DEVELOPMENT AND ANALYSIS OF NEW ITERATIVE SCHEMES FOR SOLVING NONLINEAR EQUATIONS

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A THESIS SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA, NIGERIA IN PARTIAL FULFILMENT OF THE REQUIREMENT FOR THE AWARD OF MASTER OF SCIENCE DEGREE IN MATHEMATICS.

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DECLARATION

I declare that the work in this thesis entitled 'Development and Analysis of New iterative schemes for solving nonlinear Equations' has been performed by me in the Department of Mathematics under the supervision of Dr. B. Sani and Dr. A. Mohammed. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any University.

MUHAMMAD SHAKUR NDAYAWO	……..	……
Name of Student	signature	Date

CERTIFICATION

This thesis entitled 'Development and analysis of new iterative schemes for solving nonlinear equations' by Muhammad Shakur Ndayawo, meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to my daughters Rahmat and Khadijat.

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My profound gratitude is to Almighty Allah for giving me the strength and ability to carry out this work successfully and for bringing me to this stage in my life. I would like to express my sincere and hearty gratitude to my supervisor Dr. B. Sani, for his inspired guidance and valuable discussions and who sacrificed his time and energy to go through the manuscript, I am indeed grateful for his vivid supervision and understanding.

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ABSTRACT

In this thesis, we have developed two new iterative schemes for solving nonlinear equations. The two schemes have been constructed from Taylor's series expansion and Adomian decomposition method. The two schemes have been compared with other existing iterative methods using one way analysis of variance (ANOVA). They are found to be efficient and better than some of the existing schemes. The results show that Newton-Raphson method and New scheme 1 have more advantage with a maximum of seven iterations each, while new scheme 2 has nine. Basto *et al.* and Abbasbany have equal number of thirteen iterations each. The Adomian has sixteen iterations. Thirty numerical examples are given and solved to justify the efficiency of the new iterative schemes.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

Solving nonlinear equations is an important part of numerical analysis. In recent years, interests have grown considerably in developing effective iterative methods (IM) for computing solutions for large systems in science and engineering. The development of faster and more robust IM and preconditions which can be efficiently mapped to a variety of problems is of fundamental importance in that it will be of great assistance to scientists and engineers throughout many disciplines. In numerical analysis this approach is in contrast to direct methods which attempt to solve the problem by a finite sequence of operations. Numerical analysis assumes this task, and with it the limitations of practical calculations. Numerical answers are usually tentative and, at best, known to be accurate only to within certain bounds. IM are often useful even for linear problems involving a large number of variables, where direct methods would be prohibitively expensive. Intuitively iterative methods keep on improving upon subsequent iterations. With iteration methods, the operational cost can often be reduced.

In this thesis, we study iterative methods for solving nonlinear equations, $f(x)=0,$ where $f(x)$ is any continuously differentiable real valued function. The iterative methods we try to develop for this class of equations will require knowledge of initial guess for desired roots of the equation.

Traub (1964), Numerical methods for solving nonlinear equations are divided into two categories; the interval methods and the initial point methods.. Bisection method is an example of interval methods. The initial point methods use one or more initial points as the starting values to find the approximate solution using recurrence relation. In this study, we concentrate mainly on one point initial methods. The major disadvantage of these methods however, is that their convergences are not guaranteed and the choice of initial guess requires some insight. These methods are however, usually faster than the interval methods. Secant method and Newton method are examples of initial point methods. There are several methods for solving nonlinear equations and here we introduce a few of them.

Newton or Newton-Raphson method is the most widely used method for finding roots of an equation. According to Traub (1964), it begins with an initial approximation, x_0 and generates a sequence of successive iterates $\{x_k\}_{k=0}^{\infty}$ converging quadratically to simple roots.

In Secant Method, which is a variant of Newton-Raphson's method it use finite difference to approximate the derivative of the function $y = f(x)$ close to the root by the line (secant) and requires two initial points (x_{n-1}, f_{n-1}) and (x_n, f_n), where $f_n = f(x_n)$.Taking the point of intersection of this line with the x-axis as the subsequent iterate. We get $x_{n+1} = \frac{x_n - x_{n-1}}{f_n - f_{n-1}} f_n, n = 1,2\ldots$ where x_{n-1} and x_n are two consecutive iterates. Since a secant line is defined using two points on the graph of $f(x)$, as opposed to a tangent line that requires information at only one point on the graph, we need two initial approximations x_0 and x_1. Traub (1964), the method has a super linear convergence.

The Bisection Method tries to decrease the size of the interval in which a solution exist. If the function $f(x)$ satisfies $f(a_0)f(b_0)<0$, then the equation starts with one sign at a_0 and ends with the opposite sign at b_0, and if $f(a_0)f(b_0)=0$, then either a_0 or b_0 or both are roots of $f(x) = 0$. This method consists of finding midpoint a_0 and b_0. If $m_1=\frac{1}{2}(a_0+b_0)$ is the midpoint of this interval, then the root will lie either in the interval (a_0,m_1) or in the interval (m_1,b_0) provided that $f(m_1)\neq 0$. If $f(m_1)=0$, then m_1 is the required root. Repeating this procedure, we obtain the bisection method

$$m_{n+1}=a_n+\frac{1}{2}(b_n-a_n), n=0,1,\ldots.$$

Where $$(a_{n+1})=\{(a_n,m_{n+1}) \text{ if } f(a_n)f(m_{n+1})<0 \qquad (1.1)$$

and

$$(b_{n+1})=\{(m_{n+1},b_n) \text{ if } f(m_{n+1})f(b_n)<0$$

We take the midpoint of the last interval as an approximation to the root. Traub (1964), if $f(x)$ is continuous in the interval $[a, b]$ which contains the root, the method converges.

Hafiz and Bahgat (2012), several iterative methods have been developed to solve nonlinear algebraic equations and the system of nonlinear equations. These methods have been improved using Taylor polynomials, homotopy perturbation method and Adomian decomposition methods.

The homotopy perturbation method (HPM) was developed for solving nonlinear systems, He (1999). HPM linearizes any given problem (converting it to a series of linear equations). The method gives a rapid convergence of the solution and only a few iterations lead to accurate result. In contrast to the traditional perturbation methods, this method does not require a small parameter in an equation. In this method, a homotopy with an imbedding parameter $p \in [0, 1]$ is constructed, and the imbedding parameter is considered as a "small parameter". Li (2009), when $p=0$, the system of equation usually reduces to a sufficiently simplified form, which normally admits a rather simple solution. As p increases to 1, the system under goes a sequence of deformations, the solution of each is close to that at the previous stage of deformation. When $p=1$, the system takes the original form of the equation and the final stage of deformation gives the desired solution.

Adomian (1984), developed a new method known as the Adomian decomposition method (ADM) for solving functional equations of any kind: ordinary differential equations (ODEs), algebraic, partial differential equations (PDEs), integral equations, etc. The ADM breaks any given problem into linear and nonlinear parts. The linear operator representing the linear portion of the equation is inverted and the inverse operator is then applied to both sides of the equation, before applying the initial or boundary conditions. The term that contains the independent variable alone is taken as the initial approximation. The unknown function is then decomposed into a series whose components are to be determined. The components are determined in terms of polynomials called Adomian polynomials whose successive terms are determined using a recurrent relation.

Traub (1964), classified one-point iterative methods into,

(i) One-point methods without memory

(ii) One-point methods with memory

If the value of the root is ascertained by using new data only at one point and no previous data is used, then it is called one-point iterative method without memory. An example of one-point iterative methods without memory is the Newton-Raphson method. Hence, if x_{n+1} is estimated by new data at x_n and no previous data is used, we have $x_{n+1} = \varphi(x_n)$, then φ is called one-point iteration function without memory.

If the value of the root is ascertained by using new data at one point and by using the previous data at either one or more than one points, then the iterative method is called one-point method with memory. Secant method is an example of one point iterative method with memory. If x_{n+1} is estimated by new data at x_n and the previous data at $x_{n+1},\ldots,x_{n-m}$, we have $x_{n+1} = \varphi(x_n, x_{n-1},\ldots,x_{n-m})$ and φ is called one-point iteration function with memory.

