CONSTRUCTION AND ANALYSIS OF STOPPING TIMES AND BELATED INTEGRALS ON A FILTERED PROBABILITY SPACE

BY

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DECLARATION

I hereby declare that the work in this dissertation titled "**CONSTRUCTION AND ANALYSIS OFSTOPPING TIMES AND BELATED INTEGRALS ON A FILTERED PROBABILITY SPACE"** was performed by me in the Department of Mathematics under the supervision of Professor A. A. Tijjani,Professor G. U. Garba and Professor Babangida Sani. The information derivedfrom the literature has been duly acknowledged in the text and a list of references provided. No part of this work has been presented for another degree or diploma at this or any other institution.

CERTIFICATION

This thesis titled **CONSTRUCTION AND ANALYSIS OF STOPPING TIMES AND BELATED INTEGRALS ON A FILTERED PROBABILITY SPACE**meets the regulations governing the award of the degree of Doctor of the Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to my late parents

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ABBREVIATIONS DEFINITIONS AND SYMBOLS

a.s:-	almost surely
a.e.:-	almost everywhere
a.u.:-	almost uniformly
masa:-	maximal abelian subalgebra
w.p.1 :-	with probability 1
CCR :-	canonical commutation relation
CAR :-	canonical anti-commutation relation
a':-	a factor or a commutant of the set of the operators a
$a(f)$:-	a fermion operator
Z(M) :-	the centre of the algebra M
$A(F)$:-	σ-algebra of the family F_t
F_t :-	an increasing family of F
F :-	a family of subsets of a set Ω
Ω:-	a non-empty set
∅ :-	an empty set
A^c :-	a compliment of the set A
ℬ(ϒ) :-	Borel sets
ϒ :-	set of real numbers
$ϒ^+$;-	positive real or extended real numbers
∞ :-	set of natural numbers
≤ :-	set of complex numbers
K:-	field of scalars

$\Pi(\Omega)$:-	partition of Ω
$\mu(A)$:-	a measure on a set A
P;-	a probability measure
p :-	a projection
Γ :-	a gauge space
L^{∞} :-	the set of all bounded random variables
$\uparrow$:-	increases monotonically (to)
I_A ;-	indicator of the set A
W(t) :-	a Brownian motion or a Wiener process
$X_t = X(t)$:-	a stochastic process at time t
$M^{\perp}$:-	an orthogonal set of M
M :-	a von Neumann algebra or a C^* - algebra
W^0 :-	the set of all annihilators of the se W
T^* :-	the adjoint of an operator T
tr :-	a trace
$\mathcal{H}$:-	a Hilbert space
$\mathcal{B}(\mathcal{H})$:-	the space of all bounded operators on $\mathcal{H}$
$\mathcal{H}_1 \otimes \mathcal{H}_2$:-	the tensor product of two Hilbert spaces
$\bigoplus_{\Omega} \mathcal{H}_i$:-	Hilbert spaces direct sum indexed by Ω
$\mathcal{H}^{\otimes n}$:-	n fold tensor product of $\mathcal{H}$
$\mathcal{H}^{\& m}$:-	symmetric tensor product of $\mathcal{H}$
$\mathcal{H}^{\square n}$:-	anti-symmetric tensor product of $\mathcal{H}$
: :-	an equivalence relation
° :-	a subordinate relation

! :-	modular operator
σ_t :-	automorphism operator indexed by $t \in$ T (a time horizon)
Ψ_t :-	a Clifford process indexed by t
τ :-	a stopping time
$E(A \mid B)$:-	expectation of A given B
E_{t} :-	expectation at time t
E_{t} :-	a projection-valued operator or time projection associated with τ
$X_{\mathrm{t}(v)}$:-	a quantum stoppingprocess
$\mu = \mu^{+} + \mu^{-}$:-	a signed measure
F :-	collection of all sets that are finite unions of intervals
$\lVert \mathcal{E} \rVert^{b}$:-	right semivariation of $\mathcal{E}$
μ^{b} :-	μ - flat measure
θ :-	control measure
$\exists$:-	there exists
$\forall$:-	"for all" or "for every" or "for each"
$\in$:-	"belongs to" or "is a member of"

ABSTRACT

Using an intuitive definition of a 1-0-process, a bijection is established between stopping times and adapted processes that are non-decreasing and takevalues 0 and 1. In the theory of stopping time σ-algebra and its minimal elements on a filtered probability space, the σ-algebra of the minimal elements of the stopping times is shown to coincide with the stopping time σ-algebra. A defined stochastic process relative to a stopping time is proved to be a stopped process.In the belated integral theory, it is established that if two processes are μ - flat integrable then their product is also μ-flat integrable and the integral of their product is the product of the integrals.

CHAPTER ONE

GENERAL BACKGROUND

1.1 Preamble

In the seventh century, the theory of probability began in an attempt to calculate the odds of winning in certain games of chance. Mathematicians, in the middle of twentieth century, developed general techniques for maximizing the chances of beating a casino or winning against an intelligent opponent. There is a leavable gambling problems, in which a player can halt a play at anytime, and unleavable problems, in which a player is compelled to continue playing forever, Doob, (1971). In a leavable problem, a player must choose, in addition to a strategy, a rule for stopping. In essence, a decision to stop at anytime τ will be allowed to depend on the partial history of states up to that time but not beyond it. So, a stopping time is thus a mapping from the set of histories. In probability theory, a stopping time is often defined by a stopping rule, a mechanism for deciding whether to continue or to stop a process on the basis of the present position and past events, and which will always lead to a decision to stop at some finite time.

On the other hand, an integration theory relating to the non-commutative stochastic integral is set up by using a particular field consisting of finite unions of intervals. This integration called the belated integral will be over all $\mathbb{R}^+$, since the measure involved will be bounded.

1.2 Statement of the problem

In any classical stopping time defined on a filtered probability space, there may be a defined stopping process associated with such time. Here, there is the problem of establishing the one-one correspondence between the stopping time and the associated stopping process if it exists.On the same filtered probability space, there is the problem of construction of minimal σ-algebra of stopping time and its relation with the σ-algebra of the stopping time itself.In the algebra of the continuous and bounded linear operators, commutativity is not always guaranteed. Here, there is the problem of construction of two processes whose μ-flat integrals of product is equal to the product of their μ-flat integrals.

Basically, this study is undertaken with a view to answer certain problems emanating from the literature of stopping times and their algebras. Investigation is carried out in the theories of belated integrals. For the above purposes background surveys of the theories are necessary. Primarily, from the review of the literature in chapter two and the studies of measures and processes in chapter three, results are established in chapters four and five.

1.3 Justification/ Significance of the Study

Scholars such as Sinelnikov, (2012) and other contemporaries approached the theory of 1-0-process in a different construction. It is wished that a different approach can be taken to arrive at another construction. Also,followingBarnett and Wilde, (1986)and the improved McShane's division by Toh and Chew, (1999), some analysis and construction are possible.

1.4 Aim and Objectives

The main aim of this work is to study existing theories of stopping times, their algebras and contribute to the literatures thereof through the establishment of new results. This can be achieved through the following objectives:

1. To construct a stochastic process relative to a stopping time so that such a stochastic process can be a stopped process.
2. To construct a bijection between a stopping time and a relative stopping process.
3. To analyze the algebra of stopping times,construct the algebra of its minmal elements and compare the collection with the σ-algebra as a whole.
4. To construct two essentially bounded μ-flat integrable processes evaluate the integrability of their product.

1.5 Research Methodology

The methodology adopted to realize the above aim and set objectives is such that in the first case the use of characteristic function is carried out to define a stochastic process relative to a stopping time and later it is shown that it is a stopping process. In the second, the intuitive definition of 1-0-process is used to construct the bijective relation between a stopping time and a stopping process. Similar technique is employed with the use of σ-algebra of stopping times to obtain the desired result. μ-flat integrability is considered upon an essentially bounded process and a new relationship is established.

1.6 Organization of the Dissertation

This dissertation consists of six chapters with the first concentrating basically on the statements of the problems, aim and objectives and the significance of the studies. The second chapter is primarily the literature review, definitions and other mathematical tools relevant to the research. Measures and stochastic processes are studied in chapter three. Measures of the Riemann Stieltjes type, probability measures and the various stochastic processes are studied. Definition of the stochastic processes culminated into the definition of the stochasticintegrals, sometimes called the Ito integrals are provided. The stochasticity in the Hilbert space together with the concept of Clifford calculus, where some operators are chosen to behave in somewhat the way as in the classical theory has been studied. Stochasticity in quantum sense, that is, the non-commutativity situation which is later used to define integration in the same sense is also attended to. Boson and Fermion Fock spaces whose definitions centered around the concept of direct sum and tensor product has also been discussed.The stochastic processes which are families of random variables on a probability space referred to as *commutative* processes, that is the classical case, and those which are families of (possibly unbounded) operators on Hilbert space referred to as *non-commutative* processes, the no-classical case are also studied. The constructions leading to the Ito-Clifford integrals especially by Barnett *et al* (1982) centered on the square integrable L^2 spaces are studied. The generalization of such construction by Carlen and Kree (1998) to $L^1$is also studied.

Stopping times are studied in chapter four. Both the classical and the quantum theories are considered. Constructions are put in place in the classical theories. Conditional expectations and the concept of martingales are redefined in quantum sense. In the quantum theory, it is noted that stopping times or random times are projection-valued function adapted to a filtered von Neumann algebra.

A study of belated integrals is undertaken in chapter five, where the theoretically admissible integrals that are rather narrow are widened.A consideration of the right belated variation on Borel sets rather than the usual left-handed open intervals is discussed. In this way, the theory is expanded and a result is proved.

Summary of the whole dissertation and its conclusion follow in chapter six together with recommendations. Some contributions are alsohighlighted while some areas of possible research are enlisted.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Basically the notion of functionalspaces and their convergenceis integral in the study of functional analysis. The convergence may occur in mean, in norm and may be uniform or point-wise, strongly or weakly and so on.Although these notions are wide, some few relevant ones and other theoretical surveys can be studied.

2.1.1 Functional spaces

A mapping $f : X \to C$ from a set X into the scalar field C is said to be a functional. The set of all such mappings is said to form a *functional space*. It can be recalled that a mapping from a normed space X into itself or into a normed space Y is called an *operator*. Some types of convergence in respect of functionals may be considered.

2.1.2 Uniform convergence

A sequence of functions $\{f_n\}$ is said to converge uniformly on D ⊂□ if given $\varepsilon > 0$, ∃ N™□ such that $|f_n(x) - f(x)| < \varepsilon$, for all n™ N. Geometrically the entire graph of$f_n$and $f$will lie within the strip bounded by $f + \varepsilon$ and $f - \varepsilon$ below and above respectively, irrespective of the choiceof $x$as illustrated in figure 2.1. [Ponnusamy, 2000].

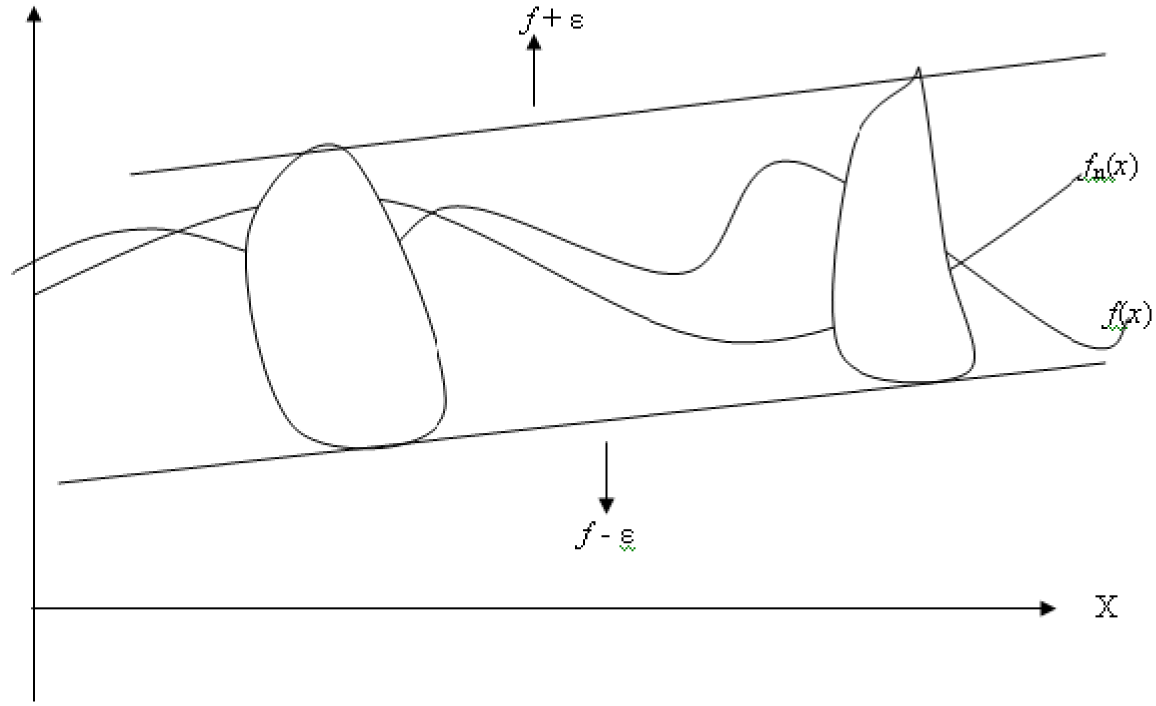


Figure 2.1 : Uniform Convergence

2.1.3 Point-wise convergence

If $\{f_n\}$ converges point-wise to f on D, then

$\forall \varepsilon > 0,$ and for each $x \in D\ \exists N(\varepsilon, x) \in \mathbb{N}$ such that $\left| f_n(x) - f(x) \right| < \varepsilon, \forall n \geq N$

2.1.4. Linear operators on a normed space

A mapping *f* from a normed linear space *L* to another such space *M*, in other words a linear self mapping will be called a linear operator if

$f(ax + by) = af(x) + bf(y)\ \forall x, y \in L,\ a, b \in \mathbb{R}$

2.1.5. Norm of a linear operator

If *f* is a linear operator the norm of *f* can be definedby

$\| f \| = \sup\left\{ \| f(x) \| : x \in L, \|x\| \le 1 \right\}$ or equivalently $\| f \| = \sup\left\{ \| f(x) \| : x \in L, \|x\| = 1 \right\}$
or $\| f \| = \sup\left\{ \frac{\| f(x) \|}{\|x\|} : x \in L, x \neq 1 \right\}$

2.1.6 Strong and weak convergence

$\{ f_n \}$ is said to converge uniformly to f in norm if and only if

$\| f_n - f \| \to 0$ and this denoted by $f_n \xrightarrow{u} f$. f_n converges strongly to f written $f_n \xrightarrow{s} f$

if and only if $f_n(x) \to f(x)$ $\forall x \in L$

A sequence or a net $\{x_n\}$ in the normed linear space *L* is said to converge weakly to

$x \in L$ if and only if $f(x_n) \to f(x)$ for every $f \in J$, where *J* is the set of all linear

functionals on *L*. Call *J* the conjugate space of *L*. The strong convergence in this sense

is the metric one, that is, $\|x_n - x\| \to 0$. The denotations of strong and weak convergence

are $x_n \xrightarrow{s} x$ and $x_n \xrightarrow{w} x$ respectively.

2.2 Operator Algebra

The theory of rings was introduced by von Neumann in 1929 with sole aim of giving a sound foundation to quantum mechanics. The theory was later renamed, von Neumann algebra. In 1930 and early 1940, J. von Neumann and F. J. Murray laid down the foundation ofthe “new’ field of mathematics called operator algebras. Since then a number of topics emanate out and some to independent fields, Ash, (1972).

2.2.1 Operators on Hilbert space

A survey of the operators on Hilbert spaces can now be discussed.

2.2.2 Adjoint operations

Let $\mathcal{B}(\mathcal{H})$ be the space of all bounded linear operators on a Hilbert space $\mathcal{H}$ .The mapping $T \mapsto \langle Tx, y \rangle \ \forall\ x, y \in \mathcal{H}$ is a bounded linear functional on $\mathcal{B}(\mathcal{H})$. By Reiz, there is a unique representation such that

$$\langle Tx, y \rangle = \langle x, T^* y \rangle \quad \forall\, x \in \mathcal{H} \,. \tag{2.1}$$

The map $T \mapsto T^*$ is called an *adjoint operation* on $\mathcal{B}(\mathcal{H})$. The adjoint operation has the properties that for any operators $S, T \in \mathcal{B}(\mathcal{H})$ and scalars α, λ:

$$(\alpha S + \lambda T)^* = \overline{\alpha} S^* + \overline{\lambda} T^* \quad \ldots\ (\textit{antilinear}) \tag{2.2}$$

$$(ST)^* = T^* S^* \quad \ldots\ (\textit{order reversing}) \tag{2.3}$$

$$T^{**} = T \quad \ldots\ (\textit{involution}) \tag{2.4}$$

$$\|T^*\| = \|T\|. \quad \ldots\ (\textit{isometry}) \tag{2.5}$$

$$\langle T^* x, y \rangle = \langle x, Ty \rangle \quad \forall x, y \in \mathcal{H} \tag{2.6}$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *self-adjoint* if $T^* = T$, *normal* if $T^*T = TT^*$ and *unitary* if $T^*T = TT^* = I$. It can therefore be seen that every self-adjoint operator is normal and every unitary is isometric. Therefore, it is clear that $\|T^*T\| = \|T\|^2$.

$$T^* = T = T^2 \qquad \ldots\text{(is a } \textit{projection}\text{)} \qquad (2.7)$$

If A is any algebra over the complex numbers, a *-operation in A satisfying (2.2) to (2.4) is sometimes referred to as an *involution*, Ash, (1972).

2.3An Algebra

A vector space V over K is called an *algebra* if and only if there exists a mapping $V \times V \to V,\ (x, y) \mapsto xy$ satisfying the following axioms for all $x, y \in V$ and $\alpha, \beta \in K$:

$$x(yz) = (xy)z \qquad (2.8)$$

$$\alpha(\beta x) = (\alpha\beta)x \qquad (2.9)$$

$$x(y+z) = xy + xz \text{ and } (x+y)z = xz + yz \qquad (2.10)$$

2.3.1 C*-algebra

A Banach algebra with identity and an involution satisfying all the properties (2.2) to (2.5) above is said to be a C^*-algebra.

Bounded linear functionals on Hilbert space are easily describable by

2.3.2 A concrete C*-algebras

From above, it is seen that C*-algebras are *involutive* Banach algebras. They are isomorphic to norm-closed sub-algebras of the algebra of all bounded operators on some complex Hilbert space $\mathcal{B}(\mathcal{H})$. So, a *concrete C*-algebra* is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ which is closed in the norm topology.

2.3.3 Non-degenerate *-algebras

IT is said that a *-algebra A of operators on Hilbert space $\mathcal{H}$ acts *non-degenerately* if

$$Tx = 0 \ \forall \ T \in A \Rightarrow x = 0 \,. \tag{2.11}$$

Since A is a *-algebra, this is equivalent to condition that the subspace

$$A\mathcal{H} = \operatorname{span}\{Tx : T \in A, x \in \mathcal{H}\}, \tag{2.12}$$

is dense in $\mathcal{H}$.

2.3.4 * -Homomorphism

A **-homomorphism* between two C*-algebras (or von Neumann algebras) M and N is a mapping $\pi : x \in M \to \pi(x) \in N$, defined for all $x \in M$ such that

$$\pi(\alpha x + \beta y) = \alpha\pi(x) + \beta\pi(y), \ \forall x, y \in M, \ \alpha, \beta \in \mathbb{C} \tag{2.13}$$

$$\pi(xy) = \pi(x)\pi(y) \quad \forall x, y \in M \tag{2.14}$$

$$\pi(x^*) = \pi(x)^* \ \forall x \in M \tag{2.15}$$

If π is onto, then

$$x \geq 0 \Rightarrow \pi(x) \geq 0 \ \text{(i.e.} \ \pi \text{ is positive preserving)} \tag{2.16}$$

If π is continuous, then

$$\|\pi(x)\| \leq \|x\| \ \forall x \in M \tag{2.17}$$

2.3.5 Representations

A *representation* of a C*-algebra M is a *-homomorphism from M to $\mathcal{B}(\mathcal{H})$. Two representations π and p on M and N respectively, are equivalent if there is a unitary operator $u \in \mathcal{B}(M, N)$ with the property:

$$u\pi(x)u^* = p(x) \quad \forall\ x \in M \tag{2.18}$$

If π is a *-isomorphism, between M and $\pi(M)$, then the pair, ($\mathcal{H}$, π) is said to be a faithful representation. $\mathcal{H}$ is called the *representation space* and π, the representation of M in $\mathcal{H}$.

A representation ($\mathcal{H}$, π) of M is said to be *proper* or *nondegenerate* if either

(i). The closed subspace [$\mathcal{H}$, $\pi(M)$] spanned by $\pi(x)\xi$, $x \in M$, $\xi \in \mathcal{H}$, coincides with the whole space or

(ii). For any nonzero $\xi \in \mathcal{H}$, there exists $x \in M$ with $\pi(x)\xi \neq 0$

otherwise, the closed subspace [$\mathcal{H}$, $\pi(M)$] is called the *essential space* of π and is denoted by $\mathcal{H}(\pi)$.

A sub-representation of a representation π on $\mathcal{H}$ is the restriction of π to a closed invariant subspace of $\mathcal{H}$. A representation is *irreducible* if it has no nontrivial closed invariant subspace. If $\pi_{\mathrm{i}}(i \in \Omega)$ is a representation on $\mathcal{H}_{\mathrm{i}}$ then the sum $\underset{\mathrm{i}}{\oplus}\pi_{\mathrm{i}}$ of π_{i} is the diagonal sum acting on $\underset{\mathrm{i}}{\oplus}\mathcal{H}_{\mathrm{i}}$, Blackadar, (2006).

2.3.6 Cyclic representation

A vector ξ in a Hilbert space $\mathcal{H}$ is said to be cyclic for a set of bounded operators in M if the set $\{x\xi : x \in M\}$ is dense in $\mathcal{H}$.

A cyclic representation of a von Neumann algebra M is defined to be the triple ($\mathcal{H}$, π, ξ) where ($\mathcal{H}$, π) is a representation of M and ξ is a vector in $\mathcal{H}$ which is cyclic for M in $\mathcal{H}$

2.3.7 State

Let M be a C*-algebra. A linear functional ϕ on M is positive, written

$$\phi \geq 0 \text{ if } \phi(x) \geq 0 \text{ whenever } x \geq 0. \tag{2.19}$$

A *state* on *A* is a linear functional of norm 1, i.e. $\|\phi\| = 1$.

So, a state is positive, linear and of norm 1. A state is *faithful* or *normal* if it satisfies the conditions of faithfulness or normality [see (2.31) and (2.32)]. The set of all states on *M*, is denoted by $\mathrm{S}(M)$ and is called the *state space* of *M*,Blackadar, (2006).

2.3.8 Lemma [Gelfand-Naimark-Segal (GNS)]

If ($\mathcal{H}$, π) is a non-degenerate representation of a von Neumann algebra (or a C*-algebra) *M*, and Ω is a cyclic vector in $\mathcal{H}$ with $\|\Omega\| = 1$, then w_Ω defined by

$\mathrm{w}_\Omega(x) = \langle \Omega, \pi(x)\Omega \rangle$ is a state over M, Ash, (1972).

Proof

$$
\begin{aligned}
\mathrm{w}_\Omega(x^*x) &= \langle \Omega, \pi(x^*x)\Omega \rangle \qquad \text{- (by definition)} \\
&= \langle \Omega, \pi(x^*)\pi(x)\Omega \rangle \\
&= \langle \Omega, \pi(x)^*\pi(x)\Omega \rangle \text{ - } [\pi(x^*) = (\pi(x))^* = \pi(x)^*] \\
&= \langle \pi(x)\Omega, \pi(x)\Omega \rangle \\
&= \|\pi(x)\Omega\|^2 \geq 0
\end{aligned}
$$

which proves that w_Ω is positive.

For any $\alpha \in$ □

$$
\begin{aligned}
\mathrm{w}_\Omega(\alpha x) &= \langle \Omega, \pi(\alpha x)\Omega \rangle \\
&= \langle \Omega, \alpha\pi(x)\Omega \rangle \text{ - (by linearity of } \pi) \\
&= \bar{\alpha}\langle \Omega, \pi(x)\Omega \rangle \\
&= \bar{\alpha}\mathrm{w}_\Omega(x) = \alpha \mathrm{w}_\Omega(x)
\end{aligned}
$$

proving that w_Ω is linear.

Now, $\mathrm{w}_\Omega(I) = \langle \Omega, \pi(I)\Omega \rangle = \langle \Omega, \Omega \rangle = \|\Omega\|^2 = 1$.

Hence the Lemma, and the state is called a *vector state*.

In fact, every state is a vector state for some non degenerate representation.

2.3.9 Remark

The origin of the term "state", which is the mathematical formulation of quantum mechanics where the states of a physical system are given by the probability distribution (unit vector in L^2-space) and observables are self-adjoint operators [Bratteli and Robinson (1987)]. The value of the observable T on the state h, is $\langle T_{\mathrm{h}}, \mathrm{h}\rangle$.

