A STUDY OF THE THEORY OF MULTISET RELATIONS AND PARTITIONS

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AISHAT OVAYIOZA SALIHU

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DECLARATION

I declare that the work in this thesis entitled “A Study of the Theory of Multiset Relations and Partitions” has been performed by me in the department of mathematics under the supervision of Professor D. Singh and Dr. A.M. Ibrahim.

The information derived from literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any university or institution.

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Name of student *Signature* *Date*

CERTIFICATION

This thesis titled "A Study of the Theory of Multiset Relations and Partitions" by Aishat Ovayioza Salihu meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

---------- Date:----------

Prof. D. Singh

Chairman, Supervisory Committee

---------- Date: ----------

Dr. A. M. Ibrahim

Member, Supervisory Committee

---------- Date: ----------

External Examiner

---------- Date: ----------

Dr. A. A. Tijjani

Head of Department

---------- Date: ----------

Prof. A. A. Joshua

Dean, Post Graduate School

DEDICATION

This work is dedicated to my Grand spiritual father Maulana Sheikh Ibrahim Nyass and all the Lovers of the Holy Prophet (S A W).

ACKNOWLEDGEMENT

All praise is due to Allah the lord of the world Who has made this work a reality. May His everlasting peace and blessings be upon His noble apostle whose light will continue to be the source of all illumination till the end of time.

I wish to first express my enormous gratitude to my supervisor, Prof. D. Singh, whose help and encouragement is overwhelming throughout my course of study. May his kindness and goodwill be mightily rewarded (Ameen).

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ABSTRACT

In this thesis, we briefly present the concept of multiset and its representation. Basic operations under multisets are defined. Structures and types of multiset relations and functions are discussed and a scheme for constructing the transitive closure of an mset relation is described. We also define operations on multiset relations and their matrix representation which is our major contribution in this thesis. Finally, we formulate the theory of multiset partitions.

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CHAPTER ONE

1.0 INTRODUCTION

The theory of set relations and partitions has been studied extensively in mathematics along with its applications in diverse fields especially in solving many real world problems; for example, in solving a complex transportation problem where one needs to see how a city A is related to a city B or in an establishment where one needs to partition staff into different sections or problems related to air traffic systems, etc.

In Cantorian set theory, usually called a standard or crisp set theory, a set is considered as "any collection into a whole M of definite and distinguished objects (called the elements of M) of our intuition or our thought" [Blizard, 1989]. One unavoidable consequence of Cantor's definition is that no element can occur more than once in a classical set. Singh (1985) argues that Cantor's assertion of excluding repeated elements does not go hand in hand with many situations arising in solving real world problems. For example, the repeated roots of $x^2 - 2x + 1 = 0$, repeated hydrogen atoms in a water molecule (H_2O), the repeated prime factors of an integer $n > 0$, etc. need to be considered significant. Once we admit repetition of objects, we have multisets.

In view of the recent developments taking place in the study of multisets, a number of important areas of research requiring multiset relations and partitions have come to the fore. Especially from a practical point of view, multisets are found useful in providing structures as they arise quite naturally in certain areas of mathematics, computer science, physics, linguistics and philosophy. In particular, multisets and partitions of a multiset have been found to be natural representations for many important combinatorial optimization problems.

In this thesis, we propose to present a comprehensive account of fundamentals of multisets, operations between multisets, define multiset relations, operation between multiset relations, properties of multiset relations and to develop a scheme for their matrix representations. Finally, we describe partitions of a set and multiset.

1.1 STATEMENT OF THE RESEARCH PROBLEM

We wish to study the structure and types of multiset relations and functions. A scheme for constructing the transitive closure of an mset relation will be described. We propose to define operations on multiset relations and their matrix representations. Finally, the theory of multiset partitions is to be formulated.

1.2 AIM AND OBJECTIVE

In order to achieve the aim and objective of this research work, we propose to study

(1) fundamentals of multisets and multiset operations;

(2) multiset relations and functions;

(3) operations on multiset relations and their matrix representations;

(4) partitions of sets and

(4) partitions of multisets.

1.3 METHODOLOGY

The literature review gives us a direction to focus our problem. We systematically study all multiset operations specially relevant to our research. Multiset relations and their matrix representations are studied in a new perspective. We also study multiset functions in order to bring a couple of distinctive features that is relevant in mathematics.

Finally, we study partitions of a set as a precursor to studying partitions of a multiset.

CHAPTER TWO

LITERATURE REVIEW

Studies undertaken on relations and partitions of sets abound. In the recent years, owing to the development of multiset theory (see Blizard (1989, 1991), Singh (2006), Singh *et al*. (2007, 2008) and Wildberger (2003). Researches have been carried out to establish analogous results in multiset context. Some existing theories on sets like relations, composition of relations, equivalence relations, partitions and functions are suitably being extended to multisets.

The theory of relations and partitions of multisets have been examined by many researchers in varying dimensions.

Bender (1974) and Bender *et al.* (1984), showed that the number of blocks in a random partition (each partition equally likely) is asymptotically normal by deriving a local limit theorem and also indicated how results of Haigh on the number of blocks of various sizes and largest block size can be extended from sets to multisets.

Clements (1988) conducted a research on multiset k-families where the generalized Macaulay theorem is extended slightly to a form that is useful in k-families.

Christian *et al.* (2004) studied a wide range of disjoint, partition and intersection constraints for set and multiset variables.

Ernasto and Stefano (2007), studied multidimensional inequality in partitions of finite multisets with thresholds where Lorenz-like pre-order, a family of functions preserving such a pre-order, with a counterpart of the Pigou-Dalton transfers defined and a version of the celebrated Hardy-wood-Polya characterization results are provided.

Recently, Girsh and Sunil (2009), following Cert *et al.* (1971), Peterson (1976) and Yager (1986) attempted to present a generalized concept of relations, function, composition of relation and equivalence relation in multiset context.

We propose to present a comprehensive study of the aforesaid developments and outline a technique for defining operations on multiset relations and thier matrix representation. Finally, we elaborate the theory of sets and multiset partitions.

CHAPTER THREE

FUNDAMENTALS OF MULTISET AND OPERATIONS UNDER MULTISET

3.0 INTRODUCTION

In this chapter, the concept of a multiset and its representation are briefly presented. We also describe the operations on a multiset and examine what is mathematically sound and what is not obtainable under the concept at this moment.

3.1 THE CONCEPT OF A MULTISET

A multiset (mset, for short) is an unordered collection of objects in which, unlike a standard (Cantorian) set, duplicates or multiples of objects are admitted. In other words, an mset is a collection in which objects may appear more than once and each individual occurrence of an object is called an element. All duplicates of an object in an mset are indistinguishable. The objects of an mset are the distinguishable or distinct elements of the mset. The distinction made between the terms object and element does enrich the multiset language. However, use of the term element alone may suffice if there does not arise any confusion (see Wilderberger, (2003) for details)

The concept of the term multiple-membership collection is as old as the concept of number (see Blizard (1989, 1991), Knuth(1981) and Singh et al (2007 & 2008) for an early history of multisets and their applications). For example, collection of repeated roots of a polynomial equation, repeated observation in a statistical sample, repeated hydrogen atoms in a water molecule (H_2O), the prime factor of an integer $n > 0$, etc., are all multisets. It is interesting to note that $\pi = 3.141\ldots$ can be identified with a

multiset containing the number 1 with multiplicity 3, $\frac{1}{10}$ with multiplicity 1, $\frac{1}{100}$ with multiplicity 4, etc.

The term multiset, as noted by knuth [1981], was first suggested by N.G.de Bruijn in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz; list, heap, bunch, bag, sample, weighted set, occurrence set and fireset (finitely repeated element set) used in different contexts but conveying synonimity with mset. Infact, prior to coinage of the term multiset, the term bag was in frequent use. Currently, multiset and bag are being used interchangeably.

The number of occurrences of an object x in an mset A, which is finite in most of the studies that involve mset, is called its multiplicity or characteristic value, denoted by $m_A(x)$ or $c_A(x)$ or simply $A(x)$. In the theory of bags, $x \in\in A$ is used in place of $m_A(x)$. a well formed formula $x \in^n y$ semantically reads that x belong n times to y. The number of distinct elements in an mset (which need not be finite) and their multiplicities jointly determine its cardinality, usually denoted by $\#A$ or $C(A)$ or $|A|$. In other words, the cardinality of an mset is the sum of the multiplicities of all its objects.

A cardinality-bounded mset space X^n can be defined as the set of all msets whose objects are drawn from a ground set X such that no object in any mset $A \in X^n$ occurs more than n times.

The set of distinct elements of an mset is called its root or support. The cardinality of the root set of an mset is called its dimension.

An mset is called regular or constant if all its objects occur with the same multiplicity and the common multiplicity is called its height. For example, $[a, b]_{3,3}$ is a regular mset of height 3.

An mset is called simple if all its elements are the same. For example,$[a]_3$ is a simple mset.it follows that the root set of a simple mset is a singleton.

A unique mset that does not contain any member is called the empty mset, denoted by $\emptyset$.

