PERISTALTIC PUMPING OF FLUID, EFFECTS OF POISEUILLE FLOW

BY

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DECLARATION

I hereby declare that this thesis has been composed by myself and that it is a record of my own research work under the supervision of Dr. L.M. Srivastava. It has not been accepted in any previous application for a higher degree. All quotations are distinguished by quotation marks and the sources of information are specifically acknowledged by means of references.

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December, 1986.

CERTIFICATION

This thesis entitled "PERISTALTIC PUMPING OF FLUID, EFFECTS OF POISEUILLE FLOW" by Ime Jimmy Uwanta meets the regulations governing the award of the degree of M.Sc (Mathematics) of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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Member, Supervisory Committee

Date.....

Member, Supervisory Committee

Date: 12/4/87

Head of Department

Date:.....

Dean, Postgraduate Schcol

DEDICATION

Dedicated to my late parents Jimmy and Rosa who did not live to reap the fruits of their labour, and my sisters Juliet and Comfort Relatives, Friends who missed me during the period and my fiancee Ema

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I wish to expressed my sincere gratitude to my supervisor Dr. L.M. SRIVASTAVA of Department of Mathematics, Ahmadu Bello University, Zaria for not only suggesting the topic for the thesis but also for his guidance and advice in various dimensions to the complexion.

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ABSTRACT

Title: Peristaltic Pumping of Fluid.

Effects of Poiseuille Flow.

When a fluid is set into motion by a steady progressive sinusoidal wave of moderate amplitude in a channel, a basic Poiseuille flow caused by a uniform pressure became present giving rise to problem of peristalsis. Peristalsis is a muscle controlled flow. Since muscles are involved as the walls are flexible or not flexible depending upon each case, the fluid is propelled by rhythmic waves of contraction called peristalsis (meaning to send around). A typical peristaltic wave is preceeded by relaxation. However, the strength and velocity of peristaltic contractions are increased by impulses carried over the walls. As the structures relax, the cycle begins again.

Peristalsis has been one of the major mechanisms for fluid transport in many biological systems. The study of mechanism of peristalsis in mechanical and physiological situations has recently become the object of scientific research since its first investigation by Latham (1966) both experimental and theoretical because of its importance to human lives.

The first chapter fives a survey of the literature review, the contributions made to date, aim and scope of

peristaltic fluid pumping to biomechanical systems. The second chapter describes the interaction effect of Poiseuille flow on peristaltic fluid-particle suspension transport in a two-dimensional mathematical model when the wall (inelastic) of the channel executes sinusoidal motion of small amplitude. The approaches and formulations in solving the mathematical model were for closed form solutions, solution for low Reynolds number, higher order Reynolds number, Stokes approximation, long wave approximation and for series solution. As the volume fraction density is negligible, it has no effect on D. In chapter three, the walls of the channel are flexible. The influence of Poiseuille flow on pure peristalsis is investigated an: the mean flow reversal is found to occur at the centre and at the boundaries. when the walls of the channel are elastic, pure peristalsis involves flow reversal only at the centre. Analysis of the problem include formulation of the problem, method of solution, pure peristalsis, elastic wall, dissinative wall, closed form solution, series solution and the results and dissicussion centring on the effect of C and D, F(y) - F(l) etc thus increase or decrease in C greatly affects other outputs. Chapter four describes the motion of micropolar fluids as having translatory degrees of freedom, the ability to rotate and can support couple stresses which affected the mode of

motion, a mark difference from other fluids. The approach used were introducing the subject micropolar fluid, analysis of the problem, solution for free pumpinp case, interaction with Toiseuille flow, Stokes approximation and solution for higher Order Reynolds number.

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CHAPTER I:

1.1 INTRODUCTION AND LITERATURE REVIEW

Peristalsis is known to physiologists to be one of the major mechanisms for fluid transport in many biological systems. In particular peristaltic mechanism may be involved in urine transport from kidney to bladder through the ureter, movement of chyme in the gastro-intestinal tract, transport of sperm in the ductus efferentes of male reproductive tracts and in the cervical canal, movement of ovum in the fallopian tubes, lymph transport in the lymphatic vessels and in the vasomotion of small blood vessels. In addition, peristaltic pumping processes occur in many practical applications involving biomechanical systems.

The study of the mechanism of peristalsis, in both mechanical and physiological situations, has recently become the object of scientific research. Since the first investigation of Latham (1966), several theoretical and experimental attempts have been made to understand peristaltic action in different situations. All such investigations seem to differ in various details. A review of much of the early literature is presented in an excellent article by Jaffrin and Shapiro (1971). A summary of most of the theoretical investigation reported so far, arranged according to the geometry, fluid nature, the Reynolds number R_e, the wavelength parameter α (the pipe radius/channel half-width divided by the wavelength),

the amplitude parameter ε (the wave amplitude divided by the pipe radius/channel half-width), and the wave shape is presented in Table 1. In addition, Fung (1971a, 1971b, 1971c) studied peristaltic flow, taking muscle action in the tube wall, into account. Some new ideas on peristaltic flow are also given in a book by Liron (1978). Considerable experimental investigations of peristaltic pumping have also been undertaken (Latham (1966); Mank (1976); Shapiro and Latham (1966); Eckstein (1970); Weinberg (1970); Yin and Fung (1971); Hung and Brown (1976); Zimmerman (1977)). Of greatest interest to this study are Zien and Ostrach (1969), C.H. Li (1970) and Liron (1976).

1.2 AIM AND SCOPE

In the brief introduction outlined above, various autors have been working on the topic peristalic transport as given in the summary of theoretical contribution to peristaltic motion (see Table 1). Since the topic is of imense and important area in human endeavours, and peristaltic fluid pumping occurs in many practical applications involving bio-mechanical systems, more research work is being undertaken.

Chapter II deals with a two-dimensional fluid-particle

Table 1: Summary of theoretical contribution to peristaltic motion

Author(s), geometry and fluid	R	α	ε	Waveshape
Latham (1966) PN	0	0	arbitrary	sinusoidal
Shapiro (1967) PN	0	0	arbitrary	sinusoidal
Burns and Parkes (1967) PAN	0	arbitrary	$\alpha\varepsilon<<1$	sinusoidal
Barton and Raynor (1968) AN	0	0, arbitrary	arbitrary $\alpha\varepsilon<<1$	sinusoidal
Shapiro et al. (1969) PAN	0	0	arbitrary	sinusoidal
Meginniss (1970) AN	0	0	arbitrary	sinusoidal
Lykoudis and Roos (1970) AN	0	0	arbitrary	sinusoidal
Roos and Lykoudis (1971) AN	0	0	arbitrary	sinusoidal
Lee et al. (1971) AN	0	0	arbitrary	non-sinusoidal
Tong and Vawter (1972) AN	0	arbitrary	arbitrary	sinusoidal
Negrin et al. (1974) AN	0	arbitrary	arbitrary	sinusoidal
Gupta and Seshadri (1976) PAN	0	0	arbitrary	sinusoidal
Shukla et al. (1980) PAN	0	0	arbitrary	sinusoidal
Srivastava and Srivastava (1982) PAN	0	0	arbitrary	sinusoidal
Srivastava et al. (1982,a,b,c) PAN	0	0	arbitrary	sinusoidal
Srivastava (1982) A (Casson fluid)	0	0	arbitrary	sinusoidal
Srivastava and Srivastava A(1983b) (Non-Newtonian)	0	0	arbitrary	sinusoidal
Hanin (1968) PN	arbitrary	0	small	lateral bending
Fung and Yin (1968) PN	arbitrary	arbitrary	small	sinusoidal
Yin and Fung (1969) AN	arbitrary	arbitrary	small	sinusoidal
Chow (1970) AN	small and arbitrary	arbitrary & small	small	sinusoidal
Raju and Devanathan (1972) A (Non-Newtonian)	small	arbitrary	small	sinusoidal
Mittra and Prasad (1973, 74) PN	small and arbitrary	arbitrary	small	sinusoidal

Author(s), geometry and fluid	R	α	ε	Waveshape
Devi and Devanathan (1975) A (Micropolar)	**small**	**arbitrary**	**small**	**sinusoidal**
Wilson et al. (1977) PN	**arbitrary**	**arbitrary**	**small**	**lateral bending**
Wilson and Perel (1979) PN	arbitrary	arbitrary	small	sinusoidal
Wilson (1979) PN	arbitrary	arbitrary	small	sinusoidal
Srivastava and Srivastava (1983a) AN	arbitrary	arbitrary	small	sinusoidal
Zien and Ostrach (1970) PN	small	small	arbitrary	sinusoidal
Li (1970) AN	small	small	arbitrary	sinusoidal
Jaffrin (1971) PN	small	small	arbitrary	arbitrary
Manton (1975) AN	small	small	arbitrary	arbitrary
Liron (1976) PAN	small	small	arbitrary	arbitrary
Mank (1976) PN	high	arbitrary	high	unsymmetric
Mahrenholtz et al. (1978) PN	high	arbitrary	high	unsymmetric
Rath (1978) N (Elastic tube)	arbitrary	-	arbitrary	sinusoidal
Kaimal (1978) AN	arbitrary	small	small	arbitrary
Wilson and Panton (1979) PN	arbitrary	small	O(1)	lateral and complex bending

P - plane two dimensional

A - axisymmetric

N - Newtonian

suspension in which the walls of the channel were considered inelastic because the travelling wave motions had no accompanying area change. Starting with the Navier-Stokes equations and using the stream function Ψ, equations of motion of the fluid were obtained. Expansion of Ψ in power series subject to certain boundary conditions gave rise to coupled fourth order equations of Orr-Sommerfield type whose solutions were got. Calculations were also made for closed form solutions [illegible] for finite values of R and K and consequently for the series solution, Stoke's approximation and higher order Reynolds number pertaining to long wave length.

Chapter III basically is treating the case when the fluid particle suspension confined to the channel as the walls exhibit some movement because it is a muscle controlled motion due to a membrance or a thin plate. Because of these movements the walls are flexible thus we refer to it as elastic. The governing equations were Navier-Stokes equations with respect to the fluid and particle phases subject to certain boundary conditions. After the nondimensionalization of some parameters and intensive calculations, solutions were obtained for pure peristalsis (free pumping K = 0). In this case the Poiseuille flow is zero. Also considered were the role played by the properties of the wall for the solution in zero damping. Other solutions were for finite and small values of R and K and the series solution.

In chapter IV, some flow properties subjected to steady and oscilatory flow through the channel in terms of the motions of the individual suspended particles and their effect on the surrounding liquid are described. The fluid-particle suspension is set into motion, the fluid stresses on the surface of the suspended particle cause them to rotate as they travel down the channel, and because of the velocity gradient in the fluid, the suspended particles are at times brought into close proximity and interaction since there is a marked change in the translational and rotational motion. Proceeding analytically, the zeroth, first and second order equations were obtained. Also considered was the free pumping and the interaction with Poiseville flow as it affects the imposed pressure gradient. The solving of higher order differential equations were also done.

To get quantitative ideas of the problems, some of the results obtained are computated and displayed by graphs and Tables.

CHAPTER II: INTERACTION OF PERISTALTIC FLUID - PARTICLE SUSPENSION IN A CHANNEL.

2.1: INTRODUCTION

In this chapter, the interaction of peristaltic fluid-particle suspension, based on Mittra and Prasad (1974) model is studied.

Mittra and Prasad (1974) discussed the effect of Poiseville flow on peristaltic transport in a two-dimensional mathematical model of peristalsis for the case when the wall of the channel excutes sinusoidal motion of small amplitude. Small amplitude wave motion in this context is characterized by the parameter wave amplitude divided by characteristic tranverse dimension being "small". That is, when the equations are properly non-dimensionalized, this becomes the perturbation parameter, typically referred to as ϵ.

One notes that a viscous incompressible fluid is filled in the channel of finite length and the walls of the channel are not flexible in which we termed as in-elastic (i.e. include wave wall motions without an accompanying area Change). Its motion is studied. A perturbation solution is found satisfying the complete Navier-Stokes equations for the case of small amplitude ratio. The mean flow is found to depend linearly on amplitude ratio when an imposed pressure

for the case of small amplitude ratio. The mean flow is found to depend linearly on amplitude ratio when an imposed pressure gradient exists. Calculations up to the second order perturbations are carried out. Closed form solutions have been obtained for limiting values of Reynolds number R and the Poiseuille flow parameter while the Stoke's Approximation, the long-wave approximation and the method of Frobenius series solution have been used.

Our model is an extension of Mittra and Prasad but differ in many respects. In this case, a particle-fluid suspension (notably blood) is employed inplace of a single phase fluid. The suspension is treated as a two phase fluid, having a fluid phase and the particle phase, resulting in six sets of simultaneous equations. When solved, the solutions are rewarding such that when C is zero, it reduces to the result got by Mittra and Prasad.

2.2 FORMULATION OF THE PROBLEMS

Consider a two - dimensional infinite channel of uniform thickness, 2d (see Fig. 1), filled with a mixture of small spherical rigid particles in an incompressible Newtonian viscous fluid. The walls of the channel are flexible, on which are imposed travelling, sinusoidal wave of small amplitude. The equations governing conservation of mass and linear momentum for both the fluid and particle phase using a continuum approach are expressed as (Drew, 1979; Srivastava and Srivastava, 1983) fluid phase:

$$(1-C)\rho_f\left[\frac{\partial u_f}{\partial t} + u_f\frac{\partial u_f}{\partial x} + v_f\frac{\partial u_f}{\partial y}\right] = -(1-C)\frac{\partial p}{\partial x} + (1-C)\mu_s(C)\nabla^2 u_f + CS(u_p - u_f) \quad (1)$$

$$(1-C)\rho_f\left[\frac{\partial v_f}{\partial t} + u_f\frac{\partial v_f}{\partial x} + v_f\frac{\partial v_f}{\partial y}\right] = -(1-C)\frac{\partial p}{\partial y} + (1-C)\mu_s(C)\nabla^2 v_f + CS(v_p - v_f) \quad (2)$$

$$\frac{\partial}{\partial x}\left\{(1-C)u_f\right\} + \frac{\partial}{\partial y}\left\{(1-C)v_f\right\} = 0 \quad (3)$$

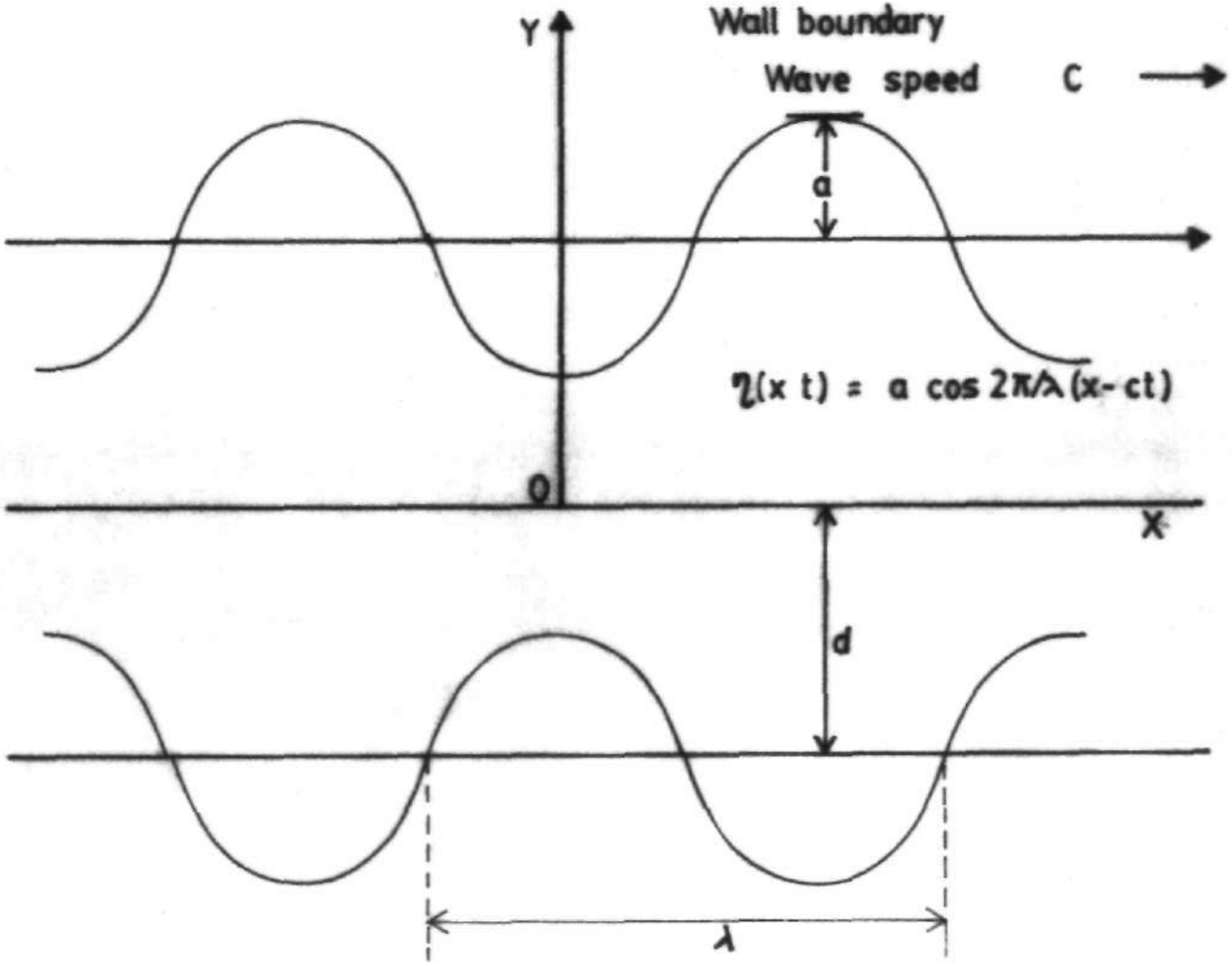
Particulate phase:

$$C\rho_p\left[\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial x} + v_p\frac{\partial u_p}{\partial y}\right] = -C\frac{\partial p}{\partial x} + CS(u_f - u_p) \quad (4)$$

$$C\rho_p\left[\frac{\partial v_p}{\partial t} + u_p\frac{\partial v_p}{\partial x} + v_p\frac{\partial v_p}{\partial y}\right] = -C\frac{\partial p}{\partial y} + CS(v_f - v_p) \quad (5)$$

$$\frac{\partial}{\partial x}\left\{C u_p\right\} + \frac{\partial}{\partial y}\left\{C v_p\right\} = 0 \quad (6)$$

In equations (1) - (6), x and y are cartesian coordinates with x measured in the direction of wave propagation and y measured in the direction normal to the mean position



PERISTALTIC MOTION OF THE WALLS

Fig. 1

of the channel walls, (u_f, v_f) denotes fluid phase velocities, (u_p, v_p) denotes particulate phase velocities, ρ_f and ρ_p be the actual densities of the materials constituting fluid and particulate phase respectively, $(1-C)\rho_f$ the fluid phase density, $C\rho_p$ the particulate phase density, p denotes the pressure, C denotes the volume fraction density of the particles, $\mu(C)$ is the particle fluid mixture viscosity (also the effective viscosity of the suspension), and S the drag coefficient of interaction for the force exerted by one phase on the other.

The number density of the particles is considered so small that the field interaction between particles may be neglected. Thus the diffusivity terms, which can model the effects of particle-particle impacts due to the Brownian motion are neglected. It is worth mentioning here that the effects of Brownian motion was considered by others including Batchelor (1974, 1976). The volume fraction density, C of the particles is chosen also constant. This is a good assumption for low concentration of small particles.

The expression for the drag coefficient for the present problem is selected as

$$S = \frac{9}{2}\frac{\mu_o}{a^2}\lambda'(C) \,, \tag{7}$$

$$\lambda'(C) = \frac{4 + 3\left[8C - 3C^2\right]^{1/2} + 3C}{\left[2 - 3C\right]^2}$$

where μ is the fluid viscosity, and a is the radius of the particle. Relation (7) represents the classical Stoke's drag

for small particle Reynolds number, modified to account for the finite particulate fractional volume through the function $\lambda'(C)$, obtained by Tam (1969).

Many empirical relations have been suggested to express the viscosity of the suspension as a function of particle concentration and viscosity of the suspending medium. Einstein was first to obtain theoretically that the viscosity of the suspension μ_s was related to that of the suspending medium μ_o for spheres in suspension by: $\mu_s = \mu_o/(1-2.5C)$. However, the Einstein formula expresses the viscosity of the suspension only when C is less than 0.05. As C increases from 0.05, the suspension viscosity varies from the Einstein equation. For the present problem, an empirical relation for the viscosity of the suspension.

$$\mu_s(C) = \mu_o \frac{1}{1-qc} \tag{8}$$

$$q = 0.070 \text{ EXP}\left[2.49C + \frac{1107}{T} \exp(-1.69C)\right]$$

where T is absolute temperature (^oK), suggested by Chorm and Kurland (1974), is used. The viscosity of the suspension expressed by this formula is found to be reasonably accurate upto C = 0.6. Chorm and Kurland (1974) tested equation (8) with a cone and plate viscometer and found to be in agreement within 10% in case of blood supension. Now introducing the stream functions for two phase as

$$u_f = \frac{\partial \Psi_f}{\partial y}, \quad v = -\frac{\partial \Psi_f}{\partial x}$$

$$u_p = \frac{\partial \Psi_p}{\partial y}, \quad v_p = -\frac{\partial \Psi_p}{\partial x} \tag{9}$$

Eliminating the pressure gradient from equations (1) and (2) and (4) and (5) to get stream function equations

$$(1-C)\rho_f\left[\frac{\partial}{\partial t}\nabla^2\Psi_f+\Psi_{fy}\nabla^2\Psi_{fx}-\Psi_{fx}\nabla^2\Psi_{fy}\right]=(1-C)\mu(C)\nabla^4\Psi_f+CS(\nabla^2\Psi_p-\nabla^2\Psi_f) \tag{10}$$

$$C\rho_p\left[\frac{\partial}{\partial t}\nabla^2\Psi_p+\Psi_{py}\nabla^2\Psi_{px}-\Psi_{px}\nabla^2\Psi_{py}\right]=CS(\nabla^2\Psi_f-\nabla^2\Psi_p)$$

where ∇^2 denotes the Laplacian operator $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$ and the second subscripts x and y stand for partial derivatives.

The boundary conditions that must be satisfied by the fluid on the walls are the no-slip and impermeability conditions. The walls of the channel are assumed to be flexible but extensible with a travelling sinusoidal wave, and displacement in channel walls are in transverse direction only. Hence boundary conditions are :

$$\left.\begin{aligned} u_f &= \Psi_{fy} = 0 \\ v_f &= -\Psi_{fx} = \pm\frac{\partial\eta}{\partial t} \\ v_f &= -\Psi_{px} = \pm\frac{\partial\eta}{\partial t} \end{aligned}\right\} \text{ on } y = \pm d \pm \eta \tag{12}$$

The transverse displacement, η, of the walls is represented as

$$\eta(x,t) = a\cos\frac{2\pi}{\lambda}(x-ct) \tag{13}$$

where a is the amplitude, λ the wavelength, and C the wave speed. The following set of non-dimensional variables and parameters are selected :

$$x' = \frac{x}{d}\ ,\quad y' = \frac{y}{d}\ ,\quad u_f' = \frac{u_f}{c}\ ,\quad v_f' = \frac{v_f}{c}\ ,$$

$$u_p' = \frac{u_p}{c}\ ,\quad v_p' = \frac{v_p}{c}\ ,\quad \eta' = \frac{\eta}{d}\ ,\quad \Psi_f' = \frac{\Psi_f}{cd}\ ,$$

$$\Psi_p' = \frac{\Psi_p}{cd}, \quad t' = \frac{ct}{d}, \quad p' = \frac{p}{\rho_f c^2}, \quad \nu = \frac{\mu_o}{\rho_f},$$

Suspension Reynolds number $R = \dfrac{\epsilon d \rho_f}{(1-C)M_s}$ (14)

wave number $\alpha = \dfrac{2\pi d}{\lambda}$

amplitude ratio $\epsilon = \dfrac{a}{d}$

suspension parameter $M = \dfrac{Sd^2}{(1-C)\mu_s}$

suspension parameter $N = \dfrac{sd^2\rho_f}{(1-C)\rho_p\mu_s}$

Thus the system of equations (10 - 13) now becomes, after dropping the primes,

$$(1-C)R\left[\frac{\partial}{\partial t}\nabla^2\Psi_f + \Psi_{fy}\nabla^2\Psi_{fx} - \Psi_{fx}\nabla^2\Psi_{fy}\right] = \nabla^4\Psi_f + CM(\nabla^2\Psi_p - \nabla^2\Psi_f) \quad (15)$$

$$CR\left[\frac{\partial}{\partial t}\nabla^2\Psi_p + \Psi_{py}\nabla^2\Psi_{px} - \Psi_{px}\nabla^2\Psi_{py}\right] = CN(\nabla^2\Psi_f - \nabla^2\Psi_p) \quad (16)$$

$$\eta = \epsilon \cos \alpha (x-t) \quad (17)$$

and
$$\left.\begin{aligned} \Psi_{fy} &= 0 \\ u_{fx} &= \mp \alpha \epsilon \sin \alpha (x-t) \\ u_{px} &= \mp \alpha \epsilon \sin \alpha (x-t) \end{aligned}\right\} \quad \text{on} \quad y = \pm (1 + \eta) \quad (18)$$

2.3 METHOD OF SOLUTION

Assuming the amplitude ratio ϵ of the wave to be small, we obtain the solution for the stream function as a power series in terms of ϵ, by expanding Ψ_f, Ψ_p and $\frac{\partial p}{\partial x}$ in the form

$$\Psi_f = \Psi_{fo} + \epsilon\Psi_{f1} + \epsilon^2\Psi_{f2} + \cdots\cdots\cdots \quad (19)$$

$$\Psi_p = \Psi_{po} + \epsilon\Psi_{p1} + \epsilon^2\Psi_{p2} + \cdots\cdots\cdots \quad (20)$$

$$\frac{\partial p}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_o + \epsilon\left(\frac{\partial p}{\partial x}\right)_1 + \epsilon^2\left(\frac{\partial p}{\partial x}\right)_2 + \cdots\cdots\cdots \quad (21)$$

In equation (21), the first term on the right hand side corresponds to the imposed gradient associated with the primary flow and the other terms correspond to the peristaltic motion, or higher imposed pressure gradient.

Substituting equations (19) and (20) in equations (15) and (16) and collecting like powers of ε, the following set of differential equations for $\Psi_{(f,p)o}$, $\Psi_{(f,p)1}$, $\Psi_{(f,p)2}$, ... are obtained.

$$(1-C)R\left[\frac{\partial}{\partial t}\nabla^2\Psi_{fo} + \Psi_{foy}\nabla^2\Psi_{fox} - \Psi_{fox}\nabla^2\Psi_{foy}\right] = \nabla^4\Psi_{fo} + CM(\nabla^2\Psi_{po} - \nabla^2\Psi_{fo})$$

$$(1-C)R\left[\frac{\partial}{\partial t}\nabla^2\Psi_{f_1} + \Psi_{foy}\nabla^2\Psi_{f1x} + \Psi_{f1y}\nabla^2\Psi_{fox} - \Psi_{fox}\nabla^2\Psi_{f1y} - \Psi_{f1x}\nabla^2\Psi_{foy}\right] \quad (22)$$

$$= \nabla^4\Psi_{f_1} + CN(\nabla^2\Psi_{p_1} - \nabla^2\Psi_{f_1}) \quad (23)$$

$$(1-C)R\left[\frac{\partial}{\partial t}\nabla^2\Psi_2 + \Psi_{foy}\nabla^2\Psi_{f2x} - \Psi_{fox}\nabla^2\Psi_{f2y} + \Psi_{f2y}\nabla^2\Psi_{fox} - \Psi_{f2x}\nabla^2\Psi_{foy}\right.$$

$$\left. + \Psi_{f1x}\nabla^2\Psi_{f1y} + \Psi_{f1y}\nabla^2\Psi_{f1x}\right] = \nabla^4\Psi_{f_2} + CM(\nabla^2\Psi_{p_2} - \nabla^2\Psi_{f_2}) \quad (24)$$

$$CR\left[\frac{\partial}{\partial t}\nabla^2\Psi_{po} + \Psi_{poy}\nabla^2\Psi_{pox} - \Psi_{pox}\nabla^2\Psi_{poy}\right] = CN(\nabla^2\Psi_{fo} - \nabla^2\Psi_{po}) \quad (25)$$

$$CR\left[\frac{\partial}{\partial t}\nabla^2\Psi_{p_1} + \Psi_{poy}\nabla^2\Psi_{p1x} + \Psi_{p1y}\nabla^2\Psi_{pox} - \Psi_{pox}\nabla^2\Psi_{p1y} - \Psi_{p1x}\nabla^2\Psi_{poy}\right] = CN(\nabla^2\Psi_{f_1} - \nabla^2\Psi_{p_1}) \quad (26)$$

$$CR\left[\frac{\partial}{\partial t}\nabla^2\Psi_{p_2} + \Psi_{poy}\nabla^2\Psi_{p2x} - \Psi_{pox}\nabla^2\Psi_{p2y} + \Psi_{p2y}\nabla^2\Psi_{pox} - \Psi_{p2x}\nabla^2\Psi_{poy} - \Psi_{p1x}\nabla^2\Psi_{p1y} + \Psi_{p1y}\nabla^2\Psi_{p1x}\right]$$

$$= CN(\nabla^2\Psi_{f_2} - \nabla^2\Psi_{p_2}) \quad (27)$$

Since ε appears both explicitly and implicitly in the boundary conditions, expanding $\Psi_{f,p}$ about the mean position of the wall displacement, first equation of (18) may be written as

$$\Psi_{fy}(\pm 1) \pm \eta\Psi_{fyy}(\pm 1) + \frac{\eta^2}{2}\Psi_{fyyy}(\pm 1) \pm O(\eta^3) = 0$$

After substituting equation (19) into equation (28) and collecting like powers of ε, the following set of boundary conditions are obtained:

$$\Psi_{foy}(\pm 1) = 0 \tag{29}$$

$$\Psi_{f_1y}(\pm 1) = \mp \cos\alpha\,(x-t)\,\Psi_{foyy}(\pm 1) \tag{30}$$

$$\Psi_{f_2y}(\pm 1) = \mp \cos\alpha\,(x-t)\Psi_{f_1yy}(\pm 1) - \tfrac{1}{2}\cos^2\alpha(x-t)\Psi_{foyyy}(\pm 1) \tag{31}$$

Similarly, the second and third equations of (19), yields the following set of boundary conditions :

$$\Psi_{fox}(\pm 1) = \Psi_{pox}(\pm 1) = 0 \tag{32}$$

$$\left.\begin{aligned}\Psi_{f_1x}(\pm 1) &= \mp \cos\alpha\,(x-t)\Psi_{foxy}(\pm 1) \mp \alpha \sin\alpha(x-t)\\ \Psi_{p_1x}(\pm 1) &= \mp \cos\alpha(x-t)\Psi_{poxy}(\pm 1) \mp \alpha \sin\alpha(x-t)\end{aligned}\right\} \tag{33}$$

$$\left.\begin{aligned}\Psi_{f_2x}(\pm 1) &= \mp \cos\alpha(x-t)\Psi_{1xy}(\pm 1) - \tfrac{1}{2}\cos^2\alpha(x-t)\Psi_{foxyy}(\pm 1)\\ \Psi_{p_2x}(\pm 1) &= \mp \cos\alpha(x-t)\Psi_{p_1xy}(\pm 1) - \tfrac{1}{2}\cos^2\alpha(x-t)\Psi_{poxyy}(\pm 1)\end{aligned}\right\} \tag{34}$$

Solving each of the two three sets of equations using the foregoing boundary conditions, subject to the steady parallel flow and transverse symmetry assumption; equations (22) and (25), under the boundary conditions (29) and (32) and for constant pressure gradient in the x-direction, yields the following classical poiseuille flow for fluid and particulate phase :

$$\Psi_{fo} = K\left(y - \tfrac{1}{3}y^3\right) \tag{35}$$

$$\Psi_{po} = K\left(y + \frac{2y}{M} - \frac{1}{3}y^3\right) \tag{36}$$

$$K = -\frac{R}{2}\left(\frac{\partial p}{\partial x}\right)_o^M \tag{37}$$

K is the Poiseuille flow parameter

Thus the effect of the particles on the fluid velocity profile is to cause an increase in the viscosity i.e. fluid viscosity μ_o is replaced by suspension viscosity $\mu_s = \mu_o|(1-qC)$ (see equation (8)), and thus for a given pressure difference less fluid will flow through the channel further, the particles lead the fluid by a relative velocity proportional to $\frac{1}{M}\left(\frac{\partial p}{\partial x}\right)_o$.

Using equations (35) and (36), the remaining two sets of equations for fluid and particulate phase along with their boundary conditions can be written as first order perturbation :

$$(1-C)R\left[\frac{\partial}{\partial t}\nabla^2\Psi_{f_1} + K(1-y^2)\nabla^2\Psi_{f_1x} + 2K\Psi_{f_1x}\right] = \nabla^4\Psi_{f_1} + CM(\nabla^2\Psi_{p_1} - \nabla^2\Psi_{f_1}) \tag{38}$$

$$CR\left[\frac{\partial}{\partial t}\nabla^2\Psi_{p_1} + K(1-y^2+\frac{2}{M})\nabla^2\Psi_{p_1x} + 2K\Psi_{p_1x}\right] = CN(\nabla^2\Psi_{f_1} - \nabla^2\Psi_{p_1}) \tag{39}$$

$$\left.\begin{aligned} \Psi_{f_1y}(\pm 1) &= \pm 2K\cos\alpha(x-t) \\ \Psi_{f_1x}(\pm 1) &= \mp\alpha\sin\alpha(x-t) \\ \Psi_{p_1x}(\pm 1) &= \mp\alpha\sin\alpha(x-t) \end{aligned}\right\} \tag{40}$$

Second order perturbation :

$$(1-C)R\left[\frac{\partial}{\partial t}\nabla^2\Psi_{f_2} + K(1-y^2)\nabla^2\Psi_{f_2x} + 2K\Psi_{f_2x} - \Psi_{f_1x}\nabla^2\Psi_{f_1y} + \Psi_{f_1y}\nabla^2\Psi_{f_1x}\right]$$
$$= \nabla^4\Psi_{f_2} + CM(\nabla^2\Psi_{p_2} - \nabla^2\Psi_{f_2}) \tag{41}$$

$$CR\left[\frac{\partial}{\partial t}\nabla^2\Psi_{p2} + K\left(1-y^2+\frac{2}{M}\right)\nabla^2\Psi_{p2x} + 2K\Psi_{p2x} \quad \Psi_{p1x}\nabla^2\Psi_{p1y}+\Psi_{p1y}\nabla^2\Psi_{1x}\right]$$

$$= CN(\nabla^2\Psi_{f_2} - \nabla^2\Psi_{p_2}) \qquad (42)$$

$$\left.\begin{aligned}
\Psi_{f2y}^{(\pm1)} &= \mp \cos\alpha(x-t)\Psi_{f_1yy}^{(\pm1)} + K\cos^2\alpha(x-t)\\
\Psi_{f2x}^{(\pm1)} &= \mp \cos\alpha(x-t)\Psi_{f_1xy}^{(\pm1)}\\
\Psi_{p_2x}(\pm1) &= \mp \cos\alpha(x-t)\Psi_{p_1xy}(\pm1)
\end{aligned}\right\} \qquad (43)$$

The equations (38) and (39) representing the first order effects, are coupled linear, fourth order, homogeneous differential equations with variable coefficients. The solution of these equations subject to the non- homogeneous boundary conditions (40) will be obtained using normal mode method. As the domain extends to infinity in the x-direction, the problem admits a solution of the form

$$\Psi_{(f,p)}(x,y,t) = \sum_{n=-\infty}^{+\infty} \Psi_{(f,p)}(x,y,t)e^{in\alpha x} \qquad (44)$$

which after substituting into equations (38) and (39) yields for each α

$$\Psi_{(f,p)n\alpha}(y,t) = \sum_{m=-\infty}^{+\infty} g_m \phi_{(f,p)m}^{(y)} e^{im\alpha t} \qquad (45)$$

Further, it is noted that the only term remaining in the expansion for α are $n = m = \pm 1$, as the present problem is of a disturbance involving a specific wave number. Thus,

$$\left.\begin{aligned}
2\Psi_{f_1}(x,y,t) &= \phi_{f_1}(y)\, e^{i\alpha(x-t)} + \phi^*_{f_1}(y)\, e^{-i\alpha(x-t)}\\
2\Psi_{p_1}(x,y,t) &= \phi_{p_1}(y)\, e^{i\alpha(x-t)} + \phi^*_{p_1}(y)\, e^{-i\alpha(x-t)}
\end{aligned}\right\} \qquad (46)$$

where the asterisk denotes the complex conjugate.

Substituting equation (46) into equations (38)-(40), O_{rr} - Sommer field type of equations is obtained :

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i(1-C)\alpha R\left[1-K(1-y^2)\right]\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f_1} - 2iK(1-C)\alpha R\phi_{f_1} = CM\left(\frac{d^2}{dy^2} - \alpha^2\right)(\phi_{f_1} - \phi_{p_1}) \quad (47)$$

$$iC\alpha R\left[1-K(1-y^2+\frac{2}{M})\right]\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi_{p_1} - 2iKC\alpha R\phi_{p_1} = CN\left(\frac{d^2}{dy^2}-\alpha^2\right)(\phi_{p_1}-\phi_{f_1}) \quad (48)$$

with boundary conditions :

$$\begin{aligned} \phi'_{f_1}(\pm 1) &= 2K \\ \phi_{f_1}(\pm 1) &= \pm 1 \\ \phi_{p_1}(\pm 1) &= \pm 1 \end{aligned} \quad (49)$$

Further, equations (41) - (43) can be satisfied by the solution of the form

$$2\Psi_{f_2}(x,y,t) = \phi_{f_{20}}(y) + \phi_{f_{22}}(y)e^{2i\alpha(x-t)} + \phi^*_{f_{22}}(y)e^{-2i\alpha(x-t)} \quad (50)$$

$$2\Psi_{p_2}(x,y,t) = \phi_{p_{20}}(y) + \phi_{p_{22}}(y)e^{2i\alpha(x-t)} + \phi^*_{p_{22}}(y)e^{-2i\alpha(x-t)}$$

A substitution of equation (50) into equations (41) - (43) leads to the differential equations :

$$\phi'''_{f_{20}}(y) = -\frac{i(1-C)\alpha R}{2}\left(\phi^{*\prime\prime}_{f_1}\phi^*_{f_1} - \phi^*_{f_1}\phi''_{f_1}\right) + CM\left(\phi''_{p_{20}} - \phi''_{f_{20}}\right) \quad (51)$$

$$0 = -\frac{iC\alpha R}{2}\left(\phi_{p_1}\phi^{*\prime\prime}_{p_1} - \phi^*_{p_1}\phi''_{p_1}\right)' + CN\left(\phi''_{p_{20}} - \phi''_{f_{20}}\right) \quad (52)$$

$$\left(\frac{d^2}{dy^2}-4\alpha^2\right)\left[\frac{d^2}{dy^2}-4\alpha^2+2i(1-C)\alpha R\right]\phi_{f_{22}} = 2i(1-C)K\alpha R(1-y^2)\left(\frac{d^2}{dy^2}-4\alpha^2\right)\phi_{f_{22}} + 4i(1-C)K\alpha R\,\phi_{f_{22}} + \frac{i(1-C)\alpha R}{2}\left(\phi'_{f_1}\phi''_{f_1} - \phi_{f_1}\phi'''_{f_1}\right) + CM\left(\frac{d^2}{dy^2}-4\alpha^2\right)(\phi_{f_{22}} - \phi_{p_{22}}) \quad (53)$$

$$2iC\alpha R\left(\frac{d^2}{dy^2}-4\alpha^2\right)\phi_{p_{22}}=2iCK\alpha R\left(1-v^2-\frac{2}{M}\right)\left(\frac{d^2}{dy^2}-4\alpha^2\right)\phi_{p_{22}}+4iCK\alpha R\ \phi_{p_{22}}$$

$$-\frac{iC\alpha R}{2}\left[\phi'_{p_1}\phi'''_{p_1}-\phi_{p_1}\phi'''_{p_1}\right]+GN\left(\frac{d^2}{dy^2}-4\alpha^2\right)\left(\phi_{p_{22}}-\phi_{f_{22}}\right) \quad (54)$$

with the boundary conditions :

$$\phi'_{f_{20}}(\pm1) = K \mp \frac{1}{2}\left[\phi''_{f_1}(\pm1)+\phi^{*}{}'_{f_1}(\pm1)\right]$$

$$\phi'_{f_{22}}(\pm1) = \frac{K}{2} \mp \frac{1}{2}\phi''_{f_1}(\pm1)$$

$$\phi_{f_{22}}(\pm1) \pm \frac{1}{4}\phi'_{f_1}(\pm1) = 0 \quad (55)$$

$$\phi_{p_{22}}(\pm1) \pm \frac{1}{4}\phi'_{p_1}(\pm1) = 0$$

Thus we obtained a set of differential equations together with the corresponding boundary conditions which are sufficient to determine the solution of the problem up to the second order in ε. Our main attention is to find out solution of differential equations for ϕ_{f_1} and ϕ_{p_1}. Although the equations (47-48) for ϕ_{f_1} and ϕ_{p_1} are coupled fourth-order ordinary differential equations with variable coefficients, it would be, perhaps, impossible to obtain solution of these differential equations for arbitrary values of R, α and K. This is just because of the moving boundary considered in the present problem. The condition of moving boundary has made the boundary condition non-homogeneous and thus the problem is not an eigenvalue problem as in all problems of hydrodynamic stability for which solutions are available in the literature. However, we can obtain the solution of the problem under various restrictions imposed on R, α and K.

2.4 CLOSED FORM SOLUTIONS FOR SMALL R AND K

Our main aim is to solve equations (47-48), (51-52) subject to their boundary conditions for values of $K \neq 0$.

Let $\phi_{f1}, \phi_{f20}, \phi_{p20}$ be expanded in increasing powers of K as:

$$\phi_{f_1} = \phi_{f10} + K\phi_{f11} + K^2\phi_{f22} + \ldots.$$

$$\phi_{p_1} = \phi_{p10} + K\phi_{p11} + K^2\phi_{p22} + \ldots. \qquad (56)$$

$$\phi_{f20} = \S_{f20} + K\S_{f21} + K^2\S_{f22} + \ldots.$$

$$\phi_{p20} = \S_{p20} + K\S_{p21} + K^2\S_{p22} + \ldots.$$

A substitution of (56) in (47) and (48), and coefficients of like powers of K collected, the following equations are obtained :

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i(1-C)\alpha R\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f10} = CM\left(\frac{d^2}{dy2} - \alpha^2\right)(\phi_{f10} - \phi_{p10}) \qquad (57)$$

$$iC\alpha R\left[\frac{d^2}{dy^2} - \alpha^2\right]\phi_{p10} = CN\left(\frac{d^2}{dy^2} - \alpha^2\right)(\phi_{p10} - \phi_{f10}) \qquad (58)$$

with boundary conditions

$$\phi'_{f10}(\pm 1) = 0$$

$$\phi_{f10}(\pm 1) = \pm 1$$

$$\phi_{p10}(\pm 1) = \pm 1 \qquad \ldots. \quad (59)$$

Also

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i(1-C)\alpha R\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f11} = i(1-C)\alpha R(1-y^2)\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f10}$$

$$- 2i(1-C)\alpha R\,\phi_{f10} + CM\left[\frac{d^2}{dy^2} - \alpha^2\right]\left(\phi_{f11} - \phi_{p_{11}}\right) \qquad \ldots \quad (60)$$

$$iC\alpha R\left[\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi_{p_{11}}-(1-y^2+\frac{2}{M})\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi_{p_{10}}\right]=2iC\alpha R\phi_{p_{10}}+$$

$$+CM\left(\frac{d^2}{dy^2}-\alpha^2\right)\left(\phi_{p_{11}}-\phi_{f_{11}}\right) \quad .... \quad (61)$$

With boundary conditions

$$\begin{aligned}\phi'_{f_{11}}(\pm1) &= 2\\ \phi_{f_{11}}(\pm1) &= 0 \qquad ... \quad (62)\\ \phi_{p_{11}}(\pm1) &= 0\end{aligned}$$

etc.

Similarly, substituting (56) in (51-52) leads to

$$\S''_{f_{20}} = -\frac{i\alpha R}{2}\left[\phi_{f_{10}}\phi^{*''}_{f_{10}}-\phi^{*}_{f_{10}}\phi''_{f_{10}}\right]'+CM\left[\S''_{f_{20}}-\S''_{p_{20}}\right] \quad ...(63)$$

$$\frac{iC\alpha R}{2}\left[\phi_{p_{10}}\phi^{*''}_{p_{10}}-\phi^{*}_{p_{10}}\phi''_{p_{10}}\right]'-CN\left(\S''_{p_{20}}-\S''_{f_{20}}\right)=0 \quad (64)$$

$$\S''_{f_{21}}=-\frac{i\alpha R}{2}\left[\phi_{f_{10}}\phi^{*''}_{f_{11}}+\phi_{f_{11}}\phi^{*''}_{f_{10}})-(\phi^{*}_{f_{10}}\phi^{*''}_{f_{11}}+\phi^{*}_{f_{11}}\phi^{*''}_{f_{10}}\right)\Big]+$$

$$+CM\left(\S''_{f_{21}}-\S''_{p_{21}}\right) \quad (65)$$

$$\frac{iC\alpha R}{2}\left[\left(\phi_{p_{11}}\phi^{*''}_{p_{11}}+\phi_{p_{11}}\phi^{*''}_{p_{10}}\right)-\left(\phi^{*}_{p_{10}}\phi''_{p_{11}}+\phi^{*}_{p_{11}}\phi''_{p_{10}}\right)\right]'-CN\left(\S''_{p_{21}}-\S''_{t_{21}}\right)=0$$

etc. (66)

With boundary conditions

$$\S'_{f_{20}}(\pm1) = -\tfrac{1}{2}\left(\phi''_{f_{10}}(\pm1)+\phi^{*''}_{f_{10}}(\pm1)\right) \quad (67)$$

$$\S'_{f_{21}}(\pm1) = 1-\tfrac{1}{2}\left(\phi''_{f_{11}}(\pm1)+\phi^{*''}_{f_{11}}(\pm1)\right) \quad (68)$$

Now eliminating $\phi_{p_{10}}$ between (57), (58) the equations reduce to

$$\left(\frac{d^2}{dy^2}-\beta^2\right)\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi_{f_{10}}=0 \quad (69)$$

$$\left(\frac{d^2}{dy^2}-\beta^2\right)\left(\frac{d^2}{dy^2}-\alpha^2\right)\phi_{p_{10}}=0 \quad (70)$$

Solutions of (69), (70) subject to the boundary conditions (59) may be obtained as

$$\phi_{f_{10}}(y) = A \sinh\alpha y + B \sinh\beta y \qquad (71)$$

$$\phi_{D_{10}}(y) = E_1 \sinh\alpha y + E_2 \sinh\beta y \qquad (72)$$

where $$\beta^2 = \alpha^2 - i\alpha RL \qquad (73)$$

$$L = 1 - C + \frac{CM}{N - i\alpha R} \qquad (74)$$

$$A = \frac{-\alpha \cosh\beta}{\alpha\cosh\alpha \sinh\beta - \beta\cosh\beta \sinh\alpha} \qquad (75)$$

$$B = \frac{\alpha\cosh\alpha}{\alpha\cosh\alpha \sinh\beta - \beta\cosh\beta \sinh\alpha} \qquad (76)$$

$$E_1 = \frac{1 - E_2 \sinh\beta}{\sinh\alpha} \qquad (77)$$

$$E_2 = - \frac{i\alpha RINB}{N - i\alpha R} \qquad (78)$$

We will now consider the effects of Reynolds number R on the general pattern of the wave, in particular to low Reynolds number. One on recall that a Reynolds number R represents the ratio of the inertia terms to the frictional terms in the equations of motion, it gives an indication of the corresponding fluid motion. A small Reynolds number implies that the viscosity is predominant. Therefore when Reynolds number R is small, and for the approximations to hold, only the first power of R is retained. Expanding β thus

$$\beta = \alpha - \frac{iRL}{2} + \frac{R^2L^2}{8\alpha} + \qquad (79)$$

Since the first power is to be retained for low Reynolds

number (79) reduces to

$$\beta = \alpha - \frac{iRL}{2} \tag{80}$$

The value of β is to be substituted in equations (75-78)

$$\left.\begin{aligned} \text{But } \sinh\beta &= \sinh\alpha - \frac{iRL}{2}\cosh\alpha \\ \cosh\beta &= \cosh\alpha - \frac{iRL}{2}\sinh\alpha \\ \beta\cosh\beta &= \alpha\cosh\alpha - \frac{iRL}{2}(\cosh\alpha + \sinh\alpha \\ \beta^2\sinh\beta &= \alpha^2\sinh\alpha - \frac{iRL}{2}(2\alpha\sinh\alpha + \alpha^2\cosh\alpha) \end{aligned}\right\} \tag{81}$$

Equations (75 - 78) become

$$A = \frac{\alpha\sinh\alpha + \left(\frac{2i\alpha}{RL} + 1\right)\cosh\alpha}{\sinh\alpha\cosh\alpha - \alpha} \tag{82}$$

$$B = -\frac{2i\alpha\cosh\alpha}{RL(\sinh\alpha\cosh\alpha - \alpha)} \tag{83}$$

$$E_1 = \frac{(1-\sinh\alpha) + \frac{iR}{2N}(NL\cosh\alpha - 2\alpha\sinh\alpha)}{\sinh\alpha} \tag{84}$$

$$E_2 = -\frac{2\alpha^2\cosh\alpha\left(1 + \frac{i\alpha R}{N}\right)}{\sinh\alpha\cosh\alpha - \alpha} \quad \ldots\ldots \tag{85}$$

For small values of R and K equations (71), (72), (67) respectively reduce to

$$\phi_{f10}(y) = A\sinh\alpha y + B\sinh\left(\alpha - \frac{iRL}{2}\right)y \quad \ldots \tag{86}$$

$$\phi_{p10}(y) = E_1\sinh\alpha y + E_2\sinh\left(\alpha - \frac{iRL}{2}\right)y \tag{87}$$

$$\S'_{f20}(\pm 1) = -\tfrac{1}{2}\left[(A + A^*)\alpha^2\sinh\alpha + B\beta^2\sinh\beta + B^*\beta^{*2}\sinh\beta^*\right]$$

$$= \frac{\alpha^2(\sinh\alpha\cosh\alpha + \alpha)}{\sinh\alpha\cosh\alpha - \alpha} \tag{88}$$

Solutions of equations (60) and (61) subject to the boundary conditions (62) are

$$\phi_{f_{11}}(y) = A_4 \sinh \alpha y + B_4 \sinh \beta y + f(y) \qquad (89)$$

$$\phi_{p_{11}}(y) = 0 \qquad (90)$$

where $f(y) = -\alpha^2(1-C)R^2LB \sinh \beta y - 2i(1-C)\alpha R\left[\frac{A}{\alpha^4}\sinh\alpha y + \frac{B}{\beta^4}\sinh \beta y\right] + \frac{1}{\beta^4}\left[\alpha^2(1-C)R + \frac{iCME_2L\alpha R}{i\alpha R - N}\right]\left[y^2\sinh\beta y - 8y\frac{\cosh\beta y}{\beta} + 20\frac{\sinh\beta y}{\beta^2}\right]$

$$+ \frac{iCM\alpha R}{i\alpha R - N}\left[-\frac{E_2L}{\beta^4}\left(1+\frac{2}{M'}\right)\sinh\beta y + 2\left(E_1\frac{\sinh\alpha y}{\alpha^4} + E_2\sinh\beta y\right)\right] \qquad (90)$$

is non-approximated value.

