A STUDY OF FUZZY NUMBERS AND METHODS FOR THEIR RANKING

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DEPARTMENT OF MATHEMATICS

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DECLARATION

I declare that the work in this dissertation entitled "**A STUDY OF FUZZY NUMBERS AND METHODS FOR THEIR RANKING**" has been performed by me in the Department of Mathematics under the supervision of Prof D. Singh and Dr. A. M. Ibrahim.The information derived from literature has been duly acknowledged in the text and a list of referencesprovided. No part of this dissertation was previously presented for another degree or diploma at this or any other Institution.

Suleiman DattiADAM ______________

Name of Student Signature and Date

CERTIFACATION

The dissertation titled "**A STUDY OF FUZZY NUMBERS AND METHODS FOR THEIR RANKING**" by Suleiman DattiADAM (MSC/SCI/34321/2012-2013) meets the regulations governing the award of the degree of Master of Sciencein mathematics of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This dissertation is dedicated to Almighty ALLAH for His grace and mercy, and to my beloved parents late Alh. Datti Garba and MalamaRabi'at Ibrahim, whose love and passion for education have won me this priceless gift.

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ABSTRACT

Fuzzy numbers have been studied extensively in mathematics along with its applications in diverse fields, especially in solving many real-life problems. This research work based on an extensive survey of the related existing works, formulates a ranking method for generalized trapezoidal fuzzy numbers by computing section-wise perimeters of the trapezoid representing a generalized fuzzy number. The perimeter of trapezoid representing a generalized fuzzy number is viewed as comprising three segments:- left and right divergences which are right angled triangles and the centre as a rectangle. In order to evaluate its degree of robustness, new similarity measures are computed and finally compared with the related results obtained under other existing similarity measure approaches by way of providing a comparison table.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Background to the Study

Zadeh (1965) formulated fuzzy set theory as a powerful tool to deal with problems involving imprecision for which other extant approaches were found incapacitated. In course of time, the concept of fuzzy sets has made a perceptible advance and a number of important concepts related to modeling complex fuzzy situations has emerged, investigated and applied. In all this, the development of fuzzy arithmetic has occupied a central position. Fuzzy arithmetic include describingfuzzy numbers, operation of fuzzy numbers, and ordering of fuzzy numbers (Dubois and prade, 1980),(Karfmann and Gupta, 1985) and (Zimmerman, 1996) for various details.

In order to seek a model in the set of real numbers R, special fuzzy sets characterized by their membership functions of the form $\mu_{\tilde{A}}: R \longrightarrow [0, 1]$ were defined which gave rise to the concept of fuzzy numbers. A fuzzy number is a fuzzy set representing a real line interval with fuzzy boundaries (Zadeh, 1965).

Fuzzy numbers can appear in different shapes such as bell shape, triangular, trapezoidal, etc. It is known that triangular fuzzy numbers being simplest in the form have most frequently been explored in application systems. However, as observed in (Zimmermann, 1996),from computational efficiency point of view, the trapezoidal fuzzy numbers are found to have an edge. Fuzzy numbers have found applications in risk analysis, decision making, approximate reasoning; fuzzy control, optimization, forecasting, etc.(Zimmermann, 1996), (Zadeh, 1965). In most of these applications, it becomes imperative to define an ordering of fuzzy numbers and, in turn, that of an efficient ranking function $\mathcal{R} : P(\mathbb{R}) \longrightarrow \mathbb{R}$or $R : P(I) \longrightarrow \mathbb{R}$ where I is the unit interval [0, 1],$P(\mathbb{R})$ is a set of fuzzy sets in $\mathbb{R}$, and $P(I)$ is the set of fuzzy sets in I. It may be noted that there is no loss of generality in restricting the domain of definition of fuzzy subsets from $\mathbb{R}$ to I (Bortolan&Degani, 1985).

The method of ranking fuzzy numbers was first proposed by (Jain, 1976) for decision making in fuzzy environment by way of representing ill-defined quantities as a fuzzy set. In decision theory, ranking of fuzzy numbers is, found uniquely important, for example, the concept of optimum or best choice is completely based on rankingor comparison. In course of time, a number of methods of ranking fuzzy numbers have been proposed. In spite of the existence of avariety of methods none has been found fully satisfactory.

1.2Statement of the Research Problem

Ranking fuzzy numbers play a very significant role in decision making, optimization,and data representation, just to mention a few. Many ranking methods have been proposed so far, however, there are yet no method that can give a satisfactory solution to every situation. Some of the methods are found counterintuitive, some are not discriminating,some use only the local information of fuzzy values and some cannot rank crisp numbers. This gives a room for proposing, developing and revising different ranking methods of fuzzy numbers that provide results close to intuition and other researches in the literature.

1.3 Aim and Objectives

The aim of this research is to investigate ranking methods of fuzzy numbers and develop a new competing approach. In order to achieve this goal, the objectives are to:

(i) conduct a critical study of existing ranking methods of fuzzy numbers,

(ii) developa new approach for ranking generalized trapezoidal fuzzy numbers,

(iii) developa similarity measure using the new approach, and

(iv) apply it to risk analysis.

1.4 Methodology

We critically study some major ranking methods of fuzzy numbers: method for ranking fuzzy numbers by considering both the mean and dispersion of alternatives (Lee and Li 1988), ranking method based on integral value index (Liou and wang, 1992), ranking method of fuzzy numbers based on the concept of *existence* (Cheng and Lee, 1994), ranking fuzzy numbers using the distance method (Cheng, 1998), approximate approach for ranking fuzzy numbers based on left and right dominance (Chen and Lu, 2001), method of ranking fuzzy numbers based on the area between the centroid point and original point (Chu and Tsao, 2002), method of ranking fuzzy numbers with preference weighting function expectations (Liu and Han, 2005), method for ranking fuzzy numbers based on adapting two dimensions dominance (Chang *et al*., 2006), method for ranking p-norm trapezoidal fuzzy numbers (Chen and Tang, 2008), method of ranking of trapezoidal fuzzy numbers based on left and right spreads (Abbasbandy and Hajjari, 2009), method for ranking generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread (Kumar *et al*., 2011a), method for ranking in perimeters of two generalized trapezoidal fuzzy numbers (Rezvani, 2012). In this dissertation we propose to develope a new method for ranking fuzzy numbers that considers the section-wise perimeters of a generalized trapezoidal fuzzy number and compute similarity measures to strengthen our approach.

1.5 Significance of the study

Methods for ranking fuzzy numbers have many applications in different fields of study. The role of ranking in decision making, optimization and engineering cannot be over emphasized. As there is no general single method developed so far, it is important to explore new techniques for ranking fuzzy numbers

1.6Definition of terms

The following definitions are largely adopted from(Chang, 1981),(Zimmermann, 1996)(Asady, 2010), (Lakashmana et al., 2011), (Kumar et al., 2011), (Rezvani, 2012), etc.

(a) Fuzzy Set

Let X bea universal set. Then the fuzzy subset $\tilde{A}$of$X$ is defined by its membership function $\mu_{\tilde{A}}: X \longrightarrow [0, 1]$ which assigns a real number $\mu_{\tilde{A}}(x)$in the interval [0, 1], to each element $x \in X$, where the value of $\mu_{\tilde{A}}(x)$at x shows the grade of membershipof xin$\tilde{A}$.

(b) Complement of a Fuzzy Set

The complement μ^c of a fuzzy subset μ of a set X is a fuzzy subset given by:

$$\mu^c(x) = 1 - \mu(x), \quad x \in X.$$

(c) α–cut of a Fuzzy Set

Given a fuzzy set $\tilde{A}$in X and any real number $\boldsymbol{\alpha} \in [0, 1]$, then the $\boldsymbol{\alpha}$ -cut or $\boldsymbol{\alpha}$ -level or cut worthy set of $\tilde{A}$, denoted by $\tilde{A}^{\alpha}$ is the crisp set $\tilde{A}^{\alpha} = \{x \in X : \mu_{\tilde{A}}(x) \geq \boldsymbol{\alpha}\}$.

The strong a-cut, denoted by $\tilde{A}^{\alpha+}$is the crisp set $\tilde{A}^{\alpha+} = \{x \in X: \mu_{\tilde{A}}(x) > \boldsymbol{\alpha}\}$.

(d) Normal Fuzzy Set

A fuzzy subset $\tilde{A}$ of universal set U is normal if and only if$sup_{x\in U}\mu_{\tilde{A}}(\mathrm{x}) = 1$.

(e) Convex Fuzzy Set A fuzzy subset $\tilde{A}$ of Universe set U is convex if and only if $\lambda x + (1-\lambda)y \geq (\mu_{\tilde{A}}(\mathrm{x}) \Lambda \mu_{\tilde{A}}(\mathrm{y})), \forall x, y \in U, \forall \lambda \in [0,1]$, where Λ denotes the minimum operator.

(f)Fuzzy Number

A fuzzy set $\tilde{A}$ defined on the universal set of a real number is said to be a fuzzy number if its membership function has the following characteristics:

(i) $\mu_{\tilde{A}} : \mathbb{R} \rightarrow [0,1]$is continuous.

(ii) $\mu_{\tilde{A}}(x) = 0 \ forall x \in (-\infty, a] \cup [d, \infty)$.

(iii) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$.

(iv) $\mu_{\tilde{A}}(x) = 1 \, forall x \in [b, c]$, where $a \leq b \leq c \leq d$.

(g) Triangular Fuzzy Number

A triangular fuzzy number $\tilde{A}$ is a fuzzy number with a piece wise linear membership function $\mu_{\tilde{A}}$ defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ \frac{(c-x)}{(c-b)}\ , & b \leq x \leq c \\ 0, & otherwise. \end{cases}$$

written as a triplet (a, b, c) where $a \leq b \leq c; a$ and c stand for the lower and upper values of the support of the fuzzy number $\tilde{A}$, respectively, and b for the modal value.

Now,$\frac{(x-a)}{(b-a)}$ and $\frac{(c-x)}{(c-b)}$ are known as left and right legs of the triangular fuzzy number (a, b, c).

(h) Complement of a Triangular Fuzzy Number

Let $\tilde{A} = (a, b, c)$ be a triangular fuzzy number.The complement $\tilde{A}^c$ of $\tilde{A}$ is defined by $\mu_{\tilde{A}}^c(x) = 1 - \mu_{\tilde{A}}(x)$. Hence the membership function $\mu_{\tilde{A}}^c$ is defined by

$$\mu_{\tilde{A}}^c(x) = \begin{cases} \frac{(a-x)}{(b-a)}, & a \leq x \leq b \\ \frac{(x-b)}{(c-b)}, & b \leq x \leq c \\ 1, & otherwise. \end{cases}$$

(i) Trapezoidal Fuzzy Number

A fuzzy number $\tilde{A} = (a, b, c. d)$ is said to be a trapezoidal fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{(x-a)}{(b-a)}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \frac{(x-d)}{(c-d)}\ , & c \leq x \leq d. \end{cases}$$

Alternatively,

Since both L and R of trapezoidal fuzzy numbers are linear, it can be represented by quadruple $(a, b, c, d) \in R^4$ and is completely characterized by $a \leq b \leq c \leq d$. In this case, the r-level sets are given by $\underline{u}^r = a + r(b - a)$ and $\overline{u}^r = d + r(d - c)$.If we have $b = c$, the quadruple reduces to a triple (a, b, c) which is a triangular fuzzy number.

(j) Generalized Fuzzy Number

A fuzzy set $\tilde{A}$ defined on the universal set of real numbers is said to be a *generalized fuzzy number* if its membership function has the following characteristics:

(i) $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$ is continuous,

(ii) $\mu_{\tilde{A}}(x) = 0 \ \ for\ all\ \ x \in (-\infty, a] \cup [d, \infty)$,

(iii) $\mu_{\tilde{A}}(x)$ is strictly increasing on $[a, b]$ and strictly decreasing on $[c, d]$,

(iv) $\mu_{\tilde{A}}(x) = w\ for\ all\ x \in [b, c]$, where $0 < w \leq 1. \forall\ a, b, c, d \in \mathbb{R}$.

Generalized fuzzy number can be represented alternatively by

$$\mu_{\tilde{A}}(x) = \begin{cases} \mu_{\tilde{A}}^L(x), & a \leq x \leq b \\ w, & b \leq x \leq c \\ \mu_{\tilde{A}}^R(x), & c \leq x \leq d \\ 0\ , & otherwise, \end{cases}$$

where$\mu_{\tilde{A}}^L(x)$: $[a, b] \to [0, w]$ and $\mu_{\tilde{A}}^R(x)$: $[c, d] \to [0, w]$ are two strictly monotonic and continuous functions from $\mathbb{R}$ to the closed interval [0, w]. A fuzzy number$\tilde{A}$ iscalled *normal* if $w = 1$, otherwise $\tilde{A}$ is said to be a *generalized* or *non-normal* fuzzy number.The image $-\tilde{A}$ of $\tilde{A}$ can be expressed by $(-a, -b, -c, -d; w)$.

(k) Generalized Trapezoidal Fuzzy Number

A fuzzy number $\tilde{A}$ is called a generalized trapezoidal fuzzy number,denoted $\tilde{A} = (a, b, c, d; w)$, if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} \dfrac{w(x-a)}{(b-a)}, & a \le x \le b \\ w, & b \le x \le c \\ \dfrac{w(x-d)}{(c-d)}, & c \le x \le d. \end{cases}$$

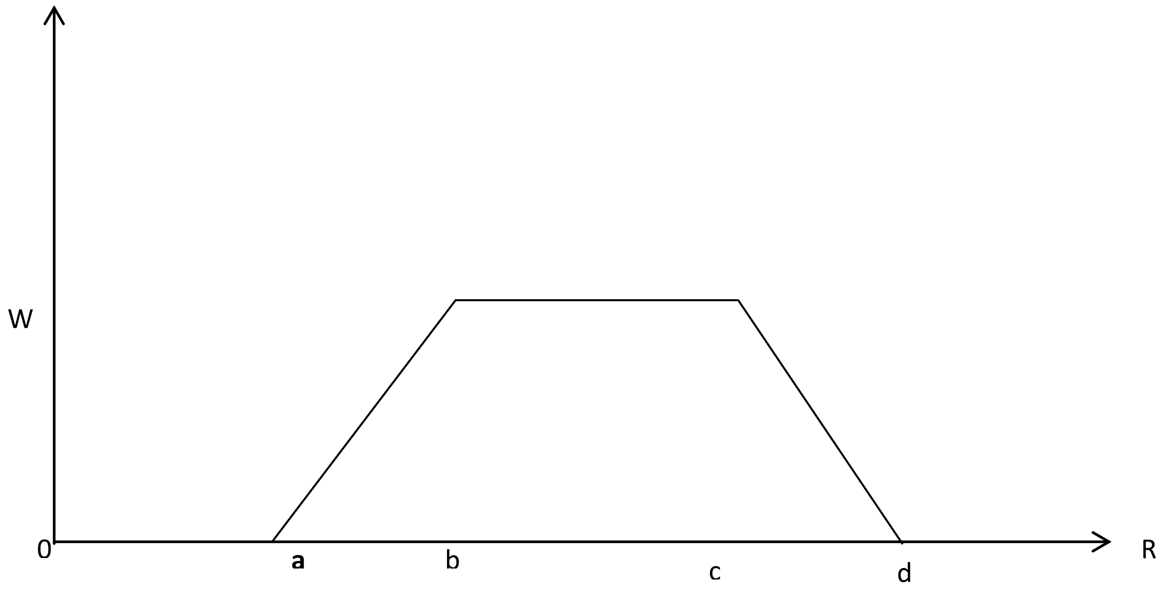


Figure 3.5: A generalized trapezoidal fuzzy number

(l) Normal and Non-normal Trapezoidal Fuzzy Number

A trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ is said to benormal if $w = 1$ while itis called non-normal if $w \neq 1$ and its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & -\infty < x \le a \\ \dfrac{w(x-a)}{(b-a)}, & a \le x \le b \\ w, & b \le x \le c \\ \dfrac{w(x-d)}{(c-d)}, & c \le x \le d \\ 0, & d \le x < \infty. \end{cases}$$

(m) Non-normal p-norm Trapezoidal Fuzzy Number

A non- normal fuzzy number $\tilde{A} = (a, b, c, d; w)$ is said to be a non- normal p-norm trapezoidal fuzzy number, denoted by $\tilde{A} = (a, b, c, d; w)_p$, if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & -\infty < x \leq a \\ w(1 - \left(\frac{x-b}{a-b}\right)^p)^{1/p}, & a \leq x \leq b \\ w, & b \leq x \leq c \\ w\,(1 - \left(\frac{x-c}{d-c}\right)^p)^{1/p}, & c \leq x \leq d \\ 0, & d \leq x < \infty \end{cases}$$

where p is a positive integer.

(n) Positive and Negative Fuzzy Numbers

A fuzzy number $\tilde{A}$ is said to be positive if $0 < a_1 \leq a_2$ holds for the support $\Gamma_{\tilde{A}} = [a_1 , a_2]of\tilde{A}$, that is, $\Gamma_{\tilde{A}}$ is the positive real line. $\forall\, a_1, a_2 \in \mathbb{R}$.

Similarly,$\tilde{A}$ is called negative if $a_1 \leq a_2 \leq 0$, and $\tilde{A}$ is called zero if $a_1 \leq 0 \leq a_2$.

CHAPTER TWO

LITERATURE REVIEW

Zadeh (1965)formulated fuzzy set theory as a powerful mathematical tool to deal with problems involving vagueness and uncertainty, which other extant approaches could not adequately address. In course of time, the theory of fuzzy sets has made a perceptible advance and a number of important concepts related to modeling complex fuzzy situations have emerged, studied, and applied. In particular, in order to seek a model in the set of real numbers$\mathbb{R}$, special fuzzy sets characterized by their membership functions were defined, of the form$\mu_{\tilde{A}} : X \rightarrow [0,1]$, which gave rise to the concept of fuzzy numbers. Thus, a fuzzy number is a fuzzy set representing a real line interval with fuzzy boundaries. More explicitly, a fuzzy number is a fuzzy set in $\mathbb{R}$, which is convex and normalized, and its membership function needs to be piecewise continuous. A non-normalized fuzzy number is called a generalized fuzzy number.

In the following, we present a brief review of various methods of ranking fuzzy numbers.

The method of ranking fuzzy numbers was first proposed by (Jain, 1976) for decision making in fuzzy situation by representing the ill-defined quantity as a fuzzy set. Soon after,

2.1 Methods of Ranking Fuzzy Numbers Using Fuzzy Index

(Yager, 1981) proposed*four indices* that could be employed for the purpose of ordering fuzzy quantities in $[0,1]$.The criteria for ranking fuzzy numbers based on the left and right indices was employed.

Lee and Li (1988) introduced a method for ranking fuzzy numbers by considering both the mean and dispersion of alternatives and came up with two groups of indices: uniform and the proportional probability distributions.

Buckley and Chanas (1989) developed a fast method of ranking of multiple alternatives using fuzzy numbers. Many alternatives were defined and considered to rank normal and non-normal triangular and trapezoidal fuzzy numbers.

Kim and Park (1990) proposed a method of ranking fuzzy numbers with *index of optimism*, denoted α, higher the value of α, higher the rank of the fuzzy number. In this method, the decision makers' views strictly depend on α (optimistic or pessimistic).

(Liou and Wang, 1992) developed a ranking method based on *integral value index*, similar to Kim and Park's method.

Garcia andLamata (2007) presented a modification of the index of (Liou & Wang, 1992) approach for ranking fuzzy numbers.The method could rank more than two fuzzy numbers simultaneously. The method used an index of optimism to reflect the decision maker's optimistic attitude, which was missing in (Liou and Wang, 1992).

(Kumar *et al.*, 2011b) also, presented a method of ranking generalized exponential fuzzy numbers using integral value approach.

Luu et al.(2013a, 2013b) proposed an improved ranking method for fuzzy numbers based on the *centroid index* which considered the cetroid point of normal and non-normal fuzzy numbers to obtain the *index of optimism*. They also developed another method using left and right indices. The main idea behind this method is to obtain the difference between left and right relative values using different decision levels.

Vincent and Dat (2014) proposed an improved ranking method for fuzzy numbers with integral values.They used the novel left, right and total integral values of the fuzzy numbers to overcome the short comings of (Liou & Wang, 1992)approach: it cannot differentiate normal and non-normal fuzzy numbers, and cannot rank properly fuzzy numbers with compensation of areas.

2.2 Methods of Ranking Fuzzy Numbers Using Area Representation

Fortemps and Roubens (1996) presented a ranking method based on *area compensation*. The compensation of area is considered in ranking both normal and non-normal triangular and trapezoidal fuzzy numbers, some interesting properties related to the area compensation procedure were developed to compare fuzzy numbers.

Abbasbandy and Asady (2006) proposed a distance based approach called *sign distance*, for ranking fuzzy numbers. It is found an improvement upon the *distance method.*

Ezzati and Saneifard (2010) introduced a method for ranking of fuzzy numbers with *continuous weighted quasi-arithmetic means*. The method can effectively rank various fuzzy numbers. The proposed model is studied for a broad class of fuzzy numbers.

Nejad and Mashinchi(2011) presented a method for ranking fuzzy numbers based on the areas on the left and the right sides of fuzzy numbers. The divergence of a fuzzy number is considered to differentiate between the left and the right area to rank fuzzy numbers.

Rezvani (2012) proposed a method for ranking in perimeters of two generalized trapezoidal fuzzy numbers. The method considered the perimeter of a generalized trapezoidal fuzzy number together with its vertical and horizontal height to rank fuzzy numbers.

Rezvani; (2013a) presented two methods of ranking generalized trapezoidal fuzzy numbers using their *areas.*

Wang *et al*. (2013) presented a method for ranking non-normal trapezoidal fuzzy numbers based on the quasi-slope and geometrical distance. The method overcomes the limitation of some approaches in ranking fuzzy numbers. The method not only makes the calculation easy, but also improved some shortcomings of the previous methods in ranking fuzzy numbers.

2.3 Methods of Ranking Fuzzy Numbers Using Centroid Approach

Cheng (1998) introduced a method for ranking fuzzy numbers using the *distance method*, where the distance of the centroid point of each fuzzy number and original point was calculated. This method could also rank more than two fuzzy numbers simultaneously, and the fuzzy numbers need not be normal. The coefficient of variation (C.V. index) is considered to improve Lee and Li's method (1988).

Chu and Tsao (2002) proposed a method of ranking fuzzy numbers based on the area between the centroid point and original point.

Chang *et al*. (2006) developed a conceptual procedure for ranking fuzzy numbers based on *adaptive two dimensions dominance*. The method was applied to rank normal, non-normal, positive and negative fuzzy numbers. Also, conceptual procedures were used to describe how to intuitively rank fuzzy numbers.

Wang and Lee (2008) proposed a method to rank fuzzy numbers by comparing the *centroid* by using horizontal and vertical coordinate values of fuzzy numbers. They also used the centroid concept in developing a ranking index called the method of ranking fuzzy numbers with an area between the centroid and original point.

