ALGEBRAIC STUDY OF RHOTRIXGROUPS

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DEPARTMENT OF MATHEMATICS,
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DECLARATION

I, declare that the work in this dissertation titled "ALGEBRAIC STUDY OF RHOTRIX GROUPS'' is a collection of my original research work and it has not been presented for any other qualification anywhere. This work was undertaken by me in the Department of Mathematics under the supervision of Dr A. Mohammed and Prof B. Sani. The Information from derived from the literature has been duly acknowledge in the text and in the list of references provided.

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Halima ADAMU

Name of student Date

CERTIFICATION

This dissertation entitled ''ALGEBRAIC STUDY OF RHOTRIX GROUPS'' by ADAMU, Halima meets the regulations governing the award of the degree of Master of Science ofAhmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

To my family and humanity in general.

ACKNOWLEDGEMENT

I begin by thanking ALLAH (SWT) for giving me guidance, wisdom, good health and the opportunity to undertake this onerous task. May His mercies and blessings be upon me and whoever comes across this work.

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ABSTRACT

In this dissertation, Algebraicproperties ofgroups ofrhotriceswere constructed and somealready known theories in group theory were verified. Cyclic rhotrixgroup was developed from the cyclic multiplication table presented. Subgroups of the rhotrix groups developed were also identified and the order of the rhotrix groups were discussed .Homomorphisms and Isomorphisms of rhotrix groups were equally presented. Finite algebraic structures that satisfy the axioms of groups were constructed using rhotrix sets as the underlying sets.Ideas in group theory were systematized into rhotrix group theory using already known concepts such as cosets, order, subgroups, Lagrange's theorem of a group and so on. The entries were derived from residue classes of modulo two,three and five.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 INTRODUCTION

Mathematics is a very rich subject which serves as an integral part of life as a whole. The application of Mathematics is related to every sphere of life. Professions like Engineering, Economics, Finance, Medicine Science, Pharmacy, just to mention but a few ,cannot do well without Mathematics.

Mathematics has a lot of branches among which is Algebra. In algebra, concepts such as linear algebra andabstract algebra are studied. Of interest in this study are Groups, which is a branch of abstract algebra. The study is intended to look at recent innovations in this area. The concept of rhotrix was introduced by Ajibade(2003) as an extension of ideas on matrix – tertions and matrix- noitrets. Ajibade presented the initial concept of the algebra and analysis ofrhotrix and established some interesting relationship between rhotrices and their hearts. A Rhotrix*R* of dimension three was defined as

$$R = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a,b,c,d,e \in \Re \right\}, \quad - \quad - \quad - \qquad (1.1)$$

where h(R) = c, the element at the perpendicular intersection of the two diagonals of a rhotrix*R* and it is called the heart of *R*.Thus,

$$R=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle,$$ is a specific rhotrix.

After rhotrix theory was introduced by Ajibade in (2003), several authors developed interest to research in the area. Construction of algebraic structures by way of using rhotrix set as the underlying set became an innovation.

Sani (2004) proposed an alternative method of rhotrix multiplication of size three. The method is row-column based and it is non-commutative but associative. This is unlike that of Ajibade which is both commutative and associative. It was shown in Sani (2004) that there exists an isomorphic relationship between the groups of all invertible rhotrices of size three and the group of all 2×2 dimensional matrices.

In this dissertation, constructions of groups of rhotrices having entries in sets of residue classes are presented as algebraic structures. The types of rhotrix groupsconsidered in this dissertation are commutative in property,and they serve as direct extension to those initiated by Mohammed (2007a).

1.2 STATEMENT OF THE PROBLEM

In the literature, it has been shown that invertible matrices of a fixed dimension form a group. Thus, invertible 2×2 matrices for instance form a group. It will therefore be very interesting to see whether invertible rhotrices of any fixed size form a group.

In this dissertation, a construction of algebraic groups using rhotrix sets as the underlying sets, with entries in residue classes of modulo two, three and five were specifically used.

1.3 RESEARCH AIMAND OBJECTIVES

The aim of this research work is to construct rhotrix groups bearing in mind the following objectives:

i. To construct finite algebraic structures that satisfy the axioms of groups using rhotrix sets as underlying sets.
ii. To systematize ideas in group theory to rhotrix group theory,using already known concepts such as cosets, order, subgroups, Lagrange's theorem of a group and so on.

1.4 RESEARCH METHODOLOGY

In the first state, a fundamental study of ideas in group and rhotrix theories was carried out. Next, Algebraic structures were examined, having satisfied the group axioms; they were systematized to rhotrixgroups structures specifically of unit hearts.

1.5 SIGNIFICANCE OF THE STUDY

The study will reveal the extent to which mathematical enrichment exercises could be developed to help in extending existing ideas.

The findings of the study will give diversified knowledge on how ideas in group theory can be extended to the young theory of rhotrices.

1.6 DEFINITIONS OF TERMS

1.6.1 A group: Let G be a non–empty set and $*$ a binary operation defined on G. The set G is said to be a group if the following axioms are satisfied:

a) For all a, b,$\in$G, a$*$b$\in$G i.e. closure axiom.

b) For all a, b c $\in G, (a * b) * c = a * (b * c)$ i.e. associativity axiom.

c) There exists $e \in G \ni$ for any $a \in G$ $a * e = e * a = a$ identity axiom.

d) For any $a \in G \ \exists \ a^{-1} \in G, \ni a * a^{-1} = a^{-1} * a = e,$ that is inverses exist

Further to this, if $a * b = b * a$ then the group is said to be commutative.

1.6.2 A rhotrix group: If a set of rhotriceswith a binary operation satisfy the axioms of a group, then the set is termed as a rhotrix group.

1.6.3 The symmetric group: Consider the set of permutations of n elements under the operation of composition of permutations. This set forms a group called the symmetric group denoted as S_n.The order of the group is n-factorial, $n!$.

1.6.4 The Cyclic group: If a set has all its elements as powers of a particular element in the set, then it forms a group called the cyclic group, denoted as Z_n if the elements are integers. In this case if the base element is say a, then all other elements in the set are a^n for some integer n. In particular, the identity element, e, is a^m for some integer m.

1.6.5 HEART OF A RHOTRIX

The element at the perpendicular intersection of the two diagonals of a rhotrix$R$ is called its heart denoted by $h(R)$.

1.6.6 EQUALITY OF RHOTRICES

Two rhotrices$R$ and S are said to be equal if both are of the same size and order, and each element of R is equal to the corresponding element of S for each pair of the entries.

1.7 ORGANIZATION OF THE DISSERTATION

Apart from chapter one, we give the following details for the remaining chapters. Chapter two is the literature review, where the past literature on the types of rhotrices and the operations on rhotrices, such as addition, scalar multiplication and multiplication (o), determinant and inverse of a rhotrix were discussed. Also, concepts discussed by earlier contributors by way of seminars and contributed papers will be looked into, so as to provide a better and wider view for understanding.

In chapter three, various theorems, rules and concepts of group theory in relation to rhotrices are presented.

In chapter four, the discussion centered on systematic presentation of existing results in classical group theory to rhotrix group theory.

Lastly, chapter five presents summary, conclusion and recommendations.

CHAPTER TWO

LITERATURE REVIEW

2.1. INTRODUCTION

The concept of rhotrix was first introduced in(Ajibade, 2003),as an extension of ideas on matrix tertions and matrix noitrets suggested by Atanassov and Shannon (1998).Ajibade(2003) discussed the initial algebra and analysis on rhotrices and also set up some relationships between rhotrices and their hearts.

The initial conceptofrhotrix introduced by Ajibadewas presented as objects which lie in some way between (2×2) -dimensional and (3×3)-dimensional matrices defined as:

$$R = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle : a, b, c, d, e \in \Re \right\} \quad - \quad - \quad - \quad (2.1)$$

for a base rhotrix of size three, where h(R) = c as the heart of R.

Extension in size of R is possible. An n−size rhotrix*R*, denoted R_n, has thenumber of entries as $\frac{1}{2}(n^2+1)$, $n \in 2Z^+ + 1$. It is worthy of note that all rhotrices are of odd size.

The heart based representation of any rhotrix R of size 3 is

$$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle .$$

2.2 RHOTRIX ALGEBRA

Let

$$R_3(\Re)=\left\{\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array}\right\rangle : a,b,c,d,e\in\Re\right\}$$

be a rhotrix set of size three over reals. The operations of rhotrix addition and multiplication are discussed in the following two subsections.