2.3.10 Weights

A weight on a C*-algebra M is a function

$$\phi : \mathrm{M}_{+} \rightarrow [0,\infty] \text{ such that } \phi(0) = 0,$$

$$\phi(a+b) = \phi(a) + \phi(b) \text{ for all } a,b \in M \tag{2.20}$$

And $\phi(\lambda a) = \lambda\phi(a)$ for $a \in M_{+}$, $\lambda > 0$. (2.21)

A weight is densely defined if $\{x \in A_{+} : \phi(x) < \infty\}$ is dense in M_{+}. ϕ will be called *faithful* if it satisfies

$$\phi(x) = 0 \Rightarrow x = 0 \tag{2.22}$$

2.3.11 Pullbacks

Suppose that M_1, M_2, N are C*-algebras and ϕ_{i} is *-homomorphismfrom

M_{i} to N_{i}. We seek a C*-algebra L and a *-homomorphism ψ_{i} from L to M_{i} making the following diagram commutes and which is universal in the sense that if Q is any

$$\begin{tikzcd} L \arrow[r, "\psi_2"] \arrow[d, "\psi_1"'] & M_2 \arrow[d, "\phi_2"] \\ M_1 \arrow[r, "\phi_1"'] & N \end{tikzcd}$$

C*-algebra and $\omega_i : Q \to M_i$ satisfying $\phi_1 \circ \omega_1 = \phi_2 \circ \omega_2$, then there is a unique *-homomorphism $\theta : Q \to L$ such that $\omega_i = \psi_i \circ \theta$. Any such L is naturally unique up to isomorphism commuting with ψ_i. Such an$L$ always exists and one way of constructing L is

$$\{(a_1, a_2) : \phi_1(a_1) = \phi_2(a_2)\} \subseteq M_1 \oplus M_2 \tag{2.23}$$

L is called the *pullback* of (M_1, M_2) along (ϕ_1, ϕ_2), Blackadar, (2006).

2.3.12 Annihilators

Let W be a subset of a vector space V. A linear functional $\phi \in V^*$ is called an annihilator of W if $\phi(w) = 0 \;\; \forall \; w \in W$, *i.e.* $\phi(W) = \{0\}$.

The set of all such mappings is denoted by W^0 and is called the annihilators of W and is a subspace of V^*. If V is finite, then, Lipschutz, (1974),

$$\text{(i) } \dim W + \dim W^0 = \dim V \quad \text{and} \quad \text{(ii) } W^{00} = W \tag{2.24}$$

2.3.13 Theorem [Lipschutz, (1974)]

Let $\mathcal{H}$ be a Hilbert space and ϕ a bounded linear functional on $\mathcal{H}$. Then there is a unique vector $\mathrm{w} \in \mathcal{H}$ with

$$\phi(x) = \langle x, \mathrm{w} \rangle \text{ for all } x \in \mathcal{H} \text{ and } \|\mathrm{w}\| = \|\phi\|. \tag{2.25}$$

2.3.14 Trace and trace class:

Let $\mathcal{H}$ be a Hilbert space and $0 \le T \in \mathcal{B}(\mathcal{H})$. The *trace* of *T* denoted by $tr(T)$ or $\tau(T)$ is defined to be

$$tr(T) = \sum_i \langle Tx_i, x_i \rangle \in [0, \infty], \tag{2.26}$$

where $\{x_i\}$ is an orthonormal basis for $\mathcal{H}$.

If $S \in \mathcal{B}(\mathcal{H})$, then

$$tr(SS^*) = tr(S^*S). \tag{2.27}$$

Also, for any unitary *U*,

$$tr(T) = tr(U^*TU) \text{ and } tr(T) \ge \|T\|. \text{ (i.e. } tr \text{ is unitarily invariant)} \tag{2.28}$$

$$tr(S+T) = tr(S) + tr(T), \quad \text{if } S, T \ge 0 \text{, and} \tag{2.29}$$

$$tr(S) \le tr(T), \text{ if } 0 \le S \le T. \tag{2.30}$$

If $\quad tr(S^*S) = 0 \;\Rightarrow\; S = 0, \qquad (2.31)$

then *tr* is said to be *faithful*

If $tr(\sup S_i) = \sup tr(S_i) \qquad \forall S \in \mathcal{B}(\mathcal{H})_+ \qquad (2.32)$

then *tr* will be called *normal*

Notice that if (T_j) is a net in $\mathcal{B}(\mathcal{H})_+$ converging weakly to *T*, then $\langle T_j x, x \rangle \to \langle Tx_i, x_i \rangle$ for each *i*. So, $tr(T) \le \liminf tr(T_j)$. $tr(T)$ is not continuous if $\mathcal{H}$ is infinite-dimensional, Blackadar, (2006).

2.3.15 Affiliation

(a) An operator *T* on a Hilbert space $\mathcal{H}$ is said to be *affiliated* to a von Neumann algebra M on $\mathcal{H}$ (written $T\eta M$) if it commutes with every unitary operator in the commutant M'.

(b) A linear set D in a Hilbert space $\mathcal{H}$ is said to be *strongly dense* in $\mathcal{H}$ with respect to von Neumann algebra M of $\mathcal{H}$ if

(i) $D\eta M$

(ii) there exists a sequence $\{R_n\}$ of closed linear manifolds associated with M such that $R_n \subset D,\ R_n^{\perp}$ is finite and $R_n^{\perp} \downarrow 0$. D is then said to be dense in $\{R_n\}$.

(c) An operator T is called *measurable* with respect to M if

(i) $T\eta M$.

(ii) T has strong dense domain.

(iii) T is closed.

(d) An operator T is *essentially measurable* with respect to M if

(i) $T\eta M$.

(ii) there exists $\{R_n\}$ of closed linear manifolds associated with M such that $R_n \subset D(T),$ the contraction of T to R_n is finite and $R_n \downarrow 0$.

(iii) $\overline{T}$, the closure of T exists. In either case T is said to be strongly defined on $\{R_n\}$, Bratteli and Robinson, (1987).

2.3.16 Factor

For any set a of operators, a' is defined to be the set of all operators commuting with every operator in a. Then it is a fact that a *-closed set a of operators is a weakly closed algebra if and only if $a = a''$

If $\mathcal{A}$ is a weakly closed algebra, then $\mathcal{A} \cap \mathcal{A}'$ is the *centre* of $\mathcal{A}$. The centre of $\mathcal{A}$ consists of only a scalar multiple of the identity if and only if the union $\mathcal{A} \cup \mathcal{A}'$ generates the algebra $\mathcal{B}(\mathcal{H})$. In this case $\mathcal{A}$ is said to be a *factor*.

So, a von Neumann *M* is called a factor if $Z(M) = \mathbb{C} I$, where $Z(M)$ is the centre of *M*. *M* is a factor if, and only M' is a factor.

2.4 von Neumann Algebra

A von Neumann algebra is a subalgebra *M* (or W*-algebra) of $\mathcal{B}(\mathcal{H})$ which is self-adjoint, contains identity and closed in the weak operator topology. That is if x_{i} is a net of operators in *M* and $\langle x_{\mathrm{i}} h, k \rangle \to \langle xh, k \rangle$ for some $x \in \mathcal{B}(\mathcal{H})$, and all $h, k \in \mathcal{H}$, then $x \in M$. In other words, a von Neumann is a *-algebra *M* of $\mathcal{B}(\mathcal{H})$ satisfying $M = M''$.

A von Neumann algebra contains an abundance of projections.Every von Neumann . algebra is a *-algebra and almost all the results and techniques obtained in the *-algebras apply to von Neumann algebra. Although, it is common to view C*-algebras as “noncommutative topological spaces”, it is usual to regard the von Neumann algebras as the “noncommutative measure spaces”, Blackadar, (2006).

2.4.1 Projection

Suppose that a vector space *V* is a direct sum of its subspaces W_{i}, that is, $V = W_1 \oplus W_2 \oplus ....\oplus W_r$. The projection of *V* into its subspaces W_k is the mapping $p: V \to V$ defined by $p(v) = w_k$, where, $v = w_1 + w_2 + ...+ w_r \in W_i$, uniquely. We can see that

$$w_k = 0w_1 + 0w_2 + ... + w_k + 0w_{k+1} + ... + 0w_r, \tag{2.33}$$

is the unique sum corresponding to $w_k \in W_k$. So, $p(w_k) = w_k$, and for all $v \in V$,

$$p^2(v) = p(p(v)) = p(w_k) = w_k = p(v) \quad \Rightarrow p^2 = p \tag{2.34}$$

An operator p in $\mathcal{B}(\mathcal{H})$ is a projection if $p^* = p$ and $p^2 = p$. If p is a projection then, $(I - p) = p^{\perp}$, i.e. the orthogonal projection of p is also a projection because

$$\left(p^{\perp}\right)^2 = \left(I - p\right)^2 = I - 2p + p^2 = I - 2p + p = I - p = p^{\perp}. \tag{2.35}$$

It is known that if M is the subspace of $\square^n$ and x is any vector in $\square^n$, then x can be resolved into a component in M and a component perpendicular to M. i.e. $x = y + z$ where $y \in M$ and z is orthogonal to every vector in M, That is to say that if M is a closed subspace of a Hilbert space $\mathcal{H}$ and $x \in \mathcal{H}$ then x has the unique representation $x = y + z$ where $y \in M$ and $z \perp M$. Further, y is the projection of x on M, Lipschutz, (1974).

The paragraph above is referred to as the *projection theorem* which is sometimes expressed by saying that $\mathcal{H}$ is the orthogonal direct sum of M and $M^{\perp}$ written

$$\mathcal{H} = M \oplus M^{\perp}. \tag{2.36}$$

It is usual to write 1 in place of the identity I in a von Neumann algebra M.

If p is a projection then p is said to be a *partial isometry*if there exists a closed subspace $K \subset \mathcal{H}$ such that

$$\|px\| = \|x\| \quad \text{(continuity)} \ \forall x \in K, \ px = 0 \, \forall x \in K^{\perp} \tag{2.37}$$

p is said to be *positive* if $\langle px, x\rangle \geq 0 \;\; \forall\, x \in \mathcal{H}$. A projection of *norm one* is a mapping p from a von Neumann algebra M onto its subalgebra N whose norm is *one*,

2.4.2 Theorem[Tomiyama (1957)]

Let M be a von Neumann algebra and N its von Neumann subalgebra. If p is a projection of norm one from M onto N, then the following conditions holds:

$$x \leq y \Rightarrow p(x) \leq p(y) \;\forall\; x, y \in M \;\; (\,p \text{ is order –preserving}) \tag{2.38}$$

$$p(axb) = ap(x)b,\; x \in M,\; \forall\; a, b \in N \tag{2.39}$$

$$p(x^*)\, p(x) \leq p(x^* x) \;\; x \in M \tag{2.40}$$

$$p(x^*) = p(x)^* \text{(i.e. } p \text{ is *-preserving)} \tag{2.41}$$

2.4.3 Equivalent projections

Projections p and q in a von Neumann algebra M are (*Murray-von Neumann*) *equivalent*, written $p \sim q$, if there is a partial isometry $u \in M$, with $u^* u = p$ and $uu^* = q$. The projection p is said to be *subordinate* to q, written $p \precsim q$, if $p \sim q' \leq q$. (These relations depend on the choice of the algebra M containing p and q). Equivalence is additive in a von Neumann algebra. If $\{p_i, q_i\}$ is a set of projections in M with $p_i \perp p_j$ and $q_i \perp q_j$ for $i \neq j$ and $p_i \sim q_i$ for all i, then

$$\sum_i p_i \sim \sum_i q_i\,, \tag{2.42}$$

(the partial isometries can be added in the strong operator topology). Similarly, if $p_i \precsim q_i$ for all i, then

$$\sum_i p_{\mathrm{i}} \P \sum_i q_{\mathrm{i}} . \tag{2.43}$$

It can be noticed that if *p* and *q* are projections in a von Neumann algebra *M*, then

$$(p \vee q) - p \sim q - (p \wedge q), \tag{2.44}$$

where, $(p \vee q) - p$ and $q - (p \wedge q)$ are the source and range projections of $q(1 - p)$ respectively.

2.4.4 Proposition [Blackadar (2006)]

Let *p* and *q* be nonzero projections in a von Neumann algebra *M*. If $qMp \neq 0$, then there are nonzero projections $p' \leq p$ *and* $q' \leq q$ with $p' \sim q'$. In other words if $qxp \neq 0$, then the source and range projection of qxp do the trick.

2.4.5 Central Projection and Support

A projection *p* is called a *central projection* on *M* if it belongs to the centre *Z*(*M*) of *M*.

Every projection *p* in von Neumann algebra *M* has a *central support* (or *central carrier*) denoted by z_p which is the smallest projection in the centre *Z*(*M*) containing *p* as a subprojection. Since *Z*(*M*) is itself a von Neumann algebra, then z_p exists and

$$1 - z_p = \vee\{q \in M : qMp = 0\} \tag{2.45}$$

Projections *p* and *q* have nonzero equivalent subprojection if and only if $z_p z_q \neq 0$. In particular, if

$$q \P p, \text{ then } z_q \P z_p . \tag{2.46}$$

2.4.6 Remark

If M is a von Neumann algebra and $x, y \in M$ with $yMx = 0$, then there is a projection $z \in Z(M)$ such that $x \in zM$ $(\text{i.e. } xz = x)$ and $y \in (1-z)M$ $(\text{ i.e. } yz = 0)$

2.4.7 Proposition [SCHRODER-BERNSTEIN]

Let p, q be projections in a von Neumann algebra M. If p ¶ q and q ¶ p, then $p \sim q$.

2.4.8 Abelian, finite and infinite projections

A projection p in a von Neumann algebra M is said to be

(i)*abelian* if pMp is commutative

(ii)*finite* if $p \sim p' \leq p \Rightarrow p' = p$

(iii) *infinite* if it is not finite

(iv)*properly infinite* if $p \sim p_1$, $p \sim p_2$ with $p_1, p_2 \leq p$ and $p_1 \perp p_2$

(i.e. if there is no nonzero finite central projections)

(v)*purely infinite* if there is no nonzero finite projection

(vi)*faithful* if no nonzero central projection of *Mannihilatesp*, or if

$z_p = I$.

A unital von Neumann algebra M is finite, infinite, and properly infinite if 1_M is

finite, infinite, and properly infinite respectively. By this definition, 0 is properly infinite (it is also finite), Blackadar, (2006).

2.4.9 Type and classification of von Neumann algebra

If M is a von Neumann algebra and p is a projection in M then there is a central projection $z \in M$ such that pz is finite and $p(1-z)$ is properly finite. In particular taking $p = 1$, there is a unique central projection $z_f \in M$ such that z_f is finite and $1-z_f$ is properly finite. Mz_f, is called the finite part of M, and $M(1-z_f)$, the infinite part.

$$M = Mz_f \oplus M(1 = z_f). \tag{2.47}$$

It is noted that z_f does not contain all finite projections of M in general. For instance, if $M = \mathfrak{B}(\mathcal{H})$ then $z_f = 0$.

Let p be a projection in a von Neumann algebra M, then$M$is classified as follows:

(i) M is finite, properly infinite and purely infinite if the identity projection I is finite,properly infinite and purely infinite respectively.

(ii) M is *semifinite* if every nonzero central projection contains a nonzero finite projection, or M is semifinite if 1_M (unital algebra M) is semifinite.

(iii) M is type I if any nonzero central projection contains a nonzero abelian projection.

(iv) M is type I_∞ if it is properly infinite and of type I.

(v) M is type II if it is semifinite and does not contain any nonzero abelian projection.

(vi) M is type II_1 if it is both finite and of type II.

(vii) M is type II_∞ if it is properly infinite and of type II.

(viii) M is *purely infinite* (or Type III) if it does not contain any nonzero finite projection or M is purely infinite if 1_M is purely infinite.

2.4.10 Commutative von Neumann algebras

A commutative von Neumann algebra is Type I.

2.4.11 Proposition[Blackadar (2006)]

Let Z be a commutative von Neumann algebra on $\mathcal{H}$. Then Z' is Type I.

Proof

$\mathcal{H}$ can be broken up into direct sum of cyclic subspaces under Z. On each subspace, a direct summand of Z acts faithfully and cyclically. If q is a projection onto one of these cyclic subspaces, then $q \in Z'$ and $qZ'q = Zq$ since Zq is a maximal abelian subalgebra (*masa*) on $q\mathcal{H}$. Thus, q is an abelian projection in Z'.

If M is a Type I von Neumann algebra on a Hilbert space $\mathcal{B}(\mathcal{H}_1)$, then M is also Type I. This is true for Type II and Type III von Neumann algebras

2.4.12 Purely atomic

A von Neumann algebra *M* is *purely atomic* if *M* contains a set $\{p_i\}$ of minimal projections with $\sum p_i = 1$. Since minimal projection is abelian, a purely atomic von Neumann algebra is Type I. But many Type I von Neumann algebras are purely atomic. For example, $L^{\infty}(X, \mu)$ if and only if it is a product of Type I factors. Also a von Neumann algebra *M* is purely atomic if and only if it has a purely atomic masa.

Tomiyama, (1957) showed that each projection of norm one of a von Neumann algebra onto its von Neumann subalgebra enjoys most of the properties of conditional expectation. Takesaki, (1972) gave sufficient conditions for the existence of conditional expectation of a projection of norm one from a von Neumann algebra onto its von Neumann subalgebra. Goldstein, (1991) and Tijjani (2003) – (using different methods) showed the extension of conditional expectation to crossed products of von Neumann algebra.Umegaki,(1956) showed the existence of a conditional expectation from a finite von Neumann algebra onto its own von Neumann subalgebra which occurred as non-commutative extension of the classical conditional expectation when the von Neumann algebra has a faithful normal semi-finite trace tr with $tr(I) = 1$.

2.4.13 Gauge space

A gauge space Γ is a triple $(\mathcal{H}, M, tr)$, composed of a Hilbert space $\mathcal{H}$, a von Neumann algebra *M* and a faithful normal trace tr. The trace, tr is unitarily invariant, in this case every unitary operator $m \in M$ and projection $p \in M$

$$tr(m^{*}pm) = tr(p) \tag{2.48}$$

Γ is said to be (metrically) finite in case $tr(I)$ is finite, where *I* is an identity operator.

The theory was well developed with advances beginning with the Tomita-Takesaki Modular Theory in the late 1960 and culminating in Cannes' classification of injective factors. Thereafter, the modern era of the subject ushered in. Voiculescu, (1985), (1995) and (2002) introduced the concepts of free probability, entropy and free products in von Neumann and connected the reduced free products C*-algebras with topology.

2.5 Modular Theory

One of the useful developments in the history of von Neumann algebra is the modular theory. It gives a precise connection between a von Neumann algebra and it's commutant.

2.5.1 Standard form representation

Let M be a von Neumann algebra and ψ a faithful semifinite weight on M. The ψ *-standard form representation* of M is the GNS representation from ψ .

A normal π on M is in standard form if it is (unitarily equivalent to) a ψ- standard form representation for some faithful normal semifinite weight ψ. A von Neumann algebra on $\mathcal{H}$ is in standard form if its identity representation is in standard form.

Let M be a von Neumann algebra in standard form on a Hilbert space $\mathcal{H}$. i.e. the identity representation is the GNS representation from a faithful normal semifinite weight ϕ. Two closed conjugate-linear densely defined operations S and F on $\mathcal{H}$, can now be defined.

Let S_0 be the densely defined operator

$$x\xi = \eta \mapsto x^*\xi \text{ for } x \in M \tag{2.49}$$

where ξ is the cyclic and separating vector associated with the faithful normal state ϕ, and F_0, the densely defined operator

$$y\xi = \eta \mapsto y^* \xi \text{ for } y \in M'. \tag{2.50}$$

If $x \in M$ and $y \in M'$, then

$$\left\langle S_0\left(x\xi\right), y\xi\right\rangle = \left\langle x^*\xi, y\xi\right\rangle = \left\langle y^* x^* \xi, \xi\right\rangle = \left\langle y^*\xi, x\xi\right\rangle = \left\langle F_0\left(y\xi\right), x\xi\right\rangle \quad . \tag{2.51}$$

Thus, $S_0^* \supseteq F_0$ *and* $F_0^* \supseteq S_0$ which means that F_0 and S_0 are closable.

Let *S* be the closure of S_0 and $F = S^* = S_0^*$. Since $S_0 = S_0^{-1}$, it follows that S and F are one-one with dense range and $S = S^{-1}$, $F = F^{-1}$.

Set $\Delta = S^* S = FS$. Then Δ is an invertible densely defined positive operator with

$$\Delta^{-1} = \left(FS\right)^{-1} = S^{-1}F^{-1} = SF\ . \tag{2.52}$$

If $S = J\Delta^{1/2}$ is the polar decomposition, then *J* is an invertible conjugate linear isometry and

$$S = S^{-1} = \Delta^{-1/2} J^{-1}. \tag{2.53}$$

But,

$$S = \left(SS^*\right)^{1/2} J = \Delta^{-1/2} J \Rightarrow J = J^{-1}, J^2 = 1. \qquad (2.54).$$

meaning that *J* is an involution.

Therefore, there is

$$J\Delta J = \Delta^{-1} \text{ and } F = J\Delta^{1/2} = \Delta^{1/2} J \tag{2.55}$$

Since, $S\xi = \xi$, then

$$F\xi = \xi,\ J\xi = \xi,\ \Delta\xi = \xi \tag{2.56}$$

Δ is called the *modular operator* for $\left(M, \phi\right)$. It is sometimes denoted by Δ_ϕ, emphasizing its dependence on ϕ.

There is also an alternative approach using the bounded operators due to Rieffel and van Daele (1977), and also Pedersen (1979). The situation is simpler if M is semifinite and ϕ is a trace. In this case, $S = F = J$ and $\Delta = 1$.For $t \in \mathbb{R}$, the operator Δ^{it} is unitary and $t \mapsto \Delta^{it}$ is a strongly continuous one parameter group of unitaries. Thus, the automorphisms σ_t defined by

$$\sigma_t(x) = \Delta^{it} x \Delta^{-it} \tag{2.57}$$

is a strongly continuous one parameter group of automorphisms of $\mathcal{B}(\mathcal{H})$ .Because of the conjugate linearity of J, from $J\Delta = \Delta^{-1}J$, it implies that $J\Delta^{it} = \Delta^{it}J \ \forall \ t \in \mathbb{R}$. Thus, the automorphisms σ_t of $\mathcal{B}(\mathcal{H})$ commute with conjugate-linear automorphism j of $\mathcal{B}(\mathcal{H})$ defined by

$$j(x) = JxJ \ . \tag{2.58}$$

2.5.2 Theorem [Tomita-Takesaki]

$$\begin{aligned} &j(M) = M' \quad (\text{so } j(M') = M) \\ &\sigma_t(M) = M \text{ and } \sigma_t(M') = M' \ \forall \ t \in \mathbb{R} \end{aligned} \tag{2.59}$$

If $x \in Z(M)$, then $j(x) = x^*$ and $\sigma_t(x) \ \forall \ t$

van Daele (1982) explained why the theorem is true. The argument concentrated on the boundedness of S, hence F and Δ. There are also some considerable additional works in general case in Laszlo (1975), Kadison (1978) and also Serban and Laszlo (1979).

2.5.3 Symmetric and standard forms

Let M be a von Neumann algebra on $\mathcal{H}$. M is *symmetric form* (sometimes called *hyperstandard* form) if there is an involution J of $\mathcal{H}$ such that

$$JMJ = M' \text{ and } JxJ = x^* \ \forall \ x \in Z(M). \tag{2.60}$$

A normal representation π is symmetric form representation if $\pi(M)$ is in symmetric form.

Takesaki, (1973) gave the construction of crossed products of von Neumann algebra defined as the automorphism $\sigma(x)=\Delta^{it}x\Delta^{-it}$, where $t\in\mathbb{R}$, the operator Δ is unitary and $t\mapsto\Delta^{it}$ is a strongly continuous one parameter group of unitaries. In this sense, this automorphism defines a strongly continuous one parameter group of automorphism of $\mathcal{B}(\mathcal{H})$.

The study of Mayne, (1982) showed the centralizing automorphisms on commutative rings. Thereafter, Bell and Martindale, (1987) and (1988), Bresar and Vukman, (1989), Hirano, *et al*, (1983) and Lanski, (1988) in different ways, developed the theories of centralizing mappings on commutative von Neumann algebras.

Turumaru, (1952) was the first to study the tensor products of C^*-algebras. Bermudez and Kalton, (2001), discussed and laid down some beautiful operators on von Neumann algebras. In view of this, the concept of tensor product to show the natural extension of operators in a set of bounded linear operators in a Hilbert space is discussed.

