DEVELOPING ECONOMIC PRODUCTION QUANTITY MODELS FOR ITEMS THAT EXHIBIT DELAY IN DETERIORATION WITH RELIABILITY CONSIDERATION

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BY

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DEDICATION

To the entire Dari's family whom I owe my successes in life.

DECLARATION

I declare that the work in this thesis entitled "Developing Economic Production Quantity (EPQ) Models for Items that Exhibit Delay in Deterioration with Reliability Consideration" has been carried out by me in the department of mathematics under the supervision of Dr. Babangida Sani and Dr. Muhammed Abdul. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any Institution.

DARI SANI

Name of Student	Sign	Date

CERTIFICATION

This thesis entitled "DEVELOPING ECONOMIC PRODUCTION QUANTITY (EPQ) MODELS FOR ITEMS THAT EXHIBIT DELAY IN DETERIORATION WITH RELIABILITY CONSIDERATION" by Sani DARI (M.Sc/SCIEN/06114/09-10) meets the regulations governing the award of the degree of Master of Science (M.Sc.) in Mathematics of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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ABSTRACT

This thesis studies some Economic Production Quantity (EPQ) models of deteriorating items that exhibit delay in deterioration with reliability consideration. Two modifications to existing models are presented; the first modification assumes a constant demand both before and after deterioration begins, while the second modification assumes a linearly time dependent demand after deterioration begins. The unit cost of production of an item is assumed to be directly related to the process reliability and inversely related to the demand rates. Numerical examples are given to illustrate the applications of the models.

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CHAPTER ONE

GENERAL INTRODUCTION

1.0. INTRODUCTION

Inventory consists of materials, commodities, products, etc, which are usually carried in stock in order to be consumed or benefited from when needed. An Economic Order Quantity (EOQ) model sometimes referred to as Economic Order Lot-size model is an inventory control model, which determines the optimal quantity to be ordered so as to meet a deterministic demand over a planned period of time in order to minimize cost. An Economic Production Quantity (EPQ) or Economic Production Lot-size model is an inventory control model which determines the optimal quantity to be produced so as to meet a deterministic demand with the objective of minimizing cost. Thus, EPQ model is an offshoot of the well known EOQ model.

In several articles in the literature on inventory models (focusing on EOQ and EPQ), it is assumed that items can be stored for a long period of time for future use without spoilage. However, it is a general knowledge that almost all items on inventory deteriorate over time. Deterioration can be referred to as depression in quality/quantity of items kept on inventory for certain purpose. An item on inventory becomes reliable, if it satisfies the probability that it will adequately perform its specified purpose, for a specified period of time, under specified environmental conditions. Thus reliability is influenced by the decisions made during the design and manufacturing of the product.

In this thesis, we study some EPQ models of deteriorating items which exhibit delay in deterioration with reliability consideration. These are items which do not start deteriorating immediately they are stored, until later. Such items include potatoes, yam, bread, cakes, to name a few. Two modifications to existing models are presented; the first modification assumes a constant demand both before and after deterioration begins, while the second assumes a linearly time dependent demand after deterioration commences.

1.1 COMPONENTS OF INVENTORY MODELS

The profit of a production is affected by the policies on inventory; as such the choice among inventory policies depends upon their relative profitability. Profitability is determined by the following factors: the costs of ordering or production set-up costs (in case of production), shortage costs, holding costs, salvage costs, revenues and discount rates.

- *Cost of Ordering*: This is the cost of placing an order to an outside supplier or releasing a production order to a manufacturing shop.

- *Set-up cost*: This is the cost incurred in preparing a machine or process for manufacturing an order. It includes the design cost, location of machinery, employee hiring, research and development expenses, and labor cost for cleaning and changing tools or holders.

- *Shortage Cost:* Shortage cost (sometimes called the *unsatisfied demand cost*) is incurred when the amount of the commodity required (demand) exceeds the available stock.

- *Holding Cost*: Holding cost (sometimes called the *storage cost*) represents all the costs of capital tied up, space, insurance, protection, and taxes attributed to storage. The holding cost can be assessed either continuously or on a period-by-period basis.

- *Salvage Value:* Salvage value of an item is the value of a leftover item when no further inventory is desired. The salvage value represents the disposal value of the item to the firm, perhaps through a discounted sale. The negative of the salvage value is called the salvage cost. If there is a cost associated with the disposal of an item, the salvage cost may be positive.

- *Revenue*: Revenue may or may not be included in the model. If both the price and the demand for the product are established by the market and so are outside the control of the company, the revenue from sales (assuming demand is met) is independent of the firm's inventory policy and may be neglected. However, if revenue is not neglected in the model, the *loss in revenue* must then be included in the shortage cost whenever the firm cannot meet the demand and the sale is lost.

- *Discount Rate*: discount rate takes into account the time value of money. When a firm ties up capital in inventory, the firm is prevented from using this money for alternative purposes.

1.2 A GENERALIZED INVENTORY MODEL

The ultimate objective of an inventory model is to answer two questions.

1. How much to order/produce?

2. When to order/produce?

The answer to the first question (how much to order/produce) is expressed in terms of what we call the order/production quantity and the second question (when-to-order/produce) is the inventory level at which a new order/product should be placed/produced usually expressed in terms of re-order point.

According to Hadley and Whitin (1963), one can summarize the total cost of a general inventory model as a function of its principal components in the following manner:

Total inventory cost = purchasing cost + setup cost (or ordering cost) + holding cost
+ shortage cost (if shortages are allowed)

1.3 TYPES OF INVENTORY MODELS

Basically, all inventory models (EOQ/EPQ) are classified into two categories:

- Deterministic model and
- Stochastic model

1.3.1 DETERMINISTIC DEMAND

This is a situation where by the demand rate is known with certainty. It can be further classified into uniform (constant) demand and time-dependent demand. The time-dependent demand is also classified into discrete time dependent demand and continuous time dependent demand. The time-dependent demand may be:

- linearly increasing given by $\mu(t) = a + bt$; $a > 0, b > 0$,
- linearly decreasing given by $\mu(t) = a + bt$; $a > 0, b < 0$,

- it may be exponentially increasing given by $\mu(t) = ae^{bt}$; $a > 0, b > 0$, or
- exponentially decreasing with $\mu(t) = ae^{bt}$; $a > 0, b < 0$ and so on.

1.3.2 STOCHASTIC DEMAND

This is a situation where by the demand rate follows a statistical distribution which may be a known probability distribution or an arbitrary probability distribution.

Inventory models can also be classified based on their current mode of supervision: periodic review and continuous review. In *periodic review*, the level of the inventory is to be checked at discrete intervals, e.g., at an interval of one month, and decisions on ordering are to be made only at these times (an interval of one month) even if the inventory level is below the reorder point between the current and preceding review times. In *continuous review*, placement of an order is done as soon as the stock level falls down to the prescribed reorder point. (Hillier and Lieberman, 2001)

1.3.3 DETERMINISTIC CONTINUOUS-REVIEW MODEL

Manufacturers/producers, retailers as well as wholesalers face a common inventory scenario; that items in stock are exhausted/drained over time and they are refilled/replaced by the arrival of new manufactured items. An inventory control model describing such situation in a production environment is the economic production quantity *(EPQ)* model, which is sometimes referred to as the production lot-size model.

In the EPQ model, the demand rate is constant. The inventory is replenished when required by producing a batch of fixed size (Q units), where all Q units are produced at the desired time. For

this case of basic economic production quantity (*EPQ)* model, the following costs are considered:

K = set-up cost per production run

c= unit cost of the item

h= inventory carrying cost in a production cycle.

The main objective is either to minimize the total inventory cost per unit time or to maximize the profit. (Paknejad *et al*., 1995)

1.3.3.1 THE BASIC EPQ MODEL

The basic EPQ model is an offshoot of the well known Economic Order Quantity (EOQ) model and the two (EPQ and EOQ) have similar assumptions. The EPQ as an inventory control model is usually based on the following assumptions:

i. Constant deterministic demand rate per unit time.

ii. If inventory level drops to 0, then the production quantity (Q) to replenish inventory is produced all at once.

iii. Planned shortages are not permitted.

The total cost of production per unit time T is computed from the following components.

Production cost per cycle $=K+cQ$.

The average inventory level during a cycle is $Q/2$ units, and the corresponding cost is $hQ/2$ per unit time. Since the cycle length is Q/a,

$$\text{Holding cost per cycle} = \frac{hQ^2}{2a}$$

Then,

$$\text{Total cost per cycle } = K + cQ + \frac{hQ^2}{2a}$$

And so, the total cost per unit time T is

$$T = \frac{K + cQ + hQ^2/(2a)}{Q/a} = \frac{aK}{Q} + ac + \frac{hQ}{2}$$

The value of Q, say Q^*, that minimizes T is found by setting the first derivative of T with respect to Q to zero (provided that the second derivative is positive).

i.e. $$\frac{dT}{dQ} = -\frac{aK}{Q^2} + \frac{h}{2} = 0$$

giving $$Q^* = \sqrt{\frac{2aK}{h}} \qquad (1.1)$$

which is the well-known *EPQ formula.* The corresponding *cycle time,* say t^*, is

$$t^* = \frac{Q^*}{a} = \sqrt{\frac{2K}{ah}} \qquad (1.2)$$

(Nahmias, 2009)

1.3.4 DETERMINISTIC PERIODIC-REVIEW MODEL

The assumptions in the basic EPQ model are not always realistic. This is why several authors modified the model over time to reflect several realistic scenarios. When the assumption of constant demand is relaxed for instance i.e. when the amounts that need to be withdrawn from

inventory are allowed to vary from period to period, the *EPQ formula* no longer ensures a minimum-cost solution, for all cycles.

Suppose planning is to be done for the next *n* periods regarding how much (if any) to produce to replenish inventory at the beginning of each of the periods. The demands for the respective periods are *known* (but *not* the same in every period) and are denoted by

r_i = demand in period i, for $i = 1,2,3,\ldots,n$

The EPQ in this case is given by

$$Q^*_i = \sqrt{\frac{2r_iK}{h}} \qquad \text{for } i = 1,2,3,\ldots,n \qquad (1.3)$$

and

$$t^*_i = \frac{Q^*_i}{r_i} = \sqrt{\frac{2K}{r_ih}} \qquad \text{for } i = 1,2,3,\ldots,n \qquad (1.4)$$

(Hadley and Whitin, 1963; Hillier and Lieberman, 2001)

1.3.5 A STOCHASTIC CONTINUOUS-REVIEW MODEL

In a stochastic continuous-review inventory system for a particular item, there are two factors to be considered, namely:

R = reorder point,

Q = order quantity.

For a retailer or wholesaler (or a manufacturer replenishing its raw materials inventory from a supplier), the purchase order for Q units of the product is the order quantity. On the other hand, for a manufacturer managing its finished products on inventory, the production run of size Q is the order quantity.

Inventory policy based on these factors (R and Q) is as follows: an order for Q more units is to be placed to replenish the inventory, if the inventory level of the product drops to R units. Such a policy is sometimes called reorder point, order quantity policy or *(R, Q)* policy. [Consequently, the overall model might be referred to as the *(R, Q)* model. Other modifications such as *(Q, R)* model, *(Q, R)* policy, and so on, are also used.] (Paknejad *et al*., 1995)

1.3.5.1 CHOOSING THE ORDER QUANTITY *Q*

The approach used in formulating Q for stochastic continuous-review model is as follows:

Total cost=setup cost + purchase cost + holding cost +shortage cost

$$T = K + cQ + \frac{hR^2}{2a} + \frac{p\left(Q-R\right)^2}{2a} \tag{1.5}$$

where p is the shortage.

