 r_2^3} - \frac{3A_2}{2n^2 r_2^3} = 0 \quad (3.23)$$

Substituting (3.22) in (3.21) gives

$$(1-\mu) - \frac{(1-\mu)q}{n^2 r_1^3} - \frac{3(1-\mu)\mathrm{A}}{2n^2 r_1^5} = 0 = (1-\mu)\left[1 - \frac{q}{n^2 r_1^3} - \frac{3\mathrm{A}}{2n^2 r_1^5}\right] = 0$$

Since $(1-\mu) \neq 0 \Rightarrow 1 - \frac{q}{n^2 r_1^3} - \frac{3A}{2n^2 r_1^5} = 0$ (3.24)

From (3.23)

$$n^2 - \frac{1}{r_2^3} - \frac{3A}{2r_2^5} - \frac{3A_2}{2r_2^5} = 0 \Rightarrow n^2 = \frac{1}{r_2^3} + \frac{3A}{2r_2^5} - \frac{3A_2}{2r_2^5}$$

with $A = A_1 = A_2 = 0$ we have

$$n^2 = \frac{1}{r_2^3} \Rightarrow r_2 = \frac{1}{n^{2/3}} \quad (3.25)$$

But $n^2 = \frac{1}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}\right)$ with $A_2 = 0$

We have $n^2 = \frac{1}{a}\left(1+\frac{3e^2}{2}\right)$(3.26)

$$n^2 = \frac{q}{r_1^3}+\frac{3A}{2r_1^5} \quad \text{With } A = 0$$

$$n^2 = \frac{q}{r_1^3}\frac{1}{a}\left(1+\frac{3e^2}{2}\right) \Rightarrow r_1^3 = \frac{q}{n^2} \Rightarrow r_1 = \frac{q^{1/3}}{n^{2/3}} \text{ (3.27)}$$

$$\frac{1}{r_2^3} = n^2 = \frac{1}{a}\left(1+\frac{3e^2}{2}\right) \text{(3.28)}$$

Using (3.28), (3.27) becomes

$$n^2 = \frac{1}{a}\left(1+\frac{3e^2}{2}\right) \Rightarrow n^{2/3} = \frac{1}{a^{1/3}}\left(1+\frac{e^2}{2}\right) \Rightarrow \frac{1}{n^{2/3}} = a^{1/3}\left(1-\frac{e^2}{2}\right)$$

$$r_1^3 = \frac{q}{n^2} \Rightarrow \frac{1}{r_1^3} = \frac{1}{r_1^3} = \frac{n^2}{q} = \frac{1}{aq}\left(1+\frac{3e^2}{2}\right)$$

From (3.27)

$$n^2 = \frac{q}{r_1^3} = \frac{1}{a}\left(1+\frac{3e^2}{2}\right)$$

$$\frac{1}{r_1^3} = \frac{1}{aq}\left(1+\frac{3e^2}{2}\right)$$

With oblateness of the secondary and of the third body r_2 will change slightly by say ε

$$\Rightarrow \varepsilon + \frac{1}{n^{2/3}} = \varepsilon + a^{1/3}\left(1-\frac{e^2}{2}\right) \text{(3.29)}$$

$$\Rightarrow r_1 = \frac{q^{1/3}}{n^{2/3}} = (aq)^{1/3}\left(1-\frac{e^2}{2}\right) \text{(3.30)}$$

and

$$\frac{1}{r_2} = \frac{1}{\varepsilon + a^{1/3}\left(1-\frac{e^2}{2}\right)}$$

$$r_2 = \varepsilon + a^{1/3}\left(1-\frac{e^2}{2}\right)$$

$$r_1 = (aq)^{1/3}\left(1-\frac{e^2}{2}\right)$$

$$\frac{1}{r_2} = \frac{1}{\varepsilon + a^{1/3}\left(1-\frac{e^2}{2}\right)} = \frac{1}{a^{1/3}\left(1-\frac{e^2}{2}\right)\left(1+\varepsilon\left(a^{1/3}\left(1-\frac{e^2}{2}\right)\right)\right)}$$

$$\frac{1}{r_2} = a^{1/3}\left(1-\frac{e^2}{2}\right)\left[1-\varepsilon a^{-1/3}\left(1-\frac{e^2}{2}\right)\right]$$

$$\frac{1}{r_2^3} = a^{-1}\left(1+\frac{3e^2}{2}\right)\left(1-3\varepsilon a^{-1/3}\left(1+\frac{e^2}{2}\right)\right) = \frac{1}{a}\left(1+\frac{3e^2}{2}-3a^{-1/3}\varepsilon\right)(3.31)$$

$$\frac{1}{r_2^5} = \frac{1}{a^{5/3}}\left(1+\frac{5e^2}{2}-5\varepsilon a^{-1/3}\right)(3.32)$$

Using (3.24)

$$n^2 - \frac{q}{r_1^3} - \frac{3A}{2r_1^5} = 0 \Rightarrow n^2 = \frac{q}{r_1^3} + \frac{3A}{2r_1^5}$$

i.e.

$$n^2 = \frac{1}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}\right)$$

Using (3.4) (3.23) (3.31) and (3.32) gives

$$n^2 = \frac{1}{r_2^3} - \frac{3A}{2r_2^5} + \frac{3A_2}{2r_2^3}$$

$$1+\frac{3e^2}{2}+\frac{3A_2}{2} = 1+\frac{3e^2}{2}-3\varepsilon a^{-1/3}+\frac{3A}{2a^{2/3}}+\frac{3A_2}{2a^{2/3}}-\frac{15\varepsilon A}{2a}-\frac{15\varepsilon A_2}{2a}$$

$$\frac{3A_2}{2}-\frac{3A}{2a^{2/3}}-\frac{3A_2}{2a^{2/3}} = -3\varepsilon\left(a^{-1/3}+\frac{5A}{2a}+\frac{5A_2}{2a}\right)$$

$$3\left(\frac{A_2}{2}-\frac{A}{2a^{2/3}}-\frac{A_2}{2a^{2/3}}\right) = -3\varepsilon\left(a^{-1/3}+\frac{5A}{2a}+\frac{5A_2}{2a}\right)$$

$$\varepsilon = -\frac{\frac{A_2}{2}-\frac{A}{2a^{2/3}}-\frac{A_2}{2a^{2/3}}}{a^{-1/3}+\frac{5A}{2a}-\frac{5A_2}{2a}} = -\frac{\frac{1}{2}\left(A_2 - Aa^{-2/3} - A_2 a^{-2/3}\right)}{a^{-1/3}\left(1+\frac{5A}{2a^{2/3}}+\frac{5A_2}{2a^{2/3}}\right)}$$

$$\varepsilon = -\frac{a^{1/3}}{2}\left(A_2 - Aa^{-2/3} - A_2a^{-2/3}\right)\left(1 - \frac{5A}{2a^{2/3}} - \frac{5A_2}{2a^{2/3}}\right)$$

$$\varepsilon = -\frac{a^{1/3}}{2}\left(A_2 - Aa^{-2/3} - A_2a^{-2/3}\right) (3.33)$$

Substituting (3.33) in (3.29), gives

$$r_2 = \varepsilon + a^{1/3}\left(1 - \frac{e^2}{2}\right) = \frac{a^{1/3}}{2}\left(A_2 - A - A_2a^{-2/3}\right) + a^{1/3}\left(1 - \frac{e^2}{2}\right)$$

$$r_2 = a^{1/3}\left(1 - \frac{e^2}{2} - \frac{A_2}{2} + \frac{A}{2a^{2/3}} + \frac{A_2}{2a^{2/3}}\right)$$

$$r_2^2 = a^{2/3}\left(1 - e^2 - A_2\left(1 - a^{-2/3}\right) + a^{-2/3}A\right) (3.34)$$

$$r_1^3 = \frac{q}{n^2} \quad n^2 = \frac{1}{a}\left(1 + \frac{3e^2}{2} + \frac{3A_2}{2}\right) \Rightarrow \frac{1}{n^2} = a\left(1 - \frac{3e^2}{2} - \frac{3A_2}{2}\right)$$

$$\frac{1}{n^{2/3}} = a^{1/3}\left(1 - \frac{e^2}{2} - \frac{A_2}{2}\right)$$

$$r_1^3 = (aq)\left(1 - \frac{3e^2}{2} - \frac{3A_2}{2}\right) \Rightarrow r_1 = \frac{q^{1/3}}{n^{2/3}}$$

$$\frac{1}{n^{2/3}} = a^{1/3}\left(1 - \frac{e^2}{2} - \frac{A_2}{2}\right)$$

$$r_1 = (aq)^{1/3}\left(1 - \frac{e^2}{2} - \frac{A_2}{2}\right)$$

$$r_1^2 = (aq)^{2/3}\left(1 - e^2 - A_2 + Aa^{-2/3}q^{-5/3}\right) \qquad (3.35)$$

$$r_1^2 = (\xi + \mu)^2 + \eta^2$$

$$r_2^2 = (\xi + \mu - 1)^2 + \eta^2$$

$$n^2 = \frac{q}{r_1^3} + \frac{3A}{2r_1^5}$$

$$\frac{1}{a}\left(1 + \frac{3e^2}{2} + \frac{3A_2}{2}\right) = \frac{q}{aq}\left(1 + \frac{3e^2}{2} - 3(aq)^{-1/3}\varepsilon\right) + \frac{3A}{2(aq)^{5/3}}\left(1 + \frac{3e^2}{2} - 5\varepsilon(aq)^{-1/3}\right)$$

$$1 + \frac{3e^2}{2} + \frac{3A_2}{2} = 1 + \frac{3e^2}{2} - 3(aq)^{-1/3}\varepsilon + \frac{3A}{2a^{2/3}q^{5/3}} - \frac{15A\varepsilon}{2(aq)^{4/3}}$$

$$\frac{3A_2}{2}-\frac{3A}{2a^{2/3}q^{5/3}}=-3\left((aq)^{-1/3}-\frac{5A}{2(aq)^{4/3}}\right)$$

$$\varepsilon_1=-\frac{\left(\frac{A_2}{2}-\frac{A}{2a^{2/3}q^{5/3}}\right)}{\left((aq)^{-1/3}-\frac{5A}{2(aq)^{4/3}}\right)}$$

$$\varepsilon_1=-\frac{1}{2}\left(A_2-Aa^{-2/3}q^{-5/3}\right)(aq)^{1/3}\left(1-\frac{5A}{2(aq)}\right)$$

$$=-\frac{(aq)^{1/3}}{2}\left(A_2-Aa^{-2/3}q^{-5/3}\right)$$

$$r_1=\varepsilon_1+(aq)^{1/3}\left(1-\frac{e^2}{2}\right)=-\frac{(aq)^{1/3}}{2}\left(A_2-Aa^{-2/3}q^{-5/3}\right)+(aq)^{1/3}\left(1-\frac{e^2}{2}\right)$$

$$=(aq)^{1/3}\left(1-\frac{e^2}{2}-\frac{A_2}{2}+\frac{A}{2a^{2/3}q^{5/3}}\right)$$

$$r_1^2=(aq)^{2/3}\left(1-e^2-A_2+Aa^{-2/3}q^{-5/3}\right)$$

$$r_1^2-r_2^2=2\xi+2\mu-1$$

Using (3.34) and (3.35), gives

$$(aq)^{2/3}\left(1-e^2-A_2+Aa^{-2/3}q^{-5/3}\right)-a^{2/3}\left(1-e^2-A_2a^{-2/3}+Aa^{-2/3}\right)=2\xi+2\mu-1$$

$$2\xi=1-2\mu+(aq)^{2/3}\left(1-e^2-A_2+Aa^{-2/3}q^{-5/3}\right)-a^{2/3}\left(1-e^2-A_2a^{-2/3}+Aa^{-2/3}\right)$$

$$\xi=\frac{1}{2}-\mu+\frac{1}{2}(aq)^{2/3}\left(1-e^2-A_2+Aa^{-2/3}q^{-5/3}\right)-\frac{1}{2}a^{2/3}\left(1-e^2-A_2a^{-2/3}+Aa^{-2/3}\right)$$
(3.36)

$$\xi+\mu=\frac{1}{2}+\frac{1}{2}\left[(aq)^{2/3}\left(1-e^2-A_2\right)\right]-a^{2/3}\left(1-e^2-A_2a^{-2/3}+Aa^{-2/3}\right)$$

$$=\frac{1}{2}\left(1+(aq)^{2/3}\left(1-e^2-A_2\right)-a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{-2/3}\right)\right)$$

$$(\xi+\mu)^2=\frac{1}{4}\left(1+2(aq)^{2/3}\left(1-e^2-A_2\right)-2a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{-2/3}\right)\right)$$

$$r_1^2=(\xi+\mu)^2+\eta^2\Rightarrow\eta^2=r_1^2-(\xi+\mu)^2$$

$$\eta^2 = (aq)^{2/3}\left(1-e^2-A_2+Aa^{-2/3}q^{-5/3}\right)-\frac{1}{4}\begin{pmatrix}1+2(aq)^{2/3}\left(1-e^2-A_2\right)\\ -2a^{2/3}\left(1-e^2-A_2a^{-2/3}+Aa^{-2/3}\right)\end{pmatrix}$$

$$\eta = \left[(aq)^{2/3}\left(1-e^2-A_2+Aa^{-2/3}q^{-5/3}\right)-\frac{1}{4}\begin{pmatrix}1+2(aq)^{2/3}\left(1-e^2-A_2+Aa^{-2/3}q^{-5/3}\right)-\\ 2a^{2/3}\left(1-e^2-A_2a^{-2/3}+Aa^{-2/3}\right)\end{pmatrix}\right]^2$$

(3.37)

Equations (3.36) and (3.37) give the positions of the triangular points $l_{4,5}$ given by

$(\xi,\pm\eta)\, l_{4,5} - (\xi \pm \eta)$

3.7 Linear Stability of the Equilibrium Points

3.7.1 Stability of the triangular equilibrium points

We examine the ability of an equilibrium point to restrain the motion of a displaced body in its vicinity. If its motion is a rapid departure from the vicinity, we call such a position an unstable one.

In evaluation of the second partial derivatives of equation (3.2), in the case of triangular points, we have

$$\Omega = \left(1-e^2\right)^{1/2}\left[\frac{\xi^2+\eta^2}{2}+\frac{1}{n^2}\left(\frac{(1-\mu)q}{r_1}+\frac{\mu}{r_2}+\frac{(1-\mu)A}{2r_1^3}+\frac{\mu A}{2r_2^3}+\frac{\mu A_2}{2r_2^3}\right)\right]$$

$$\Omega_\xi = \left(1-e^2\right)^{1/2}\left[\xi-\frac{1}{n^2}\begin{pmatrix}\frac{(1-\mu)(\xi+\mu)q}{r_1^3}+\frac{\mu(\xi+\mu-1)}{r_2^3}+\frac{3(1-\mu)(\xi+\mu)A}{2r_1^5}\\ +\frac{3\mu(\xi+\mu-1)A}{2r_2^5}+\frac{3\mu(\xi+\mu-1)A_2}{2r_2^5}\end{pmatrix}\right]$$

$$\Omega_{\xi\xi}=\left(1-e^2\right)^{1/2}\left[\begin{array}{l}1-\dfrac{1}{n^2}\left(\dfrac{(1-\mu)q}{r_1^3}+\dfrac{\mu}{r_2^3}+\dfrac{3(1-\mu)A}{2r_1^5}+\dfrac{3\mu A}{2r_2^5}+\dfrac{3\mu A_2}{2r_2^5}\right)+\\ \dfrac{1}{n^2}\left(\begin{array}{l}\dfrac{(1-\mu)(\xi+\mu)^2 q}{r_1^5}+\dfrac{3\mu(\xi+\mu-1)^2}{2r_2^5}+\dfrac{3(\xi+\mu-1)^2 A}{2r_2^5}\\ +\dfrac{15\mu(\xi+\mu-1)^2 A}{2r_2^7}+\dfrac{15\mu(\xi+\mu-1)^2 A_2}{2r_2^7}\end{array}\right)\end{array}\right]$$

$$\Omega_{\eta}=\left(1-e^2\right)^{1/2}\left[\eta\left(1-\frac{1}{n^2}\left(\frac{(1-\mu)q}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3(1-\mu)A}{2r_1^5}+\frac{3\mu A}{2r_2^5}+\frac{3\mu A_2}{2r_2^5}\right)\right)\right]$$

$$\Omega_{\eta\eta}=\left(1-e^2\right)^{1/2}\left[\begin{array}{l}1-\dfrac{1}{n^2}\left(\dfrac{(1-\mu)q}{r_1^3}+\dfrac{\mu}{r_2^3}+\dfrac{3(1-\mu)A}{2r_1^5}+\dfrac{3\mu A}{2r_2^5}+\dfrac{3\mu A_2}{2r_2^5}\right)+\\ \dfrac{\eta^2}{n^2}\left(\dfrac{3(1-\mu)q}{r_1^5}+\dfrac{3\mu}{2r_2^5}+\dfrac{15(-\mu)A}{2r_1^7}+\dfrac{15\mu A}{2r_2^7}+\dfrac{15\mu A_2}{2r_2^7}\right)\end{array}\right]$$

$$\Omega_{\xi\eta}=\left(1-e^2\right)^{1/2}\left[\frac{\eta}{n^2}\left(\begin{array}{l}\dfrac{3(1-\mu)(\xi+\mu)q}{r_1^5}+\dfrac{3\mu(\xi+\mu-1)}{r_2^5}+\dfrac{15(1-\mu)(\xi+\mu)A}{2r_1^7}\\ +\dfrac{15\mu(\xi+\mu-1)A}{2r_2^7}+\dfrac{15\mu(\xi+\mu-1)}{2r_2^7}\end{array}\right)\right]$$

$$r_1=(aq)^{1/3}\left(1-\frac{e^2}{2}-\frac{A_2}{2}+\frac{A}{2a^{2/3}q^{5/3}}\right)$$

$$\frac{1}{r_1^5}=(aq)^{-5/3}\left(1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{5A}{2a^{2/3}q^{5/3}}\right)$$

$$r_1^3=(aq)\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{3A}{2a^{2/3}q^{5/3}}\right)$$

$$\frac{1}{r_1^3}=\frac{1}{(aq)\left(1-\dfrac{3e^2}{2}-\dfrac{3A_2}{2}+\dfrac{3A}{2a^{2/3}q^{5/3}}\right)}=(aq)^{-1}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A}{2a^{2/3}q^{5/3}}\right)$$

$$n^2=\frac{1}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}\right)$$

$$\frac{1}{n^2}=a\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)$$

$$r_2 = a^{1/3}\left(1-\frac{e^2}{2}-\frac{A_2}{2}+\frac{A_2}{2a^{2/3}}+\frac{A}{2a^{2/3}}\right)$$

$$r_2^2 = a^{2/3}\left(1-e^2-A_2\left(1-a^{-2/3}\right)+Aa^{-2/3}\right)$$

$$r_2^3 = a\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\left(1-a^{-2/3}\right)+\frac{3Aa^{-2/3}}{2}\right)$$

$$\frac{1}{r_2^3} = \frac{1}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A_2}{2a^{2/3}}-\frac{3A}{2a^{2/3}}\right)$$

$$\frac{1}{r_2^5} = \frac{1}{a^{5/3}}\left(1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{5A_2}{2a^{2/3}}-\frac{5A}{2a^{2/3}}\right)$$

$$\frac{1}{r_2^7} = \frac{1}{a^{7/3}}\left(1+\frac{7e^2}{2}+\frac{7A_2}{2}-\frac{7A_2}{2a^{2/3}}-\frac{7A}{2a^{2/3}}\right)$$

$$\xi = \frac{1}{2}-\mu+\frac{1}{2}\left[(aq)^{2/3}\left(1-e^2-A_2\right)-a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{-2//3}\right)\right]$$

$$(\xi+\mu)^2 = \frac{1}{4}\left[1+2(aq)^{2/3}\left(1-e^2-A_2\right)-2a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{-2//3}\right)\right]$$

$$\xi+\mu-1 = -\frac{1}{2}+\frac{1}{2}\left[(aq)^{2/3}\left(1-e^2-A_2\right)-a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{-2//3}\right)\right]$$

$$\xi+\mu-1 = -\frac{1}{2}\left[1-(aq)^{2/3}\left(1-e^2-A_2\right)-a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{-2//3}\right)\right]$$

$$(\xi+\mu-1)^2 = \frac{1}{4}\left[1-2(aq)^{2/3}\left(1-e^2-A_2\right)+2a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{-2//3}\right)\right]$$