Two msets A and B are called equal, denoted by $A = B$ if $M_A(x) = M_B(x)$ for all objects x.

An mset A is called a submultiset (submset,for short) or a multisubset (msubset,for short) of an mset B, denoted by $A \subseteq B$, if $M_A(x) \leq M_B(x)$ for all objects x. An mset is called the parent multiset or overmultiset in relation to its submsets.For example, $[a, b]_{1,2}$ is a submset of $[a, b, c]_{1,3,1}$ and the latter is a parent mset of the former.It follows that $A = B$ if and only $A \subseteq B$ and $B \subseteq A$. Also $A \subset B$ if $A \subseteq B$ and $A \neq B$

A submset of a given mset is called whole if it contains all multiplicities of the common objects. For example,$[a, b]_{2,3}$ is a whole mset of $[a, b, c]_{2,3,4}$.

A submset of a given mset is called full if it contains all objects of the parent mset. For example,$[a, b, c]_{1,2,3}$ is a full mset of $[a, b, c]_{2,3,4}$.

The powermset of an mset A denoted by $\wp(A)$,is the mset of all submsets of A.

3.2 REPRESENTATION OF A MULTISET

Following Blizard (1989, 1991), Apostolos (2003) and Singh et al (2006, 2007 & 2008), we present the following:

3.2.1 Multiplicative form

Following Meyer and McRobbie (1982) cf Singh (2006), the use of square brackets to represent an mset has become almost standard. Thus, an mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notationally

written as [[*a, b, b, c, c, c*]] or [*a, b b, c, c, c*] or$[a, b, c]_{1,2,3}$ or$[a^1b^2c^3]$ or [*a1, b*2*, c*3], depending on one's taste and convenience.

3.2.2. Linear form

Wildberger (2003) cf Singh (2006) puts forward a linear notation for multisets, which seems quite novel, especially when negative multiplicities (integral as well as rational) are to be dealt with. For example, the mset $M = [a, b, c]_{1,2,3}$ can be written as

$M = [a] + 2[b] + 3[c]$.

Similarly, a rational mset can be represented, for example,

$$N = \frac{2}{5}[5] - \frac{1}{2}[1_8]$$

In order to accommodate negative multiplicities round brackets are used:

(a) in an mset stands for negative of *a*; for example,

[2, 4, (5), (5), 4] = [2] + 2[4] - 2[5].

In the same place, the distinction between the terms `element' and `object' occurring in an mset is made explicit as follows:

Each individual occurrence of an object *x* in an mset *A* is called an element of *A*. Thus, in the linear notation of *M* above; *b*, for example, is an object appearing twice and every occurrence of *b* is an element of *M*. It follows that the distinct elements of an mset are the objects. An object is an element if its multiplicity is unity.

According to Singh (2006), there are notations used in (Wilberger (2003), pp. 5-6) to represent data structures of set, ordered set, multiset and list, are quite instructive:

A collection containing *x, y, z*..... is denoted by $\{xyz \ldots\}$or [x_y_z_...] if it is a set; $\{x, y, z, \ldots\}$ if it is an ordered set; [*xyz:*] = [x_y_z_...] if it is a multiset; and [*x,y,z*...] if it is a list.

Note that a list is an ordered sequence of elements with repetitions allowed,

Whereas an mset is a sequence with its ordering stripped off.

3.2.3. Multiset as a Sequence

A multiset can also be represented as a sequence in which the multiplicity of an element equals the number of times the element occurs in the sequence, which is exactly Dedekind's `frequency-number'. The idea is to construct an mset as a sequence (a function with domain □, the set of natural numbers) and ignore the ordering of its elements, which can be done by taking all permutations of the domain of the sequence.

3.2.4. Multiset as a Family of Sets

A multiset can also be represented as a family of sets, which is altogether a generalization of the idea of a sequence described above. Thus, the family of sets $F = \{F_i\}, i \in I$ where $F_{i=} F_j$, if $i = j$, which identfies a repeated element, represents an mset. Clearly, such a family F is a function:$I \rightarrow \{\{F_i | i \in I\}\}$, which in turn, is a sequence if $I = \varkappa$.

3.2.5. Multiset as a Numeric-valued function

Representation of an mset as a numeric-valued or cardinal-valued function abounds, especially in the application areas. Formally, an mset is just a mapping from some ground or generic or universal set into some set of numbers.

3.2.6. Multiset as a generalized characteristic function

Similar to the representation of a set by its characteristic function (function whose range is $\{0,1\}$), a multiset or hybrid set is determined by its generalized characteristic function (whose range is the set of integers, positive, negative or zero). These are the categorical representations of both forms of multisets.

3.3 OPERATIONS UNDER MULTISETS

Let $A, B, C, \ldots$ be msets over a given generic set X

Definition 3.3.1: Union ($\cup$)

The union of two msets A and B results in a new mset denoted by $A \cup B$ such that

$\forall x \in X, M_{A \cup B}(x) = M_A(x) \cup M_B(x) = Max\{M_A(x), M_B(x)\}$

That is an object z occurring a times in A and b times in B, occurs maximum $\{a, b\}$ times in $(A \cup B)$, if such a maximum exists, otherwise the minimum of $\{a, b\}$ is taken which always exists. In other words, $(A \cup B)$ is the smallest mset C which contain both A and B i.e $A \subseteq C, B \subseteq C$.

It follows that for any given mset X there exists an mset Y which contains elements of elements of X, where the multiplicity of an element z in Y is the maximum multiplicity of z as an element of elements of X along with the above stipulation on the existence of such a maximum. The fact is denoted by $Y = \cup X$

Clearly, $Dom(\cup X) = \cup\{Dom(A), A \in X$ and that the multiplicity of z in Y is the maximum of its multiplicities as an element of elements of X if it exists, otherwise the minimum is taken.

Example:

$A = [1,1,2,2,3]$, $B = [1,2,3,3]$ then $A \cup B = [1,1,2,2,3,3]$

Also, it follows that for a finite mset X, the maximum multiplicity of elements of elements of X always exists. However, for certain infinite set like $X = \{[y]_1, [y]_2, [y]_3 \ldots\}$, the maximum multiplicity of elements of elements of X does not exist, and hence $\cup X = \{y\}$. It is obvious by the definition of the union that multiplicity of any $y \in X \neq \emptyset$ is relevant to $\cup X$, and hence $\cup X = \cup X^*$ Blizard (1989)

The above definition can be extended to an arbitrary number of msets as follows

$$\cup A_i = \{m_{\cup_i A_i}(x).x \mid m_{\cup_i A_i}(x) = max_{i \in I} m_{A_i}(x) for\ all\ x \in D\}$$

Definition 3.3.2:Intersection ($\cap$)

The intersection of two msets A and B drawn from a set X is an mset A denoted by $A \cap B$ such that $\forall x \in X$, $M_A(x) = M_{A \cap B}(x) = M_A(x) \cap M_B = Min\{M_A(x), M_B(x)\}$, being the intersection of two numbers, $A \cap B$is the largest mset C which is contained in both A and B i.e $C \subseteq A, C \subseteq B$.

That is an object x occurring a times in A and b in B, occurs minimum $\{a, b\}$ times in $A \cap B$, which always exists.

In general, for a given mset X, $Dom(\cap X) = \cap\{Dom\,(A), A \in X$ and $z \in \cap\, X$

Example:

Let $A = [2,2,3,3]$, $B = [1,2,3,3]$ then $A \cap B = [2,3,3]$

The above definition can be extended to an arbitrary number of msets as follows

$$\cap A_i = \{m_{\cap_i A_i}(x).x \mid m_{\cap_i A_i}(x) = min_{i \in I} m_{A_i}(x) for\ all\ x \in D\}$$

Note that for any mset X we have $\cap X \subseteq \cup X$

Definition 3.3.3: Addition or Sum or Merge (+ ,⊎ *or* ⊕)

Addition of two msets M_A and M_B drawn from a set X results in a new mset $A \uplus B$ such that $\forall\, x \in X, M_{A \uplus B}(x) = M_A(x) + M_B(x)$

That is , an object x occurring a times in A and b times in B , occurs $a + b$ times in $A \uplus B$. This being the direct sum or arithmetic addition of two numbers, is sometimes called the *counting law.*

Example:

$A = [1,1,2,3,3]$, $B = [1,2,3,3]$ then $A \uplus B = [1,1,1,2,2,3,3,3,3]$

The above definition can be extended to an arbitrary number of Msets as follows

$$\sum A_i = \{m_{\sum_i A_i}(x).x \mid m_{\sum_i A_i}(x) = \sum_{i \in I} m_{A_i}(x) for\ all\ x \in D\}$$

Note that in a cardinality-bounded mset space X^n, the definition of arithmetic sum needs to be modified as follows

$$m_{A+B}(x) = minimum\{n, m_A(x) + m_B(x)\}$$

It follows that the insertion of an element x into an mset A gives rise to a new mset $A' = A + x$ such that $m_{A'}(x) = m_A(x) + 1$ and $m_{A'}(y) = m_A(y) + 1$ for all $y \neq x$

It should also be noted that if X be an infinite mset, then the multiplicity of some mset $A \in \uplus X$ may not be finite. In that case the multiplicity of A in X is used.