Total cost per unit time *T(Q,R)* is given as

$$T\left(Q,R\right) = \frac{aK}{Q} + ac + \frac{hR^2}{2Q} + \frac{p\left(Q-R\right)^2}{2Q} \tag{1.6}$$

taking the partial derivative of (1.6) with respect to Q and R and set the result to zero, we have

$$R^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p}{p+h}} \quad \text{and} \quad Q^* = \sqrt{\frac{2aK}{h}}\sqrt{\frac{p+h}{p}} \tag{1.7}$$

(Ra'afat, 1991)

1.3.6 STOCHASTIC PERIODIC-REVIEW MODELS

This is a situation when we assume that the demand is uncertain. However, in contrast to the continuous-review inventory system, we now assume that the system is only being monitored periodically. At the end of each period, when the current inventory level is determined, a decision is made on how much to order (if any) to replenish inventory for the next period. Each of these decisions takes into account the planning for multiple periods into the future.

1.4 ORDER POINT AND SAFETY STOCK

The economic production quantity model indicates how many units to produce. Practitioners are also concerned with the order point. This quantity reflects the level of inventory that triggers the start of set up for additional units.

Determination of the order point is based on three factors:

- usage (quantity of inventory used or sold each day),
- lead time (is the time it takes from the start of set up to when the goods are produced), and
- safety stock (The quantity of inventory kept on hand by a company in the event of fluctuating usage or unusual delays in lead time).

$$\text{Order point} = (\text{usage per unit time} \times \text{lead time}) + \text{safety stock}$$

If usage per unit time is entirely constant and lead time is known with certainty, the order point is equal to usage per unit time multiplied by lead time:

$$\text{Order point} = \text{usage per unit time} \times \text{lead time}$$

i.e. there is no need for safety stock. Note that in the EPQ case, lead time is zero and safety stock is also zero, so the addition of these two in the basic EPQ model is one of the ways in which the model is modified to reflect some realistic situations. (Ra'afat, 1991)

1.5 THE FINITE PRODUCTION RATE MODELS WITH DETERIORATION

Misra (1975) developed the first production lot size model in which both a constant and variable rate of deterioration were considered and obtained approximate expressions for the production lot size with no backlogging. For the case of Weibull distribution deterioration, no closed expression for the lot size and the average total cost was possible. However, for the case of exponential distribution (i.e. constant deterioration rate), through a series of approximations, Misra (1975) calculated the optimal production lot size to be

$$Q = \left\{1 + \left[\frac{(C_3 d\theta)}{(C_1 p)}\right]\right\}^{-0.5} Q_E, \text{ with } Q_E = \left\{\frac{(2C_2 dp)}{[C_1(p - d)]}\right\}^{0.5},$$

where

Q_E is the production lot size for items without decay,

p is the constant production rate,

θ is the constant rate of decay,

d is the constant demand rate and

C_1, C_2, and C_3 are inventory carrying cost, ordering cost and deteriorating cost respectively.

Shah and Jaiswal (1976) derived results similar to those of Misra (1975) for a constant deterioration rate and extended the model to include backlogging. By assuming the average carrying inventory to be approximately one-half the maximum level of inventory and using the same notation with Misra, they obtained the following expression for the production lot size as a function of inventory cycle time: $Q = \left(\frac{p}{\theta}\right)\ln\left\{1 + \left(\frac{d}{p}\right)\left[\exp(\theta T) - 1\right]\right\}$, where T is the inventory cycle time.

1.6 THE INVENTORY MODELS WITH DELAY IN DETERIORATION

In the EOQ model with constant rate of deterioration, many authors assume that deterioration of the items start from the instance of their arrival in stock. As a matter of fact, many items (for example, firsthand vegetables, fruits, and some items produced in industry like bread, cakes, etc) have a span of maintaining fresh quality or original condition. During that period, there is no deterioration occurring, but after sometime, deterioration begins. Thus it is important to consider inventory problems for non-instantaneous deteriorating items. Ouyang *et al*. (2006) developed an EOQ model for non-instantaneous deteriorating items with permissible delay in payments and where the demand before deterioration starts is the same as that after deterioration begins. Musa

and Sani (2012) developed an EOQ inventory model for delayed deteriorating items under permissible delay in payments but where the demand before deterioration starts is different from that after deterioration starts. Thus, this paper is a generalization of Ouyang *et al*. (2006).

Similarly, in the context of EPQ model, many authors assume that deterioration start immediately after production, but this is not the usual situation. Some items (for example; bread, cakes, etc) have a span of maintaining their original condition. Hence, it is important to consider inventory problems of delayed deteriorating items. Sugapriya and Jeyaraman (2008a) developed a model to determine a common production cycle time for an economic production quantity model of non-instantaneous deteriorating items allowing price discount and permissible delay in payments. Sugapriya and Jeyaraman (2008b) also developed an EPQ model for non-instantaneous deteriorating items in which production and demand rate are constant, holding cost varies with time, completely deteriorated units are discarded, partially deteriorated items are sold with some discount and no shortage is allowed. Baraya and Sani (2011) developed an EPQ model for delayed deteriorating items with stock-dependent demand rate and linear time dependent holding cost.

1.7 JUSTIFICATION FOR THE RESEARCH

The economic production quantity (EPQ) model has been widely used in practice because of its simplicity. However, there are some drawbacks in the assumptions of the original EPQ model and many authors have tried to improve it with different assumptions. The assumption of the unconstrained production period length is one of these shortcomings. The classical EPQ model assumes that production period length is unconstrained. However, in real production

environment, this assumption is not always tenable because, it can often be observed that the production period length is constrained due to some technical services reasons. Hence, the inventory policy determined by the conventional model would be inappropriate.

Reliability of an item is the probability that it will adequately perform its specified purpose for a specified period of time under specified environmental conditions. Product reliability is influenced by the decisions made during the design and manufacturing of the product. This implies that reliability can be viewed as a link to integrate the different stages of manufacturing - design, engineering, production, marketing, and post sale service - in an effective manner. As such reliability is very important in the context of new products. Recently research articles, emphasize the growing importance of this subject to both consumer and producer (Cheng, 1991). Objective determination of reliability costs will help manufacturers plan operations more effectively since an accurate knowledge of reliability costs allows more accurate profit expectations which may, in turn, lead to some marketing advantages (Cheng, 1991).

In case of demand of an item, it is natural that the higher the price of an item the lower the demand and the lower the price of an item the higher the demand of such item, i.e. the unit cost of an item is inversely related to the demand of the product. In general the unit cost of production is directly proportional to the reliability of the product and inversely related to the demand of the product.

1.8 THE PROBLEM STUDIED IN THIS THESIS

EPQ model has been widely used for more than four decades as an important tool to control inventory. However, as already indicated, EPQ model did not represent the real world problem in some situations. The analysis for finding an EPQ therefore has several weaknesses. This is why,

many authors had to make extensions or modifications in several aspects of the original EPQ model. The quality assurance (reliability) is one good aspect that could be added to the EPQ model since quality assurance plays an important role in the demand of an item in the market. Quality assurance was incorporated in the models of many authors such as Cheng (1991) and Tripathy et al. (2003).

Musa and Sani (2009) developed an EOQ model for items that exhibit delay in deterioration. The non-instantaneous deterioration (delay in deterioration) is a situation where items do not start deteriorating immediately they are stocked. During this period, before deterioration sets in, depletion of inventory is dependent on demand only. As deterioration sets-in depletion is then dependent on both demand and deterioration. The items that exhibit delay in deterioration include farm produce such as fruits, potatoes etc. or even fashion goods such as cars, fabrics etc. In this thesis, we intend to make an extension of Musa and Sani (2009) but in the context of EPQ by assuming the unit cost of production of an item to be directly related to reliability (quality assurances in producing the item) and inversely related to demand rates. This is a reasonable assumption because the higher the reliability of an item the higher the price is in many cases, and the lower the reliability of the item the lower the price is in many cases. Also, the higher the price of an item the lower the demand of such item, and the lower the price of the item the higher the demand of such item in many cases.

1.9 LIMITATION

The applicability of the study is limited to items with delay in deterioration, where the unit cost of production is directly related to reliability and it is inversely related to quantity demanded.

1.10 RESEARCH METHODOLOGY

The approach we use in this study will start with the review of existing literature in both Economic Order Quantity (EOQ) model and Economic Production Quantity (EPQ) model. It will also review literature on constant demand, varying demand, constant deterioration, varying deterioration and reliability consideration in EPQ models. Mathematical modeling will then be used to derive the mathematical relationship for the required models under the stated assumptions after which numerical examples will be used to show the application of the models. Sensitivity analyses will also be conducted to see the effect of changes in some of the parameters.

1.11 RESEARCH AIMS AND OBJECTIVES

The aim of this research is to develop economic production quantity models for items which exhibit delayed deterioration with quality assurance consideration.

The objectives of this research are:

- To investigate the effect of quality assurance (reliability) in the EPQ inventory model of items that exhibit delayed deterioration;
- To develop an EPQ model of items which exhibit delayed deterioration with quality assurance consideration and constant demand;
- To develop an EPQ model of items which exhibit delayed deterioration with quality assurance consideration and linear demand (after deterioration begins).

1.12 OUTLINE OF THE THESIS

Chapter one deals with the general introduction of the thesis. It starts with the introduction of inventory control theory, components of inventory, generalized inventory model and basic classification of inventories. The chapter goes ahead to discuss the Economic Production Quantity (EPQ) model and some other models that depend on it. The justification of reliability consideration, problem studied, methodology, objectives of the study and limitations of the study are also stated in the chapter.

Chapter two surveys the existing literature on Inventory Management and Control and particularly the deteriorating inventory. It covers various inventory related problems especially the *EOQ*/EPQ where *EOQ/EPQ* models are discussed under different mathematical assumptions. These include inventory models for constant deteriorating items and inventory models for non-instantaneous (delayed) deteriorating items, inventory models for deteriorating items with process reliability, inventory models for items with constant demand, inventory models for items with varying demand, inventory models for deteriorating items with varying demand and inventory models with imperfect quality.

Chapter three contains development of the mathematical model (EPQ) on items that exhibit delay in deterioration with reliability consideration and constant demand (both before and after deterioration sets-in). It is assumed in the model that the unit cost of production is directly related to reliability of the product and inversely related to the rates of demand. The chapter then gives some numerical examples to show how the model is applied. The chapter also gives a sensitivity analysis of some important parameters.

Chapter four contains development of the EPQ model of items that exhibit delay in deterioration with reliability consideration but where the demand rate after deterioration sets-in is linearly time dependent. Numerical examples are also given to illustrate the application of the model. The chapter also gives a sensitivity analysis of some important parameters.

Chapter five gives the summary of the thesis, contribution of the thesis to research on inventory of deteriorating items and it gives a conclusion of the work. Possible areas of interest where the research could be extended are also given and this is in addition to further recommendations given.

1.13 DEFINITIONS OF SOME BASIC TERMS: The following are the definitions of some of the technical terms we will frequently use in this thesis.