2.6 Amplifications and Commutants

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert Spaces, there are natural extension of operators in $\mathcal{B}(\mathcal{H}_i)$ to operators on $\mathcal{H}_1\otimes\mathcal{H}_2$. This is to say that if $T\in\mathcal{B}(\mathcal{H}_1)$, for any $x,y\in\mathcal{H}$, we can define $T\otimes I\in\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2)$ by

$$(T\otimes I)(x\otimes y)=Tx\otimes y. \tag{2.60}$$

Similarly, for $S \in \mathcal{B}(\mathcal{H}_2)$, $I \otimes S$ is defined by

$$(I \otimes S)(x \otimes y) = x \otimes Sy. \tag{2.61}$$

The mappings $T \mapsto T \otimes I$ *and* $S \mapsto I \otimes S$ are isometries. The images of $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ are denoted by $\mathcal{B}(\mathcal{H}_1) \otimes I$ and $I \otimes \mathcal{B}(\mathcal{H}_2)$, respectively and are called *amplification*. More generally, if $S_1, T_1 \in \mathcal{B}(\mathcal{H}_1)$, $S_2, T_2 \in \mathcal{B}(\mathcal{H}_2)$, the definition can be

$$S_1 \otimes S_2 \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \text{ by } (S_1 \otimes S_2)(x \otimes y) = S_1 x \otimes S_2 y. \tag{2.62}$$

Then,

$$S_1 \otimes S_2 = (S_1 \otimes I)(I \otimes S_2) = (I \otimes S_2)(S_1 \otimes I). \tag{2.63}$$

$$(S_1 + T_1) \otimes S_2 = (S_1 \otimes S_2) + (T_1 \otimes S_2). \tag{2.64}$$

$$S_1 \otimes (S_2 + T_2) = (S_1 \otimes S_2) + (S_1 \otimes T_2). \tag{2.65}$$

$$(S_1 \otimes S_2)(T_1 \otimes T_2) = S_1 T_1 \otimes S_2 T_2. \tag{2.66}$$

$$(S_1 \otimes S_2)^* = S_1^* \otimes S_2^*. \tag{2.67}$$

$$\|S_1 \otimes S_2\| = \|S_1\| \|S_2\|. \tag{2.68}$$

If $G \subseteq \mathcal{B}(\mathcal{H})$, the *commutant* of G is $G' = \{T \in \mathcal{B}(\mathcal{H}) : ST = TS \ \forall \ S \in G\}$. G' is a closed subalgebra of $\mathcal{B}(\mathcal{H})$ containing I. The *bicommutant* $(G')'$ is denoted by G''. There is then $G \subseteq G''$, and $G_1' \supseteq G_2'$ if $G_1 \subset G_2$. It follows that $G' = (G'')'$ for any subset G of $\mathcal{B}(\mathcal{H})$, Blackadar, (2006).

2.7 Stochastic Calculus

A powerful tool in classical probability theory is the stochastic calculus. Boulem and Said (2009), suggested a probabilistic setup to model profitability and a function of a market price of a set of commodities and found the related optimal strategy to sustain it under the constraint that the project faces the abandonment risk when being non-profitable under fixed finite interval. Savas and Masahiko (2012)combined the works of Frank (2009) to say that the issue of finding strategy to sustain structural profitability of an investment project whose production activity depends on the market price of a number of underlying commodities is addressed. Previously various kinds of stochastic calculi have been introduced in quantum probability especially by Barnett *et-al*, (1982), Hudson and Parthasarathy, (1984) and also Lindsay and Maasen, (1988). The approach normally starts with the canonical commutation relation (CCR) or the canonical anti-commutation relation (CAR) over a square integrable space $L^2\left(\mathbb{R}^+, dt, \mathcal{H}\right)$, where $\mathcal{H}$ is a Hilbert space.

2.8 Non-commutative Stochastic Integral

A non-commutative stochastic integral has been developed by Barnett and Wilde, (1993) which was achieved by departing from the existence and properties of random times of the integral to its associated time projection. It is known that some amount of classical theory has a counterpart in quantum (non-commutative) context. Following Barnett *et al* (1996) some constructions have been made on conditional expectations, martingales and stochastic integrals all within the non-commutative domain. Following the development of L^{P}- stochastic calculus for the Clifford stochastic integrals, Barnett *et al*, (1982), constructed a theory of stochastic integration – the L^2- theory, which could unify the classical as well as the several different quantum theories. Correspondingly, Carlen and Kree, (1998), developed certain non-commutative L^{P} – inequalities relating to Clifford integrals and their integrands. The results were applied to extend the domain of the Clifford integrals from L^2 to L^1 integrands and gave applications to optional stopping of Clifford martingales. On separate account, Barnett, *et al*, (1982)[*]developedthe purely non-commutative (Fermions) stochastic calculus. Barnett and Voliotis, (1995) investigated the stochastic integrations and random stopping in a non-commutative filtration by embedding the analysis in the von Neumann algebra.

Many algebraic structures, stochastic integrals (which do not require the usual boundedness) relating to Fock spaces have been developed by Belavkin, (1991), Belavkin and Brown, (2010) and (2011).The family of operator-valued measures on $\mathbb{R}^{+}$ is then introduced relative to the representation, for instance expressed in terms of creation or annihilation or number or field operators. It is shown in Barnett *et al* (1982) that it is possible to develop a theory of stochastic integration with respect to these operator-valued measures. The theory is used to prove an operator form of the Ito formula - this is usually stated in the form that the weak or strong product of two stochastic integrals is sum of stochastic integrals.

A construction of quantum stochastic integrals for the integrator being a martingale in von Neumann algebra was undertaken by Luczak (2010). In the setting, the integrand being a suitable process with values in the same algebra, as densely defined operators affiliated with the algebra. In the case of finite algebra, the integrator is allowed to be an L^2 – martingale in which case the integrals are also L^2 – martingales.

2.9 Stopping Time Theory

In the concept of stopping times in classical sense, Sinelnikov, (2012), considered optimal stopping problems for a Brownian motion with negative drift on the interval (0, 1) and related it to be the best estimation of two unpredictable moments, i.e. the moments when the process attains its maximum value and the moment when the process attains its last zero. Also, discrete times and processes were extensively discussed in Szymon et al, (2013), Xing and Shige, (2011) and Knill, (2013). An example of stopping problem driven by fragmentation processes was given by Andreasand Juan, (2012). They showed that one can work with the concept of stopping times to formulate the notion of optimal stopping problems. Further, the problem can be reduced to a classical optimal stopping problem for a generalized Ornstein-Uhlenbeck process associated with Bertoin's tagged fragment. They went on to solve the latter using a classical verification technique.

2.10 Martingale Theory

The concept of martingale theory was developed by Leo Doob (1910 – 2004). The word "martingale" means a gambling system in which loosing bets are doubled. There have been developments in the theory. In the classical theory there are the works of Doob, (1971), Rao, (1984), Kopp, (1976), William, (1991), Shereve, (2004) and also, Lalonde, (2013) who reviewed the martingales stopping theorems. Scholars such as Barnett and Thakrar, (1990), gave the non-commutative approach to the theory and showed that it enjoys many of the properties of the classical theory.

2.11 Measure Theoretic Integration

A measure theoretic integral, named the belated integral was developed by Barnett and Wilde, (1986). The vector approach the duo adopted has the advantage of underlining the analogy between the classical and the quantum theory of stochastic integration. Theories relating to non-commutative stochastic integral have indicated that it is a kind of vector Riemann-Stieltjes integral. In Dunford and Schwartz, (1958), Elworthy, (1982), Wilde, (1974), Protter, (1979), Toh and Chew, (1999), Xu and Lee, (1993) and

Muldowney, (2000) all used the generalized Riemann approach to study Ito's stochastic integral. Specifically, Henstock, (1991) and Lee, (1976) used full division of [0,1] to define a stochastic integral for deterministic integrands which was a direct generalization of the classical non-stochastic case. McShane, (1974) used a belated partial division to define McShane stochastic integral. In Xu and Lee, (1993), a stochastic integral was defined by using a full belated division and a proof of the equivalence of this integral and the classical Ito integral was given. The full-belated division was obtained from Henstock, (1991).A variation approach was introduced in Toh and Chew, (1999) to study the Ito integral, which is a modification of McShane, (1974) by using partial divisions. In Chew *et al*, (2002), two generalized Riemann approach to the Ito integral were considered; namely, the Ito-Henstock and the Ito-Mcshane and established the equivalence of the Ito-Henstock integral with the classical Ito's integral. They proved the equivalence of all the three integrals thereby offering an alternative proof of equivalence in Toh and Chew, (1999).

Bartle, (1956) developed a bilinear vector integral which in the beginning appeared to be a natural setting for non-commutative stochastic integral. However, Barnett, *et al* (1982) showed that the non-commutative stochastic integral could not be obtained using Bartle's vector integral. By weakening a few of the requirements of Bartle's theory (which is on $\mathbb{R}^{+}$) and making some abstraction on some of the structures used, Barnett and Wilde, (1986) obtained an integration theory on $\mathbb{R}^{+}$. From this development, all the integrals described in their previous works including that of McShane served as special cases of this integration theory. This also reduces the Bartle's integral as a special case. In this way the distinct theories are unified.

CHAPTER THREE

MEASURES AND STOCHASTIC PROCESSES

3.1Introduction

In this chapter, a study of measure spaces and processes is undertaken with a view to make further analysis so that some constructions in chapters four and five can be effected.

3.2Measures

A measure is a set function, i.e. an assignment of a number $\mu(A)$ to each set A in a certain class. So a certain structure has to be provided for μ to have meaning.

3.3 σ - Algebra

Let Φ be a family or collection of subsets of a set Ω. Then Φ is called a *field* (or *an algebra*) if and only if $\Omega \in \Phi$ and Φ is closed under complementation and finite unions i.e.

$$\Omega \in \Phi \text{ and } \varnothing \in \Phi \tag{3.1}$$

$$\text{If } A \in \Phi, \text{ then } A^c \in \Phi \tag{3.2}$$

$$\text{If } A_{1,} A_2, ..., A_n \in \Phi, \text{ then } \bigcup_{i=1}^{n} A_i \in \Phi. \tag{3.3}$$

Φ will be closed under finite intersection because

$$\bigcap_{i=1}^{n} A_i = \left(\bigcup_{i=1}^{n} A_i^c\right)^c \in \Phi \tag{3.4}$$

If (3.3) is replaced by the closure of countable unions, i.e. if:

$$A_{1,} A_2, \ldots \in \Phi \text{ then } \bigcup_{i=1}^{\infty} A_i \in \Phi \tag{3.5}$$

Φ, will be called a σ - *algebra* (or a σ -field) and in this case, Φ will be closed under countable intersection. A σ - algebra Φ, will be closed under limits of increasing sequences because, if

$$A = \bigcup_{n=1}^{\infty} A_n \text{ , then } \bigcup_{i=1}^{n} A_i \uparrow A \text{ and conversely, if } A_n \uparrow A \text{, then } \bigcup_{n=1}^{\infty} A_n = A$$

A measure μ, on a σ - algebra Φ is a nonnegative real-valued function such that whenever $A_{1,} A_2, \ldots$ for a finite number or countably infinite collections of disjoint sets in Φ, we have

$$\mu\left(\bigcup_{n=1} A_n\right) = \sum_n \mu(A_n) \tag{3.6}$$

If $\mu(\Omega) = 1$, μ is called *probability* measure.

3.4 Borel Sets

Borel sets, denoted by $\mathcal{B}(\mathbb{R})$ can be describedas the smallest σ - field containing the class of all open intervals of $\mathbb{R}$. Since an open set is a countable union of open intervals, we can as well describe $\mathcal{B}(\mathbb{R})$ as the smallest σ - field containing all open sets of $\mathbb{R}$. A set is open if and only if its complement is closed, then $\mathcal{B}(\mathbb{R})$ is the smallest σ - field containing all closed sets of $\mathbb{R}$. Borel sets can then be generated by starting with intervals and forming complements and countable unions and intersections in all possible ways, Ash, (1972). This means that every set which can be written down and just about every set imaginable is in $\mathcal{B}(\mathbb{R})$. For instance any $(a,b)\in\mathbb{R}$,

$(a,\infty)=\bigcup_{n=1}^{\infty}(a,a+n)$, $(-\infty,a)=\bigcup_{n=1}^{\infty}(a-n,a)$ and $(-\infty,a)\cup(b,\infty)$ are Borel sets. Also, $[a,b]=((-\infty,a)\cup(b,\infty))^{c}$ is a Borel set, which means that

$[a,\infty)=\bigcup_{n=1}^{\infty}[a,a+n)$ and $(-\infty,a]=\bigcup_{n-1}^{\infty}(a-n,a]$ are Borel sets because $(a,b]=(-\infty,b]\cap(a,\infty)$

Every set which contains only one real number (a singleton) is Borel. $\{a\}=\bigcap_{n=1}^{\infty}\left(a-\frac{1}{n},a+\frac{1}{n}\right)$ and every set containing finitely many numbers is Borel.

3.5Measure Space

A *measure space* is the triple, (Ω,Φ,μ) where Ω is a set, Φ a σ-algebra of subsets of Ω and μ is a measure on Φ.

Let F_{μ} be class of sets $A\bigcup N$, where *A* ranges over Φ and *N* over all subsets of sets of measure zero in Φ. Then F_{μ} is a σ-algebra containing Φ. We can extend μ to F_{μ} by setting, $\mu(A\cup N)=\mu(A)$. The measure space $(\Omega,\mathrm{F}_{\mu},\mu)$, is called the *completion* of $(\Omega,\mathrm{F}\ ,\mu)$ and F_{μ} is the *completion* of Φ relative to μ.

3.5.1Lebesgue-Stieltjes measure

A *Lebesgue-Stieltjes measure* on $\mathbb{R}$ is a measure μ on $\mathcal{B}(\mathbb{R})$ such that $\mu(I)<\infty$ for each bounded interval *I*. A *distribution* on $\mathbb{R}$ is a map $F:\mathbb{R}\to\mathbb{R}$ that is increasing $[\text{i.e. } a<b \Rightarrow F(a)\le F(b)]$ and right continuous $\left[\text{i.e. } \lim_{x\to x_0^+} F(x)=F(x_0)\right]$

3.5.2Remarks

If *F* is a distribution function and μ, the corresponding Lebesgue-Stieltjes measure then

$$\mu(a,b]=F(b)-F(a) \tag{3.7}$$

If $f:\mathbb{R}\to\mathbb{R}$, $f\ge 0$ and f is integrable for any finite interval, and if we fix $F(0)$ arbitrary and define

$$\begin{aligned} F(x)-F(0)&=\int_0^x f(t)\,dt \quad x>0 \\ F(0)-F(x)&=\int_x^0 f(t)\,dt \quad x<0 \end{aligned} \tag{3.8}$$

Then *F* is a continuous distribution function and thus gives rise to a Lebesgue-Stieltjes integral.

Specifically,

$$\mu(a,b]=\int_a^b f(x). \tag{3.9}$$

In a particular, we may take $f(x)=1$ for all *x* and $F(x)=x$, then $\mu(a,b]=b-a$. The set function μ is called the Lebesgue measure on $\mathcal{B}(\mathbb{R})$.

3.5.3 Probability measure

If μ is replaced by *P*, and *P* is a mapping $P:\Omega \to (0,\, 1]$ such that $P(\Omega)=1$, the triple $(\Omega, F\,, P)$ is called a *probability measure* with the property that

$$P(\Omega)=1 \tag{3.10}$$

$$P(\varnothing)=0 \tag{3.11}$$

$$\text{If } \; A_1 \subset A_2, \;\; \Rightarrow P(A_2 \setminus A_1)=P(A_2)-P(A_1) \tag{3.12}$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right)=\sum_{i=1}^{\infty} P(A_i), \quad \text{if } \; A_i \cap A_j \quad \textit{for } i \neq j \tag{3.13}$$

$$P(A^c)=1-P(A) \tag{3.14}$$

$$\text{If } \; A_i \subset A_{i+1} \quad i=1,2,....., \quad \text{then } \; P\left(\bigcup_{i=1}^{\infty} A_i\right)=\lim P(A_i) \tag{3.15}$$

$$\text{If } \; A_i \supset A_{i+1} \quad i=1,2,....., \quad \text{then } \; P\left(\bigcap_{i=1}^{\infty} A_i\right)=\lim P(A_i) \tag{3.16}$$

If F contains all subsets of probability zero, then $(\Omega, F\,, P)$ is said to be *complete*. If Ω and F are both real, then $(\square\,, \square\,, P)$ is called a *Gaussian probability space*, Ash, (1972)

3.5.4 Event

If $(\Omega\,,\, F\,)$ is a measure space, then a point $\mathrm{w} \in \Omega$ is called an *event*. If $(\Omega, F\,, P)$ is a probability space, the set function *P* assigns to every event *A*, a number *P*(*A*) where $0 \le P(A) \le 1$

3.5.5 Filtered probability space

Interest may be focused in events that are concerned with some dynamical systems. It may be such that which event occurred prior to some given time, $t \in T \subseteq [0,\infty)$, a time horizon. Let $(\Omega, \mathcal{F}, P)$ be a probability space, then an increasing family of sub-σ-algebra $\{\mathcal{F}_t, t \in T\}$ or $\{\mathcal{F}_t\}_{t\geq 0}$ such that for all $0 \leq t < s < \infty$, there is the relation $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ and is called a *filteration* of the probability space $(\Omega, \mathcal{F}, P)$ if it is right continuous, i.e.

$$\mathcal{F}_{t^+} = \mathcal{F}_t \quad \text{or that } \mathcal{F}_s = \bigcap_{t>s} \mathcal{F}_t \quad \forall s \geq 0 \tag{3.17}$$

and $\mathcal{F}_0$ contains all *P*-null sets i.e. $P(A) = 0$ are contained in $\mathcal{F}_0$ for all $A \in \mathcal{F}$.

The quadruplet $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0} P)$ is called a *filtered probability space*. $\mathcal{F}_t$ is the collection of all events that had occurred prior to the time t .

3.5.6Radon-Nykodym theorem

If μ is a measure on σ-field Φ and λ is a *signed measure* (countably additive set function) on the σ-field Φ of subsets of Ω. Assume that λ is absolutely continuous with respect to μ. Then there is a Borel measurable function $g : \Omega \to \mathbb{R}$ such that

$$\lambda(A) = \int_A g d\mu \text{ for all } A \in \Phi. \tag{3.18}$$

If h is another such function then $g = h$ a.e. (almost everywhere).

where the integral is in the sense of Lebesgue. Call $\Lambda^2[a,b]$, the space of all real-valued square integrable functions on $[a,b]$, [Ponnusamy, (2000)].

3.6 Indicator

If A is a subset of $\mathbb{R}$ that is not a Borel set and I_A is an *indicator* of A, then

$I_A(w)=1$ for $w\in A$ and $I_A(\omega)=0$ for $w\notin A$, here, I_A is not Borel measurable

because $\{w: I_A(w)=1\}=A\notin \mathcal{B}(\mathbb{R})$. Indicator functions are sometimes called *characteristic functions.*

3.7 Simple Function

Let $(\Omega, \mathcal{F})$ be a measurable space. If $h:\Omega\to\overline{\mathbb{R}}$, h is said to be *simple* if and only if it is Borel measurable and takes on only finitely distinct values or that h can be written as a finite sum

$$\sum_{i=1}^{r} x_i I_{A_i}, \tag{3.19}$$

where A_i are disjoint sets in $\mathcal{F}$ and I_{A_i} is the indicator of A_i, x_i need not be distinct.

3.8 Integrals

Let h be simple such that $h=\sum_{i=1}^{r} x_i I_{A_i}$ where A_i are disjoint in $\mathcal{F}$, the operation

$$\int_{\Omega} h\,d\mu=\sum_{i=1}^{r} x_i\mu(A_i) \tag{3.20}$$

is an integral provided $+\infty$ and $-\infty$ do not both appear in the sum. If they do, the integral does not exist.

3.8.1 Square integrability

Let a and b satisfy $-\infty<a<b<\infty$. If $f\in\Lambda^2[a,b]$, then

$$\int_a^b \left| f(x) \right|^2 dx < \infty \qquad , \tag{3.21}$$

3.8.2 Ito integrals on L^2

Let $X_t = X(t)$ be a step function in $L^2[\alpha, \beta]$, say $X_t = X_{t_1}$ if $t \le t_i < t_{i+1}$, $0 < i < r-1$ where $\alpha = t_0 < t_1 < \ldots < t_r = \beta$. The random variable $\sum_{k=0}^{r-1} X_{t_k} \left[W(t_{k+1}) - W(t_k) \right]$ is denoted by $\int_\alpha^\beta X_t dw(t)$ and is called the *stochastic integral* of *X* with respect to the Brownian motion *W*. It is also called the *Ito Integral*.Let $X(t)$ be an adapted process such that $\int_0^T X^2(t) dt < \infty$, $t \in [0, T]$. Then the Ito Integral $\int_0^T X(t) dB(t)$ is defined with the following properties: if $X(t)$ and $Y(t)$ are defined and α, β are constants, then

$$\int_0^T \left(\alpha X(t) + \beta Y(t) \right) dB(t) = \alpha \int_0^T \left(X(t) \right) dB(t) + \beta \int_0^T \left(Y(t) \right) dB(t) \text{ or}$$

$$\int_0^T X(t) I_{(a,b]}(t) dB(t) = \int_a^b X(t) dB(t)$$

If the process satisfies $\int_0^T E\left(X^2(t) \right) dt < \infty$ then

$$E\left(\int_0^T X(t) dB(t) \right) = 0 \qquad \text{--} \qquad \text{(zero mean property)}$$

$$E\left(\int_0^T X(t) dB(t) \right)^2 = \int_0^T E\left(X^2(t) \right) dt \qquad \text{--} \qquad \text{(isometry property)}$$

If both $X(t)$ and $Y(t)$ satisfy $\int_0^T E\left(X^2(t) \right) dt < \infty$,

then $E\left(\int_0^T X(t) dBt \int_0^T Y(t) dBt \right) = \int_0^T E\left(X(t) . Y(t) \right) dt$, Klebaner, (2005)

3.9 Sample Path

For each point $w \in \Omega$, the map $t \to X(t, w)$ is called the corresponding *path* or *realization* or *trajectory* of the stochastic process. The time that evolves as an observer observes an experiment being ran involving a random variable $X(t)$ is in fact the sample path. i.e. $\{X(t, w)\ t \geq 0\}$ for some fixed $w \in \Omega$. Re-running the experiment means looking at another sample path.

3.10 Stochastic Processes

Let (Ω, F, P), be a probability space. Denote by $\lambda(\Omega, F, P)$, the collection of all random variables defined on (Ω, F, P) with $(\square, \square)$ as the *state* space (finite or countably infinite set).By a *stochastic process* it means a collection or family of random variables $X = \{X_t : t \in T\}$ indexed by time, $T \subseteq [0, \infty)$. Since the index *t* represents time, then X_t can be thought of as the “state” or the “position” of the process at time *t*. *T*, is called the parameter set and $(\square, \square)$, the state of the process. If *T* is countable, the process is said to be *discrete parameter process.* If *T* is not countable, the process is said to be a *continuous parameter process*. In the latter case, the usual examples are $T = \square_+ = [0, \infty)$ and $T = [a, b] = \square$.

If the family of *T*-indexed stochastic process is denoted by $\lambda^T(\Omega, F, P)$, then $X_t \in \lambda(\Omega, F, P)$ and $(t, w) \in T \times \Omega$. Thus a stochastic process is a function of two variables i.e. a function of time $t \in T$ and a random variable $w \in \Omega$. Therefore, $X : T \times \Omega \to \square$, can be written as $X(t, w)$ or $X(t)(w)$ Friedman, (1975),.

Consider the arrivals of customers at a store, and suppose the experiment is set up to measure the inter-arrival times. Let the inter-arrival times be positive random variables $X_1, X_2 ....$ Then for each $t \in [0, \infty)$, put $N_t = k$ if and only if the integer *k* is such that $X_1 + ... + X_k \le t < X_1 + ... + X_{k+1}$, and put $N_t = 0$ if $t < X_1$. Then N_t is the number of arrivals in the time interval $[0, t]$. It can be seen that for each $t \ge 0$, N_t is a random variable taking values in the set $\mathbb{N}$. Thus, $\{N_t, t \ge 0\}$ is a continuous time process with values in the state space $\mathbb{N}$. The sample paths of this process are non-decreasing, right continuous and they increase by jumps of size 1 at the points $X_1 + ... + X_k$. On the other hand, $N_t < \infty$ for all $t \ge 0$ if and only if $\sum_{k=1}^{\infty} X_k = \infty$. Thisis called an *arrival process*.

A discrete time stochastic process $\{X_n, n = 0,1,2,...\}$ with a finite number of states $S = \{1,2,3\}$ is such that: You move from state 1 to state 2 with probability 1. From state 3 you move either to 1 or to 2 with equal probability $\frac{1}{2}$, and from 2 you jump to 3 with probability $\frac{1}{3}$, otherwise stay at 2. This is an example of a *Markov chain*.

