BAYESIAN ESTIMATION OF THE SHAPE PARAMETER OF ODD GENERALIZED EXPONENTIAL-EXPONENTIAL DISTRIBUTION

BY

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DECLARATION

I declare that the work in this dissertation titled "Bayesian Estimation of the Shape Parameter of Odd Generalized Exponential-Exponential Distribution" has been carried out by me in the Department of Statistics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

_____ _____

YUSUF Bolaji Lukman Date

CERTIFICATION

This dissertation titled, "BAYESIAN ESTIMATION OF THE SHAPE PARAMETER OF ODD GENERALIZED EXPONENTIAL-EXPONENTIAL DISTRIBUTION by YUSUF Bolaji Lukman meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to ALLAH, the Almighty.

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ABSTRACT

The Odd Generalized Exponential-Exponential Distribution (*OGEED*) could be used in various fields to model variables whose chances of success or survival decreases with time. It was also discovered that the *OGEED* has higher positive skewness and has been found to have performed better than some existing distributions such as the Gamma, Exponentiated Exponential, Weibull and Pareto distributions in a real life applications. The shape parameter of the Odd Generalized Exponential-Exponential Distribution using the Bayesian method of estimation and comparing the estimates with that of maximum Likelihood by assuming two non-informative prior distributions namely; Uniform and Jeffrey prior distributions. These estimates were obtained using the squared error loss function (*SELF*), Quadratic loss function (*QLF*) and precautionary loss function (*PLF*). The posterior distributions of the *OGEED* were derived and also the Estimates and risks were also obtained using the above mentioned priors and loss functions. Furthermore, we carried out Monte-Carlo simulation using R software to assess the performance of the two methods by making use of the Biases and *MSEs* of the Estimates under the Bayesian approach and Maximum likelihood method. Our result showed that Bayesian Method using Quadratic Loss Function (*QLF*) under both Uniform and Jeffrey priors produces the best estimates of the shape parameter compared to estimates using Maximum Likelihood method, Squared Error Loss Function (*SELF*) and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors irrespective of the values of the parameters and the different sample sizes. It is also discovered that the scale parameter has no effect on the estimates of the shape parameter.

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CHAPTER ONE: INTRODUCTION

1.1 Background to the Study

In Bayesian approach, the prior information is combined with any new information that is available to form the basis for statistical inference. Statistical approaches that use prior knowledge in addition to the sample evidence to estimate the population parameters are known as Bayesian methods.

The Bayesian approach seeks to optimally merge information from two sources: (1) knowledge that is known from theory or opinion formed at the beginning of the research in the form of a prior, and (2) information contained in the data in the form of likelihood functions. Basically, the prior distribution represents our initial belief, whereas the information in the data is expressed by the likelihood function. Combining prior distribution and likelihood function, we can obtain the posterior distribution.

The one parameter Exponential distribution describes the time between events in a Poisson process. Its discrete analogue is the Geometric distribution. Apart from its usage in Poisson processes, it has been used extensively in the literature for life testing. The Exponential distribution is memoryless and has a constant failure rate; this latter property makes the distribution unsuitable for real life problems with bathtub failure rates (Singh *et al*., 2013) and inverted bathtub failure rates, hence there is need to generalize the Exponential distribution in order to increase its flexibility and capability to model some other real life problems.

There are several ways of adding one or more parameters to a distribution function which makes the resulting distribution richer and more flexible for modeling data. Some of the recent studies on the generalization of exponential distribution include the transmuted exponential distribution

(Owoloko *et al*., 2015), transmuted inverse exponential distribution (Oguntunde and Adejumo, 2015), the odd generalized exponential-exponential distribution (Maiti and Pramanik, 2015) and the Weibull-Exponential distribution (Oguntunde *et al*., 2015). Of interest to us in this dissertation is the odd generalized exponential-exponential distribution (*OGEED*) which can be use in various fields to model variables whose chances of success decreases with time whereas that of failure increases as time increases. It was also discovered that the *OGEED* has higher positive skewness and has been found to have performed better than some existing distributions such as the Gamma, Exponentiated-Exponential, Weibull and Pareto distributions.

1.2 Statement of the Problem

Despite the nature and performance of the odd generalized exponential-exponential distribution (*OGEED*) compared to other distributions, there has been no study that has used Bayesian approach to estimate the shape parameter of the *OGEED* as at the time of this literature review. Thus, it is always of interest to study the behavior and properties of an estimator of a parameter of a new distribution.

In Bayesian inference, the performance of the estimator or predictor depends on the prior distribution and also on the loss function used. This study shall use Bayesian approach to estimate the shape parameter of the *OGEED* proposed by Maiti and Pramanik (2015) under the Uniform and Jeffrey Priors using the Squared Error Loss Function (*SELF*), Precautionary Loss Function (*PLF*) and Quadratic Loss Function (*QLF*). Our choice of the two priors is because we have little or no prior knowledge about the parameter of interest and so we want a prior with minimal influence on the inference to be made. The most significant thing about Bayesian method over the frequentist or classical methods is the ability of the Bayesian analysis to make use of that additional information in the form of the prior distribution. As a result of this

incorporation, Bayesian methods do always provide stronger inferences from the same data. This is because the prior information allows the Bayesian analysis to be more responsive to the context of the data.

Besides the above, Aliyu and Yahaya (2016), Preda *et al.* (2010) and Dey (2010) in their separate studies employed extensive Monte Carlo simulations to compare the performances of the Bayes estimates with those of Maximum Likelihood Estimates (*MLEs*) in Generalized Rayleigh distribution, Modified Weibull distribution and Generalized Exponential distribution respectively. It was found that the estimates under Bayesian method were more stable than the estimates under the method of *MLE* i.e. the non-classical or Bayes estimates have smaller risks or *RMSE*.

1.3 Aim and Objectives of the Study

The aim of this study is to estimate the shape parameter of Odd Generalized Exponential-Exponential Distribution (*OGEED*) using Bayesian approach.

The specific objectives are to:

i. determine the posterior distributions under the Uniform and Jeffrey priors.
ii. obtain the Bayes estimate and Posterior risk using the Squared Error Loss Function (*SELF*), Precautionary Loss Function (*PLF*) and Quadratic Loss Function (*QLF*) under the Uniform and Jeffrey Priors.
iii. carry out a simulation study in order to find the most appropriate combination of loss functions and priors good for estimating the shape parameter of the distribution.
iv. compare Bayesian approach with the method of *MLE*.

1.4 Significance of the Study

In this research, we choose two non informative priors due to the fact that we have little knowledge about the distribution of the shape parameter. Also, our interest in the shape parameter rather than scale parameter is because the shape parameter is responsible for the outstretched or degree of its skewness and kurtosis.

This study will clearly estimate the shape parameter of the Odd Generalized Exponential-Exponential Distribution (*OGEED*) using Bayesian approach which is expected to be more stable.

This work serves as a reference material for other researchers and could be used to compare the estimate of the parameter obtained by using Bayesian method and the method of maximum likelihood estimation.

CHAPTER TWO: LITERATURE REVIEW

2.0 Introduction

Bayesian inference has found application in a wide range of activities, including Science, Engineering, Philosophy, Medicine, Sport, and Law. In the philosophy of decision theory, Bayesian inference is closely related to subjective probability, often called "Bayesian probability" in which we combine new information that is available with the prior information we have, to form the basis for the statistical procedures. Statistical approaches that use prior knowledge, possibly subjective, in addition to the sample evidence to estimate the population parameters are known as Bayesian methods.

This chapter reviewed some of the relevant works done by researchers which are directly related to this research topic. In doing so, this chapter reviewed some works under the following headings;

- Bayesian Estimation Approach
- Comparing Bayesian and Maximum Likelihood Estimates.

2.1 Bayesian Estimation Approach

Himanshu and Arun (2009) estimated the shape parameter of Generalized Pareto Distribution (GPD) using quasi, inverted gamma and uniform prior distributions under the LINEX, precautionary and entropy loss functions. The estimates were compared with that of the Bayes' estimators under the squared error loss function. In the work, it was shown that, none of the estimators under quasi prior uniformly dominates any estimator under the various loss functions.

Sanku (2010) estimated the shape parameter of generalized exponential distribution using extended Jeffrey's prior under the quadratic loss function, squared error loss function and general

entropy loss function. The estimators were compared by obtaining the risk of the estimators under the different loss functions and relative efficiency of the estimators. Monte Carlo simulation was performed in order to compare the estimators. The risk of the estimators were all found to be minima for the shape parameter of the distribution, the risk of general entropy loss function was found to be the least and as the sample size increases, the risk for all the estimators decreases as expected. Sanku (2010) recommended the use of general entropy loss function for the estimation of the shape parameter of generalized exponential distribution using the non-informative prior.

Sanku and Tanujit (2011) estimated the parameter of Rayleigh distribution using Bayesian technique. The estimates were obtained under Jeffrey's and extended Jeffrey's priors using some loss functions. Monte Carlo simulation was used to compare the estimates in terms of biases and MSE. It was observed that when sample size increases from 10 to 50, the bias and MSE both decreases quite significantly and as far as biases are concerned, the estimate under the extended Jeffrey's prior is better than the estimate under the Jeffrey's prior.

Sajid *et al.* (2012) estimated the scale parameter of Laplace model using different asymmetric loss functions comprising precautionary, weighted squared, modified (quadratic) squared loss functions. Symmetric loss function was also used. The estimators were obtained under the assumption of inverted gamma, inverted chi-squared, informative levy and Gumbel type II priors. The estimates were compared using the posterior risks under the different loss functions. It was discovered that Gumbel type II prior results are more precise than other priors and modified squared error loss function have the least posterior risk. Hence, it was concluded that Gumbel type II prior is the best prior for the estimation of the scale parameter of laplace model. Also, as discovered by many researchers, the posterior risk decreases as sample size increases.

Uniform, Jeffreys, Maxwell, Rayleigh, Chi and normal (considering the location parameter to be zero) priors under the squared error, quadratic, entropy and precautionary loss functions were used by Navid and Muhammad (2012) to derive an estimate of the scale parameter of error function distribution. Bayesian credible intervals under each prior were also derived and the estimators were evaluated and compared using simulation study. It was discovered that, the estimators based on entropy loss function under informative priors were the best when estimating the scale parameter of error function distribution.

Feroze (2012) discussed the Bayesian analysis of the scale parameter of inverse Gaussian distribution using different priors and loss function. He used both informative (exponential, gamma and chi square) and noninformative (uniformand Jeffreys) priors with eight loss functions namely; squared error loss function (SELF), quadratic loss function (QLF), entropy loss function (ELF), weighted loss function (WLF), squared logarithmic loss function (SLLF), linex loss function (LLF), precautionary loss function (PLF) and weighted balanced loss function (WBLF). It was deduced from the study that, the performance of the estimates under uniform prior is better than those under Jeffreys prior for most of the cases. While in case of informative priors (used), the performance of estimates using exponential prior was the best in terms of Bayes risks. Similarly, in comparison of informative and non informative priors, the informative priors give the better results. Although, the estimates under informative priors converge to the estimates under non-informative priors as the values of hyper-parameters approach to zero. It can also be indicated that the estimates under ELF are associated with the minimum risks using each prior.

Almutairi and Heng (2012) obtained the shape parameter of Generalized Power Distribution (GPD) via Bayesian approach under the non-informative (uniform) and informative (gamma)

priors using the squared error loss function. The Bayesian estimators were found to be dependent upon the parameters of the prior distribution. Monte Carlo simulation was carried out in order to make comparison among the estimates and it was observed that, the shape parameter obtained from GPD function also explored the fact that, the Bayesian method could be used in the process of estimating the shape parameter ranging from small to large samples.