1.2 Motivation for this study

Many methods and algorithms have been developed to solve problems of nonlinear algebraic equations over the years. Despite these efforts, no single algorithm is capable of solving any and all nonlinear problems. Depending on the system and the degree of nonlinearity, one solution scheme may be preferred over another. To keep up with recent computational challenges in the field of numerical analysis, it is imperative to develop new schemes that are capable of taking advantage of the latest advances in numerical analysis. This is what

motivates us to undertake this study. Advances such as Adomian decomposition method, Basto *et al.* method.

1.3 Problem studied in the thesis

In our present work, we have tried to develop two new iterative schemes which are based on Taylor's series and Adomian method. The first scheme truncates the Taylor's series after the third term while the second scheme truncates the series after the fourth term. Moreover in both schemes, it is assumed that $\frac{f''(x)}{f'(x)} \approx 1$.

1.4 Aim and objectives

The aim of the study is to develop and analyse new iterative schemes for solving nonlinear equations.

The objectives of the study are

(i) To review iterative schemes between 1998 and 2012 which have been developed from Adomian decomposition method, Homotopy perturbation method and variants of Newton-Raphson's method for solving nonlinear equations.

(ii) To develop new schemes that could compete with previous schemes and probably have further advantages.

(iii)To compare the new schemes with the existing known iterative schemes.

1.5 Limitation of the study

This work is limited to initial point methods and specifically, only to one point iterative methods. Moreover, for those problems whose second derivatives are far away from the first derivatives, the assumption $\frac{f''(x)}{f'(x)} \approx 1$ is a limitation.

1.6 Definitions of term

(i) Let $x^* - x_n$ be the truncation error in the *nth* iterate where x^* is the required root. If there exists a number $p \geq 1$ and a constant $c \neq 0$ such that

$$\lim_{n\to\infty} \frac{\left|x_{n+1} - x^*\right|}{\left|x_n - x^*\right|^p} = c,$$

then p is called the order of convergence of the method, Burden and Faires (2011) .

Note that;

If $p = 1$, we say that (x_n) is linearly convergent.

If $p > 1$, we have super linear convergence.

If $p = 2$, we have quadratic convergence.

1.7 Theorems used in the study

We now state the following theorems without proofs, which will be used in this study. The proofs can be found in any standard analysis text book such as Burden and Faires (2011).

Theorem 1.7.1: Mean value Theorem

If $f \in C[a,b]$ (where $C[a,b]$ is the space of continuous functions on $[a,b]$) and $f(x)$ is differentiable in (a,b), then there exists a point $c \in (a,b)$ such that $f(b)-f(a)=(b-a)f'(c)$.

Theorem 1.7.2: Fixed Point Theorem

If $g \in C[a,b]$ and $g(x) \in [a,b]$ for all $x \in [a,b]$, then g has a fixed point in $[a,b]$. If in addition, $g'(x)$ exists on (a,b) and a positive constant $k<1$ exists with $|g'(x)| \le k$, for all $x \in (a,b)$, then the fixed point in $[a,b]$ is unique.

Theorem 1.7.3: Rolle's Theorem

Suppose $f \in C[a,b]$ and f is differentiable on (a,b). If $f(a)=f(b)$, then a number c in (a,b) exists with $f'(c)=0$.

Theorem 1.7.4: Extreme Value Theorem

Suppose $f \in C[a,b]$ and $c_1, c_2 \in [a,b]$ with $f(c_1) \le f(x) \le f(c_2)$, for all $x \in [a,b]$. In addition, if f is differentiable on (a,b), then the numbers c_1 and c_2 occur either at the endpoints of $[a,b]$ or where $f'=0$.

Theorem1.7.5: Taylor's Theorem

Suppose $f \in C^n[a,b]$, (where $C^n[a,b]$ is the space of n-times continuously differentiable functions on $[a,b]$) and that f^{n+1} exists on $[a,b]$, with $x_0 \in [a,b]$. Then for every $x \in [a,b]$, there exists a number $\xi(\mathrm{x})$ between x_o and x with

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^n(x_0)}{n!}(x - x_0)^n + R_n(x)$$

Where $R_n(x) = \frac{f^{n+1}\xi(x)}{(n+1)!}(x - x_0)^{n+1}$ (called the remainder term or truncation error).

1.8 Outline of the thesis

The present thesis is structured as follows:

Chapter 1, which is the present chapter constitutes the general introduction, the aims and objectives of the study and limitations of the study. It also contains some basic definitions and theorems that are important to the study being done.

Chapter 2 is a review of previous work that is related to the work under study.

Chapter 3 is a description of the Adomian method and how it is used to develop the two new iterative methods.

Chapter 4 contains analysis of numerical results to illustrate the effectiveness of the new methods.

Chapter 5 contains summary, conclusion and recommendations. It also contains discussion of potential areas for further work.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

A numerical method for determining zeroes of a functional equation, $f(x) = 0$ is generally an iterative method that will converge to zero of the function, $f(x)$. It is simply a method which produces an approximate rather than exact solution. As one would presume, each algorithm has its advantages and disadvantages and therefore selecting the right algorithm for a given problem is never easy. Various methods for solving one variable nonlinear equations are presented in the literature. Probably the easiest numerical method for solving a nonlinear equation is the Newton (Newton-Raphson) method as already stated in chapter 1. This method have local convergent and will converge to complex zeros only if the initial guess is complex. However, it can be suitably modified to compute zeros of complex polynomials and transcendental equations. When Newton's method does converge, the convergence is quadratic, ie the order of convergence is two.

The Newton-Raphson algorithm is derived from Taylor series expansion of nonlinear equation. The higher order terms (second order and higher order derivatives of the series) are neglected assuming that the initial guess for the iterative process is closer to the solution. Hence the equation for Newton- Raphson method is one of the reduced forms of Taylor series expansion. The Newton-Raphson method is an iterative process for solving other non-linear equations. In the iterative process, the first order derivative of the non-linear equation is calculated at an initial guess of the variables for the first iteration. The change in variables is then calculated by solving linear equations that contain the first order derivative and input

vector. The change in variable is used to update the variables in each iteration. The updated variables are used in successive iteration, Ortega and Rheinboldt (1970) and are given as follows:

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ which is the Newton-Raphson method, where x_{n+1} is the new iterate and x_n is the previous one.

2.2 Generalizations of Newton's method

Several researchers generalized this method and offered methods which suggest convergence with higher orders in comparison with Newton's method. The following works are variants of Newton-Raphson method.

Jisheng *et al.* (2006) presented a new modification of Newton's method for solving non-linear equations. Analysis of convergence showed that the new method is cubically convergent. Per iteration, the new method requires two evaluations of the function and one evaluation of its first derivative. Thus, the new method is preferable if the computational costs of the first derivative are equal or more than those of the function itself. Its practical utility was demonstrated by numerical examples. Jisheng *et al.* scheme is as follows:

$$y_n = x_n + \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f(y_n)}{f'(y_n)}. \qquad (2.1)$$

Hosseini (2009) presented two one-step iteration methods to solve nonlinear equations. Several examples were presented and compared to other well known methods, showing the accuracy and fast convergence of the proposed methods. Their schemes are as follows:

i. $$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f'(x_n)} \tag{2.2}$$

ii. $$x_{n+1} = x_n - \frac{f\left(2f'^2 - ff''\right)^2}{f'\left(\left(2f'^2 - ff''\right)\left(2f'^2 - 2ff''\right) + \frac{2}{3}f^2 f f''\right)}. \tag{2.3}$$

Saeed and Khthr (2010) introduced a family of new iterative methods for solving nonlinear equations based on Newton's method. The order of convergence and corresponding error equations of the obtained iteration formulae were derived analytically and with the help of Maple. Their schemes are as follows:

a. $$x_{n+1} = x_n - \frac{12f(x_n)f'^2(x_n) + f^2(x_n)[f''(x_n) - f''(y_n)]}{6f'^2(x_n)[f'(x_n) + f'(y_n)]}, \tag{2.4}$$

$$y_n = x_n - \frac{2f(x_n)f'(x_n)}{2f'^2(x_n) - f(x_n)f''(x_n)}.$$

b. $$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)},$$

$$z_n = x_n - \frac{12f(x_n)f'^2(x_n) + f^2(x_n)\left[f''(x_n) - f''\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right]}{6f'^2(x_n)\left[f'(x_n) + f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)\right]}. \tag{2.5}$$

