A GENERAL ITERATIVE SCHEME FOR FIXED POINTS OF NONLINEAR OPERATORS

By

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Declaration page

I declare that the work in this thesis entitled A GENERAL ITERATIVE SCHEME FORFIXED POINTS OF NONLINEAR OPERATORS has been carried out by me in the Department of Mathematics under the supervision of Dr. B. Ali and Dr. A. O. Ajibade.

The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

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(Student) Signature Date

Certification

This thesis entitled 'A GENERAL ITERATIVE SCHEME FOR FIXED POINTS OF NONLINEAR OPERATORS' meets the regulations governing the award of the degree of Masters in Mathematics of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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Dedication

To my father,

Mr. Daniel Owa

My mother,

Mrs. Igunnu Owa,

and my wife,

Justina Owa

Each of whom I owe more than what I can express here.

Acknowledgement

My first gratitude goes to Almighty God for his guidance and protection throughout my life and studies. My sincere gratitude and appreciation go to my parents, my wife and my understanding children for their immense prayers, support and encouragement at all times, I pray that they live long to reap the fruits of their labour and I pray that Almighty God will reward them abundantly. Thank you, I am very grateful.

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I equally commend all staff of the Department of Mathematics for their assistance in verifying my results and advice. May Almighty God bless and reward them all.

Finally, I wish to express my thanks to my well-wishers whose encouragement has been a major support over several years.

Owa A,

August, 2014.

Abstract

This thesis examines a general iterative scheme for fixed points of nonlinear operators using the special cases of the Krasnoselskii-Mann (KM) iterative procedure for particular choices of the nonexpansive operator N. We prove many theorems and lemmas that help to see the relationship between some iterative algorithms. The iterative scheme $x^{k+1} = Tx^k =$

$(1- \alpha)x^k + \alpha Nx^k$ has been shown to be one of the schemes that can be generally used to represent some iterative schemes for finding fixed points of nonlinear operators.

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Chapter One

Introduction and Preliminaries

1.1 Introduction

Many problems in Mathematics and related field can be solved by finding fixed point of a particular operator, and algorithms for finding such points play a prominent role in a number of applications.

The article by Bauschke (1996) is fundamental to this work. This section deals with definitions of some basic terms used in the subsequent discussions while some examples are also given to make these definitions clearer.

1.2 Research questions

There are many open challenging questions on fixed point theorem that have still not been answered which include the following;

i. Is there a general algorithm for some iterative schemes finding fixed point of nonlinear operator?

ii. Are there significant relationships between nonexpansive *ne* operator N, firmly nonexpansive fne operator F and average *av* operator T?

iii. Are the iterative schemes of the complements of nonexpansive *ne*, firmly nonexpansive fne and average operator *av T* converge to fixed points of T as the main operators?

iv. Which of the properties of the operator T is sufficient to guarantee convergence of the sequence $\{T^k x\}$ to fixed point of T, whenever such fixed points exist?

1.3 **Aim and Objectives of the Study**

The aim of this thesis is to unify the several iterative algorithms for finding the fixed points of some nonlinear operators. The iterative methods considered have the form,

$$x^{k+1} = T x^{k}. \qquad \text{for } k = 0,1\ldots \qquad (1.3.1)$$

Where T is a linear or nonlinear continuous operator on a real (possibly infinite dimensional) Hilbert space H and x^0 is an arbitrary starting vector. For any operator T on H the fixed point set of T is denoted by $\text{Fix}(T) = \{z/\ Tz = z\}$.

The main objectives here are;

i. To find a common iterative scheme for some iterative algorithms for finding the fixed points of some nonlinear operators.

ii. To find the property of the operator T which is sufficient to guarantee convergence of the sequence $\{T^k x\}$ to a fixed point of T whenever such fixed points exist.

1.4 Statement of the Research Problem

In the algorithms of interest here, the operator T is selected so that the set Fix(T) contains those vectors z that possess the properties we desire that is 'find a general iterative scheme for some iterative algorithms for nonlinear operators' finding a fixed point of the some iterative schemes leads to a solution of the problem.

Some applications involve constrained optimization, in which we seek a vector x in a given convex set C that minimizes a certain function f. For suitable $\gamma > 0$ the fixed points of the operator $T = P_C(I - \gamma\nabla f)$ will solve the problem under conditions to be discussed below.

1.5 Research Methodology

This thesis is a review of the paper of Byrne (2004). Many related and necessary papers were consulted for effective outcome. These papers are thoroughly reviewed to cover a major part of this work. Lemmas, propositions and Theorems are stated and proved in order to ascertain the properties, relationship and applications of the operators used in the thesis while conclusion are drawn based on the theorems, propositions and lemmas proved.

1.6 Convergence in the normed space

Let $(X, \|.\|)$ be a normed space and let $\{x_n\}$ be a sequence in X, then $\{x_n\}$ is said to

(i) converge strongly, if $\exists\, x \in X \ni \|x_n - x\| \to 0$ as $n \to \infty$,

(ii) converge strongly if $\exists\, x \in X \ni |f(x_n) - f(x)| \to 0$ as $n \to \infty \quad \forall \quad f \in x^*$

1.7 Nonexpansive operator

An operator N on H is called nonexpansive *ne* if $\forall\, x, y \in H$, we have

$$\|Nx - Ny\| \leq \|x - y\|$$

1.8 Hilbert space

An inner product space (or a pre-Hilbert space) is said to be complete if every Cauchy sequence $\{x_n\} \in X$ converges to a point $x \in X$. A complete pre-Hilbert space is called a Hilbert space.

1.9 Fixed Point

Let X be a nonempty set and $T: X \to X$ be a map. We say that $x \in X$ is a fixed point of T if $T(x) = x$ and we denote the set of all fixed points of T by $F(T) = \{x \in X : Tx = x\}$.

Many problems of nonlinear analysis could be solved by the use of various fixed points theorems.

Example 1.9.1

1. If X = $\mathbb{R}$ and T(x) = x^2+ 5x + 4, then F (T) ={-2};

2. If X = R and T (x) = x^2 - x, then F(T) = {0, 2}.

Let X be any set and T: X → X be a map. For any given $x \in X$, we denote $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the nth iterate of x under T, For simplicity we use Tx instead of T (x).

The mapping T^n ($n \geq 0$) is called the nth iterate of T. For any $x \in X$; the sequence $\{x_n\}_{n \geq 0} \subset X$ given by $x_n = Tx_{n-1}$, n = 1, 2, 3,… is known as the Picard's iterations. For any given self-mapping T the following properties obviously hold:

i. F (T) $\subset$ F(T^n) for each $n \in N$,

ii. F(T^n) = {x}, $\Rightarrow$ F(T) = {x}.

But the reverse in (i) is not true in general, as shown by the next example.

Example 1.9.2 Let T: {1, 2, 3} → {1, 2, 3} be denoted by T(1) = 3,T (2) = 2 and T (3) = 1.

Then F (T) = {2} and FT(2) = {1, 2, 3}.

The fixed point theorems are concerned with finding conditions on the structure that the set X must be endowed as well as on the properties of the operator T: X → X, in order to obtain result on,

a. existence (and possibly uniqueness) of fixed point .

b. the data dependence of fixed points.

c. the construction of fixed point.

1.10 Operators on Hilbert Space

(i) Adjoint Operator

Let y be a fixed element in H and let T (y) be defined on H by $T(y)x = \langle T(x), y\rangle$, $\forall\, x \in H$ then $T(y)$ is a bounded linear functional on H.

Therefore, by Riez representation Theorem $\exists$ a unique vector $z \in H \ni T(y) = \langle x, z\rangle$, $\forall\, x \in H$ that is $\langle x, z\rangle = \langle T(x), y\rangle\ \forall\, x \in H$ putting $z = T^* y$ we see that $\langle x, Ty\rangle = \langle T(x), y)\rangle\ \forall\, x, y \in H$.

The mapping T^* is called the adjoint of T

(ii) Self-adjoint Operator

Let T be a bounded linear operator on a Hilbert space H. Then T is said to be self-adjoint if $T = T^*$. Note that $0 = 0^*$ and $I = I^*$, that is zero and identity operators are self-adjoints.

(iii) Normal Operator

A bounded linear operator T on a Hilbert space H is called a Normal operator if $T^* T = TT^*$

(iv) Unitary Operator

A bounded linear operator T on a Hilbert space H is called unitary, if $T^* T = TT^* = I$

1.11 Firmly nonexpansive operator

An operator F on H is called firmly nonexpansive *fne* if F is defined as

$$\langle Fx - Fy, x - y\rangle \geq \|Fx - Fy\|^2 \quad \forall\, x,y \in H$$

1.12 Monotone Operator

An operator $G: H \to H$ is monotone if, for all $x, y \in H$, we have

$$\langle Gx - Gy, x - y\rangle \geq 0$$

1.13 Strict contraction

An operator S on H is said to be a strict contraction (sc) if $\exists\ \delta \in [0, 1)$, such that $\forall$ $x, y \in H$,

$$\|Sx - Sy\| \leq \delta\, \|x - y\|$$

1.14 Average operator

An operator A is called average if $\exists\ \alpha \in (0; 1)$ and ne operator N, $\ni$ we have $A = (1- \alpha)I + \alpha N$

1.15 V-ism operator

An operator G on H is called v-inverse strongly monotone (v- ism), v > 0, ∋

$$\langle Gx - Gy, x - y\rangle \geq v \|Gx - Gy\|^2$$

1.16 Sequences in a Metric Space

Let X be a nonempty set, a sequence in X is a function from ℕ (the set of positive integers) into X. If x is a sequence in X, the image x(n) of n ∈ ℕ is usually denoted by $\{x^n\}$. It is customary to denote the sequence x by the classical symbol $\{x_n\}$.

Some time we write it as $\{x_1, x_2, x_3 \ldots, x_n \ldots\}$.

The following are examples of real sequences: $\{\frac{n}{n+1}\}$, $\{\frac{1}{n}\}$, $\{\cos \frac{n\pi}{3}\}$.

A sequence $\{x_n\}$ of real number is called

(a) strictly monotone decreasing if $x_{n+1} < x_n \; \forall \; n \in \mathbb{N}$;

(b) Monotone non-decreasing if $x_{n+1} \geq x_n \; \forall \; n \in \mathbb{N}$;

(c) Strictly monotone increasing if $x_{n+1} < x_n \; \forall \; n \in \mathbb{N}$;

(d) Monotone non-increasing if $x_{n+1} \leq x_n \; \forall \; n \in \mathbb{N}$;

1.17 Subsequences in a Metric Space

Let x: ℕ → X is a sequence on X. Let n: ℕ → ℕ be a strictly increasing function, then the function x_n: ℕ →X is called a subsequence of the original sequence x: ℕ → X.

Example 1.17.1 *The sequence of prime numbers {2, 3, 5, 7, 11,} is a subsequence of the sequence of natural numbers.*

The proof of this is shown below:

Let $\{x_n\} = \{2, 3, 5, 7, 11, \ldots\}$.

Then x1 = 2, $x_2 = 3$, $x_3 = 5$, $x_4 = 7$, e t c.

Take $n_1= 2$, $n_2= 3$, $n_3= 5$, $n_4= 7$, … . Then $\{n_k\}$ is a strictly increasing sequence of positive integers. Therefore,

$x(n_{(1)}) = 2$

$x(n_{(2)}) =3$

$x(n_{(3)}) = 5$ etc.:

Hence, {2, 3, 5, 7, 11, …}. Is a subsequence of $\{x_n\}$.

1.18 Convergence of a Sequence in a Metric Space

Let $\{x_n\}$ be a sequence in a metric space (X; d), this sequence is said to converge to a point $x \in X$ if given $\varepsilon > 0$, $\exists\ n_0 \in N$; such that $d(x_n, x) < \varepsilon \quad \forall\ n \geq n_0$

The fact that $\{x_n\}$ converges to x is expressed by writing $x_n \to x$ as $n \to 1$ or $\lim x_n = x$

A sequence $\{x_n\}$ in a metric space (X, d) is called a convergent sequence if the sequence converges to a point say $x \in X$.

1.19 Bounded Sequences

A sequence $\{x_n\}$ is said to be bounded below if there exists a real number α such that $x_n \geq \alpha$ $\forall\ n \in N$.

The number α is called a lower bound of the sequence $\{x_n\}$.

Example 1.19.1

The sequence $\{x_n\} = \{1, 2, 3, 4, \ldots\}$ is bounded below as there exists a number 1 which is less than each term of the sequence i.e.

$x_n \geq 1, \quad \forall\ n \in N.$

A sequence $\{x_n\}$ is said to be bounded above if there exists a real number β such that $x_n \leq \beta$ $\forall\ n \in N$.