Taking limit as $R \to 0$ therefore $f(y) = 0$ and

$$\phi_{f_{11}}(y) = A_4 \sinh \alpha y + B_4 \sinh \beta y \quad \ldots\ldots \quad (92)$$

where $A_4 = \dfrac{2\sinh\beta}{\alpha\cosh\alpha\sinh\beta - \beta\cosh\beta\sinh\alpha}$

$$= \frac{RL\cosh\alpha - 2i\sinh\alpha}{RL(\sinh\alpha\cosh\alpha - \alpha)} \quad \ldots\ldots \quad (93)$$

$$B_4 = \frac{2i\sinh\alpha}{RL(\sinh\alpha\cosh\alpha - \alpha)} \quad \ldots \quad (94)$$

Eliminating $\S''_{p_{20}}$ in equations (63) and (64)

$$\S''_{f_{20}} = -\frac{i\alpha R}{2}\left[\phi_{f_{10}}\phi^{*\prime\prime}_{f_{10}} - \phi^{*}_{f_{10}}\phi''_{f_{10}} - \frac{CM}{N}\left(\phi_{p_{10}}\phi^{*\prime\prime}_{p_{10}} - \phi^{*}_{p_{10}}\phi''_{p_{10}}\right)\right]' \quad \ldots \quad (95)$$

Integrating (95) thrice

$$\S'_{f_{20}} = N_0(y) + \tau_1 y^2 + \tau_2 y + \tau_3 \quad \ldots \quad (96)$$

where τ_1, τ_2, τ_3 are constants and

$$N_0(y) = \frac{\alpha^2R^2}{2}\left[\frac{1}{2}\left(A^*B - \frac{CME_1^*E_2}{N}\right)\left(\frac{\cosh(\alpha+\beta)y}{(\alpha+\beta)^2} - \frac{\cosh(\alpha-\beta)y}{(\alpha-\beta)^2}\right) + \right.$$

$$+ \tfrac{1}{2}\left(AB^{*}-CME_1E_2^{*}\right)\left[\cosh\frac{(\alpha+\beta^{*})y}{(\alpha+\beta^{*})^{2}} - \cosh\frac{(\alpha-\beta^{*})y}{(\alpha-\beta^{*})^{2}}\right]$$

$$+ \left[BB^{*} - \frac{CME_2E_2^{*}}{N}\right]\left[\cosh\frac{(\beta+\beta^{*})y}{(\beta+\beta^{*})^{2}} - \cosh\frac{(\beta-\beta^{*})y}{(\beta-\beta^{*})^{2}}\right]\Bigg] \quad \ldots \quad (97)$$

substituting (96) in (64) and on integrating

$$\S'_{p20} = T(y) - \S'_{f20} + T_o \quad \ldots \quad (98)$$

where

$$T(y) = -\frac{\alpha^2R^2L}{2N}\left[\frac{E_1^{*}E_2}{2}\{\cosh(\alpha+\beta)y-\cosh(\alpha-\beta)y\}+\frac{E_1E_2^{*}}{2}\{\cosh(\alpha-\beta^{*})y\right.$$

$$\left. - \cosh(\alpha-\beta^{*})y\}+E_2E_2^{*}\{\cosh(\beta+\beta^{*})y-\cosh(\beta-\beta^{*})y\}\right] \quad \ldots(99)$$

Also, elimination of $\S''_{p21}$ in (65) and (66) gives

$$\S''_{f21} = -\frac{i\alpha R}{2}\left[(\phi_{f10}-\phi^{*}_{f10})\phi^{*'}_{f11}+(\phi_{f11}-\phi^{*}_{f11})\phi^{*'}_{f10}\right]$$

Integrating thrice leads to $\quad \ldots \quad (100)$

$$\S'_{f21} = G^{0}(y) + h_oy^2 + h_1y + h_2 \quad \ldots \quad (101)$$

where h_o, h_1, h_2 are constants of integration and

$$G^{0}(y) = -\frac{i\alpha R}{2}\left[\frac{1}{\beta}\left[(A-2A^{*})A_4+A_4A^{*}\right]\cosh 2\alpha y+\frac{\alpha^2}{2}\left(A_4^{*}B+A^{*}B_4\right)\left\{\frac{\cosh(\alpha+\beta)y}{(\alpha+\beta)^{2}}+\frac{\cos(\alpha-\beta)y}{(\alpha-\beta)^{2}}\right\}-\right.$$

$$-\frac{\alpha^2}{2}\left(A^{*}B_4^{*}+A_4^{*}B^{*}\right)\left[\frac{\cosh(\alpha+\beta^{*})y}{(\alpha+\beta^{*})^{2}} - \frac{\cosh(\alpha-\beta^{*})y}{(\alpha-\beta^{*})^{2}}\right]+\frac{\beta^{*2}}{2}\left\{(A-A^{*})B_4^{*}+\right.$$

$$\left.(A_4-A_4^{*})B^{*}\right]\left[\frac{\cosh(\alpha+\beta^{*})y}{(\alpha+\beta^{*})^{2}} - \frac{\cosh(\alpha-\beta^{*})y}{(\alpha-\beta^{*})^{2}}\right]+\frac{\beta^{*2}}{2}\left(BB_4^{*}+B^{*}B_4\right)\left[\frac{\cosh(\beta+\beta^{*})y}{(\beta+\beta^{*})^{2}} - \frac{\cosh(\beta-\beta^{*})y}{(\beta-\beta^{*})^{2}}\right]$$

$$\ldots.(102)$$

And from (66)

$$\S'_{p21} = \S'_{f21} + N^{0} \quad \ldots \quad (103)$$

The boundary value of $\S'_{f21}$ is obtained from (68) and (89) as

$$\S'_{f21} = 1 - \frac{\alpha^2\sinh\alpha\cosh\alpha}{\sinh\alpha\cosh\alpha-\alpha} \quad \ldots \quad (104)$$

The boundary value of ϕ'_{f20} is obtained from

$$\phi'_{f20} = \S'_{f20} + K\S'_{f21} \qquad \text{at} \quad y = \pm 1 \quad \ldots \quad (105)$$

Denoting the above by D and using (88) and (104)

$$D = \frac{\alpha^2(\sinh\alpha\cosh\alpha+\alpha)}{\sinh\alpha\cosh\alpha-\alpha} + K\left[1 - \frac{\alpha^2\sinh\alpha\cosh\alpha}{\sinh\alpha\cosh\alpha-\alpha}\right] \quad \ldots \quad (106)$$

2.5 Solution for Low Reynolds number and for arbitrary K.

2.5.1 (a) Particle free fluid.

Following Pung and Yih (1968) and taking C = 0 in equations (38), (40), (41), (43) the following differential equations are recorded for the first and second order perturbation

$$R\left[\frac{\partial}{\partial t}\nabla^2\Psi_1 + K(1-y^2)\nabla\Psi_{1x} + 2K\Psi_{1x}\right] = \nabla^4\Psi_1 \quad \ldots \quad (107)$$

$$\Psi_{1y}(\pm 1) = \pm 2K\cos\alpha\,(x-t)$$

$$\Psi_{1x}(\pm 1) = \mp \alpha\sin\alpha\,(x-t) \qquad (108)$$

$$R\left[\frac{\partial}{\partial t}\nabla^2\Psi_2 + K(1-y^2)\nabla^2\Psi_{2x} + 2K\Psi_{2x} - \Psi_{1x}\nabla^2\Psi_{1y} + \Psi_{1y}\nabla^2\Psi_{1x}\right] = \nabla^4\Psi_2 \quad \ldots (109)$$

$$\left.\begin{aligned}\Psi_{2y}(\pm 1) &= \mp\cos\alpha(x-t)\Psi_{1yy}(\pm 1) + K(\cos^2\alpha(x-t)\\ \Psi_{2x}(\pm 1) &= \mp\cos\alpha\,(x-t)\Psi_{1xy}(\pm 1)\end{aligned}\right\} \qquad (110)$$

It would be convenient, in this case if we seek the solution in the following form :

$$\left.\begin{aligned}\Psi_1 &= g_1(y)\cos\alpha(x-t) + t_1(y)\sin\alpha(x-t)\\ \Psi_2 &= F_2(y) + g_2(y)\cos 2\alpha(x-t) + t_2(y)\sin 2\alpha(x-t)\end{aligned}\right\} \qquad (111)$$

where g_1, t_1, F_2, g_2, and t_2 are real functions to be determined. With equation (111), equation (107) reduce to

$$-\alpha R\left[\frac{d^2}{dy^2} - \alpha^2 - 2K\right)t_1 - K(1-y^2)\left(\frac{d^2}{dy^2} - \alpha^2\right)t_1\Big] = \left(\frac{d^2}{dy^2} - \alpha^2\right)^2 g_1 \qquad (112)$$

$$\alpha R\left[\left\{\frac{d^2}{dy^2}-\alpha^2-2K\right)g_1-K(1-y^2)\left(\frac{d^2}{dy^2}-\alpha^2\right)g_1\right] = \left(\frac{d^2}{dy^2}-\alpha^2\right)^2 t_1 \quad (113)$$

Subject to the boundary conditions

$$\left.\begin{aligned} g_1'(\pm 1) &= 2K \\ t_1'(\pm 1) &= 0 \\ g_1(\pm 1) &= 1 \\ t_1(\pm 1) &= 0 \end{aligned}\right\} \quad (114)$$

while equation (109) is obtained as

$$F_2''' + \left(\frac{d^2}{dy^2}-4\alpha^2\right)^2 g_2 + 2\alpha R\left[1-K(1-y^2)\right]\left(\frac{d^2}{dy^2}-4\alpha^2\right) t_2 - 2KRg_2'$$

$$= -\frac{\alpha R}{2}\left[t_1\left(g_1'''+g_1'''-\alpha^2(g_1'+g_1)\right)+g_1\left(t_1'''+t_1''-\alpha^2(t_1'+t_1)\right)\right] \quad (115)$$

$$\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2 t_2 - 2\alpha R\left[1-K(1-y^2)\right]\left(\frac{d^2}{dy^2}-4\alpha^2\right) g_2 - 2KRt_2'$$

$$= -\frac{\alpha R}{2}\left[t_1(t_1'''-t_1''+\alpha^2(t_1-t_1')) - (g_1 g_1''' - g_1 g_1'')\right] \quad (116)$$

$$\left(\frac{d^3}{dy^3}2KR\right)F_2' = \frac{\alpha R}{2}\left[t_1(g_1'''+\alpha^2(g_1-g_1')-g_1''')-g_1(t_1'''-t_1''+\alpha^2(t_1-t_1'))\right] \quad (117)$$

With the boundary conditions

$$\left.\begin{aligned} 2F_2'(\pm 1) + g_1''(\pm 1) &= 0 \\ t_1'(\pm 1) &= 0 \\ g_1'(\pm 1) &= 4g_2(\pm 1) \\ t_2(\pm 1) &= 0 \\ 2F_2'(\pm 1) + g_1'''(\pm 1) &= 0 \\ 2F_2'(\pm 1) + t_1''(\pm 1) &= 0 \end{aligned}\right\} \quad (118)$$

To solve for $g_1(y)$, $t_1(y)$ and $F_2(y)$ without considering $K = 0$, free pumping as done by Fung and Yih, the problem

will be solved in the form of a power series of the Reynolds number R. Hence the solutions may be expaned in the form

$$\left.\begin{aligned} g_1 &= g_{10} + R^2 g_{12} + \cdots \\ t_1 &= t_{11} R + O(R^3) \\ F_2 &= F_{20} + R^2 F_{22} + \cdots \end{aligned}\right\} \quad (119)$$

As $R \to 0$ (112), (113), (117) give

$$g_1''' - 2\alpha^2 g_1'' + \alpha^4 g_1 = 0 \quad (120)$$

$$t_1''' - 2\alpha^2 t_1'' + \alpha^4 t_1 = 0 \quad (121)$$

$$F_2''' = 0 \quad (122)$$

2.5.2 Zero Reynolds number : Stokes Approximation

Considering the case when Reynolds number is very small only $g_{10}(y)$, $F_{20}(y)$ need be solved, thus

$$g_{10}''' - 2\alpha^2 g_{10}'' + \alpha^4 g_{10} = 0 \quad (123)$$

$$F_{10}''' = 0 \quad (124)$$

Subject to
$$\left.\begin{aligned} g_{10}'(\pm 1) &= 2K \\ g_{10}(\pm 1) &= 1 \\ 2F_{20}'(\pm 1) + g_{10}''(\pm 1) &= 0 \end{aligned}\right\} \quad (25)$$

Whose solution to (123), (124) are respectively

$$g_{10} = B_1 y + q \quad (126)$$

$$F_{20} = M\left(y - \frac{y^3}{3}\right) + q_1 \quad (127)$$

where

$B_1 = 2K$, $q = 1 - 2K$, $M = -\frac{G_1}{2}$ and q_1 remains arbitrary

2.5.3 Higher Order Reynolds number

The higher order Reynolds number equations are got by substituting into equations (112- 114) and (117-118) since the interest is to calculate F_{22} then t_{11} and g_{12} must be found by comparing the coefficients of Reynolds number.

The first order perturbation

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 t_{11} = \alpha\left[[1-K(1-y^2)]\left(\frac{d^2}{dy^2}-\alpha^2\right)g_{10} - 2Kg_{10}\right] \tag{128}$$

With the boundary conditions

$$\left.\begin{aligned} t'_{11}(\pm 1) &= 0 \\ t_{11}(\pm 1) &= 0 \end{aligned}\right\} \tag{129}$$

While the second order perturbation equations are

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 g_{12} = -\left[1-K(1-y^2)\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)t_{11} \tag{130}$$

$$\left.\begin{aligned} t'_{12}(\pm 1) &= 0 \\ g_{12}(\pm 1) &= 0 \end{aligned}\right\} \tag{131}$$

$$F''_{22} = -\frac{\alpha}{2}\left[\alpha^2 t_{11}(g'_{10}-g_{10}) + g_{10}(t''_{11}-t''_{11}-\alpha^2(t'_{11}-t_{11}))\right] \tag{132}$$

$$F'_{22}(\pm 1) + \tfrac{1}{2}g''_{12} = 0 \tag{133}$$

The solutions to equations (128), (130), (132) are respectively given by

$$t_{11} = \frac{G_1}{\alpha^4} e^{\alpha^2 y}\left[y-\frac{2}{\alpha^2}\right] + \frac{G_2}{\alpha^4} e^{\alpha^2 y} + G_3 y + G_4$$

$$-\frac{1}{\alpha^3}\left[\alpha^2(1-K(1-y^2)) + 2K\right](B_1 y+q) + \frac{4K}{\alpha^4}(3B_1 y + q) \tag{134}$$

$$\phi_{12} = \frac{Q_1}{\alpha^4}\left(y-M_{21}\right)e^{\alpha^2 y}+\frac{Q_2}{\alpha^4}e^{\alpha y}+Q_5\left[G_1 e^{\alpha^2 y}\left(\left(1-\frac{1}{\alpha^2}\right)+\frac{2}{\alpha^4}\right)+G_2\left(1-\frac{1}{\alpha^2}\right)e^{\alpha^2 y}\right.$$

$$\left.-\alpha^2\left(G_3 y+G_4\right)+\frac{1}{\alpha}\left[\alpha^2\left(1-y^2\left(1-y^2\right)\right)+2K\right]\left(B_1 y+q\right)-4KB_1+2K\left(B_1 y+q\right)\right] \quad (135)$$

$$F_{22} = \int\left(S(y) - S(1) - q_2\left(y - \frac{y^2}{2} - \frac{y^3}{3}\right)\right.$$

$$\left.+\frac{y}{2}\left[\left(Q_1+Q_2\right)e^{\alpha^2}+Q_5\left[\left(G_1+G_2\right)e^{\alpha^2} - \frac{2K}{\alpha}\left(3B_1+q\right)\right]\right] \quad (136)$$

where

$$G_1 = \frac{8Kq\alpha}{M_{11}\sinh\alpha^2}\left(1-\sinh\alpha^2\right) \quad (137)$$

$$M_{11} = 2\left[-\frac{1}{\alpha^2}\left(\frac{\sinh\alpha^2}{\alpha^2}-1\right)\left(\coth\alpha^2-1\right)+\frac{\cosh\alpha^2}{\alpha^2}-\frac{2\sinh\alpha^2}{\alpha^4}+\left(\frac{1}{\alpha^2}-1\right)e^{\alpha^2}\right] \quad (138)$$

$$G_2 = -G_1\left[\frac{1}{\alpha^2}\left(\coth\alpha^2+1\right)-\frac{2}{\alpha^2}\right]+\frac{4\alpha Kq}{\sinh\alpha^2} \quad (139)$$

$$G_3 = -\frac{G_1}{\alpha^4}\left(1-\frac{1}{\alpha^2}\right)e^{\alpha^2}-\frac{G_2 e^{\alpha^2}}{\alpha^2} \quad (140)$$

$$G_4 = \frac{1}{\alpha^4}\left[G_1\left(2\frac{\cosh\alpha^2}{\alpha^2}-\sinh\alpha^2\right)-G_2\cosh\alpha^2+2Kq(\alpha-2)\right] \quad (141)$$

$$Q_1 = -\frac{e^{\alpha^2}}{e^{\alpha^2}M_{41}-\left(1+\frac{1}{\alpha^4}\right)}\left[Q_2\left(\frac{1}{\alpha^2 e^{\alpha^2}}-M_{51}\right)+Q_5\left[-\frac{G_1}{\alpha^2}\left(1+\frac{1}{\alpha^2}\right)\frac{1}{e^{\alpha^2}}+\frac{G_2}{\alpha^2 e^{\alpha^2}}+G_3\right.\right.$$

$$\left.\left.-\frac{1}{\alpha^3}\left(2KB_1+2\alpha^2K\left(B_1-q\right)+\frac{12KB_1}{\alpha^4}\right]+M_{31}\right]\right] \quad (142)$$

$$Q_2 = -\frac{1}{\sinh\alpha^2}\left[Q_1\left(\cosh\alpha^2-\frac{\sinh\alpha^2}{\alpha^4}\right)+\alpha^2 Q_5\left[\frac{G_1}{\alpha^2}\left(\cosh\alpha^2-\frac{\sinh\alpha^2}{\alpha^2}\right)+\right.\right.$$

$$\left.\left.+\frac{G_2\sinh\alpha^2}{\alpha^2}-\frac{2Kq}{\alpha}\right]\right] \quad (143)$$

$$Q_3 = Q_1 M_{41} - Q_2 M_{51} + M_{31} \quad (144)$$

$$Q_4 = \frac{1}{\alpha^7 Q_5}\left[2Q_1\left(\coth\alpha^2+M_{21}\right)-Q_2\right]\cosh\alpha^2+\frac{1}{\alpha^2}\left[G_1\left(\left(1-\frac{1}{\alpha^2}\right)\sinh\alpha^2+\frac{2}{\alpha^4}\cosh\alpha^2\right)+\frac{4K(q-B_1)}{\alpha}\right.$$

$$\left.+\left(1-\frac{1}{\alpha^2}\right)G_2\cosh\alpha^2\right] \quad (145)$$

$$\left.\begin{aligned} Q_5 &= \frac{\alpha^3 + 2}{\alpha^7} \\ M_2 &= \frac{\alpha^2 + 1}{\alpha^4} \end{aligned}\right\} \quad (146)$$

$$M_3 = -\frac{1}{\alpha^2}\left[G_1\left[\left(1-\frac{1}{\alpha^2}\right)\cosh\alpha^2 + \frac{2}{\alpha^4}\sinh\alpha^2\right] + G_2\left(1-\frac{1}{\alpha^2}\right)\sinh\alpha^2\right] \quad (147)$$

$$M_4 = \frac{M_{21}\sinh\alpha^2 - \cosh\alpha^2}{\alpha^7 Q_5} \quad (148)$$

$$M_5 = \frac{\sinh\alpha^2}{\alpha^7 Q_5} \quad (149)$$

2.5.4 (b) Suspension Fluid

We would seek solution of equations (38-43) in the following form:

$$\left.\begin{aligned} \psi_1 &= \Theta_1 \cos\alpha(x-t) + S_1 \sin\alpha(x-t) \\ \psi_{p1} &= \Theta_{p1} \cos\alpha(x-t) + S_{p1} \sin\alpha(x-t) \\ \psi_2 &= L_2 + \Theta_2 \cos 2\alpha(x-t) + S_2 \sin 2\alpha(x-t) \\ \psi_{p2} &= L_{p2} + \Theta_{p2} \cos 2\alpha(x-t) + S_{p2} \sin 2\alpha(x-t) \end{aligned}\right\} \quad (150)$$

where Θ_1, S_1, Θ_{p1}, S_{p1}, L_2, Θ_2, S_2, L_{p2}, Θ_{p2}, S_{p2} are real functions of y to be determined. Substituting equation (150) into equations (38-43), equations obtained are given below :

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2\Theta_1 + CM\left[\Theta''_{p1} - \Theta''_1 - \alpha^2(\Theta_{p1} - \Theta_1)\right] = \alpha(1-C)R\left[\left[1+K(1-y^2)\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)S_1 + 2K\,S_1\right] \quad (151)$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 S_1 + CM\left[S''_{p1} - S''_1 - \alpha^2(S_{p1} - S_1)\right] =$$

[illegible] (152)

$$\alpha(1-C)R\left[\left[1+K(1-y^2)\right]\left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta_1-2K\Theta_1\right] \tag{152}$$

ϵ^2:

$$N\left[\Theta_1''-\Theta_{p_1}''-\alpha^2(\Theta_1-\Theta_{p_1})\right]=\alpha R\left[\left[K(1-y^2+\frac{2}{M})-1\right]\left(\frac{d^2}{dy^2}-\alpha^2\right)S_{p_1}+2KS_{p_1}\right] \tag{153}$$

$$N\left[S_1''-S_{p_1}''-\alpha^2(S_1-S_{p_1})\right]=\alpha R\left[\left[1-K(1-y^2+\frac{2}{M})\right]\left(\frac{d^2}{dy^2}-\alpha^2\right)\Theta_{p_1}-2K\Theta_{p_1}\right] \tag{154}$$

$$\left.\begin{aligned}\Theta_1(\pm1)&=\pm1\\ \Theta_1'(\pm1)&=2K\\ \Theta_{p_1}(\pm1)&=\pm1\\ S_1(\pm1)&=0\\ S_1'(\pm1)&=0\\ S_{p_1}(\pm1)&=0\end{aligned}\right\} \tag{155}$$

$$\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2\Theta_2+CM\left[\Theta_{p_2}''-\Theta_2''-4\alpha^2(\Theta_{p_2}-\Theta_2)\right]$$
$$=2(1-C)\alpha R\left[\left[K(1-y^2)-1\right]\left(\frac{d^2}{dy^2}-4\alpha^2\right)S_2+2KS_2-\frac{1}{4}\left[S_1(\Theta_1''-2\alpha^2\Theta_1'-\alpha^2\Theta_1+\Theta_1'')-\Theta_1(S_1''-\alpha^2S_1')+\Theta_1S_1''\right]\right] \tag{156}$$

$$\left(\frac{d^2}{dy^2}-4\alpha^2\right)^2S_2+CM\left[S_{p_2}''-S_2''-S_2''-4\alpha^2(S_{p_2}-S_2)\right]$$
$$=2(1-C)\alpha R\left[\left[1-K(1-y^2)\right]\left(\frac{d^2}{dy^2}-4\alpha^2\right)\Theta_2-2K\Theta_2-\frac{1}{4}\left[\Theta_1\Theta_1''-\Theta_1'\Theta_1'+S_1(S_1''-2\alpha^2S_1')+S_1'S_1''\right]\right] \tag{157}$$

$$L_2''+CM\left[L_{p_2}''-L_2''\right]=-\frac{\alpha}{2}(1-C)R\left[S_1(\Theta_1''-2\alpha^2\Theta_1')+\Theta_1S_1''+\Theta_1'S_1'-S_1'\Theta_1'\right] \tag{158}$$

$$-2\alpha R\left[\left[1+K\left(1-y^2+\frac{2}{M}\right)\right]\left(\frac{d^2}{dy^2}-4\alpha^2\right)\Theta_{p_2}+2K\Theta_{p_2}+\frac{1}{4}\left[S_{p_1}S'''_{p_1}-\Theta_{p_1}\Theta'''_{p_1}+\right.\right.$$

$$\left.\left.+\Theta'_{p_1}\Theta''_{p_1}-S'_{p_1}S''_{p_1}\right]\right]=N\left[S''_2-S''_{p_2}-4\alpha^2\left(S_2-S_{p_2}\right)\right]\ldots\quad(159)$$

$$2\alpha R\left[\left[1+K\left(1-y^2+\frac{2}{M}\right)\right]\left(\frac{d^2}{dy^2}-4\alpha^2\right)S_{p_2}+2K\,S_{p_2}+\frac{1}{4}\left[S'_{p_1}\Theta''_{p_1}+\left(\Theta'_{p_1}-\Theta_{p_1}\right)S''_{p_1}-\right.\right.$$

$$\left.\left.-S_{p_1}\Theta'''_{p_1}\right]\right]=N\left[\Theta''_2-\Theta''_{p_2}-4\alpha^2\left(\Theta_2-\Theta_{p_2}\right)\right]\quad\ldots\ldots\quad(160)$$

$$2(1-C)\alpha R\left[\left\{K(1-y^2)-1\right\}\left(\frac{d^2}{dy^2}-4\alpha^2\right)S_2+2KS_2-\frac{1}{4}\left[S_1\left(\Theta'''_1-2\alpha^2\Theta'_1-\alpha^2\Theta_1+\Theta'_1\right)-\right.\right.$$

$$\left.\left.-\Theta_1\left(S'''_1-\alpha^2S'_1+\Theta'_1S''_1\right)\right]\right]=\left(\frac{d^2}{dy^2}-4\alpha^2\right)\Theta_2+CM\left(\Theta''_{p_2}-4\alpha^2\right)\left(\Theta_{p_2}-\Theta_2\right)-\Theta'''_2\Big]\quad(161)$$

$$\frac{\alpha R}{2}\left[\Theta'_{p_1}S''_{p_1}+\Theta_{p_1}S'''_{p_1}-S_{p_1}\Theta''_{p_1}-S'_{p_1}\Theta'''_{p_1}\right]=N\left(L''_2-L''_{p_2}\right)\quad(162)$$

With the boundary conditions

$$\left.\begin{aligned}
L'_2(\pm1) &= \pm\frac{\Theta''_1}{2}(\pm1)+\frac{K}{2}\\
\Theta'_2(\pm1) &= \pm\tfrac{1}{2}\Theta''_1(\pm1)+\frac{K}{2}\\
S'_2(\pm1) &= \pm\tfrac{1}{4}S''_1(\pm1)\\
S_{p_2}(\pm1) &= \pm\frac{1}{4}\Theta'_{p_1}(\pm1)\\
S_{p_2}(\pm1) &= 0\\
S'_{p_2}(\pm1) &= 0\\
\Theta_2(\pm1) &= \pm\frac{1}{4}\Theta'_1\\
S_2(\pm1) &= 0\\
S'_1(\pm1) &= 0
\end{aligned}\right\}\quad(163)$$

It can be seen that C = 0 in equations (151 - 152, 156 - 158) coresponds to equations obtained by Mittra.

To solve those real functions, it is necessary to solve

the equations in a power series of R.

If the real functions are expanded thus

$$\left.\begin{aligned} \Theta_1 &= \Theta_{10} + R^2\Theta_{12} + \ldots \\ S_1 &= RS_{11} + O(R^3) \\ \Theta_{p_1} &= T_{10} + R^2 T_{12} + \ldots \\ S_{p_1} &= RQ_{11} + O(R^3) \\ L_2 &= L_{20} + R^2 L_{22} + \ldots \\ L_{p_2} &= Q_{20} + R^2 Q_{22} + \ldots \end{aligned}\right\} \quad (164)$$

2.5.5 Stokes Approximation

As the Reynolds number is very small equation (164) is substituted in equations (151 - 163), thus

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2\Theta_{10} = -CM\left[\left(\frac{d^2}{dy^2} - \alpha^2\right)T_{10} - \left(\frac{d^2}{dy^2} - \alpha^2\right)\Theta_{10}\right] \quad \ldots (165)$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\Theta_{10} - \left(\frac{d^2}{dy^2} - \alpha^2\right)T_{10} = 0 \quad \ldots (166)$$

whose solution is

$$\Theta_{10} = A_{12}\sinh \alpha y + B_{12}\cosh \alpha y \quad \ldots (167)$$

$$T_{10} = C_{12}\sinh \alpha y + Q_{12}\cosh \alpha y \quad (168)$$

Subject to

$$\left.\begin{aligned} \Theta'_{10}(\pm 1) &= 2K \\ \Theta_1(\pm 1) &= \pm 1 \\ T_{10}(\pm 1) &= \pm 1 \end{aligned}\right\} \quad \ldots (169)$$

where

$$\left.\begin{aligned} A_{12} &= \frac{2K}{\alpha\cosh \alpha}\sec^2\alpha - \frac{\tanh \alpha}{\alpha} \\ B_{12} &= \frac{\cosh \alpha - 2K\tanh \alpha}{\alpha\cosh \alpha} \end{aligned}\right\} \quad \ldots (170)$$

$$Q_{12} = 0 \quad \text{and} \quad C_{12} = \frac{1}{\sinh\alpha}$$

$$\text{and} \quad L_{20}^{*} + CM\left[Q_{20}^{*} - L_{20}^{*} \right] = 0 \qquad (171)$$

$$L_{20}^{*} - Q_{20}^{*} = C \qquad (172)$$

Subject to

$$L_{20}^{*}(\pm 1) = \mp \tfrac{1}{2}\Theta_{10}^{*}(\pm 1) + \frac{K}{2} \qquad (173)$$

The solutions to equations are

$$L_{20}(y) = -C_{13}\left(y - \frac{y^3}{3}\right) + C_{21}\, y^2 + e_1 y + e_2 \qquad (174)$$

$$Q_{20} = L_{20}(y) + e_3\, y + e_4 \qquad (175)$$

where $C_{13}\ C_{21}\ e_{11}\ e_{21}\ e_{31}\ e_4$ are constants of integration.

2.5.6 Higher Reynolds number

The solutions to other higher Reynolds number in the suspension fluid are

$$S_{11}(y) = T_1 \sinh \alpha y + T_2 \cosh \alpha y \qquad (176)$$

with $S_{11}(\pm 1) = 0$, $S_{11}'(\pm 1) = 0$

$$\Theta_{12}(y) = P_6 \sinh \alpha y + \cosh y \qquad \ldots \qquad (177)$$

$$L_{22}(y) = -(1-C)\frac{\alpha^3}{2}\left[\frac{P_3}{8\alpha^3}\left[\frac{\sinh 2\alpha y}{2\alpha} - y\cosh 2\alpha\right] + (P_4 - P_5)\frac{\cosh 2\alpha y}{16\,\alpha^4} + \right.$$

$$\left. \frac{(P_4 - P_5)}{24} y^4 - C_{31}\left(y - \frac{y^3}{3}\right) + \frac{K}{2} \cdot \frac{\cosh \alpha y}{2} \right] + C_4 \qquad \ldots \quad (178)$$

$$Q_{22}(y) = L_{22}(y) + a_2 y + a_3 \qquad \ldots \quad (179)$$

$$Q_{11}(y) = 0 \qquad \ldots \quad (180)$$

where $P_1 = \frac{-2}{\alpha^3}\left[(N_1 A_{12} + N_2) \ldots + N_1 B_{12} \tanh \alpha \ldots \right]$

$$P_2 = \frac{2}{\alpha^3}\left[\left(-\frac{1}{\alpha}\right)(N_1 A_{12} + N_2)\tanh \alpha + N_1 B_{12}\left(\tanh^2 \alpha - \frac{1}{\alpha}\right) \right]$$

$$
\left.
\begin{aligned}
P_3 &= \alpha\left(T_3 A_{12} + T_2 B_{12}\right) - \left(T_4 B_{12} + T_2 A_{12}\right) \\
P_4 &= \tfrac{1}{2}\left[T_2\left(\alpha A_{12} - 2B_{12}\right) + \alpha T_3 B_{12}\right] \\
P_5 &= \tfrac{1}{2}\left[T_3\left(\alpha B_{12} - 2A_{12}\right) + \alpha T_2 A_{12} \right] \\
&\cdots \\
P_6 &= - \tanh \alpha
\end{aligned}
\right\} \qquad (181)
$$

$$
\left.
\begin{aligned}
N_1 &= - 2\alpha K(1-C) \\
N_2 &= - \frac{2\alpha\, CMK\, a}{N} \\
T_3 &= \frac{2}{\alpha^4}\left[N_1 A_{12} + N_2 + \frac{\alpha^4 P_1}{2} \right] \\
T_2 &= \frac{2}{\alpha^4}\left[N_1 B_{12} + \frac{\alpha^4 P_2}{2} \right]
\end{aligned}
\right\} \qquad (182)
$$

C_{31}, C_4, a_2, a_3 are constants of integration.

2.5.7 : Long Wave Approximation

If the length of the peristaltic wave is much greater than then wave number α but the Reynolds number may not be small then expanding α in series form under the same restriction on ε then equations (151 - 155, 158, 162 - 163) are valid- according to Chow (1970). Also equation (164) can be reppresented in the form

$$
\left.
\begin{aligned}
\theta_1 &= \theta^*_{10} + \alpha^2 \theta^*_{12} + \cdots \\
S_1 &= \alpha\, S^*_{11} + O(\alpha^3) \\
\theta_{p_1} &= T^*_{10} + \alpha^2 T^*_{12} + \cdots \\
S_{p_1} &= \alpha\, Q^*_{11} + O(\alpha^3) \\
L_2 &= L^*_{20} + \alpha^2 L^*_{22} + \cdots \\
L_{p_2} &= Q^*_{20} + \alpha^2 Q^*_{22} + \cdots
\end{aligned}
\right\} \qquad \cdots \qquad (183)
$$

Where θ^*_{10}, θ^*_{12}, S^*_{11}, T^*_{10}, T^*_{12}, Q^*_{11}, L^*_{20}, Q^*_{20}, M^*_{22} are functions to be determined as before without asterisk.

To get the solutions of these functions, equation (183) is substituted in the valid equations mentioned earlier and the coefficients of α in its ascending powers are equated. Though calculations are tedious and lengthy, one still get the following solutions :

α^0:

$$\theta^*_{10} = a_9\left(y - \frac{y^3}{3} - \frac{2}{3}\right) - 2K(1-y) + 1 \qquad (184)$$

$$T^*_{10} = \theta^*_{10} + a_4(y-1) \qquad (185)$$

$$L^*_{20} = a_9\left(y + \frac{y^2}{2} - \frac{y^3}{3}\right) + \frac{1}{3}Ky + a_6 \qquad (185)$$

$$Q^*_{20} = L^*_{20} + hy + h_1 \qquad (186)$$

where

$$\left.\begin{aligned} a_4 &= \frac{2}{3}a_9 - 2K - 1 \\ a_5 &= -a_4 \end{aligned}\right\} \qquad (187)$$

α':

$$S^*_{11}(y) = (1-C)R\left[\frac{a_9y^5}{60} + 2Ka_9\left(\frac{y^5}{120} - \frac{y^7}{840}\right) - 2K\left[-a_9\left(\frac{y^5}{120} - \frac{y^7}{2520} - \frac{y^4}{36}\right) - 2K\left(\frac{y^4}{24} - \frac{y^5}{120}\right) + \frac{y^4}{24}\right]\right] + \frac{CMR}{N}\left[a_9\frac{y^5}{60} - 2K\,a_9x\left(\frac{y^5}{120} - \frac{y^7}{840} + \frac{y^5}{60M}\right) - 2K\left[-a_9\left(\frac{y^5}{120} - \frac{y^7}{2520} - \frac{y^4}{36}\right) - 2K\left(\frac{y^4}{24} - \frac{y^5}{120}\right) + \frac{y^4}{24}(1+a_5) + a_4\frac{y}{120}\right]\right] + h_2\frac{y^5}{6} + h_3\frac{y^2}{2} + h_4y + h_5 \qquad (188)$$

$$Q^{*}_{11}(y) = S^{*}_{11} - \frac{K}{N}\left[\left[\frac{a_9 y^3}{3} - 2K\, a_9\left(\frac{y^2}{2} - \frac{y^4}{12} + \frac{y^2}{M}\right)\right] - 2K\left[a_9\left(\frac{y^2}{3} - \frac{y^3}{6} + \frac{y^5}{60}\right)\right.\right.$$

$$\left.\left. - 2K\left(\frac{y}{2} - \frac{y^3}{6}\right) + \frac{y^2}{2}(1 + a_5) + a_4\frac{y^3}{6}\right]\right] + a_6 y + a_7 \qquad (189)$$

Where

$$h_2 = (1-C)R\left[\frac{4K^2}{5} - \frac{2a_9}{5} - \frac{4267K\, a_9}{9555}\right] + \frac{CMR}{N}\left[-\frac{2a_9}{5} - \frac{6K\, a_9}{105} + \right.$$

$$\left. + \frac{4Ka_9}{5M} - \frac{K^2}{6} - \frac{Ka_4}{10}\right] \qquad (190)$$

$$h_3 = \frac{2}{3}(1-C)KR\left[\frac{a_9}{3} - K + \tfrac{1}{2}\right) + \frac{2CMKR}{3N}\left[\frac{a_9}{3} - K + \tfrac{1}{2}(1 + a_5)\right]$$

(191)

$$h_4 = \frac{(1-C)R}{6}\left[K^2 - \frac{a_9}{2} - \frac{13K\, a_9}{15}\right] + \frac{CMR}{6N}\left[-\frac{a_9}{2} + K\, a_9 \times\right.$$

$$\left.\left(\frac{1}{M} - \frac{1}{15}\right) + \frac{K}{2}(K + a_4)\right] - \tfrac{1}{2}h_2 \qquad (192)$$

$$l_5 = \frac{(1-C)KR}{6}\left(\frac{a_9}{3} \cdot y + 1\right) + \frac{CMKR}{6N}\left[\frac{a_9}{3} - K + \tfrac{1}{2}(1 + a_5)\right] - \tfrac{1}{2}h_3$$

(193)

$$\ell_6 = S^{*}_{11}(-1) - C^{*}_{11}(1) + \frac{K}{N}\left[\frac{a_9}{3} - 2K\left[-\frac{3a_9}{20} + \frac{K}{3} + \frac{a_4}{6}\right]\right] \qquad (194)$$

$$\ell_7 = \frac{R}{N}\left[-2Ka_9\left(\frac{5}{12} + \frac{1}{M}\right) - 2K\left[\frac{a_9}{3} - K + \tfrac{1}{2}(1 + a_5)\right]\right] - \tfrac{1}{2}\left(S^{*}_{11}(1) + S^{*}_{11}(-1)\right)$$

(195)

α^2:

$$\theta^{*}_{12}(y) = \iint\left[(1-C)R\left[S^{*}_{11} + K(\gamma(y) - f(y)) - 2\left[a_9\left(y - \frac{y^3}{3} - \frac{2}{3}\right) + 2K(1-y) - 1\right] + \right.\right.$$

$$+ CM\left[\frac{R}{N}\left[\left\{K\left[\left(1 + \frac{2}{M}\right)Q^{*}_{11} - f(y) + \ell(y)\right] - Q^{*}_{11} + 2K\ell(y) - m(y)\right\}\right] + \right.$$

$$\left.\left. + 2\left[-a_9\left(\frac{y^3}{6} - \frac{y^5}{60} - \frac{y^2}{3}\right) - 2K\left(\frac{y^2}{2} - \frac{y^3}{6}\right) + \frac{y^2}{2}\right]\right]\right] + h_8\frac{y^2}{2} +$$

$$+ h_9\, y + h_{10} \qquad (196)$$

With boundary conditions
$$\left.\begin{aligned} \theta^*_{12}(\pm 1) &= 0 \\ \theta^{*\prime}(\pm 1) &= 0 \end{aligned}\right\} \qquad (197)$$

Thus
$$\theta_{12}(y) = \int Z(y) + h_8\frac{y^2}{6} + h_9\frac{y^2}{2} + h_{10}y + h_{11} \qquad (198)$$

while L_{22} is a fourth order coupled differential equations whose solution is too complex for determination. The coupled equations are

$$Q^{*\prime\prime}_{22} - L^{*\prime}_{22} = -\frac{R}{2N}\left[Q^{*\prime}_{11}\, T^{*\prime}_{10} + T^*_{10}Q^{*\prime\prime}_{11} - Q^*_{11}\, T^{*\prime\prime}_{10} - Q^{*\prime}_{11}\, T^{*\prime}_{10}\right] \qquad (199)$$

$$L^{*\prime\prime}_{22} = \frac{CMR}{2N}\left[Q^{*\prime}_{11}\, T^{*\prime}_{10} + T^*_{10}\, Q^{*\prime\prime}_{11} - Q^*_{11}\, T^{*\prime\prime}_{10} - Q^{*\prime}_{11}\, T^{*\prime}_{10}\right] - \frac{(1-C)R_x}{2}$$

$$\left[Q^*_{11}\,\theta^*_{10} + \theta^*_{10}\, S^{*\prime\prime}_{11} + \theta^{*\prime}_{10}\, Q^{*\prime}_{11} - Q^{*\prime}_{11}\theta^{*\prime}_{10}\right] \qquad (200)$$

With boundary condition
$$L^{*\prime}_{22} = -\frac{1}{2}\,\theta^{*\prime}_{12}(\pm 1) + \frac{K}{2} \qquad (201)$$

Where
$$\gamma(y) = 3\int\left[(1-C)R\left[\frac{a_9y^6}{360} + 2Ka_9\left(\frac{y^6}{720} - \frac{y^8}{6720}\right) + 2K\left\{a_9 \times \left(\frac{y^6}{720} - \frac{y^8}{20160} - \frac{y^5}{180}\right) + 2K\left(\frac{y^5}{120} - \frac{y^6}{720}\right) - \frac{y^5}{120}\right]\right] + \frac{CMR}{N} \times \left[\frac{a_9y^6}{360} - 2Ka_9\left(\frac{y^6}{720} - \frac{y^8}{6720} + \frac{y^6}{360}\right) - 2K\left[-a_9\left(\frac{y^6}{720} - \frac{y^8}{20160} - \frac{y^5}{180}\right) - 2K\left(\frac{y^5}{120} - \frac{y^6}{720}\right) + \frac{y^5}{120}(1+a_5) + \frac{a_4y^6}{720}\right] + h_2\frac{y^4}{24} + h_3\frac{y^3}{6} + h_4\frac{y^2}{2} + h_5y\right]\right] \qquad (202)$$

$$f(y) = \int\Big[(1-C)R\Big[a_9\frac{y^6}{18} + 2Ka_9\Big(\frac{y^6}{36} - \frac{y^8}{160}\Big) + 2K\Big[a_9\Big(\frac{y^6}{36} - \frac{y^8}{480} - \frac{y^5}{15}\Big) + 2K\Big(\frac{y^5}{10} - \frac{y^6}{36}\Big) - \frac{y^5}{10}\Big]\Big] + \frac{CMR}{N}\Big[a_9\frac{y^6}{18} - 2K\,a_9x\Big(\frac{y^5}{36} - \frac{y^6}{160} + \frac{y^6}{18M}\Big) - 2K\Big[-a_9\Big(\frac{y^6}{36} - \frac{y^8}{480} - \frac{y^5}{15}\Big) -$$

$$- 2K\Big(\frac{y^5}{10} - \frac{y^6}{36}\Big) + \frac{y^5}{10}(1 + a_5) + a_4\frac{y^6}{36}\Big] + h_2\frac{y^4}{4} + h_3\frac{y^3}{3}\Big] \tag{203}$$

$$m(y) = \int\Big[-\frac{R}{N}\Big[\Big[a_9\frac{y^4}{12} - 2K\Big(\frac{y^3}{6} - \frac{y^5}{60} + \frac{y^3}{3M}\Big)\Big] -$$

$$2K\Big[-a_9\Big(\frac{y^4}{24} - \frac{y^6}{360} - \frac{y^3}{9}\Big) - 2K\Big(\frac{y^3}{6} - \frac{y^4}{24}\Big) + \frac{y^3}{6}(1+a_5) + a_4\frac{y^4}{24}\Big]\Big] +$$

$$+ \frac{a_6y^2}{2} + ya_7\Big] \tag{204}$$

$$\ell(y) = 3\int\Big[\frac{R}{N}\Big[a_9\frac{y^5}{5} - 2K\,a_9\Big[\frac{y^4}{4} - \frac{y^6}{18} + \frac{y^4}{2M}\Big]\Big] - 2K\Big[-a_9\Big(\frac{y^5}{10} - \frac{y^7}{84} - \frac{y^4}{6}\Big) -$$

$$- 2K\Big(\frac{y^4}{8} - \frac{y^5}{10}\Big) + \frac{y^4}{4}(1 + a_5) + a_4\frac{y^5}{10}\Big]\Big] \tag{205}$$

$$Z(y) = \int\Big[(1-C)R\Big[S^*_{11} + K(Y(v) - f(y) - 2\Big[a_9\Big(v - \frac{y^3}{3} - \frac{2}{3}\Big) + 2K(1-v) - 1\Big]$$

$$+ CM\Big\{\frac{R}{N}\Big[\Big[v\Big[\Big(14\frac{2}{M}\Big)Q^*_{11} - f(y) + \ell(y)\Big] - Q^*_{11} + 2K\ell(y) - m(y)\Big]\Big]$$

$$+ 2\Big[-a_9\Big(\frac{y^3}{6} - \frac{y^5}{60} - \frac{y^2}{2}\Big) - 2K\Big(\frac{y^2}{2} - \frac{y^3}{6}\Big) + \frac{y^2}{2}\Big]\Big]\Big] \tag{206}$$

$$\left.\begin{aligned} h_8 &= 3\int\Big[Z(1) - Z(-1)\Big] - \Big[Z(1) + Z(-1)\Big] \\ h_9 &= -\int Z(1) - Z(-1)\Big] \\ h_{10} &= -\Big[Z(1) + Z(-1) + \frac{h_8}{2}\Big] \\ h_{11} &= \int\Big[Z(-1) - Z(1)\Big] + Z(1) - Z(-1) \end{aligned}\right\} \tag{207}$$

2.6 Series Solution

When K and R are not small, solutions to (47), (48) are obtained by expressing $\phi_{f1}(y)$ and $\phi_{p1}(y)$ in terms of an infinite series in powers of y. Using Frobenius series method for solving differential equations with unknown coefficients, let

$$\left.\begin{aligned} \phi_{f1} &= \sum_{n=0}^{\infty} A_n y^n \\ \phi_{p1} &= \sum_{n=0}^{\infty} B_n y^n \end{aligned}\right\} \qquad (208)$$

Expanding and differentiating (208) in the following form

$$\left.\begin{aligned} \phi_{f1} &= A_0 + A_1 y + A_2 y^2 + A_3 y^3 + A_4 y^4 + \ldots. \\ \phi'_{f1} &= A_1 2A_2 y + 3A_3 y^2 + 4 A_4 y^3 + \ldots\ldots \\ \phi''_{f1} &= 2A_2 + 6 A_3 y + 12 A_4 y^2 + \ldots.. \\ \phi'''_{f1} &= 24 A_4 + 120 A_5 y + \ldots.. \end{aligned}\right\} \qquad (209)$$

and likewise

$$\left.\begin{aligned} \phi_{p1} &= B_0 + B_1 y + B_2 y^2 + B_3 y^3 + B_4 y^4 + \ldots.. \\ \phi'_{p1} &= B_1 + 2B_2 y + 3B_3 y^2 + 4 B_4 y^3 + \ldots. \\ \phi''_{p1} &= 2B_2 + 6B_3 y + 12 B_4 y^2 + \ldots.. \\ \phi'''_{p1} &= 24 B_4 + 120 B_5 y + \ldots.. \end{aligned}\right\} \qquad (210)$$

Inserting (209), (210) into (47) and (48) and collecting like powers of y and equating the coefficients of each power of y, the following are got with some coefficients remaining arbitrary.