Wang (2011) presented a third-order family of Newton-like iteration method for solving nonlinear equations. Method that is free from computation of second-order derivatives and turned out to require, per iteration, one evaluation of the function and two evaluations involving its first derivative. Analysis of this method demonstrated that its iterative solution is cubically convergent. Numerical examples were given to illustrate the efficiency of the method. The scheme is as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{\frac{1}{2}f'(x_n) + \frac{1}{2}f'\left(x_n - \frac{f(x_n)}{f'(x_n)}\right)}. \tag{2.6}$$

Zhongyong *et al.* (2011) constructed a fifth-order iterative method for solving nonlinear equations, which is a variant of Newton-Raphson method. Zhongyong *et al.* method is as follows:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

where x_n is the previous iterate and x_{n+1} is the new iterate.

$$x_{n+1} = y_n - \left[1 + \left(\frac{f(y_n)}{f(x_n)}\right)^2\right]\frac{f(y_n)}{f'(y_n)}. \tag{2.7}$$

Thukral (2012) presented a fifth-order method for finding multiple zeros of nonlinear equations. The new method requires two evaluations of functions and first derivatives. It was proved that the method has a convergence of order five. Finally, some numerical examples

were given to show the performance of the presented method, and how it compared with some known methods. Thukral scheme is as follows;

$$y_n = x_n - m\frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - m\left(1 + z_n^{\frac{2}{m}}\right)\left(\frac{f(y_n)}{f'(y_n)}\right), \tag{2.8}$$

where $z_n = f(y_n)f(x_n)^{-1}$.

2.3 The Adomian decomposition method

As stated in chapter 1, Adomian (1986), applies decomposition method to operator equation of the form $fu = g$, where f may be a nonlinear (and/or) stochastic operator and x a stochastic process on an appropriate probability space. Adomian (1986), considered basic equation in the form $\mathcal{L}u + \mathcal{N}u = g$ or $Lu + Nu = g$ if both operators are deterministic where L is a linear (deterministic) operator and N a nonlinear (deterministic) operator. Thus an ordinary quadratic equation of the form $ax^2 + bx + c = 0$ can be written as

$$Lu + Nu = g,$$

where $Nx = ax^2$, $L = b$ and $g = -c$.

So that $Lu = g - Nu$

$$\text{or} \quad bx = -c - ax^2.$$

The operation L^{-1} for the differential equation is an integral operator. In this case it is simply a divisor of b.

Hence,

$$x = \left(-\frac{c}{b}\right) - \left(\frac{a}{b}\right)x^2.$$

Which is the Adomian format, Adomian (1986). The solution x is now decomposed into components $x_0 + x_1 + \cdots$, where x_0 is taken as $-\frac{c}{b}$ and $x_1, x_2, \cdots$ are still to be determined. Thus

$$x_0 = -\frac{c}{b}.$$

So that

$$x = x_0 - \left(\frac{a}{b}\right)x^2,$$

with x_0 now known.

Adomian, (1986), the nonlinear term without the coefficient in this case is x^2 and is replaced by $\sum_{n=0}^{\infty} A_n(x_0, x_1, \cdots, x_n)$, where $A_n(x_0, x_1, \cdots, x_n)$ are known as the Adomian polynomials.

2.4 Studies based on Adomian decomposition method

The following works are based on Adomian decomposition method;

Wazwaz (1998) compared the performance of Adomian decomposition method and Taylor's series method applied to the solution of linear and nonlinear ordinary differential equations. Wazwaz showed that even though the two methods provided the solution in an infinite series, the decomposition method provided a fast convergent series of easily computable components and eliminates the cumbersome computational work needed by Taylor series method.

Wazwaz (1999) introduced a modification of the Adomian decomposition method. The method accelerates the convergence of the series solution when compared with the standard Adomian decomposition method. The modified method was shown to be computationally efficient when used to solve examples of both differential and integral equations.

Wazwaz (2000) also presented a new method for calculating Adomian polynomials for nonlinear operators. The method proved to be efficient in calculating Adomian polynomials for all forms of nonlinearity. In case of nonlinear models, the method gives reliable results in obtaining analytical approximation that converges very rapidly. The new method depends mainly on algebraic and trigonometric identities and on Taylor expansion.

Abbasbandy (2003) presented a numerical algorithm for solving nonlinear equations based on Newton–Raphson method. The modified Adomian decomposition method was also applied in constructing the numerical algorithm. The scheme is as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x)}{2[f'(x_n)]^3} - \frac{[f(x_n)]^3 [f''(x_n)]^2}{2[f'(x_n)]^5}. \qquad (2.9)$$

Biazar *et al*. (2004) used Adomian decomposition method to solve systems of differential equations of the first order and ordinary differential equation of any order by converting it into a system of differential equations of order one.

Kaya and El-Sayed (2004) presented an algorithm based on Adomian decomposition method and the algorithm is to approximate the solution of systems of nonlinear algebraic equations. Special cases of systems of nonlinear equations were solved using the algorithm and the convergence of this algorithm was shown to be rapid.

Abbasbandy (2005) also presented some numerical algorithms for solving a system of two nonlinear equations (with two variables) based on Newton's method. The modified Adomian decomposition method was applied also in constructing the numerical algorithms.

Chun (2005) presented a sequence of iterative methods for solving nonlinear equations. The Adomian decomposition method was applied to an equivalent coupled system to construct the sequence. It was shown that the order of convergence increased as the sequence progressed. Chun's scheme is as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 3\frac{f(x_{n+1})}{f'(x_n)} + \frac{f(x_{n+1})f'(x_{n+1})}{[f'(x_n)]^2} - \frac{f(x_{n+1})\left(f(x_{n+1})f''(x_{n+1}) + 2[f'(x_{n+1})]^2\right)}{2[f'(x_n)]^3}. \quad (2.10)$$

Basto *et al.* (2006) constructed a new iterative method to solve nonlinear equations. The new method is based on the proposals of Abbasbandy on improving the order of accuracy of Newton–Raphson method. The convergence of the new scheme was proved to be at least of cubic order. Basto *et al.* iterative scheme is as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2 f''(x)}{2[f'(x_n)]^3 - 2f(x_n)f'(x_n)f''(x_n)}. \quad (2.11)$$

Hosseini and Nasabzadeh (2006) developed a method to determine the rate of convergence of Adomian decomposition method. The proposed method was used to solve singular and nonsingular ordinary differential equations.

Darvishi and Barati (2006) derived a cubic method from the Adomian decomposition method to solve systems of nonlinear equations. The authors showed that the method is third-order convergent using classical Taylor's expansion but the numerical experiments conducted by them showed that the method exhibits super cubic convergence.

Chi and Feng (2008) presented an iterative method for solving nonlinear equations, which is also based on Adomian decomposition method. The property of convergence was proved and some numerical illustrations were also given to show the efficiency of the algorithm, which is as follows:

$$x_{n+1} = x_n - \frac{f(x_n)}{f[x_{n-1}, x_n]} - \frac{f^2(x_n)f[x_{n-2}, x_{n-1}, x_n]}{f[x_{n-1}, x_n]^3}. \quad (2.12)$$

Feng (2009) presented a new two-step iterative method for solving nonlinear equations. The new scheme was developed from Taylor' series expansion and Adomian decomposition method. The new iterative scheme has quadratic convergence. The iterative scheme is as follows:

$$x_{n+1} = y_n - \frac{f(y_n)}{2f'(x_n) - f'(y_n)}, \tag{2.13}$$

where $$y_n = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Karthikeyan (2010) suggested new algorithm for minimization of nonlinear functions which is based on ADM. Karthikeyan derived and stated a technique called external touch technique which was given as $x_{n+1} = x_n - \frac{f'(x_n)}{\sqrt{1 + f''^2(x_n) - f'(x_n) f'''(x_n)}}$ .The new algorithm was compared with the Newton's Algorithm and External Touch technique algorithm after which the new algorithm was found to be efficient. Karthikeyan's iterative scheme is as follows;

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = y_n - \frac{f'(y_n)}{2f''(x_n) - f''(y_n)}. \tag{2.14}$$

Oda *et al.* (2010) modified the method of computing Adomian's polynomial to find the numerical solution for non-linear systems of partial differential equations with less number of components, more accuracy and faster convergence when compared with the Adomian decomposition method.