Azam and Ahmad (2014b) estimated the scale parameter of Nakagami distribution using Bayesian approach. In their study, the scale parameter was estimated under three prior distributions, namely; Uniform, Inverse Exponential and Levy priors based on three loss functions: Squared Error Loss Function, Quadratic Loss Function and Precautionary Loss Function. The study of Azam and Ahmad (2014b) showed that the Precautionary Loss Function produces the least posterior risk when uniform prior is used while Squared Error Loss Function is the best when inverse exponential and Levy Priors are used. It was further shown that, the posterior risk is inversely proportional to the choice of values of the scale parameter using all priors.

The Bayesian estimate of the shape parameter of Power Function distribution under Exponential, Pareto, Chi-square, Quasi and extended Jeffrey prior using squared error loss, Quadratic loss and Precautionary loss functions were obtained by Azam and Ahmad (2014a). The performance of an estimators were assesed on the basis of its relative posterior risk while Monte-carlos simulation was also used to compare the performance of the estimators. It was discovered that the Precautionary loss function (PLF) produces the least posterior risk when Exponential and Pareto priors were used. Squared Error loss function (SELF) is the best when Chi-square, Quasi and extended Jeffrey were also obtained.

Wasif and Navid (2015) estimated the scale parameter of Nakagami distribution via Bayesian approach using uniform and gamma priors under the assumption of weighted, balanced weighted and precautionary loss functions. Assessment of the performance of the estimators reveal that the performance of inverse gamma prior is better than that of uniform prior and weighted balance loss function is better than weighted and precautionary loss functions.

Nasir *et al.* (2015) studied Bayesian estimation of the scale parameter of log logistic distribution using square error loss function, precautionary loss function, simple precautionary loss function and weighted loss function with two non-informative priors (uniform and Jeffery). The study showed that Bayes estimators approach to their true value and posterior risks decreases by increasing sample size. From the study, Jeffery prior performs better than the uniform prior and also precautionary loss function perform better than the other loss functions. Therefore, Jeffery with precautionary loss function provides minimum posterior risks as compared to other loss functions and priors.

Wajiha and Muhammed (2015) studied the shape parameter of Frechet distribution using the Gumbel Type – II and the Levy prior under four loss functions. It was observed that the Gumbel Type – II performed better than the Levy Prior as its posterior risks is smallest amongst the two assumed prior distributions. Weighted balance loss function was found to be better as its posterior risk was least among all the loss functions. So, Gumbel Type – II with weighted balance loss function provides less posterior risks as compare to other loss functions and priors.

2.2 Comparing Bayesian and Maximum Likelihood Estimates

Preda *et al.* (2010) obtained the Bayesian estimators of the shape and scale parameters of modified Weibull distribution using Lindley's approximation under the squared error loss

function, LINEX loss function and generalized entropy loss function. The study also compared the performances of these estimates with that of maximum likelihood for different sample sizes.

Dey (2010) obtained and compared the Bayesian estimates of the shape parameter of Generalized Exponential Distribution based on a class of non-informative prior under the assumption of quadratic loss function, squared log-error loss function and general entropy loss function (*GED*) were estimated and compared with that of maximum likelihood estimates. It was shown that, the root mean square error (*RMSE*) of the bayesian estimates is smaller than that of the classsical estimator. It was observed that the risk under general entropy loss function is the least compared to the risk under squared error loss function, square log error loss function and maximum likelihood estimates and it was further observed that, as sample size increases, the risk of the estimators decreases.

Aliyu and Yahaya (2016) studied the shape parameter of generalized Rayleigh distribution under non-informative prior. Extended Jeffrey's prior with three loss functions namely; Entropy Loss Function, Precautionary Loss Function, Squared Error Loss Function was used. The method of maximum likelihood (*MLE*) was also employed Extensive Monte Carlo simulations was carried out to compare the performances of the Bayes estimates with that of *MLEs*. It was observed that the estimate under the Entropy loss function is more stable i.e. the estimates have smaller risk or RMSE than the estimates under squared error loss function, Precautionary loss function and *MLEs*.

CHAPTER THREE: METHODOLOGY.

3.1 Odd Generalized Exponential-Exponential Distribution

This work employed the under-listed methods and techniques to estimate the shape parameter of the odd generalized exponential-exponential distribution (*OGEED*), λ.

According to Maiti and Pramanik (2015), if X is a random variable, then the probability density function (*pdf*) and the cumulative distribution function (*cdf*) of the odd generalized exponential-exponential distribution are given by;

$$f(x) = \lambda\theta e^{\theta x} e^{-\lambda\left(e^{\theta x}-1\right)} \tag{3.1}$$

and

$$F(x) = 1 - e^{-\lambda\left(e^{\theta x}-1\right)} \tag{3.2}$$

respectively. where; $x > 0$, $\theta > 0$ is a scale parameter and $\lambda > 0$ is shape parameter.

The following is a graphical representation of the *pdf* and *cdf* of the *OGEED*. Given some values of the parameters θ and λ, we provide some possible shapes for the *pdf* and the *cdf* of the *OGEED* as shown in Figure 3.1 and 3.2 below:

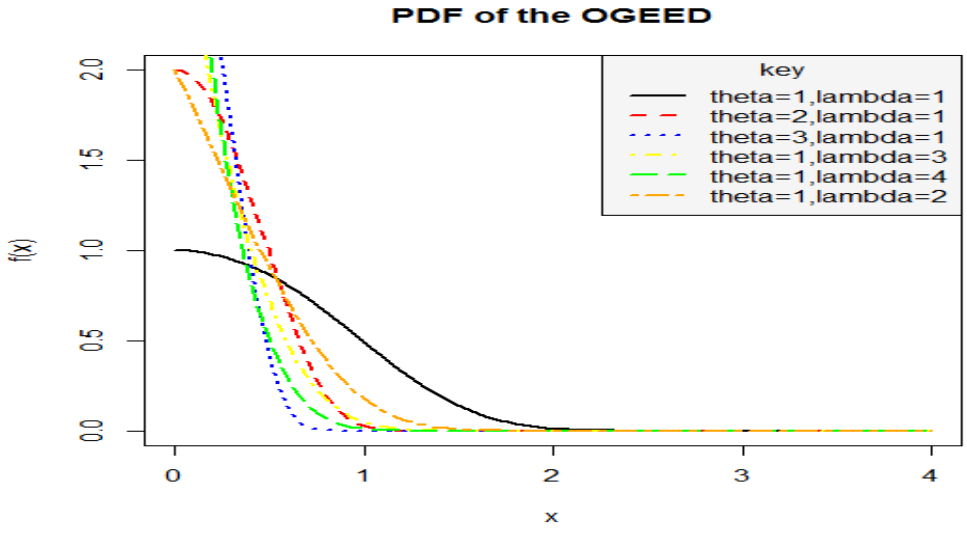

Figure 3.1: *PDF* of the *OGEED* for different values of θ and λ.

Figure **3.1** indicates that the *OGEED* is a skewed distribution and as such skewed to the right for most parameter values. This means that distribution can be very useful for datasets that are positively skewed.

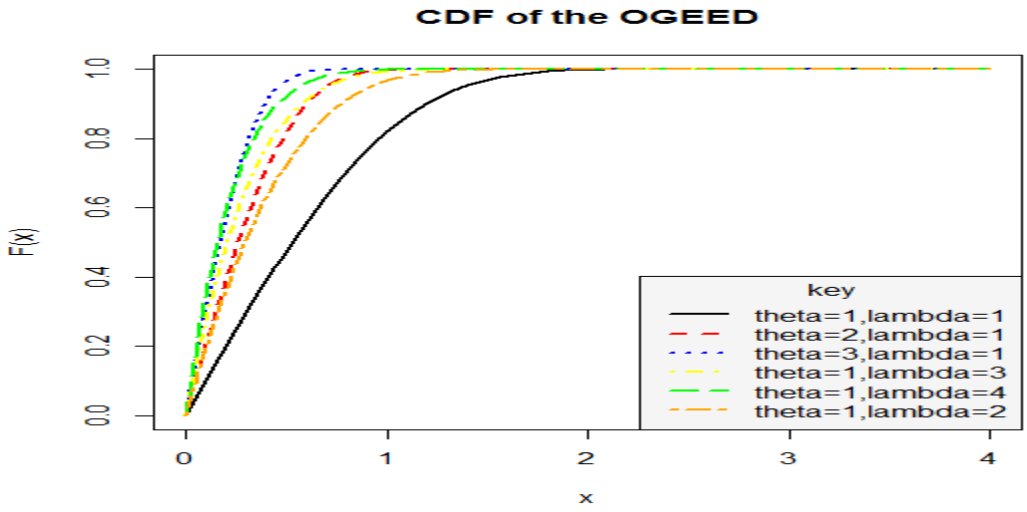

Figure 3.2: *CDF* of the *OGEED* for different values of θ and λ.

From the above *cdf* plot, the *cdf* increases when *X* increases, and approaches 1 when *X* becomes large, as expected. Hence, the *OGEED* is a valid probability density function.

3.2 Maximum Likelihood Estimation Method

Let $x_1, x_2, ..., x_n$ be a random sample from a population *X* with probability density function $f(x;\theta,\lambda)$, where θ and λ are unknown parameters and θ is assumed to be known as θ_0. The likelihood function, $L(X/\theta_\circ,\lambda)$, is defined to be the joint density of the random variables $x_1, x_2, ..., x_n$. The *pdf* of the *OGEED* is given as

$$f(x;\theta_\circ,\lambda) = \lambda\theta_\circ e^{\theta_\circ x} e^{-\lambda\left(e^{\theta_\circ x}-1\right)} \tag{3.3}$$

The likelihood function is given by;

$$L(\lambda,\theta_\circ \mid \underline{X}) = \left(\lambda\theta_\circ\right)^n e^{\theta_\circ \sum_{i=1}^{n} x_i} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_\circ x_i}-1\right)} \tag{3.4}$$

Let the log-likelihood function, $l = \log L\left(X_1, X_2, ..., X_n \mid \theta_\circ, \lambda\right)$, therefore

$$l = n\log\theta_\circ + n\log\lambda + \theta_\circ\sum_{i=1}^{n} x_i - \lambda\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)$$

Differentiating l partially with respect to λ, the parameter of interest and solving for $\hat{\lambda}$ gives;

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right) = 0 \Rightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)} \qquad (3.5)$$

3.3 Posterior Distributions

In this Bayesian framework, all the information about λ coming from the data is contained in the likelihood. Values of the parameters that correspond with the largest value of the likelihood are the parameters that are most supported by the data. The prior distribution represents our initial belief in the parameter of interest λ. This means that λ is regarded as a random variable with pdf, P(λ). Once the data has been observed, the likelihood function or simply the likelihood is constructed. The likelihood is the joint probability function of the data, but viewed as a function of the parameters, treating the observed data as fixed quantities. Assuming that the values, $x = (x_1, \ldots\ldots\ x_n)$ are obtained independently, the likelihood function is given by

$$L(\lambda|x) = P(x_1, x_2, \ldots., x_n|\lambda) = \prod_{i=1}^{n} P(X_i;|\lambda)$$

Note that the likelihood is a function of λ for a given x, while the PDF is a function of x for a given λ.