3.10.1Wiener process

A real-valued stochastic process $W(.)$ is called a *Brownian motion* or a *Wiener process* if

$W_0 = W(0) = 0$, almost surely (a.s.). $E(W_t) = 0$, i.e. the expectation (mean) of W_t is zero. The Wiener increments are independent on any non-overlapping interval, meaning that if $\left(S_1, t_1\right) \bigcap \left(s_2, t_2\right) = \varnothing$ then $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent

$$W\left(t\right) - W\left(s\right) \text{ is } N\left(0, t-s\right) \tag{3.22}$$

i.e. $W\left(t\right) - W\left(s\right)$ is normally distributed with mean zero and variance $t - s$

The correlation coefficient function $E(W_t W_s) = t \wedge s$, where $t \wedge s$ means the $\min(t, s)$

The sample paths of the Wiener process are continuous functions for $t \in \left[0\ ,\ \infty\right)$ but not differentiable anywhere.

Given, $W_u^s = \left\{W_u : 0 \le u \le s\right\}$, then $E\left(W_t / W_0^s\right) = W_s$ i.e. the martingale property holds.

The expectation of the variance given W_0^s is the variance itself,

$$\text{i.e. } E\left[\left(W_t - W_s\right)^2 / W_0^s\right] = t - s \tag{3.23}$$

For any positive integerss, *t*, the following are also valid, Hida, (1980).

$$\operatorname{var}\left(W_t\right) = E\left(W_t^2\right) = t \tag{3.24}$$

$$E\left(W_t - W_s\right) = 0 \tag{3.25}$$

$$E\left[\left(W_t - W_s\right)^2\right] = t - s \tag{3.26}$$

Weiner process has stationery increments since $W\left(t\right) - W\left(s\right)$ and $W\left(t + t'\right) - W\left(s + t'\right)$ have the same density function:Andrzej and Michael (2008)

$$g\left(t, x\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}} \tag{3.27}$$

Thus,

$$\begin{aligned} E\left[\left(W_t - W_s\right)^n\right] &= \frac{1}{2\pi\left(t-s\right)} \int_{-\infty}^{\infty} x^n e^{-\frac{x^2}{2t}} dx \\ &= \begin{cases} 1, 2, \ldots \left(n-1\right)\left(t-s\right)^{\frac{n}{2}} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases} \end{aligned} \tag{3.28}$$

3.10.2 Adapted process

Let $\{W(t)\}_{t\geq0}$ be a Brownian motion defined on (Ω, F, P) with filteration $\{\mathrm{F}_t\}_{t\geq0}$, if for any $s\geq0$, the random variable F_s is measurable relative to the σ-algebra of F_s, i.e. the value of the random variable F_s is completely determined by the Brownian motion $\{W(t)\}_{t\geq0}$ up to the time *s*, then $\{\mathrm{F}_t\}_{t\geq0}$ will be said to be adapted to $\{W(t)\}_{t\geq0}$.

3.10.3Separable process

A stochastic process $\{X_t\}_{t\geq0}$ defined on the space (Ω, F, P) is called separable if there exists a countably dense subset $S=\{s_1, s_2 ......\}$ of $T=[0\ ,\ \infty)$- called the separant or the separating set such that for any open interval I_0 and any closed interval I_c, the subset $\overline{A}=\bigcup_{t\in T\bigcap I_0}\{\mathrm{w}\in\Omega : X_t(\Omega)\in I_c\}$ of Ω differs from the event

$A=\bigcup_{s_j\in S\bigcap I_0}\{\mathrm{w}\in\Omega : X_{s_j}(\Omega)\in I_c\}$ by a subset of a null event. Equivalently, $\{X_t\}_{t\geq0}$ is separable if there is a countable sequence $S=\{s_i\}$ that is a dense subset of *I* (an interval) and a subset *A* of Ω with *P*(*A*) = 0 such that for $\omega\notin A$,

$$\{X(t,\mathrm{w})\in \mathrm{F}\ \ \forall t\in J\}=\{X(s_j,\mathrm{w})\in \mathrm{F}\ \ \forall s_j\in J\} \quad , \tag{3.29}$$

for any open subset *J* of *I*

3.10.4Stochastic equivalence

Two stochastic processes $\{X_t\}_{t\geq0}$ and $\{Y_t\}_{t\geq0}$, $t\in I$ (an interval) defined on the same probability space are said to be *stochastically equivalent* if $P\left[\{X_t\}_{t\geq0}\neq\{Y_t\}_{t\geq0}\right]=0$ or if $P\left[\{X_t\}_{t\geq0}=\{Y_t\}_{t\geq0}\right]=1$ for all $t\in I$. Call $\{X_t\}_{t\geq0}$ the version of $\{Y_t\}_{t\geq0}$. Two equivalent processes may have quite different sample paths. For instance, let η be a nonnegative random variable with continuous distribution function. Set $T=[0,\infty)$. The processes

$X_t=0$ and $Y_t=\begin{cases}0 \text{ if } \eta\neq \mathrm{t}\\ 1 \text{ if } \eta=t\end{cases}$ are equivalent but their sample paths are different.

3.10.5Indistinguishable processes

Two stochastic processes $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ are said to be *indistinguishable*

if $X.(\mathrm{w})=Y.(\mathrm{w})$ for all $\mathrm{w}\notin N$, with $P(N)=0$.Two discrete time stochastic processes which are equivalent are also indistinguishable. Also, twoequivalent stochastic processes which have right continuous sample paths are indistinguishable.

[Shreve, (2004)]

3.10.6Continuity in probability processes

A real-valued stochastic process $\{X_t\}_{t\geq 0}, t\in T$ and T is an interval of □ , is said to be *continuous in probability* if for any $\varepsilon>0$ and $\forall\, t\in T$

$$\lim_{s\to t} P\left(\left|X_t-X_s\right|>\varepsilon\right)=0$$

3.10.7 Stationery and symmetric processes

Let $\{X_t\}_{t\geq 0}$ be the stochastic process with values in K(a field of real or complex numbers). Then the process is said to be *stationery* if $\{X_t\}_{t\geq 0}$ and $\{X_{t+s}\}_{t\geq 0}, s\geq 0$ are equivalent for all $s\in$□ . It is *symmetric* if the process $\{X_t\}_{t\geq 0}$ and $\{X_{-t}\}_{t\geq 0}$ are equivalent.It will be called *weakly stochastically continuous* if $f\left(X_t\right)$ is continuous in measure as a function of $t \in \mathbb{R}$ for continuous functions f over K, Klein, (1981).

3.10.8Periodic stochastic process

A stochastic process $\left\{X_t\right\}_{t\geq 0}$ is called periodic with period β, $0<\beta<\infty$, if $X_{t+\beta}=X_t$ for all $t \in \mathbb{R}$. A non-periodic will be said to be periodic with period ∞

3.10.9Lévy process

Let $\left(\Omega, \mathrm{F}, \left\{\mathrm{F}\right\}_{t\geq 0} P\right)$ be a filtered probability space satisfying the usual hypothesis of completeness and right continuity. Let X be such that $X\left(0\right)=0$ almost surely (a.s). If X has stationery increments or strongly independent increments i.e. $X\left(t\right)-X\left(s\right)$ is independent of F_s for all $0\leq s<t<\infty_s$, then $X=\left\{x_t\right\}_{t\geq 0}$ will be called an adapted real-valued *Lévy process*, Applebaum, (2009), defined on $\left(\Omega, \mathrm{F}, P\right)$.

3.10.10Step function

A stochastic process $\left\{X_t\right\}_{t\geq 0}$ defined on $\left[\alpha\ ,\ \beta\right]$ is called a step function if there exists a partition $\alpha=t_0<t_1<........t_r=\beta$ of $\left[\alpha\ ,\ \beta\right]$ such that

$X_t=X_{t_i}$ if $t_i\leq t<t_{i+1}$, $0\leq i<r-1$

3.10.11Simple process

An adapted process $\left\{X_t\right\}_{t\geq 0}$ is said to be a simple process if it is a random step function. This means that for some sequence of time $0<t_1<t_2.....<t_m<\infty$ and random variables $\xi_0<\xi_1<\xi_2.....<\xi_{m-1}<\infty$ such that ξ_j is measurable relative to F_{ij},

$$X(s)=\sum_{j=0}^{m-1}\xi_j I_{(t_j,t_{j+1})}(s)$$

3.10.12 Additive process

An *additive process* is a family $L=(\mathcal{L}(s,t))_{0\le s\le t}$ of random variables defined on $\mathfrak{D}$ -(the total subsets of $\mathcal{H}$) such that

(i) for all $s\le t\in\mathbb{R}^{+}$, the operator $\mathcal{L}(s,t)$ is a *t*-adapted operator

(ii) for all r,s,t with $r<s<t, \mathcal{L}(r,t)=\mathcal{L}(r,s)+\mathcal{L}(s,t)$; $\mathcal{L}(t,t)=0$ on $\mathfrak{D}_t$

3.11 The Clifford Calculus

In the analogue of the classical theory, let the random variable of a stochastic process $\{X_t\}_{t\ge0}$ be replaced by operators. Though it may be observed that the operators which enter the theory are not random, and so the theory is different in spirit from for example quantum mechanics with random potentials. Quantum mechanics is a probability theory (a non Kolmogorovian model) with enough stochasticity. Recalling the classical theory where the concept of stochastic process is relative to a filtration $(\mathrm{F}_t)_{t\ge0}$, in place of Ω the emphasis is on $Ł^2(\Omega,\mu,\mathrm{F})$, so, $Ł^\infty(\Omega,\mu,\mathrm{F})$ (an abelian W*-algebra) is written in place of F . The vector $\xi(\mathrm{W})=1$ is a vector in $Ł^2(\Omega,\mu,\mathrm{F})$, cyclic for the algebra $Ł^\infty(\Omega,\mu,\mathrm{F})$. In place of the filtration $(\mathrm{F}_t)_{t\ge0}$, the emphasis is on the filtration of the W*-algebra $(L^\infty(\Omega,\mu,\mathrm{F}_t))_{t\ge0}$.A stochastic process of class L^P is then a map $X:[0,\infty)\to L^P(\Omega,\mu,\mathrm{F})$ such that $X_t\in L^\infty(\Omega,\mu,\mathrm{F})$ for all $t\in[0,\infty)$ is defined. This is to say that $\{X_t\}_{t\ge0}$ is adapted to $(\mathrm{F}_t)_{t\ge0}$.If $p\ge2, X_t$ is a vector in $Ł^2(\Omega,\mu,\mathrm{F}_t)$, it can also be regarded as an operator affiliated to $L^\infty(\Omega,\mu,\mathrm{F}_t)$, i.e. commuting with the commutant of $L^{\infty}(\Omega,\mu,\mathrm{F}_{t})$. Thus the general classical theory is formulated which looks more like a quantum theory.To formulate a quantum analogue of these concepts, F or rather $L^{\infty}(\Omega,\mu,\mathrm{F})$ is replaced by a W*-algebra $\mathcal{A}$, which in general is non abelian with a cyclic vector $\xi\in\mathcal{H}$. In place of (F_{t}) or rather $L^{\infty}(\Omega,\mu,\mathrm{F}_{t})$, an increasing family $(\mathcal{A}_{t})$ of W*-algebra with $\mathcal{A}$ being generated by $\bigcup_{t}\mathcal{A}_{t}$ is put in place. If ξ defines a trace *tr*, on $\mathcal{A}$ with $tr(A)=\langle\xi,A\xi\rangle$, then the noncommutative $L^{P}-$spaces, $L^{P}(\mathcal{A},tr)$ can be defined.

3.12Quantum Stochastic Process

From the construction above, a quantum stochastic process is the system $(\mathcal{A},\ \{\mathcal{A}_{t}\}_{t\geq 0},\xi)$ is a map $X:t\rightarrow X_{t},\ t\in[0,\infty)$ where X_{t} is affiliated to $\mathcal{A}_{t}$ i.e. commutes with commutant of $\mathcal{A}_{t}$ This means that $X:[0,\infty)\rightarrow\mathcal{A}$ with $X_{t}\in\mathcal{A}_{t}$.The natural setting for quantum theory is the theory of C*-algebra whose various representations give algebra of operators. The simple ones are the C*-algebra of the CCR for the Bosons and C*-algebra of CAR for the Fermions. Each of these algebras is determined by the choice of a complex Hilbert space *K*, " the one particle space" of wave function f, and a linear map $f\rightarrow a(f)^{*}$ from *K* to operators acting on Hilbert space $\mathcal{H}$ and obeying the CCR or CAR namely

$$a(f)^{*}a(g)\pm a(g)a(f)^{*}=\langle g,f\rangle_{K} \tag{3.30}$$

(upper sign for Fermions, lower sign for Bosons) and

$$a(f)a(g)\pm a(g)a(f)=0\ \forall f,g\in K. \tag{3.31}$$

Here, $\langle .,. \rangle_K$ denotes the scalar product in K, which is linear in the second variable and conjugate linear in the first variable. $a(f)$ will be called the *annihilation operator* for wave function $f \in K$ with the norm

$$\|a(f)\| = \|f\| \tag{3.32}$$

and its conjugate $a(f)^*$ will be called the *creation operator*.

In Fermion case (upper sign), the set of all operators $\{a(f) : f \in K\}$ generates a C*-algebra depending only on K. This is the CAR. The Boson operators $a(f)$ (lower sign) are bounded, but subject to certain conditions can also be used to define a unique C*-algebra, the CCR.Of the many representations of the CCR or CAR, the most important ones are the Fock representations where there is a vector ξ_F called the *Fock vacuum*, in the representation of $\mathcal{H}$, and ξ_F is annihilated by all $a(f)$, that is:

$$a(f)\xi_F = 0 \text{ for all } f \in K \tag{3.33}$$

3.13Symmetric and Anti-symmetric Tensor Products

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle . , . \rangle$ which is anti-linear in the first variable and let $p(\mathcal{H})$ denote the set of all projections in $\mathcal{H}$. Let S_n denote the symmetric permutation group of the set $\{n = 1, 2, ..., n\}$, thus $\sigma \in S_n$ is a bijection. For each $\sigma \in S_n$, let U_σ be defined on the product vectors in $\mathcal{H}^{\otimes n}$ (the n-fold tensor product of $\mathcal{H}$) by

$$U_\sigma h_1 \otimes ... \otimes h_n = h_1\sigma^{-1}(1) \otimes h_2\sigma^{-1}(2) \otimes ... \otimes h_n\sigma^{-1}(n), \quad n \in \square , h_i \in \mathcal{H} , \tag{3.34}$$

where σ^{-1} is the inverse of σ. Then U_σ is the scalar product of preserving map of the total set of product vectors in $\mathcal{H}^{\otimes n}$ onto itself. The mapping $\sigma \to U_\sigma$ is a homomorphism from the finite S_n into Y($\mathcal{H}^{\otimes n}$), the unitary group on $\mathcal{H}^{\otimes n}$.

The closed subspaces

$$\mathcal{H}^{\& n} = \{h \in \mathcal{H}^{\otimes n} : U_\sigma h = h \ \ \forall \sigma \in S_n\} \tag{3.35}$$

$$\mathcal{H}^{!\ n} = \{h \in \mathcal{H}^{\otimes n} : U_\sigma h = \varepsilon(\sigma) h \ \ \forall \sigma \in S_n\}, \tag{3.36}$$

where $\varepsilon(\sigma) = \pm 1$, according to whether the permutation is even or odd. The subspaces are called the *n-fold symmetric* and *anti-symmetric tensor products* of $\mathcal{H}$ respectively.

3.14Boson and Fermion Fock Spaces

If the statistical feature on the dynamics of a single particle are described by states on *p*($\mathcal{H}$) and the dynamics of *n* such identical particles are described by states on *p*($\mathcal{H}^{\otimes n}$) for *n* = 2, 3,…, then such a particle is called *Boson*. On the other hand, if it is described by states on *p*($\mathcal{H}^{\otimes n}$) for every *n*, then such a particle is called *Fermion*.

Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^{\otimes n}$, $\mathcal{H}^{\& n}$, and $\mathcal{H}^{!\ n}$ be the *n*-fold tensor products, the symmetric tensor products and anti-symmetric tensor products of $\mathcal{H}$ respectively, where 0-fold product is the 1-dimensional complex plain and the 1-fold product in $\mathcal{H}$ is itself in all the three cases. That is $\mathcal{H}^{\otimes.0} = \mathcal{H}^{\& 0.} = \mathcal{H}^{!\ 0} = \mathbb{C}$ - a one particle subspace. The spaces,

$$\Gamma_{fr}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n}, \qquad \Gamma_s(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\& n}, \qquad \Gamma_\sigma(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^{!\ n} \tag{3.37}$$

are respectively called the *free* (or *Maxwell-Boltzman*), the *symmetric* (or *Boson*) and the *anti-symmetric* (or *Fermion*) Fock spaces over $\mathcal{H}$. The *n*th direct summand in each case is called the *n-particle* subspaces.

It can be observed that the one particle space K is not the same as the space $\mathcal{H}$ on which the operators act. In fact, $\mathcal{H}$ is the "Fock space over K" that is the indefinite direct sum,

$$\mathbb{C} \oplus K \oplus (K \oplus K) \oplus (K \oplus K \oplus K) \oplus ... \tag{3.38}$$

of zero, one-, two-, … and many particle wave functions, symmetrised for Bosons or anti-symmetrised for Fermions. A simplest model of quantum stochastic process can be obtained by choosing

$K = L^2(\mathbb{R}^+, dt)$. The parameter t on which one particle wave functions depend, is interpreted as time. To construct the objects arising from the discussions above, put $\mathcal{A}_t$ = W*-algebra generated by $\{a(f) : f \in K_t\}$ where $K_t = L^2([0,t], dt)$-the set of all wave functions zero outside $[0,t]$. For cyclic vector ξ, choose the Fock vacuum ξ_F. The set up now is $(\mathcal{A}, \{\mathcal{A}_t\}_{t\geq 0}, \xi_F)$.

In the Boson case, all $\{X_t\}_{t\geq 0}$ commute and generate an abelian subalgebra $\mathrm{F} \subseteq \mathcal{B}(\mathcal{H})$ and sub-filtration $\{\mathrm{F}_t\}_{t\geq 0}$.The process X_t when $a(f)$ are Fermion operators is called the *Clifford process*, sometimes denoted by Ψ_t. Ψ_t is integrable in a Fermion field $\psi(t)$ over [0, t] such that

$$\Psi_t = \int_0^t \psi(s)\, ds = \psi(\chi_{[0,t]}). \tag{3.39}$$

This is a self-adjoint operator for each *t*. Being a Fermion field, its increments anti-commute rather than commute, i.e.

$$(\Psi_s - \Psi_t)(\Psi_u - \Psi_v) = -(\Psi_u - \Psi_v)(\Psi_s - \Psi_t) \text{ if } s < t < u < v \tag{3.40}$$

The Fermion field is defined by $\psi(f) = a(f) + a(f)^*$

Let $\mathcal{A}_t$ be a von Neumann algebra generated by $\{\Psi_s, 0 \le s \le t\}$. The family $\{\mathcal{A}_t\}_{t\ge 0}$ is the analogue of the “filtration” of a stochastic process. It is continuous to the right and left, i.e. $\mathcal{A}_t = W^*\{\mathcal{A}_s, s < t\} =$ von Neumann algebra generated by $\{\mathcal{A}_s, s < t\}$

$\mathcal{A}_t = \bigcap_{s>t} \mathcal{A}_s$; $\mathcal{A} = W^*\{\mathcal{A}_t, t \ge 0\}$; $\mathcal{A}_s \subseteq \mathcal{A}_t$ if $s \le t$. $\mathcal{A}_s$ is said to be the “noise up to time *t*”

3.15Clifford Operator Algebra

Let $\mathscr{C}$ denote the W^* - algebra generated by the bounded operators $\{\psi(z) : z \in \mathscr{H}\}$. $\mathscr{C}$ is called the *weakly closed Clifford operator-algebra* of $(\mathscr{H}, J)$. The *fock vacuum* (no particle vector) is the vector $\xi_F = 1 \in \mathscr{H}^{!0} \subset \Gamma_\sigma(\mathscr{H})$. It is known that ξ_F is cyclic for $\mathscr{C}$ and $tr(.) = (\xi_F, \xi_F)$ is a faithful central state of $\mathscr{C}$, so that $(\Gamma_\sigma(\mathscr{H}), \mathscr{C}, tr)$ is a *regular probability gauge space*.For $1 \le p < \infty$, $L^p(\mathscr{C})$ is the completion of $\mathscr{C}$ with respect to the norm

$$\|u\|_p = \tau\left(|u|^p\right)^{\frac{1}{p}} = \left\langle \xi_F, \left(u^* u\right)^{\frac{p}{2}} \xi_F \right\rangle^{\frac{1}{p}} \qquad . \tag{3.41}$$

The elements of $L^p(\mathscr{C})$ can be identified with operators (possibly unbounded) on $\Gamma_\sigma(\mathscr{H})$. $L^\infty(\mathscr{C})$ is by definition $\mathscr{C}$ equipped with its operator norm. The map $u \mapsto \mu\xi_F$ from $\mathscr{C}$ into $\Gamma_\sigma(\mathscr{H})$ extends to a unitary operator

$$D : L^2(\mathscr{C}) \to \Gamma_\sigma(\mathscr{H}), \tag{3.42}$$

and is called the *duality transformation*.The action of $\mathscr{C}$ on $\Gamma_\sigma(\mathscr{H})$ under *D* becomes, left multiplication on $L^2(\mathscr{C})$ so that $(\Gamma_\sigma(\mathscr{H}), \mathscr{C}, tr)$ is well defined;

Barnett, *et al* (1982).If $\mathscr{B}$ is a W^*- subalgebra of $\mathscr{C}$, then $L^P(\mathscr{B})$ is the completion of $\mathscr{B}$ with respect to $\|.\|_P$ and so can be considered as the dual space of $L^P(\mathscr{C})$ Tomiyama, (1957)

3.16The L^P -Martingale

An L^P- martingale (martingales and related properties are discussed in chapter iv) adapted to the family $\{\mathscr{C}_t : t \in \mathbb{R}_+\}$ is a collection $\{X_t : t \in \mathbb{R}_+\}$ of operators on $\Gamma_\sigma(\mathscr{H})$ such that $X_t \in L^P(\mathscr{C}_t)$ for $t \in \mathbb{R}_+$ and $E_s(X_t) = X_s \ \forall\ 0 \le s \le t.$

It can be noticed that if $X \in L^P(\mathscr{C})$, then $\{E_t(X) : t \in \mathbb{R}_+\}$ is an L^p- martingale. For $1 < p < \infty$, it is shown that an L^p- martingale X_t is of this form if and only if $\sup_t \|X_t\|_p < \infty$, for which if $p = 1$, $\{X_t\}$ is required to be weakly relatively compact. L^p- martingale are also discussed in Barnett, *et al* (1982), Barnett, (1981), Gross,. (1972), Kunze, (1958), Segal, (1953), Stinespring, (1959) and Yeadon, (1975).

3.17 Lemma[Barnett *et al*, (1982)]

Let $\{X_t : t \in \mathbb{R}^+\}$ be an L^p–martingale, and suppose that $0 \le s \le t$. Then $m\big(f(X_t - X_s)g\big) = 0$ for any $f \in L^q(\mathscr{C}_s)$, $g \in L^r(\mathscr{C}_s)$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. In particular, $m(f(\psi_t - \psi_s)g) = 0$, for any $f \in L^q(\mathscr{C}_s)$, $g \in L^r(\mathscr{C}_s)$ with $\frac{1}{q} + \frac{1}{r} = 1$.

Proof

We have

$$\begin{aligned} m\left(f\left(X_t - X_s \right) g \right) &= m\left(gf\left(X_t - X_s \right) \right) \\ &= m\left(gfE_s\left(X_t - X_s \right) \right) \\ &= 0 \end{aligned}$$

The particular case follows since $\psi_t \in L^\infty$ for all t .

3.18Definite Parity

An element $h \in L^2\left(\mathscr{C}\right)$ is said to be even or odd if $\beta h = h$ or $-h$, respectively where $\beta : L^2\left(\mathscr{C}\right) \to L^2\left(\mathscr{C}\right)$ is a self-adjoint unitary operator. *h*is said to have a*definite parity* if *h* is either even or odd. If $f : \left[t_0\ , t\right] \to L^2\left(\mathscr{C}\right)$, we say that *f* is even or odd if $\beta f\left(s\right) = f\left(s\right)$ or $-f\left(s\right)$, respectively for all $s \in \left[t_0\ , t\right]$. Also *f*has definite parity if *f* is even or odd.

Writing $h \in L^2\left(\mathscr{C}\right)$ as $h = \frac{1}{2}\left(h + \beta h\right) + \frac{1}{2}\left(h - \beta h\right)$, it is seen that *h* is the sum of an even element and an odd element of $L^2\left(\mathscr{C}\right)$. (0 is defined asboth even or odd), Yeadon, (1975).

3.19 Lemma [Barnett *et al*, (1982)]

Let $0 \le t_\circ \le s \le t$, and suppose $g \in L^2\left(\mathscr{C}_{t_0}\right)$ has definite parity. Then

$$\left(\psi_t - \psi_s\right) g = \pm g\left(\psi_t - \psi_s\right)$$

CHAPTER FOUR

STOPPING TIMES ON FILTERED PROBABILITY SPACE

4.1 Introduction

It is known that some amount of classical theory has a counterpart in quantum (non-commutative) context. Surveys on the classical stopping times have been undertaken and some constructions have been made.Other constructions and analysis on martingales and stochastic processes are also considered.