For example, if $X = \{[z]_1, [z]_2, [z]_3 \ldots\}$ then $\uplus X = \cup X = \{z\}$, Blizard (1989).

The following properties can be easily established for union, intersection, and sum of multisets:

1. Commutativity: $A \sqcup B = B \sqcup A$, $A \uplus B = B \uplus A$

$A \cap B = B \cap A$;

2. Associativity: $A \sqcup (B \sqcup C) = (A \sqcup B) \sqcup C$, $A \uplus (B \uplus C) = (A \uplus B) \uplus C$

$A \cap (B \cap C) = (A \cap B) \cap C$;

3. Idempotency: $A \sqcup A = A$

$A \cap A = A$;

but $A \uplus A \neq A$

4. Distributivity: $A \sqcup (B \cap C) = (A \sqcup B) \cap (A \sqcup C)$

$A \cap (B \sqcup C) = (A \cap B) \sqcup (A \cap C)$;

$A \uplus (B \sqcup C) = (A \uplus B) \sqcup (A \uplus C)$

$A \uplus (B \cap C) = (A \uplus B) \cap (A \uplus C)$;

For any mset A and B

$$|A \uplus B| = |A| + |B| = |A \cup B| + |A \cap B|$$

$\uplus$ distributes over both $\cup$ and $\cap$ but not the other way round, thus, $\uplus$ is stronger than $\cup,\cap$. Blizard (1989)

Definition 3.3.4: Difference ($-$ *or* $\setminus$ *or* Θ), Symmetric Difference (Δ) and complementation ($\sqcup$ or c)

Difference: let A and B be two msets and $B \subseteq A$. The arithmetic difference or subtraction of B from A, denoted by $A - B$, is the mset$C \subseteq A$ such that

$m_C(x) = m_A(x) - m_B(x)$ for all objects x. In general

$m_C(x) = m_A(x) - m_{A \cap B}(x) = \max\{m_A(x) - m_B(x), 0\}$ for all objects x. It is also called the relative complement of B in A

It follows that the deletion of an element x from an mset A gives rise to a new mset

$A' = A - x$ such that $m_{A'}(x) = \{m_A(x) - 1, 0\}$.

It should be noted that in multiset context, an element may occur both in an mset and its complement. Accordingly, some of the consequences of the aforementioned definition may not satisfy the laws of classical set theory. For example, $A = [a, b]_{4,5}$ and $B = [a, b]_{2,3}$, then $A - B = [a, b]_{2,2} \subset B$

Symmetric Difference: the symmetric difference of A and B, denoted by $A \Delta B$, is the mset C such that $m_C(x) = |m_{AA}(x) - m_B(x)|$ for all objects x

Complementation: Let $A \in X^n$. The complement of A, denoted by $\bar{A}$ or A^C, in X^n is a member X^n such that $m_{\bar{A}}(x) = n - m_A(x)$ for all objects x in X. Alternatively, let $\Im = \{A_1, A_2 \ldots\}$ be a family of msets. The maximum mset $\mathbb{Z}$ is defined such that

$m_{\mathbb{Z}}(x) = max\ m_A(x), A \in \Im$ for all objects x and all $A \in \Im$. Then ,$\bar{A} = \mathbb{Z} - A$ such that $m_{\bar{A}}(x) = m_{\mathbb{Z}}(x) - m_A(x)$, for all objects x Petrovsky (1994 and 1997)

Note that the Mset space X^n is closed with respect to $\cup, \cap\ and\ \uplus$ (with modification mentioned in definition 3.3.8), but Demorgan laws do not hold in totality.

Definition 3.3.5: Multiplication on Scalar

kA, where k is a natural number, is defined as $m_{kA}(x) = k.m_A(x)$, for all objects x. It follows that $kA = A + A + \cdots + A - k$ times.

Definition 3.3.6: Arithmetic Multiplication

The arithmetic multiplication of msets $A_1, A_2, \ldots$, denoted by $\prod_{i \in I} A_i$, where I is the index set, is defined as $m_{\prod_{i \in I} A_i}(x) = \prod_{i \in I} m_{Ai}(x)$, for all objects x. It follows that $A^n = A \cdot A \cdot \ldots \cdot A - n$ times can be defined as $m_{A^n}(x) = (m_A(x))^n$, for all objects x.

Definition 3.3.7: Cartesian (or Direct) Product

Let $\left({}^{m}/_{x}, {}^{n}/_{y}\right)/k$ denote that x occurs m times, y occurs n times and the ordered pair (x, y) occurs k times. Also, in ordered to comprehend the ordered $n-tuple$

$\left({}^{m_1}/_{x_1}, {}^{m_2}/_{x_2}, \ldots, {}^{m_n}/_{x_n}\right)/k$, let $C_i(x_1, x_2, \ldots x_i, \ldots x_n)$ denote the count of the i^{th} coordinate in the tuple for $i = 1,2, \ldots, n$

The Cartesian product $A_1 \times A_2 \times \ldots \times A_n$ of nonempty msets $A_1, A_2, \ldots, A_n$ can be defined as mset of all ordered n −tuples $(x_1, x_2, \ldots, x_n)$ where $x_i \in^{P_i} A_i, i = 1,2, \ldots, n$ and $m_{A_1 \times A_2 \times \ldots \times A_n}(x_1, x_2, \ldots x_n) = \prod P_i, i = 1,2, \ldots, n$

Follows that $(\times A)^n = A \times A \times \ldots \times A - n$ times can be defined as $m_{(\times A)^n}(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} m_A(x_1), x_1 \in A, i = 1,2, \ldots, n$. Thus for any two msets A and B, $A \times B = \{\left({}^{m}/_{x}, {}^{n}/_{y}\right)/mn : x \in^{m} A, y \in^{n} B\}$. That is

$m_{A \times B}(x, y) = m_A(x) \cdot m_B(y)$.

For example, let $A = \left[{}^{1}/_{x}, {}^{2}/_{y}\right]$ and $B = \left[{}^{2}/_{x}, {}^{3}/_{z}\right]$ then

$A \times B = \{\left({}^{1}/_{x}, {}^{2}/_{x}\right)/2, \left({}^{1}/_{x}, {}^{3}/_{z}\right)/3, \left({}^{2}/_{y}, {}^{2}/_{x}\right)/4, \left({}^{2}/_{y}, {}^{3}/_{z}\right)/6\}$ and

$$A \times A = \{\left({}^{1}/_{x}, {}^{1}/_{x}\right)/1, \left({}^{1}/_{x}, {}^{2}/_{y}\right)/2, \left({}^{2}/_{y}, {}^{1}/_{x}\right)/2, \left({}^{2}/_{y}, {}^{2}/_{y}\right)/4\}$$

We have the following distributive laws holding for direct product

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

$$A \times (B \cap C) = (A \times B) \cap (A \times C)$$

$$A \times (B \uplus C) = (A \times B) \uplus (A \times C)$$

However, in general, the commutative and associative laws do not hold for direct products. i.e. generally,

$$A \times B \neq B \times A$$

$$(A \times B) \times C \neq A \times (B \times C)$$

Definition 3.3.8 . The insertion of an element x into an mset A results in a new mset A' denoted by $A' = A \uplus x$ such that $M_{A'}(x) = M_A(x) + 1$ and $M_{A'}(y) = M_A(y) \forall y \neq x$

Definition 3.3.9 . The removal of an element x from an mset A results in a new mset A' denoted by $A' = A \; x$ such that

$M_{A'}(x) = Max\{M_A(x) - 1,0\}$ and $M_{A'}(y) = M_A(y) \; \forall y \neq x$

Notation 3.3.10:

Let A be an mset from X with x appearing n times in M. It is denoted by $x \in^n A$. $A = \{{}^{k_1}/_{x_1}, {}^{k_2}/_{x_2} \ldots\ldots {}^{k_n}/_{x_n}\}$ where A is an mset with x_1 appearing k_1 times, x_2 appearing k_2 times and so on. $[A]_x$ denotes that the element x belongs to the mset A and $|[A]_x|$ denotes the cardinality of an element x in A

CHAPTER FOUR

MULTISET RELATIONS AND FUNCTIONS

4.0 INTRODUCTION

Relations and functions are fundamental concepts in discrete mathematics used to define how sets of objects or elements of multiset relate to other sets of objects or other elements of multisets. They provide important modelling tools in almost all areas of scientific developments. In particular, they are used in relational database management systems, task scheduling systems, methods of solving various optimization problems etc,. In this chapter, which contains our main research, we wish to present multiset relations, operations on multiset relations and their matrix representation. We formulate a scheme for constructing transitive closure of a multiset relation. We also present multiset functions.

4.1 DEFINITIONS

DEFINITION 4.1.1: Multiset Relation

Let X^n be a cardinality-bounded mset space defined on a set X and A be a member of X^n. In consonance with the terminologies used for explicating the notion of relations in set, we say that any submset R of $A \times A$ is called an mset relation on A (or simply, a relation on A if the context is clear), symbolized as $R: A \to A$,where every member of R has a count $C_1(x, y)$ and $C_2(x, y)$.Also m/x is R −related to n/y is symbolized as $(m/x, n/y) \in R$ or $m/x \, R \, n/y$.

