A STUDY ON COMMUTATIVITY THEOREMS FOR RINGS AND NEAR-RINGS

BY

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DEPARTMENT OF MATHEMATICS

AHMADU BELLO UNIVERSITY, ZARIA, NIGERIA

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DECLARATION

I declare that the work in this thesis entitled "ASTUDY ON SOME COMMUTATIVITY THEOREMS FOR RINGS AND NEAR-RINGS" has been performed by me in the Department of Mathematics, A.B.U Zaria under the supervision of Professor M.A. Khan, Professor D. Singh and Dr. A. M. Ibrahim. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

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CERTIFICATION

This thesis entitled" A STUDY ON SOME COMMUTATIVITY THEORMS FOR RINGS AND NEAR-RINGS" by MADUGU Abdu meets the regulations governing the award of the degree of Doctor of Philosophy of the Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This thesis is dedicated to my lovely family.

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ABSTRACT

This research work deals with commutativity theorems for rings and near-rings with their various generalizations in the setting of some special classes of rings as well as derivations in rings and near-rings. We present their fundamentals and also prove some commutativity results for semi prime rings and ring with unity.We then extended the results to s-unital rings. Some results on commutativity of rings and near-rings are generalized and justified by examples. Posner's first theorem isestablished for σ −prime rings. Jordan left and right σ −derivations as well as generalized Jordan left and right σ −derivations on σ −rings are investigated. Further more,distributive near-ring, pseudo-abelian near-ring and distributively generated near-rings are introduced andthese help in developing certain class of derivations on near-rings. Finally, permuting $4-(\sigma,\tau)$ derivations on prime near-rings are generalized to permuting $n-(\sigma,\tau)$ derivations.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

An algebraic system is a non-empty set together with one or more binary operations defined on the set. This system subsequently considered shall be divided in to structures according to the axioms, which they satisfy. The division leads to the branch of mathematics known as abstract algebra (viz: semi group,group, ring, near-ring and so on).Sets and binary operations (functions) are the fundamental and main ingredients of algebraic systems. Examplesof algebraic systems are setof all natural numbers (or counting numbers) $\mathbb{N}$, set of allintegers $\mathbb{Z}$,set of all rational numbers $\mathbb{Q}$, set of all real numbers $\mathbb{R}$ and set of all complex numbers $\mathbb{C}$. Each of these sets is an important algebraic system under addition $(+)$ and multiplication $(\cdot)$.

A ring is a non empty set with two binary operations, namely addition (+) and multiplication $(\cdot)$, satisfying a collection of axioms. These axioms require addition to satisfy the axioms for an abelian group while multiplication is associative. The two operations are connected by the (left and right) distributive laws. Some examples of rings are set of integers, rational numbers, real numbers, complex numbers, 2×2 real (complex) matrices and so on.

An additively written group$(N_R,+)$(not necessary abelian) equipped with a binary operation $(\cdot):N_R\times N_R\to N_R$ defined by . $(x,y)\to xy$ suchthat $(xy)z=x(yz)$ and

$x(y+z) = xy + xz \; \forall \; x, y, z \in N_R$ is called a left near-ring (respectively, a right near-ring if $(y+z)x = yx + zx \; \forall \; x, y, z \in N_R$). A near-ring N_R is said to be zero symmetric if $0x = 0$ for all $x \in N_R$.

For example, let V be a linear space with a basis $e_1, e_2, \ldots, e_n$ over a field F of characteristic different from two. Define a mapping $\cdot : V \times V \to V$ by $vw = 0$ for all $v, w \in V$ with $v \neq \pm e_1$ and $e_1 w = w, (-e_1)w = -w$. With respect to this multiplication V is a left zero-symmetric near-ring. In view of the above multiplication, $e_1(e_2 + e_3) = e_2 + e_3$. On the other hand, neither $e_2 + e_3 = e_1$ nor $e_2 + e_3 = -e_1$ since $e_1, e_2, \ldots e_n$ is linearly independent, and hence $(e_2 + e_3)e_1 = 0$. Since right distributive law fails V is not a ring.

The commencement of the twentieth century marks the beginning of the investigations of classes of rings which turn out to be commutative under certain types of hypotheses. Although, the famous Weddurburn theorem (1905) on commutativity of finite division rings was proved in the early twentieth century, yet it was during the past four decades that striking results with conditions imposed on a ring, render the ring commutative or almost commutative. An early example is the Weddurburn theorem on finite division rings which plays an important role in commutativity theorems.

Much significant contributions were made in (Herstein, 1968) by generalizing some classical commutativity theorems and obtaining various conditions which rendered rings to be commutative or nearly commutative. For instance, let $Z\langle x_1, x_2, \ldots, x_n \rangle$ denote the ring of polynomials over the set of integers $\mathbb{Z}$, in non-commuting variables $x_1, x_2, \ldots, x_n$. Let R be a ring (not

necessarily unital, not necessarily commutative). We say $f \in Z\langle x_1, x_2, ..., x_n \rangle$ is a polynomial identity (PI) on R if $f(a_1, a_2, ..., a_n) = 0 \quad \forall a_i \in R$. R is called PI- ring if it satisfies a non-trivial polynomial identity. Commutative rings, for example, all satisfy the polynomial identity $xy - yx = 0, \forall x, y \in R$.

Commutativity theorems provide conditions on a ring R which imply that R is commutative.For example, let R be a ring for which $\forall a \in R \; \exists$ an integer $n(a) > 1$ with $a^{n(a)} = a$. Then R is commutative,according to Jacobson's theorem (Jacobson, 1945).Many of the commutativity theorems discussed in (Herstein, 1968) assert that if R satisfies a certain polynomial identities then R is commutative.

There is no clear-cut way to allow cancellation among the elements of rings, which is always permissible in case of groups. This restricted the progress of rings till the general structure theory (called structure of rings) was developedby Jacobson (1964). Commutativity theorems are part of the study of polynomial identities in noncommutative rings. These theorems assert that under certain conditions, the ring at hand must be commutative. There are several results dealing with conditions under which a ring is commutative. Generally such conditions are placed on the ring itself or its subsets, or commutators.

Herstein (1953) proved that commutativity follows if one assumes for every x there existsa polynomial p with integral coefficients such that $x^2 p(x) - x$ is central. Martindale (1958) went still further by weakening the assumption to the existence of an element x such that $x^2 a - x$ is central.

If every part of a ring R satisfies:all commutators arecentral, all nilpotent elements are central, andall one-sided ideals are two sided, then R is commutative. It is then natural to look at the extension of commutativity of rings through the following properties:

a. To prove commutativity of R, a ring with unity 1, if R is n-torsion free and $g(X), h(X) \in Z(X)$ such that

$x^p [x^n, y] y^q = f(y)[x, y^t g(y)]^k h(y) \forall x \in R$,where $p \geq 0, q \geq 0, t \geq 0, n > 1, k > 1$ are fixed integers.

b. For each $x, y \in R$, there exist polynomials $f(t),\ g(t) \in Z[t]$and integers

$r = r(x,y) > 1, s = s(x,y) > 1, p = p(x,y) \geq 0, q = q(x,y) \geq 0, m \geq 1,$

$t \geq 2$with$(r,s) = 1$ such that $x^p [x^r, y] = g(y)[x, y^t f(y)]^m h(y)$ and

$x^q [x^s, y] = \tilde{g}(y)[x, y^t f(y)]^m \tilde{h}(y)$.

Remark1.1

In problem a;

(i). one can examineevidently that there are dual properties found by replacing $x^p [x^n, y] y^q$ by $y^p [x^n, y] x^q$ in definition (a)., and also by replacing $x^p [x^r, y]$ and $x^q [x^s, y]$ by $[x^r, y] x^p$ and $[x^s, y] x^q$ in (b).

(ii). rings with unity and s-unital rings are considered to satisfy properties (a) and (b) above for the development of commutativity theorems.

1.2 Statement of the Problem

The introduction of rings theory is a boostto modern algebra.However, the inadmissibility of cancellation law impeded the early development of ring theory until the structure theory of rings was propoed by Jacobson (1964).Many researchers have contributed to the concepts of commutativity theorems of rings and near rings in recent time. Prominent among the contributors are Jing and Lu (2003), Oukhtite *(*2011*)*, Pinter-Lucke (2007), Vukman (2007), Dhara and Shrama (2009), *Khan et al.* (2013), Huang (2011), Ashraf and Siddeeque (2013) and so on. Despite their giant strides, the notion of permuting $4-(\sigma,\tau)$derivation as well as its generalization to permuting $n-(\sigma,\tau)$derivations in commutative ring and near ring are still unexplored.

Motivated by these, we introducethe notion of permuting $4-(\sigma,\tau)$derivation as well as permuting $n-(\sigma,\tau)$derivation in near-rings, where n is a positive integer.We study some properties of the derivations and show that additive commutativity of a near ring N_R satisfies certain identities involving permuting $n-(\sigma,\tau)$ derivations of a prime near ring. We alsogive illustrations to justify the notions in various theorems, and investigate the polynomial identities with constraints such as commutators, torsion free conditions on rings and near-rings to establish commutativity theorems.

1.3 Justification of the Study

Commutativity theorems for rings and near-rings with their applications have been studied by lots of Mathematicians. The essence of this research is to encourage the pursuit of research on applications of ring theory in diverse areas; such as toconcentrate on the interdisciplinary efforts involved in the pursuit of information

technology and coding theory. Boolean rings are well-known for their applications in various branches of mathematics as well as in computer science. All such rings are in fact commutative rings. Near-ring theory is applicable in computer science, digital computing, data communications, and so on. Hence the study of commutativity of rings is an important problem in non commutative rings.

1.4 Aim and Objectives of the Study

The aim of this research work is to investigate some results on commutativity of semi prime rings, rings with unity, s-unital rings and permuting $4-(\sigma,\tau)$derivation as well as permuting $n-(\sigma,\tau)$ derivations on prime near-rings.

In order to achieve the above aim,the objectives considered are to:

(i). extend the related results for one sided s-unital rings and n-torsion free rings,

(ii). establish the resultsof Jordan right σ −derivation and generalized Jordan right σ −derivation on $\sigma-$ rings,

(iii). introduce the notion of permuting$4-(\sigma,\tau)$derivation as well as permuting $n-(\sigma,\tau)$derivation in near-rings,

(iv). show that additive commutativity of a near ring N_R satisfies certain identities involving permuting $n-(\sigma,\tau)$ derivations of a prime near ring,

(v). give examples to justify the notions of permuting$4-(\sigma,\tau)$derivation and permuting $n-(\sigma,\tau)$derivations,

(vi). extend Posner's first theorem to σ −prime rings of characteristic different fromtwo,

(vii). examine polynomial identities with constraints such as commutators, torsion free conditions on rings and near-rings thereby establishing commutativity theorems.

1.5 Methodology of the Study

In order to achieve the objectives outlined above, we conducted an up-to date critical review of literature dealing with various fundamental aspects of commutativity of rings and near-ring, and define some terms on derivations in near-rings which helpin formulatingour results.

1.6 Definition of Terms

a. A map $d: R \to R$ is called a derivation of ring R if it satisfies the following properties:(i). $d\ (x\ +\ y)\ =\ d(x)\ +\ d(y)$; and

(ii). $d(x\ y)\ =\ yd(x) +\ x\ d(y)$, for all $x, y\ \in\ R$.

The example of a non trivial derivation is the usual differentiation on the ring $F[x]$ of polynomials defined over a field F.

b. For a fixed a in R, define $d_a: B \to B$ such that $d_a(x)\ =\ [a, x]$ for all $x\ \in R$.

Then it can be shown that d_a is a derivation of R. This d_a is called an inner derivation of R determined by `a' and usually it is denoted by I_a.Clearly, every inner derivation of a ring R is a derivation. But the converse needs not be true in general.

For example, take the ring $B = \left\{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} | \alpha, \beta, \gamma \in Z \right\}$ with respect to matrix addition and matrix multiplication. Define a map $d: B \to B$ by

$$d\begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & -\gamma \\ 0 & 0 & 0 \end{pmatrix}.$$

Remark 1.2

(i). It is easy to check that d is a derivation of B, which is not an inner derivation of B.

(ii). An additive mapping $d: B \to B$ is called a Jordan derivation if $d(x^2) = xd(x) + xd(x)$ for all $x \in B$.

(iii). Every derivation is a Jordan derivation but the converse needs not be true in general.

c. Let N_R be a non empty set equipped with two binary operations say '$+$' and '$\cdot$', then N_R is called a left near ring (respectively a right near-ring) if:

(i). $(N_R, +)$ is a group (not necessarily abelian),

(ii). $(N_R, \cdot)$ is a semi group,(i.e. $N_R$satisfies closure and assotiative law with regards to multiplication),

(iii). $x\,(y + z) = x\,y + x\,z \;\forall\, x, y, z \in N_R$ (respectively, $(y + z)x = y\,x + zx\,,\; \forall\, x, y, z \in N_R$).

The examples of such structure are the following:

(i). Let $(\mathbb{C}, +)$ be the usual group of complex numbers with respect to ordinary addition of complex numbers. Let us define `$*$' in $\mathbb{C}$ as $x * y = |x|\, y$ for all $\forall\, x, y \in \mathbb{C}$.Then $(\mathbb{C}, +, *)$ is a left near ring which is not a right near ring.

(ii). Let $(G, +)$ be a non abelian group. Consider S, the set of all functions

From$G$ to G. Then $(S, +, \cdot)$ is a right near-ring, but not a left near-ring,

with regard to the operation of addition `+' and multiplication ` ·' defined as:

$(f + g)(x) = f(x) + g(x) \forall\ x \in G$, and

$(f\ g)\ (x) = f\ (g(x)) \forall\ x \in G$,where$f, g \in S$.

A left near- ring N_R is called zero symmetric if 0x = 0 for all $x \in N_R$.

Throughout the discussion, N_R will denote a zero symmetric left near-ring

with center $Z\ (N_R)$ unless otherwise mentioned.

d. Let N_R be a near-ring. Then

(i). An additive mapping $f : N_R \to N_R$ is called a right generalized derivation of N_R if there exists a derivation d of N_R such that

$f(xy) = f(x)y + xd(y)\ for\ all\ x, y \in N_R$.

(ii). An additive mapping $f : N_R \to N_R$ is called a left generalized derivation of N_R if there exists a derivation d of N_R such that

$f(xy) = d(x)y + xf(y)\ for\ all\ x, y \in N_R$.

(iii). An additive mapping $f : N_R \to N_R$ is called a generalized derivation ofN_R if ∃ a derivation d of$N_R$such that $f(xy) = f(x)y + xd(y)$ for all $x, y \in N_R$ and $f(xy) = d(x)y + xf(y)$ for all $x, y \in N_R$.

For example, let S be any zero symmetric left near-ring. Take

$S_1 = \left\{\begin{pmatrix} 0 & \alpha \\ 0 & \beta \end{pmatrix} | 0, \alpha, \beta \in S\right\}$. Then S_1 is a zero symmetric left near-ring with regard to the matrix addition and multiplication.

e. Define $F: S_1 \to S_1$ and $D: S_1 \to S_1$ by $F\begin{pmatrix}0 & \alpha\\ 0 & \beta\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & \beta\end{pmatrix}$ and

$D\begin{pmatrix}0 & \alpha\\ 0 & \beta\end{pmatrix} = \begin{pmatrix}0 & \alpha\\ 0 & 0\end{pmatrix}$. It can be checked that F is a right generalized derivation of S_1 with associated derivation D of S_1 but it is not a left generalized derivation of S_1 with associated derivation D of S_1.

For example, consider $S_2 = \left\{\begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix} | 0, \alpha, \beta \in S\right\}$. Then S_2 is a zero symmetric left near-ring with respect to the matrix addition and multiplication.

f. Define $F: S_2 \to S_2$ and $D: S_2 \to S_2$ by $F\begin{pmatrix}\alpha & \beta\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \beta\\ 0 & 0\end{pmatrix}$ and

$D\begin{pmatrix}0 & \alpha\\ 0 & \beta\end{pmatrix} = \begin{pmatrix}0 & \alpha\\ 0 & 0\end{pmatrix}$.

It can be easily seen that F is a left generalized derivation of S_2 with associated derivation D of S_2 but F is not a right generalized derivation of S_2 with associated derivation D of S_2.

For example, consider $S_3 = \left\{\begin{pmatrix}0 & \alpha & \beta\\ 0 & 0 & 0\\ 0 & 0 & \gamma\end{pmatrix} \middle| 0, \alpha, \beta, \gamma \in S\right\}$. Then S_3 is a zero symmetric left near-ring with regard to the matrix addition and multiplication.

g. Define $F: S_3 \to S_3$ and $D: S_3 \to S_3$ as F $\begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

$D\begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.Then it can be easily seen that F is a

generalized derivation of S_3 with associated derivation D of S_3.