Suresh *et al*. (2012) proposed a method of ranking generalized trapezoidal fuzzy number using centroid of centroids.

Rezvani(2013b) proposed a ranking method using Euclidean distance between the incentre of centroids.The method split the plane figure of a triangular fuzzy number into three parts and calculated the centroid of each part followed by the incentre of centroids, there by finding the Euclidean distance.

2.4 Methods of Ranking Fuzzy Numbers Using Preference Function

Dubois and Prade (1983) presented a method of ranking fuzzy numbers in the setting of *possibility theory*, which seems to be the best in terms of *dominance*. The index here does not force any particular choice.

Botolan and Degani (1985) reviewed some of the methods for ranking fuzzy subsets. These methods include: (Jain, 1976), (Yager, 1981) and (Dubois and Prade, 1983).

Delgado *et al*. (1988) presented a procedure for ranking fuzzy numbers using fuzzy relations. Different fuzzy relations were stated in order to rank fuzzy numbers. The method evolved as one of the most powerful ranking method due to the use of relations.

(Yuan, 1991) presented a criterion for evaluating fuzzy ranking methods. Four criteria for evaluating fuzzy ranking methods were investigated: fuzzy preference representations, rationality of fuzzy orderings, distinguish abilities, and robustness of the method.

Kwang and Lee (1999) considered the *overall possibility distributions* of fuzzy numbers. Different distributions were considered in order to look at the possibility to rank fuzzy numbers satisfactorily.

Modarres and Nezhad (2001) developed a ranking method based on *preference function*. Each fuzzy number is measured point by point.At each point, the most preferred number is identified. This method was especially designed to evaluate alternatives in multi-criteria or multi-attribute decision making. The method is intuitive and can be used to discriminate between numbers easily due to point wise ranking.

2.5 Methods of Ranking Fuzzy Numbers Using Maximizing and Minimizing Sets

Raj and Kumar (1999) developed a ranking alternative with fuzzy weights using maximizing and minimizing sets. The maximum and minimum valus of both left and the right side of a fuzzy number were considered.

Chou *et al*. (2011) proposed a method for ranking fuzzy numbers using maximizing set and minimizing set. Unlike other similar methods, this method considered two left and two right utilities instead of one-one. It also takes into account the decision maker's optimistic attitude of fuzzy numbers.

Ezzati et al. (2012)presented an approach for ranking symmetric fuzzy numbers. The method is based on the new *magnitude* concepts to state the ranking order for both triangular and trapezoidal fuzzy numbers.

2.6 Method of Ranking Fuzzy Numbers Using Fuzzy Risk analysis

(Chen and Chen, 2009) presented a method for fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads.

This is similarto the method of Chen and Wang (2009) that studied fuzzy risk analysis based on ranking fuzzy numbers.

2.7 Methods of Ranking Fuzzy Numbers Using L-R Representation

Chen and Lu (2001) developed an approximate approach for ranking fuzzy numbers based on left and right dominance. The approach only required a few left and right spreads at some α-levels of fuzzy numbers to determine the respective dominance of a fuzzy number over the other. The total dominance is then determined by combining the left and right dominance based on a decision maker's optimistic perspectives. Such dominance is useful in ranking the fuzzy numbers when membership functions cannot be easily acquired.

Area ranking method of fuzzy numbers based on positive and negative ideal point was introduced by (Wang and Luo, 2009). The method considered both normal and non-normal fuzzy numbers. The positive and negative ideal points were used in ranking fuzzy numbers. Another method for ranking L-R fuzzy numbers based on *deviation degree* was contributed by (Wang et al., 2009) and (Asady, 2010).

Abbasbandy and Hajjari (2009) introduced an approach for ranking trapezoidal fuzzy numbers based on *the left and right spreads at some α-levels of trapezoidal fuzzy numbers*. The computation is shown to be easier and simpler than the other methods.

Kumar *et al*. (2011a) proposed a method for ranking generalized trapezoidal fuzzy numbers based on rank, mode, divergence and spread.The main advantage of the approach is that, it provides the correct ordering of both non-normal and normal trapezoidal fuzzy numbers.

Kumar *etal* (2011c) proposed a new approach for ranking L-R type generalized fuzzy numbers. The method considered the left and right sides of both triangular and trapezoidal fuzzy numbers.

Parandi *et al*.(2014) developed a method for ranking normal and non-normal fuzzy numbers. The method was based on left and right areas of triangular and trapezoidal fuzzy numbers.

2.8 Overview on Fuzzy Arithmetic Operation

Wang*etal*. (2005) developed fuzzy mathematical structure for fuzzy inference and cognitive computation. The mathematical models of fuzzy numbers and their algebraic properties were shown to be applicable to rigorous modeling of fuzzy entities.

Bede and Fodor (2006) developed the *product type* operations between fuzzy numbers and their application in geology. The multiplicative used here is based on Zadeh's extension principle and the triangular and trapezoidal approximation is studied and compared.

A computational method for fuzzy arithmetic operations was discussed by (AktherandAhmad 2009). The algorithm of the developed method with a numerical example is also provided. Using this method, four basic arithmetic operations between any two TFNs can be evaluated without complexity.

Another work on fuzzy arithmetic with parametric L-R fuzzy numbers was developed by (Stefanini andSorini, 2009). In this work, they suggested and described a new family of parametric representations for L-R fuzzy numbers and used them for fuzzy calculations and arithmetics.

Dutta et al. (2011) discussed a comparative study of a fuzzy arithmetic with and without using α-cut method. The method is general and simpler enough to deal with different types of fuzzy arithmetic, including exponentiation, extracting nth root, and taking logarithm.

After studying the existing literature of ranking methods of fuzzy numbers and their various shortcomings, this work intends to develop a new method of ranking fuzzy numbers that can compete with different existing approaches both in its simplicity and ease of application to risk analysis.

CHAPTER THREE

FUNDAMENTALS OF FUZZY NUMBERS

As metionedealier, all the fundamental notions described in this chapter are abstracted largely from (Chang, 1981),(Zimmermann, 1987)(Asady, 2010), (Lakashmana et al., 2011), (Kumar et al., 2011), (Rezvani, 2012), etc.

3.1 Fuzzy Number

The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X (Zadeh, 1965). This function can be generalized to a function $\mu_{\tilde{A}}$ such that the values assigned to the elements of the universal set X fall within a specified range i.e.,$\mu_{\tilde{A}} : X \longrightarrow [0,1]$. The assigned value indicates the membership grade of the element in the set $\tilde{A}$. The function $\mu_{\tilde{A}}$ is called the membership function and the set

$\tilde{A} = \{(x,\ \mu_{\tilde{A}}(x)) : x \in \mathrm{X}\}$defined by $\mu_{\tilde{A}}(x)$ for each $x \in \mathrm{X}$ is called a fuzzy set.

An interval defined on real number $\mathbb{R}$ is said to be a subset of $\mathbb{R}$.For instance an interval$\tilde{A} = [a\ ,c]$, $c \in \mathbb{R}$,for $a < c$, can be expressed by membership function as

$$\mu_{\tilde{A}}(\mathrm{x}) = \begin{cases} 0, & x < a \\ 1, & a \leq x \leq c \\ 0, & x > c \end{cases}$$

If $a = c$, then the interval indicates a point. That is, $[a, a]$= a as shown in figure 3.1 below.

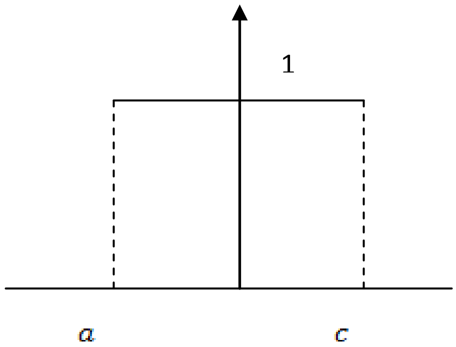
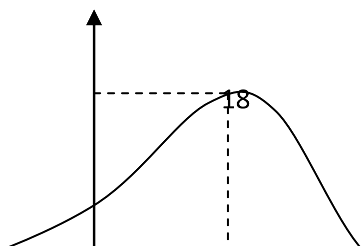


Fig 3.1 Interval $\tilde{A} = [a, c]$

The boundary of this interval is ambiguous since the interval is also a fuzzy set. Generally, a fuzzy interval is represented by two end points a and c and a peak point b as $[a, b, c]$. The α -cut operations can be also applied to the fuzzy numbers. If we denoteα -cut interval for fuzzy number $\tilde{A}$as $\tilde{A}_\alpha$, the obtained interval $\tilde{A}_\alpha$ is defined as $\tilde{A}_\alpha = [a^{(\alpha)}, c^{(\alpha)}]$. The intervals are clearly seen in figure 3.2 below.



A fuzzy number is defined on a fuzzy set which is

(i) Convex,

(ii) Normalized,

(iii) its membership function is piece-wise continuous, and

(iv) it is defined in the real number.

The condition of normalization implies that maximum membership value is 1, while the convex condition is that the line by α -cut is continuous and α -cut intervals satisfy the following relation:

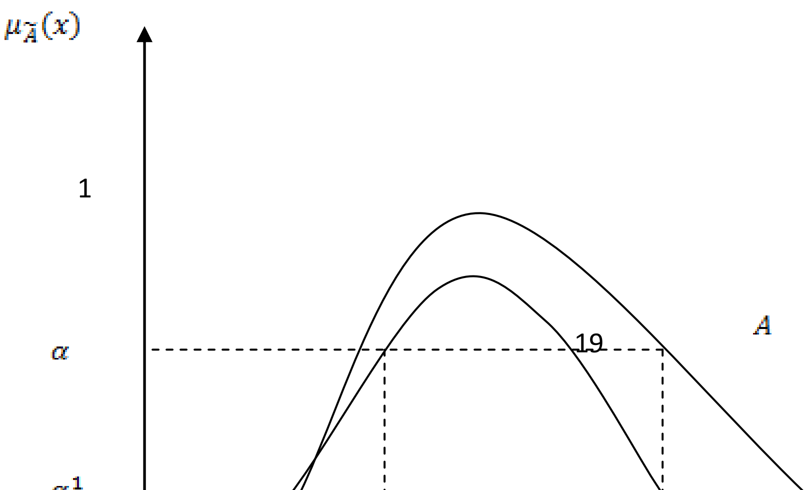
$$\tilde{A}_\alpha = [a^{(\alpha)}, c^{(\alpha)}]$$

$$(\alpha' < \alpha) \Rightarrow a^{(\alpha')} \leq a^{(\alpha)},\ c^{(\alpha')} \geq c^{(\alpha)}.$$

The convex condition may also be written as

$$(\alpha' < \alpha) \Rightarrow (\tilde{A}_\alpha \subset \tilde{A}_{\alpha'}).$$

The pictorial structure of α-cut fuzzy number is as shown in figure 3.3



$a^{(0)} a^{\alpha^1} a^{\alpha} c^{\alpha} c^{0} c^{\alpha^1}$

$\widetilde{A}_{\alpha} = [a^{\alpha}, c^{\alpha}]$

$\widetilde{A}_{\alpha^1} = [a^{(\alpha^1)}, \quad c^{(\alpha^1)}]$

Fig 3.3 Anα-cut of a fuzzy number (Kwang, 2002)

3.2 Representation of GeneralizedTriangular Fuzzy Numbers(TFN)

α- cut of a Triangular Fuzzy Number (TFN) which defines a set of closed intervals is given as

$[(b-a)\alpha + a, (b-c)\alpha + c], \ \forall\alpha \in [0,1], \ a, b, c \in R$(3.1)

(Ronald and Robert, 1997)

The standard arithmetic operations and their definitions based on the triplet on fuzzy sets $\tilde{A}$ and $\tilde{B}$areas follows:

Let $\tilde{A} = (a_1, b_1, c_1)and\tilde{B} = (a_2, b_2, c_2)$, where$a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$, therefore,

$$\tilde{A} \oplus \tilde{B} = (a_1 + a_2, \quad b_1 + b_2, \quad c_1 + c_2)$$

$$\tilde{A} \ominus \tilde{B} = (a_1 - c_2, \quad b_1 - b_2, \quad c_1 - a_2)$$

$$\tilde{A} \otimes \tilde{B} = (a_1 . a_2, \quad b_1 . b_2, \quad c_1 . c_2)$$

$$\tilde{A} \oslash \tilde{B} = \left(\frac{a_1}{c_2}, \quad \frac{b_1}{b_2}, \quad \frac{c_1}{a_2}\right)$$

These are binary operations on real numbers. A binary operation (*) in $\mathbb{R}$ is called increasing if for $x_1 > y_1$ and $x_2 > y_2$, then $x_1 * x_2 > y_1 * y_2$, and is called decreasing if $x_1 * x_2 < y_1 * y_2$.(Zimmermann, 1996).

Remark 3.1

i. If $\tilde{A}$ and $\tilde{B}$ are fuzzy numbers, then for any binary increasing (decreasing) function (*),$\tilde{A} * \tilde{B}$ is a fuzzy number.

ii. Given $\tilde{A} = (a_1, b_1, c_1)$ and $\tilde{B} = (a_2, b_2, c_2)$,

$\tilde{C} = \tilde{A} \otimes \tilde{B} = (a_1 a_2,\ b_1 b_2,\ c_1 c_2)$.

iii. The standard operators $\oplus$ and $\otimes$ are binary increasing operators, while the operators $\ominus$ and $\oslash$ are neither strictly increasing nor decreasing.

The above expression of multiplication can be referred to as *standard approximation*, the actual result is found by rewriting the membership function to define a set of closed intervals as in expression (3.1), then the expressions defining the closed intervals are operated on using interval arithmetic.

Now, for any two TFNs we have

$$\tilde{A} = (a_1, b_1, c_1) = [(b_1 - a_1)\alpha + a_1,\ \ (b_1 - c_1)\alpha + c_1]$$

$$\tilde{B} = (a_2, b_2, c_2) = [(b_2 - a_2)\alpha + a_2,\ \ (b_2 - c_2)\alpha + c_2]. \tag{3.2}$$

Also, from (3.2) the product can be calculated as

$\tilde{C} = \tilde{A} \otimes \tilde{B} = \big[\big((b_1 - a_1)\alpha + a_1\big) \times (b_2 - a_2)\alpha + a_2\big), (b_1 - c_1)\alpha + c_1 \times$

$(b_2 - c_2)\alpha + c_2)]$.

That is,

$$\tilde{C} = [(b_1 - a_1)(b_2 - a_2)\alpha^2 + (b_2 - a_2)a_1\alpha + (b_1 - a_1)a_2\alpha + a_1 a_2\ ,\ \ (b_1 - c_1)$$

$$(b_2 - c_2)\alpha^2 + (b_1 - c_1)c_2\alpha + (b_2 - c_2)c_1\alpha + c_1c_2]. \quad (3.3)$$

3.2.1Operations of Generalized Triangular Fuzzy Numbers and Their Properties

This is the process of generating a triangular fuzzy number with depicting three point representations from approximated value of α- cut operation.(Zimmermann, 1996)

Example 3.1

Let's consider the α- cut of two fuzzy numbers $\tilde{A} = (1,2,4)$ and

$\tilde{B} = (2,4,6)$. We have

$$\tilde{A}_\alpha = [(2-1)\alpha + 1, \quad -(4-2)\alpha + 4] = [\alpha + 1, \quad -2\alpha + 4],$$

$$\tilde{B}_\alpha = [(4-2)\alpha + 2, -(6-4)\alpha + 6] = [2\alpha + 2, -2\alpha + 6].$$

Now,$\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ are two crisp intervals and the multiplication of $\tilde{A}_\alpha$ with$\tilde{B}_\alpha$, $\alpha \in [0,1]$,can be performed as follows:

$$\tilde{A}_\alpha \otimes \tilde{B}_\alpha = [\alpha + 1, -2\alpha + 4] \otimes [2\alpha + 2, -2\alpha + 6]$$

$$= [(\alpha + 1)(2\alpha + 2), \quad (-2\alpha + 4)(-2\alpha + 6)]$$

$$= [2\alpha^2 + 4\alpha + 2, \quad 4\alpha^2 - 20\alpha + 24].$$

When $\alpha = 0$, $\tilde{A}_0 \otimes \tilde{B}_0 = [2, 24] \quad \forall \alpha \in [0,1]$.

When $\alpha = 1$, $\tilde{A}_1 \otimes \tilde{B}_1 = [2 + 4 + 2, \quad 4 - 20 + 24] = [8,8] = 8, \quad \forall \alpha \in [0,1]$.

The results obtained can be expressed as a triangular fuzzy number which is an approximation of$\tilde{A} \otimes \tilde{B} \cong (2,8,24)$.

3.3 Representation of Generalized Trapezoidal Fuzzy Numbers (GTFN)

Generalized trapezoidal fuzzy numbers are represented by a quadriple$(a, b, c, d; w)$. The value of w is the distinguishing factor between normal and non-normal generalized trapezoidal fuzzy numbers i.e., $w = 1$ for normalized trapezoidal fuzzy numbers and w <1 for non-normalized trapezoidal fuzzy numbers.

3.3.1 Operations on Generalized Trapezoidal Fuzzy Numbers (GTFN) and Their Properties

If $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ are any two generalized trapezoidal Fuzzy numbers, then

(i) $\tilde{A}(+)\tilde{B} = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2; \min(w_1, w_2))$

(ii) $\tilde{A}(-)\tilde{B} = (a_1 - d_2, b_1 - c_2, c_1 - b_2, d_1 - a_2; \min(w_1, w_2))$

(iii) $K\tilde{A} = (ka_1, kb_1, kc_1, kd_1; w_1); k > 0$

(iv) $K\tilde{A} = (kd_1, kc_1, kb_1, ka_1; w_1); k < 0.$

For any two generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$, we have the following:

(a) Closure property:

(i) $\tilde{A}(+)\tilde{B}$ is also a generalized trapezoidal fuzzy number

(ii) $\tilde{A}(-)\tilde{B}$ is also a generalized trapezoidal fuzzy number

(iii) $\tilde{A} \otimes \tilde{B}$, $\tilde{A}(/)\tilde{B}$ and inverses are not closed, in general

(iv) Minimum and maximum of fuzzy numbers are not closed in general

(b) Approximation of trapezoidal shape

In many cases, the results obtained from multiplication or division operations are approximated trapezoidal shape.

3.4 Operations of fuzzy numbers using Interval

The operations of fuzzy numbers can be generalized from that of crisp intervals.

Suppose $\tilde{A} = [a_1,\ a_2]$ and $\tilde{B} = [b_1,\ b_2]$, $\forall\, a_1, a_2, b_1, b_2 \in \mathbb{R}$ are numbers expressed as intervals, then the following are some main operations:

(i) Addition

$$[a_1, a_2](+)[b_1, b_2] = [a_1 + b_1\ , a_2 + b_2].$$

(ii) Subtraction

$$[a_1, a_2](-)[b_1, b_2] = [a_1 - b_2 \,, a_2 - b_1].$$

(iii) Multiplication

$$[a_1, a_2](\cdot)[b_1, b_2] = [(a_1 \cdot b_1) \wedge (a_1 \cdot b_2) \wedge (a_2 \cdot b_1) \wedge (a_2 \cdot b_2) \,,$$

$$(a_1 \cdot b_1) \vee (a_1 \cdot b_2) \vee (a_2 \cdot b_1) \vee (a_2 \cdot b_2)].$$

(iv) Division

$$[a_1, a_2](/)[b_1, b_2] = [(a_1/b_1) \wedge (a_1/b_2) \wedge (a_2/b_1) \wedge (a_2/b_2),$$

$$(a_1/b_1) \vee (a_1/b_2) \vee (a_2/b_1) \vee (a_2/b_2)]$$

where$b_1 \neq 0$ and $b_2 \neq 0$

(v) Inverse intervals;

$$[a_1, a_2]^{-1} = [\frac{1}{a_1} \wedge \frac{1}{a_2}, \quad \frac{1}{a_1} \vee \frac{1}{a_2}]$$

where$a_1 \neq 0$ and $a_2 \neq 0$.(Kwang, 2002)

Some additional and alternative definitions of operations using intervals are as follows:

(i) Multiplication

$$[a_1, a_2](\cdot)[b_1, b_2] = [a_1 \cdot b_1 \,, \quad a_2 \cdot b_2]$$

(ii) Division

$[a_1, a_2](/)[b_1, b_2] = [a_1/b_2, \; a_2/b_1 \,]$,Where $b_1 \neq 0$ and $b_2 \neq 0$

(iii) Inverse intervals

$[a_1, a_2]^{-1} = [{}^{1}/_{a_2}, \quad {}^{1}/_{a_1}]$, where$a_1 \neq 0$ and $a_2 \neq 0$.

(iv) Minimum

$$[a_1, a_2](\wedge)[b_1, b_2] = [a_1 \wedge b_1, \quad a_2 \wedge b_2]$$

(v) Maximum

$$[a_1, a_2](\wedge)[b_1, b_2] = [a_1 \vee b_1, \quad a_2 \vee b_2]$$

Example 3.2

If $\tilde{A}$ and $\tilde{B}$ are two intervals, $\tilde{A} = [3,\ 5], \tilde{B} = [-2,\ 7]$. Then the following hold:

$$\tilde{A}(+)\tilde{B} = [3-2, \qquad 5+7] = [1, 12].$$

$$\tilde{A}(-)\tilde{B} = [3-7, \qquad 5-(-2)] = [-4,7].$$

$$\tilde{A}(\cdot)\tilde{B} =$$

$$[(3\cdot(-2) \wedge (3\cdot 7) \wedge (5\cdot(-2)) \wedge (5\cdot 7), (3\cdot(-2)) \vee (3\cdot 7) \vee (5\cdot(-2)) \vee (5.7)]$$

$$= [-10, 35].$$

$$\tilde{A}(/)\tilde{B} = [(3/(-2)) \wedge (3/7) \wedge (5/(-2)) \wedge (5/7),$$

$$(3/(-2)) \vee (3/7) \vee (5/(-2)) \vee (5/7)] = [-2.5, 5/7].$$

$$\tilde{B}^{-1} = [-2,7]^{-1} = \left[\frac{1}{(-2)} \wedge \frac{1}{7}, \quad \frac{1}{(-2)} \vee \frac{1}{7}\right] = \left[\frac{-1}{2}, \qquad \frac{1}{7}\right].$$

3.5 Operations of Fuzzy Numbers Using $\alpha - cut$ Intervals

We refer to $\alpha - cut$ interval of afuzzy number $\tilde{A} = [a_1, a_2]$ as acrisp set

$$\tilde{A}_\alpha = \left[a_1^{(\alpha)}, a_2^{(\alpha)}\right], \forall\alpha \in [0,1], \qquad a_1, a_2, a_1^{(\alpha)}, a_2^{(\alpha)} \in \mathbb{R}.$$

Similarly, $\tilde{B}_\alpha = \left[b_1^{(\alpha)}, b_2^{(\alpha)}\right]$, $\forall\alpha \in [0,1]$, $b_1, b_2, b_1^{(\alpha)}, b_2^{(\alpha)} \in \mathbb{R}$ is an $\alpha - cut$ interval resulted from $\tilde{B} = [b_1, b_2]$, $b_1, b_2 \in \mathbb{R}$.