Fomally, $R = \{(m/x, n/y)/mn: (m/x, n/y) \in^{mn} R\}$.

Note that $(m/x, n/y) \in^{mn} R$ actually means that ordered pair (x, y) occurs mn times in R.

DEFINITION 4.1.2: Domain and Range of an mset relation

The domain and range of the mset relation R on A is defined as follows:

$DomR = \{x \in^r A: \exists\ y \in^s A\ such\ that\ {}^r/_x\, R\, {}^s/_y\}$Where

$C_{DomR}(x) = Sup\{C_1(x,y): x \in^r A\}$, which always exist for cardinality bounded msets.

$RanR = \{y \in^s M: \exists\ x \in^r A\ such\ that\ {}^r/_x\, R\, {}^s/_y\}$Where

$C_{ranR}(x) = Sup\{C_2(x,y): y \in^s A\}$, which always exist for cardinality bounded msets.

Note that, for cardinality-bounded msets, the supremum (sup) is simply the maximum (max).

For example: let $A = [{}^8/_x, {}^{11}/_y, {}^{15}/_z]$ be an mset and

$R =$

$\{({}^2/_x, {}^4/_y)/8, ({}^5/_x, {}^3/_x)/15, ({}^7/_x, {}^{11}/_z)/77, ({}^8/_y, {}^6/_x)/48, ({}^{11}/_y, {}^{13}/_z)/143$

$({}^7/_z, {}^7/_z)/49, ({}^{12}/_z, {}^{10}/_y)/120, ({}^{14}/_z, {}^5/_x)/70\}$

is an mset relation defined on A then $DomR = \{{}^7/_x, {}^{11}/_y, {}^{14}/_z\}$ and

$$RanR = \{{}^6/_x, {}^{10}/_y, {}^{13}/_z\}$$

DEFINITION 4.1.3: Inverse of an mset relation

The inverse of an mset relation R denoted by R^{-1} is defined as

$R^{-1} = \{({}^n/_y, {}^m/_x)/mn: ({}^m/_x, {}^n/_y) \in^{mn} R\}$

Note that,akin to the existence of the inverse of a relation on sets, the inverse of an mset relation always exists as an mset relation.

Example: $let\ A = [^4/_x, ^7/_y, ^9/_z]$ and

$$R = \left\{(^2/_x, ^3/_y)/6, \left(^3/_x, ^5/_z\right)/15, (^3/_y, ^4/_z)/12, \left(^4/_z, ^5/_y\right)/20, (^3/_z, ^3/_x)/9\right\}$$

be an mset relation on$A$ then

$$R^{-1} = \left\{(^3/_y, ^2/_x)/6, \left(^5/_z, ^3/_x\right)/15, (^4/_z, ^3/_y)/12, \left(^5/_y, ^4/_z\right)/20, (^3/_x, ^3/_z)/9\right\}.$$Is also an mset relation on$A$

4.2 OPERATIONS ON MULTISET RELATIONS

Relations on msets are simply msets. That is, subsets of ordered pairs of the Cartesian product of an set. It therefore make sense to use the usual multiset operations (intersection ∩, union ∪, sum ⊎ and difference $-$ or ⊖) to combine relations to create new relations. Hence, if R_1 and R_2 are two mset relations on $A \in X^n$, then $R_1 \cup R_2$, $R_1 \uplus R_2$, $R_1 \cap R_2$, $R_1 - R_2$ the compliment of R_1 or R_2 , $R_1 \Delta R_2$ e.t.c are also mset relations on A. For example, $R_1 \cup R_2$ can be defined as the mset consisting of all distinct pairs of R_1 and R_2 and the common pairs appearing only once or equivalently, appearing with its maximum multiplicity.

For illustration, let R_1 and R_2 be two msets relations defined on $A \in X^n$ given

$$A = [^4/_x, ^7/_y, ^9/_z]$$

$R_1 =$

$\{(2/x, 3/y)/6, (3/x, 5/z)/15, (3/y, 4/z)/12, (4/z, 5/y)/20, (3/z, 3/x)/9\}$ and

$R_2 =$

$\{(2/x, 3/y)/6, (4/x, 4/z)/16, (4/z, 5/y)/20, (6/z, 3/x)/18, (3/z, 4/z)/12\}$ then

$$R_1 \cup R_2 = \{(2/x, 3/y)/6, (4/x, 5/z)/20, (5/y, 4/z)/20, (3/y, 4/y)/12$$

$(4/z, 5/y)/20, (6/z, 3/x)/18, (3/z, 4/z)/12\}$ and

$$R_1 \cap R_2 = \{(2/x, 3/y)/6, (3/x, 4/z)/12, (3/z, 3/x)/9\}$$

4.3 COMPOSITION OF MULTISET RELATIONS

It is a known fact that transitive closures of relations have applications in certain areas such as networks, syntactic analysis, fault detection and diagnosis in switching circuits e.t.c., which require computing compositions of relations. Below shows some details for constructing a composition of mset relations.

Let A, B, C be three msets and R be an mset relation from A to B and S be an mset relation from B to C. The composition of R and S denoted by $S \circ R$ (or $R \circ S$, depending on the convention one follows) is an mset relation from A to C and is defined as follows:

If m/x is in A and k/z in C then $m/x \, (S \circ R) \, k/z$ if and only if there is some $m/x \, R \, n/y$ and $n/y \, S \, k/z$ such that $C_1(x, z) =$ Max $\{Min\{C_1(x, y), C_1(y, z)\}\}$ and $C_2(x, z) =$ Max$\{Min\{C_2(x, y), C_2(y, z)\}\}$ meaning that

m = max {min {m,n}} and k = max {min {n,k}}

For example:

Let $A = [{}^{4}/_{x}, {}^{5}/_{y}, {}^{3}/_{z}]$

$B = \left[{}^{5}/_{a}, {}^{3}/_{b}, {}^{4}/_{c}\right]$

$C = [{}^{4}/_{p}, {}^{2}/_{r}, {}^{3}/_{t}]$ be given msets

Let R be an mset relation from A to B and S be an mset relation from B to C defined as

$$R = \{({}^{3}/_{x}, {}^{2}/_{a})/6, \left({}^{2}/_{x}, {}^{2}/_{b}\right)/4, ({}^{3}/_{y}, {}^{3}/_{c})/9 \left({}^{2}/_{z}, {}^{1}/_{b}\right)/2, ({}^{1}/_{z}, {}^{3}/_{a})/3\}$$

$$S = \{({}^{3}/_{a}, {}^{1}/_{r})/3, ({}^{2}/_{a}, {}^{2}/_{t})/2, \left({}^{2}/_{b}, {}^{3}/_{p}\right)/6 ({}^{2}/_{c}, {}^{2}/_{p})/4, 2\}$$

then

$$SoR = \{({}^{2}/_{x}, {}^{2}/_{p})/4, ({}^{3}/_{x}, {}^{1}/_{r})/3, ({}^{2}/_{x}, {}^{2}/_{t})/4 ({}^{2}/_{y}, {}^{2}/_{p})/4, ({}^{1}/_{y}, {}^{2}/_{c})/2$$
$$({}^{2}/_{z}, {}^{1}/_{p})/2, ({}^{1}/_{z}, {}^{1}/_{r})/1 ({}^{1}/_{z}, {}^{2}/_{t})/2,\}$$

It is easy to verify that the stipulations made above in order that composite relation is defined hold for the example described above. In particular, $Dom(SoR) \subseteq DomR$ and $Ran(SoR) \subseteq RanS$. on the same line, for a given relation R on an mset $A, RoR, RoRoR, \ldots$ can be composed. In particular, RoR, denoted by R^2, can be defined and, in turn , RoR^2, denoted by R^3, can be defined and so on. This eventually leads to obtaining transitive closure of R as it is done in the case of relation on set viz., $R^+ = R \cup R^2 \cup R^3 \ldots$

On a final note, the composition of mset relations is not commutative. For example, let $A = [{}^{4}/_{a}, {}^{3}/_{b}, {}^{5}/_{c}]$ be a given mset.

Let $R = \{({}^{3}/_{a}, {}^{2}/_{a})/6, ({}^{2}/_{a}, {}^{3}/_{c})/6, ({}^{2}/_{b}, {}^{2}/_{c})/2, ({}^{3}/_{c}, {}^{3}/_{c})/9\}$ and

$S = \{({}^{2}/_{a}, {}^{2}/_{b})/4, ({}^{2}/_{b}, {}^{3}/_{a})/4, ({}^{3}/_{c}, {}^{2}/_{b})/6\}$ be a relation on A

Then $SoR = ({}^{2}/_{a}, {}^{2}/_{b})/4, ({}^{2}/_{b}, {}^{2}/_{b})/4, ({}^{3}/_{c}, {}^{2}/_{b})/6\}$ and

$RoS = \{({}^{2}/_{b}, {}^{3}/_{c})/6, ({}^{2}/_{b}, {}^{2}/_{a})/4, ({}^{2}/_{c}, {}^{2}/_{c})/4\}$ proving that $SoR \neq RoS$

It can also be easily proved that $(SoR)^{-1} = R^{-1}oS^{-1}$

4.4 SOME PROPERTIES OF MULTISET RELATIONS

Similar to defining some particular properties of a relation on a set, we can define them on an mset as well. Let R be an mset relation on A

(i). *Void mset Relation*: as any relation on an mset A is a subset of $A \times A$ and the empty mset $\emptyset$ is a unique submset of $A \times A$, it is called the Void mset relation. However, the way an mset relation is defined in 4.1.2, the emergence of empty mset relation is excluded.