2.2.1 Addition of rhotrices

Theaddition of rhotricesas introduced in Ajibade(2003) for rhotrices of size three is followed here. The operation of addition of two or more rhotrices is found by adding the corresponding elements of the rhotrices. Let R and S be two base rhotrices, then their sum is gotten by adding the corresponding a_{ij} of R and b_{ij} of S.

Example,

$$R=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array}\right\rangle \text{and } S=\left\langle \begin{array}{ccc} & f & \\ g & h(S) & j \\ & k & \end{array}\right\rangle \text{ then}$$

$$R+S=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array}\right\rangle+\left\langle \begin{array}{ccc} & f & \\ g & h(S) & j \\ & k & \end{array}\right\rangle$$

$$=\left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(S) & d+j \\ & e+k & \end{array}\right\rangle, \quad - \quad - \quad - \qquad (2.2)$$

2.2.2 Properties ofRhotrix Addition

The following properties hold for any rhotrices A, B, C.

i. *Existence of additive identity*: Given a zerorhotrix O of the same size, we have $A+O = O + A = A$.

ii. *Existence of additive inverse* : There exists a unique rhotrix$(-A)$ such that $A+(-A)=O$ which is the additive inverse of A, where O is a zerorhotrix having the same size as A.

iii. *Commutativity*: $A+B = B + A$.

iv. *Associativity*: $(A + B) + C = A + (B + C)$.

2.2.3 Commutative Rhotrix Multiplication

The multiplication ofrhotrices as introduced inAjibade(2003) was defined forrhotrices of size three, recorded as follows:

Let

$R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ and $S = \left\langle \begin{matrix} & f & \\ g & h(S) & j \\ & k & \end{matrix} \right\rangle$ be any two rhotrices, then

$$R \circ S = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & f & \\ g & h(S) & j \\ & k & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & ah(S) + fh(R) & \\ bh(S) + gh(R) & h(R)h(S) & dh(S) + jh(R) \\ & eh(S) + kh(R) & \end{matrix} \right\rangle .$$

It was noted in (Ajibade, 2003) that $(R_3(\Re),+)$ is an Abelian group, with identity,

$$O = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle.$$

Some established theorems in relation to rhotrices and their hearts as recorded from (Ajibade, 2003) arepresented asbelow:

Theorem 2.2.3.1

A rhotrix*R*is invertible if and only if $h(R) \neq 0$.

Theorem 2.2.3.2

For anyrhotrix $R \neq O, R^2 = O$ *if and only if* $h(R) = 0$.

The multiplication of rhotrices of size three defined inAjibade(2003) was later extended to rhotrices of size *n* by Mohammed (2011), as follows:

Let

$$A(n)=\left\langle \begin{matrix} & & & & a_1 & & & & \\ & & & a_2 & a_3 & a_4 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{\left\{\frac{w+1}{2}\right\}-n/2} & \cdots & \cdots & \cdots & a_{\left\{\frac{w+1}{2}\right\}} & \cdots & \cdots & \cdots & a_{\left\{\frac{w+1}{2}\right\}+n/2} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_{w-3} & a_{w-2} & {}_{w-1} & & & \\ & & & & a_w & & & & \end{matrix} \right\rangle$$

and

$$B(n)=\left\langle \begin{matrix} & & & & b_1 & & & & \\ & & & b_2 & b_3 & b_4 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{\left\{\frac{w+1}{2}\right\}-n/2} & \cdots & \cdots & \cdots & b_{\left\{\frac{w+1}{2}\right\}} & \cdots & \cdots & \cdots & b_{\left\{\frac{w+1}{2}\right\}+n/2} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & b_{w-3} & b_{w-2} & b_{w-1} & & & \\ & & & & b_w & & & & \end{matrix} \right\rangle$$

Then,

$$C(n)=A(n)\circ B(n)$$

$$
= \left\langle \begin{array}{ccccccc}
 & & & a_1 & & & \\
 & & a_2 & a_3 & a_4 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
a_{\left\{\frac{w+1}{2}\right\}-n/2} \; \cdots \; \cdots \; \cdots & & & a_{\left\{\frac{w+1}{2}\right\}} & & & \cdots \; \cdots \; \cdots \; a_{\left\{\frac{w+1}{2}\right\}+n/2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & a_{w-3} & a_{w-2} & {}_{w-1} & & \\
 & & & a_w & & &
\end{array} \right\rangle \circ \left\langle \begin{array}{ccccccc}
 & & & b_1 & & & \\
 & & b_2 & b_3 & b_4 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
b_{\left\{\frac{w+1}{2}\right\}-n/2} \; \cdots \; \cdots \; \cdots & & & b_{\left\{\frac{w+1}{2}\right\}} & & & \cdots \; \cdots \; \cdots \; b_{\left\{\frac{w+1}{2}\right\}+n/2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & b_{w-3} & b_{w-2} & b_{w-1} & & \\
 & & & b_w & & &
\end{array} \right\rangle
$$

$$
= \left\langle \begin{array}{c}
a_1 h(B) + b_1 h(A) \\
a_2 h(B) + b_2 h(A) \quad a_3 h(B) + b_3 h(A) \quad a_4 h(B) + b_4 h(A) \\
\cdots \; \cdots \; \cdots \; \cdots \; \cdots \\
\cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \\
a_{\left\{\frac{n+1}{2}\right\}-n/2} h(B) + b_{\left\{\frac{n+1}{2}\right\}-n/2} h(A) \quad \cdots \quad \cdots \quad \cdots \quad h(A)h(B) \quad \cdots \quad \cdots \quad \cdots \quad a_{\left\{\frac{n+1}{2}\right\}+n/2} h(B) + b_{\left\{\frac{n+1}{2}\right\}+n/2} h(A) \\
\cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \; \cdots \\
\cdots \; \cdots \; \cdots \; \cdots \; \cdots \\
a_{w-3} h(B) + b_{w-3} h(A) \quad a_{w-2} h(B) + b_{w-2} h(A) \quad a_{w-1} h(B) + b_{w-1} h(A) \\
a_w h(B) + b_w h(A)
\end{array} \right\rangle
$$

Where $n \in 2Z^{+}+1$, $w = \frac{1}{2}\left(n^2+1\right)$, $h(A) = a_{\left\{\frac{w+1}{2}\right\}}$, $h(B) = b_{\left\{\frac{w+1}{2}\right\}}$ *and* $n/2$ is the integer obtained on division of *n* by 2.

Remark

The method of rhotrix multiplication defined in Ajibade(2003) was given a generalization in Mohammed(2011) and then named as heart-based method for rhotrix multiplication.

2.2.4 Properties of heart based method for rhotrix multiplication

i. *Rhotrix multiplication is commutative*.

ii. *Rhotrix multiplication is associative*: that is, given any three rhotrices R, S and Q, we have $(RoS)oQ = Ro(SoQ)$.

iii. *Rhotrixmultiplication is distributive over addition*: *i.e.*given any two rhotricesR and S, we have $\mathrm{Ro}(S+T) = RoS + RoT$.

iv. *Existence of an identity rhotrix*: For any rhotrix A, we have

$A \circ I = I \circ A = A,$ where$I$ is the identity rhotrix.

2.2.5 Identity and Inverse element of a rhotrix

Ajibade(2003) determined the identity of a rhotrix $R \in R_3(\Re)$ as $I = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle$

,

Also, the inverse of a rhotrix $R \in R_3(\Re)$ is

$$R^{-1} = \frac{-1}{\left(h(R)\right)^2} \left\langle \begin{matrix} & a & \\ b & -h(R) & d \\ & e & \end{matrix} \right\rangle,$$

where $h(R) \neq 0$.

Mohammed $(2007a)$ presented various imaginations of rhomboid arrays forming algebraic structures such as groups, semigroups, monoids, rings, integral domains and field using rhotrix set as the underlying set. Special types of rhotricessuch as symmetric rhotrices, diagonal rhotrices, upper and lower triangular rhotrices, zero heartrhotrices,unit heart rhotrices,hearty rhotrices, odd and even heart rhotriceswere also identified to form various algebraic structures in Mohammed(2007a).

Mohammed $(2007b)$ established the theorem on rhotrix exponent rule and extended the results to special series and polynomial equations over rhotrices by way of systematization.

The following theorems were recorded from Mohammed $(2007a)$

Theorem 1

Let $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ be any rhotrix of size 3, then for any integer value m,

$R^m = \left(h(R)\right)^{m-1} \left\langle \begin{matrix} & ma & \\ mb & h(R) & md \\ & me & \end{matrix} \right\rangle$, *in particular*, R^0 *and* R^{-1} are the identity and inverse of R respectively, provided $h(R)$ is non-zero.

Theorem 2

Let p be any element of a group G consisting of all invertible rhotrices of the same size, if $H = \{P^m : m \in Z\}$, then H is a cyclic group generated by p.