The operations between $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ can be expressedas follows:

$$\left[a_1^{(\alpha)}, a_2^{(\alpha)}\right](+)\left[b_1^{(\alpha)}, b_2^{(\alpha)}\right] = \left[a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}\right]$$

$$\left[a_1^{(\alpha)}, a_2^{(\alpha)}\right](-)[b_1^{(\alpha)}, b_2^{(\alpha)}\ ] = \ [a_1^{(\alpha)} - b_2^{(\alpha)},\ a_2^{(\alpha)} + b_1^{(\alpha)}\ ]$$

These operations can also be applicable to multiplication and division in the same manner.

3.6 Operations of Fuzzy Numbers Using Membership Function

Given $\tilde{A}$ and $\tilde{B}$ as fuzzy numbers, we have the following:

(i) Addition: $\tilde{A}(+)\tilde{B} = \mu_{\tilde{A}(+)\tilde{B}}(z) = \bigvee_{Z=x+y} (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y))$

(ii) Subtraction: $\tilde{A}(-)\tilde{B} = \mu_{\tilde{A}(-)\tilde{B}}(z) = \bigvee_{Z=x-y} (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y))$

(iii) Multiplication: $\tilde{A}(\cdot)\tilde{B} = \mu_{\tilde{A}(\cdot)\tilde{B}}(z) = \bigvee_{Z=x\cdot y} (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y))$

(iv) Division: $\tilde{A}(/)\tilde{B} = \mu_{\tilde{A}(/)\tilde{B}}(z) = \bigvee_{Z=x/y} (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y))$

(v) Minimum: $\tilde{A}(\wedge)\tilde{B} = \mu_{\tilde{A}(\wedge)\tilde{B}}(z) = \bigvee_{Z=x\wedge y} (\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y))$

(vi) Maximum: $\tilde{A}(\vee)\tilde{B} = \mu_{\tilde{A}(\vee)\tilde{B}}(z) = \bigvee_{Z=x\vee y} \big(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)\big)$.

$$\forall x, y, z \in \mathbb{R}.$$

Example 3.3

Let $\tilde{A} = \{(2,1), (3, 0.5)\}$, $\tilde{B} = \{(3,1), (4, 0.5)\}$ be two fuzzy numbers and let $x \in \tilde{A}$, $y \in \tilde{B}$, $z \in (\tilde{A}(+)\tilde{B})$. Then

For $z < 5$, $\mu_{\tilde{A}(+)\tilde{B}}(z) = 0$

(i) $z = 5$, the result from $x + y = 2 + 3$

$$\mu_{\tilde{A}}(2) \wedge \mu_{\tilde{B}}(3) = 1 \wedge 1 = 1$$

$$\mu_{\tilde{A}(+)\tilde{B}}(5) = \bigvee_{5=2+3} (1) = 1$$

(ii) $z = 6$, the results from $x + y = 3 + 3$ *or* $x + y = 2 + 4$

$$\mu_{\tilde{A}}(3) \wedge \mu_{\tilde{B}}(3) = 0.5 \wedge 1 = 0.5$$

$$\mu_{\tilde{A}}(2) \wedge \mu_{\tilde{B}}(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_{\tilde{A}(+)\tilde{B}}(6) = \bigvee_{\substack{6=3+3\\6=2+4}} (0.5, 0.5) = 0.5$$

(iii) $z = 7$, results from $x + y = 3 + 4$

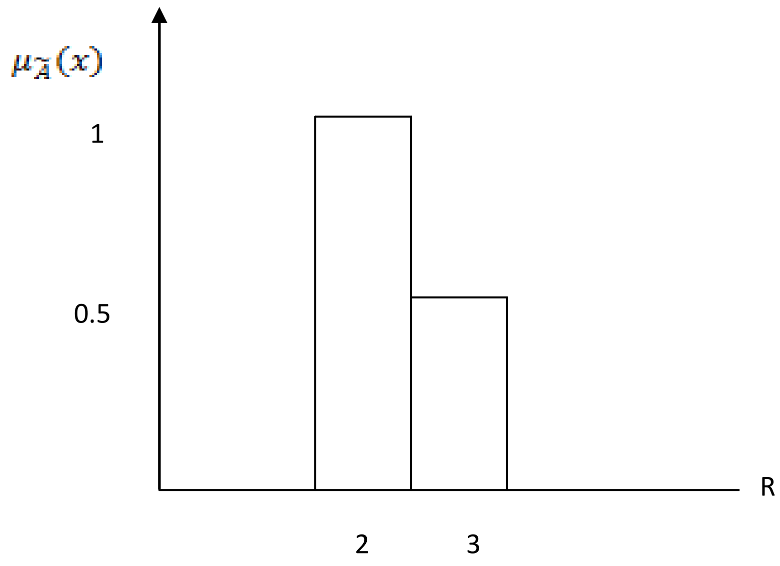
$$\mu_{\tilde{A}}(3) \wedge \mu_{\tilde{B}}(4) = 0.5 \wedge 0.5 = 0.5$$

$$\mu_{\tilde{A}(+)\tilde{B}}(7) = \bigvee_{7=4+4} (0.5) = 0.5$$

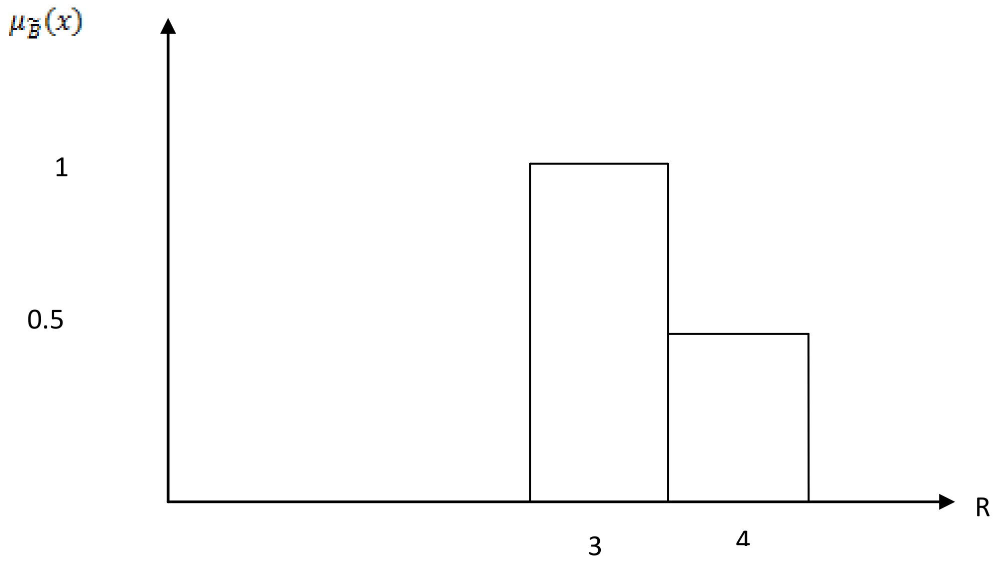
(iv) For $z > 7, \mu_{\tilde{A}(+)\tilde{B}}(z) = 0$

$\tilde{A}(+)\tilde{B} = \{(5,1),(6,0.5),(7,0.5)\}$.

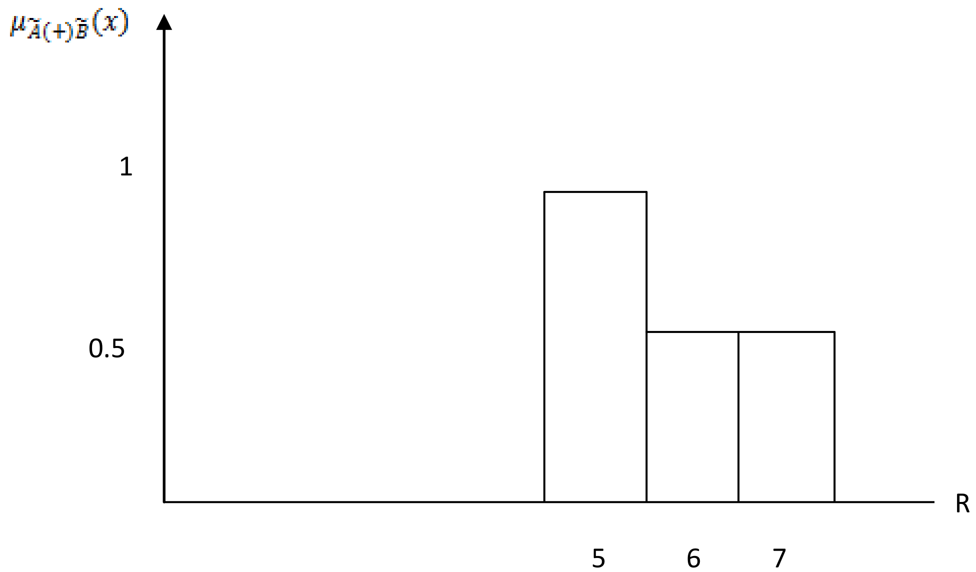
Hence, $\tilde{A}(+)\tilde{B}$ can also be represented diagrammatically as follows:



(a) Fig 3.5 Fuzzy set $\tilde{A}$



(b) Fig. 3.6 Fuzzy Number $\tilde{B}$



(c) Fig. 3.7 Fuzzy set $\tilde{A}(+)\tilde{B}$

Example 3.4

Let $\tilde{A}$ and $\tilde{B}$ of example (3.3) and let $x \in \tilde{A},\ y \in \tilde{B}, z \in \left(\tilde{A}(-)\tilde{B}\right)$.The$\tilde{A}(-)\tilde{B}$ is definedas follows:

(i) For $z < -2,\ \mu_{\tilde{A}(-)\tilde{B}}(z) = 0$

(ii) $z = -2,$ results from $x - y = 2 - 4$

$$\mu_{\tilde{A}}(2) \wedge \mu_{\tilde{B}}(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_{\tilde{A}(-)\tilde{B}}(-2) = 0.5$$

(iii) $z = -1,$ results from $x - y = 2 - 3\ or\ x - y = 3 - 4$

$$\mu_{\tilde{A}}(2) \wedge \mu_{\tilde{B}}(3) = 1 \wedge 1 = 1$$

$$\mu_{\tilde{A}}(3) \wedge \mu_{\tilde{B}}(4) = 0.5 \wedge 0.5 = 0.5$$

$$\mu_{\tilde{A}(-)\tilde{B}}(-1) = \bigvee_{\substack{-1 = 2-3 \\ -1 = 3-4}} (1, 0.5) = 1$$

(iv) $z = 0$,results from $x - y = 3 - 3$

$$\mu_{\tilde{A}}(3) \wedge \mu_{\tilde{B}}(3) = 0.5 \wedge 1 = 0.5$$

$$\mu_{\tilde{A}(-)\tilde{B}}(0) = 0.5$$

(v) For $z \geq 1,\ \mu_{\tilde{A}(-)\tilde{B}}(z) = 0$

Hence, $\tilde{A}(-)\tilde{B} = \{(-2, 0.5), (-1, 1), (0, 0.5)\}$. Schematically, the following:

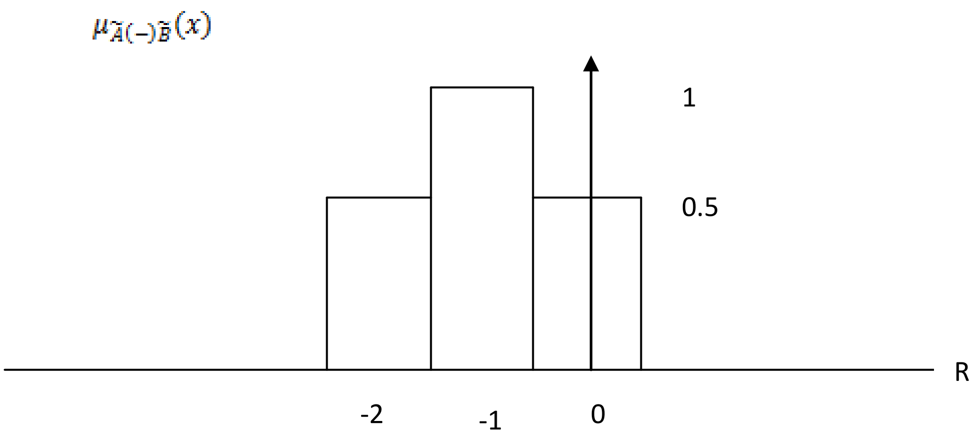


Fig. 3.8 Fuzzy Number $\tilde{A}$ (-)$\tilde{B}$

Example 3.5

Let$\tilde{A}$ and $\tilde{B}$ of example (3.3), and let$x \in \tilde{A},\ y \in \tilde{B}, z \in (\tilde{A}(\vee)\tilde{B})$. Then$(\tilde{A}(\vee)\tilde{B})$ is defined by $\mu_{\tilde{A}(\vee)\tilde{B}}(z)$ as follows:

(i) $z \leq 2, \mu_{\tilde{A}(\vee)\tilde{B}}(z) = 0$

(ii) $z = 3$ from $x \vee y = 2 \vee 3$ and $x \vee y = 3 \vee 3$

$$\mu_{\tilde{A}}(2) \wedge \mu_{\tilde{B}}(3) = 1 \wedge 1 = 1$$

$$\mu_{\tilde{A}}(3) \wedge \mu_{\tilde{B}}(3) = 0.5 \wedge 1 = 0.5$$

$$\mu_{\tilde{A}(\vee)\tilde{B}}(3) = \bigvee_{\substack{3 = 2\vee 3 \\ 3 = 3\vee 3}} (1, 0.5) = 1$$

(iii) $z = 4,$ from $x \vee y = 2 \vee 4$ *and* $x \vee y = 3 \vee 4$

$$\mu_{\tilde{A}}(2) \wedge \mu_{\tilde{B}}(4) = 1 \wedge 0.5 = 0.5$$

$$\mu_{\tilde{A}}(3) \wedge \mu_{\tilde{B}}(4) = 0.5 \wedge 0.5 = 0.5$$

$$\mu_{\tilde{A}(\vee)\tilde{B}}(4) = \bigvee_{\substack{4 = 2\vee 4 \\ 4 = 3\vee 4}} (0.5, 0.5) = 0.5$$

(iv) $z > 5$ impossible $\mu_{\tilde{A}(\vee)\tilde{B}}(z) = 0,$ and $\tilde{A}(\vee)\tilde{B}$ is defined to be

$$\tilde{A}(\vee)\tilde{B} = \{(3,1), (4, 0.5)\}$$

3.7 The Extension Principle (EP)

Let$\tilde{A}$ and $\tilde{B}$ be fuzzy numbers in $\mathbb{R}$ and let $(*)$ be a binary operation defined in R, then the operation $(*)$ can be extended to the fuzzy numbers as follows:

$$\tilde{A} * \tilde{B} = \int_{for\ x, y \in \mathbb{R}.} (\mu_{\tilde{A}}(x) \wedge (\mu_{\tilde{A}}(y) / x * y) \tag{3.5}$$

where$(*)$can be any of$\{+,\ -,\ \times$ and $\div\}$.(Mizumoto and Tanaka, 1979)

Thus, EP can be also used to define operations of fuzzy numbers.

3.8 Convex and Normal Fuzzy Numbers

A fuzzy number $\tilde{A}$ in $\mathbb{R}$ is said to be *convex* if for any real number $x, y, z \in \mathbb{R}$ with $x \le y \le z$; $\mu_{\tilde{A}}(y) \ge \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{A}}(z)$where$\wedge$ stands for *min.*

Also, a fuzzy number $\tilde{A}$ is called *normal* if

$$\max_{x} \mu_{\tilde{A}}(x) = 1.$$

A fuzzy number which is normal and convex is referred to as a normal convex fuzzy number. (Mizumoto and Tanaka, 1979)

3.8.1$\alpha -$ Level Sets of a Fuzzy Number

An α-level set of a fuzzy number $\tilde{A}$ is a *non- fuzzy* set, denoted by $\tilde{A}_{\alpha}$, is defined by

$$\tilde{A}_{\alpha} = \{x \setminus \mu_{\tilde{A}}(x) \ge \alpha\}, \quad 0 < \alpha \le 1$$

It follows that, $\alpha_1 \le \alpha_2 \Rightarrow \tilde{A}_{\alpha_1} \supseteq \tilde{A}_{\alpha_2}$.

The following equivalence hold:

(i) We can obtain$\tilde{A} = \tilde{B}$ for any α using α-level sets, and given two equal α-level sets, we can obtain $\tilde{A} = \widetilde{B}$.

(ii) Let a fuzzy number $\tilde{A}$ be convex, then$\tilde{A}_{\alpha}$ is a convex set (or an *interval*) in $\mathbb{R}$ and vice versa.

(iii) A fuzzy number $\tilde{A}$ may bedecomposed into its level-sets through the resolutionidentity.$\tilde{A} =$

$$\int_0^1 \alpha \tilde{A}_{\alpha} \tag{3.6}$$

where$\alpha\tilde{A}_{\alpha}$ is the product of a scalar α with the set $\tilde{A}_{\alpha}$ and $\int$ is the union of $\tilde{A}_{\alpha}'s$ with α ranging from 0 to 1.

3.8.2 Support of a Fuzzy Number

The support $\Gamma_{\tilde{A}}$ of a fuzzy number $\tilde{A}$ is defined as a special case of level-set by the following definition

$$\Gamma_{\tilde{A}} = \{x \backslash\, \mu_{\tilde{A}}(x) > 0\}.$$

3.8.3 Operations of Fuzzy Numbersusing α–Level-sets

Let $\tilde{A}_\alpha$ and $\tilde{B}_\alpha$ be α-level sets of a convex fuzzy numbers $\tilde{A}$and $\tilde{B}$, respectively.Then theα-level sets are intervals in $\mathbb{R}$, which are special convex fuzzy numbers whose grades are unity at x belonging to $\tilde{A}_\alpha$ and zero elsewhere.

In other words, the α-level set $(\tilde{A}+\tilde{B})_\alpha$ is the sum of α-level sets $\tilde{A}_\alpha$and$\tilde{B}_\alpha$. By using the resolution identity (3.6), we can express $\tilde{A}+\tilde{B}$ as

$$\tilde{A}+\tilde{B} = \int_0^1 \alpha\, (\tilde{A}+\tilde{B})_\alpha = \int_0^1 \alpha\, (\tilde{A}_\alpha + \tilde{B}_\alpha)$$

Similarly, we can obtain

$$\tilde{A}-\tilde{B} = \int_0^1 \alpha \left(\tilde{A}-\tilde{B}\right)_\alpha = \int_0^1 \alpha \left(\tilde{A}_\alpha - \tilde{B}_\alpha\right),$$

$\tilde{A}\times\tilde{B} = \int_0^1 \alpha\, (\tilde{A}\times\tilde{B})_\alpha = \int_0^1 \alpha\left(\tilde{A}_\alpha \times \tilde{B}_\alpha\right)$,and

$\tilde{A}\div\tilde{B} = \int_0^1 \alpha\, (\tilde{A}\div\tilde{B})_\alpha = \int_0^1 \alpha\, (\tilde{A}_\alpha \div \tilde{B}_\alpha)$.

3.9 Algebraic Properties of Fuzzy Numbers

This section discusses the Algebraic properties of fuzzy numbers under the operations ($+$, $-$, $\times$and $\div$).

Theorem 3.2:

For any fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$, we have the following properties:

(i) Associative laws

$$(\tilde{A} + \tilde{B}) + \tilde{C} = \tilde{A} + (\tilde{B} + \tilde{C})$$

$$(\tilde{A} \times \tilde{B}) \times \tilde{C} = \tilde{A} \times (\tilde{B} \times \tilde{C})$$

(ii) commutative laws

$$\tilde{A} + \tilde{B} = \tilde{B} + \tilde{A}$$

$$\tilde{A} \times \tilde{B} = \tilde{B} \times \tilde{A}$$

(iii) identity laws

$$\tilde{A} + 0 = \tilde{A}$$

$$\tilde{A} \times 1 = \tilde{A}$$

where 0 and 1 are zero and unity, respectively, in the ordinary sense.

Remark 3.2

(i) We note that if $\tilde{A}$ is reduced to a real number, $-\tilde{A}$ and $1/\tilde{A}$ are the inverse of $\tilde{A}$ under $+$ and $\times$, respectively.

(ii) For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, if one is convex and the other one is non-convex, then the execution result of $\tilde{A}$ and $\tilde{B}$ under $+, -, \times$ and $\div$ may be convex or non-convex.

Theorem 3.3

If $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are positive convex fuzzy numbers, then

$\tilde{A} \times (\tilde{B} + \tilde{C}) = (\tilde{A} \times \tilde{B}) + (\tilde{A} \times \tilde{C})$ (distributive law).

(Mizumoto and Tanaka, 1979)

Proof

Let α-level sets of positive convex fuzzy numbers $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ be $\tilde{A}_\alpha = [a_1, a_2]$, $\tilde{B}_\alpha = [b_1, b_2]$ and $\tilde{C}_\alpha = [c_1, c_2]$, respectively. In turn each level set is an interval in $\mathbb{R}$ and $0 < a_1 \le a_2$, $0 < b_1 \le b_2$ and $0 < c_1 \le c_2$.

Therefore, for each $0 < \alpha \le 1$,

$$[\tilde{A} \times (\tilde{B} + \tilde{C})]_\alpha = \tilde{A}_\alpha \times (\tilde{B}_\alpha + \tilde{C}_\alpha)$$

$$= [a_1, a_2] \times ([b_1, b_2] + [c_1, c_2])$$

$$= [a_1, a_2] \times ([b_1 + c_1, \quad b_2 + c_2])$$

$$= [a_1(b_1 + c_1), a_2(b_2 + c_2)] \ .$$

The right hand member will be

$$[(\tilde{A} \times \tilde{B}) + (\tilde{A} \times \tilde{C})]_\alpha = \left(\tilde{A}_\alpha \times \tilde{B}_\alpha\right) + (\tilde{A}_\alpha \times \tilde{C}_\alpha)$$

$$= ([a_1, a_2] \times [b_1, b_2]) + ([a_1, a_2] \times [c_1, c_2])$$

$$= [a_1 b_1, a_2 b_2] + [a_1 c_1, \quad a_2 c_2] \ .$$

$$= [a_1 b_1 + a_1 c_1 \, , a_2 b_2 + a_2 c_2]$$

$$= [a_1(b_1 + c_1) \, , \ a_2(b_2 + c_2)]$$

$$= [\tilde{A} \times (\tilde{B} + \tilde{C})]_\alpha.$$

Thus, using the resolution identity (3.6), we can obtain

$$\tilde{A} \times \left(\tilde{B} + \tilde{C}\right) = \left(\tilde{A} \times \tilde{B}\right) + (\tilde{A} \times \tilde{C})$$

Remark 3.5: When α-level set is an empty set Φ, the following holds.

$$\tilde{A}_\alpha + \Phi = \Phi,$$

$$\tilde{A}_\alpha \times \Phi = \Phi.$$

CHAPTER FOUR

METHODS OF RANKING FUZZY NUMBERS

In this chapter, we describe various ranking methods of fuzzy numbers that have appeared in the literature.