Subsequently, we replace N_R by N to denote near-ring for simplicity.

1.7 Organization of the Thesis

This thesis is divided into six chapters. Chapter one covers the general introduction of the thesis and Chapter two dicusses the review of selected literature. The fundamentals of commutative theorems for rings are explicated in Chapter three while derivations of some special classes of $\sigma -$ rings are covered in Chapter four. Chapter five gives the results of some special classes of near-rings. Chapter six summarises, concludes and gives direction for future research.

1.8 Limitation of the study.

This research work is limited to the study of commutativity theorems for semi prime ring, ring with unity 1, s-unital rings and n-(s, t) permutting derivations on prime near-rings. It studies how non commuatative rings mentioned above under suitable constraints become commutative rings. In addition the notion of n-(s, t) permutting derivations on prime near-rings has been generalized to any positive integer n but to work it out for larger n ($n \geq 10$), one needs a computer.

CHAPTER TWO

LITERATURE REVIEW

The origin of commutativity theorems for rings could be traced to the work of Wedderburn (1905) titled "A finite division ring is necessarily a field" in Transaction American Mathematical society. This result has attracted the attention of most mathematicians because it was so unexpected,interrelating two seemingly unrelated things, namely the number of elements in certain algebraic systems and the multiplication of the system.For algebraists, the mentioned Wedderburn theorem served as a jumping-off point for a large area of research in commutativity of rings in the 1950s.

Jacobson (1945)proved that Algebraic division algebra over a finite field is commutative.During the last several decades, there have been many results concerning conditions that force a ring to be commutative. There are now more than 400 papers in which conditions are given that determine commutativity for a ring or for special type of rings such as prime rings, semi-prime rings, rings with identity 1 and s-unital rings. These results were stimulated by the famous result ofJacobson (1964) which stated that if for every x in a ring R there exists a positive integer $n(x)$ such that $x^{n(x)} = x$, then R is commutative.

Kaplansky (1948) proved that if a division ring D satisfies any polynomial identity then it is finite dimensional over its center. Kaplansky (1951) asserted that, if R be is a ring with center $Z(R)$ and a positive integer $n(x) > 1$ is such that $x^{n(x)} \in Z(R)$for every $x \epsilon R$, then If R in addition is semi simple then R is commutative.

Herstein (1953)proved that if a ring R satisfying the property that for every $x \in R$, there exists an integer $n(x) > 1$ such that $x^{n(x)} - x \in Z(R)$,then R is commutative. Faith (1960) showed that, if$D$ is a division ring and $A \neq D$ a subring of D, and suppose for every $x \in D, x^{n(x)} \in A$, where $n(x) \geq 1$ depends on x,then D is commutative.

Herstein (1968) proved a classical result which states that: If in a ring R there exists a positive integer n such that $x^n - x$ is in the center, then R is commutative.This result was further strengthened by weakening the condition so that the exponent "n" depends on x and is not global.

Bell (1968)proved that if for every pair in a ring there exists an integer $n>1$ such that $[x,y]^n = [x,y]$, then the ring is commutative.Gupta (1970) proved that if a division ring satisfies the polynomial identity $xy^2x = yx^2y\,(or\,[xy,yx]=0)$, it is commutative. Bell (1975) showed that a ring R is commutative if for each pair of elements x, y in R, there exists integers $m, n \geq 1$ such that $xy = y^m x^n$.

Awtar (1984) proved that a semiprime ring R is commutative if R satisfies the property $xy^2x - yx^2y \in Z(R)(or\ [xy,yx] \in Z(R))$, the center of R. This result can also be proved with the help of Posner's theorem on polynomial identity when either$R$ has characteristic different from 2, or R has characteristic 2.

Psomopoulos (1984) proved thatif R is an s-unital ring, and m and n are fixed positive integers, where $[x^n y - y^m x, x] = 0$ for all x and y in R, then R is commutative.Khan *et al.*(1986) asserted that if R is a semi prime ring satisfying the condition $[(xy)^2 - xy, x] = 0$ for all x, y in R, then R is

commutative.Searcoidand MacHale (1986) proved that aring R become necessarily commutative under generalised Boolean identity namely $(xy)^2 = xy$ for all $x, y \; in \; R$.

Khan and Quadri. (1988) proved that an associative ring R is commutative if a ring with identity 1 satisfying the property, that for each pair of elements $x, y \in R$, there exist integers $m \geq 1, n \geq 1$ such that $[xy - y^m x^n, x] = 0$.Kezlan (1982) proved that if n is a positive integer and a ring R with unity 1 satisfying the property that for each pair of elements x, y in R, there exists an integer $m = m(x, y) > 1$ such that $[xy - y^m x^n, x] = 0$, then R is commutative.

Khan*et al.* (1995) showed that a semi prime ring R must be commutative and if for each $x \in R, \exists$ an integer $n = n(x) > 1$ such that$(xy)^n - x^n y^n$ is central for each $y \; in \; R$.

Khan et al.(1995) proved that, if$R$ is a ring with no non zero nil ideals. If for each pair of elements $x, y \in R,$ there exists an integer $n = n(x, y) \geq 1$ such that

$[(xy)^n - x^n y^n, x] = 0 = [(yx)^n - x^n y^n, x]$,then$R$ is commutative.

Pinter –Lucke (2007) presented a survey article which states that there have been many results concerning conditions that force a ring to be commutative. These results were stimulated by Jacobson's (1964) famous result and were extensively developed by Herstein. Pinter presented the survey by organizing the results according to whether they come from variations on Herstein's conditions, or they

depend on general polynomial conditions,or they depend on the presence of a derivation, or whether a ring has special properties that make commutativity more

easily accessible. The most recent conditions concern product sets and lead to results in a new area of inquiry.

Posner (1957) studied the derivation on prime rings and proved the commutativity of R if $\mathcal{S}$ is nonzero derivations on a prime ring R such that $x\delta(x)-\delta(x)x$ is in the centre of R for all $x \in R$. Posner's theorem has been generalized by a number of authors in several ways.

Awtar (1984) first gave the short proof of the Posner's theorem and continued the study of the Posner's theorem on Lie ideals and Jordan ideals of R and proved the following:

(i) Suppose R is a prime ring of characteristics different from 2 and 3 and L a Lie ideal of R. Let d be a nonzero derivation on R such that for all u in L, $ud(u)-d(u)u$ is in the center of R. Then L is contained in the center of R. Moreover, if R has characteristic 3 and L is a Lie ideal with u^2 in L for each $u \in L$, then L is in the center of R.

(ii) Let R be a prime ring of characteristics not equal to 2 and J a Jordan ideal of R. If d is a nonzero derivation on R with $ud(u)-d(u)u$ in the center of R for all u in J, then J is contained in the center of R.

(iii) If R is a prime ring of characteristics not equal to 2 and U is a Lie (Jordan) ideal and a sub ring of R with $ud(u)-d(u)u$ in the center of R for all $u \, in \, U$, where d is a nonzero derivation on R, then U is commutative. It is remarked that, the assumptions on Lie (Jordan) ideal and subring on U are essential.

Ahmed (1976) had studied derivations on Lie and Jordan ideals of a prime ring R and proved: (i) Let R be a prime ring of characteristic not 2, and let U be a Lie (or Jordan) ideal of R. If d is a derivation defined on U, and if a is an element of the sub ring $T(U)$, generated by U, or a is an element of R, according as U is a Lie (or Jordan) ideal of R, such that $a\,du = 0$, for all $u \in U$, then either $a = 0$ or $du = 0$. (ii) Let d_1, d_2 be derivations defined for all $u \in U$, and also for u^2 and u^3 if U is a Lie ideal of R, such that the iterate $d_1 d_2$ is also a derivation, satisfying the same conditions as d_1, d_2. Let $dx(u) \in U$, whether U is a Lie (or Jordan ideal) of R. Then, at least, one of $d_1(u)$ and d_2(u) is zero, for all $u \in U$.

Bresar (1991) introduced the concept of a generalized derivation as follows: an additive mapping $H: B \to B$ is said to be generalized derivation if there exists an associated derivation d: $B \to B$ such that $H(xy) = H(x)y + xd(y)$ holds for all pairs $x, y \in B$. An additive mapping $H: B \to B$ is said to be a generalized Jordan derivation if there exists a Jordan derivation d: $B \to B$ such that $H(x^2) = H(x)x + xd(x)$ for all $x \in B$.

Vukman (2007) established that every generalized derivation on a two torsion free semiprime ring is a generalised derivation. Bresar (1991) presented the notion of Jordan triple derivation as follows: an additive mapping $d: B \to B$ is called Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in B$. In the same paper, Bresar proved that any Jordan triple derivation on a 2-torsion free semiprime ring is a derivation.

Motivated by the definition of generalized Jordan derivation, Jing and Lu (2003) introduced the concept of generalized Jordan triple derivation as: an additive mapping $H: B \to B$ is said to be generalized Jordan triple derivation on B if there exists a Jordan triple derivation $d: B \to B$ such that$H(xyx) = H(\mathrm{x})\mathrm{yx} + \mathrm{xd(y)x} + \mathrm{xyd(x)}$ holds for all $\mathrm{x}, \mathrm{y} \in \mathrm{B}$.

Braser (1991) defined the notion of right derivation as: An additive mapping $d: B \to B$ is called a right derivation (resp. right Jordan derivation) if $d(\mathrm{xy}) = \mathrm{d(x)y} + d(\mathrm{y})\mathrm{x}$ (respectively Jordan right derivation in case $d(\mathrm{x}^2) = 2\mathrm{d(x)x}$ holds for all $x, y \in B$. An additive mapping $d: B \to B$ is said to be a left derivation (resp. Jordan left derivation) if $d(\mathrm{xy}) = \mathrm{d(x)y} + \mathrm{d(y)x}$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in B$. Obviously, every right(resp. left) derivation on a ring B is a Jordan right (resp. left) derivation but the converse needs not be true in general.

Dhara and Shrama (2009) proved that, if$n \geq 1$ is a fixed integer and let B be an$(n+1)!$ −torsion free ring. If $H: B \to B$ and $d: B \to B$ are two additive mappings satisfying $H(x^{n+1}) = H(x)x^n + xd(x)x^{n-1} + x^2d(x)x^{n-2} + \cdots + x^nd(x)$ for all $x \in B$, then d is a Jordan derivation and H is a generalized Jordan derivation.

Several authors have established numerous results pertaining to the commutativity of σ-prime rings and also acknowledging derivations which are centralizing or commuting on some appropriate subsets of this ring.

Oukhtite et al. (2006)proved certain results concerning derivations of σ-prime rings. Khan et al. (2010) extended the result inσ-prime rings. Huang (2011) sketched the proofs of the known results concerning generalized derivations of prime rings to σ-prime rings.Near-fields were the first near-rings considered in the literature. Dickson

(1905) altered the multiplication in the field in order to get examples of one-sided distributive fields (near-fields) showing that the second distributive law does not follow from remaining axioms for a (skew) field.

Zassenhauss (1935) determined all finite near-fields, which have order p^n. Ore (1931) and Furtwangler-Taussy (1936) started axiomatic study in the thirties of the last century which we now call near-ring. The first authors to use the name near-ring were Zassenhauss (1935), Blackett (1950) and Jordan (1951). The late fifties of the last century brought the start of a rapid development of the theory of near-rings. The study of derivation in near-rings was initiated by Bell and Mason (1987) where theydeduced many results regarding the behavior of near-ring N as a commutative ring. Later, several authors viz: Maksa (1987), Khan (2003), Golbasiand Aydin (2004), Park (2009)and so on generalized the notion of derivation in different directions viz. (σ,τ)-derivation, left generalized derivation, right generalized derivation, generalized derivation, symmetric-bi-derivation and permuting-tri-derivation and so on.

Motivated by the notion of permuting$n-$derivation given by Park (2009) in rings.Ashraf and Siddeeque (2013) proved that, if N is an n!-torsion free near ringand suppose there exists a permuting n-additive mapping $d:N\times N\times ...\times N\to N$ such that $(N,N,...,N)\subseteq Z$. *If* $\delta(x)=0$ for all $x\in N$, where δ is the trace of d. Then $d=0$.

In this context, we introduce the notion of permuting $(\sigma,\tau)-4$-derivation and permuting (σ,τ)-n-derivations in near-rings, and constructed examples to justify these facts. We further investigate some properties involving permuting (σ,τ)-n-derivations of a prime near-ring N which force N to be a commutative ring. Through findings in

the literature, we wish to continue our investigation on derivations in different types of rings and near-rings. In this sequel, we investigate the relationship between the commutativity of rings and the existence of certain specific derivation of rings and near-rings.We extend the Posner's theorem to σ-prime rings for possible results.

CHAPTER THREE

FUNDAMENTALS OF COMMUTATIVITY THEOREMS FOR RINGS

We know that in a commutative ring R, the commutators $xy - yx$ are central. It is natural and interesting to question whether a ring in which all the commutators are central, needs be commutative? In general, the commutators need not be centralized. The ring of 3×3 strictly upper triangular matrices over a field F is one of the examples of those rings which satisfy the condition but it is not commutative. It is, rather, surprising that this problem could not be investigated till 1962 when Herstein proved that a division ring D in which $xy - yx$ is central for every pair x, y of D must be commutative. We discussthe techniques involve in commutativity of a semi prime ring.

3.1 Ring-theoretic concepts

Let R be an associative ring. For all $x, y \in R$; the symbol $[x, y]$ represents Lie product$x\,y - y\,x$ and $x \circ y$denotes Jordan product $x\,y + y\,x$.

Definition 3.1.1

Let R be a ring. If there exists a positive integer n such that $n\,x = 0$for all $x \in R$; then in this case the smallest positive integer with this property is calledthe characteristic of the ring R and is denoted by char (R). If no such positive integer exists, then R is said to be of characteristic zero.

Definition 3.1.2

A ring R is called Boolean ring if all its elements are idempotents, i.e., $x^2 = x$ for all $x \in R$.

Remark 3.1.1

Every Boolean ring is a commutative ring if it has characteristic 2.

Definition 3.1.3

An element x of a ring R is called nilpotent if $x^m = 0$ for some positive integer m.

Remarks 3.1.2

(i) For any idempotent $e \in R, ex - exe \text{ and } xe - exe$ are nilpotents in R.

(ii) Every nilpotent element is necessarily a divisor of zero.

(iii) If R is a prime ring with no non-zero nilpotent element, then R has no zero divisor.

However, if $ab = 0$, then $(ab)x = 0$. This implies that $a(bx) = 0$ for allx $\in R$ *i*.e.,

$(bx)a = 0$for all $x \in R$ and hence $bRa = \{0\}$. Since$R$is a prime, either $a = 0$ *or* $b = 0$ then R has no zero divisors.

Definition 3.1.4

Let S be a non empty subset of R, then the centralizer $C(S)$ of S in R is defined by $C(S) = \{x \in R \mid sx = xs \ \forall \ s \in S\}$.

Definition 3.1.5

An ideal A of R is called a nilpotent ideal if $A^n = (0)$ for some positive integer n. If every element of A is nilpotent, then A is said to be a nil ideal.

Remark 3.1.3

Every nilpotent ideal is necessary a nil ideal but the converse needs not be true.

Definition 3.1.6

The center of a ring R is defined to be the set of all those elements of R that commute with every element of R and is denoted as Z, (i.e. $Z = \{x \in R | xr = yr \ \forall \ r \in R\}$).

Remark 3.1.4

(i) A ring R is commutative if and only if $Z(R) = R$.

(ii) The centre of a prime ring is free from zero divisors.

(iii) The centre of semi prime ring contains no non zero nilpotent elements.

Definition 3.1.7

An element $x \in R$ is called n-torsion free if $nx = 0$ implies $x = 0$.Further if $nx = 0$ implies $x = 0$ for all $x \in R$; then R is called an n-torsion free ring.

Definition 3.1.8

A ring R is called a prime ring if zero ideal is a prime ideal of R.

Remark 3.1.5

A ring R is a prime ring if and only if any one of the following holds:

(i) If A and B are ideals of R such that $AB = \{0\}$, then $A = \{0\}$or $B = \{0\}$.

(ii) If $a, b \in R$ such that $aRb = \{0\}$, then $a = 0$ or $b = 0$.

Definition 3.1.9

A ring R which has no nonzero nilpotent ideal is said to be a semiprime ring.

Remarks 3.1.6

(i) A ring R is semiprime if and only if for any $a \in R$ such that$aRa = \{0\}$ implies that $a = 0$.

(ii) Every prime ring is a semiprime ring but its converse is not true. The ring $Z_6$of residue classes modulo 6 is a semiprime ring but not a prime ring.