Theorem 3

Let P and P^* be any two rhotrices of the same size, such that P is a unit heart rhotrix having equal corresponding elements with P^*, but $h(P) \neq h(P^*)$.

If $G = \{P : h(P) = 1\}$ is a collection of all P of the same size, and

$G^* = \{P^* : h(P^*) = 0\}$is a collection of all P^* of the same size.Then:

(a) $\langle G^*, \circ \rangle$ *and* $\langle G, + \rangle$ are groups

(b) The mapping $\theta : G \to G^*$ defined by

$\theta(P) = P^*$ is an isomorphism.

The rhotrix exponent rule for base rhotrix in Mohammed(2007a), was thereafter established, characterized and systematized for expressing special series and polynomial equations over baserhotricesby Mohammed (2007b). The rhotrix exponent rule was generalized in Mohammed(2011) as

$$R_n^{\ m} = \left(h(R)\right)^{m-1} \left\langle \begin{array}{ccccccccccc} & & & & & mr_1 & & & & & \\ & & & & mr_2 & mr_3 & mr_4 & & & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ mr_{\left\{\frac{w+1}{2}\right\}-n/2} & \cdots & \cdots & \cdots & & h(R) & & \cdots & \cdots & \cdots & mr_{\left\{\frac{w+1}{2}\right\}+n/2} \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \cdots & \cdots & & & \\ & & & & mr_{r_{w-3}} & mr_{r_{w-2}} & mr_{r_{w-1}} & & & & \\ & & & & & mr_{r_w} & & & & & \end{array} \right\rangle,$$

where $n \in 2Z^{+}+1,\ w=\frac{1}{2}\left(n^{2}+1\right),\ h(R)=r_{\left\{\frac{w+1}{2}\right\}}$.

The study of linear systems over rhotrices was initiated by Aminu(2009). In that study,Aminu investigated the systems of linear equations arising from the rhotrix equation $A \circ B = C$, where the rhotrices A, B and C are of the same size.

For the solvability of linear equations arising from the rhotrix equation A o B= C, the following propositions establishing a necessary and sufficient condition were recorded.

2.2.7 Proposition

According to Aminu (2009) if A, B and C be three rhotrices of size with entries in $\Re$. Then the system of linear equations resulting from A o B =C has

i) A unique solution if and only $h(R)\neq 0$ *and* $h(C)\neq 0$

ii) An infinite solution if and only if $h(A)=h(C)=0$

iii) No solution if and only if $h(A)=0$ *and* $h(C)\neq 0$

Tudunkaya and Makunjuola (2010) presented the method of constructing finite fields over rhotrices whose cardinality was given by way of concrete examples.

Usaini and Tudunkaya (2011a) constructed certain fields of fractions over rhotrices by extending the work of Mohammed (2009).The construction which was done step by step and at each step, a particular algebraic property was shown. Usaini and

Tudunkaya (2012) later discovered that the field as constructedin(Mohammed, 2009) was only possible for set of hearty rhotrices of the same size as in(Mohammed, 2007a).

Mohammed and Tella (2012) presented the categorized rhotrix sets and rhotrix spaces over real and complex fields. The idea was to systematize waysof characterizing rhotrices over the field of numbers and their expressions as rhotrix set spaces, stimulated through their work.

Mohammed and Tijjani (2011) defined metric or distance function between elements in a rhotrix set to the set of real numbers. The idea was extended to construction of metric topological spaces in their work.

Tudunkaya and Makunjuola (2012a), proposed a certain quadratic extension as an extension to the work of Mohammed(2007b). Thereafter, rhotrix polynomial and polynomial rhotrices were proposed by Tudunkaya (2013).

Usaini and Tudunkaya (2011b) presented some notes on rhotrices and the construction of finite fields. Tudunkaya and Makunjuola (2012b) worked on properties of certain finite fields constructed over rhotrices.

2.3 NON CUMMUTATIVE RHOTRIX MULTIPLICATION

An alternative multiplication of rhotrices was proposed by Sani, (2004). Since rhotrices lie somewhere between 2×2 and 3×3 matrices, Sani defined their multiplication similarly to that of matrices. This is the row-column multiplication which was extended to rhotrices.

Sani (2004) defined the rows and columns of a rhotix as follows:

$$R=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \text{then the rows are } \begin{matrix} b & a \\ e & d \end{matrix} \text{ while the columns are } \begin{matrix} a & d \\ b & e \end{matrix}.$$

The multiplication of any two rhotrices using the row − column multiplication method is given as follows;

Let R and Q be two rhotrices of the same size. If

$$R=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \text{and } Q=\left\langle \begin{matrix} & f & \\ g & h(Q) & i \\ & j & \end{matrix} \right\rangle, \text{ then}$$

$$R\circ Q=\left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & f & \\ g & h(Q) & i \\ & j & \end{matrix} \right\rangle = \left\langle \begin{matrix} & af+dg & \\ bf+eg & h(R)h(Q) & ai+dj \\ & bi+ej & \end{matrix} \right\rangle$$

This multiplication, like that of matrices is non commutative, but itis associative.

2.3.1 Identity Element With Respect To The Non-Commutative Rhotrix Multiplication Method

Sani(2004) determined the identity, inverse, determinant and transpose of the rhotrix as follows:

$$I=\left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle,$$

$$R^{-1}=\left\langle \begin{matrix} & \frac{e}{ae-bd} & \\ \frac{-b}{ae-bd} & \frac{1}{h(R)} & \frac{-d}{ae-bd} \\ & \frac{a}{ae-bd} & \end{matrix} \right\rangle$$

$$\det(R)=h(R)(ae-bd), \text{ and}$$

$$R^{T}=\left\langle \begin{matrix} & a & \\ d & h(R) & b \\ & e & \end{matrix} \right\rangle$$

Sani (2007) extended the non-commutative rhotrix multiplication method to rhotrices having size n as follows:

$$R(n)\circ S(n)=\left\langle a_{i_1 j_1}, \quad c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, \quad d_{l_2 k_2} \right\rangle = \left\langle \sum_{i_2 j_1=1}^{w} \left(a_{i_1 j_1}, \quad b_{i_2 j_2} \right), \sum_{l_2 k_1=1}^{w} \left(c_{l_1 k_1}, \quad d_{l_2 k_2} \right), \right\rangle.$$

Where $w=\frac{1}{2}(n+1)$, and

$$R(n)=\left\langle a_{ij}, \quad c_{lk} \right\rangle = \left\langle \begin{matrix} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{w1} & \ldots & \ldots & \ldots & \ldots & \ldots & a_{1w} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{ww-2} & c_{w-1w-2} & a_{w-1w-1} & c_{w-2w-1} & c_{w-2w} & & \\ & a_{ww-1} & c_{w-1w-1} & a_{w-1w} & & & \\ & & a_{ww} & & & & \end{matrix} \right\rangle$$

while

$$S(n)=\langle a_{ij}, \quad c_{lk}\rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_{21} & d_{11} & b_{12} & & & \\ & & b_{31} & d_{21} & b_{22} & d_{12} & b_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{w1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & b_{1w} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & b_{ww-2} & d_{w-1w-2} & b_{w-1w-1} & d_{w-2w-1} & b_{w-2w} & & & \\ & & b_{ww-1} & d_{w-1w-1} & b_{w-1w} & & & & \\ & & & b_{ww} & & & & & \end{array} \right\rangle$$

The elements a_{ij}, $(i, j = 1,2,...,w)$ *and* $c_{lk}, (l,k = 1,2,...,w-1)$ being the major and minor entries of $R(n)$ respectively. Similarly the elements $b_{ij}, (i, j = 1,2,...,w)$ *and* $d_{lk}, (l,k = 1,2,...,w-1)$ are the major and minor entries of $S(n)$ respectively.

Sani(2007)established some interesting relationships between n-size rhotrices and invertible $w \times w$ dimensional matrices, where $w = \frac{1}{2}(n+1)$, $n \in 2Z^{+}+1$. Sani(2007) further generalized the definition of identity, inverse, determinant ,and transpose of rhotrix $R(n)$ of size n $(provided\ R(n) \neq 0)$.

Kaurangini and Sani (2007) constructed a special form of rhotrix termed as Hilbert rhotrix. The concept of Hilbert matrix and its relationship with special rhotrices of dimensions 3×3 *and* 2×2 was also discussed.