From (47)

$$A_4 = \frac{A_2}{12}\left[2\alpha^2 + i(1-C)\alpha R(K-1) + CM\right] + \frac{A_0}{24}\left[2i(1-C)K\alpha R - \alpha^4 + i(1-C)\alpha^3 R - CM - \alpha^2 K\right] + CM\left[\alpha^2 B_0 - 2B_2\right] \quad (211)$$

$$A_5 = \frac{A_3}{120}\left[2\alpha^2 - i(1-C)(1-K)\alpha R\right] + \frac{A_1}{120}\left[2i(1-C)\alpha KR - \alpha^4 - i(1-C)(K-1)\alpha^3 R - CM\alpha^2\right] + CM\left[\alpha^2 B_1 - 6B_3\right] \quad (212)$$

$$A_6 = \frac{A_4}{30}\left[2\alpha^2 - i(1-C)(1-K)\alpha R + CM\right] + \frac{A_2}{360}\left[2i(1-C)K\alpha R - \alpha^4 - i(1-C)\alpha R\left[(2-\alpha^2)K - \alpha^2\right] - CM\alpha^2\right] + \frac{A_0}{360} i(1-C)X \alpha^3 KR + \frac{CM}{360}\left[\alpha^2 B_2 - 12 B_4\right] \quad (213)$$

whose recurrence formula is

$$A_{n+6} = \frac{\left[2\alpha^2 - i(1-C)(1-K)\alpha R + CM\right]}{(n+6)\ (n+5)} A_{n+4} + \frac{\left[2i(1-C)K\alpha R - \alpha^4 - i(1-C)\alpha R\left[(2-\alpha^2)K - \alpha^2\right] - CM\alpha^2\right]}{(n+6)\ (n+5)\ (n+4)\ (n+3)} A_{n+2} + \frac{i(1-C)\alpha^3 KR\ A_n}{(n+6)\ (n+5)\ (n+4)\ (n+3)} + \frac{CM}{(n+6)\ (n+5)\ (n+4)\ (n+3)}\left[\alpha^3 B_{n+2} - 12\ B_{n+4}\right] \quad (214)$$

For equation (48)

$$B_2 = \frac{\frac{1}{2}\left[2iK\alpha R + i\alpha^3 R\left(1 + \frac{2K}{M} - \alpha^2 N\right)\right] B_0 + \frac{N}{2}\left(\alpha^2 A_0 - 2A_2\right)}{i\alpha R\left(1 - \frac{2K}{M} - K\right) - N} \qquad (215)$$

$$B_3 = \frac{B_1}{6}\left[\frac{2iK\alpha R + i\alpha^3 R\left(1 - \frac{2K}{M} - K\right) - \alpha^2 N}{i\alpha R\left(1 - \frac{2K}{M} - K\right) - N}\right] + \frac{N}{6}\left[\frac{\alpha^2 A_1 - 6A_3}{i\alpha R\left(1 - \frac{2K}{M} - K\right) - N}\right]$$

$$B_4 = \frac{A_2}{12}\,\frac{\alpha^2 N}{i\alpha R\left(1 - \frac{2K}{M} - K\right) - N} - A_4\,\frac{N}{i\alpha R\left(1 - \frac{2K}{M} - K\right) - N} +$$

$$+ \frac{B_2}{12}\left[\frac{i\alpha^3 R\left(1 - \frac{2K}{M} - K\right) - \alpha^2 N - 2i\alpha KR}{i\alpha R\left(1 - \frac{2K}{M} - K\right) - N}\right]$$

$$+ \frac{B_0}{12}\,\frac{i\alpha^3 KR}{i\alpha R\left(1 - \frac{2K}{M} - K\right) - N} \qquad (217)$$

The recurrence formula for (217) is

$$B_{n+4} = \frac{\alpha^2 N}{(n+4)(n+3)S_1} A_{n+2} - \frac{N}{S_1} A_{n+4} + \frac{\alpha^2 S_1 - (n+2)i\alpha KR}{(n+4)(n+3)S_1} B_{n+2} +$$

$$+ \frac{i\alpha^3 KR}{(n+4)(n+3)S_1} B_n \qquad (218)$$

Where

$$S_1 = i\alpha R\left(1 - \frac{2K}{M} - K\right) - N \qquad (219)$$

An examination of (214), (218) together with equations (211), (212), (215) and (216) reveal that all A_n, $n \geq 4$ can be expressed in terms of A_0, A_1, A_2, A_3, B_0, B_1, B_2, B_3

while all B_n, $n \geq 2$ can be similarly expressed in terms of A_1, A_2, A_3, B_0, B_2. The boundary condition (49) demands that $\phi_{f1}(y)$ and $\phi_{p1}(y)$ be antisymmetric with respect to y leading to

$$\begin{aligned} A_0 &= A_2 = 0 \\ B_0 &= B_2 = 0 \end{aligned} \qquad (220)$$

Since all A_n, $n \geq 6$ from equation (214) are expressed in terms of A_1, A_3, B_1, B_3 and D_n, $n \geq 4$ from equation (218) are expressed in terms of B_1, the solution

$$\phi_{f1}(y) = A_1 \sum_{n=0}^{\infty} a_{1n}\, y^n + A_3 \sum_{n=0}^{\infty} a_{3n}\, y^n + B_1 \sum_{n=0}^{\infty} b_{1n}\, y^n + \\ + B_3 \sum_{n=0}^{\infty} b_{3n}\, y^n \quad \ldots\ (221)$$

$$\phi_{p1}(y) = A_1 \sum_{n=0}^{\infty} a'_{1n}\, y^n + A_3 \sum_{n=0}^{\infty} a'_{3n}\, y^n + B_1 \sum_{n=0}^{\infty} b'_{1n}\, y^n + \\ + B_3 \sum_{n=0}^{\infty} b'_{3n}\, y^n \quad \ldots\ (222)$$

Equations (221) - (222) can be expressed as shown below since B_3 is a function of A_1, A_3, B_1.

$$\phi_{f1}(y) = A_1 \sum_{n=0}^{\infty} a_{1n}\, y^n + A_3 \sum_{n=0}^{\infty} a_{3n}\, y^n + B_1 \sum_{n=0}^{\infty} b_{1n}\, y^n \Bigg\}$$

$$\phi_{p_1}(y) = A_1 \sum_{n=0}^{\infty} a'_{1n}\, y^n + A_3 \sum_{n=0}^{\infty} a'_{3n}\, y^n + B_1 \sum_{n=0}^{\infty} b'_{1n}\, y^n \Bigg\} \quad \ldots\ldots \quad (223)$$

where

$$A_n = a_{1n} A_1 + a_{3n} A_3 + b_{1n} B_1$$

$$B_n = a'_{1n} A_1 + a'_{3n} A_3 + b'_{1n} B_1$$

$a_{j,n}$, $a'_{j,n}$, $b_{j,n}$, $b'_{j,n}$ $j = 1, 3$ and $n = 0, 1 \ldots\ldots$

denote the coefficients of A_j, B_j in the expression for A_n, B_1 determined for $n \geq 6$ and $n \geq 4$ respectively. Moreover, for $n = 0, 1, \ldots\ldots$ a_{jn}, b_{jn} are given by (211), (212) and (215), (216). So that A_0, A_1, A_2, A_3 and B_0, B_1, B_2, B_3 are independent of each other therefore

$$\left.\begin{aligned} &a_{10} = a_{12} = a_{13} = a_{14} = a_{30} = a_{31}\ a_{32} = a_{34} = 0 \\ &a'_{10} = a'_{12} = a'_{13} = a'_{14} = a'_{30} = a'_{31} = a'_{32} = a'_{34} = 0 \\ &b_{10} = b_{12} = b_{13} = b_{14} = 0 \\ &b'_{10} = b'_{12} = b'_{13} = b'_{14} = 0 \end{aligned}\right\} \quad (224)$$

$$\left.\begin{aligned} a_{11} &= a_{33} = 1 \\ a'_{11} &= a'_{33} = 1 \\ b_{11} &= 0 \\ b'_{11} &= 1 \end{aligned}\right\} \qquad (225)$$

The boundary conditions (49) using (223) reduce to

$$\left.\begin{aligned} A_1 \sum_{n=0}^{\infty} n a_{1n} + A_3 \sum_{n=0}^{\infty} n a_{3n} + B_1 \sum_{n=0}^{\infty} n b_{1n} &= 2K \\ A_1 \sum_{n=0}^{\infty} a_{1n} + A_3 \sum_{n=0}^{\infty} a_{3n} + B_1 \sum_{n=0}^{\infty} b_{1n} &= 1 \\ A_1 \sum_{n=0}^{\infty} a'_{1n} + A_3 \sum_{n=0}^{\infty} a'_{3n} + B_1 \sum_{n=0}^{\infty} b'_{1n} &= 1 \end{aligned}\right\} \qquad (226)$$

In this case

$$D = K - \tfrac{1}{2}\left(\phi'_{f_1}(\pm 1) + \phi^{*\prime}_{f_1}(\pm 1)\right) \qquad \ldots \qquad (227)$$

is obtained from (55c), thus from the above using (221), D is finally obtained as shown below

$$D = K - \tfrac{1}{2}\Big[A_1 \sum_{n=0}^{\infty} n(n-1)a_{1n} + A_3 \sum_{n=0}^{\infty} n(n-1)a_{3n} + B_1 \sum_{n=0}^{\infty} n(n-1)b_{1n} + A_1^* \sum_{n=0}^{\infty} n(n-1)a^*_{1n} + A_3^* \sum_{n=0}^{\infty} n(n-1)a^*_{3n} + B_1^* \sum_{n=0}^{\infty} n(n-1)b^*_{1n}\Big]$$

(228)

where asterisks * denotes complex conjugates to A_1, a_{1n}, A_3, a_{3n}, B_1, b_{1n} ,. Integrating (51) twice where it represents a steady inertia component in the presence of particle phase due to convective acceleration corresponding to ϕ_{p_1} and ϕ_{p_1}

$$\phi''_{f20}(y) = -\frac{i\alpha R}{2}\Big[(1-C)\left(\phi_{f1}(y)\ \phi^{*\prime}_{f1}(y) - \phi'_{f1}(y)\ \phi^{*}_{f1}(y)\right) + \frac{CM}{N}\left(\phi_{p_1}(y)\ \phi^{*\prime}_{p_1}(y) - \phi^{*}_{p_1}(y)\ \phi'_{p_1}\right)\Big] + C'_1\, y + C_2 \quad (229)$$

where C'_1, C_2 are constants of integration.

Redefining $\phi_{f1}(y)$ as

$$\phi_{f1}(y) = \sum_{n=0}^{\infty} X_m\, y^m \quad (230)$$

then

$$\left.\begin{aligned} \phi'_{f1}(y) &= \sum_{n=0}^{\infty} X_n\, y^{n-1}\ , \\ \phi^{*}_{f1} &= \sum_{n=0}^{\infty} X^{*}_m\, y^m \\ \text{and}\quad \phi^{*\prime}_{f1}(y) &= \sum_{n=0}^{\infty} n\, X^{*}_n\, y^{n-1} \end{aligned}\right\} \quad (231)$$

and $\phi_{p1}(y)$ defined as

$$\phi_{p1}(y) = \sum_{m=0}^{\infty} Z_m\, y^m \quad (232)$$

$$\left.\begin{aligned} \text{So} \quad \phi'_{p_1} &= \sum_{n=0}^{\infty} n\, Z_n\, y^{n-1}, \quad \phi^*_{p_1} = \sum_{m=0}^{\infty} Z^*_m\, y^m \\ \text{and} \quad \phi^{*\prime}_{p_1} &= \sum_{n=0}^{\infty} n\, Z^*_n\, y^{n-1} \end{aligned}\right\} \tag{233}$$

Where

$$\left.\begin{aligned} X_m &= A_1\, a_{1m} + A_3\, a_{3m} + B_1\, b_{1m} \\ X_n &= A_1\, a_{1n} + A_3\, a_{3n} + B_1\, b_{1n} \\ Z_m &= A_1\, a'_{1m} + A_3\, a'_{3m} + B_1\, b'_{1m} \\ Z_n &= A_1\, a'_{1n} + A_3\, a'_{3n} + B_1\, b'_{1n} \end{aligned}\right\} \tag{234}$$

and X^*_m, X^*_n, Z^*_m, Z^*_n are complex conjugates to X_m, X_n, Z_m, Z_n respectively.

Substituting (230) - (233) and integrating (229) once

$$\phi'_{f20}(y) = F(y) + C_1\, y^2 + C_2\, y + C_3 \tag{235}$$

where C_1, C_2, C_3 are constants of integration

$$F(y) = -\frac{i\alpha R}{2}\left[(1 - C)\left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} X_m\, X^*_n\, \frac{n}{m+n}\, y^{m+n} + \right.\right.$$

$$\left.\left. - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} X_n\, X^*_m\, \frac{m}{m+n}\, y^{m+n} \right] + \frac{CM}{N}\left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_m\, Z^*_n \times \right.\right.$$

$$\frac{n}{m+n} y^{m+n} - \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} Z_n Z_m^* \frac{n}{m+n} y^{m+n} \Big] \Big] \qquad (236)$$

But $\phi'_{f20}(\pm 1) = D$ at the boundaries demands that $C_2 = 0$

$$\therefore \quad C_1 + C_3 = D - F(1) \qquad (237)$$

Therefore $\phi'_{f20}(y) = F(y) - F(1) + D - C_1 (1 - y^2)$ (238)

From equation (52)

$$\phi''_{p20}(y) = \frac{i\alpha R}{2N} \left(\phi_{p1} \phi^{*\prime}_{p1} - \phi^*_{p1} \phi'_{p1} \right)' + \phi''_{f20}(y)$$

Integrating once

$$\phi'_{p20}(y) = G(y) + F(y) - F(1) + D - C_1(1 - y^2) + C_4 \qquad (240)$$

Where $G(y) = \frac{i\alpha R}{2N} \left[\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n(n-1) Z_m Z_n^* y^{m+n-2} - \right.$

$$\left. - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} n(n-1) Z_m^* Z_n y^{m+n-2} \right] \qquad (241)$$

Since the mean pressure gradient K is not zero, the time mean average of the axial velocity is given by

$$\bar{U}_f(y) = K(1-y^2) + \frac{\varepsilon^2}{2} \left[F(y) - F(1) + D - C_1(1-y^2) \right] \qquad (242)$$

$$\bar{U}_p(y) = K(1-y^2) + \frac{\varepsilon^2}{2} \left[G(y) + F(y) - F(1) + D - C_1 \left(1-y^2 - \frac{C_4}{C_1}\right) \right]$$

(243)

To calculate the A_1, A_3 and B_1, equation (226) can be re-written as :

$$\left.\begin{aligned} A_1\,\alpha_1 + A_3\,\alpha_2 + B_1\,\alpha_3 &= 2K \\ A_1\,\alpha_4 + A_3\,\alpha_5 + B_1\,\alpha_6 &= 1 \\ A_1\,\alpha_7 + A_3\,\alpha_8 + B_1\,\alpha_9 &= 1 \end{aligned}\right\} \qquad (244)$$

where

$$\alpha_1 = \sum_{n=0}^{\infty} n\,a_{1n} = 1 + 5a_{15} + 7\,a_{17} + \ldots\ldots$$

$$\alpha_2 = \sum_{n=0}^{\infty} n\,a_{3n} = 3 + 5\,a_{35} + 7\,a_{37} + \ldots\ldots$$

$$\alpha_3 = \sum_{n=0}^{\infty} n\,b_{1n} = 5\,b_{15} + 7\,b_{17} + \ldots\ldots$$

$$\alpha_4 = \sum_{n=0}^{\infty} a_{1n} = 1 + a_{15} + a_{17} + \ldots\ldots$$

$$\alpha_5 = \sum_{n=0}^{\infty} a_{3n} = 1 + a_{35} + a_{37} +$$

$$\alpha_6 = \sum_{n=0}^{\infty} b_{1n} = b_{15} + b_{17} + \ldots\ldots$$

$$\alpha_7 = \sum_{n=0}^{\infty} a'_{1n} = 1 + a'_{16} + a'_{17} + \ldots\ldots$$

$$\alpha_8 = \sum_{n=0}^{\infty} a'_{3n} = 1 + a'_{35} + a'_{37} + \ldots\ldots$$

$$\alpha_9 = \sum_{n=0}^{\infty} b'_{1n} = 1 + b'_{15} + b'_{17} + \ldots\ldots$$

Solving (244)

$$A_1 = \frac{\left\{\left[2K\,\alpha_5-\alpha_2\right]\left[\alpha_6\,\alpha_8-\alpha_5\alpha_9\right] + \left[\alpha_5-\alpha_8\right]\left[\alpha_3\alpha_5 - \alpha_2\,\alpha_6\right]\right\}}{\left[\alpha_1\alpha_5-\alpha_2\alpha_4\right]\left[\alpha_6\alpha_8 - \alpha_5\alpha_9\right]-\left[\alpha_4\alpha_8-\alpha_5\alpha_7\right]\left[\alpha_3\alpha_5 - \alpha_2\,\alpha_6\right]} \tag{245}$$

$$B_1 = \left\{2K\,\alpha_5 - \alpha_2 - A_1\left[\alpha_1\,\alpha_5-\alpha_2\alpha_4\right]\right\} \Big/ \left\{\alpha_3\alpha_5 - \alpha_2\,\alpha_6\right\} \tag{246}$$

$$A_3 = \frac{2K - \left[A_1\,\alpha_1 + B_1\,\alpha_3\right]}{\alpha_2} \tag{247}$$

Other coefficients in A_n, n = 1, 3, 5, 7, ...29 are given below :

$$a_{15} = P_2 + P_1$$

where $$P_2 = \frac{1}{120}\left[i\alpha(1-C)R\left\{2-(K-1)\alpha^2\right\} - \alpha^2(\alpha^2 - CM)\right]$$

$$P_1 = -\frac{CMN\,\alpha^2}{S_1}$$

$$b_{15} = CM(\alpha^2 + U_1)$$

$$U_1 = -\frac{(2iK\alpha R + \alpha^2 S_1)}{S_1}$$

$$a_{35} = \frac{1}{120}\left[2\alpha^2 - i(1-C)(1-K)\alpha R\right] + P_3$$

$$P_3 = \frac{6CMN}{S_1}$$

$$a'_{15} = \frac{(\alpha^2 S_1 - 3i\alpha KR)P_1 - 20\, a_{15}}{20\, S_1}$$

$$a'_{35} = \frac{\alpha^2 N S_1^2 - 20\, N\, a_{35} + (\alpha^2 S_1 - 3i\alpha KR)P_3}{20\, S_1}$$

$$b'_{15} = \frac{(\alpha^2 S_1 - 3i\alpha KR)U_1 + i\alpha^3 KR - 20\, Nb_{15}}{20\, S_1}$$

$$a_{17} = Q\, a_{15} + V_1 + \frac{CM}{840}(\alpha^3 P_1 - 12\, a'_{15})$$

$$a'_{37} = Q\, a_{35} + H + \frac{CM}{840}(\alpha^3 P_3 - 12\, a'_{35})$$

$$b_{17} = Q\, b_{15} + \frac{CM}{840}(\alpha^3 U_1 - 12\, b'_{15})$$

$$Q = \frac{2\alpha^2 - i(1-C)(1-K)\alpha R + CM}{42}$$

$$H = \frac{i(1-C)\alpha R\left\{2K - [(2-\alpha^2)K - \alpha^2\right\} - CM\alpha^2}{840}$$

$$V_1 = \frac{i(1-C)\,\alpha^3 KR}{840}$$

$$a'_{17} = \frac{1}{42\, S_1}\left[\alpha^2 N\, a_{15} - 42\, N\, a_{17} + (\alpha^2 S_1 - 5iKR)a'_{15} + i\alpha^3 KRP_1\right]$$

$$b'_{17} = \frac{1}{42\, S_1}\left[\alpha^2 Nb_{15} - 42\, Nb_{17} + (\alpha^2 S_1 - 5iKR)b'_{15} + i\alpha^3 KRU_1\right]$$

$$a_{19} = \frac{7Q\, a_{17}}{12} + \frac{Q\, a_{15}}{72} + \frac{CM}{3024}\left(\alpha^2\, a'_{15} - 12\, a'_{17}\right)$$

$$a_{39} = \frac{7Q\ a_{37}}{12} + \frac{Q\ a_{35}}{72} + \frac{CM}{3024}\left(\alpha^3\ a'_{35} - 12\ a'_{39}\right) + \frac{5H}{18}$$

$$b_{19} = \frac{7Qb_{17}}{12} + \frac{Qb_{15}}{72} + \frac{CM}{3024}\left(\alpha^3\ b'_{15} - 12\ b'_{17}\right)$$

$$a'_{19} = \frac{1}{72S_1}\left[\alpha^2 Na_{17} - 72\ Na_{19} + (\alpha^2 S_1 - 7i\alpha KR)a'_{17} + i\alpha^3 KR\ a_{15}\right]$$

$$a'_{39} = \frac{1}{72S_1}\left[\alpha^2 Na_{37} - 72\ Na_{39} + (\alpha^2 S_1 - 7i\alpha KR)a'_{37} + i\alpha^3 KRa_{35}\right]$$

$$b'_{19} = \frac{1}{72S_1}\left[\alpha^2\ Nb_{17} - 72\ Nb_{19} + (\alpha^2 S_1 - 7i\alpha KR)b'_{17} + i\alpha^3 KR\ b_{15}\right]$$

$$a_{111} = \frac{42\ Q\ a_{19}}{110} + \frac{84\ H\ a_{17}}{792} + \frac{84\ V_1\ a_{15}}{792} + \frac{CM}{7920}\left[\alpha^2 a'_{17} - 12\ a'_{19}\right]$$

$$a_{311} = \frac{42\ Q\ a_{39}}{110} + \frac{84\ Ha_{37}}{792} + \frac{84\ V_1\ a_{35}}{792} + \frac{CM}{7920}\left[\alpha^2 a'_{37} - 12\ a'_{39}\right]$$

$$b_{111} = \frac{42\ Qb_{19}}{110} + \frac{84}{792}\left(Hb_{17} + V_1 b_{15}\right) + \frac{CM}{7920}\left[\alpha^2 b'_{17} - 12\ b'_{19}\right]$$

$$a'_{111} = \frac{1}{110S_1}\left[\alpha^2 Na_{19} - 110N\ a_{111} + (\alpha^2 S_1 - 9i\alpha KR)a'_{19} + i\alpha^3 KR\ a'_{17}\right]$$

$$a'_{311} = \frac{1}{110S_1}\left[\alpha^2 Na_{39} - 110\ Na_{311} + (\alpha^2 S_1 - 9i\alpha KR)a'_{39} + i\alpha^3 KR\ a'_{37}\right]$$

$$b'_{111} = \frac{1}{110S_1}\left[\alpha^2\ Nb_{19} - 110\ N\ b_{111} + (\alpha^2 S_1 - 9i\alpha KR)b'_{19} + i\alpha^3 KRb'_{17}\right]$$

$$a_{119} = \frac{7Q\ a_{111}}{26} + \frac{7}{143}\left(Ha_{19} + V_1 a_{17}\right) + \frac{CM}{17160}\left[\alpha^2 a'_{19} - 12\ a'_{111}\right]$$

$$a_{313} = \frac{7Q\ a_{311}}{26} + \frac{7}{143}\left(Ha_{39} + V_1 a_{37}\right) + \frac{CM}{17160}\left[\alpha^2 a'_{39} - 12\ a'_{311}\right]$$

$$b'_{123} = \frac{1}{506S_1}\left[\alpha^2 Nb_{121} - 506Nb_{123} + (\alpha^2 S_1 - 21i\alpha KR)b'_{121} + i\alpha^3 KR\, b_{119}\right]$$

$$a_{125} = \frac{7Q\, a_{123}}{100} + \frac{7}{2530}\left(Ha_{121} + V_1 a_{119}\right) + \frac{CM}{1200.253}\left[\alpha^2 a'_{121} - 12a'_{123}\right]$$

$$a_{325} = \frac{7Q\, a_{323}}{100} + \frac{7}{2530}\left(Ha_{321} + V_1 a_{319}\right) + \frac{CM}{1200.253}\left[\alpha^2 a'_{321} - 12a'_{323}\right]$$

$$b_{125} = \frac{7Q\, b_{123}}{100} + \frac{7}{2530}\left(Hb_{121} + V_1 a_{119}\right) + \frac{CM}{1200.253}\left[\alpha^2 b'_{121} - 12\, b'_{123}\right]$$

$$a'_{125} = \frac{1}{600S_1}\left[\alpha^2 Na_{123} - 600Na_{125} + (\alpha^2 S_1 - 23i\alpha KR)a'_{123} + i\alpha^3 KR\, a'_{121}\right]$$

$$a'_{325} = \frac{1}{600S_1}\left[\alpha^2 Na_{323} - 600Na_{325} + (\alpha^2 S_1 - 23i\alpha KR)a'_{323} + i\alpha^3 KRa'_{321}\right]$$

$$b'_{125} = \frac{1}{600S_1}\left[\alpha^2 Nb_{123} - 600Nb_{125} + (\alpha^2 S_1 - 23i\alpha KR)b'_{123} + i\alpha^3 KRb'_{121}\right]$$

$$a_{127} = \frac{7Q\, a_{125}}{117} + \frac{7}{135.26}\left(Ha_{123} + V_1 a_{121}\right) + \frac{CM}{27.26.600}\left[\alpha^2 a'_{123} - 12a'_{325}\right]$$

$$a_{327} = \frac{7Qa_{325}}{117} + \frac{7}{135.26}\left(Ha_{323} + V_1 a_{321}\right) + \frac{CM}{27.26.600}\left[\alpha^2 a'_{323} - 12a'_{325}\right]$$

$$b_{127} = \frac{7Q\, b_{125}}{117} + \frac{7}{135.26}\left(Hb_{123} + V_1 b_{121}\right) + \frac{CM}{27.26.600}\left[\alpha^2 b'_{123} - 12\, b'_{125}\right]$$

$$a'_{127} = \frac{1}{650S_1}\left[\alpha^2 Na_{125} - 650\, Na_{127} + (\alpha^2 S_1 - 25i\alpha KR)a'_{125} + i\alpha^3 KRa'_{123}\right]$$

$$a'_{327} = \frac{1}{650S_1}\left[\alpha^2 Na_{325} - 650Na_{327} + (\alpha^2 S_1 - 25i\alpha KR)a'_{325} + i\alpha^3 KRa'_{323}\right]$$

$$b'_{127} = \frac{1}{650S_1}\left[\alpha^2 Nb_{125} - 650Nb_{127} + (\alpha^2 S_1 - 25i\alpha KR)b'_{325} + i\alpha^3 KR\, b_{327}\right]$$

$$b_{113} = \frac{7Q\, b_{111}}{26} + \frac{1}{145}\left(Hb_{19} + V_1 b_{17}\right) + \frac{CM}{17160}\left[\alpha^3 b'_{19} - 12\, b'_{111}\right]$$

$$a'_{113} = \frac{1}{156S_1}\left[\alpha^2 Na_{111} - 156Na_{113} + (\alpha^2 S_1 - 11iKR)a'_{111} + i\alpha^3 KRa_{19}\right]$$

$$a'_{313} = \frac{1}{156S_1}\left[\alpha^2 Na_{311} - 156Na_{313} + (\alpha^2 S_1 - 11i\alpha KR)a'_{311} + i\alpha^3 KRa'_{39}\right]$$

$$b'_{113} = \frac{1}{156S_1}\left[\alpha^2 Nb_{111} - 156Nb_{113} + (\alpha^2 S_1 - 11i\alpha KR)b'_{111} + i\alpha^3 KRb'_{19}\right]$$

$$a_{115} = \frac{Q\, a_{113}}{5} + \frac{1}{39}\left(Ha_{111} + V_1 a_{19}\right) + \frac{CM}{32760}\left[\alpha^2 a'_{111} - 12\, a'_{113}\right]$$

$$a_{315} = Q\, a_{313} + \frac{1}{39}\left(Ha_{311} + V_1 a_{39}\right) + \frac{CM}{32760}\left[\alpha^2\, a'_{311} - 12\, a'_{313}\right]$$

$$b_{115} = \frac{Q\, b_{113}}{5} + \frac{1}{39}\left(Hb_{111} + V_1\, b_{19}\right) + \frac{CM}{32760}\left[\alpha^2 b'_{111} - 12\, b'_{113}\right]$$

$$a'_{115} = \frac{1}{210S_1}\left[\alpha^2 Na_{113} - 210N\, a_{115} + (\alpha^2 S_1 - 13i\alpha KR)a'_{113} + i\alpha^3 KR\, a'_{111}\right]$$

$$a'_{315} = \frac{1}{210S_1}\left[\alpha^2 Na_{313} - 210a_{315} + (\alpha^2 S_1 - 13i\alpha KR)a'_{313} + i\alpha^3 KR\, a'_{311}\right]$$

$$b'_{115} = \frac{1}{210S_1}\left[\alpha^2 Nb_{113} - 210Nb_{115} + (\alpha^2 S_1 - 13i\alpha KR)b'_{113} + i\alpha^3 KR\, b'_{111}\right]$$

$$a_{117} = \frac{210}{136}\, a_{115} + \frac{1}{68}\left(H\, a_{113} + V_1 a_{111}\right) + \frac{CM}{17.16.15.14} \times$$

$$\left[\alpha^2\, a'_{113} - 12\, a'_{115}\right]$$

$$a_{317} = \frac{210\, a_{315}}{136} + \frac{1}{68}(Ha_{313} + V_1 a_{311}) + \frac{CM}{17.16.15.14}\left[\alpha^2 a'_{313} - 12a'_{315}\right]$$

$$b_{117} = \frac{210 b_{115}}{136} + \frac{1}{68}(H\, b_{113} + V_1 b_{111}) + \frac{CM}{17.16.15.14}\left[\alpha^2 b'_{113} - 12\, b'_{115}\right]$$

$$a'_{117} = \frac{1}{272S_1}\left[\alpha^2 Na_{115} - 272Na_{117} + (\alpha^2 S_1 - 15i\alpha KR)a'_{115} + i\alpha^3 KR\, a'_{113}\right]$$

$$a'_{317} = \frac{1}{272S_1}\left[\alpha^2 Na_{315} - 272Na_{317} + (\alpha^2 S_1 - 15i\alpha KR)a'_{315} + i\alpha^3 KR\, a'_{313}\right]$$

$$b'_{117} = \frac{1}{272S_1}\left[\alpha^2 Nb_{115} - 272Nb_{117} + (\alpha^2 S_1 - 15i\alpha KR)b'_{115} + i\alpha^3 KRb'_{113}\right]$$

$$a_{119} = \frac{70\, a_{117}}{57} + \frac{35}{19.17.6}(H\, a_{115} + V_1 a_{113}) + \frac{CM}{19.18.17.16} \times$$

$$\left[\alpha^2\, a'_{115} - 12\, a'_{117}\right]$$

$$a_{319} = \frac{70\, a_{317}}{57} + \frac{35}{19.17.6}(H\, a_{315} + V_1 a_{313}) + \frac{CM}{19.18.17.16} \times$$

$$\left[\alpha^2\, a'_{315} - 12\, a'_{317}\right]$$

$$b_{119} = \frac{70\, b_{117}}{57} + \frac{35}{19.17.6}(Hb_{115} + V_1 b_{113}) + \frac{CM}{19.18.17.16} \times$$

$$\left[\alpha^2\, b'_{115} - 12\, b'_{117}\right]$$

$$a'_{119} = \frac{1}{342S_1}\left[\alpha^2 Na_{117} - 342Na_{119} + (\alpha^2 S_1 - 17i\alpha KR)a'_{117} + i\alpha^3 KRa'_{115}\right]$$

$$a'_{319} = \frac{1}{342S_1}\left[\alpha^2 Na_{317} - 342Na_{319} + (\alpha^2 S_1 - 17i\alpha KR)a'_{317} + i\alpha^3 KRa'_{315}\right]$$

$$b'_{119} = \frac{1}{342S_1}\left[\alpha^2 Nb_{117} - 343Nb'_{319} + (\alpha^2 S_1 - 17i\alpha KR)b'_{117} + i\alpha^3 KRb'_{115}\right]$$

$$a_{121} = \frac{Q\, a_{119}}{10} + \frac{1}{171}(Ha_{117} + V_1 a_{115}) + \frac{CM}{420.342}\left[\alpha^2 a'_{117} - 12\, a'_{115}\right]$$

$$a_{321} = \frac{Q\, a_{319}}{10} + \frac{1}{171}(Ha_{317} + V_1 a_{315}) + \frac{CM}{420.342}\left[\alpha^2 a'_{317} - 12\, a'_{319}\right]$$

$$b_{121} = \frac{Q\, b_{119}}{10} + \frac{1}{171}(Hb_{117} + V_1 b_{115}) + \frac{CM}{420.342}\left[\alpha^2\, b'_{117} - 12\, b'_{119}\right]$$

$$a'_{121} = \frac{1}{420S_1}\left[\alpha^2 Na_{119} - 420Na_{121} + (\alpha^2 S_1 - 19i\alpha KR)a'_{119} + i\alpha^3 KRa'_{117}\right]$$

$$a'_{321} = \frac{1}{420S_1}\left[\alpha^2 Na_{319} - 420Na_{321} + (\alpha^2 S_1 - 19i\alpha KR)a'_{319} + i\alpha^3 KR\, a'_{317}\right]$$

$$b'_{121} = \frac{1}{420S_1}\left[\alpha^2 Nb_{119} - 420Nb_{121} + (\alpha^2 S_1 - 19i\alpha KR)b'_{119} + i\alpha^3 KR\, b'_{117}\right]$$

$$a_{123} = \frac{210\, a_{121}}{253} + \frac{1}{253}(Ha_{119} + V_1 a_{117}) + \frac{CM}{506.420}\left[\alpha^2 a'_{119} - 12a'_{121}\right]$$

$$a_{323} = \frac{210\, a_{321}}{253} + \frac{1}{253}(Ha_{319} + V_1 b_{117}) + \frac{CM}{506.420}\left[\alpha^2 a'_{319} - 12\, a'_{321}\right]$$

$$b_{123} = \frac{1}{253}\left[210\, b_{121} + Hb_{119} + V_1 b_{117} + \frac{CM}{840}\left\{\alpha^2 b'_{119} - 12\, b'_{121}\right\}\right]$$

$$a'_{123} \quad \frac{1}{506S_1}\left[\alpha^2 Na_{121} - 506Na_{123} + (\alpha^2 S_1 - 2i\alpha KR)a'_{121} + i\alpha^3 KR\, a'_{119}\right]$$

$$a'_{323} = \frac{1}{506S_1}\left[\alpha^2 Na_{321} - 506Na_{323} + (\alpha^2 S_1 - 21\, i\alpha KR)a'_{321} + i\alpha^3 KRa'_{319}\right]$$

$$a_{129} = \frac{3Q\, a_{127}}{58} + \frac{35}{29.27.26}\left(Ha_{125} + V_1 a_{123}\right) + \frac{CM}{29.28.27.26}$$

$$\left[\, a^2\, a'_{125} - 12\, a'_{127} \,\right]$$

$$a_{329} = \frac{3Q\, a_{327}}{58} + \frac{35}{29\ 27.26}\left(H\, a_{325} + V_1 a_{323}\right) + \frac{CM}{29.28.27.26}$$

$$\left[\, a2\, a_{325} - 12\, a'_{327} \,\right]$$

$$b_{129} = \frac{3Q\, b_{127}}{58} + \frac{35}{29.27.26}\left(Hb_{125} + V_1 b_{123}\right) + \frac{CM}{29.28.27.26}$$

$$\left[\, a^2\, b'_{125} - 12\, b_{127} \,\right]$$

2.7 Numerical Results and Discussion

Numerical calculations show that the mean axial velocity distribution (averaged over a time) due to peristaltic motion in the case of free pumping is dominated by the constant D. The constant D, which defines the boundary value of ϕ'_{f20} in equation (99), has its origin in the radial gradient of the first-order axial velocity distribution, as can be seen in equation (67). This comes from the offset boundary conditions i.e. to the second- order non linear effect of the fact that the nonslip condition applies to the wavy wall and not to the mean position of the wall. The constant D is a function of the wave number α and the Reynolds number R, but its variation with α and R is slight. For small values of R and K we can express

$$\phi_{f1} = \phi_{f10} + K\phi_{f11}$$

where ϕ_{f10} and ϕ_{f11} are obtained from (86) and (92) respectively. The values of the constant D for various values of α and K are tabulated in Table 2 . The first term in (106) represents the value of D at K = 0 and small R. A comparison of this value of D has been made in Table 3 with the values calculated by Yang and Yih's expression. In Table 2 D is decreasing with increase in Poiseuille flow parameter K while it is increasing with increase in the wave number α. Since α and R are very small, C has no significant effect on D but where effect of C is predominant is in the series solution. One notes that theoretically when C = 0 in the above chapter, the equations assume

the form of Mittra model but differ in many respect in that ours is a combination of particle and fluid phases.

Table 2 Value of D for small R and K

K =	0.1	0.2	0.3	0.4	0.5	0.6
α			D			
0.1	2.81941	2.77576	2.73211	2.68845	2.64480	2.60115
0.2	2.95,61	2.90510	2.35260	2.80009	2.74759	2.69501
0.3	2.97723	2.92106	2.86489	2.80872	2.75255	2.69638
0.4	3.00501	2.94373	2.38241	2.82110	2.75978	2.69846
0.5	3.03643	2.96872	2.90101	2.83331	2.76560	2.69789
0.6	3.07659	3.00098	2.92537	2.84976	2.77415	2.69854
0.7	3.12511	3.04011	2.95510	2.87010	2.78509	2.70006
0.8	3.18427	3.08882	2.99277	2.89673	2.80068	2.70464
0.9	3.25442	3.14576	3.03711	2.92846	2.81980	2.71115
1.0	3.33573	3.21280	3.08987	2.96693	2.84400	2.72106

Table 3 Comparison of D for small R with D of Fung, Yih's Model (K = 0)

α	$D(R \leq 1)$	Values of D as obtained by Fung and Yih
0.1	3.00432	3.00401
0.2	3.01620	3.01610
0.3	3.03655	3.03651
0.4	3.06562	3.06560
0.5	3.10288	3.10387
0.6	3.15195	3.15197
0.7	3.21060	3.21062
0.8	3.28063	3.28068
0.9	3.36303	3.36307
1.0	3.45673	3.45382

CHAPTER III: THE INFLUENCES OF WALL PROPERTIES AND POISEUILLE FLOW OF A FLUID - PARTICLE SUSPENSION IN PERISTALSIS (ELASTIC)

3.1. INTRODUCTION.

This chapter treats the influence of wall properties and Poiseuille flow of a fluid-particle suspension in peristalsis (Elastic) based on Mittra and Prasad (1973) model.

Mittra and Prasad (1973) discussed that since theoretical results depend upon solutions of Navier-Stokes equations, an assumption of either inertia-free flow or flow with small amplitude was found convenient. In this chapter, a two-dimensional peristaltic flow of a Newtonian viscous fluid-particle suspension is filled in a channel of infinite length but finite depth (width). When the fluid is set into motion by a steady progressive sinusoidal wave of moderate amplitude the walls considered as flexible (Elastic) a basic Poiseuille flow caused by a uniform pressure became present given rise to problem of peristalsis. (Note here that no effect will be made as to elaborate how the wave originated nor to consider loss or gain in energy of the system). Peristalsis is a muscle controlled flow. Since muscles are involved as the walls are flexible, the fluid is propelled by rhythmic waves of

contraction called peristalsis (to send around). A typical peristaltic wave is preceeded by relaxation. However, the strength and the velocity of peristaltic contractions are increased by impulses carried over the walls. As the structures relax, the cycle begins again. The action of the muscles are governed by a set of equations related to the wall displacement. Its mechanism is assumed in the form of a sinusoidal wave of moderate amplitude imposed on the flexible walls of the channel.

The chapter follows same pattern and methods of solution as the Mittra and Prasad. The two governing equations were solved for fluid phase and the particle phase taking into consideration the action of the walls since there may be a forced oscillation of the fluid due to a harmonic travelling wave. Solutions regarding pure peristalsis, closed forms solutions, effect of damping and series solutions were obtained.

3.2 Formulation of the problem

Consider a two dimensional infinite channel of uniform thickness 2d (Fig.2) filled with a mixture of small spherical rigid particles in an incompressible Newtonian viscous fluid. The walls of the channel are flexible, considered as either thin elastic plates or membranes on which are imposed travelling, sinusoidal waves of moderate amplitude. The equations governing both the fluid and particle phase are expressed as

fluid phase:

$$(1 - C)\rho_f\left[\frac{\partial u_f}{\partial t} + u_f \frac{\partial u_f}{\partial x} + v_f \frac{\partial u_f}{\partial y}\right] = - (1 - C)\frac{\partial p}{\partial x} + (1- C)\mu_s x$$

$$(C)\nabla^2 u_f + CS(u_p - u_f) \qquad (1)$$

$$(1- C)\rho_f\left[\frac{\partial v_f}{\partial t} + u_f\frac{\partial v_f}{\partial x} + V_f \frac{\partial v_f}{\partial y}\right] = - (1- C)\frac{\partial p}{\partial y} + (1- C)\, x$$

$$\mu_s(C)\nabla^2 v_f + CS(v_p - v_f) \qquad (2)$$

$$\frac{\partial}{\partial x}\left\{(1- C)u_f\right\} + \frac{\partial}{\partial y}\left\{(1- C)v_f\right\} = 0 \quad \ldots \qquad (3)$$

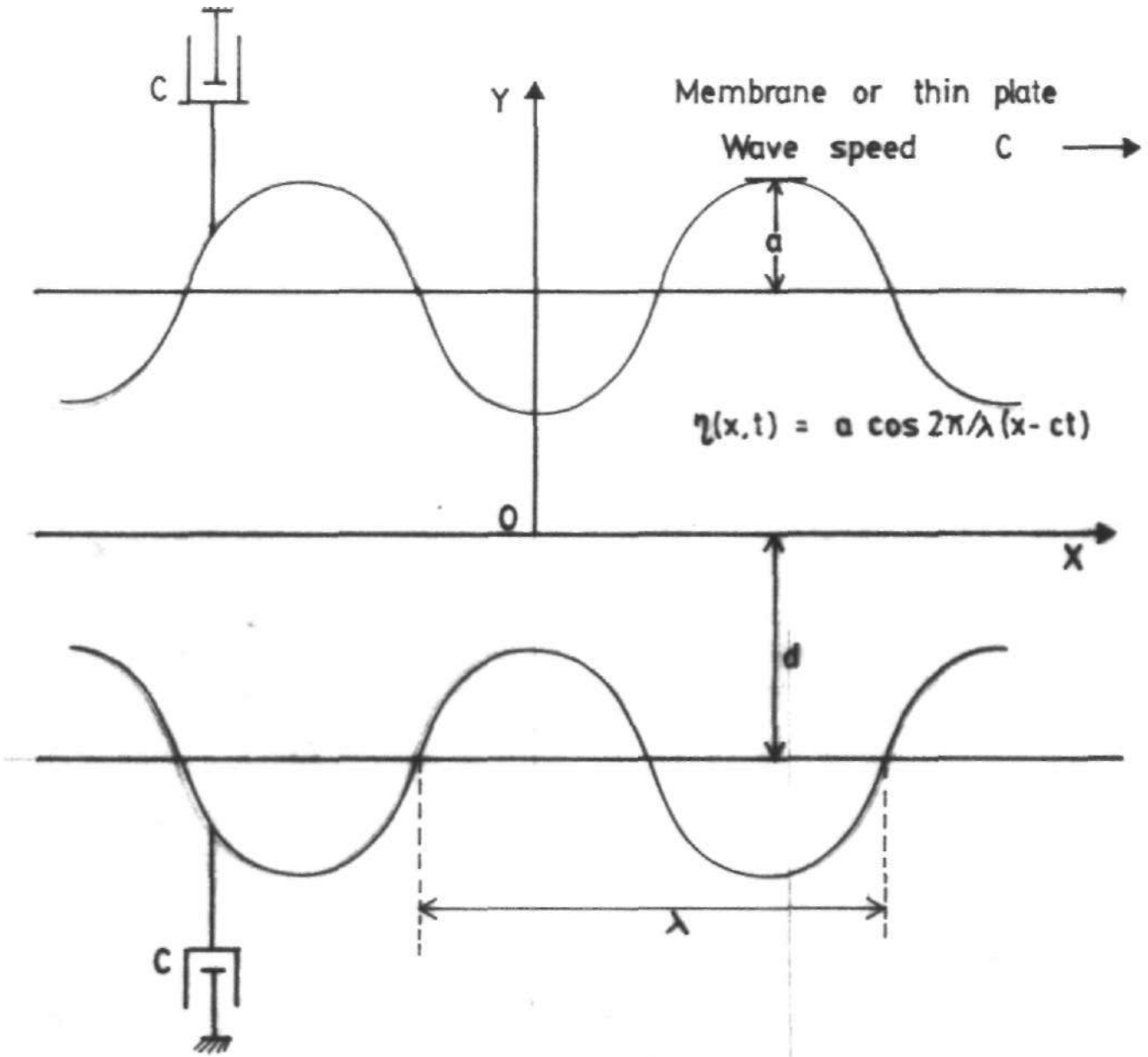
Particulate phase:

$$C\rho_p\left[\frac{\partial u_p}{\partial t} + u_p\frac{\partial u_p}{\partial x} + v_p \frac{\partial u_p}{\partial y}\right] = - C\frac{\partial p}{\partial x} + CS(u_f - u_p) \quad (4)$$

$$C\rho_p\left[\frac{\partial v_p}{\partial t} + u_p \frac{\partial v_p}{\partial x} + v_p \frac{\partial v_p}{\partial y}\right] = - C \frac{\partial p}{\partial y} + CS(v_f - v_p) \quad (5)$$

$$\frac{\partial}{\partial x}\left\{C\, u_p\right\} + \frac{\partial}{\partial y}\left\{C v_p\right\} = 0 \qquad (6)$$

Note here that all symbols used in the above equations have been explained in chapter 2.



PERISTALTIC MOTION OF THE WALLS

Fig. 2

Introducing the stream functions for the two phase

$$\left.\begin{aligned} u_f &= \frac{\partial \psi_f}{\partial y}, \quad v_f = -\frac{\partial \psi_f}{\partial x} \\ u_p &= \frac{\partial \psi_p}{\partial y}, \quad v_p = -\frac{\partial \psi_p}{\partial x} \end{aligned}\right\} \tag{7}$$

Eliminating the pressure gradient from equations (1) and (2) and (4) and (5), the equations of motion of a viscous fluid in terms of steam functions are

$$(1-C)\rho_f\left[\frac{\partial}{\partial t}\nabla^2\psi_f + \psi_{fy}\nabla^2\psi_{fx} - \psi_{fx}\nabla^2\psi_{fy}\right] = (1-C)\mu(C)\nabla^4\psi_p +$$

$$+ CS(\nabla^2\psi_p - \nabla^2\psi_f). \tag{8}$$

$$C\rho_p\left[\frac{\partial}{\partial t}\nabla^2\psi_p + \psi_{py}\nabla^2\psi_{px} - \psi_{px}\nabla^2\psi_{py}\right] = CS(\nabla^2\psi_f - \nabla^2\psi_p) \tag{9}$$

In the above equations, x and y are cartesian coordinates with x measured in the direction of wave propagation and y measured in the direction normal to the mean position of the elastic walls. The velocity components in the directions of increasing x and y are u_f, u_p and v_f, v_p and the origin is at the centre line of the channel, ρ_f, ρ_p are densities of fluid and particles respectively, p is the pressure while ν the Kinetic viscosity. The subscripts indicate partial differentiation and ∇^2 is the two-dimensional Laplace operator: $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

Dynamic boundary conditions are imposed on the fluid by the symmetric motion of the flexible walls. In reality, the

ureteral wall is exicited by the muscles whose tension will control its deformation. The field equations representing the action of these muscles will involve variables, which through continuity conditions, will be related to those of the walls and the fluid. To simplify analysis, assume that they yield solutions in terms of displacement variable η_1 (x,t) at the middle plane of the wall so that

$$\eta_1(x, t) = \eta(x, t)$$

where $\eta(x,t)$ is the displacement of the wall of the channel. Further η_1 is assumed to be in the form of a sinusoidal wave of amplitude a, wave length λ and wave speed c.

Therefore

$$\eta = a \cos \frac{2\pi}{\lambda} (x - ct) \qquad (10)$$

The governing equations of motion of the flexible walls may be expressed as

$$\mathcal{L}(\eta) = p - p_0 \qquad (11)$$

where $\mathcal{L}$ is an operator, which is used to represent the motion of a classical thin plate or that of a stretched membrane with viscous damping forces such that

$$\mathcal{L} \equiv - T\frac{\partial^2}{\partial x^2} + m \frac{\partial^2}{\partial t^2} + H \frac{\partial}{\partial t} \text{(membrane)} \qquad (12)$$

$$\mathcal{L} \equiv W \frac{\partial^4}{\partial x^4} + m\frac{\partial^2}{\partial t^2} + H \frac{\partial}{\partial t} \text{(thin plate)} \qquad (13)$$

where

T is the tension in the membrane

w the flexural rigidity of the plate

m the mass per unit area

H the coefficient of viscous damping forces

p the interaction presure due to the motion of the fluid on the wall.

p_o the pressure on the outside surface of the wall due to the tensior in the muscles,

This tension may be obtained through the constitutive relation of the muscles when the displacements are known, but $p_o = 0$ will be assumed. It is assumed that the walls are inextensible so that only their lateral motions normal to the undeformed positions occur. The horizontal displacement, thus is assumed to be zero, which results into

$$u_f \equiv \psi_{fy} = 0$$
$$u_p \equiv \psi_{py} = 0 \qquad \text{at} \quad y = \pm d \pm \eta \qquad (14)$$

Continuity of stresses requires that at the interfaces of the walls and the fluid, p must be the same as that which acts on the fluid at $y = \pm d \pm \eta$. By using x - momentum equation, this leads to

$$(1-C)\frac{\partial}{\partial x}\int(\eta) \equiv (1-C)\frac{\partial p}{\partial x} = \rho_f \nu \nabla^2 \psi_{fy} - \rho_f (1-C)(\psi_{fyt} + \psi_{fy}\psi_{fyx} - \psi_{fx}\psi_{fyy}) + (\psi_{py} - \psi_{fy}) \text{ at } y = \pm d \pm \eta \qquad (15)$$

and the boundary condition of the particle assumed as

$$C\frac{\partial}{\partial x}\int(\eta) = -C\left[\psi_{pyt} + \psi_{py}\psi_{fyx} - \psi_{pyy}\psi_{fx}\right] + (\psi_{fy} - \psi_{py}) \qquad (16)$$

It may be mentioned that (14), (16) are the counterparts of the boundary conditions known as no-slip and no-seperation

conditions, respectively, in hydrodynamic stability with rigid boundaries.