- *Backlogging*: This is the process of holding customer orders to be filled later when they cannot be settled immediately because of stockouts.
- *Backorder*: A customer order that cannot be filled when presented, and for which the customer is prepared to wait for some time.
- *Backorder cost*: The cost of handling the backorder (special handling, follow-ups etc.) plus whatever loss of goodwill occurs as a result of having to backorder an item.
- *Demand rate*: This is also called the usage rate. It is the number of units demanded by customers of production departments per unit of time. The demand may be constant (static) or variable (dynamic).

- *Instantaneous inventory receipt*: This is the inventory that is received or obtained at one point in time and not over a period of time.

- *Inventory Turnover (or stock turn)*: This is a ratio showing how many times a company's inventory is sold and replaced over a period.

- *Lead time*: This is the time between ordering a replenishment of an item and actually receiving the item into inventory. The lead-time can be either deterministic (constant or variable) or probabilistic.

- *Instantaneous delivery:* If the lead time of an item is zero, then we have a special case of instantaneous delivery where there is no need for placing an order in advance. This occurs in many cases in production industries when the production run is so planned that new items produced become available just as old items finish. This is clearly seen in a bakery for instance. In our own study, we consider the lead time to be zero; therefore we have a case of *Instantaneous delivery.*

- *Inventory cycle*: This is made up of the activities of sensing a need for ordering materials, placing an order, lead time for getting the material delivered, receiving the material and using it.

- *Inventory level*: This refers to the amount of materials on hand in inventory that is ready for use, i.e. the current amount of a product that a business has in stock.

- *Inventory carrying (holding) cost***:** This is the cost a business incurs over a certain period of time, to hold and store its inventory.

- *Order quantity*: This is the quantity of material produced each time inventory is replenished.

- *Planned shortages*: This is a situation where stock outs are planned.

- *Set-up cost*: This is the cost incurred in preparing a machine or processing for manufacturing an order. It includes the design cost, moving of machinery, employee hiring, research and development expenses, and labor cost for cleaning and changing tools or holders.

- *Time horizon*: The period over which the inventory level will be controlled is called the time horizon. It can be finite or infinite depending on the nature of demand.

CHAPTER TWO

LITERATURE REVIEW

2.0 INTRODUCTION

This chapter deals mainly with the literature on inventory management, with particular interest to the deteriorating inventory items of which this thesis is based. Some relevant works of authors on this subject are given:

2.1 THE BASIC ECONOMIC ORDER QUANTITY

F.W Harris in 1913 first developed *EOQ* model with the derivation of the square root formula

$$q = \sqrt{\frac{2c_0 D}{h}}$$

where c_o, h and D are ordering cost, the holding cost, and demand rate respectively. Harris developed the formula by determining a quantity q which balances the annual carrying costs with the annual ordering costs; $\left(\frac{q}{2}\right)h = \left(\frac{D}{q}\right)c_o$; while considering a constant demand and no deterioration. (Ra'afat, 1991)

2.2 EPQ OR LOT SIZE INVENTORY MODELS WITH CONSTANT DETERIORATION

This class of models extends the basic EOQ model by applying it in a production environment. It is clearly noted that there is some similarity between (*i*) a retailing unit which orders for items

and sells them to customers directly and (*ii*) a production unit which produces items in house and sells them also directly to customers such as bakery. Misra (1975) developed the first production lot size model in which both a constant and variable rate of deterioration were considered and obtained approximate expressions for the production lot size with no backlogging. For the case of Weibull distribution deterioration, no closed expression for the lot size and the average total cost was possible. However, for the case of exponential distribution (i.e. constant deterioration rate), through a series of approximations, Misra obtained an expression for the lot size and also determined the average total cost. Shah and Jaiswal (1976) derived results similar to those of Misra (1975) for a constant deterioration rate and extended the model to include backlogging. By assuming the average carrying inventory to be approximately one-half the maximum level of inventory, they obtained an expression for the production lot size as a function of inventory cycle time. Aggarwal and Bahari-Hashani (1991) developed an EPQ model assuming that items are deteriorating at a constant rate, the demand rate is known and decreases negative exponentially, no shortages are allowed, and the production rate is known but can vary from one period to another over a finite planning period. Pakkala and Achary (1992) considered a production-inventory model of deteriorating items with two storage facilities and a constant demand rate, shortages are allowed. Yang and Wee (2003), considered a multi-lot-size production-inventory system for deteriorating items with constant production and demand rates.

2.3 INVENTORY MODEL FOR NON-INSTANTANEOUS DETERIORATING ITEMS

In all the models above, the authors assumed that deterioration of the items start from the instance of their arrival in stock. As a matter of fact, many items (for example, firsthand

vegetables, fruits, and some items produced in industry like bread, cakes, etc) have a span of maintaining fresh quality or original condition. During that period, there is no deterioration occurring. Ouyang *et al*. (2006) developed an EOQ model for non-instantaneous deteriorating items with permissible delay in payments and where the demand before deterioration starts is the same as that after deterioration begins. Sugapriya and Jeyaraman (2008a) developed a model to determine a common production cycle time for an economic production quantity model of non-instantaneous deteriorating items allowing price discount and permissible delay in payments. Sugapriya and Jeyaraman (2008b) also developed an EPQ model for non-instantaneous deteriorating item in which production and demand rate are constant, holding cost varies with time, completely deteriorated units are discarded, partially deteriorated items are allowed to carry discount and no shortage is allowed. Musa and Sani (2009) developed an EOQ inventory model for some tropical items that exhibit delay in deterioration. Geetha and Uthayakumar (2010) developed an EOQ model for non-instantaneous deteriorating inventory items with partial backlogging**.** Monika and Shon (2010) developed an EOQ model for two levels of storage for non-instantaneous deteriorating items with stock dependent demand, time varying partial backlogging and under permissible delay in payments. Baraya and Sani (2011) developed an EPQ model for delayed deteriorating items with stock-dependent demand rate and linear time dependent holding cost. Musa and Sani (2012) also developed an EOQ inventory model for delayed deteriorating items under permissible delay in payments but where the demand before deterioration starts is different from that after deterioration starts. Thus, this paper is a generalization of Ouyang *et al*. (2006).

2.4 INVENTORY MODEL WITH PROCESS RELIABILITY(QUALITY ASSURANCE)

Many authors modified inventory policies by considering the issue of process reliability and quality improvement in EOQ/EPQ models. Tapiero et al. (1987) developed an EOQ model for reliability, pricing and quality control. Cheng (1991) developed a model that integrates quality considerations with EPQ. The author assumed that unit production cost increases with increases in process reliability and quality assurance expenses. Paknejad *et al*. (1995) presented a quality-adjusted lot-sizing model with stochastic demand and constant lead-time. Specifically, they investigated the case of continuous review (s,Q) models in which an order of size Q is placed each time the inventory position (based on non defective items) reaches the order point s. Results indicated that as the probability of defective items increases, for a given constant lead-time, the optimal lot size and the optimal reorder point both increase significantly. Further, for a given defective probability, the lot size and the reorder point increase substantially as the lead-time increases. This indicates that improving quality when lead-time is constant should result in smaller lot sizes. Tripathy and Pattnaik (2009) developed an EPQ model to obtain an optimal ordering policy wherein demand exceeds supply, all items are subjected to inspection and defective items are discarded. The unit cost of production is inversely related to both process reliability and demand rate. Under reasonable conditions, maximum positive cost savings are generated when the process reliability increases. Mahapatra *et al*. (2012) developed an EPQ model with demand dependent unit production cost in fuzzy environment. Flexibility and reliability consideration were introduced in the production process. The model was developed under fuzzy goal and fuzzy restrictions on budgetary cost.

2.5 DETERIORATING INVENTORY MODELS WITH VARYING DEMAND RATE

The assumption of constant demand rate is not always applicable to many inventory items such as style goods, tasty foods etc. Many products experience a period of rising demand during their growth phase which may decline due to the introduction of a competitor or change in consumer preference. This phenomenon prompted many researchers to develop deteriorating inventory models with time varying demand patterns. Continuous-time and discrete-time demand rates have been considered in the literature where most of the continuous-time inventory models considered either linearly increasing/decreasing demand or exponentially increasing/decreasing demand patterns, and so on.

Aggarwal and Bahari-Hashani (1991), developed an EPQ model with constant deteriorating rate, the demand rate is known and decreases negative exponentially, no shortages are allowed, and the production rate vary from one period to another over a finite planning period. Pakkala and Achary (1992), developed an EPQ model for deteriorating items with two storage facilities and a constant demand rate, Shortages are allowed. Balkhi (2001) developed an EPQ model for deteriorating items where demand and production rate are known and continuous functions of time (e.g. linear function). A finite planning horizon was considered and shortages were allowed and fully backordered. A mathematical model was developed and the conditions under which the total cost of inventory system achieves its unique global minimum were obtained. Teng and Chang (2005) developed an EPQ model for deteriorating items when the demand rate is dependent on both the stock level and the selling price per unit, and in which the deteriorating rate was constant. The author also provided the necessary conditions to decide the optimal solution which maximized profit of the EPQ model. From the model, decision makers could get aid in developing the optimal price and in production run time decisions. Manna and Chaudhuri

(2006) developed a production-inventory model for deteriorating items with ramp-type demand pattern. The demand pattern was a two-phase ramp-type function modeled by a linear function of time in the first phase and a constant in the second phase. Zaid and Ali (2009) developed an EPQ model for deteriorating items in which each of the production, the demand and the deterioration rates, as well as all cost parameters are assumed to be general functions of time. Besides, shortages are allowed but are partially backordered; inflation and time value of money are all taken into account. Abdul and Murtala (2011) developed an EPQ model for deteriorating items with increasing-steady-decreasing demand pattern under the effect of inflation and time value of money. Shortages are allowed with partial backlogging of demand and a two-parameter Weibull-distribution function is used for the rate of deterioration. Sarkar and Chakrabarti (2013) developed an EPQ model for the lot-size inventory system with finite production rate, taking into consideration the effect of decay and the condition of permissible delay in payments. Deterioration is assumed to follow Weibull distribution and demand rate is assumed to be exponential.

2.6 INVENTORY MODELS WITH IMPERFECT QUALITY

The EPQ model employs mathematical modeling to balance production setup cost and holding cost, to assist producers in determining economic production lot size that minimizes the overall production- inventory costs (Hadley and Whitin, 1963; Hillier and Lieberman, 2001). Classic EPQ model implicitly assumes that all items produced are of perfect quality. But in real-life production systems, due to many controllable and/or uncontrollable factors, generation of defective items is inevitable. Thus, many studies have been carried out to enhance EPQ model by

addressing the imperfect production related issues (Nahmias, 2009). Shih (1980) examined two inventory models to the case where the proportion of defective units in the accepted lot is a random variable with known probability distribution. Optimal solutions to the modified system were developed and comparisons with the traditional models were also presented via numerical examples. Rosenblatt and Lee (1986) studied an EPQ model that deals with imperfect quality. They assumed that at some random point in time, the process might shift from an in-control to an out-of-control state, and a fixed percentage of defective items are produced. Approximate solutions for obtaining an optimal lot size were developed in Rosenblatt and Lee (1986). Zhang and Gerchak (1990) considered joint lot sizing and inspection policy in an EOQ model with random yield. Kim *et al*. (2001) studied the optimal production run length and inspection schedules in a deteriorating production process. They assumed that a production process is subject to a random deterioration from the in-control state to the out of control state and thus produces some proportion of defective items. By minimizing the overall production inventory costs, an optimal production run length and an optimal number of inspections were derived and unique properties of the proposed model were discussed. Kuang-Ku *et al*; (2010) developed an integrating cost reduction delivery policy into an imperfect production system with repairable items. El-Kassar *et al*., (2012) developed an EPQ model with imperfect quality raw material needed for production. Two scenarios were considered. In the first, the imperfect quality items of the raw material are sold at a discounted price at the end of the screening period. In the second scenario, the imperfect quality items are kept in stock until the end of the inventory cycle and returned to the supplier when the next order is received.