The challenge posed by Ajibade(2003) on whether a rhotrix could be transformed to matrix and vice-versa was resolved by Sani(2008), when he proposed the method of coupled matrices, ie converting a rhotrix into a special kind of matrix. This is by rotating therhotrix R of size $n \in 2Z^{+1}+1$ through 45° in an anti-clockwise direction.

$$R_n = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{t1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1t} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & a_{n-2} & c_{t-1t-2} & & \cdots & & c_{t-1t-1} & a_{t-2t} & \\ & & a_{n-1} & & c_{t-1t-1} & & a_{t-1t} & & \\ & & & & a_{tt} & & & & \end{array} \right\rangle$$

where $t = \frac{n+1}{2}$ so if $\boldsymbol{n = 3}$, then $\boldsymbol{t = 2}$, if $\boldsymbol{n = 5, t = 3}$. Accordingly, R_3 *and* R_5 are given by

$$R_3^{T/2} = \left\langle \begin{array}{ccc} & a_{11} & \\ a_{21} & c_{11} & a_{12} \\ & a_{22} & \end{array} \right\rangle = \begin{bmatrix} a_{11} & & a_{12} \\ & c_{11} & \\ a_{21} & & a_{22} \end{bmatrix}$$

$$R_5^{T/2} = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{11} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{11} & \\ a_{21} & & a_{22} & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix}$$

Where $T/2$ indicates a rotation through 45° in an anti-clockwise direction. Thus, in general for a coupled matrix, we have: $R_n^{T/5} = \left\langle a_{ij}, \quad c_{kl} \right\rangle^{T/2} = \left[a_{ij}, \quad c_{kl} \right] = [Ac](n)$.This rotation results into a coupled matrix.The coupled matrix consists of two matrices of dimensions $w \times w$ *and* $(w-1)\times(w-1)$.

Two coupled matrices $[Ac]_n$ and $[Bd]_n$ can be multiplied by completing the missing spaces with zeros, a filled coupled matrix results after the multiplication by removing the zeros earlier used in completing the free spaces.

Sani(2008) recorded the following:

2.3.2 TheoremSani(2008)

If a coupled matrix $[Ac]_n$ is completed with zeros, then its determinant is the product of the determinant of the matrices $[A]_{w\times w}$ and $[C]_{(w-1)\times(w-1)}$, where $w = \frac{1}{2}(n+1)$.

Remark:

The determinant of a coupled matrix $[Ac]_n$ can be gotten in like manner as the determinant of a rhotrix of size n, i.e. $\det[Ac]_n$ = $\det[A]_n \det[C]_{n-1}$.

Aminu$(2010a)$ constructed a system of equations of the form

$R_n(X)=b,$ where R_n is an n-dimensional rhotrix and X the unknown n-dimensional rhotrix and b the resulting rhotrix vector. Aminu further stated that any system of equations of this form is called a system of n rhotrix equations.

Aminu $(2010a)$ recorded the following theorem.

2.3.3 THEOREMAminu(2010a)

Let $R_n=\langle a_{ij}, c_{lk}\rangle$ be an n-dimensional rhotrix. For the system of equation $R_n\langle x^{nj}\rangle=\langle b^{nj}\rangle$ to be solvable, a necessary and sufficient condition corresponding to the system of equations, $Ax^{wj}=b^{wj}$ is solvable, where

$A=(a_{ij})\in\Re^{w\times w}, x^{wj}, b^{wj}\in\Re^{w\times w}$ *and* $w=\frac{1}{2}(n+1)$.

Aminu (2010a) equally presented concepts of rhotrix eigenvector and eigenvalue problems. The rhotrix eigenvalue problem (REP) is given by $R_n=\langle a_{ij}, c_{kl}\rangle$ find all

$\lambda\in\Re\ (eigenvalues)$ and an n-dimensional rhotrix column vector

$\langle x^{nj}\rangle, \langle x^{nj}\rangle\neq 0\ (eigenvectors)$ such that $R_n\langle x^{nj}\rangle=\langle b^{nj}\rangle$

The following result was formulatedby Aminu$(2010a)$ in order to solve REPs.

2.3.4 THEOREMAminu(2010a)

Let $R_n=\langle a_{ij}, c_{kl}\rangle$ be an n-dimensional rhotrix. Then, $\lambda\in\Re$, is a rhotrix eigenvalue of R_n if and only if $\det(A-\lambda I)=0$, where $A=(a_{ij})\in\Re^{w\times w}$ and $w=\frac{1}{2}(n+1)$.

2.3.4.1 CorollaryAminu(2010a)

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an n-dimensional rhotrix and $A = (a_{ij}) \in \Re^{w \times w}$ be the matrix generated from $R(n)$, with $w = \frac{1}{2}(n+1)$. Then $\langle v^{nj} \rangle$ is the rhotrix eigenvector corresponding to the eigenvalue λ if the system $(A - \lambda I) = 0$.

Sharma and Kanwar(2011) presented a note on the relationship between invertible rhotrices and associated invertible matrices.

Absalom et al. (2011) adopted the row-column base method of multiplication of rhotices of size n as proposed by Sani(2007) for the implementation of algorithm designs.

Aminu$(2012b)$, extended his work on equations over rhotrices to the concept of Cayley-Hamilton theorems which are well-known in linear algebra. The formulated and proved concepts in matrices were extended to rhotrices. These formulated and proved concepts were considered by Sharma and Kanwar$(2012a)$ in the Cayley Hamilton theorem for rhotrices.

Sharma and Kanwar$(2012b)$ used the concept of adjoint of a matrix and presented the concept of adjoint of a rhotrix and its basic properties. Some analogous results of matrices in the context of rhotrices were proved.

Usaini$(2012a)$ presented the concept of involution to rhotrices and presented their properties. The method of constructing involuntary rhotrices were also indicated in his work. Sharma and Kanwar(2013) presented some theorems of involution in the context of rhotrices by considering extensions on the involutoryrhotrix.

Aminu (2010c) extended the concept of linear mappings to rhotrices, where the linear mapping $T: U \rightarrow V$, such that U and V are rhotrix vector spaces were considered and their properties presented.

2.3.5 SOME PROPERTIES OF ISOMORPHIC MAPPINGS

If the mapping f:$G \rightarrow G^1$ is an isomorphism. Then

(i) Identities correspond: ie if e is the identity of G then f$(e) = e^1$ is the identity of G^1 which implies image of the identity of G is the identity of G^1.

(ii) Inverse correspond i.e the image of the inverse of an element of A is the inverse of the image of that element ie f$(A^{-1}) = [f(A)]^{-1}$.

(iii) The order of an element and its image are the same ie $0(a) = 0[f(a)]$

(iv) $f^{-1}$is also an isomorphic mapping.

(v) The product of two isomorphisms is also an isomorphism.

CHAPTER THREE

FORMATION OF RHOTRIX GROUPS

3.1 INTRODUCTION

Let G be a non- empty set equipped with a binary operation denoted by '$*$' If for any $a,b \in G$, a$*$b or more conveniently ab represents the element of G obtained by applying the said binary operation between the elements a and b of G taken in order. The algebraic structure $\boldsymbol{G}^*$ is a group if the binary operation $*$ satisfies the following axioms.

a) For any a,b$\in \boldsymbol{G}$ a$*$b$\in \boldsymbol{G}$ implies closure axiom.

b) $(\boldsymbol{a} * \boldsymbol{b}) * \boldsymbol{c} = \boldsymbol{a} * (\boldsymbol{b} * \boldsymbol{c})$for all a, b, c $\in \boldsymbol{G}$ implies associativity axiom.

c) If there exists e $\in \boldsymbol{G}$ such that for all a$\in \boldsymbol{G}$ e$*$a=a$*$e=a. implies identity axiom.

d) If for all a$\in \boldsymbol{G}$,there exists b $\in \boldsymbol{G}$ such that a$*$b=b$*$a=e implies inverse axiom.

A group is said to be abelian or commutative if in addition to all of the above axioms, it also satisfies ab=ba, for all a,b$\in \boldsymbol{G}$.

3.2 RHOTRIX GROUP

A rhotrix set $\hat{G}$ for which the law of composition is defined and on which a binary operation, $*$ is defined forms a rhotrix group if the following conditions are satisfied.

(I) Closure; to every ordered pair A,B of $\hat{G}$, there exists a unique element C of $\hat{G}$ written as C=A$*$B=AB

(II) If A,B,C are any three rhotrices of $\hat{G}$, then $(AB)C = A(BC)$ so that either side may be denoted as ABC. Then the associative axiom is satisfied.

(III) Unit element: The rhotrix set $\hat{G}$ contains an element 1, called the unit element (or the identity or neutral element) such that for every element A of $\hat{G}$ A1=1A=A.

(IV) Inverse element: Corresponding to every element A of $\hat{G}$,there exists in $\hat{G}$ an element A^{-1} such that $AA^{-1} = A^{-1}A = 1$.