To obtain the posterior distribution $P(\lambda/x)$, the probability distribution of the parameters once the data has been observed, we apply Bayes' Theorem

$$p(\lambda \mid \underline{X}) = \frac{p(\lambda) L(\lambda \mid \underline{X})}{\int_0^{\infty} p(\lambda) L(\lambda \mid \underline{X}) d\lambda} \tag{3.6}$$

where $p(\lambda)$ and $L(\lambda \mid \underline{X})$ are the prior distribution and the Likelihood function respectively.

3.3.1 Posterior Distribution of the Shape Parameter under the Assumption of Uniform Prior

The uniform prior as a non-informative prior relating to the shape parameter λ is defined as:

$p(\lambda) \propto 1; 0 < \lambda < \infty$

Recall that the posterior distribution of the shape parameter λ is defined in equation (3.6).

Also, recall that the likelihood function of the *OGEED* is given in equation (3.4) while for the shape parameter in particular is given as:

$$L(\lambda \mid \underline{X}) \propto \lambda^n e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_i x_i} - 1\right)} \tag{3.7}$$

Now, let

$$K_1 = \int_0^{\infty} p(\lambda) L(\lambda \mid \underline{X}) d\lambda \tag{3.8}$$

Substituting for $p(\lambda)$ and $L(\lambda \mid \underline{X})$; we have:

$$K_1 = \int_0^{\infty} \lambda^n e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_i x_i} - 1\right)} d\lambda \tag{3.9}$$

Also, using integration by substitution method in equation (3.9) where limit is that: as $\lambda \to 0, u \to 0$ and also as $\lambda \to \infty, u \to \infty$; we obtain the following:

Let $u = \lambda\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right) \Rightarrow \lambda = \dfrac{u}{\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.9) and simplifying; we have:

$$K_1 = \int_0^\infty \left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)} \right)^n e^{-u} \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)}$$

$$K_1 = \frac{1}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^{n+1}} \int_0^\infty u^n e^{-u} du$$

Recall that $\int_0^\infty t^{k-1} e^{-t} dt = \Gamma(k)$ and that $\int_0^\infty t^k e^{-t} dt = \int_0^\infty t^{k+1-1} e^{-t} dt = \Gamma(k+1)$

Hence;

$$K_1 = \frac{1}{\left(\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)\right)^{n+1}} \int_0^\infty u^n e^{-u} du = \frac{1}{\left(\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)\right)^{n+1}} \int_0^\infty u^{n+1-1} e^{-u} du \qquad (3.10)$$

Therefore:

$$K_1 = \frac{\Gamma(n+1)}{\left(\sum_{i=1}^{n}\left(e^{\theta_\circ x_i} - 1\right)\right)^{n+1}} \qquad (3.11)$$

Substituting for K_1, $p(\lambda)$ and $L(\lambda \mid \underline{X})$ in equation (3.6) and simplifying; we obtain the posterior distribution under uniform prior as follows:

$$p(\lambda \mid \underline{X}) = \frac{\lambda^n e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_o x_i}-1\right)}}{\frac{\Gamma(n+1)}{\left(\sum_{i=1}^{n}\left(e^{\theta_o x_i}-1\right)\right)^{n+1}}}$$

$$p(\lambda \mid \underline{X}) = \frac{\lambda^n e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_o x_i}-1\right)}}{\left(\sum_{i=1}^{n}\left(e^{\theta_o x_i}-1\right)\right)^{-(n+1)}\Gamma(n+1)} \tag{3.12}$$

3.3.2 Posterior Distribution of the Shape Parameter under the Assumption of Jeffrey's Prior

The Jeffrey's prior as a non-informative prior relating to the shape parameter λ of the *OGEED* is defined as:

$$p(\lambda) \propto \frac{1}{\lambda}; 0 < \lambda < \infty \tag{3.13}$$

The posterior distribution of the shape parameter λ for a given data using Jeffrey's prior is given in equation (3.6).

Also, let

$$K_2 = \int_0^{\infty} p(\lambda) L(\lambda \mid \underline{X}) d\lambda \tag{3.14}$$

Substituting for $p(\lambda)$ and $L(\lambda \mid \underline{X})$; we have:

$$K_2 = \int_0^{\infty} \lambda^{n-1} e^{-\lambda} \sum_{i=1}^{n} (e^{\theta_o x_i} - 1)\, d\lambda \tag{3.15}$$

Using integration by substitution method in equation (3.15); we obtain the following:

Let $u = \lambda \sum_{i=1}^{n} (e^{\theta_o x_i} - 1) \Rightarrow \lambda = \dfrac{u}{\sum_{i=1}^{n} (e^{\theta_o x_i} - 1)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n} (e^{\theta_o x_i} - 1)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n} (e^{\theta_o x_i} - 1)}$$

Substituting for λ and $d\lambda$ in equation (3.16) and simplifying; we have:

$$K_2 = \int_0^{\infty} \left(\frac{u}{\sum_{i=1}^{n} (e^{\theta_o x_i} - 1)} \right)^{n-1} e^{-u} \frac{du}{\sum_{i=1}^{n} (e^{\theta_o x_i} - 1)}$$

$$K_2 = \frac{1}{\left(\sum_{i=1}^{n} (e^{\theta_o x_i} - 1) \right)^{n}} \int_0^{\infty} u^{n-1} e^{-u} du \qquad (3.16)$$

Also recall that $\int_0^{\infty} t^{k-1} e^{-t} dt = \Gamma(k)$ and that $\int_0^{\infty} t^{k} e^{-t} dt = \int_0^{\infty} t^{k+1-1} e^{-t} dt = \Gamma(k+1)$

Hence;

$$K_2 = \frac{\Gamma(n)}{\left(\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1 \right) \right)^{n}} \qquad (3.17)$$

Substituting for K_2, $p(\lambda)$ and $L(\lambda \mid \underline{X})$ in equation (3.14) and simplifying; we obtain the posterior distribution under Jeffrey's prior as follows:

$$p(\lambda \mid \underline{X}) = \frac{\lambda^{-1} \lambda^{n} e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)}}{\frac{\Gamma(n)}{\left(\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)\right)^{n}}}$$

$$p(\lambda \mid \underline{X}) = \frac{\lambda^{n-1} \left(\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)\right)^{n} e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)}}{\Gamma(n)} \tag{3.18}$$

3.4 Bayesian Estimation

In statistics and decision theory, a loss function is a function that maps an event into a real number intuitively representing some cost associated with the event. Typically it is used for parameter estimation and that event in question is some function of the difference between estimated and true values for an instance of data.

3.4.1 Bayesian Estimation under Uniform Prior Using Three Loss Functions

Here, we estimate the shape parameter of the *OGEED* under three loss functions using the posterior distribution obtained from the uniform prior. A Loss function, $L(\lambda, \lambda_{SELF})$ is that which describes the losses incurred by making an estimate $\hat{\lambda}$ of the true value of the parameter is λ.

3.4.1.1 Under Squared Error Loss Function (SELF)

The squared error loss function proposed by Legendre (1805) and Gauss (1810) relating to the shape parameter λ is defined from Azam and Ahmad (2014b) as

$$L(\lambda, \lambda_{SELF}) = (\lambda - \lambda_{SELF})^2 \tag{3.19}$$

where λ_{SELF} is the estimator of the parameter λ under SELF.

The derivation of Bayes estimator using SELF under uniform prior is obtained as:

$$\lambda_{SELF} = E(\lambda) = E(\lambda \mid \underline{X})$$

$$E(\lambda \mid \underline{X}) = \int_0^{\infty} \lambda p(\lambda \mid \underline{X}) d\lambda \tag{3.20}$$

Now recall that for uniform prior,

$$p(\lambda \mid \underline{X}) = \frac{\lambda^n e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)}}{\left(\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)\right)^{-(n+1)} \Gamma(n+1)} = \frac{\lambda^n \left(\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)\right)^{n+1} e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)}}{\Gamma(n+1)}$$

Substituting for $p(\lambda \mid \underline{X})$ in equation (3.20); we have:

$$E(\lambda \mid \underline{X}) = \int_0^{\infty} \lambda \frac{\lambda^n \left(\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)\right)^{n+1} e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)}}{\Gamma(n+1)} d\lambda$$

$$E(\lambda \mid \underline{X}) = \frac{\left(\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)\right)^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \lambda^{n+1} e^{-\lambda \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)} d\lambda \tag{3.21}$$

Also, using integration by substitution method in equation (3.21); we obtain the following:

Let $u = \lambda \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n} \left(e^{\theta_0 x_i} - 1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.21) and simplifying; we have:

$$E\left(\lambda \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}}{\Gamma\left(n+1\right)}\int_0^{\infty}\left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right)^{n+1} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$E\left(\lambda \mid \underline{X}\right)=\frac{1}{\Gamma\left(n+1\right)\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\int_0^{\infty} u^{n+2-1}e^{-u}du$$

Again recall that $\int_0^{\infty} t^{k-1}e^{-t}dt=\Gamma\left(k\right)$ and that $\int_0^{\infty} t^{k}e^{-t}dt=\int_0^{\infty} t^{(k+1)-1}e^{-t}dt=\Gamma\left(k+1\right)$

Hence;

$$\lambda_{SELF}=E\left(\lambda \mid \underline{X}\right)=\frac{\Gamma\left(n+2\right)}{\Gamma\left(n+1\right)\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$\lambda_{SELF}=E\left(\lambda \mid \underline{X}\right)=\frac{n+1}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} \tag{3.22}$$

3.4.1.2 Under Quadratic Loss Function (QLF)

The quadratic loss function is defined from Azam and Ahmad (2014b) as

$$L\left(\lambda,\lambda_{QLF}\right)=\left(\frac{\lambda-\lambda_{QLF}}{\lambda}\right)^2 \tag{3.23}$$

where λ_{QLF} is the estimator of the parameter λ under QLF.

The derivation of Bayes estimator using QLF under uniform prior is obtained as:

$$\lambda_{QLF}=\frac{E\left(\lambda^{-1}\right)}{E\left(\lambda^{-2}\right)}=\frac{E\left(\lambda^{-1} \mid \underline{X}\right)}{E\left(\lambda^{-2} \mid \underline{X}\right)}$$

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \int_0^{\infty} \lambda^{-1} p\left(\lambda \mid \underline{X}\right) d\lambda \tag{3.24}$$

Now recall that for uniform prior,

$$p\left(\lambda \mid \underline{X}\right) = \frac{\lambda^n e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{-(n+1)} \Gamma\left(n+1\right)} = \frac{\lambda^n \left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1} e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma\left(n+1\right)}$$

Substituting for $p\left(\lambda \mid \underline{X}\right)$ in equation (3.24); we have:

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \int_0^{\infty} \lambda^{-1} \frac{\lambda^n \left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1} e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma\left(n+1\right)} d\lambda$$

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}}{\Gamma\left(n+1\right)} \int_0^{\infty} \lambda^{n-1} e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} d\lambda \tag{3.25}$$

Using integration by substitution in equation (3.25); we obtain the following:

Let $u = \lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.25) and simplifying; we have:

$$E\left(\lambda^{-1}\mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}}{\Gamma(n+1)}\int_0^{\infty}\left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right)^{n-1} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$E\left(\lambda^{-1}\mid \underline{X}\right) = \frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}{\Gamma(n+1)}\int_0^{\infty} u^{n-1}e^{-u}du$$

$$E\left(\lambda^{-1}\mid \underline{X}\right) = \frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma(n)}{\Gamma(n+1)} \tag{3.26}$$

Similarly;

$$E\left(\lambda^{-2}\mid \underline{X}\right) = \int_0^{\infty}\lambda^{-2}p\left(\lambda\mid \underline{X}\right)d\lambda \tag{3.27}$$