The number β is called an upper bound of the sequence $\{x_n\}$.

Example 1.19.2

For example, the sequence $\{x_n\} = \{-1, -2, -3, -4 \ldots\}$ is bounded above as there exists a number -1 which is greater than each term of the sequence i.e. $x_n \leq -1, \quad \forall\ n \in N$.

A sequence $\{x_n\}$ is said to be bounded if it is bounded both above and below. i.e. a sequence $\{x_n\}$ is bounded if there exist two real numbers α and β such that $\alpha \leq x_n \leq \beta \quad \forall\ n \in N$.

If we choose M = max $\{|\alpha||\beta|\}$, then the sequence $\{x_n\}$is bounded provided $|x_n| \leq M$

$\forall\ n \in N$.

Example 1.19.3

if $x_n = (\frac{1}{n})$, then $\{x_n\} = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$, where $x_n \leq 0$ and $x_n \geq 1$, $\forall\ n \in N$. i.e. $0 \leq x_n \leq 1$.

$\forall\ n \in N$.

Hence, the sequence $\{x_n\}$ is bounded.

1.20 Attracting mapping

Definition 1.21.1 *Suppose D is a closed convex nonempty set, T: D→D is nonexpansive, and F is a closed convex nonempty subset of D. we say that T is attracting w:r:t: F if for every* $x \in D|_F$,

$f \in F$, *then we have* $\quad \|Tx - f\| < \|x - f\|$,

In other words, every point in F attracts every point outside F. A more quantitative and stronger version is the following. We say that T is strongly attracting w,r,t, F if there is some k > 0 s.t for every $x \in D, f \in F$

$$k\|x - Tx\|^2 \leq \|x - f\|^2 - \|Tx - f\|^2,$$

Alternatively, if emphasis is laid on k explicitly, we say that T is k-attracting w.r.t F. In several instances, F is Fix T, in this case, we simply speak of *attracting, strongly attracting, or k - attracting mapping.*

1.21 Outline of the thesis

This thesis contains five other chapters after this chapter. The outlines of chapter two to six are as follows:

Chapter 2: This contains the reviewed relevant literature that has direct bearing with the thesis 'a general iterative scheme for signal enhancement and image modification'

Chapter 3: In this chapter, we proved lemmas, theorems, definitions and propositions; we also compare them in order to see their significant relationship

Chapter 4: This chapter contains the analysis of some of the algorithms that are popularly used in signal enhancement and image modification which are relevant to the work.

Chapter 5: Here, we present the main result of the thesis through a theorem which is proved to see how the objectives are accomplished.

Chapter 6: We hereby give the conclusion based on the main result presented in chapter five and also present the summary of the entire work.

Chapter two

2.0 Literature Review

2.1 **A Superior Implementation of the Algebraic Reconstruction Technique Algorithm**

In recent times, many scholars worked extensively on signal processing and image reconstruction; on its formations, importance and applications.

Anderson and Kak (1984), worked on a superior implication of Algebraic Reconstruction Technique (ART) and agreed that the implementation of ART generates a good reconstruction in only one iteration, it also appears to have a computational advantage over the more traditional implementation of ART.

Potential applications of this implementation include image reconstruction in conjunction with ray tracing for ultrasound and microwave tomography in which the curved nature of the rays lead to non-uniform ray density across the image.

The two types of algebraic approach to image reconstruction; ART-type and simultaneous image Reconstruction Technique (SIRT)-type mutually exclusive advantages, ART-type methods based directly on the sequential scheme of the Kaczmarz procedure and enjoy a rapid convergence in the sense of the root-mean-square error criterion, Censor and Elfving (1994).

The problems associated with the sequential ART algorithm arise because of inconsistencies in the set of equations representing the forward process. The discrete formulation does not exactly represent the line integrals of the original continuous image function. Byrne (2002) found that, the procedure for solving a system of consistent equations has been demonstrated to converge towards a generalized inverse, according to Eggermont (1990), the resulting vector is likely to represent a very noisy looking image.

They also opined that a significant reduction in the noise can be achieved if we apply all error correction terms for rays in a particular projection simultaneously rather than in the conventional sequential fashion. Furthermore, the good performance of the basic ART algorithm by the root-mean-squared error criterion has prompted them to seek to improved reconstructions in the set of the equations representing the ray integrals as well as in the reconstruction procedure itself.

As the number of iterations of an algorithm increases, the continued convergence will ultimately depend on the accuracy of the discrete representation of the forward projection process

Byrne (2004).

2.2 The Multiple-sets Split Feasibility Problem and its Applications for Inverse Problems

According to Censor and Borrfeld (1999), the multiple-sets split feasibility problem requires finding a point closest to a family of closed convex sets in one space such that its image under a linear transformation will be closest to another family of closed convex sets in the image space, their study on the above, offers both intensity-modulated radiation therapy (IMRT) and for other inverse problems, is a mathematically valid frame work for applying projection algorithms to inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operator's range.

Given closed convex sets $C_i \subseteq R^N$, $i = 1, 2, \ldots, t.$ and closed convex sets $Q_j \subseteq R^{M,}$ $j = 1,; 2,.., r,$ in the N- and M- dimensional Euclidean spaces, respectively, the multiple-sets split feasibility problem, proposed and studied here, is to find a vector x* for which

$$x^* \in C := \bigcap_{t=1}^{t} C_i \text{ and that } Ax^* \in Q := \bigcap_{j=1}^{r} Q_j \qquad (2.2.1)$$

where A is a given M x N real matrix. This can serve as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in the operator's range.

The multiple-sets split feasibility problem extends the well-known convex feasibility problem, when there are no matrix A and sets Q_j present at all, or put differently, when $Q = R^M$.

The systems of linear equations, linear inequalities or convex inequalities, according to Borwein and Lewis (2000), are all encompassed by the convex feasibility problem which has broad applicability in many areas of mathematics and the physical and engineering sciences.

These include, among others, optimization theory and image reconstruction from projections in computerized tomography.

2.3 Projection Methods and their advantages

Bortfeld and Dugundji (1970) confirmed that Projections onto sets are used in a wide variety of methods in optimization theory but not every method that uses projections really belongs to the class of projection methods.

Projection methods are iterative algorithms that use projections onto sets while relying on the general principle that when a family of (usually closed and convex) sets is presented then projections onto the given individual sets are easier to perform than projections onto other sets (intersections, image sets under some transformation, etc.) that are derived from the given individual sets.

Projection algorithms employ projections onto convex sets in various ways. They may use different kinds of projections and, sometimes, even use different projections within the same algorithm.

They serve to solve a variety of problems which are either of the feasibility or the optimization types. They have different algorithmic structures, of which some are particularly suitable for parallel computing, and they demonstrate nice convergence properties and/or good initial behaviour patterns.

This class of algorithms has witnessed great progress in recent years and its member algorithms have been applied with success to fully-discretized models of problems in image reconstruction and image processing, Stark and Censor (1998).

Apart from theoretical interest, the main advantage of projection methods which makes them successful in real-world applications is computational.

They commonly have the ability to handle huge-size problems of dimensions beyond which other, more sophisticated currently available methods cease to be efficient. The multiple-sets split feasibility problem and its applications for inverse problems algorithm are the projections onto the given individual sets (assumed and actually easy to perform) and the algorithmic structure is either sequential or simultaneous (or in-between).

Sequential algorithmic structures cater for the row-action approach, Byrne (1995), while simultaneous algorithmic structures favour parallel computing platforms, Gordon (1970).

2.4 The Split Feasibility Problem

The special case when there is only one set in each space, i.e. t = r = 1 in (2.2.1), was proposed by Elfving (1994) and termed the split feasibility problem (because of the limitation to one set in each space we will call this from now on the two-sets split feasibility problem).

There we used our simultaneous multiprojections algorithm, Censor (1994) to obtain an iterative algorithm whose iterative step has the form

$$x^{k+1} = A^{-1}(I + AA^T)^{-1}(AP_C(x^k) + AA^T P_Q(Ax^k)), \qquad (2.4.1)$$

to solve the two-sets split feasibility problem. Here x^k and x^{k+1} are the current and the next iteration vectors, respectively, T stands for matrix transposition, I is the unit matrix and P_C and P_Q denote the orthogonal projections onto C and Q, respectively.

That solution was restricted to the case when M = N and to the feasible case, i.e., when $Q \cap A(C) \neq \emptyset$. Byrne and Censor (2002) investigated this further. Recognizing the potential difficulties with calculating inverses of matrices, or equivalently, solving a linear system in each iterative step, particularly when the dimensions are large, Byrne (1997) devised the CQ algorithm which uses the iterative step

$$x^{k+1} = P_C(x^k + \gamma A^T(P_Q - I)Ax^k) \qquad (2.4.2)$$

Where $\gamma \in (0, 2/L)$ and L is the largest eigenvalue of the matrix A^TA. One might wonder why not solve one of the convex feasibility problems of finding a point in $Q \cap A(C)$ or of finding a point in $C \cap A^{-1}(Q)$ instead of using the CQ-algorithm.

Examples of situations when this would not be recommended can occur when, due to the underlying specific data of the real-world problem, it is not easy to perform projections onto the sets $A(_C)$ and/or $A^{-1}(Q)$.

Other examples might occur when the dimensions M and N are very different from each other and choosing one of those convex feasibility problems would cost us in calculating projections for one of the sets in a much larger dimensional space than if we perform projections in each space separately onto the given individual sets. Similar arguments apply to the multiple-sets split feasibility problem.

2.5 Cimmino's Method and the Algebraic Reconstruction Technique

Gerchberg (1974) opined that to illustrate the simultaneous and sequential methods just described, we consider the problem of solving a system of linear equations Ax = b, where A is a real M by N matrix. For m = 1,… ,M let a^m be the mth column of A^T, so that

$$b_m = \langle a^m, x\rangle, \text{ and let } C_m = \{\omega \mid \langle a^m, \omega\rangle = b_m\}. \qquad (2.5.1)$$

Assume for notational convenience that the rows of A have length one.

Then we have

$$P_{Cm}\, x = x + (b_m \quad \langle a^m; x\rangle)a^m \quad .$$

The simultaneous algorithm now takes the form

$$x^{k+1} = x^k + \frac{1}{M}\sum_{m=1}^{M} \quad (b_m - \langle a^m, xk\rangle)a^m$$

or

$$x^{k+1} = x^k + \frac{1}{M} A^T(b - Ax^k\rangle).$$

This method is sometime called Cimmino's method according to Censor and Elfving (1994). The sequential method has the iterative step

$$x^{k+1} = x^k + (b_{m(k)} - \langle a^{m(k)}, x^k\rangle)\, a^{m(k)} \text{ for } m(k) = k(\text{mod}M) + 1.$$

This method, according to Byrne (2002) and Herman (1970) is called the algebraic reconstruction technique (ART). It has been shown by Tanabe (1971) that the product of the metric projections has fixed points in this case, also confirmed by Gubin *et al.* (1967).

When the system Ax = b has multiple solutions both Cimmino's method and the ART converge to that solution closest to the starting vector x^0.

2.6 Bandlimited Extrapolation Methods

The bandlimited extrapolation problem as an illustration of alternating POCS, according to Byrne (2002), the resulting iterative algorithm is the Gerchberg-Papoulis (GP) method based on Gerchberg (1974) and Rockafellar (1970). As we shall see, the method can be implemented in a noniterative manner, leading to more general linear and nonlinear extrapolation procedures that have been used for image and array processing.

The continuous formulation of the bandlimited extrapolation problem is the following: let f (t) and F (ω) be a Fourier transform pair, where t and ω are real variables and

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{i\omega t}\, dt, \tag{2.6.1}$$

$$f(t) = \int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t}\, d\omega /2\pi, \tag{2.6.2}$$

We assume that $F(\omega) = 0$, for $|\omega| > \Omega$, where Ω is a positive quantity.

The function f (t) is then said to be Ω bandlimited. If we know f (t) for t in some bounded interval of the real line, then these data determine $F(\omega)$ uniquely, by analyticity; the extension of f (t) to complex z, given by

$$f(z) = \int_{-\infty}^{\infty} f(\omega)\, e^{-i\omega t}\, d\omega /2\pi, \tag{2.6.3}$$

can be differentiated under the integral sign, since the limits of integration are definite. Therefore, the known values of f (t) determine f (z) for all other values of z; we can, in theory, extrapolate f outside the data window.

In practice, we have only definitely many values of f (t) and these are typically noisy. We shall not address the noise problem here, except to say that it is usually handled by including regularization in the solving of each of the systems of linear equations we encounter in what follows.

The definitely many values of if , say $f(t_1),\ldots, f(t_N)$, may be obtained at irregularly spaced sample points $\{t_n\}$ but often correspond to uniformly spaced sampling points $\{t_n = a + n\Delta\}$, we consider the latter case here.