(iii) A prime ideal is necessary a semiprime ideal but the converse is not true.For example in the ring Z of integers, ideal$2 \cap 3 = 6$ is semi prime which is not prime

Definition 3.1.10

An additive mapping $x \to \sigma(x)$ of R into itself is called an involution on R if it satisfies the conditions: (i) $\sigma(\sigma(x)) = x$, (ii) $\sigma(xy) = \sigma(y)\sigma(x)$ for all $x, y \in R$. A ring R equipped with an involution σ is called a ring with involution or a σ −ring.

Example 3.1.1

Let $B = \{\alpha + \beta i + \gamma j + \delta k | \alpha, \beta, \gamma, \delta \in R\}$be the ring of real quaternion.

Define $b \rightarrow b\text{^} *$ of Q into itself as $b\text{^} * = \alpha - \beta i - \gamma j - \delta k$ where $b = \alpha + \beta i + \gamma j + \delta k \in B$.

It can be easily checked that σ is an involution of B. Therefore B is a $\sigma -$ring.

Definition 3.1.11

Let R be $\sigma -$ring. Then an element $x \in R$ is called a symmetricelement of R if $\sigma(x) = x$ and an element $y \in R$ is called a skew symmetric element of Rif $\sigma(y) = -y$. The set of all symmetric and skew symmetric elements of R is denoted by$S_a\sigma(R)$.

Definition 3.1.12

Let R be a $\sigma -$ ring. Then an ideal I of R is called a$\sigma -$ideal if $\sigma(I)$ = I.

Example 3.1.2

Let $R = R[x] \times B$, where $R[x]$ is the polynomial ring over the ring R ofreal numbers and B is the ring of real quaternion. Defineσ: $R \rightarrow R\ as\ \sigma(f(x), b) = f(-x, \bar{b})$, where $f(x) \in R[x]$ and $\bar{b}$= $\alpha - \beta i - \gamma j - \delta k$ where b= $\alpha + \beta i + \gamma j + \delta k \in B$. Clearly, σ is an involution of R and the set $I = R[x] \times \{0\}$is an $\sigma -$ideal of R.

Definition 3.1.13

A ring R with involution σ is said to be $\sigma-$prime if $aRb = aR\sigma(b) = \{0\}$where $a, b \in R$(equivalently$aRb = \sigma(a)Rb = \{0\}$) implies that either $a = 0 \; or \; b = 0$.

Example 3.1.3

Let $Z \times Z$ be σ-ring, where $\sigma$is defined on $Z \times Z \; by \; \sigma(x,y) = (y,x)$: It can be easily proved that $Z \times Z$ is a σ-prime ring.

Definition 3.1.14

A non void set U of a ring R is called a Lie (resp. Jordan) sub ring R if U is an additive subgroup of R and for $x, y \in R \Rightarrow [x, y](resp\, x \circ y)$ is also in U.

Definition 3.1.15

An additive subgroup U of a ring R is said to be a Lie (resp. Jordan) ideal of R if whenever $u \in U \; and \; r \in R$ then $[u, r](resp(u \circ r))$ is also in U .

Example 3.1.4

Let.$R = \left\{ \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \middle| \, a, b, c \, \epsilon Z_2 \right\}$. Then it can be easily seen that $U = \left\{ \begin{pmatrix} a & 0 \\ b & 01 \end{pmatrix} \middle| \, a, b \; \epsilon Z_2 \right\}$. is a Lie ideal of R and

$J = \left\{ \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} \middle| \, b, \epsilon Z_2 \right\}$.is a Jordan ideal of R .

Definition 3.1.16

By $GF(q)$, we mean the Galois field of (finite field) with q elements and $(GF(q))_2$ the ring of all 2×2 matrices over $GF(q)$ and their elements are

$$e_{11}=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, e_{12}=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, e_{21}=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, e_{22}=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

Theorem 3.1.1 (Herstein, 1953)

Let R be a ring, and let $n>1$ be an integer. Suppose that $x^n - x \in Z(R) \forall\, x \in R$. Then R is a commutative ring.

Theorem 3.1.2(Bell and Kappe 1989)

Let R be a ring satisfying a polynomial identity of the form $[x, y] = p(x, y)$, where $p(X,Y) \in Z\langle X,Y\rangle$. Then the ring of polynomials in two non-commuting indeterminates over ring Z of integers has the following equivalent properties:

(i) $p(x, y)$ is the kernel of the natural homomorphism from $Z\langle X, Y\rangle$ *to* $Z(X, Y)$

(ii) the ring of polynomials in two non-commuting in determinates;

(iii) each monomial of $p(X,Y)$has total degree of at least 3

(iv) each monomial of $p\,(X,Y)$ has X -degree at least 2, or Y -degree at least1.

(v) R is commutative.

Corollary 3.1.1

Let l,m,n be non negative integers and R be a ring with unity 1. If for each $x, y \in R$, $\exists$ a polynomial $b(X)$ *in* $X^2Z[X]$ *such that* $[yx^m - x^n b(y)x^l, x] = 0$, then R is commutative.

Theorem 3.1.3 (Kezlan, 1982)

Let f be a polynomial in n noncommuting indeterminates $x_1, x_2, ..., x_n$ with relatively prime integral coefficients. Then the following statements are equivalent.

(i) For any ring R satisfying the polynomial identity $f = 0$, $C(R)$ is a nil ideal.

(ii) For every prime p, $(GF(p))_2$ fails to satisfy $f = 0$.

(iii) Every semi prime-ring R satisfying $f = 0$ is commutative.

Theorem3.1.4 (Streb, 1989)

Let R be a noncommutative ring. Then there exists a factorsubring of R which is of the (i), (ii),(iii),(iv), and (v) types below:

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}$, p a prime.

(i)$_{(a)}$ $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime. (i)$_{(b)}$ $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}$, p a prime.

(ii) $m_\sigma(F) = \{\begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} | \, a, b \in F\}$, where F is of finite dimension withanontrivial automorphism σ.

(iii) A noncommutative division ring.

(iv) $S = \langle 1 \rangle + T, T$ is a noncommutative radical subring of S, which must be a domain.

(v) $S = \langle 1 \rangle + T, T$ isa noncommutative subring of$S$ such that$T[T,T] = [T,T]T = 0$.

Our method of proof uses some iteration techniques which are based on the following theorem due to Tong (1984).

Theorem 3.1.5(Tong, 1984)

Let R be a ring with unity 1. For all $x, y \in R$ define I_p^k recursively as

$I_0^k(x) = x^k$,and$I_p^k(x) = I_{p-1}^k(x+1) - I_{p-1}^k(x)$, if $p \geq 1$, then

$I_{p-1}^k(x) = \frac{1}{2}(k-1)k! + k!\,x$;and$I_j^k(x) = 0$ for $j > k$.

3.2 Results on Semiprime Rings

A well known result in ring theory of Herstein (1961) which gives condition on the commutators $xy - yx$ of a ring that is necessary and sufficient for commutativity of R is as follows: If, for each $x, y \in R$, there exists a natural number $n = n(x,y) > 1$ depending on x and y, and R satisfying the condition $[x,y]^n = [x,y]$.

Bell in (1973) proved that a ring R is generated by n^{th} power of its elements and satisfying the polynomial identity $[x^n, y] = [x, y^n]$ for all $x, y \in R$ and a fixed integer $n > 1$.

Harmanci (1977) proved the commutativity of a ring with unity 1 satisfying $[x^k, y] = [x, y^k], k = n, n+1$.

Gupta (1980) extended it for the commutativity of a semi prime ring R with unity 1 when R satisfies $\left[x^m, y\right]-\left[x, y^m\right] \in Z(R)$ and $\left[x^{m+1}, y\right]-\left[x, y^{m+1}\right] \in Z(R)\, x, y \in R$.

Khan et al. (2013) established the commutativity of semiprime ring R satisfying the following ring property: There exists a positive integer $m > 1$ such that$[x^n, y^m]k \pm [x^m, y], x] = 0$, $\forall x, y \in R$ and k, n are natural numbers.

One can ask a natural question: "Is this result valid for case of Jordan product,

$(x^n o\, y^m)^k \pm (x^m y)^k$ $where$ $(x\, o\, y) = x\, y + y\, x$? We answer this question and prove the result for semiprime ring without identity.

For any $x, y \in R$ and k and n are natural numbers, we define the following ring properties:

(P_1) there exists a positive integer $m > 1$ such that

$[(x^n o y^m)^k \pm (x^m \text{oy}), x] = 0$;

(P_2) there exists a positive integer $m > 1$ such that

$[(x^n \text{oy}^m)^k \pm (x^m \text{oy}), \text{y}] = 0$.

Theorem 3.2.1

Let R be a semiprime ring. Then the following statements are equivalent:

(a) R satisfies (P_1),

(b) R satisfies (P_2),

(c) $R$is a commutative ring.

Proof. Since a semiprime ring R is isomorphic to a sub direct sum of prime rings R_α, each of α, which as a homomorphic imageof R satisfies the hypothesis (P_1)or (P_2) placed on R. So, we may assume that the ring R is prime satisfying one of the conditions (a) and (b). Given that$R$ satisfies (c), then (c)$\Rightarrow$(a) and (c)$\Rightarrow$(b) as a commutative ring R satisfies (a) and (b). Next, we claim that (a) $\Rightarrow$ (c) and (b) $\Rightarrow$(c).

In order to end theabove proof, we break it in two steps.

Step 3.2.1. If R be a prime ring satisfying one of the conditions (a) and (b)

then R has no nilpotent elements of R form an ideal.

Now, assume w is an element of$R$ with $w^2 = 0$but $w \neq 0$. Let R satisfy (a). Replacingwx for x and $w\ x\ w$ for y and using the fact that $t^2 = 0$, we have

$[((wx)^m o\ wxw) \pm((wx)^n o\ (wxw)^m)^k, wx] = [(wx)^m\ wxw, wx] = (wx)^{m+3} = 0\ \forall x \in R.$

Next, if R satisfies (b), then by setting wxw for x and $xw$for y, we obtain

$[(wxw\ o\ (xw)^m) \pm((wxw)^n o\ (xw)^m)^k, xw] = [\ wxw\ (xw)^m, xw] = (wx)^{m+3} = 0\ \forall\ x \in R.$

If $w\,R \neq 0$, then this shows that $w\,R$ is nonzero nil right ideal satisfying $t^{m+3} = 0\ \forall t \in w\,R$. By an application of a result of Levitzki (Herstein, 1968), rules this out and hence $w\,R = \{0\}$, and primeness of R yields $w = 0$.

Step 3.2.2 Suppose that R is a prime ring satisfying one of the conditions (a) and (b) of Theorem 3.2.1.Then the commutator ideal of R is nil.

If again R satisfy (a).Then we have

$[(x^n o\ y^m)^k \pm (x^m o\ y), x] = 0.$

This is a polynomial identity with co-prime integral coefficients. But no ring of 2×2 matrices over GF (p), p a prime, satisfies the property (a), by taking $x = e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $y = e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. In view of Theorem 3.1.3, the commutator ideal of R is nil.

From Step 3.2.2, the commutator ideal of R is nil. In view of Step 3.2.1, R has no nonzero nilpotent elements and hence R is commutative. This shows (a) $\Rightarrow$ (c).

Using the same way as above, we can prove (b) $\Rightarrow$ (c).

Remarks 3.2.1

(i) The ring of 3×3 strictly upper triangular matrices over some field provides an example to show that the semiprime in Theorem 3.2.1 is not superfluous.

(ii) The ring of $n\times n$ strictly upper triangular matrices over an associative ring satisfies the hypothesis of Theorem 3.2.1, but need not be commutative for $n > 2$. So, Theorem 3.2.1 is not valid for arbitrary rings.

3.3 Some Results on Ring with Unity

A well known result due to Herstein (1968) asserts that a ring R satisfying the polynomial identity $(x + y)^n = x^n + y^n$ must have a nil commutator ideal. Clearly, a ring R satisfying the polynomial identity $(x + y)^n = x^n + y^n$ also satisfies $[x^n, y] = [x, y^n]$. Motivated by this observation, Bell (1978) proved commutativity of n-torsion free ring with unity 1 satisfying the identity $[x^n, y] = [x, y^n]$.

In this context, we define the following ring property.

(P_3) For every $y \in R$, there exist polynomials $f(X), g(X), \text{and } h(X) \in \mathbf{Z}[X]$ such that

$x^p[x^n, y]\, y^q = f(y)\,[x, y^t g(y)]^k\, h(y)$ where $p \geq 0$, $q \geq 0$, $t \geq 0$, $n > 1$, $k > 1$ are fixed.

$Q(n)$ For any positive integer n and $x, y \in R$, $n[x, y] = 0$ implies $[x, y] = 0$.

Remark 3.3.1

Several authors have investigated commutativity of rings satisfying various special cases of the above property. This result extended many well-known commutativity theorems.

We again begin with the following known results.

Lemma 3.3.2 (Jacobson, 1964)

If $x, y \in R$ such that $[x, y]$ commutes with x, then for any $r \geq 1$, $[x^r, y] = r\, x^{r-1}\, [x, y]$.

Lemma 3.3.3 (Bell, 1982)

Let Rbe a ring with unity 1 and $f: R \to R$ any polynomial function of two variables with the property $f(x+1, y) = f(x, y)$ for all $x, y \in R$. If, for all $x, y \in R$, $x^m f(x, y) = 0$ or $f(x, y)\, x^m = 0$ for a fixed integerm, then necessarily $f(x, y) = 0$.

Lemma 3.3.4(Bell, 1989)

Let R be a ring (may be without unity 1) and for each $x, y \in R$, there exists a polynomial $f(X) \in X\,\mathbf{Z}[X]$ such that $[x, y] = [x, y]\, f(x)$. Then R is a commutative ring

Theorem 3.3.1

Let R be a ring with unity 1 and R satisfies the properties (P_3),$Q(n)$ and atleast one of p, q is zero. Then R is a commutative ring.

We prove the following results which will be used in proving Theorem 3.3.1.

Lemma 3.3.1

Let R be a ring satisfying(P_3) with at least one of p, q being zero. Then the commutator ideal is nil.

Proof. Replacingxwithx + y in the condition (P_3) and using (P_3), we obtain

$$(x+y)^p [(x+y)^n, y] y^q = x^p[x^n, y] y^q \quad (3.3.1)$$

Equation (3.3.1) is a polynomial identity and if $p = 0$ in (3.3.1), then we observe that $x = -e_{11} + e_{21}$ and $y = e_{11}$fail to satisfy the equality in the ring of 2×2 matrices over GF (p), p a prime. In view of Kezlan (1982) result, we get the commutator ideal ofRto be nil.

Next, if $q = 0$ in $(3.3.1)$, then $x = -e_{11} + e_{12}$, and $y = e_{11}$yields the required result.

Lemma 3.3.2

Let R be a ring with unity 1 satisfying(P_3) and R has the property $Q(n)$ with $q = 0$. Then $N(R) \subseteq Z(R)$.

Proof Let$b \in N(R)$. Then there exists a positive integer s such that

$b^s \in Z(R)$,for all $s \geq r, r$ is minimal. $(3.3.2)$

If $r = 1$, then the result is trivial. Suppose $r > 1$. Replacing x withb^{r-1} in (P_3), we find that $b^{(r-1)p}[b^{(r-1)n}, y] y^q = f(y) [b^{(r-1)}, y^t g(y)]^k h(y)$.

In view of $(3.3.2)$ and the fact that $(r-1)n \geq r$, it gives

$f(y) [b^{(r-1)}, y^t g(y)]^k h(y) = 0.$ $(3.3.3)$

Again, replacing x with $1 + b^{r-1}$ in (P_3) and using $(3.3.3)$, we get

$$(1+ b^{(r-1)})^p[(1 +b^{(r-1)})^n, y] y^q = 0. \qquad (3.3.4)$$

Since $(1+ b^{(r-1)})$ is invertible, $(3.3.4)$ becomes

$$[(1 +b^{(r-1)})^n, y] y^q = 0 \text{ for all } y \in R. \qquad (3.3.5)$$

Replace y by $1 + y$ in $(3.3.5)$ and use Lemma 3.3.3 to get

$$[(1 +b^{(r-1)})^n, y] = 0 \text{ for all } y \in R.$$

From $(3.3.2)$, the last equation implies that $[1 + n\, b^{(r-1)}, y] = 0$ which implies that $n\,[b^{(r-1)}, y] = 0$ for all $y \in R$.

By an application of the property $Q(n)$ we get $[b^{(r-1)}, y] = 0$ and $b^{(r-1)} \in Z(R)$.

This contradicts the minimality of r in (3.3.2) and hence $r = 1$ and $b \in Z(R)$.