Now recall that for uniform prior,

$$p\left(\lambda\mid \underline{X}\right) = \frac{\lambda^{n}e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{-(n+1)}\Gamma(n+1)} = \frac{\lambda^{n}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n+1)}$$

Substituting for $p\left(\lambda\mid \underline{X}\right)$ in equation (3.27); we have:

$$E\left(\lambda^{-2}\mid \underline{X}\right) = \int_0^{\infty}\lambda^{-2}\frac{\lambda^{n}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n+1)}d\lambda$$

$$E\left(\lambda^{-2} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^{n+1}}{\Gamma\left(n+1\right)} \int_0^{\infty} \lambda^{n-2} e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)} d\lambda \tag{3.28}$$

Using integration by substitution in equation (3.28); we obtain the following:

$$\text{Let } u = \lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.28) and simplifying; we have:

$$E\left(\lambda^{-2} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^{n+1}}{\Gamma\left(n+1\right)} \int_0^{\infty} \left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}\right)^{n-2} e^{-\lambda \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)} \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

$$E\left(\lambda^{-2} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^{2}}{\Gamma\left(n+1\right)} \int_0^{\infty} u^{n-1-1} e^{-u} du$$

$$E\left(\lambda^{-2} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^{2} \Gamma\left(n-1\right)}{\Gamma\left(n+1\right)} \tag{3.29}$$

But recall that

$$\lambda_{QLF} = \frac{E\left(\lambda^{-1}\right)}{E\left(\lambda^{-2}\right)} = \frac{E\left(\lambda^{-1} \mid \underline{X}\right)}{E\left(\lambda^{-2} \mid \underline{X}\right)}$$

This implies that

$$\lambda_{QLF} = \frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma(n)}{\Gamma(n+1)} \div \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2 \Gamma(n-1)}{\Gamma(n+1)}$$

$$\lambda_{QLF} = \frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma(n)}{\Gamma(n+1)} \times \frac{\Gamma(n+1)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2 \Gamma(n-1)}$$

$$\lambda_{QLF} = \frac{\Gamma(n)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)\Gamma(n-1)}$$

$$\lambda_{QLF} = \frac{n-1}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)} \tag{3.30}$$

3.4.1.3 Using Precautionary Loss Function (PLF)

The precautionary loss function (PLF) introduced by Norstrom (1996) is an asymmetric loss function and is defined from Azam and Ahmad (2014b) as

$$L\left(\lambda_{PLF}, \lambda\right) = \frac{\left(\lambda_{PLF} - \lambda\right)^2}{\lambda} \tag{3.31}$$

where λ_{PLF} is the estimator of the parameter λ under PLF.

Similarly, the derivation of Bayes estimator using PLF under uniform prior is obtained:

$$\lambda_{PLF} = \left\{E\left(\lambda^2\right)\right\}^{\frac{1}{2}} = \left\{E\left(\lambda^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \sqrt{E\left(\lambda^2 \mid \underline{X}\right)}$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \int_0^{\infty} \lambda^2 p\left(\lambda \mid \underline{X}\right) d\lambda \tag{3.32}$$

Now recall that for uniform prior,

$$p\left(\lambda \mid \underline{X}\right) = \frac{\lambda^{n} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{-(n+1)}\Gamma\left(n+1\right)} = \frac{\lambda^{n}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma\left(n+1\right)}$$

Substituting for $p\left(\lambda \mid \underline{X}\right)$ in equation (3.32); we have:

$$E\left(\lambda^2 \mid \underline{X}\right) = \int_0^{\infty} \lambda^2 \frac{\lambda^{n}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma\left(n+1\right)} d\lambda$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}}{\Gamma\left(n+1\right)} \int_0^{\infty} \lambda^{n+2} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} d\lambda \qquad (3.33)$$

Similarly, using integration by substitution method in equation (3.33); we obtain the following:

Let $u = \lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.33) and simplifying; we have:

$$E\left(\lambda^2 \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}}{\Gamma\left(n+1\right)}\int_0^{\infty}\left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right)^{n+2} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$E\left(\lambda^2 \mid \underline{X}\right)=\frac{1}{\Gamma\left(n+1\right)\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}}\int_0^{\infty}u^{n+3-1}e^{-u}du$$

$$E\left(\lambda^2 \mid \underline{X}\right)=\frac{\Gamma\left(n+3\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n+1\right)}$$

$$\lambda_{PLF}=\left\{E\left(\lambda^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}}=\sqrt{\frac{\Gamma\left(n+3\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n+1\right)}}$$

$$\lambda_{PLF}=\left\{E\left(\lambda^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}}=\frac{\left[\left(n+2\right)\left(n+1\right)\right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)} \qquad (3.34)$$

It is very clear that the relationship: $\lambda_{PLF} > \lambda_{SELF} > \lambda_{MLE} > \lambda_{QLF}$ holds for all parameter values and λ_{QLF} under the uniform prior appears to be the minimum.

3.4.2 Bayesian Estimation under Jeffrey's Prior Using Three Loss Functions

This section presents the estimation of the shape parameter of the *OGEED* under three loss functions using the posterior distribution obtained from Jeffrey's prior.

3.4.2.1 Using Squared Error Loss Function (SELF)

The derivation of Bayes estimator using SELF under Jeffrey's prior is obtained as:

$$\lambda_{SELF}=E\left(\lambda\right)=E\left(\lambda \mid \underline{X}\right)$$

$$E(\lambda \mid \underline{X}) = \int_0^{\infty} \lambda p(\lambda \mid \underline{X}) d\lambda \tag{3.35}$$

Now recall that for Jeffrey's prior,

$$p(\lambda \mid \underline{X}) = \frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)}$$

Substituting for $p(\lambda \mid \underline{X})$ in equation (3.36); we have:

$$E(\lambda \mid \underline{X}) = \int_0^{\infty} \lambda \frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)} d\lambda$$

$$E(\lambda \mid \underline{X}) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n}{\Gamma(n)} \int_0^{\infty} \lambda^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} d\lambda \tag{3.36}$$

Also, using integration by substitution method in equation (3.36); we obtain the following:

Let $u = \lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.36) and simplifying; we have:

$$E\left(\lambda \mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^n}{\Gamma(n)}\int_0^{\infty}\left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$E\left(\lambda \mid \underline{X}\right)=\frac{1}{\Gamma(n)\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\int_0^{\infty}u^{n+1-1}e^{-u}du$$

$$\lambda_{SELF}=E\left(\lambda \mid \underline{X}\right)=\frac{\Gamma(n+1)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma(n)}$$

$$\lambda_{SELF}=E\left(\lambda \mid \underline{X}\right)=\frac{n}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} \tag{3.37}$$

3.4.2.2 Under Quadratic Loss Function (QLF)

The derivation of Bayes estimator using QLF under Jeffrey's prior is obtained as:

$$\lambda_{QLF}=\frac{E\left(\lambda^{-1}\right)}{E\left(\lambda^{-2}\right)}=\frac{E\left(\lambda^{-1} \mid \underline{X}\right)}{E\left(\lambda^{-2} \mid \underline{X}\right)}$$

$$E\left(\lambda^{-1} \mid \underline{X}\right)=\int_0^{\infty}\lambda^{-1}p\left(\lambda \mid \underline{X}\right)d\lambda \tag{3.38}$$

Now recall that for Jeffrey's prior,

$$p\left(\lambda \mid \underline{X}\right)=\frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)}$$

Substituting for $p\left(\lambda \mid \underline{X}\right)$ in equation (3.38); we have:

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \int_0^{\infty} \lambda^{-1} \frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)} d\lambda$$

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n}{\Gamma(n)} \int_0^{\infty} \lambda^{n-2} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} d\lambda \qquad (3.39)$$

Using integration by substitution in equation (3.39); we obtain the following:

Let $u = \lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.39) and simplifying; we have:

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n}{\Gamma(n)} \int_0^{\infty} \left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}\right)^{n-2} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}{\Gamma(n)} \int_0^{\infty} u^{n-2} e^{-u} du$$

$$E\left(\lambda^{-1} \mid \underline{X}\right) = \frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\Gamma(n-1)}{\Gamma(n)} \qquad (3.40)$$

Similarly;

$$E\left(\lambda^{-2} \mid \underline{X}\right) = \int_0^{\infty} \lambda^{-2} p\left(\lambda \mid \underline{X}\right) d\lambda \tag{3.41}$$

Now recall that for Jeffrey's prior,

$$p\left(\lambda \mid \underline{X}\right) = \frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)}$$

Substituting for $p\left(\lambda \mid \underline{X}\right)$ in equation (3.41); we have:

$$E\left(\lambda^{-2} \mid \underline{X}\right) = \int_0^{\infty} \lambda^{-2} \frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)} d\lambda$$

$$E\left(\lambda^{-2} \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n}{\Gamma(n)} \int_0^{\infty} \lambda^{n-3} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} d\lambda \tag{3.42}$$

Using integration by substitution in equation (3.42); we obtain the following:

Let $u = \lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.42) and simplifying; we have:

$$E\left(\lambda^{-2}\mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n}}{\Gamma(n)}\int_0^{\infty}\left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right)^{n-3} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$E\left(\lambda^{-2}\mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}}{\Gamma(n)}\int_0^{\infty}u^{n-2-1}e^{-u}du$$

$$E\left(\lambda^{-2}\mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma(n-2)}{\Gamma(n)} \tag{3.43}$$

But recall that

$$\lambda_{QLF}=\frac{E\left(\lambda^{-1}\right)}{E\left(\lambda^{-2}\right)}=\frac{E\left(\lambda^{-1}\mid \underline{X}\right)}{E\left(\lambda^{-2}\mid \underline{X}\right)}$$

This implies that

$$\lambda_{QLF}=\frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma(n-1)}{\Gamma(n)}\div\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma(n-2)}{\Gamma(n)}$$

$$\lambda_{QLF}=\frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma(n-1)}{\Gamma(n)}\times\frac{\Gamma(n)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma(n-2)}$$

$$\lambda_{QLF}=\frac{\Gamma(n-1)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)\Gamma(n-2)}$$

$$\lambda_{QLF}=\frac{n-2}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} \tag{3.44}$$

3.4.2.3 Using Precautionary Loss Function (PLF)

Similarly, the derivation of Bayes estimator using PLF under Jeffrey's prior is obtained as:

$$\lambda_{PLF} = \left\{E\left(\lambda^2\right)\right\}^{\frac{1}{2}} = \left\{E\left(\lambda^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \sqrt{E\left(\lambda^2 \mid \underline{X}\right)}$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \int_0^{\infty} \lambda^2 p\left(\lambda \mid \underline{X}\right) d\lambda \tag{3.45}$$

Now recall that for Jeffrey's prior,

$$p\left(\lambda \mid \underline{X}\right) = \frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)}$$

Substituting for $p\left(\lambda \mid \underline{X}\right)$ in equation (3.45); we have:

$$E\left(\lambda^2 \mid \underline{X}\right) = \int_0^{\infty} \lambda^2 \frac{\lambda^{n-1}\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}}{\Gamma(n)} d\lambda$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n}{\Gamma(n)} \int_0^{\infty} \lambda^{n+1} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)} d\lambda \tag{3.46}$$

Similarly, using integration by substitution method in equation (3.46); we obtain the following:

Let $u = \lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.46) and simplifying; we have:

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^n}{\Gamma(n)} \int_0^{\infty} \left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}\right)^{n+1} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)} \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)}$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{1}{\Gamma(n)\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^2} \int_0^{\infty} u^{n+2-1} e^{-u} du$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{\Gamma(n+2)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^2 \Gamma(n)}$$

$$\lambda_{PLF} = \left\{E\left(\lambda^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \sqrt{\frac{\Gamma(n+2)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^2 \Gamma(n)}}$$

$$\lambda_{PLF} = \left\{E\left(\lambda^2 \mid \underline{X}\right)\right\}^{\frac{1}{2}} = \frac{\left[n(n+1)\right]^{\frac{1}{2}}}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)} \qquad (3.47)$$

It is also clear that λ_{MLE} is the same as λ_{SELF} under Jeffrey's prior and the relationship: $\lambda_{PLF} > \lambda_{SELF} > \lambda_{MLE} > \lambda_{QLF}$ holds for all parameter values and λ_{QLF} under the Jeffrey's prior appears to be the minimum.