(ii). *Identity mset relation*: the identity on an mset A denoted by I_A, is the mset of all pairs having equal coordinates in $A \times A$. Such a relation keeps every element of A fixed.

(iii). *Universal mset relation*: The universal mset relation on an mset A is simply the multiset $A \times A$ itself. Follows that I_A is the diagonal of $A \times A$.

(iv). *Reflexive mset relation*: R is called reflexive if $({}^{m}/_{x}, {}^{m}/_{x}) \in R$ for every ${}^{m}/_{x}$ in A. Note that I_A is contained in any R on A which is reflexive. Follows that if R is reflexive, then $R \cap R^{-1} \neq \emptyset$.

(v).Symmetric mset relation: R is called symmetric if $({}^{m}/_{x}, {}^{n}/_{y}) \in R$, then $({}^{n}/_{y}, {}^{m}/_{x}) \in R$ for all ${}^{m}/_{x}, {}^{n}/_{y} \in A$. Follows that R is symmetric if and only if $R \cap R^{-1} \neq \emptyset$

(vi). *Transitive mset relation*: R is called transitive on A if $({}^{m}/_{x}, {}^{n}/_{y}) \in R$ and $({}^{n}/_{y}, {}^{p}/_{z}) \in R$, then $({}^{m}/_{x}, {}^{p}/_{z}) \in R$, for all ${}^{m}/_{x}, {}^{n}/_{y}, {}^{p}/_{z} \in A$

(vii). *Equivalence mset relation*: R is called an equivalence relation on A if it is reflexive, symmetric and transitive on A.

For example, let $A = [{}^{4}/_{x}, {}^{5}/_{y}, {}^{6}/_{z}]$ and

$R =$

$\{({}^{4}/_{x}, {}^{4}/_{x})/16, ({}^{4}/_{x}, {}^{6}/_{z})/24, ({}^{6}/_{z}, {}^{4}/_{x})/24, ({}^{5}/_{y}, {}^{5}/_{y})/25, ({}^{6}/_{z}, {}^{6}/_{z})/36, \},$

then R is an equivalence relation on A. Blizard (1989) and Girish and Sunil (2009)

4.5 MATRIX REPRESENTATION OF MULTISET RELATIONS

Similar to the representation of a relation on a set by a matrix or graph, an mset relation can also be represented.

Let X^n be a cardinality-bounded mset space generated by the elements of a given generic set X, and let R be an mset relation on $A \in X^n$. If we assume that the elements

of X appear in a certain order, then R can be represented by a matrix whose elements are mn if $({}^{m}/_{x}, {}^{n}/_{y}) \in R$ and $0s$ if $({}^{m}/_{x}, {}^{n}/_{y}) \notin R$. Note that a relation R from mset A to mset B will have $|A|$ rows and $|B|$ columns; mn in row i and column j of the matrix means that the relation R holds for the ith element ${}^{m}/_{x}$ of A and jth element ${}^{n}/_{y}$ of B; "0" means the relation does not hold.

For example let $A = [{}^{4}/_{x}, {}^{5}/_{y}, {}^{6}/_{z}]$ and R an mset relation defined on A by $R =$

$$\left\{({}^{4}/_{x}, {}^{4}/_{x})/16, ({}^{4}/_{x}, {}^{6}/_{z})/24, ({}^{6}/_{z}, {}^{4}/_{x})/24, \left({}^{5}/_{y}, {}^{5}/_{y}\right)/25, ({}^{6}/_{z}, {}^{6}/_{z})/36\right\}$$

the matrix representation of R is as follows

$$\begin{bmatrix} 16 & 0 & 24 \\ 0 & 25 & 0 \\ 24 & 0 & 36 \end{bmatrix}$$

It may be noted that if A contains 3 objects, then the relation matrix is 3×3.It can also be represented in a tabular form as follows:

	x	y	z
x	16	0	24
y	0	25	0
z	24	0	36

As in the case of matrix representation of a relation on a set A it is possible to draw some of the properties of an mset relation from its matrix. For example, if a relation R

on an mset A is reflexive, then all the diagonal entries in its matrix must be non zero squared numbers. However, this is only a necessary condition as can be checked by following the definition. On the other hand, if R is irreflexive, then all the diagonal entries must be zero. R may be neither reflexive nor irreflexive. Similarly, if a relation is symmetric, then both $\binom{m}{x}, \binom{n}{y}$ and $\binom{n}{y}, \binom{m}{x}$ must appear. For R being antisymmetric, no such dual pairs appear.However, as in the case of a relation in a set, it is not straightforward to determine from the matrix of an mset relation R, whether it is transitive.It may also be noted that if the ground set X is large and R defined on $A \in X^n$ is required to possess many properties, both the graphical and matrix representations of R become unwieldy. However, the matrix representation in contrast to its graph can be obtained on a computer, simply because computers are not good in looking at pictures.

It is straightforward to see that aforesaid matrix representation of an mset relation in an mset A can as well be extended to a composition of two relations on A. For an illustration, let us consider the example discussed at the end of the section 4.3. We have as follows:

$$M_R = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix}, M_S = \begin{bmatrix} 0 & 0 & 6 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$M_{SoR} = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

Similarly

$$M_{RoS} = \begin{bmatrix} 0 & 0 & 0 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 6 & 0 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

Note that, akin to the technique used for constructing the matrix of a composite set relatrions, Boolean product of the matrices of the two mset relations is computed. For example, the entry in the second row and third column of $M_{R\circ S}$ is given by {(6 and 6) or (4 or 4) or (0 and 9)}= {6 or 4 or 0}= 6.

Note that the motivation behind representing the occurrence of $(m/x, n/y)$ by the number mn in the aforesaid matrix representation is just to indicate that an mset relation are in consideration ,otherwise the same $0-1$ matrix used for representing relations in sets does work.

4.6 CLOSURES OF MULTISET RELATIONS

If a relation R on a multiset M fails to have a certain property, we may be able to extend R to a relation R^* on M that does have that property. R^* is called the closure of R. i.e. Inducing a property on a relation is called its closure. Intuitively, a closure of an mset relation R denoted by R^* is the smallest extension of R to achieve a certain property.

Given an mset A and a relation R on A. We can calculate the reflexive, symmetric and transitive closure of R using the following formular.

(1) Reflexive closure can be calculated by adding (a,a) to R for every $a \in A$ unless (a,a) is already in R

For example $let\ A =$ $[5/x, 7/y, 5/z]$ be an mset and

$$R = \left\{\left(5/x, 3/y\right)/15, \left(4/x, 3/z\right)/12, \left(7/y, 5/z\right)/35, \left(5/z, 5/z\right)/25\right\}$$

Clearly, R is not reflexive. Thus we have closure of R which is reflexive by adding (a, a) to R for every $a \in A$ except (a, a) is already in R

$R^* =$

$$\left\{\left({}^{5}/_{x}, {}^{5}/_{x}\right)/25, \left({}^{5}/_{x}, {}^{3}/_{y}\right)/15, ({}^{4}/_{x}, {}^{3}/_{z})/12, ({}^{7}/_{y}, {}^{7}/_{y})/49, \left({}^{7}/_{y}, {}^{5}/_{z}/35\right), \left({}^{5}/_{z}, {}^{5}/_{z}\right)/25\right\}$$

(2) The symmetric closure of R can be calculated by adding (b, a) to R for every $(a, b) \in R$ unless (b, a) is already in R. That is we can create symmetric closures using the inverse of a relation i.e $R \cup R^{-1}$ where

$R^{-1} = \{(b, a); (a, b) \in R\}^{\cdot\cdot}$

For example:

Let $A = [{}^{4}/_{x}, {}^{5}/_{y}, {}^{6}/_{z}]$ and $R = \left\{({}^{4}/_{x}, {}^{6}/_{z})/24, \left({}^{5}/_{y}, {}^{5}/_{z}\right)/25, ({}^{6}/_{z}, {}^{4}/_{x})/24\right\}$.

Then the symmetric closure of R is

$$R^* = \left\{({}^{4}/_{x}, {}^{6}/_{z})/24, \left({}^{5}/_{y}, {}^{5}/_{z}\right)/25, ({}^{6}/_{z}, {}^{4}/_{x})/24, \left({}^{5}/_{z}, {}^{5}/_{y}\right)/25\right\}$$

(3) Transitive closure of an mset relation R on an mset A can be calculated by finding the union of all compositions of R with itself up to n times where n is the number of elements in A.