The above set of equations are nondimensionalised by introducing the following parameters.

$$\left.\begin{aligned} x' &= \frac{x}{d}, \quad y' = \frac{y}{d}, \quad u_f' = \frac{u_f}{c}, \quad v_f' = \frac{u_f}{c}, \\ u_p' &= \frac{u_p}{c}, \quad v_p' = \frac{v_p}{c}, \quad \eta' = \frac{\eta}{d}, \quad \psi_f' = \frac{\psi_f}{cd}, \\ \psi_p' &= \frac{\psi_p}{cd}, \quad t' = \frac{ct}{d}, \quad p' = \frac{p}{\rho_f c^2}, \nu = \frac{\mu_o}{\rho_f} \end{aligned}\right\} \tag{17}$$

$$\left.\begin{aligned} &\text{Suspension Reynolds number} \quad R = \frac{cd\rho_f}{(1-C)\mu_s} \\ &\text{Wave number} \quad \alpha = \frac{2\pi d}{\lambda} \\ &\text{amplitude ratio} \quad \varepsilon = \frac{a}{d} \\ &\text{Suspension parameter} \quad M = \frac{9\mu d^2}{(1-C)\mu_s} \\ &\text{Suspension parameter} \quad N = \frac{9d^2\rho_f}{(1-C)\rho_p\mu_s} \end{aligned}\right\} \tag{18}$$

Droping the primes (8, 9, 10, 14 - 16) yield the following

$$(1-C)\left[\frac{\partial}{\partial t}\nabla^2\psi_f + \psi_{fy}\nabla^2\psi_{fx} - \psi_{fx}\nabla^2\psi_{fy}\right] = \frac{1}{R}\left[\nabla^2\nabla^2\psi_f + CM(\nabla^2\psi_p - \nabla^2\psi_f)\right] \tag{19}$$

$$C\left[\frac{\partial}{\partial t}\nabla^2\psi_p + \psi_{py}\nabla^2\psi_{px} - \psi_{px}\nabla^2\psi_{py}\right] = \frac{CN}{R}\left[\nabla^2\psi_f - \nabla^2\psi_p\right] \tag{20}$$

$$\eta = \varepsilon \cos \alpha (x - t) \tag{21}$$

$$\left.\begin{aligned} \psi_{py} &= C \\ \psi_{py} &= 0 \end{aligned}\right\} \quad \text{at} \quad y = \pm (d + \eta) \tag{22}$$

and

$$(1-C)\frac{\partial}{\partial x}\left[\quad(\eta)\right] = \frac{1}{R}\nabla^2\Psi_{fy} - (1-C)\left[\Psi_{fyt} + \Psi_{fy}\Psi_{fyx} - \Psi_{fx}\Psi_{fyy}\right] + q_2\left[\Psi_{py} - \Psi_{fy}\right] \quad (23)$$

$$C\frac{\partial}{\partial x}\left[\quad(\eta)\right] = -C\bar{\rho}\left[\Psi_{pyt} + \Psi_{pyt}\Psi_{fyx} - \Psi_{px}\Psi_{pyy}\right] + q_2\left[\Psi_{fy} - \Psi_{py}\right] \quad (24)$$

at $y = \pm(1+\eta)$

where $q_2 = \dfrac{CSd}{c\rho_f}$, $\bar{\rho} = \dfrac{\rho_n}{\rho_f}$ (25)

$$\frac{\partial}{\partial x}\left[\quad\right] = -\frac{K_3}{R^2}\frac{\partial^3}{\partial x^3} + m'\frac{\partial^3}{\partial x \partial t^2} + \frac{K_2}{R}\frac{\partial^2}{\partial x \partial t} \quad \text{(membrane)} \quad (26)$$

$$\frac{\partial}{\partial x}\left[\quad\right] = \frac{K_4}{R^2}\frac{\partial^5}{\partial x^5} + m'\frac{\partial^3}{\partial x \partial t^2} + \frac{K_2}{R}\frac{\partial^2}{\partial x \partial t} \quad \text{(thin plate)} \quad (27)$$

The following nondimensional quantities related to the wall are introduced in (26) and (27).

$$K_2 = \frac{Hd}{\mu}, \quad K_3 = \frac{T\rho_f d}{\mu^2}, \quad K_4 = \frac{W\rho_f d^3}{\mu^2}, \quad m' = \frac{m}{\rho_f d} \quad (28)$$

3.3 Method of solution

Expanding the the stream functions Ψ_f, Ψ_p and $\frac{\partial p}{\partial x}$ in power series of ε, assuming ε to be small

$$\Psi_f = \Psi_{fo} + \varepsilon\Psi_{f1} + \varepsilon^2\Psi_{f2} + \cdots \quad (29)$$

$$\Psi_p = \Psi_{po} + \varepsilon\Psi_{p1} + \varepsilon^2\Psi_{p2} + \cdots \quad (30)$$

$$\frac{\partial p}{\partial x} = \left(\frac{\partial p}{\partial x}\right)_o + \varepsilon\left(\frac{\partial p}{\partial x}\right)_1 + \varepsilon^2\left(\frac{\partial p}{\partial x}\right)_2 + \cdots \quad (31)$$

Substituting (29) and (30) in (19) and (20) and collecting like powers of ε

(29) becomes

$$\varepsilon^0 : (1-C)\left[\frac{\partial}{\partial t}\nabla^2\Psi_{fo} + \Psi_{foy}\nabla^2\Psi_{fox} - \Psi_{fox}\nabla^2\Psi_{foy}\right] = \frac{1}{R}\left[\nabla^2\nabla^2\Psi_{fo} + CM(\nabla^2\Psi_{po} - \nabla^2\Psi_{fo})\right] \quad (32)$$

$$\varepsilon^1 : (1-C)\left[\frac{\partial}{\partial t}\nabla^2\Psi_{f1} + \Psi_{foy}\nabla^2\Psi_{f1x} + \Psi_{f1y}\nabla^2\Psi_{fox} - \Psi_{fox}\nabla^2\Psi_{f1y} - \Psi_{f1x}\nabla^2\Psi_{foy}\right] = \frac{1}{R}\left[\nabla^2\nabla^2\Psi_{f1} + CM(\nabla^2\Psi_{p1} - \nabla^2\Psi_{f1})\right] \quad (33)$$

$$\varepsilon^2 : (1-C)\left[\frac{\partial}{\partial t}\nabla^2\Psi_{f2} + \Psi_{foy}\nabla^2\Psi_{f2x} + \Psi_{f1y}\nabla^2\Psi_{f1x} + \Psi_{f2y}\nabla^2\Psi_{fox} - \Psi_{fox}\nabla^2\Psi_{f2y} - \Psi_{f1x}\nabla^2\Psi_{f1y} - \Psi_{f2x}\nabla^2\Psi_{foy}\right] = \frac{1}{R}\left[\nabla^2\nabla^2\Psi_{f2} + CM(\nabla^2\Psi_{p2} - \nabla^2\Psi_{f2})\right] \quad (34)$$

(20) becomes

$$\varepsilon^0 : C\left[\frac{\partial}{\partial t}\nabla^2\Psi_{po} + \Psi_{poy}\nabla^2\Psi_{pox} - \Psi_{pox}\nabla^2\Psi_{poy}\right] = \frac{CN}{R}\left[\nabla^2\Psi_{fo} - \nabla^2\Psi_{po}\right] \quad (35)$$

$$\varepsilon^1 : C\left[\frac{\partial}{\partial t}\nabla^2\Psi_{p1} + \Psi_{poy}\nabla^2\Psi_{p1x} + \Psi_{p1y}\nabla^2\Psi_{pox} - \Psi_{pox}\nabla^2\Psi_{p1y} - \Psi_{p1x}\nabla^2\Psi_{poy}\right] = \frac{CN}{R}\left[\nabla^2\Psi_{f1} - \nabla^2\Psi_{p1}\right] \quad (36)$$

$$\varepsilon^2 : C\left[\frac{\partial}{\partial t}\nabla^2\Psi_{p2} + \Psi_{poy}\nabla^2\Psi_{p2x} + \Psi_{p1y}\nabla^2\Psi_{p1x} + \Psi_{p2y}\nabla^2\Psi_{pox} - \Psi_{pox}\nabla^2\Psi_{p2y} - \Psi_{p1x}\nabla^2\Psi_{p1y} - \Psi_{p2x}\nabla^2\Psi_{poy}\right] = \frac{CN}{R}\left[\nabla^2\Psi_{f2} - \nabla^2\Psi_{p2}\right] \quad (37)$$

(23) becomes

$$\varepsilon^0 : 0 = \frac{1}{R}\nabla^2\Psi_{fo} - (1-C)\left[\Psi^{(\pm 1)}_{foyt} + \Psi_{foy}\Psi_{foyx} - \Psi_{fox}\Psi_{foyy}\right] + a_2\left[\Psi_{poy} - \Psi_{foy}\right] \quad (38)$$

$$\varepsilon^1: \quad (1-C)\delta_1 e^{i\alpha(x-t)} = \frac{1}{R}\nabla^2\psi_{f1y} - (1-C)\left[\psi_{f1yt} + \psi_{foy}\psi_{f1yx} + \psi_{f1y}\psi_{foyx} - \psi_{fox} \right.$$
$$\left. \psi_{f1yy} - \psi_{f1x}\psi_{foyy}\right] + q_2\left[\psi_{poy} - \psi_{foy}\right] \qquad (39)$$

$$\varepsilon^2: \quad 0 = \frac{1}{R}\nabla^2\psi_{f2y} - (1-C)\left[\psi_{f2yt} + \psi_{foy}\psi_{f2yx} + \psi_{f1y}\psi_{f1yx} + \right.$$
$$\left. + \psi_{f2y}\psi_{foyx} - \psi_{fox}\psi_{f2yy} - \psi_{f1x}\psi_{f1yy} - \psi_{f2x}\psi_{foyyy}\right]$$
$$+ q_2\left[\psi_{p2y} - \psi_{f2y}\right] \qquad (40)$$

and (24)

$$\varepsilon^1: \quad -C\,\delta_1\, e^{i\alpha(x-t)} = -\bar{\rho}C\left[\psi_{p1yt} + \psi_{poy}\psi_{f1yx} + \psi_{p1y}\psi_{foyx} - \right.$$
$$\left. - \psi_{pox}\psi_{p1yy} - \psi_{p1x}\psi_{poyy}\right] + q_2\left[\psi_{f1y} - \psi_{p1y}\right] \qquad (41)$$

$$\varepsilon^2: \quad 0 = -\bar{\rho}C\left[\psi_{p2yt} + \psi_{poy}\psi_{p2yx} + \psi_{p1y}\psi_{f1yx} + \psi_{p2y}\psi_{foyx} - \right.$$
$$\left. - \psi_{pox}\psi_{p2yy} - \psi_{p1x}\psi_{p1yy} - \psi_{p2x}\psi_{poyy}\right] + q_2\left[\psi_{f2y} - \psi_{p2y}\right] \qquad (42)$$

Expanding equations (14) about the mean position of the wall

$$\left.\begin{aligned} \psi^{(\pm1)}_{fy} \pm \eta\,\psi^{(\pm1)}_{foy} + \frac{\eta^2}{2}\psi^{(\pm1)}_{fyyy} + \ldots = 0 \\ \psi^{(\pm1)}_{py} \pm \eta\,\psi^{(\pm1)}_{poy} + \frac{\eta^2}{2}\psi^{(\pm1)}_{pyyy} + \ldots = 0 \end{aligned}\right\} \qquad (43)$$

and substituting (21), (29), (30) into (43) and collecting like powers of ε

$$\left.\begin{aligned} &\psi^{(\pm1)}_{foy} = 0 \\ &\psi^{(\pm1)}_{f1y} + \psi^{(\pm1)}_{foyy}\cos\alpha(x-t) = 0 \\ &\psi^{(\pm1)}_{f2y} + \psi^{(\pm1)}_{f1yy}\cos\alpha(x-t) + \tfrac{1}{2}\psi^{(\pm1)}_{foyyy}\cos^2\alpha(x-t) \\ &\qquad = 0 \end{aligned}\right\} \qquad (44)$$

A well-known solutions of the Navier- Stokes equations (32) and (35) are given by the classical Poiseville flow as

$$\left.\begin{aligned} \Psi_{fo} &= K\left(y - \frac{y^3}{3}\right) \\ \Psi_{po} &= K\left(y - \frac{1}{3}y^3 + \frac{2y}{M}\right) \end{aligned}\right\} \qquad (45)$$

where

$$2K = -R\left(\frac{\partial p}{\partial x}\right)$$

The uniform pressure gradient $\frac{\partial p}{\partial x}$ in the x-direction is constant. In addition to the Poiseville flow, the travelling wave motion in the wall will create, in a steady state situation, a peristaltic mean flow. This mean flow will depend upon the physical quantities relevant to the motion of the system. The case of pure persistalsis, K = 0 has been termed "free pumping" by Fung and Yih (1968). Indeed, their studies were concerned with only this case. In the present study, an extensive investigation is presented to include the effects of wall properties for both cases k = 0 and $K \neq 0$.

Using equation (45), the first order perturbation equations (33), (36) reduce to

$$(1-C)\left[\frac{\partial}{\partial t}\nabla^2\psi_{f1} + K(1-y^2)\nabla^2\psi_{f1X} + 2K\,\psi_{f1X}\right] = \frac{1}{R}\left[\nabla^2\nabla^2\psi_{f1} + CM(\nabla^2\psi_{p1} - \nabla^2\psi_{f1})\right]$$

$$C\left[\frac{\partial}{\partial t}\nabla^2\psi_{p1} + K\left(1-y^2+\frac{2}{M}\right)\nabla^2\psi_{p1x} + 2K\psi_{p1X}\right] = \frac{CN}{R}\left[\nabla^2\psi_{f1} - \nabla^2\psi_{p1}\right] \qquad (46)$$

(47)

Subject to the boundary conditions

$$\Psi_{f_1 y}(\pm 1) = \mp 2K \cos\alpha \ (\alpha - t) \tag{48}$$

$$(1-C)\,\delta_1 e^{i\alpha(x-t)} = \frac{1}{R}\nabla^2\Psi_{f_1 y} - (1-C)\left[\Psi_{f_1 yt} + \frac{2K}{M}\Psi_{f_1 yx} + 2K\Psi_{f_1 x}\right] + q_2\left[\Psi_{p_1 y} - \Psi_{f_1 y}\right] \tag{49}$$

$$C\,\delta_1\, e^{i\alpha(x-t)} = -\bar{\rho}C\left[\Psi_{p_1 y} + \frac{2K}{M}\Psi_{f_1 yx} + 2K\Psi_{p_1 x}\right] + q_2\left[\Psi_{f_1 y} - \Psi_{p_1 y}\right] \tag{50}$$

while the second order perturbation equations (34), (37) also tend to

$$(1-C)\left[\frac{\partial}{\partial t}\nabla^2\Psi_{f_2} + \Psi_{f_1 y}\nabla^2\Psi_{f_1 x} + K(1-y^2)\nabla^2\Psi_{f_2 x} - \Psi_{f_1 x}\nabla^2\Psi_{f_1 y} + 2K\Psi_{f_2 x}\right] = \frac{1}{R}\left[\nabla^4\Psi_{f_2} + CM(\nabla^2\Psi_{p_2} - \nabla^2\Psi_{f_2})\right] \tag{51}$$

$$C\left[\frac{\partial}{\partial t}\nabla^2\Psi_{p_2} + K\left(1-y^2+\frac{2}{M}\right)\nabla^2\Psi_{p_2 x} + \Psi_{p_1 y}\nabla^2\Psi_{p_1 x} - \Psi_{p_1 x}\nabla^2\Psi_{p_1 y} + 2K\Psi_{p_2 x}\right] = \frac{CN}{R}\left[\nabla^2\Psi_{f_2} - \nabla^2\Psi_{p_2}\right] \tag{52}$$

and its boundary conditions

$$\Psi_{f_2 y}(\pm 1) + \Psi_{f_1 yy}(\pm 1)\cos\alpha(x-t) - K\cos^2\alpha(x-t) = 0 \tag{53}$$

$$0 = \frac{1}{R}\nabla^2\Psi_{f_2 y} - (1-C)\left[\Psi_{f_2 yt} + \frac{2K}{M}\Psi_{f_2 yx} + \Psi_{f_1 y}\Psi_{f_1 yx} - \Psi_{f_1 x}\Psi_{f_1 yy} + 2K\Psi_{f_2 x}\right] + q_2\left[\Psi_{p_2 y} - \Psi_{f_2 y}\right] \tag{54}$$

$$0 = -\bar{\rho}C\left[\Psi_{p_2 yt} + \frac{2K}{M}\Psi_{p_2 yx} + \Psi_{p_1 y}\Psi_{f_1 yx} - \Psi_{p_1 x}\Psi_{p_1 yy} + 2K\Psi_{p_2 x}\right] + q_2\left[\Psi_{f_2 y} - \Psi_{p_2 y}\right] \tag{55}$$

Although the mean peristaltic motion is caused by a single harmonic wave of wave number α, the details of flow are quite complex because of the nonlinearity in the governing equations. The velocity field in the channel consists of all harmonics and solutions to equations (46) - (50) may assume the form

$$\left.\begin{aligned} \Psi_{f1} &= \tfrac{1}{2}\left[\, \phi_{f1}\, e^{i\alpha(x-t)} + \phi^{*}_{f1}\, e^{-i\alpha(x-t)} \,\right] \\ \Psi_{p1} &= \tfrac{1}{2}\left[\, \phi_{p1}\, e^{i\alpha(x-t)} + \phi^{*}_{p1}\, e^{-i\alpha(x-t)} \,\right] \end{aligned}\right\} \qquad (56)$$

where the asterisk denotes the complex conjugate .

Substituting (56) into (46) - (50) yield the Orr - somerfield type of equation.

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i\alpha(1-C)R\left[1-K(1-y^2)\right]\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f1} - 2i\alpha(1-C)KR\phi_{f1} = CN\left(\frac{d^2}{dy^2} - \alpha^2\right)(\phi_{f1} - \phi_{p1}) \qquad (57)$$

$$i\alpha C\left[1-K\left(1-y^2+\frac{2}{M}\right)\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{p1} - 2i\alpha CKR\,\phi_{p1} = CN\left(\frac{d^2}{dy^2} - \alpha^2\right)(\phi_{p1} - \phi_{f1}) \qquad (58)$$

With the boundary conditions

$$\phi'_{f1}(\pm 1) = 2K \qquad (59)$$

$$(1-C)\delta_1 = \frac{1}{2R}\left(\phi'''_{f1}(\pm 1) - \alpha^2\phi_{f1}(\pm 1)\right) - i(1-C)\alpha\left[-\tfrac{1}{2}\phi'_{f1}(\pm 1) + K\left(\frac{\phi'_{f1}(\pm 1)}{M} + \phi_{f1}(\pm 1)\right)\right] - \frac{q_2}{2}\left[\,\phi'_{f1}(\pm 1) - \phi'_{p1}(\pm 1)\,\right] \qquad (60)$$

$$\delta_1 = -\,i\alpha\bar{\rho}\left[-\tfrac{1}{2}\phi'_{p1}(\pm 1) + K\left(\frac{\phi'_{f1}}{M} + \phi_{p1}(\pm 1)\right)\right] + \frac{q_2}{2}\left[\phi'_{f1}(\pm 1) - \phi'_{p1}(\pm 1)\right] \qquad (61)$$

Also equations (51) - (55) can be satisfied by the solution of the form

$$\left.\begin{aligned} \Psi_{f_2} &= \tfrac{1}{2}\left[\phi_{f_{20}} + \phi_{f_{22}}\, e^{2i\alpha(x-t)} + \phi^*_{f_{22}}\, e^{-2i\alpha(x-t)} \right] \\ \Psi_{p_2} &= \tfrac{1}{2}\left[\phi_{p_{20}} + \phi_{p_{22}}\, e^{2i\alpha(x-t)} + \phi^*_{p_{22}}\, e^{-2i\alpha(x-t)} \right] \end{aligned}\right\} \tag{62}$$

giving

$$\phi'''_{f_{20}} = -\frac{i\alpha(1-C)R}{2}\left[\phi_{f_1}\phi^{*\prime\prime}_{f_1} - \phi^*_{f_1}\phi''_{f_1}\right]' + CM\left[\phi''_{f_{20}} - \phi''_{p_{20}}\right] \tag{63}$$

$$0 = -\frac{i\alpha CR}{2}\left[\phi_{p_1}\phi^{*\prime\prime}_{p_1} - \phi^*_{p_1}\phi''_{p_1}\right]' + CN\left[\phi''_{p_{20}} - \phi'_{f_{20}}\right] \tag{64}$$

$$\left[\frac{d^2}{dy^2} - 4\alpha^2 + 2i\alpha(1-C)R\right]\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\phi_{f_{22}} = 2i\alpha(1-C)K\alpha R \times$$

$$(1-y^2)\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\phi_{f_{22}} + 4i\alpha(1-C)KR\,\phi_{f_{22}} + \frac{i(1-C\alpha R}{2} \times$$

$$\left(\phi'_{f_1}\,\phi''_{f_1} - \phi_{f_1}\,\phi'''_{f_1}\right) + CM\left(\frac{d^2}{dy} - 4\alpha^2\right)\left[\phi_{f_{22}} - \phi_{p_{22}}\right] \tag{65}$$

$$2i\alpha CR\left[1 - K\left(1 - y^2 + \frac{2}{M}\right)\right]\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\phi_{p_{22}} = \frac{i\alpha CR}{2}\left[\phi'_{p_1}\phi^{*\prime\prime}_{p_1} - \phi_{p_1}\phi^{*\prime\prime\prime}_{p_1}\right] +$$

$$+ CN\left(\frac{d^2}{dy^2} - 4\alpha^2\right)\left(\phi_{p_{22}} - \phi_{f_{22}}\right) + 4i\alpha KRC\phi_{p_{22}} \tag{66}$$

with boundary conditions

$$\phi'_{f_{20}}(\pm 1) = K - \tfrac{1}{2}\left[\phi''_{f_1}(\pm 1) + \phi^{*\prime\prime}_{f_1}(\pm 1)\right] \tag{67}$$

$$\phi'_{f_{22}}(\pm 1) + \tfrac{1}{2}\phi''_{f_1}(\pm 1) - \frac{K}{2} = 0 \tag{68}$$

$$\phi'_{p_{22}}(\pm1) + \tfrac{1}{2}\phi''_{p_1}(\pm1) - \frac{K}{2} = 0 \qquad (69)$$

$$\frac{1}{R}\phi'''_{f_{20}}(\pm1) + \frac{(1-C)}{2}i\alpha\left[\phi_{f_1}(\pm1)\,\phi^{*\prime\prime}_{f_1}(\pm1) - \phi^{*}_{f_1}(\pm1)\,\phi''_{f_1}(\pm1)\right] +$$

$$+ q_2\left[\phi'_{p_{20}}(\pm1) - \phi'_{f_{20}}(\pm1)\right] = 0 \qquad (70)$$

$$\frac{1}{2R}\left[\phi'''_{f_{22}}(\pm1) - 4\alpha^2\phi'_{f_{22}}(\pm1)\right] - i\alpha(1-C)\left[\left(1-\frac{2K}{M}\right)\phi'_{f_{22}}(\pm1) +\right.$$

$$\left. + \frac{1}{4}\left(\phi'^{2}_{f_1}(\pm1) - \phi_{f_1}(\pm1)\,\phi''_{f_1}(\pm1)\right) + 2K\phi_{f_{22}}(\pm1)\right] +$$

$$+ \frac{q_2}{2}\left[\phi_{p_{22}}(\pm1) - \phi'_{f_{22}}(\pm1)\right] = 0 \qquad (71)$$

$$- i\alpha\bar{\rho}C\left[\tfrac{1}{2}\left(\phi^{*\prime}_{f_1}(\pm1)\,\phi'_{f_1}(\pm1) - \phi'_{p_1}(\pm1)\,\phi'_{f_1}(\pm1) + \phi_{p_1}(\pm1)\,\phi^{*\prime\prime}_{f_1}(\pm1)\right.\right.$$

$$\left.\left. + \phi^{*}_{p_1}(\pm1)\,\phi''_{p_1}(\pm1)\,\phi''_{p_1}(\pm1)\right)\right] + q_2\left[\phi'_{f_{20}}(\pm1) - \phi'_{p_{20}}(\pm1)\right]$$

$$= 0 \qquad (72)$$

$$- i\alpha\bar{\rho}C\left[-\tfrac{1}{2}\phi'_{f_{22}}(\pm1) + \frac{2K}{M}\phi_{p_{22}} + \frac{1}{4}\left(\phi'_{p_1}(\pm1)\phi'_{f_1}(\pm1) -\right.\right.$$

$$\left.\left. - \phi_{p_1}(\pm1)\phi''_{f_1}(\pm1) + 2K\phi_{p_{22}}\right)\right] + \frac{q_2}{2}\left[\phi'_{f_{22}}(\pm1) - \phi'_{p_{22}}(\pm1)\right]$$

$$= 0 \qquad (73)$$

Note that in equation (49) downwards δ_1 is given by

$$\left.\begin{aligned} \delta_1 &= \frac{iK_3\alpha^3}{R} + \frac{K_2\alpha^2}{R} - im\,\alpha^3 \text{ (membrane)} \\ \delta_1 &= \frac{iK_4\alpha^5}{R^2} + \frac{K_2\alpha^2}{R} - im\,\alpha^3 \text{ (thin plate)} \end{aligned}\right\} \qquad (74)$$

3.4 Pure Peristalsis

Pure peristalsis occurs when K = 0; this has been termed "free pumping" by Fung and Yik. In this case the Poiseuille

flow is zero and the first order equations reduce to

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i\alpha(1-C)R\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f_1} = CM\left(\frac{d^2}{dy^2} - \alpha^2\right)\times$$

$$(\phi_{f_1} - \phi_{p_1}) \qquad (75)$$

$$i\alpha CR\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{p_1} = CM\left(\frac{d^2}{dy^2} - \alpha^2\right)(\phi_{p_1} - \phi_{f_1}) \qquad (76)$$

$$\left.\begin{aligned} \phi'_{f_1}(\pm 1) &= 0 \\ \phi'_{p_1}(\pm 1) &= -\frac{2i\delta_1}{\bar{\rho}q_2(1+\alpha)} \\ \phi'''_{f_1}(\pm 1) &= 2R(1-C)\delta_1 \end{aligned}\right\} \qquad (77)$$

Solving (75) and (76) subject to (77) the solutions are

$$\phi_{f_1}(y) = A_{11}\sinh\alpha y + B_{11}\sinh\beta y \qquad (78)$$

$$\phi_{p_1}(y) = C_{11}\sinh\alpha y + E_{11}\sinh\beta y \qquad (79)$$

where $$\beta = \left[\alpha^2 - i\alpha(1-C)R + CM\left(1 + \frac{N}{i\alpha R - N}\right)\right]^{\frac{1}{2}} \qquad (80)$$

$$A_{11} = -\frac{2i\delta_1}{T_1^2\alpha^2\cosh\alpha} \qquad (81)$$

$$B_{11} = \frac{2i\delta_1}{T_1^2\alpha\beta\cosh\beta} \qquad (82)$$

$$C_{11} = -\frac{1}{\alpha\cosh\alpha}\left[\frac{2i\delta_1}{\alpha\beta(1+\alpha)} + E_{11}\beta\cosh\beta\right] \qquad (83)$$

$$E_{11} = \frac{B_{11}N}{N - i\alpha R} \qquad (84)$$

$$T_1^2 = 1 + \frac{CM}{(1-C)(N-i\alpha R)} \qquad (85)$$

Eliminating ϕ_{p20} from equations (63) and (64)

$$\phi'''_{f20} = -\frac{i\alpha R}{2}\left[(1-C)(\phi_{f1}\phi''^{*}_{f1} - \phi^{*}_{f1}\phi''_{f1}) + \frac{CM}{N}(\phi_{p1}\phi''^{*}_{p1} - \phi^{*}_{p1}\phi''_{p1})\right] \quad (86)$$

Integrating thrice

$$\phi'_{f20} = F(y) + C_1 y^2 + C_2 y + C_3 \quad (87)$$

where $F(y) = \alpha^2\frac{(1-C)^2R^2}{2}\left[\frac{T_1^2}{2}\left[A^*_{11} B_{11} + \frac{CM}{N(1-C)} C^*_{11}E_{11}\right] X \right.$

$$\left[\frac{\cosh(\alpha+\beta)y}{(\alpha+\beta)^2} - \frac{\cosh(\alpha-\beta)y}{(\alpha-\beta)^2} \right] + \frac{T_1^{*2}}{2}\left[A_{11} B^*_{11} + \right.$$

$$\left. + \frac{CM}{N(1-C)}\right]\left[\frac{\cosh(\alpha+\beta^*)y}{(\alpha+\beta^*)^2} - \frac{\cosh(\alpha-\beta^*)y}{(\alpha-\beta^*)^2} \right] +$$

$$+ \frac{(T^2+T_1^*)}{2}\left[B_{11} B^*_{11} + \frac{CM}{N(1-C)} E_{11} E^*_{11}\right]\left[\frac{\cosh(\beta+\beta^*)y}{(\beta+\beta^*)^2} - \right.$$

$$\left.\left. - \frac{\cosh(\beta-\beta^*)y}{(\beta-\beta^*)^2} \right]\right] \quad (88)$$

Since we are interested in the real part of $F(y)$ then it can be written as

$$F(y) = \alpha^2R^2(1-C)^2 Re\left[\frac{T_1^2}{2}\left[A^*_{11} B_{11} + \frac{CM}{N(1-C)} C^*_{11} E_{11}\right]\left[\frac{\cosh(\alpha+\beta)y}{(\alpha+\beta)^2} - \right.\right.$$

$$\left.\left. \frac{\cosh(\alpha-\beta)y}{(\alpha-\beta)^2}\right]\right] + \alpha^2R^2(1-C)^2\frac{(T_1^2+T_1^{*2})}{4}\left[B_{11} B^*_{11} + \frac{CM}{N(1-C)} E_{11}E^*_{11}\right] X$$

$$\left[\frac{\cosh(\beta+\beta^*)y}{(\beta+\beta^*)^2} - \frac{\cosh(\beta-\beta^*)y}{(\beta-\beta^*)^2} \right] \quad (89)$$

$$C_1 = -\frac{i\alpha^2CR}{4}\left[i\alpha R(1-C)(\bar{\rho} - \frac{M}{N})\left[2R_eT_1^2 C^*_{11} E_{11} \sinh\alpha \sinh\beta \right.\right.$$

$$\left.\left. \sinh\beta + (T_1^2 + T_1^{*2})E_{11}E^*_{11} \sinh\beta \sin\beta^*\right] +\right.$$

$$+ \bar{\rho}\left[\alpha^2 C_{11} A^*_{11} \cosh^2\alpha + \alpha\beta^* C_{11} B^*_{11} \cosh\alpha \cosh\beta^* + \right.$$

$$\left. + \alpha\beta A^*_{11} E_{11} \cosh\alpha \cosh\beta + \beta\beta^* B^*_{11} E_{11} \cosh\beta \cosh\beta^*\right]\Big] \quad (90)$$

To determine D, the boundary condition demands that

$$\phi'_{f_{20}}(\pm 1) = D$$

$$\therefore \quad D = -\tfrac{1}{2}\left[\phi'_{f_1}(\pm 1) + \phi^{*\prime}_{f_1}(\pm 1)\right] \quad (91)$$

$$= -\tfrac{1}{2}\left[(A_{11} + A^*_{11})\alpha^2 \sinh\alpha + B_{11}\beta^2 \sinh\beta + B^*_{11}\beta^{*2}\sinh\beta^*\right]$$

$$D = -\tfrac{1}{2}\left[(A^*_{11} + A_{11})\alpha^2 \sinh\alpha + 2R_e B\,\beta^2 \sinh\beta\right] \quad (92)$$

At this point using (87)

$$\left.\begin{aligned} D &= F(1) + C_1 + C_2 + C_3 \\ D &= F(1) + C_1 - C_2 + C_3 \end{aligned}\right\} \quad (93)$$

$$C_3 = D - F(1) - C_1, \quad C_2 = 0 \quad (94)$$

$$\therefore \quad \phi'_{f_{20}}(y) = F(y) - F(1) + D - C_1(1-y^2) \quad (95)$$

To calculate $\phi'_{p_{20}}$, equation (64) is used therefore

$$\phi'_{p_{20}} = \frac{i\alpha R}{2N}\left[\phi_{p_1}\phi^{*\prime}_{p_1} - \phi^*_{p_1}\phi'_{p_1}\right]' + \phi'_{f_{20}} \quad (96)$$

Integrating once

$$\phi'_{p_{20}} = G(y) + H_1 \quad (97)$$

subject to $\phi'_{p_{20}}(\pm 1) = \frac{iC\bar{\rho}}{2}\left[\phi'_{p_1}(\pm 1)\,\phi'_{f_1}(\pm 1) + \phi_{p_1}(\pm 1)\,X\right.$

$$\left.\phi^{*\prime}_{p}(\pm 1) - \phi^*_{p_1}(\pm 1)\,\phi'_{p_1}(\pm 1)\right]$$

$$+ \phi'_{f_{20}}(\pm 1) \quad (98)$$

where, $G(y) = I_1(y) + F(y) - F(1) + D - C_1(1 - y^2)$ (99)

$$I_1(y) = \frac{i\alpha R}{2N}\left[(\alpha^2-\beta^2)C^*_{11}E_{11}\sinh\alpha y\sinh\beta y + (\beta^{*2}-\alpha^2)C_{11}E^*_{11}\times\right.$$

$$\left.\sinh\alpha y\sinh\beta^* y + (\beta^{*2}-\beta^2)E_{11}E^*_{11}\sinh\beta y\sinh\beta^* y\right] \quad (100)$$

However,

$$\phi'_{p20}(\pm 1) = \frac{i\bar{\rho}C}{2}\left[\frac{NI_1(1)}{\alpha R} + I(1)\right] + D \equiv D_1 \quad (101)$$

Where

$$I(y) = \alpha^2 A^*_{11}C_{11}\cosh^2\alpha y + C_{11}B^*_{11}\alpha\beta^*\cosh\alpha y\cosh\beta^* y + \alpha\beta E_{11}A^*_{11}\cosh\alpha y\cosh\beta y + \beta\beta^* E_{11}B^*_{11}\cosh\beta y \times \cosh\beta^* y \quad (102)$$

Using (97) and (101)

$$H_1 = D_1 - G(1) \quad (103)$$

Therefore (97) becomes

$$\phi'_{p20} = G(y) - G(1) + D_1 \quad (104)$$

The peristaltic mean flow are obtained as

$$\bar{U}_f(y) = \frac{\varepsilon^2}{2}\phi'_{f20} = \frac{\varepsilon^2}{2}\left[F(y) - F(1) + D - C_1(1-y^2)\right] \quad (105)$$

$$\bar{U}_p(y) = \frac{\varepsilon^2}{2}\phi'_{p20} = \frac{\varepsilon^2}{2}\left[G(y) - G(1) + D_1\right] \quad (106)$$

The solutions formally coincide with that of Fung and Yik however, the results differ in many respects and are discussed below. Interestingly enough, when the equation of motion of the wall is satisfied C_1 is determined completely. In the cited reference C_1 remained arbitrary and its determination was attributed to the considerations of conditions at the ends of the channel.

3.5 Elastic Wall ($K_2 = 0$)

In order to clearly see the role played by the properties of the wall, we consider zero damping, $K_2 = 0$ and obtain

$$A_{11} = \frac{2\alpha\delta}{T^2R^2\cosh\alpha} \tag{107}$$

$$B_{11} = -\frac{2\alpha^2\delta}{T^2R^2\beta\cosh\beta} \tag{108}$$

$$C_{11} = -\frac{1}{\alpha\cosh\alpha}\left[-\frac{2\alpha^2\delta}{(1+\alpha)\beta R^2} + E_{11}\beta\cosh\beta\right] \tag{109}$$

and C_1 same as (90) with A_{11}, B_{11}, C_{11} defined by equation (107 - 109), where $\delta = K_3 - mR^2$(membrane) .

The quantities D and D_1 simplify to

$$D = -\left[\frac{\delta\alpha^3}{T^2T^{*2}R^2}\tanh\alpha\,(T^2+T^{*2}) + R_eB_{11}\beta^2\sinh\beta\right] \tag{110}$$

$$D_1 = \frac{i\bar{\rho}C}{2}\left[\frac{NI_1(1)}{\alpha R} + I(1)\right] + D \tag{111}$$

The mean velocity distributions are given by the following expression in which F(y), (G(y) are obtained from (89) and (99) respectively

$$\bar{U}_f(y) = \frac{\epsilon^2}{2}\left[F(y) - F(1) + D - C_1(1-y^2)\right] \tag{112}$$

$$\bar{U}_p(y) = \frac{\epsilon^2}{2}\left[G(y) - G(1) + D_1\right] \tag{113}$$

In order to investigate flow reversal we first consider $R < 1$ so that explicity expressions for D, D_1 and $F(0) - F(1)$, $G(0) - G(1)$ may be obtained. When one considers the limit $R \to 0$, one finds that the differential equations are regular and yield finite solutions independent of R but the boundary conditions possess singularity. A finite solution can be obtained in the limit $R \to 0$ by considering expansion in powers of R A_{11} is complex and B_{11} may be expanded as

$$B_{11} = - \frac{A_{11}\alpha \cosh \alpha}{\beta \cosh \beta}$$

$$= - A_{11} \left[1 + \frac{iRLb_1}{2} + R^2 L^2 \left(a_1 - \frac{b_1^2}{4}\right) + \ldots \right] \quad (114)$$

noting that

$$\beta = \alpha - \frac{iRL}{2} + \frac{R^2L^2}{8\alpha} + \quad (115)$$

$$\left.\begin{aligned} \cosh \beta &= \cosh \alpha - \frac{iRL}{2} \sinh \alpha + \frac{R^2L^2}{8\alpha}(\sinh \alpha - \alpha \cosh \alpha) \\ \sinh \beta &= \sinh \alpha - \frac{iRL}{2}\cosh \alpha + \frac{R^2L^2}{2} (\cosh \alpha - \alpha \sinh \alpha) \end{aligned}\right\} \quad (116)$$

$$\left.\begin{aligned} \text{where } a_1 &= \frac{1}{8} \left(1 + \frac{\tanh \alpha}{\alpha} - \frac{1}{\alpha^2}\right) \\ b_1 &= \tanh \alpha + \frac{1}{\alpha} \end{aligned}\right\} \quad (117)$$

Estimating other quantities recoded in D, D_1, C_{11}, C_1, $I_1(\pm 1)$, $I(\pm 1)$ are given below

$$B_{11}\beta^2 \sinh \beta = - A_{11}\alpha^2 \sinh \alpha \left[1 - iRL\left(d_1 - \frac{b_1}{2}\right) + R^2L^2\left(a_1 - \frac{b_1^2}{4} + \frac{b_1 d_1}{2} - c_1\right)\right] \quad (118)$$

where

$$\left.\begin{aligned} c_1 &= \frac{1}{8}\left(1 + \frac{3 \coth\alpha}{\alpha}\right) \\ d_1 &= \frac{1}{\alpha} + \frac{\coth\alpha}{2} \end{aligned}\right\} \quad (119)$$

$$E_{11}\beta \cosh \beta = - A_{11}\alpha^3 \sinh \alpha \cosh \alpha \left[1 - \frac{iRL}{2}\left[1 + \tanh\alpha + 2\left(d_1 - \frac{b_1}{2} - \frac{\alpha}{NL}\right)\right] - \frac{R^2L^2}{8\alpha}\left[3 \tanh \alpha - \alpha - \frac{1}{\alpha} - \frac{8\alpha^2}{NL}\left(d_1 - \frac{b_1}{2} + \frac{1}{2}\right) - 8\alpha\left[a_1 - \frac{b_1}{4} + \frac{b_1 d_1}{2} - c_1 + \frac{1}{2L}(1+\tanh \alpha)\left(d_1 - \frac{b_1}{2}\right)\right] + \frac{8\alpha^3}{N^2L^2}\right]\right] \quad (120)$$

$$E_{11}\,E_{11}^{*}\sinh\beta\,\sinh\beta^{*} = A_{11}\,A_{11}^{*}\left[\sinh^2\alpha + R^2\left[\frac{L^2}{4\alpha}\left\{\sinh\alpha\right.\right.\right.$$

$$\left.\left.\left.\times(\cosh\alpha - \alpha\sinh\alpha) - \cosh^2\alpha\right\} + \sinh^2\alpha\left\{2L^2\left(a_1 - \frac{b_1^2}{8}\right) - \frac{\alpha^2}{N^2}\right\}\right]\right] \quad (121)$$

Then (109) becomes

$$C_{11} = -\frac{1}{\alpha\cosh\alpha}\left[-\frac{2\alpha\delta}{(1+\alpha)}\left(\frac{1}{R^2} + \frac{iL}{2\alpha R} - \frac{3L^2}{8\alpha^2}\right) - A_{11}\alpha^3\sinh\alpha\,\times\right.$$

$$\left.\cosh\alpha\left(1 - \frac{iRLb_4}{2} - \frac{R^2L^2b_5}{8\alpha}\right)\right] \quad (122)$$

where

$$\left.\begin{aligned} b_4 &= 1 + \tanh\alpha + 2\left(d_1 - \frac{b_1}{2} - \frac{\alpha}{NL}\right) \\ b_5 &= 3\tanh\alpha - \alpha - \frac{1}{\alpha} - \frac{8\alpha^2}{NL}\left(d_1 - \frac{b_1}{2} + \tfrac{1}{2}\right) + \frac{8\alpha^3}{N^2L^2} - \\ &\quad - 8\alpha\left[a_1 - \frac{b_1}{4} + \frac{b_1d_1}{2} - c_1 + \frac{1}{2L}(1+\tanh\alpha)\left(d_1 - \frac{b_1}{2}\right)\right] \end{aligned}\right\} \quad (123)$$

$$C_{11}^{*}\,E_{11}\sinh\beta = \frac{A_{11}}{\alpha\cosh\alpha}\left[-\frac{2\alpha\delta\sinh\alpha}{(1+\alpha)}\left(\frac{1}{R^2} - \frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right) - A_{11}\times\right.$$

$$\alpha^3\sinh^2\alpha\cosh\alpha + i\left\{\frac{\delta L\sinh\alpha}{(1+\alpha)R} - \frac{RLb_4}{2}A_{11}\alpha^3\sinh^2\alpha\cosh\alpha\right.$$

$$-\frac{2\alpha\delta b_3}{(1+\alpha)R} - \frac{RLb_4}{2}A_{11}\alpha^3\sinh^2\alpha\cosh\alpha - \frac{2\alpha\delta b_3}{(1+\alpha)R} - A_{11}\alpha^3Rb_3\times$$

$$\left.\left.\sinh\alpha\cosh\alpha\right\} + \frac{R^2L}{2}\left\{\frac{b_5\sinh\alpha}{4\alpha} + b_3b_4\right\}A_{11}\alpha^3\sinh\alpha\cosh\alpha\right] \quad (124)$$

where

$$\left.\begin{aligned} b_2 &= \frac{L^2}{8\alpha}(\cosh\alpha - \alpha\sinh\alpha) + \frac{L}{2}a_2\cosh\alpha + \\ &\quad \sinh\alpha\left[L^2\left(a_1 - \frac{b_1^2}{4}\right) - \frac{\alpha\,a_2}{N}\right] \\ b_3 &= a_2\sinh\alpha - \frac{L}{2}\cosh\alpha \\ a_2 &= \frac{\alpha}{N} + \frac{b_1L}{2} \end{aligned}\right\} \quad (125)$$

$$R_e C^*_{11} E_{11} \sinh\beta = \frac{A_{11}}{\alpha \cosh\alpha}\left[\frac{2\alpha\delta \sinh\alpha}{(1+\alpha)}\left(\frac{1}{R^2} - \frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right) - \right.$$

$$\left. A_{11}\alpha^3 \sinh^2\alpha \cosh\alpha + \frac{R^2L}{2}\left(b_5 \frac{\sinh\alpha}{4\alpha} + b_3 b_4\right) A_{11}\alpha^3 \sinh\alpha\right] \quad (126)$$

$$C_{11} B^*_{11}\beta^* \cosh\beta^* = A^*_{11}\left[-\frac{2\alpha\delta}{1+\alpha}\left(\frac{1}{R^2} - \frac{3L^2}{8\alpha^2} + \frac{L^2}{4\alpha}(b_1 + s_1) + V_1L^2\right)\right.$$

$$- A_{11}\alpha^3 \sinh\alpha \cosh\alpha + iL\left[-\frac{2\alpha\delta}{1\pm\alpha}\left(\frac{1}{2\alpha R} - \frac{1}{2R}(b_1+s_1) + \frac{RL^2}{16\alpha^2} \times\right.\right.$$

$$\left.\left.(b_1 + S_1) + \frac{L^2V_1R}{2\alpha}\right) + A_{11}\alpha^3 \sinh\alpha \cosh\alpha \left(\frac{Rb_4}{2} + \frac{R}{2}(b_1+s_1)\right)\right]$$

$$\left. + R^2L^2\left[A_{11}\alpha^3 \sinh\alpha \cosh\alpha\left(\frac{b_5}{8\alpha} + \frac{b_4}{4}(b_1 + s_1) - V_1\right) + \frac{3\delta L^2V_1}{4\alpha(1+\alpha)}\right]\right] \quad (127)$$

$$E_{11}\beta\beta^* B^*_{11} \cosh\beta \cosh\beta^* = \alpha^2 A_{11}A^*_{11}\cosh^2\alpha \left[1 + iR\left(d_2 - \frac{L}{2}(b_1 + s_1)\right)\right.$$

$$\left. + R^2\left[V_2 + L^2V_1 + \frac{L}{2}(b_1 + s_1)d_2\right]\right] \quad (128)$$

Where

$$\left.\begin{aligned} S_1 &= 1 + \tanh\alpha \\ S_2 &= 3\tanh\alpha - \alpha - \frac{1}{\alpha} \\ V_1 &= d_1 - \frac{b_1^2}{4} - \frac{b_1 s_1}{4} - \frac{s_2}{8\alpha} \\ d_2 &= \frac{\alpha}{N} - \frac{L s_1}{2} + \frac{Lb_1}{2} \\ V_2 &= \frac{\alpha S_1 L}{2N} - \frac{\alpha^2}{N^2} - \frac{L^2 C_2}{8\alpha} + \frac{Lb_1}{2}\left(\frac{LS_1}{2} - \frac{\alpha}{N}\right) + L^2\left(a_1 - \frac{b_1^2}{4}\right) \end{aligned}\right\} \quad (129)$$

Using (114) - (119) equation (110) turns out to be

$$D = -\left[\left(\frac{T_1^{*2} + T_1^2}{2T_1^{*2}} - 1\right) - R^2L^2\left(a_1 - \frac{b_1}{4} + \frac{b_1d_1}{2} - C_1\right)\right]A_{11}\alpha^2 \sinh\alpha \quad (130)$$