3.5 Posterior Risks under the Priors and Loss Functions

The posterior risks of the Bayes estimators under the three loss functions from both uniform and Jeffrey's prior are obtained as follows:

3.5.1 Posterior Risks under the Uniform Prior

The posterior risks of the Bayes estimators under the three loss functions from the Uniform prior are as follows:

3.5.1.1 Using Squared Error Loss Function (SELF)

Using the Squared error loss function (SELF), the posterior risk, $p(\lambda_{SELF})$ is defined from Azam and Ahmad (2014b) as:

$$P(\lambda_{SELF}) = E(\lambda^2 \mid \underline{X}) - \{E(\lambda \mid \underline{X})\}^2 \tag{3.48}$$

where

$$E(\lambda^2 \mid \underline{X}) = \int_0^\infty \lambda^2 p(\lambda \mid \underline{X}) d\lambda \tag{3.49}$$

Now recall that for uniform prior,

$$p(\lambda \mid \underline{X}) = \frac{\lambda^n e^{-\lambda \sum_{i=1}^n (e^{\theta_0 x_i} - 1)}}{\left(\sum_{i=1}^n (e^{\theta_0 x_i} - 1)\right)^{-(n+1)} \Gamma(n+1)} = \frac{\lambda^n \left(\sum_{i=1}^n (e^{\theta_0 x_i} - 1)\right)^{n+1} e^{-\lambda \sum_{i=1}^n (e^{\theta_0 x_i} - 1)}}{\Gamma(n+1)}$$

Substituting for $p(\lambda \mid \underline{X})$ in equation (3.49); we have:

$$E(\lambda^2 \mid \underline{X}) = \int_0^\infty \lambda^2 \frac{\lambda^n \left(\sum_{i=1}^n (e^{\theta_0 x_i} - 1)\right)^{n+1} e^{-\lambda \sum_{i=1}^n (e^{\theta_0 x_i} - 1)}}{\Gamma(n+1)} d\lambda$$

$$E(\lambda^2 \mid \underline{X}) = \frac{\left(\sum_{i=1}^n (e^{\theta_0 x_i} - 1)\right)^{n+1}}{\Gamma(n+1)} \int_0^\infty \lambda^{n+2} e^{-\lambda \sum_{i=1}^n (e^{\theta_0 x_i} - 1)} d\lambda \tag{3.50}$$

Also, using integration by substitution method in equation (3.50); we obtain the following:

$$\text{Let } u = \lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right) \Rightarrow \lambda = \frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$\frac{du}{d\lambda} = \sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)$$

$$d\lambda = \frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

Substituting for λ and $d\lambda$ in equation (3.50) and simplifying; we have:

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{n+1}}{\Gamma(n+1)}\int_0^{\infty}\left(\frac{u}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right)^{n+2} e^{-\lambda\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\frac{du}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{1}{\Gamma(n+1)\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2}\int_0^{\infty} u^{n+3-1}e^{-u}du$$

$$E\left(\lambda^2 \mid \underline{X}\right) = \frac{\Gamma(n+3)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2(n+1)} \tag{3.51}$$

while

$$E\left(\lambda \mid \underline{X}\right) = \frac{\Gamma(n+2)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma(n+1)} \tag{3.52}$$

Therefore, the posterior risk under uniform prior using the squared error loss function is:

$$P\left(\lambda_{SELF}\right)=\frac{\Gamma\left(n+3\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2\Gamma\left(n+1\right)}-\left\{\frac{\Gamma\left(n+2\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n+1\right)}\right\}^2 \tag{3.53}$$

$$P\left(\lambda_{SELF}\right)=\frac{\left(n+2\right)\left(n+1\right)-\left(n+1\right)^2}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2}$$

$$P\left(\lambda_{SELF}\right)=\frac{\left(n+2\right)\left(n+1\right)-\left(n+1\right)^2}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2}$$

$$P\left(\lambda_{SELF}\right)=\frac{n+1}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2} \tag{3.54}$$

3.5.1.2 Using Quadratic Loss Function (QLF)

Using the Quadratic loss function (QLF), the posterior risk, $p\left(\lambda_{QLF}\right)$ is defined from Azam and Ahmad (2014b) as:

$$P\left(\lambda_{QLF}\right)=1-\frac{\left\{E\left(\lambda^{-1}\mid \underline{X}\right)\right\}^2}{E\left(\lambda^{-2}\mid \underline{X}\right)} \tag{3.55}$$

where

$$E\left(\lambda^{-1}\mid \underline{X}\right)=\frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n\right)}{\Gamma\left(n+1\right)} \tag{3.56}$$

and;

$$E\left(\lambda^{-2}\mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2\Gamma\left(n-1\right)}{\Gamma\left(n+1\right)} \tag{3.57}$$

Therefore, the posterior risk under uniform prior using the Quadratic loss function is given as:

$$P\left(\lambda_{QLF}\right)=1-\frac{\left\{E\left(\lambda^{-1}\mid \underline{X}\right)\right\}^{2}}{E\left(\lambda^{-2}\mid \underline{X}\right)}=1-\left\{\left(\frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n\right)}{\Gamma\left(n+1\right)}\right)^{2}\div\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n-1\right)}{\Gamma\left(n+1\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=1-\left\{\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\left[\Gamma\left(n\right)\right]^{2}}{\left[\Gamma\left(n+1\right)\right]^{2}}\div\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n-1\right)}{\Gamma\left(n+1\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=1-\left\{\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\left[\Gamma\left(n\right)\right]^{2}}{\left[\Gamma\left(n+1\right)\right]^{2}}\times\frac{\Gamma\left(n+1\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n-1\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=1-\left\{\frac{\left[\Gamma\left(n\right)\right]^{2}}{\Gamma\left(n+1\right)\Gamma\left(n-1\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=\frac{\Gamma\left(n+1\right)\Gamma\left(n-1\right)-\left[\Gamma\left(n\right)\right]^{2}}{\Gamma\left(n+1\right)\Gamma\left(n-1\right)}$$

$$P\left(\lambda_{QLF}\right)=\frac{1}{n} \tag{3.58}$$

3.5.1.3 Precautionary Loss Function (PLF)

Using the Precautionary loss function (PLF), the posterior risk, $p\left(\lambda_{PLF}\right)$ is defined from Azam and Ahmad (2014b) as:

$$P\left(\lambda_{PLF}\right)=2\left\{\lambda_{PLF}-E\left(\lambda\mid \underline{X}\right)\right\} \tag{3.59}$$

where

$$\lambda_{PLF}=\left\{E\left(\lambda^2\mid \underline{X}\right)\right\}^{\frac{1}{2}}=\sqrt{\frac{\Gamma\left(n+3\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2\Gamma\left(n+1\right)}}=\left\{\frac{\Gamma\left(n+3\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^2\Gamma\left(n+1\right)}\right\}^{\frac{1}{2}}$$

and;

$$E\left(\lambda\mid \underline{X}\right)=\frac{\Gamma\left(n+2\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n+1\right)}$$

Therefore, the posterior risk under uniform prior using the Precautionary loss function is given as:

$$P\left(\lambda_{PLF}\right)=2\left\{\frac{\left\{\Gamma\left(n+3\right)\right\}^{\frac{1}{2}}}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\left\{\Gamma\left(n+1\right)\right\}^{\frac{1}{2}}}-\frac{\Gamma\left(n+2\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n+1\right)}\right\}$$

$$P\left(\lambda_{PLF}\right)=2\left\{\frac{\left\{\Gamma\left(n+3\right)\Gamma\left(n+1\right)\right\}^{\frac{1}{2}}-\Gamma\left(n+2\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n+1\right)}\right\}$$

$$P\left(\lambda_{PLF}\right)=2\left\{\frac{\left\{\left(n+2\right)\left(n+1\right)\right\}^{\frac{1}{2}}-\left(n+1\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right\} \tag{3.60}$$

3.5.2 Posterior Risks under Jeffrey's Prior

The posterior risks of the Bayes estimators under the three loss functions from the Jeffrey's prior are as follows:

3.5.2.1 Using Squared Error Loss Function (SELF)

Using the Squared error loss function (SELF), the posterior risk, $p(\lambda_{SELF})$ is defined from Azam and Ahmad (2014b) as:

$$P(\lambda_{SELF}) = E(\lambda^2 \mid \underline{X}) - \{E(\lambda \mid \underline{X})\}^2 \tag{3.61}$$

where

$$E(\lambda^2 \mid \underline{X}) = \frac{\Gamma(n+2)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^2 \Gamma(n)} \tag{3.62}$$

while

$$E(\lambda \mid \underline{X}) = \frac{\Gamma(n+1)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\Gamma(n)} \tag{3.63}$$

Therefore, the posterior risk under Jeffrey's prior using the squared error loss function is:

$$P(\lambda_{SELF}) = \frac{\Gamma(n+2)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^2 \Gamma(n)} - \left\{\frac{\Gamma(n+1)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\Gamma(n)}\right\}^2 \tag{3.64}$$

$$P(\lambda_{SELF}) = \frac{\Gamma(n+2)\Gamma(n) - \left(\Gamma(n+1)\right)^2}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^2 \left(\Gamma(n)\right)^2}$$

$$P(\lambda_{SELF}) = \frac{n}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i} - 1\right)\right)^2} \tag{3.65}$$

3.5.2.2 Using Quadratic Loss Function (QLF)

Using the Quadratic loss function (QLF), the posterior risk, $p(\lambda_{QLF})$ is defined from Azam and Ahmad (2014b) as:

$$P\left(\lambda_{QLF}\right)=1-\frac{\left\{E\left(\lambda^{-1}\mid \underline{X}\right)\right\}^{2}}{E\left(\lambda^{-2}\mid \underline{X}\right)} \tag{3.66}$$

where

$$E\left(\lambda^{-1}\mid \underline{X}\right)=\frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n-1\right)}{\Gamma\left(n\right)} \tag{3.67}$$

and;

$$E\left(\lambda^{-2}\mid \underline{X}\right)=\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n-2\right)}{\Gamma\left(n\right)} \tag{3.68}$$