4.1 Liouand Wang's Ranking Approach

Liou and Wang(1992) proposed an approach for ranking fuzzy numbers with an *index of optimism*$\alpha \in [0,1]$ by combining the left and right integral values.

Vincent and Dat, (2014) defined the left and right integral values of fuzzy number $\tilde{A}$ as:

$I_L(\tilde{A}) = \int_0^1 g_{\tilde{A}}^L(y)dyandI_R(\tilde{A}) = \int_0^1 g_{\tilde{A}}^R(y)dy.$ (4.1)

Liou and Wang also defined the total integral value with an index of optimism $\alpha \in [0,1]$ as

$$I_T^\alpha(\tilde{A}) = \alpha I_R(\tilde{A}) + (1-\alpha)I_L(\tilde{A})$$

$$= \alpha \int_0^1 g_{\tilde{A}}^R(y)dy + (1-\alpha)\int_0^1 g_{\tilde{A}}^L(y)dy \quad (4.2)$$

where$g_{\tilde{A}}^L(y)$ and $g_{\tilde{A}}^R(y)$ are respectively the inverse function of $f_{\tilde{A}}^L(x)$ and $f_{\tilde{A}}^R(x)$. The index of optimism α represents the degree of optimism of a decision maker. A larger α indicates a higher degree of optimism. For $\alpha = 0$ and$\alpha = 1$, the values of $I_T^\alpha(\tilde{A})$ represent the view point of pessimistic and optimistic decision makers, respectively. For a moderate decision maker, with$\alpha = 0.5$, the total integral value of each fuzzy number$\tilde{A}$ becomes$I_T^{0.5}(\tilde{A}) = \left(\frac{1}{2}\right)\left[I_R(\tilde{A}) + I_L(\tilde{A})\right]$. The greater is $I_T^\alpha(\tilde{A})$, the bigger the fuzzy number $\tilde{A}_i$ and the higher its ranking order.

Vincent and Dat(2014) observed that (Liou and Wang, 1992) had some short comings. They noted, in particular, that:

(i) Examples given by Liou and Wang in their ranking approach do not satisfy the reasonable properties for the ordering of fuzzy numbers.

(ii)Liou and Wang's method cannot differentiate normal and non- normal triangular and trapezoidal fuzzy numbers.

(iii)For the case of trapezoidal and triangular fuzzy numbers, Liou and Wang proved that the total integral values of normal and non-normal fuzzy numbers are the same.

The following example shows failure of Liou and Wang's method.

Example 4.1

Consider the triangular fuzzy numbers $\tilde{A} = (3,5,7;1)$ and $\tilde{B} = (3,5,7;0.8)$ as shown below:

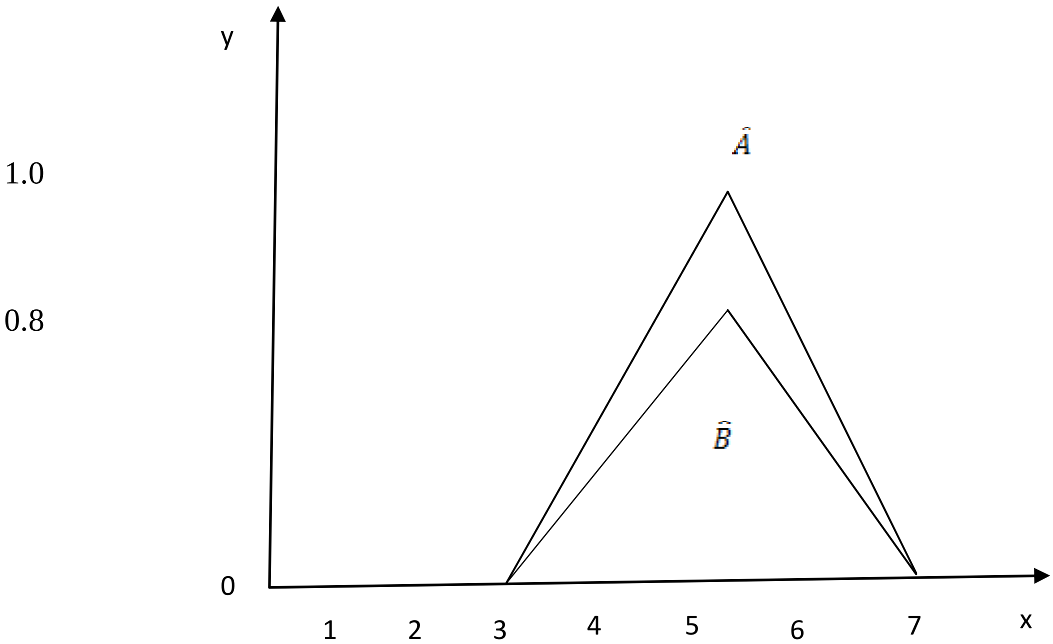


Figure 4.1 A fuzzy number $\tilde{A}$ and $\tilde{B}$ in example (4.1)

Intuitively,$\tilde{A} > \tilde{B}$. However, usingLiou and Wang's method, the total integral values of the triangular fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are, respectively,$I_T^\alpha(\tilde{A}) = 4 + 2\alpha$and $I_T^\alpha(\tilde{B}) = 4 + 2\alpha$.Thus $\tilde{A}$ and $\tilde{B}$ have the same ranking order for every $\alpha \in [0,1]$.

In continuation, (Garcia andLamata, 2007) showed that fuzzy numbers having the same compensation areas cannot differentiate by Liou and Wang's method. For example, fuzzy numbers $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2;\ w_2)$ with $I_L(\tilde{A}) = I_L(\tilde{B})$ and $I_R(\tilde{A}) = I_R(\tilde{B})$ cannotbe differentiated using Liou and Wang' method.In this regard, (Vincent andDat, 2014) proposed a revised ranking method based on the novel integral values and median value of fuzzy numbers as follows:

Definition 4.1 Suppose there are n fuzzy numbers $\tilde{A}_i,\ i = 1,2, \ldots n$, each with the left membership function $f^L_{\tilde{A}_i}$ and the right membership function $f^R_{\tilde{A}_i}$.The novel left and right integral values of $\tilde{A}_i$ are defined by

$$S_L(\tilde{A}_i) = w_i(b_i, X_{min}) - \int_{ai}^{bi} f^L_{\tilde{A}_i}(x)dx, \text{ and} S_R(\tilde{A}_i) = w_i(c_i, X_{min}) + \int_{ci}^{di} f^R_{\tilde{A}_i}(x)dx \tag{4.3}$$

where$X_{min} = infP, P = \cup_{i=1}^{n} p_i,\ p_i = \left\{ \frac{X}{f_{\tilde{A}_i}(x)} > 0 \right\},\ w_i = Sup_x f_{\tilde{A}_i}(x),$

both$S_L(\tilde{A}_i)$ and $S_R(\tilde{A}_i) \geq 0$.

Therefore, $\tilde{A}_i > \tilde{A}_k$ if and only if $S_L(\tilde{A}_i) > S_L(\tilde{A}_k)$ and $S_R(\tilde{A}_i) > S_R(\tilde{A}_k)$,

$\tilde{A}_i < \tilde{A}_k$if and only if $S_L(\tilde{A}_i) < S_L(\tilde{A}_k)$ and $S_R(\tilde{A}_i) < S_R(\tilde{A}_k)$, and

$\tilde{A}_i \sim \tilde{A}_k$if and only if $S_L(\tilde{A}_i) = S_L(\tilde{A}_k)$ and $S_R(\tilde{A}_i) = S_R(\tilde{A}_k)$ both hold.

The novel integral value with index of optimism $\alpha \in [0,1]$ is defined as

$$S_T^{\alpha}(\tilde{A}_i) = \alpha S_R(\tilde{A}_i) + (1 - \alpha) S_L(\tilde{A}_i). \tag{4.4}$$

Considering the normal trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; 1)$, for $\alpha \in [0,1]$, the total integral value can be obtained by

$$S_T^{\alpha}(\tilde{A}) = \left(\frac{1}{2}\right)[\alpha(c + d) + (1 - \alpha)(b + a) - 2X_{min}]. \tag{4.5}$$

Also, the total integral value of normal triangular fuzzy number $\tilde{A} = (a, b, d; 1)$ when$b = c$ is given by

$$S_T^{\alpha}(\tilde{A}) = \left(\frac{1}{2}\right)[\alpha d + b + (1 - \alpha) - 2X_{min}]. \tag{4.6}$$

When$X_{min} = 0$, both (4.5) and (4.6) are the same as in (Liou and Wang, 1992).

The proposed method uses $S_T^{\alpha}(\tilde{A}_i)$ to rank fuzzy numbers.Therefore, for any two distinct fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the following properties hold:

(i) if $S_T^{\alpha}(\tilde{A}) < S_T^{\alpha}(\tilde{B})$, then $\tilde{A} < \tilde{B}$,

(ii) if $S_T^{\alpha}(\tilde{A}) > S_T^{\alpha}(\tilde{B})$, then $\tilde{A} > \tilde{B}$, and

(iii) if $S_T^{\alpha}(\tilde{A}) = S_T^{\alpha}(\tilde{B})$, then $\tilde{A} \sim \tilde{B}$.

4.2 Chen's Maximizing Set and Minimizing Set Ranking Method

Chen (1985) proposed a method which employed the criteria of total utility through maximizing set and minimizing set to rank fuzzy numbers. The method is described as follows:

Definition 4.2: Suppose that $\tilde{A}_i$, $i = 1,2, \dots n$ are n fuzzy numbers, each with a triangular membership $f_{\tilde{A}_i}(x) = 0$.The*maximizing* set M is a fuzzy subset with membership function f_M given as

$$f_M(x) = \begin{cases} [(X - X_{min})/(X_{max} - X_{min})]^k, & X_{min} \leq X \leq X_{max} \\ 0, & otherwise. \end{cases} \tag{4.7}$$

The*minimizing* set G is a fuzzy subset with membership function f_G given as

$$f_G(x) = \begin{cases} [(X_{max} - X)/(X_{max} - X_{min})]^k, & X_{min} \leq X \leq X_{max} \\ 0, & otherwise. \end{cases} \tag{4.8}$$

where$X_{min} = inf S$, $S = \bigcup_{i=1}^{n} S_i$, $S_i = \{X / f_{\tilde{A}_i}(x) > 0\}$, $X_{max} = Sup\, S$, and k is a constant whose value can be varied to suit the application.

The *right utility* value of each alternative $\tilde{A}_i$ is defined as

$U_M(i) = Sup_X\left(f_M(x) \wedge f_{\tilde{A}_i}(x)\right),\ i = 1,2, \dots n.$(4.9)

The *left utility* value of each alternative $\tilde{A}_i$ is defined as

$U_G(i) = Sup_X\left(f_G(x) \wedge f_{\tilde{A}_i}(x)\right),\ i = 1,2, \dots n.$(4.10)

The *total utility* value or *ordering* value of each fuzzy number $\tilde{A}_i$ is defined as

$$U_T(i) = \frac{[U_M(i) + 1 - U_G(i)]}{2}\ , i = 1,2, \dots n. \quad (4.11)$$

The n fuzzy numbers $\tilde{A}_i$can, therefore, be ranked according to their total utilities.

In order words, the following holds:

(i) $\tilde{A}_i < \tilde{A}_j$ if and only if $U_T(\tilde{A}_i) < U_T(\tilde{A}_j)$,

(ii) $\tilde{A}_i > \tilde{A}_j$ if and only if $U_T(\tilde{A}_i) > U_T(\tilde{A}_j)$, and

(iii) $\tilde{A}_i \sim \tilde{A}_j$ if and only if $U_T(\tilde{A}_i) = U_T(\tilde{A}_j)$.

Chou et al.(2011) showed Chen's method of ranking fuzzy numbers with maximizing set and minimizing set have some shortcomings, by taking an example in (4.2) below,

Example 4.2: Consider the two normal triangular fuzzy numbers $\tilde{A} = (3,6,9;1)$ and $\tilde{B} = (5,6,7;1)$ as shown in the diagram below:

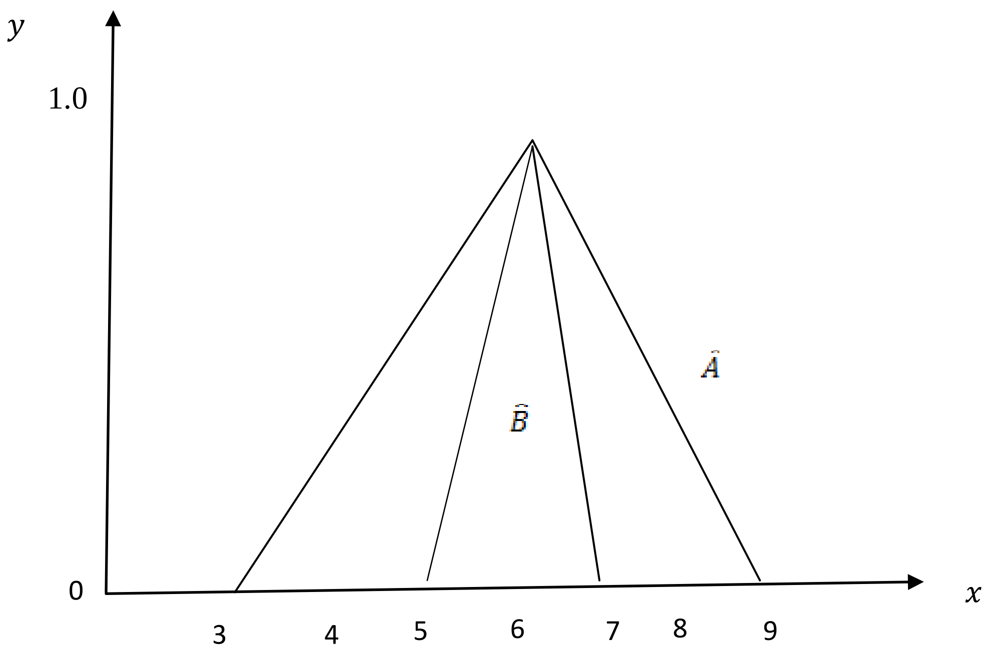


Figure 4.2 the fuzzy numbers $\tilde{A}$ and $\tilde{B}$ of example 4.2

Clearly,$\tilde{A}$ and $\tilde{B}$ both have the same total utilities 0.5 and, therefore, they cannot be distinguished by Chen's ranking method.

To overcome the short comings of Chen's ranking method,(Chou et al., 2011) proposed a ranking method that ranks fuzzy numbers by pair-wise comparison.The revised method considered not only the left and right utilities, but also the decision maker's optimistic attitude. This method uses the same maximizing set and minimizing set technique and defined the total utility value of each fuzzy number$\tilde{A}_i$ with index of optimism α which represents the degree of optimism of a decision makers as in(Liou and Wang, 1992),(Klim and Park, 1990), and (Wang and Luo, 2009). A larger α indicates a higher degree of optimism. More specifically, when $\alpha = 0$, the total utility value $U_T^0(\tilde{A}_i)$ representing a pessimistic decision maker's view point is equal to the total utility value of $\tilde{A}_i$. Conversely, for an optimistic decision maker i.e.,$\alpha = 1$, the total utility value $U_T^0(\tilde{A}_i)$ is equal to the total right utility value of$\tilde{A}_i$.

4.3 Chen and Tang's Methodfor Ranking Non- Normal Triangular and Trapezoidal Fuzzy Numbers

Chen and Tang (2008) pointed out the shortcomings of (Liou and Wang, 1992) and proposed their ownmethod of rankingthat worked on the basis of two non-normal trapezoidal fuzzy numbers with different heights. However, but they further proposed how to deal with non-normal trapezoidal fuzzy numbers with equal heights.

Proposition 4.3

Let $\tilde{A} = (a, b, c, d; w)$ and $\tilde{B} = (a, e, d; w)$ be non-normal trapezoidal and triangular fuzzy numbers, respectively, where$-\infty < a \leq b \leq e \leq c \leq d < \infty$.Then the following results hold: (Chen and Tang, 2008).

$$(i) I_L(\tilde{B}) \geq I_L(\tilde{A}) (ii) I_R(\tilde{A}) \geq I_R(\tilde{B})$$

$$(iii) I_T^{\alpha}(\tilde{A}) > I_T^{\alpha}(\tilde{B}) \ if \ e < c\alpha + (1-\alpha)b$$

$$(iv) I_T^{\alpha}(\tilde{A}) = I_T^{\alpha}(\tilde{B}) \ if \ e = c\alpha + (1-\alpha)b \ and$$

$$(v) I_T^{\alpha}(\tilde{A}) < I_T^{\alpha}(\tilde{B}) \ if \ e > c\alpha + (1-\alpha)b$$

where$I_L(\tilde{A}) = \frac{w(a+b)}{2}$, $I_L(\tilde{B}) = \frac{w(a+e)}{2}$, $I_R(\tilde{A}) = \frac{w(c+d)}{2}$, $I_R(\tilde{B}) = \frac{w(e+d)}{2}$,

$I_T^{\alpha}(\tilde{A}) = \frac{w}{2}[\alpha(c+d) + (1-\alpha)(a+b)]$, $I_T^{\alpha}(\tilde{B}) = \frac{w}{2}[\alpha d + e + (1-\alpha)a]$.

The above results are correct only if either both $\tilde{A}$ and $\tilde{B}$ are normal or both non-normal fuzzy numbers with equal heights.

Remark 4.3

It may be observed that the method of (Chen and Tang, 2008), like that of (Liou and Wang, 1992) and many other methods, failed to rank non-normal fuzzy numbers of different heights.

It can be seen in the following cases:

Let $\tilde{A} = (5,7,9,10; 0.4)$ and $\tilde{B} = (5,8,10; 0.2)$ be non-normal trapezoidal and non-normal triangular fuzzy numbers, then $I_L(\tilde{A}) > I_L(\tilde{B})$which is a contradiction.

Again, Let $\tilde{A} = (0,1,4,5; 0.9)$ and $\tilde{B} = (0,3,5; 0.1)$ be non-normal trapezoidal and non-normal triangular fuzzy numbers, then$\alpha = 0.3$, $e > c\alpha + (1-\alpha)b$,but$I_T^{0.3}(\tilde{A}) > I_T^{0.3}(\tilde{B})$ which is a contradiction.

Nevertheless, in most of real-life problems, we need to compare non-normal fuzzy numbers with different heights.

Note that, the following results arefound useful to compare two fuzzy numbers$\tilde{A}$and $\tilde{B}$: to a great extant:

(i) If $I_T^{\alpha}(\tilde{A}) > I_T^{\alpha}(\tilde{B})$, then $\tilde{A} > \tilde{B}$

(ii) If $I_T^{\alpha}(\tilde{A}) < I_T^{\alpha}(\tilde{B})$, then $\tilde{A} < \tilde{B}$

(iii) If $I_T^{\alpha}(\tilde{A}) = I_T^{\alpha}(\tilde{B})$, then $\tilde{A} \sim \tilde{B}$

Kumar *et al.*(2011) pointed out that the method of (Chen and Tang, 2008) suffers from the same shortcomings discussed above and proposed a new method which could rank both normal and non-normal p-norm trapezoidal fuzzy numbers.

This method worksusing ranking steps adopted in (Chen and Tang, 2008), but in a modified form to obtain the value of $I_T^{\alpha}(\tilde{A})$ and$I_T^{\alpha}(\tilde{B})$ as described below:

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)_P$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)_P$ be two non-normal p-norm trapezoidal fuzzy numbers. Find the values of $I_T^{\alpha}(\tilde{A})$ and$I_T^{\alpha}(\tilde{B})$.

Step 1:Find $w = \min(w_1, w_2)$.

Step 2:$I_T^{\alpha}(\tilde{A}) = \frac{1}{2}\int_w^0 \{L^{-1}(x) + R^{-1}(x)\}dx$where

$L^{-1}(x) = b_1 + (a_1 - b_1)\left(1 - \left(\frac{x}{w}\right)^p\right)^{1/p}$ and$R^{-1}(x) = c_1 + (d_1 - c_1)\left(1 - \left(\frac{x}{w}\right)^p\right)^{\frac{1}{p}}$.

(4.12)

$$I_T^{\alpha}(\tilde{A}) = w\left\{\left(\alpha(d_1 - c_1) + (1-\alpha)(a_1 - b_1)\right)\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{p \times \Gamma\left(\frac{2}{p}+1\right)} + \alpha c_1 + (1-\alpha)b_1\right\}, \text{ and}$$

$$I_T^{\alpha}(\tilde{B}) = w\left\{\left(\alpha(d_2 - c_2) + (1-\alpha)(a_2 - b_2)\right)\frac{\Gamma\left(\frac{1}{p}+1\right)\Gamma\left(\frac{1}{p}\right)}{p \times \Gamma\left(\frac{2}{p}+1\right)} + \alpha c_2 + (1-\alpha)b_2\right\} \quad (4.13)$$

Step 3: Check $I_T^{\alpha}(\tilde{A}) > I_T^{\alpha}(\tilde{B})$ *or* $I_T^{\alpha}(\tilde{A}) < I_T^{\alpha}(\tilde{B})$ *or* $I_T^{\alpha}(\tilde{A}) = I_T^{\alpha}(\tilde{B})$

Case (i) If $I_T^{\alpha}(\tilde{A}) > I_T^{\alpha}(\tilde{B})$ then $\tilde{A} > \tilde{B}$, $\forall \alpha \in [0,1]$,

Case (ii) If $I_T^{\alpha}(\tilde{A}) < I_T^{\alpha}(\tilde{B})$ then $\tilde{A} < \tilde{B}$, $\forall \alpha \in [0,1]$, and

Case (iii) If $I_T^{\alpha}(\tilde{A}) = I_T^{\alpha}(\tilde{B})$then$\tilde{A} \sim \tilde{B}$, $\forall \alpha \in [0,1]$.

Using the examples considered above to show that (Chen and Wang, 2008) failed to be consistent, it can be seen below that such contradiction arises in (Kumar et al., 2011).

Let $\tilde{A} = (5,7,9,10; 0.4)$ and $\tilde{B} = (5,8,10; 0.2)$ be non-normal trapezoidal and non-normal triangular fuzzy numbers then

Step 1: $\min(0.4, 0.2) = 0.2$

Step 2: $I_L(\tilde{A}) = 1.2$ and $I_L(\tilde{B}) = 1.3$

$\Rightarrow I_L(\tilde{A}) < I_L(\tilde{B})$.

Let $\tilde{A} = (0,1,4,5; 0.9)$ and $\tilde{B} = (0,3,5; 0.1)$ be non-normal trapezoidal and non-normal triangular fuzzy numbers, for $\alpha = 0.3,\ e > c\alpha + (1-\alpha)b$.