4.2 Martingale

As stated in 2.10 the word “martingale” means a gambling system in which loosing bets are doubled. The concept may probably be understood from the idea of conditional expectations.

4.2.1 Conditional expectation.

The conditional expectation or conditional mean of *Y* given *X*is defined by

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y f(y \mid x)\, dy \tag{4.1}$$

where $f(y/x)$ is the conditional density function of *Y* given *X*. Similarly, the conditional variance of *Y* given *X*can be definedas

$$E\left[(Y-\mu_2)^2 \mid X = x\right] = \int_{-\infty}^{\infty} (y-\mu_2)^2 f(y \mid x)\, dy \tag{4.2}$$

where $\mu_2 = E(Y \mid X = x)$. If *X* and *Y* are independent, then

$$E(Y \mid X = x) = E(Y) \tag{4.3}$$

and

$$E(Y)=\int_{-\infty}^{\infty}E(Y\mid X=x)\,f_1(x)\,dx \qquad (4.4)$$

where $f_1(x)$ is the marginal density function of the random variable *X*. Spiegel *et al*(2000).For some given sets *A*, *B* and *C*, and random variables $X, X_n \in L^1$, conditional expectation has the following properties:

(i) the map $X \to E[X \mid B] \geq 0$ is linear

(ii) For $X \geq 0 \Rightarrow E[X \mid B] \geq 0$ -positivity

(iii) For $C \subset B \subset A \Rightarrow E\big[E[X \mid B] \mid C\big] = E[X \mid C]$ - tower property

(iv) For $|X_n| \leq X, E[\liminf_{n\to\infty} X_n \mid B] \leq \liminf_{n\to\infty} E[X_n \mid B]$- conditional Fatou

(v) For $|X_n| \leq X, X_n \to X$ $a.e \Rightarrow E[X_n \mid B] \to E[X \mid B]$ $a.e$ - conditional dominated convergence

(vi) If h is convex, $E\big[h(X) \mid B\big] \geq h\big(E[X \mid B]\big) \Rightarrow \big\|E[X \mid B]\big\|_p \leq \|X\|_p$ --- conditional Jensen

(vii) For $z \in L^{\infty}(B) \Rightarrow E[zX \mid B] = zE[X \mid B]$ - extracting knowledge

(viii) For X independent of $C \Rightarrow E[X \mid C] = E[X]$ - independence

Suppose $X_1, X_2 ..., X_n$ is a sequence of random variables representing the winnings in a game after *n* trials in succession. Having survived the first *n* trials, the expected fortune after trials n+1 will be

$$E(X_{n+1} \mid X_1, X_2, ..., X_n) \qquad (4.5)$$

If this equals X_n, the game is "fair" since the expected gain after trials *n*+1 is

$$E\left(X_{n+1} - X_n \mid X_1, X_2, ..., X_n\right) = X_n - X_n = 0. \tag{4.6}$$

If

$$E\left(X_{n+1} \mid X_1, X_2, ..., X_n\right) \geq X_n, \tag{4.7}$$

then the game is "favourable" and if

$$E\left(X_{n+1} \mid X_1, X_2, ..., X_n\right) \leq X_n, \tag{4.8}$$

the game is "unfavourable".

From (4.5) – (4.8), it can be seen that a random process $X = \{X_k\}$ which models a player's fortune in a fair game is his expected fortune at time *k* given the history up to this point is equal to his current fortune. $E\left[X_{k+1} \mid X_1, ..., X_k\right] = X_k$. This is equivalently, (4.5) and this in turn implies for all *k*,

$$E\left(X_{k+1}\right) = E\left(X_k\right) = E\left(X_{k-1}\right) = ... = E\left(X_1\right) = E\left(X_0\right) \tag{4.9}$$

So, player's expected fortune at any time is equal to his starting expected fortune. The question is whether the game remains fair when stopped at a randomly chosen time. Specifically, if τ is a random stopping time and X_τ denotes the game stopped at this time, the question is whether $E\left(X_t\right) = E\left(X_0\right)$ as well?In general, the answer is no as pointed out by Doyle and Snell, (1984). The duo envisaged a situation where the player is allowed to go into debt by any amount to play for an arbitrarily long time. In such a situation, the player may inevitably come out ahead. There are conditions which will guarantee fairness. Doyle and Snell give them in theorem 4.2.2 below.

Let $\{X_1, X_2 ..., X_n\}$ be a sequence of integrable random variables on a probability space $\left(\Omega, \mathrm{F}, P\right)$ and $\mathrm{F}_1 \subset \mathrm{F}_2 \subset \mathrm{F}_3 \subset ...$, an increasing sequence of sub-σ - fields. Assume

$$X_n : \left(\Omega, \mathrm{F}_n\right) \to \left(\mathbb{R}, \mathcal{B}(\mathbb{R})\right).$$

The sequence $\{X_n\}$ is said to be a *martingale* relative to F_n or $\{X_n, \mathrm{F}_n\}$ is a martingale if and only if

$$E\left(X_{n+1} \mid \mathrm{F}_n\right) = X_n \text{ almost surely (a.s.)}$$

It is a *submartingale* if and only if

$$E\left(X_{n+1} \mid \mathrm{F}_n\right) \geq X_n, \text{ a.s.}$$

and a *supermartingale* if and only if

$$E\left(X_{n+1} \mid \mathrm{F}_n\right) \leq X_n \text{ a.s.}$$

The condition $E\left(X_{n+1} \mid \mathrm{F}_n\right) = X_n$ a. s. is equivalent to

$$\int_A E\left(X_{n+1} \mid \mathrm{F}_n\right) dp = \int_A X_n dp \quad \forall A \in \mathrm{F}_n .$$

Also,

$$\int_A E\left(X_{n+1} \mid \mathrm{F}_n\right) dp = \int_A X_{n+1} dp \quad \forall A \in \mathrm{F}$$,by definition of conditional expectation.

Therefore $\{X_n, \mathrm{F}_n\}$ is a martingale if and only if

$$\int_A \left(X_n\right) dp = \int_A X_{n+1} dp \quad \forall A \in \mathrm{F}_n ,$$

a submartingale if and only if

$$\int_A \left(X_n\right) dp \leq \int_A X_{n+1} dp \quad \forall A \in \mathrm{F}_n$$

and a supermartingale if and only if

$$\int_A \left(X_n\right) dp \geq \int_A X_{n+1} dp \quad \forall A \in \mathrm{F}_n .$$

So, $E(X_n)$ is constant in a martingale, increases in a submartingale and decreases in a super martingale, Goldstein, (1991).

4.2.2 Theorem [Martingale stopping theorem]

A fair game that is stopped at random time will remain fair to the end of the game if it is assumed that;

(a) There is a finite amount of money in the world

(b) A player must stop if he wins all of this money or goes into debt by this amount

Lalonde (2013), provides a more formal statement of the above theorem and call it Doob's Optional Stopping Theorem and also provided an alternative proof. In Doob's exposition, the concepts of Williams, (1991) and Doob, (1971), were basically used.

4.2.3 Examples

(a) Assume all random variables are integrable.

If $X_n = X$, then, $\{X_n\}$ is a martingale; if $X_1 \le X_2 \le ...$, then, $\{X_n\}$ is submartingale and if $X_1 \ge X_2 \ge ...$, then, $\{X_n\}$ is supermartingale.

(b) Let $Y_1, Y_2, ...$ be independent variables with zero mean and set

$Y_n = \sum_{k=1}^{n} Y_k$, $\mathrm{F}_n = \mathrm{F}\ (Y_1, Y_2, ..., Y_n)$. Then $\{Y_n, \mathrm{F}_n\}$] is a martingale.

It is clear that the requirements for a martingale are a probability space $(\Omega, \mathrm{F}\ , P)$, a sequence of σ-algebras $\mathrm{F}_0 \subset \mathrm{F}_1 \subset ... \subset \mathrm{F}_\mathrm{n} \subset \mathrm{F}$ and a sequence of variables $X_0, X_1, ..., X_n$, that is a stochastic process. The conditions for a martingale are that each X_k is F_k-measurable (i.e. if the information in F_k is known, so is the value of X_k).

This is to say that $\{X_\mathrm{k}\}$ is adapted to the filtration $\{\mathrm{F}_\mathrm{k}\}$ and for each k, $E[X_{\mathrm{k}+1} \mid \mathrm{F}_\mathrm{k}] = X_\mathrm{k}$, meaning that martingales tend to go neither up nor down, Williams, (1991). Martingales are allegory of 'life', Gut, (2004). Supermartingales tend to go down, $E[X_{\mathrm{k}+1} \mid \mathrm{F}_\mathrm{k}] \le X_\mathrm{k}$ and sub-martingales tend to go up, $E[X_{\mathrm{k}+1} \mid \mathrm{F}_\mathrm{k}] \ge X_\mathrm{k}$. Gusak *et al* , (2010) and Strook, (1998).

4.2.4 Theorem[Jan, (2013)]

Let $X = \{X_0, X_1, ..., X_n\}$ be a submartingale. The following maximal inequalities of Doob hold;

(i) $$P\left(\max_{0 \le j \le n} X_j \ge \lambda\right) = \frac{1}{\lambda} E\left[X_n\right]$$

(ii) $$P\left(\max_{0 \le j \le n} \left|X_j\right| \ge \lambda\right) = \frac{2}{\lambda} E\left[5X_n - 2X_0\right]$$ and if X is a martingale, then

(iii) $$P\left(\max_{0 \le j \le n} \left|X_j\right| \ge \lambda\right) = \frac{1}{\lambda} E\left[\left|X_n\right|\right]$$

4.2.5 Theorem[Doob-Meyer's decomposition for discrete sub-martingale]

Let $\{X_j : j \in \square\}$ be a submartingale. Then there exists a unique martingale $M = \{M_k : k \in \square\}$ together with a unique predictable increasing process A_k with $A_0 = 0$ such that

$$X_k = M_k + A_k \text{ for } k \in \square \, .$$

The proofs are in Doob, (1994) and Jan, (2013).

4.3Markov Processes

In the case of Markov process, the requirements are the subsets $A \in \mathcal{B}(\square)$ and a function $g : \square \to \square$ such that $g^{-1} : \mathcal{B}(\square) \to \mathcal{B}(\square)$. The conditions for a Markov process are the probability space (Ω, F, P), a filtration $\{\mathrm{F}_\mathrm{k}\}_{k=0}^{n}$ under F and a stochastic process $\{X_\mathrm{k}\}_{k=0}^{n}$ on (Ω, F, P). The process will then be called Markov if $\{X_\mathrm{k}\}$ is adapted to $\{\mathrm{F}_\mathrm{k}\}$ and for each $k = 0,1,2,...,n-1$, the distribution of $X_{\mathrm{k}+1}$ conditioned on F_k is the same as $X_{\mathrm{k}+1}$ conditioned on X_k. This is to say that in the Markov property, there are agreements of distributions expectations of all functions (example, Laplace and characteristic functions). For these reasons, in every case of the Markov properties, $g(X)$ could just as well be written for some function g. [Soren *et al*, (2013)]. For instance, if $A \in \mathcal{B}(\square)$, then $P\left[X_{k+1} \in A \mid \mathrm{F}_k\right] = g\left(X_k\right)$ with g a function depending on the set A.The Markov property says that if the state of a system at a particular time t known, additional information regarding the behavior of the system at times $s < t$ has no effect on the estimate of the probable development of the system at $s > t$. i.e. the future is independent of the past when the present is known. This property is characterized as a property of a "certain process" called the *Markov process*. Other discussions on Markov processes may be found in Ioannis and Steven (1988), William (1991), Zonfei and Zonghu (2010) and Lei and Lunart (2012).

4.4Random Walk

Consider a d-dimensional lattice 2^d where each point has 2d neighbours. A walker starts at the origin $0 \in 2^d$ and makes in each time a random step into one of the 2d directions. Define a sequence of identically independently distributed (IID) random variables X_k which take values in $I = \left\{ e \in z^d : |e| = \sum_{i=1}^{d} |e_i| = 1 \right\}$, and which have the uniform distribution defined by $P\left[X_\mathrm{k} = e\right] = \dfrac{1}{2d}$ for all $e \in I$ .The random variable

$S_\mathrm{k} = \sum_{\mathrm{i}=1}^{k} X_\mathrm{i}$ with $S_0 = 0$, describes the position of the walker at time k. The discrete random stochastic process S_k is called the *random walk* on the lattice 2^d. Define the set $A_\mathrm{k} = \{S_\mathrm{k} = 0\}$ and the random variable $Y_\mathrm{k} = 1_{A_k}$ .If the walker has returned to the position $0 \in 2^d$ at time *k*, then $Y_k = 1$, otherwise $Y_k = 0$. Then $B_k = \sum_{i=1}^{k} Y_i$ accounts for the number of visits of the origin 0 of the walker up to time *k* and $B_k = \sum_{i=1}^{\infty} Y_i$ counts the number of visits at the origin. The expression $E[B] = \sum_{k=0}^{\infty} P[S_k = 0]$, tells us how many times the walker is expected to return to the origin, Szymon, *et al* (2013)

4.4.1Remark

A process $A = \{A_k : k \in \square\}$ is called predictable if A_k is measurable for F_{k-1} $\forall k$. The theorem below in an appropriate form is also true for continuous time process, i.e. $X = \{X_t : t \geq 0\}$.

Discussion onthe concept of stopping time, may begin with an example in a finite probability space (Ω, F, P), a sequence of σ-algebras $F_0 \subset F_1 \subset ... \subset F_n \subset F$ and a sequence of variables $X_0, X_1, ..., X_n$

4.4.2 Example

Let $\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ be the set ofallpossibleoutcomesof threecointosses.Let F bethesetof allsubsetsof Ω. Somesetsin F are $\varnothing$, $\{HHH, HHT, HTH, HTT\}$, $\{TTT\}$,and Ω itself. Supposeacoinhas probability $\frac{1}{3}$ for *H*and $\frac{2}{3}$ for *T*.Fortheindividualelementsof Ω, there are then

$$P\{HHH\}=\left(\frac{1}{3}\right)^3 \qquad P\{HHT\}=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) \qquad P\{HTH\}=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)$$
$$P\{HTT\}=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \qquad P\{THH\}=\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) \qquad P\{THT\}=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$$
$$P\{TTH\}=\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2 \qquad P\{TTT\}=\left(\frac{2}{3}\right)^3$$

For a particular event,

$$P\{HHH,HHT,HTH,HTT\}=\left(\frac{1}{3}\right)^3+2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)+\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2=\frac{1}{3},$$

whichisanother wayof sayingthattheprobabilityof Honthefirst toss is $1/3$.

Herearesomeimportant σ-algebras ofsubsetsof theset Ω.

$$\begin{aligned}
&F_0=\{\varnothing,\Omega\},\\
&F_1=\{\varnothing,\Omega,\{HHH,HHT,HTH,HTT\},\{THH,THT,TTH,TTT\}\},\\
&F_2=\{\varnothing,\Omega,\{HHH,HHT,\},\{HTH,HTT,\}\{THT,TTH,\},\{TTH,TTT\},\text{all sets by their unions}\},\\
&F_3=F\ =\ \text{The set of all subsets of }\Omega.=\ =
\end{aligned}$$

If the sets above are re-represented as

$$\begin{aligned}
&A_H=\{HHH,HHT,HTH,HTT\}=\{H\text{ on the first toss}\},\\
&A_T=\{THH,THT,TTH,TTT\}=\{T\text{ on the first toss}\},\\
&A_{HH}=\{HHH,HHT\}=\{HH\text{ on the first two tosses}\},\\
&A_{HT}=\{HTH,HTT\}=\{HT\text{ on the first two tosses}\},\\
&A_{TH}=\{THH,THT\}=\{TH\text{ on the first two tosses}\},\\
&A_{TT}=\{TTH,TTT\}=\{TT\text{ on the first two tosses}\},
\end{aligned}$$

then

$$F_2=\{\varnothing,\Omega,A_{HH},A_{HT},A_{TH},A_{TT},A_H,A_T,A_{HH}\cup A_{TH},A_{HH}\cup A_{TT},A_{HT}\cup A_{TH},A_{HT}\cup A_{TT},A_{HH}^c,A_{HT}^c,A_{TT}^c\}$$

σ–algebrasareinterpretedasarecordof information, Battacharya and Waymire (2007).Supposethecoinistossedthreetimes,andyou arenottoldtheoutcome,butyouaretold,foreveryset in $\mathcal{F}_1$ whetherornottheoutcomeisinthat set.Forexample,youwouldbetoldthattheoutcomeisnotin $\varnothing$ and isin Ω.Moreover, youmight betold that theoutcomeisnotin A_H butisin A_T .Ineffect,you havebeentold thatthefirsttoss wasa T,and nothingmore.The σ-algebra $\mathcal{F}_1$ issaidtocontainthe"informationof thefirsttoss", whichisusuallycalledthe"informationupto time 1".Similarly, $\mathcal{F}_2$ containsthe"informationof the first two tosses", which is the "information up to time 2". The σ-algebra $\mathcal{F}_3=\mathcal{F}$ contains the "full information" about the outcome of all three tosses. $\mathcal{F}_0$ (the trivial σ-algebra) contains no information, this is because, knowingwhethertheoutcome of an event of thethreetossesis(orisnot)in $\varnothing$ and whetheritis(orisnot)in Ωtellsyounothingabout the event. The information obtained from example 4.4.2 will enable one to define the concept of the classical stopping time with respect to the adapted σ-algebra.

4.5Stopping Time

Let $\left(\Omega,\mathcal{F},\{\mathcal{F}\}_{t\geq 0}P\right)$ be filtration of a probability space and X be a real-valued random variable on the probability space $\left(\Omega,\mathcal{F},P\right)$, then X is called an $\mathcal{F}_t$ *-stopping time* if and only if for each $t\in T$, the set $\{w\in\Omega \mid X(w)\leq t\}\in\mathcal{F}_t$. This is also called a *random* or *Markov time*, Rao, (1984). So a stopping time is a map $X:\Omega\to[0,\infty)$ such that

$$\{w\in\Omega \mid X(w)\leq t\}\in\mathcal{F}_t \quad .$$

From the statement above, it is seen that astopping time is a random variable $t:\Omega\mapsto\{0,1,2,...,n\}\cup\{\infty\}$ with the property that

$$\{w \in \Omega : \tau(w) = t\} \in F_t \quad \forall t = 0,1,2,...,n,\infty$$

Usually, $T = \{0,1,2,...,n\}$ is set as the index set for the time variable *t*, and the σ-algebra F_t is the collection of events which are observable up to and including time *t*. The condition that τ is a stopping time means that the outcome of the event $\{\tau \le t\}$ is known at time *t*. Kruglov, (2011).In discrete time situation, where $T = \{0,1,2,...,n\}$, the condition that $\{\tau \le t\} \in F_t$ means that at any time $t \in T$ it is known based on information up to time *t* if it has been stopped already or not which is equivalent to the requirement that $\{\tau = t\} \in F_t$. This is not true for continuous time case in which *T* is an interval of the real numbers and hence uncountable due to the fact that σ-algebras are not in general closed under taking uncountable unions of events.

A random variable τ is a stopping time for a stochastic process $\{X_t\}$ if it is a stopping time for the natural filtration of *X*, that is $\{\tau \le t\} \in \sigma(X_s : s \le t)$.

The first time that an adapted process X_t, hits a given value or set of values is a stopping time. The inclusion of ∞ into the range of τ is to cover all cases where X_t never hits the given values, that is, $\tau = \infty$ if the random event never happens.

In other words, let $\{X_t, t \ge 0\}$ be a stochastic process. *A stopping time* with respect to $\{X_t, t \ge 0\}$ is a random time such that for each $t \ge 0$, the event $\{\tau = t\}$ is completely determined by (at most) the total information known up the time $t, \{X_0, X_1, ..., X_t\}$.

4.5.1Properties of stopping times

(i) If τ and γ are stopping times, so are $\tau \vee \gamma$ and $\tau \wedge \gamma$. In consequence the following relations hold:

$$\{\mathrm{t} \vee \gamma \le t\} = \{\mathrm{t} \le t\} \cap \{\gamma \le t\}$$
$$\{\mathrm{t} \wedge \gamma \le t\} = \{\mathrm{t} \le t\} \cup \{\gamma \le t\}$$

(ii) Given t , a stopping time, define the σ-field

$$\mathrm{F}_{\mathrm{t}} = \{A \cap \{\mathrm{t} \le t\} \in \mathrm{F}_t \text{ for all } t \ge 0\}, A \in \Omega.$$

F_{t} is a σ-field because it contains the empty set, it is stable by complements due to the fact that

$A^c \cap \{\mathrm{t} \le t\} = \left(A \cup \{\mathrm{t} > t\}\right)^c = \left(\left(A \cap \{\mathrm{t} \le t\}\right) \cup \{\mathrm{t} > t\}\right)^c$, and is stable by countable intersections.

(iii) If t and γ are stopping times, such that, $\mathrm{t} \le \gamma$ then $\mathrm{F}_{\mathrm{t}} \subset \mathrm{F}_{\gamma}$.

In consequence, if $A \in \mathrm{F}_{\mathrm{t}}$

then $A \cap \{\gamma \le t\} = \left[A \cap \{\mathrm{t} \le t\}\right] \cap \{\gamma \le t\} \in \mathrm{F}_{\mathrm{t}} \qquad \forall\, t \ge 0$

(iv) Let $\{X_t\}$ be an adapted stochastic process, and t be a stopping time. Assume that $\mathrm{t} < \infty$. If the parameter is continuous, it is assumed that the trajectories of the process $\{X_t\}$ are right continuous. Then the random variable $X_{\mathrm{t}}(\mathrm{w}) = X_{\mathrm{t}(\mathrm{w})}(\mathrm{w})$ is F_{t} -measurable. Capuzzo and Ferreti, (2001).

In the context of gambling, in which X_t denotes the total earnings after the gambling at time *t*, a stopping time t is thus a rule that tells exactly at what time to stop gambling. The decision to stop after a given gamble can only depend (at most) on the "information" known at that time (not on the future information). This is to say that a gambler stops the betting process when he/she reaches a certain goal. The time it takes to reach this goal is generally not a deterministic one. Rather, it is a random variable depending on the current result of the bet, as well as the combined information from all previous bets.

For instance a gambler has =N=1,000 and plays the slot machine at =N=1 per day. The gambler stops playing when his capital is depleted. The number $\mathrm{t} = n_1$ of plays that it takes the gambler to stop is a stopping time.The gambler stops playing when his/her capital reaches =N=2,000. The number $\mathrm{t} = n_2$ of plays that it takes the gambler to stop is a stopping time.The gambler stops when his/her capital reaches =N=2,000 or is depleted, whichever comes first. The number $\mathrm{t} = \min(n_1, n_2)$ of plays that it takes the gambler to stop is a stopping time. Since winnings and losses can alternately fluctuates, the gambler may experience many number of plays for which the capital neither depletes nor reaches =N=2,000. The number $\mathrm{t} = \max(n_1, n_2) \cup \{\infty\}$ of plays it takes the gambler to stop without necessarily hitting any of the given goals is a stopping time.

4.5.2Hitting time(First passage)

Suppose a stochastic process $\{X_t : t \geq 0\}$ has a discrete state space and let *i* be fixed state. Let $\mathrm{t} = \min(t \geq 0 : X_t = i)$.

This is called the *first passagetime* of the process into state *i*. It is also called the *hitting time* of the process to state *i*. *A*can be set to be the collection of states such that $A = \{1, 2, 3, 7\}$ or $A = \{2, 4, 6, ...\}$ and τ is the first passage (hitting time) into the set *A*. Bass,(2010),(2011).

$$\mathrm{t} = \min\{t \geq 0 : X_t \in A\}. \qquad (4.10)$$

4.5.3Hitting times are stopping times

In a gambler's ruin problem, that is a gambling stops when either $X_t = N$ or $X_t = 0$ (capital depleted), whichever happens first; the first passage time to the set $A = \{0, N\}$

Now, $\{\tau = 0\} = \{X_0 \in A\}$, hence this depends only on X_0, and for $t \geq 1$,

$\{\tau = t\} = \{X_0 \notin A, X_1 \notin A, ..., X_{t-1} \notin A, X_t \in A\}$ and thus depends on $\{X_0, X_1, ..., X_t\}$.

When $A = \{j\}$, this reduces to hitting time to state *j* and

$$\{\tau = t\} = \{X_0 \neq j, X_1 \neq j, ..., X_{t-1} \neq j, X_t = j\}$$

In gambler's ruin problem with $X_0 = j \in \{1, 2, ..., N-1\}$, $X_n = j + \Delta_1 ... + \Delta_n$ (simple random walk) until the set $A = \{0, N\}$ is hit.

Thus $\tau = \min\{t \geq 0, X_t \in A\}$ is a stopping time with respect to both $\{X_t\}$ and $\{\Delta_t\}$. Thus, shown that hitting times are stopping times. [Decreusfond and Lunart, (2008)]•

Furthermore, for $t \geq 2$,

$$\begin{aligned}\{\tau = t\} &= \{X_0 \notin A, X_1 \notin A, ..., X_{t-1} \notin A, X_t \in A\} \\ &= \{j + \Delta_1 \notin A, ... j + \Delta_1 + ... \Delta_{t-1} \notin A, j + \Delta_1 + ... + \Delta_t \in A\}\end{aligned}$$

and thus is completely determined by both $\{X_1, ..., X_t\}$ and $\{\Delta_1, ..., \Delta_t\}$. This means that if the initial condition is known $X_0 = j$, then $\{X_t\}$ and $\{\Delta_t\}$ contain the same information.