Therefore, the posterior risk under Jeffrey's prior using the Quadratic loss function is given as:

$$P\left(\lambda_{QLF}\right)=1-\frac{\left\{E\left(\lambda^{-1}\mid \underline{X}\right)\right\}^{2}}{E\left(\lambda^{-2}\mid \underline{X}\right)}=1-\left\{\left(\frac{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n-1\right)}{\Gamma\left(n\right)}\right)^{2}\div\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n-2\right)}{\Gamma\left(n\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=1-\left\{\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\left\{\Gamma\left(n-1\right)\right\}^{2}}{\left\{\Gamma\left(n\right)\right\}^{2}}\div\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n-2\right)}{\Gamma\left(n\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=1-\left\{\frac{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\left\{\Gamma\left(n-1\right)\right\}^{2}}{\left\{\Gamma\left(n\right)\right\}^{2}}\times\frac{\Gamma\left(n\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n-2\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=1-\left\{\frac{\left\{\Gamma\left(n-1\right)\right\}^{2}}{\Gamma\left(n\right)\Gamma\left(n-2\right)}\right\}$$

$$P\left(\lambda_{QLF}\right)=\frac{\Gamma\left(n\right)\Gamma\left(n-2\right)-\left[\Gamma\left(n-1\right)\right]^{2}}{\Gamma\left(n\right)\Gamma\left(n-2\right)}$$

$$P\left(\lambda_{QLF}\right)=\frac{1}{n-1} \tag{3.69}$$

3.5.2.3 Precautionary Loss Function (PLF)

Using the Precautionary loss function (PLF), the posterior risk, $p\left(\lambda_{PLF}\right)$ is defined from Azam and Ahmad (2014b) as:

$$P\left(\lambda_{PLF}\right)=2\left\{\lambda_{PLF}-E\left(\lambda \mid \underline{X}\right)\right\} \tag{3.70}$$

where

$$\lambda_{PLF}=\left\{E\left(\lambda^{2} \mid \underline{X}\right)\right\}^{\frac{1}{2}}=\sqrt{\frac{\Gamma\left(n+2\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n\right)}}=\left\{\frac{\Gamma\left(n+2\right)}{\left(\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\right)^{2}\Gamma\left(n\right)}\right\}^{\frac{1}{2}}$$

and;

$$E\left(\lambda \mid \underline{X}\right)=\frac{\Gamma\left(n+1\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n\right)}$$

Therefore, the posterior risk under Jeffrey's prior using the Precautionary loss function is given as:

$$P\left(\lambda_{PLF}\right)=2\left\{\frac{\left\{\Gamma\left(n+2\right)\right\}^{\frac{1}{2}}}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\left\{\Gamma\left(n\right)\right\}^{\frac{1}{2}}}-\frac{\Gamma\left(n+1\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n\right)}\right\}$$

$$P\left(\lambda_{PLF}\right)=2\left\{\frac{\left\{\Gamma\left(n+2\right)\Gamma\left(n\right)\right\}^{\frac{1}{2}}-\Gamma\left(n+1\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)\Gamma\left(n\right)}\right\}$$

$$P\left(\lambda_{PLF}\right)=2\left\{\frac{\left\{n\left(n+1\right)\right\}^{\frac{1}{2}}-\left(n\right)}{\sum_{i=1}^{n}\left(e^{\theta_0 x_i}-1\right)}\right\} \tag{3.71}$$

3.6 Simulation study

Simulation is a numerical technique for conducting experiments on the computer. Monte Carlo simulation is a computer experiment involving random sampling from probability distributions. Usually, when statisticians talk about "simulations," they mean "Monte Carlo simulations". This is done by making use of the quantile function from the distribution of interest. This function is derived by inverting the *cdf* of any given continuous probability distribution. It is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question. Hyndman and Fan (1996) defined the quantile function for any distribution in the form

$Q(u) = F^{-1}(u)$ where Q(u) is the quantile function of $F(x)$ for $0 < u < 1$

Taking $F(x)$ to be the *cdf* of the *OGEED* and inverting it as above will give us the quantile function as follows.

$$F(x) = 1 - e^{-\lambda\left(e^{\theta_\circ x} - 1\right)}$$

Inverting $F(x) = u$

$$F(x) = 1 - e^{-\lambda\left(e^{\theta_\circ x} - 1\right)} = u \tag{3.72}$$

Simplifying equation (3.72) above, we obtain:

$$1 - u = e^{-\lambda\left(e^{\theta_\circ x} - 1\right)}$$

$$\ln(1-u) = -\lambda\left(e^{\theta_\circ x} - 1\right)$$

$$\left(1 - \frac{\ln(1-u)}{\lambda}\right) = e^{\theta_\circ x}$$

$$\ln\left(1 - \frac{\ln(1-u)}{\lambda}\right) = \theta_\circ x$$

$$x = \frac{1}{\theta_{\circ}} \ln\left(1 - \frac{\ln(1-u)}{\lambda}\right)$$

$$Q(u) = X_q = \frac{1}{\theta_{\circ}} \ln\left(1 - \frac{\ln(1-u)}{\lambda}\right) \tag{3.73}$$

In this dissertation, we used R software to generate random numbers from the odd generalized exponential-exponential distribution with different values of shape and scale parameters to investigate the performance of the estimators of the three loss functions under non-informative priors using the quantile function above. The Maximum Likelihood estimates, Bayesian estimates and posterior risks were obtained under a reasonably high number of replications, different sample sizes were used to investigate the performance of the estimators in relation to their Biases and mean squared errors as well as the sample sizes.

The performance of the two methods was evaluated using the following performance measures:

Bias $= \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\lambda}_i - \lambda\right)$ and

Mean Square Error, $MSE = \frac{1}{N}\sum_{i=1}^{N}\left(\hat{\lambda}_i - \lambda\right)^2$

Where $\hat{\lambda}$ is the estimate of λ (the shape parameter) from the *ith* simulated data and N is the number of Monte Carlo samples.

Note: the estimate with the lowest Bias and MSE would be considered as the best.

CHAPTER FOUR: RESULTS, ANALYSIS AND DISCUSSION

4.1 Simulation Studies to Compare Uniform and Jeffrey Prior

This chapter compared the estimators previously studied or derived in chapter three using simulated data from the odd generalized Exponential-Exponential distribution to obtain numerical estimates of the shape parameter which has enable the researcher to measure the loss attached to the value of the estimated parameter against the true value of the shape parameter. In this study we used the Bias and MSE to assess the performance of each estimator under priors using the three loss functions (SELF QLF and PLF). For this purpose we used different values of shape parameter λ and different constant values of scale parameter θ_0 under different sample sizes. We used R software to generate random sample of size n = (20, 55, 75, 100) from *OGEED* by using $\lambda \in$ {2; 3} and $\theta_0 \in$ {1.5; 4.5}. The following tables present the results of our simulation study by listing the estimates of the shape parameter under the appropriate estimation methods such as the Maximum Likelihood Estimation (*MLE*), Squared Error Loss Function

(*SELF*), Quadratic Loss Function (*QLF*), and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey prior.

Table 4.1: Estimates, Biases and Mean Squared Errors based on the replications and sample sizes where $\lambda = 2$ and $\theta_{\circ} = 1.5$.

Sample	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**		
Sizes			**SELF**	**QLF**	**PLF**	**SELF**	**QLF**	**PLF**
20	**Estimate**	2.1790	2.2880	2.0700	2.3418	2.1790	1.9611	2.2328
	BIAS	0.1790	0.2880	0.0700	0.3418	0.1790	-0.0890	0.2328
	MSE	0.2575	0.3160	0.2224	0.3536	0.2575	0.2107	0.2835
55	**Estimate**	2.4858	2.5310	2.4406	2.5535	2.4858	2.3955	2.5083
	BIAS	0.4858	0.5310	0.4406	0.5535	0.4858	0.3955	0.5083
	MSE	0.0789	0.0857	0.0750	0.0900	0.0789	0.0738	0.0819
75	**Estimate**	1.8580	1.8827	1.8332	1.8951	1.8580	1.8084	1.8703
	BIAS	-0.1420	-0.1173	-0.1668	-0.1049	-0.1420	-0.1916	-0.1297
	MSE	0.0563	0.0598	0.0542	0.0622	0.0563	0.0536	0.0579

100	**Estimate**	2.1790	2.2008	2.1572	2.2116	2.1790	2.1354	2.1898
	BIAS	0.1790	0.2008	0.1572	0.2116	0.1790	0.1354	0.1898
	MSE	0.0423	0.0443	0.0412	0.0456	0.0423	0.0408	0.0432

From Table 4.1 where $\lambda = 2$ and $\theta_{\circ} = 1.5$ we can see that the estimator using *QLF* is better than the other estimators under both Uniform and Jeffrey priors with smaller values of Bias and MSE irrespective of the variation in the samples. Hence, we can in general say that Bayesian estimation (using *QLF* under Uniform and Jeffrey prior) for this parameter is better than Method of Maximum Likelihood estimation (*MLE*) for the chosen parameter values irrespective of small, medium or large sample sizes.

Table 4.2: Estimates, Biases and Mean Squared Errors based on the replications and sample sizes where $\lambda = 2$ and $\theta_{\circ} = 4.5$.

Sample	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**		
Sizes			**SELF**	**QLF**	**PLF**	**SELF**	**QLF**	**PLF**
20	**Estimate**	2.1790	2.2880	2.0700	2.3418	2.1790	1.9611	2.2328
	BIAS	0.1790	0.2880	0.0700	0.3418	0.1790	-0.0890	0.2328
	MSE	0.2575	0.3160	0.2224	0.3536	0.2575	0.2107	0.2835
55	**Estimate**	2.4858	2.5310	2.4406	2.5535	2.4858	2.3955	2.5083
	BIAS	0.4858	0.5310	0.4406	0.5535	0.4858	0.3955	0.5083
	MSE	0.0789	0.0857	0.0750	0.0900	0.0789	0.0738	0.0819
75	**Estimate**	1.8580	1.8827	1.8332	1.8951	1.8580	1.8084	1.8703
	BIAS	-0.1420	-0.1173	-0.1668	-0.1049	-0.1420	-0.1916	-0.1297
	MSE	0.0563	0.0598	0.0542	0.0622	0.0563	0.0536	0.0579

100	**Estimate**	2.1790	2.2008	2.1572	2.2116	2.1790	2.1354	2.1898
	BIAS	0.1790	0.2008	0.1572	0.2116	0.1790	0.1354	0.1898
	MSE	0.0423	0.0443	0.0412	0.0456	0.0423	0.0408	0.0432

Table 4.2 where $\lambda = 2$ and $\theta_{\circ} = 4.5$ also gives the similar result found in Table 4.1 with lower values of Bias and MSE for the estimators using QLF under both Uniform and Jeffrey prior. This result indicates that *QLF* under Jeffrey prior produces the best estimator followed by *QLF* under Uniform prior and these performances are found to be consistent irrespective of the different sample sizes used. It is also observed that the values in Table 4.2 are equal to those in Table 4.1 (for $\theta_{\circ} = 4.5$ and $\theta_{\circ} = 1.5$ respectively) which is an indication that the scale parameter values have no effect on the estimates obtained.

Table 4.3: Estimates, Biases and Mean Squared Error and based on the replications and sample sizes where $\lambda = 3$ and $\theta_{\circ} = 1.5$.