2.5 Studies based on Homotopy perturbation method

Awawdeh (2010) presented iterative methods for solving systems of nonlinear algebraic equations which are based on Homotopy Analysis Method (HAM). The new methods are

second and third order iterative methods which were shown to be more efficient than some of the classical methods such as Newton's, method.

Noor (2010) used the modified homotopy perturbation method to develop a number of iterative methods for solving nonlinear equations. By modifying the given nonlinear equation as an equivalent coupled system of equations using the Taylor's series and method of He (1999), the method was used to express any given nonlinear equation as a sum of linear and nonlinear equations. Noor also showed the equivalence between the Adomian decomposition method and the homotopy perturbation method. Several examples were presented and used to compare the method with other methods, showing the accuracy and fast convergence of the presented method.

Recently Sehati *et al.* (2012) presented several new iterative methods for solving nonlinear algebraic equations incorporating homotopy perturbation method. The iterative formulas were based on the He's homotopy perturbation method. It was shown that the new methods led to four algorithms which are of fifth, seventh, tenth and fourteenth order of convergence. These methods result in real or complex simple roots of certain nonlinear equations. The merit of the new algorithms is that, in case a nonlinear equation has complex roots, it can give complex solutions even when the initial approximation is chosen real. Several examples were presented and compared to other methods, showing the accuracy and fast convergence of the presented methods which are as follows:

i. $$x_{n+1} = y_n - \frac{f(y_n)}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}},$$

where $y_n = x_n - \dfrac{f'(x_n) + \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}{f''(x_n)}$.

ii. $$x_{n+1} = y_n + z_n + \frac{z_n\left\{f'(x_n) - f'(y_n) + \left(y_n - x_n - \frac{1}{2}z_n\right)f'^2(x_n)f''(x_n)\right\}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}, \qquad (2.15)$$

where $z_n = -\dfrac{f(y_n)}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}$.

iii. $$x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}. \qquad (2.16)$$

iv. $$x_{n+1} = w_n - \frac{f(w_n)}{f'(w_n)},$$

where $w_n = y_n + z_n + \dfrac{z_n\left\{f'(x_n) - f'(y_n) + \left(y_n - x_n - \frac{1}{2}z_n\right)f'^2(x_n)f''(x_n)\right\}}{\sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}$.

CHAPTER THREE

CONSTRUCTION OF THE NEW SCHEMES

3.1 Introduction

In this section we construct two new methods for solving nonlinear algebraic and transcendental equations, $f(x)=0$. As mentioned earlier, there is no general methods for finding solutions of nonlinear equations. Researchers are continuously trying to develop easy methods to accurately and efficiently solve problems of nonlinear equations. The new schemes we have developed give results faster than the Adomian's decomposition method and equally or faster than other methods derived from Adomian's and other methods. The convergences of the new schemes are proved to be of cubic order. Several examples are presented and the new schemes are used in solving the examples. The examples are also solved with other existing methods, namely Abbasbany's, Adomian decomposition method, Basto *et al.* and Newton-Raphson method, which showed the accuracy and fast convergence of these new schemes. With the assumption that $\frac{f''}{f'} \approx 1$, the schemes obviously are free of second derivative and this reduces the computational cost. This assumption is in particular true for the function e^x and some functions such as x^3+4x^2+8x+8 at $x=-\frac{2}{3}$, x^3-6x-4 at $x=-0.732$ and $3x-\ln x-16$ at $x=-0.434$, see appendix. Of course there are other situations where the assumption is far away from being true.

The schemes start with an initial guess and then generate a sequence of approximations which improve the solution of a problem at each step.

3.2 The present work

Suppose we consider the nonlinear equation $f(x)=0$, such that α is a root of the equation and f is a continuous function on an interval containing α. For the new scheme 1, we start from Taylor's series around x and truncate the series after the third term, while for the new scheme 2, we truncate after the fourth term of the series. Assuming that the initial guess for the iterative process is closer to the solution, then we apply ADM. The ADM involves breaking the given equation into linear and nonlinear parts. The linear operator representing the linear portion of the equation is inverted and the inverse operator is then applied to the nonlinear part. The nonlinear part is decomposed into a series of Adomian polynomials. This method gives a solution in the form of a series whose terms are determined by a recursive relationship using these Adomian polynomials. To explain clearly the Adomian approach, consider the equation $Fy=f$

where F is a nonlinear differential operator involving both y and f are functions of t. Rewriting the equation we get

$$Ly+Ry+Ny=f\,, \tag{3.1}$$

Which is in the operator form.

where L is an operator representing the linear portion of F which is invertible, Adomian (1986). R is the remainder of the linear portion, and N is a nonlinear operator representing the nonlinear terms in F. Applying the operator L^{-1}, to equation (3.1), we get

$$L^{-1}Ly = L^{-1}f - L^{-1}Ry - L^{-1}Ny\,.$$

Taken F to be the differential operator and L is linear, L^{-1} would represent integration and with any given conditions, $L^{-1}Ly$ will give an equation for y incorporating these conditions, Adomian (1986). We get

$$y(t) = g(t) - L^{-1}Ry - L^{-1}Ny ,$$

where $g(t)$ represents the function generated by integrating f and using the given conditions. Assume that the unknown function can be written as an infinite series

$$y(t) = \sum_{n=0}^{\infty} y_n(t) .$$

If $y_0 = g(t)$ and the unknown functions are determined by a recursive relationship defined below. Which is found by decomposing the nonlinear term into a series of polynomials, A_n. The nonlinear term is written as

$$Ny = \sum_{n=0}^{\infty} A_n ,$$

where A_n are functions called Adomian polynomials depending on $y_0, y_1, \cdots, y_n$, Adomian (1986).

To determine the Adomian polynomials, a grouping parameter, λ is introduced as follows:

$$y(t) = \sum_{n=0}^{\infty} \lambda^n y_n$$

and

$$Ny(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n .$$

Then the $A_n's$ can be generated by using the expression

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty} \lambda^i y_i \right) \right]_{\lambda=0}, \text{ where } n = 0,1,2,\cdots . \qquad (3.2)$$

Adomian (1986), noted that λ is not a "smallness parameter".

Adomian (1988), illustrated the first few polynomials for all kind of nonlinearity as,

$$\begin{aligned}
A_0 &= f(y_0), \\
A_1 &= y_1 f'(y_0), \\
A_2 &= y_2 f'(y_0) + \frac{1}{2} y_1^2 f''(y_0), \\
A_3 &= y_3 f'(y_0) + y_1 y_2 + \frac{1}{3!} y_1^3 f'''(y_0), \\
&\vdots
\end{aligned}$$

3.3 Construction of the new schemes

3.3.1 New iterative scheme 1

Consider the Taylor's series expansion,

$$f(x+h)=f(x)+hf'(x_0)+h^2\frac{f''(x_0)}{2!}+\cdots$$

we obtain

$$f(x-h)=f(x)-hf'(x)+h^2\frac{f''(x)}{2!}-h^3\frac{f'''(x)}{3!}+\cdots,$$

Truncating the higher order values of the series from the third order and equating to zero, we obtain

$$f(x)-hf'(x)+\frac{h^2}{2}f''(x)=0 \tag{3.3}$$

Looking for a small h, such that

$$hf'(x)=f(x)+\frac{h^2}{2}f''(x) \tag{3.4}$$

$$h=\frac{f(x)}{f'(x)}+\frac{h^2f''(x)}{2f'(x)} \tag{3.5}$$

We now use Adomian decomposition method to get

$$h = c + N(h), \tag{3.6}$$

where $c = \dfrac{f(x)}{f'(x)}$ and $N(h) = \dfrac{h^2 f''(x)}{2f'(x)} = \dfrac{h^2}{2}$ (on the assumption that $\dfrac{f''(x)}{f'(x)} \approx 1$).