Proof of Theorem 3.3.1

Combining Results $3.3.1$ and 3.3.2, we find that

$$C(R) \subseteq N(R) \subseteq Z(R). \qquad (3.3.6)$$

Replacing x by $x + 1$ in the property (P_3), we get

$$(1 +x)^p[(1 + x)^n, y] y^q = f(y) [1 + x, y^t g(y)]^k h(y) = f(y) [x, y^t g(y)]^k h(y)$$
$$= x^p[x^n, y] y^q.$$

This implies that $((1 +x)^p[(1 + x)^n, y] - x^p[x^n, y]) y^q = 0.$

Replacing y by $y + 1$ in the last equation and using Lemma 3.3.3, we get

$(1+x)^p[(1+x)^n, y] = x^p[x^n, y]$.

By an application of Lemma 3.3.2 and $(3.3.6)$, yields

$n(1+x)^{p+n-1}[(1+x), y] = n\,x^{p+n-1}[x, y]$.

This implies that

$n[x, y]\{(1+x)^{p+n-1} - x^{p+n-1})\} = 0. (3.3.7)$

Using the property $Q(n)$ in $(3.3.7)$, we get

$[x, y]\{(1+x)^{p+n-1} - x^{p+n-1})\} = 0.$

This polynomial identity can be re-written in the form

$[x, y] = [x, y]\,h(x)$ for some $h(X) \in XZ[X]$. By an application of Lemma 3.3.4, we get the required result.

Theorem 3.3.2

Let $n > 1$, $k > 0$, $p \geq 0$, $q \geq 0$, $t \geq 0$ be fixed non negative integers and let R be a ring with unity 1 in which for every $y \in R$ there exist polynomials f(X), g (X), and h(X) $\in \mathbf{Z}[X]$ such that $x^p[x^n, y]\,x^q = f(y)\,[x, y^t g(y)]^k\,h(y)$, for all $x \in R$. Further, if at least one of p, q is zero and R satisfies the property Q (n), then R is commutative.

We prove the following results which will be used in proving Theorem 3.3.2.

Lemma 3.3.4

Let n >1, k >0, p ≥ 0, q ≥ 0, t ≥ 0 be fixed non negative integers and let R be a ring in which for every y ∈ R there exist polynomials f(X), g (X), and h(X) ∈**Z**[X] such that $x^p[x^n, y]\, x^q = f(y)\,[x, y^t\, g(y)]^k\, h(y)$, for all x ∈ R If at least one of p, q is zero, then the commutator ideal is nil.

Proof Replacing x by x + y in the condition (P) and using (P), we get

$(x+y)^p[(x+y)^n, y](x+y)^q = x^p[x^n, y]\, x^q$

This is a polynomial identity and if p = 0, then we find that

$x = -e_{11} + e_{21}$ and $y = e_{11}$ fail to satisfy the equality in the ring of 2 × 2 matrices over GF (p), p a prime.

In view of Theorem 3.1.3, we get the required result.

Next, if $q = 0$, then select $x = -e_{11} + e_{12}$, and $y = e_{11}$ to get the required result.

Lemma 3.3.5

Let $n > 1, k > 0, p \geq 0, q \geq 0, t \geq 0$ be fixed non negative integers and let R be a ring with unity 1 in which for every y ∈ R there exist polynomials f(X), g (X), and h(X) ∈**Z**[X] such that $x^p[x^n, y]\, x^q = f(y)\,[x, yt\, g(y)]^k h(y)$, for all $x \in R$. Moreover, if R satisfies the property Q(n), then $N(R) \subseteq Z(R)$.

Proof Let $u \in N(R)$. Then there exists a positive integer r such that $u^t \in Z(R)$, for all $t \geq r$, where r is minimal. (3.3.8)

If r = 1, then the result is trivial. Therefore, assume that r > 1. Replacing x by u^{r-1} in the given property, we get

$u^{(r-1)p}[u^{(r-1)n}, y]\, u^{q(r-1)} = f(y)\,[u^{(r-1)}, y^t g(y)]^k\, h(y)$.

By application of (3.3.8) and the fact that $(r-1)n \geq r$, we get

$$f(y)\,[u^{(r-1)}, y^t g(y)]^k\, h(y) = 0. \qquad (3.3.9)$$

Again, replacing x by $1 + u^{r-1}$ in the hypothesis, we get

$$(1+ u^{(r-1)})^p[\,(1 + u^{(r-1)})^n, y]\,(1+ u^{(r-1)})^q = f(y)\,[u^{(r-1)}, y^t g(y)]^k\, h(y).$$

Thus in view of (3.3.9), we find that

$$(1+ u^{(r-1)})^p[\,(1 + u^{(r-1)})^n, y]\,(1+ u^{(r-1)})^q = 0.$$

But since $(1+ u^{(r-1)})$ is invertible, the last relation implies that

$$[(1 + u^{(r-1)})^n, y] = 0 \text{ for all } y \in R. \qquad (3.3.10)$$

Combining (3.3.8) and (3.3.10), we get

$$0 = [(1 + b^{(r-1)})^n, y] = [1 + n\, u^{(r-1)}, y] = 0.$$

This implies that $n\,[u^{(r-1)}, y] = 0$ for all y in R.

By an application of the property Q(n) we get $[u^{(r-1)}, y] = 0$. This implies $u^{(r-1)} \in Z(R)$.

This contradicts the minimality of r in (3.3.8) and hence r = 1and $u \in Z(R)$.

Proof of Theorem 3.3.2

Combining Results 3.4.1 and 3.4.2, we find that

$C(R) \subseteq N(R) \subseteq Z(R)$. (3.3.11)

Replacing x by x +1 in the property (P_3) of page 31, we get

$(1+x)^p[(1+x)^n, y](1+x)^q = f(y)[1+x, y^t g(y)]^k h(y) = f(y)[x, y^t g(y)]^k h(y)$

$= x^p[x^n, y] x^q$

By application of Lemma 3.3.2 and (3.3.11) we get

$n[x, y](1+x)^{p+q+n-1}) = n[x,y] x^{p+q+n-1})$

This implies that

$n[x, y]\{(1+x)^{p+q+n-1}) - x^{p+q+n-1}\} = 0,$

by an application of Q(n) we get

$[x, y]\{(1+x)^{p+q+n-1} - x^{p+q+n-1})\} = 0.$

This polynomial identity can be re-written in the form

$[x, y] = [x, y] h(x)$ for some $h(X) \in XZ[X]$.

Hence, by Lemma 3.3.4, R is commutative.

Corollary 3.3.1 (Khan, 2010)

Let $n > 1$, $s = \geq 0$, $t = \geq 0$, be fixed non-negative integers and let R be a ring with unity 1 in which for every y in R there exist integers $p = p(y) \geq 0$, $q = q(y) \geq 0$, $m = m(y) \geq 0$ such that $x^s[x, y] x^t = \pm y^p[x^n, y^m]^r y^q$. Then R is commutative if at least one of s or t is zero.

Corollary 3.3.2

Let R be a ring with unity 1 in which for each y in R there exist

$x^p[x^n, y]x^q = y^r[x, y^m]y^s \forall\ x \in R$, where $n > 1$, $m>1$, $p \geq 0$, $q \geq 0$, $r \geq 0$, $s \geq 0$ are fixed non negative integers. Then R is commutative if R satisfies the property Q(n) and at least one of p and q is zero.

Remark 3.3.2

As a consequence of Corollary 3.3.2 with $p = q = r = s = 0$and $m = n$, then we get the following result.

Corollary 3.3.3(Bell,1978)

Let R be an n-torsion free ring with unity 1 and $n > 1$ be a fixed positive integer. Let R satisfy $[x^n, y] = [x, y^n]$. Then R is commutative.

Corollary 3.3.4

Let R be a ring with unity 1 in which for each y in R there exist integers m and f (t), g(t), h(t) in Z [t] such that $x^m[x, y] = f(y)[x, y^2g(y)]h(y)$. Then R is commutative (and conversely).

Remark 3.3.3

Theorem 3.3.1 is false for rings without identity 1 because any nilpotent ring of index ≤ 4 and nil ring of index 2 will easily satisfy either (P) or ((P_3) but such rings need not be commutative (see for reference (Gupta, 1970)).

Remark 3.3.4

The following example strengthens the existence of the property Q(n) in the hypotheses of Theorem 3.3.1 and Theorem 3.3.2.

Example 3.3.2

Take $R\left\{\begin{bmatrix} \alpha & \beta & \gamma \\ 0 & \alpha & \delta \\ 0 & 0 & \alpha \end{bmatrix} \middle| \alpha, \beta, \gamma, \delta \in GF(2)\right\}$. Then R is a non commutative ring with unity satisfying the property $[x^2, y] = [x, y^2]$.

Remark 3.3.5

The following example shows that the existence of unity 1 in the hypotheses of Theorem 3.3.1 and theorem 3.3.2 are essential.

Corollary 3.3.5(Khan, 2010)

Let R be a ring with unity 1 in which $\forall\, x, y \in R$ there exist integers $n \geq 0, s \geq 0, t \geq 0, p \geq 0, r > 0, m > 1$ such that $x^s[x, y]\, y^t = \pm\, y^p[x^n, y^m]^r\, y^q$.

Then R is commutative if at least one of p or q is zero.

Corollary 3.3.6

Let R be a ring with unity 1which satisfies the property

$x^p[x^n, y]\, y^q = y^r[x, y^m]\, y^s \ \forall\, x, y \in R$, for fixed integers $n > 1,\ p \geq 0, q \geq 0.$

Then R is commutative if R satisfies the property $Q(n)$ and at least one of p or$q$ is zero.

Corollary 3.3.7

Let R be a ring withunity 1 in which for each $y \in R$ there exist integers m and $f(t), g(t), h(t) \in Z[t]$ such that $x^m[x, y] = f(y)[x, yg(y)]h(y)$ Then R is commutative (and conversely).

Remark 3.3.6

The following example shows that the existence of unity 1 in the hypothesis of Theorem3.3.2is essential. 3

Example 3.3.3

Let D_m be the ring of all $m \times m$ matrices over a division ring D and $B_n = \{(b_{ij}) \in D_m | b_{ij} = 0, i \geq j\}$.Then B_3 is a non commutative ring of index3 which satisfies the identity $[x^m, y] = [x, y^m]$ for all $x, y \in B_3$ and $m \geq 2$.

Remark 3.3.7

From the above example one can observe that Theorems 3.3.1 and 3.3.2 cannot be extended to arbitrary rings.

3.4 Some Results on s-unital Rings

In this section, we investigate the commutativity of certain class ofrings that is the class containing rings with unity (s-unital ring, torsion free ring) which may be a kind of methodology, which is a technique of Streb's classification with Chacron's condition. For each $x, y \in R$ there exist $b(t), g(t) \in t^2 Z[t]$ such that $[x - g(x), y - b(x)] = 0$. where $b(t), g(t)$ are polynomials identities in $Z[t]$.

As above consequences, Theorems 3.3.1and 3.3.2 require that the ring contain a unity, but a ring need not have a unit and researchers still want to be able to work with them. In an effort to have a class of rings that are close to having a unit, the concept of s −unital was developed.

Definition 3.4.1

A ring R is called left (resp. right) s −unital if $x \in Rx$ $(resp.\ x \in xR)$ for each $x \in R$. Further, R is called s −unital if it is both left as well as right s −unital, that is, $x \in xR \cap Rx$ for each $x \in R$.

Definition 3.4.2

If R is s-unital (resp. left, or right s-unital),then for any finite subset F of R, there existsanelement $e \in R$ such that $ex = xe = x$ (respectively, $ex = x$or $xe = x$) $\forall\, x \in F$.Such an element e is called the pseudo (resp. Pseudo left or pseudo right) identity of F in R.

Remark 3.4.1

The existence of non-commutative R with R^2 being central rules out the possible generalization of the Theorems 3.3.1and 3.3.2 for arbitrary rings.

In this sequel, Example 3.3.3 strengthens the existence of unity 1 in the hypothesis of the Theorems3.3.1, and 3.3.2; nevertheless the same theorem may be extended to one sided s-unital rings. There are several examples in the existing literature (see.(Hirano et al, 1982)), which shows that these classes of rings are the generalizations of the class of rings with unity 1, particularly a number of commutativity theorems have been extended to one sided s −unital rings.

We begin with the following theorem.

Theorem 3.4.1

Let R be a left s −unital ring satisfying (P_3) with $q = 0$. If R satisfies the property $Q(n)$, then R is commutative.

Proof Let R be a left s −unital ring satisfying $x^p[x^n, y]y^q = f(y)\,[x, y^t g(y)]^k\, h(y)$ for some $f(X), g(X)$, and $h(X) \in \boldsymbol{Z}[X]$. Choose an element $e \in R$ such that $ex = x, ey = y$. Replacing x by e in(P_3), we have $e^p[e^n, y] = f(y)\,[e, y^t g(y)]^k\, h(y)$ for some $f(X), g(X), h(X) \in \boldsymbol{Z}[X]$ and fixed integers $n > 1, k > 0, p \geq 0, t > 0$. This implies that $y = ye^n \in yR$. Hence R is right s −unital and then s −unital. In view of Proposition 1 of (Hirano et al., 1982) we can assume that R has unity 1 and hence by Theorem 3.1, we get the required result.

Similarly, we can prove the following result.

Theorem 3.4.2

Let R be a right $s-$unital ring satisfying (P_3) with $p = 0$. If R satisfies the property $Q(n)$, then R is commutative.

Corollary 3.4.1

Let $m > 1, k > 0, l = m + t, p, q, r, n, s, t$ be fixed non negative integers and R a left (right) $s-$unital ring in which for every $y \in R$ there exists nonnegativeintegers $r = r(y), s = s(y), m = m(y)$ suchthat

$$x^p[x^n, y] = y^r[x^n, y^l]^k y^s$$

$\left(\text{resp.}\ [x^n, y]x^p = y^r[x^n, y^l]^k y^s\right)$. Then R is is a commutative ring.

Remark 3.4.2

The following example shows that a right (resp. left) $s-$ unital ring with the conditions $x^p[x^n, y] = \pm y^r[x^n, y^l]^k y^s (resp.\ [x^n, y]x^p = \pm y^r[x^n, y^l]^k y^s)$ need not be commutative.

Example 3.4.1

Let

$$R_1 = \left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}, \begin{pmatrix}0 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right\}$$

$$(resp.\ R_2 = \left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\right\}$$

be a subring of 2×2 matrices over $GF(2)$. Then R_1 (resp. R_2) is a noncommutative right (resp. left) $s-$unital ring

satisfyingthecondition $x^p[x^n,y]=\pm y^r[x^n,y^l]^k y^s$(resp$[x^n,y]x^q=\pm y^r[x^n,y^l]^k y^s$) for any fixed positive integers $n>1, m>1, l=m+t,\ p,\ q,\ r,\ s,t$. Indeed, both R_1 and R_2 have the property $Q(n)$ for odd n.

3.5 Results for Some Special Classes of Rings

In this section, we discuss the commutativity of somespecial class of rings (that is, ring with unity, s-unital ring, torsion free ring), satisfying one of the following properties:

(i) $\{1-p(yx^m)\}[yx^m-b(x^m y),x]\{1-p(yx^m)\}$and

$\{1-p(xy^m)\}[y^m x-b(x^m y),y]\{1-p(xy^m)\}$

(ii) $y^r[x^n,y]=g(x)[x^q f(x),y]^k h(x)$or $[x^n,y]y^r=g(x)[x^q f(x),y]^k h(x)$.

where $b(X),b'(X)\in X^2Z[X]$,$p(X),p'(X)$,$q(X),q'(X)\in Z[X]$,and $m\geq 0, n\geq 1,\ r\geq 0, q\geq 2, k>1$are integers.Further, we extend these results to the case when the integral exponents in the underlying conditions are no longer fixed; rather they depend on the pair of ring elements x,y for their values.

Now consider the following conditions:

(c_1)For each $x,y\in R$, a fixed n integer $m\geq 1$ and there exist polynomials

$b(X),b'(X)\in X^2Z[X]$and$p(X)$,$p'(X),q(X),q'(X)\in Z[X]$such that

$\{1-p(yx^m)\}[yx^m-yx^m b(x^m y),x]\{1-q(yx^m)=0;$and

$\{1-p`(yx^m)\}[y^m x - y^m xb(xy^m), y]\{1-q`(xy^m)\} = 0,$

(c_2) For each $x, y \in R$, there is an integer $m \geq 0$ and the polynomials $b(X), b'(X) \in X^2 Z[X]$and (X) ,$p'(X), q(X), q'(X) \in Z[X]$ such that

$\{1-p(yx^m)\}[yx^m - yx^m b(x^m y), x]\{1-q(yx^m)\} = 0$; and

$\{1-p`(xy^m)[y^m x - y^m xb`(xy^m), y]\{1-q`(xy^m) = 0.$

(c_3) Foreach $x, y \in R$ thereexistpolynomials

$f(X), g(X), h(X)\tilde{f}(X), \tilde{g}(X), \tilde{h}(X) \in Z[X]$such that

$[x^n, y^m] = g(x)[x^q f(x), y]h(x)$and $[x^n, y^m] = \tilde{g}(x)\left[x^q \tilde{f}(x), y\right]\tilde{h}(x)$,where $m > 1, q \geq 2, n \geq 1$ are fixed integers.