Calculation of $\Omega_{\xi\xi}$

$$\Omega_{\xi\xi} = \left(1-e^2\right)^{1/2}\left[\begin{array}{l} 1-\frac{1}{n^2}\left(\frac{(1-\mu)q}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3(1-\mu)A}{2r_1^5}+\frac{3\mu A}{2r_2^5}+\frac{3\mu A_2}{2r_2^5}\right)+ \\ \frac{1}{n^2}\left(\begin{array}{l}\frac{(1-\mu)(\xi+\mu)^2 q}{r_1^5}+\frac{3\mu(\xi+\mu-1)^2}{2r_2^5}+\frac{3(\xi+\mu-1)^2 A}{2r_2^5} \\ +\frac{15\mu(\xi+\mu-1)^2 A}{2r_2^7}+\frac{15\mu(\xi+\mu-1)^2 A_2}{2r_2^7}\end{array}\right)\end{array}\right]$$

$$\frac{(1-\mu)q}{r_1^3} = \frac{(1-\mu)}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A}{2a^{2/3}q^{5/3}}\right)$$

$$\frac{\mu}{r_2^3} = \frac{\mu}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A_2}{2a^{2/3}}-\frac{3A}{2a^{2/3}}\right)$$

$$\frac{3(1-\mu)A}{2r_1^3}=\frac{3(1-\mu)A}{2(aq)^{5/3}}$$

$$\frac{3\mu A}{2r_2^5}=\frac{3\mu A}{2(aq)^{5/3}}$$

$$\frac{3\mu A_2}{2r_2^5}=\frac{3\mu A_2}{2(aq)^{5/3}}$$

$$\left\{\frac{(1-\mu)q}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3(1-\mu)A}{2r_1^5}+\frac{3\mu A}{r_2^5}+\frac{3\mu A_2}{r_2^5}\right\}=\frac{(1-\mu)}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A}{2a^{2/3}q^{5/3}}\right)+$$

$$\frac{\mu}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A}{2a^{2/3}}+\frac{3A_2}{2a^{2/3}}\right)+\frac{3(1-\mu)A}{2(aq)^{5/3}}+\frac{3\mu A}{2a^{5/3}}+\frac{3\mu A_2}{2q^{5/3}}$$

$$\Rightarrow 1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3\mu A_2}{2a^{2/3}}+\frac{3\mu A_2}{2a^{2/3}}+\frac{3(1-\mu)A}{2a^{2/3}q^{5/3}}+\frac{3\mu A}{2a^{2/3}}+\frac{3\mu A_2}{2q^{5/3}}+\frac{3(1-\mu)A}{2a^{2/3}q^{5/3}}$$

$$\frac{1}{n^2}\left(\frac{(1-\mu)q}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3(1-\mu)A}{2r_1^3}+\frac{3\mu A}{2r_2^3}+\frac{3\mu A_2}{2r_2^3}\right)$$

$$\frac{1}{n^2}=a\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)$$

$$=\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)\frac{1}{a}\left(1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3\mu A_2}{2a^{2/3}}-\frac{3\mu A}{2a^{2/3}}+\frac{3(1-\mu)A}{2a^{2/3}q^{5/3}}+\frac{3\mu A}{2a^{2/3}}+\frac{3\mu A_2}{2q^{2/3}}\right)$$

$$=1+\frac{3(1-\mu)A}{2a^{2/3}q^{5/3}}-\frac{3(1-\mu)A}{2a^{2/3}q^{5/3}}$$

$$\frac{3(1-\mu)(\xi+\mu)^2q}{n^2r_1^5}=\frac{3(1-\mu)q\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)}{4(aq)^{5/3}}\begin{pmatrix}1+2(aq)^{2/3}\left(1-e^2-A_2+Aa^{2/3}q^{5/3}\right)\\ -2a^{2/3}\left(1-e^2-A_2+A_2a^{-2/3}+Aa^{2/3}\right)\\ \left(1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{3e^2}{2}-\frac{3A_2}{2}-\frac{5A}{2a^{2/3}q^{5/3}}\right)\\ \left(1-e^2-A_2-\frac{5A}{2a^{-2/3}q^{5/3}}\right)\end{pmatrix}$$

$$\frac{3(1-\mu)(\xi+\mu)^2q}{n^2r_1^5}=\frac{3(1-\mu)q}{4(aq)^{2/3}}\left(\left(1+e^2+A_2+\frac{5A}{2a^{-2/3}q^{-5/3}}\right)\begin{pmatrix}1+2(aq)^{2/3}\begin{pmatrix}1-e^2-A_2\\ +Aa^{-2/3}q^{5/3}\end{pmatrix}\\ -2a^{2/3}\begin{pmatrix}1-e^2-A_2\\ +A_2a^{-2/3}+Aa^{2/3}\end{pmatrix}\end{pmatrix}\right)$$

$$\frac{3(1-\mu)(\xi+\mu)^2 q}{n^2 r_1^5}=\frac{3(1-\mu)}{4(aq)^{2/3}}+\frac{3(1-\mu)e^2}{4(aq)^{2/3}}-\frac{3(1-\mu)A_2}{2(aq)^{2/3}}+\frac{3(1-\mu)}{2}-$$
$$\frac{3(1-\mu)}{2q^{2/3}}-\frac{9(1-\mu)A}{4a^{2/3}q^{5/3}}+\frac{15(1-\mu)A}{4a^{2/3}q^{7/3}}-\frac{15(1-\mu)A}{8a^{4/3}q^{7/3}}$$

$$\frac{3\mu(\xi+\mu-1)^2}{n^2 r_2^5}=\frac{3\mu a}{4a^{5/3}}\left(1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{5A_2}{2a^{2/3}}-\frac{5A}{2a^{2/3}}\right)\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)$$
$$\left(1-2(aq)^{2/3}\left(1-e^2-A_2\right)+2a^{2/3}\left(1-e^2-A_2+A_2a^{2/3}+Aa^{-2/3}\right)\right)$$

$$\frac{3\mu(\xi+\mu-1)^2}{n^2 r_2^5}=\frac{3\mu}{4a^{2/3}}\begin{bmatrix}1+e^2-2A_2-3A+2a^{-2/3}-2(aq)^{2/3}-\frac{5A_2}{2a^{2/3}}+5A_2q^{2/3}+5Aq^{2/3}\\-\frac{5A}{2a^{2/3}}+\frac{2A}{q}\end{bmatrix}$$

$$\frac{3\mu(\xi+\mu-1)^2}{n^2 r_1^5}=\frac{3\mu}{4a^{2/3}}+\frac{3\mu e^2}{4a^{2/3}}-\frac{3\mu A_2}{2a^{2/3}}-\frac{9\mu A}{4a^{2/3}}+\frac{3\mu}{2}-\frac{3\mu q^{2/3}}{2}-\frac{15\mu A_2}{8a^{4/3}}+\frac{15\mu A_2 q^{2/3}}{4a^{2/3}}$$
$$+\frac{15\mu A q^{2/3}}{4a^{2/3}}+\frac{3\mu A}{2a^{2/3}q}-\frac{15\mu A}{8a^{4/3}}$$

$$\frac{15(1-\mu)(\xi+\mu)^2 A}{2n^2 r_1^7}=\frac{15(1-\mu)A}{8(aq)^{7/3}}\left(1+\frac{7e^2}{2}+\frac{7A_2}{2}\right)\begin{bmatrix}1+2(aq)^{2/3}\left(1-e^2-A_2+Aa^{2/3}q^{5/3}\right)-2a^{2/3}\\\left(1-e^2-A_2+Aa^{2/3}\right)a\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)\end{bmatrix}$$

$$\frac{15(1-\mu)(\xi+\mu)^2 A}{2n^2 r_1^7}=\frac{15(1-\mu)A}{8a^{4/3}q^{7/3}}+\frac{15(1-\mu)A}{4a^{2/3}q^{5/3}}-\frac{15(1-\mu)A}{4a^{2/3}q^{7/3}}$$

$$\frac{15\mu(\xi+\mu-1)^2 A}{2n^2 r_1^7}=\frac{15\mu A}{8a^{7/3}}\left(1+\frac{7e^2}{2}+\frac{7A_2}{2}-\frac{7A}{2a^{2/3}}\right)\begin{bmatrix}\begin{pmatrix}1-2(aq)^{2/3}+2(aq)^{2/3}e^2+\\2(aq)^{2/3}A_2+2a^{2/3}-2a^{2/3}e^2\\-2a^{2/3}A_2+2A_2+2A\end{pmatrix}\\a\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)\end{bmatrix}$$

$$\frac{15\mu(\xi+\mu-1)^2 A}{2n^2 r_1^7}=\frac{15\mu A}{8a^{4/3}}-\frac{15\mu A q^{2/3}}{4a^{2/3}}+\frac{15\mu A}{4a^{2/3}}$$

$$\frac{15\mu(\xi+\mu-1)^2 A_2}{2n^2 r_2^7}=\frac{15\mu A_2}{8a^{7/3}}a\left(1-\frac{3e^2}{2}-\frac{3A_2}{2}\right)\left(1+\frac{7e^2}{2}+\frac{7A_2}{2}-\frac{7A_2}{2a^{2/3}}-\frac{7A}{2a^{2/3}}\right)$$
$$\left(1-2(aq)^{2/3}+2a^{2/3}\right)$$

$$\frac{15\mu(\xi+\mu-1)^2 A_2}{2n^2 r_2^7} = a\left(\frac{15\mu A_2}{8a^{7/3}} - \frac{15\mu A_2 q^{2/3}}{4a^{5/3}} + \frac{15\mu A_2}{4a^{5/3}}\right)$$

$$\frac{1}{n^2}\left(\begin{array}{l}\frac{3(1-\mu)(\xi+\mu)^2 q}{r_1^5} + \frac{3\mu(\xi+\mu-1)^2}{r_2^5} + \frac{15(1-\mu)(\xi+\mu)^2 A}{2r_1^7} + \frac{15\mu(\xi+\mu-1)^2 A}{2r_2^7} + \\ \frac{15\mu(\xi+\mu-1)^2 A_2}{2r_2^7}\end{array}\right)$$

$$\Rightarrow \frac{3\mu}{4a^{2/3}} + \frac{3(1-\mu)}{4(aq)^{2/3}} + \frac{3(1-\mu)e^2}{4(aq)^{2/3}} + \frac{3\mu e^2}{4a^{2/3}} + \frac{3}{2} - \frac{3}{2q^{2/3}} - \frac{3(1-\mu)A_2}{4(aq)^{2/3}} + \frac{9\mu A_2}{4a^{2/3}} + \frac{3\mu A}{2a^{2/3}}$$
$$-\frac{3(1-\mu)A}{2(aq)^{2/3}} + \frac{3(1-\mu)A}{2a^{2/3}q^{5/3}} + \frac{15A}{2a^{2/3}} - \frac{21\mu A}{4a^{2/3}q} - \frac{15(1-\mu)A}{4a^{2/3}q^{7/3}}$$

$$\Omega_{\xi\xi} = \left(1-e^2\right)^{-1/2}\left\{\begin{array}{l}\frac{3\mu}{4a^{2/3}} + \frac{3(1-\mu)}{4(aq)^{2/3}} + \frac{3}{2} - \frac{3}{2q^{2/3}} + \left(\frac{3(1-\mu)}{4(aq)^{2/3}} + \frac{3\mu}{4a^{2/3}}\right)e^2 + \\ \left(\begin{array}{l}\frac{3\mu}{2a^{2/3}} - \frac{3(1-\mu)}{2(aq)^{2/3}} + \frac{3(1-\mu)}{2a^{2/3}q^{5/3}} \\ +\frac{15}{2a^{2/3}} - \frac{21\mu}{4a^{2/3}q} - \frac{15(1-\mu)}{4a^{2/3}q^{7/3}}\end{array}\right)A + \left(-\frac{3(1-\mu)}{4(aq)^{2/3}} + \frac{9\mu}{4a^{2/3}}\right)A_2\end{array}\right\}$$

(3.38)

$$\Omega_{\xi\xi} = K + Le^2 + MA + NA_2$$

Calculation of $\Omega_{\eta\eta}$

$$+\frac{3}{2q^{2/3}} - \frac{3\mu}{2q^{2/3}} + \frac{3\mu}{2} + \frac{3e^2}{2q^{2/3}} - \frac{3\mu e^2}{2q^{2/3}} + \frac{3\mu e^2}{2} + \frac{3A_2}{2q^{2/3}} - \frac{3\mu A_2}{2q^{2/3}} + \frac{3\mu A_2}{2} + \frac{3}{2} + \frac{3\mu}{2}$$
$$+\frac{3\mu q^{2/3}}{2} + \frac{3e^2}{2}$$

$$-\frac{3\mu e^2}{2} + \frac{3\mu e^2 q^{2/3}}{2} + \frac{3A_2}{2} - \frac{3\mu A_2}{2} + \frac{3\mu A_2 q^{2/3}}{2} - \frac{3e^2}{2q^{2/3}} + \frac{3\mu e^2}{2q^{2/3}} - \frac{3\mu e^2}{2} - \frac{3e^2}{2} + \frac{3\mu e^2}{2}$$
$$-\frac{3\mu e^2 q^{3/2}}{2} -$$

$$-\frac{3A_2}{2q^{2/3}}+\frac{3\mu A_2}{2q^{2/3}}-\frac{3\mu A_2}{2}-\frac{3A_2}{2}+\frac{3\mu A_2}{2}-\frac{3\mu A_2 q^{2/3}}{2}+\frac{3A_2}{2(aq)^{2/3}}-\frac{3\mu A_2}{2(aq)^{2/3}}$$

$$+\frac{3\mu A_2}{2a^{2/3}}+\frac{3A}{2a^{2/3}q^{5/3}}-\frac{3\mu A}{2a^{2/3}q^{5/3}}+\frac{3\mu A}{2a^{2/3}q}+\frac{3A}{2(aq)^{2/3}}-\frac{3\mu A}{2(aq)^{2/3}}+\frac{3\mu A}{2a^{2/3}}$$

$$\Omega_{\eta\eta}=\left(1-e^2\right)^{-1/2}\left\{\begin{array}{l}\frac{3\mu}{4a^{2/3}}-\frac{3(1-\mu)}{4(aq)^{2/3}}+\frac{3}{2}-\frac{3\mu q^{2/3}}{2}+\frac{3(1-\mu)}{2q^{2/3}}+\left(-\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}\right)e^2+\\ \left(\frac{3\mu}{2a^{2/3}q}+\frac{3(1-\mu)}{2(aq)^{2/3}}+\frac{3(1-\mu)}{2a^{2/3}}+\frac{3\mu}{2a^{2/3}}\right)A+\left(\frac{3(1-\mu)}{4(aq)^{2/3}}+\frac{3\mu}{4a^{2/3}}\right)A_2\end{array}\right\}$$