A rhotrix group $\hat{G}$, which has the additional property that for every two of its elements $A, B \in \hat{G}$, AB=BA is called an Abelianrhotrix group or commutative rhotrix group.

3.3 Definition: The rhotrix elements A, B are said to commute $(ortobepermutable)$ if AB = BA. For example the value 1, commutes with every element and A commutes with A^{-1}.

3.4 Definition : If for any three rhotrix sets A,B,C we have AB=CB which implies A=C, then we say the right cancellation law is satisfied. Furthermore,if BA= BC, implying A=C, then the left cancellation law is satisfied.

3.5 Definition: If $\hat{G}$ consists of a finite number of rhotrix elements, then this number is called the order of $\hat{G}$; otherwise $\hat{G}$ is said to be of infinite order. The order of $\hat{G}$ whether finite or infinite, will be denoted by $|\hat{G}|$.

Although, the notation of multiplication is commonly used for the composition of rhotrix group elements, it is sometimes convenient to adopt other notations ,such as $A \circ B$ to express the composition of A and B.

3.6 Theorem

Let $G_3 = \left\{ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle : a,b,d,e \in Z_3, Z_3 = \{0,1,2\} \right\}$ then $(G_3, \circ)$ is a rhotrix group.

Refer to appendix A for the complete entries of G_3

Proof

Let A, B, C be elements of the set G_3 such that

$$A = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle, B = \left\langle \begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix} \right\rangle, C = \left\langle \begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix} \right\rangle$$

i. Then, test for closure:

$$A\circ B=B\circ A=\left\langle\begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle\circ\left\langle\begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix}\right\rangle=\left\langle\begin{matrix} & a+a' & \\ b+b' & 1 & d+d' \\ & e+e' & \end{matrix}\right\rangle$$

$$=\left\langle\begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix}\right\rangle\circ\left\langle\begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle=\left\langle\begin{matrix} & a'+a & \\ b'+b & 1 & d'+d \\ & e'+e & \end{matrix}\right\rangle\in G$$

ii. Test for associativity i.e

$(A\circ B)\circ C=A\circ(B\circ C)$

$$\Rightarrow\left(\left\langle\begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle\circ\left\langle\begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix}\right\rangle\right)\circ\left\langle\begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix}\right\rangle=\left\langle\begin{matrix} & a+a' & \\ b+b' & 1 & d+d' \\ & e+e' & \end{matrix}\right\rangle\circ\left\langle\begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix}\right\rangle=\left\langle\begin{matrix} & a+a'+a'' & \\ b+b'+b'' & 1 & d+d'+d'' \\ & e+e'+e'' & \end{matrix}\right\rangle$$

and

$$\left\langle\begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle\circ\left(\left\langle\begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix}\right\rangle\circ\left\langle\begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix}\right\rangle\right)=\left\langle\begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle\circ\left\langle\begin{matrix} & a'+a'' & \\ b'+b'' & 1 & d'+d'' \\ & e'+e'' & \end{matrix}\right\rangle=\left\langle\begin{matrix} & a+a'+a'' & \\ b+b'+b'' & 1 & d+d'+d'' \\ & e+e'+e'' & \end{matrix}\right\rangle$$

$$=\left\langle\begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle\circ\left\langle\begin{matrix} & a'+a'' & \\ b'+b'' & 1 & d'+d'' \\ & e'+e'' & \end{matrix}\right\rangle=\left\langle\begin{matrix} & a+a'+a'' & \\ b+b'+b'' & 1 & d+d'+d'' \\ & e+e'+e'' & \end{matrix}\right\rangle$$

Hence, $(A\circ B)\circ C=A\circ(B\circ C)$.Therefore the operation is associative.

iii. Test for identity:

$$A \circ I = I \circ A = A$$

$$ie\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$$

iv. Test for inverses:

$$A \circ A^{-1} = A^{-1} \circ A = I \, .$$

Let $A = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$, $A^{-1} = \dfrac{1}{h(A)^2} \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & -a & \\ -b & 1 & -d \\ & -e & \end{matrix} \right\rangle$

Therefore

$$A \circ A^{-1} = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & -a & \\ -b & 1 & -d \\ & -e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & -a & \\ -b & 1 & -d \\ & -e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle \in G.$$

Hence $\langle G_3, \circ \rangle$ is a group.

The order of the group is 81, because there are 81 entries in the group, see appendix A.

3.7 Definition:

A subset K of a group $\left\langle \hat{G}, \circ \right\rangle$ such that K is also a group under the binary operation of $\left\langle \hat{G}, \circ \right\rangle$ is called a subgroup of $\hat{G}$ *iff* :

(i) $\forall a,b \in K, \; ab \in K$

(ii) $\forall a \in K, \; a^{-1} \in K$

3.7.1 Example:

Some of the subgroups of $\langle G_3, \circ \rangle$ are;

(A)

$$\left(\left\langle\begin{matrix} & a & \\ 0 & 1 & d \\ & e & \end{matrix}\right\rangle\right)=\left\{\begin{matrix}\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \\ , \\ \left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle\end{matrix}\right\}$$

(B). $$\left(\left\langle\begin{matrix} & a & \\ b & 1 & 0 \\ & e & \end{matrix}\right\rangle\right)=\left\{\begin{matrix}\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \\ , \\ \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle\end{matrix}\right\}$$

(C)

$$\left(\left\langle\begin{matrix} & 0 & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle\right)=\left\{\begin{matrix}\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \\ \\ \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\end{matrix}\right\}$$

(D).

$$\left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & 0 & \end{matrix} \right\rangle\right) = \left\{ \begin{matrix} \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle \\ \\ \left\langle \begin{matrix} & 2 & \\ 2 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 2 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \end{matrix} \right\}$$

Lagrange's theorem:

If G is a finite group and K a subgroup of G then Lagrange's theorem says: the order of the subgroup of K divides the order of the group G.

The case of our subgroups above, each of them has order 9 and so clearly that number divides 81 which verifies Lagrange's theorem in this case.

The order of an element $R \in \langle \hat{G}, \circ \rangle$ is 3, except the identity.

Example:

$$(i)\ \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle^{3} = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle.$$

$$(ii)\ \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle^{3} = \left(\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle \right)$$

$$= \left(\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 2 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle \right) = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle.$$

$$(iii)\ \left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle = \left(\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle \right).$$

$$= \left(\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle \right) = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

3.9 THEOREM

The set

$$G_5 = \left\{ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle : a, b, d, e, \in Z_5, Z_5 = \{0,1,2,3,4\} \right\}$$ is arhotrix group.

The complete entries in the set G_5 are in appendix B.

Proof

Let J, K, L be elements of G_5 such that

$$J = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle, K = \left\langle \begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix} \right\rangle, I = \left\langle \begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix} \right\rangle$$

Then

(i) $$J \circ K = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+a' & \\ b+b' & 1 & d+d' \\ & e+e' & \end{matrix} \right\rangle$$

Also

$$K \circ J = \left\langle \begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a'+a & \\ b'+b & 1 & d'+d \\ & e'+e & \end{matrix} \right\rangle$$ i.e closure property is satisfied.

(ii) Associativity

$$(J \circ K)\circ I = \left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix} \right\rangle\right) \circ \left\langle \begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix} \right\rangle$$

$$= \left\{\left\langle \begin{matrix} & a+a' & \\ b+b' & 1 & d+d' \\ & e+e' & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix} \right\rangle\right\} = \left\langle \begin{matrix} & a+a'+a'' & \\ b+b'+b'' & 1 & d+d'+d'' \\ & e+e'+e'' & \end{matrix} \right\rangle$$

Also

$$J \circ (K \circ I) = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \circ \left\{\left\langle \begin{matrix} & a' & \\ b' & 1 & d' \\ & e' & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a'' & \\ b'' & 1 & d'' \\ & e'' & \end{matrix} \right\rangle\right\}$$

$$= \left\{\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a'+a'' & \\ b'+b'' & 1 & d'+d'' \\ & e'+e'' & \end{matrix} \right\rangle\right\} = \left\langle \begin{matrix} & a+a'+a'' & \\ b+b'+b'' & 1 & d+d'+d'' \\ & e+e'+e'' & \end{matrix} \right\rangle$$

Hence associativity property has been satisfied.