2.7 OTHER EPQ INVENTORY MODELS

Wee and Law (1999) developed a production inventory model for a deteriorating item with price-dependent demand. They also considered shortages and time value of money. The deterioration rate was assumed a random variable that follows the Weibull distribution. An optimization procedure was presented and applied to search for optimal production policies that maximize the total net present value of profit.

Giri *et al*. (2005) developed an EPQ model where demand rate increases with time, the production rate is finite and adjustable in each cycle over an infinite planning horizon, and shortages are permitted and partially backordered. They suggested a procedure to approximately find the minimum total cost of the system over a finite time horizon.

Panda *et al*. (2009) developed a single item Economic Production Quantity (EPQ) model with ramp-type demand function. The model determines the optimal production stopping time to maximize total unit profit of the system.

Pradhan and Tripathy (2012) developed an EPQ model for an Item with three parameter Weibull deterioration rate and price discount. Here production and demand rate are considered to be constant and the holding cost per unit is assumed to be constant with respect to time. Completely deteriorated units are discarded and partially deteriorated items are allowed to carry a discount. Shortages are not allowed.

CHAPTER THREE

AN EPQ MODEL FOR ITEMS THAT EXHIBIT DELAY IN DETERIORATION WITH RELIABILITY CONSIDERATION AND CONSTANT DEMAND

3.0 INTRODUCTION:

This chapter uses the technique of mathematical modeling to derive the EPQ model for deteriorating inventory that exhibits delay in deterioration where the unit cost of production of an item is assumed to be directly related to reliability consideration and inversely related to the rates of demand. The cycle length is obtained and numerical solutions are given for some problems to show how the model is applied. Some sensitivity analysis is also carried out on some parameters.

3.1 NOTATION AND ASSUMPTIONS

NOTATION

μ_1 The demand rate (units per unit time) during the period before deterioration sets in

μ_2 The demand rate (units per unit time)after deterioration sets in

Q The production quantity (units in a production run)

X The production cycle length (time unit)

C The unit cost of the item (in Naira)

A The set-up cost per production run (Naira per production run)

i The inventory carrying charge (excluding interest charges)

β The rate of deterioration

X_1 The time deterioration sets in

X_2 The difference between the production cycle length and the time deterioration sets in

Y_0 The initial inventory

Y_d The inventory level at the time deterioration begins

H The holding cost or inventory carrying cost in a production cycle

r Process Reliability of the items $(0 \leq r \leq 1)$

ASSUMPTIONS

(i) Instantaneous production

(ii) Unconstrained suppliers capital

(iii) Backorders not allowed

(iv) All items are inspected and defective ones are discarded

(v) Demand of product exceeds its supply

(vi) The unit cost of production of the item C is directly related to the process reliability r (quality assurance) and inversely related to the demand rates μ_1 & μ_2 . This relationship is specifically assumed to be:

$$C \propto r^k \left(\mu_1 \frac{X_1}{X} + \mu_2 \frac{X_2}{X} \right)^{-b}$$

$$\Rightarrow \quad C \propto (1-r)^{-k} \left(\mu_1 \frac{X_1}{X} + \mu_2 \frac{X_2}{X} \right)^{-b}$$

$$\therefore \quad C = a(1-r)^{-k} \left(\mu_1 \frac{X_1}{X} + \mu_2 \frac{X_2}{X} \right)^{-b} \qquad (3.0)$$

where a, b, k are non-negative real numbers to be chosen based on some known data so as to provide the best fit of the estimated cost function.

$(\boldsymbol{vii})$ $X_1 \leq X$

3.2 THE MATHEMATICAL MODEL

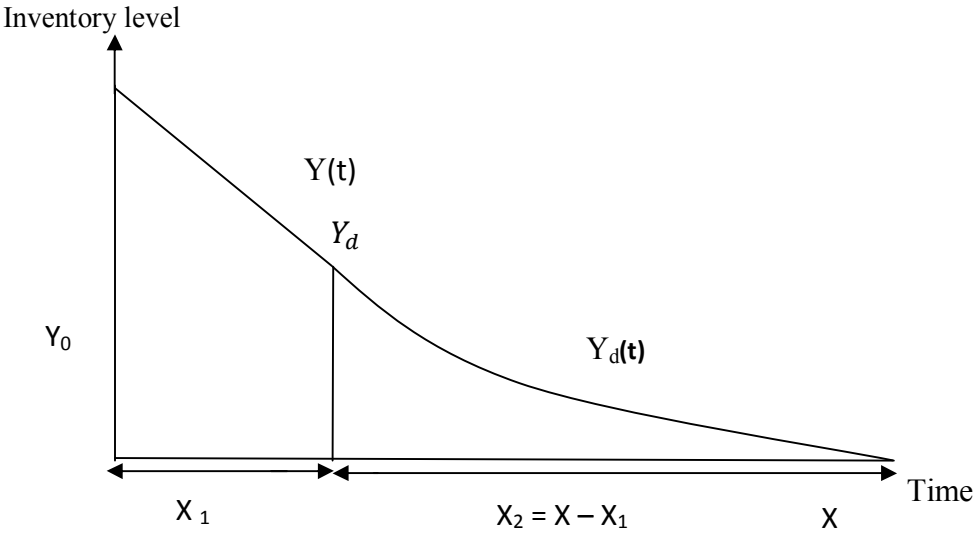


Fig. 3.1: Inventory movement in a delayed deterioration situation with $X_1 \leq X$
(Musa and Sani, 2012)

Let $Y(t)$ and $Y_d(t)$ be the inventory level before and after deterioration sets in respectively. $Y(t)$ depends only on demand and the differential equation that describes the situation before deterioration sets in is given by

$$\frac{dY(t)}{dt} = -\mu_1 \qquad \text{where } 0 \leq t \leq X_1 \tag{3.1}$$

Solving (3.1), we get

$$Y(t) = -\mu_1 t + E \qquad \text{(where } E \text{ is a constant)} \tag{3.2}$$

applying the boundary condition $[Y(t) = Y_0 \text{ at } t = 0]$ in (3.2), we get

$$Y_0 = E \tag{3.3}$$

substituting (3.3) in (3.2), we obtain

$$Y(t) = -\mu_1 t + Y_0 \tag{3.4}$$

applying the boundary condition $[Y(t) = Y_d \text{ at } t = X_1]$ in (3.4), we get

$$Y_d = -\mu_1 X_1 + Y_0$$

$$Y_0 = Y_d + \mu_1 X_1 \tag{3.5}$$

substituting (3.5) in (3.4), we obtain

$$Y(t) = Y_d + (X_1 - t)\mu_1 \tag{3.6}$$

After deterioration sets in, depletion of inventory will depend on both demand and deterioration and the differential equation which describes the situation is given by

$$\frac{dY_d(t)}{dt} + \beta Y_d(t) = -\mu_2 \quad \text{where } X_1 \le t \le X \tag{3.7}$$

Equation (3.7) is a linear differential equation and so using the integrating factor,

$$Y_H = e^{\beta t}$$

we get its solution as

$$Y_d(t) = Fe^{-\beta t} - \frac{\mu_2}{\beta} \qquad \text{(where } F \text{ is a constant)} \tag{3.8}$$

applying the boundary condition $[Y_d(t) = Y_d \text{ at } t = X_1]$ in (3.8), we get

$$Y_d = Fe^{-\beta X_1} - \frac{\mu_2}{\beta}$$

$$\Rightarrow \qquad F = \left(Y_d + \frac{\mu_2}{\beta}\right)e^{\beta X_1}$$

substituting F in (3.8), we get

$$Y_d(t) = \left(Y_d + \frac{\mu_2}{\beta}\right)e^{\beta X_1}e^{-\beta t} - \frac{\mu_2}{\beta}$$

$$\Rightarrow \; Y_d(t) = \left(Y_d + \frac{\mu_2}{\beta}\right)e^{-\beta(t - X_1)} - \frac{\mu_2}{\beta}$$

$$\Rightarrow \; Y_d(t) = \frac{\mu_2}{\beta}\left(e^{(X_1 - t)\beta} - 1\right) + Y_d e^{(X_1 - t)\beta} \tag{3.9}$$

Applying the boundary condition [$Y_d(t) = 0$ at $t = X$] in (3.9), we get

$$0 = \frac{\mu_2}{\beta}\left(e^{(X_1 - X)\beta} - 1\right) + Y_d e^{(X_1 - X)\beta}$$

$$\Rightarrow \quad Y_d e^{(X_1-X)\beta} = -\frac{\mu_2}{\beta}\left(e^{(X_1-X)\beta}-1\right)$$

$$\therefore \quad Y_d = -\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right) \tag{3.10}$$

Substituting (3.10) in (3.6), we obtain

$$Y(t) = -\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right)+(X_1-t)\mu_1 \tag{3.11}$$

Substituting (3.10) in (3.9), we get

$$Y_d(t) = \frac{\mu_2}{\beta}\left(e^{(X_1-t)\beta}-1\right)-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right)e^{(X_1-t)\beta}$$

$$\Rightarrow \quad Y_d(t) = \frac{\mu_2}{\beta}e^{(X_1-t)\beta}-\frac{\mu_2}{\beta}-\frac{\mu_2}{\beta}e^{(X_1-t)\beta}+\frac{\mu_2}{\beta}e^{(-X_1+X+X_1-t)\beta}$$

$$\Rightarrow \quad Y_d(t) = -\frac{\mu_2}{\beta}+\frac{\mu_2}{\beta}e^{(X-t)\beta}$$

$$\therefore \quad Y_d(t) = \frac{\mu_2}{\beta}\left(e^{(X-t)\beta}-1\right) \tag{3.12}$$

The number of deteriorated items $d(X_2)$ is the difference between the total demand in X_2 i.e. $X_2\mu_2$ and the total inventory at the time deterioration sets in i.e. Y_d ,

$$\therefore \quad d(X_2) = Y_d - X_2\mu_2$$

$$\Rightarrow \quad d(X_2) = -\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right)-X_2\mu_2$$

and so

$$d(X_2) = -\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}+X_2\beta\right) \tag{3.13}$$