For example

Let $A = [{}^{3}/_{x}, {}^{4}/_{y}, {}^{2}/_{z}]$ and

$$R = \{({}^{3}/_{x}, {}^{3}/_{x})/9, ({}^{2}/_{x}, {}^{3}/_{y})/6, ({}^{3}/_{y}, {}^{2}/_{z})/6, ({}^{4}/_{y}, {}^{2}/_{x})/8\}$$

Then the transitive closure of

$$R = R \cup R^2 \cup R^3$$

$$\therefore R^* = \left\{ \begin{matrix} ({}^3/_x, {}^3/_x)/9, ({}^2/_x, {}^2/_y)/4, ({}^3/_y, {}^2/_x)/6, ({}^2/_x, {}^2/_z)/4, \\ ({}^2/_y, {}^2/_y)/4, ({}^2/_y, {}^2/_z)/4 \end{matrix} \right\}$$

4.7. MULTISET FUNCTIONS

Akin to defining a set function, the underlying assumption for defining an mset function between two msets is not to allow mapping of indistinguishable elements to distinguishable elements; however images of distinct elements of the domain need not be distinct elements of the range. In turn, it amounts to defining a function between the root sets of the two msets involved along with some consistency conditions observed.

4.7.1. Definition : mset Functions

Let A and B be msets. An mset relation f from A to B sometimes denoted as

$f: A \to B$ *or* $A \xrightarrow{f} B$, is said to be an mset function if for every element ${}^m/_x \in Dom\, f$ there exists exactly one element ${}^n/_y \in ran\, f$ such that $({}^m/_x, {}^n/_y) \in f$ satisfying the consistency condition that the pair (x, y) occurs only $C_1(x, y)$ times.

For example, let $A = \{{}^3/_x, {}^5/_y\}$ and $B = \{{}^2/_p, {}^6/_r\}$ be mset. Then,

$f = \{({}^3/_x, {}^6/_r)/3, ({}^5/_y, {}^2/_p)/5\}$ is an mset function from A to B.

4.7.2. Definition : Injective mset Functions

The mset function $f: A \to B$ is an injection (one-one) if

(i) $f: A^* \to B^*$ is an injection, and

(ii) $\forall z(z \in A^* \Rightarrow m_A(x) \leq m_B(f(z)))$

Thus, distinct elements of the domain under an injective mset functions are mapped into distinct elements of the range. That is ${}^{m_1}/_{x_1} \neq {}^{m_2}/_{x_2} \in Dom\, f \Rightarrow f({}^{m_1}/_{x_1}) \neq f({}^{m_2}/_{x_2}) \in ran\, f$. In other words f if one – one if no two elements in $Dom\, f$ have the same image under f with

$C_1(x, y) \leq C_2(x, y)$ for all $(x, y) \in f$.

For example

$f: \{{}^{3}/_{x}\} \to \{{}^{10}/_{y}\}$ defined by $f = \{({}^{3}/_{x}, {}^{10}/_{y})/3\}$ is an injection

4.7.3. Definition: Surjective mset Functions

The mset function $f: A \to B$ is a surjection (onto) if

$f: A^* \to B^*$ is a surjection, and

(i) $\forall z(z \in A^* \Rightarrow m_A(x) \geq m_B(f(z)))$

In other words, an mset function f is onto if

$ran\, f = co - Dom\, f\ with\ C_1(x, y) \geq C_2(x, y)$ for all $(x, y) \in f$

Note, as mentioned earlier that the distinct elements of $Dom\, f$ need not be mapped under f to distinct elements of $ran\, f$

For example,

$f: \{{}^{5}/_{x}\} \to \{{}^{4}/_{y}\}$ defined by $f = \{({}^{5}/_{x}, {}^{4}/_{y})/5\}$ is a surjection

4.7.4. Definition : Bijective mset Functions

The mset function $f: A \to B$ is a bijection (one –one and onto) if

(i) $f: A^* \to B^*$ is a bijection, and

(ii) $\forall z(z \in A^* \Rightarrow m_A(x) = m_B(f(z)))$

In other words, an mset function f is bijective if f s both one-one and onto with $C_1(x, y) = C_2(x, y)$ for every pair $(x, y) \in f$

For example

$f: \{{}^3/_x, {}^5/_y\} \to \{{}^5/_p, {}^3/_r\}$ defined by $f = \{({}^3/_x, {}^3/_r)/3, ({}^5/_y, {}^5/_p)/5\}$ is a bijection

Note also that an mset function may neither be an injection nor a surjection. For example,

$f: \{{}^6/_x, {}^2/_y\} \to \{{}^3/_p, {}^4/_r\}$ defined by $f = \{({}^6/_x, {}^3/_p)/6, ({}^2/_y, {}^4/_r)/2\}$ is such an mset function.

4.7.5. Definition: Constant mset Function

An mset function f is called a "constant" function if all the elements of $Dom\ f$ are mapped into a single element of a $ran\ f$ with $C\ ran\ f(x) = 1$

4.7.6. Definition: Identity Function

An mset function on an mset A is said to be an identity function on A , denoted by I_M or $I: A \to A$ if $I(^m/_x) = {}^m/_x$ with $C_1(x,x) = C_2(x,x)$

4.7.7. Definition : Composite mset Functions

Let A, B, C be msets. Let $f: A \to B$ and $g: B \to C$ be two mset functions. The composition or composite or relative product of f and g, denoted by gof (or fog, depending on the convention one follows) is the mset function defined by $(gof)(x) = g(f(x))$ with $C_1(x,z) = \min\{C_1(x,y), C_1(y,z)\}$ and $C_2(x,z) = \min\{C_2(x,y), C_2(y,z)\}$.

Sometimes if gof is defined , it is called the "left-composition" of g with f or g is called "left-composable".

Moreover, for mset functions $f: A \to A$ and $g: A \to A$ the composite mset functions $fog, gof, fof, gog\ etc$ can be computed. Like mset relations, the composition of mset function is assocaitve.

Note that , unlike for defining composition of relations, the underlying assumption for defining composition of functions is $ran\ f \subseteq dom\ g$; otherwise gof is empty.

Example

Let $f = \{(^3/_x, {}^3/_p)/3, (^4/_y, {}^5/_p)/4, (^3/_z, {}^6/_q)/3\}$

$g = \{(^5/_p, {}^7/_t)/5, (^6/_q, {}^8/_u)/6\}$

Then $gof = \{(^3/_x, {}^3/_t)/3, (^4/_y, {}^5/_t)/4, (^3/_z, {}^6/_u)/3\}$

From above

$Dom\, f = \{^3/_x, ^4/_y, ^3/_z\}$ and $Ran\, f = \{^5/_p, ^6/_q\}$; $Dom\, g = \{^5/_p, ^6/_q\}$

And $Ran\, g = \{^7/_t, ^8/_q\}$

Since $f \subseteq Dom\, g$, is verified that gof is well defined.

Moreover, $Dom(gof) = \{^3/_x, ^4/_y, ^3/_z\}$ and $Ran\,(gof) = \{^5/_t, ^6/_u\}$ this shows that $Dom(gof) \subseteq Dom\, f$ and $Ran\,(gof) \subseteq Ran\, g$ granting the consistency conditions for gof to be defined as composite relation.

4.7.8. Definition: Inverse mset Functions

Let A and B be msets. An mset function $f: A \to B$ is called invertible, it is inverse (or converse) mset relation denoted by f^{-1} is also an mset function. For example, let f be the mset function illustrated in Definition 4.7.1, thus

$f^{-1} = \{(^6/_r, ^3/_x)/3, \left(^2/_p, ^5/_y\right)/5\}$ is an mset function.

It can be easily seen that not every mset function is invertible.

4.8 Propositions

It is encouraging to see that similar to the results holding for set functions, the following hold for mset functions as well.

4.8.1.Proposition

Let $f: A \to B$ be an mset function and let A_1 and A_2 be any nonempty msubsets of A.Then the following holds

(i) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$

(ii) $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$

(iii) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$[the inequality holds only if f is an injection]

(iv) $f(A_1 - A_2) \supseteq f(A_1) - f(A_2)$

4.8.2. Proposition

Let $f: A \to B$ be an mset function. Then the following hold:

(i) $f^{-1}: A \to B$ be an mset function if and only if f is bijective

(ii) If $f^{-1}: A \to B$ be an mset function, then it is also bijective

(iii) $(f^{-1})^{-1}$ is the mset function

(iv) $I_B o f = f$

(v) $f o I_A = f$

(vi) $f^{-1} o f = I_A$

(vii) $f o f^{-1} = I_B$

(viii) If $f: A \to B$ and $g: B \to A$ be invertible mset functions, then the mset function gof is invertible and $(gof)^{-1} = f^{-1} o g^{-1}$

The proofs for the aforesaid sets of prepositions follow from the respective definitions. (See [Pet 76, 81], [Rag 86], [Bil, 89] and [GJ, 2009]).

It is worth noting that some of the consequences of the foresaid formulation of mset functions contrasts with those that hold for set functions. For example;

(1) mset of equal cardinality need not have bijections between them. For example, the mset $\{3/x, 2/y\}$ and $\{4/p, 1/r\}$ are of equal cardinality, but no bijection between

them exist, because the multiplicities of the elements in the domain msets are not equal to the multiplicities of the elements in the range msets.