(3.39)

$$\Omega_{\eta\eta}=O+Pe^2+QA_2+RA$$

Calculation of $\Omega_{\xi\xi}\Omega_{\eta\eta}$

Multiply equation (3.39) by (3.40)

$$\left(\Omega_{\xi\xi}\Omega_{\eta\eta}\right)=\left(K+Le^2+MA_2+NA\right)\left(O+Pe^2+QA_2+RA\right)$$

$$KO+KPe^2+KQA_2+KRA+LOe^2+MOA_2+NOA$$

$$KO+\left(KP+LO\right)e^2+\left(KQ+MO\right)A_2+\left(KR+NO\right)A$$

$$KO\Rightarrow\left\{\frac{3}{2}-\frac{3}{2q^{2/3}}+\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{4(aq)^{2/3}}\right\}\left\{-\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{2q^{2/3}}+\frac{3}{2}+\frac{3\mu q^{2/3}}{2}\right\}$$

$$KO\Rightarrow\frac{9(1-\mu)}{4q^{2/3}}+\frac{9}{4}+\frac{9\mu q^{2/3}}{4}+\frac{9(1-\mu)}{8a^{2/3}q^{4/3}}+\frac{9\mu}{8(aq)^{2/3}}-\frac{9(1-\mu)}{4q^{4/3}}-\frac{9}{4q^{2/3}}-\frac{9\mu q^{2/3}}{4q^{2/3}}$$

$$-\frac{9\mu(1-\mu)}{8a^{4/3}q^{2/3}}$$

$$-\frac{9\mu^2}{16a^{4/3}}+\frac{9\mu(1-\mu)}{8(aq)^{2/3}}+\frac{9\mu^2q^{2/3}}{8a^{2/3}}+\frac{9(1-\mu)^2}{16(aq)^{4/3}}+\frac{9(-\mu)^2}{8a^{2/3}q^{4/3}}+\frac{9\mu(1-\mu)q^{2/3}}{8(aq)^{2/3}}$$

$$KP\Rightarrow\left\{\frac{3}{2}-\frac{3}{2q^{2/3}}+\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{4(aq)^{2/3}}\right\}\left\{-\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}\right\}$$

$$KP\Rightarrow\frac{-9(1-\mu)}{8(aq)^{2/3}}-\frac{9\mu}{8a^{2/3}}+\frac{9(1-\mu)}{8a^{2/3}q^{4/3}}+\frac{9\mu}{8(aq)^{2/3}}-\frac{9\mu(1-\mu)}{8a^{4/3}q^{2/3}}-\frac{9\mu^2}{16a^{4/3}}-\frac{9(1-\mu)^2}{16(aq)^{4/3}}$$

$$KQ \Rightarrow \left\{\frac{3}{2}-\frac{3}{2q^{2/3}}+\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{4(aq)^{2/3}}\right\}\left\{\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}\right\}$$

$$KQ \Rightarrow \frac{9(1-\mu)}{8(aq)^{2/3}}+\frac{9\mu}{8a^{2/3}}-\frac{9\mu}{8(aq)^{2/3}}+\frac{9\mu(1-\mu)}{16a^{4/3}q^{2/3}}+\frac{9\mu^2}{16a^{4/3}}+\frac{9(1-\mu)^2}{16(aq)^{4/3}}+\frac{9\mu(1-\mu)}{16a^{4/3}q^{2/3}}$$

$$KR \Rightarrow \left\{\frac{3}{2}-\frac{3}{2q^{2/3}}+\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{4(aq)^{2/3}}\right\}\left\{\frac{3(1-\mu)}{2a^{2/3}q^{5/3}}+\frac{3\mu}{2a^{2/3}q}+\frac{3(1-\mu)}{2(aq)^{2/3}}+\frac{3\mu}{2a^{2/3}}\right\}$$

$$KR \Rightarrow \frac{9(1-\mu)}{4a^{2/3}q^{4/3}}+\frac{9\mu}{4a^{2/3}q}+\frac{9(1-\mu)}{4(aq)^{2/3}}+\frac{9\mu}{4a^{2/3}}-\frac{9(1-\mu)}{8a^{2/3}q^{4/3}}-\frac{9\mu}{4a^{2/3}}-\frac{9(1-\mu)}{4a^{4/3}q^{7/3}}-\frac{9\mu}{4a^{2/3}q^{5/3}}$$
$$-\frac{9\mu}{4(aq)^{2/3}}+\frac{9\mu(1-\mu)}{4a^{4/3}q^{5/3}}+\frac{9\mu^2}{8a^{4/3}q}$$

$$LO \Rightarrow \left\{\frac{3(1-\mu)}{4(aq)^{2/3}}+\frac{3\mu}{4a^{2/3}}\right\}\left\{-\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{2q^{2/3}}+\frac{3}{2}+\frac{3\mu q^{2/3}}{2}\right\}$$

$$LO \Rightarrow \frac{-9(1-\mu)^2}{16(aq)^{4/3}}-\frac{9\mu(1-\mu)}{8a^{4/3}q^{2/3}}+\frac{9(1-\mu)^2}{8a^{2/3}q^{4/3}}+\frac{9(1-\mu)}{8(aq)^{2/3}}-\frac{9\mu(1-\mu)}{8a^{2/3}}-\frac{9\mu^2}{16a^{4/3}}+\frac{9\mu(1-\mu)}{8(aq)^{2/3}}$$
$$+\frac{9\mu}{8a^{2/3}}+\frac{9\mu^2q^{2/3}}{8a^{2/3}}$$

$$MO \Rightarrow \left\{\frac{9\mu}{4a^{2/3}}-\frac{3(1-\mu)}{4(aq)^{2/3}}\right\}\left\{-\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{2q^{2/3}}+\frac{3}{2}+\frac{3\mu q^{2/3}}{2}\right\}$$

$$NO \Rightarrow \left\{\frac{3\mu}{2a^{2/3}}-\frac{3(1-\mu)}{2(aq)^{2/3}}+\frac{3(1-\mu)}{2a^{2/3}q^{5/3}}+\frac{15}{4a^{2/3}q}-\frac{21\mu}{4a^{2/3}q}-\frac{15(1-\mu)}{4a^{2/3}q^{7/3}}\right\}$$
$$\left\{-\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{2q^{2/3}}+\frac{3}{2}+\frac{3\mu q^{2/3}}{2}\right\}$$

$$NO \Rightarrow -\frac{9\mu^2}{8a^{4/3}}+\frac{9\mu(1-\mu)}{4(aq)^{2/3}}+\frac{9\mu}{4a^{2/3}}+\frac{9\mu^2q^{2/3}}{4a^{2/3}}+\frac{9(1-\mu)^2}{8(aq)^{4/3}}-\frac{9(1-\mu)^2}{4a^{2/3}q^{4/3}}-\frac{9(1-\mu)}{4(aq)^{2/3}}-\frac{9\mu(1-\mu)}{4a^{2/3}}$$
$$-\frac{9(1-\mu)^2}{8a^{4/3}q^{7/3}}+\frac{45\mu(1-\mu)}{16a^{4/3}q^{5/3}}+\frac{9(1-\mu)^2}{4a^{2/3}q^{7/3}}+\frac{63(1-\mu)}{8a^{2/3}q^{5/3}}+\frac{9\mu(1-\mu)}{4a^{2/3}q}-\frac{45(1-\mu)}{16a^{4/3}q^{5/3}}-\frac{45\mu}{16a^{4/3}q}+\frac{45}{8a^{2/3}q}$$
$$+\frac{45\mu}{8a^{2/3}q^{4/3}}+\frac{63\mu^2}{16a^{2/3}q}-\frac{27\mu(1-\mu)}{2a^{2/3}q^{5/3}}-\frac{63\mu^2}{8a^{2/3}q^{4/3}}+\frac{45(1-\mu)^2}{16a^{4/3}q^3}+\frac{45\mu(1-\mu)}{16a^{4/3}q^{7/3}}+\frac{45(1-\mu)^2}{8a^{2/3}q^3}-\frac{45(1-\mu)}{8a^{2/3}q^{4/3}}$$

Combining all

$$\left(\Omega_{\xi\xi}\Omega_{\eta\eta}\right) \Rightarrow KO + KPe^{2} + KQA_{2} + KRA + LOe^{2} + MOA_{2} + NOA$$

$$KP + LO \Rightarrow \left\{ \frac{-9(1-\mu)}{8(aq)^{2/3}} - \frac{9\mu}{8a^{2/3}} + \frac{9(1-\mu)}{8a^{2/3}q^{4/3}} + \frac{9\mu}{8(aq)^{2/3}} - \frac{9\mu(1-\mu)}{8a^{4/3}q^{2/3}} - \frac{9\mu^{2}}{16a^{4/3}} - \frac{9(1-\mu)^{2}}{16(aq)^{4/3}} \right\} +$$

$$\left\{ \begin{array}{l} \frac{-9(1-\mu)^{2}}{16(aq)^{4/3}} - \frac{9\mu(1-\mu)}{8a^{4/3}q^{2/3}} + \frac{9(1-\mu)^{2}}{8a^{2/3}q^{4/3}} + \frac{9(1-\mu)}{8(aq)^{2/3}} - \frac{9\mu(1-\mu)}{8a^{2/3}} - \frac{9\mu^{2}}{16a^{4/3}} + \frac{9\mu(1-\mu)}{8(aq)^{2/3}} + \frac{9\mu}{8a^{2/3}} \\ + \frac{9\mu^{2}q^{2/3}}{8a^{2/3}} \end{array} \right\}$$

$$KP + LO \Rightarrow \left\{ \begin{array}{l} \frac{9(1-\mu)}{8a^{2/3}q^{4/3}} + \frac{9\mu}{8(aq)^{2/3}} - \frac{9\mu(1-\mu)}{4a^{4/3}q^{2/3}} - \frac{9\mu^{2}}{8a^{4/3}} + \frac{9(1-\mu)^{2}}{8(aq)^{4/3}} + \frac{9(1-\mu)^{2}}{8a^{2/3}q^{4/3}} + \\ \frac{9\mu(1-\mu)}{8a^{2/3}} - \frac{9\mu(1-\mu)}{8(aq)^{2/3}} + \frac{9\mu^{2}q^{2/3}}{8a^{2/3}} \end{array} \right\}$$

$$KR + NO \Rightarrow \left\{ \begin{array}{l} \frac{9(1-\mu)}{4a^{2/3}q^{4/3}} + \frac{9\mu}{4a^{2/3}q} + \frac{9(1-\mu)}{4(aq)^{2/3}} + \frac{9\mu}{4a^{2/3}} - \frac{9(1-\mu)}{8a^{2/3}q^{4/3}} - \frac{9\mu}{4a^{2/3}} - \frac{9(1-\mu)}{4a^{4/3}q^{7/3}} \\ - \frac{9\mu}{4a^{2/3}q^{5/3}} - \frac{9\mu}{4(aq)^{2/3}} + \frac{9\mu(1-\mu)}{4a^{4/3}q^{5/3}} + \frac{9\mu^{2}}{8a^{4/3}q} \end{array} \right\} +$$

$$\left\{ \begin{array}{l} \frac{9\mu^{2}}{8a^{4/3}} + \frac{9\mu(1-\mu)}{4(aq)^{2/3}} + \frac{9\mu}{4a^{2/3}} + \frac{9\mu^{2}q^{2/3}}{4a^{2/3}} + \frac{9(1-\mu)^{2}}{8(aq)^{4/3}} - \frac{9(1-\mu)^{2}}{4a^{2/3}q^{4/3}} - \frac{9(1-\mu)}{4(aq)^{2/3}} - \frac{9\mu(1-\mu)}{4a^{2/3}} \\ - \frac{9(1-\mu)^{2}}{8a^{4/3}q^{7/3}} + \frac{45\mu(1-\mu)}{16a^{4/3}q^{5/3}} + \frac{9(1-\mu)^{2}}{4a^{2/3}q^{7/3}} + \frac{63(1-\mu)}{8a^{2/3}q^{5/3}} + \frac{9\mu(1-\mu)}{4a^{2/3}q} - \frac{45(1-\mu)}{16a^{4/3}q^{5/3}} \\ - \frac{45\mu}{16a^{4/3}q} + \frac{45}{8a^{2/3}q} + \frac{45\mu}{8a^{2/3}q^{4/3}} + \frac{63\mu^{2}}{16a^{2/3}q} - \frac{27\mu(1-\mu)}{2a^{2/3}q^{5/3}} - \frac{63\mu^{2}}{8a^{2/3}q^{4/3}} + \frac{45\mu(1-\mu)}{16a^{4/3}q^{7/3}} + \\ \frac{45(1-\mu)^{2}}{8a^{2/3}q^{3}} + \frac{45(1-\mu)}{8a^{4/3}q^{7/3}} \end{array} \right\}$$

$$KR+NO \Rightarrow \left\{ \begin{matrix} \frac{81(1-\mu)}{8a^{2/3}q^{5/3}} - \frac{45\mu}{8a^{2/3}q} + \frac{9\mu}{2a^{2/3}} - \frac{63(1-\mu)}{8a^{2/3}q^{7/3}} - \frac{9\mu}{4a^{2/3}q^{5/3}} - \frac{9(1-\mu)}{4a^{2/3}q^{4/3}} - \frac{9\mu}{4(aq)^{2/3}} \\ + \frac{81\mu(1-\mu)}{16a^{4/3}q^{5/3}} + \frac{81\mu^2}{16a^{4/3}q} + \frac{9\mu(1-\mu)}{4a^{4/3}q^{2/3}} - \frac{9(1-\mu)^2}{4(aq)^{4/3}} + \frac{9\mu(1-\mu)}{4(aq)^{2/3}} + \frac{9\mu^2 q^{2/3}}{4a^{2/3}} - \\ \frac{9(1-\mu)^2}{4a^{2/3}q^{7/3}} - \frac{9\mu(1-\mu)}{4a^{2/3}} + \frac{9(1-\mu)^2}{4a^{2/3}q^{7/3}} + \frac{9\mu(1-\mu)}{4a^{2/3}q} - \frac{45(1-\mu)}{16a^{4/3}q^{5/3}} - \frac{45}{16a^{4/3}q} \\ + \frac{45}{8a^{2/3}q} + \frac{45\mu}{8a^{2/3}q^{4/3}} - \frac{27\mu(1-\mu)}{2a^{2/3}q^{5/3}} - \frac{63\mu^2}{8a^{2/3}q^{4/3}} + \frac{45(1-\mu)^2}{16a^{4/3}q^3} + \\ \frac{45\mu(1-\mu)}{16a^{4/3}q^{7/3}} + \frac{45(1-\mu)^2}{8a^{2/3}q^3} \end{matrix} \right\}$$

$$KQ+MO \Rightarrow \left\{ \begin{matrix} \frac{9\mu}{2a^{2/3}} - \frac{9(1-\mu)}{8a^{2/3}q^{4/3}} - \frac{9\mu}{8(aq)^{2/3}} - \frac{9\mu^2}{8a^{4/3}} + \frac{9(1-\mu)^2}{8(aq)^{4/3}} + \frac{27\mu(1-\mu)}{8(aq)^{2/3}} + \frac{27\mu^2 q^{2/3}}{8a^{2/3}} \\ - \frac{9(1-\mu)^2}{8a^{2/3}q^{4/3}} - \frac{9\mu(1-\mu)}{8a^{2/3}} \end{matrix} \right\}$$

Now combining all

$$\left(\Omega_{\xi\xi}\Omega_{\eta\eta}\right) = \left(1-e^2\right)^{-1} \left\{ \frac{9(1-\mu)}{4q^{2/3}} + \frac{9}{4} + \frac{9\mu q^{2/3}}{4} + \frac{9(1-\mu)}{8a^{2/3}q^{4/3}} + \frac{9\mu}{8(aq)^{2/3}} - \frac{9(1-\mu)}{4q^{4/3}} - \frac{9}{4q^{2/3}} \right.$$

$$-\frac{9\mu}{4}$$

$$-\frac{9\mu(1-\mu)}{8a^{4/3}q^{2/3}} - \frac{9\mu^2}{16a^{4/3}} + \frac{9\mu(1-\mu)}{8(aq)^{2/3}} + \frac{9\mu^2 q^{2/3}}{4a^{2/3}} - \frac{9(1-\mu)^2}{16(aq)^{4/3}} + \frac{9(1-\mu)^2}{8a^{2/3}q^{4/3}} + \frac{9\mu(1-\mu)}{8a^{2/3}} +$$

$$\left[\begin{matrix} \frac{9(1-\mu)}{8a^{2/3}q^{4/3}} + \frac{9\mu}{8(aq)^{2/3}} - \frac{9\mu(1-\mu)}{4a^{4/3}q^{2/3}} - \frac{9\mu^2}{8a^{4/3}} - \frac{9(1-\mu)^2}{8(aq)^{4/3}} - \frac{9(1-\mu)^2}{8(aq)^{4/3}} + \frac{9(1-\mu)^2}{8a^{2/3}q^{4/3}} \\ + \frac{9\mu(1-\mu)}{8a^{2/3}} + \frac{9\mu(1-\mu)}{8(aq)^{2/3}} + \frac{9\mu^2 q^{2/3}}{4a^{2/3}} \end{matrix} \right] e^2$$

$$+$$

$$\left[\begin{matrix} \frac{9\mu}{2a^{2/3}} - \frac{9(1-\mu)}{8a^{2/3}q^{4/3}} - \frac{9\mu}{8(aq)^{2/3}} - \frac{9\mu^2}{8a^{4/3}} + \frac{27\mu(1-\mu)^2}{8(aq)^{4/3}} + \frac{27\mu^2 q^{2/3}}{8a^{2/3}} - \frac{9(1-\mu)^2}{8a^{2/3q^{4/3}}} \\ + \frac{9\mu(1-\mu)}{8a^{2/3}} \end{matrix} \right] A_2$$

$$\left[\begin{array}{l}\frac{81(1-\mu)}{8a^{2/3}q^{5/3}}-\frac{45\mu}{8a^{2/3}q}+\frac{9\mu}{2a^{2/3}}-\frac{63(1-\mu)}{8a^{2/3}q^{7/3}}-\frac{9\mu}{4a^{2/3}q^{5/3}}-\frac{9(1-\mu)}{8a^{2/3}q^{4/3}}-\frac{9\mu}{4(aq)^{2/3}}\\ +\frac{81\mu(1-\mu)}{16a^{4/3}q^{5/3}}+\frac{81\mu^2}{16a^{4/3}q}+\frac{9\mu(1-\mu)}{4a^{4/3}q^{2/3}}+\frac{9(1-\mu)^2}{4(aq)^{4/3}}+\frac{9\mu(1-\mu)}{4(aq)^{2/3}}+\frac{9\mu^2q^{2/3}}{4a^{2/3}}\\ -\frac{9(1-\mu)^2}{4a^{2/3}q^{4/3}}-\frac{9\mu^2q^{2/3}}{4a^{2/3}}-\frac{9(1-\mu)^2}{4a^{2/3}q^{4/3}}-\frac{9\mu(1-\mu)}{4a^{2/3}}+\frac{9(1-\mu)^2}{4a^{2/3}q^{7/3}}+\frac{9\mu(1-\mu)}{4a^{2/3}q}\\ -\frac{45(1-\mu)}{16a^{4/3}q^{5/3}}-\frac{45\mu}{16a^{4/3}q}+\frac{45}{8a^{2/3}q}+\frac{45\mu}{8a^{2/3}q^{1/3}}-\frac{27\mu(1-\mu)}{2a^{2/3}q^{5/3}}-\frac{63\mu^2}{8a^{2/3}q^{1/3}}\\ \frac{45(1-\mu)^2}{16a^{4/3}q^3}+\frac{45\mu(1-\mu)}{16a^{4/3}q^{7/3}}-\frac{45(1-\mu)^2}{8a^{2/3}q^3}\end{array}\right]A\Bigg\}$$