(iii) Test for identity

There exists $I = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle \ni$ for all $J \in G_5$.

$$J \circ I = I \circ J = J \text{ i.e.} \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle = \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$$

$$J \in G_5$$

(iv) Test for inverses: For any

$$\exists\, J^{-1}\ in\ \ G_5\ \ \ni J\circ J^{-1}=J^{-1}\circ J=I$$

$$J=\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle,\quad J^{-1}=\frac{-1}{h\left(J\right)^2}\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle=\left\langle \begin{matrix} & -a & \\ -b & 1 & -d \\ & -e & \end{matrix} \right\rangle$$

$$J\circ J^{-1}=J^{-1}\circ J=\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle\circ\left\langle \begin{matrix} & -a & \\ -b & 1 & -d \\ & -e & \end{matrix} \right\rangle=\left\langle \begin{matrix} & -a & \\ -b & 1 & -d \\ & -e & \end{matrix} \right\rangle\circ\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$$

$$=\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle\in G_5$$

Hence all the axioms of a group have been satisfied by G_5Therefore G_5 is a group.

The order of (G_5, o) is 625. The 625 entries in (G_5, o)appear in appendix B.

Some of the subgroups of $\left\langle G_5,\circ\right\rangle$ are:

(A)

$$\left\{\begin{matrix}
\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 3 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 4 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 3 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 4 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 2 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 3 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 4 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 1 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 2 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 3 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 4 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 1 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 2 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 3 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle,\left\langle \begin{matrix} & 4 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle
\end{matrix}\right\}$$

(B)

$$
\left\{
\begin{array}{l}
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \\
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle, \\
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle, \\
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 3 \\ & 0 & \end{array} \right\rangle, \\
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 4 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 4 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 4 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 3 & \\ 0 & 1 & 4 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 4 & \\ 0 & 1 & 4 \\ & 0 & \end{array} \right\rangle,
\end{array}
\right\}
$$

(C)

$$\left\{\begin{matrix}
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \\
, \\
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle,
\end{matrix}\right\}$$

(D)

$$\left\{\begin{array}{l}
\left\langle\begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 1 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 3 & \\ 1 & 1 & 1 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 4 & \\ 1 & 1 & 1 \\ & 2 & \end{array}\right\rangle, \\
\left\langle\begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 2 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 3 & \\ 1 & 1 & 2 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 4 & \\ 1 & 1 & 2 \\ & 2 & \end{array}\right\rangle, \\
\left\langle\begin{array}{ccc} & 0 & \\ 1 & 1 & 2 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 0 & \\ 1 & 1 & 3 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 3 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 3 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 3 & \\ 1 & 1 & 3 \\ & 2 & \end{array}\right\rangle, \\
\left\langle\begin{array}{ccc} & 4 & \\ 1 & 1 & 3 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 0 & \\ 1 & 1 & 4 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 4 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 4 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 3 & \\ 1 & 1 & 4 \\ & 2 & \end{array}\right\rangle, \\
\left\langle\begin{array}{ccc} & 4 & \\ 1 & 1 & 4 \\ & 2 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \\ & 3 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 3 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 2 & \\ 1 & 1 & 0 \\ & 3 & \end{array}\right\rangle,
\left\langle\begin{array}{ccc} & 3 & \\ 1 & 1 & 0 \\ & 3 & \end{array}\right\rangle,
\end{array}\right\}$$

(E)

$$\left\{\begin{matrix}
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,
\end{matrix}\right\}$$

It is clear that Lagranges's theorem is satisfied because the order of (G_5,o) is 625 and the order of the subgroups of (G_5,o) is 25.

The order of an element in (G_5,o) is gotten by multiplying the element by itself continuously until the result is identity. For instance,

$$\left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle^5 = \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle \circ \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle \circ \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle \circ \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle \circ \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle = \left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle$$

This implies the order of the element $\left\langle \begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle$ is 5.

Again

$$\left\langle \begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle^5 = \left\langle \begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle.$$

Actually all elements in G_5 except the identity have order 5. From the proofs of

2

Theorems 3.6 and 3.7, it is clear that for any fixed natural number $n \geq 2$, the

following theorem holds.

3.10 THEOREM

For any fixed natural number, n≥2 , n prime, the set

$$G_n = \left\{ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle : a, b, d, e \in Z_n \ where \ Z_n = \{0, 1, 2, ..., n-1\} \right\},$$

is a group.

Proof

Same method of proof as in theorems 3.6 and 3.7.

Thus for G_2

$$G_2 = \left\{ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle : a, b, d, e, \in Z_2 \ \ where\ Z_2 = \{0,1,2\} \right\}$$

is a group. It has entries as follows

$$G_2 = \left\{ \left\langle \begin{matrix} & 0 & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle, \left\langle \begin{matrix} & a & \\ 0 & 1 & d \\ & e & \end{matrix} \right\rangle, \left\langle \begin{matrix} & a & \\ 1 & 1 & d \\ & e & \end{matrix} \right\rangle, \left\langle \begin{matrix} & a & \\ b & 1 & 0 \\ & e & \end{matrix} \right\rangle, \left\langle \begin{matrix} & a & \\ b & 1 & 1 \\ & e & \end{matrix} \right\rangle, \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & 1 & \end{matrix} \right\rangle \right\}$$

So that the complete entries are:

$$\left\{ \begin{matrix} \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \\ \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \end{matrix} \right\}$$

The order of (G_2, o) is 16.

The order of every element $R \in \langle G_2, \circ \rangle$ is 2.

$$i.e.\ \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle^2 = \left(\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle \right) = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

also

$$\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle^2 = \left(\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle \right) = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

CHAPTER FOUR

ISOMORPHISMS OF RHOTRIX GROUPS

4.0 INTRODUCTION

Rhotrix is an innovation by way of extension from matrix tersions and matrix noitrets. Rhotrix is a new area of study relating to linear mathematical algebra. From our results so far, there are sets of rhotrix algebraic structure which satisfy the axioms of group and so they form a rhotrix group.

We intend to demonstrate by way of example, the result in (Mohammed, 2007a, theorem 1) which said there is an isomorphism between group of unit heart rhotrices of the same size and zero heart rhotrices of the same size. Furthermore, some important properties of isomorphic mappings would be discussed.

4.1 EXAMPLE OF ISOMORPHISMIC GROUP OF RHOTRICES

Let P and P^* be any two rhotrices of the same size, such that P is a unit heart rhotrix having equal corresponding elements with P^*, but $h(P) \neq h(P^*)$. If $G = \{P : h(P) = 1\}$ is a collection of all P of the same size, and $G^* = \{P^* : h(P^*) \neq 1\}$ is a collection of all P^* of the samr size, then

(a) $\langle G, \circ \rangle$ *and* $\langle G^*, + \rangle$ *are groups*

(b) *the mapping* $\theta : G \leftarrow G^*$ *defined by* $\theta(P) = P^*$

is an isomorphism

Proof

(a) Let $P = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$ and $Q = \left\langle \begin{matrix} & f & \\ g & 1 & j \\ & k & \end{matrix} \right\rangle$

be any two elements of the group G. Then

$P \circ Q = \left\langle \begin{matrix} & a+f & \\ b+g & 1 & d+j \\ & e+k & \end{matrix} \right\rangle$ is a unit heartrhotrix belonging to G.Therefore, G

is closed with respect to multiplication. Since for any $P \in G,\ P \circ I = I \circ P = P.$

Also,for any $P \in G$,

there exists a corresponding inverse $P^{-1} = \left\langle \begin{matrix} & -a & \\ -b & 1 & -d \\ & -e & \end{matrix} \right\rangle \in G,$ such that

$P \circ P^{-1} = P^{-1} \circ P = I$ belongs to G.So elements of G contain inverse elements.

Also associative law holds in G, since for any

$P, Q, S \in G,\ (P \circ Q) \circ S = P \circ (Q \circ S) \in G.$ Hence G is a group on all unit heart

rhotrices of the same size under multiplication.

Nest, we show that G^* is a group under adition.

Let $P^* = \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle$ *and* $Q^* = \left\langle \begin{matrix} & f & \\ g & 0 & j \\ & k & \end{matrix} \right\rangle$ be any two elements of the group G^*. Then

$$P+Q = \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & 0 & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+f & \\ b+g & 0 & d+j \\ & e+k & \end{matrix} \right\rangle,$$ is a zero heart rhotrix belonging to G^*. So G^* is closed under adition. The identity element $0 = \left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix} \right\rangle$ belongs to G^*, since for any $P^* \in G, P^* + 0 = 0 + P^* = P^* \in G,$

Corresponding to each $P^* \in G,$ there is an additive inverse elment

$-P^* = \left\langle \begin{matrix} & -a & \\ -b & 0 & -d \\ & -e & \end{matrix} \right\rangle$ *such that* $P^* + -P^* = -P^* + P^* = 0 \in G^*.$ Finally, associative law hoids in G^*, since for any $P^*, Q^*, S^* \in G^*$ *we have* $(P^* + Q^*) + S^* = P^* + (Q^* + S^*)$ belongs to G^*. Hence G^* is a group on all zero heart rhotrices of the same size under addition.