While (121) - (126) give

$$I_1(\pm 1) = -\frac{\alpha^2(L-C)R}{2N}\left[\frac{2A_{11}T_1^2 \sinh\alpha}{\alpha\cosh\alpha}\left[-\frac{2\alpha\delta \sinh\alpha}{1+\alpha}\left(\frac{1}{R^2} - \frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right)\right.\right.$$

$$\left.\left. - A_{11}\alpha^3 \sinh^2\alpha \cosh\alpha + \frac{R^2L}{2}\left(b_5\frac{\sinh\alpha}{4\alpha} + b_3b_4\right) A_{11}\alpha^3 \sinh\alpha \cosh\alpha\right] +\right.$$

$$+ A_{11} A_{11}^{*}(T_1^2 + T_1^{*2})\Big[\sinh^2\alpha + R^2\Big[\frac{L}{4\alpha}\Big(\sinh\alpha\,(\cosh\alpha - \alpha\sinh\alpha) - \cosh^2\alpha\Big) + \sinh^2\alpha\Big(2L^2\Big(a_1 - \frac{b_1^2}{8}\Big) - \frac{\alpha^2}{N^2}\Big)\Big]\Big]\Big] \qquad (131)$$

and (127) - (129)

$$I(\pm 1) = - A_{11}^{*}\alpha\cosh\alpha\Big[- \frac{2\alpha\delta}{1+\alpha}\Big(\frac{1}{R^2} + \frac{iL}{2\alpha R} - \frac{3L^2}{8\alpha^2}\Big) - A_{11}\alpha^3 \times$$

$$\sinh\alpha\cosh\alpha\Big(1 - \frac{iRLb_4}{2} - \frac{R^2L^2b_5}{8\alpha}\Big)\frac{2\alpha\delta}{1+\alpha}\Big(\frac{1}{R^2} - \frac{3L^2}{8\alpha^2} +$$

$$+ \frac{L^2}{4\alpha}\Big(b_1 + s_1) + V_1L^2\Big) - A_{11}\alpha^3\sinh\alpha\cosh\alpha$$

$$+ iL\Big[- \frac{2\alpha\delta}{1+\alpha}\Big\{\frac{1}{2\alpha R} - \frac{L}{2R}(b_1 + s_1) + \frac{RL^2}{16\alpha^2}(b_1 + s_1) + \frac{L^2V_1R}{2\alpha}\Big\} +$$

$$\frac{RA_{11}\alpha^3\sinh\alpha}{2}\cosh\alpha\Big[b_4 + b_1 + s_1\Big]\Big]$$

$$+ R^2L^2\Big[A_{11}\alpha^3\sinh\alpha\cosh\alpha\Big(\frac{b_5}{8\alpha} + \frac{b_4}{4}(b_1 + s_1) - V_1\Big) + \frac{3\delta L^2V_1}{4\alpha(1+\alpha)}\Big]$$

$$- A_{11}\alpha\cosh\alpha\Big[R^2V_1L^2 + \frac{R^2L}{2}(b_1 + s_1)d_2 - \frac{iRL}{2}(b_1 + s_1)\Big]\Big] \qquad (132)$$

and

$$C_1 = \frac{i\alpha RC}{4}\Big[i\alpha R(1-C)\Big(\bar{\rho} - \frac{M}{N}\Big)\Big\{\frac{2A_{11}T_1^2}{\alpha}\tanh\alpha\Big[- \frac{2\alpha\delta\sinh\alpha}{1+\alpha} \times$$

$$\Big(\frac{1}{R} - \frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\Big) - A_{11}\alpha^3\sinh^2\alpha\cosh\alpha + \frac{R^2L}{2}\Big(b_5\frac{\sinh\alpha}{4\alpha}$$

$$+ b_3b_4\Big)A_{11}\alpha^3\sinh\alpha\cosh\alpha\Big]$$

$$+ (T_1^2 + T_1^{*2})A_{11}A_{11}^{*}\Big[\sinh^2\alpha + R^2\Big[\frac{L^2}{4\alpha}(\sinh\alpha\,(\cosh\alpha - \alpha\sinh\alpha) - \cosh^2\alpha)$$

$$+ \sinh^2\alpha\Big(L^2\Big(k_1^2 - 2k_2\Big) - \frac{\alpha^2}{N^2}\Big)\Big]\Big]\Big\} + \bar{\rho}\Big\{-A_{11}^{*}\alpha\cosh\alpha\Big[-\frac{2\alpha\delta}{1+\alpha}\Big(\frac{1}{R^2} + \frac{iL}{2\alpha R} - \frac{3L^2}{8\alpha^2}\Big)$$

$$- A_{11}\alpha^3\sinh\alpha\cosh\alpha\Big(1 - \frac{iRLb_4}{2} - \frac{R^2L^2b_5}{8\alpha}\Big)\Big] + \alpha A_{11}^{*}\cosh\alpha\Big[-\frac{2\alpha\delta}{1+\alpha}\Big(\frac{1}{R^2} - \frac{3L^2}{8\alpha^2} + \frac{L^2}{4\alpha} \times$$

$$(b_1 + s_1) + v_1 L^2) - A_{11}\alpha^3 \sinh\alpha\cosh\alpha + iL\left[-\frac{2\alpha\delta}{1+\alpha}\left(\frac{1}{2\alpha R} - \frac{1}{2R}(b_1+s_1) + \frac{RL^2}{16\alpha^2}(b_1+S_1)\right.\right.$$

$$\left.\left. + \frac{L^2 V_1 R}{2\alpha}\right) + \frac{RA_{11}\alpha^3 \sinh\alpha\cosh\alpha}{2}\left(b_4+b_1+S_1\right)\right] + R^2L^2\left[A_{11}\alpha^3\sinh\alpha\cosh\alpha \times\right.$$

$$\left.\left(\frac{b_5}{8\alpha} + \frac{b_4}{4}(b_1+s_1) - V_1\right) + \frac{3\delta L^2 V_1}{4\alpha(1+\alpha}\right]\right]$$

$$+ \alpha^2 A_{11}A_{11}^*\cosh^2\alpha\left[1+iR\left(d_2 - \frac{L}{2}(b_1+s_1)\right) + R^2\left(V_2 + L^2V_1 + \frac{L}{2}(b_1+s_1)d_2\right)\right]$$

$$- A_{11}A_{11}^*\alpha^4 \sinh\alpha\cosh^2\alpha\left(1 - \frac{iRLb_4}{2} - \frac{R^2L^2b_5}{8\alpha}\right)\Big\}\Big] \qquad (133)$$

where

$$\left.\begin{aligned} k_1 &= -\tfrac{1}{2}\left(\tanh\alpha + \frac{1}{\alpha}\right) \\ k_2 &= k_1^2 + \frac{1}{8\alpha^2} - \frac{\tanh\alpha}{4\alpha} - \frac{1}{8} \end{aligned}\right\} \qquad (134)$$

Taking the limit as $R \to 0$, equations (130), (111) reduce to

$$D = \frac{\delta\alpha^3}{4T_1^2}\left[-\frac{1}{\alpha} + \left(2 + \frac{1}{\alpha^2}\right)\tanh\alpha + \frac{\tanh^2\alpha}{\alpha} - 2\tanh^3\alpha\right] \qquad (135)$$

$$D_1 = -\frac{3i\alpha V_2\bar{\rho}CL^4\delta^2}{4(1+\alpha)T_1^{*2}} + D \qquad (136)$$

While (133) and (103) respectively become

$$C_1 = \frac{i\alpha^2C\delta^2}{2}\left[i\alpha(1-C)\left(\bar{\rho} - \frac{M}{N}\right)\left[4\frac{\tanh\alpha}{\cosh\alpha}\left(-\frac{\sinh\alpha}{1+\alpha}\left[-\frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right] - \frac{\alpha^3\sinh^2\alpha}{T_1^2}\right)\right.\right.$$

$$+ \frac{2(T_1^2+T_1^{*2})\alpha}{T_1^2\,T_1^{*2}\cosh^2\alpha}\left[\frac{L^2}{4\alpha}\left(\sinh\alpha\left[\cosh\alpha - \alpha\sinh\alpha\right] - \cosh^2\alpha\right) + \sinh^2\alpha\left[L^2\right.\right.$$

$$\left.\left. + \sinh^2\alpha\left[L^2\left(K_1^2 - 2K_2\right) - \frac{\alpha^2}{N^2}\right]\right]\right] + \frac{\bar{\rho}iL^3}{2T_1^2\delta}\left(V_1 + \alpha_2\right)\Big]\Big] \qquad (137)$$

$$H_1 = -\frac{3i\alpha V_1\bar{\rho}CL^4\delta^2}{4(1+\alpha)T_1^2} + \frac{\alpha^3\delta^2(1-C)}{NT_1^2\cosh^2\alpha}\left[2\sinh\alpha\left[-\frac{2}{1+\alpha}\left(-\frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right)\right.\right.$$

$$\left. + \frac{\alpha^3 L}{T^2}\left(\frac{b_5\sinh\alpha}{4\alpha} + b_3b_4\right)\right] + \frac{(T_1^2+T_1^{*2})\alpha L^2}{2T_1^{*2}}\left[\cosh^2\alpha + 4\sinh^2\alpha\left(K_1^2 - 2K_2 - \frac{\alpha^2}{N^2L^2}\right)\right.$$

$$\left.\left. + \frac{\sinh\alpha}{\alpha}(\cosh\alpha - \alpha\sinh\alpha)\right]\right] \qquad (138)$$

The constant terms of velocity are proportional to δ and that D_2, $D_1 > 0$. The mean flow reversals if exist therefore depend upon $F(0) - F(1)$ and $G(0) - G(1)$, and in this case take place at the centre of the channel. To really compute these values, one needs to go back to equations (89) and (99)

$$F(0) = \alpha^2(1-C)^2R^2R_e\left[T_1^2\left(A_{11}^*B_{11} + \frac{CM}{N(1-C)}C_{11}^*E_{11}\right)\left(\frac{1}{(\alpha+\beta)^2} - \frac{1}{(\alpha-\beta)^2}\right)\right]$$

$$+ \alpha^2R^2\frac{(1-C)^2(T_1^2+T_1^{*2})}{4}\left[B_{11}B_{11}^* + \frac{CM}{N(1-C)}E_{11}E_{11}^*\right]\left[\frac{1}{(\beta+\beta^*)^2} - \frac{1}{(\beta-\beta^*)^2}\right] \quad (139)$$

$$F(1) = \alpha^2R^2(1-C)^2R_e\left[\frac{T_1^2}{2}\left(A_{11}^*B_{11} + \frac{CM}{N(1-C)}C_{11}^*E_{11}\right)\left[\frac{\cosh(\alpha+\beta)}{(\alpha+\beta)^2} - \frac{\cosh(\alpha-\beta)}{(\alpha-\beta)^2}\right]\right]$$

$$+ \alpha^2\frac{R^2(1-C)^2(T_1^2+T_1^{*2})}{4}\left[B_{11}B_{11}^* + \frac{CM}{N(1-C)}E_{11}E_{11}^*\right]\left[\frac{\cosh(\beta+\beta^*)}{(\beta+\beta^*)^2} - \frac{\cosh(\beta-\beta^*)}{(\beta-\beta^*)^2}\right] \quad (140)$$

On using

$$\frac{\cosh(\alpha+\beta)}{(\alpha+\beta)^2} = \frac{1}{2} + \frac{1}{(\alpha+\beta)^2} + \frac{(\alpha+\beta)^2}{4!} + \frac{(\alpha+\beta)^4}{6!} + \frac{(\alpha+\beta)^6}{8!} + \ldots \quad (141)$$

etc.

$$F(0) - F(1) = -\frac{\alpha^2R^2(1-C)^2T_1^2}{4}\left[R_e\left[A_{11}^*B_{11} + \frac{CM}{N(1-C)}C_{11}^*E_{11}\right]\left[(\alpha+\beta)^2 - (\alpha-\beta)^2\right] + \frac{(T_1^2+T_1^{*2})}{2}\left[B_{11}B_{11}^* + \frac{CM}{N(1-C)}E_{11}E_{11}^*\right]\left[(\beta+\beta^*)^2 - (\beta-\beta^*)^2\right]\right]$$

$$+ \alpha^2\frac{R^2(1-C)T_1^2}{2}\left[R_e\left[A_{11}^*B_{11} + \frac{CM}{N(1-C)}C_{11}^*E_{11}\right]\left[\frac{(\alpha+\beta)^4}{6!} + \frac{(\alpha+\beta)^6}{8!} + \ldots\right]\right.$$

$$\left. + \frac{(T^2+T^{*2})}{2}\left[B_{11}B_{11}^* + \frac{CM}{N(1-C)}E_{11}E_{11}^*\right]\left[\frac{(\beta+\beta^*)^4}{6!} + \frac{(\beta+\beta^*)^6}{8!} + \ldots\right]\right] \quad (142)$$

in the above certain terms which involve order of R more than R^4 have been neglected.

R

Noting the following

$$R_e\ A_{11}^*\ B_{11}(\alpha+\beta)^n = A_{11}A_{11}^*(2\alpha)^n\left[-1+R^2L^2\left[K_2-K_1\frac{n\alpha_1}{2\alpha}-\left(\frac{n\alpha_2}{2\alpha}-\right.\right.\right.$$
$$\left.\left.\left.-\frac{n(n-1)\alpha_1^2}{8\alpha^2}\right)\right]+\ldots\right] \quad (143)$$

$$R_e C_{11}^*(\alpha+\beta)^n = \frac{A_{11}(2\alpha)^n}{\alpha\cosh\alpha}\left[M_1+R^2\left[A_{11}M_3+\frac{3\delta L^2M_2}{4\alpha(1+\alpha)}+\left(\frac{nL^2\alpha_2}{2\alpha}+\right.\right.\right.$$
$$\left.\left.\left.+\frac{n(n-1)L^2\alpha_1^2}{8\alpha^2}\right)M_1\right]+\ldots\ldots\right] \quad (144)$$

$$\left.\begin{aligned} B_{11}B_{11}^*(\beta+\beta^*)^n &= A_{11}A_{11}^*(2\alpha)^n\left[1+R^2L^2\left(\frac{n\alpha_2}{\alpha}+K_1^2-2K_2\right)+\ldots\right] \\ E_{11}E_{11}^*(\beta+\beta^*)^n &= A_{11}A_{11}^*(2\alpha)^n\left[1+R^2\left[L^2(K_1^2-2K_2)-\frac{\alpha^2}{N^2}+\frac{n\alpha_2}{\alpha}\right]+\ldots\right] \end{aligned}\right\} \quad (145)$$

$$\left.\begin{aligned} R_e\ A_{11}\ B_{11}(\alpha-\beta)^2 &= A_{11}^2(2\alpha)^2\frac{\alpha_2}{2\alpha}R^2L^2 \\ R_e\ C_{11}^*\ E_{11}(\alpha-\beta)^2 &= \frac{A_{11}}{\alpha\cosh\alpha}(2\alpha)^2\frac{\alpha_2}{2\alpha}M_1R^2 \end{aligned}\right\} \quad (146)$$

$$\left.\begin{aligned} B_{11}\ B_{11}^*(\beta-\beta^*)^2 &= -(2\alpha)^2\ A_{11}\ A_{11}^*\ \frac{R^2L^2}{(2\alpha)^2} \\ E_{11}\ E_{11}^*(\beta-\beta^*)^2 &= -(2\alpha)^2\ A_{11}\ A_{11}^*\ \frac{R^2L^2}{(2\alpha)^2} \end{aligned}\right\} \quad (147)$$

where

$$\left.\begin{aligned} \alpha_1 &= -\tfrac{1}{3},\quad \alpha_2 = \frac{1}{8\alpha} \\ M_1 &= -\frac{2\alpha\delta}{1+\alpha}\left[\frac{1}{R^2}-\frac{3L^2}{8\alpha^2}+\frac{La_2}{2\alpha}+M_2\right] \end{aligned}\right\} \quad (148)$$

$$\left.\begin{aligned} M_2 &= L^2(a_1-K_1^2)-\frac{\alpha q_2}{N} \\ M_3 &= \alpha^2\sinh\alpha\cosh\alpha\left[\frac{L^2b_5}{\alpha_2}+\frac{Lb_4a_2}{2}-M_2\right] \end{aligned}\right\} \quad (149)$$

Substituting equations (143) - (149) into (142)

$$F(0)-F(1) = -\frac{\alpha^2R^2(1-C)^2T_1^2}{48}\left[A_{11}A_{11}^*(2\alpha)^2\left[-1+R^2L^2\left[-\frac{K_1\alpha_1}{\alpha}-\left(\frac{3\alpha_2}{2\alpha}-\frac{1}{4}\frac{\alpha_1^2}{\alpha^2}\right)\right]\right]+\ldots\right.$$

$$+\frac{T_1^2+T_1^{*2}}{2}\left[1+R^2L^2\left[\frac{2\alpha_2}{\alpha}+K_1^2-2K_2+\frac{\alpha_1^2}{\alpha^2}\right]+\ldots\right]\Bigg]$$

$$+\frac{CM}{N(1-C)}\left[\frac{A_{11}(2\alpha)^2}{\alpha\cosh\alpha}\left[M_1+R^2\left[A_{11}M_3+\frac{3\delta L^2M_2}{4\alpha(1+\alpha)}+\right.\right.\right.$$

$$\left.\left.\left.+L^2M_1\left[\frac{\alpha_2}{\alpha}+\frac{\alpha_1^2}{4\alpha^2}-\frac{\alpha_2}{2\alpha L^2}\right]\right]\right]$$

$$+\frac{T_1^2+T_1^{*2}}{2}\left[A_{11}A_{11}^*(2\alpha)^2\left[1+R^2\left(\left[K_1^2-2K_2+\frac{\alpha_1^2}{\alpha^2}\right]L^2-\frac{\alpha^2}{N^2}+\frac{2\alpha_2}{\alpha}\right)\right]\right]$$

$$+\frac{\alpha^2R^2(1-C)^2T_1^2}{2}\sum_{n=4,6,8}\frac{(2\alpha)^n}{(n+2)!}\left[A_{11}A_{11}^*\left[-1+R^2L^2\left\{K_2-K_1\frac{n\alpha_1}{2\alpha}-\right.\right.\right.$$

$$\left.\left.\left.-\left(\frac{n\alpha_2}{2\alpha}-\frac{n(n-1)}{8}\frac{\alpha_1^2}{\alpha^2}\right)\right\}+\ldots.\right]\right]$$

$$+\frac{i}{2}(T_1^2+T_1^{*2})A_{11}A_{11}^*\left[1+R^2L^2\left(\frac{n\alpha_2}{\alpha}+K_1^2-2K_2\right)+\ldots..\right]$$

$$+\frac{CM}{N(1-C)}\left\{\frac{A_{11}}{\alpha\cosh\alpha}\left[M_1+R^2\left[A_{11}M_3+\frac{3L^2\delta M_2}{4\alpha(1+\alpha)}+L^2M_1\left[\frac{n\alpha_2}{2\alpha}+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\frac{n(n-1)}{8}\cdot\frac{\alpha_1^2}{\alpha^2}+\ldots..\right]\right]\right]$$

$$+\left(\frac{T_1^2+T_1^{*2}}{2}\right)A_{11}A_{11}^*\left[1+R^2\left\{L^2(K_1^2-2K_2)-\frac{\alpha^2}{N^2}+\frac{n\alpha_2}{\alpha}\right\}+\ldots..\right]\right]\Bigg\}\Bigg] \qquad \ldots.(150)$$

putting $A_{11}=\dfrac{2\alpha\delta}{R^2T^2\cosh\alpha}$ and its conjugate, and taking limit as $R\to 0$.

$$F(0)-F(1)=-\alpha^4\frac{(1-C)^2\delta^2}{2\cosh^2\alpha}\left[\frac{2\alpha}{3T_1^{*2}}\left[\left[-K_1\frac{\alpha_1}{\alpha}-\left(\frac{3\alpha_2}{2\alpha}-\frac{1}{4}\frac{\alpha_1^2}{\alpha^2}\right)+\right.\right.\right.$$

$$\left.\left.\left.\frac{T_1^2+T_1^{*2}}{2}\left[2\frac{\alpha_2}{\alpha}+K_1^2-2K_2+\frac{\alpha_1^2}{\alpha^2}\right]\right]L^2\right]+\frac{CM}{N(1-C)}\left[\frac{2\alpha}{3}\left[-\frac{1}{1+\alpha}\left[1-\right.\right.\right.\right.$$

$$\left.\left.\left.\left.L^2\left[\frac{\alpha_2}{\alpha}+\frac{\alpha_1^2}{4\alpha^2}-\frac{\alpha_2}{2L^2}\right]\right]\left[-\frac{3L^2}{8\alpha^2}+\frac{La_2}{2\alpha}+M_2\right]+\frac{M_3}{T_1^2\cosh\alpha}\right]+\right.$$

$$+\frac{T_1^2+T_1^{*2}}{2}\cdot\frac{2\alpha^2}{3}\left\{\left\{\left(K_1^2-2K_2+\frac{\alpha_1^2}{\alpha^2}\right)L^2-\frac{\alpha^2}{N^2}+\frac{2\alpha_2}{\alpha}\right\}\right]$$

$$+\sum_{n=4,6,8}\frac{(2\alpha)^n}{(n+2)!}\left[\frac{L^2}{T_1^{*2}}\left[K_2-K_1\frac{n\alpha_1}{2\alpha}-\left(\frac{n\alpha_2}{2\alpha}-n\frac{(n-1)}{8}\frac{\alpha_1^2}{\alpha^2}\right)+\left(\frac{T_1^2+T_1^{*2}}{2}\right)\left(\frac{n\alpha_2}{\alpha}+K_1^2-2K_2\right)\right]\right.$$

$$+\frac{CM}{N(1-C)}\left\{-\frac{1}{1+\alpha}\cdot\left[\frac{1}{\alpha}-L^2\left(\frac{n\alpha_2}{2\alpha}+\frac{n(n-1)\alpha_1^2}{8\alpha^2}\right)\right]\left[-\frac{3L^2}{8\alpha^2}+\frac{La_2}{2\alpha}+M_2\right]+\frac{M_3}{T_1^2}\cosh\alpha\right.$$

$$\left.\left.+\left(\frac{T_1^2+T_1^{*2}}{2T_1^{*2}}\right)\left[L^2\left(K_1^2-2K_2\right)-\frac{\alpha^2}{N^2}+\frac{n\alpha_2}{\alpha}\right]\right\}\right]\right] \qquad (151)$$

$$G(0)-G(1)=F(0)-F(1)-I_1(1) \qquad (152)$$

But

$$I_1(1)=-\alpha^2\frac{(1-C)R^2}{2N}\left[2\,R_eT_1^2C_{11}^*E_{11}\sinh\alpha\sinh\beta+\right.$$

$$\left.(T_1^2+T_1^{*2})\,E_{11}\,E_{11}^*\sinh\beta\sinh\beta^*\right] \qquad (153)$$

where

$$R_eC_{11}^*E_{11}\sinh\beta=\frac{A_{11}}{\alpha\cosh\alpha}\left[-\frac{2\alpha\delta\sinh\alpha}{1+\alpha}\left(\frac{1}{R^2}-\frac{3L^2}{8\alpha^2}+b_2+\frac{Lb_3}{2\alpha}\right)-A_{11}\alpha^3\sinh^2\alpha\times\right.$$

$$\left.\cosh\alpha+\frac{R^2L}{2}\left(\frac{b_5\sinh\alpha}{4\alpha}+b_3\,b_4\right)A_{11}\alpha^3\sinh\alpha\cosh\alpha\right] \qquad (154)$$

$$E_{11}E_{11}^*\sinh\beta\sinh\beta^*=A_{11}\,A_{11}^*\left[\sinh^2\alpha+\frac{R^2L^2}{4}\left[\cosh^2\alpha+4\sinh^2\alpha\times\right.\right.$$

$$\left.\left.\left(K_1^2-2K_2-\frac{\alpha^2}{N^2L^2}\right)+\frac{\sinh\alpha}{\alpha}(\cosh\alpha-\alpha\sinh\alpha)\right]\right] \qquad (155)$$

substituting (153) into (152) and taking limit as $R\to 0$

$$G(0)-G(1)=F(0)-F(1)+\frac{\alpha^2(1-C)\delta^2}{2NT_1^2\cosh^2\alpha}\left[8\alpha\sinh\alpha\left\{-\frac{1}{1+\alpha}\times\right.\right.$$

$$\left.\left.\left(-\frac{3L^2}{8\alpha^2}+b_2+\frac{Lb_3}{2\alpha}\right)+\frac{L\alpha^3}{2T_1^2}\left(\frac{b_5\sinh\alpha}{4}+b_3\,b_4\right)\right\}+\right.$$

$$+ \left(T_1^2 + T_1^{*2}\right) \frac{\alpha^2 L^2}{T_1^{*2}} \left[\cosh^2\alpha + 4\sinh^2\alpha\left(K_1^2 - 2K_2 - \frac{\alpha^2}{N^2L^2}\right) + \right.$$

$$\left. + \frac{\sinh\alpha}{\alpha}\left(\cosh\alpha - \alpha\sinh\alpha\right)\right] \qquad (156)$$

Investigations into (151) and (156) show that $F(0) - F(1) \leq 0$ and $G(0) - G(1) \leq 0$. The magnitude of velocity distributions attributed to $F(0) - F(1)$ and $G(0) - G(1)$ are proportional to δ^2. Flow reversals occur at the centre as said earlier and the criteria

$$\delta_0 = \frac{D}{F(0)-F(1)} \qquad (157)$$

$$\delta_{01} = \frac{D_1}{G(0)-G(1)} \qquad (158)$$

3.6 Dissipative wall

For nonvanishing damping δ_1 and δ are to be re-defined, recalling that

$$\delta_1 = \frac{iK_3\alpha^3}{R^2} + \frac{K_2\alpha^2}{R} - im\alpha^3$$

The $$\delta_1 = i\left[\delta - \frac{iK_2\alpha^2}{R}\right] \qquad (159)$$

where $$\delta = \frac{K_3\alpha^3}{R^2} - m\alpha^3$$

The following that contains δ are re-defined

$$A_{11} = \frac{2}{\alpha^2 T^2 \cosh\alpha}\left(\delta - \frac{iK_2\alpha^2}{R}\right) \qquad (160)$$

$$B_{11} = -\frac{2}{\alpha\beta T_1^2 \cosh\beta}\left(\delta - \frac{iK_2\alpha^2}{R}\right) \qquad (161)$$

$$E_{11} = -\frac{2N}{(N^2+\alpha^2R^2)\alpha\beta T_1^2\cosh\beta}\left[N\delta + K_2\alpha^3 + i\alpha\left(R\delta - \frac{NK_2\alpha}{R}\right)\right], \qquad (162)$$

$$C_{11} = \frac{2}{\alpha\cosh\alpha}\left[\frac{1}{\alpha\beta(1+\alpha)}\left(\delta - \frac{iK_2\alpha^2}{R}\right)\gamma + \frac{N}{(N^2+\alpha^2R^2)\alpha T_1^2}\left[N\delta + K_2\alpha^3 + \right.\right.$$

$$+ i\alpha\left(R\delta - \frac{NK_2\alpha}{R}\right)\Bigg]\Bigg] \qquad (163)$$

$$C_1 = \frac{i\alpha^2 C}{2}\left[\delta^2 - \frac{2iK_2\alpha^2\delta}{R} - \frac{K_2^2\alpha^4}{R^2}\right]\left[i\alpha(1-C)\left(\bar{\rho} - \frac{M}{N}\right)\left[4\frac{\tanh\alpha}{\cosh\alpha}\times\right.\right.$$

$$\left(-\frac{\sinh\alpha}{1+\alpha}\left[-\frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right] - \frac{\alpha^3\sinh^2\alpha}{T_1^2}\right) +$$

$$+ 2\frac{(T_1^2 + T_1^{*2})}{T_1^2 T_1^{*2}\cosh^2\alpha}\left[\frac{L^2}{4\alpha}\left(\sinh\alpha\left[\cosh\alpha - \alpha\sinh\alpha\right] - \cosh^2\alpha\right)\right.$$

$$\left.\left. + \sinh^2\alpha\left[L^2\left(K_1^2 - 2K_2\right) - \frac{\alpha^2}{N^2}\right]\right]\right] + \frac{\bar{\rho}iL^3(V_1 + \alpha_2)}{2T_1^2\left(\delta - \frac{iK_2\alpha^2}{R}\right)}\Bigg] \qquad (164)$$

$$H_1 = \left[\delta^2 - \frac{2iK_2\alpha^2\delta}{R} - \frac{K_2^2\alpha^4}{R^2}\right]\left[-\frac{3i\alpha V_1\bar{\rho}CL^4}{4(1+\alpha)T_1^2} + \frac{\alpha^3(1-C)}{NT_1^2\cosh^2\alpha}\left[2\sinh\alpha\times\right.\right.$$

$$\left[-\frac{2}{1+\alpha}\left(-\frac{3L}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right)\right.$$

$$\left.\left. + \frac{\alpha^3 L}{T_1^2}\left(\frac{b_5\sinh\alpha}{4\alpha} + b_3 b_4\right)\right] + \frac{(T_1^2 + T_1^{*2})\alpha L^2}{2T_1^{*2}}\left[\cosh^2\alpha + 4\sinh^2\alpha\times\right.\right.$$

$$\left.\left.\left.\left(K_1^2 - 2K_2 - \frac{\alpha^2}{N^2L^2}\right) + \frac{\sinh}{\alpha}\left(\cosh\alpha - \alpha\sinh\alpha\right)\right]\right]\right] \qquad (165)$$

For this case D and D_1 may be expressed as

$$D = D_e + D_d \qquad (166)$$

$$D_1 = D_{e_1} + D_{d_1} \qquad (167)$$

where D_e is given by (110) and D_{e_1} same as D_e

$$D_d = \frac{K_2\alpha}{R}\ \mathrm{Im}\left(\beta\tanh\beta\right) \qquad (168)$$

$$D_{d_1} = \frac{i\bar{\rho}C}{2}\left[-\frac{2(1-C)A_{11}T_1^2\tanh\alpha K_2}{\alpha(1+\alpha)R^2}\left\{\frac{1}{R^2} - \frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right\}\sinh\alpha\right.$$

$$\left. + \frac{2K_2\alpha^4 A_{11}^*\cosh\alpha}{(1+\alpha)R}\left[\frac{2}{R^2} + \frac{iL}{2\alpha R} - \frac{3L^2}{4\alpha^2} + \frac{L^2}{4\alpha}\left(b_1 + s_1 + V_1L^2 + \frac{1}{2\alpha R}\right.\right.\right. -$$

$$- \frac{1}{2R}(b_1 + s_1) + \frac{RL^2}{16\alpha^2}(b_1 + s_1) + \frac{L^2 V_1 R}{2\alpha}\Big]\Big] \quad (169)$$

For $R \leq 1$

$$D_d = \frac{K_2 L}{2}\left(\tanh^2\alpha - \frac{\tanh\alpha}{\alpha} - 1\right) \quad (170)$$

$$D_{d1} = \frac{i\bar{\rho}C}{2}\Big[-\frac{2(1-C)A_{11}T_1^2 K_2 \tanh\alpha \sinh\alpha}{\alpha(1+\alpha)} \times$$

$$\left(1 - \frac{3L^2}{8\alpha^2} + b_2 + \frac{Lb_3}{2\alpha}\right) +$$

$$+ \frac{2K_2\alpha^4 A_{11}\cosh\alpha}{1+\alpha}\Big[2 + \frac{iL}{2\alpha} - \frac{3L}{4\alpha^2} + \frac{L^2}{4\alpha}(b_1+s_1) + V_1L^2 + \frac{1}{2\alpha} - \tfrac{1}{2}(b_1+s_1)$$

$$+ \frac{L^2}{16\alpha^2}(b_1 + s_1) + \frac{L^2 V_1}{2\alpha}\Big]\Big] \quad (171)$$

3.7 Interaction of Poiseuille flow with Peristaltic motion

For the presence of the basic flow, first order perturbation in the stream function results into a linear fourth order differential equation with variable coefficients. Closed form solutions to this type of equation in terms of known functions are not possible but the solution can be obtained for small K in the closed form. The method of solution will be discussed for small and finite values of K.

3.8 Closed form solution for $K << 1$

Let ϕ_{f1}, ϕ_{p1}, ϕ_{f20} and ϕ_{p20} in equations (57) - (61), (63) - (64), (67) be expanded in ascending powers of K in the form

$$\phi_{f1} = \phi_{f10} + K\phi_{f11} + K^2\phi_{f12} + \cdots\cdots$$

$$\left.\begin{aligned}\phi_{p1} &= \phi_{p10} + K\phi_{p11} + K^2\phi_{p12} + \\ \phi_{f20} &= \chi_{f20} + K\chi_{f21} + K^2\chi_{f22} + \ldots\ldots \\ \phi_{p20} &= \chi_{p20} + K\chi_{p21} + K^2\chi_{p22} + \ldots\end{aligned}\right\} \qquad (172)$$

The equations become

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i\alpha R(1-C)\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f10} = CM\left(\frac{d^2}{dy^2} - \alpha^2\right)\left(\phi_{f10} - \phi_{p10}\right)$$

$$i\alpha CR\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{p10} = -CN\left(\frac{d^2}{dy^2} - \alpha^2\right)\left(\phi_{f10} - \phi_{p10}\right) \qquad (174)$$

Subject to the boundary conditions

$$\left.\begin{aligned}\phi'_{f10}(\pm 1) &= 0 \\ \phi'_{p10}(\pm 1) &= -\frac{2i\delta_1}{(1+\alpha)\alpha\beta} \\ \phi'_{f1}(\pm 1) &= 2R(1-C)\delta_1\end{aligned}\right\} \qquad (175)$$

and

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i\alpha R(1-C)\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f11} - i\alpha(1-C)R(1-Y^2)\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{f10} -$$

$$- 2i\alpha(1-C)R\phi_{f10} = CM\left(\frac{d^2}{dy^2} - \alpha^2\right)\left(\phi_{f11} - \phi_{p11}\right) \qquad (176)$$

$$i\alpha CR\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{p11} - i\alpha CR\left(1-y^2 + \frac{2}{M}\right)\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{p10} -$$

$$2i\alpha CR\phi_{p10} = -CN\left(\frac{d^2}{dy^2} - \alpha^2\right)\left(\phi_{f11} - \phi_{p11}\right) \qquad (177)$$

subject to the boundary conditions

$$\phi'_{f11}(\pm 1) = 2 \qquad (178)$$

$$\frac{1}{2R}\left(\phi''_{f11} - 2\alpha^2\right) - (1-C)i\alpha\left[\phi_{f10}(\pm1)-1\right] + \frac{q_2}{2}\left[\phi'_{p11}(\pm1)-2\right]$$

$$= (1-C)\delta_1 \qquad (179)$$

$$-i\alpha\bar{\rho}\left[-\tfrac{1}{2}\phi'_{p11}(\pm1)+\phi_{p1\frac{1}{2}}(\pm1)\right] + \frac{q_2}{2}\left[2 - \phi'_{p11}(\pm1)\right] = \delta_1 \qquad (180)$$

For closed form solutions of small values of R to be taken into account, β can be expanded thus

$$\beta = \alpha - \frac{iRL}{2} + \frac{R^2L}{8\alpha} + \ldots .$$

When the Reynolds number is small, it is sufficient in this case that only the first power of R be retained. Thus

$$\beta = \alpha - \frac{iRL}{2} \qquad (181)$$

The solutions to equations (172) and 174) subject to (175) are given below

$$\phi_{f10} = A_{11}\sinh\alpha y + B_{11}\sinh\beta y \qquad (182)$$

$$\phi_{p10} = C_{11}\sinh\alpha y + E_{11}\sinh\beta y \qquad (183)$$

where

$$\left.\begin{aligned} A_{11} &= \frac{2\alpha\delta}{T_1^2R^2\cosh\alpha} \\ B_{11} &= -\frac{2\alpha^2\delta}{R^2T_1^2\beta\cosh\beta} \end{aligned}\right\} \qquad (184)$$

$$\left.\begin{aligned} E_{11} &= \frac{NB_{11}}{N-i\alpha R} \\ C_{11} &= -\frac{1}{\alpha\cosh\alpha}\left[-\frac{2\alpha^2\delta}{(1+\alpha)\beta R^2} + E_{11}\,\beta\cosh\beta\right] \end{aligned}\right\} \qquad (185)$$

For small values of P

$$\phi_{10} = A_{11} \sinh \alpha y + B_{11} \sinh (\alpha - \frac{iRL}{2}) y \qquad (186)$$

$$\phi_{p_{10}} = C_{11} \sinh \alpha y + E_{11} \sinh(\alpha - \frac{iRL}{2}) y \qquad (187)$$

and

$$B_{11} \sinh(\alpha - \frac{iRL}{2}) y = - A_{11} \left[\sinh \alpha y + \frac{iRL}{2} (b_1 \sinh \alpha y - y \cosh \alpha y + \ldots \right] \qquad (188)$$

$$C_{11} = A_{11} \left[1 + \frac{T_1^2 \alpha}{1+\alpha} + iR \left[U_2 - \frac{L}{2} \left(U_1 - \frac{T_1^2}{1+\alpha} \right) \right] + \ldots . \right] \qquad (189)$$

$$E_{11} \sinh(\alpha - \frac{iRL}{2}) y = - A_{11} \left[\sinh \alpha y + iR \left(U_2 \sinh \alpha y - \right. \right.$$

$$\left. \left. - \frac{Ly \cosh \alpha y}{2} \right) + \ldots \ldots \ldots \right] \qquad (190)$$

where

$$U_1 = 1 + \tanh \alpha$$

$$U_2 = \frac{\alpha}{N} + \frac{Lb_1}{2}$$

substituting (188) - (190) in equations (186) and (187), they turn out to be

$$\phi_{f10}(y) = - \frac{A_{11} iRL}{2} \left[b_1 \sinh \alpha y - y \cosh \alpha y \right] \qquad (191)$$

$$\phi_{p_{10}} = A_{11} \left[\frac{T_1^2 \alpha \sinh \alpha y}{1 + \alpha} + \frac{iRL}{2} \left[y \cosh \alpha y + \left(\frac{T_1^2}{1+\alpha} - U_1 \right) \times \right. \right.$$

$$\left. \left. \sinh \alpha y \right] \right] \qquad (192)$$

Also, on solving equations (176) and (177) subject to (178) - (180) under the assumption of small values of R and as well as in the limit as $R \to 0$

$$\phi_{f11} = \frac{\sinh\alpha y}{\alpha\cosh\alpha}\left[2 - \left[Y_1 + \frac{4\alpha\delta}{T_1^2}\left(\alpha^2 Y_2 + \frac{2LT_1^2CM\delta}{\alpha^2(1+\alpha)}\right)\right] - \frac{1}{2}\left[Y_1 + \frac{4\alpha\delta}{T_1^2}\times\right.\right.$$

$$\left.\left(\alpha^2 Y_2 - \frac{2T_1^2LCM\delta}{\alpha^2(1+\alpha)}\right) + \frac{2LCM\delta}{\alpha^2(1+\alpha)}\left(\frac{1}{\alpha} + 2\alpha\delta\right)\right] - \frac{1}{2\alpha^2}\left[Y_3 +\right.$$

$$\left. + \frac{4\alpha\delta}{T_1^2}\left(\alpha^2 Y_4 - 2\frac{LCM\delta}{1+\alpha}\right) + 2\frac{LCM\delta}{1+\alpha}\left(1 - \frac{4\alpha\delta}{T_1^2}\right)\right]\right] + \sinh\alpha y \times$$

$$\left[\frac{1}{2\alpha\cosh\alpha}\left[Y_1 + \frac{2\alpha\delta}{T_1^2}\left[2\alpha^2 Y_2 + \frac{CM\delta T_1^2}{\alpha(1+\alpha)}\left(b_1 + \frac{L}{\alpha^2} - \frac{4L}{\alpha}\right) +\right.\right.\right.$$

$$\left.\left.\left. + 4\left(b_1 + \frac{L}{\alpha^2}\right)\frac{T_1^2CM\delta}{1+\alpha}\right]\right] - \frac{1}{2\alpha^3\cosh\alpha}\left[Y_3 + \frac{2\alpha^2CM\delta}{1+\alpha}\left(b_1 + \frac{L}{\alpha^2}\right) +\right.\right.$$

$$\left.\left. + \frac{4\alpha\delta}{T_1^2}\left[\alpha^2 Y_4 - \frac{2LCM\delta}{1+\alpha} + \frac{2\alpha^2CM\delta}{1+\alpha}\left(b_1 + \frac{L}{\alpha^2}\right)\right] - Y_4 L + \left(b_1 + \frac{L}{\alpha^2}\right)\times\right.\right.$$

$$\left.\left.\left(\alpha^3 Y_4 - \frac{2LCM\delta}{1+\alpha}\right)\right]\right] - Ly\frac{\cosh\alpha y}{2}\left[\frac{1}{\alpha\cosh\alpha}\left[\frac{2CM\delta}{\alpha(1+\alpha)} +\right.\right.$$

$$\left.\left. + \frac{8\alpha CM\delta^2}{1+\alpha}\right] - \frac{2CM\delta(T_1^2+4\alpha\delta)}{\alpha(1+\alpha)T_1^2\cosh\alpha}\right] + Y_5 + \frac{\alpha\delta}{T_1^2\cosh\alpha}\left[\alpha^2 Y_6 -\right.$$

$$\left. - \frac{CML\,T_1^2\sinh\alpha y}{\alpha^5(1+\alpha)}\right| \qquad (193)$$

While

$$\phi_{p11} = J_8\sinh\alpha y - \frac{1}{CN}\left[\frac{2\alpha^2\delta CL}{T_1^2\cosh\alpha}\left[\sinh\alpha y - \frac{1}{\alpha^2}\left(\alpha y^2\sinh\alpha y -\right.\right.\right.$$

$$\left.\left.\left. 4\alpha y\cosh\alpha y + 6\sinh\alpha y\right)\right] + \frac{2iC}{\alpha}\left[-\frac{2\delta\sinh\alpha y}{\cosh\alpha}\times\right.\right.$$

$$\left.\left.\left[\frac{iL}{2\alpha(1+\alpha)} - \alpha^3\frac{iLb_4\sinh\alpha}{2T_1^2} + \frac{i\alpha}{1+\alpha} + \frac{i\alpha^4\sinh\alpha}{NT_1^2}\right]\right.\right.$$

$$\left.\left. - \frac{2\alpha\delta}{T_1^2\cosh\alpha}\left[i\left(\frac{L}{\alpha^2} + U_2\right) + \frac{i\alpha}{N} - \frac{iLy}{2}\cosh\alpha y\right]\right]\right]$$

$$.- CN\left[-\frac{iL}{\alpha^2}\left[\frac{\sinh\alpha y}{2\alpha\cosh\alpha}\left(-\frac{4iCM\delta}{1+\alpha}\left(\frac{1}{\alpha}+4\alpha\delta\right)\right)+2\sinh\alpha y\right.\right.$$

$$\frac{iCM\delta(T_1^2+4\alpha\delta)}{\alpha(1+\alpha)T_1^2\cosh\alpha}+Y_5-\frac{4iCM\sinh\alpha y}{\alpha(1+\alpha)\cosh\alpha}+\frac{2\alpha\delta}{T_1^2\cosh\alpha}\left[2\alpha^2Y_6-\right.$$

$$\left.\left.\frac{2CMLT_1^2\sinh\alpha y}{\alpha^5(1+\alpha)}\right]+\frac{8\delta CM\sinh\alpha y}{\alpha(1+\alpha)N\cosh\alpha}-\frac{1}{\alpha^2}\left[Y_5-\frac{y^2}{2}-\frac{4iCM\sinh\alpha y}{\alpha^3(1+\alpha)\cosh\alpha}\right.\right.$$

$$\left.\left.\left.+\frac{2\alpha\delta}{T_1^2\cosh\alpha}\left(\alpha^2Y_6y^2-\frac{2LCM\sinh\alpha y}{(1+\alpha)\alpha^7}\right)+\frac{8\delta CM\sinh\alpha y}{\alpha(1+\alpha)N\cosh\alpha}\right]\right]\right] \quad (194)$$

where

$$Y_1=2L\delta(1-C)\left[\alpha+b_1+\frac{12}{\alpha^2}+\frac{3}{\alpha}+1-\frac{(6+\alpha)}{\alpha}\tanh\alpha\right.$$

$$\left.-\frac{2CM\alpha\delta}{T_1^2}\left\{-\frac{L}{\alpha^2}\left[2\left(\frac{1}{M}-\frac{6}{\alpha^2}\right)+\frac{6\tanh\alpha}{\alpha}\right]-\frac{L}{\alpha^3}\left(\tanh\alpha+\frac{T_1^2}{1+\alpha}-U_1-\frac{3}{\alpha}\right)\right\}\right] \quad (195)$$

$$Y_2=\frac{(1-C)L}{\alpha^4}\left[b_1-\alpha-1+\left(\frac{10}{\alpha}-1\right)\left(\tanh\alpha-\frac{5}{\alpha}\right)\right]$$

$$-\frac{2CM\delta}{T_1^2\alpha^2}\left[-L\left(\frac{2}{M}+\frac{50}{\alpha^2}-\frac{10}{\alpha}\tanh\alpha\right)-\frac{L}{\alpha}\left(\frac{T_1^2}{1+\alpha}-U_1-\frac{5}{\alpha}+\tanh\alpha\right)\right] \quad (196)$$

$$Y_3=\frac{2\alpha\delta L(1-C)}{T_1^2}\left[b_1-\alpha+1-\frac{(2+\alpha)}{\alpha}\tanh\alpha+\right.$$

$$\left.+\frac{(2+\alpha)}{\alpha^2}-\frac{2\alpha^2CM\delta}{T_1^2}\left[-L\left[2\left(\frac{1}{M}-\frac{1}{\alpha^2}\right)+2\frac{\tanh\alpha}{\alpha}\right]-\frac{L}{\alpha}\left[\frac{T_1^2}{1+\alpha}-U_1\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{\alpha}+\tanh\alpha\right]\right]\right] \quad (197)$$

$$Y_4=\frac{(1-C)L}{\alpha^2}\left[b_1-\alpha-1+5\tanh\alpha+\frac{15}{\alpha}\right]-$$

$$-\frac{2CM\delta}{T_1^2}\left[-L\left(\frac{2}{M}-\frac{12}{\alpha^2}+\frac{6\tanh\alpha}{\alpha}\right)-L\left(\frac{T_1^2}{1+\alpha}-U_1-\frac{3}{\alpha}+\tanh\alpha\right)\right] \quad (198)$$

$$Y_5=\frac{2\alpha^2\delta L(1-C)}{T_1^2\cosh\alpha}\left[(b_1-\alpha)\frac{\sinh\alpha y}{\alpha^4}+M-M_{11}\right]-\frac{2\alpha^2CM\delta}{T_1^2\cosh\alpha}\times$$

$$\left[-\alpha L\left[\left(1+\frac{2}{M}\right)\frac{\sinh\alpha y}{\alpha^4}-M\right]-L\left[M_{11}+\left(\frac{T_1^2}{1+\alpha}-U_1\right)\frac{\sinh\alpha y}{\alpha^4}\right]\right] \quad (199)$$

$$Y_6=\alpha(1-C)L\left[(b_1-\alpha)\frac{\sinh\alpha y}{\alpha^6}-(M_{13}+M_4)-CM\left[-\left(1+\frac{2}{M}\right)L\frac{\sinh\alpha y}{\alpha^5}\right.\right.$$

$$\left.\left.+\alpha LM_{13}-L\left[M_4+\left(\frac{T_1^2}{1+\alpha}-U_1\right)\frac{\sinh\alpha y}{\alpha^6}\right]\right]\right] \quad (200)$$

$$\left.\begin{aligned} M&=\frac{1}{\alpha^4}\left[y^2\sinh\alpha y-\frac{4}{\alpha}\left(2y\cosh\alpha y-\frac{5\sinh\alpha y}{\alpha}\right)\right]\\ M_{11}&=\frac{1}{\alpha^4}\left[y\cosh\alpha y-\frac{4\sinh\alpha y}{\alpha}\right] \end{aligned}\right\} \quad (201)$$

$$\left.\begin{aligned} M_{13}&=\frac{1}{\alpha^6}\left[y^2\sinh\alpha y-\frac{2}{\alpha}\left(6y\cosh\alpha y-\frac{31\sinh\alpha y}{\alpha}\right)\right]\\ M_4&=\frac{1}{\alpha^6}\left[y\cosh\alpha y-\frac{6\sinh\alpha y}{\alpha}\right] \end{aligned}\right\} \quad (202)$$

$$J_6=\frac{1}{\alpha\cosh\alpha}\left[-\frac{2}{\alpha(i\bar{\rho}-1)}+\frac{1}{CN}\left\{\frac{2\alpha^2LC\delta}{T_1^2\cosh\alpha}\left[-\alpha\left(1+\frac{2}{M}\right)\cosh\alpha\right.\right.\right.$$

$$\left.\left.\left.-\frac{1}{\alpha^2}(2\sinh\alpha-\alpha\cosh\alpha-2\cosh\alpha)\right]\right.\right.$$

$$\left.-\frac{2iC}{\alpha}\left[-\frac{2\alpha\delta}{1+\alpha}\left[\frac{i\alpha}{N}+\frac{iL}{2\alpha}+\frac{(1+\alpha)\alpha^2\sinh\alpha}{T_1^2}\right]\right]-\frac{4i\alpha C\delta}{T_1^2}\times\right.$$

$$\left.\left[V_2-\frac{L^2}{2\alpha^2}-\frac{Ld_2}{2\alpha}-\frac{L}{N}-\frac{\alpha^2}{N^2}\right]\right\}-CN\left[-\frac{iL\cosh\alpha}{2\alpha^2}\times\right.$$

$$\left.\left[\frac{4iCM\delta}{1+\alpha}\left(1+\frac{4\alpha\delta}{T_1^2}-\frac{1}{\alpha}-4\alpha\delta+2i\alpha\delta L\left(\frac{1}{\alpha}-\tanh\alpha\right)\right)\right]\right. \quad (203)$$

3.9 Series solution

Closed form solutions for R and K are not possible. Using Frobenius series method for solving differential

equations with unknown coefficients,

let
$$\left.\begin{aligned} \phi_{f1} &= \sum_{n=o}^{\infty} A_n\, y^n \\ \phi_{p1} &= \sum_{n=o}^{\infty} B_n\, y^n \end{aligned}\right\} \qquad (204)$$

be the solutions of equations (57), (58) when substituted and comparing the coefficients of like powers of y then

$$A_4 = \frac{A_2}{12}\left[2\alpha^2 + i(1-C)\alpha R(K-1) + CM\right] + \frac{A_o}{24}\left[2i(1-C)K\alpha R - \alpha^4 + i(1-C)\alpha^3 R - CM - \alpha^2 K\right] + CM\left[\alpha^2 B_o - 2B_2\right] \qquad (205)$$

$$A_5 = \frac{A_3}{120}\left[2\alpha^2 - i(1-C)(1-K)\alpha R\right] + \frac{A_1}{120}\left[2i(1-C)\alpha KR - \alpha^4 - i(1-C)\chi (K-1)\alpha^3 R - CM\,\alpha^2\right] + CM\left[\alpha^2 B_1 - 6B_3\right] \qquad (206)$$

$$A_6 = \frac{A_4}{30}\left[2\alpha^2 - i(1-C)(1-K)\alpha R + CM\right] + \frac{A_2}{360}\left[2i(1-C)K\alpha R - \alpha^4 - i(1-c)\alpha R\left[(2-\alpha^2)K - \alpha^2\right] - CM\alpha^2\right] + \frac{A_o}{360} i(1-C)\alpha^3 KR + \frac{CM}{360}\left[\alpha^2 B_2 - 12\,B_4\right] \qquad (207)$$

and the recurrence formula is

$$A_{n+6} = \frac{\left[2\alpha^2 - i(1-C)(1-K)\alpha R + CM\right]}{(n+6)(n+5)} A_{n+4} + \frac{\left[2i(1-C)K\alpha R - \alpha^4 - i(1-C)\alpha R\left[(2-\alpha^2)K - \alpha^2\right] - CM\alpha^2\right]}{(n+6)(n+5)(n+4)(n+3)} A_{n+2} + \frac{i(1-C)\alpha^3 KR}{(n+6)(n+5)(n+4)(n+3)} A_n + \frac{CM}{(n+6)(n+5)(n+4)(n+3)}\left[\alpha^2 B_{n+2} - 12 B_{n+4}\right] \qquad (208)$$