The cost associated with storage of the whole inventory until it is sold or used, i.e. inventory carrying cost H, and is given by:

$$H = iC\int_0^{X_1} Y(t)dt + iC\int_{X_1}^{X} Y_d(t)\,dt \tag{3.14}$$

Substituting (3.0), (3.11) and (3.12) in (3.14), we obtain

$$H = ia(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu_2\frac{X_2}{X}\right)^{-b}\int_0^{X_1}\left(-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right)+(X_1-t)\mu_1\right)dt$$

$$+ia(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu_2\frac{X_2}{X}\right)^{-b}\int_{X_1}^{X}\frac{\mu_2}{\beta}\left(e^{(X-t)\beta}-1\right)dt =$$

$$ia(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu_2\frac{X_2}{X}\right)^{-b}\left\{\int_0^{X_1}\left(-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right)+(X_1-t)\mu_1\right)dt+\frac{\mu_2}{\beta}\int_{X_1}^{X}\left(e^{(X-t)\beta}-1\right)dt\right\}$$

$$=ia(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu_2\frac{X_2}{X}\right)^{-b}\left\{\left[-\frac{\mu_2}{\beta}\left(t-te^{-(X_1-X)\beta}\right)+\left(X_1t-\frac{t^2}{2}\right)\mu_1\right]_0^{X_1}+\frac{\mu_2}{\beta}\left[-\frac{e^{(X-t)\beta}}{\beta}-t\right]_{X_1}^{X}\right\}$$

$$=ia(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu_2\frac{X_2}{X}\right)^{-b}\left\{-\frac{\mu_2}{\beta}\left(X_1-X_1e^{-(X_1-X)\beta}\right)+\left(X_1^2-\frac{X_1^2}{2}\right)\mu_1+\frac{\mu_2}{\beta}\left(-\frac{1}{\beta}-X+\frac{e^{(X-X_1)\beta}}{\beta}+X_1\right)\right\}$$

so that after further simplification, we obtain

$$H=\left(\left(1+\frac{1}{\beta X_1}\right)e^{-(X_1-X)\beta}+\frac{X_1\mu_1\beta}{2\mu_2}-\frac{1}{\beta X_1}-\frac{X}{X_1}\right)\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu_2X_2\right)^{-b}\mu_2X_1}{\beta X^{-b}} \qquad (3.15)$$

The total variable cost is the sum of set-up cost, cost of deteriorated items and inventory carrying cost, i.e.

$$\text{Total variable cost (Z)} = A+Cd\left(X_2\right)+H \qquad (3.16)$$

$$\therefore \quad Z=A+a(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu_2\frac{X_2}{X}\right)^{-b}\left(-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}+X_2\beta\right)\right)$$

$$+\left\{\left(1+\frac{1}{\beta X_1}\right)e^{-(X_1-X)\beta}+\frac{X_1\mu_1\beta}{2\mu_2}-\frac{1}{\beta X_1}-\frac{X}{X_1}\right\}\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu_2X_2\right)^{-b}\mu_2X_1}{\beta X^{-b}} \qquad (3.17)$$

The total variable cost per unit time is given as

$$Z(X) = \frac{\textit{Total variable cost per production run}}{\textit{The production cycle length}}$$

$$\Rightarrow \qquad Z(X)=\frac{A}{X}-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2X_2\right)^{-b}\mu_2}{\beta X^{-b}}\left(\frac{1}{X}\right)+\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2X_2\right)^{-b}\mu_2}{\beta X^{-b}}\left(\frac{e^{-(X_1-X)\beta}}{X}\right)$$

$$-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2X_2\right)^{-b}\mu_2}{X^{-b}}\left(\frac{X-X_1}{X}\right)$$

$$+\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{e^{-(X_1-X)\beta}}{X}\right)+\frac{X_1\mu_1\beta}{2\mu_2}\left(\frac{1}{X}\right)-\frac{1}{\beta X_1}\left(\frac{1}{X}\right)-\left(\frac{1}{X}\right)\frac{X}{X_1}\right\}\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu_2X_2\right)^{-b}\mu_2X_1}{\beta X^{-b}}$$

since $X_2=X-X_1$,

$$\Rightarrow \qquad Z(X)=\frac{A}{X}-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{1}{X^{-b+1}}\right)$$

$$+\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{e^{-(X_1-X)\beta}}{X^{-b+1}}\right)-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{1}\left(\frac{X-X_1}{X^{-b+1}}\right)$$

$$+\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{e^{-(X_1-X)\beta}}{X}\right)+\frac{X_1\mu_1\beta}{2\mu_2}\left(\frac{1}{X}\right)-\frac{1}{\beta X_1}\left(\frac{1}{X}\right)-\left(\frac{1}{X}\right)\frac{X}{X_1}\right\}\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2X_1}{\beta X^{-b}} \quad (3.18)$$

To obtain the value of X which minimizes the total variable cost per unit time, we take the derivative of $Z(X)$ with respect to X and then set the result to zero. The value of X satisfying that equation gives the minimum provided $\frac{d^2(Z)}{dX^2}>0$. The derivative is given as:

$$\frac{dZ(X)}{dX}=-\frac{A}{X^2}-\frac{a(b-1)(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{1}{X^{-b+2}}\right)+\frac{ab\mu_2{}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta}\left(\frac{1}{X^{-b+1}}\right)$$

$$+\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{\beta X^{-b+1}+(b-1)X^{-b}}{X^{2(-b+1)}}\right)e^{-(X_1-X)\beta}-\frac{ab\mu_2{}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta}\left(\frac{e^{-(X_1-X)\beta}}{X^{-b+1}}\right)$$

$$-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{1}\left(\frac{X^{-b+1}+(b-1)(X-X_1)X^{-b}}{X^{2(-b+1)}}\right)+\frac{ab\mu_2{}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{1}\left(\frac{X-X_1}{X^{-b+1}}\right)$$

$$+\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{\beta X-1}{X^2}\right)e^{-(X_1-X)\beta}-\frac{X_1\mu_1\beta}{2\mu_2}\left(\frac{1}{X^2}\right)+\frac{1}{\beta X_1}\left(\frac{1}{X^2}\right)\right\}\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2X_1}{\beta X^{-b}}$$

$$+\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{e^{-(X_1-X)\beta}}{X}\right)+\frac{X_1\mu_1\beta}{2\mu_2}\left(\frac{1}{X}\right)-\frac{1}{\beta X_1}\left(\frac{1}{X}\right)-\left(\frac{1}{X}\right)\frac{X}{X_1}\right\}\frac{iab(1-r)^{-k}\left\{-\mu_2\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-1}+X^{-1}\right\}\mu_2X_1}{X^{-b}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{b}\beta}=0$$

$$\Rightarrow \quad -\frac{A}{X^2}-\frac{a(b-1)(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{1}{X^{-b+2}}\right)+\frac{ab{\mu_2}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta}\left(\frac{1}{X^{-b+1}}\right)$$
$$+\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{X^{-b}\left(X\beta+(b-1)\right)}{X^{2(-b+1)}}\right)e^{-(X_1-X)\beta}-\frac{ab{\mu_2}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta}\left(\frac{e^{-(X_1-X)\beta}}{X^{-b+1}}\right)$$
$$-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{1}\left(\frac{X^{-b}\left(bX-X_1(\mathrm{b}-1)\right)}{X^{2(-b+1)}}\right)+\frac{ab{\mu_2}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{1}\left(\frac{X-X_1}{X^{-b+1}}\right)$$
$$+\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{\beta X-1}{1}\right)e^{-(X_1-X)\beta}-\frac{X_1\mu_1\beta}{2\mu_2}+\frac{1}{\beta X_1}\right\}\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2X_1}{\beta X^{-b+2}}$$
$$+\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{e^{-(X_1-X)\beta}}{1}\right)+\frac{X_1\mu_1\beta}{2\mu_2}-\frac{1}{\beta X_1}-\frac{X}{X_1}\right\}\frac{iab(1-r)^{-k}\left\{-\mu_2\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-1}+X^{-1}\right\}\mu_2X_1}{X^{-b+1}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{b}\beta}=0$$

$$\Rightarrow \quad \frac{A}{X^2}+\frac{a(b-1)(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{1}{X^{-b+2}}\right)-\frac{ab{\mu_2}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta}\left(\frac{1}{X^{-b+1}}\right)$$
$$-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{\left(X\beta+(b-1)\right)}{X^{-b+2}}\right)e^{-(X_1-X)\beta}+\frac{ab{\mu_2}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta}\left(\frac{e^{-(X_1-X)\beta}}{X^{-b+1}}\right)$$
$$+\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{1}\left(\frac{\left(bX-X_1(\mathrm{b}-1)\right)}{X^{-b+2}}\right)-\frac{ab{\mu_2}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{1}\left(\frac{X-X_1}{X^{-b+1}}\right)$$
$$-\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{\beta X-1}{1}\right)e^{-(X_1-X)\beta}-\frac{X_1\mu_1\beta}{2\mu_2}+\frac{1}{\beta X_1}\right\}\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2X_1}{\beta X^{-b+2}}$$
$$-\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{e^{-(X_1-X)\beta}}{1}\right)+\frac{X_1\mu_1\beta}{2\mu_2}-\frac{1}{\beta X_1}-\frac{X}{X_1}\right\}\frac{iab(1-r)^{-k}\left\{-\mu_2\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-1}+X^{-1}\right\}\mu_2X_1}{\beta X^{-b+1}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{b}}=0$$

Multiplying through by $X^{(-b+2)}$, we get

$$\frac{A}{X^b}+\frac{a(b-1)(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}-\frac{ab\mu_2{}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta X^{-1}}$$
$$-\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{\beta}\left(\frac{\left(X\beta+(b-1)\right)}{1}\right)e^{-(X_1-X)\beta}+\frac{ab\mu_2{}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{\beta X^{-1}}\left(\frac{e^{-(X_1-X)\beta}}{1}\right)$$
$$+\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2}{1}\left(\frac{\left(bX-X_1(b-1)\right)}{1}\right)-\frac{ab\mu_2{}^2(1-r)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b-1}}{1}\left(\frac{X-X_1}{X^{-1}}\right)$$
$$-\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{\beta X-1}{1}\right)e^{-(X_1-X)\beta}-\frac{X_1\mu_1\beta}{2\mu_2}+\frac{1}{\beta X_1}\right\}\frac{ia\left(1-r\right)^{-k}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-b}\mu_2X_1}{\beta}$$
$$-\left\{\left(1+\frac{1}{\beta X_1}\right)\left(\frac{e^{-(X_1-X)\beta}}{1}\right)+\frac{X_1\mu_1\beta}{2\mu_2}-\frac{1}{\beta X_1}-\frac{X}{X_1}\right\}\frac{iab(1-r)^{-k}\left\{-\mu_2\left(\mu_1X_1+\mu_2(X-X_1)\right)^{-1}+X^{-1}\right\}\mu_2X_1}{\beta X^{-1}\left(\mu_1X_1+\mu_2(X-X_1)\right)^{b}}=0 \quad (3.19)$$

Now, if other parameters are given, equation (3.19) can be used to determine the Xwhich minimizes the total variable cost. The EPQ of the corresponding X will be computed from:

EPQ = demand before deterioration starts + demand after deterioration starts + number of deteriorated items

$$= \mu_1X_1+\mu_2X_2+d(X_2)$$

Substituting (3.13), we have

$$EPQ=\mu_1X_1+\mu_2\left(X-X_1\right)-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}+X_2\beta\right)$$

$$=\mu_1X_1+\mu_2\left(X-X_1\right)-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}+\left(X-X_1\right)\beta\right)$$

$$=\mu_1X_1+\mu_2\left(X-X_1\right)-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right)-\mu_2\left(X-X_1\right)$$

$$\therefore \quad EPQ=\mu_1X_1-\frac{\mu_2}{\beta}\left(1-e^{-(X_1-X)\beta}\right) \quad (3.20)$$

(Dari and Sani, 2013)

3.3 RESULTS OBTAINED FROM THE MODEL

In this section, we apply the model developed in the last section using some examples to determine the best cycle length. In all the examples, $\frac{d^2Z(X)}{dX^2}$ was found to be greater than zero showing that the cost function is convex with regard to all the examples.