(3.40)

Calculation of $\Omega_{\xi\xi}+\Omega_{\eta\eta}$

Add equation (3.38) and (3.39) to obtain

$$\Omega_{\xi\xi}=\left\{\begin{array}{l}\frac{3}{2}-\frac{3a}{2q^{2/3}}+\frac{3\mu}{4a^{2/3}}+\left(\frac{3(1-\mu)}{4(aq)^{2/3}}+\frac{3\mu}{4a^{2/3}}\right)e^2+\left(\frac{9\mu}{4a^{2/3}}-\frac{3(1-\mu)}{4(aq)^{2/3}}\right)A_2\\ +\left(\frac{3\mu}{2a^{2/3}}-\frac{3(1-\mu)}{2(aq)^{2/3}}+\frac{3(1-\mu)}{4a^{2/3}q^{5/3}}+\frac{15}{4a^{2/3}q}+\frac{21\mu}{4a^{2/3}q}-\frac{15(1-\mu)}{4a^{2/3}q^{7/3}}\right)A\end{array}\right\}$$

$$\Omega_{\eta\eta}=\left\{\begin{array}{l}\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}+\frac{3(1-\mu)}{2q^{2/3}}+\frac{3}{2}+\frac{3\mu q^{2/3}}{2}+\left(\frac{3(1-\mu)}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}\right)e^2\\ +\left(\frac{3\mu}{4(aq)^{2/3}}-\frac{3\mu}{4a^{2/3}}\right)A_2+\left(\begin{array}{l}\frac{3\mu}{2a^{2/3}}-\frac{3(1-\mu)}{2(aq)^{2/3}}+\frac{3(1-\mu)}{2a^{2/3}q^{5/3}}+\frac{15}{4a^{2/3}q}+\\ \frac{21\mu}{4a^{2/3}q}-\frac{15(1-\mu)}{4a^{2/3}q^{7/3}}\end{array}\right)A\end{array}\right\}$$

$$\Omega_{\xi\xi}+\Omega_{\eta\eta}=\left\{3-\frac{3}{2q^{2/3}}+\frac{3(1-\mu)}{2q^{2/3}}+\frac{3\mu q^{2/3}}{2}+0e^{2}+\left(\frac{3\mu}{a^{2/3}}\right)A_{2}+\left(\begin{array}{l}\frac{3}{a^{2/3}}+\frac{3(1-\mu)}{a^{2/3}q^{5/3}}\\ +\frac{15}{4a^{2/3}q}-\frac{15\mu}{4a^{2/3}q}\\ -\frac{15(1-\mu)}{4a^{2/3}q^{7/3}}\end{array}\right)A\right\}$$

$$\Omega_{\xi\xi}+\Omega_{\eta\eta}-4=\left\{-1-\frac{3}{2q^{2/3}}+\frac{3(1-\mu)}{2q^{2/3}}+\frac{3\mu q^{2/3}}{2}+\left(\frac{3\mu}{a^{2/3}}\right)A_{2}+\left(\begin{array}{l}\frac{3}{a^{2/3}}+\frac{3(1-\mu)}{a^{2/3}q^{5/3}}+\\ \frac{21}{4a^{2/3}q}-\frac{21\mu}{4a^{2/3}q}-\\ \frac{15(1-\mu)}{4a^{2/3}q^{7/3}}\end{array}\right)A\right\}$$

$$\Omega_{\xi\xi}+\Omega_{\eta\eta}=\left(1-e^{2}\right)^{-1/2}3\left\{\left[1-\frac{1}{2q^{2/3}}+\frac{(1-\mu)}{2q^{2/3}}+\frac{\mu q^{2/3}}{2}\right]+\frac{\mu A_{2}}{a^{2/3}}+\left[\begin{array}{l}\frac{\mu}{2a^{2/3}}+\frac{(1-\mu)}{a^{2/3}q^{2/3}}+\\ \frac{5}{4a^{2/3}q}-\frac{5\mu}{4a^{2/3}q}-\\ \frac{5(1-\mu)}{4a^{2/3}q^{2/3}}\end{array}\right]A\right\}$$

$$=\left(1-e^{2}\right)^{-1/2}3\left\{\left[1-\frac{1}{2}+\frac{(1-\mu)}{2}+\frac{\mu}{2}\right]+\mu\left(1+\frac{2\alpha}{3}\right)A_{2}+\left[\begin{array}{l}+\mu\left(1+\frac{2\alpha}{3}\right)(1-\mu)\left(1+\frac{2\alpha}{3}\right)\\ +\frac{5}{4}\left(1+\frac{2\alpha}{3}\right)-\frac{5\mu}{4}\left(1+\frac{2\alpha}{3}\right)\\ \frac{5(1-\mu)}{4}\left(1+\frac{2\alpha}{3}\right)\end{array}\right]A\right\}$$

$$=\left(1-e^{2}\right)^{-1/2}3\left\{\left[1+\mu A_{2}+\left(\mu+(1-\mu)+\frac{5}{4}-\frac{5\mu}{4}-\frac{5(1-\mu)}{4}\right)A\right]\right\}$$

$$=\left(1-e^{2}\right)^{-1/2}3\left\{1+\mu A_{2}+A\right\}\qquad =3\phi_{1}$$

Substituting these values in the characteristic equations (3.12) and restricting ourselves only to the linear terms in e^2, a A, A_2, and q where for simplicity, we let $a=1-\alpha$ and $q=1-\beta$ and we ignore the second and higher order terms of e^2, a, A, A_2, q and their products, we obtain

$$4\left(\lambda^{2}\right)^{2}+4\left(4-3\phi_{1}\right)\lambda^{2}+27\mu(1-\mu)+4\phi_{2}=0 \qquad (3.41)$$

where

$$B = 4 - 3\phi_1$$

$$C = 27\mu(1-\mu) + 4\phi_2 = 0$$

$$\phi_1 = \left(1 - e^2\right)^{-1/2} \left\{1 + \mu A_2 + A\right\}$$

and

$$\phi_2 = 3\mu(1-\mu)\left\{\alpha + \frac{\beta}{2} + 3(A_2 + A) + \frac{15e^2}{2}\right\} \qquad (3.42)$$

Equation (3.41) is a quadratic equation in λ^2 which yields

$$\lambda^2 = \frac{-(4-3\phi_1) \pm \left[(4-3\phi_1)^2 - 27\mu(1-\mu) - 4\phi_2\right]^{1/2}}{2}$$

For stable motion, we require λ to be purely imaginary. Thisimplies that, the motion of the infinitesimal must be bounded and periodic, such that $\lambda^2 < 0$.This implies that $3\phi_1 - 4 \le 0$ and the discriminant

$$\Delta = (4-3\phi_1)^2 - 27\mu(1-\mu) - 4\phi_2 \qquad (3.43)$$

which means

$$\Delta = (4-3\phi_1)^2 - 27\mu(1-\mu) - 4\phi_2 > 0$$

This yield

$$0 < e \le \left[1 - \frac{9}{16}(1 + \mu A_2 + A)^2\right]^{1/2} \qquad (3.44)$$

When $A = A_2 = 0$, equation (3.44) becomes

$$0 < e \le \frac{\sqrt{7}}{4}$$

The characteristics roots will be either real or complex conjugate if equation (3.44) is not satisfied. In case of complex roots, the positive real part indicates instability of the inves tigated triangular points.

Now, making use of equations (3.44) we have

Let $B = 4-\left(\Omega_{\xi\xi}+\Omega_{\eta\eta}\right) = 4-3\left(1+\mu A_2 + A + \frac{e^2}{2}\right)$

$$\Rightarrow B = 1-3\mu A_2 - 3A - \frac{3e^2}{2}$$

$$\therefore B^2 = \left(1-3\mu A_2 - 3A - \frac{3e^2}{2}\right)^2$$

$$= 1-6\mu A_2 - 6A - 3e^2$$

thus

$$\Delta = B^2 - 4C = 1-6\mu A_2 - 6A - 3e^2 - 4\left(\frac{27\mu(1-\mu)}{4}+\varphi_2\right) \tag{3.46}$$

$$= -4\left(\frac{27\mu}{4}-\frac{27\mu^2}{4}+\left(3\mu-3\mu^2\right)\left(\alpha+\frac{\beta}{2}+3A_2+3A+\frac{15e^2}{4}\right)\right)$$

$$= -27\mu+27\mu^2+4\left(\begin{array}{l} 3\mu\alpha+\frac{3\mu\beta}{2}+9\mu A_2+9\mu A+\frac{45\mu e^2}{4}-3\mu^2\alpha-\frac{3\mu^2\beta}{2}-9\mu^2 A_2 \\ -9\mu A-\frac{45\mu^2 e^2}{4} \end{array}\right)$$

$$= -27\mu+27\mu^2-12\mu\alpha-6\mu\beta-36\mu A_2-36\mu A-45\mu e^2+12\mu^2\alpha+6\mu^2\beta+36\mu^2 A_2$$
$$+36\mu A+45\mu^2 e^2$$

$$\Delta = A_2-6A-3e^2-27\mu+27\mu^2-12\mu\alpha-6\mu\beta-36\mu A_2-36\mu A-45\mu e^2+12\mu^2\alpha+$$
$$6\mu^2\beta+36\mu^2 A_2+36\mu A+45\mu^2 e^2$$

$\Delta = 1-27\mu+27\mu^2$ If $\Delta = 0$, the $\mu = \dfrac{27\pm\sqrt{27^2-4\times 27}}{2\times 27} = \dfrac{27\pm 27\sqrt{1-\frac{4}{27}}}{54}$

$$= \frac{1\pm\sqrt{\frac{23}{27}}}{2}$$

Here we have

$$\left(27+12\alpha+6\beta+36A_2+36A+45e^2\right)\mu^2-\left(27+12\alpha+6\beta+36A+45e^2+45A_2\right)\mu$$
$$+\left(1-6A-3e^2\right)=0$$

Let $a=27+12\alpha+6\beta+36A_2+36A+45e^2$

$$b=-27-12\alpha-6\beta-42A_2-36A-45e^2$$

$$c=1-6A-3e^2$$

$$b^2=\left(-27\left(1+\frac{12\alpha}{27}+\frac{6\beta}{27}+\frac{36A}{27}+\frac{45e^2}{27}+\frac{42A_2}{27}\right)\right)$$

$$b^2=729\left(1+\frac{24\alpha}{27}+\frac{12\beta}{27}+\frac{72A}{27}+\frac{90e^2}{27}+\frac{84A_2}{27}\right)$$

$$b^2=729+648\alpha+324\beta+1944A+243e^2+2268A_2$$

$$4ac=4\left(27+12\alpha+6\beta+36A_2+36A+45e^2\right)\left(1-6A-3e^2\right)$$

$$4ac=4\left(27+12\alpha+6\beta+36A_2+36A+45e^2-162A-81e^2\right)$$

$$4ac=4\left(27+12\alpha+6\beta+36A_2-126A-36e^2\right)$$

$$=108+48\alpha+24\beta+144A_2-504A-144e^2$$

$$b^2-4ac=729+648\alpha+1944A+243e^2+2268A_2-108-48\alpha-144A_2+504A+144e^2$$

$$=621+600\alpha+300\alpha+2448A_2+2574e^2+2124A_2$$

$$=621\left\{1+\frac{600\alpha}{621}+\frac{300\beta}{621}+\frac{2448\mathrm{A}}{621}+\frac{2574e^2}{621}+\frac{2124\mathrm{A}_2}{621}\right\}$$

$$\left(b^2-4ac\right)^{1/2}=\left\{621\left\{1+\frac{600\alpha}{621}+\frac{300\beta}{621}+\frac{2448\mathrm{A}}{621}+\frac{257e^2}{621}+\frac{2124\mathrm{A}_2}{621}\right\}\right\}^{1/2}$$

$$=\sqrt{621\left\{1+\frac{1}{2}\left\{\frac{600\alpha}{621}+\frac{300\beta}{621}+\frac{2448\mathrm{A}}{621}+\frac{2574e^2}{621}+\frac{2124\mathrm{A}_2}{621}\right\}\right\}}$$

$$\left(b^2-4ac\right)^{1/2}=\sqrt[3]{69}\left\{1+\frac{300\alpha}{621}+\frac{150\beta}{621}+\frac{1224\mathrm{A}}{621}+\frac{1287e^2}{621}+\frac{1062\mathrm{A}_2}{621}\right\}$$

Let $\mu_C = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-b}{2a} \pm \dfrac{\sqrt{b^2-4ac}}{2a} = \dfrac{1}{2}\left\{\dfrac{-b}{a} \pm \dfrac{\sqrt{b^2-4ac}}{a}\right\}$

$$\mu_C = \frac{27\left\{1+\frac{4\alpha}{9}+\frac{2\beta}{9}+\frac{4\mathrm{A}}{3}+\frac{5e^2}{3}+\frac{14\mathrm{A}_2}{9}\right\}}{2\times 27\left\{1+\frac{4\alpha}{9}+\frac{2\beta}{9}+\frac{4\mathrm{A}_2}{3}+\frac{4\mathrm{A}}{3}+\frac{5e^2}{3}\right\}} \pm \frac{3\sqrt{69}\left\{\begin{array}{l}1+\frac{300\alpha}{621}+\frac{150\alpha}{621}+\frac{1224\mathrm{A}}{621}\\ +\frac{1287e^2}{621}+\frac{1062\mathrm{A}_2}{621}\end{array}\right\}}{2\times 27\left\{1+\frac{4\alpha}{9}+\frac{2\beta}{9}+\frac{4\mathrm{A}_2}{3}+\frac{4\mathrm{A}}{3}+\frac{5e^2}{3}\right\}}$$

$$= \left\{\frac{\frac{1}{2}\left\{1+\frac{4\alpha}{9}+\frac{2\beta}{9}+\frac{4\mathrm{A}}{3}+\frac{5e^2}{3}+\frac{4\mathrm{A}_2}{3}\right\}}{\left\{1+\frac{4\alpha}{9}+\frac{2\beta}{9}+\frac{4\mathrm{A}_2}{3}+\frac{4\mathrm{A}}{3}+\frac{5e^2}{3}\right\}^{-1}}\right\} \pm \frac{\sqrt{69}}{2\times 9}\,\frac{\left\{\begin{array}{l}1+\frac{100\alpha}{207}+\frac{50\beta}{207}+\frac{408\mathrm{A}}{207}+\frac{429e^2}{207}\\ +\frac{354\mathrm{A}_2}{207}\end{array}\right\}}{\left\{1+\frac{4\alpha}{9}+\frac{2\beta}{9}+\frac{4\mathrm{A}_2}{3}+\frac{4\mathrm{A}}{3}+\frac{5e^2}{3}\right\}^{-1}}$$

$$\mu_c = \frac{1}{2}\left\{1+\frac{4\alpha}{9}+\frac{2\beta}{9}+\frac{4A}{3}+\frac{5e^2}{3}+\frac{4A_2}{3}\right\}\left\{1-\frac{4\alpha}{9}-\frac{2\beta}{9}-\frac{4A_2}{3}-\frac{4A}{3}-\frac{5e^2}{3}\right\}$$

$$\pm\frac{\sqrt{69}}{2\times 9}\left\{\begin{array}{l}1+\frac{100\alpha}{207}+\frac{50\beta}{207}+\frac{408A}{207}+\frac{429e^2}{207}\\ +\frac{354A_2}{207}\end{array}\right\}\left\{1-\frac{4\alpha}{9}-\frac{2\beta}{9}-\frac{4A_2}{3}-\frac{4A}{3}-\frac{5e^2}{3}\right\}$$

$$\mu_c = \frac{1}{2}\left\{1-\frac{4\alpha}{9}-\frac{2\beta}{9}-\frac{4A_2}{3}-\frac{4A}{3}-\frac{5e^2}{3}+\frac{4\alpha}{9}+\frac{2\beta}{9}\frac{4A}{3}+\frac{5e^2}{3}+\frac{4A_2}{3}\right\}$$

$$\pm\frac{\sqrt{69}}{2\times 9}\left\{\begin{array}{l}1-\frac{4\alpha}{9}-\frac{2\beta}{9}-\frac{4A_2}{3}-\frac{4A}{3}-\frac{5e^2}{3}+\frac{100\alpha}{207}+\frac{50\beta}{207}+\frac{408A}{207}+\frac{429e^2}{207}\\ +\frac{354A_2}{207}\end{array}\right\}$$

$$\mu_C = \frac{1}{2}\{1\} \pm \frac{\sqrt{69}}{2\times 9}\left\{1+\frac{8\alpha}{207}+\frac{4\beta}{207}+\frac{26\mathrm{A}_2}{69}+\frac{44\mathrm{A}}{69}+\frac{28e^2}{69}\right\}$$

$$= \frac{1}{2}\{1\} \pm \frac{\sqrt{69}}{2\times 9}+\frac{4\sqrt{69}\alpha}{1863}+\frac{2\sqrt{69}\beta}{1863}+\frac{13\mathrm{A}_2\sqrt{69}}{621}+\frac{22\mathrm{A}\sqrt{69}}{621}+\frac{14e^2\sqrt{69}}{621}$$

$$= \frac{1}{2}\left\{1-\frac{\sqrt{69}}{9}\right\}+\frac{4\sqrt{69}\alpha}{1863}+\frac{2\sqrt{69}\beta}{1863}+\frac{13\mathrm{A}_2\sqrt{69}}{621}+\frac{22\mathrm{A}\sqrt{69}}{621}-\frac{14e^2\sqrt{69}}{621}$$

$$
= \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right) - \frac{4\sqrt{69}\alpha}{1863} \times \frac{\sqrt{69}}{\sqrt{69}} - \frac{2\sqrt{69}\beta}{1863} \times \frac{\sqrt{69}}{\sqrt{69}} - \frac{13\mathrm{A}_2\sqrt{69}}{621} \times \frac{\sqrt{69}}{\sqrt{69}} - \frac{22\mathrm{A}\sqrt{69}}{621} \times \frac{\sqrt{69}}{\sqrt{69}}
$$

$$
+ \frac{14e^2\sqrt{69}}{621} \times \frac{\sqrt{69}}{\sqrt{69}}
$$

$$
= \frac{1}{2}\left(1 - \frac{\sqrt{69}}{9}\right) - \frac{4\alpha}{27\sqrt{69}} - \frac{2\beta}{27\sqrt{69}} - \frac{13\mathrm{A}_2}{9\sqrt{69}} - \frac{22\mathrm{A}}{9\sqrt{69}} - \frac{14e^2}{9\sqrt{69}} \qquad (3.47)
$$

Equation (3.44) and equation (3.46) are the necessary conditions for the stability of the triangular points. The critical value μc given by equation (3.47) is the solution of the quadratic equation $\Delta = 0$. It represents the effects of the various parameters involved on the size of the region of stability. A close examination shows that they all cause a reduction in the size of the region confirming their destabilizing tendencies.