While

$$B_2 = \frac{1}{2}\,\frac{\left[2i\alpha KR+i\alpha^3R\left(1-\frac{2K}{M}-K\right)-\alpha^2N\right]}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N}\,B_0 + \frac{N}{2}\,\frac{\left[\alpha^2A_0-2A_2\right]}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N} \tag{209}$$

$$B_3 = \frac{B_1}{6}\,\frac{\left[2iK\alpha R+i\alpha^3R\left(1-\frac{2K}{M}-K\right)-\alpha^2N\right]}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N} + \frac{N}{6}\,\frac{\left[\alpha^2A_1-6A_3\right]}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N} \tag{210}$$

$$B_4 = \frac{A_2}{12}\,\frac{\alpha^2N}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N} - A_4\,\frac{N}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N} +$$

$$\frac{B_2}{12}\,\frac{\left[i\alpha^3R\left(1-\frac{2K}{M}-K\right)-\alpha^2N-2i\alpha KR\right]}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N} + \frac{B_0}{12}\,\frac{i\alpha^3KR}{i\alpha R\left(1-\frac{2K}{M}-K\right)-N} \tag{211}$$

The recurrence formula for (211) is

$$B_{n+4} = -\frac{\alpha^2N}{(n+4)(n+3)S_1}A_{n+2} - \frac{N}{S_1}A_{n+4} + \frac{\alpha^2S_1-(n+2)i\alpha KR}{(n+4)(n+3)S_1}B_{n+2} +$$

$$+ \frac{i\alpha^3KR}{(n+4)(n+3)S_1}B_n \tag{212}$$

Where

$$S_1 = i\alpha R\left(1-\frac{2K}{M}-K\right)-N$$

The recurrence relations of equations (208), (212) and an examination of (205) - (212) show that all A_n, $n \geq 4$ and B_n, $n \geq 2$ can be expressed in terms of A_0, A_1, A_2, A_3 B_0, B_1. The boundary condition (59) require that $\phi_{f1}(y)$ be antisymmetric with respect to y thus

$$A_o = A_2 = 0$$
$$B_o = B_2 = 0 \qquad (213)$$

Also examining (208) and (212) reveal that all A_n, $n \geq 6$ and B_n $n \geq 4$ respectively may be expressed in terms of A_1, A_3, B_1, B_3 . The solution may be written as

$$\left.\begin{aligned} \phi_{f1}(y) &= A_1 \sum_{n=o}^{\infty} a_{1n} y^n + A_3 \sum_{n=o}^{\infty} a_{3n} y^n + B_1 \sum_{n=o}^{\infty} b_{1n} y^n \\ \phi_{p1}(y) &= A_1 \sum_{n=o}^{\infty} a'_{1n} y^n + A_3 \sum_{n=o}^{\infty} a'_{3n} y^n + B_1 \sum_{n=o}^{\infty} b'_{1n} y^n \end{aligned}\right\} \qquad (214)$$

Since B_3 is a function of A_1, A_3, B_1 . Where a_{jn}, a'_{jn}, b_{jn}, b'_{jn}, $j = 1, 3$ and $n = 0, 1, 2, \ldots\ldots$ denotes the coefficients of A_j, B_j in the expression for A_n, $n \geq 6$ and B_n, $n \geq 4$.

where

$$\left.\begin{aligned} A_n &= a_{1n} A_1 + a_{3n} A_3 + b_{1n} B_1 \\ B_n &= a'_{1n} A_1 + a'_{3n} A_3 + b'_{1n} B_1 \end{aligned}\right] \qquad (215)$$

Moreover, for $n = 0, 1, \ldots 5$ a_{jn}, b_{jn} are given by (205), (206) and (209), (210) so that A_o, A_1, A_2, A_3 and B_o, B_1, B_2, B_o are independent of each other, thus

$$\left.\begin{aligned} a_{10} &= a_{12} = a_{13} = a_{14} = a_{30} = a_{31} = a_{32} = a_{34} = 0 \\ a_{10} &= a'_{12} = a'_{13} = a'_{14} = a'_{30} = a'_{31} = a'_{32} = a'_{34} = 0 \\ b_{10} &= b_{12} = b_{13} = b_{14} = 0 \\ b'_{10} &= b'_{12} = b'_{13} = b'_{14} = 0 \end{aligned}\right\}$$

(216)

$$\left.\begin{aligned} a_{11} &= a_{33} = 1 \\ a'_{11} &= a'_{33} = 1 \\ b_{11} &= 0 \\ b'_{11} &= 1 \end{aligned}\right\} \qquad (217)$$

The boundary conditions (59) - (61) reduce to

$$A_1 \sum_{n=0}^{\infty} n\, a_{1n} + A_3 \sum_{n=0}^{\infty} n\, a_{3n} + B_1 \sum_{n=0}^{\infty} n\, b_{2n} = 2K \qquad (218)$$

$$A_1\left[\frac{1}{2R}\left\{\sum_{n=0}^{\infty} n(n-1)(n-2)a_{1n} - \alpha^2 \sum_{n=0}^{\infty} a_{1n}\right\} - i(1-C)\alpha\left[-\tfrac{1}{2}\sum_{n=0}^{\infty} a_{1n} + K\left\{\frac{1}{M}\sum_{n=0}^{\infty} n\, a_{1n} + \sum_{n=0}^{\infty} a_{1n}\right\}\right] - \frac{q_2}{2}\left[\sum_{n=0}^{\infty} n a'_{1n}\right]\right] + A_3\left[\frac{1}{2R}\left\{\sum_{n=0}^{\infty} n(n-1)(n-2)a_{3n} - \alpha^2 \sum_{n=0}^{\infty} a_{3n}\right\} - i(1-C)\alpha\left[-\tfrac{1}{2}\sum_{n=0}^{\infty} a_{3n} + K\left\{\frac{1}{M}\sum_{n=0}^{\infty} n a_{3n} + \sum_{n=0}^{\infty} a_{3n}\right\}\right] - \frac{q_2}{2}\left[\sum_{n=0}^{\infty} n\, a_{3n} - \sum_{n=0}^{\infty} n\, a'_{3n}\right]\right]$$

$$+ B_1\left[\frac{1}{2R}\left\{\sum_{n=0}^{\infty} n(n-1)(n-2)b_{1n} - \alpha^2 \sum_{n=0}^{\infty} b_{1n}\right\} - i(1-C)\alpha\left[-\tfrac{1}{2}\sum_{n=0}^{\infty} b_{1n} + K\left\{\frac{1}{M}\sum_{n=0}^{\infty} n\, b_{1n} + \sum_{n=0}^{\infty} b_{1n}\right\}\right] - \frac{q_2}{2}\left[\sum_{n=0}^{\infty} n\, b_{1n} - \sum_{n=0}^{\infty} n\, b'_{1n}\right]\right] = (1-C)\delta_1 \qquad (219)$$

$$A_1\left[-i\alpha\bar{\rho}\left\{-\tfrac{1}{2}\sum_{n=0}^{\infty} n\, a'_{1n} + K\left\{\frac{1}{M}\sum_{n=0}^{\infty} n\, a_{1n} + \sum_{n=0}^{\infty} a'_{1n}\right\}\right\} + \frac{q_2}{2}\left[\sum_{n=0}^{\infty} n\, a_{1n} - \sum_{n=0}^{\infty} n\, a'_{1n}\right]\right] + A_3\left[-i\alpha\bar{\rho}\left\{-\tfrac{1}{2}\sum_{n=0}^{\infty} n a'_{3n} + K\left\{\frac{1}{M}\sum_{n=0}^{\infty} n\, a_{3n} + \sum_{n=0}^{\infty} a'_{3n}\right\}\right\} + \frac{q_2}{2}\left[\sum_{n=0}^{\infty} n\, a_{3n} - \sum_{n=0}^{\infty} n\, a'_{3n}\right]\right]$$

$$+ B_1 \left[- i\alpha\bar{\rho} \left\{ - \frac{1}{2} \sum_{n=o}^{\infty} n b_{1n}' + K \left(\frac{1}{M} \sum_{n=o}^{\infty} n\, b_{1n} + \sum_{n=o}^{\infty} b_{1n}' \right) \right\} + \right.$$

$$\left. + \frac{q_2}{2} \left[\sum_{n=o}^{\infty} n\, b_{1n} - \sum_{n=o}^{\infty} n\, b_{1n}' \right] \right] = \delta_1 \qquad (220)$$

Equations (218) - (220) can be written as

$$\left.\begin{aligned} A_1 \alpha_1 + A_3 \alpha_2 + B_1 \alpha_3 &= 2K \\ A_1 \alpha_4 + A_3 \alpha_5 + B_1 \alpha_6 &= (1 - C)\delta_1 \\ A_1 \alpha_7 + A_3 \alpha_8 + B_1 \alpha_9 &= \delta_1 \end{aligned}\right\} \qquad (221)$$

where

$$\alpha_1 = \sum_{n=o}^{\infty} n\, a_{1n}$$

$$\alpha_2 = \sum_{n=o}^{\infty} n\, a_{3n}$$

$$\alpha_3 = \sum_{n=o}^{\infty} n\, b_{1n}$$

$$\alpha_4 = \sum_{n=o}^{\infty} \left[\frac{1}{2R} \left\{ n(n-1)(n-2) a_{1n} - \alpha^2 a_{1n} \right\} - \right.$$

$$\left. - i(1-C)\alpha \left[- \frac{a_{1n}}{2} + K \left\{ \frac{n}{M} a_{1n} + a_{1n} \right\} \right] - \frac{q_2}{2} \left[n a_{1n} - n a_{1n}' \right] \right]$$

$$\alpha_5 = \sum_{n=o}^{\infty} \left[\frac{1}{2R} \left\{ n(n-1)(n-2) a_{3n} - \alpha^2 a_{3n} \right\} - i(1-C)\alpha \left[- \frac{a_{3n}}{2} + \right. \right.$$

$$\left. \left. + K \left\{ \frac{n}{M} a_{3n} + a_{3n} \right\} \right] - \frac{q_2}{2} \left[n\, a_{3n} - n\, a_{3n}' \right] \right]$$

$$\alpha_6 = \sum_{n=o}^{\infty} \left[\frac{1}{2R} \left\{ n(n-1)(n-2) b_{1n} - \alpha^2 b_{1n} \right\} - i(1-C)\alpha \left[- \frac{b_{1n}}{2} + \right. \right.$$

$$\left. \left. + K \left\{ \frac{n}{M} b_{1n} + b_{1n} \right\} \right] - \frac{q_2}{2} \left[n b_{1n} - n\, b_{1n}' \right] \right]$$

C_1', C_2 are constants of integration

Redefining $\phi_{f1}(y)$ as

$$\phi_{f_1}(y) = \sum_{m=o}^{\infty} X_m \, y^m \tag{227}$$

then

$$\left.\begin{aligned} \phi^*_{f1}(y) &= \sum_{m=o}^{\infty} X^*_m \, y^m \\ \phi'_{f1}(y) &= \sum_{n=o}^{\infty} n \, X_n \, y^{n-1} \\ \phi^{*\prime}_{f1} &= \sum_{n=o}^{\infty} n \, X^*_n \, y^{n-1} \end{aligned}\right\} \tag{228}$$

and $\phi_{p1}(y)$ re-defined as

$$\phi_{p_1}(y) = \sum_{m=o}^{\infty} Z_m \, y^m \tag{229}$$

$$\left.\begin{aligned} \text{so} \quad \phi^*_{p1}(y) &= \sum_{m=o}^{\infty} Z^*_m \, y^m \\ \phi'_{p1}(y) &= \sum_{n=o}^{\infty} n Z_n \, y^{n-1} \\ \phi^{*\prime}_{p1} &= \sum_{n=o}^{\infty} n \, Z^*_n \, y^{n-1} \end{aligned}\right\} \tag{230}$$

so that

$$\left.\begin{aligned} X_m &= A_1 \, a_m + A_3 \, a_{3m} + B_1 \, b_{1m} \\ X_n &= A_1 \, a_{1n} + A_3 \, a_{3n} + B_1 \, b_{1n} \\ Z_m &= A_1 \, a'_{1m} + A_3 \, a'_{3m} + B_1 \, b'_{1m} \\ Z_n &= A_1 \, a'_{1n} + A_3 \, a'_{3n} + B_1 \, b'_{1n} \end{aligned}\right\} \tag{231}$$

where X^*_m, X^*_n, Z^*_m, Z^*_n are complex conjugates to X_m, X_n, Z_m Z_n respectively.

Substituting equations (227) - (230) in (226) and integrating once gives

$$\phi'_{f20}(y) = F(y) + C_1 y^2 + C_2 y + C_3 \qquad (232)$$

where

$$F(y) = \frac{-i\alpha R}{2}\left[(1-C)\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X_m X_n^* \frac{n}{m+n} y^{m+n} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} X_m^* X_n \frac{n}{m+n} y^{m+n}\right] + \frac{CM}{N}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Z_m Z_n^* \frac{n}{m+n} y^{m+n} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Z_m^* Z_n^* \frac{n}{m+n} y^{m+n}\right]\right] \qquad (233)$$

The boundary condition $\phi'_{f20}(\pm 1) = D$ at the boundaries and demands that C_2 be zero. Therefore

$$C_1 + C_3 = D - F(1) \qquad (234)$$

and

$$\phi'_{f20} = F(y) - F(1) + D - C_1(1 - y^2) \qquad (235)$$

Integrating (64) once

$$\phi'_{p20}(y) = G(y) + F(y) - F(1) + D - C_1(1 - y^2) + C_4 \qquad (236)$$

where,

$$G(y) = \frac{i\alpha R}{2N}\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n(n-1) Z_m Z_n^* y^{m+n-2} - \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} n(n-1) Z_m^* Z_n y^{m+n-2}\right] \qquad (237)$$

Our solutions for the mean speed (average over time) flow are given as :

$$\bar{U}_{f1} = \frac{\varepsilon^2}{2}\phi'_{f20} = \frac{\varepsilon^2}{2}\left[F(y) - F(1) + 0 - G(1 - y^2)\right] \qquad (238)$$

$$\bar{U}_p(y) = \frac{\varepsilon}{2}^2 \phi'_{p_{20}} = \frac{\varepsilon}{2}^2 \left[G(y) + H(y) - F(1) + D - C_1(1-y^2) + C_4 \right] \tag{239}$$

The constant C_1 must be determined from considerations of conditions at the ends of the channel in series form using the solutions of equations (227) - (230) into (86) subject to the boundary conditions (70) and (72) that reduce to

$$\frac{1}{R} \phi'''_{f_{20}}(\pm 1) + \frac{i\alpha}{2} \left\{ (1-C) \left[\phi_{f_1}(\pm 1) \, \phi^{*\prime\prime}_{f_1} - \phi^*_{f_1}(\pm 1) \, \phi''_{f_1}(\pm 1) \right] - \right.$$

$$\bar{\rho} \, C \left[\phi^{*\prime}_{f_1}(\pm 1) \, \phi'_{f_1}(\pm 1) \quad - \phi_{p_1}(\pm 1) \, \phi^{*\prime}_{p_1}(\pm 1) + \right.$$

$$\left. \left. - \phi_{p_1}(\pm 1) \, \phi^{*\prime}_{f_1} + \phi^*_{p_1}(\pm 1) \, \phi'_{p_1}(\pm 1) \right] \right\} = 0 \tag{240}$$

Integrating equation (86) once subject to the boundary condition (240) then

$$C_1 = \frac{i\alpha CR}{4} \left\{ \frac{M}{N} \left[\sum_{m=o}^{\infty} \sum_{n=o}^{\infty} Z_m \, Z^*_n \, n(n-1) \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} \times \right. \right.$$

$$\left. Z^*_m \, Z_n \, n(n-1) \right] + \bar{\rho} \left[\sum_{n=o}^{\infty} n^2 X^*_n \, X_n - \sum_{n=o}^{\infty} n^2 \, X_n \, Z_n - \right.$$

$$\left. \left. \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} Z_m \, Z^*_n \, n(n-1) + \sum_{m=o}^{\infty} \sum_{n=o}^{\infty} Z^*_m \, Z_n \, n(n-1) \right] \right\} \tag{241}$$

3.91 Numerical Results and Discussion

For pure peristalsis (K = 0) Tables 4 and 5 show the distribution of F(y) - F(1) which controls the direction of peristaltic mean flow across the cross section for various values of α and R and for $K_3 = 1$. F(y) - F(1) is seen to be maximum near the centre of the channel and remains constant over a certain range. This range widens with increasing R hence flow reversal occurs in that F(y) - F(1) becomes negative. The values of D for various α and R ($K_3 = 1$) are given in Table 6 showing D is always positive. A plot of some of these values in Figures 3, 4, 5 will be used to analize the effects of C (the volume fraction density) on Poiseuille flow which is our main contribution to the thesis. For dissipating walls, viscous damping forces in the boundary walls is represented by K_2. For non zero, $C_1 \neq 0$, it gives rise to Poiseuille type flow. Some values of D for this case are shown in Fig. 6; as the wave number α becomes very small damping occurs showing that there are some disturbances during motion. For influence of Poiseuille flow on peristalsis, values of D obtained by series solution for finite R(1, 10, 20, 100) are listed in Table 7 for various values of α and K = 0.1, 0.2 at K_2 = 1,100, and 1000. A study of the results indicate that there exists a particular Poiseuille parameter K at which D is negative suggesting flow reversal occurs at the walls.

To discuss the effect of C, in tables 4 and 5 when C is very small F(y) - F(1) is positive, as C increases F(y) - F(1) decreases. In Table 6, as C increases D decreases, this arises from the fact that the fluid density ρ_f and the particle density ρ_p of the body interact. In Table 8 the values of C_1 is always positive which increases as C increases. This shows that as the volume fraction density between the two densities one of them being greater than the other, its result is positive. Figures 3, 4, 5 show various combinations of C on the effect of damping on α.

Table 4: Effect of damping on $F(y) - F(1)$ ($K_2 = 0$, $R = 1$, $K_3 = 1$).

	C = 0			C = .2		
α =	0.1	0.3	0.5	0.1	0.3	0.5
y			F(y)-F(1)			
	$x10^{-10}$	$x10^{-7}$	$x10^{-5}$	$x10^{-6}$	$x10^{-4}$	$x10^{-3}$
0	-0.1104	-2.3940	-1.1410	0.1025	0.2546	0.2122
0.2	-0.4083	-2.3820	-1.1360	0.1024	0.2542	0.2119
0.4	-0.3798	-2.2160	-1.0570	0.0999	0.2484	0.2077
0.6	-0.2819	-1.6460	-0.7887	0.0893	0.2229	0.1883
0.8	-0.1141	-0.0667	-0.3195	0.0607	0.1526	0.1316
1.0	0	0	0	0	0	0

Table 5 : D for pure peristalsis ($K_2 = 0$, $K_3 = 1$, $m = 0$)

	R = 0.1					R = 1				
α	C = 0	.1	.2	.3	.4	C = 0	.1	.2	.3	.4
1	0.245133	0.222114	0.198766	0.175083	0.151095	0.227137	0.205523	0.183720	0.161679	0.139402
2	0.683238	0.619077	0.554000	0.487991	0.421131	0.632370	0.571985	0.511268	0.44999	0.388012
3	0.856240	0.775837	0.694284	0.611563	0.527776	0.820385	0.742305	0.663803	0.584478	0.504261
4	1.036610	0.939273	0.840543	0.740399	0.638963	1.014043	0.917779	0.820959	0.723054	0.623991
5.	1259932	1.141627	1.021629	0.899910	0.776622	1.243769	1.125803	1.007174	0.887185	0.765733
6.	1.562282	1.361220	1.218141	1.073011	0.926008	1.489343	1.348090	1.206128	1.062520	0.917131
7	1.750415	1.586053	1.419343	1.250242	1.078959	1.739434	1.574403	1.408661	1.241028	1.071267
8.	2.000010	1.812211	1.621729	1.428517	1.232811	1.990411	1.801461	1.611905	1.420137	1.225925
10	2.499926	2.265181	2.027090	1.785564	1.540962	2.492232	2.255265	2.016110	1.778185	1.535118
	R = 10					R = 100				
1	0.032850	0.029347	0.025983	0.022687	0.019420	0.001269	0.001123	0.000978	0.000850	0.000734
2.	0.124945	0.111206	0.098544	0.086321	0.074217	0.006538	0.005843	0.005014	0.004273	0.003689
3.	0.270634	0.239470	0.212133	0.186291	0.160785	0.017005	0.015327	0.013155	0.010976	0.009379
4.	0.460030	0.404694	0.358249	0.315291	0.273042	0.033377	0.030234	0.026090	0.021488	0.018059
5	0.680265	0.595063	0.526334	0.464180	0.403262	0.056060	0.050915	0.044182	0.036180	0.029819
6	0.921025	0.801260	0.706057	0.625702	0.545253	0.065133	0.077574	0.067617	0.055306	0.044783
7	1.174992	1.016909	0.897607	0.794762	0.694628	0.121266	0.110323	0.096477	0.079007	0.062961
8.	1.437152	1.237680	1.091086	0.967904	0.848387	0.164036	0.149216	0.130785	0.107341	0.084452
10	1.973682	1.684659	1.480500	1.317985	1.161667	0.270104	0.245414	0.215015	0177832	0.137446

Table 7: Values of D obtained by method $K_3 = 1$.

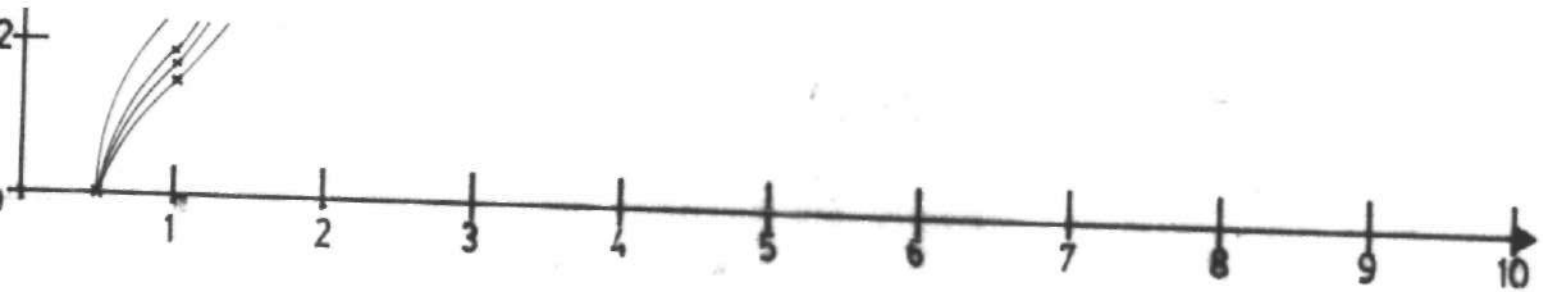
$K_3 = 1$

r =1	1.0		10.0		20.0		100.0	
Π = α	0.1	0.2	0.1	0.2	0.1	0.2	0.1	0.2
0.1	0.0991	0.1090	0.0912	0.1940	0.0732	0.1840	-0.2580	0.0468
0.2	0.0969	0.1950	0.0656	0.1780	-0.0013	0.1510	-0.4990	-0.2080
0.3	0.0975	0.1920	0.0224	0.1490	-0.1160	0.0813	-1.3900	0.5300
0.4	0.1080	0.1970	-0.0380	0.0939	-0.2470	-0.0655	-0.7930	-0.6430
0.5	0.1360	0.2160	-0.1110	-0.0071	-0.3610	-0.3020	-0.9120	-0.6330
0.6	0.1860	0.2560	-0.1890	-0.1680	-0.4100	-0.5290	-1.0800	-0.7180
			$K_3 = 100$					
0.1	0.0974	0.1930	0.0733	0.1740	0.0721	0.1830	-0.2590	0.0439
0.2	0.1530	0.2070	-0.0455	0.0653	-0.0234	0.1330	-0.5080	-0.2200
0.3	0.6640	0.5750	-0.2090	-0.0905	-0.1930	-0.0077	-1.1000	0.3740
0.4	2.3300	1.9400	-0.3200	-0.2320	-0.3790	-0.2710	-0.7600	-0.5800
0.5	5.9000	5.1500	-0.2450	-0.2530	-0.4540	-0.5500	-0.3880	-0.5920
0.6	11.90000	10.6000	0.2100	0.0434	-0.0191	-0.3290	-1.0700	-0.6830
			$K_3 = 1000$					
0.1	0.0822	0.1430	-0.0898	-0.0023	0.0618	0.1800	-0.2660	0.0377
0.2	0.6620	0.3140	-1.0600	-0.9550	-0.2240	-0.0318	-0.5950	-0.3380
0.3	5.8100	4.1600	-2.3100	-2.2700	-0.9010	-0.8170	1.5100	-1.1400
0.4	22.5000	17.8000	-2.8900	-3.2000	-1.5900	-2.1400	-0.4600	-0.0054
0.5	58.3000	49.0000	-1.4700	-2.4800	-1.3000	-2.8000	-0.4600	-0.0054
0.6	119.0000	105.000	3.8400	1.9800	3.5400	-1.4900	-0.9240	-0.3570

Table 8: Values of C_1 for pure peristalsis (non - series).

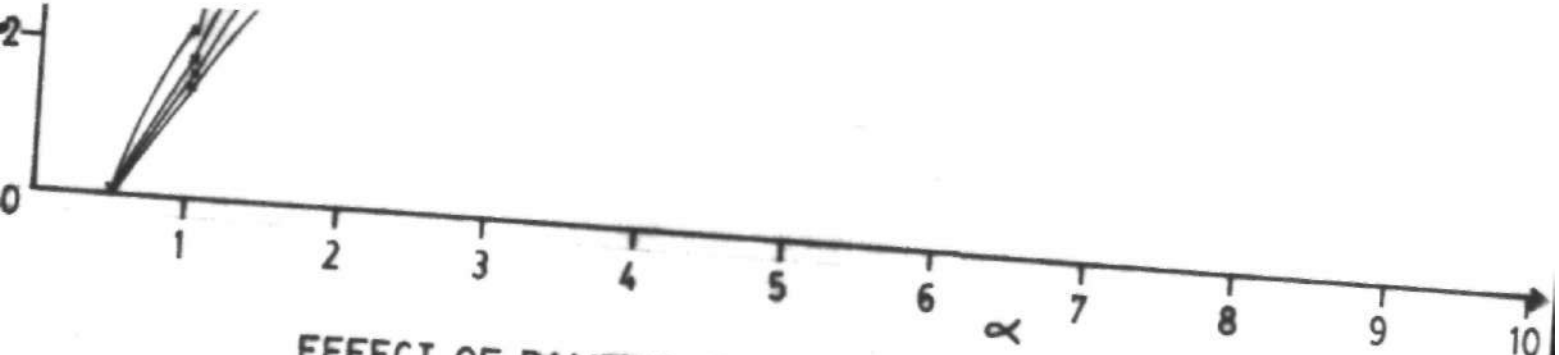
R = 1

	$K_2 = 0$				$K_2 = .1$			
α	C = .1	.2	.3	.4	.1	.2	.3	.4
.10	0.4336E-07	0.7688E-07	0.1006E-06	0	0.8671E-07	0.1538E -06	0.2012E-06	0
.50	0.1098E-03	0.1952E-03	0.2559E-03	0	0.1142E-03	0.2031E- 03	0.2662E-02	0
1.00	0.1081E-02	0.1921E-02	0.2507E-02	0	0.1092E-02	0.1940E-02	0.2532E-02	0
2.00	0.5561E-02	0.9506E-02	0.1172E-01	0	0.5575E-02	0.9529E-02	0.1175E-01	0
3.00	0.1633E-01	0.2773E-01	0.3361E-01	0	0.1635E-01	0.2776E-01	0.3365E-01	0
			R = 100					
.1	0.2275E-12	0.3646E-12	0.4211E-12	0	0.2275E-08	0.3647E-08	0.4221E-08	0
.5(	0.1971E-08	0.3365E-08	0.4005E-08	0	0.7905E-06	01350E-05	0.1606E-05	0
1.0(	0.5960E-07	0.1165E-06	0.1480E-06	0	0.5919E-05	0.1176E 04	0.1495E-04	0
2.0(	0.8651E-06	0.2359E-05	0.3609E-05	0	0.2249E-04	0.6135E-04	0.9384E-04	0
3.0(	0.2981E-05	0.1004E-04	0.1852E-04	0	0.3610E-04	01215E-03	0.2243E-03	0



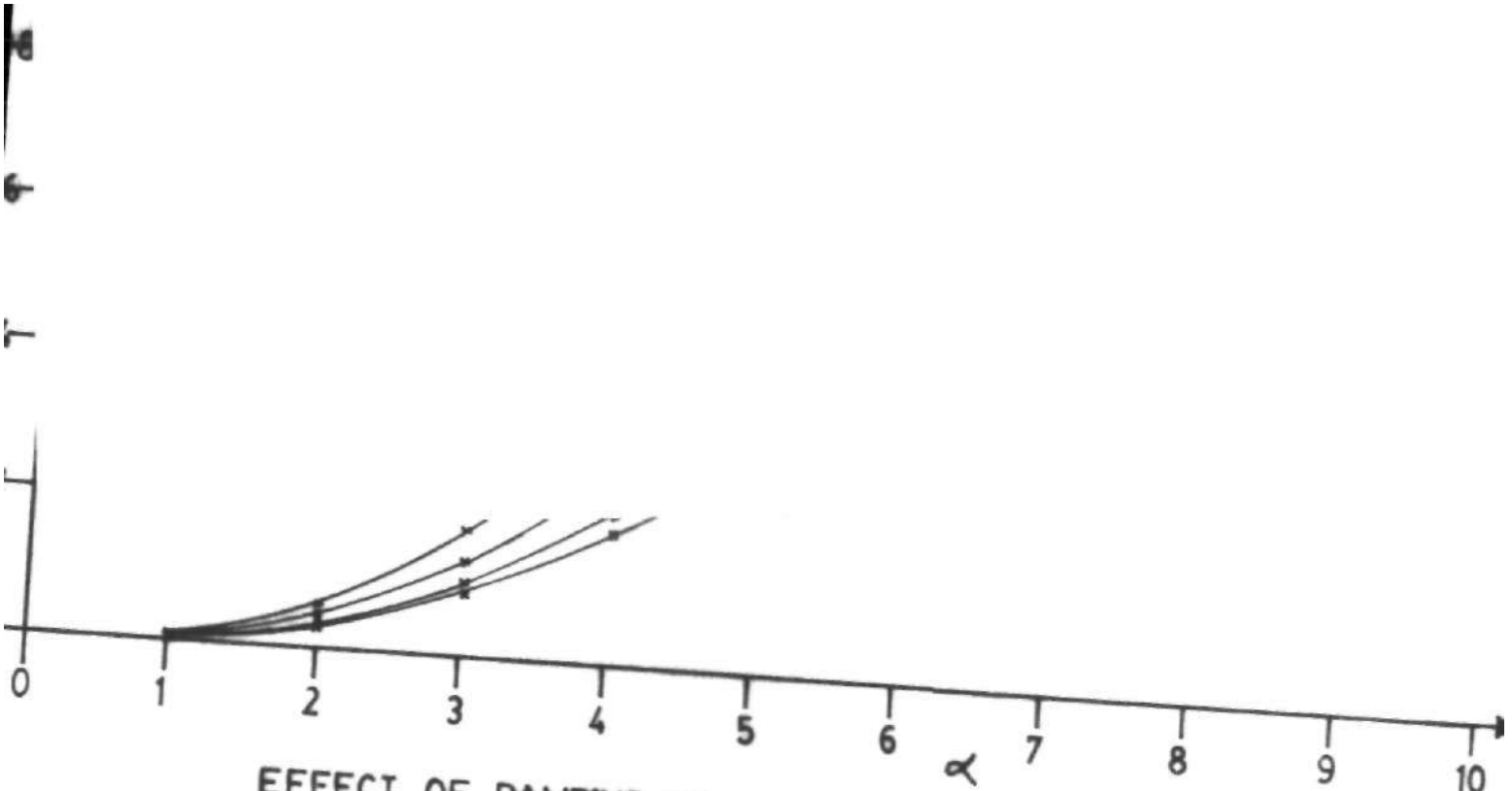
EFFECT OF DAMPING ON α (K_p=0, K_v=1, ω_n=0, R=0·1)

Fig 3



EFFECT OF DAMPING ON ∝ (K_a=0, K_s=1, M=0, R=1)

Fig. 4



EFFECT OF DAMPING ON α ($K_2=0$, $K_3=1$, $M=0$, $R=100$)

Fig. 5

EFFECT OF DAMPING ON α ($K_s=1, M=0, K=0$)

Fig. 6

CHAPTER IV: MICROPOLAR PERISTALTIC TRANSPORT OF A FLUID - PARTICLE SUSPENSION

4.1: INTRODUCTION.

The basis for this chapter is an extension of Chow's paper on "Peristaltic transport in circular cylindrical pipe" to a micropolar peristaltic transport of a fluid- particle suspension in two-dimensional domain model.

Chow described the flow of an incompressible viscous fluid through a circular pipe for the case of an axisymmetric wave travelling along the wall when the initial motion is assumed to be a Hagen-Poiseuille flow.

To understand the approach to micropolar fluids many authors like Eringen (1966), Stokes (1966), Cowin (1974) have proposed micro-continuum theory to study the particle size effects in the flow of suspensions. From the theory of simple microfluids, Rao and Rapu (1980) noted that it takes of the effect arising from the local structure of fluid elements and their intrinsic motions consequent to these effects. Apart from the usual translatory degrees of freedom, the fluid element has additional degrees of freedom enabling it to undergo intrinsic rotation as well as deformation and the latter features are reckoned by the gyration vector fields. The simple version leads to the micropolar fluids in which local motion of a fluid element is a rigid rotations (i.e. the spin and rotation give rise to micropolar). Thus Parvathamma and Devanathan (1983) noted that micropolar fluids

can support couple stresses and body couples and also exhibits microrational effects and microrotational inertia.

The fluid departs from Navier-Stoke's theory because of sustenance of couple stress and the non-symmetry of the stress tensor. Our fluid is taken as blood. Blood under certain flow situations can be represented by a polar fluid. It consists of a suspension of a variety of cells in an aqueous solution of organic and inorganic substances (called plasma) Lens and Fung (1970) and that in the microcirculation, the radii of the blood cells are larger than the radii of the blood vessels and hence clearly behaves like a non-Newtonian fluid. At low shear flow rates, the rheological properties of the blood are very complex. A viscous incompressible fluid is assumed to be confirmed in a channel which symmetric travelling transverse waves are imposed. The motion is studied The solution is represented by a power series expansion to the second order of amplitude - to - radius ratio, ε^2 which is to be much less than unity ($\varepsilon^2 << 1$). Another assumption is that the tangential motion of the wall is negligible. Therefore an expansion solution is valid to within $O(\varepsilon^2)$.

We started the analysis parallel to Fung and obtained the Zeroth, first and second order equations. The solutions for free pumping (the zero pressure gradient) and the close form solutions obtained in which the higher order solutions are concerned. Solutions were obtained in relation to Stokes Approximation, a situation in which the Reynolds number is small enough to be considered inertia-free and solutions for higher order Reynolds number.

4.2 Formulation of the Problem.

The general balence equations in the absence of body forces, couples, and thermal effects for a micropolar fluid given by Kline and Allen (1970) are

$$\dot{\rho} + \rho V_{i,i} = 0 \tag{1}$$

$$t_{ji,j} = \rho \dot{V}_i \tag{2}$$

$$h_{K[ij],K} + t_{[ij]} = \frac{\cdot}{\rho I_{[iK} W_{Kj]}} \tag{3}$$

where for an arbitrary second - order tensor **A**

$$A_{(ij)} = \tfrac{1}{2}(A_{ij} + A_{ji}) \; ; \quad A_{[ij]} = \tfrac{1}{2}(A_{ij} - A_{ji})$$

In the foregoing ρ is the mass density, V the velocity, t the Cauchy stress, U the couple stress, I the substructure inertia coefficients, $W_{[ij]}$ the substructure angular velocity, and $W_{(ij)}$ is a measure of substructure deformation rate. The dot denotes the material derivative and the comma partial differetiation.

The inertia tensor I satisfies

$$\dot{I}_{ij} = I_{iK} W_{Kj} + I_{Kj} W_{Ki} \tag{4}$$

For the case of rigid spherical substructure

$$I_{ij} = k^2 \delta_{ij} \tag{5}$$

Equations (4) and (5) require that the substructure deformation rates $W_{(ij)}$ vanish. The constant k in equation (5) measures the average substructure radius of gyration.

Constitive equations for incompressible fluids with rigid spherical substructure have been discussed extensively by Allen and Kline (1970s), (1960) and are summarised next

$$\left.\begin{aligned} t_{ij} &= -p\,\delta_{ij} + 2h V_{(i,j)} - 2h_1 (V_{[i,j]} + W_{[ij]}) \\ h_{i[Kj]} &= n_1 W_{[Kj],i} + n_2 (W_{[ij],K} - W_{[iK],j}) + \\ &\quad n_3 (\delta_{iK} W_{[mj],m} - \delta_{ij} W_{[mK],m}) \end{aligned}\right\} \quad (6)$$

The coefficient h is the usual viscosity, h_1 the vortex viscosity, the n's have been spin viscosity, and p is undermined pressure.

The entropy production inequality requires that

$$h \geq 0, \quad h_1 \geq 0, \quad n_1 + n_2 + n_3 \geq 0, \; n_1 \geq 0, \; -n_1 \leq n_2 \leq n_1/2 \quad (7)$$

At this point, attention will be restricted to two- dimensional unsteady motion with the velocity, spin, and pressure fields having the form

$$V_i = (U(x,y,t);\, V(x,y,t);\, 0), \; W_{[ij]} = (W_{[12]} = W = W(x,y);\, 0;\, 0),$$

$$p = p(x, y, t) \quad (8)$$

Consider the flow of a micropolar fluid moving in axi symmetric form in a flexible and extensible two dimen-

sional channel. The fluid is set into motion by imposing an axisymmetric sinusoidal travelling waves of small amplitude a. The appropriate equation describing the flow are obtained using equations (5), (6) and (8) in balance equations (1) - (3) as

$$\rho\left(\frac{\partial u}{\partial t} + UU_x + VU_y \right) = - \frac{\partial p}{\partial x} + (h + h_1)\nabla_u^2 + 2h_1\, \omega_y \quad (9)$$

$$\rho\left(\frac{\partial v}{\partial t} + UV_x + VV_y \right) = - \frac{\partial p}{\partial y} + (h + h_1)\nabla^2 V - 2h_1\, \omega_x \quad (10)$$

$$\rho K^2\left(\frac{\partial \omega}{\partial t} + U\omega_x + V\omega_y \right) = hk^2\nabla^2\omega - h_1(U_y - V_x + 2\omega) \quad (11)$$

and the equation of continuity (12)

$$U_x + V_y = 0$$

in which $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator and where x and y subscripts denote differentiation with respect to the x and y coordinates respectively. In addition use has been made of the relationship

$$hk^2 = n_1 + n_2 + n_3 \quad (13)$$

which arises from the requirement that substructure spin identically equal to the local fluid vorticity should be a possible solution of the balance equations. This requirement appears to be a natural one and has the desirable

result of reducing the number of constitutive coefficients.

The fluid is subjected to boundary conditions imposed by the symmetric motion of the flexible walls. Let the vertical displacements of the upper and lower walls be η and $-\eta$ then

$$\eta(x, y) = a \cos\frac{2\pi}{\lambda}(x - ct) \tag{14}$$

where a is the amplitude, λ the wavelength and c the wave speed. Introducing dimensionless variables

$$\left.\begin{aligned} &\frac{x'}{d}, \; y' = \frac{y}{d}, \; U' = \frac{U}{C}, \; V' = \frac{V}{C}, \; \omega' = \frac{\omega d}{c}, \; t' = \frac{ct}{d} \\ &P' = \frac{p}{\rho c^2}, \; \nu' = \frac{\nu}{cd}, \; R = \frac{dc}{\nu+\nu_1}, \; K' = \frac{K}{d} \end{aligned}\right\} \tag{15}$$

where $\nu = \frac{h}{\rho}$, $\nu_1 = \frac{h_1}{\rho}$

Using equation (15) in equations (9) - (12) yield the following after dropping the accents.

$$R\left[\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + V\frac{\partial u}{\partial y}\right] = -R\frac{\partial p}{\partial x} + \nabla^2 u + \frac{2\sigma}{1+\sigma}\frac{\partial \omega}{\partial y} \tag{16}$$

$$R\left[\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + V\frac{\partial v}{\partial y}\right] = -R\frac{\partial p}{\partial y} + \nabla^2 V - \frac{2\sigma}{1+\sigma}\frac{\partial \omega}{\partial x} \tag{17}$$

$$(1+\sigma)R\left[\frac{\partial \omega}{\partial t} + U\frac{\partial \omega}{\partial x} + V\frac{\partial \omega}{\partial y}\right] = \nabla_\omega^2 - \bar{\alpha}^2\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + 2\omega\right] \tag{18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{19}$$

where $\sigma = \frac{h_1}{h}$, $\bar{\alpha} = \frac{1}{K}$

Introducing the steam function Ψ as

$$U = \frac{\partial \Psi}{\partial y} , \quad V = -\frac{\partial \Psi}{\partial x} \tag{20}$$

and a substitution of equation (20) in equations (6) - (19) gives

$$R\left[\frac{\partial^2 \Psi}{\partial t \partial y} + \frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2}\right] = -R\frac{\partial p}{\partial x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial \Psi}{\partial y} + \frac{2\sigma}{1+\sigma}\frac{\partial \omega}{\partial y} \tag{21}$$

$$R\left[\frac{\partial^2 \Psi}{\partial t \partial x} + \frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial x \partial y}\right] = R\frac{\partial p}{\partial y} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\frac{\partial \Psi}{\partial x} + \frac{2\sigma}{1+\sigma}\frac{\partial \omega}{\partial x} \tag{22}$$

$$(1+\sigma)R\left[\frac{\partial \omega}{\partial t} + \frac{\partial \Psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial \omega}{\partial y}\right] = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\omega - \bar{\alpha}^2\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + 2\omega\right) \tag{23}$$

Assume that there is no horizontal displacement in the wall then the relevant boundary conditions of the problem are the classical non slip boundary on the velocity and a new constant spin condition are due to Ariman et al at the wall are given by

$$\left.\begin{aligned} \Psi_y &= 0, \quad \Psi_x = \mp \alpha \varepsilon \sin \alpha(x-t) \\ \omega_y &= 0 \end{aligned}\right\} \text{ at } y = \pm(1+\eta) \quad (24)$$

where $\varepsilon = \frac{a}{d}$, $\alpha = \frac{2\pi d}{\lambda}$, $\eta = \varepsilon \cos (x - t)$

4.3 Analysis

The solution for Ψ is obtained as a power series in terms of the amplitude of the wave ε by expanding Ψ, ω and p in the following form

$$\left.\begin{aligned} \Psi &= \Psi_o + \varepsilon\Psi_1 + \varepsilon^2\Psi_2 + \ldots\ldots \\ \omega &= \omega_o + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots\ldots \\ p &= p_o + \varepsilon p_1 + \varepsilon^2 p_2 + \ldots\ldots \end{aligned}\right\} \quad (25)$$

also the boundary condition (24) can be expanded as

$$\left.\begin{aligned} \Psi_y(\pm 1) &\pm \eta\Psi_{yy}(\pm 1) + \frac{\eta^2}{2}\Psi_{yyy}(\pm 1) + \ldots 0 \\ \Psi_x(\pm 1) &\pm \eta\Psi_{xy}(\pm 1) + \frac{\eta^2}{2}\Psi_{yyy}(\pm 1) + \ldots 0 \\ \omega_y(\pm 1) &\pm \eta\,\omega_{yy}(\pm 1) + \frac{\eta^2}{2}\omega_{yyy}(\pm 1) + \ldots 0 \end{aligned}\right\} \quad (26)$$

Eliminating pressure gradient between equations (21) and (22), one obtains

$$R\left[\frac{\partial}{\partial t}\nabla^2\Psi + \Psi_y\nabla^2\Psi_x - \Psi_x\nabla^2\Psi_y\right] - \frac{2\sigma}{1+\sigma}\nabla^2\omega = \nabla^2\nabla^2\Psi \quad (27)$$

and equation (23) may be rewritten as

$$(1+\sigma)R\left[\frac{\partial\omega}{\partial t}+\frac{\partial\Psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\Psi}{\partial x}\frac{\partial\omega}{\partial y}\right]+\bar{\alpha}^2(\nabla^2\Psi+2\omega)=\nabla^2\omega \quad (28)$$

Substitution of equations (25) and (26) into (27) - (28) and (24) respectively yield the following set of differential equations along with the boundary conditions for zeroth, first and second order perturbations.