Let $H_m = h_0 + h_1 + \cdots + h_m = h_0 + A_0 + A_1 + \cdots + A_{m-1}$ denote the $(m+1)$terms approximation of h.

For $m = 0$,

$$H_0 = h_0 = c = \frac{f(x)}{f'(x)}.$$

If α is a root then

$$\alpha = x - h \approx x - H_0 = x - \frac{f(x)}{f'(x)}$$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$$

For $m = 1$,

$$h \approx H_1 = h_0 + h_1 = h_0 + A_0$$

$$h_0 = c = \frac{f(x)}{f'(x)}$$

$$A_0 = N(h_0) = \frac{h_0^2}{2} = \frac{f^2(x)}{2f'^2(x)}$$

$$\alpha = x - h_1 \approx x - H_1 = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)}{2f'^2(x)}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)}{2f'^2(x_n)}$$

For $m = 2$,

$$h \approx H_2 = h_0 + h_1 + h_2 = h_0 + A_0 + A_1$$

$$h_0 = c = \frac{f(x)}{f'(x)}\ ,\ h_1 = \frac{f^2(x_n)}{2f'^2(x_n)}$$

$$N'(h) = \frac{d}{dh}N(h) = h$$

$$A_1 = N'(h_0)h_1 = \frac{f(x)}{f'(x)} \times \frac{f^2(x)}{2f'^2(x)} = \frac{f^3(x)}{2f'^3(x)}$$

$$\alpha = x - h_2 \approx x - H_2 = x - \frac{f(x)}{f'(x)} - \frac{f^2(x)}{2f'^2(x)} - \frac{f^3(x)}{2f'^3(x)}$$

The new iterative scheme 1 is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2}{2[f'(x_n)]^2} - \frac{[f(x_n)]^3}{2[f'(x_n)]^3} \qquad (3.7)$$

3.3.1.1 Convergence analysis for new the scheme 1

Consider the iteration function g as

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} - \frac{[f(x)]^3}{2[f'(x)]^3}$$

The following theorem for convergence of the scheme holds.

3.3.1.2 Theorem

Let x^* be solution of the equation $f(x)=0,\ f \in C^2$. If $f'(x^*) \neq 0$ then there exists an interval I containing x^* such that for $x_0 \in I$, the iterative scheme (3.7) converges to the only solution of $f(x)=0$ belonging to I.

Proof

The statements $f(x^*)=0,\ f'(x^*) \neq 0 \Rightarrow \left[f'(x)^*\right]^2 \neq 0, \left[f'(x)^*\right]^3 \neq 0$ coupled with the fact that $f \in C^2$ implies that $g(x^*) = x^*$ and g is continuous and differentiable at $x = x^*$.

Thus $|g'(x^*)| = 0 < 1$. There exists an $\varepsilon > 0$ such that for $x \in (x^* - \varepsilon, x^* + \varepsilon)$, $|g'(x)| < 1$, and by the fixed point theorem, Burden and Faires (2011) the iterative scheme $x_{n+1} = g(x)$ converges to the unique solution in that interval.

3.3.1.3 Theorem

Consider the nonlinear equation $f(x)=0$. Suppose $f \in C^4$. Then for the iterative method (3.7), the convergence is at least of order 3.

Proof

Consider the iterative method (3.7)

$$g(x)=x-\frac{f(x)}{f'(x)}-\frac{[f(x)]^2}{2[f'(x)]^2}-\frac{[f(x)]^3}{2[f'(x)]^3}.$$

Using maple17 software, to find g', g'' and g''', we get

$$g'(x)=\frac{\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}-\frac{\{f(x)\}}{\left\{\frac{d}{dx}f(x)\right\}}+\frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}-\frac{3}{2}\frac{\{f(x)\}^2}{\left\{\frac{d}{dx}f(x)\right\}^2}+\frac{3}{2}\frac{\{f(x)\}^3\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}$$

$$\Rightarrow\ g'(x)=0$$

$$g''(x) = \frac{\left\{\frac{d^2}{dx^2} f(x)\right\}}{\left\{\frac{d}{dx} f(x)\right\}} - \frac{2\{f(x)\}\left\{\frac{d^2}{dx^2} f(x)\right\}^2}{\left\{\frac{d}{dx} f(x)\right\}^3} + \frac{\{f(x)\}\left\{\frac{d^3}{dx^3} f(x)\right\}}{\left\{\frac{d}{dx} f(x)\right\}^2} - 1$$
$$+ \frac{3\{f(x)\}\left\{\frac{d^2}{dx^2} f(x)\right\}}{\left\{\frac{d}{dx} f(x)\right\}^2} - \frac{3\{f(x)\}^2\left\{\frac{d^2}{dx^2} f(x)\right\}^2}{\left\{\frac{d}{dx} f(x)\right\}^4} + \frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3} f(x)\right\}}{\left\{\frac{d}{dx} f(x)\right\}^3}$$
$$- \frac{3\{f(x)\}}{\left\{\frac{d}{dx} f(x)\right\}} + \frac{15}{2}\frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2} f(x)\right\}}{\left\{\frac{d}{dx} f(x)\right\}^3} - \frac{6\{f(x)\}^3\left\{\frac{d^2}{dx^2} f(x)\right\}^2}{\left\{\frac{d}{dx} f(x)\right\}^5}$$
$$+ \frac{3}{2}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3} f(x)\right\}}{\left\{\frac{d}{dx} f(x)\right\}^4}$$

$$\Rightarrow\ g''(x) = \frac{f''(x)}{f'(x)} - 1 \approx 0,$$

Since $\frac{f''(x)}{f'(x)} \approx 1$

$$\therefore g''(x) = 0$$

$$g'''(x) =$$

$$-3 - \frac{12\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{81}{2}\frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{6\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^4}$$

$$- \frac{6\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} + \frac{12\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^5}$$

$$- \frac{9\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{30\{f(x)\}^3\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^6}$$

$$- \frac{18\{f(x)\}^3\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5} + \frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$

$$+ \frac{3}{2}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} - \frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^2} + \frac{\{f(x)\}\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}$$

$$+ \frac{2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}} + \frac{5\{f(x)\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} + \frac{12\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$

$$+ \frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}} + \frac{18\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}$$

$$\Rightarrow\ g'''(x) = -3 - 3\left\{\frac{f''(x)}{f'(x)}\right\}^2 + 2\left\{\frac{f'''(x)}{f'(x)}\right\} + 3\left\{\frac{f''(x)}{f'(x)}\right\}$$

$$= 2\left\{\frac{f'''(x)}{f'(x)}\right\} - 3$$

$$\therefore g'''(x) = 2\left\{\frac{f'''(x)}{f'(x)}\right\} - 3$$

$$g'(x^*) = g''(x^*) = 0 \tag{3.8}$$

and

$$g'''(x^*) = 2\left\{\frac{f'''(x)}{f'(x)}\right\} - 3 \tag{3.9}$$

From Taylor limited expansion of $g(x_n)$ around x^*, we get, for min $(x_n, x^*) < \xi_n < \max(x_n, x^*)$

$$x_{n+1} - x^* = g(x_n) - g(x^*)$$

$$= g'(x^*)(x_n - x^*) + \frac{g''(x^*)}{2}(x_n - x^*)^2 + \frac{g'''(\xi_n)}{6}(x_n - x^*)^3 \tag{3.10}$$

From equation (3.8) and (3.10) for $x_n \neq x^*$, we get

$$\frac{(x_{n+1} - x^*)}{(x_n - x^*)^3} = \frac{g'''(\xi_n)}{6}$$

The statement $f \in C^4$ implies that $g \in C^2$ in the neighborhood of interest of $x = x^*$. Hence for $g'''(x^*) \neq 0$

$$\lim_{n\to\infty}\frac{(x_{n+1}-x^*)}{(x_n-x^*)^3}=\frac{g'''\left(\lim_{n\to\infty}\xi_n\right)}{6}=\frac{g'''\left(x^*\right)}{6}\neq 0.$$

$\Rightarrow$ The new scheme 1 is of order three, See Ndayawo and Sani (2013).