CHAPTER FOUR

LOCATIONS AND LINEAR STABILITY OF THE COLLINEAR EQUILIBRIUM POINTS

4.1 Introduction

In this chapter, we determine the positions and stability of the collinear equilibrium points.

4.2Locations of the Collinear equilibrium Points

To obtain the collinear points, we obtain the first derivative of equation (3.2) with respect to ξ, η and ζ respectively, and equate them to zero. That is $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$. Since the collinear points lie only, on the $\xi - axis$, it implies that $\zeta = \eta = 0$ on the system i.e.

$$\Omega_\xi = \left(1-e^2\right)^{-1/2}\left[\xi - \frac{1}{n^2}\left\langle \frac{(1-\mu)(\xi+\eta)q}{r_1^3} + \frac{\mu(\xi+\mu-1)}{r_2^3} + \frac{3(1-\mu)(\xi+\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu(\xi+\mu-1)\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2(\xi+\mu-1)}{2r_2^5}\right\rangle\right] \tag{4.1}$$

$$\Omega_\eta = \left(1-e^2\right)^{-1/2}\left[\eta\left(1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3(1-\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2}{2r_2^5}\right\}\right)\right] \tag{4.2}$$

$$\Omega_\zeta = \left(1-e^2\right)^{-1/2}\left[\zeta\left(\frac{(1-\mu)q}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3(1-\mu)\mathrm{A}}{2r_1^5} + \frac{3\mu\mathrm{A}}{2r_2^5} + \frac{3\mu\mathrm{A}_2}{2r_2^5}\right)\right] \tag{4.3}$$

Since, $\eta = \zeta = 0$. Using equation (3.3), the first equation (4.1), reduces to

$$\xi n^2 - \frac{(1-\mu)(\xi+\mu)q}{|\xi+\mu|^3} - \frac{\mu(\xi+\mu-1)}{|\xi+\mu-1|} - \frac{3(1-\mu)(\xi+\mu)A}{2|\xi+\mu|^5} - \frac{3\mu(\xi+\mu-1)A}{2|\xi+\mu-1|^5} - \frac{3\mu A_2(\mu+\mu-1)}{2|\xi+\mu-1|^5} = 0 \tag{4.4}$$

And equation (3.3) becomes

$$r_1^2 = (\xi+\mu)^2$$
$$r_2^2 = (\xi+\mu-1)^2 \tag{4.5}$$

Solving equation (4.4) using equation (4.5) with $\xi_1 = -\mu \, and \, \xi_2 = 1-\mu$

$$2\xi n^2(\xi-\xi_1)^4(\xi-\xi_2)^4 - 2(1-\mu)(\xi-\xi_1)^2(\xi-\xi_2)^4 - 2\mu(\xi-\xi_1)^4(\xi-\xi_2)^2 -3(1-\mu)A(\xi-\xi_1)^4 - 3\mu A(\xi-\xi_1)^4 - 3\mu A_2(\xi-\xi_1)^4 = 0 \tag{4.6}$$

The orbital plane on ξ-axis is divided into three parts; $\xi > \xi_2$, $\xi_1 < \xi < \xi_2$, and $\xi_1 > \xi$ with respect to the primaries, which corresponds to the collinear points $L_i\,(1,2,3)$.

Case 1: Position of $L_1\left(\xi > \xi_2\right)$

Let the collinear point L_1 be on the right hand side of the smaller primary at a distance ρ from it on the ξ-axis. Then

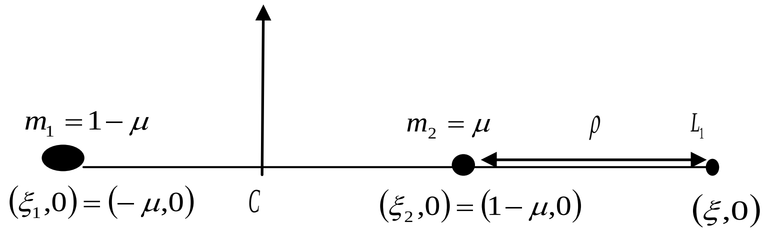

Figure 4.1: Position of the collinear equilibrium point L_1

In the interval $\left(\xi > \xi_2\right)$, we let $\xi - \xi_2 = \rho$, $\xi - \xi_1 = 1 + \rho \Rightarrow \xi = 1 + \rho + \xi_1$ (4.7).

Using equation (4.6), $r_1 = 1 + \rho$, $r_2 = \rho$. Since the distance between the primaries is unity.

Thus, by substituting equation (4.7) in the equation (4.6) we obtain

$$
\begin{aligned}
&2n^2(1+\rho-\mu)(1+\rho)^4\rho^4 - 2(1-\mu)(1+\rho)^2\rho^4 q - 2\mu\,(1+\rho)^4\rho^2 - 3(1-\mu)A\rho^4 \\
&-3\mu\,(1+\rho)^4 A - 3\mu(1+\rho)^4 A_2 = 0
\end{aligned}
\tag{4.8}
$$

Hence, expanding equation (4.8) yields

$$
\begin{aligned}
&2n^2\rho^9 + 2n^2(5-\mu)\rho^8 + 2n^2\left(2(5-2\mu)\right)\rho^7 + 2\left(2n^2(5-3\mu) - (q-\mu q+\mu)\right)\rho^6 \\
&+\left(2n^2(5-4\mu) - 4(q-\mu q+2\mu)\right)\rho^5 + \left(2n^2(1-\mu) - 2(q-\mu q+6\mu) - 3(1-\mu)A + A_2\right)\rho^4 \\
&-\left(4(2\mu+6\mu A+6\mu A_2)\right)\rho^3 - \left(2(\mu+9\mu A+9\mu A_2)\right)\rho^2 \\
&-\left(4(3\mu A+3\mu A_2)\right)\rho - \left(3(\mu A+\mu A_2)\right) = 0
\end{aligned}
\tag{4.9}
$$

Cases 2: Position of $L_2(\xi_1 < \xi < \xi_2)$

Let the collinear point L_2 be on the left hand side of the smaller primary at a distance ρ from it on the ξ-axis.

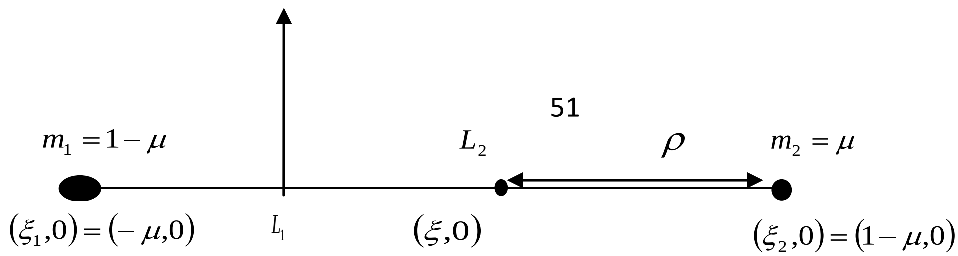

Figure 4.2: Position of the collinear equilibrium point L_2

In the interval $L_2\left(\xi_1 < \xi < \xi_2\right)$, we let $\xi_2 - \xi = \rho\,; \xi - \xi_1 = 1 - \rho \Rightarrow \xi = 1 - \rho - \mu\,;$ and $r_1 = 1 - \rho\,,\ r_2 = \rho$ (4.10) Using equation (4.10) in equation (4.6) yields

$$2n^2(1-\rho-\mu)(1-\rho)^4\rho^4 - 2(1-\mu)(1-\rho)^2\rho^4 q + 2\mu(1-\rho)^4\rho^2 - 3(1-\mu)A\rho^4 + 3\mu A(1-\rho)^4$$
$$+3\mu A_2(1-\rho)^4 = 0$$

(4.11)

Hence, expanding equation (4.11) yields

$$-2n^2\rho^9 + 2n^2(5-\mu)\rho^8 - 2n^2(2(5-2\mu))\rho^7 + 2\left(2n^2(5-3\mu) - 2(q-\mu q+\mu)\right)\rho^6$$
$$-\left(2n^2(5-4\mu) - 4(\mu q - q + 2\mu)\right)\rho^5 + \left(2n^2(1-\mu) - 2(q-\mu q-6\mu) + 3(2\mu A + A_2 - A)\right)\rho^4$$
$$-\left(4(2\mu + 3\mu A + 3\mu A_2)\right)\rho^3 + \left(2(\mu + 9\mu A + 9\mu A_2)\right)\rho^2$$
$$-\left(12(\mu A + \mu A_2)\right)\rho + \left(3(\mu A + \mu A_2)\right) = 0$$

(4.12)

Case 3: Position of $L_3\left(\xi_1 > \xi\right)$

Let the collinear point L_3 be on the left hand side of the bigger primary at a distance $1-\rho$ from it on the ξ-axis.

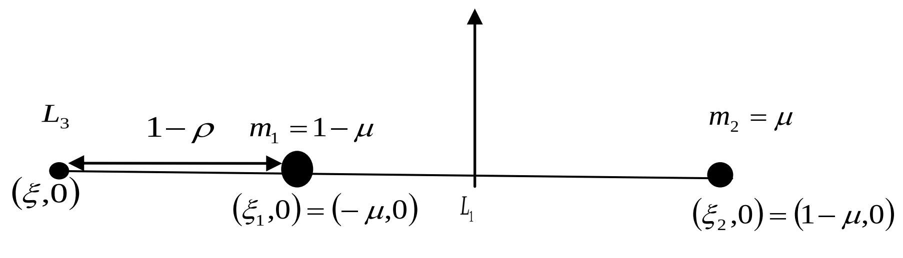

Figure 4.3: Position of the collinear equilibrium point L_3

Lastly, in the interval $L_3\left(\xi_1 > \xi\right)$, we let $\xi_1 - \xi = 1-\rho; \xi_2 - \xi = 2-\rho$, and

$$r_1 = 1-\rho; r_2 = 2-\rho \tag{4.13}$$

Using equation (4.13) in equation (4.6) yields

$$\begin{aligned}&2n^2\left(\rho-\mu-1\right)\left(1-\rho\right)^4\left(2-\rho\right)^4+2\left(1-\mu\right)\left(1-\rho\right)^2\left(2-\rho\right)^4 q+2\mu\ \left(2-\rho\right)^2\left(1-\rho\right)^4\\&+3\left(1-\mu\right)A\left(2-\rho\right)^4+3\mu A\left(1-\rho\right)^4+3\mu A_2\left(1-\rho\right)^4=0\end{aligned} \tag{4.14}$$

Hence, expanding equation (4.14) yields

$$\begin{aligned}&2n^2\rho^9-2n^2\left(13+\mu\right)\rho^8+4n^2\left(37+6\mu\right)\rho^7-2\left(2n^2\left(121-31\mu\right)-\left(q-\mu q+\mu\right)\right)\rho^6\\&+2\left(n^2\left(501+180\mu\right)-2\left(5q-5\mu q+4\mu\ \right)\right)\rho^5-\begin{pmatrix}2n^2\left(681+321\mu\right)-2\left(41\mu q-41q+26\mu\ \right)\\+3\left(A+A_2\right)\end{pmatrix}\\&\ )\rho^4+\left(2n^2\left(609+360\mu\right)-88\left(2q-2\mu q+\mu\ \right)-12\left(2A-\mu A+\mu\ A_2\right)\right)\rho^3\\&-\left(2n^2\left(345+249\mu\right)-2\left(104\mu q-104q-41\mu\ \right)-18\left(4A-3\mu A+\mu A_2\right)\right)\rho^2\\&+\left(2n^2\left(112+96\mu\right)-2\left(64q-64\mu q+20\mu\ \right)-96A-12\left(\mu A\ +\mu A_2\ \right)\right)\rho\\&-2\left(2n^2\left(8\left(1+\mu\right)\right)-4\left(4q-4\mu q+\mu\ \right)-48\left(1-\mu\right)A\right)+3\left(\mu A\ +\mu A_2\right)=0\end{aligned}$$

(4.15)

4.3Stability of the collinear equilibrium points

To study the stability of the collinear equilibrium points$L_i(1,2,3)$, we consider the characteristic equation (3.12) of the system given by:

$$\lambda^4-\left(\Omega^0_{\xi\xi}+\Omega^0_{\eta\eta}-4\right)\lambda^2+\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}-\left(\Omega^0_{\xi\eta}\right)^2=0$$

Here, we obtain the points corresponding to the collinear points by taking the second partial derivatives of equation (3.1), with $\eta=0$. Thus we have

$$\Omega^0_{\xi\xi}=\left[1+\frac{2}{n^2}\left\{\frac{(1-\mu)q}{|\xi+\mu|^3}+\frac{\mu}{|\xi+\mu-1|^3}+\frac{3(1-\mu)A}{|\xi+\mu|^5}+\frac{3\mu A}{|\xi+\mu-1|^5}\right\}\right]$$

$$\Omega^0_{\eta\eta} = \left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q}{|\xi+\mu|^3} + \frac{\mu}{|\xi+\mu-1|^5} + \frac{3(1-\mu)A}{2|\xi+\mu|^5} + \frac{3\mu A}{2|\xi+\mu-1^5|} + \frac{3\mu A_2}{2|\xi+\mu-1|^5}\right\}\right]$$

$\Omega^0_{\zeta\zeta} = 0$ (4.16)

Since all the parameters in $\Omega^0_{\xi\xi}$ are positive and $\Omega^0_{\xi\xi}$ contains only positive terms, it impliesthat $\Omega^0_{\xi\xi} > 0$ (4.17) 4.3.1 Stability of $L_1 (\xi > \xi_2)$

In the first interval, we have $r_1 = (\xi + \mu) \Rightarrow \xi = (r_1 - \mu)$ and $r_2 = (\xi + \mu - 1)$ (4.18)

Substituting equation (4.18) in equation (4.4) yields

$$\frac{(1-\mu)q}{r_1^2} = -\xi n^2 - \frac{\mu}{r_2^2} - \frac{3(1-\mu)A}{2r_1^4} - \frac{3\mu A}{2r_2^4} - \frac{3\mu A_2}{2r_2^4} \quad (4.19)$$

Substituting equation (4.19) in the second equation of (3.42) gives

$$\Omega^0_{\eta\eta} = \left(1-e^2\right)^{-1/2}\left[1 - \frac{1}{n^2}\left\{\begin{array}{l}\frac{1}{r_1}\left(-\xi n^2 - \frac{\mu}{r_2^2} - \frac{3(1-\mu)A}{2r_1^4} - \frac{3\mu A}{2r_2^4} - \frac{3\mu A_2}{2r_2^4}\right) + \frac{\mu}{r_2^3} + \frac{3(1-\mu)A}{2r_1^5} \\ + \frac{3\mu A}{2r_2^5} + \frac{3\mu A_2}{2r_2^5}\end{array}\right\}\right] \quad (4.20)$$

Which Implies that

$$\Omega^0_{\eta\eta} = \left(1-e^2\right)^{-1/2}\left[-\frac{\mu}{r_1} + \frac{1}{n^2}\left\{\frac{\mu}{r_2^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{3\mu A}{2r_2^4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{3\mu A_2}{2r_2^4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right\}\right] \quad (4.21)$$

Thus, $\Omega^0_{\eta\eta} < 0$. Since $\frac{\mu}{r_2^2} < 0, \frac{3\mu A}{r_2^4} < 0, \frac{3\mu A_2}{2r_2^4}, 0$ and $A, A_2 << 1, r_1 > 1, r_2 < 1$

4.3.2 Stability of $L_2 (\xi_1 < \xi < \xi_2)$

In the second interval, we have; $r_1 = (\xi + \mu) \Rightarrow \xi = (r_1 - \mu)$ and $r_2 = -(\xi + \mu - 1)$ (4.22)

Substituting equation (4.22) in equation (4.4) yields

$$\frac{(1-\mu)q}{r_1^2} = \xi n^2 + \frac{\mu}{r_2^2} - \frac{3(1-\mu)A}{2r_1^4} + \frac{3\mu A}{2r_2^4} + \frac{3\mu A_2}{2r_2^4} \tag{4.23}$$

Substituting equation (4.23) in the second equation of (3.42) gives

$$\Omega_{\eta\eta}^0 = \left(1-e^2\right)^{-1/2}\left[\frac{\mu}{r_1} - \frac{1}{n^2}\left\{\frac{\mu}{r_2^2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \frac{3\mu A_2}{2r_2^4}\left(\frac{1}{r_1}+\frac{1}{r_2}\right) + \frac{3\mu A}{2r_2^4}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\right\}\right]$$

(4.24)

Thus, $\Omega_{\eta\eta}^0 < 0$. Since $\frac{\mu}{r_2^2} < 0, \frac{3\mu A}{r_2^4} < 0, \frac{3\mu A_2}{2r_2^4}, 0$ and $A, A_2 << 1, r_1 > 1, r_2 < 1$

4.3.3Stability of $L_3\left(\xi_1 > \xi\right)$

In the last interval, we have; $r_1 = -(\xi + \mu) \Rightarrow \xi = -(r_1 + \mu)$ and $r_2 = -(\xi + \mu - 1)$

(4.25)

Subtituting equation (4.25) in equation (4.4) yields

$$\Omega_{\eta\eta}^0 = \left(1-e^2\right)^{-1/2}\left[-\frac{\mu}{r_1} + \frac{1}{n^2}\left\{\frac{\mu}{r_2^2}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) + \frac{3\mu A}{2r_2^4}\left(\frac{1}{r_1}-\frac{1}{r_2}\right) + \frac{3\mu A_2}{2r_2^4}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\right\}\right] \tag{4.27}$$

Thus, $\Omega_{\eta\eta}^0 < 0$.Since $\frac{\mu}{r_2^2} < 0, \frac{3\mu A_2}{2r_2^4} < 0, \frac{3\mu A}{2r_2^4} < 0$ and $A, A_2 << 1, r_1 > 1, r_2 < 1$

Clearly, $\Omega_{\xi\xi}^0 > 0, \Omega_{\eta\eta}^0 < 0$, and $\Omega_{\xi\eta}^0 = 0$.