3.3.1 NUMERICAL EXAMPLES

In each example the values were substituted in equation *(3.19)*, and the appropriate period X, which minimized the cost function was determined, where $a = 10, b = 1$ and $k = 6$. Table 3.1 gives a summary of the results obtained.

Table 3.1: Complete results of Ten Numerical examples on the application of the model

S/N	A $(naira)$	μ_1 $(units)$	μ_2 $(units)$	r	i	X_1 $(days)$	β	X $(days)$	Z $(naira)$	EPQ $(units)$
1	2500	2000	1200	0.90	0.13	7	0.40	14	101462.1	61
2	2700	2500	1000	0.91	0.15	14	0.50	17	120365.7	104
3	2650	3000	1600	0.88	0.14	21	0.30	31	51542.96	217
4	2400	3500	2000	0.87	0.12	28	0.45	38	36977.83	324
5	3000	3200	800	0.89	0.11	35	0.35	41	58586.68	320
6	2300	2400	1100	0.86	0.10	42	0.51	53	25627.45	310
7	1900	3300	1400	0.85	0.16	49	0.55	59	22629.26	482
8	2600	2000	1300	0.84	0.17	56	0.60	67	23621.13	346
9	3500	2300	500	0.92	0.01	63	0.20	66	52952.8	401
10	1500	1800	700	0.83	0.18	70	0.32	85	14478.15	374

3.3.2 SENSITIVITY ANALYSIS

We have carried out a sensitivity analysis on the effect of changes in the system parameters ($a, b,$ and k) on the total variable cost per unit time $Z(X)$, using the following example which is example 1 in table 3.1:

Table 3.2: Example 1 *in table 3.1*

a	b	k	A	μ_1	μ_2	r	i	X_1	β	X	Z	EPQ
10	1	6	2500	2000	1200	0.90	0.13	7	0.40	14	101462.1	61

Table 3.3: Sensitivity Analysis (of data from table 3.2)

System parameter	% change in the system parameter	% change in Z
a	-2	-0.71521
	-1	-0.3576
	1	0.357604
	2	0.715208
b	-2	5.685805
	-1	2.738079
	1	-2.54334
	2	-4.9058
k	-2	-8.63336
	-1	-4.61439
	1	5.298033
	2	11.38099

From the table above, it can be observed that:

- $\mathrm{Z(x)}$ is sensitive to changes in the parameter.
- $\mathrm{Z(x)}$ is more sensitive to changes in k and is less sensitive to a.

CHAPTER FOUR

AN EPQ MODEL FOR DELAYED DETERIORATING ITEMS WITH RELIABILITY CONSIDERATION AND LINEAR DEMAND

4.0 INTRODUCTION

In chapter three we developed a model for deteriorating inventory that exhibits delay in deterioration where the unit cost of production of an item is assumed to be directly related to the process reliability and inversely related to the rates of demand. The deterioration time and demand was assumed to be a constant both before and after deterioration sets-in. In this chapter we extend the model in chapter three under the additional assumption that the demand after deterioration sets-in is linearly dependent on time.

4.1 NOTATION AND ASSUMPTIONS

NOTATION

Same as stated under 3.1 above.

ASSUMPTIONS

The assumptions are also the same as in 3.1 but with the additional assumption that:

1. the demand after deterioration begins is linearly dependent on time given as;
   $\mu_2 = c_2 + c_1 t$; where c_2 and c_1 are constants

2. $C \propto r^k \left(\mu_1 \frac{X_1}{X} + \mu \frac{X_2}{X} \right)^{-b}$,where $\mu = \frac{1}{X_2} \int_{X_1}^{X} \mu_2(t)\,dt$

$$\Rightarrow \quad C \propto (1-r)^{-k} \left(\mu_1 \frac{X_1}{X} + \mu \frac{X_2}{X} \right)^{-b}$$

$$\therefore \quad C = a(1-r)^{-k} \left(\mu_1 \frac{X_1}{X} + \mu \frac{X_2}{X} \right)^{-b} \qquad (4.0)$$

Where *a, b,* and *k* are non-negative real numbers to be chosen based on some known data so as to provide the best fit of the estimated cost function.

4.2 THE MATHEMATICAL MODEL

The diagram that depicts this model is the same as in fig. 3.1 except that the demand after deterioration begins is assumed to be linear while the demand before deterioration is constant. Thus the curve in the second phase will be sharper or flatter depending upon whether the demand is linearly increasing or decreasing respectively. Next, let $Y(t)$ and $Y_d(t)$ be the inventory level before and after deterioration sets in respectively. $Y(t)$ depends only on demand and the differential equation that describes the situation before deterioration sets in is given by the same equation as in (3.1) and the solution after applying the boundary condition $Y(0) = Y_0$ *and* $Y(X_1) = Y_d$, we get same equation as in (3.6).

After deterioration sets in, depletion of inventory will depend on both demand and deterioration and the differential equation which describes the situation is given by the same equation as in (3.7), i.e.

$$\frac{dY_d(t)}{dt} + \beta Y_d(t) = -\mu_2(t) \quad \text{where } X_1 \le t \le X$$

However, μ_2 is linearly dependent on time and so could be assumed to be $\mu_2 = c_2 + c_1 t$ where c_2 and c_1 are constants.

$$\therefore \quad \frac{dY_d(t)}{dt} + \beta Y_d(t) = -(c_2 + c_1 t) \tag{4.1}$$

Equation (4.1) is a linear differential equation and so using the integrating factor,

$$Y_H = e^{\beta t}$$

we get its solution as

$$Y_d(t) = Fe^{-\beta t} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta}t \qquad \text{(where } F \text{ is a constant)} \tag{4.2}$$

applying the boundary condition $[Y_d(t) = Y_d \text{ at } t = X_1]$ in (4.2), we get

$$Y_d = Fe^{-\beta t} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1 X_1}{\beta}$$

$$\Rightarrow \quad F = \left(Y_d - \frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) e^{\beta X_1}$$

substituting F in (4.2), we get

$$Y_d(t) = \left(Y_d - \frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) e^{\beta X_1} e^{-\beta t} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta}t$$

$$\Rightarrow \; Y_d(t) = \left(Y_d - \frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) e^{\beta(X_1 - t)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta}t$$

$$\Rightarrow \; Y_d(t) = Y_d e^{\beta(X_1 - t)} - \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) e^{\beta(X_1 - t)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta}t \tag{4.3}$$

Also from *fig3.1,* we have the boundary condition $Y_d(t) = 0$ at $t = X$. By substituting this in (4.3), we obtain

$$0 = Y_d e^{\beta(X_1 - X)} - \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) e^{\beta(X_1 - X)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1 X}{\beta}$$

$$\Rightarrow \quad Y_d e^{\beta(X_1 - X)} = \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) e^{\beta(X_1 - X)} - \frac{c_1 - c_2\beta}{\beta^2} + \frac{c_1 X}{\beta}$$

$$\therefore \qquad Y_d = \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right) e^{-\beta(X_1 - X)} \tag{4.4}$$

Substituting (4.4) in $Y(t) = Y_d + (X_1 - t)\mu_1$ [equivalent to (3.6)], we get

$$Y(t) = \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right) e^{-\beta(X_1 - X)} + (X_1 - t)\mu_1 \tag{4.5}$$

from (4.4) and (4.3), we obtain

$$Y_d(t) = \left(\left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right) e^{-\beta(X_1 - X)}\right) e^{\beta(X_1 - t)} - \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) e^{\beta(X_1 - t)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta} t$$

$$\Rightarrow Y_d(t) = \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) e^{\beta(X_1 - t)} - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right) e^{-\beta(X_1 - X - X_1 + t)} - \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) e^{\beta(X_1 - t)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta} t$$

$$\Rightarrow Y_d(t) = \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) e^{\beta(X_1 - t)} - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right) e^{-\beta(t - X)} - \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) e^{\beta(X_1 - t)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta} t$$

$$\therefore \qquad Y_d(t) = -\left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right) e^{-\beta(t - X)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta} t \tag{4.6}$$

The number of deteriorated items $d(X_2)$ is the difference between the total demand in X_2 i. e. $\int_{X_1}^{X} (c_2 + c_1 t)\, dt$ and the total inventory at the time deterioration sets in i. e. Y_d ,

$$\therefore \qquad d(X_2) = Y_d - \int_{X_1}^{X} \mu_2(t)\, dt$$

$$= Y_d - \int_{X_1}^{X} (c_2 + c_1 t)\, dt$$

$$= \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2} \right) e^{-\beta(X_1 - X)} - \left(tc_2 + \frac{c_1}{2} t^2 \right) \Big|_{X_1}^{X}$$

$$= \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2} \right) e^{-\beta(X_1 - X)} - \left[t\left(c_2 + \frac{c_1}{2} t \right) \right]_{X_1}^{X}$$

and so

$$d(X_2) = \left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2} \right) e^{-\beta(X_1 - X)} - (X - X_1)\left(c_2 + \frac{c_1}{2}(X - X_1) \right) \quad (4.7)$$

The cost associated with storage of the whole inventory until it is sold or used, i.e. inventory carrying cost H, is given by:

$$H = iC\int_0^{X_1} Y(t)dt + iC\int_{X_1}^{X} Y_d(t)\,dt \quad (4.8)$$

substituting (4.0), (4.5) and (4.6) in (4.8), we obtain

$$H = ia(1-r)^{-k}\left(\mu_1 \frac{X_1}{X} + \mu \frac{X_2}{X} \right)^{-b} \int_0^{X_1} \left(\left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2} \right) e^{-\beta(X_1 - X)} + (X_1 - t)\mu_1 \right) dt$$
$$+ ia(1-r)^{-k}\left(\mu_1 \frac{X_1}{X} + \mu \frac{X_2}{X} \right)^{-b} \int_{X_1}^{X} \left(- \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2} \right) e^{-\beta(t - X)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta} t \right) dt$$

$$= ia(1-r)^{-k}\left(\mu_1 \frac{X_1}{X} + \mu \frac{X_2}{X} \right)^{-b} \left\{ \int_0^{X_1} \left(\left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2} \right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2} \right) e^{-\beta(X_1 - X)} + (X_1 - t)\mu_1 \right) dt \right.$$
$$\left. + \int_{X_1}^{X} \left(- \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2} \right) e^{-\beta(t - X)} + \frac{c_1 - c_2\beta}{\beta^2} - \frac{c_1}{\beta} t \right) dt \right\}$$

$$=ia(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu\frac{X_2}{X}\right)^{-b}\left\{\left[\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)t-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^2}\right)te^{-\beta(X_1-X)}+\left(X_1t-\frac{t^2}{2}\right)\mu_1\right]_0^{X_1}\right.$$
$$\left.+\left[\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^3}\right)e^{-\beta(t-X)}+\left(\frac{c_1-c_2\beta}{\beta^2}\right)t-\frac{c_1}{2\beta}t^2\right]_{X_1}^{X}\right\}$$

$$=ia(1-r)^{-k}\left(\mu_1\frac{X_1}{X}+\mu\frac{X_2}{X}\right)^{-b}\left\{\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)X_1-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^2}\right)X_1e^{-\beta(X_1-X)}+\left(X_1^2-\frac{X_1^2}{2}\right)\mu_1\right.$$
$$\left.+\left[\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^3}\right)+\left(\frac{c_1-c_2\beta}{\beta^2}\right)X-\frac{c_1}{2\beta}X^2-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^3}\right)e^{-\beta(X_1-X)}-\left(\frac{c_1-c_2\beta}{\beta^2}\right)X_1+\frac{c_1}{2\beta}X_1^2\right]\right\}$$

so that after further simplification, we obtain

$$H=\left(\left(\frac{c_1-c_2\beta}{\beta^2}\right)\left(X-\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}+\frac{1}{\beta}\right)-\frac{c_1}{2}\left(X_1^2+X^2\right)+\frac{c_1X}{1}\left(X_1e^{-(X_1-X)\beta}-\frac{1}{\beta}\left(1-e^{-(X_1-X)\beta}\right)\right)\right.$$
$$\left.+\frac{\beta X_1^2\mu_1}{2}\right)\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu X_2\right)^{-b}}{\beta X^{-b}} \qquad (4.9)$$

The total variable cost is the sum of set-up cost, cost of deteriorated items and inventory carrying cost, i.e.