Sample sizes	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**		
			SELF	**QLF**	**PLF**	**SELF**	**QLF**	**PLF**
20	**Estimate**	3.2685	3.4319	3.1051	3.5126	3.2685	2.9416	3.3492
	BIAS	0.2685	0.4319	0.1051	0.5126	0.2685	-0.0584	0.3492
	MSE	0.5794	0.7110	0.5005	0.7955	0.5794	0.4741	0.6379
55	**Estimate**	3.7288	3.7966	3.6610	3.8303	3.7288	3.5932	3.7625
	BIAS	0.7288	0.7966	0.6610	0.8303	0.7288	0.5932	0.7625
	MSE	0.1776	0.1927	0.1687	0.2026	0.1776	0.1661	0.1843
75	**Estimate**	2.7870	2.8241	2.7498	2.8426	2.7870	2.7126	2.8055
	BIAS	-0.2130	-0.1759	-0.2502	-0.1574	-0.2130	-0.2874	-0.1945

	MSE	0.1267	0.1347	0.1220	0.1399	0.1267	0.1206	0.1302
100	**Estimate**	3.2685	3.3011	3.2358	3.3175	3.2685	3.2031	3.2848
	BIAS	0.2685	0.3011	0.2358	0.3175	0.2685	0.2031	0.2848
	MSE	0.0953	0.0997	0.0926	0.1027	0.0953	0.0918	0.0973

Again from Table 4.3 for $\lambda = 3$ and $\theta_{\circ} = 1.5$, it is obvious that QLF (under Jeffrey prior) yielded the best estimate for the shape parameter despite the changes in the sample sizes. This performance is still seconded by *QLF* (under Uniform prior). This is also true despite the variations in sample sizes.

Table 4.4: Estimates, Biases and Mean Squared Errors based on the replications and sample sizes where $\lambda = 3$ and $\theta_{\circ} = 4.5$.

Sample sizes	**Measures**	**MLE**	**Uniform Prior**			**Jeffrey's Prior**		
			SELF	**QLF**	**PLF**	**SELF**	**QLF**	**PLF**
20	**Estimate**	3.2685	3.4319	3.1051	3.5126	3.2685	2.9416	3.3492
	BIAS	0.2685	0.4319	0.1051	0.5126	0.2685	-0.0584	0.3492
	MSE	0.5794	0.7110	0.5005	0.7955	0.5794	0.4741	0.6379
55	**Estimate**	3.7288	3.7966	3.6610	3.8303	3.7288	3.5932	3.7625
	BIAS	0.7288	0.7966	0.6610	0.8303	0.7288	0.5932	0.7625
	MSE	0.1776	0.1927	0.1687	0.2026	0.1776	0.1661	0.1843
75	**Estimate**	2.7870	2.8241	2.7498	2.8426	2.7870	2.7126	2.8055
	BIAS	-0.2130	-0.1759	-0.2502	-0.1574	-0.2130	-0.2874	-0.1945
	MSE	0.1267	0.1347	0.1220	0.1399	0.1267	0.1206	0.1302
100	**Estimate**	3.2685	3.3011	3.2358	3.3175	3.2685	3.2031	3.2848
	BIAS	0.2685	0.3011	0.2358	0.3175	0.2685	0.2031	0.2848
	MSE	0.0953	0.0997	0.0926	0.1027	0.0953	0.0918	0.0973

The above Table (Table 4.4 where $\lambda = 3$ and $\theta_\circ = 4.5$**)** also shows that Jeffrey prior with QLF gave us better estimates for the shape parameter and gives a similar pattern of the result found in Table 4.3 with lower values of Bias and MSE for the estimators using QLF under both Uniform and Jeffrey prior. This result indicates that QLF under Jeffrey prior produces the best estimator followed by QLF under Uniform prior and these performances are found to be consistent irrespective of the different sample size used. It is also observed that the values in Table 4.4 are similar to those in Table 4.3(for $\theta_\circ = 4.5$ and $\theta_\circ = 1.5$ respectively) which is also an indication that the scale parameter values have no effect on the estimates obtained.

4.2 Discussion of Results

When looking at all the results presented in the above Tables, we can conclude that Bayes estimates using Quadratic loss function (*QLF*) under Jeffrey prior as well as *QLF* using Uniform prior are associated with minimum Biases and MSEs when compared to those obtained using *MLE, SELF* and PLF under Jeffrey prior and Uniform prior irrespective of the parametric values as well as the allocated sample sizes of n=20, 55, 75 and 100. More so, we observed that the values in Table 4.2 are similar to those in Table 4.1 while that of Table 4.4 and Table 4.3 are also the same (for $\theta_\circ = 4.5$ and $\theta_\circ = 1.5$ respectively) which is an indication that the scale parameter values have no effect on the estimates of the shape parameter obtained.

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

This work estimated the shape parameter of the Odd Generalized Exponential-Exponential Distribution using the Bayesian method of estimation and comparing the estimates with that of maximum Likelihood by assuming two non-informative prior distributions namely; Uniform and Jeffrey prior distributions. These estimates were obtained under the squared error loss function (*SELF*), Quadratic loss function (*QLF*) and precautionary loss function (*PLF*). The posterior distribution of the *OGEED* was derived and also the Estimates and risks were also obtained using the above mentioned priors and loss functions. Furthermore, we carried out Monte-Carlo simulation using R software to assess the performance of the two methods by making use of the biases and *MSEs* of the Estimates under the Bayesian approach and Maximum likelihood method.

5.2 Conclusion

In this research, we obtained Bayesian estimators of the shape parameter of Odd Generalized Exponential-Exponential distribution. The Posterior distributions of this parameter are derived by using Uniform and Jeffrey priors. Bayes estimators and their risks have been derived by using three loss functions under the two prior distributions. The three loss functions taken up are Squared Error Loss Function (SELF), Quadratic Loss Function (QLF) and Precautionary Loss Function (PLF). The performance of these estimators is assessed on the basis of their relative Biases and mean square errors. Monte Carlo Simulations are used to compare the performance of the estimators. It is discovered that using the *QLF* (under Jeffrey prior) produces the least measures of bias and MSE, followed by the *QLF* (under Uniform prior) then the *SELF*, *MLE*

and lastly the *PLF* under both Uniform and Jeffrey priors irrespective of the parameter values and difference in sample size. Most importantly, we found that Bayesian Method using Quadratic Loss Function (*QLF*) under both Uniform and Jeffrey priors produces the best estimates of the shape parameter compared to estimates using Maximum Likelihood method, Squared Error Loss Function (*SELF*) and Precautionary Loss Function (*PLF*) under both Uniform and Jeffrey priors irrespective of the values of the parameters and the different sample sizes. It is also discovered that the scale parameter has no effect on the estimates of the shape parameter.

5.3 Recommendations

Based on our findings from the results of this study, we recommend that;

i. Bayesian method using Quadratic Loss Function should be used under Jeffrey prior for the estimation of the shape parameter of the *OGEED* irrespective of the parametric values or the sample size.

ii. When estimating the shape parameter in question, the researcher should also consider Quadratic Loss Function under Uniform prior.

5.4 Contribution to Knowledge

This researcher has contributed to knowledge in the following ways;

i. This study was able to derive the posterior distributions of the shape parameter of *OGEED* assuming Uniform and Jeffrey priors using *SELF*, *QLF* and *PLF*.

ii. The Estimates, Biases, Mean Square Errors and Posterior risks of the said posterior distributions were also obtained in this study mathematically and through the use of Monte-Carlo simulation techniques.

iii. Following the comparative analysis of this study, we were able to establish that the shape parameter of the *OGEED* can best be obtained using the method of Bayesian estimation using Quadratic Loss Function under the assumption of Jeffrey and Uniform prior.

5.5 Areas of Further Research

Subsequent researches in line with this study can look at the following;

i. The estimation of the same parameter using some other priors and loss functions.

ii. Compare some other classical methods besides MLE with Bayesian approach base on different prior distributions and loss functions on the same parameter of the same distribution.

iii. Estimate the scale parameter of the *OGEED* using some appropriate methods.

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APPENDIXES

```
Library (Newdistns)
Loading required package: Adequacy Model
set. Seed (111)
terna=function(n,a,b,itr)
{
amles=c()
auses=c()
auqes=c()
aupes=c()
ajses=c()
ajqes=c()
ajpes=c()
msemles=c()
mseuses=c()
mseuqes=c()
mseupes=c()
msejses=c()
msejqes=c()
msejpes=c()
biasmles=c()
biasuses=c()
biasuqes=c()
biasupes=c()
biasjses=c()
biasjqes=c()
```

```
biasjpes=c()
puses=c()
puqes=c()
pupes=c()
pjses=c()
pjqes=c()
pjpes=c()
for(i in 1:itr)
 {
U=runif(n,0,1)
x=(log(1-(log(1-U))/a))/b
amle=(n/(sum(exp(b*x)-1)))
amles[i]=amle
amler=mean(amles)
msemle=(amle-a)^2
msemles[i]=msemle
msemler=(sum(msemles))/10000
biasmle=abs(amle-a)
biasmles[i]=biasmle
biasmler=(sum(biasmles))/10000
ause=((gamma(n+2))/(gamma(n+1)*sum(exp(b*x)-1)))
auses[i]=ause
auser=mean(auses)
mseuse=(ause-a)^2
mseuses[i]=mseuse
mseuser=(sum(mseuses))/10000
```

```
biasuse=abs(ause-a)
biasuses[i]=biasuse
biasuser=(sum(biasuses))/10000
puse=((gamma(n+3)*gamma(n+1))-(gamma(n+2))^2)/(gamma(n+1)*sum(exp(b*x)-1))^2
puses[i]=puse
auqe=(gamma(n))/(gamma(n-1)*sum(exp(b*x)-1))
auqes[i]=auqe
auqer=mean(auqes)
mseuqe=(auqe-a)^2
mseuqes[i]=mseuqe
mseuqer=(sum(mseuqes))/10000
biasuqe=abs(auqe-a)
biasuqes[i]=biasuqe
biasuqer=(sum(biasuqes))/10000
puqe=((gamma(n+1)*gamma(n-1))-(gamma(n))^2)/(gamma(n+1)*gamma(n-1))
puqes[i]=puqe
aupe=((gamma(n+3))/(gamma(n+1)*(sum(exp(b*x)-1))^2))^0.5
aupes[i]=aupe
auper=mean(aupes)
mseupe=(aupe-a)^2
mseupes[i]=mseupe
mseuper=(sum(mseupes))/10000
biasupe=abs(aupe-a)
biasupes[i]=biasupe
biasuper=(sum(biasupes))/10000
pupe=2*((((gamma(n+3)*gamma(n+1))^0.5)-(gamma(n+2)))/(gamma(n+1)*sum(exp(b*x)-1)))
```

```
pupes[i]=pupe

ajse=(gamma(n+1))/(gamma(n)*sum(exp(b*x)-1))

ajses[i]=ajse

ajser=mean(ajses)

msejse=(ajse-a)^2

msejses[i]=msejse

msejser=(sum(msejses))/10000

biasjse=abs(ajse-a)

biasjses[i]=biasjse

biasjser=(sum(biasjses))/10000

pjse=((gamma(n+2)*gamma(n))-(gamma(n+1))^2)/(gamma(n)*sum(exp(b*x)-1))^2

pjses[i]=pjse

ajqe=(gamma(n-1))/(gamma(n-2)*sum(exp(b*x)-1))

ajqes[i]=ajqe

ajqer=mean(ajqes)

msejqe=(ajqe-a)^2

msejqes[i]=msejqe

msejqer=(sum(msejqes))/10000

biasjqe=abs(ajqe-a)

biasjqes[i]=biasjqe

biasjqer=(sum(biasjqes))/10000

pjqe=((gamma(n)*gamma(n-2))-(gamma(n-1))^2)/(gamma(n)*gamma(n-2))

pjqes[i]=pjqe

ajpe=((gamma(n+2))/(gamma(n)*(sum(exp(b*x)-1))^2))^0.5

ajpes[i]=ajpe

ajper=mean(ajpes)
```

```
msejpe=(ajpe-a)^2

msejpes[i]=msejpe

msejper=(sum(msejpes))/10000

biasjpe=abs(ajpe-a)

biasjpes[i]=biasjpe

biasjper=(sum(biasjpes))/10000

pjpe=2*((((gamma(n+2)*gamma(n))^0.5)-(gamma(n+1)))/(gamma(n)*sum(exp(b*x)-1)))

pjpes[i]=pjpe

 result=cbind(a,b,amler,auser,auqer,auper,ajser,ajqer,ajper)

estresult=cbind(amle,ause,auqe,aupe,ajse,ajqe,ajpe)

biasresult=cbind(biasmler,biasuser,biasuqer,biasuper,biasjser,biasjqer,biasjper)

mseresult=cbind(msemler,mseuser,mseuqer,mseuper,msejser,msejqer,msejper)

postresult=cbind(puse,puqe,pupe,pjse,pjqe,pjpe)

colnames(estresult)=c("MLEs","BSELF-PU","BQLF-UP","BPLF-UP","BSELF-JP","BQLF-JP","BPLF-
JP")

colnames(postresult)=c("PSELF-UP","PQLF-UP","PPLF-UP","PSELF-JP","PQLF-JP","PPLF-JP")

colnames(biasresult)=c("BIAS-MLEs","BIAS-PSELF-UP","BIAS-PQLF-UP","BIAS-PPLF-UP","BIAS-
PSELF-JP","BIAS-PQLF-JP","BIAS-PPLF-JP")

colnames(result)=c("a","b","MLE","PSELF-UP","PQLF-UP","PPLF-UP","PSELF-JP","PQLF-
JP","PPLF-JP")

}

return(list(result=result,amles=amles,auses=auses,auqes=auqes,aupes=aupes,ajses=ajses,ajqes=ajqes,ajpe
s=ajpes,msemles=msemles,mseuses=mseuses,mseuqes=mseuqes,mseupes=mseupes,msejses=msejses,ms
ejqes=msejqes,msejpes=msejpes,biasmles=biasmles,biasuses=biasuses,biasuqes=biasuqes,biasupes=biasu
pes,biasjses=biasjses,biasjqes=biasjqes,biasjpes=biasjpes,puses=puses,puqes=puqes,pupes=pupes,pjses=p
jses,pjqes=pjqes,pjpes=pjpes,mseresult=mseresult,biasresult=biasresult,estresult=estresult,postresult=post
result))

}

>set.seed(111)

> terna(20,2,1.5,10000)
```

```
> $mseresult
      msemler mseuser   mseuqer   mseuper   msejser   msejqer   msejper
[1,] 0.2575327 0.31602 0.2224369 0.3535598 0.2575327 0.2107325 0.2834962

$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] 0.1789795     0.2879724    0.0699758     0.341822     0.1789795
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    -0.0889615    0.2328173

$estresult
       MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 2.178986 2.287936 2.070037 2.341777 2.178986 1.961088 2.232796

> set.seed(111)
> terna(20,2,4.5,10000)
$mseresult
      msemler mseuser   mseuqer   mseuper   msejser   msejqer   msejper
[1,] 0.2575327 0.31602 0.2224369 0.3535598 0.2575327 0.2107325 0.2834962

$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] 0.1789795     0.2879724    0.0699758     0.341822     0.1789795
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    -0.0889615    0.2328173
```