Since, $\Omega_{\xi\xi}^0\Omega_{\eta\eta}^0 - \left(\Omega_{\xi\eta}^0\right)^2 < 0$ the discriminant of equation (18) is positive and its characteristic roots can be expressed as:

$$\lambda_{1,3} = -\frac{\left\{-4+\Omega^0_{\eta\eta}+\Omega^0_{\xi\xi}-\sqrt{16-8\Omega^0_{\eta\eta}+\left(\Omega^0_{\eta\eta}\right)^2+4\left(\Omega^0_{\xi\eta}\right)^2-8\Omega^0_{\xi\xi}-2\Omega^0_{\eta\eta}\Omega^0_{\xi\xi}+\left(\Omega^0_{\xi\xi}\right)^2}\right\}^{1/2}}{\sqrt{2}}$$

$$\lambda_{2,4} = \frac{\left\{-4+\Omega^0_{\eta\eta}+\Omega^0_{\xi\xi}-\sqrt{16-8\Omega^0_{\eta\eta}+\left(\Omega^0_{\eta\eta}\right)^2+4\left(\Omega^0_{\xi\eta}\right)^2-8\Omega^0_{\xi\xi}-2\Omega^0_{\eta\eta}\Omega^0_{\xi\xi}+\left(\Omega^0_{\xi\xi}\right)^2}\right\}^{1/2}}{\sqrt{2}} \tag{4.28}$$

Hence, we conclude that, the collinear equilibrium points are unstable (due to the nature of the characteristic roots of equations (4.28), despite the introduction of perturbations on account of oblateness, eccentricity and luminosity.

CHAPTER FIVE

RESULTS AND DISCUSSION

5.1 Introduction

This chapter obtains numerically and graphically the analytical results in chapters three and four with the help of the binary system CEN X-3 and PRS JI518 +4904 and assumed values of some potential, the positions of the equilibrium points, regions of stability of the triangular equilibrium points and the stability of collinear points. The relevant numerical data of the binaries obtained from NASA ADS and Singh and Umar (2013a) are given in table 5.1.

Table 5.1: Relevant Numerical data

Binary System	**Masses (M_{SUN}) $M_1(M_{sun})$**	**$M_2(M_{sun})$**	**Luminosity (L_{sun})**	**Mass ratio (μ)**	**Radiation pressure (q)**
CEN X-3	20.5	1.21	0.634	0.05533	0.999968

5.2 Numerical Results of the Triangular Equilibrium Points

This section computes numerically the positions of the triangular equilibrium points $L_{4,5}$ for the binary system CEN X-3 using equation (3.20). The locations of the triangular points for CEN X-3are presented in Tables 5.2-5.5. The circular cases $(e=0, a=1, q=0)$are first shown, then, oblateness factor of the smaller primary is introduced increasingly. Subsequently, the effect of increasing radiation of the bigger primary is showcased for Cen X-3. The effects of eccentricity and oblateness on the locations of the triangular points of CEN X-3 are shown graphically in Figures 5.1 & 5.2. Figures 5.3 & 5.8 show the effects of the various parameters on the size of the regions of stability for an arbitrary system within the interval of stability.

Table 5.2: Effect of oblateness on $L_{4,5}$of CEN X-3 for e=0.3 and a=0.9 A=0.01

A_2	ξ	$\pm\eta$	μ_c
0.0	0.444691	0.779914	0.0169406
0.00001	0.444686	0.779911	0.0169389
0.0001	0.444641	0.779886	0.0169232
0.001	0.444191	0.77937	0.0167667

0.01	0.439691	0.777138	0.0152017
0.1	0.394692	0.751698	0.00044845
0.15	0.369693	0.737185	-0.0091429
0.2	0.344693	0.722381	-0.0178375

Table 5.3: Effect of oblateness on $L_{4,5}$ of CEN X-3 for e=0.3, a=0.9 and A_2=0.1

A	ξ	$\pm\eta$	μ_C
0.0	0.394692	0.745016	0.00249432
0.00001	0.394692	0.745023	0.00249137
0.0001	0.394692	0.745084	0.00246489
0.001	0.394692	0.745687	0.00220004
0.01	0.394692	0.751698	-0.00044845
0.1	0.394694	0.809352	-0.0269333
0.15	0.394694	0.839674	-0.0416472
0.2	0.394695	0.868938	-0.056361

Table 5.4: Effect of eccentricity on $L_{4,5}$ of CEN X-3 for a=0.9

A	A_2	e	ξ	$\pm\eta$	μ_c
0.01	0.01	0.1	0.43969	0.823721	0.0301831
0.01	0.01	0.2	0.439691	0.806568	0.0245651
0.01	0.01	0.3	0.439691	0.777138	0.0152017
0.01	0.01	0.4	0.439692	0.733957	0.00209303
0.01	0.01	0.5	0.439693	0.674387	-0.014761
0.01	0.01	0.6	0.439694	0.593516	-0.0353603
0.01	0.01	0.7	0.439695	0.480708	-0.059705

Table 5.5: Effect of radiation pressure on $L_{4,5}$ of CEN X-3 for a=0.9 and e=0.3

A	A_2	q	ξ	$\pm\eta$	μ_c
0.01	0.01	1.0000	0.4397	0.777144	0.0152017
0.01	0.01	0.9999	0.439673	0.777126	0.0152017
0.01	0.01	0.9888	0.436619	0.775159	0.0152017
0.01	0.01	0.9555	0.427394	0.769186	0.0152017
0.01	0.01	0.8999	0.411774	0.758964	0.0152017
0.01	0.01	0.8555	0.399093	0.750564	0.0152017
0.01	0.01	0.7999	0.382938	0.739723	0.0152017
0.01	0.01	0.7555	0.369803	0.730791	0.0152017
0.01	0.01	0.7000	0.353071	0.719252	0.0152017

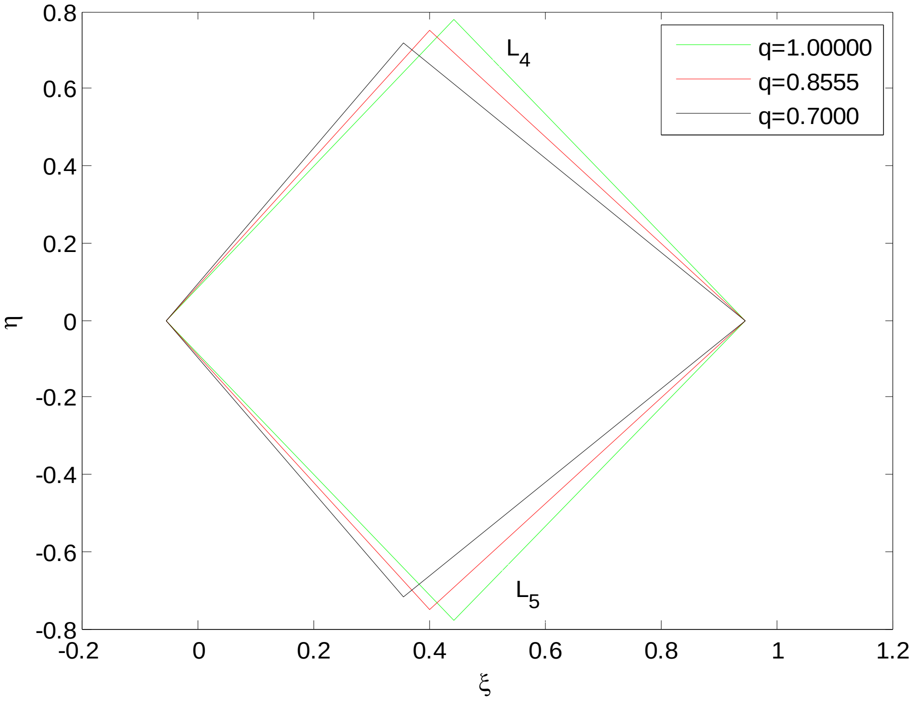

Figure 5.1: Effect of radiation pressure on $L_{4,5}$ of CEN X-3 for $a = 0.9, A = 0.01, and\ A_2 = 0.01$

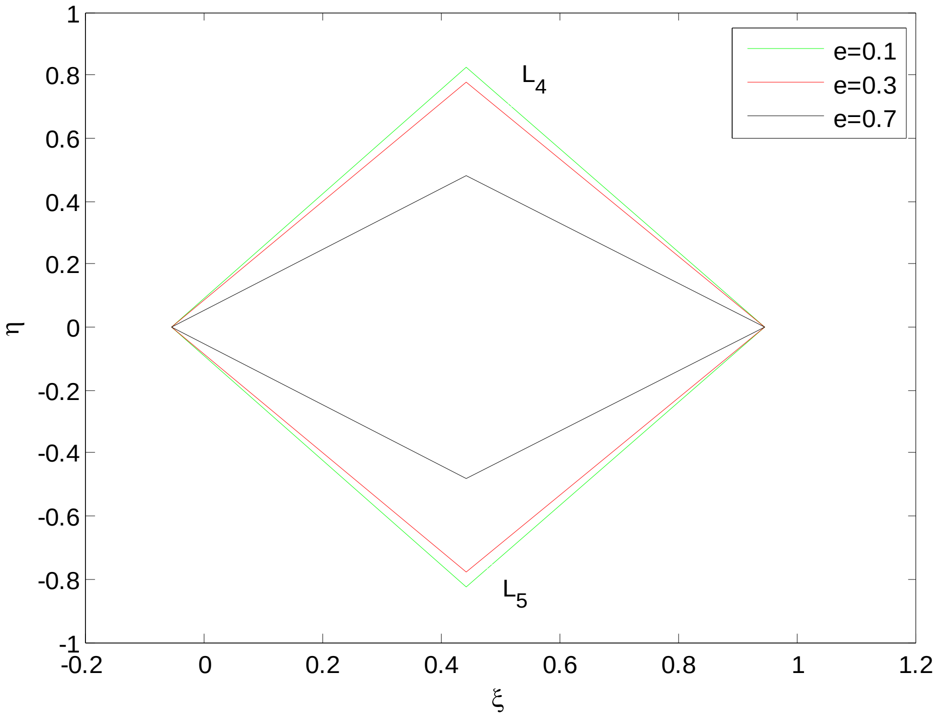

Figure 5.1: Effect of eccentricity on $L_{4,5}$ of CEN X-3 for $a = 0.9, A = 0.01$ *and* $A_2 = 0.01$

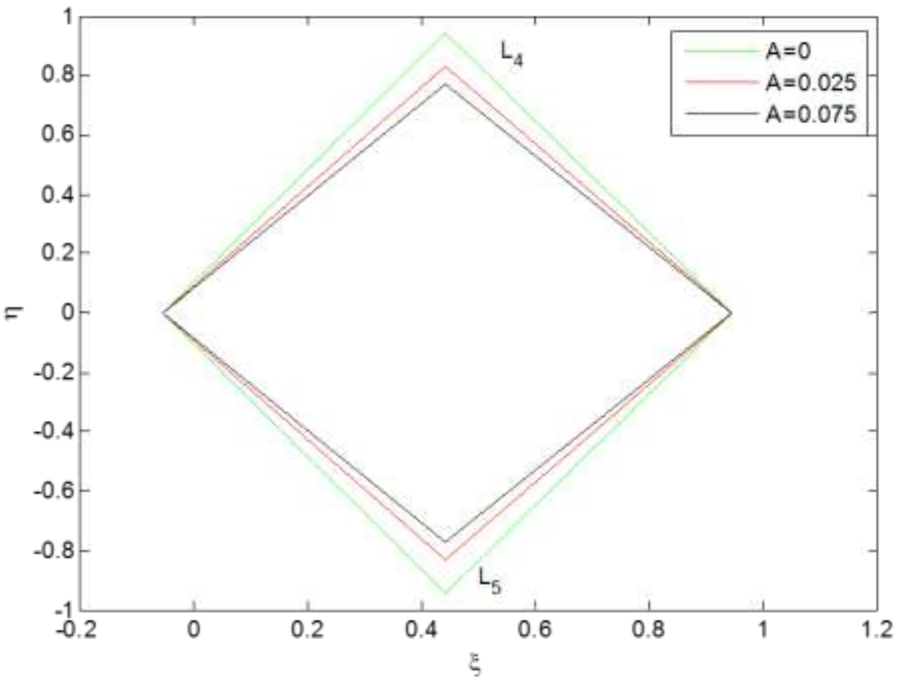

Figure 5.3: Effect of oblateness of the third body on $L_{4,5}$ of CEN X-3 for $a = 0.9, A = 0.01$ *and* $A_2 = 0.01$

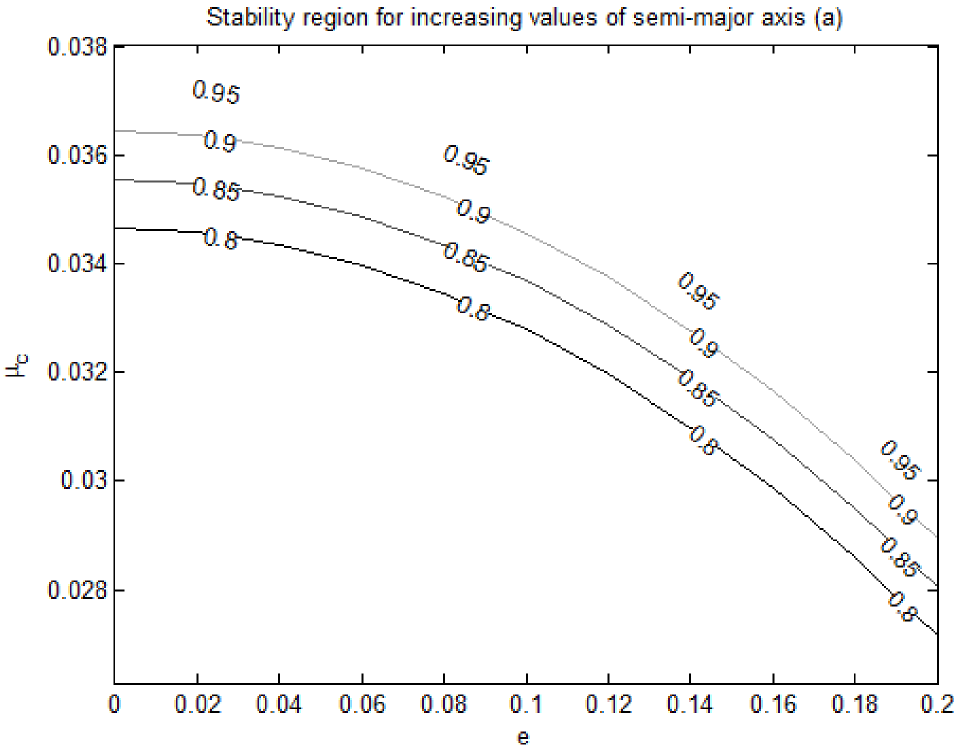

Figure 5.4: Effect of semi-major axis (a) on the stability region for A=0.001,$\boldsymbol{q} = \mathbf{0.99998}$ $\boldsymbol{and}$ $\boldsymbol{A_2} = \mathbf{0}$

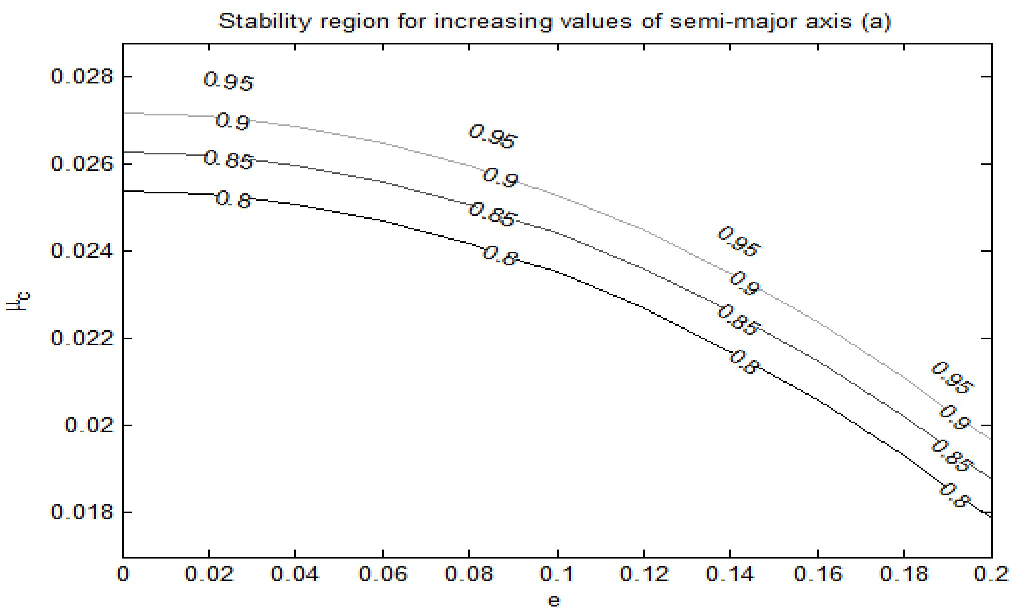

Figure 5.5: Effect of semi-major axis (a) on the stability region for A=0.001,$\boldsymbol{q} = \mathbf{0.99998}$, $\boldsymbol{and}$ $\boldsymbol{A_2} = \mathbf{0.00225}$