For the remainder of this section we assume that the function $F(\omega)$ is supported on the interval $[-\Omega, \Omega]$, for some $\Omega < \pi$. The sequence of Fourier coefficients of F is denoted f. Our data are the Fourier coefficients f (n), for $n \in \{M; M + 1,\ldots, N\}$, forming the vector *d*.

For any function $G(\omega)$ let $\Omega G(\omega)$ be the function that equals $G(\omega)$ for $|\omega| \leq \Omega$ and equals zero otherwise. For any sequence of Fourier coefficients $g = \{g(n)\}$ let Dg denote the sequence whose

terms are g(n) for n ∈ {M, M + 1,… N} and zero otherwise. Let $\eta g = G$ be the operator taking a sequence of Fourier coefficients g into the function

$$G(\omega) = \sum_{n=-\infty}^{+\infty} g(n) \exp(in\,\omega), \qquad \text{for } \omega \in (-\pi, \pi).$$

Let $H = L^2(-\pi, \pi)$, $C_1 = L_2(-\omega, \omega)$ and C_2 be the set of all members $G(\omega)$ of H whose Fourier coefficients satisfy g(n) = f (n) for n = M, M +1, …, N .

The metric projection of a function $G(\omega) \in H$ onto C_1 is $G(\omega)$. The metric projection onto C_2 is implemented by passing from $G(\omega)$ to the sequence of its Fourier coefficients $\eta^{-1}G = g$, then replacing those coefficients for n = M, M + 1, …, N with f (n) and calculating the resulting Fourier series; that is, the metric projection of G onto C_2 is $\eta(Df + (I - D)\eta^{-1}G)$.

We begin the GP iteration with the function $F^0(\omega) = 0$ for all $\omega \in (-\pi, \pi)$. For k = 0, 1,…, having calculated F^K with f^k its sequence of Fourier coefficients, we dene F^{k+1} by

$$F^{k+1} = \mathbf{\Omega}\eta\,(Df + (I - D)\eta^{-1}F^k),$$

It would appear that, in order to implement this algorithm, we must calculate the entries of the sequence $\{(I - D)\eta^{-1}F^k)\}$ for all integers *n* not in the set {M, M + 1, …, N }; this is not the case, fortunately. Note that $Fk^{+1} - F^k = \mathbf{\Omega}\eta D(f - f^k) = \mathbf{\Omega}\eta a^k$ where the entries of the sequence $D(f - f^k) = a^k$ are zero, except for n = M,…, N. Since $F^0 = 0$ it follows that each F^k has the form $F^k = \mathbf{\Omega}\eta b^k$, for some sequence b^k with $b^k(n) = 0$ for *n* not in the set {M, M + 1,…, N }.

From this we conclude that the limit F^∞ has the form $F^\infty 1(\omega) = \mathbf{\Omega} \sum_{n=M}^{N} c_n \exp(in\,\omega)$, for appropriate c_n. The coefficients c_n can then be determined by equating the Fourier coefficients of both sides of this equation. To do this we must solve the definite system of linear equations

$$f(m) = \sum_{n=M}^{N} c_n \frac{\sin \Omega(m-n)}{\pi(m-n)}, \qquad (2.6.4)$$

where m = M,…, N . This, of course, can also be done iteratively, if we desire. A different approach is frequently used, resulting in a slightly different extrapolation. This second approach

formulates the problem entirely in terms of finite vectors and interprets the Fourier transform as a linear transformation between finite vectors, as is done with the fast Fourier transform (FFT) algorithm.

From the discussion above, we see that for an arbitrary data vector d and an arbitrary choice of the band [$-\boldsymbol{\Omega}$, $\boldsymbol{\Omega}$] in [$-\pi, \pi$] there is a function $F_{\boldsymbol{\Omega}}(\omega)$ supported on [$-\boldsymbol{\Omega}$, $\boldsymbol{\Omega}$] that is consistent with the data in the vector d. The function $F_{\boldsymbol{\Omega}}$ has the form

$$F_{\boldsymbol{\Omega}}(\omega) = \boldsymbol{\Omega} \sum_{n=M}^{N} c_n \exp(in\omega), \qquad (2.6.5)$$

The coefficients c_n solve the equation (2.4.4). To perform data extrapolation one now evaluates the Fourier transform of $F_{\boldsymbol{\Omega}}$ at the desired points. Note that this method applies equally to uniformly and nonuniformly spaced data and is easily extended to higher dimensions. This noniterative implementation of the GP extrapolation is not new; it was presented in Bregman (1967), and has been rediscovered several times since then according to Tanabe (1971).

The form of the estimator in equation (2.4.5) suggests an extension, called the PDFT estimator, involving the use of a prior estimate, $P(\omega) \geq 0$, of the magnitude function $|F(\omega)|$. Suppose now that the data that comprise the vector d are the values $f(t_n)$, n = 0, 1,…,N , for some possibly nonequispaced points t_n. The PDFT *estimate*, FPDFT (ω), has the form

$$FPDFT(\omega) = P(\omega)\, a_n \exp(itn\omega), \qquad (2.6.6)$$

where the coefficients a_n solve the system of equations $P_a = d$, with P the matrix whose entries are $p(t_m - t_n)$, the Fourier transform of $P(\omega)$ evaluated at the points $t_m - t_n$, $m, n = 1, \ldots, N$.

This estimate, which can also be viewed as a data extrapolation method, was first discussed according to Byrne (1995). The PDFT was applied to image processing by Byrne (1997) and to phase retrieval by the same person Byrne (1998). Variants of the PDFT those are nonlinear in the data and related to maximum entropy and maximum likelihood estimation were discussed

extensively by Gordon *et al.* (1970). The PDFT estimate of *F* is the unique function G consistent with the data that minimizes the weighted energy

$$\int \quad |G(\omega)|^2 P(\omega)^{-1} d\,\omega. \qquad (2.6.7)$$

If the data are equispaced and = then the PDFT becomes the well-known discrete time Fourier transform (DFT). The PDFT can be implemented iteratively by discretizing the function to be estimated and representing the Fourier transform by means of a matrix.

The prior P (ω) then becomes a finite vector of eights. The ART method can then be used to calculate a minimum weighted norm solution. This approach is particularly useful for large data sets encountered in image reconstruction.

2.7 The Landweber algorithms

It is easy to find important examples of the SFP: Anderson and Kak (1984), if $C \subseteq \mathbb{R}^N$ and $Q = \{b\}$ then solving the SF P amounts to solving the linear system of equations $Ax = b$; if C is a proper subset of R^N, such as the nonnegative cone, then we seeks solutions of $Ax = b$ that lie within C, if there are any.

The SFP is currently of some interest in dynamic PET medical image reconstruction, for reasons discussed in detail by Byrne (2002). Generally, we cannot solve the SFP in closed form and iterative methods are needed.

A number of well-known iterative algorithms, such as the Landweber (1951) i.e. With x^0 arbitrary and k = 0,1,… let

$$x^{k+1} = x^k + \gamma A^T(b - Ax^k), \qquad (2.7.1)$$

and projected Landweber methods in its own case: For general nonempty closed convex set C we obtain the projected Landweber method for finding a solution of $Ax = b$ in C: For x^0 arbitrary and k = 0, 1,…, let

$$x^{k+1} = P_C(x^k + \gamma A^T(b - Ax^k)), \qquad (2.7.2)$$

are particular cases of the CQ algorithm. From the convergence theorem for the CQ algorithm it follows that the Landweber Algorithm converges to a solution of Ax = b and the projected Landweber algorithm converges to a solution of Ax = b in C, whenever such solutions exist.

When there are no solutions of the desired type, the Landweber algorithm converges to a least squares approximate solution of Ax = b, while, by corollary (4.1.1), the projected Landweber method will converge to a minimizer, over the set C, of the function $\|b - Ax\|$, whenever such a minimizer exists. The GP iterative procedure for bandlimited extrapolation and super-resolution is an example of the Landweber algorithm.

Another example of the Landweber method is the simultaneous algebraic reconstruction technique (SART), as narrated by Byrne and Censor (2001) for solving Ax = b, for nonnegative matrix A. Let A be an M by N matrix with nonnegative entries. Let $A_{i+} > 0$ be the sum of the entries in the ith row of A and $A_{+j} > 0$ be the sum of the entries in the jth column of A. Consider the (possibly inconsistent) system Ax = b.

The SART algorithm has the following iterative step:

$$x_j^{k+1} = x_j^k + \frac{1}{A+j} \sum_{i=1}^{M} (b_i - (Ax^k)i)/A_{i+}. \qquad (2.7.3)$$

We make the following changes of variables:

$$B_{ij} = A_{ij} /(A_{i+})^{1/2} (A_{+j})^{1/2},$$

$$Z_j = x_j(A_{+j})^{1/2} \quad \text{and} \quad c_i = b_i/(A_{i+})^{1/2}.$$

Then the SART iterative step can be written as

$$z^{k+1} = z^k + B^T(c - Bz^k). \qquad (2.7.4)$$

This is a particular case of the Landweber algorithm, with $\gamma = 1$. The convergence of SART follows from

theorem 3.1.1, once we know that the largest eigenvalue of B^TB is less than two; in fact, it is shown to be one by De pierro (1991).

2.8 The Geometric Properties of Banach Spaces and Nonlinear Iterations

According to Trieste, Hilbert spaces have the nicest geometric properties, among all infinite dimensional Banach spaces. The availability of the inner product, the fact that the proximity map or nearest point map of a real Hilbert space H on to a closed convex subset K of H Lipschitzian with constant 1, and the following two identities

$$\|x + y\|^2 = \|x\|^2 + 2\langle x - y\rangle + \|y\|^2, \quad (2.8.1)$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \quad (2.8.2)$$

Which hold for all x, y ∈ H, are some of the geometric properties that characterize inner product spaces and also make certain problems posed in Hilbert spaces more manageable than those in general Banach spaces. However, as has been rightly observed by Papoulis (1975), '... many, and probably most, mathematical objects and models do not naturally live in Hilbert spaces'.

Consequently, according to Aubin (1993) to extend some of the Hilbert spaces techniques to more general Banach spaces, analogues of the identities (2.8.1) and (2.8.2) have to be developed. For its development, the duality map which has become a most important tool in nonlinear functional analysis plays a central role. Byrne and Fiddy (1987), obtained the following analogue of (2.8.1)

for l_p spaces, $l < p < \infty$.

$$\|x + y\|^2 \leq (p-1)\|x\|^2 + \|y\|^2 + 2\langle x, j(y)\rangle, \quad 2 \leq p < \infty,$$

$$(p-1)\|x + y\|^2 \leq \|x\|^2 + \|y\|^2 + 2h\langle x, j(y)\rangle, \quad 1 \leq p < \infty,$$

Analogues of (2.8.2) were also obtained by Bynum. Zeidler (1990), obtained an analogue of (2.8.1) in uniformly smooth Banach spaces.

A point $x \in K$ is said to be a fixed point of T if $T x = x$. Now, consider the differential equation $du/dt + Au(t) = 0$ which describes an evolution system where A is an accretive map from Banach space E to itself. In Hilbert spaces, accretive operators are called monotone.

At equilibrium state, $du/dt = 0$ and so a solution of $Au = 0$ describes the equilibrium or stable state of the system. This is very desirable in many applications in, for example, ecology, economic, physics, to name a few.

Consequently, considerable research efforts have been devoted to methods of solving the equation $Au = 0$ when A is accretive. Since generally A is nonlinear, there is no closed form solution of this equation. The standard technique is to introduce an operator T defined by

$T := I - A$ where I is the identity map on E. Such a T is called a *pseudo - contraction* (or is called *pseudo – contractive*), by Byrne and Fitzgerald (1983). It is then clear that any zero of A is a fixed point of T.

As a result of this, the study of fixed point theory for pseudo-contractive maps has attracted the interest of numerous scientists and has become a flourishing area of research, within a long period of time, for numerous mathematicians. A very important subclass of the class of pseudo-contractive mappings is that of nonexpansive mappings. Where $T: K \to K$ is called nonexpansive if

$$\|T x - T y\| \leq \|x - y\|$$

holds for arbitrary $x, y \in K$. Apart from being an obvious generalization of the contraction mappings. Nonexpansive maps are important, as has been observed by McLachlan (1970), mainly for the following two reasons:

(i) Nonexpansive maps are intimately connected with the monotonicity methods developed since the early 1960's and constitute one of the first classes of nonlinear mappings for which fixed point theorems were obtained by using the *ne* geometric properties of the underlying Banach spaces instead of compactness properties.