Step 1:$\min(0.9,\ 0.1) = 0.1$

Step 2:$I_T^{0.3}(\tilde{A}) = 0.17$ and $I_T^{0.3}(\tilde{B}) = 0.225$

$\Rightarrow I_T^{0.3}(\tilde{A}) < I_T^{0.3}(\tilde{B})$.

4.4 Lee and Li's Method of Ranking Fuzzy Numbers

Lee and Li (1988) proposed the uses of generalized mean and standard deviation based on the probability measures of fuzzy event to rank fuzzy numbers. This method ranks fuzzy numbers based on two different criteria, namely

(i) Uniform Distribution: $f(\tilde{A}) = 1/|\tilde{A}|, \tilde{A} \in U$. The mean $\bar{x}_u(\tilde{A})$ and standard deviation $\delta_u(\tilde{A})$ are defined by (Chang et al., 2006)as follows:

$$\bar{x}_u(\tilde{A}) = \frac{\int_{S(\tilde{A})} x\mu_{\tilde{A}}(x)dx}{\int_{S(\tilde{A})} \mu_{\tilde{A}}(x)dx} \tag{4.14}$$

$$\delta_u(\tilde{A}) = \left[\left(\frac{\int_{S(\tilde{A})} x^2\mu_{\tilde{A}}(x)dx}{\int_{S(\tilde{A})} \mu_{\tilde{A}}(x)dx}\right) - \left(\bar{x}_u(\tilde{A})\right)^2\right]^{1/2} \tag{4.15}$$

where$S(\tilde{A})$ is the support of a fuzzy number $\tilde{A}$.

(ii) Proportional Distribution: $f(\tilde{A}) = k\mu_{\tilde{A}}(x)$, $\tilde{A} \in P$ where k is the proportional constant, then we have

$$\bar{x}_p(\tilde{A}) = \frac{\int_{S(\tilde{A})} x^2 \mu_{\tilde{A}}(x)dx}{\int_{S(\tilde{A})} (\mu_{\tilde{A}}(x))^2 dx} \tag{4.16}$$

$$\delta_p(\tilde{A}) = \left[\left(\frac{\int_{S(\tilde{A})} x^2 (\mu_{\tilde{A}}(x))^2 dx}{\int_{S(\tilde{A})} (\mu_{\tilde{A}}(x))^2 dx} \right) - \left(\bar{x}_u(\tilde{A}) \right)^2 \right]^{1/2} \tag{4.17}$$

4.5. Methods of Ranking Fuzzy Numbers by Computing the Centroid of Fuzzy Numbers

Several methods of ranking by considering the centroid of a fuzzy number have been proposed. In this section, we discuss some representative works on this matter.

4.5.1 Cheng'sDistance Method (Centroid index)

Cheng(1998) proposed a new approach for ranking fuzzy numbers by *distance method.*The method states that the distance from the original point to the centroid point denoted $(\bar{x}_0, \bar{y}_0)$ is computed further, the coefficient of variation (C.V. index) was introduced in order to improve it. Cheng's(C.V. index) is defined as follows:

C.V.$= \delta(standard\ error)/\ (mean)$ where $\mu \neq 0$and $\delta > 0$.

Details of Cheng,s Distance Method:

$\tilde{A} = (a, b, c, d; 1)$. The centroid point $(\bar{x}_0, \bar{y}_0)$ for a fuzzy number $\tilde{A}$, defined by (Cheng, 1998), is as follows:

$$x_0(\tilde{A}) = \frac{\int_a^b (x f_{\tilde{A}}^L) dx + \int_b^c x dx + \int_c^d (x f_{\tilde{A}}^R) dx}{\int_a^b (f_{\tilde{A}}^L) dx + \int_b^c dx + \int_c^d (f_{\tilde{A}}^R) dx}, \quad y_0(\tilde{A}) = \frac{\int_0^1 (y g_{\tilde{A}}^L) dy + \int_0^1 (y g_{\tilde{A}}^R) dy}{\int_0^1 (g_{\tilde{A}}^L) dy + \int_0^1 (g_{\tilde{A}}^R) dy}$$

Clearly, the inverse functions of $f_{\tilde{A}}^L$ and $f_{\tilde{A}}^R$ are $g_{\tilde{A}}^L = a + (b - a)y$ and

$g_{\tilde{A}}^{R} = d + (c-d)y$. Now,

$$\bar{x}_0(\tilde{A}) = \frac{w\int_a^b \left[x\frac{x-a}{b-a}\right]dx + w\int_b^c xdx + w\int_c^d \left[x\frac{x-d}{c-d}\right]dx}{w\int_a^b \frac{x-a}{b-a}dx + w\int_b^c dx + w\int_c^d \frac{x-d}{c-d}dx}$$

$$= \frac{\int_a^b (xf_{\tilde{A}}^{L})dx + \int_b^c xdx + \int_c^d (xf_{\tilde{A}}^{R})dx}{\int_a^b (f_{\tilde{A}}^{L})dx + \int_b^c dx + \int_c^d (f_{\tilde{A}}^{R})dx} \tag{4.18}$$

$$\bar{y}_0(\tilde{A}) = \frac{w[\int_0^1 (y)g_{\tilde{A}}^{L}(y)\,dy + \int_0^1 (y)g_{\tilde{A}}^{R}(y)\,dy]}{\int_0^1 g_{\tilde{A}}^{L}(y)\,dy + \int_0^1 g_{\tilde{A}}^{R}(y)\,dy} \tag{4.19}$$

The ranking index $R(\tilde{A})$ of a fuzzy number $\tilde{A}$ is defined by

$$R(\tilde{A}) = \sqrt{(\bar{x}_0)^2 + (\bar{y}_0)^2}. \tag{4.20}$$

Larger the value of $R(\tilde{A})$better the ranking of a fuzzy number $\tilde{A}$.

Cheng used the ranking indexed defined in (4.20) to state the following stipulations for ranking fuzzy numbers. Let$\tilde{A}_i, \tilde{A}_j \in S$ where $S = \{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\}$.

(i) If $R(\tilde{A}_i) < R(\tilde{A}_j),\ then\ \tilde{A}_i < \tilde{A}_j$,

(ii) If $R(\tilde{A}_i) > R(\tilde{A}_j),\ then\ \tilde{A}_i > \tilde{A}_j$, and

(iii) If $R(\tilde{A}_i) = R(\tilde{A}_j),\ then\ \tilde{A}_i \sim \tilde{A}_j$.

4.5.2 Chu and Tsao's Method of Ranking by Computing the Centroid of Fuzzy Numbers

Chu andTsao(2002) proposed a method of ranking fuzzy numbers by computing the area between the centroid point and original point. In this work, the inverse function of $\mu_{\tilde{A}}^{L}$ defined by$\mu_{\tilde{A}}^{L}: [a,b] \to [0,w]$ need to be continuous and strictly increasing and the inverse function of $\mu_{\tilde{A}}^{R}$ defined by$\mu_{\tilde{A}}^{R}: [c,d] \to [0,w]$ also need to be continuous and strictly decreasing. The inverse fuctions$g_{\tilde{A}}^{L}$ and $g_{\tilde{A}}^{R}$ of $\mu_{\tilde{A}}^{L}$and $\mu_{\tilde{A}}^{R}$, respectively, were also considered. Since $\mu_{\tilde{A}}^{L}: [a,b] \to [0,w]$ is continuous and strictly increasing, $g_{\tilde{A}}^{L}: [0,w] \to$

$[a, b]$ is also continuous and strictly increasing. Similarly$\mu_{\tilde{A}}^{R}: [c, d] \rightarrow [0, w]$ is continuous and strictly decreasing and thus $\mu_{\tilde{A}}^{R}: [0, w] \rightarrow [c, d]$ is continuous and strictly decreasing as well. The value of the centroid point defined in this method is the same as defined by (Cheng, 1998) and (Wang and Lee, 2008).

The area between the centroid point $\left(\bar{x}(\tilde{A}), \bar{y}(\tilde{A})\right)$ and original point (0, 0) of a fuzzy number $\tilde{A}$ was defined as

$$S(\tilde{A}) = \bar{x}(\tilde{A}).\bar{y}(\tilde{A}) \tag{4.21}$$

where$\bar{x}(\tilde{A})$ and$\bar{y}(\tilde{A})$ indicate the values of the distance from the centroid point to original point on horizontal axis and vertical axis, respectively, for fuzzy number$\tilde{A}$.

The fuzzy numbers were ranked according to the areas covered.Larger the area, larger the fuzzy number.The following relations were defined for ranking two or more fuzzy numbers:

(i) If $S(\tilde{A}) > S(\tilde{B}),\ then\ \tilde{A} > \tilde{B},$

(ii) If $S(\tilde{A}) < S(\tilde{B}),\ then\ \tilde{A} < \tilde{B}$, and

(iii) If $S(\tilde{A}) = S(\tilde{B}),\ then\ \tilde{A} \sim \tilde{B}$

Wang and Lee(2008) pointed out some shortcomings of Chu and Tsao's method. The following example was considered to show that Chu and Tsao's method is counterintuitive.

Let $\tilde{A} = (1,2,3; 1)$ and $\tilde{B} = (9,10,11; 0.1)$be two triangular fuzzy numbers. Clearly,$\tilde{A}$ is smaller than$\tilde{B}$. However, the ranking outcome of Chu and Tsao's method is contrary to this. Here,

$\bar{x}(\tilde{A}) = 2$ and$\bar{y}(\tilde{A}) = 0.5$.

Thus the area of the fuzzy number $\tilde{A}$ between the centroid point $\left(\bar{x}(\tilde{A}), \bar{y}(\tilde{A})\right)$ and original point (0, 0) is $S(\tilde{A}) = 2 \times 0.5 = 1$.

Similarly, for $\tilde{B}$,

$\bar{x}(\tilde{B}) = 10$, $\bar{y}(\tilde{B}) = 0.05$, and

The area of the fuzzy number $\tilde{B}$ between the centroid point $\left(\bar{x}(\tilde{B}), \bar{y}(\tilde{B})\right)$ and original point (0, 0) is $S(\tilde{B}) = 10 \times 0.05 = 0.5$.

Accordingly,$\tilde{A}$ is bigger than $\tilde{B}$ as $S(\tilde{A}) > S(\tilde{B})$.

4.5.4 Luu's Ranking Method based on the Centroid Index of Fuzzy Numbers

Luu et al.(2012) proposed a centroid index ranking method based upon the centroid formulae of (Wang et al., 2006) and (Shieh, 2007), The method focused on the cetroid point of a fuzzy number corresponding to horizontal axis $\bar{x}$ as well as the vertical axis $\bar{y}$ and claimed a refinement on other related ranking methods. The centroid point $(\bar{x}, \bar{y})$ for a fuzzy number $\tilde{A}$ is defined by (Luu et al., 2012) and (Shieh, 2007) as follows:

$$\bar{x}_{\tilde{A}} = \frac{\int_{-\infty}^{\infty} x\tilde{A}(x)dx}{\int_{-\infty}^{\infty} \tilde{A}(x)dx} \quad and \quad \bar{y}_{\tilde{A}} = \frac{\int_{0}^{w} \alpha\left|\tilde{A}^{\alpha}\right| d\alpha}{\int_{0}^{w}\left|\tilde{A}^{\alpha}\right| d\alpha} \tag{4.22}$$

where$\tilde{A}$ is a fuzzy number with $\sup_{x \in R} \tilde{A}(x) = \bar{w}$ and $|\tilde{A}^{\alpha}|$is the length of the α-cut $\tilde{A}^{\alpha}$, $0 > \alpha \leq 1$, and $\left|\tilde{A}\right| = \tilde{A}_{R}^{\alpha} - \tilde{A}_{L}^{\alpha}$. If $\tilde{A}$ is a crisp set with $\tilde{A}(x_0) = \bar{w}$,and $\tilde{A}(x) = 0$, if $x \neq x_0$, then its centroid point is defined by $(x_0,\ \bar{w})$.

For a trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; \bar{w})$, the centroid point $(\bar{x}_{\tilde{A}},\ \bar{y}_{\tilde{A}})$ is defined as

$\bar{x}_0(\tilde{A}) = \frac{1}{3}\left[a + b + c + d - \frac{dc - ab}{(d+c)-(a+b)}\right]and\bar{y}_0(\tilde{A}) =$

$$\frac{\bar{w}}{3}\left[1 + \frac{c-b}{(d+c)-(a+b)}\right] \tag{4.23}$$

(Luu*et al.*, 2012) developed theircentriod index as follows:

Suppose $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ are fuzzy numbers, we first calculate the centroid point of all fuzzy numbers $\tilde{A}_i = (\bar{x}_{\tilde{A}},\ \bar{y}_{\tilde{A}}),\ i = 1,2,\ \ldots, n$; then define $G = (x_{min}, y_{min})$ such that $x_{min} = infS,\ S = \bigcup_{i=1}^{n} S_i,\ S_i = \{x \backslash f_{\tilde{A}_i}(x) > 0\}$, while $y_{min} = infY,\ Y = \bigcup_{i=1}^{n} Y_i$, $Y_i = \{y \backslash Y_{\tilde{A}_i}(x) \leq \bar{w}\}$. The distance between the centroid point $\tilde{A}_i = (\bar{x}_{\tilde{A}_i},\ \bar{y}_{\tilde{A}_i}), i = 1,2,$ $\ldots, n$.

$$D(\tilde{A}_i, G) = \sqrt{(\bar{x}_{\tilde{A}_i} - x_{min})^2 + (\bar{y}_{\tilde{A}_i}, \frac{\bar{w}}{3} y_{min})^2} \tag{4.24}$$

If $\tilde{A}_i, \tilde{A}_j$ are two fuzzy numbers, their ranking order is defined as follows:

(i) $\tilde{A}_i < \tilde{A}_j \Leftrightarrow D(\tilde{A}_i, G) < D(\tilde{A}_j, G)$,

(ii) $\tilde{A}_i > \tilde{A}_j \Leftrightarrow D(\tilde{A}_i, G) > D(\tilde{A}_j, G)$ and

(iii) $\tilde{A}_i \sim \tilde{A}_j \Leftrightarrow D(\tilde{A}_i, G) = D(\tilde{A}_j, G)$.

Consider the following sets of fuzzy numbers, adapted from (Luu et al., 2012),(Chen and Sanguansat, 2011) and many others, in order to make a comparison between different ranking methods:

Set 1: $\tilde{A} = (0.1,0.3,0.5; 0.8)$ and $\tilde{B} = (0.1,0.3,0.5; 1)$.

Set 2: $\tilde{A} = (0.1,0.2,0.4,0.5; 1)$ and $\tilde{B} = (0.1,0.3,0.5; 1)$.

Set 1: In figure (4.3) and table(4.1) below, the comparison results of different ranking methods are shown using the fuzzy numbers in set 1.It is clearly seen from table 1 that (Yager, 1980)'s approach lead to an incorrect ranking order i.e.,$\tilde{A} \sim \tilde{B}$ whereas (Murakami et al., 1983), (Cheng, 1998), (Chu and Tsao, 2002), (Chen and Chen, 2007), (Chen and Chen, 2009), (Chen andSanguansat, 2011), (Luu et al., 2012) approaches get the same ranking order i.e.,$\tilde{A} < \tilde{B}$, and is the correct one since is consistent with intuition.

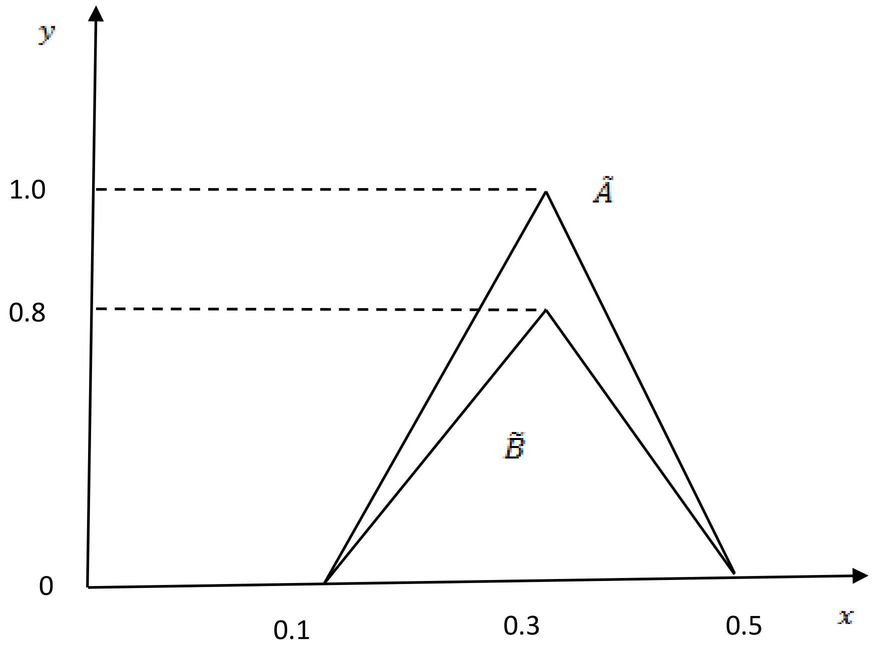


Figure 4.3: Fuzzy Numbers $\tilde{A}$ *and* $\tilde{B}$ in set 1

Table 4.1 Comparative Results of Set 1

Ranking approaches	$\tilde{A}$	$\tilde{B}$	Ranking
yager (1980)	0.3	0.3	$\tilde{A} \sim \tilde{B}$
Murakami et al. (1983)	0.233	0.3	$\tilde{A} < \tilde{B}$
Cheng (1998)	0.461	0.583	$\tilde{A} < \tilde{B}$
Chu and Tsao (2002)	0.12	0.15	$\tilde{A} < \tilde{B}$
Chen and Chen (2007)	0.367	0.446	$\tilde{A} < \tilde{B}$
Chen and Chen 2009	0.206	0.258	$\tilde{A} < \tilde{B}$
Chen and Sanguansat (2011)	0.282	0.3	$\tilde{A} < \tilde{B}$
Luu et al. (2012)	0.196	0.244	$\tilde{A} < \tilde{B}$

Set 2: Figure (4.4) shows the graphs of the two fuzzy numbers of set 2 and the comparative results for different existing methods of ranking fuzzy numbers is presented in table(4.2). It is clearly seen from the table that ((Yager, 1980), (Cheng, 1998), (Chu and Tsao, 2002) and (Chen andSanguansant)) cannot differentiate$\tilde{A}$and$\tilde{B}$. Moreover, (Murakami et al., 1983), (Chen and Chen, 2007) and (Chen and Chen, 2009), they all have $\tilde{A} < \tilde{B}$ as the ranking order which is unreasonable and not consistent with human intuition due to the fact that the centre of gravity of $\tilde{A}$ is larger than that of $\tilde{B}$ on the y-axis.

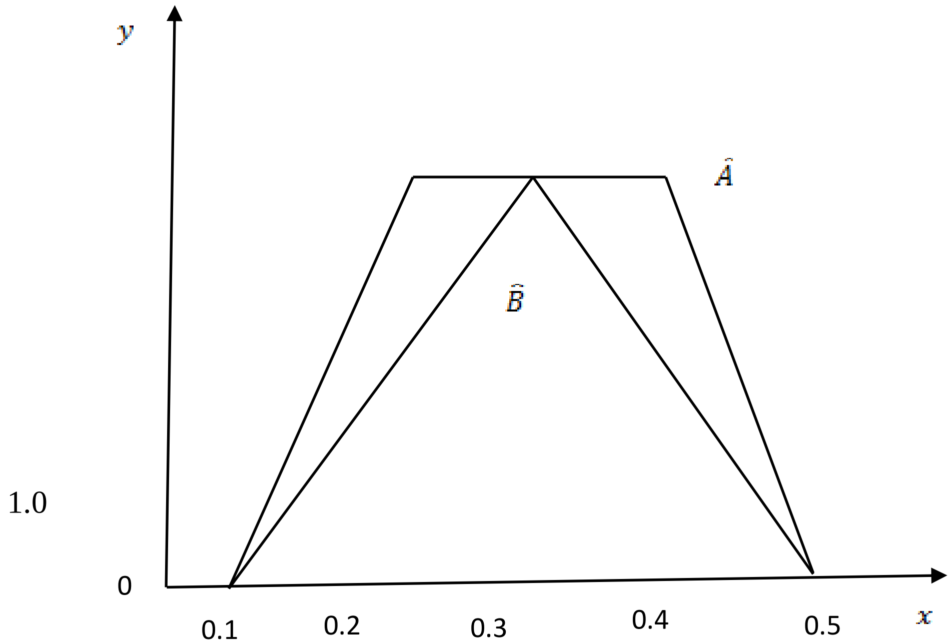


Figure 4.4: Fuzzy numbers $\tilde{A}$ *and* $\tilde{B}$ in Set 2

Table 4.2: Comparative Results of Set 2

Ranking approaches	$\tilde{A}$	$\tilde{B}$	Ranking
yager [26]	0.3	0.3	$\tilde{A} \sim \tilde{B}$
Murakami et al. [18]	0.33	0.417	$\tilde{A} < \tilde{B}$
Cheng[15]	0.583	0.583	$\tilde{A} \sim \tilde{B}$
Chu and Tsao[17]	0.15	0.15	$\tilde{A} \sim \tilde{B}$
Chen and Chen [24]	0.424	0.446	$\tilde{A} < \tilde{B}$
Chen and Chen [25]	0.254	0.258	$\tilde{A} < \tilde{B}$
Chen and Sanguansat [23]	0.3	0.3	$\tilde{A} \sim \tilde{B}$
Luu et al. [20]	0.333	0.222	$\tilde{A} > \tilde{B}$

This shows that no method of ranking fuzzy numbers is found full-proof and hence further research is awaiting.

4.5.5 Babu's Method of Ranking Generalized Fuzzy Numbers Using Centroid of Centroids

Still on the centroid of fuzzy numbers, (Suresh et al., 2012) considered a generalized trapezoidal fuzzy numbersdivided into three different triagles and computed the centroid of centroids of the three triangles.the centroid $G_{\tilde{A}}(\bar{x}_0, \ \bar{y}_0)$ of generalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$is obtained as

$$G_{\tilde{A}}(\bar{x}_0, \quad \bar{y}_0) = \left(\frac{a + 2b + 5c + d}{9}, \frac{4w}{9}\right) \tag{4.25}$$

For triangular fuzzy number $\tilde{A} = (a, b, d; w)$, the centroid is obtained as

$$G_{\tilde{A}}(\bar{x}_0, \quad \bar{y}_0) = \left(\frac{a + 7b + d}{9}, \frac{4w}{9}\right) \quad (4.26)$$

The ranking function of the generalized trapezoidal fuzzy number which maps the set of all fuzzy numbers to a set of real numbers is defined as

$$R(\tilde{A}) = \bar{x}_0 \times \bar{y}_0 = \frac{a + 2b + 5c + d}{9} \times \frac{4w}{9}. \quad (4.27)$$

This is the area between the centroid of centroids$G_{\tilde{A}}(\bar{x}_0, \ \bar{y}_0)$.