(c_4) Foreach $x, y \in R$,thereexistpolynomials

$f(X), g(X), h(X)\tilde{f}(X), \tilde{g}(X), \tilde{h}(X) \in Z[X]$such that

$y^r[x^n, y] = g(x)[x^q f(x), y]h(x)$and $[x^n, y]y^r = \tilde{g}(x)\left[x^q \tilde{f}(x), y\right]\tilde{h}(x)$,where $n \geq 1, q \geq 2, r \geq 1$ are fixed integers.

(CH) For each $x, y \in R$there exist $g(t), h(t) \in t^2 Z[t]$such that $[x - g(x), y - h(x)] = 0.$

(H) For each $x, y \in R$there exist $g(t), h(t) \in t^2 Z[t]$such that $[x - g(x), y] = 0.$

Herstien (1955) proved that if R is a ring satisfying the property (H) then R is commutative.

It is natural to consider the related properties; $[xy - p(xy), x] and [xy - q(xy), x] = 0$ for some $p(X), q(X) in X^2 Z[X]$ depending on the ring elements x, y.

Putcha and Yaqub (1973) remarked that if for every $x, y \in R$, there exists a polynomial $p(X) \in X^2 Z[X]$ such that $xy - p(xy)$ is central, then R^2 must be central.

It is a natural question one can ask, that what can we say about the commutativity of ring R, if the underlying conditions are replaced by $[yx^m - x^n b(y)x^{'}] = 0?$. We prove rather a more general result by establishing that a ring with unity satisfying the property (c_1) is commutative. Further we shall consider property (c_2), where integral exponents are allowed to vary with the pair of ring elements x, y and also the ring satisfying the Chacron's condition (CH). Finally, we extend the properties $(c_3)$and (c_4) to s-unital rings by using the iteration technique.

Consider the following types of rings.

(i) $\begin{pmatrix} GF(p) & GF(p) \\ 0 & 0 \end{pmatrix}, p$ a prime; $\begin{pmatrix} 0 & GF(p) \\ 0 & GF(p) \end{pmatrix}, p$ a prime; $\begin{pmatrix} GF(p) & GF(p) \\ 0 & GF(p) \end{pmatrix}, p$ a prime.

(ii) $m_\sigma(F) = \{\begin{pmatrix} a & b \\ 0 & \sigma(a) \end{pmatrix} | a, b \in F\}$, where F is finite with a non trivial automorphism σ.

(iii) A noncommutative division ring.

(iv) $S = \langle 1 \rangle + T, T$ a noncommutative radical sub ring of S, must be a domain.

(v) $S = \langle 1 \rangle + T, T$ a noncommutative subring of S, T[T,T]=[T,T]T=0.

In 1989, Streb gave a nice classification for non commutative rings which yields a powerful tool for obtaining a number of commutativity theorems. It follows from Streb (1989) that if R is a non commutative ring with unity, then there exists a factor sub ring of R which is of type (i),(ii),(iii),(iv) or (v).This observation gives the following proposition that plays an important role in our discussion.

Proposition 3.5.1(Komatsu and Tominaga, 1989)

Let p be a property which is inherited by factor sub rings. If no ring of type (i), (ii), (iii), (iv), or (v) satisfies p then every ring with unity 1 satisfying p is commutative.

Now we state the following known results.

Lemma 3.5.2 (Komatsu and Tominaga, 1989)

If R is a non commutative ring satisfying(CH), then there exists a factor subring of R which is of type (i), or (ii).

Lemma 3.5.3 (Herstein, 1955)

Let R be a ring in which for all $x, y \in R$, there exists polynomial $f(X) \in X^2 Z[X]$ such that $[x - f(x), y] = 0,$ then R is commutative.

Theorem 3.5.1

Let R be a ring with unity 1 satisfying (c_1). Then R is a commutative ring.

Proof In view of Proposition 3.3.1, it is enough to prove that R cannot be of type (i), (ii), (iii), (iv), (v). Choose some polynomials $b(X) \in X^2 Z[X]$ $p(X), q(X) \in X\, Z[X]$ such that $\{1 - p(yx^m)\}[yx^m - yx^m b(x^m y), x]\{1 - q(yx^m)\} = 0$.

Taking $w \in R \; such that \; b(x^m y) = x^m w \; and \; b(yx^m) = wx^m$. For some $b(X) \, in \, X^2 Z[X]$ and $p`(X), q`(X), b`(X) \, in \, XZ[X]$, we find that

$$\{1 - p`(b(yx^m))\}[b(x^m y) - b(yx^m)b`(b(yx^m)), x]\{1 - q`(b(yx^m))\} = 0 \quad (3.5.1)$$

and

$$\{1 - p`(wx^m)\}[x^m w - wx^m b`(wx^m), x]\{1 - q`(wx^m)\} = 0 . \quad (3.5.2)$$

Combining (3.5.1) and (3.5.2), we get

$$\{1 - p`(b(yx^m))\{1 - p(yx^m)\}[yx^m - yx^m b`(b(yx^m)), x]\{1 - q(yx^m)\}\{1 - q`(b(yx^m))\} = 0 .$$

This implies that

$$\{1 - p(yx^m)\}[yx^m - yx^m b(yx^m), x]\{1 - q(yx^m)\} = 0 . \quad (3.5.3)$$

Let R be a ring of type (i) suppose that R satisfies the property (c_1). Then it satisfies (3.5.3). Taking $x = e_{11}$ and $y = e_{21}$, we find that $\{1 - p(e_{21}e^m{}_{11})\}[e_{11}, e_{21}e^m{}_{11} - e_{21}e^m{}_{11}b(e_{21}e^m{}_{11})]\{1 - q(e_{21}e^m{}_{11})\} \neq 0,$ a contradiction.

Let the ring $R = M_\sigma(F)$ be of type (ii). If R satisfies the property (c_1), then choose $\begin{pmatrix} \alpha & 0 \\ 0 & 0(\alpha) \end{pmatrix}, (\alpha \neq \sigma(\alpha))$ and $y = e_{21}$, we get

$$\{1 - p(yx^m)\}[x, yx^m - yx^m b(yx^m)]\{1 - q(yx^m)\} = e_{12}(\alpha - \sigma(\alpha)) \neq 0 .$$

Thus R cannot be of type (ii).

Let R be a ring of type (iii). Suppose that R satisfies the property (c_1). Let u be a unit in R, that is $u \in U$, and for arbitrary element $y \in R$ we obtain polynomials $b(X) \in X^2Z[X]$ *and* $p(X), q(X) \in XZ[X]$ such that $\{1-p(yu^{-m}u^m)\}[yu^{-m}u^m - yu^{-m}u^m b(yu^{-m}u^m), u]\{1-q(yu^{-m}u^m)\} = 0$.

This implies that $\{1-p(y)\}[y-yb(y),u]\{1-q(y)\} = 0$.

This shows that either $1-g(y) = 0, 1-h(y) = 0, or\,[y-yf(y),u] = 0.$

In all the cases, R is commutative by Lemma 3.5.2, a contradiction.

Let R be of type (iv). Let R satisfy (c_1). Then a careful scrutiny of the proof of type (iii) gives that there exist x and $u \in U$ and arbitrary $y \in R$ such that either $[x, y-b(y)] = 0, y-yp(y) = 0\, or\, y-yq(y) = 0$ for all

$b(X) \in X^2Z[X]$ and $p(X), q(X) \in XZ[X]$. But in the present case if $t_1, t_2 \in T$ then $u = 1+t_1$ is a unit and there exist $b(X) \in X^2Z[X]$ and $p(X), q(X) \in XZ[X]$ such that either $[t_1 - q(t_2), 1+t_1] = 0, t_2 - t_2 p(t_2) = 0\, or\, t_2 - t_2 q(t_2) = 0$. Thus, in every case T is commutative by Lemma 3.5.3,a contradiction.

Further, let R be of type (v). Let $t_1, t_2 \in T$ such that $[t_1, t_2] \neq 0$ suppose that R satisfies (c_1). Then there exist polynomials $b(X) \in X^2Z[X], p(X), q(X) \in XZ[X]$ such that

$\{1-p(t_2(1+t_1)^m)\}[1+t_1, t_2(1+t_1)^m - t_2(1+t_1)^m b(t_2(1+t_1)^m)] = 0$.

This implies that $[t_1, t_2] = 0$, a contradiction.

Hence, one can see that no ring of type (i),(ii),(iii),(iv) or (v) satisfies (c_1) and an application of Proposition 3.5.1, R is commutative.

Corollary 3. 5.1

Let l, m, n be fixed non negative integers and R be a ring with unity 1. If for each $x, y \in R$, there exists a polynomial $b(X) \in X^2 Z[X]$ such that $[yx^m - x^n b(y)x^l, x] = 0$, then R is commutative.

Corollary 3.5.2

Let l, m, n be non negative integers and R be a ring with unity 1. If for each $x, y \in R$, there exists a polynomial $b(X) \in X^2 Z[X]$ such that $[yx^m - x^n b(y)x^l, x] = 0$, then R is commutative.

Remark 3.5.1

Given an integral exponent m with property (c_1) which is allowed to vary with the pair of elements x and y, that is, if R satisfies the property (c_2), then a careful scrutiny of the proof of Theorem 3.5.1 asserts that R has no factor subring of type (i), or (ii).

Further, if R satisfies the property of Chacrons (CH), in view of Lemma 3.5.3, we get the following result.

Theorem 3.5.2

Suppose R is a ring with unity 1satisfying(CH). Moreover, if R satisfies the property (c_2) and (c_3) then R is commutative (and conversely).

The following example demonstrates that in Theorem 3.5.1 and Theorem 3.5.2 the existence of both the conditionsare not superfluous (even if the ring R has unity 1).

Example 3.5.1

Let $R=\{\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} | a,b,c,d \in GF(2)\}$. Then R is a noncommutative ring with unity satisfying the condition $y^r[x,y^4]=x^r[x^4,y]x^8$ where r,s and $t$may be any non-negative integers.

In view of Example 3.5.1, one can ask a natural question:what additional conditions are needed to establish the commutativity of an arbitrary ring R if we simply assume $y^t[x,y^m]=g(x)[x^q f(x),y]h(x)$ and $[x,y^m]y^t=g`(x)[x^q f`(x),y]h`(x)$.

Naturally, it is a tempting conjecture that an m- torsion free ring with unity 1 satisfying any of the above properties must be commutative. Thus, under certain appropriate constraints on the commutators involved in the underlying condition, we can prove some interesting cases of the conjecture. Let $r>1, q\geq 2, n\geq 1$be fixed integers, then wedefine the properties below.

(a) For each $x,y \in R$, there exist polynomials$f(X),g(X),h(X) \in Z[X]$ such that$[x^n,y^r]=g(x)[x^q f(x),y]h(x)$ *or* $[x^n,y^r]=g`(x)[x^q f`(x),y]h`(x)$.

(b) For each$x,y \in R$, there exist polynomials$f(X),g(X),h(X) \in Z[X]$such that $y^r[x^m,y]=g(x)[x^q f(x),y]h(x)$or $[x^m,y]y^r=g(x)[x^q f(x),y]h(x)$.

To prove the commutativity of a ring with the above properties, we need some extra conditions on commutators in R such as the following property:

Q (m) For any $x, y \in R,\ m[x, y] = 0,$ where m is some positive integer.

An iteration technique was developed by Tong (1984). The method of Tong's proof originates the following generalization of the commutative law: $xy^m x^m y = x^{m+1} y^{n-1}$.

Indeed, we begin with the following lemmas:

Lemma 3.5.4(Tong, 1984)

Let R be a ring with unity 1, and $I_0^k(x) = x^k$ *if* $p \geq 1$, let $I_p^k(x) = I_{p-1}^k(x+1) - I_{p-1}^k(x)$.

For all $x \, in \, R,$ then $I_{k-1}^k(x) = \frac{1}{2}(k-1)k! + k!x;$

$I_k^k = k!$ *and* $I_j^k(x) = 0$ *for* $j > k$**.**

Lemma 3.5.5(Tong, 1984)

Let R be an s-unital ring and let k be a finite subset of R such that $f = f(K) \in R$, a pseudo identity of K. Let t be a positive integer. For all $x \in K$ define I_p^q recursively as $I_0^q(x) = x^q$, *and* $I_p^q(x) = I_{p-1}^q(f` + x) - I_{p-1}^q(x)$, if $p \geq 1$. Further,if $f` = f$, then

$I_{q-1}^q(x) = \frac{1}{2}(q-1)q!\, f + q!xI_p^q = q!\, f$ *and* $I_s^q(x) = 0, s > q$**.** (3.5.4)

Let $f \neq f$. Then

$I_{q-1}^q(x) = f^{t_1} + ... + f^{t_{(q-1)q!\frac{1}{2}}} + q!xI_q^q = q!\, f`$ *and* $I_s^q(x) = 0$ *for* $s > q$. (3.5.5)

Our aim is to extend the properties (a) and (b) of page 52 to s-unital rings by using the iteration technique.

We prove the following theorem.

Theorem 3.5.3

Let R be an s-unital ring satisfying any of the properties (a) and (b). Moreover, if R satisfies $Q((\max\{r,n\})!)$, then R is commutative.

Proof Let R satisfy (a).Then first we shall use induction on y^r. From Lemma 3.5.4 $I_k(x) = I_k^r(x)$ *for* $k = 0,1,2,3,4,...$ The condition (a) can be written as

$$[x^n, I_0(y)] = g(x)[f(x), y]h(x) \qquad (3.5.6)$$

Let $e \in R$ be a pseudo identity of $\{x, y\} \subseteq R$ and let t be a non-negative integer. Clearly $[x, y] = [x, e^t + y] = [e^t + x, y]$

Replacing y by $(e^t + y)$ in (3.5.6) and using Lemma 3.5.5 we get $[x^n, I_0(y) + I_1(y)] = g(x)[x^q f(x), y]h(x)$. Using(3.5.6), we get

$$[x^n, I_0(y)] = 0 \text{ for all } x, y \in R. \qquad (3.5.7)$$

Replacing y *by* $(e^t + x)$ and using Lemma 3.5.5, we obtain

$$[x^n, I_1(y+1)] = [x^n, I_1(y) + I_2(y)] = 0.$$

Again using (3.5.6), we get $[x^n, I_2(y)] = 0$.

Replacing y *by* $(e^t + x)$ and then iterating $(r-1)$ times, we find that

$$[x^n, I_{r-1}(y)] = 0. \qquad (3.5.8)$$

By Lemma 3.5.5, equation (3.5.8)gives $r![x^n, y] = 0$.

Finally, replacing x by $(e^t + x)$ and using the similar techniques as above we obtain $r!n![x, y] = 0$ and hence by the property $Q((max\{r, n\})!)$, weobtain the commutativity of R.

Let R satisfy (b). Then using the iteration technique as above, we find that $I_0(y)[x^n, y] = g(x)[x^q f(x), y]h(x)$ *or* $[x^n y]I_0(y) = g(x)[x^q f(x), y]h(x)$.

Replacing y by $(e^t + x)$ and using Lemma 3.5.5, we obtain $I_1(y)[x^n, y] = 0$ or $[x^n, y]I_1(y) = 0$.

Proceeding in the same line as above, we finally get $I_r(y)[x^n, y] = 0$ *or* $[x^n, y]I_r(y) = 0$.

Thus in both cases we get $r![x^n, y] = 0$.

Using the same way of replacing x by $(e^t + x)$ and iterating $(n-1)$ times we obtain $r![x^n, y] = 0$,

by the property $Q((max\{r, n\})!)$ we obtain the commutativity of R.

CHAPTER FOUR

DERIVATIONS ON SOME SPECIAL CLASS OF σ −RINGS

4.1 Definitions of Some Rings and Derivations

Definition 4.1.1

An additive mapping $x \to \sigma(x)$ on a ring B is called an involution if $\sigma(\sigma(x)) = x$ and $\sigma(xy) = \sigma(y)\sigma(x)$ hold for all $x, y \in B$. A ring equipped with an involution is called a ring with involution or σ −ring.

Example 4.1.1

Let $Q = \{x + yi + z j + wk \mid x, y, z, w \in B\}$ be the ring of real quaternions.

Define $\alpha \mapsto \beta$ of Q into itself as $\beta = x - yi - z j - wk$ where $\alpha = x + yi + z j + wk \in Q$.

It can be easily seen that `σ' is an involution of Q. Therefore Q is a σ −ring.