(ii) Nonexpansive mappings appear in applications as the transition operators for initial value problems of differential inclusion of the form $0 \in du/dt + T(t)u$, where the operators $\{T(t)\}$ are, in general, set-valued and are *accretive or dissipative and minimally continuous*.

If K is a closed nonempty subset of a Banach space and T: K→ K is nonexpansive, it is known that *T* may not have a fixed point (unlike the case if *T* is a strict contraction), and even when it has, the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$, $n \geq 1$ (the so-called Picard sequence) may fail to converge to such a fixed point. This can be seen by considering an anticlockwise rotation of the unit disc of $\mathbb{R}^2$ about the origin through an angle of say, $\frac{\pi}{4}$. This map is nonexpansive with the origin as the unique fixed point, but the Picard sequence fail to converge with any starting point $x^0 \neq 0$. Krasnosel'skii, however, showed that in this example, if the Picard iteration formula is replaced by the following formula. $x^0 \in K, x^{n+1} = \frac{1}{2}(x^n + Tx^n), \quad n \geq 0$ (2.8.3)

Then the iterative sequence converges to the unique fixed point. In general, if E is a normed linear space and T is a nonexpansive mapping, the following generalization of (1) which has proved successful in the approximation of a fixed point of T (when it exists) was given by Censor *et al.* (1998). $x^0 \in K, x^{n+1} = (1- \lambda)x^n + \lambda Tx^n), n \geq 0, \lambda \in (0, 1)$ (2.8.4)

However, the most general iterative formula for approximation of fixed points of nonexpansive mappings, which is called the Mann iteration formula (in the light of Mann by Byrne (1999)), is the following: $x^0 \in K, x^{n+1} = (1- \gamma_n)x_n + \gamma_n Tx_n), n \geq 0,$ (2.8.5)

where $\{\gamma_n\}$ is a real sequence in the interval (0,1) satisfying the following conditions:

(i) $\lim \lambda_n = 0$

(ii) $\sum_{n=1}^{\infty} \lambda_n = \infty$

The recursion formula (2.8.4) is consequently called the Krasnoselskii Mann (KM) formula for finding fixed points of ne (nonexpansive) mappings. The following quotation indicates part of the interest in iterative approximation of fixed points of nonexpansive mappings.

Many well-known algorithms in signal processing and image reconstruction are iterative in nature. A wide variety of iterative procedures used in signal processing and image reconstruction and elsewhere are special cases of the KM iteration procedure, for particular choices of the ne operator according to Youla (1987).

The study of the Krasnoselkii-Mann iterative procedures for the approximation of fixed points of nonexpansive mappings and fixed points of some of their generalizations, and approximation of zeros of accretive-type operators have been a flourishing area of research for many Mathematicians.

2.9 Interior point optimization algorithms

The entropy-based methods discussed in the previous section are interior point methods in that the vectors that occur in the calculations always lie within the positive cone of $\mathbb{R}^J$, Byrne (2001).

A more general method of this sort is the interior point algorithm (IPA) by Bertsekas (1997).

The IPA is designed to minimize a convex differentiable function f over the domain of a second convex differentiable function *h*. The iterative step of the IPA is to solve

$$\nabla h(x^{k+1}) = \nabla h(x^k) - \nabla \gamma f(x^k), \quad \text{for } x^{k+1}, \qquad (2,9.1)$$

where $\gamma > 0$ is chosen so that the function $h - \gamma f$ is convex. Note the similarities between this iterative step and the iterative step $x^{k+1} = P_C(x^k - \gamma \nabla f(x^k))$ of the algorithm we considered earlier.

Other conditions are required for convergence as discussed by Byrne (2001). Applications of the IPA to medical imaging were discussed by Leahy and Byrne (2000).

Chapter Three

SOME NONLINEAR OPERATORS AND THEIR RELATIONS

3.1 Introduction

An operator N on Hilbert space H is called nonexpansive ne if x, y ∈ H such that $\|Nx - Ny\| \leq \|x - y\|$. As we shall see, the fact that a *ne* operator N has fixed points is not sufficient to guarantee convergence of the orbit sequence $\{N^k x\}$, additional conditions are needed. An operator S on H is said to be a strict contraction (sc) if there is a $\sigma \in (0,1)$ such that, for all $x, y \in H$,

$$\|Sx - Sy\| \leq \sigma \|x - y\|$$

the well-known Banach-Picard's theorem by Vardi *et al.* (1985) assures us that the operator S has a unique fixed point, to which the orbit sequence $\{S^k x\}$ converges, for any starting point x^o. Requiring the operator to be a *sc* is quite restrictive; most of the operators we are interested in here have multiple fixed points, so are not sc. The Krasnoselskii-Mann (KM) theorem suggests strongly that we should concentrate on average *av* operators. An operator A is called average if there is $\alpha \in (0, 1)$ and *ne* operator N such that

$$A = (1- \alpha)I + \alpha N.$$

An operator G on H is called v-inverse strongly monotone *v - ism* (also called co-coercive) if there is $v > 0$ such that,

$\langle Gx - Gy, x - y \rangle \geq v\|Gx - Gy\|^2$, according to Byrne (1995) (3.1.1)

Theorem 3.1.1 *If the complement of operator N denoted by G = I- N is* $\frac{1}{2}$ *- ism then, the main operator is ne*

Proof

$$\langle Gx - Gy, x - y\rangle \geq \frac{1}{2} \|Gx - Gy\|^2$$

$$\langle Gx - Gy, x - y\rangle - \frac{1}{2} \|Gx - Gy\|^2 \geq 0$$

$$2\langle (I - N)x - (I - N)y, x - y\rangle - \|(I - N)x - (I - N)y\|^2 \geq 0$$

$$2\langle x - Nx - y + Ny, x - y\rangle - \|x - Nx - y + Ny\|^2 \geq 0$$

$$2\langle Ny - Nx + x - y, x - y\rangle$$

$$- \langle Ny - Nx + x - y, Ny - Nx + x - y \geq 0$$

$$2\langle Ny - Nx, x - y + x - y, x - y\rangle$$

$$-\langle Ny - Nx, Ny - Nx + Ny - Nx, x - y + Ny - Nx, x - y + x - y, x - y\rangle \geq 0$$

$$2(\langle Ny - Nx, x - y\rangle + \langle x - y, x - y\rangle)$$

$$- \langle Ny - Nx, Ny - Nx\rangle - \langle Ny - Nx, x - y\rangle - \langle Ny - Nx, x - y\rangle - \langle x - y, x - y\rangle \geq 0$$

$$2\langle Ny - Nx, x - y\rangle + 2\|x - y\| - \|Ny - Nx\|^2 - 2\langle Ny - Nx, x - y\rangle - \|x - y\|^2 \geq 0$$

$$\|x - y\|^2 - \|Ny - Nx\|^2 \geq 0$$

$$\|Ny - Nx\| \leq \|x - y\|$$

Theorem 3.1.2 *If G is v- ism and* $\gamma > 0$ *then the operator* γG *is v-ism*

Proof

let

$\langle Gx - Gy, x - y\rangle \geq v\|Gx - Gy\|^2$ hold:

But

$$\langle \gamma Gx - \gamma Gy, x - y \rangle = \gamma \langle Gx - Gy, x - y$$

$$\langle \gamma Gx - \gamma Gy, x - y \rangle - \gamma v \| Gx - Gy \|^2 \geq 0$$

$$\gamma \langle Gx - Gy, x - y \rangle - \gamma v \| \frac{1}{\gamma} x \gamma Gx - \frac{1}{\gamma} x \gamma Gy \|^2 \geq 0$$

$$\gamma \langle Gx - Gy, x - y \rangle - \frac{\gamma v}{\gamma 2} \| Gx - Gy \|^2 \geq 0$$

$$\gamma \langle Gx - Gy, x - y \rangle - \frac{v}{\gamma} \| Gx - Gy \|^2 \geq 0$$

Theorem 3.1.3

Let N be a ne operator on H. For k = 0,1,... let $\alpha_k \in (0, 1)$. *Then the sequence* $\{x^k\}$ *defined by the iterative scheme* $x^{k+1} = (1- \alpha_k)x^k + \alpha_k N x^k$, *converges weakly to a fixed point of N, provided* $\sum_{k=0}^{\infty} (1- \alpha_k) = +\infty$, *whenever such fixed points exist.*

An operator G : H → H is monotone in De pierro (1991) if, for all x, y ∈ H such that

$$\langle Gx - Gy, x - y \rangle \geq 0 \qquad (3.1.2)$$

to illustrate, suppose that g is a convex, differentiable real-valued function on H.

Then

$$\langle \nabla g(y), x - y \rangle \leq g(x) - g(y)$$

and $$\langle \nabla g(x), y - x \rangle \leq g(y) - g(x)$$

$$- \langle -\nabla g(y), x - y \rangle \leq g(x) \quad g(y)$$

$$- \langle \nabla g(x), x - y \rangle \leq g(y) - g(x)$$

$$- \langle \nabla g(x) - \nabla g(y), x - y \rangle \leq 0$$

$$\langle \nabla g(x) - \nabla g(y), x - y \rangle \geq 0$$

Therefore is established that the derivative of convex function g is a monotone operator. If ĉ minimizes the function g over the closed convex set C, then $\langle \nabla g(\hat{c}), c - c \rangle \geq 0$ for all c ∈ C.

For general monotone operator G the variational inequality problem (VIP) with respect to G and C, denoted VIP (G, C), is to find $\hat{c} \in C$ with $\langle \nabla g(\hat{c}), c - \hat{c} \rangle \geq 0$ for all $c \in C$ according to Youla (1987). Subject to certain restrictions on G and γ, the sequence denoted by the iterative step

$$x^{k+1} = P_C(I - \gamma G)x^k$$

will converge to a solution of the VIP (G; C), if solutions exist. For each $x \in H$ the metric projection $P_C x$ is that member of C closest to x and is characterized as the unique element of C for which

$\langle c - P_C x, P_C x - x \rangle \geq 0$ for all $c \in C$. Therefore $\hat{c} = PC(\hat{c} \quad G\,\hat{c})$ if and only if

$$\langle c - \hat{c}, \hat{c} - (\hat{c} - G\,\hat{c}) \rangle = \gamma \langle c - \hat{c}, G\,\hat{c} \rangle \geq 0;$$

To see this, since $\hat{c} = P_C(I - \gamma G)\,\hat{c}$,

then $\langle c - \hat{c}, \hat{c} - (\hat{c} - G\,\hat{c}) \rangle = \gamma \langle c - \hat{c}, G\,\hat{c} \rangle \geq 0$,

Becomes

$$\langle c - P_C(I - \gamma G)\,\hat{c}, P_C(I - \gamma G)\,\hat{c} - (P_C(I - \gamma G)\,\hat{c} - \gamma G)\hat{c}) \geq 0$$

$$\langle c - P_C(I - \gamma G)\hat{c}, - \gamma G)\hat{c} \rangle \geq 0$$

$$\langle c - \hat{c}, - \gamma G)\hat{c} \rangle \geq 0$$

$$\gamma \langle c - \hat{c}, - G)\hat{c} \rangle \geq 0 \quad \forall\, c \in C$$

Consequently, the vector $\hat{c}$ solves the VIP (G, C) if and only if $\hat{c}$ is a fixed point of the operator $P_C(I - \gamma G)$. This is the motivation for considering the iteration $xk^{+1}\, P_C(I - \gamma G)x^k$.

As we shall see now, in seeking fixed points for an operator *T* it is helpful to consider properties of its complement, *I* - *T*.

The following identity relates an operator *T* to its complement G = *I* – *T*,

$$\square\, x - y\square^2 - \square T\, x - Ty\square^2 = 2\langle Gx - Gy, x - y \rangle - \square Gx - Gy\square^2 \qquad (3.1.3)$$

This can be seen as follows, let $G = I - T$ then taking the right hand side become,

$$2\langle Gx - Gy, x - y\rangle - \|Gx - Gy\|^2$$

$$= 2\langle (I - T)x - (I - T)y, x - y\rangle - \|(I - T)x - (I - T)y\|^2$$

$$= 2\langle x - Tx - y + Ty, x - y\rangle - (\langle x - Tx - y + Ty, x - Tx - y + Ty\rangle)$$

$$= 2(\langle Ty - Tx, x - y\rangle + \langle x - y, x - y\rangle) - [\langle Ty - Tx, Ty - Tx\rangle$$

$$+ \langle Ty - Tx, x - y\rangle + \langle Ty - Tx, x - y\rangle + \langle x - y, x - y\rangle]$$

$$= 2\langle Ty - Tx, x - y\rangle + 2\|x - y\|^2 - \|Tx - Ty\|^2 - 2\langle Ty - Tx, x - y\rangle - \|x - y\|^2$$

$$= \|x - y\|^2 - \|Ty - Tx\|^2$$

$$= \|x - y\|^2 - \|Tx - Ty\|^2$$

Hence:

$$\|x - y\|^2 - \|Tx - Ty\|^2 = 2\langle Gx - Gy, x - y\rangle - \|Gx - Gy\|^2.$$

Lemma 3.1.4 Let Q be *av* and N be *ne*. Then operator T defined by $T = (1- \alpha)Q + \alpha N$ is *av* for some $\alpha \in (0, 1)$.