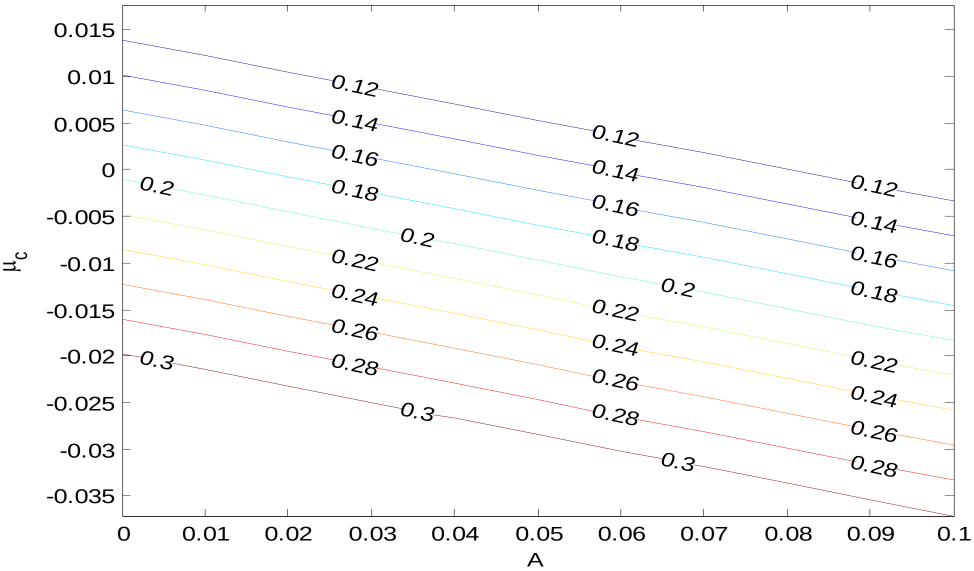

Figure 5.6: Effect of eccentricity on the stability region for A=0.00, $\boldsymbol{a = 0.9\ and\ = A_2 = 0.01}$

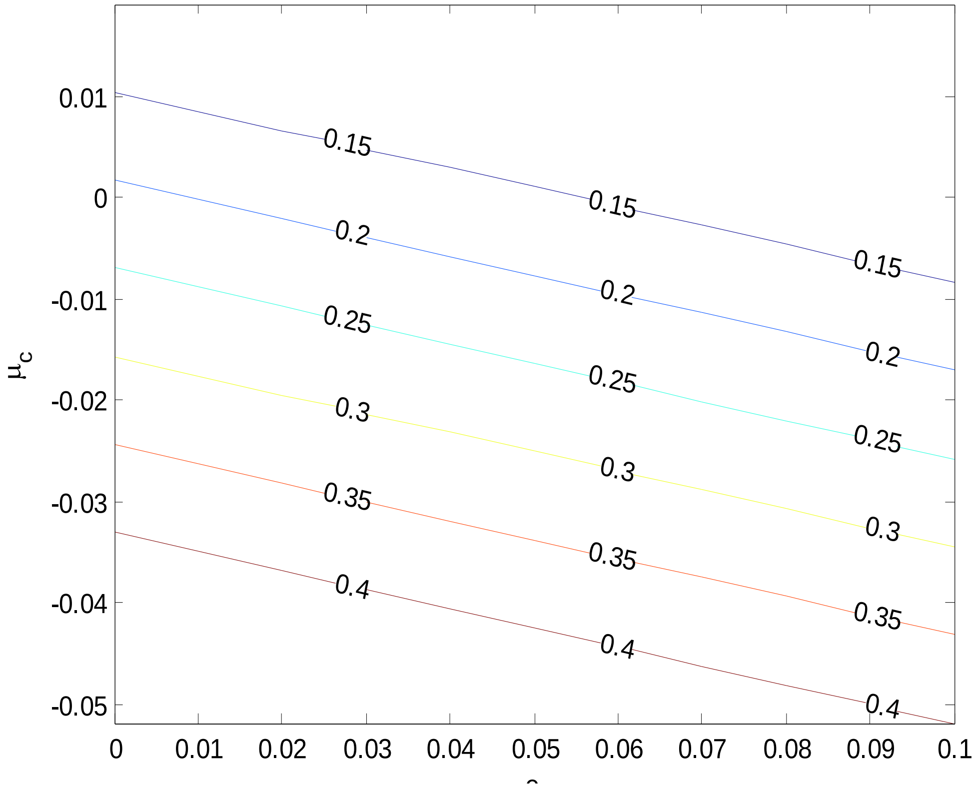

Figure 5.7: Effect of oblateness on the stability region for e=0.3, $\boldsymbol{a = 0.9\ and\ = A_2 = 0.01}$

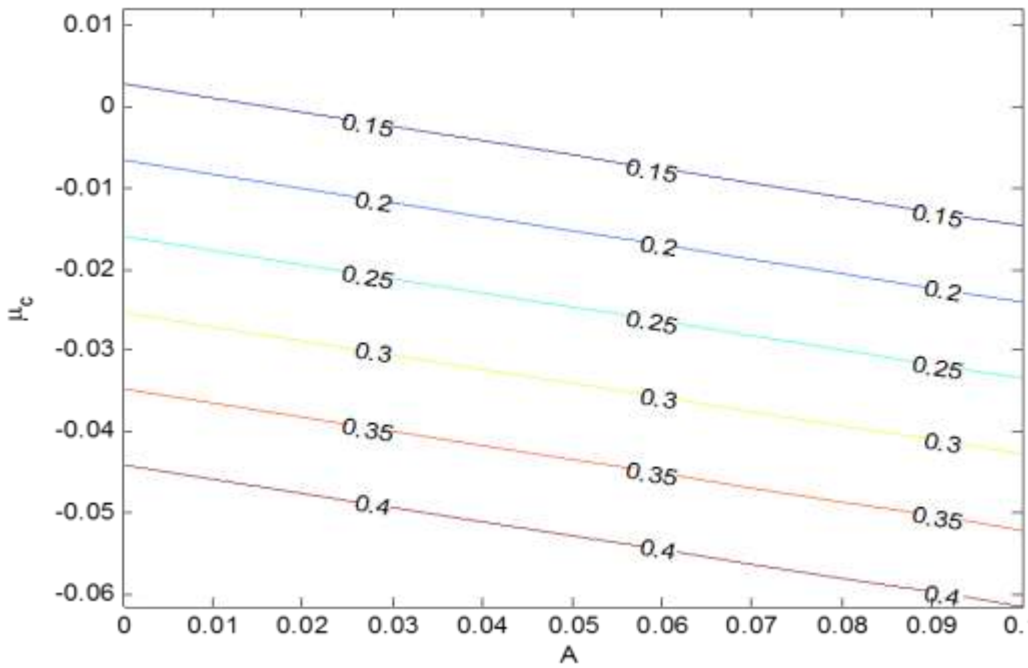

Figure 5.8: Effect of eccentricity on the stability region for A=0.01, $\boldsymbol{a = 0.9\ and\ A_2 = 0.01}$

5.2.1 Discussion of the Triangular Equilibrium points

The motion of a test particle under the influence of a radiating primary and an oblate secondary has been investigated in the elliptic restricted problem of three bodies. The equations of motion are given in equations 3.1-3.4. While the locations and nature of stability of the triangular equilibrium points (equations 3.37 & 3.48) reveal the effects of the various parameters on the dynamics of the system.

In Tables 5.2-5.5, the effects of oblateness, radiation and eccentricity on the locations of the triangular points of CEN X-3 are investigated. It is observed that for high values of eccentricity ($e \geq 0.6$) and low values of semi-major axis ($a \leq 0.3$) the triangular points cease to exist agreeing with Singh and Umar (2013b). Moreover, as oblateness and radiation factors increase, there is a shift towards the bigger primary and towards the line joining the primaries in Cen X-3.

For the stability of the dynamical system, $0 < \mu < \mu_C$ is required (equation 3.48). In this range of stability, the effects of oblateness, eccentricity, and radiation pressure on the size of the region of stability are shown in Figures 5.1-5.8. A gradual reduction in the size of the region of stability is seen with increase in all the parameters leading to instability of the system. This confirms the work of Singh and Umar 2012 a, b. From Table 5.1, the X-ray binaries under consideration have mass ratios outside this range of stability and as such all have unstable triangular points.

Our results agree with Singh and Umar (2013a) in the absence of the triaxiality of the secondary and with Narayan *et al.* (2015) when radiation of the primary and oblateness of the primary are neglected in their work and in ours respectively. This is despite the fact that the nature of the equations of motion differ. The condition of stability for eccentricity $0 < e \le \frac{\sqrt{7}}{4}$ coincides in all works.

5.3 Numerical Results of the Collinear Equilibrium Points

We obtain numerically, the roots and positions of the collinear points $L_i(i=1, 2, 3)$ of the high mass X-ray binary (HMXB) CEN X-3. The radiation pressure factor q of the bigger primary is computed using $q=1-(AkL/a\rho m)$ on the basis of Stefan-Boltzmann's law with $k=1$; where $m\ and\ L$ are the mass and luminosity respectively; $a\ and\ \rho$ are the radius and density of a moving test particle; k is the radiation pressure efficiency of a star; $A=(3/16\pi CG)$. In the C.G.S. system $A=2.9838\times10^{-5}$.The effects of radiation, oblateness, eccentricity and semi-major axis are given in Tables5.6 – 5.9. $\rho L_i\ (i=1,2,3)$ are the roots of the characteristic equation which correspond to the position of the collinear point $L_i(i=1,2,3)$.These effects are shown graphically in Figures 5.9-5.22.

Table 5.6: Effect of oblateness of the third bodyfor $A_2 = 0.1\ and\ e = 0.3$ on L_1 of CEN X-3

A	ρL_1	L_1	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\boldsymbol{\Omega_{\xi\xi}\Omega_{\eta\eta}}$
0.0	0.31192	1.25662	12.5888	-2.63618	-33.1865
0.00001	0.311925	1.25663	12.5889	-2.63615	-33.1862
0.0001	0.311973	1.25667	12.5888	-2.63579	-33.1813
0.001	0.312452	1.25715	12.5873	-2.63216	-33.1319
0.01	0.317096	1.2618	12.5714	-2.59741	-32.653
0.1	0.353696	1.2984	12.3361	-2.34619	-28.9429
0.15	0.369307	1.31401	12.1958	-2.24961	-27.4358
0.2	0.382913	1.32761	12.0617	-2.1698	-26.1715

Table 5.7: Effect of oblateness of the secondaryfor $A = 0.1\ and\ e = 0.3$ on L_1 of CEN X-3

$\boldsymbol{A_2}$	ρL_1	L_1	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\boldsymbol{\Omega_{\xi\xi}\Omega_{\eta\eta}}$
0.0	0.252645	1.19735	12.6053	-3.25283	-41.0028
0.00001	0.25266	1.19736	12.6061	-3.25274	-41.0044
0.0001	0.25279	1.19749	12.6146	-3.25211	-41.0241
0.001	0.254076	1.19878	12.6961	-3.24572	-41.2079

0.01	0.26517	1.20987	13.3528	-3.19685	-42.6869
0.1	0.317096	1.2618	15.9962	-3.13255	-50.1088
0.15	0.331331	1.27603	16.8064	-3.17422	-53.3474
0.2	0.341777	1.28648	17.4875	-3.22749	-56.4407

Table 5.8: Effect of eccentricity for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L_1 of CEN X-3

e	ρL_1	L_1	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
0.9	0.198229	1.14293	36.5528	-9.77998	-357.485
0.8	0.208242	1.15294	24.6692	-6.58868	-162.537
0.7	0.219083	1.16378	19.2094	-5.11185	-98.1954
0.6	0.230614	1.17531	15.8825	-4.20218	-66.741
0.5	0.242536	1.18724	13.6131	-3.57316	-48.6419
0.4	0.254319	1.19902	11.9962	-3.11806	-37.4048
0.3	0.26517	1.20987	10.8469	-2.78957	-30.2582
0.2	0.274077	1.21878	10.0723	-2.56512	-25.8367
0.1	0.279981	1.21878	9.62358	-2.43376	-23.4215

Table 5.9: Effect of radiation for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L_1 of CEN X-3

Q	ρL_1	L_1	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
1.0000	0.265171	1.20987	10.8468	-2.78953	-30.2575
0.9999	0.265167	1.20987	10.8472	-2.78969	-30.2605
0.9888	0.264677	1.20938	10.9053	-2.8091	-30.6342
0.9555	0.263223	1.20792	11.081	-2.86775	-31.7779
0.8999	0.260839	1.20554	11.3805	-2.96755	-33.7722
0.8555	0.258973	1.20367	11.6253	-3.04888	-35.4443
0.7999	0.256684	1.20138	11.9388	-3.15274	-37.6401
0.7555	0.254892	1.19959	12.1949	-3.23735	-39.4793
0.7000	0.252699	1.1974	12.5217	-	-

				3.34503	41.8856

Table 5.10: Effect of semi major axis for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L_1 of CEN X-3

a	ρL_1	L_1	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
0.9	0.26517	1.20987	10.8469	-2.78957	-30.2582
0.8	0.237523	1.18222	14.2093	-3.77175	-53.5938
0.7	0.211565	1.15627	19.1825	5.18481	-99.4578
0.6	0.187011	1.13171	26.7954	-7.28654	-195.246
0.5	0.163478	1.10818	39.0487	-10.5742	-412.91
0.4	0.140462	1.08516	60.3294	-16.1316	-973.209
0.3	0.117242	1.06194	102.116	-26.7819	-2734.86
0.2	0.0925924	1.03729	204.482	52.3515	-10705.0
0.1	0.279981	1.22468	1.94286	-0.29340	-0.57003

Table 5.11: Effect of oblateness of the third body for $A_2 = 0.1\ and\ e = 0.3$ on L_2 of Cen x-3

A	ρL_2	L_2	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
0.0	0.33407	0.61063	13.7783	-3.74623	-51.6168
0.00001	0.33407	0.61063	13.7792	-3.74645	-51.6232
0.0001	0.334067	0.610633	13.7876	-3.74855	-51.6836
0.001	0.334041	0.610659	13.8712	-3.76943	-52.2865
0.01	0.333799	0.610901	14.7067	-3.97822	-58.5065
0.1	0.332332	0.612368	22.3361	-6.00446	-137.727
0.15	0.331896	0.612804	27.7331	-7.23423	-200.628
0.2	0.331584	0.613116	32.3915	-8.39876	-272.049

Table 5.13: Effect of eccentricity for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L_2 of CEN X-3

e	ρL_2	L_2	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
0.9	0.276528	0.668172	13.8353	-4.34309	-60.0877
0.8	0.278875	0.665825	13.2931	-4.1084	-54.6135
0.7	0.282831	0.661869	12.5142	-	-

				3.76384	47.6135
0.6	0.288448	0.656252	11.6439	-3.36263	-39.1543
0.5	0.295777	0.648923	10.8314	-2.9563	-32.0208
0.4	0.304844	0.639856	10.2198	-2.58797	-26.4486
0.3	0.315616	0.629084	9.97602	-2.29509	-22.8959
0.2	0.327974	0.616726	10.4179	-2.12844	-22.1738
0.1	0.341707	0.602993	12.7364	-2.25756	-28.7531

Table 5.14: Effect of radiation pressure for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L_2 of CEN X-3

q	ρL_2	L_2	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\boldsymbol{\Omega_{\xi\xi}\Omega_{\eta\eta}}$
1.0000	0.282827	0.661873	12.5146	-3.76401	-47.105
0.9999	0.282839	0.661861	12.5132	-3.76349	-47.0935
0.9888	0.284174	0.660526	12.3659	-3.70678	-45.8378
0.9555	0.288305	0.656395	11.9311	-3.53875	-42.2213
0.8999	0.295658	0.649042	11.2292	-3.26531	-36.6668
0.8555	0.301979	0.64271	11.6909	-3.05362	-32.6459
0.7999	0.310523	0.634177	10.0457	-2.79724	-28.1002
0.7555	0.317904	0.626796	9.55425	-2.59976	-24.8388
0.7000	0.327914	0.616786	41.1381	-2.36233	-97.1815

Table 5.15: Effect of semi major axis for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L_2 of CEN X-3

a	ρL_2	L_2	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\boldsymbol{\Omega_{\xi\xi}\Omega_{\eta\eta}}$
0.9	0.282831	0.661869	12.5142	-3.76384	-47.1014
0.8	0.299364	0.645336	10.3181	-2.78082	-28.6927
0.7	0.33928	0.620772	8.46926	-1.8884	-15.9971
0.6	0.360172	0.584528	7.17597	-1.1551	-

					8.28897
0.5	0.410987	0.533713	6.61074	-0.62746	-4.14802
0.4	0.476725	0.467975	6.85358	-0.29182	-2.00003
0.3	0.556166	0.388534	8.06778	-0.08665	-0.69911
0.2	0.649487	0.295213	5.72795	-0.01423	-0.08152
0.1	0.762051	0.182649	20.6302	-0.19328	3.98741

Table 5.16: Effect of oblateness for $A_2 = 0.1\ and\ e = 0.3$ on L_3 of CEN X-3

A	ρL_3	L_3	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\boldsymbol{\Omega_{\xi\xi}\Omega_{\eta\eta}}$
0.0	0.160563	-0.894737	3.68341	-0.060336	-0.222242
0.00001	0.160557	-0.894743	3.68346	-0.060335	-0.222244
0.0001	0.160505	-0.894795	3.68398	-0.060332	-0.222263
0.001	0.159988	-0.895312	3.68915	-0.060299	-0.222453
0.01	0.154917	-0.900383	3.73911	-0.059974	-0.224253
0.1	0.111907	-0.943393	4.11764	-0.057334	-0.23608
0.15	0.092245	-0.963054	4.26693	-0.056188	-0.239752
0.2	0.074628	-0.980672	4.38986	-0.055192	-0.242288