4.5.4Independent stopping time

Let $\{X_t : t \geq 0\}$ be a stochastic process and suppose that τ is any random time that is independent of the process. Then τ is a stopping time. Here $\{\tau = t\}$ does not depend on $\{X_t : t \geq 0\}$ (past or future). An example of this is that before a gambler starts gambling, it is already decided decides that it will stop after the 10$^{\text{th}}$ gamble regardless of all else. i.e. $P\{\tau = 10\}$ .Another example is that if every day after looking at the stock price, a trader throws a fair die, and decides to sell the stock the first time the die shows up an odd number. In this case τ is independent of the stock pricing

4.5.5Non-stopping times (Last exit time)

Here an example of a rat in the open maze problem can be considered in which the rat eventually reaches freedom (state 0) and never returns into the maze. Suppose that the rat starts off in cell 1, $X_0 = 1$. Let τ denote the last time the rat visits cell 1 before leaving the maze:

$$\tau = \max\{t \geq 0 : X_t = 1\} \tag{4.11}$$

The future must be known in order to determine such a time. For instance the event $\{\tau = 0\}$ tells us that in fact the rat never returned to cell 1

$$\{\tau = 0\} = \{X_0 \neq 1, X_1 \neq 1, X_2 \neq 1 ...\}$$

So, this depends on all of the future, not just X_0. This is not a stopping time. In general, a *last exit time* (the last time a process hits a given state or set of states) is not a stopping time for in order to know the last has happened or occurred, the future must be known.

4.5.6 Other stopping times

If τ is a stopping time with respect to $\{X_t\}$, then we can see that the event $\{\tau \leq t\}$ can only depend at most on $\{X_0, X_1, ..., X_t\}$. Stopping by time τ can only depend on the information up to time *t*. This statement can be proved: $\{\tau \leq t\}$ is the union of $n+1$ events that is $\{\tau \leq t\} = \bigcup_{j=0}^{n} \{\tau = j\}$ By definition of stopping time, each $\{\tau = j\}, j \leq t$

depends at most on $\{X_0, X_1, ..., X_j\} \subseteq \{X_0, X_1, ..., X_t\}$ .This means that the union is also contained in $\{X_0, X_1, ..., X_t\}$. Consider $\{\tau < t\}$, by writing it as $\{\tau \le t-1\}$ thus is determined by $\{X_0, X_1, ..., X_{t-1}\}$. Another consideration is $\{\tau > t\}$, which is equivalent to $\{\tau \le t\}^c$. Since $\{\tau \le t\}$ is completely determined by $\{X_0, X_1, ..., X_t\}$, so also its complement. For instance, if $\{t \ge 0 : X_t = j\}$ is a hitting time then $\{\tau > t\} = \{X_0 \ne j, X_1 \ne j, ..., X_t \ne j\}$ and hence only depends on $\{X_0, X_1, ..., X_t\}$.

Possibly the most popular example of stopping time is the hitting time of a stochastic process, that is that first time at which a certain pre-specified set is hit by the considered process.The reverse of the above statement is as well true: for any stopping time, there exists an adapted stochastic process and a Borel-measurable set such that the corresponding hitting time will be exactly this stopping time. The statement and other constructions of stopping times are also in Lutz (2012), Jukka (2012), Persi and Laurent (2009), Neil (1980) and Nudley and Gutman (1977).

4.6The 1-0 – Process

A stochastic process can be choesn which will assume value 1 until just before the stopping time is reached, from which on it will be 0. The 1-0-*process*, as it is called therefore, first hits the Borel set {0} at the stopping time. As such a 1-1 relation can established between stopping times and adapted càdlàg (4.6.1) processes that are non-decreasing and take the values 0 and 1. A 1-0-process as described above is akin to a random traffic light which can go through three possible scenarios over time (with 'green' standing for '1' and 'red' standing for '0') as follows:

(i) Stays red forever (stopped immediately)

(ii) Green at the beginning, then turns red and stays red for the rest of the time (stopped at some stage)

(iii) Stays green forever (never stopped)

For the adaptedness of 1-0-process, it can only change based on information up to the corresponding point in time. This intuitive interpretation of stopping time as the time when such a ‘traffic light’ lights seems to be easier to understand than the concept of random time which is ‘known once it has been reached’

4.6.1Càdlàg

Càdlàg (a French word standing for: *continue à droite, limitée à gauche*) meaning right- continuous with left-handed limit (RCLL).

If for all ω ϵ Ω the paths $X(\mathrm{w}): T \rightarrow \square$ have the property RCLL, a real-valued stochastic process will be called càdlàg.

4.6.2Stopping process

An adapted càdlàg process $X = (X_t)_{t \in T}$ on $(\Omega, \mathrm{F}, \mathrm{F}_t, P)$ is a stopping process if

$$X_t(\mathrm{w}) \in \{0,1\} \qquad (\mathrm{w} \in \Omega, t \in T) \tag{4.12}$$

and

$$X_s \geq X_t \qquad (s \leq t, s, t \in T) \tag{4.13}$$

For a finite or infinite discrete time axis given by $T = \{t_k : k \in \square, t_k \geq t_j \text{ if } k \geq j\}$, an adapted process satisfying (4.12) and (4.13) is automatically càdlàg.

4.6.3Definition

For a stopping process X on $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$, define

$$\tau^X(\omega) = \begin{cases} +\infty & \text{if } X_t(\omega) = 1 \ \forall t \in T \\ \min\{t \in T : X_t(\omega) = 0\} & \text{otherwise} \end{cases} \tag{4.14}$$

The minimum in the lower case exists because of the càdlàg property of each path.

By definition

$$X_t = 1_{\{\tau^X > t\}} \qquad (t \in T) \tag{4.15}$$

and by adaptedness of X and that $\{\tau^X > t\} = \{\tau^X \le t\}^c$ which implies that

$$\{\tau^X \le t\} \in \mathcal{F}_t \quad (t \in T) \tag{4.16}$$

Therefore τ^X is a stopping time for a stopping process X. τ^X is the first time of X hitting the Lebesgue-measure set {0}.Following Fischer, (2012), a minimal element of stopping time σ-algebra can be formulated.

4.6.4Stopping time σ-algebra

Let τ be a stopping time on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Define the *stopping time σ-algebra* with respect to τ as

$$\mathcal{F}_\tau = \{B \in \mathcal{F}_t : B \cap \{\tau \le t\} \in \mathcal{F}_t \qquad \forall t \in T\}: \tag{4.17}$$

it can be seen that $\mathcal{F}_\tau$ is a σ-algebra.

4.6.5Minimal elements

For a measurable space $(\Omega, \mathcal{F})$, the set of the *minimal elements in the σ-algebra* $\mathcal{F}_\tau$ can be defined as

$$\mathcal{A}(\mathcal{F}) = \{A \in \mathcal{F} : A \ne \emptyset \text{ and if } B \in \mathcal{F} \text{ and } B \subset A, \text{ then } B = A\} \tag{4.18}$$

Equation (4.18) means that $A \in \mathcal{A}(\mathcal{F})$ cannot be split in $\mathcal{F}$ which is why elements of $\mathcal{F}$ are also referred to as 'atoms' of $\mathcal{F}$. For $|\mathcal{F}| < \infty$, it may be noticed that $\mathcal{A}(\mathcal{F})$ is a partition of Ω and that

$$\mathcal{F} = \sigma(\mathcal{A}(\mathcal{F})) \tag{4.19}$$

since any nonempty $B \in \mathcal{F}$ can be written as a finite union of elements in $\mathcal{A}(\mathcal{F})$.

4.6.6Definition

For $t \in T \cup \{+\infty\}$, the set of minimal elements in $\mathcal{F}_t$ can be denoted as

$$\mathcal{A}_t = \mathcal{A}(\mathcal{F}_t) = \{A \in \mathcal{F}_t : A \neq \varnothing \text{ and if } B \subset \mathcal{F}_t \text{ and } B \subset A, \text{ then } B = A\} \tag{4.20}$$

With the definition

$$\mathcal{A}_t^t = \{A \in \mathcal{A}_t : A \subset \{t = t\}\} \quad (t \in T \cup \{+\infty\}) \tag{4.21}$$

$$\Rightarrow \mathcal{A}_t = \bigcup_{t \in T \cup \{+\infty\}} A_t^t \tag{4.22}$$

It is seen that $\mathcal{A}_t^t$ are disjoint for $t \in T \cup \{+\infty\}$.

4.6.7Proposition

For $|\mathcal{F}_\infty| < +\infty,\ \sigma(t) \subset \mathcal{F}_t$ and in general $\sigma(t) \neq \mathcal{F}_t$

The assertion of this proposition can be proved using the stochastic tree (Figure 4.1).

A discrete time finite space filtration can be represented as a stochastic tree. In the setting, the set of all paths of maximal length represents Ω, where $\Omega = \{w_1, w_2, ..., w_8\}$.

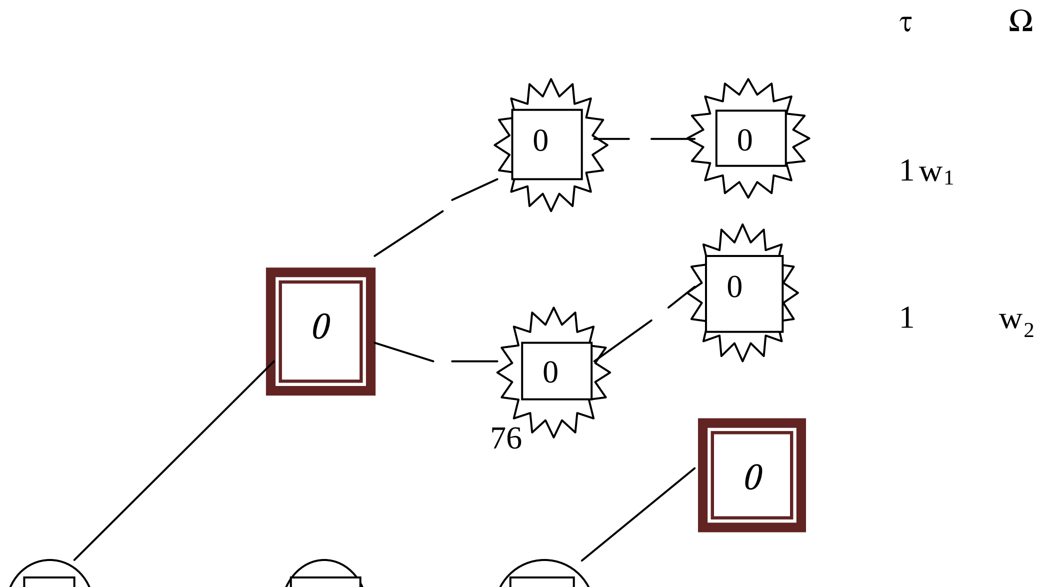
A path up to some node at time *t* represents the set of those paths of full length that have this path up to some time *t* in common.The paths up to time *t* represent the minimal elements ('atoms') $\mathcal{A}_t$ of $\mathcal{F}_t$

Here $T = \{0,1,2,3\}$ and $F_3 = F_\infty = P(\Omega)$ (where $P(\Omega) =$ the partition of Ω). In the presentation (fig 4.1), for instance

$$A_1 = \{\{w_1, w_2\}, \{w_3, w_4\}, \{w_5, ..., w_8\}\} \tag{4.23}$$

$$A_2 = \{\{w_1, w_2\}, \{w_3, w_4\}, \{w_5, w_6\}, \{w_7, w_8\}\} \tag{4.24}$$

$$A_3 = \{\{w_1, w_2\}, \{w_3\}, \{w_4\}, \{w_5, w_6\}, \{w_7\}, \{w_8\}\} \tag{4.25}$$



3 ω_3

X^t 3 w_4

2 w_5

2 w_6

3 w_7

3 w_8

t

0 1 2 3

Figure 4.1 Stochastic Tree

In (fig. 4.1) a stopping time t and the corresponding stopping process X^t are given by the values of the nodes of the tree. The intuitive interpretation of the time as the random time at which X^t jumps to ('hits') 0, becomes clear (with bolded boxed zeros). The paths up to those bolded boxed zeros represent the minimal elements A_t of the stopping time σ-algebra F_t .

This follows from the fact that for any $A \in \mathrm{A}_{\mathrm{t}}$ then $A \subset \{\mathrm{t} = t\}$ and therefore, $X_t^{\mathrm{t}}(A) = 0$, but $X_s^{\mathrm{t}}(A) = 1$ for $s < t$. So, in the given tree example, the' frontier of first zeros' or 'precisers', that is, the paths leading up to it describes the stopping σ-algebra (as well as the stopping time itself). So, A_{t} contains the information in the system that can be explored up to time t . In the case of the example;

$$\mathrm{A}_{\mathrm{t}} = \{\{\mathrm{w}_1, \mathrm{w}_2\}, \{\mathrm{w}_3\}, \{\mathrm{w}_4\}, \{\mathrm{w}_5, \mathrm{w}_6\}\{\mathrm{w}_7\}, \{\mathrm{w}_8\}\} \tag{4.26}$$

It is seen that the minimal elements of $\sigma(\mathrm{t})$ denoted by $\mathrm{A}(\sigma(\tau))$ are given by

$$\mathrm{A}(\sigma(\mathrm{t})) = \{\{\mathrm{w}_1, \mathrm{w}_2\}, \{\mathrm{w}_5, \mathrm{w}_6\}, \{\mathrm{w}_3, \mathrm{w}_4, \mathrm{w}_7, \mathrm{w}_8\}\} \tag{4.27}$$

Therefore, $\mathrm{A}_{\mathrm{t}} \neq \mathrm{A}(\sigma(\mathrm{t}))$, hence, $\sigma(\mathrm{t}) \neq \mathrm{F}_{\mathrm{t}}$ as stated in the proposition.

4.7Stopped Process

Given a stochastic process X_k which is adapted to the filtration F_k and let t be a stopping time with respect to F_k.

Define the random variable

$$X_{\mathrm{t}(\mathrm{w})} = \begin{cases} X_{\mathrm{t}(\mathrm{w})}, & \mathrm{t}(\mathrm{w}) < \infty \\ 0 & \text{otherwise} \end{cases} \qquad \text{or equivalently, } X_{\mathrm{t}} = \sum_{k=0}^{\infty} X_k 1_{\{\mathrm{t} = k\}} .$$

The process $X_k^{\mathrm{t}} = X_{\mathrm{t} \wedge k}$ is called a *stopped process*. It is equivalent to X_{t} for $\mathrm{t} \leq k$ and equals X_k for $\mathrm{t} > k$.

As stated earlier, classical theories in stopping have some counterparts in the quantum theory. Alook at some of these analogies can start with the stopping time projections.

4.8 Stopping Time and Time Projections

Let M be a von Neumann algebra and let (B_α) be a filtration of M. A non-commutative stopping time adapted to the filtration (B_α) of von Neumann algebra M is a map $\tau : [0,\infty] \rightarrow \mathrm{B}_\alpha^{proj}$ with $\tau(\mathrm{w}) = 0$ and $\tau(\infty) = I$. (I is an identity).

$\tau(\alpha) \in \mathrm{B}_\alpha^{proj}$ for $\alpha \in (0,\infty)$ and if $\alpha_1 \leq \alpha_2$, then $\tau(\alpha_1) \leq \tau(\alpha_2)$. It is seen that τ is an increasing, projection-valued process starting at zero and terminating at I (or at infinity).

4.8.1 Definition

Let $\tau = (q_t)$ and $\sigma = (p_t)$ be stopping times.

Then $\sigma \leq \tau \Leftrightarrow q_t \leq p_t \ \forall \in \mathbb{R}^+$ and $\sigma \wedge \tau = (p_t \vee q_t)$, $\sigma \vee \tau = (p_t \wedge q_t)$,

where $\tau \vee \sigma$ and $\tau \wedge \sigma$ denote *supremum* and *infimum*, respectively of τ and σ. If $t \in \mathbb{R}^+$, then this deterministic time is identified with a random variable (p_β) , say, by setting

$$p_\beta = \begin{cases} 0, & 0 \leq \beta \leq t \\ I, & \beta > t \end{cases}$$

Let $p \in [1,\infty)$ be fixed and $X = (X_\alpha)$ be an $L^p-\text{process}$, then X is said to be closed by X_∞ if $X_\infty = \lim_{\alpha\rightarrow\infty} X_\alpha$ and $E_\alpha(X_\infty) = X_\alpha$: where E_σ are expectations.

4.8.2 Quantum stopped process

Let $\tau = (q_t)$ be a random time, $\upsilon \in \mathrm{Fin}([0,\infty])$ (where $\mathrm{Fin}([a,b])$ is a collection of all finite partitions of $[a,b]$) , say $\upsilon = \{t_1, t_2, ..., t_n\}$, then define

$$X_{\mathrm{t}(\upsilon)} = X_{\mathrm{t}\{t_i\}} = \sum_{i=1}^{n} X_{t_i}\left(q_{t_i} - q_{t_{i-1}}\right) = \sum_{\upsilon} X_{t_i}\left(q_{t_i} - q_{t_{i-1}}\right)$$
$$= \sum_{\upsilon} X_{t_i} \Delta q_{t_i} \qquad (4.28)$$

If $\lim_{\upsilon} X_{\mathrm{t}(\upsilon)}$ exists in $\|.\|_p$, denote it by X_{t} and it will be called *stopped process* associated with X and t .It can therefore be seen that $X_{\mathrm{t}(\upsilon)} \in L^p\left(\mathrm{B}_\infty\right)$ and $\upsilon \to X_{\mathrm{t}(\upsilon)}$ is a net

4.8.3 Theorem [Barnett, *et al* (1996)]

Let $X = \left(X_t\right)$ be a weakly relatively compact $L^1 - \text{martingale}$ and $\mathrm{t} = \left(p_t\right)$ a stopping time. Suppose that X_{t} exists. Then for each $t \in \mathbb{R}^+$, $X_{\mathrm{t}\wedge t}$ exists and the process $X^{\mathrm{t}} = \left(X_{\mathrm{t}\wedge t}\right)$ is a martingale.

Proof

Since X is weakly, relatively compact, there exists $X_\infty \in L^1$ such that $\lim_{t\to\infty} X_t = X_\infty$ and $X_t = E_t\left(X_\infty\right)$ i.e. closed at infinity.

Consider $\upsilon \in \mathrm{Fin}\left(\left[0,\infty\right]\right)$ and let $t \in \mathbb{R}^+$ be fixed. We can assume that $\alpha \in \upsilon$. Suppose $\upsilon = \{t_i\}$ and $t = t_r$. Now,

$$X_{\mathrm{t}\wedge t(\upsilon)} = \sum X_{t_i} \Delta p_{t_i} + X_t\left(I - p_t\right) \qquad \text{- (by definition)}$$

$$E_t\left(X_{\mathrm{t}(\upsilon)}\right)=\sum_{i\le r}X_{t_i}\Delta p_{t_i}+E_t\left(\sum_{i>r}X_{t_i}\Delta p_{t_i}\right)$$

Also, $$=\sum_{i\le r}X_{t_i}\Delta p_{t_i}+E_t\left(\sum_{i>r}E_{t_i}\left(X_\infty\Delta p_{t_i}\right)\right)$$

$$=\sum_{i\le r}X_{t_i}\Delta p_{t_i}+\sum_{i>r}E_{t_i}\left(X_\infty\right)\Delta p_{t_i} \qquad \text{- (tower property)}$$

$$=X_{\mathrm{t}\wedge t(\upsilon)}.$$

Since E_t is L^1- continuous, we have

$$\lim_{\upsilon}X_{\mathrm{t}\wedge t(\upsilon)}=\lim_{\upsilon}E_t\left(X_{\mathrm{t}(\upsilon)}\right)=E_t\left(X_{\mathrm{t}}\right).$$

This means that $X_{\mathrm{t}\wedge t}$ exists and is equal to $E_t\left(X_{\mathrm{t}}\right)$

Therefore, $X_{\mathrm{t}\wedge t}$ is a martingale .

4.9 Stopping for L^2-Martingale

If $X=\left(X_t\right)$ is an L^2 – martingale adapted to $\left(\mathrm{B}_\alpha\right)$ and suppose that $\left(X_t\right)$ is a bounded subset of L^2. Then there is $X_\infty\in L^2\left(\mathrm{B}_\alpha\right)$ such that $X_t=E_t\left(X_\infty\right)$ and $\lim_{t\to\infty}E_t\left(X_t\right)=X_\infty$.

$\left(X_t\right)$ is L^2 – right continuous, i.e. $\left(\bigcup_{t\in[0,\infty]}X_t\right)=X$ and $\bigcap_{\alpha>t}X_\alpha=X_t$

If $\mathrm{t}=\left(q_t\right)$ is a random stopping time and $\upsilon\in\mathrm{Fin}\left(\left[0,\infty\right]\right)$, then

$$X_{\mathrm{t}(\upsilon)}=\sum_{\upsilon}X_{t_i}\Delta q_{t_i}=\sum_{\upsilon}E_tX_\infty\Delta q_{t_i}$$
$$=E_{\mathrm{t}(\upsilon)}\left(X_\infty\right)$$

where

$$E_{\mathrm{t}(\upsilon)}(.)=\sum_{\upsilon}E_{t_i}(.)\Delta q_{t_i}$$

E_t is the conditional expectation for $L^2\left(\mathrm{A}\right)$ onto $L^2\left(\mathrm{B}_\alpha\right)$.

The properties of $E_{\mathrm{t}(\upsilon)}$ are given by

4.9.1Theorem [Barnett, *et al* (1996)] and [modified by Tijjani, (2001)]

Let $\mathrm{t} = (q_t)$ be a stopping time. Then,

(a) For each $\upsilon \in \mathrm{Fin}([0,\infty])$, the operator $E_{\mathrm{t}(\upsilon)}$ is a self-adjoint projection on $L^2(\mathrm{A})$.

(b) If $\sigma = (p_t)$ is a stopping time such that $\sigma \le \mathrm{t}$ and $\upsilon \in \mathrm{Fin}([0,\infty])$, then $E_{\sigma(\upsilon)} \le E_{\mathrm{t}(\upsilon)}$

(c) Let $\upsilon_1, \upsilon_2 \in \mathrm{Fin}([0,\infty])$ be such that $\upsilon_1 \le \upsilon_2$. Then $E_{\mathrm{t}(\upsilon_2)} \le E_{\mathrm{t}(\upsilon_1)}$.

From the theorem above, it follows that for each stopping time $\mathrm{t} = (q_t)$ and $\upsilon \in \mathrm{Fin}([0,\infty])$, the projections $E_{\mathrm{t}(\upsilon)}$ form a deceasing net. So, its infimum exists.

Let $E_{\mathrm{t}(.)} = \inf\limits_{\upsilon} E_{\mathrm{t}(\upsilon)}(.) = \inf\limits_{\upsilon} \sum\limits_{i} E_{t_i}(.)\Delta q_{t_i}$ and call E_{t} the time projection associated with t

4.9.2Corollary [Barnett, *et al* (1996)]

Let σ, t be stopping times such that $\sigma \le \mathrm{t}$, then $E_\sigma \le E_{\mathrm{t}}$.

Proof

Let $\upsilon \in \mathrm{Fin}([0,\infty])$. Then, $E_{\sigma(\upsilon)} \le E_{\mathrm{t}(\upsilon)}$ by (4.33) (b).

Thus, $E_\sigma = \lim\limits_{\upsilon} E_{\sigma(\upsilon)} \le \lim\limits_{\upsilon} E_{\mathrm{t}(\upsilon)} = E_{\mathrm{t}}$

4.9.3Theorem [Barnett, *et al* (1996)]

Let (X_t) be a bounded L^2 –martingale such that

$E_t(X) = X_t$ and $\lim_{t\to\infty} X_t = X$ for some $X \in L^2(\mathrm{A})$. Then for a stopping time $\mathrm{t} = (q_t)$, and $\upsilon \in \mathrm{Fin}([0,\infty])$, $\lim_{\upsilon} X_{\mathrm{t}(\upsilon)} = X_{\mathrm{t}}$ exists and $X_{\mathrm{t}} = E_{\mathrm{t}}(X)$.

Proof

Since $X_{\mathrm{t}(\upsilon)} = E_{\mathrm{t}(\upsilon)}(X)$ and $\lim_{\upsilon} E_{\mathrm{t}(\upsilon)}(X) = E_{\mathrm{t}}(X)$, then $\lim_{\upsilon} X_{\mathrm{t}(\upsilon)} = X_{\mathrm{t}}$ exists and $X_{\mathrm{t}} = E_{\mathrm{t}}(X)$. That is $E_{\mathrm{t}}(X) = \lim_{\upsilon} \sum E_{t_i}(X)\Delta q_{t_i} = \lim_{\upsilon} \sum X_{t_i}\Delta q_{t_i} = X_{\mathrm{t}}$

The operator E_{t} occurring in the proof of the above theorem is called the *stopped operator* while the random variable X_{t} is called the stopped process.