(2) Schreder-Bornstein theorem fails.

For example, let $A = \{{}^{2}/_{x_1}, {}^{4}/_{x_2}, {}^{6}/_{x_3} \ldots\ldots\}$ and $B = \{{}^{1}/_{y_0}, {}^{3}/_{y_1}, {}^{5}/_{y_2} \ldots\ldots\}$ be msets.

The function $f: A^* \to B^*$ defined by $f(x_n) = y_n$ makes $f: A \to B$ an injection so that $A \leq B$.

The function $g: B^* \to A^*$ defined by $g(y_n) = X_{n+1}$ makes $g: B \to A$ an injection so that $B \leq A$

However, there cannot be a bijection $h: A \to B$ since all multiplicities in A are even and those in B are odd.

See [Hic 80], [Bli, 89] and [GJ 2008] for further elaborations.

CHAPTER FIVE

THEORY OF SET AND MULTISET PARTITIONS.

5.0 INTRODUCTION

The mathematics of partitions plays a significant role in mathematics, computer science(especially in finite state machine and finite-state acceptors, etc.) and linguistics (regular grammars), just to mention a few.

In this chapter, we present the theory of set partitions and multiset partitions.

5.1. Theory of set partitions

Definition 5.1.1. Covering and Partition of a set

Let S be a given set and $A = \{A_1, A_2, \dots, A_m\}$ be a family of sets where each $A_i, i = 1, 2, \dots, m$ is a nonempty subset of S and $\bigcup_{i=1}^{m} A_i = S$.

Then the set A is said to be a "covering" of S, and the sets $A_1, A_2, \dots A_m$ are to cover S. If, in addition, the members of A are mutually disjoint, then A is said to be a "partition" of S, and the sets $A_1, A_2, \dots A_m$ are called the "blocks" of the partition. In other words, a partition of S is a disjoint union of nonempty sets whose union is S. It follows from above that every partition is also a covering.

For example, Let $S = \{1,2,3\}, A = \{\{1,2\},\{2,3\}\}, B = \{\{1\},\{1,3\}\}, C = \{\{1\},\{2,3\}\}$, $D = \{\{1,2,3\}\}$ and $E = \{\{1\},\{2\},\{3\}\}$. Then, the set A is a covering and not a partition of S , the sets C, D and E are partitions of S;the set B is neither a covering nor a partition of S; D is the smallest partition of S having only one block, the set itself; and the set E is the largest partition having blocks containing only single

elements. Infact, for any finite set S with n (distinct) elements, every partition of S if finite, the smallest partition of S consists of S itself as the only block, and the largest partitions of S consists of n blocks each containing only single elements. Note that as partitions are defined as sets, it follows that certain set theoretic relations can be defined on the class of partitions of a set.

Definition 5.1.2: the set of R- relatives where R is a relation on a set X.

The set of $R-$ relatives (or R-mates) of an element $x \in X$, denoted by $R(x)$ is the set of all $y \in X$ such that $x\ R\ y$. In the same vein,R-relative set of all $x \in X$, denoted by $R(X)$, is simply the range of $R\ = \{y \mid y \in X\ \wedge\ \exists\ x \in X$ such that $x\ R\ y\ \}$.

Definition 5.1.3: $R-$Equivalence classes

Let R be an equivalence relation on a set X. For $x \in X$, the set$[x]_R \subseteq X$, defined by

$[x]_R = \{y \mid y \in X\ \ xRy\}$ is called an R-equivalence class generated by x.

Thus, $[x]_R$, also written as $^{x}/_{R}$, is the class of all $R-$relatives of $x \in X$. The family of all equivalence classes in this case, also called the "quotient set of X by R", or "X modulo R" is denoted is denoted by $^{X}/_{R}$

Theorem 5.1.1: Fundamental theorem of partition

Every equivalence relation in set gives rise to a unique partition of the set and conversely, every partition induces an equivalence relation on that set. The blocks of the partition corresponds to the $R-$equivalence classes.

See [Christian *et al*, 2004] for details.

Definition 5.1.4: Complete intersection (or the minterms) and some partitions of the universal set.

Any formula (or expression) made up of n −distinct variables which consist of conjunctions (or intersections) of either a variable or its negation (and not both) is called a Boolean conjuction (minterm or complete intersection).

For example, let A and B be any two subsets of the universal set E. Then $\cap B$, $A \cap \sim B$, $\sim A \cap B$ and $\sim A \cap \sim B$ are all possible minterms generated by the subsets A and B. They are mutually disjoint and their union is E

In general, if $A_1, A_2, \ldots A_n$ are any n subsets of the universal set E, then the complete intersections or minterms generated by these n subsets are 2^n. They are mutually disjoint and their union is E. We note that in order to make use of the aforesaid notion, it would be convenient to consider the elements of E ordered in some way as it is usually done in representing a set on a computer. For example, the minterms generated by three subsets A, B and C of the universal set E can be written as

$$I_o = \sim A \cap \sim B \cap \sim C, \quad I_1 = \sim A \cap \sim B \cap C, \quad I_2 = \sim A \cap B \cap \sim C$$

$$I_3 = \sim A \cap B \cap C$$

$$I_4 = A \cap \sim B \cap \sim C \qquad I_5 = A \cap \sim B \cap C \quad I_6 = A \cap B \cap \sim C$$

$$I_{2^3-1} = A \cap B \cap C$$

In general $I_o, I_1, \ldots I_{2^n-1}$ can be used to denote all the mutually disjoint minterms generated by n −subsets of the universal set E such that = $\bigcup_{k=0}^{2^n-1} I_k$.

Thus, the complete intersection or the minterms generated by a finite number of sets define a partition of the universal set E and they are called the blocks of the partition of E.

It is worth to note here that in order to obtain a minterm represented by I_K, k = $0,1,\ldots 2^n - 1$ we need to write the subscript k as a binary integer containing n digits (the number of subsets generating the minterms)

For example, in order to write I_7 in the above example with three subsets of E, we need to write `7` as "111", a binary integer containing three digits corresponding to the number of subsets.

5. 2 Theory of Multiset Partitions

Definition on 5. 2. 1 The mset of *R*–relatives

We describe it as a generalization of our definition 5.2.2

Let *R* be an mset relation on an mset *A*. Let $^m/_x \in A$. The mset of *R*–relatives (or *R*–mates) of an element $^m/_x \in A$, denoted by *R*($^m/_x$), is the mset of all $^n/_y \in A$ such that there exists some *k* satisfying $^k/_x$ R $^n/_y$. For example, let *A* = $\left\{ {}^5/_x \, , \, {}^7/_y , \, {}^9/_z \right\}$ be an mset and

R =

$\left\{ ({}^2/_x , \, {}^3/_y), (\, {}^4/_x \, , \, {}^3/_z), \left(\, {}^4/_y \, , \, {}^5/_z \right), \left({}^6/_y \, , \, {}^5/_x \right), (\, {}^3/_z \, , \, {}^6/_y), (\, {}^4/_z , \, {}^4/_z), \left({}^5/_z , \, {}^3/_x \right) \right\}$

be an mset relation on *A*. Then, dom *R* = $\left\{ {}^4/_x , \, {}^6/_y , \, {}^5/_z \right\}$ and ran *R* = $\left\{ {}^5/_x , \, {}^6/_y , \, {}^5/_z \right\}$.

In this case, $R(\, {}^2/_x) = R(\, {}^4/_x) = \{ \, {}^3/_y , \, {}^3/_z \}$,

$R(\,^{4}/_{y}) = R(\,^{6}/_{y}) = \{\,^{5}/_{x}, \,^{5}/_{z}\}$, and

$R(\,^{3}/_{z}) = R(\,^{4}/_{z}) = R(\,^{5}/_{z}) = \{\,^{3}/_{x}, \,^{6}/_{y}, \,^{4}/_{z}\}$.

The aforesaid definition can be suitably extended to defining the mset of all the *R*–mates of a submset of a given mset as follows:

Let $A_1 \subseteq A$. Then $R(A_1)$, the *R*–relative mset of A_1, is the mset containing all those $^{n}/_{y} \in A$, which are *R*–related to some $^{k}/_{x} \in A_1$. Follows that $R(A_1)$ is the union of all $R(\,^{m}/_{x})$ where $^{m}/_{x} \in A_1$. In fact, $R(A_1) \subseteq$ ran *R*, and equality holds if and only if both A_1 and *A* have the same root set (i.e., they contain the same objects with different multiplicities). Note that for any submset A_k of *A* the mset $R(A_k)$ is defined with respect to the relation R on *A*.

For example, let us consider the above example. Let $A_1 = \{\,^{2}/_{x}, \,^{4}/_{y}, \,^{5}/_{z}\} \subseteq A$, where $A_1^* = A^*$. Then, $R(A_1) = \{\,^{5}/_{x}, \,^{6}/_{y}, \,^{5}/_{z}\}$ = The union of all $R(\,^{m}/_{x})$ for $^{m}/_{x} \in A_1 = R(\,^{2}/_{x}) \cup R(\,^{4}/_{y}) \cup R(\,^{5}/_{z}) = R(A) =$ ran *R*.