Table 5.17: Effect of oblateness for $A = 0.1\ and\ e = 0.3$ on L_3 of CEN X-3

$\boldsymbol{A_2}$	ρL_3	L_3	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\boldsymbol{\Omega_{\xi\xi}\Omega_{\eta\eta}}$
0.0	0.119008	-0.936292	3.44778	-0.057927	-0.199722
0.00001	0.119012	-0.936288	3.44778	-0.057928	-0.199723
0.0001	0.119047	-0.936253	3.44779	-0.057930	-0.199732
0.001	0.119397	-0.935903	3.44787	-0.057955	-0.199823
0.01	0.12287	-0.93243	3.4487	-0.058204	-0.200729
0.1	0.154842	-0.900458	3.45679	-0.060573	-0.209388
0.15	0.170721	-	3.46109	-	-

		0.884528		0.061804	0.213913
0.2	0.185468	0.869832	3.46528	-0.062983	-0.218257

Table 5.18: Effect of eccentricity for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L$_3$ of CEN X-3

e	ρL_3	L_3	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
0.9	0.0897869	-0.96551	3.29911	-0.05334	-0.17598
0.8	0.10277	-0.95253	3.35309	-0.05512	-0.18485
0.7	0.12287	-0.93243	3.4487	-0.05820	-0.20072
0.6	0.148238	-0.90706	3.59618	0.062767	-0.22572
0.5	0.176968	-0.87833	3.81452	-0.06921	-0.26403
0.4	0.207388	-0.84791	4.14029	-0.07836	-0.32446
0.3	0.238202	-0.81709	4.65181	-0.09199	-0.42795
0.2	0.268497	-0.78680	5.55465	-0.11482	-0.63781
0.1	0.297687	-0.75761	7.67235	-0.16571	-1.27138

Table 5.19: Effect of radiation for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L$_3$ of CEN X-3

q	ρL_3	L_3	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
1.0000	0.122861	-0.93243	3.4487	-0.05820	-0.20072
0.9999	0.122888	-0.93241	3.4487	-0.05820	-0.20073
0.9888	0.125932	-0.92936	3.44865	-0.05836	-0.20129
0.9555	0.135186	-0.92011	3.44852	-0.05887	-0.20302
0.8999	0.15107	-0.90423	3.44831	-0.05975	-0.20606
0.8555	0.164169	-0.89111	3.44818	-0.06050	-0.20863
0.7999	0.181137	-0.87416	3.44807	-0.06150	-0.21206
0.7555	0.195176	-0.86012	3.44807	-0.06235	-0.21499
0.7000	0.213393	-0.84190	3.44823	-0.06348	-0.21892

Table 5.20: Effect of semi major axis for $A = 0.01, A_2 = 0.01, and\ e = 0.3$ on L_3 of CEN X-3

a	ρL_3	L_3	$\Omega_{\xi\xi}$	$\Omega_{\eta\eta}$	$\Omega_{\xi\xi}\Omega_{\eta\eta}$
0.9	0.122949	-0.93235	3.44937	-0.05820	-0.20078
0.8	0.189662	-0.86563	3.76973	-0.06307	-0.23777
0.7	0.259134	-0.79616	4.1886	-0.06912	-0.28955
0.6	0.331885	-0.72341	4.75936	-0.07690	-0.36603
0.5	0.408629	-0.64667	5.582	-0.08732	-0.48745
0.4	0.49041	-0.56489	6.86756	-0.10210	-0.70121
0.3	0.578869	-0.47643	9.14703	-0.12495	-1.14298
0.2	0.676919	-0.37838	14.2067	-0.16572	-2.35441
0.1	0.791169	-0.26413	32.9741	-0.26364	-8.69356

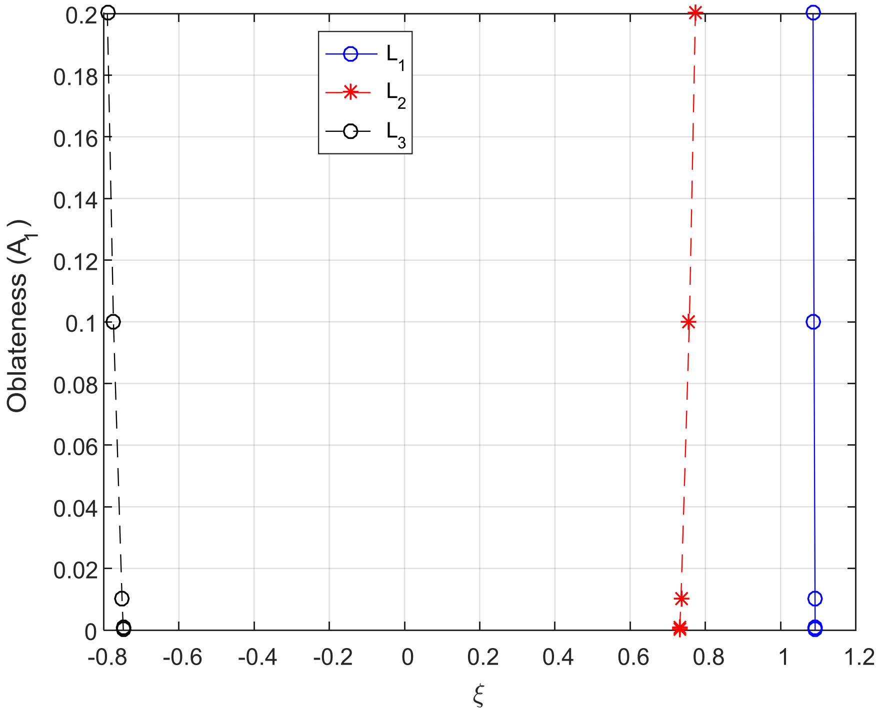

Figure 5.9: Effect of oblateness of the third body on the collinear points $L_i (i = 1,2,3)$ of CEN X-3

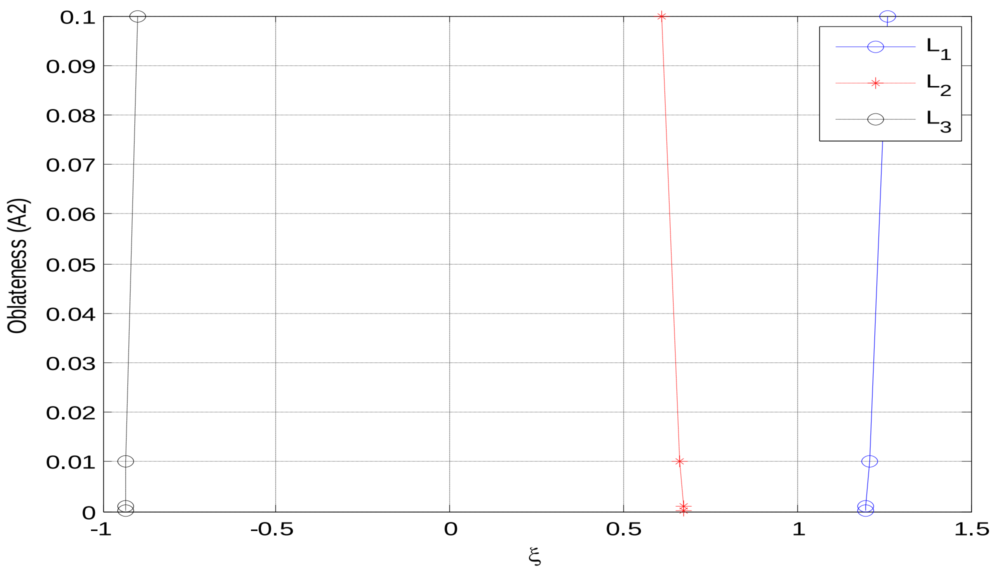

Figure 5.10: Effect of oblatenessof the smaller primary on the collinear points$L_i(i = 1,2,3)$ of CEN X-3

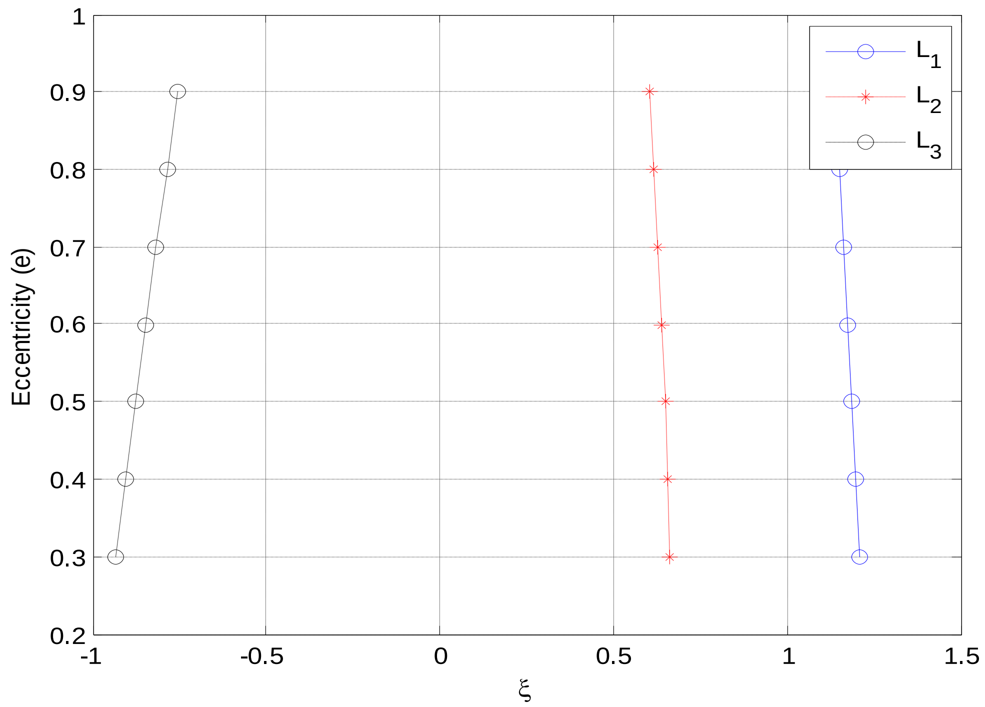

Figure 5.11: Effect of eccentricity on the collinear points$L_i(i = 1,2,3)$ of CEN X-3

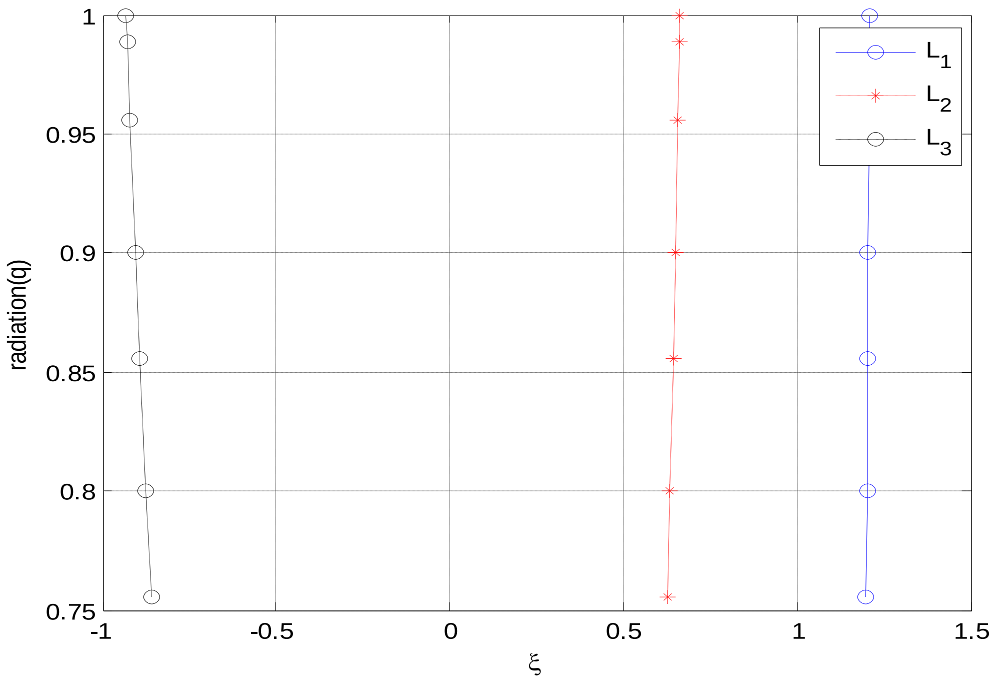

Figure 5.12: Effect of radiation pressure on the collinear points $L_i\,(i = 1,2,3)$ of CEN X-3

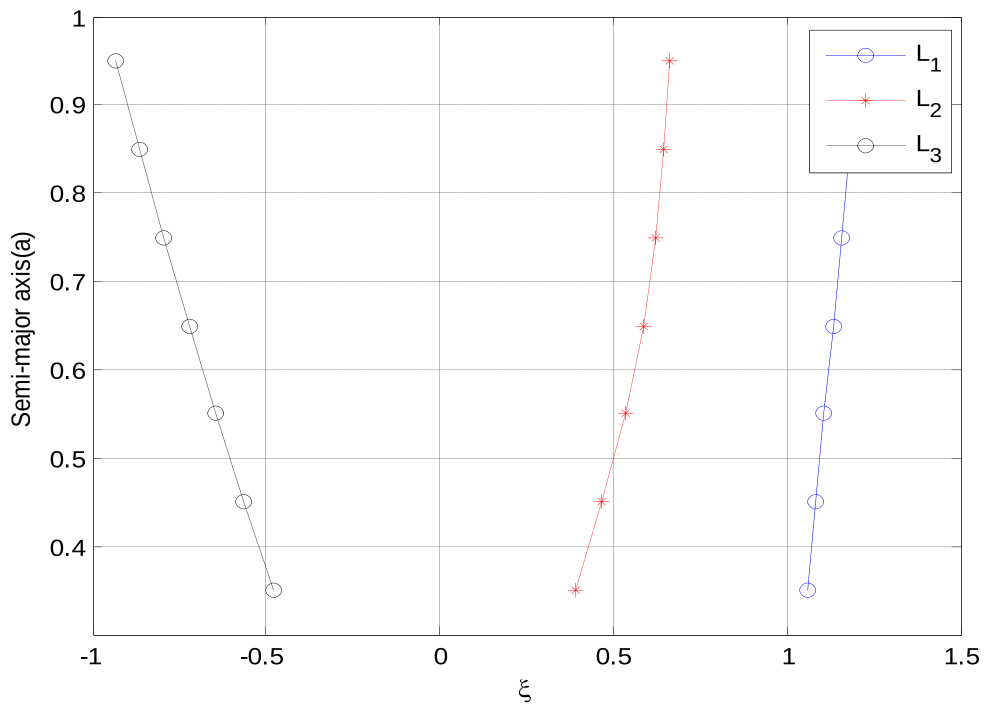

Figure 5.13: Effect of semi major axis on the collinear points $L_i(i = 1,2,3)$ of CEN X-3

5.3.1 Discussionof the Collinear Equilibrium points

The positions and linear stability of the collinear points of the ER3BP under the effects of radiation of the primary and oblateness of the secondary have been investigated. A numerical investigation of these effects on the positions of the collinear point $L_{1, 2, 3}$ of CEN X-3 shows that for CEN X-3, radiation and eccentricity cause a shift away from the smaller primary while semi-major axis and oblateness tend to generate a shift towards the smaller primary (Figures 5.9-5.13). Increase in radiation and eccentricity cause a shift towards the bigger primary; increase in oblateness and semi-major results in a shift away from the bigger primary on $L_{2, 3}$. The stability behavior of the system remains unaffected by the introduction of these parameters. The collinear points are unstable in the Lyapunov sense.

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Summary

This dissertation has investigated the dynamics of an oblate infinitesimal particle in the vicinity of a radiating primary and an oblate secondary in the framework of ER3BP. The analytical results were obtained and applied to the X-ray binary CEN X-3, to obtain the numerical results with the help of Mathematica software. The equations of motion were adopted from Singh and Umar 2013b and are affected by the eccentricity of the orbits, radiation of the primary body and oblateness of the secondary body.

The locations of the five equilibrium points of this problem have been determined, two triangular and three collinear equilibrium points and are dependent on the potentials.

The linear stability of the equilibrium points (triangular and collinear equilibrium points) of this problem have been studied and are found to be affected by the eccentricity of the orbits, radiation of the primary body and oblateness of the secondary and third bodies.

6.2 Conclusion

This dissertation has focused on the use of analytical and numerical techniques to obtain the locations and linear stability of the equilibrium points.

The locations and nature of stability of the triangular equilibrium points reveal the effects of the various parameters on the dynamics of the system. The locations of these triangular equilibrium points are significantly dependent on the parameters of the system. The stability of the triangular equilibrium points coincides with result obtained by Szebehely (1967) and are conditionally stable for $0 < \mu < \mu_C$ and unstable for $\mu_C \leq$

$\mu \leq \frac{1}{2}$,where the critical mass $\mu_C = 0.03852 \ldots$. The effect of the parameters on the region of stability for Cen x-3 studied shows a gradual reduction in the size of the stability regionleading to the instability of CEN X-3 due to $0 < \mu < \mu_C$. Despite the fact that, the nature of the equations of motion of this problem differ from Singh and Umar (2012a, 2013b) and Narayan *et al.*(2015), the condition of stability for eccentricity $0 < e \leq \frac{\sqrt{7}}{4}$ coincides in all cases.

The locations of the collinear equilibrium points are significantly affected by the parameters of the system, while the stability behavior of the system remains unaffected by the introduction of those parameters. Hence, the collinear points are unstable, while the triangular points are conditionally stable.

6.3 Recommendations

This dissertation can be extended for future research in the following ways:

1. Investigation on the "Out-of –plane '' points of the problem
2. Investigation of the problem with variable masses
3. Investigation of the nonlinear stability of theproblem

6.4 Contribution to knowledge

1. The inclusion in this study of radiation pressure, and oblateness of secondary and of the third body, eccentricity of the orbit has contributed a lot to the dynamics of space in elliptic R3BP.

2. It has established the effects of these parameters on certain stellar and solar systems.

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