Proof Let $Q = (1-\beta)I + \beta M$ for some $\beta \in (0; 1)$ and *ne* operator M. Let $1- \gamma = (1- \alpha)(1- \beta)$. Then we have

$$T = (1- \alpha)[(1- \beta)I + \beta M] + \alpha N$$

$$= (1- \alpha)(1- \beta)I + (1- \alpha)\beta M + \alpha N$$

$$= (1- \gamma)I + \gamma[(1- \alpha)\beta \gamma^{-1}M + 1\ \alpha \gamma^{-1}N]$$

$$= (1- \gamma)I + \gamma U;$$

where $U = [(1- \alpha)\beta \gamma^{-1}M + 1\ \alpha \gamma^{-1}N]$:

Since the operator $U = [(1- \alpha)\beta \gamma^{-1}M + 1\ \alpha \gamma^{-1}N]$ is easily shown to be *ne* and the convex combination of two ne operators is again ne, T is *av*.

To see that U =[(1- α)β γ^{-1}M +1 α γ^{-1}N] is ne operator i.e.

$\|Ux - Uy\| \leq \|x - y\|$

but $1 - \gamma = (1 - \alpha)(1 - \beta)$

$1 - \gamma = (1 - \alpha - \beta) + \alpha\beta$

$\gamma = \alpha + \beta - \alpha\beta$

we compute this as follows

$\|Ux - Uy\|$

$= \| [(1 - \alpha)\beta \gamma^{-1}M + \alpha \gamma^{-1}N]x - [(1 - \alpha)\beta \gamma^{-1}M + \alpha \gamma^{-1}N]y \|$

$= \| (1 - \alpha)\beta \gamma^{-1}M(x) + \alpha\gamma^{-1}N(x) - (1 - \alpha)\beta \gamma^{-1}M(y) - \alpha \gamma^{-1}N(y) \|$

$\leq \| (1 - \alpha)\beta\gamma^{-1}M(x) - (1 - \alpha)\beta\gamma^{-1}M(y) \| + \|\alpha\gamma^{-1}N(x) - \alpha)\gamma^{-1}N(y)\|$

$\leq (1 - \alpha)\beta\gamma^{-1}\|M(x) - M(y)\| + \alpha\gamma^{-1}\|N(x) - \alpha)\gamma^{-1}N(y)\|$

$= [(1 - \alpha)\beta\gamma^{-1} + \alpha\gamma^{-1}\| x - y\|$

$= \gamma^{-1} [(1 - \alpha)\beta + \alpha\| x - y\|$

$\leq \frac{[(1-\alpha)\beta + \alpha\| x - y\|}{\gamma}$

$= \frac{[(1-\alpha)\beta + \alpha\| x - y\|}{\alpha + \beta - \alpha\beta}$

$= \|x - y\|$

$\Rightarrow \|Ux - Uy\| \leq \|x - y\|$

Lemma 3.1.5 An operator A is *av* if and only if its complement

$G = I - A$ is *v* - ism for some $v > \frac{1}{2}$

Proof We assume first that A is *av*. Then there is $\alpha \in (0, 1)$ and ne operator N such that

$A = (1 - \alpha)I + \alpha N$,

and so G = I - A = α (I - N). Since N is ne, I - N is $\frac{1}{2}$ - ism and G = α (I - N) is $\frac{1}{2\alpha}$ - ism.

Conversely, assume that G is v - ism for some v > $\frac{1}{2}$.

Let $\alpha = \frac{1}{2v}$ and write A = (1- α)I + αN for N = I - $\frac{1}{\alpha}$ G. Since I - N = I - $\frac{1}{\alpha}$ G, I - N is αv - ism.

Consequently I - N is $\frac{1}{2}$ - ism and N is *ne*. Therefore, A is *av*.

3.2 **Firmly nonexpansive operators**

Proposition 3.2.1 If operator F on H is firmly nonexpansive fne then, the relation

$$\langle Fx - Fy, x - y\rangle - \|Fx - Fy\|^2 = \langle Gx - Gy, x - y\rangle - \|Gx - Gy\|^2 \qquad (3.2.1)$$

hold where G = I – F.

Proof Let G = I - F

Then

$$\langle Gx - Gy, x - y\rangle - \|Gx - Gy\|^2$$

$$= \langle (I - F)x - (I - F)y, x - y\rangle - \|(I - F)x - (I - F)y\|^2$$

$$= \langle x - Fx - y + Fy, x - y\rangle - \langle x - Fx - y + Fy, x - Fx - y + Fy\rangle$$

$$= \langle Fy - Fx + x - y, x - y\rangle - \langle Fy - Fx + x - y, Fy - Fx + x - y\rangle$$

$$\leq \langle Fy - Fx, x - y\rangle + \langle x - y, x - y\rangle - \langle Fy - Fx, Fy - Fx\rangle$$

$$- \langle Fy - Fx, x - y\rangle - \langle Fy - Fx, x - y\rangle - \langle x - y, x - y\rangle$$

$$\leq \langle Fy - Fx, x - y\rangle + \|x - y\|^2 - \|Fy - Fx\|^2 - 2\langle Fy - Fx, x - y\rangle - \|x - y\|^2$$

$$\leq \|Fy - Fx\|^2 + \langle Fy - Fx, x - y\rangle$$

This implies that $\langle Fx - Fy, x - y\rangle - \|Fy - Fx\|^2$, and the relationship is established.

Lemma 3.2.2 *If the complement of operator F is fne then, the operator is fne.*

Proof

Assume the complement of *F fne*, then

$$\langle (I - F)\, x - (I - F)y,\ x - y\rangle - \|(I - F)x - (I - F)y\|^2 \geq 0$$

$$\langle x - Fx - y + Fy,\ x - y\rangle - \langle x - Fx - y + Fy,\ x - Fx - y + Fy\rangle \geq 0$$

$$\langle Fy - Fx + x - y,\ x - y\rangle - \langle Fy - Fx + x - y,\ Fy - Fx + x - y\rangle \geq 0$$

$$\langle Fy - Fx, x - y\rangle + \langle x - y, x - y\rangle$$

$$- \langle Fy - Fx, Fy - Fx\rangle - \langle Fy - Fx, x - y\rangle - \langle Fy - Fx, x - y\rangle - \langle x - y, x - y\rangle \geq 0$$

$$\langle Fy - Fx, x - y\rangle + \|x - y\|^2 - \|Fy - Fx\|^2 - 2\langle Fy - Fx, x - y\rangle - \|x - y\|^2 \geq 0$$

$$- \langle Fy - Fx, x - y\rangle - \|Fy - Fx\|^2 \geq 0$$

$$\langle Fy - Fx, x - y\rangle \geq \|Fy - Fx\|^2$$

Hence, An operator F is *fne* if its complement I - F is *fne.*

Lemma 3.2.3 If F is *fne* then F is av.

Proof

For any operator F with G = *I* - *F* we have by proposition 3.2.1

$\langle Fx - Fy, x - y\rangle - \|Fx - Fy\|^2 = \langle Gx - Gy, x - y\rangle - \|Gx - Gy\|^2$

The left-hand side is nonnegative if and only if the right-hand side is. Finally,

If F is *fne* then $I - F$ is *fne*, so $I - F$ is v- ism for $v = 1$.

Therefore F is *av* by Lemma 3.1.5

Corollary 3.2.4

Let $T = (1-\alpha)F + \alpha N$ for some $\alpha \in (0, 1)$. If F is fne and N is ne then T is av.

Proof Since the metric projection of x onto C is characterized by the inequality

$$\langle c - P_C x, P_C x - x\rangle \geq 0 \quad \text{for all } c \in C,$$

we have

$$\langle P_C y - P_C x, P_C x - x\rangle \geq 0 \qquad (3.2.2)$$

and

$$\langle P_C x - P_C y, P_C y - y\rangle \geq 0 \qquad (3.2.3)$$

Adding, 3.2.2 and 3.2.3 we get

$$\langle P_C x - P_C y, P_C y - P_C x + x - y\rangle \geq 0$$

$$\langle P_C x - P_C y, P_C y - P_C x\rangle + \langle P_C x - P_C y, x - y\rangle \geq 0$$

$$-\langle P_C x - P_C y, P_{Cx} - P_C y\rangle + \langle P_C x - P_C y, x - y\rangle \geq 0$$

$$\langle P_C x - P_C y, x - y\rangle \geq \langle P_C x - P_C y, P_C y - P_C x\rangle$$

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2$$

the operator P_C is *fne*, and therefore also av.

The product of finitely many ne operators is again ne, while the product of finitely many fne operators, even metric projections, need not be *fne*. It is a helpful fact that the product of finitely many av operators is again av. If $A = (1-\alpha)I + \alpha N$ is av and B is av then $T = AB$ has the form

$T = (1)B + NB$.

Since B is av and NB is ne, it follows from Lemma 3.1.5 that T is av. Summarizing, we have

Proposition 3.2.5

If A and B are av, then T = AB is av. Combining this proposition with Theorem 3.1.3 we obtain Dolidze's theorem by Chidume (1963).

Theorem 3.2.6 Let G be v - ism and $\gamma \in (0, 2v)$. Then, for any x, the sequence $\{(P_C(I - \gamma G))k^{x}\}$ converges weakly to a solution of VIP(G, C),whenever solutions exist.

Proof Since the operator γG is $\frac{1}{2\alpha}$ - ism then, its complement $(I - \gamma G)$ also P_C are *av* and for the fact that, the product of finitely many ne operators is again ne then $P_C(I - \gamma G)$ is *av*. It is possible for Fix(AB) to be nonempty while Fix(A) ⋂ Fix(B) is empty;

However, if the latter is nonempty, it must coincide with Fix(AB) as discussed by Berterro and Boccacci (1998).

Proposition 3.2.7 Let A and B be av operators and suppose that

Fix(A) ⋂ Fix(B) is nonempty. Then Fix(A) ⋂ Fix(B) = Fix(AB) = Fix(BA).

Proof

Let I - A be v_A - ism and I - B be v_B - ism, where both v_A and v_B are taken greater than $\frac{1}{2}$.

Let z be in Fix (A) ⋂ Fix (B) and *x* in F ix(BA).

Then

$$\|z - x\|^2 \geq \|z - Ax\|^2 + (2v_A - 1)\|Ax - x\|^2$$

$$\geq \|z - BAx\|^2 + (2v_B - 1)\|BAx - Ax\|^2 + (2v_A - 1)\|Ax - x\|^2$$

$$= \|z - x\|^2 + (2v_B - 1)\|BAx - Ax\|^2 + (2v_A - 1)\|Ax - x\|^2$$

Therefore $\|Ax - x\| = 0$ and $\|BAx - Ax\| = \|Bx - x\| = 0$.

If $A_1, \ldots, A_M$ are av operators, then so are the operators $A = \frac{1}{M}\sum_{m=}^{M} A_m$ and $B = A_M A_{M-1} \ldots A_1$.

The orbit sequence $\{A^k x\}$ will converge weakly whenever Fix(A) is nonempty; such an iterative

scheme is sometime called a simultaneous method If the operators A_m have common fixed points, then the orbit sequence $\{B^k x\}$ will converge weakly to such a common fixed point; such methods are sometimes called sequential methods.

If Fix(B) is nonempty, but the A_m have no common fixed point, then the sequence $\{B^k x\}$ converges to a fixed point z^0 such that, with $z^1=A_{1z,}$ $z^2= A_{2z-1},\ldots,$ $z^{M-1}= A_{M-1}z^{M-2}$, we have $A_M z^{M-1}= z^0$. Such a set of M vectors is called a limit cycle.

3.3 Prototype of a strongly attracting mapping

Lemma 3.3.1 Suppose D is a closed convex nonempty set, T: D → D be firmly nonexpansive with fixed points, and

$\alpha \in (0, 2)$, R := (1- α)Id + αT and fix x ∈ D, f ∈ Fix T, then

(i) Fix R = Fix T

(ii) $\langle x - f, x - Tx\rangle \geq \| x - Tx\|^2$ and $\langle x - Tx, Tx - f\rangle \geq 0$

(iii) $\| x - f \|^2 - \|R x - f \|^2 = 2\alpha\langle x - f, x - Tx\rangle - \alpha^2 \| x - Tx\|^2$

(iv) R is (2- α)/α-acttracting: $\| x - f \|^2 - \|R x - f \|^2 \geq (2-\alpha)/\alpha \|R x - f \|^2 = (2-\alpha)/\alpha \|x - Tx \|^2$

Proof .