(b) We are to show that the mapping $\theta : G \to G^*$ *defined by* $\theta(P) = P^*$ is an isomorphism.

Let P and Q be any two elements of G,

$$\theta(P\circ Q)=\theta\left\{\left\langle \begin{matrix} & a+f & \\ b+g & 1 & d+j \\ & e+k & \end{matrix}\right\rangle\right\}=\left\langle \begin{matrix} & a+f & \\ b+g & 0 & d+j \\ & e+k & \end{matrix}\right\rangle$$

$$=P^{*}+Q^{*}=\theta(P)+\theta(Q)$$

Therefore, θ is a homomorphism. Thus is 1-1 and onto since

$$\ker\theta=\left\{I\in G:\theta(I)=\theta\left(\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle\right)=\left\langle \begin{matrix} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{matrix}\right\rangle=0\in G\right\}$$

$$\operatorname{Im}\theta=\left\{P^{*}\in G^{*}:P^{*}=\theta(P)\ \textit{for each corresponding}\ P\in G\right\}.$$

Thus, the image of θ is the whole of G^{*}. Hence θ is an isomorphism.

Let also

$$G=\left\{\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix}\right\rangle : a,b,d,e\in Z_3\right\}$$

and

$$\hat{G}'=\left\{\left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix}\right\rangle : a,b,d,e\in Z_3\right\}$$

With ‘’o’’ as the binary operation in G and ‘’+’’ being also yhe binary operation in $\hat{G}$.

Then

$\theta : \hat{G} \to \hat{G}'$ defined by $\theta : \left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \right) = \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle$ is an isomorphism .

PROOF

For one-one

Let $\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle$ *and* $\left\langle \begin{matrix} & f & \\ g & 1 & j \\ & k & \end{matrix} \right\rangle$

be arbitrary elements of the group of unit heart rhotrices with entries in Z_3, where

$a \neq f,\ b \neq g, d \neq j\ and\ e \neq k$

then,

$$\theta\left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \right) = \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle \in \hat{G}$$

and

$$\theta\left(\left\langle \begin{matrix} & f & \\ g & 1 & j \\ & k & \end{matrix} \right\rangle \right) = \left\langle \begin{matrix} & f & \\ g & 0 & j \\ & k & \end{matrix} \right\rangle \in \hat{G}$$

Which are clearly different elements of each other.

To show onto:

Let $\left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle \in \hat{G}'$ then thereexist $\left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle \in \hat{G}$ such that

$$\theta\left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle\right) = \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle. \quad \Rightarrow \theta \textit{ is onto.}$$

Test for homomorphism

$$R_1,\ R_2 \in G$$
$$\theta : G \to G'$$
$$\theta(R_1), \theta(R_2) \in G'.$$

Let $R_1 = \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle \in G$ and $R_2 = \left\langle \begin{matrix} & f & \\ g & 1 & j \\ & k & \end{matrix} \right\rangle \in G$ bearbitrarily elements.

Then

$$\theta(R_1 \circ R_2) \Rightarrow \theta\left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & 1 & j \\ & k & \end{matrix} \right\rangle\right)$$

$$= \theta\left(\left\langle \begin{matrix} & a+f & \\ b+g & 1 & d+j \\ & e+k & \end{matrix} \right\rangle\right) = \left\langle \begin{matrix} & a+f & \\ b+g & 0 & d+j \\ & e+k & \end{matrix} \right\rangle.$$

On the other hand

$\theta R_1 \circ \theta R_2 =$

$$= \theta\left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle\right) + \theta\left(\left\langle \begin{matrix} & f & \\ g & 1 & j \\ & k & \end{matrix} \right\rangle\right)$$

$$= \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & 0 & j \\ & k & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & a+f & \\ b+g & 0 & d+j \\ & e+k & \end{matrix} \right\rangle$$

$\therefore \theta(R_1 + R_2) = \theta R_1 + \theta R_{2.}$

Therefore the function defined by

$\theta : G \to G'$

Such that

$$\theta\left(\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle\right) = \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle$$

is an isomorphism.

4.2 THE MULTIPLICATION TABLE FOR A RHOTRIX GROUP

4.4.1 Let $\hat{G}$ be a finite rhotrix group of order four such that

$$\hat{G} = \left\{ \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & -1 & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & -i & 0 \\ & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccc} & 0 & \\ 0 & i & 0 \\ & 0 & \end{array} \right\rangle \right\}$$

Since $i = \sqrt{-1},\ i^2 = -1,\ -i \cdot i = -\left(i^2\right) = -(-1) = 1.$

Thus the multiplication table is

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

With

$$I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,\ A = \left\langle \begin{array}{ccc} & 0 & \\ 0 & -1 & 0 \\ & 0 & \end{array} \right\rangle,\ B = \left\langle \begin{array}{ccc} & 0 & \\ 0 & -i & 0 \\ & 0 & \end{array} \right\rangle,\ and\ C = \left\langle \begin{array}{ccc} & 0 & \\ 0 & i & 0 \\ & 0 & \end{array} \right\rangle.$$

4.3 CYCLIC RHOTRIX GROUPS

4.3.1THEOREM

Let $\left\langle \hat{C}, \circ \right\rangle = \left\{ X^m : m \in Z \right\}$ be a cyclic group under multiplication generated by

$X=\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle, a,b,d,e,\in \hat{C}$ and let $\langle Z,+\rangle$ be agroup of integers under addition. Then

$\left\langle \hat{C},\circ \right\rangle$ Is isomorphic to $\langle Z,+\rangle$.

Proof

Consider the rhotrix set

$$\hat{C}=\left\{X^m : m\in \mathbf{Z}\right\}=\left\{\ldots, X^{-2}, X^{-1}, X^0, X^1, X^2, \ldots,\right\}$$

With $X=\left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle : a,b,d,e\in \hat{C}$. Multiplication is defined by the rule

$X^r\, X^s = X^{r+s}\;\; r,s=(0,\pm 1,\pm 2,\ldots)$ and our convention of $X^0=\left\langle \begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle = I, \hat{C}$ becomes an Abelian group which is called the infinite cyclic group generated by X. This group is isomorphic with the additive group of integers,that is with the set $\mathbf{Z}=\{0,\pm 1,\pm 2,\ldots,\}$ in which the composition of r with s is defined as r+s. The correspondence which establishes the isomorphism is given by $X^r\theta = r$.

On the other hand, if X is assumed to satisfy the equation $X^m = X^0$ where m is a positive integer greater than unity and it is assumed that $X^k \neq X^0$ when $0 < 1 < m$ the set $\hat{C}=\left\{X^0, X, X^2, \ldots, X^{m-1}\right\}$ of distinct symbols form an Abelian group of order m under the rule $X^r X^s = X^{r+s}\left(r, s = 0, 1, ..., m-1\right)$, where r+s has to be reduced to its least non-negative residue modulo m. This group is called cyclic group of order m, generated by X. It is isomorphic to the additive group of residue modulo m.

Consider $Z_3 = \{0, 1, 2\}$ and the rhotrix group

$$G_3 = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & 1 & d \\ & e & \end{array} \right\rangle : a, b, d, e, \in Z_3, Z_3 = \{0, 1, 2\} \right\} \text{ with for instance}$$

$$A = \left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 2 \\ & 1 & \end{array} \right\rangle, \ B = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 1 & 1 \\ & 2 & \end{array} \right\rangle, \ I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle. \text{ Then}$$

$$A \circ B = I = \left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle \ and$$

$$A \circ A = A^2 = B = \left\langle \begin{array}{ccc} & 1 & \\ 2 & 1 & 1 \\ & 2 & \end{array} \right\rangle, B \circ B = B^2 = A = \left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 2 \\ & 1 & \end{array} \right\rangle$$

Hence the group

$$G_3 = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & 1 & d \\ & e & \end{array} \right\rangle : a, b, d, e, \in Z_3 \right\} \text{ is cyclic.}$$

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.0 SUMMARY

In this thesis, we have shown that already known algebra structures can be systematized into rhotrix groups. As the facts about groups are developed, we shall often observe analogies with facts we already know, and will serve as mathematical model on all spheres of studies irrespective of the course.

Experience provides ample verification of the assumption that in learning mathematics it is pedagogically essential to develop concepts carefully, and to proceed from familiar to the unfamiliar , from specific to general and from concrete to abstract.