Definition 4.1.2

A ring R with involution σ is said to be σ −prime if aRb = aR$\sigma(b)$= {0},where a, b ∈ R (equivalently aRb = $\sigma(a)$Rb = {0}, where a, b ∈R) implies that either a = 0 or b = 0.

Remark 4.1.1

Clearly, every prime ring having an involution σ is a σ -prime but the converse needs not be true in general.

An ideal I of R is called aσ −ideal of R if σ(I) = I.

Remark 4.1.2

Let R be a σ-prime ring, a∈R such that aRa = {0}. This implies that

aRaR$\sigma(a)$= {0}. Now σ −primeness of R insures that a = 0 or aR $\sigma(a)$ = {0}.

Now aR $\sigma(a)$ = {0}together with aRa = {0}gives us a = 0.

This indicates that every σ-prime ring is semiprime.

Definition 4.1.3

Let B be a σ −ring. An additive mapping $d: B \to B$ is said to be right σ-derivation if $d(xy) = d(x)\sigma(y) + d(y)x$ for all $x, y \in B$.

(i) An additive mapping $H: B \to B$ is said to be a generalized right σ-derivation if there exists a right σ-derivation d $\ni H(xy) = H(x)\sigma(y) + d(y)x \forall$ x,y ∈ B.

Definition 4.1.4

An additive mapping $d: B \to B$ is called a σ-derivation (resp. Jordan σ −derivation) if $d(xy) = d(x)\sigma(y) + xd(y)$ (resp.d$(x^2) = d(x)\sigma(x) + xd(x)$) for all $x, y \in B$.

Definition 4.1.5

An additive mapping $H: B \to B$ is said to be a generalized σ-derivation (resp. generalized Jordan σ-derivation) if there exists a σ-derivation (Jordan σ-

derivation)such that$H(xy) = H(x)\sigma(y) + xd(y)$(resp. $H(x^2) = H(x)\sigma(x) + xd(x)$) for all $x, y \in B$. Motivated by the definition of σ-derivation and generalized σ-derivation we introduce the notion of right σ-derivation and generalized right σ-derivation as follows:

Braser (1991) defined the notion of right derivation as follows: An additive mapping d: $B \to B$ is called a right derivation (resp. right Jordan derivation) if $d(xy) = d(x)y + d(y)x$ (respectively Jordan right derivation in case $d(x^2) = 2d(x)x$) holds for all $x, y \in B$. An additive mapping d: $B \to B$ is said to be a left derivation (resp. Jordan left derivation) if $d(xy) = d(x)y + d(y)x$ (resp. $d(x^2) = 2xd(x)$) holds for all $x, y \in B$. Obviously, every right(resp. left) derivation on a ring B is a Jordan right (resp. left) derivation but the converse needs not be true in general.

Dhara and Shrama (2009) proved the following theorem

Theorem 4.1.1

Let $n \geq 1$ be a fixed integer and let B be an $(n+1)!$-torsion free ring. If $H: B \to B$ and $d: B \to B$ are two additive mappings satisfying $H(x^{n+1}) = H(x)x^n + xd(x)x^{n-1} + x^2d(x)x^{n-2} + \cdots + x^nd(x)$ for all $x \in B$. Then d is a Jordan derivation and H is a generalized Jordan derivation.

Remark 4.1.3

The concept of generalized right σ-derivation cover the concept of right σ-derivation. A generalized right σ-derivation with d = 0 includes the concept of left σ-centraliser (that is, an additive mapping $T: B \to B$ satisfies $T(xy) = T(x)\sigma(y))\forall\, x, y \in B$.

From the definition of right σ-derivation and generalized right σ-derivation on a σ −ring, it is easy to introduce the concepts of Jordan right σ −derivation and generalized Jordan right σ-derivation on a σ-ring B as follows:

Definition 4.1.6

An additive mapping $d: B \to B$ is said to be Jordan right σ-derivation if $d(x^2) = d(x)\sigma(x) + d(x)x$ for all $x \in B$.

Definition 4.1.7

An additive mapping $H: B \to B$ is said to be a generalized Jordan right σ-derivation if there exists a right σ-derivation d such that $H(x^2) = H(x)\sigma(x) + d(x)x$ for all $x \in B$.

Suppose $H: B \to B$ and $d: B \to B$ are two additive mappings satisfying

$$H(x^{n+1}) = H(x)[\sigma(x)]^n + [\sigma(x)]^{n-1}d(x)x + [\sigma(x)]^{n-2}d(x)x^2 + \cdots + d(x)x^n, \forall\, x \in B. \quad (*)$$

In view of Theorem 4.1.1, it is natural to ask for the additive mappings satisfying (*), i.e., that Jordan right σ-derivation and generalized Jordan right σ-derivation.

Dhara et al. (2011) proved the identity $H(x^{m+n+1}) = H(x)x^{m+n} + x^m d(x)x^n$ for x in a non-central Lie ideal of a prime ring B, where both H and d are generalized

derivations of B and then determined the relationship between the structures of H and d.

We improve Theorem 4.1.1 in the setting ofσ-ring and also that has the same flavour as Dhara et al. (2011). In fact, we will find a condition that leads H and d to be respectively, a generalized Jordan right σ −derivation and a Jordan right σ −derivation(i.e., it is shown that additive mappings H, d: B→ B on an $(n+1)!-$ torsion free σ-ring B satisfying (*), implies $H(x^2) = H(x)\sigma(x) + d(x)x$ and $d(x^2) = d(x)\sigma(x) + d(x)x$ for all x ∈ B).

Lemma 4.1.2

Let M be a ring with involution σ, then $\sigma(e) = e$, where e is the identity element of M.

Proof. Since$e = \sigma\big(\sigma(e)\big) = \sigma(\sigma(e)e) = \sigma(e)e = \sigma(e)$. This yields the required result.

Theorem 4.1.3

Suppose M is an$(n+1)!$ −tortion free σ-ring with identity element e and $n \geq 1$ is a fixed integer. If $G, \delta: M \to M$ are two additive mappings satisfying

$G(x^{n+1}) = [\sigma(x)]^n G(x) + [\sigma(x)]^{n-1} x\delta(x) + [\sigma(x)]^{n-2} x^2\delta(x) + \cdots + x^n\delta(x)$for all $x \in M$, then δ is a Jordan left σ −derivation and G is a generalized Jordan left σ −derivation on M, where σ is an involution on M.

Proof. Given that

$$G(x^{n+1}) = [\sigma(x)]^n G(x) + [\sigma(x)]^{n-1} x\delta(x) + [\sigma(x)]^{n-2} x^2\delta(x) + \cdots + x^n\delta(x)$$

$= [\sigma(x)]^n G(x) + \sum_{j=1}^{n} [\sigma(x)]^{n-j} x^j \delta(x)$ for all $x \in M$. (4.1.1)

Replacing x by e in (4.1.1), we get

$G(e^{n+1}) = [\sigma(e)]^n G(e) + \sum_{j=1}^{n} [\sigma(e)]^{n-j} e^j \delta(e)$

Since e is the identity element in M and by lemma (4.1.2), we have

$$G(e^n e) = eG(e) + \sum_{j=1}^{n} [\sigma(e)]^{n-j} e^j \delta(e)$$

$$\Rightarrow G(e) = G(e) + \sum_{j=1}^{n} e^j \delta(e)\, e$$

$$\Rightarrow G(e) = G(e) + \delta(e) \sum_{j=1}^{n} e^j$$

$$\Rightarrow G(e) = G(e) + \delta(e)[e^1 + e^2 + \cdots + e^n]$$

$$\Rightarrow G(e) = G(e) + \delta(e)[e + e + \cdots + e]$$

$$\Rightarrow G(e) = G(e) + ne\delta(e)$$

$$\therefore G(e) = G(e) + n\delta(e)$$

This implies that $n\delta(e) = 0$, and since M is n −torsion free, we get $\delta(e) = 0$.

Replacing x by $x + ke$ in (4.1.1), where k be anypositive integer, we have

$G((x + ke)^{n+1})$

$$= (\sigma(x + ke))^n G(x + ke) + \sum_{j=1}^{n} (x + ke)^j \delta(x + ke)(\sigma(x + ke))^{n-j}$$

Since G is an operator, we have

$$G((x+ke)^{n+1}) = (\sigma(x)+\sigma(ke))^n(G(x)+G(ke))$$

$$+\sum_{j=1}^{n}(x+ke)^j\delta(x+ke)(\sigma(x)+\sigma(ke))^{n-j}$$

$$\Rightarrow G((x+ke)^{n+1}) = (\sigma(x)+ke)^n(G(x)+kG(e)) +$$

$$\sum_{j=1}^{n}(x+ke)^j\delta(x+ke)(\sigma(x)+ke)^{n-j}$$

Again δ is an operator, so we get

$$G((x+ke)^{n+1}) = (\sigma(x)+ke)^n(G(x)+kG(e)) +$$

$$\sum_{j=1}^{n}(x+ke)^j(\delta(x)+k\delta(e))(\sigma(x)+ke)^{n-j}$$

But $\delta(e)=0$, then

$G((x+ke)^{n+1}) = (\sigma(x)+ke)^n(G(x)+kG(e)) + \sum_{j=1}^{n}(x+ke)^j\delta(x)(\sigma(x)+$

$ke)^{n-j}$ (4.1.2)

By expanding (4.1.2), we have

$$\left(x^{n+1}+\binom{n+1}{1}x^{n}k+\binom{n+1}{2}x^{n-1}k^{2}+\cdots+k^{n+1}e\right)$$

$$=\left(G(x)+kG(e)\right)\left\{\left(\sigma(x)\right)^{n}+\binom{n}{1}\left(\sigma(x)\right)^{n-1}k+\binom{n}{2}\left(\sigma(x)\right)^{n-2}k^{2}\right.$$

$$\left.+\cdots+k^{n}e\right\}+\sum_{j=1}^{n}\{x^{j}+\binom{j}{1}x^{j-1}k+\cdots+\binom{j}{j-2}x^{2}k^{j-2}$$

$$+\binom{j}{j-1}xk^{j-1}+k^{j}e\}\delta(x)\{\left(\sigma(x)\right)^{n-j}+\binom{n-j}{1}\left(\sigma(x)\right)^{n-j-1}k$$

$$+\cdots+\binom{n-j}{n-j-2}\left(\sigma(x)\right)^{2}k^{n-j-2}+\binom{n-j}{n-j-1}\sigma(x)k^{n-j-1}$$

$$+k^{n-j}e\}G(x^{n+1})+G(\binom{n+1}{1}x^{n}k+\binom{n+1}{2}x^{n-1}k^{2}+\cdots$$

$$+k^{n+1}e)$$

$$=\left(G(x)+kG(e)\right)\left(\sigma(x)\right)^{n}$$

$$+\left(G(x)+kG(e)\right)\left\{\binom{n}{1}\left(\sigma(x)\right)^{n-1}k+\binom{n}{2}\left(\sigma(x)\right)^{n-2}k^{2}+\cdots+k^{n}e\right\}$$

$$+\sum_{J=1}^{n}x^{j}\delta(x)\left\{\left(\sigma(x)\right)^{n-j}+\binom{n-j}{1}\left(\sigma(x)\right)^{n-j-1}k+\cdots\right.$$

$$\left.+\binom{n-j}{n-j-2}\left(\sigma(x)\right)^{2}k^{n-j-2}+\binom{n-j}{n-j-1}\sigma(x)k^{n-j-1}+k^{n-j}e\right\}$$

$$+\sum_{j=1}^{n}\{\binom{j}{1}x^{j-1}k+\cdots+\binom{j}{j-2}x^{2}k^{j-2}+\binom{j}{j-1}xk^{j-1}$$

$$+k^{j}e\}\delta(x)\{\left(\sigma(x)\right)^{n-j}+\binom{n-j}{1}\left(\sigma(x)\right)^{n-j-1}k+\cdots$$

$$+\binom{n-j}{n-j-2}\left(\sigma(x)\right)^{2}k^{n-j-2}+\binom{n-j}{n-j-1}\sigma(x)k^{n-j-1}+k^{n-j}e\}$$

$$\therefore G(x^{n+1}) + G(\binom{n+1}{1} x^n k + \binom{n+1}{2} x^{n-1}k^2 + \cdots + k^{n+1}e)$$

$$= G(x)\big(\sigma(x)\big)^n + kG(e)\big(\sigma(x)\big)^n$$

$$+ \big(G(x) + kG(e)\big)\left\{\binom{n}{1}\big(\sigma(x)\big)^{n-1}k + \binom{n}{2}\big(\sigma(x)\big)^{n-2}k^2 + \cdots + k^n e\right\}$$

$$+ \sum_{j=1}^{n} x^j\, \delta(x)(\sigma(x))^{n-j}$$

$$+ \sum_{J=1}^{n} x^j \delta(x)\left\{\binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k + \cdots\right.$$

$$\left. + \binom{n-j}{n-j-2}\big(\sigma(x)\big)^2 k^{n-j-2} + \binom{n-j}{n-j-1}\sigma(x)k^{n-j-1} + k^{n-j}e\right\}$$

$$+ \sum_{j=1}^{n}\{\binom{j}{1}x^{j-1}k + \cdots + \binom{j}{j-2}x^2k^{j-2} + \binom{j}{j-1}xk^{j-1}$$

$$+ k^j e\}\delta(x)\{\big(\sigma(x)\big)^{n-j} + \binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k + \cdots$$

$$+ \binom{n-j}{n-j-2}\big(\sigma(x)\big)^2 k^{n-j-2} + \binom{n-j}{n-j-1}\sigma(x)k^{n-j-1}$$

$$+ k^{n-j}e\}G(x^{n+1})$$

$$+ G\left(\binom{n+1}{1}x^n k + \binom{n+1}{2}x^{n-1}k^2 + \cdots + k^{n+1}e\right)$$

$$= G(x)\big(\sigma(x)\big)^n + \sum_{j=1}^{n} x^j\, \delta(x)(\sigma(x))^{n-j} + kG(e)\big(\sigma(x)\big)^n$$

$$+ \big(G(x) + kG(e)\big)\left\{\binom{n}{1}\big(\sigma(x)\big)^{n-1}k + \binom{n}{2}\big(\sigma(x)\big)^{n-2}k^2 + \cdots + k^n e\right\}$$

$$+ \sum_{J=1}^{n} x^j \delta(x)\left\{\binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k + \cdots\right.$$

$$\left. + \binom{n-j}{n-j-2}\big(\sigma(x)\big)^2 k^{n-j-2} + \binom{n-j}{n-j-1}\sigma(x)k^{n-j-1} + k^{n-j}e\right\}$$

$$+ \sum_{j=1}^{n}\{\binom{j}{1}x^{j-1}k + \cdots + \binom{j}{j-2}x^2k^{j-2} + \binom{j}{j-1}xk^{j-1}$$

$$+ k^j e\}\delta(x)\{\big(\sigma(x)\big)^{n-j} + \binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k + \cdots$$

$$+ \binom{n-j}{n-j-2}\big(\sigma(x)\big)^2 k^{n-j-2} + \binom{n-j}{n-j-1}\sigma(x)k^{n-j-1} + k^{n-j}e\}$$

Now, using (4.1.1) we have

$$G(x^{n+1}) + G\left(\binom{n+1}{1}x^n k + \binom{n+1}{2}x^{n-1}k^2 + \cdots + k^{n+1}e\right)$$

$$= G(x^{n+1}) + kG(e)\big(\sigma(x)\big)^n$$

$$+ \big(G(x) + kG(e)\big)\left\{\binom{n}{1}\big(\sigma(x)\big)^{n-1}k + \binom{n}{2}\big(\sigma(x)\big)^{n-2}k^2 + \cdots + k^n e\right\}$$

$$+ \sum_{J=1}^{n} x^j \delta(x)\left\{\binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k + \cdots\right.$$

$$\left. + \binom{n-j}{n-j-2}\big(\sigma(x)\big)^2 k^{n-j-2} + \binom{n-j}{n-j-1}\sigma(x)k^{n-j-1} + k^{n-j}e\right\}$$

$$+ \sum_{j=1}^{n}\{\binom{j}{1}x^{j-1}k + \cdots + \binom{j}{j-2}x^2 k^{j-2} + \binom{j}{j-1}xk^{j-1}$$

$$+ k^j e\}\delta(x)\{\big(\sigma(x)\big)^{n-j} + \binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k + \cdots$$

$$+ \binom{n-j}{n-j-2}\big(\sigma(x)\big)^2 k^{n-j-2} + \binom{n-j}{n-j-1}\sigma(x)k^{n-j-1} + k^{n-j}e\}$$

$$\therefore G\left(\binom{n+1}{1}x^n k+\binom{n+1}{2}x^{n-1}k^2+\cdots+k^{n+1}e\right)$$

$$=kG(e)\big(\sigma(x)\big)^n$$

$$+\big(G(x)+kG(e)\big)\left\{\binom{n}{1}\big(\sigma(x)\big)^{n-1}k+\binom{n}{2}\big(\sigma(x)\big)^{n-2}k^2+\cdots+k^n e\right\}$$

$$+\sum_{J=1}^{n}x^j\delta(x)\left\{\binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k+\cdots\right.$$

$$\left.+\binom{n-j}{n-j-2}\big(\sigma(x)\big)^2k^{n-j-2}+\binom{n-j}{n-j-1}\sigma(x)k^{n-j-1}+k^{n-j}e\right\}$$

$$+\sum_{j=1}^{n}\{\binom{j}{1}x^{j-1}k+\cdots+\binom{j}{j-2}x^2k^{j-2}+\binom{j}{j-1}xk^{j-1}$$

$$+k^j e\}\delta(x)\{\big(\sigma(x)\big)^{n-j}+\binom{n-j}{1}\big(\sigma(x)\big)^{n-j-1}k+\cdots$$

$$+\binom{n-j}{n-j-2}\big(\sigma(x)\big)^2k^{n-j-2}+\binom{n-j}{n-j-1}\sigma(x)k^{n-j-1}+k^{n-j}e\}$$