(i) is immediate

(ii) since T is firmly nonexpansive, we obtain

$$\|Tx - f \|^2 \leq \langle Tx - f, x - f\rangle$$

$$\Leftrightarrow \|Tx - f \|^2 \leq \langle Tx - f, x - f\rangle - \alpha^2 \| x - Tx\|^2$$

$$\Leftrightarrow \|Tx - x\|^2 + \|x - f \|^2 + 2\langle Tx - x, x - f\rangle \leq \langle Tx - f, x - f\rangle$$

$$\Leftrightarrow \|Tx - x\|^2 \leq \langle x - Tx, x - f\rangle = \langle x - Tx, (x - Tx) + (Tx - f)\rangle$$

$$0 \leq \langle x - Tx - f, Tx - f\rangle$$

(iii) Is a direct calculation

$$\| x - f \|^2 - \| Rx - f \|^2$$

$$= \| x - f \|^2 - \|(1-\alpha)(x - f) + \alpha(Tx - f \|^2$$

$$= \| x - f \|^2 - [(1-\alpha)\|(x - f) \|^2 + \alpha^2 \|Tx - f \|^2 + 2\alpha(1-\alpha)\langle x - f, Tx - f \rangle]$$

$$= 2\alpha \| x - f \|^2 - \alpha^2 \|x - f \|^2 - \alpha^2 \|Tx - f \|^2 + 2\alpha^2 \langle x - f, Tx - f \rangle - 2\alpha \langle x - f, Tx - f \rangle$$

$$= 2\alpha \langle x - f, (x\text{-}f) - (Tx - f) \rangle - 2\alpha \langle x - f, Tx - f \rangle - \alpha^2 [\|x - f \|^2 + \|Tx - f \|^2 - 2\langle x - f, Tx - f \rangle]$$

$$= 2\alpha \langle x - f, x - Tx \rangle - \alpha^2 \|x - Tx \|^2.$$

(iv): By (ii), (iii) and the definition of **R**, we get

$$\| x - f \|^2 - \| R x - f \|^2 = 2\alpha \langle x - f, x - Tx \rangle - \alpha^2 \| x - Tx \|^2$$

$$\geq 2\alpha \| x - Tx \|^2 - \alpha^2 \| x - Tx \|^2$$

$$= \alpha(2-\alpha) \| x - Tx \|^2$$

$$= (2-\alpha)/\alpha \| x - Rx \|^2$$

Note that (i) and (ii) are actually true for an arbitrary nonexpansive mapping T: this will, however, not be needed in what follows. Since projections are firmly nonexpansive, we immediately obtain the following result,

Corollary 3.3.2 *If P is the projection onto some closed convex nonempty set S and* $\alpha \in (0, 2)$, *then* $R := (1-\alpha)Id + \alpha P$ *is* $(1\text{-}\alpha)/\alpha$- *attracting w.r.t. S for* $x \in X$, $s \in S$,

$$\| x - s \|^2 - \| R x - s \|^2 \geq (2-\alpha) d^2 \quad (x, s)$$

3.4 A system of generalized equilibrium problems

According to Stark and Yang (1998). Let H be a real Hilbert space with inner product < ., .> and norm ||.||, respectively and let C be a closed convex subset of H. Let F be a bi-function from C x

C into the set of real number $\mathbb{R}$. Let A: C $\rightarrow$ H be a nonlinear mapping. Then, we consider the following equilibrium problem; Finding z $\in$ C such that

$$F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall\, y \in C \tag{3.4.1}$$

The set of such z $\in$ C is denoted by *EP* i.e

$$EP = \{z \in C: F(z, y) + \langle Az, y - z\rangle \geq 0, \quad \forall\, y \in C\}$$

If A$\equiv$ 0, *EP* is denoted by EP (*F*)

If F $\equiv$0, *EP* is denoted by VIP(C, A),

where A: C $\rightarrow$ H is also an inverse-strongly monotone mapping and then obtain a strong convergence theorem.

let F_1, F_2: C x C $\rightarrow$ R be two monotone bi-function and $\lambda, \mu \geq 0$ be two constants, then by Narayanan *et al.* (2001).

let A_1, $A_2 \rightarrow$ H be α-inverse-strongly monotone and β - inverse-strongly monotone, respectively we consider the following problem for finding.

(x*,y*) $\in$ C x C $\ni$

$$F_1(x^*, z) + \langle A_1 y^*, z - x^*\rangle + \tfrac{1}{\lambda}\langle z - x^*, x^* - y^*\rangle \geq 0, \quad \forall\, z \in C$$

$$F_2(y^*, z) + \langle A_2 x^*, z - y^*\rangle + \tfrac{1}{\mu}\langle z - y^*, y^* - x^*\rangle \geq 0, \quad \forall\, z \in C \tag{3.4.2}$$

which is called a system of generalized equilibrium problems, in particular

If $\lambda = \mu$ and A_1, A $\equiv$ 0, then (3.2.2) reduces to finding (x*,y*) $\in$ C x C $\ni$

$$\lambda F_1(x^*, z) + \langle z - x^*, x^* - y^*\rangle \geq 0, \quad \forall\, z \in C$$

$$\lambda\, F_2(y^*, z) + \langle z - y^*, y^* - x^*\rangle \geq 0, \quad \forall\, z \in C \tag{3.4.3}$$

moreover, if F_1, F_2 $\equiv$ 0 then the problem (3.4.3) reduces to finding (x*,y*) $\in$ C x C $\ni$

$$\langle A_1 y^*, z - x^*\rangle + \tfrac{1}{\lambda}\langle z - x^*, x^* - y^*\rangle \geq 0, \quad \forall\, z \in C$$

$$\langle A_2 x^*, z - y^*\rangle + \tfrac{1}{\mu}\langle z - y^*, y^* - x^*\rangle \geq 0, \quad \forall\, z \in C \tag{3.4.4}$$

which implies that

$$\langle A_1 y^*, z - x^* \rangle + \tfrac{1}{\lambda}\langle x^* - y^*, z - x^* \rangle \geq 0, \quad \forall\, z \in C$$

$$\langle A_2 x^*, z - y^* \rangle + \tfrac{1}{\mu}\langle y^* - z^*, z - y^* \rangle \geq 0, \quad \forall\, z \in C \qquad (3.4.5)$$

Then

$$\langle A_1 y^* + \tfrac{1}{\lambda}(x^* - y^*), z - x^* \rangle \geq 0, \quad \forall\, z \in C$$

$$\langle A_2 x^* + \tfrac{1}{\mu}(y^* - z^*), z - y^* \rangle \geq 0, \quad \forall\, z \in C \qquad (3.4.6)$$

Hence

$$\langle \lambda A_1 y^* + (x^* - y^*), z - x^* \rangle \geq 0, \quad \forall\, z \in C$$

$$\langle \mu A_2 x^* + (y^* - z^*), z - y^* \rangle \geq 0, \quad \forall\, z \in C \qquad (3.4.7)$$

Now

Proof note that (x^*, y^*) is a solution of (3.4.1)

$$F_1(x^*, z) + \langle A_1 y^*, z - x^* \rangle + \tfrac{1}{\lambda}\langle z - x^*, x^* - y^* \rangle \geq 0, \quad \forall\, z \in C$$

$$F_2(y^*, z) + \langle A_2 x^*, z - y^* \rangle + \tfrac{1}{\mu}\langle z - y^*, y^* - x^* \rangle \geq 0, \quad \forall\, z \in C$$

But H is a real Hilbert space

$$F_1(x^*, z) + \tfrac{1}{\lambda}\langle z - x^*, x^* - y^* \rangle + \tfrac{1}{\lambda}\langle z - x^*, \lambda A_1 y^* \rangle \geq 0, \quad \forall\, z \in C$$

$$F_2(y^*, z) + \tfrac{1}{\mu}\langle z - y^*, y^* - x^* \rangle + \tfrac{1}{\mu}\langle z - y^*, \mu A_2 x^* \rangle \geq 0, \quad \forall\, z \in C \qquad (3.4.8)$$

$$\Leftrightarrow \quad F_1(x^*, z) + \tfrac{1}{\lambda}\langle z - x^*, x^* - (y^* - \lambda A_1 y^*) \rangle \geq 0, \quad \forall\, z \in C$$

$$F_2(y^*, z) + \tfrac{1}{\mu}\langle z - y^*, y^* - (x^* - \mu A_2 x^*) \rangle \geq 0, \quad \forall\, z \in C \qquad (3.4.9)$$

$$\Leftrightarrow \quad x^* \in T^1 \lambda(y^* - \lambda A_1 y^*), \quad \forall\, z \in C$$

$$y^* \in T^2 \mu(x^* - \mu A_2 x^*), \quad \forall\, z \in C \qquad (3.4.10)$$

But $T^1 \lambda$ and $T^2 \mu$ are single-valued

$$\Leftrightarrow \quad x^* = T^1 \lambda(y^* - \lambda A_1 y^*), \quad \forall\, z \in C$$

$$y^* = T^2\mu(x^* - \mu A_2 x^*), \quad \forall z \in C \tag{3.4.11}$$

$$x^* = T^1\lambda[(T^2\mu(x^* - \mu A_2 x^*) - \lambda A_1 T^2\mu(x^* - \mu A_2 x^*)] \tag{3.4.12}$$

$$y^* = T^2\mu[(T^1\lambda(y^* - \lambda A_1 y^*) - \mu A_2 T^1\lambda(y^* - \lambda A_1 y^*)] \tag{3.4.13}$$

CLAIM:

For $\lambda \in (0,2\alpha)$, $\mu \in (0, 2\beta)$: $(1 - \lambda A_1)(1 - \mu A_2)$ and G are all nonexpansive.

now proof of the claim.

Proof It is enough to prove that $1 - \lambda A1$ and G are nonexpansive. Let $\lambda \in (0,2\alpha)$ with $\alpha > 0$, since A_1 is α -ism, we have for $x; y \in C$.

$$\begin{aligned}
&\|(1 - \lambda A_1)x - (1 - \lambda A_1)y\|^2 \\
&= \|(x - y) - \lambda(A_1 x - A_1 y)\|^2 \\
&= \|x - y\|^2 - 2\lambda\langle x - y, A_1 x - A_{1y}\rangle + \lambda^2 \|A_1 x - A_1 y\|^2 \\
&\leq \|x - y\|^2 - 2\lambda\alpha \|A_1 x - A_1 y\|^2 + \lambda^2 \|A_1 x - A_1 y\|^2 \\
&= \|x - y\|^2 - \lambda(\lambda - 2\alpha)\|A_1 x - A_1 y\|^2 \\
&\leq \|x - y\|^2
\end{aligned}$$

$\Rightarrow (1 - \lambda A_1)$ is nonexpansive and since $T^1\lambda$ and $T^2\mu$ are nonexpansive, then

$$\begin{aligned}
\|Gx - Gy\| &\leq \|x - y\| \\
&= \|\{T^1\lambda[(T^2\mu(x - \mu A_2 x) - \lambda A_1 T^2\mu(x - \mu A_2 x)] - [(T^2\mu(y - \mu A_2 y) - \lambda A_1 T^2\mu(y - \mu A_2 y)]\}\| \\
&= \|(I - \lambda A)(T^2\mu(x - \mu A_2 x) - (I - \lambda A)(T^2\mu(y - \mu A_2 y)\| \\
&\leq \|(T^2\mu(x - \mu A_2 x) - (T^2\mu(y - \mu A_2 y)\| \\
&\leq \|(I - \mu A_2)x - (I - \mu A_2)y\| \\
&\leq \|x - y\|
\end{aligned}$$

that is G is nonexpansive. Remarks: If C is a closed bounded convex set of H then the solution of the problem (3.4.2) always exists.

Chapter Four

Some Iterative Schemes Used for Finding Fixed Points of Nonlinear Operators

4.1 Introduction

Many iterative procedures used in fixed points of nonlinear operators and elsewhere are special cases of the KM iterative procedure, for particular choices of the *ne* operator N.

4.2 Constrained Optimization Algorithms

Algorithms for signal enhancement and image modification are often iterative constrained optimization procedures designed to minimize a convex differentiable function f over a closed convex set $C \subseteq H$.

If the gradient operator ∇f is λ- Lipschitz continuous, that is, for each $x, y \in H$ we have

$$\| \nabla f(x) - \nabla f(y) \| \leq \lambda \| x - y \| \quad (4.2.1)$$

then the operator ∇f is $\frac{1}{\lambda}$ – ism. According to Byrne and Fitzgerald (1982) If $\gamma \in (0, \frac{2}{\lambda})$, then the operator $G = \gamma \nabla f$ is $\frac{1}{2\lambda}$ – ism and the operators $A = I - \gamma \nabla f$ and $P_C A$ are av.