In this work we developed rhotrix groups using residue classes of modulo two, three and five as our underlying sets, and then generalize it to modulo n. The multiplication table for rhotrix group and rhotrix maps were developed using already known theorems. The subgroups of the rhotrix groups developed were also identified and the order of each element of the different residue classes were shown.

5.1 CONCLUSION

From this work, the rhotrix groups we developed was used to verify Langrange's theorems. It was shown that cyclic rhotrix groups could be developed from the cyclic rhotrix multiplication table presented. It was equally shown that there exists homomorphisms and isomorphisms between certain rhotrix groups.

5.2 RECOMMENDATIONS

I wish to recommend thatother theorems of groups be verified using rhotrices. The study of rhotrix algebra should be encouraged at all levels of both learning and research. In institutions of higher learning, it should be included in and studied alongside matrices. To researchers in science related fields, this innovation is a challenge and they should try to incorporate it in their various disciplines.

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APPENDIX A

$$\text{The set } G_3 = \left\{ \begin{array}{l} \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle : a,b,d,e \in Z_3, \textit{where}\, Z_3 = \{0,1,2\}, \\ \textit{the residue class} \text{ mod } 3. \end{array} \right\}$$

Has entries as follows:

$$
\left\{
\begin{array}{l}
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle, \\
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{array} \right\rangle,,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 2 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 2 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 2 \\ & 2 & \end{array} \right\rangle \\
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 0 \\ & 2 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 1 \\ & 2 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 1 \\ & 2 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 0 & 1 & 1 \\ & 2 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 2 \\ & 2 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 2 \\ & 2 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle, \\
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 2 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 2 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{array} \right\rangle, \\
\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 0 \\ & 1 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 1 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 0 & \\ 1 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\left\langle \begin{array}{ccc} & 2 & \\ 1 & 1 & 2 \\ & 0 & \end{array} \right\rangle,
\end{array}
\right\}
$$

$$
\left\{
\begin{array}{l}
\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,
\end{array}
\right\}
$$

APPENDIX B

The set $G_5 = \left\{ \left\langle \begin{matrix} & a & \\ b & 1 & d \\ & e & \end{matrix} \right\rangle : a, b, d, e \in Z_5,\ where\ Z_5 = \{0,1,2,3,4\} \text{ is the residue class mod } 5. \right\}$ has entries as follows:

$$
\left\{
\begin{array}{l}
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle
\end{array}
\right\}
$$

$$
\left\{\begin{array}{l}
\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 0 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle,
\end{array}\right\}
$$

$$\Big\{ , \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle$$

$$\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1, & \\ 1 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \Big\}$$

$$\left\{ \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix} \right\rangle, \right.$$

$$\left\langle \begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1, & \\ 1 & 1 & 3 \\ & 2 & \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} & 2 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 2 & \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} & 1 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix} \right\rangle,$$

$$\left\langle \begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 3 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle,$$

$$\left. \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \right\}$$

$$\left\{ \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \right.$$

$$\left\langle\begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 1 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 4 & \\ 1 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 1 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 1 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 1 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 2 & \\ 1 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 1 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 1 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 0 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 3 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,$$

$$\left\langle\begin{matrix} & 1 & \\ 2 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle,$$

$$\left. \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 0 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 1 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 2 \\ & 1 & \end{matrix}\right\rangle, \right\}$$

$$\Bigg\{\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 2 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 2 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 2 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 3 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 3 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 3 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 3 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 3 \\ & 1 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 4 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 4 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 4 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 4 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 4 \\ & 1 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 0 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 0 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 0 \\ & 3 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 0 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 0 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 1 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 1 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 1 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 1 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 1 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 2 \\ & 3 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 2 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 2 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 2 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 2 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 3 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 3 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 3 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 3 \\ & 3 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 3 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 4 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 4 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 4 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 4 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 4 \\ & 3 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 0 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 0 \\ & 4 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 0 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 0 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 0 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 1 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 1 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 1 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 1 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 1 \\ & 4 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 2 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 2 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 2 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 2 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 2 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 3 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 3 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 3 \\ & 4 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 3 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 3 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 4 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 4 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 4 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 4 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 4 \\ & 4 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 0 \\ & 0 & \end{array}\right\rangle,$$

$$\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 0 \\ & 0 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 0 \\ & 0 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 0 \\ & 0 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 4 & \\ 2 & 1 & 0 \\ & 0 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 0 & \\ 2 & 1 & 1 \\ & 0 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 1 & \\ 2 & 1 & 1 \\ & 0 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 2 & \\ 2 & 1 & 1 \\ & 0 & \end{array}\right\rangle,\left\langle\begin{array}{ccc} & 3 & \\ 2 & 1 & 1 \\ & 0 & \end{array}\right\rangle,\Bigg\}$$

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\left\{
\begin{array}{l}
\left\langle\begin{matrix} & 4 & \\ 2 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 2 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 2 & \\ 2 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 3 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 2 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 2 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 2 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 2 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 2 & 1 & 4 \\ & 0 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 3 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 3 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 3 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 3 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 0 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 2 & \\ 3 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 1 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 2 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 3 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 3 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 3 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 4 \\ & 4 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 1 & \\ 3 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 2 & \\ 3 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 3 & \\ 3 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 4 & \\ 3 & 1 & 0 \\ & 0 & \end{matrix}\right\rangle, \left\langle\begin{matrix} & 0 & \\ 3 & 1 & 1 \\ & 0 & \end{matrix}\right\rangle,
\end{array}
\right\}
$$

$$\left\{\begin{array}{l}
\left\langle\begin{matrix}&1&\\3&1&1\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&1\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&1\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&1\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&2\\&0&\end{matrix}\right\rangle\left\langle\begin{matrix}&1&\\3&1&2\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&2\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&2\\&0&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&4&\\3&1&2\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&3\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&3\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&3\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&3\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&3\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&4\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&4\\&0&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&2&\\3&1&4\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&4\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&4\\&0&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&0\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&0\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&0\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&0\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&0\\&1&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&0&\\3&1&1\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&1\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&1\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&1\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&1\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&2\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&2\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&2\\&1&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&3&\\3&1&2\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&2\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&3\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&3\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&3\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&3\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&3\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&4\\&1&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&1&\\3&1&4\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&4\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&4\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&4\\&1&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&0\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&0\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&0\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&0\\&2&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&4&\\3&1&0\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&1\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&1\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&1\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&1\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&1\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&2\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&2\\&2&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&2&\\3&1&2\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&2\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&2\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\3&1&3\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&3\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&3\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&3\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&3\\&2&\end{matrix}\right\rangle,\\
\left\langle\begin{matrix}&0&\\3&1&4\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\3&1&4\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\3&1&4\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&3&\\3&1&4\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&4&\\3&1&4\\&2&\end{matrix}\right\rangle,\left\langle\begin{matrix}&0&\\4&1&4\\&4&\end{matrix}\right\rangle,\left\langle\begin{matrix}&1&\\4&1&4\\&4&\end{matrix}\right\rangle,\left\langle\begin{matrix}&2&\\4&1&4\\&4&\end{matrix}\right\rangle,
\end{array}\right\}$$

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\left\{
\begin{array}{l}
\left\langle \begin{matrix} & 3 & \\ 4 & 1 & 4 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 4 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 0 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 1 & \\ 4 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 1 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 4 & \\ 4 & 1 & 2 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 3 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 3 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 3 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 3 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 3 \\ & 4 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 2 & \\ 4 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 0 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 1 \\ & 0 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 0 & \\ 4 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 2 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 3 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 3 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 3 \\ & 0 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 3 & \\ 4 & 1 & 3 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 3 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 4 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 4 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 4 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 4 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 4 \\ & 0 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 1 & \\ 4 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 4 & \\ 4 & 1 & 1 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 2 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle, \\
\left\langle \begin{matrix} & 2 & \\ 4 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 3 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 0 & \\ 4 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 1 & \\ 4 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 2 & \\ 4 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 3 & \\ 4 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle, \left\langle \begin{matrix} & 4 & \\ 4 & 1 & 4 \\ & 1 & \end{matrix} \right\rangle,
\end{array}
\right\}
$$

$$\left\{\begin{array}{l}
\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 0 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 1 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 2 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 3 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 4 \\ & 2 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 0 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 1 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 2 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 3 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 0 & \\ 4 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 1 & \\ 4 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 2 & \\ 4 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle, \\
\left\langle\begin{matrix} & 3 & \\ 4 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,\left\langle\begin{matrix} & 4 & \\ 4 & 1 & 4 \\ & 3 & \end{matrix}\right\rangle,
\end{array}\right\}$$