In this method, the following parameters were defined in order to compare two or more generalized trapezoidal fuzzy numbers:

(i) Mode(M)= $\frac{1}{2}\int_0^w (b + c)dx = \frac{w}{2}(b + c)$

(ii) Spread(S) = $\int_0^w (d - a)dx = w(d - a)$

(iii)Left spread(LS) = $\int_0^w (b - a)dx = w(b - a)$

(iv)Right spread(RS) = $\int_0^w (d - c)dx = w(d - c)$

The following scheme of comparison is adopted:

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_1)$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_2)$ be two generalized trapezoidal fuzzy numbers:

Step 1.If $R(\tilde{A}) > R(\tilde{B})$, then $\tilde{A} > \tilde{B}$.

If $R(\tilde{A}) < R(\tilde{B})$, then$\tilde{A} < \tilde{B}$.

If $R(\tilde{A}) = R(\tilde{B})$,then$\tilde{A} = \tilde{B}$, and comparison is not possible then go to step 2.

Step 2. If $M(\tilde{A}) > M(\tilde{B})$, then $\tilde{A} > \tilde{B}$.

If $M(\tilde{A}) < M(\tilde{B})$, then $\tilde{A} < \tilde{B}$.

If $M(\tilde{A}) = M(\tilde{B})$,then $\tilde{A} = \tilde{B}$, and comparison is not possible, then go to step 3.

Step 3. If $S(\tilde{A}) > S(\tilde{B})$, then,$\tilde{A} > \tilde{B}$.

If $S(\tilde{A}) < S(\tilde{B})$, then,$\tilde{A} < \tilde{B}$.

If $S(\tilde{A}) = S(\tilde{B})$,then $\tilde{A} = \tilde{B}$, and comparison is not possible, then go to step 4.

Step 4.If $LS(\tilde{A}) > LS(\tilde{B})$, then$\tilde{A} > \tilde{B}$.

If $LS(\tilde{A}) < LS(\tilde{B})$, then,$\tilde{A} < \tilde{B}$.

If $LS(\tilde{A}) = LS(\tilde{B})$,then$\tilde{A} = \tilde{B}$, and comparison is not possible, then go to step 5

If $w_1 > w_2$, then $\tilde{A} > \tilde{B}$.

If $w_1 < w_2$, then$\tilde{A} < \tilde{B}$.

If $w_1 = w_2$,then$\tilde{A} \approx \tilde{B}$.

4.5.6Rezvani's Method of Ranking Generalized Trapezoidal Fuzzy Numbers with Euclidean distance BetweentheIncentre of Centroids

Rezvani(2013) considered the balancing point (i.e., centroid) of a trapezoid by dividing the trapezoid into three triangles and computed the incentre of the centroids of the three triangles.

Ranking function of the generalized trapezoidal fuzzy number $\tilde{A} = (a, b, c, d; w)$ which maps the set of all fuzzy numbers to a set of real numbers is defined as

$$R(\tilde{A}) = \sqrt{\overline{x}_0^2 + \overline{y}_0^2}\,. \tag{4.28}$$

This is the euclidean distance from the incentre of the centroids.

The following scheme was adopted in order to rank fuzzy numbers $\tilde{A}$and $\tilde{B}$.

Let $\tilde{A} = (a_1, b_1, \alpha_1, \beta_1; w_1)$ and $\tilde{B} = (a_2, b_2, \alpha_2, \beta_2; w_2)$ be two generalized trapezoidal fuzzy numbers. Then

Step 1

Find α, β and γrepresenting the centroids of the three triangles into which a trapezoid is divided.

Step 2

$$I(\overline{x}_0, \overline{y}_0)$$

Step 3

Find $(\tilde{A}) = \sqrt{\overline{x}_0^2 + \overline{y}_0^2}$.

(i) If $R(\tilde{A}) > R(\tilde{B})$, then$\tilde{A} > \tilde{B}$

(ii) If $R(\tilde{A}) < R(\tilde{B})$, then$\tilde{A} < \tilde{B}$, and

(iii) If $R(\tilde{A}) = R(\tilde{B})$, then$\tilde{A} \sim \tilde{B}$.

4.6Abbasbandy and Hajjari's Method for Ranking Generalized Trapezoidal Fuzzy Numbers in Parametric Form

Abbasbandy and Hajjari (2009) proposed a new method of ranking generalized trapezoidal fuzzy numbers in parametric form by defining the magnitude of the trapezoidal fuzzy numbers.

Definition 4.6: For an arbitrary trapezoidal fuzzy number $\tilde{A} = (x_0, y_0, \sigma, \beta)$ with parametric form,$\tilde{A} = \left(L_{\tilde{A}}^{-1}(r), R_{\tilde{A}}^{-1}(r)\right)$, the magnitude of the trapezoidal fuzzy number $\tilde{A}$ is definedas

$$Mag(\tilde{A}) = \frac{1}{2}\left(\int_0^1 (L_{\tilde{A}}^{-1}(r), R_{\tilde{A}}^{-1}(r) + x_0 + y_0) f(r) dr\right) \quad (4.29)$$

where the function $f(r)$ is a non-negetive and increasing function on $[0,1]$ with $f(0) = 0, f(1) = 1$and$\int_0^1 f(r)dr = 1/2$. Obviously, function $f(r)$ can be considered as a weighting function. The scheme adopted is as follows:

For any two trapezoidal fuzzy numbers $\tilde{A}$and $\tilde{B} \in E$, the ranking of $\tilde{A}$ and $\tilde{B}$ by mag(.) on E is defined by

(i) $Mag(\tilde{A}) > Mag(\tilde{B})$ if and only if $\tilde{A} > \tilde{B}$,

(ii) $Mag(\tilde{A}) < Mag(\tilde{B})$ if and only if $\tilde{A} < \tilde{B}$

(iii) $Mag(\tilde{A}) = Mag(\tilde{B})$if and only if $\tilde{A} \sim \tilde{B}$.

Example 4.6 Let us consider four fuzzy numbers $\tilde{A} = (1,5,1), \tilde{B} = (1/4,2,1), \tilde{C} = (2,9,1) and\ \tilde{D} = (0,1,2,0)$ as in (Asady andZendehnam, 2007) and (Abbasbandy andAsady, 2006). From the above method,$Mag(\tilde{A}) = 0.6666, Mag(\tilde{B}) = 0.1666, Mag(\tilde{C}) = 1.3334,\ and\ Mag(\tilde{D}) = 0.3334$, which gives the ranking order $\tilde{B} < \tilde{D} < \tilde{A} < \tilde{C}$, and is inconsonance with intuition. Note that the results obtained by*distance minimization method* of (Asady andZendehnam, 2007) is foundunreasonable on this example.

4.7 Abbasbady and Asady's Method ofRanking Fuzzy Numbers Using Sign Distance

Abbasbandy and Asady(2006) proposed a method of ranking fuzzy numbers using*sign distance* which is associated with the *metric* D in E. The membership function of $a \in R$ is $\tilde{A}_a(x) = 1$ if $x = a$, and $\tilde{A}_a(x) = 0$if $x \neq a$. Hence if a= 0, we have

$$\tilde{A}_0(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad (4.30)$$

This method considered$\tilde{A}_0$ as a fuzzy origin and since $\tilde{A}_0 \in E$, left fuzziness σ and right fuzziness $\beta$are 0. Thus, for each $\tilde{A} \in E$ we have

$$D_p(\tilde{A},\tilde{A}_0)=\left[\int_0^1\left(\left|L_{\tilde{A}}^{-1}(r)\right|^p+\left|R_{\tilde{A}}^{-1}(r)\right|^p\right)dr\right]^{\frac{1}{p}},\quad (p\geq 1) \qquad (4.31)$$

Definition 4.7 Let $\gamma(\tilde{A}): E\to\{-1,1\}$ be a function that is defined as

$$\gamma(\tilde{A})=Sign\left[\int_0^1\left(L_{\tilde{A}}^{-1}(r)+R_{\tilde{A}}^{-1}(r)\right)dr\right]\ \forall\tilde{A}\in E \qquad (4.32)$$

where$\gamma(\tilde{A})=\begin{cases}1, & if\ Sign\left[\int_0^1\left(L_{\tilde{A}}^{-1}(r)+R_{\tilde{A}}^{-1}(r)\right)dr\right]\geq 0\\ -1, & if\ Sign\left[\int_0^1\left(L_{\tilde{A}}^{-1}(r)+R_{\tilde{A}}^{-1}(r)\right)dr\right]<0.\end{cases}$

Remark 4.7

(i) If inf $supp(\tilde{A})\geq 0\ or\ \ inf\ L_{\tilde{A}}^{-1}(r)\geq 0,$ then $\gamma(\tilde{A})=1$

(ii) If sup $supp(\tilde{A})<0\ or\ \ sup\ R_{\tilde{A}}^{-1}(r)<\ 0,$ then $\gamma(\tilde{A})=-1$

Definition 4.7.1: For $\tilde{A}\in E,\ dp(\tilde{A},\tilde{A}_0)=\gamma(\tilde{A})D_p(\tilde{A},\tilde{A}_0)$,is called the*sign distance*.

The following steps are defined to give a ranking process in this method. The scheme for comparison is as follows:

For $\tilde{A}_i$ and $\tilde{A}_j\in E$, define the ranking of $\tilde{A}_i$ and $\tilde{A}_j$ by dp on E as:

(i) $dp(\tilde{A}_i,\tilde{A}_0)>dp(\tilde{A}_j,\tilde{A}_0)$if and only if$\tilde{A}_i>\tilde{A}_j$,

(ii) $dp(\tilde{A}_i,\tilde{A}_0)<dp(\tilde{A}_j,\tilde{A}_0)$if and only if$\tilde{A}_i<\tilde{A}_j$,

(iii) $dp(\tilde{A}_i,\tilde{A}_0)=dp(\tilde{A}_j,\tilde{A}_0)$if and only$\tilde{A}_i\sim\tilde{A}_j$.

4.8Ezzati&Saneifard's Method for Ranking Fuzzy Numbers Using Continuous Weighted Quasi-Arithmetic Means

Definition 4.8 (Ezzati andSaneifard, 2010): Let f be a continuous strictly monotonic mapping on [a,b]. For aggregated elements vector $X=(x_1,x_2,\dots,x_n)\in[a,b]^n$ and weighted vector $W = (w_1, w_2, \dots, w_n)$ with $w_i \in [0,1]^n$ with$\sum_{i=1}^{n} w_i = 1$. a weighted quasi-arithmetic mean is defined as the aggregated operator.

$$M_{w.f}(x_1, x_2, \dots, x_n) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right) \tag{4.33}$$

This can also be represented as

$$M_{w.f} = f^{-1}\left(\frac{\int_a^b w(x) f(x) dx}{\int_a^b w(x)\, dx}\right) \quad or \quad M_f(\tilde{A}) = f^{-1}\left(\frac{\int_a^b \mu_{\tilde{A}}(x) x dx}{\int_a^b \mu_{\tilde{A}}(x)\, dx}\right)$$

$$= C.O.G(\tilde{A}) \quad (4.34)$$

where f^{-1} is the inverse function of f and $W = (w_1, w_2, \dots, w_n)$ is a weighted vector of dimension n. $f(x)$ is a continuous strictly monotonic mapping on $[a, b]$ and is a continuous weighting function with $w(x) \geq 0$ and $\int_a^b w(x) \neq 0$.

This method uses $f(x) = x^r, r \neq 0$, and computes

$$M_f(\tilde{A}) = \left(\frac{\int_a^b \mu_{\tilde{A}}(x) x^r\, dx}{\int_a^b \mu_{\tilde{A}}(x)\, dx}\right)^{1/r} \tag{4.35}$$

This is called power function generator quasi-arithmetic mean.

In particular, for a trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$,

$$M_f(\tilde{A}) = \left(\frac{2(a^{r+2} - b^{r+2})(c - d) - 2(c^{r+2} - d^{r+2})(a - b)}{(r + 2)(r + 1)(a - b)(c - d)(a + b - c - d)}\right)^{1/r} \tag{4.36}$$

The scheme for comparison is as follows:

Let $\tilde{A}, \tilde{B} \in F$.Then

(i) $M_f(\tilde{A}) > M_f(\tilde{B})$if and only is$\tilde{A} > \tilde{B}$

(ii) $M_f(\tilde{A}) < M_f(\tilde{B})$if and only if$\tilde{A} < \tilde{B}$

(iii) $M_f(\tilde{A}) = M_f(\tilde{B})$if and only if$\tilde{A} \sim \tilde{B}$.

4.9 Methods of Ranking by L-R Representation

Many ranking methods have been proposed using left and right representations:

left and right indices, left and right dominance, left and right areas, etc. The main contributors on this line are (Wang, et al., 2009), (Allahviranloo et al., 2011), (Chen and Lu, 2011) and (Luu, et al., 2012).

4.9.1Allahviranloo's Method of Ranking Fuzzy Numbers Using New Weighted Distance

Allahviranloo et al.(2011) proposed the ranking method of fuzzy numbers associated with the metric D in F, where F denotes the space of fuzzy numbers.It is assumed that the fuzzy number $\tilde{A} \in F$ is represented by means of the following L-R representation:

$$\tilde{A} = \bigcup (\alpha, \tilde{A}_\alpha), \quad \alpha \in [0,1]$$

where$\tilde{A}_\alpha = [L_{\tilde{A}}(\alpha),\ R_{\tilde{A}}(\alpha)] \subset (-\infty, \infty)$. Here $L: [0,1] \to (-\infty, \infty)$ is a monotonically non-decreasing and $R: [0,1] \to (-\infty, \infty)$ is a monotonically non-decreasing left-contineous functions. The functions $L(.)$ and $R(.)$ denote the left and right sides of a fuzzy numbers, respectively.

The following values constitute the weighted averaged representative and weighted width, respectively, of a fuzzy number $\tilde{A}$:

(i) $\quad I(\tilde{A}) = \int_0^1 \left(cL_{\tilde{A}}(\alpha) + (1-c)R_{\tilde{A}}(\alpha)\right) d\alpha$

(ii) $\quad D(\tilde{A}) = \int_0^1 \left(R_{\tilde{A}}(\alpha) - L_{\tilde{A}}(\alpha)\right) f(\alpha) d\alpha \qquad (4.37)$

Here, $0 \le c \le 1$ isaoptimism/pessimism coefficient in conducting operations on fuzzy numbers. The function $f(\alpha)$ is non-negetive and increasing function on [0,1] with $f(0) = 0, f(1) = 1$ and $\int_0^1 f(\alpha) d\alpha = \frac{1}{2}$. The function $f(\alpha)$ is called weighting function, and it is choosen according to the actual situation. Here, $f(\alpha) = \alpha$is assumed.**Definition 4.9.2Let**$\nabla: F \to \{-1,1\}$ be a function that is defined by

$$\forall \tilde{A} \in F: \nabla(\tilde{A}) = \begin{cases} 1, & \text{when } I(\tilde{A}) \geq 0 \\ -1, & \text{when } I(\tilde{A}) < 0 \end{cases} \tag{4.38}$$

It has the following properties:

(i) If $inf supp(\tilde{A}) \geq 0 \;\; or \; inf L_{\tilde{A}}(\alpha) \geq 0$, then, $\nabla(\tilde{A}) = 1$

(ii) If $sup supp(\tilde{A}) < 0 \;\; or \; sup R_{\tilde{A}}(\alpha) < 0$, then, $\nabla(\tilde{A}) = -1$

Definition 4.9.3 For arbitrary fuzzy numbers $\tilde{A}$and $\tilde{B}$, the quantity

$$TRD(\tilde{A},\tilde{B}) = \sqrt{\left[I(\tilde{A}) - I(\tilde{B})\right]^2 + \left[D(\tilde{A}) - D(\tilde{B})\right]^2} \tag{4.39}$$

is called the TRD distance between the fuzzy numbers $\tilde{A}$ and $\tilde{B}$. The membership function of $a \in R$ is $\tilde{A}_a(x) = 1 \;\; if \; x = a$, and $\tilde{A}_a(x) = 0 \;\; if \; x \neq a$. Hence if $a = 0$, then

$$\tilde{A}_0(x) = \begin{cases} 1, & when \; x = 0 \\ -1, & when \; x \neq 0 \end{cases} \tag{4.40}$$

This method defined

$$TRD(\tilde{A},\tilde{A}_0) = \sqrt{\left[I(\tilde{A})\right]^2 + \left[D(\tilde{A})\right]^2} \tag{4.41}$$

Definition 4.9.4 For each $\tilde{A} \in F$ and with optimism/pessimismcoefficient equal to 0.5, $TR(\tilde{A}, \tilde{A}_0) = \;\nabla(\tilde{A}) TRD(\tilde{A},\tilde{A}_0)$, called the *weighted distance*.

The scheme for ranking is as follows:

For$\tilde{A}$, $\tilde{B} \in F$,

(i) $TR(\tilde{A},\tilde{A}_0) > TR(\tilde{B},\tilde{A}_0) \Leftrightarrow \;\tilde{A}>\tilde{B}$

(ii) $TR(\tilde{A},\tilde{A}_0) < TR(\tilde{B},\tilde{A}_0) \Leftrightarrow \;\tilde{A}<\tilde{B}$, and

(iii) $TR(\tilde{A},\tilde{A}_0) TR = (\tilde{B},\tilde{A}_0) \Leftrightarrow \;\tilde{A}\sim\tilde{B}$.

4.9.2 Chen and Lu's Method of Ranking Based on Left and Right Dominance

Chen and Lu(2011) proposed an *approximate* approach for ranking fuzzy numbers based on left and right dominance. This method considers only a few left and right spreads at some α-levels of fuzzy numbers to determine the respective dominance. Note that this method cannot be applied in ranking non-normal fuzzy numbers.

In this method, the lower and theupper limits of the $K^{th}\alpha$-cut ofa fuzzy number $\tilde{A}_i$ are defined by

$$l_{i,k} = \underset{x \in R}{inf}\{x|\mu_{\tilde{A}_i}(x) \geq \alpha_k\} \ , \quad r_{i,k} = \underset{x \in R}{sup}\{x|\mu_{\tilde{A}_i}(x) \geq \alpha_k\} \tag{4.42}$$

where$l_{i,k}$ and $r_{i,k}$ are the left and right spreads, respectively. The left (right) dominance $D^L_{i,j}$ $(D^L_{i,j})$ of $\tilde{A}_i$ over $\tilde{A}_j$ is defined by

$$D^L_{i,j} = \frac{1}{n+1}\sum_{k=0}^{n}(l_{i,k}, l_{j,k}) \quad and \quad D^R_{i,j} = \frac{1}{n+1}\sum_{k=0}^{n}(r_{i,k}, r_{j,k}) \tag{4.43}$$

where$n+1\alpha$-cuts are used to calculate the dominance.Then the n total dominances of $\tilde{A}_i$ over $\tilde{A}_j$ with index of optimism $\beta \in [0,1]$ can be defined as the convex combination of $D^L_{i,j}$ and $D^R_{i,j}$ by

$$D_{i,j}(\beta) = \beta D^R_{i,j} + (1-\beta)D^L_{i,j}$$

$$= \beta\left[\frac{1}{n+1}\sum_{k=0}^{n}(r_{i,k}, r_{j,k})\right] + (1-\beta)\left[\frac{1}{n+1}\sum_{k=0}^{n}(l_{i,k}, l_{j,k})\right]$$

$$= \frac{1}{n+1}\left\{\left[\beta\sum_{k=0}^{n} r_{i,k} + (1-\beta)\sum_{k=0}^{n} l_{i,k}\right] - \left[\beta\sum_{k=0}^{n} r_{j,k} + (1-\beta)\sum_{k=0}^{n} l_{j,k}\right]\right\} \tag{4.44}$$

The scheme for ranking is as follows:

(i) If $D_{i,j}(\beta) < 0$, then$\tilde{A}_i < \tilde{A}_j$.

(ii) If $D_{i,j}(\beta) > 0$, then $\tilde{A}_i > \tilde{A}_j$.

(iii) If $D_{i,j}(\beta) = 0$, then, $\tilde{A}_i \sim \tilde{A}_j$.

Chang et al.(2006) proposed a method titled *A conceptual procedure for ranking fuzzy numbers based on adaptive two-dimension dominance*, to improve the method of (Chen andLu, 2011) to include ranking of non-normal fuzzy numbers as well.

A trapezoidal fuzzy number $\tilde{A}_i = (a, b, c, d; L_H, R_H)$ and a triangular fuzzy number $\tilde{A}_i = (a, b, d; L_H, R_H)$ can be ranked using this method, as follows:

$$D_i = \frac{1}{2}\{\alpha[\beta R_H(i) + (1-\beta)L_H(i)] + (1-\alpha)[\beta R_S(i) + (1-\beta)L_S(i)]\} \qquad (4.45)$$

where D_i is the index of i^{th} fuzzy number, α the weighting of expert opinion by DM as signed, β: is the optimistic index of decision maker, $R_H(i)$ the right height of i^{th} fuzzy number, $L_H(i)$ the left height of i^{th} fuzzy number, $R_S(i)$ the right spread of i^{th} fuzzy number $\left(i.e., R_S(\tilde{A}) = c + d\right)$ and $L_S(i)$ the left spread of i^{th} fuzzy number $\left(i.e., L_S(\tilde{A}) = a + b\right)$.

The larger the value of D_i, the following results hold:

(i) Larger $D_i \Longrightarrow$ better ranking.

(ii) Larger $\alpha \Longrightarrow$ heavier weights of experts' opinion. Equivalently, smaller $\alpha \Longrightarrow$ lesser the influences of y-axis.

(iii) Larger β (optimistic index of Dm) $\Longrightarrow$ right spread influence is higher.

(iv) Smaller $\beta \Longrightarrow$ left spread influence is higher.

4.9.3 Luu's Ranking Method Using Left and Right Indices

In this method (Luu, et al., 2012) provided some improvement over the left and right indices method. The subtractions of the left relative values from the right relative values are considered based on different decision levels γ associated with the decision makers optimism attitude α of fuzzy numbers.

The left and the right indices are defined as follows:

$x_L(\tilde{A}_i) = f_{\tilde{A}_i^L}(x) \wedge \gamma, \quad i = 1,2,\dots,n,\ 0 < \gamma < w$ and $\quad x_R(\tilde{A}_i) = f_{\tilde{A}_i^R}(x) \wedge \gamma,\ i = 1,2,\dots,n,\ 0 < \gamma < w.$

The subtraction of the left relative values from the right relative values of each fuzzy number $\tilde{A}_i$ with index of optimism α is defined as follows:

$$D_\alpha^\gamma(\tilde{A}_i) = \alpha[x_L(\tilde{A}_i) - x_{min}] - (1-\alpha)[x_{max} - x_R(\tilde{A}_i)] \tag{4.46}$$

where$x_{min} = infS,\ x_{max} = SupS,\ S = \bigcup_{i=1}^{n} S_i,\ S_i = \{x \backslash f_{\tilde{A}_i}(x) > 0\}$.