To see above, i.e the operator ∇f is $\frac{1}{\lambda}$ – ism it is enough to show that ∇f is λ -Lipschitz continuous.

Proof Let operator G be λ -Lipschitz continuous i.e

$$\| G(x) - G(y) \| \leq \lambda \| x - y \| \quad [4.2.1]$$

But $G = I - \nabla f$, then we have

$$\| G(x) - G(y) \|^2 \leq \lambda^2 \| x - y \|^2$$

$$\| G(x) - G(y) \|^2 = \| (I - \nabla f)(x) - (I - \nabla f)(y) \|^2$$

$$= \| x - \nabla f x - y + \nabla f y \|^2$$

$$= \langle \nabla fy - \nabla fx + x - y, \nabla fy - \nabla fx + x - y \rangle$$

$$\leq \| \nabla fx - \nabla fy \|^2 + 2\langle \nabla fy - \nabla fx, x - y \rangle + \| x - y \|^2$$

But $-\| \nabla fx - \nabla fy \|^2 \leq \langle \nabla fy - \nabla fx, x - y \rangle$

$$\leq \| \nabla fx - \nabla fy \|^2 - 2\| \nabla fx - \nabla fy \|^2 + \| x - y \|^2$$

$$\leq \| x - y \|^2 - \| \nabla fx - \nabla fy \|^2$$

Also $\| \nabla fx - \nabla fy \|^2 \leq \| x - y \|^2 \leq \lambda^2 \| x - y \|^2$

$$\| \nabla fx - \nabla fy \|^2 \leq \lambda^2 \| x - y \|^2$$

$$\| \nabla fx - \nabla fy \|^2 \leq \lambda^2 \| x - y \|^2$$

4.3 Orthogonal Projection onto Sets C and Q

This is iterative algorithm that converges to a solution of the SFP, for any starting vector x^0, whenever the SF P has solutions.

It can be seen if we let F be nonempty set and $\hat{c}$ a member of F, then $\hat{c} = P_C(S\hat{c})$ and

$\| \hat{c} - x^{k+1} \| = \| P_C(S\hat{c}) - P_C(Sx^k) \| \leq \| S\hat{c} - Sx^{k+1} \|$.

We shall show that

$\| S\hat{c} - Sx^{k+1} \| = \| \hat{c} - x^{k+1} \|$ from the definition of Sx we have

$$\| S\hat{c} - Sx^{k+1} \| = \| \hat{c} - x^k + \gamma A^T (P_Q - I) A\hat{c} - \gamma A^T (P_Q - I) Ax^k \|^2$$

Expanding the terms in the right hand side we get,

$$\| S\hat{c} - Sx^{k+1} \|^2 = \| \hat{c} - x^k \|^2 + 2\gamma \langle A\hat{c} - Ax^k, P_Q Ax^k - P_Q A\hat{c} \rangle$$

$$+ \gamma^2 \| A^T (P_Q - I) A\hat{c} - A^T (P_Q - I) Ax^k \|^2$$

$$\leq \| \hat{c} - x^k \|^2 - 2\gamma \| A\hat{c} - Ax^k \|^2 + \langle A\hat{c} - Ax^k, P_Q A\hat{c} - P_Q Ax^k \rangle$$

$$+ \gamma^2 L \| (P_Q - I) A\hat{c} - (P_Q - I) Ax^k \|^2$$

Using

$$\| (P_Q - I) A\hat{c} - (P_Q - I) Ax^k \|^2 = \| P_Q A\hat{c} - P_Q Ax^k \|^2 - 2\langle A\hat{c} - Ax^k, P_Q A\hat{c} - P_Q Ax^k \rangle + \| A\hat{c} - Ax^k \|^2$$

In the line above, we find that

$$\|S\hat{c} - Sx^{k+1}\|^2 \le \|\hat{c} - x^k\|^2 - (2\gamma - \gamma^2 L)\|A\hat{c} - Ax^k\|^2$$
$$+ \gamma^2 L\|P_Q A\hat{c} - P_Q Ax^k\| - 2\gamma\langle A\hat{c} - A x^k, P_Q - P_Q Ax^k\rangle$$
$$+ (2\gamma - \gamma^2 L)\langle A\hat{c} - A x^k, P_Q - P_Q Ax^k\rangle$$

From inequality

$$\langle P_C z - P_C x, z - x\rangle \ge \|P_C z - P_C x\|^2. \quad (4.3.1)$$

And from Cauchy's inequality and the nonexpansiveness of the projection PQ, we obtain

$$\langle A\hat{c} - A x^k, P_Q - P_Q Ax^k\rangle \ge \|A\hat{c} - Ax^k\|^2$$

Since $(2\gamma - \gamma^2 L) \ge 0$. It follows that

$$\|S\hat{c} - Sx^k\|^2 \le \|\hat{c} - x^k\|^2$$

More precisely, we have

$$\|\hat{c} - x^k\|^2 - \|\hat{c} - x^{k+1}\|^2 \ge \gamma^2 L(\langle A\hat{c} - A x^k, P_Q A\hat{c} - P_Q Ax^k\rangle - \|P_Q A\hat{c} - P_Q Ax^k\|^2)$$
$$+ (2\gamma - \gamma^2 L)(\|A\hat{c} - Ax^k\|^2 - \langle A\hat{c} - A x^k, P_Q A\hat{c} - P_Q Ax^k\rangle).$$

Therefore, the sequence $\{\|\hat{c} - x^k\|^2\}$ is decreasing (so the sequence $\{x^k\}$ is bounded), also

$$\{\langle A\hat{c} - A x^k, P_Q A\hat{c} - P_Q Ax^k\rangle - \|P_Q A\hat{c} - P_Q Ax^k\|^2\} \to 0$$
$$\{\|A\hat{c} - Ax^k\|^2 - \langle A\hat{c} - A x^k, P_Q A\hat{c} - P_Q Ax^k\rangle\} \to 0$$

Since both sequences are non-negative. Let x* be an arbitrary cluster point of the sequence $\{x^k\}$ then, we have

$$\langle A\hat{c} - A x^*, P_Q A\hat{c} - P_Q A x^*\rangle = \|P_Q A\hat{c} - P_Q A x^*\|^2$$

And

$$\langle A\hat{c} - A\,x^*, P_Q\,A\hat{c} - P_QA\,x^*\rangle = \| A\hat{c} - A\,x^* \|^2$$

So that $\| A\hat{c} - A\,x^* \|^2 = \| P_QA\hat{c} - P_QA\,x^* \|^2$

(4.3.2)

then it follows from inequality $\langle c - P_Cx, P_Cx - x\rangle \geq 0$ (4.3.3)

$$\| P_QA\hat{c} - A\hat{c} \|^2 = \| P_QAx^* - A\,x^* \|^2$$

So that x* is in the set *F*, replacing the generic $\hat{c} \in F$ with x*, we see that the sequence $\{\| x^* - x^k\|^2\}$ is decreasing; but a subsequence converges to zero, so the entire sequence converges to zero.

4.4 The convex feasibility problem

Let $C_1 \ldots C_M$ be closed nonempty convex subsets of Hilbert space H. The CFP is to find a member of the intersection of the sects $C_1, C_2, \ldots, C_M$, if such elements exist.

Problems in image modification are sometime stated in this way, with the elements of the Hilbert space $H = R^N$ corresponding to vectorized images and the convex sets representing various constraints to be placed on the reconstructed images. The proximity function associated with $C_1, \ldots, C_M$ is

$$f(x) = \frac{1}{2M}\sum_{m=1}^{M} \|P_Cx - x\|^2.$$

(4.4.1)

The gradient of *f* is [2]

$$\nabla f(x) = x - \frac{1}{M}\sum_{m=1}^{M} P_Cx. \quad (4.4.2)$$

A minimizer of *f* is of $\nabla f(x)$, which is a fixed point of the av operator A given by

$$A = \frac{1}{M}\sum_{m=1}^{M} P_Cx.$$

If the intersection C of the sets $C_1, \ldots, C_M$ is nonempty, then Fix(A) = C and the orbit sequence $\{A^kx\}$ converges weakly to a member of C; if C is empty, the sequence converges weakly to a

minimizer of f. In the latter case the limit need not be a member of any of the Cm. If we wish to minimize f relative to vectors x in a closed nonempty convex set K, then we use the iterative scheme defined by $x^{k+1} = P_K(Ax^k)$

The operator PKA is also *av*, so this sequence converges, for any starting vector x^0, provided f has a minimum relative to K.

4.5 The split feasibility problem

The split feasibility problem (SFP) is to find $c \in C$ with $Ac \in Q$, if such points exist, where A is a real M by N matrix and C and Q are nonempty, closed convex sets in R^N and R^M, respectively. The algorithm for solving the SFP can be established by proving the following proposition.

Proposition 4.5.1

The vector $\hat{c} \in C$ is a fixed point of the map T, that is $T\hat{c} = \hat{c}$, if and only if $\hat{c}$ minimizes the function $\| P_Q(Ac) - (Ac) \|$ over $c \in C$

Proof Assume that $\hat{c}$ minimizes the function $\| P_Q(Ac) - (Ac) \|$ over $c \in C$ then,

$$\| P_Q(A\hat{c}) - (A\hat{c}) \| \leq \| P_Q(Ac) - (Ac) \| \leq \| q - (Ac) \| \quad \forall \quad c \in C \tag{4.5.1}$$

and $q \in Q$, choosing $q = P_Q(A\hat{c})$.

We find that

$$\| P_Q(A\hat{c}) - (A\hat{c}) \| \leq \| (Ac) - P_Q(Ac) \| \quad \forall \quad c \in C \ ,$$

which tells us that

$$A\hat{c} \leq (A\hat{c}) - P_Q(A\hat{c})$$

since the inequality

$$\langle c - P_C x, P_C x - x \rangle \geq 0,$$

then this gives us

$\langle Ac - A\hat{c}, A\hat{c} - P_Q(A\hat{c}) \rangle \geq 0 \quad \forall \quad c \in C,$

from

$\|c - S\hat{c}\|^2 = \|c - \hat{c}\|^2 + 2\gamma\langle Ac - A\hat{c}, A\hat{c} - P_Q A\hat{c} \rangle$ + terms without c,

it follows that $\hat{c}$ minimizes the function $\|c - S\hat{c}\|^2$ over $\forall \ c \in C$, or that $\hat{c} = P_C(S\hat{c}) = T\hat{c}$.

Now assume that

$T\hat{c} = \hat{c}$ then, $\hat{c} = P_C(S\hat{c})$, so that, by inequality $\langle c - P_C x, P_C x - x\rangle \geq 0$we have $\langle c - \hat{c}, \hat{c} - S\hat{c} \rangle \geq 0$,

$\forall \ c \in C$ therefore,

$$\langle Ac - A\hat{c}, A\hat{c} - P_Q(A\hat{c}) \rangle \geq 0 \quad \forall \quad c \in C, \tag{4.5.2}$$

we also have

$$\langle P_Q(A\hat{c}) - P_Q(Ac), A\hat{c} - P_Q(A\hat{c}) \rangle \geq 0 \quad \forall \quad c \in C, \tag{4.5.3}$$

Adding (4.5.2) and (4.5.3) , we obtain

$$\langle P_Q(Ac) - (Ac), P_Q(A\hat{c}) - A\hat{c}\rangle \geq \|P_Q(A\hat{c}) - A\hat{c}\|^2$$

Applying the Cauchy inequality, we have

$$\|\langle P_Q(Ac) - (Ac)\|^2 \geq \|P_Q(A\hat{c}) - A\hat{c}\|^2 \tag{4.5.4}$$

the inequality $\langle_C - P_C x, P_C x - x\rangle \geq 0$ tells us that, for any c, x and z, we have

$\langle P_C z - P_C x, P_C x - x\rangle \geq 0$ and $\langle P_C z - P_C x, z - P_C z\rangle \geq 0$

Adding, we obtain $\langle P_C z - P_C x, P_C x - P_C z - x + z \rangle \geq 0$ (4.5.5)

or

$$\langle P_C z - P_C x, z - x\rangle \geq \|P_C z - P_C\|^2 \tag{4.5.6}$$

Therefore the operator P_C is firmly nonexpansive; from the Cauchy inequality we can conclude that

$$\|P_C x - P_C z\|^2 \leq \|x - z\|^2.$$

That is, the operator P_C is nonexpansive. Infact, we can say somewhat more.