3.3.2 New iterative scheme 2

Consider the Taylor's expansion series for higher order, we obtain

$$f(x-h)=f(x)-hf'(x)+\frac{h^2}{2!}f''(x)-\frac{h^3}{3!}f'''(x)+\cdots.$$

Truncating for higher order values of the series from the fourth order and equating to zero, we obtain

$$f(x)-hf'(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x)=0. \tag{3.11}$$

Looking for a small h ,

$$hf'(x)=f(x)+\frac{h^2}{2}f''(x)-\frac{h^3}{6}f'''(x), \tag{3.12}$$

$$\Rightarrow h=\frac{f(x)}{f'(x)}+\frac{h^2f''(x)}{2f'(x)}-\frac{h^3f'''(x)}{6f'(x)}\ . \tag{3.13}$$

Applying Adomian decomposition method, we get

$$h = c + N(h),$$

where $c = \dfrac{f(x)}{f'(x)}$ and $N(h) = \dfrac{h^2}{2} - \dfrac{h^3 f'''(x)}{6f'(x)}$, on the assumption that $\dfrac{f''(x)}{f'(x)} \approx 1$

For $m = 0$,

$$H_0 = h_0 = c = \frac{f(x)}{f'(x)}.$$

If α is the root of the equation then

$$\alpha = x - h \approx x - H_0 = x - \frac{f(x)}{f'(x)}$$

and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x)}.$$

For $m = 1$,

$$h \approx H_1 = h_0 + h_1 = h_0 + A_o$$

$$A_0 = N(h_0) = \frac{h^2}{2} - \frac{h^3 f'''(x)}{6f'(x)} = \frac{f'^2(x)}{2f'^2(x)} - \frac{f^3(x) f'''(x)}{6f'^4(x)}$$

$$h \approx H_1 = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} + \frac{[f(x)]^3 f'''(x)}{6[f'(x)]^4}.$$

The new iterative scheme 2 is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{[f(x_n)]^2}{2[f'(x_n)]^2} + \frac{[f(x_n)]^3 f'''(x_n)}{6[f'(x_n)]^4} \tag{3.14}$$

See Ndayawo and Sani (2013).

3.3.2.1 Convergence analysis for new scheme 2

Consider the iteration function g as

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} + \frac{[f(x)]^3 f'''(x_n)}{6[f'(x)]^4}.$$

The following theorem for convergence of the scheme holds.

3.3.2.2 Theorem

Let x^* be solution of the equation $f(x) = 0,\ \ f \in C^4$. If $f'(x^*) \neq 0$ then there exists an interval I containing x^* such that for $x_0 \in I$, the iterative scheme (3.14) converges to the only solution of $f(x) = 0$ belonging to I.

Proof

The statements $f(x^*) = 0,\ \ f'(x^*) \neq 0 \Rightarrow \ [f'(x)^*]^2 \neq 0, [f'(x)^*]^4 \neq 0$ coupled with the fact that $f \in C^4$ implies that $g(x^*) = x^*$ and g continuous differentiable at $x = x^*$.

Thus $|g'(x^*)| = 0 < 1$. There exists an $\varepsilon > 0$ such that for $x \in (x^* - \varepsilon, x^* + \varepsilon)$, $|g'(x)| < 1$, and by the fixed point theorem, Burden and Faires (2011) the iterative scheme $x_{n+1} = g(x)$ converges to the unique solution in that interval.

The following theorem on the order of convergence holds.

3.3.2.3 Theorem

Consider the nonlinear equation $f(x) = 0$. Suppose $f \in C^6$. Then for the iterative method (3.14), the convergence is at least of order 3.

Proof

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{[f(x)]^2}{2[f'(x)]^2} + \frac{[f(x)]^3 f'''(x_n)}{6[f'(x)]^4}.$$

Using maple17 software, to find g', g'' and g''', we get

$$g'(x) = \frac{\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} - \frac{\{f(x)\}}{\left\{\frac{d}{dx}f(x)\right\}} + \frac{\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} + \frac{1}{2}\frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$
$$+ \frac{1}{6}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} - \frac{2}{3}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5}$$

$$\Rightarrow g'(x) = 0$$

$$g''(x)=\frac{\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}}-\frac{2\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^3}+\frac{2\{f(x)\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}-1$$
$$+\frac{3\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2}-\frac{3\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^4}+\frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}$$
$$+\frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3}-\frac{7}{2}\frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}$$
$$+\frac{1}{6}\frac{\{f(x)\}^3\left\{\frac{d^5}{dx^5}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4}-\frac{4}{3}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5}$$
$$+\frac{10}{3}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^6}-\frac{2}{3}\frac{\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^5}$$

$$\Rightarrow\ g''(x)=\frac{f''(x)}{f'(x)}-1\approx 0,$$

Since $\dfrac{f''(x)}{f'(x)}\approx 1$

$$\therefore g''(x)=0$$

$$
\begin{aligned}
g'''(x) = & \\
& -\frac{12\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{9\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{3\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}} \\
& + \frac{3}{2}\frac{\{f(x)\}^2\left\{\frac{d^5}{dx^5}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} - \frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^2} + \frac{4\{f(x)\}\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} \\
& + \frac{1}{6}\frac{\{f(x)\}^3\left\{\frac{d^6}{dx^6}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} + \frac{6\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^4} \\
& - \frac{15\{f(x)\}\left\{\frac{d^2}{dx^2}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} + \frac{12\{f(x)\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^5} \\
& - \frac{21}{2}\frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^4} - \frac{11}{2}\frac{\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^4} \\
& - \frac{2\{f(x)\}^3\left\{\frac{d^5}{dx^5}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5} - \frac{8}{3}\frac{\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^5} \\
& + \frac{\{f(x)\}^2\left\{\frac{d^4}{dx^4}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^3} + \frac{5\{f(x)\}\left\{\frac{d^3}{dx^3}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^2} + \frac{3\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}} \\
& + \frac{24\{f(x)\}^2\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^5} + \frac{10\{f(x)\}^3\left\{\frac{d^4}{dx^4}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^2}{\left\{\frac{d}{dx}f(x)\right\}^6} \\
& - \frac{20\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}\left\{\frac{d^2}{dx^2}f(x)\right\}^3}{\left\{\frac{d}{dx}f(x)\right\}^7} + \frac{10\{f(x)\}^3\left\{\frac{d^3}{dx^3}f(x)\right\}^2\left\{\frac{d^2}{dx^2}f(x)\right\}}{\left\{\frac{d}{dx}f(x)\right\}^6}
\end{aligned}
$$

$$
\Rightarrow g'''(x) = 3\frac{f'''(x)}{f'(x)} \neq 0
$$

$$g'(x^*) = g''(x^*) = 0 \tag{3.15}$$

and

$$g'''(x^*) = \frac{f'''(x^*)}{f'(x^*)} \tag{3.16}$$

From Taylor limited expansion of $g(x_n)$ around x^*, we get, for $\min(x_n, x^*) < \xi_n < \max(x_n, x^*)$

$$= g'(x^*)(x_n - x^*) + \frac{g''(x^*)}{2}(x_n - x^*)^2 + \frac{g'''(\xi_n)}{6}(x_n - x^*)^3 \tag{3.17}$$

From equation (3.15) and (3.17) and for $x_n \neq x^*$, we get

$$\frac{x_{n+1} - x^*}{(x_n - x^*)^3} = \frac{g'''(\xi_n)}{6}.$$

The statement $f \in C^6$ implies that $g \in C^4$ in the neighborhood of interest of $x = x^*$. Hence for $g'''(x^*) \neq 0$

$$\lim_{n\to\infty} \frac{x_{n+1} - x^*}{(x_n - x^*)^3} = \frac{g'''(\lim_{n\to\infty} \xi_n)}{6} = \frac{g'''(x^*)}{6} \neq 0.$$

$\Rightarrow$ The new scheme 2 is of order three, See Ndayawo and Sani (2013).