If $\tilde{A}_i$ is a generalized trapezoidal fuzzy number, i.e.,$\tilde{A}_i = (a_i, b_i, c_i, d_i; w_i)$, then the left and right indices for fuzzy number $\tilde{A}_i$ can be determined by the following equations:

$x_L(\tilde{A}_i) = \gamma(b_i - a_i)/w_i + a_i,$ and

$$x_R(\tilde{A}_i) = \gamma(c_i - d_i)/w_i + d_i,\ 0 < \gamma < w. \tag{4.47}$$

The subtraction of the left relative values from the right relative values of each fuzzy number $\tilde{A}_i$ with index of optimismα is as follows:

$$D_\alpha^\gamma(\tilde{A}_i) = \alpha[\gamma(b_i - a_i)/w_i + a_i - x_{min}] - (1-\alpha)[x_{max} - d_i - \gamma(c_i - d_i)w_i \tag{4.48}$$

The same procedure works for triangular fuzzy numbers when $b = c$.

It follows that, the fuzzy number $\tilde{A}_i$ is larger if the right relative value$x_{max} - x_R(\tilde{A}_i)$ is smaller and the left relative value $x_L(\tilde{A}_i) - x_{min}$ is larger.

The comparison scheme is as follows:

For any two arbitrary fuzzy numbers $\tilde{A}_i$ and $\tilde{A}_j$,

(i) If $D_\alpha^\gamma(\tilde{A}_i) < D_\alpha^\gamma(\tilde{A}_j)$, then$\tilde{A}_i$<$\tilde{A}_j$

(ii) If $D_\alpha^\gamma(\tilde{A}_i) > D_\alpha^\gamma(\tilde{A}_j)$, then$\tilde{A}_i$>$\tilde{A}_j$

(iii) If $D_\alpha^\gamma(\tilde{A}_i) = D_\alpha^\gamma(\tilde{A}_j)$,then $\tilde{A}_i \sim \tilde{A}_j$.

It may be observed if $\gamma \geq w$, then $D_{\alpha}^{\gamma}(\tilde{A}_i) = 0$.

4.9.4 Wang's Method of Ranking L-R Fuzzy Numbers Using Deviation Degree

Another method of ranking L-R fuzzy numbers by using deviation degree was proposed by (Wang, et al., 2009). The details of this method are described below:

$\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$in E,where $S(\tilde{A}_i),\ i = 1,2, \ldots n, S = \cap_{i=1}^{n} S(\tilde{A}_i)$ are the support sets$x_{min} = infS$,and $x_{max} = SupS$. Then maximal and minimal reference sets $\tilde{A}_{min}$ and $\tilde{A}_{max}$ are defined by

$$\mu_{\tilde{A}_{min}}(x) = \begin{cases} \dfrac{x_{max} - x}{x_{max} - x_{min}}, & if\ x \in S \\ 0, & otherwise \end{cases}$$

and$\mu_{\tilde{A}_{max}}(x) = \begin{cases} \frac{x - x_{min}}{x_{max} - x_{min}}, & if\ x \in S \\ 0, & otherwise \end{cases}$.

The*left* and *right* deviation degrees of $\tilde{A}_i,\ i = 1, \ldots, n$ are defined as follows:

$$d_i^L = \int_{x_{min}}^{t_i} \left(\mu_{\tilde{A}_{min}}(x) - L_{\tilde{A}}^{-1}(x)\right) dx$$

and$d_i^R = \int_{u_i}^{x_{max}} \left(\mu_{\tilde{A}_{max}}(x) - R_{\tilde{A}}^{-1}(x)\right) dx$ (4.49)

wheret_i and $u_i,\ i = 1, \ldots, n$ are the abscissas of the cross over points of $L_{\tilde{A}_i}$ and $\mu_{\tilde{A}_{min}}$ and $R_{\tilde{A}_i}$ and $\mu_{\tilde{A}_{max}}$ respectively.

Definition 4.9.5 Let$\tilde{A}_i = (a_i, b_i, c_i, d_i; w)$ in E. The*expectation* value of its centroid is defined as follows:

$M_i = \frac{\int_{a_i}^{d_i} x \mu_{\tilde{A}_i}(x) dx}{\int_{a_i}^{d_i} \mu_{\tilde{A}}(x) dx}and\lambda_i =$

$\frac{M_i - M_{min}}{M_{max} - M_{min}}$ (4.50) where$M_{max} = Max\{M_1, M_2, \dots, M_n\}$ and $M_{min} = Min\{M_1, M_2, \dots, M_n\}$. Based on the above relations, the *ranking index* values of fuzzy numbers $\tilde{A}_i,\ i = 1, \dots, n$ are given by

$$d_i = \begin{cases} \dfrac{d_i^L \lambda_i}{1 + d_i^R (1 - \lambda_i)}, & M_{max} \neq M_{min}, \quad i = 1, \dots, n \\ \dfrac{d_i^L}{1 + d_i^R}, & M_{max} = M_{min}, \quad i = 1, \dots, n. \end{cases} \tag{4.51}$$

The scheme of ranking is as follows:

(i) $\tilde{A}_i > \tilde{A}_j \Leftrightarrow d_i > d_j$,

(ii) $\tilde{A}_i < \tilde{A}_j \Leftrightarrow d_i < d_j$

(iii) $\tilde{A}_i \approx \tilde{A}_j \Leftrightarrow d_i = d_j$.

Asady(2010) revised the method of (Wang, et al, 2009) and suggested a D (.) operator for ranking of fuzzy numbers as follows:

Definition 4.9.6 If $\tilde{A}$and $\tilde{B}$ are two fuzzy numbers, the ranking order based on the revised method is defined by the following situations:

(i) If $D(\tilde{A}) < D(\tilde{B})$, then$\tilde{A} < \tilde{B}$

(ii) If $D(\tilde{A}) > D(\tilde{B})$, then$\tilde{A} > \tilde{B}$

(iii) If $D(\tilde{A}) = D(\tilde{B})$, then $\tilde{A} \approx \tilde{B}$

Also, if$\gamma_{\tilde{A}} \neq \gamma_{\tilde{B}}$, and $D^*(\tilde{A}) < D^*(\tilde{B})$then $\tilde{A} < \tilde{B}$;

if$\gamma_{\tilde{A}} \neq \gamma_{\tilde{B}}$, $D^*(\tilde{A}) > D^*(\tilde{B})$then $\tilde{A} > \tilde{B}$;

else$\tilde{A} \approx \tilde{B}$,

Where $D(\tilde{A}) = \frac{D_{\tilde{A}}^L}{1 + D_{\tilde{A}}^R}$, $D^*(\tilde{A}) = \frac{D_{\tilde{A}}^L \gamma}{1 + D_{\tilde{A}}^R \gamma}$, $D_{\tilde{A}}^L = \int_0^1 \left(R_{\tilde{A}}^{-1}(x) + L_{\tilde{A}}^{-1}(x) - 2x_{min}\right) dx$

and $D_{\tilde{A}}^R = \int_0^1 \left(2x_{max} - R_{\tilde{A}}^{-1}(x) - L_{\tilde{A}}^{-1}(x)\right) dx$.

4.9.5Nejad and Mashinchi's Methodof Ranking Fuzzy Numbers Based on L-R Sides of Fuzzy Numbers

Nejad and Mashinchi(2011) pointed out the drawbacks of Wang's method described above and presented their ranking method as follows:

Definition 4.9.7 Let $\tilde{A}_i = (a_i, b_i, c_i, d_i; w), i = 1,2, \dots, n$ be fuzzy numbers in E, $a_{min} = \min\{a_1, a_2, \dots, a_n\}$ and $d_{max} = \min\{d_1, d_2, \dots, d_n\}$. The areas S_i^L and S_i^R of the left and right sides of the fuzzy number $\tilde{A}_i$ are defined as

$$S_i^L = \int_0^w \left(L_{\tilde{A}}^{-1}(r) - a_{min}\right)dr,$$

and $S_i^R = \int_0^w \left(d_{max} - R_{\tilde{A}}^{-1}(r)\right)dr.$ (4.52)

Based on the above definitions, the proposed ranking index is defined by

$$S_i = \frac{S_i^L \lambda_i}{1 + S_i^R(1 - \lambda_i)} \quad (4.53)$$

The ranking order is defined as follows:

(i) $\tilde{A}_i > \tilde{A}_j \Leftrightarrow S_i > S_j$.

(ii) $\tilde{A}_i < \tilde{A}_j \Leftrightarrow S_i < S_j$.

(iii) $\tilde{A}_i \approx \tilde{A}_j \Leftrightarrow S_i = S_j$.

4.9.6Parandi's Method of Ranking Normal and Non-normal Fuzzy Numbers Based on L-R Areas

Parandi, et al.(2004) developed a ranking method, quite similar to the above method of (Nejad and Mashinchi, 2011) as follows:

Definition 4.9.8 Let$\tilde{A}_i = (a_i, b_i, c_i, d_i; w_i)$,

$i = 1, \dots, n$,be fuzzy numbers belonging to E.

Let $K = \max\{|a_i|, |b_i|, |c_i|, |d_i| \in \tilde{A}_i,\ i = 1, \dots, n\}, K \in \mathbb{N} - \{0\}$. for fuzzy number inE,

defined$\tilde{A}_i^*$, $i = 1, \dots, n, \tilde{A}_i^* = \left(\frac{a_i}{k}, \frac{b_i}{k}, \frac{c_i}{k}, \frac{d_i}{k}; w_i\right) = (a_i^*, b_i^*, c_i^*, d_i^*; w_i), i =$

$1, \dots, n$ (4.54)

Let $\tilde{A}_i^*, i = 1, \dots, n$be fuzzy numbers then

$w^* = \min\{w_i, i = 1, \dots, n\}, a_{min}^* = \min\{a_i^*,\ i = 1, \dots, n\},$ and

$$d_{max}^* = \max\{d_i^*, \quad i = 1, \dots, n\}.$$

The areas S_i^L and S_i^R of the left and right sides of the fuzzy numbers $\tilde{A}_i^*$ in this method are defined as follows:

$$S_i^L = \int_0^{w^*} (g_{\tilde{A}_i^*}^L(y) - a_{min}^*) dy \quad and \quad S_i^R \int_0^{w^*} \left(d_{max}^* - g_{\tilde{A}_i^*}^R(y)\right) dy \qquad (4.55)$$

Based on the above areas, the proposed ranking index of $\tilde{A}_i^*$ is defined as follows:

$$S_i = \frac{S_i^L}{1 + S_i^R}, \quad i = 1, \dots, n \qquad (4.56)$$

The scheme of ranking is the same as in (4.53) described above.

Consider the two triangular fuzzy numbers $\tilde{A} = (0.2, 0.5, 0.8)$ and $\tilde{B} = (0.4, 0.5, 0.6)$and their representations as in figure (4.5). Clearly, the ranking order by (Nejad andMashinchi, 2011) is $\tilde{A} < \tilde{B}$. The images of two fuzzy numbers $\tilde{A}\ and\ \tilde{B}$ are $\tilde{A} = (-0.8, -0.5, -0.2),\ \tilde{B} = (-0.6, -0.5, -0.4)$, respectively, and then the ranking order is $-\tilde{B} < -\tilde{A}$. On the other hand, ranking order for $\tilde{A}\ and\ \tilde{B}$ and their images by Wang's method and by the revised method of Asady are $\tilde{A} \approx \tilde{B},\ -\tilde{A} \approx -\tilde{B}$, respectively. This shows that the methods of ranking L-R fuzzy numbers by (Asady, 2010), (Wang, et al., 2009) and (Nejad andMashinchi, 2011) are reasonable as shown in figure 4.5 below.

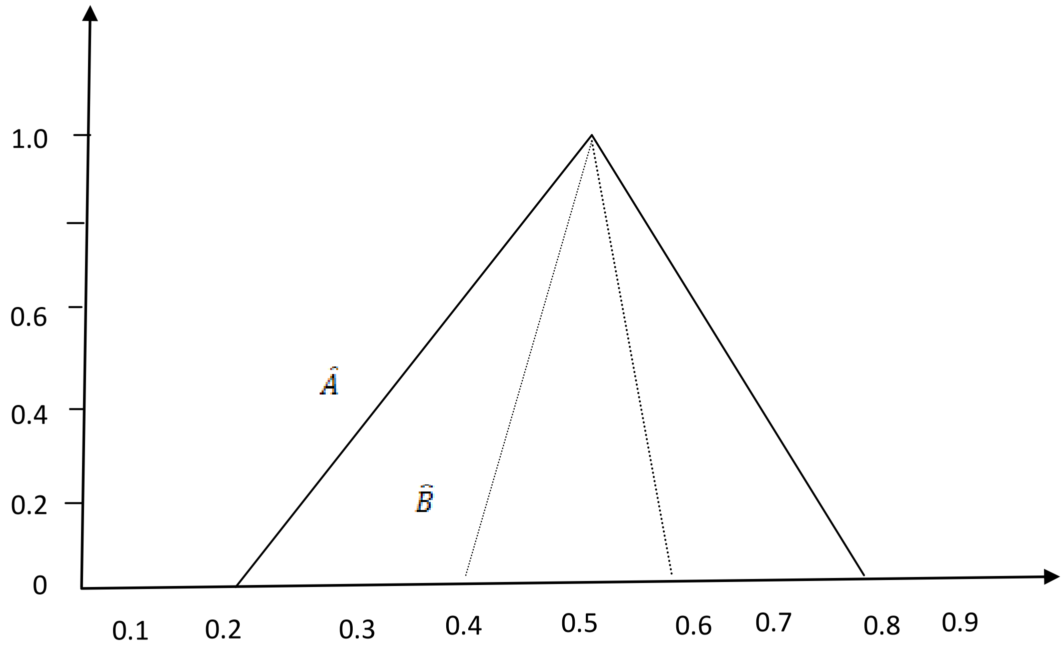


Figure 4.5: Representation of Fuzzy Numbers of the case above.

CHAPTER FIVE

A PROPOSED METHOD OF RANKING FUZZY NUMBERS

In the previous chapter, many ranking methods of fuzzy numbers have been presented, and demonstrated systematically that none of them could be taken as a general method which would rank fuzzy numbers in a diverse situation. Accordingly, the effort needs to be made to develop a simple and intuitively sound ranking method which would serve the purpose in a large number of possible situations.

In the following, a method which is quite simple and encompassing a large number of cases moderately have been developed.

Definition 5.1

A generalized trapezoidal fuzzy number$(a, b, c, d; w), a \leq b \leq c \leq d$ can be viewed as comprising three segments A, B, and C (figure 5.1).

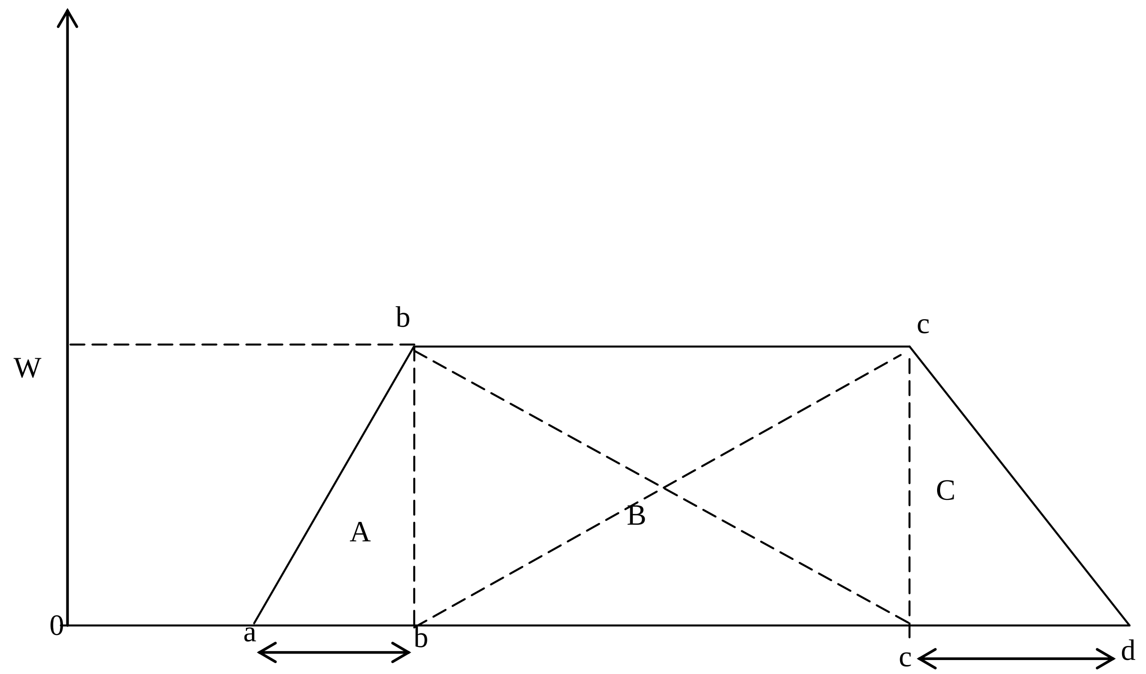


Figure 5.1: A generalized trapezoidal fuzzy number

Section-wise perimeters of the trapezoid have been computed in this work, instead of its perimeter as in (Rezvani, 2012), viz., perimeters of the three segments A, B, and Cseparately and adding them together and denote it byP^+. As a matter of fact, our generalized approach differs from (Rezvani, 2012) in respect of computing perimeter of the trapezoid representing a generalized trapezoidal fuzzy number.The diagramabove is in the generalized form of (Rezvani, 2012), who considered the formula for perimeter of a generalized trapezoidal fuzzy number

$\tilde{A} = (a_1, b_1, c_1, d_1; w_{\tilde{A}}), 0 \leq a_1 \leq b_1 \leq c_1 \leq d_1 \leq 1, 0 < w_{\tilde{A}} \leq 1,$ as

$P(\tilde{A}) = \sqrt{(a_1 - b_1)^2 + w_{\tilde{A}}^2} + \sqrt{(d_1 - c_1)^2 + w_{\tilde{A}}^2} + (c_1 - b_1) + (d_1 - a_1).$

In our generalized method, the formula for the perimeter of $\tilde{A}$ is

$$P^+(\tilde{A}) = \sqrt{(a_1 - b_1)^2 + w_{\tilde{A}}^2} + \sqrt{(d_1 - c_1)^2 + w_{\tilde{A}}^2} + (c_1 - b_1) + (d_1 - a_1) + 2w_{\tilde{A}}.$$

Similarly,

$$P^+(\tilde{B}) = \sqrt{(a_2 - b_2)^2 + w_{\tilde{B}}^2} + \sqrt{(d_2 - c_2)^2 + w_{\tilde{B}}^2} + (c_2 - b_2) + (d_2 - a_2) + 2w_{\tilde{B}}.$$

The areas $A(\tilde{A})$and $A(\tilde{B})$ of two generalized trapezoidal fuzzy numbers are calculated as in (Wen etal., 2011) and (Rezvani, 2012) as

$$A(\tilde{A}) = \frac{1}{2} w_{\tilde{A}} (c_1 - b_1 + d_1 - a_1)$$

$$A(\tilde{B}) = \frac{1}{2} w_{\tilde{B}} (c_2 - b_2 + d_2 - a_2)$$

The scheme of comparing generalized trapezoidal fuzzy numbers is as follows:

Let $\tilde{A} = (a_1, b_1, c_1, d_1; w_{\tilde{A}})$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_{\tilde{B}})$ be two generalized trapezoidal fuzzy numbers,$0 \leq a_1 \leq b_1 \leq c_1 \leq d_1 \leq 1$ and$0 \leq a_2 \leq b_2 \leq c_2 \leq d_2 \leq 1,$ and $P^+(\tilde{A})$ and $P^+(\tilde{B})$ are the section-wise perimeters of two generalized trapezoidal fuzzy numbers$\tilde{A}$and $\tilde{B}$. Then

(i) If $P^{+}(\tilde{A}) < P^{+}(\tilde{B})$, then $\tilde{A} < \tilde{B}$,

(ii) If $P^{+}(\tilde{A}) > P^{+}(\tilde{B})$, then $\tilde{A} > \tilde{B}$,

(iii) If $P^{+}(\tilde{A}) \sim P^{+}(\tilde{B})$, then $\tilde{A} \sim \tilde{B}$.

5.2 Similarity Measures under New Approach

The concept of *similarity* has been quantified in various forms in linguistics, mathematics as well as other sciences. In particular, computing the degree of similarity between two trapezoidal fuzzy numbers has played an important role (see (Chen and Chen, 2009), (Xiaoyan*etal*., 2011),(Wen *et al*., 2011) and(Rezvani, 2012, 2015)) just to mention a few.

The approach followed in (Wen *et al*., 2011) and in (Rezvani, 2012, 2015) is considered in generalized form in this work. Computing similarity measures, the factors are: horizontal center of gravity, perimeter, height, and the area of two fuzzy numbers.

Suppose that $\tilde{A} = (a_1, b_1, c_1, d_1; w_{\tilde{A}})$ and $\tilde{B} = (a_2, b_2, c_2, d_2; w_{\tilde{B}})$ are two generalized trapezoidal fuzzy numbers, $0 \le a_1 \le b_1 \le c_1 \le d_1 \le, 0 \le a_2 \le b_2 \le c_2 \le d_2 \le 1$ and $0 < w_{\tilde{A}},\ w_{\tilde{B}} \le 1$. Then the degree of similarity $S(\tilde{A}, \tilde{B})$ between the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is obtained as follows:

$$S(\tilde{A}, \tilde{B}) = \left[1 - \left|x'_{\tilde{A}} - x'_{\tilde{B}}\right|\right] \times [1 - |w_{\tilde{A}} - w_{\tilde{B}}|] \times \frac{\min\left(P^{+}(\tilde{A}), P^{+}(\tilde{B})\right) + \min\left(A(\tilde{A}), A(\tilde{B})\right)}{\max\left(P^{+}(\tilde{A}), P^{+}(\tilde{B})\right) + \max\left(A(\tilde{A}), A(\tilde{B})\right)} \quad (5.1)$$

The horizontal centers of gravity of the generalized trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ respectively denoted by $x'_{\tilde{A}}$ and $x'_{\tilde{B}}$, are calculated as in (Wen *etal*., 2011) and (Rezvani, 2012, 2015):

$$x'_{\tilde{A}} = \frac{y'_{\tilde{A}}(c_1 + b_1) + (d_1 + a_1)(w_{\tilde{A}} - y'_{\tilde{A}})}{2w_{\tilde{A}}}$$

$$x'_{\tilde{B}} = \frac{y'_{\tilde{B}}(c_2 + b_2) + (d_2 + a_2)(w_{\tilde{B}} - y'_{\tilde{B}})}{2w_{\tilde{B}}}$$

$$y'_{\tilde{A}} = \begin{cases} \frac{w_{\tilde{A}}\left(\frac{c_1-b_1}{d_1-a_1}\right)}{6}, & if a_1 \neq d_1 \ \ and\ 0 < w_{\tilde{A}} \leq 1 \\ \frac{w_{\tilde{A}}}{2}, & if a_1 = d_1 \ \ and\ 0 < w_{\tilde{A}} \leq 1 \end{cases}$$

$$y'_{\tilde{B}} = \begin{cases} \frac{w_{\tilde{B}}\left(\frac{c_2-b_2}{d_2-a_2}\right)}{6}, & if a_2 \neq d_2 \ \ and\ 0 < w_{\tilde{A}} \leq 1 \\ \frac{w_{\tilde{B}}}{2}, & if a_2 = d_2 \ \ and\ 0 < w_{\tilde{A}} \leq 1. \end{cases}$$

The scheme for comparing fuzzy numbers is as follow:

If$S(\tilde{A}, \tilde{B})$ is closer to 1, then the similarity between $\tilde{A}$ and $\tilde{B}$ is higher, and on the same scale, other results may be characterized.