This can be written as

$kf_1(\sigma(x),e)+k^2f_2(\sigma(x),e)+k^3f_3(\sigma(x),e)+\cdots+k^nf_n(\sigma(x),e)=0$, for all $x\epsilon M$, where $f_j(\sigma(x),e)$ are the coefficients of $k^{j}{}'s$ for all $j=1,2,3,\ldots,n.$

Now, replacing k by $1,2,3,\ldots,n$ in turn and considering the resulting system of n homogeneous equations, we get that the resulting matrix of the system which is a Van der Monde matrix

$$\begin{pmatrix}1 & 1 & \cdots 1\\ \vdots & \vdots & \cdots \vdots\\ n & n^2 & \cdots n^n\end{pmatrix}$$

Since the determinant of the matrix is equal to the product of positive integers, each of which is less then n, and since M is $(n+1)!-$ torsion free, it follows immediately that $f_j(\sigma(x),e)=0$ for all $x\epsilon M$ and $j=1,2,3,\ldots,n.$ Now, $f_j(\sigma(x),e)=0$implies

that$(n+1)G(x) = G(x) + n\sigma(x)G(e) + n\delta(x)$ for all $x \in M$. This yields that $nG(x) = n\sigma(x)G(e) + n\delta(x)$. Since $M$is n −torsion free, we get $G(x) = \sigma(x)G(x) + \delta(x)$ for all $x \in M$.

Again, $f_{n-1}(\sigma(x), e) = 0$ which gives

$n(n+1)G(x^2) = 2n\sigma(x)G(x) + n(n-1)(\sigma(x))^2G(x) + n(n+1)x\delta(x) + n(n-1)\sigma(x)\delta(x)$for all$x\epsilon M$. Since $M$is n −torsion free, then we obtain

$(n+1)G(x^2) = 2\sigma(x)G(x) + (n-1)\big(\sigma(x)\big)^2G(e) + (n+1)x\delta(x) + (n-1)\sigma(x)\delta(x)$for all $x\epsilon M$.

Since we have $G(x) = \sigma(x)G(e) + \delta(x)$, then putting it in the above relation, we find that

$$(n+1)G(x^2) = 2\{\sigma(x)G(e) + \delta(x)\}\sigma(x) + (n-1)\big(\sigma(x)\big)^2G(e) + (n+1)x\delta(x) + (n-1)\sigma(x)\delta(x)$$

$$= 2\big(\sigma(x)\big)^2G(e) + 2\sigma(x)\delta(x) + n\big(\sigma(x)\big)^2G(e) - \big(\sigma(x)\big)^2G(e) + nx\delta(x) + x\delta(x) + n\sigma(x)\delta(x) - \sigma(x)\delta(x)$$

$$= (n+1)\big(\sigma(x)\big)^2G(e) + (n+1)\sigma(x)\delta(x) + (n+1)x\delta(x)$$

Since M is $(n+1)$ −torsion free, then$G(x^2) = \big(\sigma(x)\big)^2G(e) + \sigma(x)\delta(x) + x\delta(x)$ for all $x \in M$, (4.1.3)

and also we have $G(x) = \sigma(x)G(e) + \delta(x)$ for all $x \in M$.

Replacing x by x^2 in the previous relation, we obtain

$G(x^2) = \big(\sigma(x)\big)^2 G(e) + \delta(x^2)$(4.1.4)

Equating (4.1.3) and (4.1.4), we find that

$\delta(x^2) = \sigma(x)\delta(x) + x\delta(x)$for all $x \epsilon M$.

(4.1.5)Now, by (4.1.3), we can write

$$G(x^2) = \big(\sigma(x)\big)^2 G(e) + \sigma(x)\delta(x) + x\delta(x)$$

$= \big(\sigma(x)G(e) + \delta(x)\big)\sigma(x) + x\delta(x)$for all $x \epsilon M$.

Using $G(x) = \sigma(x)G(e) + \delta(x)$ in the above equation, we get

$G(x^2) = \sigma(x)G(x) + x\delta(x)$for all $x \epsilon M$. Hence,δ is a Jordan left σ −derivation and$G$ is a generalized Jordan left σ −derivation on M.That is, we get the required result.

4.2 Posner's First Theorem for σ −prime Rings

Definition 4.2.1

Let I be a nonzero ideal of B. Then an additive mapping $D: I \to B$ is called a derivation from I to B if $D(x\,y) = D(x)y + x\,D(y)$ holds for all $x, y \in I$.Posner (1957)initiated the study of derivations in rings and proved two striking results.

These results have been generalized by many authors in different directions.Posner's first theorem (1957) states that if R is a prime ring of characteristic not 2 and the iterate of two derivations on R is also a derivation, then at least one of them is zero.

In this section we extend this result to σ −prime rings of characteristic different from 2.

Theorem 4.2.1

Let B be a σ −prime ring of characteristic not 2, I a nonzero σ −ideal and $D_1: I \to B$ and $D_2: I \to B$ are derivations such that the product map $D_1 D_2: I \to B$ is also aderivation. If at least one of D_1 and D_2 commutes with σ, then D_1= 0 or D_2= 0.

For developing the proof of the Theorem 4.2.1, we begin with the following results:

Lemma 4.2.1

Take a σ −prime ring B with characteristic of B $\neq 2$. Then B is 2-torsion free.

Proof. Take x ∈ B with 2x = 0. This implies that 2xyz = 0 for all y, z ∈ B.That is,

xB(2z) = {0}for all y ∈ B. Since characteristic of B is different from 2 and $B \neq \{0\}$, this yields a nonzero element s∈ B such that $2s \neq \{0\}$. Now we conclude that

$xB(2s) = \{0\} = xB\sigma(2s)$. Using σ −primeness of B, we find that $x = 0$ and hence B is 2-torsion free.

Lemma 4.2.2

Let B be a σ −prime ring and I a nonzero σ −ideal of R: If $D: I \to B$

is a derivation such that D commutes with σ. if b is an element of B and b D(x) = 0

(resp. D(x)b = 0) for all x ∈I, then either b = 0 or D = 0.

ProofReplacing$x$ by xy, where $y \in I$ in the given relation $bD(x) = 0$, we find that

$$bD(x)y + b\,x\,D(y) = 0.$$

Using given relation, gives

$$b\ x\ D(y)\ =\ 0 \text{for all } x, y \in I.$$

Replacing x by xs where $s \in B$ in the latter relation, we obtain

$$bx\ s\ D(y)\ =\ 0.$$

or $bx\ BD(y)\ =\ \{0\}$for all $x, y \in I$.

Since D commutes with σ and I is a σ −ideal, we find that

bxBD(y) = {0}= bxB $\sigma(D(y))$ for all x, y ∈ I.

Now σ −primeness of B gives

D(y) = 0 or bx = 0 for all x ∈ I.

Substituting wx where w ∈ R for x in the relation bx = 0, we get

bwx = 0.

That is

bBx = {0}for all x ∈ I.

Since I is a σ −ideal of R, we also have

bBx = bB $\sigma(x)$ = {0}.

By σ −primeness of R and I ≠ {0} imply that b = 0.

In the same way, we can show that D(x)b = 0 for all x ∈ I implies b = 0 or D = 0.

Proof of the Theorem 4.2.1 We split the proof in two steps:

Step .1: Assume that D_1commutes with σ,given that the map $D_1D_2: I \to B$ is a derivation. It istrivial that$D_2 \subseteq I$and$D_1D_2(xy) = D_1D_2(x)y + xD_1D_2(y)$ for all x, y $\in$ I.

But $D_1D_2: I \to B$ are derivations, so we have

D_1D_2 (x y) = $D_1(D_2(xy))$ = D_1D_2 (x)y + D_2(x) D_1(y) + D_1 (x) D_2 (y) + xD_1D_2 (y).

By above relations we conclude that

D_2(x) D_1(y) + D_1 (x) D_2 (y) = 0 for all x; y $\in$ I. (4.2.1)

Replacing x by xD_2 (z), where z $\in$I in the relation (4.2.1) we find that

D_2 (xD_2 (z)) D_1 (y) + D_1(x D_2 (z)) D_2 (y) = 0 for all x, y, z $\in$ I.

This gives

D_2(x) D_2(z) D_1(y) + x D_2^2(z) D_1(y) + D_1(x) D_2 (z) D_2 (y) + x D_1D_2(z) D_2 (y) = 0.

In view of equation (4.2.1) and using the fact that D_2 (I) $\subseteq I$,we find that

(D_2 (D_2 (z)) D_1 (y) + D_1 (D_2 (z)) D_2 (y)) = 0.

This implies that

D_2 (x) D_2 (z) D_1 (y) + D_1 (x) D_2 (z) D_2 (y) = 0 for all x; y; z $\in$ I. (4.2.2)

Using the relation (4.2.1) and Result 4.2.1, the relation (4.2.2) reduces to

D_1 (x) D_2 (z) D_2 (y) = 0 for all x, y, z $\in$ I.

In view of Lemma 4.2.2 we get either

D_1= 0 or D_2 (z) D_2 (y) = 0 for all y, z ∈ I.

If D_1= 0 then nothing to do.However, if

$D_1 \neq 0$,we have

D_2 (z) D_2 (y) = 0 for all y, z ∈ I. (4.2.3)

Replacing y by yz in (4.2.3) and using (4.2.3), we get

D_2(z)yD_2 (z) = 0 for all y, z ∈ I. (4.2.4)

Replacing y by t y, where t ∈ B, in (4.2.4), we obtain

D_2 (z)ByD_2 (z) = {0}. That is, yD_2 (z)B yD_2 (z) = {0} for all y, z ∈ I.

Since R is a σ −prime ring, it is semiprime also and hence we find that

yD_2 (z) = 0 for all y, z ∈ I. (4.2.5)

Replacing y by y w, where w ∈ B, in (4.2.5), we get

ywD_2 (z) = 0 i.e.; yBD_2 (z) = {0}for all y, z ∈ I.

Since I is a σ −ideal of B, so we have

$\sigma(y)BD_2$ (z) = {0}for all y, z ∈ I.

In view of σ-primeness of B and I ≠ {0}implies that $D_2 = 0$.

Step 2: Let us assume that D_2commutes with σ. From Step 1, we have

D_1 (x) D_2 (z) D_2 (y) = 0 for all x, y, z ∈ I.

Using Lemma 4.2.2, we obtaineither

D_2= 0 or D_2 (x) D_2 (z) = 0 for all x, z ∈ I.

If D_2= 0 then nothing to do, or

let $D_2 \neq 0$, we have

D_1 (x) D_2 (z) = 0 for all x, z ∈ I.

Again by Result 4.2.2 we find that either D_1= 0 or D_2 = 0. This completes the proof.

The following example indicates that the hypothesis of σ −primeness is crucial in

Theorem 4.2.1.

Example 4.2.1

Let $B = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} | 0, \alpha, \beta, \gamma \in Z \right\}$, where Z is the set of integers.

Take the map $\begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} = \sigma \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}$ of B into itself such that

$$\sigma \begin{pmatrix} \alpha & 0 & 0 \\ \beta & \gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is easy to verify that σ is an involution of the ring B, where characteristic of B $\neq 2$.

Next, let $I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} | 0, \beta \in Z \right\}$. Then I is a nonzero σ-ideal of B.

Let the map $D_1: I \to B$ and $D_2: I \to B$ be defined by

and $D_2\begin{pmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -\beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Clearly, D_1 and D_2 are derivations and σ commutes with D_1.It can be also shown that the map $D_1D_2: I \to B$ is a derivation and B is not a σ −prime ring. But neither D_1= 0 nor D_2= 0.

The following example demonstrates that the hypothesis of "characteristic B$\neq 2$" is essential in the Theorem 4.2.1.

Example 4.2.2

Let $B = Z_2[X] \times Z_2[X]$, where $Z_2[X]$ is the polynomial ringover Z_2. Let the map (p(x), q(x)) $\mapsto \sigma$(f(x), g(x)) of B into itself such that σ(p(x), q(x)) = (p(x), q(x)).Clearly, σ is an involution of B and B is a σ-prime ring. If I = [x^2] is the ideal of Z_2[X] generated by $x^2 \in Z_2$[X], then I = I ×I is a nonzeroσ −ideal of B.Set $d_1, d_2: I \to B$ such that d_1 (p(x), q(x)) = (D(p(x)), D(q(x)) and d_2(p(x), q(x)) = (D(f(x)), 0), where D is the usual differentiation in Z_2[X]. Obviously, d_1,d_2 and $d_1d_2: I \to B$ are derivations. Moreover, B is a ring of characteristic 2 and $d_1\sigma = \sigma\, d_1$.However $d_1 \neq 0$ and $d_2 \neq 0$.Taking I = B in Theorem 4.2.1, we get the corollary below.

Corollary 4.2.1

Let R be a σ −prime ring of characteristic not equal to 2 and D_1, D_2 derivations of B such that the iterate D_1D_2 is also a derivation of B. If at least one of D_1 and D_2 commutes with σ, then $D_1 = 0 \ or \ D_2 = 0$.

Using the above theorem we can obtain Posner's first theorem.

Corollary 4.2.2 (Posner, 1957)

Let *B* be a prime ring of characteristic not equal to 2 and

D_1, D_2derivations of B such that the iterate D_1D_2 is also a derivation, then at least one ofD_1, D_2 is zero.

Remark 4.2.3

Take a prime ring B with char (B)≠ 2. Clearly,$B = B \times B^{\circ}$ is a σ-prime ring of characteristic not 2. Choose I = B, which is a nonzero σ -ideal of B. Define d_1, d_2byd_1(x, y) = $(D_1(x), D_1(y))$ and d_2 (x; y) = $(D_2(x), D_2(y))$.

From the hypothesis it can be easily found that$d_1, d_2: I \to B$ are derivations and the product $d_1d_2: I \to B$ is also a derivation. As $d_1\sigma = \sigma d_1$, by Theorem 4.2.1 we obtain either $d_1 = 0$ or d_2= 0, in turn we find that either $D_1 = 0 \ or \ D_2 = 0$.

4.3 Result on Jordan right $\boldsymbol{\sigma}$ −derivation and generalized Jordan right $\boldsymbol{\sigma}$ −derivation $\boldsymbol{\sigma}$ −ring.

We begin with

Theorem 4.3.1

Let $m \geq 1$ be a fixed integer and B be $(m+1)!$ −torsion free σ-ring with identity element e. Suppose that $H: B \to B$ and $d: B \to B$are two additive mappings satisfying the property

$$H(x^{n+1}) = H(x)[\sigma(x)]^n + [\sigma(x)]^{n-1}d(x)x + [\sigma(x)]^{n-2}d(x)x^2 + \cdots + d(x)x^n, \forall\, x \in B.$$

Then H is a generalized Jordan right σ-derivation and d is a Jordan right σ-derivation on B.

ProofThe given condition can be written as

$$H(x^{n+1}) = H(x)(\sigma(x))^m + \sum_{k=1}^{m} d(x)x^k(\sigma(x))^{m-1} \text{ for all } x \in B. \quad (4.3.1)$$

We first claim that $\sigma(e) = e$ where e is the identity of Band by definition of σ −ring, we have $e = \sigma(\sigma(e)) = \sigma(\sigma(e)e) = e\sigma(e) = \sigma(e)$.

Replacing x by e in (4.3.1), we find that

$$H(e) = H(e)(\sigma(e))^m + \sum_{k=1}^{m} d(e)e^k(\sigma(e))^{m-1}$$

By our claim, $\sigma(e) = e$, using relation 4.5.2 we find that

$$H(e) = H(e) + d(e)(e^1 + e^2 + \cdots + e^m).$$

This implies that $d(e)m = 0$. Since B is m-torsion free, we obtain $d(e) = 0$.