CHAPTER FOUR

ANALYSIS OF RESULTS

4.1Introduction

We present some numerical examples to illustrate the efficiency and the accuracy of the new developed iterative methods for solving problems of nonlinear algebraic equations. To demonstrate the performance of the new methods, we solved thirty examples of different nature. As mentioned earlier, the methods used for comparison with the new developed iterative methods are Newton-Raphson method, Abbasbanby, Basto *et al.* and Adomian method. The iterative methods for this class of equations will require knowledge of initial guess for desired roots of equations. Adomian method was used to find the initial point x_0. The comparison was carried out in terms of the number of iterations obtained from the different methods used, using one way analysis of variance (ANOVA). In each case, the comparison was done only for those methods which converge for the particular numerical example. Note also that methods such as the Karthikeyan (2010) method did not converge for most of the 30 problems and so it was not used in the comparison. Similarly, other methods were not used in the comparison for one reason or another. In all cases, tolerance level for the error was taken as $e = 10^{-7}$ for a method to converge.

The results for the comparison are given on table 4.1, many of which were picked from several references.

4.2 Thirty Examples of Different Nature

1. $e^{2x} + 2x + 0.1 = 0$ Rao (2006)
2. $2x^3 - x^2 - 7x + 6 = 0$ Rao (2006)
3. $x^3 - 9x + 1 = 0$ Gupta *et al.* (2009)
4. $x^3 - 3x - 1.06 = 0$ Rao (2006)
5. $x^3 - 6x + 4 = 0$ Rao (2006)
6. $2x - 3\sin x - 5 = 0$ Rao (2006)
7. $x^3 - 3x + 1 = 0$ Rao (2006)
8. $3x - \ln x - 16 = 0$ Rao (2006)
9. $\cos x - 2x + 3 = 0$ Rao (2006)
10. $x + \ln x - 2 = 0$ Rao (2006)
11. $x^3 - x - 0.1 = 0$ Rao (2006)
12. $x^4 - 12x + 7 = 0$ Rao (2006)
13. $x + \sin x + 0.5 = 0$ Gupta *et al.* (2009)
14. $x - 2 - e^{-x} = 0$ Basto *et al.* (2006)
15. $x^3 + 4x^2 + 8x + 8 = 0$ Basto *et al.* (2006)
16. $3x - \cos x - 1 = 0$ Gupta *et al.* (2009)
17. $2x - \ln x - 7 = 0$ Rao (2006)
18. $x^2 - 1.25x + 0.25 = 0$ Adomian and Rach (1985)
19. $x^2 + 5.15x + 2.37 = 0$ Adomian and Rach (1985)
20. $x^4 - 11x + 8 = 0$ Gupta *et al.* (2009)
21. $x^3 - 4x + 2 = 0$ Rao (2006)
22. $\ln x - x + 3 = 0$ Martin *et al.* (2000)
23. $e^x - x - 3 = 0$ Martin *et al.* (2000)
24. $2x^2 - 12x + 11 = 0$ Martin *et al.* (2000)
25. $x^3 - 4x - 1 = 0$ Adomian (1984)
26. $x^4 - 5x^2 + x + 3 = 0$
27. $x^3 + 4x - 9 = 0$ Gupta *et al.* (2009)
28. $x^6 - 8x^4 + x^3 + 16x^2 - 4x - 2 = 0$
29. $x^7 - 3x^6 + 3x^5 - 13x^4 + 43x^3 - 57x^2 + 33x - 7 = 0$
30. $x^5 - 10x^4 - 6x^3 + 68x^2 - 83x + 30 = 0$

Table 4.1Comparison between Number of Iterations for Thirty Different Examples

Problems	Initial starting point, x_0	Methods used/ Number of Iterations						
		NR	BS	ABB	NS1	NS2	AD	
1	-0.05	3	2	5	2	1	NC	
2	0.86	4	2	NC	3	4	4	
3	0.11	1	2	2	1	1	6	
4	-0.35	2	1	2	2	1	8	
5	0.67	3	2	5	3	2	3	
6	2.5	3	2	NC	3	4	7	
7	0.33	3	2	3	2	2	3	
8	5.3	2	2	3	4	4	1	
9	1.5	2	2	2	2	1	1	
10	2	3	2	1	4	4	NC	
11	0.58	2	4	7	2	3	4	
12	-0.1	2	1	1	1	1	2	
13	-0.5	3	2	3	3	3	5	
14	2	3	2	2	3	3	6	
15	-1	1	4	4	7	6	6	
16	0.33	3	7	3	3	3	3	
17	3.5	3	2	2	3	3	2	
18	0.2	3	2	2	2	3	4	
19	-0.2	2	1	5	4	3	2	
20	0.73	3	3	4	3	3	2	
21	0.5	2	2	2	3	3	6	
22	3	3	3	NC	3	7	3	
23	-3	2	1	2	3	2	4	
24	0.92	3	2	3	3	9	3	
25	-0.25	1	1	2	2	4	3	
26	-3	4	2	3	3	5	NC	
27	2.25	3	2	2	2	2	NC	
28	-0.5	6	5	5	5	5	NC	
29	0.21	7	7	4	6	6	16	
30	0.4	5	13	13	5	7	NC	

Note that Adomian method was used to find the initial starting point (x_0).

Key: NR=Newton-Raphson, BS=Basto *et al.*, ABB=Abbasbandy, NS1=New scheme 1, NS2= New scheme 2, AD=Adomian. NC=Not converging.

4.3 Summary of Results Obtained for Some Solved Examples

We now present the number of iterations and solutions obtained for some examples from individual methods used. Note that the margin of error allowed is $e = 10^{-7}$ and NC means not converging.

Problem 1

$e^{2x} + 2x + 0.1 = 0$

$x_0 = -0.05$

Required solution = -0.31585

Method	No of iterations	Solutions obtained
Newton Raphson	3	-0.31585
BASTO *et al.*	2	-0.31585
ABBASBANDY	5	-0.31585
NEW SCHEME 1	2	-0.31585
NEW SCHEME 2	2	-0.31585
ADOMIAN	-	NC

The new schemes 1 and 2 perform better than Abbasbandy and Newton-Raphson's method.

Problem 6

$2x - 3\sin x - 5 = 0$

$x_0 = 2.5$

Required solution = 2.883237

Method	No of iterations	Solutions obtained
Newton Raphson	3	2.883237
BASTO *et al.*	2	2.883237
ABBASBANDY	NC	NC
NEW SCHEME 1	3	2.883237
NEW SCHEME 2	4	2.883237
ADOMIAN	7	2.642555

The new scheme 1 and Newton-Raphson's method have equal number of iterations

Problem 7

$x^3 - 3x + 1 = 0$

$x_0 = 0.33$

Required solution = 0.347296

Method	No of iterations	Solutions obtained
Newton Raphson	3	0.347296
BASTO *et al.*	2	0.347296
ABBASBANDY	3	0.347296
NEW SCHEME 1	2	0.347296
NEW SCHEME 2	2	0.347296
ADOMIAN	3	0.34332155

The new schemes 1 and 2 compete with Basto *et al.* method and better than the other methods.

Problem 16

$3x - \cos x - 1 = 0$

$x_0 = 0.33$

Required solution = 0.607102

Method	No of iterations	Solutions obtained
Newton Raphson	3	0.607102
BASTO *et al.*	7	0.648681
ABBASBANDY	3	0.607102
NEW SCHEME 1	3	0.607102
NEW SCHEME 2	3	0.607102
ADOMIAN	3	0.6088097

The new schemes 1 and 2 perform better than Basto *et al* and equally as good as the other methods.

Problem 28

$x^6 - 8x^4 + x^3 + 16x^2 - 4x - 2 = 0$

$x_0 = -0.5$

Required solution = -0.2541

Method	No of iterations	Solutions obtained
Newton Raphson	6	-0.2541
BASTO *et al.*	5	-0.2541
ABBASBANDY	5	-0.2541
NEW SCHEME 1	5	-0.2541
NEW SCHEME 2	5	-0.2541
ADOMIAN	-	NC

The new schemes 1 and 2 perform better than Newton-Raphson's method and comparably at par with others except for Adomian method which does not converge.