Lemma 4.5.2 For any closed nonempty convex set $C \in 2\ R^N$ the inequality $\|P_C x - P_C z\| \leq \|x - z\|$ holds, with equality only if

$$\|P_C x - x\|^2 \leq \|P_C z - z\|^2.$$

Lemma 4.5.3 The derivative operator ∇f is λ-Lipschitz continuous for $\lambda = \rho(A^T A)$, therefore it is v - ism for $v = \frac{1}{\lambda}$ that is $\|\nabla f x - \nabla f y\| \leq \lambda \|x - y\|$.

Proof Consider the estimates

$$\begin{aligned}\|\nabla f x - \nabla f y\|^2 &= \|A^T(I - P_Q)Ax - A^T(I - P_Q)Ay\|^2 \\ &= \|A^T[(I - P_Q)Ax - (I - P_Q)Ay]\|^2 \\ &\leq \|A^T\| \, x \, \|(I - P_Q)Ax - (I - P_Q)Ay\|^2 \\ &\leq \lambda \, \|(I - P_Q)Ax - (I - P_Q)Ay\|^2\end{aligned}$$

Also

$$\begin{aligned}&\|(I - P_Q)Ax - (I - P_Q)Ay\|^2 \\ &= \|P_Q Ay - P_Q)Ax + Ax - Ay\|^2 \\ &= \langle P_Q Ay - P_Q)Ax + Ax - Ay,\ P_Q Ay - P_Q)Ax + Ax - Ay \rangle \\ &\leq \|P_Q Ay - P_Q)Ax\|^2 + 2\langle P_Q Ay - P_Q)Ax + Ax - Ay\rangle + \|Ax - Ay\|^2\end{aligned}$$

Therefore

$$\|(I - P_Q)Ax - (I - P_Q)Ay\|^2 = \|P_Q Ay - P_Q Ax\|^2 + 2\langle P_Q Ay - P_Q Ax + Ax - Ay\rangle + \|Ax - Ay\|^2.$$

Since $\langle P_Q Ax - P_Q Ay, Ax - Ay\rangle \geq \|P_Q Ax - P_Q Ay\|^2$

We have $\|(I - P_Q)Ax - (I - P_Q)Ay\|^2 = \|Ax - Ay\|^2 + \|P_Q Ay - P_Q Ax\|^2 - 2\|P_Q Ay - P_Q Ax\|^2.$

$$= \|Ax - Ay\|^2 - \|P_Q Ay - P_Q Ax\|^2.$$

Hence $\|\nabla f x - \nabla f y\|^2 \leq \lambda \|Ax - Ay\|^2 - \lambda \|P_Q Ay - P_Q Ax\|^2$

$\| \nabla f x - \nabla f y \|^2 \leq \lambda \| Ax - Ay \|^2 - \lambda^2 \| x - y \|^2$

$\| \nabla f x - \nabla f y \|^2 \leq \lambda^2 \| x - y \|^2$

$\| \nabla f x - \nabla f y \| \leq \lambda \| x - y \|^2$

This completes the proof

Also if $\lambda \in (0, \frac{2}{\lambda})$ then $B = P_C(1 - \lambda A^T(I - P_Q)A)$ is *av* and, by Dolidze's Theorem 3.1.9, the orbit sequence $\{B^k x\}$ converges weakly to a fixed point of B, whenever such points exist.

If z is a fixed point of B, then $z = P_C(z - \lambda A^T(I - P_Q)Az))$. Therefore, for any $c \in C$ we have

$$\langle c - z , z - (z - \lambda A^T(I - P_Q)A)z \rangle \geq 0$$

This tells us that $$\langle c - z , \lambda A^T(I - P_Q)A)z \rangle \geq 0 \qquad (4.5.3.1)$$

which means, according to the characterization in the inequality $\langle C - P_C z, P_C z - z \rangle \geq 0$, that z minimizes $f(x)$ relative to the set C.

The CQ algorithm employs the relaxation parameter γ in the interval $(0, \frac{2}{\lambda})$, where L is the largest eigenvalue of the matrix $A^T A$. Choosing the best relaxation parameter in any algorithm is a nontrivial procedure.

Chapter Five

The Main Result

5.1 Introduction

From the various discussions in the previous chapters we can consider $x^{k+1} = (1- \alpha)x^k + \alpha Nx^k$ as a common iterative scheme for fixed points of some linear operators, the result in this thesis is hereby presented as theorem as follows.

5.2 Main Result

Theorem 5.2.1

Let T be an average operator on H and fixed point of T is nonempty, the Picard's scheme $\{T^k x\}$ converges weakly to a member of Fixed point of T , for any starting point $x \in H$.

Proof

Let z be a fixed point of ne operator N and let $T = (1- \alpha)I + \alpha N$, so the iterative step become

$x^{k+1}=Tx^k = (1- \alpha)x^k + \alpha Nx^k$ (5.1.1)

The identity in equation (3.1.3) is the key to proving this Theorem

Using $Tz = z$ and $(I - T)z = 0$ and setting $G = I – T$ we have

$$\| z - x^k\|^2 - \|Tz - x^{k+1}\|^2 = 2\langle Gz - G x^k, z - x^k \rangle - \|Gz - G x^k\|^2 \qquad [3.1.3]$$

Setting $Tx^k = x^{k+1}$, $G = I - T$ then

$$2\langle Gz - G x^k, z - x^k \rangle - \|Gz - G x^k\|^2 k$$

$$= 2\langle (I - T)z - (I - T)x^k, z - x^k \rangle - \|(I - T)z - (I - T) x^k\|^2$$

$$= 2\langle z - Tz - x^k + Tx^k, z - x^k \rangle - \| z - Tz - x^k + Tx^k\|^2$$

$$= 2\langle Tx^k – Tz + z - x^k, z - x^k \rangle - \langle Tx^k – Tz + z - x^k, Tx^k – Tz + z - x^k \rangle$$

$$= 2(\langle Tx^k – Tz, z - x^k \rangle + \langle z - x^k, z - x^k \rangle) - (\langle Tx^k – Tz, Tx^k – Tz \rangle$$

$$+\langle Tx^k – Tz, z - x^k \rangle + \langle Tx^k – Tz + z - x^k \rangle + \langle z - x^k, z - x^k \rangle$$

$$= 2\langle Tx^k - Tz, z - x^k \rangle + 2\| z - x^k \|^2 - \| x^k - Tz\|^2 + 2\langle Tx^k - Tz, z - x^k \rangle - \| z - x^k\|^2$$

$$= \| z - x^k \|^2 - \| x^k - Tz\|^2$$

$$= \| z - x^k \|^2 - \| Tz - x^{k+1}\|^2.$$

Since, by Lemma 3.1.3, G is $\frac{1}{2\alpha}$ - ism, we have

$$\| z - x^k \|^2 - \| z - x^{k+1}\|^2 \geq (\tfrac{1}{\alpha} - 1) \| x^k - x^{k+1}\|^2 \quad (5.1.2)$$

to see this: since G is $\frac{1}{2\alpha}$ - ism, then

$$\langle Gz - G x^k, z - x^k \rangle \geq \frac{1}{2\alpha} \|Gz - G x^k \|^2$$

so that

$$\| z - x^k \|^2 - \|Tz - x^{k+1}\|^2 = 2\langle Gz - G x^k, z - x^k \rangle - \|Gz - G x^k \|^2$$

$$\geq 2x\frac{1}{2\alpha} \|Gz - G x^k \| - \|Gz - G x^k \|^2$$

$$= (\tfrac{1}{\alpha} - 1) \|(I - T)x^k\|^2$$

$$= (\tfrac{1}{\alpha} - 1) \| x^k - x^{k+1}\|^2$$

Consequently, the sequence $\{x^k\}$ is bounded, the sequence $\{\| z - x^k \|^2\}$ is decreasing and the sequence $\{\| x^k - x^{k+1}\|^2\}$ converges to zero. Let x^* be a cluster point of $\{x^k\}$. Then we have $Tx^* = x^*$, so we may use x^* in place of the arbitrary fixed point z. It follows then that the sequence $\{\| x^* - x^k \|^2\}$ is decreasing; since a subsequence converges to zero, the entire sequence converges to zero.

Note that (i) The sequence $\{x^*\}$ is bounded if $\exists\ M \geq 0\ \| x^k\| \leq M \quad \forall\ k \in \mathbb{N}$

To see (i) above, that is boundedness of the sequence $\{x^*\}$

$(\tfrac{1}{\alpha} - 1) \| x^k - x^{k+1}\|^2 \leq \| z - x^k \|^2 + \| z - x^{k+1} \|^2$

summing this from k=1 to m for some m ∈ ℕ we get

$(\frac{1}{\alpha}-1)\sum_{k=1}^{M} \quad \| x^k - x^{k+1} \|^2 \leq \| z - x^1 \|^2 - \| z - x^2 \|^2 + \| z - x^3 \|^2 + \ldots \| z - x^M \|^2 - \| z - x^{M+1} \|^2.$

$$\leq \| z - x^1 \|^2 \leq \infty$$

Thus

$$(\frac{1}{\alpha}-1)\sum_{k=1}^{M} \quad \| x^k - x^{k+1} \|^2 < \infty$$

since this is true for all m ∈ N, we obtain

$$(\frac{1}{\alpha}-1)\sum_{k=1}^{\infty} \quad \| x^k - x^{k+1} \|^2 < \infty$$

which implies $\lim_{k\to\infty} \quad \| x^k - x^{k+1} \|^2 < \infty$

and so $\{x^k\}$ is bounded, necessary condition.

To see cluster point, by Bolzano Weierstrass Theorem, $\{\| x^k - x^{k+1} \|^2\}$ has a convergent subsequence therefore $\{\|z - x^k\|^2\}$ is a bounded sequence in R.

Then, $\exists \{\| x - x^k \|^2\}_{k=1}^{\infty} \subseteq \{\| x^k - x^{k+1} \|^2\}_{k=1}^{\infty}$ which converges.

Let $\| z - x^* \|^2$ be a limit such that

$$\lim_{k\to\infty} \quad \| z - x^k_n \|^2 = \| z - x^* \|^2$$

then x^* is a cluster point of $\{x^k\}_{k=1}^{\infty}$

now $T x^* = x^*$

this is true for

$$\lim_{k\to\infty} \quad \| x^k_n - x^k_{n+1} \|^2 = 0$$

$$\lim_{k\to\infty} \quad \| x^k_n - T x^k_n \|^2 = 0$$

$$\| x^* - T x^* \|^2 = 0$$

$$T x^* = x^*$$

Chapter Six

Summary, Conclusion and Recommendations

6.1 Summary

This work critically looked at a general iterative scheme for fixed points of nonlinear operators, after proving some theorems, propositions and lemmas that are relevant to the thesis. We have shown that the average operator T with iterative step $x^{k+1} = T\,x^k = (1-\alpha)x^k + \alpha N x^k$ is a common scheme that can represent some of the iterative algorithms being used for finding fixed points of nonlinear operators. This iterative algorithm which is a particular case of the KM approach for finding fixed points of *ne* operators on Hilbert space is extensively used in signal enhancement and image modification. These operators on Hilbert space include GP bandlimited extrapolation, the Landweber methods for finding constrained solutions of linear systems of equations, simultaneous and sequential methods for solving the convex feasibility problems.

The algorithm has broad applicability in many areas of mathematics and physical and engineering sciences. It is also useful in medical imaging which include image reconstruction in conjunction with ray tracing for ultrasound and microwave tomography.

6.2 Conclusion

i. Based on the discussions under existing, current and main result, we conclude that the iterative procedure defined by $x^{k+1} = T\,x^k = (1-\alpha)x^k + \alpha N x^k$, is a general iterative scheme for fixed point of nonlinear operators.

ii The iterative schemes of the complements of nonexpansive *ne*, firmly nonexpansive *fne* and average operator *av T* converge to fixed points of *T* as the main operators v.

iii. The strict contraction (sc) operator S defined by iterative step

$x^0 \in K$; $x^{k+1} = (1-\lambda_n)x^n + \lambda_n T x^n$; $n \geq 0$ where $\lambda_n \in (0; 1)$ called Mann iterative, is the property of av operator T that is sufficient to guarantee convergence of the sequence $\{T^k x\}$ to a fixed point of *T*, whenever such fixed point exists.

iv. There are a lot of significant relationships between nonexpansive *ne* operator N, firmly nonexpansive *fne* operator F and average *av* operator T, because each of the operators can perform the functions of another if the necessary conditions are attached to them.

6.3 Recommendations

This thesis discussed a general iterative scheme for fixed points of some nonlinear operators extensively and not the applications. We therefore recommend that a new work be carried out by looking at the application of this general scheme.

The iterative scheme is very useful in the area of Mathematics and other related fields. The use of this scheme can be extended to the area of communication and other disciplines in order to popularize the importance of this algorithm both locally and internationally.

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