Total variable cost (Z)

$$=A+Cd\left(X_2\right)+H \qquad (4.10)$$

$$\therefore \quad Z=A+\frac{a(1-r)^{-k}\left(\mu_1X_1+\mu X_2\right)^{-b}}{X^{-b}}\left(\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^2}\right)e^{-\beta(X_1-X)}-\left(X-X_1\right)\left(c_2+\frac{c_1}{2}\left(X-X_1\right)\right)\right)$$
$$+\left(\left(\frac{c_1-c_2\beta}{\beta^2}\right)\left(X-\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}+\frac{1}{\beta}\right)-\frac{c_1}{2}\left(X_1^2+X^2\right)+\frac{c_1X}{1}\left(X_1e^{-(X_1-X)\beta}-\frac{1}{\beta}\left(1-e^{-(X_1-X)\beta}\right)\right)\right.$$
$$\left.+\frac{\beta X_1^2\mu_1}{2}\right)\frac{ia(1-r)^{-k}\left(\mu_1X_1+\mu X_2\right)^{-b}}{\beta X^{-b}} \qquad (4.11)$$

The total variable cost per unit time is therefore given as

$$Z(X) = \frac{\textit{Total variable cost per production run}}{\textit{The production cycle length}}$$

$$= \frac{A}{X} + \frac{a(1-r)^{-k}\left(\mu_1 X_1 + \mu X_2\right)^{-b}}{X^{-b+1}}\left(\left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right)e^{-\beta(X_1 - X)} - (X - X_1)\left(c_2 + \frac{c_1}{2}(X - X_1)\right)\right)$$
$$+\left(\left(\frac{c_1 - c_2\beta}{\beta^2}\right)\left(X - \left(X_1 + \frac{1}{\beta}\right)e^{-(X_1 - X)\beta} + \frac{1}{\beta}\right) - \frac{c_1}{2}\left(X_1^{\,2} + X^2\right) + \frac{c_1 X}{1}\left(X_1 e^{-(X_1 - X)\beta} - \frac{1}{\beta}\left(1 - e^{-(X_1 - X)\beta}\right)\right)\right.$$
$$\left. + \frac{\beta X_1^{\,2}\mu_1}{2}\right)\frac{ia(1-r)^{-k}\left(\mu_1 X_1 + \mu X_2\right)^{-b}}{\beta X^{-b+1}}$$

However, $\mu = \frac{1}{X_2}\int_{X_1}^{X}\mu_2(t)dt = \frac{1}{X_2}\int_{X_1}^{X}(c_2 + c_1 t)dt = \frac{1}{X_2}\left[tc_2 + \frac{c_1 t^2}{2}\right]_{X_1}^{X} = \frac{[X - X_1]}{X_2}\left(c_2 + \frac{c_1}{2}(X - X_1)\right)$ as indicated in equation (4.0)

$$\Rightarrow \quad Z(X) =$$

$$\frac{A}{X} + \frac{a(1-r)^{-k}\left(\mu_1 X_1 + X_2\left\{\frac{[X - X_1]}{X_2}\left(c_2 + \frac{c_1}{2}(X - X_1)\right)\right\}\right)^{-b}}{X^{-b+1}}\left(\left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right)e^{-\beta(X_1 - X)} - (X - X_1)\left(c_2 + \frac{c_1}{2}(X - X_1)\right)\right)$$
$$+\left(\left(\frac{c_1 - c_2\beta}{\beta^2}\right)\left(X - \left(X_1 + \frac{1}{\beta}\right)e^{-(X_1 - X)\beta} + \frac{1}{\beta}\right) - \frac{c_1}{2}\left(X_1^2 + X^2\right) + \frac{c_1 X}{1}\left(X_1 e^{-(X_1 - X)\beta} - \frac{1}{\beta}\left(1 - e^{-(X_1 - X)\beta}\right)\right)\right.$$
$$\left. + \frac{\beta X_1^{\,2}\mu_1}{2}\right)\frac{ia(1-r)^{-k}\left(\mu_1 X_1 + X_2\left\{\frac{[X - X_1]}{X_2}\left(c_2 + \frac{c_1}{2}(X - X_1)\right)\right\}\right)^{-b}}{\beta X^{-b+1}}$$

$$\Rightarrow \quad Z(X) = \frac{A}{X} + \left\{\left(\left(\frac{c_1 - c_2\beta - c_1\beta X_1}{\beta^2}\right) - \left(\frac{c_1 - c_2\beta - c_1\beta X}{\beta^2}\right)e^{-\beta(X_1 - X)} - (X - X_1)\left(c_2 + \frac{c_1}{2}(X - X_1)\right)\right)\right.$$
$$+\left(\left(\frac{c_1 - c_2\beta}{\beta^2}\right)\left(X - \left(X_1 + \frac{1}{\beta}\right)e^{-(X_1 - X)\beta} + \frac{1}{\beta}\right) - \frac{c_1}{2}\left(X_1^2 + X^2\right) + \frac{c_1 X}{1}\left(X_1 e^{-(X_1 - X)\beta} - \frac{1}{\beta}\left(1 - e^{-(X_1 - X)\beta}\right)\right)\right.$$
$$\left.\left. + \frac{\beta X_1^{\,2}\mu_1}{2}\right]\frac{i}{\beta}\right\}\frac{a(1-r)^{-k}\left(2\mu_1 X_1 + [X - X_1]\left(2c_2 + c_1(X - X_1)\right)\right)^{-b}}{2^{-b}X^{-b+1}} \qquad (4.12)$$

To obtain the value of X which minimizes the total variable cost per unit time, we take the derivative of $Z(X)$ with respect to X and then set the result to zero. The value of X satisfying equation (4.12) gives the minimum provided $\frac{d^2 Z(X)}{dX^2} > 0$. The derivative is given as:

$$\frac{dZ(X)}{dX}=-\frac{A}{X^2}+\left\{\left(-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta}\right)e^{-(X_1-X)\beta}+\frac{c_1}{\beta}e^{-(X_1-X)\beta}-\left(c_2+c_1\left[X-X_1\right]\right)\right)\right.$$

$$+\left[\left(\frac{c_1-c_2\beta}{\beta}\right)\left(1-\left(X_1+\frac{1}{\beta}\right)\beta e^{-(X_1-X)\beta}\right)-c_1X+c_1\left(X_1e^{-(X_1-X)\beta}-\frac{1}{\beta}\left(1-e^{-(X_1-X)\beta}\right)\right)+c_1X\left(\beta X_1e^{-(X_1-X)\beta}+e^{-(X_1-X)\beta}\right)\right]\frac{i}{\beta}\Bigg\}\frac{a(1-r)^{-k}\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-b}}{2^{-b}X^{-b+1}}$$

$$+\left\{\left(\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^2}\right)e^{-\beta(X_1-X)}-\left(X-X_1\right)\left(c_2+\frac{c_1}{2}\left(X-X_1\right)\right)\right)+\left[\left(\frac{c_1-c_2\beta}{\beta^2}\right)\left(X-\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}+\frac{1}{\beta}\right)-\frac{c_1}{2}\left(X_1^2+X^2\right)+\frac{c_1X}{1}\left(X_1e^{-(X_1-X)\beta}-\frac{1}{\beta}\left(1-e^{-(X_1-X)\beta}\right)\right)\right.\right.$$

$$\left.\left.+\frac{\beta X_1^2\mu_1}{2}\right]\frac{i}{\beta}\right\}\frac{a(1-r)^{-k}\left\{-bX^{-b+1}\left[2c_2+2c_1\left(X-X_1\right)\right]\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-b-1}+(b-1)X^{-b}\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-b}\right\}}{2^{-b}X^{2(-b+1)}}$$

Simplifying and setting it to zero gives:

$$\Rightarrow\qquad -\frac{A}{X^2}+\left\{\left(-\frac{c_1}{\beta}e^{-(X_1-X)\beta}+\left(c_2+c_1X\right)e^{-(X_1-X)\beta}+\frac{c_1}{\beta}e^{-(X_1-X)\beta}-\left(c_2+c_1(X-X_1)\right)\right)+\left[\frac{c_1}{\beta}\right.\right.$$

$$-c_1X_1e^{-(X_1-X)\beta}-\frac{c_1}{\beta}e^{-(X_1-X)\beta}-c_2+c_2\beta\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}-c_1X+c_1X_1e^{-(X_1-X)\beta}-\frac{c_1}{\beta}+\frac{c_1}{\beta}e^{-(X_1-X)\beta}+\frac{c_1X\beta X_1}{1}e^{-(X_1-X)\beta}$$

$$\left.\left.+\frac{c_1X}{1}e^{-(X_1-X)\beta}\right]\frac{i}{\beta}\right\}\frac{a(1-r)^{-k}\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-b}}{2^{-b}X^{-b+1}}+\left\{\left(\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^2}\right)e^{-\beta(X_1-X)}-\left(X-X_1\right)\left(c_2+\frac{c_1}{2}\left(X-X_1\right)\right)\right)\right.$$

$$+\left[\left(\frac{c_1-c_2\beta}{\beta^2}\right)\left(X-\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}+\frac{1}{\beta}\right)-\frac{c_1}{2}\left(X_1^2+X^2\right)+\frac{c_1X}{1}\left(X_1e^{-(X_1-X)\beta}-\frac{1}{\beta}\left(1-e^{-(X_1-X)\beta}\right)\right)\right.$$

$$\left.\left.+\frac{\beta X_1^2\mu_1}{2}\right]\frac{i}{\beta}\right\}\frac{a(1-r)^{-k}X^{-b}\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-b}\left\{-2bX\left[c_2+c_1\left(X-X_1\right)\right]\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-1}+(b-1)\right\}}{2^{-b}X^{2(-b+1)}}=0$$

$$\Rightarrow\qquad \frac{A}{X^2}-\left\{\left(\left(c_2+c_1X\right)e^{-(X_1-X)\beta}-\left(c_2+c_1\left(X-X_1\right)\right)\right)+\left[c_2\left(\beta\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}-1\right)\right.\right.$$

$$\left.\left.+\frac{c_1X}{1}\left(\left(\beta X_1+1\right)e^{-(X_1-X)\beta}-1\right)\right]\frac{i}{\beta}\right\}\frac{a(1-r)^{-k}\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-b}}{2^{-b}X^{-b+1}}-\left\{\left(\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)\right.\right.$$

$$-\frac{(c_1-c_2\beta-c_1\beta X)}{\beta^2}e^{-(X_1-X)\beta}-\left[X-X_1\right]\left(c_2+\frac{c_1}{2}(X-X_1)\right)\Bigg)+\left[\left(\frac{c_1-c_2\beta}{\beta^2}\right)\left(X-\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}+\frac{1}{\beta}\right)-\frac{c_1}{2}\left(X_1^2+X^2\right)+\frac{c_1X}{1}\left(X_1e^{-(X_1-X)\beta}-\frac{1}{\beta}\left(1-e^{-(X_1-X)\beta}\right)\right)\right.$$

$$\left.\left.+\frac{\beta X_1^2\mu_1}{2}\right]\frac{i}{\beta}\right\}\frac{a(1-r)^{-k}\left\{-2bX\left[c_2+c_1\left(X-X_1\right)\right]\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-1}+(b-1)\right\}}{2^{-b}X^{-b+2}\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{b}}=0$$