5.3 Results and Comparison

In this subsection, similar to other methods, the same seven sets of fuzzy numbers are considered for comparing the new approach with other existing approaches. The results are presented in a tabular form in table 5.1.

Example 1.

Seven sets of fuzzy numbers are used and the results are compared with many existing methods in table (5.1).

Set 1: Let $\tilde{A} = (0.2,0.4,0.6,0.8; 0.35)$ and $\tilde{B} = (0.1,0.2,0.3,0.4; 0.7)$.

$$P^+(\tilde{A}) = 2.31 \text{and} P^+(\tilde{B}) = 3.21$$

Since $P^+(\tilde{A}) < P^+(\tilde{B}),\ \tilde{A} < \tilde{B}$ h$olds$.

Set 2: Let $\tilde{A} = (0.1,0.2,0.4,0.5; 1)$ and $\tilde{B} = (0.1,0.3,0.3,0.5; 1)$.

$$P^+(\tilde{A}) = 4.61 \text{and} P^+(\tilde{B}) = 4.44$$

Since, $P^+(\tilde{A}) > P^+(\tilde{B})\tilde{A} > \tilde{B}$ h$olds$.

Set 3: Let $\tilde{A} = (0.1,0.2,0.4,0.5; 1)$ and $\widetilde{B} = (1,1,1,1; 1)$.

$$P^+(\tilde{A}) = 4.61 \text{ and} P^+(\tilde{B}) = 4.0$$

Since $P^+(\tilde{A}) > P^+(\tilde{B})$, $\tilde{A} > \tilde{B}$ h$olds$.

Set 4: Let$\tilde{A} = (-0.5, -0.3, -0.3, -0.1; 1)$ and $\tilde{B} = (0.1,0.3,0.3,0.5; 1)$.

$$P^+(\tilde{A}) = 4.44 \text{and} P^+(\tilde{B}) = 4.44$$

Since $P^+(\tilde{A}) \sim P^+(\tilde{B})$, $\tilde{A} \sim \tilde{B}$ h$olds$.

Set 5: Let$\tilde{A} = (0.3,0.5,0.5,1; 1)$ and $\tilde{B} = (0.1,0.6,0.6,0.8; 1)$.

$$P^+(\tilde{A}) = 4.84 \text{ and} P^+(\tilde{B}) = 4.84$$

Since $P^+(\tilde{A}) \sim P^+(\tilde{B}), \tilde{A} \sim \tilde{B}$ h$olds$.

Set 6: Let $\tilde{A} = (0,0.4,0.6,0.8; 1)$, $\tilde{B} = (0.2,0.5,0.5,0.9; 1)$ and $\tilde{C} = (0.1,0.6,0.7,0.8; 1)$.

$P^+(\tilde{A}) = 5.1, P^+(\tilde{B}) = 4.82$ and$P^+(\tilde{C}) = 4.92$.

Since, $P^+(\tilde{A}) > P^+(\tilde{C}) > P^+(\tilde{B})$, $\tilde{A} > \tilde{C} > \tilde{B}$ h$olds$.

Set 7: Let $\tilde{A} = (0.1,0.2,0.4,0.5; 1)$ and $\tilde{B} = (-2,0,0,2; 1)$.

$$P^+(\tilde{A}) = 4.61 \text{and} P^+(\tilde{B}) = 10.47.$$

Since $P^+(\tilde{A}) < P^+(\tilde{B})$, $\tilde{A} < \tilde{B}$ h$olds$.

Table.5.1 Comparison of the ranking results for different approaches

Approaches	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7
(Cheng, 1998)	$\tilde{A} < \tilde{B}$	$\tilde{A} \sim \tilde{B}$	Error	$\tilde{A} \sim \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	Error
(Chu&Tsao, 2002)	$\tilde{A} < \tilde{B}$	$\tilde{A} \sim \tilde{B}$	Error	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	Error
(Chen& Chen, 2007)	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$
(Abbasbandy&Hajjari, 2009)	Error	$\tilde{A} \sim \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$
(Chen& Chen, 2009)	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < B < \tilde{C}$	$\tilde{A} > \tilde{B}$
(Kumar*etal*., 2011)	$\tilde{A} > \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$
(Babu *et al*., 2012)	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} < \tilde{B}$	$\tilde{A} = \tilde{B} < \tilde{C}$	$\tilde{A} > \tilde{B}$
(Rezvani, 2012)	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} > \tilde{B} > \tilde{C}$	$\tilde{A} < \tilde{B}$
Proposed Method	$\tilde{A} < \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} > \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} \sim \tilde{B}$	$\tilde{A} > \tilde{C} > \tilde{B}$	$\tilde{A} < \tilde{B}$

Remark 5.1

It is clearly seen from table (5.1) that our new method of ranking generalized trapezoidal fuzzy numbers using section-wise perimeters, like many other methods, can rank both normal and non-normal generalized trapezoidal fuzzy numbers. It is distinctive features of our method that it performs perceptively betterthan (Rezvani, 2012) and many others in respect of non-normal fuzzy numbers, though it is at par in the case of normal fuzzy numbers.

Example 5.2

In this example, the similarity measures of five sets of non-normalized trapezoidal fuzzy numbers are computed using our new approach and compared with many other existing approaches in table 5.2.

Set1: Let $\tilde{A} = (0.1,0.2,0.2,0.3;1)\ and\ \tilde{B} = (0.1,0.2,0.2,0.3;0.7)$

$$S(\tilde{A},\tilde{B}) = [1 - |0.2 - 0.2|] \times [1 - |1 - 0.7|]$$
$$\times \frac{\min(4.208,\ 3.014) + \min(0.1,\ 0.07)}{\max(4.208,\ 3.014) + \max(0.1,\ 0.07)}$$
$$= 0.303.$$

Set 2: Let $\tilde{A} = (0.1,0.2,0.2,0.3;1)\ and\ \tilde{B} = (0.2,0.2,0.2,0.2;0.7)$

$$S(\tilde{A},\tilde{B}) = [1 - |0.2 - 0.2|] \times [1 - |1 - 0.7|] \times \frac{\min(4.208,\ 2.8) + \min(0.1,\ 0)}{\max(4.208,\ 2.8) + \max(0.1,\ 0)}$$
$$= 0.454.$$

Set 3: Let $\tilde{A} = (0.1,0.4,0.4,0.7;0.825)\ and\ \tilde{B} = (0.3,0.4,0.4,0.5;1)$

$$S(\tilde{A},\tilde{B}) = [1 - |0.4 - 0.4|] \times [1 - |0.825 - 1|]$$
$$\times \frac{\min(4.005,\ 4.209) + \min(0.247,\ 0.1)}{\max(4.005,\ 4.209) + \max(0.247,\ 0.1)}$$
$$= 0.759$$

Set 4: Let $\tilde{A} = (0.2,0.3,0.5,0.6;0.7905)$ and $\tilde{B} = (0.3,0.4,0.4,0.5;1)$

$$S(\tilde{A},\tilde{B}) = [1 - |0.4 - 0.4|] \times [1 - |0.79205 - 1|]$$

$$\times \frac{\min(3.781,\ 4.209) + \min(0.237\ \ 0.1)}{\max(3.781,\ 4.209) + \max(0.237,\ 0.1)}$$

$$= 0.700.$$

Set 5: Let $\tilde{A} = (0.1,0.2,0.3,0.4;1)$ and $\tilde{B} = (0.1,0.2,0.3,0.4;0.8)$

$$S(\tilde{A},\tilde{B}) = [1 - |0.25 - 0.25|] \times [1 - |1 - 0.8|]$$

$$\times \frac{\min(4.409,\ 3.612) + \min(0.2\ \ 0.16)}{\max(4.409,\ 3.612) + \max(0.2,\ 0.16)}$$

$$= 0.654$$

Table 5.2 Comparison of non-normalized trapezoidal fuzzy numbers using our approach vis-à-vis some other existing approaches.

Sets	Chen (2009)	Lee (2012)	Chen (2009)	Wei (2009)	Hejazi (2011)	Wen (2011)	Rezvani (2012)	Proposed method
Set 1	1	1	0.7	0.7209	0.5113	0.5104	0.467	0.303
Set 2	0.95	0.75	0.9048	0.6215	0.383	0.4242	0.47	0.454
Set 3	0.9	0.8333	0.7425	0.814	0.6261	0.7321	0.66	0.759
Set 4	0.9	0.75	0.8911	0.838	0.6448	0.7432	0.603	0.700
Set 5	1	1	0.8	0.8248	0.6681	0.6659	0.66471	0.654

5.4 Application of the Revised Method to Risk Analysis

Definition 5.4.1 Arithmetic operations between generalized fuzzy numbers

These definitions of generalized fuzzy numbers are borrowed from (Zimmermann, 1996), (Chen and Chen, 2009), (Hsieh, 1999), and(Xiaoyan *et al*., 2011).

Let $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$ and $\tilde{B} = (b_1, b_2, b_3, b_4; w_{\tilde{B}})$ be any two generalized trapezoidal fuzzy numbers.

Generalized fuzzy numbers addition $\oplus$:

$$\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3, a_4; w_{\tilde{A}}) \oplus (b_1, b_2, b_3, b_4; w_{\tilde{B}})$$

$$= (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4; \min(w_{\tilde{A}}, w_{\tilde{B}}))(5.1)$$

Generalized fuzzy numbers subtraction $\ominus$:

$$\tilde{A} \ominus \tilde{B} = (a_1, a_2, a_3, a_4; w_{\tilde{A}}) \ominus (b_1, b_2, b_3, b_4; w_{\tilde{B}})$$

$$= (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1; \min(w_{\tilde{A}}, w_{\tilde{B}})).(5.2)$$

Generalized fuzzy numbers multiplication $\otimes$:

$$\tilde{A} \otimes \tilde{B} = (a_1, a_2, a_3, a_4; w_{\tilde{A}}) \otimes (b_1, b_2, b_3, b_4; w_{\tilde{B}})$$

$$= (a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, a_4 \times b_4; \min(w_{\tilde{A}}, w_{\tilde{B}})).(5.3)$$

Generalized fuzzy numbers division $\oslash$:

$$\tilde{A} \oslash \tilde{B} = (a_1, a_2, a_3, a_4; w_{\tilde{A}}) \oslash (b_1, b_2, b_3, b_4; w_{\tilde{B}})$$

$$= (a_1/b_4, a_2/b_3, a_3/b_2, a_4/b_1; \min(w_{\tilde{A}}, w_{\tilde{B}})).(5.4)$$

Alternatively, *division* is defined, called *division by ratio* by

$$\tilde{A} \oslash \tilde{B} = (a_1, a_2, a_3, a_4; w_{\tilde{A}}) \oslash (b_1, b_2, b_3, b_4; w_{\tilde{B}})$$

$$= (a_1/b_1, a_2/b_2, a_3/b_3, a_4/b_4; \min(w_{\tilde{A}}, w_{\tilde{B}})).$$

5.5. Fuzzy Risk Analysis Based on New Similarity Measures

The new similarity measure between generalized fuzzy numbers was used in order to deal with the fuzzy risk analysis problems. Let us consider the structure of risk analysis shown in Fig.5.2. Following (Chen and Chen, 2009) and (Xiaoyan, *et al*., 2011), each subcomponent A_i is evaluated by two evaluating items, R_i and W_i where R_i denotes the *probability of failure* of the subcomponent $A_i$andW_i denotes the *severity of loss* of the sub-component A_i, $1 \leq i \leq 3$ .In effect, all of the evaluating items are *linguistic* terms (i.e., *high*, *medium*, *low*…,etc.) to represent the valuesof R_i and W_i. Table (5.3) illustrates the linguistic terms andtheir corresponding generalized trapezoidal fuzzy numbers. Assume that there is a component Aconsistingof n subcomponents $A_1, A_2, \dots, A_n$, and that eachsubcomponent is evaluated by two evaluating items *probability of failure*$R_i$and *severity of loss*W_i . The computational steps for evaluating fuzzy risk analysis are as shown below:

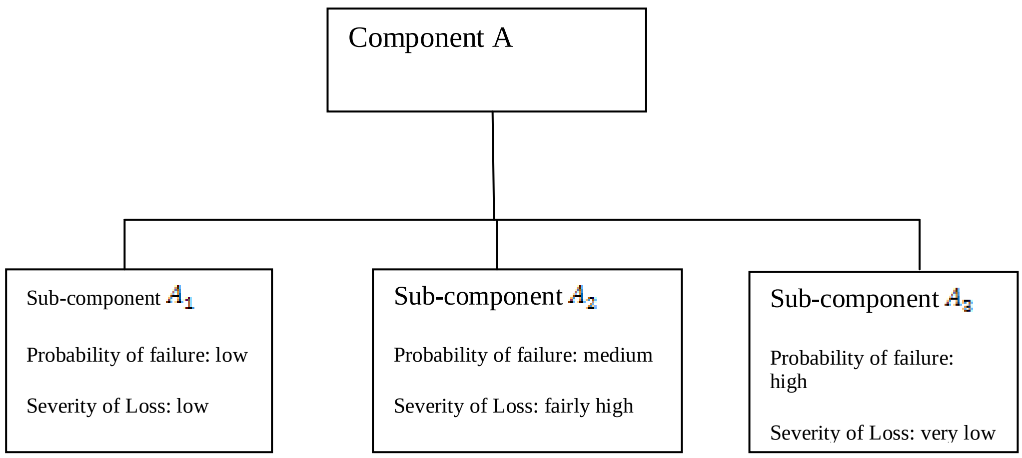


Figure 5.2. Structure of fuzzy risk analysis

Step1the generalized trapezoidal fuzzy numbers' arithmetic operations to integrate the evaluating items R_i and W_i of each subcomponentA_i,$1 \leq i \leq n$, to get the *total risk*$\tilde{R}$ of the component A is defined by

$$\tilde{R} = \left(\sum_{i=1}^{n} W_i \otimes R_i\right) \oslash (\sum_{i=1}^{n} W_i).$$

In our work, for simplicity, we interpret the probability of failure R_i and the severity of loss $W_i$representing the linguistic terms in a truncated model to get the total risk $\tilde{R}$ as follows:

$$\tilde{R} = \frac{low \otimes low \oplus medium \otimes fairly\ high \oplus high \otimes very\ low}{low \oplus fairly\ high \oplus very\ low} \quad (5.5)$$

Step 2: Use the revised similarity measures in equation(5.1) to evaluate the degree of similarity between the fuzzy number $\tilde{R}$and each linguistic term shown in Table (5.3). Table (5.3) shows the generalized trapezoidal fuzzy numbers representing different linguistic terms adapted from (Xiaoyan, et al., 2011).

Example 5.3

Consider the structure of risk analysis shown in Fig. (5.2), where the component *A* consists of three sub-components $A_1, A_2, and\ A_3$, and we want to evaluate the probability of failure $\tilde{R}$of the component *A*. Table (5.4) Shows the linguistic values of the two evaluating items R_i and $W_i$of the sub-components $A_1, A_2,$ andA_3, respectively, where the linguistic values are represented by generalized trapezoidal fuzzy numbers as shown in eqn (5.5).

Table 5.3A Nine Member Linguistic Term Set.

Linguistic Terms	Generalized Fuzzy Numbers
Absolutely low	(0.0,0.0,0.0,0.0;1.0)
Very low	(0.0,0.0,0.02,0.07;1.0)
Low	(0.04,0.1,0.18,0.23;1.0)
Fairly low	(0.17,0.22,0.36,0.42;1.0)
Medium	(0.32,0.41,0.58,0.65;1.0)
Fairly high	(0.58,0.63,,0.80,0.86;1.0)
High	(0.72,0.78,0.92,0.97,1.0)

Very high	(0.93,0.98,1.0,1.0;1.0)
Absolutely high	(1.0,1.0,1.0,1.0;1.0)

The generalized trapezoidal fuzzy number of each linguistic term stated in table (5.3) above can be transformed using the graded mean integration of $\tilde{A}$ and $\tilde{B}$ as proposed by (Xiaoyan, *et al*., 2011).

Assume that there are two generalized trapezoidal fuzzy numbers $\tilde{A} = (a_1, a_2, a_3, a_4; w_{\tilde{A}})$and $\tilde{B} = (b_1, b_2, b_3, b_4; w_{\tilde{B}})$. The graded mean integrations of $\tilde{A}$ and $\tilde{B}$ are as follows:

$$P(\tilde{A}) = \frac{a_1 + 2a_2 + 2a_3 + a_4}{6} \quad (5.6)$$

$$P(\tilde{B}) = \frac{b_1 + 2b_2 + 2b_3 + b_4}{6} \quad (5.7)$$

We can also re-write the trapezoidal fuzzy numbers $\tilde{A}$ and $\tilde{B}$ as

$$\tilde{A} = \left(P(\tilde{A}); w_{\tilde{A}}\right) \quad (5.8)$$

and

$$\tilde{B} = \left(P(\tilde{B}); w_{\tilde{B}}\right). \quad (5.9)$$

Table 5.4A Nine Member Linguistic Term Set by using the graded mean integration

Linguistic Terms	Generalized Fuzzy Numbers
Absolutely low	(0;1.0)
Very low	(0.0183;1.0)
Low	(0.138;1.0)

Fairly low	(0.2917;1.0)
Medium	(0.4917;1.0)
Fairly high	(0.7166;1.0)
High	(0.8480;1.0)
Very high	(0.9816;1.0)
Absolutelyhigh	(1.0;1.0)

Using (5.1), (5.3) and (5.4),we can evaluate the value of $\tilde{R}$ from (5.5) and table (5.3) as a single generalized trapezoidal fuzzy number as

$\tilde{R} = (0.16,0.27,0.70,1.09;1.0)$,which represents the probability of failure that can be used together with the respective linguistic terms in table (5.3).In order to calculate the degree of similarity between the generalized trapezoidal fuzzy number $\tilde{R}$ and each linguistic term using the new similarity measure approach (5.1) we have:

$$S(\tilde{R}, low) = \left[1 - \left|x'_{\tilde{R}} - x'_{low}\right|\right] \times [1 - |w_{\tilde{R}} - w_{low}|] \times \frac{\min\left(P^+(\tilde{R}), P^+(low)\right) + \min\left(A(\tilde{R}), A(low)\right)}{\max(P^+(\tilde{R}), P^+(low)) + \max(A(\tilde{R}), A(low))}.$$

That is,

$$S(\tilde{R}, low) = [1 - |0.614 - 0.135|] \times [1 - 0] \times \frac{\min(5.439,4.273) + \min(0.68,0.135)}{\max(5.439,4.273) + \max(0.68,0.135)} = 0.3753.$$

In this way, the similarity measures between the generalized trapezoidal fuzzy number $\tilde{R}$ and each linguistic term can be evaluated as follows:

$$S(\tilde{R}, absolutely\ low) = 2.527, \qquad S(\tilde{R}, very\ low) = 0.2835,$$

$$S(\tilde{R}, low) = 0.3753,$$

$S(\tilde{R}, fairly\ low) = 0.5102$, $S(\tilde{R}, medium) = 0.6778$,

$S(\tilde{R}, fairly\ \text{high}) = 0.6842$, $S(\tilde{R}, \text{high}) = 0.5762$,

$S(\tilde{R}, very\ \text{high}) = 0.4380$,

$S(\tilde{R}, absolutely\ \text{high}) = 0.4013$.

Finally, the following table of comparison of our new approach with many other similar approaches is as follows:

Table 5.5 The Results of the New Approach with Different Existing Approaches.

Linguistic terms	Chen and Chen‘s (2009)	Deng's (2004)	Wei's (2009)	Xiaoyan's (2011)	New method
Absolutely low	0.1565	0.2821	0.3235	0.6981	0.2527
Very low	0.1962	0.3127	0.3494	0.7006	0.2835
Low	0.3226	0.4704	0.4571	0.7648	0.3753
Fairly low	0.5092	0.7157	0.5921	0.8666	0.5102
Medium	0.7056	0.9072	0.6538	0.9559	0.6778
Fairly high	0.5828	0.5160	0.5960	0.7867	0.6842
High	0.4545	0.3525	0.5267	0.7131	0.5762
Very high	0.2937	0.2038	0.3977	0.6510	0.4380
Absolutely	0.2391	0.1818	0.3703	0.6433	0.4013

high					

Remark 5.3

The fuzzy number $\tilde{R} = (0.16,0.27,0.70,1.09; 1.0)$ can also be expressed using (5.6) and (5.8) and the result is $\tilde{R} = (0.5316; 1.0)$ which is much closer to the linguistic term *medium*, and coincides with the result of (Xiaoyan, *et al*., 2011). Therefore the probability of failure of the component A is medium.

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Summary

In this work, an up-to-date review of the existing literatureon ranking methods of fuzzy numbers was presented in chapters 1 and 2. In chapter 3,fundamentals of fuzzy numbers: notion of fuzzy set, algebraic properties of fuzzy numbers and the basis of fuzzy numbers' operations were discussed.A critical study of different ranking methods of fuzzy numberswith their various shortcomings and comparisons were studied in chapter 4.In chapter 5, we developed a generalizedmethod of ranking generalized trapezoidal fuzzy numbers by taking section-wise perimeters of the trapezoid. Also, we defined the similarity measures using our generalized method.Application of the new method to risk analysis was also demonstrated.

6.2 Conclusion

The method described in this work is a generalization of the method of ranking generalized trapezoidal fuzzy numbers described in (Rezvani, 2012, 2015). Its performance is evaluated on a set of standard examples used in the literature. The generalized approach is evaluated on both normal and non-normal trapezoidal fuzzy numbers, together with a revised similarity measure.On normal fuzzy numbers, performance of the modified approach described in this work is found marginally better than that of (Rezvani, 2012, 2015). Nevertheless, on non-normal cases, the modified approach is found to perform perceptively better. Finally the modified approach is applied to risk analysis and the results are compared with that of other methods.

6.3 Recommendations

In view of the significance of the subject matter of this dissertation, it is recommended that, in line with our work, other possible modifications and / or developing a prototype general method for ranking generalized fuzzy numbers could be undertaken which would help in solving a number of real-life problems in engineering, sciences, and management. For instance, cases such as orthocenter, inCentre and circumcentre could be considered in ranking both normal and non-normal trapezoidal and triangular fuzzy numbers.

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