4.9.4Theorem [Barnett, *et al* (1996)]

Let t and σ be two stopping times such that $\mathrm{t} \le \sigma$ and let (X_t) be a bounded L^2-martingale such that $E_t(X) = X_t$ and $\lim_{t\to\infty} X_t = X$ for some $X \in L^2(\mathrm{A})$.

Then $E_{\mathrm{t}}(X_\sigma) = X_{\mathrm{t}}$.

Proof

Since $\mathrm{t} \le \sigma$, we have $E_{\mathrm{t}} \le E_\sigma$ Thus, $E_{\mathrm{t}}(X_\sigma) = E_{\mathrm{t}}(E_\sigma(X_\sigma)) = X_{\mathrm{t}}$

4.9.5Remark

For a deterministic *t*, E_t can be considered as an L^2-martingale of the original conditional expectation and we see that for an L^2 bounded martingale (X_t), the limit *X*stopped by t is the action of $E_{\mathrm{t}}(.)$ on *X*.

Since $E_{\mathrm{t}}(.) = \inf E_{\mathrm{t}(\upsilon)}(.)$, we see that E_{t} has some features of an expectation viz:

For $\mathrm{t} = (q_t)$

(i) E_{t} is an L^2 – contraction

(ii) E_{t} is self-adjoint

(iii) E_{t} is a projection.

It is known [Barnett and Thakrar, (1987)] that E_{t} is not adjoint preserving. For this reason E_{t} is not a conditional expectation.

However, if it is restricted to B_{t}, some time- commutant process in a closed set, i.e. $E_{\mathrm{t}} \mid B_{\mathrm{t}}$, it can be shown that it is adjoint preserving.

4.10Results and Discussion

In the classical stopping times and stopping processes, the use of the concept (4.6) and the definitions (4.6.1), (4.6.2), (4.6.3) and (4.7) are utilized to state and prove the following result:

4.10.1 Theorem [Fulatan, (2015)]

For a stopping time τ, the stochastic process $X^{\mathrm{t}} = \left(X_t^{\mathrm{t}}\right)_{t \in T}$ defined by

$$X_t^{\mathrm{t}} = 1_{\{\mathrm{t} > t\}} \qquad \left(t \in T\right)$$

is a stopping process.

Proof

By adaptedness of *X*, it means that $\left\{\mathrm{t}^X > t\right\} = \left\{\mathrm{t}^X \le t\right\}^c$ and that $\left\{\mathrm{t} \le t\right\} \in \mathrm{F}_{\mathrm{t}} \quad \left(t \in T\right)$, which implies that X^{t} is an adapted process. From the definition, it is noticed that $X_s^{\mathrm{t}} = 1_{\left\{\mathrm{t}^X > s\right\}} \ge 1_{\left\{\mathrm{t}^X > t\right\}} = X_t^{\mathrm{t}}$ for $s \le t$,because of the càdlàg property of X^{t}.

Now,

$\lim_{t\downarrow \tau(\omega)}\left(X_t^{\tau}(\omega)\right) = X^{t}_{\tau(\omega)} = 0$ since X^{τ} is càdlàg and so a stopping process.

In another development, following definitions (4.6.4), (4.6.5), and (4.6.6), theorem (4.10.2) is also formulated and proved

4.10.2 Theorem

The elements of $\mathcal{A}_\tau$ are minimal elements of $\mathcal{F}_\tau$, that is $\mathcal{A}_\tau \subset \mathcal{A}(\mathcal{F}_\tau)$. If $|\mathcal{F}| < \infty$, then

$\mathcal{A}_\tau$ is the set of all minimal elements of $\mathcal{F}_\tau$, that is

$\mathcal{A}_\tau = \mathcal{A}(\mathcal{F}_\tau)$ and $\mathcal{F}_\tau = \sigma(\mathcal{A}_\tau) = \sigma\left(\mathcal{A}(\mathcal{F}_\tau)\right)$.

Proof

(i) $\underline{\mathcal{A}_\tau \subset \mathcal{F}_\tau}$:

Let $A \in \mathcal{A}_\tau$. So, for some $s \in T \cup \{+\infty\}$, $A \in \mathcal{A}_s$ and $A \subset \{\tau = s\}$, therefore $A \in \mathcal{F}_s$.

Assume $t < s$ for some $t \in T$. Then

$A \cap \{\tau \leq s\} = A \cap \{\tau = s\} \cap \{\tau \leq t\} = \varnothing \in \mathcal{F}_t$. For $t \in T$, assume $t \geq s$, then

$A \cap \{\tau \leq s\} = A \cap \{\tau = s\} \cap \{\tau \leq t\} = A \cap \{\tau = s\} = A \in \mathcal{F}_s \subset \mathcal{F}_t$.

Hence, $A \cap \{\tau \leq t\} \in \mathcal{F}_t \quad \forall\, t \in T \qquad \Rightarrow \mathcal{A}_\tau \subset \mathcal{F}_\tau$

(ii) $\underline{A \subset \mathcal{A}_\tau, B \in \mathcal{F}_\tau \text{ and } B \subset A \Rightarrow B = A}$:

Assume that $A \in \mathcal{A}_\tau^\infty$ and $B \in \mathcal{F}_\tau$ with $B \subset A$.

So, $A \in \mathcal{A}_\infty \Rightarrow B = A$ since $B \in \mathcal{F}_\infty = \mathcal{F}$. Assume that $A \in \mathcal{A}_\tau^t$ for some $t \in T$ and $B \subset \mathcal{F}_\tau$ with $B \subset A$. Therefore $A \in \mathcal{A}_t$ and $B \subset A \subset \{\tau = t\}$, hence $B \cap \{\tau = t\} = B$.

As $B \in \mathrm{F}_{\mathrm{t}}$, then $B \cap \{\mathrm{t} \leq t\} \in \mathrm{F}_{\mathrm{t}}$ since

$\{\mathrm{t} = t\} \in \mathrm{F}_{\mathrm{t}}$, $B \cap \{\mathrm{t} \leq t\} \cap \{\mathrm{t} = t\} = B \cap \{\mathrm{t} = t\} = B \in \mathrm{F}_{\mathrm{t}}$

But $A \in \mathrm{A}_{\mathrm{t}}$, therefore, $B = A$. (i) and (ii) proved the first statement of the theorem.

Now assume $|\mathrm{F}_{\infty}| < +\infty$. A_{t} is then a partition of **Ω**, because any two distinct sets in A_{t} are disjoint. Since $\{\mathrm{t} = t\} \in \mathrm{F}_{\mathrm{t}}$ for some $t \in T \cup \{+\infty\}$, we have $\bigcup \mathrm{A}_{\mathrm{t}}^{\mathrm{t}} = \{\mathrm{t} = t\}$, so, $\bigcup \mathrm{A}_{\mathrm{t}} = \Omega$. This proves the second statement and the third follows from equation (4.6.6).

From the intuitive description of 1-0-process and the definition (4.6.3), the result (4.10.3) below is formulated and proved as well:

4.10.3 Theorem

$$f : X \to \mathrm{t}^{X} \tag{4.29}$$

is a bijection between the stopping processes and the stopping times on $(\Omega, \mathrm{F}, \mathrm{F}_{\mathrm{t}}, P)$ such that

$$f^{-1} : \mathrm{t} \to X^{\mathrm{t}} \tag{4.30}$$

Hence $\mathrm{t}^{X^{\mathrm{t}}} = \mathrm{t}$ (4.31)

Proof

f maps processes X to stopping times τ^X as seen in (4.29). Under f, different stopping processes lead to different stopping times as seen in (4.14). Then f is therefore an injection.

For any stopping time τ, andfrom (4..14) and (4.29) there is the relation

$$\tau^{X^{\tau}}(\omega) = \begin{cases} +\infty & \text{if } 1_{\{\tau > t\}}(\omega) = 1 \qquad \forall t \in T \\ \min\left\{ t \in T : 1_{\{\tau > t\}}(\omega) = 0 \right\} & \text{otherwise} \end{cases} .$$

Since $1_{\{\tau > t\}}(\omega) = 0 \Leftrightarrow \tau(\omega) \le t$, the right hand side of (4.30) is $\tau(\omega)$. This means that

$\tau(\omega) = \tau^{X^{\tau}}(\omega)$, thus $f$issurjectve and this proves (4.29).

Now, from (4.15), it is obvious that

$$X_t^{\tau^X} = 1_{\left\{\tau^X > t\right\}} = X_t \quad \left(t \in T\right)$$

and this proves (4.30).

This theorem shows that there is a one-one correspondence between stopping times τ and an adapted càdlàg process $X = \left(X_t\right)_{t \in T}$ on $\left(\Omega, \mathcal{F}, \mathcal{F}_t, P\right)$.

CHAPTERFIVE

THE BELATED INTEGRALS

5.1 Introduction

Theories relating to non-commutative stochastic integral have indicated that it is a kind of vector Riemann-Stieltjes integral. In Dunford and Schwartz, (1958), Elworthy, (1982), McShane, (1974) and Wilde, (1974) all viewed the theories as such. This is because in 1956, Bartle developed a bilinear vector integral which in the beginning appeared to be a natural setting for non-commutative stochastic integral. However, Barnett, *et al* (1982) showed that the non-commutative stochastic integral could not be obtained using Bartle's vector integral

The theory of general quantum stochastic integrals (especially in non Fock spaces) deals mainly with the setup which can be described as follows: For a von Neumann algebra *M*with filtration $\{\mathrm{B}_\alpha\}_{\alpha\geq 0}$, there is a corresponding process $\left(X\left(t\right):t\geq 0\right)$ with values in $L^p\left(M\right)$and a corresponding process $\left(f(t):t\geq 0\right)$ also, with values in $L^q\left(M\right)$. $\frac{1}{p}+\frac{1}{q}=\frac{1}{2}$, $2\leq p,q\leq +\infty$, where $L^p\left(M\right)$ and $L^q\left(M\right)$are appropriate L^p - spaces. Among the various specific assumptions about *X* and *f*, is that *X* is a martingale in which case a definition of a stochastic integral can be given such that $\int_a^b fdX$ and $\int_a^b dXf$ are elements of $L^2\left(M\right)$. The advantage of this is that for any reasonable definition of the integral, it may be approximated by the sums of the form $\sum f\left(t_{i-1}\right)\left[X\left(t_i\right)-X\left(t_{i-1}\right)\right]$ for $\int_a^b fdX$ and

$\sum\left[X(t_i)-X(t_{i-1})\right]f(t_{i-1})$ for $\int_a^b dXf$. A product of elements from $L^p(M)$ and $L^q(M)$, is an element of $L^r(M)$ where, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$. In particular, the approximation holds in the case $r=2$ (i.e., in Hilbert space). However, the approximation can be exploited in another natural setting. For instance if is assumed that the algebra$M$ acts in a Hilbert space $\mathcal{H}$ and if f and X take their values in$M$ then the integral sum belongs to$M$too, and a question may be asked about its behavior as elements of $\mathcal{H}$. This again leads to an approximation of the integral evaluated at some points of $\mathcal{H}$, i.e., there again comes the notion of the integrals $\int_a^b fdX$ and $\int_a^b dXf$ as operators on $\mathcal{H}$.

In the existing theories of quantum integration, especially those of Lebesgue type, the classes of theoretically admissible integrals are rather narrow and lack any concrete examples of processes that can be integrated. On the other hand, for integrals of Riemann-Stieltjes type such examples have been provided. For instance, it was shown when predictable processes can be integrated with respect to quantum random time.

Some preliminary results that enter the theory can now be stated together with some definitions associated with the theory.

5.2 Integrals of Ito type

Ito and Stratonovich have given different meanings of stochastic integrals. In this sense, the Ito integral can be defined by

$$\int HdX$$

where H and X are stochastic processes of some class. This integral is regarded as a process usually denoted by $H*X$. Thus,

$$\left(H*X\right)(t,\mathrm{w})=\left(\int_{(0,t]}H_s dX_s\right)(\mathrm{w}) \tag{5.1}$$

So, the integral is defined over the interval (0, t]. The value of the integral at time 0 will be 0.

Using differential notation, (5.1) can be written as

$$\mathrm{d}\left(H*X\right)=HdX \tag{5.2}$$

in which the integrand is H and the integrator is X. the requirements on H is that it be previsible while the integrator X must be a semimartingale. For a stopping time $\tau > 0$, H_τ is known immediately before time τ because $H_\tau = \lim_{s\to\tau} H_s$. This is the reason that H is previsible. If H is an adapted L-process, then H_s is known to the observer at time s. If s and τ are stopping times where $\tau > s$, the simplest L-process is the process

$$H=1_{(s,\tau\,]} \tag{5.3}$$

That is $$H\left(t,\mathrm{w}\right)=\begin{cases}1 & \text{if } s(\mathrm{w})<t<\tau\left(\mathrm{w}\right)\\ 0 & \text{otherwise}\end{cases}$$

Let M be a martingale, then for H as defined in (5. 3), the definition of $H*M$ is

$$\left(H*M\right)_t=M_{\tau\wedge t}-M_{s\wedge t}$$

5.3 Local Martingale

A process M is a local martingale if M_0 is F_0-measurable and there is an increasing sequence of stopping times $\left(\tau_n\right)$ with $\tau_n\to\infty$ such that each stopping process $\left\{M_{\tau_n\wedge t}-M_0 : t\geq 0\right\}$ is a martingale.

5.4 Fundamental Theorem

If H is a locally bounded previsible process and M is a local martingale, then $H*M$ exists and is a local martingale. It may be noted that in a discrete setting for instance where $(\Omega, \mathcal{F}, \mathcal{F}_n, P)$ is a discrete time set-up, let M be an associated martingale. Let H be a bounded process previsible in the sense that

$$Z_{n-1} = H_n \in b\mathcal{F}_{n-1} \quad (n \in \mathbb{N}) \tag{5.4}$$

where $H*M$ is the space of bounded previsible processes.

Define

$$(H*M) = \sum_{k=1}^{n} H_k (M_k - M_{k-1}), \qquad (H*M) = 0$$

Then, $H*M$ is a martingale.

To see this, it is familiar that M is a martingale if $E[M_n - M_{n-1} \mid \mathcal{F}_{n-1}] = 0 \quad n \in \mathbb{N}$. But, since $Z_{n-1} \in b\mathcal{F}_{n-1}$, it is a fact that

$$\begin{aligned} E\left[(H*M)_n - (H*M)_{n-1} \mid \mathcal{F}_{n-1}\right] &= E\left[Z_{n-1}(M_n - M_{n-1}) \mid \mathcal{F}_{n-1}\right] \\ &= Z_{n-1} E\left[M_n - M_{n-1} \mid \mathcal{F}_{n-1}\right] \\ &= 0 \end{aligned}$$

As mentioned earlier, the integrator X will be a semimartingale. This means that X may be written in the form

$$X = X_0 + M + A \tag{5.5}$$

Where X_0 is $\mathcal{F}_0$- measurable, M is a local martingale null at zero and A is an adapted process with paths of finite variation, also null at zero.

5.5 Integrator

An adapted process X is called an integrator if the following hypothesis (a) and (b) hold:

(a) (H_n) is a sequence in bε with $\|H_n\| \to 0$

(b) $P\left[\left|(H*X)_t\right|>\varepsilon\right]\to 0$ as $n\to\infty, \forall t,\ \forall\varepsilon>0$

In this setting, bε consists of bounded previsible processes of the form

$$H=\sum_{j=1}^{n} Z_j\left(s_j, t_j\right] \tag{5.6}$$

where $n\in\mathbb{N}$, $\{s_j : j=1,2,...,n\}$ and $\{t_j : j=1,2,...,n\}$ are sequences of stopping times such that $0\le s_j\le t_j\le ...\le s_n\le t_n$ and $Z_j\in b\mathcal{F}\left(s_j\right)$ for each *j*.

For $H\in b\varepsilon$, define $H=\sup_{t,w}\left|H(t,w)\right|$. If *X* is an adapted process, there is defined

$$\left(H*X\right)_t=\sum_{j=1}^{n} Z_j\left(X_{t_j\wedge t}-X_{s_j\wedge t}\right) \tag{5.7}$$

Since it is known that a stochastic process is thought of as random variable, then $Z:T\times\Omega\to\mathbb{R}^n$ is one example.In general, the sample paths $z(.,w)$ are not bounded variations, that is the supremum of $\sum_{j=1}^{n}\left|z\left(t_{j+1},w\right)-z\left(t_j,w\right)\right|$ over partition $\prod$ of $[a,b]$ is not finite and there need bea generalization of the classical definition of integrals.

5.6Signed Measure and Bounded Variation

A *signed measure* is a mapping $\mu:\mathcal{F}\to\mathbb{R}\bigcup\{+\infty\}$ with the same additive property as an unsigned measure. A signed measure is in a unique way the difference of two positive measures $\mu=\mu^+-\mu^-$, with μ^+ and μ^- as positive measures such that there exists an element $A\in\mathcal{F}$ with the properties that $\mu^+(A)=0$ and $\mu^-(\Omega-A)=0$.

A complex measure μ is bounded or of *bounded variation* if

$$\|\mu\|=\sup\left\{\left|\mu(A)\right|: A\in\mathcal{F}\right\}<\infty \tag{5.8}$$

A property is said to hold a.e. or a.s. or w.p.1, if it holds for all $w \in \Omega - \Omega_0$ where Ω_0 is a set of measure zero.

The topological *dual* $X^{'}$ of a real topological space X is the set of all real-valued continuous linear mappings on *X*. If $\langle x^{'}, x \rangle$ denotes the duality on *X*, then $\langle x^{'}, x \rangle \in \mathbb{R}$

Two processes with the same state space *M* are said to be *isonomous* or equivalent if they have the same finite dimensional distributions. Stochastic equivalence implies isonomy but not vice versa.

5.6.1Mcshane partitions

Following Mcshane, a Riemann type definition suitable to some of our discussions are given:.

1. There is a definition of an integral which begins with the definition of a partition

$\prod = \{t_1, t_2, ..., t_{m+1}; \mathrm{t}_{\ 1}, \mathrm{t}_{\ 2}, \ldots, \mathrm{t}_{\ \mathrm{m}}\}$ of $[a,b]$ such that

$a = t_1, t_2, ..., t_{m+1} = b$.

$\bullet_{\ j} \in T$ is said to be:

(a) Belated if $\bullet_{\ j} \leq t_j$ for each *j*

(b) Cauchy if $\bullet_{\ j} = t_j$ for each *j*

(c) Riemann if $t_j \leq \bullet_{\ j} \leq t_{j+1}$ for each *j*

(d) Mesh of $\prod = \max_j \left(t_{j+1} - \left(t_j, \bullet_{\ j}\right)\right)$ for each *j* .

2. The definition of an integral implies the notation of convergence.

Let $f_n : f : \Omega \to M,\ \ n = 1, 2, ...$ be measurable. Several kinds of convergence may occur as $n \to \infty$:

(a) It is said that f_n converges *a.e* to f if $f_n(\mathrm{w}) \to f(\mathrm{w})$ except for ω in a set of measure zero.

(b) For *M* a metric space, it is said that f_n converges *in measure* to f if for each $\delta > 0$, $\mu\{\mathrm{w} : d(f_n(\mathrm{w}), f(\mathrm{w})) > \delta\} \to 0$, *d* being a metric on *M*.

(c) For *M* a Banach space, it is said that f_n converges in P^{th} *- mean* or in L^p to f if $\| f_n - f \|_{L^p} = \left(\int_{\omega} | f_n(\mathrm{w}) - f(\mathrm{w}) |^p \, d\mu(\mathrm{w}) \right)^{\frac{1}{p}} \to 0$ as $n \to \infty$. If $p = 2$, we say that f_n converges *in mean*.

(d) It is said that f_n converges *in law* to f if the measure $f_n\mu$ on M tends to the measure $f\mu$ in the sense that, $\int_M hd(f_n\mu) \to \int_M hd(f\mu)$ for $h : M \to \mathbb{R}$ bounded and continuous.

The basic idea of the Ito integral is to define a stochastic integral for belated partition, $\prod$ that is $\int_a^b A(t,\mathrm{w})\,dz(t,\mathrm{w}) = \sum_{j=1}^{n} A(t_j,\mathrm{w})\left(z(t_{j+1},\mathrm{w}) - z(t_j,\mathrm{w})\right)$.

5.6.1Belated semivariation

In setting up this integration, the requirement is a field F consisting of sets which are finite unions of intervals. The intervals may be of some given types such as left- closed and right- open, etc. F can be considered as the field of subsets of $[0,T]$ where $T > 0$ comprising of finite unions of subintervals of $[0,T]$. The intervals may be with or without a particular endpoint.

Let μ be a finitely additive set function on F with values in a linear space *Y*. Let

X be a normed space and $(X_\alpha), \alpha \in [0,T]$ an increasing family of subspaces of X, Barnett, *et al* (1982). This is to say that

$(X_\alpha), \alpha \in [0,T]$ satisfy:

(i) If $\alpha_1 \le \alpha_2$ then $X_{\alpha_1} \subseteq X_{\alpha_2}$

(ii) $X_\infty = \bigcup_\alpha X_\alpha$ is finite

(iii) $\bigcap_{\beta > \alpha} X_\beta = X_\alpha$

A bilinear multiplication is given by $X \times Y \to Z$ where Z is a Banach space. This theory is developed by Bartle (1956) using the semivariation of μ with respect to X which for $\mathcal{E} \in \mathrm{F}$ is defined as

$$\|\mathcal{E}\| = \sup\left|\sum x_i \mu \mathcal{E}_i\right| \tag{5.9}$$

where the supremum is taken over all partitions of $\mathcal{E}_i$ in F into a finite number of disjoint sets $\mathcal{E}_i$ and finite collection $\{x_i\}$ with $|x_i| \le 1$.

5.6.2 Right-belated semivariation.

The *right – belated semivariation* of μ with respect to (X_α) is denoted by $\|\mathcal{E}\|^b$ and defined as

$$\|\mathcal{E}\|^b = \sup\left|\sum x_i \mu \mathcal{E}_i\right| \tag{5.10}$$

where the supremum is taken over all partitions of $\mathcal{E}$ into a finite number of disjoint intervals $\mathcal{E}_i$ in F , and all collections $\{x_i\}$ with $x_i \in X_{\inf \mathcal{E}_i}$ for all i and $|x_i| \le 1$ It is sometimes written as $\|\mathcal{E}\|_\mu^b$ rather than $\|\mathcal{E}\|^b$ to emphasize the dependence of $\|.\|^b$ upon μ

The discussion will be restricted to $[0,T]$ rather than $\mathbb{R}^{+}$ in order to avoid technical difficulties associated with the infinite measure space.

It may be noted that the bilinear multiplication $X \times Y \to Z$ is represented above as *xy*. The order *xy* or *yx* may not necessarily matter, except perhaps there is a natural bilinear multiplication $X \times Y \to Z$ where $Z^{'}$ is a Banach space (possibly different from *Z*). In this situation, probably working with the left – belated semivariation of μ with respect to (X_α) denoted by ${}^{b}\|\mathcal{E}\|$ may be considered and defined as

$$^{b}\|\mathcal{E}\| = \sup\left|\sum \mu\mathcal{E}_i x_i\right| \tag{5.11}$$

The supremum is taken over as in (5.10)

In investigating some properties of the right – belated semivariation, a consideration of the concept of measurability is relevant.

5.6.3 Lemma [Barnett and Wilde, (1986)]

Let $\mathcal{E} \in \mathrm{F}$, then

(a) $\|\mathcal{E}\|^{b} \leq \|\mathcal{E}\|$

(b) $\|.\|$ is monotone and subadditive in F

Proof

(a) Let $\{\mathcal{E}_i\} \subset \mathrm{F}$ be a partition of $\mathcal{E}$ and $\{x_i\} \subset X$ be such that $|x_i| \leq 1$ and $x_i \in X_{\inf \mathcal{E}_i}$ .for

all *i*. Then by definition of $\|\mathcal{E}\|$, $\left|\sum x_i \mu\mathcal{E}_i\right| \leq \|\mathcal{E}\|$.

Taking supremum, one has, $\sup\left|\sum x_i \mu\mathcal{E}_i\right| \leq \|\mathcal{E}\| \Rightarrow \quad \|\mathcal{E}\|^{b} \leq \|\mathcal{E}\|$, which proves (a)

(b) Let $\mathcal{E}, \mathcal{F}, \in$ F and suppose that $\mathcal{E} \cap \mathcal{F} = \varnothing$. Let $\{\mathcal{G}_i\}$ be a partition of $\mathcal{E} \cup \mathcal{F}$ into

a finite number of disjoint intervals. Let $\{x_i\}$ be such that $x_i \in X_{\inf \mathcal{G}_i}$ and $|x_i| \le 1$, Set $\mathcal{E}_i = \mathcal{E} \cap \mathcal{G}_i$, $\mathcal{F}_i = \mathcal{F} \cap \mathcal{G}_i$ for all *i*. For a fixed *i*, $\mathcal{E}_i = \bigcup_j \mathcal{E}_{i\,j}$ where $\{\mathcal{E}_{i\,j}\}$ is a finite number of disjoint intervals. notice that $x_j \in X_{\inf \mathcal{E}_{i\,j}}$ for each *j*. Similarly, $\mathcal{F}_i = \bigcup_j \mathcal{F}_{i\,j}$ and $x_i \in X_{\inf \mathcal{F}_{i\,j}}$ for each *j*.