Multiplying through by $\dfrac{2^{-b}X^{-b+2}}{a(1-r)^{-k}\left(2\mu_1X_1+\left[X-X_1\right]\left(2c_2+c_1\left(X-X_1\right)\right)\right)^{-b}}$, we get

$$\frac{2^{-b}A(1-r)^{k}}{aX^{b}\left(2\mu_1 X_1+[X-X_1](2c_2+c_1(X-X_1))\right)^{-b}}-\left\{\left((c_2+c_1X)e^{-(X_1-X)\beta}-(c_2+c_1(X-X_1))\right)+\left[c_2\left(\beta\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}-1\right)\right.\right.$$
$$\left.\left.+\frac{c_1X}{1}\left((\beta X_1+1)e^{-(X_1-X)\beta}-1\right)\right]\frac{i}{\beta}\right\}\frac{1}{X^{-1}}-\left\{\left(\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)-\frac{(c_1-c_2\beta-c_1\beta X)}{\beta^2}e^{-(X_1-X)\beta}-[X-X_1]\left(c_2+\frac{c_1}{2}(X-X_1)\right)\right)\right.$$
$$\left.+\left[\left(\frac{c_1-c_2\beta}{\beta^2}\right)\left(X-\left(X_1+\frac{1}{\beta}\right)e^{-(X_1-X)\beta}+\frac{1}{\beta}\right)-\frac{c_1}{2}\left(X_1^2+X^2\right)+\frac{c_1X}{1}\left(X_1e^{-(X_1-X)\beta}-\frac{1}{\beta}\left(1-e^{-(X_1-X)\beta}\right)\right)\right.\right.$$
$$\left.\left.+\frac{\beta X_1^2\mu_1}{2}\right]\frac{i}{\beta}\right\}\frac{a(1-r)^{-k}\left\{-2bX\left[c_2+c_1(X-X_1)\right]\left(2\mu_1X_1+[X-X_1](2c_2+c_1(X-X_1))\right)^{-1}+(b-1)\right\}}{1}=0 \qquad (4.13)$$

If other parameters are given, equation (4.13) can be used to determine the best X which minimizes the total variable cost.The EPQ of the corresponding X will be computed from:

EPQ = demand before deterioration starts + demand after deterioration starts + number of deteriorated items

$$= \mu_1X_1+\int_{X_1}^{X}(c_2+c_1t)dt+d(X_2)$$
$$= \mu_1X_1+Y_d$$
$$EPQ=\mu_1X_1+\left(\frac{c_1-c_2\beta-c_1\beta X_1}{\beta^2}\right)-\left(\frac{c_1-c_2\beta-c_1\beta X}{\beta^2}\right)e^{-(X_1-X)\beta} \qquad (4.14)$$

using equation (4.4).

4.3 RESULTS OBTAINED FROM THE MODEL

In the previous section, we developed a mathematical model for items that exhibit delay in deterioration with reliability consideration and linear time dependent demand. Numerical examples are given in table 4.1 below to determine the best cycle length. In all the examples,

$\frac{d^2Z(X)}{dX^2}$ was found to be greater than zero showing that the cost function is convex in all the examples.

4.3.1 NUMERICAL EXAMPLES

In each example the values were substituted in equation (4.13) and the period X, which minimized the equation was determined. The table below gives the results of ten different numerical examples having different parameters as indicated:

$a = 40,\ b = 2.5,\ k = 9.5,\ c_2 = 500$ and $c_1 = 50$.

Table 4.1: Complete results of Ten Numerical examples on the application of the model

Examples	A $(naira)$	μ_1 $(units)$	r	i	X_1 $(days)$	β	X $(days)$	Z $(units)$	EPQ $(units)$
1	3500	2100	0.93	0.002	7	0.42	10	262361	44
2	3300	2150	0.92	0.003	14	0.41	18	144245	88
3	3000	2300	0.91	0.0009	21	0.43	32	87464	148
4	2700	2050	0.89	0.0012	28	0.44	53	50347	192
5	2500	1500	0.88	0.0013	35	0.46	62	34032	182
6	2400	1120	0.87	0.0014	42	0.47	71	27210	170
7	2300	2000	0.9	0.001	49	0.45	59	20263	282
8	1700	1000	0.86	0.0015	56	0.48	85	15826	195
9	1500	900	0.85	0.0016	63	0.49	98	12033	206
10	1400	700	0.84	0.0018	70	0.5	107	10600	188

4.3.2 SENSITIVITY ANALYSIS

Table 4.2 gives results of the effect of changes in the system parameters in example 6 above; i.e

$A = 2400,\ \rho_1 = 1120, r = 0.87, i = 0.0014, L_1 = 0.115068, \lambda = 0.47, L = 0.194521, Z = 27210, EPQ = 170,$ $a = 40, b = 2.5, k = 9.5, c_2 = 500$ and $c_1 = 50,$ what we get is as follows:

Table 4.2: Sensitivity Analysis

Parameters	% change in parameter	% change in Z	% change in EPQ
a	-2	-0.39206	0
	-1	-0.19603	0
	1	0.196028	0
	2	0.392057	0
b	-2	7.890924	0
	-1	3.612583	0
	1	-3.05042	0
	2	-5.62617	0
k	-2	-6.29927	0
	-1	-3.45391	0
	1	4.192634	0
	2	9.281984	0
c_2	-2	-0.01043	-0.47626
	-1	-0.00498	-0.23813
	1	0.004516	0.238129
	2	0.008576	0.476257
c_1	-2	-0.14765	-0.00738
	-1	-0.07382	-0.00369
	1	0.073818	0.003692
	2	0.147632	0.007384

For any EPQ Model, it is usually possible to accurately estimate the set-up cost (A) and inventory holding cost (H) using historical accounting and production data. However, the parameters a, b, k, c_2 and c_1 used to define the unit cost of production; production quantity and the total cost of production in our EPQ Model have to be largely estimated subjectively. As a result we found: (i) Z is sensitive to changes in a, b, k, c_2 and c_1. (ii) EPQ is sensitive to changes in c_2 and c_1 only.

The values of a, b, k, c_2 and c_1 for which the above example remain optimal are $499 \leq c_2 \leq 501$, $49.9 \leq c_1 \leq 50.1$, $9.4999 \leq k \leq 9.5002, 2.49993 \leq b \leq 2.50007, 39.98 \leq a \leq 40.02$ respectively.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 SUMMARY

This thesis presents two mathematical models on the inventory of deteriorating items which do not start deteriorating immediately they are stocked. The unit cost of production of an item is assumed to be directly related to the process reliability and inversely related to the rates of demand. Items that have this property include farm produce, bread, cake, fashion goods and so on.

The first model developed (in chapter three) is an Economic Production Quantity (EPQ) model for items that exhibit delay in deterioration with reliability consideration assuming a constant demand rate both before and after deterioration sets-in. The unit cost of production is directly related to the process reliability and inversely related to the demand rates. The second model developed (in chapter four) is an Economic Production Quantity (EPQ) model for items that exhibit delay in deterioration with the same reliability consideration but now on the assumption that the demand after deterioration sets-in is linearly time dependent and the demand is constant before deterioration sets in. Several examples are given to illustrate the applicability of the models. Sensitivity analyses were also carried out to see the effect on parameter changes.

5.2 CONCLUSION

In this thesis, we present two mathematical models on the inventory of deteriorating items which do not start deteriorating immediately they are stocked. One of our models is an extension/generalization of Musa and Sani (2009) by assuming that the unit cost of production of an item is directly related to the process reliability and inversely related to the rates of demand. The model determines the EPQ and the best cycle length.

The results of sensitivity analysis carried out on the effect of changes in the system parameters on the first model show that:

- Z(X) is sensitive to changes in system parameters *a, b* and *k*, where a, b and k are non-negative real numbers as already indicated in (3.0),
- Z(X) is more sensitive to changes in k and is less sensitive to changes in a.

In the case of the second model, the results of sensitivity analysis carried out on some system parameters also show that:

- Z(X) is sensitive to a, b, k, c_2 and c_1, where c_2 and c_1 are constants as also already indicated under assumption (4.1),
- Z(X) is more sensitive to changes in k and is less sensitive to changes in c_2 and c_1.
- EPQ is sensitive to c_2 and c_1 only

In general, it can be observed from both models that the total variable cost per unit time Z(X) is more sensitive to the changes in k. This is probably because the higher the value of k is, the higher the reliability consideration (quality assurance) which implies that the total inventory cost per unit time (i.e. Z(X)) will increase. On the other hand, the lower the value of k is, the lower

the reliability consideration (quality assurance) which implies that the total inventory cost per unit time (i.e. Z(X)) will decrease.

5.3 RECOMMENDATIONS

From the conclusion in the last section, it is clear that for both models, Z(X) is sensitive to changes in k, as such; the value of k should be controlled in order to minimize the total variable cost per unit time Z(X) appropriately. EPQ is only sensitive to changes in c_2 and c_1, hence the values of c_2 and c_1 should be controlled in order to minimize the EPQ value in the second model, if that is required.

The importance of any model developed, is for it to be implemented by practitioners. It is very much recommended for any practitioner whose product exhibit some level of delayed deterioration to actually implement the models developed in this thesis.

Also, there are various disciplines that go hand in hand with inventory. It is recommended that models developed in this thesis be used by these other disciplines in their approach. These related disciplines include forecasting, ordering, linear programming, statistics and so on.

5.4 RESEARCH EXTENSIONS

Areas of possible research extension include finding an exact solution for the two mathematical models. The applicability of the model in manufacturing company/industry will also be an important extension of this model.

Further research can also be done by considering the total variable cost per unit time Z(X) to be dependent on both cycle length X and reliability *r*, i.e. Z(*r*,X) where the cycle length X and the reliability consideration *r* are not constant.

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