Problem 29

$x^7 - 3x^6 + 3x^5 - 13x^4 + 43x^3 - 57x^2 + 33x - 7 = 0$

$x_0 = 0.21$

Required solution = 0.593686

Method	No of iterations	Solutions obtained
Newton Raphson	7	0.593686
BASTO *et al.*	7	0.593686
ABBASBANDY	4	0.593686
NEW SCHEME 1	6	0.593686
NEW SCHEME 2	6	0.593686
ADOMIAN	-	NC

The new schemes 1 and 2 have six iterations. Thus, each of them is better than Basto *et al.* and Newton-Raphson's method having seven iterations each.

4.4 Results obtained from ANOVA

An analysis of variance test was carried out on the results obtained in table 4.1 and the following conclusion can be drawn considering the confidence interval of 95%. Since $p = 0.028 < 0.05$, we conclude that there is significant difference between the number of iterations obtained for the different methods (see Table A). We investigated further, using the Duncan Multiple range (Post Hoc Test) which is a multiple comparison test displayed in Table B.

Table A: Significant difference effects

One way ANOVA

Iteration

	Sum of Squares	df	Mean Square	F	Sig.
Between Groups	63.543	5	12.709	2.581	0.028
Within Groups	817.359	166	4.924		

Table B: Performance of different homogeneous subsets

Duncan

		Subset for alpha = 0 .05	
Methods	N	1	2
Basto *et al.*	30	2.833333	
N Raphson	30	2.900000	
NS1	30	3.066667	
Abbasbandy	27		3.407407
NS 2	30		3.666667
Adomian	25		4.680000

From the Duncan Multiple range, we can deduce that in the first homogeneous subset, we have Basto *et al.*, Newton Raphson and New Scheme 1 with the least number of iterations. In the second homogeneous subset, we have Abbasbandy, the New Scheme 2 and Adomian method with higher number of iterations. Full details are shown by the descriptive statistics which is displayed in Table C.

Table C: Descriptive statistics on the Number of Iterations

	N	Mean	Std. Deviation	Std. Error	95% Confidence Interval for Mean		Min	Max
					Lower Bound	Upper Bound		
N Raphson	30	2.9000	1.322224	0.241404	2.4063	3.3937	1	7
Basto *et al.*	30	2.8333	2.450663	0.447428	1.9182	3.7484	1	13
Abbasbandy	27	3.4074	2.390032	0.459962	2.4619	4.3529	1	13
NS1	30	3.0667	1.337350	0.244166	2.5673	3.5660	1	7
NS 2	30	3.6667	1.825742	0.333333	2.9849	4.3484	1	9
Adomian	25	4.6800	3.508561	0.701712	3.2317	6.1283	1	16

Based on the test results obtained from Table C above it can be seen that the standard deviation of Newton-Raphson method is 1.322224 which is the lowest followed by that of new scheme 1 with 1.33735 then new scheme 2 with 1.825742. Abbasdandy's method has standard deviation of 2.390032 followed by Basto *et al.* which has 2.450663. Adomian method has a standard deviation of 3.508561 which is the highest. Also from Table C, analysing the standard error, Newton-Raphson method has the least error of 0.241404 followed by that of the new scheme 1 and new scheme 2 with 0.244166 and 0.333333 respectively, which are less than that of Basto's 0.447428. Abbasbandy has standard error of 0.459962 which is followed by that of Adomian with 0.701712, as the highest. The Newton-Raphson method and new scheme 1 displayed their elegance by having a maximum of seven iterations each, followed by the new scheme 2 with the maximum number of nine iterations. Basto *et al.* and Abbasbany have equal number of high number of iterations of thirteen. Lastly, Adomian method has sixteen iterations.

The analysis above shows how good the two new schemes are as compared to the previous good methods. Two other advantages of the new schemes are that

(i) They are easy to compute since they have no higher derivatives.

(ii) They do not contain long expressions.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

Determining the zeroes of nonlinear function is not always direct. It generally takes the form of constructing one or several sequences $\{x_n\}$ of complex (or pure real) numbers supposed to converge to a zero of the function. Iterative method developed should give approximations to all roots of a functional equation, Traub (1964). Many iterative methods developed will converge only if the starting value x_0 is sufficiently close to a zero of the equation. These are said to be locally convergent. Iterative methods that do not require a sufficiently close starting value are globally convergent. Generally, iterative methods with high order of convergence, converges more rapidly than that with a lower order. In this study, we have developed two new iterative methods which converge locally and whose order of convergence is three. The study is summarised below.

In this study we present two new iterative schemes for solving nonlinear equations of the form $f(x)=0.$ We started by discussing on the motivation for the study as well as the actual problem studied. We also discussed aims and objectives of the study. Basic definition and theorems which will be helpful throughout our study were then presented. In the literature we discussed the basic concept of iterative methods regarding roots of nonlinear equations. Various methods which were developed by several researchers such as the Newton-Raphson method and it variants, Adomian approach and some of the iterative methods developed based on the Adomian method, as far back from 1998 to the most recent of 2012 were also presented. Our two new schemes for solving nonlinear equations were then presented. Later

30 different examples of different nature were presented and the two new schemes applied on the examples to determine the number of iterations to reach solutions by the schemes.

Some existing iterative methods namely Adomian method, Abbasbandy, Basto *et al.* and Newton-Raphson's method were also applied to solve these examples and record the number of iterations. We then carried out one way analysis of variance test (ANOVA) to make comparison between our schemes and the other iterative methods used.

5.2 Conclusion

The number of iterations to get a solution using the two new schemes for the 30 examples is generally accommodating. It was noted that the two new schemes perform equally or better than some of the good existing methods in solving both algebraic and transcendental equations. The results from ANOVA show that there is significant difference between the numbers of iterations obtained for the different methods. The results show that Newton-Raphson method and New scheme 1 have more advantage with a maximum of seven iterations each, while new scheme 2 has nine. Basto *et al.* and Abbasbany have equal number of thirteen iterations each. The Adomian has sixteen iterations. This shows clearly that the new schemes 1 and 2 perform equally better and as efficiently as the best existing methods.

5.3 Recommendations

At the end of this study we come up with the following recommendations:

(i) Further studies with more examples should be carried out to make definite conclusions on the results obtained.

(ii) As we used the approximation $\frac{f''}{f'} \approx 1$, further studies could be carried out using other ratios $\frac{f''}{f'} \approx c$ (where c is any constant) to obtain other schemes which could be good as well.

(iii) Further studies could be carried out on the two new schemes developed to see whether they could be applied in solving complex equations.

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Appendix

The following examples give the situations where $\frac{f''}{f'}=1$.

(i) $f(x)=e^x$

$f'(x)=e^x$

$f''(x)=e^x$

$\Rightarrow \frac{f''(x)}{f'(x)}=1$

(ii) $f(x)=x^3+4x^2+8x+8$

$f'(x)=3x^2+8x+8$

$f''(x)=6x+8$

$\therefore \frac{f''(x)}{f'(x)}=\frac{6x+8}{3x^2+8x+8}$

So that for $\frac{6x+8}{3x^2+8x+8}=1$

We must have $6x+8=3x^2+8x+8$

$\Rightarrow 3x^2+2x=0$

$x(3x+2)=0$

$\therefore x=0 \text{ or } x=-\frac{2}{3}$

(iii) $f(x)=x^3-6x+4$

$f'(x)=3x^2-6$

$f''(x)=6x$

$$\therefore \frac{f''(x)}{f'(x)} = \frac{6x}{3x^2 - 6},$$

So that for $\frac{6x}{3x^2 - 6} = 1$

We must have $3x^2 - 6x - 6 = 0$

$\therefore x = -0.732$ or $x = 2.732$

(iv) $f(x) = 3x - \ln x - 16$

$$f'(x) = 3 - \frac{1}{x}$$

$$f''(x) = \frac{1}{x^2}$$

$$\therefore \frac{f''(x)}{f'(x)} = \frac{1}{3x^2 - x}$$

So that for $\frac{1}{3x^2 - x} = 1$

We must have $1 = 3x^2 - x$

$\Rightarrow 3x^2 - x - 1 = 0$

$\therefore x = -0.434$ or $x = 0.768$