$estresult

MLEs BSELF-PU BQLF-UP BPLF-UP BSELF-JP BQLF-JP BPLF-JP

[1,] 2.178986 2.287936 2.070037 2.341777 2.178986 1.961088 2.232796

> set.seed(111)

> terna(20,3,1.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer msejper

[1,] 0.5794486 0.7110451 0.500483 0.7955096 0.5794486 0.4741482 0.6378664

$biasresult

BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP

[1,] 0.2685193 0.4319086 0.1050637 0.512583 0.2685193

BIAS-PQLF-JP BIAS-PPLF-JP

[1,] -0.0584422 0.3492226

$estresult

MLEs BSELF-PU BQLF-UP BPLF-UP BSELF-JP BQLF-JP BPLF-JP

[1,] 3.268479 3.431903 3.105055 3.512665 3.268479 2.941631 3.349195

>set.seed(111)

> terna(20,3,4.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer msejper

[1,] 0.5794486 0.7110451 0.500483 0.7955096 0.5794486 0.4741482 0.6378664

```
$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] 0.2685193     0.4319086    0.1050637     0.512583     0.2685193
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    -0.0584422     0.3492226

$estresult
       MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 3.268479 3.431903 3.105055 3.512665 3.268479 2.941631 3.349195



> set.seed(111)
> terna(55,2,1.5,10000)
$mseresult
      msemler    mseuser    mseuqer    mseuper   msejser    msejqer    msejper
[1,] 0.0789306 0.08565584 0.07499169 0.09004229 0.0789306 0.07383913 0.08192978

$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] 0.4858276     0.5310134    0.4406237    0.5534935     0.4858276
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    0.3954917    0.5083366

$estresult
       MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
```

[1,] 2.485846 2.531043 2.440648 2.553541 2.485846 2.395451 2.508342

> set.seed(111)

> terna(55,2,4.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer msejper

[1,] 0.0789306 0.08565584 0.07499169 0.09004229 0.0789306 0.07383913 0.08192978

$biasresult

BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP

[1,] 0.4858276 0.5310134 0.4406237 0.5534935 0.4858276

BIAS-PQLF-JP BIAS-PPLF-JP

[1,] 0.3954917 0.5083366

$estresult

MLEs BSELF-PU BQLF-UP BPLF-UP BSELF-JP BQLF-JP BPLF-JP

[1,] 2.485846 2.531043 2.440648 2.553541 2.485846 2.395451 2.508342

>set.seed(111)

> terna(55,3,1.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer msejper

[1,] 0.1775938 0.1927256 0.1687313 0.2025952 0.1775938 0.166138 0.184342

```
$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] 0.7288414     0.7965702    0.6609856    0.8302902     0.7288414
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    0.5931875    0.7625049

$estresult
       MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 3.728768 3.796564 3.660973 3.830312 3.728768 3.593177 3.762514



> set.seed(111)
> terna(55,3,4.5,10000)

$mseresult
      msemler   mseuser   mseuqer   mseuper   msejser  msejqer  msejper
[1,] 0.1775938 0.1927256 0.1687313 0.2025952 0.1775938 0.166138 0.184342

$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] 0.7288414     0.7965702    0.6609856    0.8302902     0.7288414
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    0.5931875    0.7625049

$estresult
       MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
```

[1,] 3.728768 3.796564 3.660973 3.830312 3.728768 3.593177 3.762514

>set.seed(111)

> terna(75,2,1.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer

[1,] 0.05629553 0.05984857 0.05421975 0.06217084 0.05629553 0.05362123

msejper

[1,] 0.05788151

$biasresult

BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP

[1,] -0.1419787 -0.1172742 -0.1667523 -0.1048582 -0.1419787

BIAS-PQLF-JP BIAS-PPLF-JP

[1,] -0.1915592 -0.1296605

$estresult

MLEs BSELF-PU BQLF-UP BPLF-UP BSELF-JP BQLF-JP BPLF-JP

[1,] 1.857976 1.882749 1.833203 1.895095 1.857976 1.80843 1.870321

> set.seed(111)

> terna(75,2,4.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer

[1,] 0.05629553 0.05984857 0.05421975 0.06217084 0.05629553 0.05362123

msejper

[1,] 0.05788151

$biasresult

BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP

[1,] -0.1419787 -0.1172742 -0.1667523 -0.1048582 -0.1419787

BIAS-PQLF-JP BIAS-PPLF-JP

[1,] -0.1915592 -0.1296605

$estresult

MLEs BSELF-PU BQLF-UP BPLF-UP BSELF-JP BQLF-JP BPLF-JP

[1,] 1.857976 1.882749 1.833203 1.895095 1.857976 1.80843 1.870321

>set.seed(111)

> terna(75,3,1.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer msejper

[1,] 0.126665 0.1346593 0.1219944 0.1398844 0.126665 0.1206478 0.1302334

$biasresult

BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP

[1,] -0.2130181 -0.1759112 -0.2502284 -0.1573872 -0.2130181

```
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    -0.2874388    -0.1944907

$estresult
       MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 2.786963 2.824123 2.749804 2.842642 2.786963 2.712644 2.805482



>set.seed(111)
> terna(75,3,4.5,10000)
$mseresult
     msemler   mseuser   mseuqer   mseuper msejser   msejqer   msejper
[1,] 0.126665 0.1346593 0.1219944 0.1398844 0.126665 0.1206478 0.1302334



$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] -0.2130181    -0.1759112    -0.2502284    -0.1573872    -0.2130181
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    -0.2874388    -0.1944907

$estresult
       MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 2.786963 2.824123 2.749804 2.842642 2.786963 2.712644 2.805482
```

```
>set.seed(111)
> terna(100,2,1.5,10000)
$mseresult
      msemler    mseuser    mseuqer    mseuper    msejser    msejqer
[1,] 0.04233566 0.04432995 0.04116456 0.04563232 0.04233566 0.04081665
      msejper
[1,] 0.04322743

$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,] 0.1789646     0.2007886    0.1572304    0.2115958     0.1789646
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    0.1354226    0.1897636

$estresult
     MLEs BSELF-PU  BQLF-UP  BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 2.178975 2.200765 2.157186 2.211633 2.178975 2.135396 2.189843


>set.seed(111)
> terna(100,2,4.5,10000)
$mseresult
      msemler    mseuser    mseuqer    mseuper    msejser    msejqer
[1,] 0.04233566 0.04432995 0.04116456 0.04563232 0.04233566 0.04081665
      msejper
```

[1,] 0.04322743

$biasresult

BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP

[1,] 0.1789646 0.2007886 0.1572304 0.2115958 0.1789646

BIAS-PQLF-JP BIAS-PPLF-JP

[1,] 0.1354226 0.1897636

$estresult

MLEs BSELF-PU BQLF-UP BPLF-UP BSELF-JP BQLF-JP BPLF-JP

[1,] 2.178975 2.200765 2.157186 2.211633 2.178975 2.135396 2.189843

>set.seed(111)

> terna(100,3,1.5,10000)

$mseresult

msemler mseuser mseuqer mseuper msejser msejqer

[1,] 0.09525524 0.09974238 0.09262026 0.1026727 0.09525524 0.09183746

msejper

[1,] 0.09726171

$biasresult

BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP

[1,] 0.268497 0.301133 0.2358455 0.3174937 0.268497

BIAS-PQLF-JP BIAS-PPLF-JP

[1,] 0.2031338 0.2848455

```
$estresult
      MLEs BSELF-PU  BQLF-UP BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 3.268463 3.301148 3.235778 3.31745 3.268463 3.203094 3.284765


>set.seed(111)
> terna(100,3,4.5,10000)
$mseresult
      msemler    mseuser    mseuqer   mseuper    msejser    msejqer
[1,] 0.09525524 0.09974238 0.09262026 0.1026727 0.09525524 0.09183746
      msejper
[1,] 0.09726171

$biasresult
    BIAS-MLEs BIAS-PSELF-UP BIAS-PQLF-UP BIAS-PPLF-UP BIAS-PSELF-JP
[1,]  0.268497     0.301133    0.2358455    0.3174937      0.268497
    BIAS-PQLF-JP BIAS-PPLF-JP
[1,]    0.2031338    0.2848455

$estresult
      MLEs BSELF-PU  BQLF-UP BPLF-UP BSELF-JP  BQLF-JP  BPLF-JP
[1,] 3.268463 3.301148 3.235778 3.31745 3.268463 3.203094 3.284765
```