However, if the root set of an msubset A_k of *A* is a proper subset of the root set of *A,* then $R(A_k) \subsetneq$ ran *R*.

For example, let $A_2 = \{\,^{4}/_{x}, \,^{6}/_{y}\} \subseteq A$. Then

$R(A_2) = R(\,^{4}/_{x}) \cup R(\,^{6}/_{y}) = \{\,^{3}/_{y}, \,^{3}/_{z}\} \cup \{\,^{5}/_{x}, \,^{5}/_{z}\} = \{\,^{5}/_{x}, \,^{3}/_{y}, ^{5}/_{z}\} \subsetneq$ ran *R*.

Definition 5. 2. 2 Multiset Partitions

A partition of a nonempty mset *A* is a collection π of nonempty msubsets of *A* such that

i. each element of A belongs to one of the elements in π, and

ii. $A_1 \cap A_2 = \emptyset$ for any distinct elements A_1 and A_2 of π.

The elements in π are called the *cells* or *blocks* of π.

For example, let $A = \{ {}^{5}/_{x}, {}^{7}/_{y}, {}^{9}/_{z} \}$. Then $\pi = \{ \{ {}^{5}/_{x}, {}^{7}/_{y} \}, \{ {}^{9}/_{z} \} \}$ is a partition of A.

The cells of π are $\{ {}^{5}/_{x}, {}^{7}/_{y} \}$ and $\{ {}^{9}/_{z} \}$.

Proposition 5. 2. 1

Every partition of an mset A gives rise to an equivalence relation on A

Proof

Let π be a partition of A. Let us define an mset relation R on A as follows:

${}^{m}/_{x}\ R\ {}^{n}/_{y}$ if and only if ${}^{m}/_{x}$ and ${}^{n}/_{y}$ belong to the same block. It is easy to see that R, by definition, is reflexive: ${}^{m}/_{x}\ R\ {}^{m}/_{x}$ for each ${}^{m}/_{x} \in A$;

Symmetric: ${}^{m}/_{x}\ R\ {}^{n}/_{y}$ implies ${}^{m}/_{x}$ and ${}^{n}/_{y}$ belong to the same block of π and that implies ${}^{n}/_{y}\ R\ {}^{m}/_{x}$; and

Transitive: ${}^{m}/_{x}$ R ${}^{n}/_{y} \wedge {}^{n}/_{y}\ R\ {}^{p}/_{z}$ implies ${}^{m}/_{x}$, ${}^{n}/_{y}$ and ${}^{p}/_{z}$ belong to the same block of π and that implies ${}^{m}/_{x}\ R\ {}^{p}/_{z}$.

Such an equivalence mset relation R is said to be determined by the partitioned π.

Considering the example taken above, the partition π determines the equivalence mset relation

$R = \{ ({}^{5}/_{x}, {}^{5}/_{x}), ({}^{5}/_{x}, {}^{7}/_{y}), ({}^{7}/_{y}, {}^{5}/_{x}), ({}^{7}/_{y}, {}^{7}/_{y}), ({}^{9}/_{z}, {}^{9}/_{z}) \}$.

Remark 5. 2. 1

An alternative formulation of an mset partition π induced by an equivalence mset relation R on A can be given as follows: Using the definition 5.3.1, it is easy to see that for any equivalence mset relation R on an mset A, if $x \in^m A$ and $y \in^n A$, then $^m/_x \; R \; ^n/_y$ if and only if $R(^m/_x) = R(^n/_y)$.

Hence, $\pi = \{R(^m/_x) \setminus x \in^m A\}$.

That is, π turns out to be a collection of all distinct R–relative msets that are represented by elements of A.In the example taken above, the blocks $\{^5/_x, ^7/_y\}$ and $\{^9/_z\}$ of the partition π are respectively represented as

$R(^5/_x) = R(^7/_y) = \{^5/_x, ^7/_y\}$, and $R(^9/_z) = \{^9/_z\}$.

Note that the aforesaid representation of the blocks of a partition is not unique.

Definition 5. 2. 3 m- Equivalence classes.

Let R be an equivalence mset relation on an mset A. Then the msets $R(^m/_x)$ are said to be the m–equivalence classes determined by R and is denoted by $^A/_R$. The partitions of A are called quotient msets of A. For example, the partition π described in the example taken above is the quotient mset of A that, in turn, determines R.

CHAPTER 6

SUMMARY OF THE WORK AND FUTURE DIRECTION

6.1 SUMMARY OF THE WORK

In chapter 1, motivation, objective and methodology are presented. In chapter 2, an up-to-date literature review is presented. In chapter 3, basic notions required to conduct the research are introduced. In chapter 4, we present the theory of multiset relations and functions. In particular, we have developed a matrix representation method for various multiset relations. In chapter 5, the theory of multiset partitions is described. In this part of the thesis, the theory of set partition, providing some very useful techniques, is described in order to help in developing the theory of multiset partitions.

Summarily, Chapters 4 and 5 constitute the main content of the thesis.

6.2 SOME DIRECTIONS FOR FUTURE WORK

A number of problems related to the study of dynamics of exploiting a particular transformation on families of finite sets and msets are yet to be investigated (see [DT 03] and [Lat 00] for some initial details).

Despite the fact that no closed-formula solution for enumerating the partitions of a random set is available (see[SM 10],for instance), researches are being conducted (see[Ben 74,84],[Cle 88] and [Bli 93]) to obtain a nice asymptotic formula for enumerating the total number of partitions of a multiset. Partitioning of multisets can be exploited to define multiset orderings [JL 82]

REFERENCE

[1] Apostolos, S.(2001), *Mathematics of Multisets*, Multiset processing, LNCS 2233, 347-358.

[2] Bender, E. A. (1974), Partitions Of Multisets, *Discrete Mathematics,* Vol. 9,301-311.

[3] Bender, E. A., Devitt, J. S.and Richmond, L. B.(1984), Partitions of multisets II, Discrete Mathematics,VOl. 50, 1-8,North-Holland.

[4] Blizard, W. (1989), Multiset theory, *Notre Dame Journal of formal logic* ,Vol. 30, 36-66 .

[5] Blizard, W.(1991), The development of multiset theory, *Modern Logic*,Vol.1, 319-352

[6] Blizard, W.(1993), Dedekind multisets and function shells, *Theoretical Computer Science*,Vol.110 ,79-98.

[7] Cerf, E.F., Gostelow, K. and Volausky, S.(1971), Formal control and low properties of a model of computation, Report ENG 7178,Computer science Department, University of California, Los Angeles,CA, 81.

[8] Christian, B., Emmanuel, H., Brahim, H., and Toby, W.(2004). *Disjoint, Partition and Intersection Constraints for Set and Multiset Variables*, Proc. Apes-84.

[9] Clements, G. F.(1988), On multisets k-families, *Discrete Mathematics,* Vol. 69, 153-164.

[10] Ernesto, S., and Stefano, V.(2007),*On Multidimensional inequality in partitions of*

Multisets, A Journal of Economic theory, No 504 .

[11] Girish, K. P. and Sunil, J. J.(2008), *Relations and Functions in Multiset context*, Information Sciences, Vol. 179, Issue 6 ,758-768.

[12] Hickman, J. L.(1980), A Note on the concept of Multiset. A Bulletin of the Australian Mathematical Society Vol. 22, 211-217.

[13] Jouannad, J. P. and Lescanne, P. (1982), On Multiset Orderings, Information Processing Letters, 15: 57-63

[14] Peter, H. and Urban, L. (2006), Permutations of the Natural Numbers with Prescribed Difference Multisets,Electronic Journal of Combinatorial Number Theory, Vol.6 , No 03.

[15] Peterson,J L. (1976), Computional Sequence Sets, Journal of Computer system Sciences, Vol 13(1),1-24.

[16] Singh, D.(2006). *Multiset Theory, A New Paradigm of Science,* Senate Inaugural lecture series, Ahmadu Bello University, Zaria publication, 1-34.

[17] Singh, D., Ibrahim, A. M., Yohanna, T. and Singh, J. N.(2007), *An Overview of the Applications of multisets*, Novi Sad Journal of mathematics Vol. 37 No 2, 73-92.

[18] Singh, D., Ibrahim, A. M., Yohanna, T. and Singh, J. N.(2008), *A Systemization of Fundamentals of Multisets*, Lecturas Mathematicas,Vol. 29, 33-48.

[19] Singh, D., Dunari, M. H., and Singh, J. N. (2010), A Note on Covering And Partition of Finite set, Journal of Mathematical Sciences and Education, VN.5, No.2, 1-5

[20] Singh, D., Ibrahim, A. M., Salihu, A. O and Singh, J. N.(2011), Multiset relations and Their matrix representation, Journal of Vijanana Parishad of India, Accepted for Publication.

[21] Wildberger, N. J.(2003), A new look at multiset, School of Mathematics, UNSW Sydney 2052, Australia.

[22] Yager, R. R (1996), On the theory of bags, International journal of General system, Vol 13, 23-37.