Zeroth order equations :

$$R\left[\frac{\partial}{\partial t}\nabla^2\Psi_o+\Psi_{oy}\nabla^2\Psi_{ox}-\Psi_{ox}\nabla^2\Psi_{oy}\right]-\frac{2\sigma}{1+\sigma}\nabla^2\omega_o=\nabla^2\nabla^2\Psi_o \quad (29)$$

$$(1+\sigma)R\left[\frac{\partial}{\partial t}\omega_o+\Psi_{oy}\omega_{ox}-\Psi_{ox}\omega_{oy}\right]+\bar{\alpha}^2(\nabla^2\Psi_o+2\omega_o)=\nabla^2\omega_o \quad (30)$$

$$\Psi_{oy}(\pm1)=\Psi_{ox}(\pm1)=\omega_{oy}(\pm1)=0 \quad (31)$$

first order equations :

$$R\left[\frac{\partial}{\partial t}\nabla^2\Psi_1+\Psi_{oy}\nabla^2\Psi_{1x}+\Psi_{1y}\nabla^2\Psi_{ox}-\Psi_{ox}\nabla^2\Psi_{1y}-\Psi_{1x}\nabla^2\Psi_{oy}\right]$$

$$-\frac{2\sigma}{1+\sigma}\nabla^2\omega_1=\nabla^2\nabla^2\Psi_1 \quad (32)$$

$$(1+\sigma)R\left[\frac{\partial}{\partial t}\omega_1+\Psi_{oy}\omega_{1x}+\Psi_{1y}\omega_{ox}-\Psi_{ox}\omega_{1y}-\Psi_{1x}\omega_{oy}\right]$$

$$+\bar{\alpha}^2(\nabla^2\Psi_1+2\omega_1)=\nabla^2\omega_1 \quad (33)$$

$$\left.\begin{aligned} \Psi_{1y}(\pm 1) \pm \Psi_{oyy}(\pm 1) \cos\alpha\,(x-t) &= 0 \\ \Psi_{1x}(\pm 1) \pm \Psi_{oxy}(\pm 1) \cos\alpha\,(x-t) &= \mp\,\alpha \sin\alpha(x-t) \\ \omega_{1y}(\pm 1) \pm \omega_{oyy}(\pm 1) \cos\alpha\,(x-t) &= 0 \end{aligned}\right\} \quad (34)$$

Second order equations :

$$R\Big[\frac{\partial}{\partial t}\nabla^2\Psi_2 + \Psi_{2y}\nabla^2\Psi_{ox} + \Psi_{1y}\nabla^2\Psi_{1x} + \Psi_{oy}\nabla^2\Psi_{2x} - \Psi_{ox}\nabla^2\Psi_{2y} - \Psi_{1x}\nabla^2\Psi_{1y} - \Psi_{2x}\nabla^2\Psi_{oy}\Big] - \frac{2\sigma}{1+\sigma}\nabla^2\omega_2 = \nabla^2\nabla^2\Psi_2 \quad (35)$$

$$(1+\sigma)R\Big[\frac{\partial}{\partial t}\omega_2 + \Psi_{2y}\omega_{ox} + \Psi_{oy}\omega_{2x} + \Psi_{1y}\omega_{1x} - \Psi_{ox}\omega_{2y} - \Psi_{1x}\omega_{1y} - \Psi_{2x}\omega_{oy}\Big] + \bar{\alpha}^2(\nabla^2\Psi_2 + 2\omega_2) = \nabla^2\omega_2 \quad (36)$$

$$\left.\begin{aligned} \Psi_{2y}(\pm 1) \pm \Psi_{1yy}(\pm 1) \cos\alpha(x-t) + \tfrac{1}{2}\Psi_{oyyy}(\pm 1) \cos^2\alpha(x-t) &= 0 \\ \Psi_{2x}(\pm 1) \pm \Psi_{1xy}(\pm 1) \cos\alpha(x-t) + \tfrac{1}{2}\Psi_{oxyy}(\pm 1) \cos^2\alpha(x-t) &= 0 \\ \omega_{2y}(\pm 1) \pm \omega_{1yy}(\pm 1) \cos\alpha\,(x-t) + \tfrac{1}{2}\omega_{oyyy}(\pm 1) \cos^2\alpha(x-t) &= 0 \end{aligned}\right\} \quad (37)$$

Equations (29) - (31) together with the condition of symmetry and a uniform pressure gradient in the x-direction $(\partial p/\partial x)_o$ = constant, yield the following classical Hagen-Poiseville flow in the two dimensional channel for micropolar fluid as

$$\Psi_o = K\left[y - \frac{1}{3}y^3 - \frac{2\sigma}{1+\sigma}\,\frac{\bar{\alpha}y\cosh\bar{\alpha} - \sinh\bar{\alpha}y}{\bar{\alpha}^3\cosh\bar{\alpha}}\right] \tag{38}$$

$$\omega_o = K\left[y - \frac{\sinh\bar{\alpha}\,y}{\bar{\alpha}\cosh\bar{\alpha}}\right] \tag{39}$$

where $K = \frac{1}{2h}\left(-\frac{\partial p}{\partial x}\right)_o$

Using equations (38) - (39) into equations (32) - (37), the first order reduce to

$$R\left[\frac{\partial}{\partial t}\nabla^2\Psi_1 + K\left[1 - y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\nabla^2\Psi_{1x} - 2K\left[\frac{\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}} - 1\right]\Psi_{1x}\right]$$

$$- \frac{2\sigma}{1+\sigma}\nabla^2\omega_1 = \nabla^2\nabla^2\Psi_1 \tag{40}$$

$$(1+\sigma)R\left[\frac{\partial}{\partial t}\omega_1 + K\left[1-y^2 - \frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}\cosh\bar{\alpha}}\right]\omega_{1x} - K\left[1 - \frac{\cosh\bar{\alpha}\,y}{\cosh\bar{\alpha}}\right]\Psi_{1x}\right] + \bar{\alpha}^2(\nabla^2\Psi_1 + 2\omega_1) = \nabla^2\omega_1 \tag{41}$$

$$\left.\begin{aligned} &\Psi_{1y}(\pm 1) \pm 2K\left[\frac{\sigma}{1+\sigma}\,\frac{\sinh\bar{\alpha}}{\bar{\alpha}\cosh\bar{\alpha}} - 1\right]\cos\alpha\,(x-t) = 0 \\ &\Psi_{1x}(\pm 1) = \mp\alpha\sin\alpha\,(x-t) \\ &\omega_{1y}(\pm 1) \mp K\alpha\,\frac{\sinh\bar{\alpha}}{\cosh\bar{\alpha}}\cosh(x-t) = 0 \end{aligned}\right\} \tag{42}$$

while the second order

$$R\left[\frac{\partial}{\partial t}\nabla^2\Psi_2 + \Psi_{1y}\nabla^2\Psi_{1x} + K\left[1-y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\nabla^2\Psi_{2x} -\right.$$

$$- \Psi_{1x} \nabla^2 \Psi_{1y} - 2K \left[\frac{\sigma}{1+\sigma} \frac{\cosh \bar{\alpha} y}{\cosh \bar{\alpha}} - 1 \right] \Psi_{2x} \Bigg]$$

$$- \frac{2\sigma}{1+\sigma} \nabla^2 \omega_2 = \nabla^2 \nabla^2 \Psi_2 \tag{43}$$

$$(1+\sigma)R \Bigg[\frac{\partial}{\partial t} \omega_2 + K \Bigg[1 - y^2 - \frac{2\sigma}{1+\sigma} \frac{\cosh \bar{\alpha} - \cosh \bar{\alpha} y}{\bar{\alpha}^2 \cosh \bar{\alpha}} \Bigg] \omega_{2x} +$$

$$+ \Psi_{1y} \omega_{1x} - \Psi_{1x} \omega_{1x} - K \Bigg[1 - \frac{\cosh \bar{\alpha} y}{\cosh \bar{\alpha}} \Bigg] \Psi_{2x} \Bigg] +$$

$$+ \bar{\alpha}^2 (\nabla^2 \Psi_2 + 2\omega_2) = \nabla^2 \omega_2 \tag{44}$$

$$\left.\begin{aligned} &\Psi_{2y}(\pm 1) \pm \Psi_{1yy}(\pm 1) \cos \alpha (x-t) + K \Bigg[\frac{\sigma}{1+\sigma} \frac{\cosh \bar{\alpha}}{\cosh \bar{\alpha}} - 1 \Bigg] \times \\ &\qquad \times \cos^2 \alpha (x - t) = 0 \\ &\Psi_{2x}(\pm 1) \pm \Psi_{1xy}(\pm 1) \cos \alpha (x - t) = 0 \\ &\omega_{2y}(\pm 1) \pm \omega_{1yy}(\pm 1) \cos \alpha (x-t) - K \frac{\bar{\alpha}^2 \cosh \bar{\alpha}}{2 \cosh \bar{\alpha}} \times \\ &\qquad \cos^2 \alpha (x - t) = 0 \end{aligned}\right\} \tag{45}$$

Equations (40) - (42) can be satisfied by the solution of the form

$$\left.\begin{aligned} \Psi_1 &= \tfrac{1}{2} \Big[\phi_1(y) \, e^{i\alpha(x-t)} + \phi_1^*(y) \, e^{-i\alpha(x-t)} \Big] \\ \omega_1 &= \tfrac{1}{2} \Big[\Omega_1(y) e^{i\alpha(x-t)} + \Omega_1^*(y) e^{-i\alpha(x-t)} \Big] \end{aligned}\right\} \tag{46}$$

where the asterisk denotes the complex conjugate. A substitution into equations (40) - (42) yield the following :

$$\left[\frac{d^2}{dy^2} - \alpha^2 + i\alpha R\left\{1 - K\left[1-y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\right\}\right] x$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_1 - 4i\alpha RK\left[\frac{\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha}\,y}{\cosh\bar{\alpha}} - 1\right]\phi_1$$

$$+ \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_1 = 0 \qquad (47)$$

$$\left[\frac{d^2}{dy^2} - \alpha^2 - 2\bar{\alpha}^2 + i\alpha(1+\sigma)R\left\{1 - K\left[1 - y^2 - \right.\right.\right.$$

$$\left.\left.\left. - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}\,y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\right\}\right]\Omega_1 - \bar{\alpha}^2\left[\frac{d^2}{dy^2} - \alpha^2\right.$$

$$\left. - \frac{i\alpha(1+\sigma)K}{\bar{\alpha}^2}\left(1 - \frac{\cosh\bar{\alpha}\,y}{\cosh\bar{\alpha}}\right)R\right]\phi_1 = 0 \qquad (48)$$

$$\left.\begin{aligned}\phi_1'(\pm 1) &= \pm K\left[\frac{\sigma}{1+\sigma}\,\frac{\sinh\bar{\alpha}}{\bar{\alpha}\cosh\bar{\alpha}} - 1\right]\\ \phi_1(\pm 1) &= \pm 1\\ \Omega_1'(\pm 1) &= \pm\frac{\alpha K}{2}\,\frac{\sinh\bar{\alpha}}{\cosh\bar{\alpha}}\end{aligned}\right\} \qquad (49)$$

The equations governing ϕ_1^* and Ω_1^* are conjugate to (47)-(49) and need not be considered Equations (43) -(45) can be satisfied by the solution of the form

$$\left.\begin{aligned}\Psi_2 &= \tfrac{1}{2}\left[\phi_{20}(y) + \phi_{22}(y)e^{2i\alpha(x-t)} + \phi_{22}^*(y)e^{-2i\alpha(x-t)}\right]\\ \Omega_2 &= \tfrac{1}{2}\left[\omega_{20}(y) + \omega_{22}(y)e^{2i\alpha(x-t)} + \omega_{22}^*(y)\,e^{-2i\alpha(x-t)}\right]\end{aligned}\right\} \qquad (50)$$

Since the mean flow is concerned with the part of the solutions ϕ_{20} and ω_{20}, then

$$\phi_{20}''' + \frac{2\sigma}{1+\sigma}\,\omega_{20}'' = -\frac{i\alpha R}{2}\left(\phi_1\,\phi_1^{*\prime\prime} - \phi_1^{*}\,\phi_1''\right) \tag{51}$$

$$\omega_{20}'' - 2\bar{\alpha}^2\,\omega_{20} - \bar{\alpha}^2\,\phi_{20}'' = 0 \tag{52}$$

$$\left.\begin{aligned} &\phi_{20}''(\pm 1) \pm \tfrac{1}{2}\left(\phi_1''(\pm 1) + \phi_1^{*\prime\prime}(\pm 1)\right) + K\left[\frac{\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha}}{\cosh\bar{\alpha}} - 1\right] = 0 \\ &\omega_{20}'(\pm 1) - \frac{K\,\alpha^2\cosh\bar{\alpha}}{2\cosh\bar{\alpha}} = 0 \end{aligned}\right\} \tag{53}$$

4.4 Solution for the free pumping case

The solution for the free pumping case can be obtained by taking $\left(\frac{\partial p}{\partial x}\right)_0 = 0$ i.e $K = 0$. In this case, there will be no flow if wall motion stops. Thus equations (47) - (49) become

$$\left(\frac{d^2}{dy^2} - \beta^2\right)\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_1 + \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_1 = 0 \tag{54}$$

$$\left(\frac{d^2}{dy^2} - \beta_1^2\right)\Omega_1 - \bar{\alpha}^2\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_1 = 0 \tag{55}$$

$$\left.\begin{aligned} \phi_1'(\pm 1) &= 0 \\ \phi_1(\pm 1) &= \pm 1 \\ \Omega_1(\pm 1) &= 0 \end{aligned}\right\} \tag{56}$$

Since equations (54) and (55) are subject to equation (56), their solutions are

$$\phi_1(y) = A_{41} \sinh \alpha y + B_{41} \sinh \beta y \tag{57}$$

$$\Omega_1(y) = 0 \tag{58}$$

where

$$\left.\begin{aligned} \beta^2 &= \alpha^2 - i\alpha R \\ \beta_1^2 &= \alpha^2 + 2\bar{\alpha}^2 - i\alpha(1+\sigma)R \end{aligned}\right\} \tag{59}$$

$$\left.\begin{aligned} A_{41} &= -\frac{\bar{\beta}\cosh\beta}{\alpha\cosh\alpha\sinh\beta - \beta\cosh\beta\sinh\alpha} \\ B_{41} &= \frac{\alpha\cosh\alpha}{\alpha\cosh\alpha\sinh\beta - \beta\cosh\beta\sinh\alpha} \end{aligned}\right\} \tag{60}$$

while (51) -(53) reduce to

$$\phi_{20}'' + \frac{2\sigma}{1+\sigma}\omega_{20}' = \frac{i\alpha R}{2}\left(\phi_1\phi_1^{*\prime} - \phi_1^*\phi_1'\right)$$

$$\omega_{20}' - 2\bar{\alpha}\omega_{20} - \bar{\alpha}^2\phi_{20}' = 0$$

$$\left.\begin{aligned} \phi_{20}'(\pm 1) \omega \pm \tfrac{1}{2}\left(\phi_1''(\pm 1) + \phi_1^{*\prime}(\pm 1)\right) &= 0 \\ \omega_{20}'(\pm 1) &= 0 \end{aligned}\right\} \tag{61}$$

Equation (43) reveals that there vanishes the first order term of micro rotation, when the fluid is originally stationary which is analogous to the property observed by Devi (1975) whereas equation (57) is same as obtained by Fung and Yih (1968) as expected.

Solving (51) and (52) subject to (61)

$$\omega_{20} = A_{42} \cosh \bar{\alpha}_1 y + B_{42} \sinh \bar{\alpha}_1 y + F_1(y) - \frac{1}{\alpha_1^2}(M_{15} + M_{16}) \tag{62}$$

Applying boundary condition (61), $A_{42} = 0$, $M_{16} = 0$

$$B_{42} = \frac{-C_{12}}{\alpha_1 \cosh \alpha_1} \tag{63}$$

where $C_{12} = F_1'(1) - \frac{1}{\alpha_1^2} M_{15}$

$$\bar{\alpha}_1 = \bar{\alpha} \sqrt{\frac{2\sigma}{1+\sigma}}$$

Finally

$$\omega_{20} = C_{12} \left[y - \frac{\sinh \alpha_1 y}{\alpha_1 \cosh \alpha_1} \right] + F_1(y) - yF_1'(1) \tag{64}$$

where

$$F_1(y) = \frac{\bar{\alpha}^2 \alpha^2 R^2}{2} \left[\frac{A^* B}{2} \left[\frac{\sinh (\alpha+\beta)y}{(\alpha+\beta)\left[(\alpha+\beta)^2 - \bar{\alpha}_1^2\right]} - \frac{\sinh (\alpha-\beta)y}{(\alpha-\beta)\left[(\alpha-\beta)^2 - \bar{\alpha}_1^2\right]} \right] \right.$$

$$+ \frac{AB^*}{2}\left[\frac{\sinh(\alpha+\beta^*)y}{(\alpha+\beta^*)\left[(\alpha+\beta^*)^2-\bar{\alpha}_1^2\right]} - \frac{\sinh(\alpha-\beta^*)y}{(\alpha-\beta^*)\left[(\alpha-\beta^*)^2-\bar{\alpha}_1^2\right]}\right]$$

$$+ BB^*\left[\frac{\sinh(\beta+\beta^*)y}{(\beta+\beta^*)\left[(\beta+\beta^*)^2-\bar{\alpha}_1^2\right]} - \frac{\sinh(\beta-\beta^*)y}{(\beta-\beta^*)\left[(\beta-\beta^*)^2-\bar{\alpha}_1^2\right]}\right]\Bigg] \quad (65)$$

and $$\phi'_{20} = -\frac{2\sigma C_{12}}{\bar{\alpha}_1^2(1+\sigma)}\left[1 - \frac{\cosh\bar{\alpha}_1 y}{\cosh\bar{\alpha}_1}\right] - \frac{2\sigma}{1+\sigma}\left\{\int F_1(y) - \int F_1(1)\right\} +$$

$$+ D + F(y) - F(1) - C_2(1-y^2) \quad (66)$$

with

$$F(y) = \frac{\alpha^2R^2}{2}\left[\frac{A_{41}E^*_{41}}{2}\left[\frac{\cosh(\alpha+\beta)y}{(\alpha+\beta)^2} - \frac{\cosh(\alpha-\beta)y}{(\alpha-\beta)^2}\right] + \right.$$

$$+ \frac{A_{41}E^*_{41}}{2}\left[\frac{\cosh(\alpha+\beta^*)y}{(\alpha+\beta^*)^2} - \frac{\cosh(\alpha-\beta^*)y}{(\alpha+\beta^*)^2}\right] + B_{41}B^*_{41}\left[\frac{\cosh(\beta+\beta^*)y}{(\beta+\beta^*)^2}\right.$$

$$\left.\left.\frac{\cosh(\beta-\beta^*)y}{(\beta-\beta^*)^2}\right]\right] \quad (67)$$

$$D = -\frac{1}{2}\left[(A_{41}+A^*_{41})\alpha^2\sinh\alpha + \beta^2B_{41}\sinh\beta + \beta^{*2}B^*_{41}\sinh\beta^*\right]$$

and are same as that obtained by Fung and Yih. The solution (66) also involves an arbitrary constant which is proportional to the mean pressure gradient. The mean pressure gradient

$$\overline{\frac{\partial p}{\partial x}} = \varepsilon^2\left(\overline{\frac{\partial p}{\partial x}}\right)_2 = \frac{\varepsilon^2}{R}\left(\omega''_{20} - \alpha_1^2\omega'_{20}\right) + \frac{\varepsilon^2}{4}\left(\phi_1\phi_1^{*\prime\prime} - \phi_1^*\phi_1''\right) + O(\varepsilon^3)$$

$$= \frac{\varepsilon^2}{R}C_{12} + O(\varepsilon^3)$$

i.e. $$\left(\overline{\frac{\partial p}{\partial x}} \right)_2 = \frac{C_{12}}{R}$$

Similarly,

$$\frac{\partial p}{\partial x} = \varepsilon^2 \left(\overline{\frac{\partial p}{\partial x}} \right)_2 = \frac{\varepsilon^2}{2R} \left(\phi_{20}^{\prime\prime\prime} + \frac{2\sigma}{1+\sigma} \omega_{20}^{\prime} \right) + \frac{\varepsilon^2}{4} i\alpha \left(\phi_1 \phi_1^{*\prime\prime} - \phi_1^{*} \phi_1^{\prime\prime} \right)$$

$$+ O(\varepsilon^3) = \frac{\varepsilon^2}{R} C_2 + O(\varepsilon^3)$$

i.e. $$\left(\overline{\frac{\partial p}{\partial x}} \right)_2 = \frac{C_2}{R}$$

Thus $C_{12} = C_2$, apart from the factor $\frac{\varepsilon^2}{R}$, the time average of the pressure gradient accompanying the peristaltic motion. Note that the time - averaged pressure gradient distribution $\left(\overline{\frac{\partial p}{\partial x}} \right)_2$ turns out to be uniform over the cros-section; i.e. it is independent of the y- coordinate.

In the case when $\sigma = 0$, equation (66) reduces to that of equation (46) of Fung and Yih. The constant $C_{12} = C_2$ must be determined from considerations of conditions at the ends of the channel. With $C_{12} = C_2$ specified, our solution for the mean speed (average over time) of flow is :

$$\bar{U} = \frac{\varepsilon^2}{2} \phi_{20}^{\prime} = \frac{\varepsilon^2}{2} \left[F(y) - F(1) + D - C_2(1-y^2) - \frac{2\sigma C_{12}}{(1+\sigma)\bar{\alpha}_1^2} \times \right.$$

$$\left. \left[1 - \frac{\cosh \bar{\alpha}_1 y}{\cosh \bar{\alpha}_1} \right] - \frac{2\sigma}{1+\sigma} \left[\int F_1(y) - \int F_1(1) \right] \right] \quad (68)$$

It may be observed as a special case that when the second order mean pressure gradient due to peristaltic motion vanishes, $C_{12} = C_2 = 0$. Further, it may be noted that existence of $C_{12} = C_2$ means there is a Poiseville flow induced entirely by the peristaltic motion. Thus the solution is complete up to the second order. Extension to the higher orders is of course possible.

4.5 Interaction with Poiseuille flow

In order to observe the interaction with Poiseuille (i.e. the case when $K \neq 0$), we further return to equations (32) - (34) and by substitution of equations (38) and (39) in these equations yield the following equations for the first order perturbation along with the boundary conditions

$$R\left[\frac{\partial}{\partial t}\nabla^2\psi_1 + K\left[1-y^2 - \frac{2\sigma}{1+\sigma}\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\nabla^2\psi_{1x} + \right.$$

$$\left. + 2K\left[1 - \frac{\sigma}{1+\sigma}\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\psi_{1x}\right] = \nabla^4\psi_1 + \frac{2\sigma}{1+\sigma}\nabla^2\omega_1 \qquad (69)$$

$$(1+\sigma)R\left[\frac{\partial}{\partial t}\omega_1 + K\left[1 - y^2 - \frac{2\sigma}{1+\sigma}\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\omega_{1x} - \right.$$

$$\left. - K\left[1 - \frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\right] = \nabla^2\omega_1 - (1+\sigma)\bar{\alpha}^2\omega_1 - \frac{(1+\sigma)}{2}\bar{\alpha}^2\nabla^2\psi_1$$

$$\left.\begin{aligned} &\psi_{1y}(\pm 1) - 2K\left[1 - \frac{\sigma}{1+\sigma}\frac{\sinh\bar{\alpha}}{\bar{\alpha}\cosh\bar{\alpha}}\right]\cos\alpha(x-t) = 0 \\ &\psi_{1x}(\pm 1) = \mp\alpha\sinh\alpha\ (x-t) \end{aligned}\right\} \qquad (70)$$

$$\omega_{1y}(\pm 1) - K\bar{\alpha}\frac{\sinh\bar{\alpha}}{\cosh\bar{\alpha}}\cosh\alpha(x-t) = 0 \Bigg\} \qquad (71)$$

In view of the boundary conditions (71), the solutions for Ψ_1 and ω_1 may be assumed as

$$\Psi_1 = \bar{\phi}_1(y)\cos\alpha(x-t) + \bar{\phi}_2(y)\sin\alpha(x-t)$$
$$\omega_1 = \bar{\Omega}_1(y)\cos\alpha(x-t) + \bar{\Omega}_2(y)\sin\alpha(x-t) \qquad (72)$$

Substitution of (72) into equations (69) and (70), yield the following equations for $\bar{\phi}_1$, $\bar{\phi}_2$ and $\bar{\Omega}_1$, $\bar{\Omega}_2$.

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\phi}_1 + \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\Omega}_1 = \alpha R\left(1+K\left[1-y^2 - \frac{2\sigma}{1+\sigma}\frac{\cosh\bar{\alpha}-\cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\right)\times$$
$$\left(\frac{d^2}{dy} - \alpha^2\right)\bar{\phi}_2 + 2\alpha RK\left[1 - \frac{\sigma}{1+\sigma}\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right] \qquad (73)$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2\bar{\phi}_2 + \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\Omega}_2 = \alpha R\left[K\left[1-y^2 -\right.\right.$$
$$\left.\left. - \frac{2\sigma}{1+\sigma}\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right] - 1\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\phi}_1 + 2\alpha RK\left[1 - \frac{\sigma}{1+\sigma}\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\bar{\phi}_1 \qquad (74)$$

and

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\Omega}_1 - (1+\sigma)\bar{\alpha}^2\bar{\Omega}_1 - \frac{(1+\sigma)}{2}\bar{\alpha}^2\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\phi}_1 = -\alpha(1+\sigma)R$$
$$\left[1 - K\left[1-y^2 - \frac{2\sigma}{1+\sigma}\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\right]\bar{\Omega}_2 -$$
$$- \alpha(1+\sigma)RK\left[1 - \frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\bar{\phi}_2 \qquad (75)$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\Omega}_2 - (1+\sigma)\bar{\alpha}^2\,\bar{\Omega}_2 - \frac{(1+\sigma)}{2}\bar{\alpha}^2\left(\frac{d^2}{dy^2} - \alpha^2\right)\bar{\phi}_2 =$$

$$\alpha(1+\sigma)R\left[1-K\left[1-y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\right]\bar{\Omega}_1 +$$

$$+ \alpha(1+\sigma)\cdot RK\left[1 - \frac{\cosh\bar{\alpha}\,y}{\cosh\bar{\alpha}}\right]\bar{\phi}_1 \qquad (76)$$

It may be noted that equations (73) - (76) are complicated enough and cannot be solved in closed form even for Newtonian fluid. Hence following analysis of Chow and introducing further valid simplications namely, assuming that the Reynolds number for the flow is very small. Therefore expressing ϕ_1, $\bar{\phi}_2$ and $\bar{\Omega}_1$, $\bar{\Omega}_2$ in terms of R as

$$\left.\begin{aligned}
\bar{\phi}_1 &= \phi_{10} + R^2\phi_{12} + \cdots\cdots \\
\bar{\phi}_2 &= R\phi_{21} + R^3\phi_{22} + \cdots\cdots \\
\bar{\Omega}_1 &= \Omega_{10} + R^2\Omega_{12} + \cdots \\
\bar{\Omega}_2 &= R\Omega_{21} + R^3\Omega_{22} + \cdots
\end{aligned}\right\} \qquad (77)$$

It is understood that ϕ_{10}, ϕ_{12}, Ω_{10}, Ω_{12} etc. are functions of y alone. Substituting (77) into (75) yields the following equations for ϕ_{10} and Ω_{10}

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2\phi_{10} + \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_{10} = 0 \qquad (78)$$

$$\left[\frac{d^2}{dy^2} - \alpha^2 - (1+\alpha)\bar{\alpha}^2\right]\Omega_{10} - \frac{(1\mp\sigma)}{2}\bar{\alpha}^2\left[\frac{d^2}{dy^2} - \alpha^2\right]\phi_{10} = 0 \qquad (79)$$

The boundary conditions for ϕ_{10} and Ω_{10} are obtained as

$$\left.\begin{aligned} \phi_{10}(\pm 1) &= \pm 1 \\ \phi'_{10}(\pm 1) &= 2K\left[1 - \frac{\sigma}{1+\sigma}\,\frac{\sinh\bar{\alpha}}{\bar{\alpha}\cosh\bar{\alpha}}\right] \\ \Omega'_{10}(\pm 1) &= K\bar{\alpha}\,\frac{\sinh\bar{\alpha}}{\cosh\bar{\alpha}} \end{aligned}\right\} \qquad (80)$$

On solving (78) and (79) one finds the expression for ϕ_{10} and Ω_{10} as

$$\phi_{10} = B_2 \sinh \alpha y - \delta \sinh \alpha_1 y \qquad (81)$$

$$\Omega_{10} = A_2 \sinh \alpha_1 y + A_3 \sinh \alpha y \qquad (82)$$

Where

$$\left.\begin{aligned} B_2 &= \frac{1 + \alpha_1 \delta \sinh \alpha_1}{\sinh \alpha} \\ A_3 &= \frac{2\sigma K \bar{\alpha} \sinh \bar{\alpha} - \alpha_1(1+\sigma)\bar{\alpha}^2 \delta \cosh \bar{\alpha} \cosh \alpha_1}{2\sigma\alpha \cosh \alpha} \\ \delta &= \frac{2\sigma A_2}{(1+\sigma)\bar{\alpha}^2} \end{aligned}\right\} \qquad (83)$$

Similarly, substitution of (77) in equations (74) and (76) gives the following equations for ϕ_{21} and Ω_{21}

$$\left(\frac{d}{dy^2} - \alpha^2\right)^2 \phi_{21} + \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_{21} = \alpha\left[K\left[1 - y^2 - \frac{2\sigma}{1+\sigma} *\right.\right.$$

$$\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\Big] - 1\Big]\Big(\frac{d^2}{dy^2} - \alpha^2\Big)\phi_{10} + 2\alpha\lambda\Big[1 - \frac{\sigma\cosh\bar{\alpha}y}{1+\sigma\cosh\bar{\alpha}}\Big]\phi_{10}$$

(84)

$$\Big(\frac{d^2}{dy} - \alpha^2\Big)\Omega_{21} - (1+\sigma)\bar{\alpha}^2\Omega_{21} - \frac{(1+\sigma)}{2}\bar{\alpha}^2\Big(\frac{d^2}{dy^2} - \alpha^2\Big)\phi_{21} =$$

$$\alpha(1+\sigma)\Big[1 - K\Big[1 - y^2 - \frac{2\sigma}{1+\sigma}\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\Big]\Big]\Omega_{10}$$

$$+ \alpha(1+\sigma)K\Big[1 - \frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\Big]\phi_{10} \qquad (85)$$

The boundary conditions for ϕ_{21} and Ω_{21} are

$$\left.\begin{aligned}\phi_{21}(\pm1) &= 0\ ,\\ \phi'_{21}(\pm1) &= 0\\ \Omega'_{21}(\pm1) &= 0\end{aligned}\right\} \qquad (86)$$

On Solving (84), (85) under the boundary conditions (86), one finds that the solution for ϕ_{21} and Ω_{21} are

$$\phi_{21} = D_{13}\sinh\alpha y + D_{14}\cosh\alpha y + F_2(y) \qquad (87)$$

$$\Omega_{21} = V_3\sinh\alpha y + V_4\sinh\alpha_1 y + F_3(y) \qquad (88)$$

Where

$$D_{13} = -\frac{F'_2(1)}{\alpha\cosh\alpha}\ ,\ D_{14} = -\frac{F_2(1) + F_2(-1)}{2\cosh\alpha} = 0 \qquad (89)$$

noting that $F_2(y)$ is an odd function

$$(\alpha^3 B_2 \cosh \alpha y - \alpha_1^3 \delta \cosh \alpha_1 y) + 2\bar{\alpha} \cosh \bar{\alpha} y(\alpha^2 B_2 \sinh \alpha y - \alpha_1^2 \delta \sinh \alpha_1 y) + \bar{\alpha}^2 \sinh \bar{\alpha} y(\alpha B_2 \cosh \alpha y - \alpha_1 \delta \cosh \alpha_1 y)\Big]$$

$$+ \left(1 - \frac{\cosh \bar{\alpha} y}{\cosh \bar{\alpha}}\right)(\alpha^4 B_2 \sinh \alpha y - \alpha_1^4 \delta \sinh \alpha_1 y) - \frac{2\bar{\alpha} \sinh \bar{\alpha} y}{\cosh \bar{\alpha}} \times$$

$$(\alpha^3 B_2 \cosh \alpha y - \alpha_1^3 \delta \cosh \alpha_1 y) - \frac{\bar{\alpha}^2 \cosh \bar{\alpha} y}{\cosh \bar{\alpha}}(\alpha^2 B_2 \sinh \alpha y - \alpha_1^2 \delta \sinh \alpha_1 y) + P_2 \left\{- K\Big[2 + \bar{\alpha}^2 P_1 \cosh \bar{\alpha} y\Big] \sinh \alpha_1 y\right.$$

$$- 2K \alpha_1 (2y + \bar{\alpha} P_1 \sinh \bar{\alpha} y) \cosh \alpha_1 y + \alpha_1^2 \left\{K\Big[1 - y^2 + P_1 \times (\cosh \bar{\alpha} - \cosh \bar{\alpha} y)\Big] \sinh \alpha_1 y\right\}\Big\} + P_3 \left\{\Big[1 + P_4 \cosh \bar{\alpha} y\Big] \times\right.$$

$$(\alpha^2 B_2 \sinh \alpha y - \alpha_1^2 \delta \sinh \alpha_1 y) + 2\bar{\alpha} P_4 \sinh \bar{\alpha} y \times$$

$$(\alpha B_2 \cosh \alpha y - \alpha_1 \delta \cosh \alpha_1 y) + \bar{\alpha}^2 P_4 \cosh \bar{\alpha} y(B_2 \sinh \alpha y - \delta \sinh \alpha_1 y)\Big\}\Big] \qquad (90)$$

$$\left.\begin{aligned} P_o &= \alpha(1 + \sigma) \\ P_1 &= -\frac{2\sigma}{\bar{\alpha}^2(1+\sigma)\cosh \bar{\alpha}} \\ P_3 &= \bar{\alpha}^2 K P_o \\ P_4 &= -\frac{\sigma \alpha}{P_o \cosh \alpha} \\ P_5 &= \frac{\alpha_1^2 + \alpha^2}{\alpha^2 \alpha_1^2} \end{aligned}\right\} \qquad (91)$$

$$F_2(y)$$

$$= -\frac{2}{\alpha^2(1+\sigma)}\Bigg[V_3 \frac{(1+\sigma)\bar{\alpha}^2}{\alpha^2} \sinh \alpha y + \frac{\bar{\alpha}^2\sigma}{\alpha_1^2} V_4 \sinh \alpha_1 y - F_3(y) +$$

$$P_o K\Bigg[\frac{1}{\alpha^2} B_2 \sinh \alpha y - \frac{\delta}{\alpha_1^2} \sinh \alpha_1 y - \frac{1}{2(\cosh \bar{\alpha}}\Bigg[B_2 \times$$

$$\left(\frac{\sinh (\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh (\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right) - \delta \left(\frac{\sinh(\alpha_1+\bar{\alpha}y)}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha}y)}{(\alpha_1-\bar{\alpha})^2} \right)\Bigg]\Bigg]$$

$$+ P\Bigg[[1-K(1+P_1 \cosh \bar{\alpha})] \left[\frac{A_2}{\alpha_1^2} \sinh \alpha_1 y + \frac{A_3}{\alpha_1^2} \sinh \alpha_1 y \right] +$$

$$\frac{KP_1}{2}\Bigg[A_2 \left(\frac{\sinh (\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh (\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right)$$

$$+ A_3 \left(\frac{\sinh (\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right)\Bigg] - K\Bigg[\frac{A_2}{\alpha_1^2}\left(y^2 \sinh \alpha_1 y - \right.$$

$$\left. - 4\frac{\cosh \alpha_1 y}{\alpha_1} \right) + \frac{A_3}{\alpha^2}\left(y^2 \sinh \alpha y - 4 \frac{\cosh \alpha y}{\alpha} \right)\Bigg]\Bigg] + (\alpha_1^2+\bar{\alpha}^2\sigma)\times$$

$$\Bigg[\frac{1}{\alpha_1^2\alpha^2}\Bigg[P_o \Bigg[- \alpha^2 \left\{ [1-K(1+P_1 \cosh \bar{\alpha})] \left(A_2 \frac{\sinh \alpha_1 y}{\alpha_1^2} + \right.\right.\right.$$

$$\left. + A_3 \frac{\sinh \alpha y}{\alpha^2} \right) + \frac{KP_1}{2}\Bigg[A_2 \left(\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right) +$$

$$+ A_3 \left(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right)\Bigg]$$

$$- K\Bigg[\frac{A_2}{\alpha_1^2}\left(y^2 \sinh \alpha_1 y - 4 \frac{\cosh \alpha_1 y}{\alpha_1} \right) + \frac{A_3}{\alpha^2}\left(y^2 \sinh \alpha y - 4\frac{\cosh \alpha y}{\alpha} \right)\Bigg]\Bigg]$$

$$\left.\begin{aligned} V_4 &= \frac{\alpha \cosh \alpha\, F_3(1) - \sinh \alpha\, F_3'(1)}{\alpha_1 \sinh \alpha \cosh \alpha_1 - \alpha \cosh \alpha \sinh \alpha_1} \\ V_3 &= -\frac{V_4 \sinh \alpha_1 + F_3(1)}{\sinh \alpha} \end{aligned}\right\} \qquad (92)$$

$$F_3(y) = \frac{1}{\alpha_1^2\, \alpha^2}\Big[P_o\Big[-\alpha^2\Big\{1-K\Big[1-y^2 + P_1(\cosh\bar\alpha - \cosh\bar\alpha y)\Big]\Big\}\times$$

$$(A_2 \sinh \alpha_1 y + A_3 \sinh \alpha y) + \Big\{1-K\Big[1-y^2 +$$

$$P_1(\cosh\bar\alpha - \cosh\bar\alpha y)\Big\}\Big\{\alpha_1^2 A_2 \sinh \alpha_1 y + \alpha^2 A_3 \sinh \alpha y\Big\} +$$

$$+ 2K\Big[2y + \bar\alpha P_1 \sinh \bar\alpha y \qquad \Big]\Big[\alpha_1 A_2 \sinh \alpha_1 y +$$

$$+\alpha A_2 \sinh \alpha y\Big] + K\Big[2 + \bar\alpha^2 P_1 \cosh \bar\alpha y\Big]\Big[A_2 \sinh \alpha_1 y +$$

$$A_3 \sinh \alpha y\Big]\Big] + P_o K\Big[\alpha^2\Big[L - \frac{\cosh \bar\alpha y}{\cosh \bar\alpha}\Big]\Big[B_2 \sinh \alpha y -$$

$$\delta \sinh \alpha_1 y\Big] - \bar\alpha^2 \frac{\cosh \bar\alpha y}{\cosh \bar\alpha}\Big[B_2 \sinh \alpha y - \delta \sinh \alpha_1 y\Big]$$

$$- \frac{2\bar\alpha \sinh \bar\alpha y}{\cosh \bar\alpha}\Big[\alpha B_2 \sinh \alpha y - \alpha_1 \delta \cosh \alpha_1 y\Big] + \Big[1 - \frac{\cosh \bar\alpha y}{\cosh \bar\alpha y}\Big] \times$$

$$\Big[\alpha^2 B_2 \sinh \alpha y - \alpha_1^2 \delta \sinh \alpha_1 y\Big]\Big] + P_2\Big\{K\Big[1-y^2 +$$

$$P_1(\cosh\bar\alpha - \cosh\bar\alpha y)\Big] - 1\Big\} \sinh \alpha_1 y + P_3\Big[1 + P_4 \cosh \bar\alpha y\Big] \times$$

$$\Big[B_2 \sinh \alpha y - \delta \sinh \alpha_1 y\Big]\Big]$$

$$+ P_5\Big[-\alpha^2\Big\{\Big[1-K\Big[1-y^2 + P_1(\cosh\bar\alpha - \cosh\bar\alpha y)\Big]\Big]\alpha_1^2 A_2 \sinh \alpha_1 y +$$

$$\alpha^2 A_3 \sinh \alpha y\Big] + 2K\Big[2y + P_1 \bar\alpha \sinh \bar\alpha y\Big]\Big[\alpha_1 A_2 \cosh \alpha_1 y + \alpha A_3 \times$$

$$\sinh \alpha y\Big] + K\Big[2 + \bar{\alpha}^2 P_1 \cosh \bar{\alpha}y\Big]\Big[A_2 \sinh \alpha_1 y + A_3 \sinh \alpha y\Big]\Big\}$$

$$+ \Big[1 - K\Big[1-y^2 + P_1(\cosh \bar{\alpha} - \cosh \bar{\alpha}y)\Big]\Big]\Big[\alpha_1^4 A_2 \sinh \alpha_1 y +$$

$$+ \alpha^4 A_3 \sinh \alpha y\Big] + 3K\Big[2y + \bar{\alpha} P_1 \sinh \bar{\alpha}y\Big]\Big[\alpha_1^3 A_2 \cosh \alpha_1 y +$$

$$+ \alpha^3 A_3 \cosh \alpha y + 7K\Big[2 + \bar{\alpha}^2 P_1 \cosh \bar{\alpha}y\Big]\Big[\alpha_1^2 A_2 \sinh \alpha_1 y +$$

$$+ \alpha^2 A_3 \sinh \alpha y\Big] + 4KP_1 \bar{\alpha}^3 \sinh \bar{\alpha}y\Big[\alpha_1 A_2 \cosh \alpha_1 y + \alpha A_3 \cosh \alpha y\Big]$$

$$+ 2K\Big[[2y + P_1 \bar{\alpha} \sinh \bar{\alpha}y]\Big[\alpha_1^3 A_2 \cosh \alpha_1 y + \alpha^3 A_3 \cosh \alpha y\Big]$$

$$+ 2\Big[2 + P_1 \bar{\alpha}^2 \cosh \bar{\alpha}y\Big]\Big[\alpha_1^2 A_2 \sinh \alpha_1 y + \alpha^2 A_3 \sinh \alpha y\Big]$$

$$+ P_1 \bar{\alpha}^3\Big[\alpha_1 A_2 \cosh \alpha_1 y + \alpha A_3 \cosh \bar{\alpha}y\Big] + K\Big[2 \bar{\alpha}^3 P_1 \sinh \bar{\alpha}y \times$$

$$\Big[\alpha_1 A_2 \cosh \alpha_1 y + \alpha A_3 \cosh \alpha y\Big] + (2 + \bar{\alpha}^2 P_1 \cosh \bar{\alpha}y) \times$$

$$(\alpha_1^2 A_2 \sinh \alpha_1 y + \alpha^2 A_3 \sinh \alpha y) + \bar{\alpha}^4 P_1 (A_2 \sinh \alpha_1 y + A_3 \sinh \alpha y)$$

$$\times \cosh \bar{\alpha}y\Big] + PK\Big[-\alpha^2\Big\{\Big(1 - \frac{\cosh \bar{\alpha}y}{\cosh \bar{\alpha}}\Big)(\alpha^2 B_2 \sinh \alpha y - \alpha_1^2 \delta \sinh \alpha_1 y)$$

$$- \frac{2\bar{\alpha} \sinh \bar{\alpha}y}{\cosh \bar{\alpha}}(\alpha B_2 \cosh \alpha y - \alpha_1 \delta \cosh \alpha_1 y) - \frac{\bar{\alpha}^2 \sinh \bar{\alpha}y}{\cosh \bar{\alpha}} \times$$

$$(B_2 \sinh \alpha y - \delta \sinh \alpha_1 y)\Big\} - \frac{\bar{\alpha}^2}{\cosh \bar{\alpha}}\Big[\cosh \bar{\alpha}y(\alpha^2 B_2 \sinh \alpha y -$$

$$\alpha_1^2 \delta \sinh \alpha_1 y) + 2\bar{\alpha} \sinh \bar{\alpha}y (\alpha B_2 \cosh \alpha y - \alpha_1 \delta \cosh \alpha_1 y)$$

$$+ \bar{\alpha}^2 \cosh \bar{\alpha}y(B_2 \sinh \alpha y - \delta \sinh \alpha_1 y)\Big] - \frac{2\bar{\alpha}}{\cosh \bar{\alpha}}\Big[\sinh \bar{\alpha}y \times$$

$$+ \left(\alpha_1^2 + \bar{\alpha}^2\sigma\right)\left[\frac{Y}{\alpha_1^2\alpha^2}\left[P_0\right] \cdot \alpha^2\left\{\left[1-K(1+P_1\cosh\bar{\alpha})\right]\right.\right.$$

$$\left\{A_2\frac{\sinh\alpha_1 y}{\alpha_1^2} + \frac{A_3\sinh\alpha y}{\alpha^2}\right\} + \frac{KP_1}{2}\left[A_2\left\{\sinh(\alpha_1+\bar{\alpha})y + \right.\right.$$

$$\left.\left. + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} + A_3\left\{\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\}\right]\right.$$

$$\left. - K\left[\frac{A_2}{\alpha_1^2}\left\{y^2\sinh\alpha_1 - \frac{4\cosh\alpha_1 y}{\alpha_1}\right\} + \frac{A_3}{\alpha^2}\left\{y^2\sinh\alpha y - \frac{4\cosh\alpha y}{\alpha}\right\}\right]\right]$$

$$+ \left[1 - K(1+P_1\cosh\bar{\alpha})\right](A_2\sinh\alpha_1 y + A_3\sinh\alpha y) - K\left[A_2 \times\right.$$

$$\left\{y^2\sinh\alpha_1 y - \frac{4\cosh\alpha_1 y}{\alpha_1}\right\} + A_3\left\{y^2\sinh\alpha y - \frac{4\cosh\alpha y}{\alpha}\right\}\left.\right]$$

$$+ \frac{KP_1}{2}\left[\alpha_1^2 A_2\left\{\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \cdots + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\right\}\right.$$

$$\left. + \alpha^2 A_3\left\{\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\}\right] + 2K\left[2\left\{\frac{A_2}{\alpha_1} \times\right.\right.$$

$$\left.\left\{y\sinh\alpha_1 y - 2\frac{\cosh\alpha_1 y}{\alpha_1}\right\} + \frac{A_3}{\alpha}\left\{y\sinh\alpha y - 2\frac{\cosh\alpha y}{\alpha}\right\}\right\}$$

$$+ \frac{\bar{\alpha}P_1}{2}\left[\alpha_1 A_2\left\{\frac{\cosh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} - \frac{\cosh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\right\} + \alpha A_3 \times\right.$$

$$\left.\left\{\frac{\cosh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} \cdots - \frac{\cosh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\}\right] +$$

$$+ K\left[2\left\{\frac{A_2}{\alpha_1^2}\sinh\alpha_1 y + \frac{A_3}{\alpha^2}\sinh\alpha y\right\} + \frac{\bar{\alpha}^2 P_1}{2}\left[A_2\left\{\sinh(\alpha_1+\bar{\alpha})y + \right.\right.\right.$$

$$+ \sinh(\alpha_1 - \bar{\alpha})y) + A_3 \left\{ \frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right\} \Bigg] \Bigg] \Bigg]$$

$$+ PK \Bigg[-\alpha^2 \left\{ \frac{1}{\alpha^2} B_2 \sinh\alpha y - \frac{\delta}{\alpha_1^2} \sinh\alpha_1 y - \frac{1}{2\cosh\bar{\alpha}} \times \right.$$

$$\Bigg[B_2 \left\{ \frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right\}$$

$$\left. - \delta \left\{ \frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right\} \Bigg] \right\} \Bigg] - \frac{\bar{\alpha}^2}{2\cosh\bar{\alpha}} \Bigg[B_2 \left\{ \frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} \right.$$

$$+ \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \Big\} - \delta \left\{ \frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sin(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right\} \Bigg] - \frac{\bar{\alpha}}{\cosh\bar{\alpha}} \times$$

$$\Bigg[\alpha B_2 \left\{ \frac{\cosh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} - \frac{\cosh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right\} - \alpha_1 \delta \left\{ \frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} - \right.$$

$$\left. - \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right\} \Bigg] + B_2 \sinh\alpha y - \delta \sinh\alpha_1 y - \frac{1}{2\cosh\bar{\alpha}} \times$$

$$\Bigg[\alpha^2 B_2 \left\{ \frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right\} - \alpha_1 \delta \left\{ \frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \right.$$

$$\left. + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right\} \Bigg] + P_2 \Bigg[\left[-1 + K(1 + P_1\cosh\bar{\alpha}) \right] \frac{1}{\alpha_1^2} \sinh\alpha_1 y$$

$$- \frac{P_0 K}{2} \left\{ \frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right\} - \frac{K}{\alpha_1^2} \left\{ y^2 \sinh\alpha_1 y \right.$$

$$\left. - \frac{4\cosh\alpha_1 y}{\alpha_1} \right\} \Bigg] + P_3 \Bigg[\frac{1}{\alpha^2} B_2 \sinh\alpha y - \frac{\delta}{\alpha_1^2} \sinh\alpha_1 y + \frac{P_4}{2} \times$$

$$\left[B_2 \left\{ \frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right\} - \delta \left\{ \frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right\} \right] \Bigg] \Bigg]$$

$$+ P_5 \Bigg[-\alpha^2 \left\{ \left[1 - K(1 + P_1\cosh\bar{\alpha}) \right] (A_2 \sinh\alpha_1 y + A_3 \sinh\alpha y) \right.$$

$$+\frac{KP_1}{2}\Big[\alpha_1^2 A_2\Big(\frac{\sinh(\alpha_1+\bar\alpha)y}{(\alpha_1+\bar\alpha)^2}+\frac{\sinh(\alpha_1-\bar\alpha)y}{(\alpha_1-\bar\alpha)^2}\Big)+\alpha^2 A_3\Big(\frac{\sinh(\alpha+\bar\alpha)y}{(\alpha+\bar\alpha)^2}+$$

$$\frac{\sinh(\alpha-\bar\alpha)y}{(\alpha-\bar\alpha)^2}\Big)\Big]+K\Big[A_2\Big(y^2\sinh\alpha_1 y-\frac{4\cosh\alpha_1 y}{\alpha_1}+$$

$$+A_3\Big(y^2\sinh\alpha y-\frac{4\cosh\alpha y}{\alpha}\Big)\Big]+2K\Big[2\Big\{\frac{A_2}{\alpha_1}\Big(y\cosh\alpha_1 y-\frac{2\sinh\alpha_1 y}{\alpha_1}\Big)$$

$$+\frac{A_3}{\alpha}\Big(y\sinh\alpha y-\frac{2\cosh\alpha y}{\alpha}\Big)\Big\}+\frac{P_1\bar\alpha}{2}\Big[\alpha_1 A_2\Big\{\frac{\sinh(\alpha_1+\bar\alpha)y}{(\alpha_1+\bar\alpha)^2}-$$

$$\frac{\sinh(\alpha_1-\bar\alpha)y}{(\alpha_1-\bar\alpha)^2}\Big)+\alpha A_3\Big(\frac{\cosh(\alpha-\bar\alpha)y}{(\alpha+\bar\alpha)^2}-\frac{\cosh(\alpha-\bar\alpha)y}{(\alpha-\bar\alpha)^2}\Big)\Big]\Big]$$

$$+K\Big[2\Big(\frac{A_2}{\alpha_1^2}\sinh\alpha_1 y+\frac{A_3}{\alpha^2}\sinh\alpha y\Big)+\frac{\bar\alpha^2 P_1}{2}\Big[A_2\Big(\frac{\sinh(\alpha_1+\bar\alpha)y}{(\alpha_1+\bar\alpha)^2}+$$

$$\sinh(\alpha_1-\bar\alpha)y\Big)+A_3\Big(\sinh\frac{(\alpha+\bar\alpha)}{(\alpha+\bar\alpha)^2}+\sinh\frac{(\alpha-\bar\alpha)y}{(\alpha-\bar\alpha)^2}\Big)\Big]\Big]\Big\}+\Big[1-K(1+P_1\cosh\bar\alpha)\Big]x$$

$$\Big[\alpha_1^2 A_2\sinh\alpha_1 y+\alpha^2 A_3\sinh\alpha y\Big]+K\Big[\alpha_1^2 A_2\Big(y^2\sinh\alpha_1 y-$$

$$4\frac{\cosh\alpha_1 y}{\alpha_1}\Big)+\alpha^2 A_3\Big(y^2\sinh\alpha y-\frac{4\cosh\alpha y}{\alpha}\Big)\Big]+\frac{KP_1}{2}\Big[\alpha_1^4 A_2 x$$

$$\Big(\frac{\sinh(\alpha_1+\bar\alpha)y}{(\alpha_1+\bar\alpha)^2}+\frac{\sinh(\alpha_1-\bar\alpha)y}{(\alpha_1-\bar\alpha)^2}\Big)+\alpha^4 A_3\Big(\frac{\sinh(\alpha+\bar\alpha)y}{(\alpha-\bar\alpha)^2}+$$

$$3K\Big[2\Big\{\alpha_1 A_2\Big(y\cosh\alpha_1 y-2\frac{\sinh\alpha_1 y}{\alpha_1}\Big)+\alpha A_3\Big(y\cosh\alpha y-2\frac{\sinh\alpha y}{\alpha}\Big)\Big\}$$

$$+\frac{\bar\alpha P_1}{2}\Big[\alpha_1^2 A_2\Big(\frac{\sinh(\alpha_1+\bar\alpha)y}{(\alpha_1+\bar\alpha)^2}-\frac{\sinh(\alpha_1-\bar\alpha)y}{(\alpha_1-\bar\alpha)^2}\Big)+\alpha^3 A_3\Big(\frac{\sinh(\alpha+\bar\alpha)y}{(\alpha+\bar\alpha)^2}$$

$$-\frac{\sinh(\alpha-\bar\alpha)y}{(\alpha-\bar\alpha)^2}\Big)\Big]\Big]+7K\Big[2(A_2\cosh\alpha_1 y+A_3\sinh\alpha y)+\frac{\bar\alpha^2 P_1}{2}x$$

$$\left[\alpha_1^2 A_2 \left\{ \frac{\sinh(\alpha_1 + \alpha)y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1 - \bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right\} + \alpha^2 A_3 \left\{ \frac{\sinh(\alpha + \bar{\alpha})y}{(\alpha+\bar{\alpha})^2} - \right.\right.$$

$$\left.\left. \frac{\sinh(\alpha - \bar{\alpha})y}{(\alpha-\bar{\alpha})^2} \right\} \right] + 2K\bar{\alpha}^3 P_1 \left[\alpha_1 A_2 \left\{ \frac{\sinh(\alpha_1 + \bar{\alpha})y}{(\alpha_1 + \bar{\alpha})^2} - \frac{\sinh(\alpha_1 - \bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2} \right\} \right.$$

$$\left. + \alpha A_3 \left\{ \frac{\sinh(\alpha + \bar{\alpha})y}{(\alpha + \bar{\alpha})^2} - \frac{\sinh(\alpha - \bar{\alpha})y}{(\alpha - \bar{\alpha})^2} \right\} \right] + 2K \left[2 \left[\alpha_1 A_2 \left\{ y \cosh \alpha_1 y - \right.\right.\right.$$

$$\left.\left.\left. - 2 \frac{\sinh \alpha_1 y}{\alpha_1} \right\} + \alpha A_3 \left\{ y \cosh \alpha y - 2 \frac{\sinh \alpha y}{\alpha} \right\} \right] \right]$$