Now, $\{\mathcal{E}_{i\,j}\}$ is a partition of $\mathcal{E}$ into intervals and $\{\mathcal{F}_{i\,j}\}$ is a partition of $\mathcal{F}$ in similar manner. So,

$$\begin{aligned}
\left|\sum_i x_i \mu \mathcal{G}_i\right| &= \left|\sum_i x_i \mu\left(\mathcal{G}_i \cap (\mathcal{E} \cup \mathcal{F})\right)\right| \\
&= \left|\sum_i \sum_j x_i \mu \mathcal{E}_{i\,j} + \sum_i \sum_j x_i \mu \mathcal{F}_{i\,j}\right| \\
&\le \left|\sum_i \sum_j x_i \mu \mathcal{E}_{i\,j}\right| + \left|\sum_i \sum_j x_i \mu \mathcal{F}_{i\,j}\right| \\
&\le \|\mathcal{E}\|^b + \|\mathcal{F}\|^b \\
\Rightarrow \|\mathcal{E} \cup \mathcal{F}\|^b &\le \|\mathcal{E}\|^b + \|\mathcal{F}\|^b
\end{aligned}$$

Thus, proving the subadditivity

To prove the monotonocity, take $\mathcal{E}, \mathcal{F}, \in$ F and suppose that $\mathcal{E} \subseteq \mathcal{F}$ Let $\{\mathcal{E}_i\}$ be a partition of $\mathcal{E}$ into intervals and $\{x_i\}$ be such that $x_i \in X_{\inf \mathcal{E}_i}$ and $|x_i| \le 1$ for all *i*. Let $\{\mathcal{G}_i\}$ be a partition of $\mathcal{F} \setminus \mathcal{E}$ into intervals. Then,

$$\left|\sum x_i \mu \mathcal{E}_i\right| = \left\|\sum x_i \mu \mathcal{E}_i + 0.\mu \mathcal{G}_i\right\| \le \|\mathcal{F}\|^b$$

Taking supremum on the left, we get $\|\mathcal{E}\|^b \le \|\mathcal{F}\|^b$

$\|.\|^b$ can be extended to all subsets of $[0,T]$.

5.6.4μ - Flat null.

Let $A \subseteq [0,T]$. Define $\|A\|^b$ by $\|A\|^b = \inf\left\{\|\mathcal{E}\|^b : \mathcal{E} \in \mathrm{F}, \mathcal{E} \supseteq A\right\}$.

One can see that $\|.\|^b$ extended in this way is monotone and subadditive.

If $\|A\|^b = 0$, then A is called μ^b - null ("μ - *flat null*"). A property $P(s)$ is said to hold $\mu^b -$a.e if $\|s: \text{ not } P(s)\|^b = 0$.

5.6.5Integral of a simple process

Following the discussions 3.10.11, the following analogous definitions can be stated:

(a) A map $f:[0,T] \to X$ is a process if $f(s) \in \bar{X}_s, s \in [0,T]$ where $\bar{X}_s$ is the closure of X_s in *X*.

(b) A process is elementary if it has the form $x\chi_I(s)$, where *I* is an interval and $\|I\|^b < \infty, \quad x \in X_{\inf I}$

(c) a process is simple if it is a linear combination of elementary processes with distinct supports

(d) For a simple process f, we define the integral in the usual way. That is suppose

$f(s) = \sum_i x\chi_I(s)$. Then for $\mathcal{E} \in \mathrm{F}$

$$\int_i f d\mu = \sum_i x_i \mu(\mathcal{E} \cap I_i)$$

The integral of a simple process is independent of its representation as a sum of elementary processes. Let the set of simple processes on $\mathcal{E} \in \mathrm{F}$ be denoted by $\mathrm{S}(\mathcal{E})$,then the following results hold:

5.6.6Lemma

(a) For $\mathcal{E} \in$ F **,** S $(\mathcal{E})$ is a linearspace under point-wise operations.

(b) For a fixed $\mathcal{E} \subset$ F , the map $f \to \int_{\mathcal{E}} f d\mu \in z$ is a linear mapping, where $f \in$ S $([0,T))$

(c) For a fixed $f \in$ S $([0,T])$, the map $f \to \int_{\mathcal{E}} f d\mu \in z$ is an additive function, where $\mathcal{E} \in$ F .

(d) If $f \in$ S $([0,T])$ and $f \in$ S $([0,T])$ and $(\exists\, M \in \square^{+}) : (\forall\, s \in \mathcal{E} \in \mathcal{F}), |f(s)| < M$, then $\left|\int_{\mathcal{E}} f d\mu\right| \leq M \|\mathcal{E}\|^{b}$

This Lemma is a counterpart of Bartle's Lemma1 Bartle, (1956).

5.6.7Convergence

Relative to the definitions 2.1.2 – 2.1.3, the following can be stated

(a) Let $f, g_1, g_2, \ldots$ be functions: $[0,T] \to X$. It is said that (g_n) converges to *f* μ^b -a.e (μ - flat a.e) on a set $\mathcal{E} \in$ F if there exists $A \subseteq \mathcal{E}$ with $\|A\|^{b} = 0$ and $\forall\, s \in \mathcal{E} \setminus A,\ g_n(s) \to f(s)$ in norm.

(b) Let $f, g_1, g_2, \ldots$ be functions: $[0,T] \to X$. It is said that (g_n) converges to*f* μ^b -almost uniformly (μ - flat a.u) on a set $\mathcal{E} \in$ F if $\forall\, \varepsilon > 0$, there exists $A \subseteq [0,T]$ with $\|A\|^{b} < \varepsilon$ and the convergence is uniform on $\mathcal{E} \setminus A$.

(c) Let $f, g_1, g_2, \ldots$ be functions: $[0,T] \to X$. It is said that (g_n) converges to f in μ^b – measure (μ - flat measure) on a set $\mathcal{E} \in$ F if $\forall\, \varepsilon > 0$,

$\left\|\left\{s \in \mathcal{E} : \left|f(s) - g_n(s)\right| \geq \varepsilon\right\}\right\|^b \to 0$ as $n \to \infty$ $\quad$ ($\|.\|^b \to 0$, approaches μ - flat null)

(d) A function $f : [0,T] \to X$ is said to be μ^b – measurable if there exists a sequence of simple processes converging to *f* in μ^b – measure on $[0,T]$.

5.6.8Control measure

An extended real-valued countably additive set function measure θ defined on F which, if it exists, assures the conditions of countable subadditivity of $\| \ \|_\mu^b$, is called a *control measure*. The existence of such a measure will allow the extension of $\| \ \|_\mu^b$ from F to θ - measure subset of $(0,T]$ in a manner different from that of (5.8). This will widen the class of integrable functions.

5.6.9 Proposition [Barnett and Wilde (1986)]

Suppose that μ is finitely additive on F. Let θ be a countably additive positive real-valued set function on F and $f : \mathbb{R}^+ \to \mathbb{R}^+$ some functions continuous at $0 \in \mathbb{R}$. Further, suppose that $\forall \mathcal{E} \in$ F, $\|\mathcal{E}\|^b \leq f\left(\theta(\mathcal{E})\right)$ Then $\|.\|$ is countably subadditive on F

Proof

Let $\mathcal{E}, \mathcal{E}_1, \mathcal{E}_2 ...$ be elements of F with $\mathcal{E} = \bigcup_n \mathcal{E}_n \in$ F. It can be seen that

$\mathcal{E} = \left(\mathcal{E} \setminus \bigcup_{n=1}^{m} \mathcal{E}_n\right) \cup \left(\bigcup_{n=1}^{m} \mathcal{E}_n\right)$, then by finite additivity

$$\|\mathcal{E}\|^b \leq \left\|\mathcal{E}_n \setminus \bigcup_{n=1}^{m} \mathcal{E}_n\right\|^b + \left\|\bigcup_{n=1}^{m} \mathcal{E}_n\right\|^b \leq f\left(\theta\left(\mathcal{E}_n \setminus \bigcup_{n=1}^{m} \mathcal{E}_n\right)\right) + \sum_{n=1}^{m} \|\mathcal{E}_n\|^b$$

Now, $\mathcal{E}_n \setminus \bigcup_{n=1}^{m} \mathcal{E}_n \downarrow \varnothing$ as $m \uparrow$ $\theta(.)$ is continuous from above at $\varnothing$ and f is continuous at zero. So, $\|.\|^b$ is countably additive on F.

5.7 Outer Set

On the above proposition, it is seen that countable subadditivity can collapse outside F i.e. $O(\mathrm{F})$, where $O(\mathrm{F})$ is the outer set of F, defined by $O(\mathrm{F}) = \left\{ \mathcal{E} : \mathcal{E} = \bigcup_n \mathcal{E}_n,\ \mathcal{E} \uparrow, \mathcal{E}_n \in \mathrm{F} \right\}$ and means outer set in $[0,T]$. If $X = Y = Z = [0,T]$ and μ is a Lebesgue measure on F, then $\mathcal{E} = □ \cap [0,T]$ is countable, $\mathcal{E} = \{x_n : n \in □\}$ so to say. Here, $\mathcal{E} = \bigcup_n \{x_n\}$ and $\|\{x_n\}\|_\mu^b = 0$, thus $\sum_n \|\{x_n\}\|_\mu^b = 0$. However, the smallest element of F containing $\mathcal{E}$ is $[0,T]$, thus, $\|\mathcal{E}\|_\mu^b = T$. It is clear that $\|\mathcal{E}\|_\mu^b$ is not countably subadditive outside F.

In the integration theory, $\|\mathcal{E}\|^b$ will be written forthe belated semivariation of a set $\mathcal{E} \subseteq [0,T]$, which may be defined either as already defined or if control measure exists, as : let $A \subseteq [0,T]$. $\|A\|_\mu^b = \left\{ \inf \|\mathcal{E}\|_\mu^b : \mathcal{E} \supseteq A,\ \mathcal{E} \in O(\mathrm{F}) \right\}$

5.8μ - Flat Integration

(a). Let $f : [0,T] \to X$ be μ^b – measurable and let (f_n) be a sequence of simple processes converging to f in μ^b – measure. We say that f is μ^b – integrable over $[0,T]$ if (f_n) can be chosen so that (i). $\forall \varepsilon > 0,\ \exists\ \delta > 0 : \mathcal{E} \in \mathrm{F}$ and $\|\mathcal{E}\|^b < \delta,$ then $\forall n \left| \int_{\mathcal{E}} f_n d\mu \right| < \varepsilon$

(ii). $\forall \varepsilon > 0,\ \exists\ \delta > 0 : \mathcal{F} \in \mathrm{F}$ with $\|\mathcal{F}\|^b < \infty$ and if $\mathcal{G} \in \mathrm{F},\ \mathcal{G} \subseteq [0,T] \setminus \mathcal{F}$
then $\forall n \left| \int_{\mathcal{G}} f_n d\mu \right| < \varepsilon$

(b) Let $\mathcal{E} \in \mathrm{F}$ and f be $\mu^b -$ measurable. We say that f is $\mu^b -$ integrable over $\mathcal{E}$ if $f\chi_{\mathcal{E}}$ is $\mu^b -$ integrable over $[0,T]$.

5.8.1Theorem [Barnett and Wilde (1986)]

Let f be $\mu^b -$integrable and let (f_n) be a simple process defining f. For each $\mathcal{E} \in \mathrm{F}$, $\left(\int_{\mathcal{E}} f_n d\mu \right)$ is a Cauchy sequence in *Z*. The limit is denoted by $\int_{\mathcal{E}} fd\mu^b$ and is called the $\mu -$*belated integral* of f over $\mathcal{E}$. This limit exists uniformly for $\mathcal{E} \in \mathrm{F}$.
The $\mu^b -$ belated integral of f over is independent of the sequence of simple processes that define f.

Proof

Since (f_n) converges to f in $\mu^b -$ measure, it is Cauchy in $\mu^b -$ measure.

So, $\forall\ \eta > 0, \forall t > 0\ \exists\ N(t) \in$ such that $m,n \geq N(t)$, then $\left\| \{ s : f_n(s) - f_m(s) \geq \eta \} \right\|^b < t$

Let $\varepsilon > 0$ and choose $\delta > 0$ and $\mathcal{F}_\varepsilon$ as defined earlier. If η is taken to be $\eta = \dfrac{\varepsilon}{1 + \|\mathcal{F}_\varepsilon\|^b}$ and $t = \delta$ then there exists $N \in$ such that whenever $m,n \geq N$ are fixed, the set $\mathcal{G} = \left\{ s : \left| f_n(s) - f_m(s) \right| \geq \dfrac{\varepsilon}{1 + \|\mathcal{F}_\varepsilon\|^b} \right\} \in \mathrm{F}$ and $\|\mathcal{G}\|^b < \delta$.

Since f is $\mu^b -$ measurable,and since $\mathcal{E}$ can be written as a disjoint union of sets in F i.e.

$$\mathcal{E} = (\mathcal{E} \cap \mathcal{G}) \cup \big((\mathcal{E} \setminus \mathcal{G}) \cap \mathcal{F}_\varepsilon\big) \cup \big(\mathcal{E} \setminus (\mathcal{G} \cup \mathcal{F}_\varepsilon)\big),$$

then by the triangle inequality

$$\left|\int_{\mathcal{E}} f_n d\mu - \int_{\mathcal{E}} f_m d\mu\right| \leq \left|\int_{\mathcal{E}\cap\mathcal{G}} f_n d\mu\right| + \left|\int_{\mathcal{E}\cap\mathcal{G}} f_m d\mu\right| + \left|\int_{\mathcal{E}\setminus(\mathcal{G}\cap\mathcal{F}_\varepsilon)} f_n d\mu\right| + \left|\int_{\mathcal{E}\setminus(\mathcal{G}\cap\mathcal{F}_\varepsilon)} f_m d\mu\right| + \left|\int_{(\mathcal{E}\setminus\mathcal{G})\setminus\mathcal{F}_\varepsilon} (f_n - f_m) d\mu\right|$$

Since $\|\mathcal{E} \cap \mathcal{G}\|^b \leq \|\mathcal{G}\|^b < \delta$, then the terms involving $\mathcal{E} \cap \mathcal{G}$ are strictly less than ε.

Also, $\mathcal{E} \setminus (\mathcal{G} \cup \mathcal{F}_\varepsilon) \subseteq \mathcal{E} \setminus \mathcal{F}_\varepsilon$, so the terms involving $\mathcal{E} \setminus (\mathcal{G} \cup \mathcal{F}_\varepsilon)$ are strictly less than ε by definition and the fact that $(\mathcal{E} \setminus \mathcal{G}) \cap \mathcal{F}_\varepsilon \subseteq [0,T] \setminus \mathcal{G}$,then

$$\left|\int_{(\mathcal{E}\setminus\mathcal{G})\setminus\mathcal{F}_\varepsilon} (f_n - f_m) d\mu\right| \leq \left(\frac{1}{\varepsilon} + \|\mathcal{F}_\varepsilon\|^b\right) . \left\|(\mathcal{E} \setminus \mathcal{G}) \cap \mathcal{F}_\varepsilon\right\|^b < \varepsilon,$$ - by the monotonocity of $\|.\|^b$.

This shows that $\left(\int_{\mathcal{E}} f_n d\mu\right)$ is Cauchy in Z and that the limit exists uniformly for $\mathcal{E} \in$ F

Now if the supposition is that (f_n) and (g_n) are sequences of simple processes defining *f*, then the sequence $(f_1, g_1, f_2, g_2, \ldots)$ will satisfy:

$$\int_{\mathcal{E}} f d\mu^b = \lim_n \int_{\mathcal{E}} f_n d\mu = \lim_n \int_{\mathcal{E}} g_n d\mu .$$

5.8.2Theorem [Barnett and Wilde (1986)]

(a) If $\mathcal{E} \in$ F the map $f \in I^b(\mathcal{E}) \to \int_{\mathcal{E}} f d\mu^b \in Z$ is linear

(b) If $f \in I^b[0,T]$, the map $f \in \mathcal{E} \to \int_{\mathcal{E}} f d\mu^b \in Z$ is an additive function

(c) If $f \in I^b[0,T]$, then $\lim_{\|\mathcal{E}\|^b \to 0} \left| \int_{\mathcal{E}} f d\mu^b \right| = 0$

(d) If $f \in I^b[0,T], \forall \varepsilon > 0 \; \exists \; \mathcal{E}_\varepsilon \in \mathrm{F} : \mathcal{G} \subseteq [0,T] \setminus \mathcal{E}_\varepsilon$, then $\left| \int_{\mathcal{G}} f d\mu^b \right| < \varepsilon$

5.8.3 Essential boundedness

Let $f : [0,T] \to X$, it will be said that *f* is $\mu^b -$ essentially bounded on $\mathcal{E} \subset [0,T]$ if

$ess \sup_{s \in \mathcal{E}} |f(s)| = \inf_{\mathcal{N}} \left(\sup_{s \in \mathcal{E} \setminus \mathcal{N}} |f(s)| \right) < \infty$ where the infimum is taken over all $\mu^b -$ null sets.

5.8.4Theorem [Barnett and Wilde (1986)]

A $\mu^b -$ essentially bounded $\mu^b -$ measurable function f is integrable over each $\mathcal{E} \in \mathrm{F}$ with $\|\mathcal{E}\|^b < \infty$ Moreover,

$$\left| \int_{\mathcal{E}} f d\mu^b \right| < ess \sup_{s \in \mathcal{E}} |f(s)| . \|\mathcal{E}\|^b .$$

5.8.5Theorem[Barnett and Wilde (1986)]

Let $[0,T] \to X$ and $f_1, f_2, \ldots$ be elements of $I^b[0,T]$. Suppose that

(i) (f_n) converges to *f* in $\mu^b -$ measure

(ii) $\forall \; \varepsilon > 0 \; \exists \; \delta > 0 : \mathcal{E} \in \mathrm{F}$ and $\|\mathcal{E}\|^b < \delta,$ then $\forall \; n \left| \int_{\mathcal{E}} f_n d\mu^b \right| < \varepsilon$

(iii) $\forall \; \varepsilon > 0 \; \exists \; \mathcal{F} \in \mathrm{F} : \|\mathcal{F}\|^b < \infty,$ and if $\mathcal{G} \in \mathrm{F}$and $\mathcal{G} \subseteq [0,\mathrm{T}] \setminus \mathcal{F}$ then $\forall \; n \left| \int_{\mathcal{G}} f_n d\mu^b \right| < \varepsilon$

Then $f \in I^b[0,T]$ and for $\mathcal{E} \in \mathrm{F}$

$$\int_{\mathcal{E}} fd\mu^b = \lim_n \int_{\mathcal{E}} f_n d\mu^b$$

Moreover, this limit is uniform for $\mathcal{E} \in \mathrm{F}$.

Let M be a von Neumann algebra and $(\mathrm{B}_\alpha)_{\alpha\geq 0}$, its right continuous filtration with faithful normal state ϕ and suppose that the spaces $L^2(\mathrm{B}_\alpha)_{\alpha\geq 0}$ and $L^2(M)$ are separable. Then the belated integral based on the theory of bilinear vector integrals may be considered and the following result can be formulated:

5.9Theorem

Suppose *f* is an essentially bounded $\mu^b -$ integrable process and *g* a $\mu^b -$ integrable process. Then *f . g* is also integrable and

$$\int f.g \ d\mu^b = \int f \ d\mu^b . \int g \ d\mu^b$$

Proof

Suppose $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are sequences of $\pi(\mathrm{A})$ -adapted simple processes for which $f_n \to f$, $g_n \to g$ $\mu^{\mathrm{b}} -$a.e. and $ess\sup_n \|f_n\| \leq ess\sup\|f\| + \epsilon$ for some $\epsilon > 0$.

Since we know that $\int fd\mu^b$ is a bounded operator, $\int f_n d\mu \to \int fd\mu^b$ in the strong operator topology and

$$\sup_n \left\|\int f_n d\mu\right\| \leq ess\sup\|f\| + \epsilon$$
$$\left\|\int f_n d\mu\right\| \leq ess\sup\|f\|$$

Then

$$\left\| f.g(t) - f_n g_n(t) \right\| \le \left\| f(t)g(t) - f_n(t)g(t) \right\| + \left\| f_n(t)g(t) - f_n(t)g_n(t) \right\|$$
$$\le \left\| f(t) - f_n(t) \right\| \left\| g(t) \right\| + \left(ess\sup \| f \| + \epsilon \right) . \left\| g(t) - g_n(t) \right\|$$
$$\to 0 \text{ as } n \to \infty$$

for $\mu^b -$ almost all t, since $f_n \to f$ and $g_n \to g \quad \mu^b -$a.e.

Also,

$$\text{domain}\left(\int fd\mu^b . \int gd\mu^b \right) = \text{domain}\left(\int gd\mu^b \right)$$
$$\supseteq \{ \xi_n : n \ge 1 \}$$

because $\int fd\mu$ is a bounded operator and that

$$\left\| \int fd\mu^b . \int gd\mu^b \right\|_\Omega^2 \le ess\sup \| f \| \left\| \int gd\mu \right\|_\Omega^2$$

So, $\qquad \int fd\mu^b . \int gd\mu^b \in \mathcal{H}$

But

$$\left\| \int fd\mu^b . \int gd\mu^b - \int f_n d\mu . \int g_n d\mu \right\|_\Omega \to 0 \text{ as } n \to \infty$$

and

$$\int f_n . g_n d\mu = \int f_n d\mu \; . \int g_n d\mu$$

This means that $f \,.\, g$ is $\mu^b -$integrable and

$$\int f \,.\, gd\mu^b = \int fd\mu^b . \int gd\mu^b$$

5.10 Results and discussions.

Following the available definitions and the existing results, we have been able to come up withTheorem (5.9) and also proved it.There can be more results perhaps, if one looks at other algebraic structures on the belated integrals,especially another refinement of McShane divisions.

CHAPTER SIX

SUMMARY CONCLUSION AND RECOMMENDATION

6.1 Summary

At the beginning of this dissertation, the objectives were set followed by a review of the literature together with definition of terms. Theories of interest are studied and analyzed and some useful constructions are put in place. The analysis finally culminated into the construction of the product of belated integrals of some processes.

In the construction and analysis ofstopping times and belated integrals, the following results numbered (a) to (d) are established:

(a) Theorem [Stopping Time Process]

For a stopping time τ, the stochastic process $X^{\mathrm{t}} = \left(X_t^{\mathrm{t}}\right)_{t \in T}$ defined by

$$X_t^{\mathrm{t}} = 1_{\{\mathrm{t} > t\}} \qquad \left(t \in T\right)$$

is a stopping process.

(b)Theorem [Bijection between Stopping Time and Stopping Process]

$$f : X \to \mathrm{t}^{X}$$

is a bijection between the stopping processes and the stopping times on the filtered probability space $\left(\Omega, \mathrm{F}, \mathrm{F}_{\mathrm{t}}, P\right)$ such that

$$f^{-1} : \mathrm{t} \to X^{\mathrm{t}}$$

Hence $\mathrm{t}^{X^{\mathrm{t}}} = \mathrm{t}$

(c) Theorem [On Minimal Elements of σ-Algebra]

The elements of A_t are minimal elements of F_t, that is $\mathrm{A}_t \subset \mathrm{A}(\mathrm{F}_t)$. If $|\mathrm{F}| < \infty$, then A_t is the set of all minimal elements of F_t,that is

$\mathrm{A}_t = \mathrm{A}(\mathrm{F}_t)$ and $\mathrm{F}_t = \sigma(\mathrm{A}_t) = \sigma(\mathrm{A}(\mathrm{F}_t))$.

(d)Theorem [On the Product of μ-Flat Integrable Functions]

Suppose that f is an essentially bounded $\mu^b -$ integrable process and g a $\mu^b -$ integrable process. Then $f\,.\,g$ is also integrable and

$$\int f.g \; d\mu^b = \int f \, d\mu^b . \int g \; d\mu^b$$

6.2Conclusion

It is worth mentioning at this point that the objectives of this dissertation have been achieved through the establishment of results numbered (a) to (d) above under section 6.1. The classical aspect ofstopping times especiallycan be applied in almost all aspects of human endeavour. For instance once a process is understood and can be adapted to the sigma algebra of certain defined set then a stopping time to the process may be formulated. Fortunately, this adaptedness is also applicable even in the non-classical case, i.e., the non-commutative case.Closely connected with the stopping time is the concept of martingale which started as a game theory but now understood to be applicable in business market and even in quantum theories.

The concept of belated integrals which is to do with the partition of intervals [0, 1] in a constructive way seems to be a frontier in quantum theory.

6.3Recommendations

From the constructions and analyses in this work, it may be recommended that:

(i) Theories of stopping times can be made to stop an algebraic process such as the full or partial transformation semigroups.

(ii)Reconstruction of interval with some refinements can be employed to widen the scope of the belated integrals, thuswidening the theory.

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