$$+ \frac{\bar{\alpha} P_1}{2} \left[\alpha_1^3 A_2 \left\{ \frac{\sinh(\alpha_1 + \bar{\alpha})y}{(\alpha_1 + \bar{\alpha})^2} - \frac{\sinh(\alpha_1 - \bar{\alpha})y}{(\alpha_1 - \alpha)^2} \right\} + \alpha^3 A_3 \left\{ \frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} \right.\right.$$

$$\left.\left. - \frac{\sinh(\alpha - \bar{\alpha})y}{(\alpha - \bar{\alpha})^2} \right\} \right] + 4 \left[2(A_2 \sinh \alpha_1 y + A_3 \sinh \alpha y) + \frac{P_1 \bar{\alpha}^2}{2} \times \right.$$

$$\left[\alpha_1^2 A_2 \left\{ \frac{\sinh(\alpha_1 + \bar{\alpha})y}{(\alpha_1 + \bar{\alpha})^2} + \frac{\sinh(\alpha_1 - \bar{\alpha})y}{(\alpha_1 - \bar{\alpha})^2} \right\} + \alpha^2 A_3 \left\{ \frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} \right.\right.$$

$$\left.\left. + \frac{\sinh(\alpha - \bar{\alpha})y}{(\alpha - \bar{\alpha})^2} \right\} \right] + \frac{\bar{\alpha}^3 P_1}{2} \left[\alpha_1 A_2 \left\{ \frac{\sinh(\alpha_1 + \bar{\alpha})y}{(\alpha_1 + \bar{\alpha})^2} - \frac{\sinh(\alpha_1 - \bar{\alpha})y}{(\alpha_1 - \bar{\alpha})^2} \right\} \right.$$

$$\left. + \alpha A_3 \left\{ \frac{\sinh(\alpha + \bar{\alpha})y}{(\alpha + \bar{\alpha})^2} - \frac{\sinh(\alpha - \bar{\alpha})y}{(\alpha - \bar{\alpha})^2} \right\} \right] \left.\right] + K \left[\bar{\alpha}^3 P_1 \left[\alpha_1 A_2 \times \right.\right.$$

$$\left\{ \frac{\sinh(\alpha_1 + \bar{\alpha})y}{(\alpha_1 + \bar{\alpha})^2} - \frac{\sinh(\alpha_1 - \bar{\alpha})y}{(\alpha_1 - \bar{\alpha})^2} \right\} + \alpha A_3 \left\{ \frac{\sinh(\alpha + \bar{\alpha})y}{(\alpha + \bar{\alpha})^2} \right.$$

$$\left. - \frac{\sinh(\alpha - \bar{\alpha})y}{(\alpha - \bar{\alpha})^2} \right\} + \frac{\bar{\alpha}^4 P_1}{2} \left[A_2 \left\{ \frac{\sinh(\alpha_1 + \bar{\alpha})y}{(\alpha_1 + \bar{\alpha})^2} + \frac{\sinh(\bar{\alpha}_1 - \bar{\alpha})y}{(\alpha_1 - \alpha)^2} \right\} \right.$$

$$\left. \left. + A_3 \left\{ \frac{\sinh(\alpha + \bar{\alpha})y}{(\alpha + \bar{\alpha})^2} - \frac{\sinh(\alpha - \bar{\alpha})y}{(\alpha - \alpha)^2} \right\} \right] \right] + PK \left[- \alpha^2 \left\{ \left[\phi_{21}(y) - \right.\right.\right.$$

$$\frac{1}{2\cosh\bar{\alpha}}\Big[\alpha^2 B_2\Big(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\alpha)^2}+\frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\alpha)^2}\Big)-\delta\alpha_1^2\times$$

$$\Big(\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\alpha)^2}+\frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\Big)\Big]-\frac{\bar{\alpha}}{\cosh\bar{\alpha}}\Big[\alpha B_2\Big(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2}$$

$$-\frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\Big)-\alpha_1\delta\Big(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\alpha)^2}-\frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\alpha)^2}\Big)\Big]$$

$$-\frac{\bar{\alpha}^2}{2\cosh\bar{\alpha}}\Big[B_2\Big(\frac{\cosh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2}-\frac{\cosh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\Big)-\delta$$

$$-\delta\Big(\frac{\cosh(\alpha_1+\bar{\alpha})y}{(\alpha_1-\alpha)^2}-\frac{\cosh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\Big)\Big]\Big\}-\frac{\bar{\alpha}}{2\cosh\bar{\alpha}}\Big\{(\bar{\alpha}+2)\times$$

$$\Big[\alpha^2 B_2\Big(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2}-\frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\Big)-\alpha_1^2\delta\Big(\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2}$$

$$+\frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\Big)\Big]+(\bar{\alpha}+2)\Big[\alpha B_2\Big(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2}-\frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\Big)$$

$$-\alpha_1\delta\Big(\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2}-\frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\Big)\Big]+\bar{\alpha}^3\Big[B_2\Big(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2}$$

$$+\frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\alpha)^2}\Big)-\delta\Big(\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2}+\frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\Big)\Big]$$

$$+2(1+\bar{\alpha}^2)\Big[\alpha^3 B_2\Big(\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\alpha)^2}-\frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\Big)-\alpha_1^3\delta\times$$

$$\Big(\frac{\sin(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2}-\frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\Big)\Big\}+\Big[\alpha^2 B_2\sinh\alpha y-\alpha_1^2\delta\sinh\alpha_1 y$$

$$- \frac{1}{2\cosh\bar{\alpha}}\Bigg[\alpha^4 B_2 \left\{\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\} - \alpha_1^4 \times$$

$$\left\{\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\alpha)^2}\right\}\Bigg]\Bigg] + P_2\Bigg\{-K\Bigg[\frac{2}{\alpha_1}\sinh\alpha_1 y \cdot$$

$$+ \frac{\bar{\alpha}^2 P_1}{2}\left\{\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\right\} - 2K\alpha_1\Bigg[\frac{2}{\alpha_1^2}\Bigg\{y\cosh\alpha_1 y \cdot$$

$$- 2\frac{\sinh\alpha_1 y}{\alpha_1}\Bigg\} + \frac{\bar{\alpha}P_1}{2}\left\{\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} - \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\right\} \cdot$$

$$+ \alpha_1^2\Bigg\{\frac{K}{\alpha_1^2}\left[1 + P_1\cosh\bar{\alpha}\right]\sinh\alpha_1 y - \frac{K}{\alpha_1^2}\Bigg\{y^2\sinh\alpha_1 y - 4\frac{\cosh\alpha_1 y}{\alpha_1}$$

$$- \frac{KP_1}{2}\left\{\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha_1-\bar{\alpha})y}{(\alpha_1-\bar{\alpha})^2}\right\}\Bigg\}\Bigg\} + P_3\Bigg\{B_2\sinh\alpha y$$

$$- \delta\sinh\alpha_1 y + \frac{P_4}{2}\Bigg[\alpha^2 B_2\left\{\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\}$$

$$- \alpha_1^2\delta\left\{\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha_1+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\}\Bigg] + \bar{\alpha}P_4\Bigg[\alpha B_2\left\{\frac{\sinh(\bar{\alpha}+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2}\right.$$

$$\left. - \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\} - \alpha_1\delta\left\{\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} - \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\}\Bigg] +$$

$$+ \frac{\bar{\alpha}^2 P_4}{2}\Bigg[B_2\left\{\frac{\sinh(\alpha+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} + \frac{\sinh(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\} - \delta\left\{\frac{\sinh(\alpha_1+\bar{\alpha})y}{(\alpha+\bar{\alpha})^2} \cdot\right.$$

$$\left. + \sinh\frac{(\alpha-\bar{\alpha})y}{(\alpha-\bar{\alpha})^2}\right\}\Bigg]\Bigg]\Bigg]\Bigg]\Bigg] \qquad (93)$$

If in place of equation (50) we define the following

$$\left.\begin{aligned} \Psi_2 &= J_2(y) + V_2(y)\cos 2\alpha\,(x-t) + Q_2(y)\sin 2\alpha\,(x-t) \\ \omega_2 &= Z_2(y) + T_2(y)\cos 2\alpha(x-t) + N_2(y)\sin 2\alpha(x-t) \end{aligned}\right\} \quad (94)$$

then using (94), (72) in (43) - (45) and (71), the equations reduce to

$$\alpha R\left[\left\{\left(\frac{d^2}{dy^2} - 4\alpha^2\right)V_2 + \frac{1}{2}\left\{\bar{\phi}_1'\left(\bar{\phi}_1''' - 2\alpha^2\bar{\phi}_1'\right) + \bar{\phi}_2''\bar{\phi}_2' + \bar{\phi}_1\bar{\phi}_1''' - \right.\right.\right.$$

$$\left.\left.\left. - \bar{\phi}_2\bar{\phi}_1'''\right\}\right] - \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - 4\alpha^2\right)N_2 = \left(\frac{d^2}{dy^2} - 4\alpha^2\right)Q_2 \quad (95)$$

$$\alpha R\left[-2\left(\frac{d^2}{dy^2} - 4\alpha^2\right)Q_2 + \frac{1}{2}\left\{\bar{\phi}_1'\bar{\phi}_2'' - \bar{\phi}_2''\left(\phi_1'' - 2\alpha^2\bar{\phi}_1\right) - \bar{\phi}_1\phi_1''' - \bar{\phi}_2\phi_1''\right\}\right] -$$

$$- \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - 4\alpha^2\right)T_2 = \left(\frac{d^2}{dy^2} - 4\alpha^2\right)V_2 \quad (96)$$

$$\frac{\alpha R}{2}\left[\bar{\phi}_1'\bar{\phi}_2'' + \;\; + \bar{\phi}_2''\left(\bar{\phi}_1'' - 2\alpha^2\bar{\phi}_1\right) - \bar{\phi}_1\bar{\phi}_2''' - \bar{\phi}_2\phi_1'''\right] - \frac{2\sigma}{1+\sigma}Z_2'' = J_2'' \quad (97)$$

$$(1+\sigma)\alpha R\left[2\left(T_2'' - 2\alpha^2 T_2\right) + \frac{1}{2}\left(\bar{\phi}_2''\bar{\Omega}_2 - \bar{\phi}_1\bar{\Omega}_1 - \bar{\phi}_1\bar{\Omega}_1' - \bar{\phi}_2\bar{\Omega}_2'\right)\right] +$$

$$+ \bar{\alpha}\left[Q_2'' - 4\alpha^2 Q_2 + 2N_2\right] = N_2'' - 4\alpha^2 N_2 \quad (98)$$

$$(1+\sigma)\alpha R\left[-2\left(N_2'' - 4\alpha^2 N_2\right) + \frac{1}{2}\left(\bar{\phi}_1\bar{\Omega}_2 - \bar{\phi}_2\bar{\Omega}_1' + \bar{\phi}_2\bar{\Omega}_1 + \bar{\phi}_1\bar{\Omega}_2'\right)\right] +$$

$$+ \bar{\alpha}\left[V_2'' - 4\alpha^2 V_2 + 2T_2\right] = T_2'' - 4\alpha^2 T_2 \quad (99)$$

$$\left(1+\sigma\right)\frac{\alpha R}{2}\left[\bar{\phi}_1\,\bar{\Omega}_2 - \phi_2'\bar{\Omega}_1' - \bar{\phi}_2'\,\bar{\Omega}_1 - \bar{\phi}_1\,\bar{\Omega}_2'\right] + \bar{\alpha}\left[J_2'' + 2\,Z_2\right] = Z_2'' \tag{100}$$

and its boundary conditions

$$\left.\begin{aligned}
&J_2'(\pm1) \pm \tfrac{1}{2}\bar{\phi}_1''(\pm1) = 0\\
&V_2'(\pm1) \pm \tfrac{1}{2}\bar{\phi}_1'''(\pm1) = 0\\
&Q_2'(\pm1) \pm \tfrac{1}{2}\bar{\phi}_2''(\pm1) = 0\\
&V_2(\pm1) \mp \tfrac{1}{4}\bar{\phi}_1''(\pm1) = 0\\
&Q_2(\pm1) = 0\\
&Z_2'(\pm1) \pm \tfrac{1}{2}\bar{\Omega}_1''(\pm1) = 0\\
&T_2'(\pm1) \pm \tfrac{1}{2}\bar{\Omega}''(\pm1) = 0
\end{aligned}\right\} \tag{101}$$

$$N_2'(\pm1) \pm \tfrac{1}{2}\Omega_2''(\pm1) = 0$$

To solve J_2 , Z_2 , the real functions need be expanded thus in terms of R

$$J_2 = J_{20} + R^2\,J_{22} + \ldots\ldots$$

$$Z_2 = Z_{20} + R^2\,Z_{22} + \ldots\ldots$$

Note here that V_2, Q_2, T_2, N_2 are not considered.

4.6 Stokes Approximation

As Reynolds number R is very small, the solutions to equations (78) - (79) using equation (80)

$$\phi_{10} = B_2 \sinh \alpha y - \delta \sinh \alpha_1 y \tag{103}$$

$$\Omega_{10} = A_2 \sinh \alpha_1 y + A_3 \sinh \alpha y \tag{104}$$

which coincides with equations (81) and (82).
Equations (97) and (100) reduce to

$$\bar{\alpha}\left(J_{20}'' + 2 Z_{20}\right) = Z_{20}' \tag{105}$$

$$J_{20}'' = - \frac{2\sigma}{1+\sigma} Z_{20}' \tag{106}$$

Subject to the boundary conditions

$$\begin{aligned} J_{20}'(\pm 1) \pm \tfrac{1}{2} \bar{\phi}_{10}''(\pm 1) &= 0 \\ Z_{20}'(\pm 1) \pm \tfrac{1}{2} \Omega_{10}'(\pm 1) &= 0 \end{aligned} \tag{107}$$

Solving equations (105) and (106)

$$Z_{20} = B_4 \frac{\cosh \theta y}{\theta^2} + B_4' \tag{108}$$

$$J_{20} = \frac{2\sigma B_4 \cosh \theta y}{\theta^4 (1+\sigma)} - S_3 y^2 + S_4 - S_2 \left(y - \frac{y^3}{3}\right) \tag{109}$$

where $$S_3 = - \left[\frac{\alpha^2 B_2 \sinh \alpha}{2} - \frac{2 \sigma B_4 \sinh \theta}{\theta^3 (1+\sigma)} \right] \tag{110}$$

$$B_4 = - \frac{\theta \left(\alpha_1^2 A_2 \sinh \alpha_1 + \alpha^2 A_3 \sinh \alpha\right)}{2 \sinh \theta} \tag{111}$$

$$\theta = \sqrt{\frac{2\bar{\alpha}}{1+\sigma}} \tag{112}$$

In the following, a perturbation solution will be presented to see the effect of convective inertia term at higher Reynolds number

4.7 Solution for Higher - order Reynolds Number

R :

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)^2 \phi_{21} + \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_{21} = \alpha\left[K\left\{1-y^2 - \frac{2\sigma}{1+\sigma} \times \right.\right.$$

$$\left.\left.\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2 \cosh\bar{\alpha}}\right\} - 1\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{10} + 2\alpha K\left[1 - \frac{\sigma}{1+\sigma} \times\right.$$

$$\left.\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\phi_{10} \tag{113}$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_{21} - (1+\sigma)\bar{\alpha}^2\Omega_{21} - \frac{(1+\sigma)\bar{\alpha}^2}{2}\left(\frac{d}{dy^2} - \alpha^2\right)\phi_{21}$$

$$= \alpha(1+\sigma)\left[1 - K\left\{1-y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right\}\right]\Omega_{10} +$$

$$+ \alpha(1+\sigma)K\left[1 - \frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\phi_{10} \tag{114}$$

with boundary conditions

$$\left.\begin{aligned} \phi_{21}(\pm 1) &= 0 \\ \phi'_{21}(\pm 1) &= 0 \\ \Omega'_{21}(\pm 1) &= 0 \end{aligned}\right\} \tag{115}$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{12} + \frac{2\sigma}{1+\sigma}\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_{12} = \alpha\left[1 + K\left(1-y^2 - \frac{2\sigma}{1+\sigma} \times \right.\right.$$

$$\left.\left.\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right)\right]\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{21}$$

$$+ 2\alpha K\left[1 - \frac{\sigma}{1+\sigma}\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\phi_{21} \tag{116}$$

$$\left(\frac{d^2}{dy^2} - \alpha^2\right)\Omega_{12} - (1+\sigma)\bar{\alpha}^2\,\Omega_{12} - \frac{(1+\sigma)}{2}\bar{\alpha}^2\left(\frac{d^2}{dy^2} - \alpha^2\right)\phi_{12} =$$

$$-\alpha(1+\sigma)\left[1 - K\left(1-y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right)\right]\Omega_{21}$$

$$-\alpha(1+\sigma)K\left[1 - \frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\phi_{21} \tag{117}$$

$$\left.\begin{aligned} \phi'_{12}(\pm 1) &= 0 \\ \phi_{12}(\pm 1) &= 0 \\ \Omega'_{12}(\pm 1) &= 0 \end{aligned}\right\} \tag{118}$$

R^2 :

$$\frac{(1+\sigma)\alpha}{2}\left[\phi_{10}\Omega_{21} + \phi_{21}\Omega_{10}' - \phi_{21}'\Omega_{10}\right] + \frac{1}{2}\left[J_{22}'' + 2Z_{22}\right] = Z_{22}'' \tag{119}$$

$$\frac{\alpha}{2}\left[\phi_{10}'\phi_{21}' + \phi_{21}\left(\phi_{10}'' - 2\alpha^2\phi_{10}\right) - \phi_{10}\phi_{21}''' - \phi_{21}\phi_{10}'''\right] -$$

$$\frac{2\sigma}{1+\sigma} Z_{22}'' = J_{22}'' \tag{120}$$

and boundary conditions

$$\left.\begin{aligned} J_{22}'(\pm 1) \pm \tfrac{1}{2}\phi_{12}''(\pm 1) &= 0 \\ Z_{22}'(\pm 1) \pm \tfrac{1}{2}\Omega_{12}''(\pm 1) &= 0 \end{aligned}\right\} \tag{121}$$

The solutions to equations (113) and (114) are same as that of (87) and (88). While the solutions to equations (116), (117) are

$$\Omega_{12} = J_9 \sinh \alpha_1 y + F_4(y) + J_{91} \cosh \alpha_1 y \tag{122}$$

$$\phi_{12} = J_{10} \sinh \alpha y + F_8(y) + J_{11} \cosh \alpha y \tag{123}$$

Where $J_{91} = 0$

$J_{11} = 0$

$$\left.\begin{aligned} J_9 &= -\frac{F_4'(1)}{\alpha_1 \cosh \alpha_1} \\ J_{10} &= -\frac{F_5(1)}{\sinh \alpha} \end{aligned}\right\} \qquad (124)$$

$$F_4(y) = -\frac{\bar{\alpha}^2(1+\sigma)}{\bar{\alpha}\,\alpha_1^2}\left[\alpha\left(1 + K\left[1-y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right]\right)\times\right.$$

$$\left(F_2'(y) - \alpha^2 F_2(y)\right) + 2\alpha K\left[1 - \frac{\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha}}{\cosh\bar{\alpha}}\right]\times$$

$$\left[D_{13}\sinh\alpha y + D_{14}\cosh\alpha y + F_2(y)\right]$$

$$-\alpha(1+\sigma)\left[1 - K\left(1-y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\right)\right]\left[V_3\sinh\alpha y +\right.$$

$$\left. + V_4\sinh\alpha_1 y + F_3(y)\right] - \alpha(1+\sigma)K\left[1 - \frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\times$$

$$\left[D_{13}\sinh\alpha y + D_{14}\cosh\alpha y + F_2(y)\right] + \frac{1}{\alpha_1^2}\left\{\alpha\left[F_2'''(y) -\right.\right.$$

$$-\alpha^2 F_2'(y) + K\left[F_2'''(y) - \alpha^2 F_2'(y) - 2\left(F_2''(y) - \alpha^2 F_2(y)\right) -\right.$$

$$-2y\left(F_2'''(y) - \alpha^2 F_2'(y)\right) - 2y\left(F_2''(y) - \alpha^2 F_2'(y)\right) - y^2\times$$

$$\left(F_2'''(y) - \alpha^2 F_2'(y)\right) - \frac{2\sigma}{(1+\sigma)\bar{\alpha}^2\cosh\bar{\alpha}}\left[-\bar{\alpha}\sinh\bar{\alpha}y\left(F_2'''(y) -\right.\right.$$

$$-\alpha^2 F_2'(y)\right) + \left(\cosh\bar{\alpha} - \cosh\bar{\alpha}y\right)\left(F_2'''(y) - \alpha^2 F_2'(y)\right)$$

$$\left.\left.\left. - \bar{\alpha}^2\cosh\bar{\alpha}y\left(F_2''(y) - \alpha^2 F_2(y)\right) - \bar{\alpha}\sinh\bar{\alpha}y\times\right.\right.\right.$$

$$\left.\left.\left.\left(F_2'(y) - \alpha^2 F_2(y)\right)\right]\right]\right] + 2\alpha K\left[\alpha^2\left(D_{13}\sinh\alpha y + D_{14}\cosh\alpha y\right) +\right.$$

$$+ F_2''(y) - \frac{\sigma}{(1+\sigma)\cosh\bar{\alpha}}\Big[\bar{\alpha}^2\cosh\bar{\alpha}y\{D_{13}\sinh\alpha y + D_{14}\cosh\alpha y +$$

$$F_2(y)\} + \bar{\alpha}\sinh\bar{\alpha}y\Big[\alpha\,(D_{13}\cosh\alpha y + D_{14}\sinh\alpha y) + F_2'(y)\Big] +$$

$$+ \bar{\alpha}\sinh\bar{\alpha}y\Big[\alpha\,(D_{13}\cosh\alpha y + D_{14}\sinh\alpha y) + F_2'(y)\Big] + \cosh\bar{\alpha}y \times$$

$$\Big[\alpha(D_{13}\cosh\alpha y + D_{14}\sinh\alpha y) + F_2'(y)\Big]\Big]\Big]$$

$$- \alpha(1+\sigma)\Big\{\Big[1 - K\Big(1 - y^2 - \frac{2\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\Big)\Big]\Big[\alpha^2 V_3\sinh\alpha y$$

$$+ \alpha_1^2 V_4\sinh\alpha_1 y + F_3'(y)\Big] - 2K\Big[2\Big(-y + \frac{\sigma\sinh\bar{\alpha}y}{\bar{\alpha}(1+\sigma)\cosh\bar{\alpha}}\Big)\Big(\alpha V_3\cosh\alpha y +$$

$$+ \alpha_1 V_4\cosh\alpha_1 y + F_3'(y)\Big) + \Big[-1 + \frac{\sigma\cosh\bar{\alpha}y}{(1+\sigma)\cosh\bar{\alpha}}\Big]\Big[V_3\sinh\alpha y +$$

$$+ V_4\sinh\alpha_1 y + F_3(y)\Big]\Big]\Big\} - \alpha(1+\sigma)K\Big\{\alpha^2(D_{13}\sinh\alpha y + D_{14}\cosh\alpha y$$

$$+ F_2''(y) - \frac{1}{\cosh\bar{\alpha}}\Big[\bar{\alpha}^2\cosh\bar{\alpha}y\,(D_{13}\sinh\alpha y + D_{14}\cosh\alpha y + F_2(y))$$

$$+ \bar{\alpha}\sinh\bar{\alpha}y\Big[\alpha(D_{13}\cosh\alpha y + D_{14}\sinh\alpha y) + F_2'(y)\Big] + \bar{\alpha}\sinh\bar{\alpha}y \times$$

$$(D_{13}\sinh\alpha y + D_{14}\cosh\alpha y + F_2(y)) + \cosh\bar{\alpha}y\Big[\alpha(D_{13}\cosh\bar{\alpha}y +$$

$$D_{14}\sinh\alpha y) + F_2'(y)\Big]\Big]\Big\}\Big\}\Big] \qquad (125)$$

$$F_5(y) = -\frac{1}{\alpha^2}\Big[-\frac{2\sigma}{1+\sigma}\Big\{F_4''(y) - \alpha^2 F_4(y)\Big\} + \alpha\Big\{1 + K\Big[1 + y^2 - \frac{2\sigma}{1+\sigma} \times$$

$$\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2\cosh\bar{\alpha}}\Big]\Big\}(F_4''(y) - \alpha^2 F_2(y)) + 2\alpha K\Big\{1 - \frac{\sigma}{1+\sigma}\,\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\Big\} \times$$

$$(D_{13}\sinh\alpha y + D_{14}\cosh\alpha y + F_2(y) + \frac{1}{\alpha^2}\Big[-\frac{2\sigma}{1+\sigma}\,(F_4''(y) -$$

$$- \alpha^2 F_4'(y)) + \alpha \left[1 + K\left(1 - y^2 - \frac{2\sigma}{1+\sigma}\frac{\cosh\bar{\alpha} - \cosh\bar{\alpha}y}{\bar{\alpha}^2 \cosh\bar{\alpha}}\right)\right]\left[F_2''(y) -\right.$$

$$\left. - \alpha^2 F_2'(y)\right] + 4\alpha K\left[- y + \frac{\sigma \sinh\bar{\alpha}y}{\bar{\alpha}(1+\sigma)\cosh\bar{\alpha}}\right] \times$$

$$\left[F_2'''(y) - \alpha^2 F_2'(y)\right] + 2\alpha K\left[- 1 + \frac{\sigma \cosh\bar{\alpha}y}{(1+\sigma)\cosh\bar{\alpha}}\right]\left[F_2''(y) - \alpha^2 F_2(y)\right]$$

$$+ 2\alpha K\left[1 - \frac{\sigma}{1+\sigma}\frac{\cosh\bar{\alpha}y}{\cosh\bar{\alpha}}\right]\left[\alpha^2(D_{13}\sinh\alpha y + D_{14}\cosh\alpha y) + F_2''(y)\right] -$$

$$- \frac{4\alpha\sigma\bar{\alpha}K \sinh\bar{\alpha}y}{(1+\sigma)\cosh\bar{\alpha}}\left[\alpha(D_{13}\cosh\alpha y + D_{14}\sinh\alpha y) + F_2'(y)\right] -$$

$$\left. - \frac{2\alpha\bar{\alpha}^2 K \cosh\bar{\alpha}y}{(1+\sigma)\cosh\bar{\alpha}}\left[D_{13}\sinh\alpha y + D_{14}\cosh\alpha y + F_2(y)\right]\right] \quad (126)$$

The solutions to (119), (120) subject to (121) are

$$Z_{22} = A_{42}\frac{\sinh\Theta y}{\Theta^2} + B_{42}\frac{\cosh\Theta y}{\Theta^2} + F_6(y) \quad (127)$$

$$J_{22} = F_7(y) + C_{15}y + C' \quad (128)$$

where

$$\left.\begin{aligned} A_{42} &= -\frac{\Theta^3(2F_6'(1) + F_4''(1))}{2\cosh\Theta} \\ B_{42} &= -\frac{\alpha_1^2\Theta^3 J_9 \sinh\alpha_1}{2\sinh\Theta} \end{aligned}\right\} \quad (129)$$

$$C_{15} = -\left[F_7'(1) + \tfrac{1}{2}(\alpha^2 J_{10}\sinh\alpha + F_5''(1))\right] \quad (130)$$

while C' is an arbitrary constant

$$F_6(y) = -\frac{1}{\Theta^2}\left[\frac{1}{\Theta^2}\left\{\frac{\alpha}{2}\left[\left\{-\alpha_1\delta\cosh\alpha_1 y + B_2\cosh\alpha y\right\} \times\right.\right.\right.$$

$$\left\{ \alpha^2 (D_{13} \sinh \alpha y + D_{14} \cosh \alpha y) + F_2'(y) \right\} - \alpha^2 \left[\alpha (D_{13} \cosh \alpha y + D_{14} \sinh \alpha y) + F_2'(y) \right] \left[B_2 \sinh \alpha y - \delta \sinh \alpha_1 y \right]$$

$$- \left[B_2 \sinh \alpha y - \delta \sinh \alpha_1 \right] \left[\alpha^3 (D_{13} \cosh \alpha y + D_{14} \sinh \alpha y + F_2'''(y) \right] -$$

$$\left[D_{13} \sinh \alpha y + D_{14} \cosh \alpha y + F_2(y) \right] \left[\alpha^3 B_2 \cosh \alpha y - \alpha_1^3 \delta \cosh \alpha_1 y \right]$$

$$+ \frac{(1+\sigma)\alpha}{2\bar{\alpha}} \left[\left\{ B_2 \sinh \alpha y - \delta \sinh \alpha_1 y \right\} \left\{ V_3 \sinh \alpha y + V_4 \sinh \alpha_1 y + F_3(y) \right\} + \left\{ D_{13} \sinh \alpha y + D_{14} \cosh \alpha y + F_2(y) \right\} \left\{ \alpha B_2 \cosh \alpha y - \alpha_1 \delta \cosh \alpha_1 y \right\} - \left\{ \alpha (D_{13} \cosh \alpha y + D_{14} \sinh \alpha y + F_2'(y) \right\} \left\{ B_2 \sinh \alpha y - \delta \sinh \alpha_1 y \right\} \right]$$

$$+ \frac{(1+\sigma)\alpha}{2\bar{\alpha}} \left[(B_2 \sinh \alpha y - \delta \sinh \alpha_1 y) (V_3 \sinh \alpha y + V_4 \sinh \alpha_1 y + F_3(y)) + (D_{13} \sinh \alpha y + D_{14} \cosh \alpha y + F_2(y)) (\alpha B_2 \cosh \alpha y - \alpha_1 \delta \cosh \alpha_1 y) - \left\{ \alpha (D_{13} \cosh \alpha y + D_{14} \sinh \alpha y) + F_2'(y) \right\} \left\{ B_2 \sinh \alpha y - \delta \sinh \alpha_1 y \right\} \right] + \frac{\alpha}{2} \left[\left[D_{13} \left\{ \frac{\sinh (\alpha_1 + \alpha) y}{(\alpha_1 + \alpha)^2} - \frac{\sinh (\alpha_1 - \alpha) y}{(\alpha_1 - \alpha)^2} \right\} + D_{14} \left\{ \frac{\cosh (\alpha_1 + \alpha) y}{(\alpha_1 + \alpha)^2} + \frac{\cosh (\alpha_1 - \alpha) y}{(\alpha_1 - \alpha)^2} \right\} \right] P'$$

$$- \alpha_1 \delta \left[F_2(y) \cosh \alpha_1 y - 2 \alpha_1 \int F_2(y) \sinh \alpha_1 y \, dy + \alpha_1^2 \iint F_2 \right.$$

$$(y)\ \cosh \alpha_1\ y\ dy \Big] + \frac{\alpha^3 B_2}{2}\Big[D_{13} \sinh 2\alpha y + D_{14}(1+\cosh 2\alpha y)\Big] +$$

$$+ \alpha B_2\Big[F_2(y)\ \cosh\alpha y - 2\alpha \int F_2(y)\ \sinh\alpha y\ dy + \alpha^2 \iint F_2(y) \times$$

$$\cosh\alpha y\ dy \Big] - \alpha^2\Big[-\frac{\alpha\delta D_{13}}{2}\Big[\frac{\sinh(\alpha_1+\alpha)y}{(\alpha_1+\alpha)^2} + \frac{\sinh(\alpha_1-\alpha)y}{(\alpha_1-\alpha)^2}\Big]$$

$$+ \frac{\alpha B_2 D_{13}}{2} \sinh 2\alpha y - \frac{\alpha\delta D_{14}}{2}\Big[\frac{\cosh(\alpha_1+\alpha)y}{(\alpha_1+\alpha)^2} - \frac{\cosh(\alpha_1-\alpha)y}{(\alpha_1-\alpha)^2}\Big]$$

$$+ \alpha \frac{D_{14} B_2}{2} (\cosh 2\alpha y - 1) - \delta\Big[\int F_2(y)\ \sinh\alpha y\ dy - \alpha \times$$

$$\iint F_2(y)\ \cosh \alpha y\ dy\Big] + B_2\Big[\int F_2(y)\ \sinh\alpha_1 y\ dy - \alpha \iint F_2(y) \times$$

$$\cosh \alpha\ y\ dy \Big]\Big] - \Big[-\frac{\alpha^3\delta}{2}\Big[D_{13}\Big\{\frac{\sinh(\alpha_1+\alpha)y}{(\alpha_1+\alpha)^2} + \frac{\sinh(\alpha_1-\alpha)y}{(\alpha_1-\alpha)^2}\Big\}$$

$$+ D_{14}\Big\{\frac{\cosh(\alpha_1+\alpha)y}{(\alpha_1+\alpha)^2} - \frac{\cosh(\alpha_1-\alpha)y}{(\alpha_1-\alpha)^2}\Big\}\Big] + \frac{\alpha^3 B_2}{2}\Big[D_{13} \sinh 2\alpha y +$$

$$+ D_{14}(\cosh 2\alpha y - 1)\Big] - \delta\Big[F_2'(y)\ \sinh \alpha_1 y - 2\alpha_1 F_2(y)\ \cosh \alpha_1 y$$

$$+ 3\alpha_1^2 \int F_2(y)\ \sinh \alpha_1 y\ dy -$$

$$- \alpha_1^3 \iint F_2(y)\ \cosh \alpha_1 y\ dy\Big] + B_2\Big[F_2'(y)\ \sinh\alpha y - 2\alpha F_2(y) \times$$

$$\cosh \alpha y + 3\alpha^2 \int F_2(y)\ \sinh\alpha y\ dy - \alpha^3 \iint F_2(y)\ \cosh\alpha y\ dy\Big]\Big]\Big]$$

(131)

$$F_7(y)$$

$$= \frac{1}{\alpha}\Big[\frac{1}{\theta^2}\Big[A_{42}\ \sinh\theta y + B_{42}\ \cosh\theta y\Big] + F_6(y) - \frac{(1+\sigma)\alpha}{2} \times$$

$$\left[\frac{B_2}{2} \left[V_3 \left(\frac{\cosh 2\alpha y}{4\alpha^2} - \frac{y^2}{2} \right) + V_4 \left(\frac{\cosh 2\alpha_1 y}{4\alpha_1^2} - \frac{y^2}{2} \right) \right] + \right.$$

$$+ \iint F_3(y) \left(B_2 \sinh \alpha y - \delta \sinh \alpha_1 y \right) dy + \frac{\alpha_1 D_{13} A_2}{2} \times$$

$$\left[\frac{\cosh (\alpha_1 + \alpha) y}{(\alpha_1 + \alpha)^2} - \frac{\sinh (\alpha_1 - \alpha) y}{(\alpha_1 + \alpha)^2} \right] + A_3 \frac{D_{13} \sinh 2\alpha y}{8\alpha} +$$

$$+ \frac{\alpha_1 D_{13} A_2}{2} \left[\frac{\cosh (\alpha_1 + \alpha) y}{(\alpha_1 + \alpha)^2} + \frac{\cosh (\alpha_1 - \alpha) y}{(\alpha_1 - \alpha)^2} \right] + \frac{\alpha D_{14} A_3}{2} \times$$

$$\left[\frac{\cosh 2\alpha y}{4\alpha^2} + \frac{y^2}{2} \right] + \iint F_2(y) \left(\alpha_1 A_2 \cosh \alpha_1 y + \alpha A_3 \cosh \alpha y \right)$$

$$- \left[\frac{\alpha D_{13} A_2}{2} \left[\frac{\sinh (\alpha_1 + \alpha) y}{(\alpha_1 + \alpha)^2} + \frac{\sinh (\alpha_1 - \alpha) y}{(\alpha_1 - \alpha)^2} \right] + \frac{A_3 D_{13}}{8\alpha} \times \right.$$

$$\sinh 2\alpha y + \frac{\alpha D_{14} A_2}{2} \left[\frac{\cosh (\alpha_1 + \alpha) y}{(\alpha_1 + \alpha)^2} - \frac{\cosh (\alpha_1 - \alpha) y}{(\alpha_1 - \alpha)^2} \right]$$

$$+ A_2 \left[\int F_2(y) \sinh \alpha_1 y \, dy - \alpha_1 \iint F_2(y) \cosh \alpha_1 y \, dy \right] + A_3 \times$$

$$\left. \left[\int F_2(y) \sinh \alpha y \, dy - \alpha \iint F_2(y) \cosh \alpha y \, dy \right] \right] \right]$$

$$- 2 \left[\frac{1}{\theta^4} \left[A_{42} \sinh \theta y + B_{42} \cosh \theta y \right] + \int F_6(y) \right] \qquad (132)$$

where $F' = - \frac{\alpha_1 \alpha^2 \delta}{2}$ and

$$p^{*} = v_4\left[\frac{\cosh(\alpha_1+\alpha)y}{(\alpha_1+\alpha)^2}\ \frac{\cosh(\alpha_1-\alpha)y}{(\alpha_1-\alpha)^2}\right] - \frac{\delta}{2}\left[v_3\left[\frac{\cosh(\alpha_1+\alpha)y}{(\alpha_1+\alpha)^2}\right.\right.$$

$$\left.\left.\frac{\cosh(\alpha_1-\alpha)y}{(\alpha_1-\alpha)^2}\right] + v_4\left[\frac{\cosh 2\alpha_1 y}{4\alpha_1^2} - \frac{y^2}{2}\right]\right]$$

REFERENCES

1. Allen, S.J., Desilva, C.N. and Kline, K.A. (1967) Physics of Fluids, 10 : 2551.

2. Allen, S.J. and Kline, K.A. (1970). Transactions of the Society of Rheology, 12 : 457.

3. Ariman, T. Turk, M. A. and Sylvester N.D. (1964). Letters in Applied and Engineering Sciences, 2 : 21.

4. Barton, C and Raynor, S. (1966). Peristaltic flow in tubes. Bulletin of Mathematical Biophysics, 30 . 663 - 680.

5. Batcher, G.K. (1974). Transport properties of Two-phase materials with Random Structure. Annual Review of Fluid Mechanics, 6 : 227 - 255.

6. Batcher, G.K. (1976). Brownion Diffusion of particles with Hydrodynamics Interaction. Journal of Fluid Mechanics, 74 : 1 - 29.

7. Brown, T.D. and Hung, T.K. (1974). Computational and experimental investigations of two dimensional non-

linear peristaltic flows. Journal of Fluid Mechanics, 83 : 249 - 273.

8. Bugliarello, G. and Sevilla, J. (1970). Velocity distribution and other characteristic of steady and pulsatile blood flow in fine glass tube. Biorheology, 7 : 85 - 107.

9. Buck, R.C. and Buck, E.F. (1976). Introduction to Differential Equations. Houghton Mifflin Company, Boston, U.S.A.

10. Burn, J.C. and Parkes, T. (1967). Peristaltic Motion. Journal of Fluid Mechanics, 29 . 731 - 743.

11. Chaturani, P. and Mahajan S.P. (1982). Poiseuille flow of micropolar fluid with non-zero couple stress at boundary with applications to blood flow. Biorheology 19(4): 507 - 518.

12. Chaturani, P. and Pralhad, R.N. (1985). Blood flow in tapered tubes with Biorheological Applications. Biorheology, 22(4) : 303 - 314.

13. Chow , S.T. (1970). Peristaltic transport in a Circular cylindrical pipe. Transactions ASME Journal of Applied Mechanics, 37(1): 901 - 906.

14 Condiff, D.W. and Dahter, J.S. (1964). Physics fluids, 7 : 842.

15. Cowin, S.C. (1974). The theory of polar fluids Advances in Applied Mechanics, 14 : 279 - 347.

16. Devi, G. and Devanathan, R. (1975). Peristaltic motion of a micropolar fluid. In : Proceedings of Indian Academy of Sciences, 81A(4): 149 - 163.

17. Drew, D.A. (1979). Stability of a stokes layer of a Dusty Gas. Physical Fluid, 19: 2081.

18. Eringen, A.C. (1964). International Journal of Engineering Science 2, 205.

19. Eringen, A.C. (1966). Theory of Micropolar fluids. Journal of Mathematical Mechanics, 16 : 1 - 18.

20. Fitzwater, M.A. (1980). The lateral migration of a Sphere in a micropolar liquid. International Journal of Engineering Science, 18(2) : 1421 - 1430.

21. Fung, Y.C. and Yih, C.S. (1968). Peristaltic transport. Transactions ASME Journal of Applied Mechanics, 35(1) : 669 - 674.

22. Fung, Y.C. (1971a). Biomechanics- A survey of the blood flow problem. In Advances in Applied Mechanics 11 (Edited by Yih, C.S.), pp. 65 - 130. Academic Press, New York.

23. Fung, Y.C. (1971b). Peristaltic Pumping, A bioengineering Model. In Urodynamics: hydrodynamics of the Ureter and renal pelvis (Edited by Boyarsky, S.) Academic Press, New York.

24. Fung, Y.C. (1971c). Muscle controlled flow. In : Proceedings 12th Midwestern Mechanics Conference, University of Notre Dame, South Bend, Indiana.

25 Gupta, B.B. and Sheshadri, V. (1976). Peristaltic pumping in non-uniform tubes. Journal of Biomechanics. 9 : 105 - 109.

26. Hanin, M. (1968). The flow through a channel due to transversally oscillating walls. Israel Journal of

Technology, 6 : 67 - 71

27. Hug, T.K. and Brown, T.D. (1976). Solid-particle motion in two-dimensional peristaltic flows. Journal of Fluid Mechanics, 73 : 77 - 96.

28. Haynes, H.R. (1960). Physically basis of the dependencies of blood viscosity on tube radius. American Journal of Physiology, 198 : 1193 - 1200.

29. Jaffrin, M.Y. (1971). Inertia and stream line curvature effects on peristaltic pumping. In: Proceedings 7th Annual Meeting Society Engineering Sciences.

30. Jaffrin, M.Y. and Shapiro, A.H. (1971). Peristaltic pumping. Annual Review of Fluid Mechanics, 3: 13 - 36. Annual Review Inc., Palo Alto Publications, California, U.S.A.

31. Kaimal, M.R. (1978). Peristaltic pumping of a Newtonian fluid with particles suspended in it at low Reynolds number under long wavelength approximation. Transactions ASME Journal of Applied Mechanics, 43 : 32 - 36.

32 Kline, K.A. and Allen, S.J. (1970). Physics Fluids 13 : 263.

33. Latham, T.W. (1966). Fluid motion in a peristaltic pump M.S. Thesis Massachusetts Institute of Technology, Cambridge.

34. Lee, J.S. and Fung, Y.C. (1971). Flow in non-uniform small blood vessels. Microvasc. Res., 3: 272 - 279.

35. Lee, H.S., Fung, Y.C. and Lowenstein, C.B. (1971). Peristaltic carrying and mixing of chyme in small intestine. Journal of Biomechanics, 4: 297 - 315.

36. Lew, H.S. and Fung, Y.C. (1970). Plug Effect of Erythrocytes in Capillary Blood vessels Biophysics Journal, 10 : 80.

37. Li C.H. (1970). Peristaltic transport in circular cylindrical tubes, Journal of Biomechanics, 3: 513 -525.

38. Liron, N. (1976). On peristaltic flow and its efficiency. Bulletin of Mathematical Biology , 38 : 573 - 523.

39. Liron, N. (1978). A new look at the peristalsis and its functions. In: Horizons in Biochemistry and Bio-

physics. 5 : pp. 161 - 182. Addison-Wesley (Advance Book Programme), Reading, Massachusetts.

40. Lykondis, P.S. and Roos, R. (1970). The fluid mechanics of the ureter from a lubrication theory point of view. Journal of Fluid Mechanics, 43 : 661 - 674.

41 Mahrenholtz, O.H., Mank, M.G. and Zimmermann, R.U. (1978). The influence of wave form on peristaltic transport. Biorheology, 15 : 501 - 510.

42. Mank, M.G. (1976). Berechung der peristaltischen flussigkeits forderung Mitder method der finiten elemrnt, Diss. Honnover.

43. Manton, M.J. (1975). Long wavelength peristaltic pumping at low Reynolds number. Journal of Fluid Mechanics, 68 : 467 - 476.

44. Marion, J.B. (1970). On Perturbations. In Classical Dynamics of particles and systems. pp. 165 - 174. Academic Press, New York.

45. Meginniss, J.R. (1970). An analytical investigation of flow and hemolysis in peristaltic type blood pumps, M.S. Thesis. Massachusetts Institute of Technology, Cambridge.

46. Mittra, T.K. and Prasad, S.N. (1973). On the influence of wall properties and Poiseuille flow in Peristalsis. Journal of Biomechanics, 6 : 681 - 693.

47. Mittra, T.K. and Prasad, S.N. (1974). Interaction of peristaltic motion with Poiseuille flow. Bulletin of Mathematical Biology, 36 : 127 - 141.

48. Negrin, M.P., Shack, W.J. and Lardner, T.J. (1974). A note on peristaltic pumping. Transactions ASME Journal of Applied Mechanics, 96: 520 - 521.

49 Parvathamma, S. and Devonathan R. (1983). Microcontinuum Approach to the Pulsatile flow in Tubes with and without longitudinal vibration. Bulletin of Mathematical Biology, 45 (5) : 721 - 737.

50. Paul, A. S and Sridharon, K. (1980). The penny-shaped

Creck problem in micropolar Thermoelasticity. International Journal of Engineering Science, 18(13) : 1431 - 1448.

51 Powers, D.L. (1972). Boundary Value Problems Academic Press, New York.

52 Raju, K.K. and Devanathan, R. (1972). Peristaltic motion of a non- Newtonian fluid. Rheology Acta, 11 : 170 - 178.

53. Rao, S.K.L. and Raju, K.V. (1980). Variational algorithm for the stability of the flow micropolar fluids with stretch. International Journal of Engineering Science, 18(12) : 1411 - 1420.

54. Rath, J.J. (1978). Ein bertrag Zur berechung einer peristaltischen stromung in elastischen lectungen. Acta Mechanica, 31 : 1 - 12.

55. Roos, R. and Lykoudis, P.S. (1971). The fluid mechanics of ureter with an inserted catheter. Journal of fluid Mechanics, 46 : 625 - 630.

56. Shapiro, A. H. (1967). Pumping and retrograde diffusion in peristaltic waves. Proceeding Workshop in Ureteral Reflux in children, Nat. Acad. Sci., Nat. Resource Council. PP. 109 - 126.

57. Shapiro, A.H., Jaffrin, M.Y. and Weinberg, S.L. (1969). Peristaltic pumping with long wavelength at low Reynolds number. Journal of Fluid Mechanics, 37 : 799 - 825.

58. Shapiro, A.H. and Latham, T. W. (1966). On peristaltic pumping (abstract). In : Proceedings American Conference on Engineering Medicine Biology, 8 : 147, Holden - Day, San Francisco.

59. Shukla, J.B., Parihar, R.S., Rao, B.R.P. and Gupta, S.P. (1980). Effects of peripheral layer viscosity on peristaltic transport of a bio- fluid. Journal of Fluid Mechanics, 97 : 225 - 237.

60. Spiegel, M.R. (1971). Advanced Mathematics for Engineers and Scientists, MiGraw - Hill, New York, U. S. A.

61. Srivastava, L.M. (1982). Peristaltic transport of blood : Casson Model - I (communicated)

62. Srivastava, L.M. and Srivastava, V.P. (1982). Peristaltic transport of a two- layered model of physiological fluid. Journal of Biomechanics, 15 : 257 - 265.

63. Srivastava, L.M., Srivastava, V.P. and Sinha, S.N. (1982a). Peristaltic transport of a physiological fluid, Part - I, flow in non - uniform geometry, Biorheology 19 :

64. Srivastava, L.M., Srivastava, V.P. and Sinha, S.N. (1982b). Peristaltic transport of a physiological fluid, Part - II, Flow in uniform geometry. Biorheology, 19 :

65. Srivastava, L.M., Srivastava, V.P. and Sinha, S.N. (1982c). Peristaltic transport of a physiological fluid, Part - III, Applications. Biorheology, 19 :

66. Srivastava, L.M. and Srivastava, V.P. (1983 a). Interaction of peristaltic flow with pulsatile flow

in a circular cylindrical tube. Journal of Biomechanics (To appear).

67. Srivastava, L.M. and Srivastava, V.P. (1983 b). Peristaltic transport of a non - Newtonian fluid and Applications (communicated).

68. Srivastava, L.M. and Srivastava, V.P. (1983). On Two- phase Model of Pulsatile Blood Flow with Entrance Effects. Biopheology, 20: 761 - 177.

69. Srivastava, L.M. and Srivastava, V.P. (1984). Peristaltic Transport of Blood Casson Model II. Journal of Biomechanics, 17(11) : 821 - 823.

70. Stokes, V.K. (1966). Couple stresses in fluids. Physics Fluids, 9 : 1709 - 1715.

71 Tam, C.K.W. (1969). The Drag on a cloud of spherical particles in Low Reynolds Number Flow. Journal of Fluid Mechanics, 38 : 537 - 546.

72. Tong, P. and Vawter, D. (1972). An analysis of peristaltic pumping. Transactions ASME Journal of Applied Mechanics, 39 : 857 - 862.

78 Wilson, D.E. and Panton, R L. (1979). Peristaltic transport due to finite amplitude and contraction waves. Journal of fluid Mechanics, 90: 145 - 159.

79. Wilson, D.E. and Perel, D. (1979). Interaction of pulsatile and peristaltic induced flows. Paper no. 79 - WA/Bio - 3, In: Proceedings A S M E Bioengineering symposium, New York.

80 Wylie, W.C.R. (1979). Differential Equations, MiGraw Hill Inc. U.S.A.

81. Yin, F. and Fung, Y.C. (1969). Peristaltic waves in circular cylindrical tubes. Transactions A S M E Journal of Applied Mechanics, 36 : 579 - 587.

82. Yin, F.C.P. and Fung, Y.C. (1971). Comparison of theory and experiment in peristaltic transport. Journal of Fluid Mechanics, 47 : 93 - 112.

83. Zien, T. and Ostrach, S. (1970). A long wave approximation to peristaltic motion. Journal of Biomechanics, 3 : 63 - 75.

84. Zimmerman, R.J. (1977). Der Einflu : β - der Wellenform Auf. Die Peristaltischen Forderung, Studienarbeit (unveroffentlicht), Ionnover.