Substituting x by x + e p, p is a positive integer, in (4.3.1) and using $d(e) = 0$, we get

$$H((x + e\,p)^{m+1}) = H(x + ep)(\sigma(x + ep))^m + \sum_{k=1}^{m} d(x + ep)(x + ep)^k(\sigma(x + ep))^{m-1}$$

=(H(x) +H(e) p) $(\sigma(x)+ep)^m$ +$\sum_{k=1}^{m} d(x)(x+ep)^k(\sigma(x)+ep)^{m-1}$(4.3.2)

On expanding (4.3.2), we have

$$H\left\{x^{m+1}+\binom{m+1}{1}x^m p+\binom{m+1}{2}x^{m-1}p^2+\cdots+ep^{m+1}\right\}$$

$$=(H(x)+H(e)p)\left\{(\sigma(x))^m+\binom{m}{1}(\sigma(x))^{m-1}p+\binom{m}{2}(\sigma(x))^{m-2}p^2+\cdots+ep^m\right\}$$

$$+\sum_{k=1}^{m}\left\{d(x)\left(x^k+\binom{k}{1}x^{k-1}p+\cdots+\binom{k}{k-2}x^2p^{k-2}+\binom{k}{k-1}xp^{k-1}+\right.\right.$$
$$\left.+e\,p^k\right)\left((\sigma(x))^{m-k}+\binom{m-k}{1}(\sigma(x))^{m-k-1}p+\cdots+\right.$$
$$\left.\left.\binom{m-k}{m-k-2}(\sigma(x))^2p^{m-k-2}++\binom{m-k}{m-k-1}\sigma(x)p^{m-k-1}+ep^{m-k}\right)\right\}$$

Using (4.3.1), we find that

$$H\left\{\binom{m+1}{1}x^n p+\binom{m+1}{2}x^{m-1}p^2+\cdots+e\,p^{m+1}\right\}$$

$$=H(e)p(\sigma(x))^m+(H(x)+H(e)p)\left\{\binom{m}{1}(\sigma(x))^{m-1}p+\binom{m}{2}(\sigma(x))^{m-2}p^2+\cdots+\right.$$
$$\left.\binom{m}{m-2}(\sigma(x))^2p^{m-2}+\binom{m}{m-1}\sigma(x)p^{m-1}+e\,p^m\right\}$$
$$+$$
$$+\sum_{k=1}^{m}d(x)\left\{\binom{n-k}{1}(\sigma(x))^{n-k-1}p+\cdots+\binom{m-k}{m-k-2}(\sigma(x))^2p^{m-k-2}+\right.$$
$$\left.\binom{m-k}{m-k-1}\sigma(x)p^{m-k-1}+e\,p^{m-k}\right\}$$
$$+\sum_{k=1}^{m}\left\{d(x)\binom{k}{1}x^{k-1}p+\cdots+\binom{k}{k-2}x^2p^{k-2}++\binom{k}{k-1}xp^{k-1}+\right.$$
$$\left.e\,p^k\right\}\left\{(\sigma(x))^{m-k}+\binom{m-k}{1}(\sigma(x))^{m-k-1}p+\cdots++\binom{m-k}{m-k-2}(\sigma(x))^2p^{m-k-2}+\right.$$
$$\left.\binom{m-k}{m-k-1}\sigma(x)p^{m-k-1}+e\,p^{m-k}\right\}.$$

Since p is different from zero, then it can be written as

$f_0(\sigma(x),e)+pf_1(\sigma(x),e)+p^2f_2(\sigma(x),e)+p^3f_3(\sigma(x),e)+\cdots+p^mf_m(\sigma(x),e)=0$
,

for all $x \in B$, where $f_k(\sigma(x),e)$ are the coefficients of p^k's for all $k=0,1,2,3,\ldots,m.$

Replacing p by $1,2,3,\ldots,m+1$ in turn and considering the resulting system of m homogeneous equations, we get theresulting matrix of the system as a Vander monde matrix

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 2^2 & \ldots & 2^m \\ 1 & 3 & 3^2 & \ldots & 3^m \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 1 & m+1 & (m+1)^2 & \ldots & (m+1)^m \end{pmatrix}$$

Therefore, the determinant of the matrix is equal to the product of positive integers, each of which is less than $m+1$, and since B is $(m+1)!-$ torsion free, this yields

$f_k(\sigma(x),e)=0$for all $x \in M$ and $k=0,1,2,3,\ldots,m.$

Next, $f_k(\sigma(x),e)=0$ implies that

$$(m+1)H(x)=H(x)+H(e)m\,\sigma(x)+m\,d(x)\text{for all } x \in M.$$

This gives

$$mH(x)=m\,H(e)\sigma(x)+md(x)\,.$$

Since B is m −torsion free, we find that

$$H(x)=H(e)\sigma(x)+d(x)\text{for all } x \in B \qquad (4.3.3)$$

Now, $f_{m-2}(\sigma(x), e) = 0$ implies that

$$m(m+1)H(x^2) = 2m\,H(x)\sigma(x) + m(m-1)H(e)(\sigma(x))^2 + m(m+1)d(x)x +$$

$$m(m-1)d(x)\sigma(x) \text{for all } x \in B.$$

Since B is m −torsion free, this becomes

$$(m+1)H(x^2) = 2H(x)\sigma(x) + (m-1)H(e)\big(\sigma(x)\big)^2 + (m+1)d(x)x +$$

$$(m-1)d(x)\sigma(x) \text{for all } x \in B.$$

Using (4.3.3) in the above relation, we find that

$$(m+1)H(x^2) = 2\{H(e)\sigma(x) + d(x)\}\sigma(x) + (m-1)H(e)\big(\sigma(x)\big)^2 + (m+1)d(x)x +$$
$$+ (m-1)d(x)\sigma(x)$$
$$= 2H(e)(\sigma(x))^2 + 2d(x)\sigma(x) + H(e)(\sigma(x))^2 - H(e)(\sigma(x))^2$$

$$+md(x)x + d(x)x + md(x)\sigma(x) - d(x)\sigma(x)$$

$$= (m+1)H(e)(\sigma(x))^2 + (m+1)d(x)\sigma(x) + (m+1)d(x)x.$$

But B is $(m+1)$ −torsion free, also

$$H(x^2) = H(e)\big(\sigma(x)\big)^2 + d(x)\sigma(x) + d(x)x \text{for all } x \in B, \quad (4.3.4)$$

Replacing x by x^2 in (4.3.3), we obtain

$$H(x^2) = H(e)\big(\sigma(x)\big)^2 + d(x^2) \quad (4.3.5)$$

Combining (4.3.4) and (4.3.5), we get

$$d(x^2) = d(x)\sigma(x) + d(x)x \forall\ x \in B. \qquad (4.3.6)$$

In view of (4.3.4), we have

$$\mathrm{H}(x^2) = H(e)\big(\sigma(x)\big)^2 + d(x)\sigma(x) + d(x)x$$

$$= \big(H(e)\sigma(x) + d(x)\big)\sigma(x) + d(x)x \forall x \in B.$$

Using (4.3.3) in the above equation, we obtain

$$H(x^2) = H(x)\sigma(x) + d(x)x \forall x \in B.$$

This completes the proof.

CHAPTER FIVE

RESULTS ON SOME SPECIAL CLASSES OF NEAR-RINGS

5.1 Near-ring Theoretic Concepts

A long standing result due to Herstein (1968) asserts that a periodic ring is commutative if its nilpotent elements are central.Ligh (1989) has raised a question whether similar result willhold for distributively generated (d-g) near-rings. Answering that, Bell (1973), gave an affirmative answer and proved that if N is a (d-g) near- ring with its nilpotent elements lying in the center,and that the set I of nilpotent elements forms an ideal and N/I is periodic, then N must be commutative. Herstein (1968) asserts that if a ring R satisfies the identity $[x^n, y] = [x, y]$, then R is commutative.

Bell (1973) showed that if it is a distributively generated (d-g) near-ring with an identity, and if for each x, y $\in$ N, there exists n>1, depending on *x* and y such that

$$(x\,y - y\,x)^n = x\,y - y\,x$$

then N is a commutative ring. Bell also pointed out that the conclusion will not hold if one drops the existence of an identity. One can ask a natural question: Is a distributively generated (d-g) near-ring N commutative without an identity, such that if for each *x*, *y* in N , there exists *n* >1 , depending on *x* and *y*,satisfy$(x\,y + y\,x)^n = x\,y + y\,x$?

One of the purposes of this section is to give an affirmative answer to this question and also examine the consequence of dropping the hypothesis that N has an identity. It will be shown that Nturns out to be a commutative near-ring.

In this sequel, if for each x, y ∈ N, there exist m = m(x, y) > 1, n = n(x, y) > 1 and k > 0, then we define the near-ring property as

$$(x^m o\ y^n)^k = (x^m\ o\ y^n) \quad (5.1)$$

It is not always trivial to obtain near-ring theoretic analogues of ring theoretic results. Many of them do not hold in general either. As an example, we recognize that a Boolean ring is necessarily commutative but there exist Boolean near rings which are non commutative. For example: Let a set $N = \{0, a\}$ with operations of addition'+' and multiplication'·'defined by

+	0	a
0	0	a
a	a	o

and

·	0	a
0	0	a
a	0	a

. Clearly, $(N, +, \cdot)$ is a Boolean near ring but not commutative

Definition 5.1.1

Let N be a non empty set equipped with two binary operations say

$' + '$and$' \cdot '$. N is called a left near ring (resp. right near-ring) if

(i) (N, +) is a group(not necessarily abelian);

(ii) (N,·) is a semi group; and

(iii) x(y + z) = xy + xz for all x, y, z ∈ N (respectively, (y + z)x = yx + zx ∀x,y, z ∈ N).

Example 5.1.1.

(i) Let (C,+) be usual group of complex numbers with regard to ordinaryaddition of complex numbers. Let us define '*' in C as follows: a * b = $|a|b$ for all a, b $\in$ C. Then (C, +, $*$) is a left near ring which is not a right near ring.

(ii) Let (G, +) be a non abelian group. Consider S, the set of all functions from G to G. Then (S,+, ·) is a right near-ring with regard to the operation `+' and ′·' defined as

(f + g)(x) = f(x) + g(x) for all x $\in$ G, and (fg)(x) = f(g(x))for all x $\in$ G where

f, g $\in$ S.But it is not a left near-ring.

Definition 5.1.2

A left near-ring N is called zero symmetric if 0x = 0 for all x $\in$ N.

Remark 5.1.1

The near-rings C and S discussed in the above Examples 5.1.1 (i)& (ii) are zero symmetric.

Definition 5.1.3

A left near-ring N is called prime near-ring if xNy = {0}, where

x, y $\in$ N, implies x = 0 or y = 0.

It is called semiprime ring if xNx = {0}, where x $\in$ N, implies x = 0.

Definition 5.1.4

The multiplicative center of near-ring N, usually denoted by Z and defined as $Z = \{x \in N \mid xy = yx \ \ \forall y \in N\}$.

The additive center of N is defined as $Z^+ = \{x \in N \mid x + y = y + x \text{ for all } y \in N\}$.

Definition 5.1.5

Let N be a near-ring. Then N is called a distributive near-ring if

$(y + z)x = y x + z x$ for all $x, y, z \in N$.

Definition 5.1.6

Let N be a near-ring. Then N is called a pseudo-abelian near-ring if

$xy + zt = zt + xy$ for all $x, y, z, t \in N$.

Definition 5.1.7

Let N be a near-ring. Then N is called a commutative near-ring if

$xy = yx$ for all $x, y \in N$.

Remark 5.1.2

Every commutative near-ring is a distributive near-ring but the converse is not true. For justification, consider a non abelian group (G, +), a noncommutative ring (R,+, ·) and N = G×R. Define componentwise addition `+' in N and multiplication *in Nby (a, b) * (c, d) = (0, bd), where (a, b), (c, d) $\in$ N.

It can be easily demonstrated that $(N,+,\cdot)$ is a distributive near-ring but not commutative.

Definition 5.1.8

A near-ring R is said to be distributively generated (d-g) near-ring ifit contains a multiplicative sub semi group of distributive elements which generate the additive group (R,+).

Remark 5.1.3

The near-ring generated additively by all the endomorphisms of a group (G,+) not necessarily abelian is distributively generated near-ring.

Definition 5.1.9

Let N be a near-ring and K a nonempty subset of N. Then a normal subgroup (K, +) of (N, +) is a left ideal (resp. a right ideal) of N if $xk \in K$ (resp $(x + k)y - xy \in K$)holds for all x, $y \in N$ and for all $k \in K$. K is called an ideal of N if it is both a left ideal as well as a right ideal of N.

Definition 5.1.10

Let N be a near-ring. Then a non empty subset U of N is said to be a

semigroup left (resp. right) ideal of N if $NU \subseteq U$ (resp. $UN \subseteq U$) and if U is both a

semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of N.

Remark 5.1.4

For any x; y ∈ N, the symbol [x, y] will denote the multiplicative

commutator xy-yx, while (x,y) will indicate the additive commutator $x + y - x - y$ and $x \circ y$ will represent the anti-commutator $xy + yx$. If σ and τ are automorphisms ofN, then the symbol $[x,y]_{\sigma,\tau}$ will denote the (σ,τ) −commutator $x\sigma(y) - \tau(y)x$.

Definition 5.1.11

A near-ring Nis an orthogonal sum of sub near-rings Aand Bdenoted by N= A⊕ B if AB= BA= {0}and each element of *N* has a unique representation n of the form a+b,$a \in A$ and b ∈ B.

Definition 5.1.12

A near-ring N is said to be zero-symmetric if 0x = 0 (left-distributive of N guarantees x0 = 0) for all x ∈N. In what follows all near-rings are left near-rings with multiplicative center Z.

Definition 5.1.13

A near-ring N is called D-near-ring if every non-zero homomorphic image T of N satisfies the following conditions:

(i) T has a non-zero right distributive element.

(ii) Additive group (T, +) of T is abelian implies that (T, +,·) is a ring.

Remark 5.1.5

From the above definitions, it is clear that all distributive and (d-g) near-rings are examples of D-near-rings. However, there are D-near-rings, which are not necessarily (d-g) near-rings. However, Example 2.5 of Section 6 of (Frolich, 1958) shows that the class of D-near-rings is larger than that of the (d-g) near-rings.

Definition 5.1.14

A near-ring *N* is called zero-symmetricif 0 x = 0 for all *x* in Nand zero-commutative if xy = 0 implies y x = 0 for all x, y ∈R (recall left distributivity yields x0 = 0).

5.2 Some Results on Near-rings

Bell and Ligh (1989) studied the direct sum decomposition of rings satisfying the related properties $xy = (xy)^2 p(x,y)$ or $xy = (yx)^2 p(x,y)$, for $p(X,Y) \in Z(X,Y)$, the ring of polynomials in two non-commuting indeterminate and remarked that in case of near- rings the analogous hypotheses do not quite yield direct sum decomposition.

Theorem 5.2.1

Let N be a D-near-ring satisfying condition (5.1). If idempotent elements of N are central, then N = A ⊕ B, where A is a sub ring and B is a sub near-ring with trivial multiplication.

Before proving this result, we state the following known results.

Lemma 5.2.1(Bell, 1973)

Let N be a zero symmetric near- ring satisfying the following properties:

(i) For each x in N, there exists a positive integer n(x) >1, such that $x^{n(x)} = x$

(ii) Every nontrivial homomorphic image of N contains a nonzero central idempotent.Then (N,+) is commutative.

Lemma 5.2.2(Bell andKappe, 1989)

Let N be a near- ring with unity 1.Then, for every$x \in N$,< x >= x <1, x >.

Lemma 5.2.3(Reddy and Murthy, 1985)

Let N be a near ring in which idempotents are multiplicatively central. If e and f are any idempotents, there exists an idempotent g such that $g\,e = e$and $g\,f = f$.

In order to prove our theorem, we establish the following results

Lemma 5.2.4

Let N be a D - near ring satisfying (5.1). If N* $\subset$Z (N), then N\N* is a periodic and a commutative ring.

ProofSinceN* $\subset$Z(N), N* is an ideal. Consider the near- ring N\N*. Take a nonzero distributive element x of N\N*(where N\N* can be written as a sub direct product of near rings without zero divisors, so one may assume N\N*has no nonzero divisors). It is noticed that x = x f(x) for some f(x) $\in$< x >by property (5.1)when y = x. This implies that e = f (d) is a nonzero idempotent. Since e(er –r) = 0 for every r $\in$N\N*, so e is a left identity in N\N*. Taking arbitrary a, b $\in$N\N* and using the fact that e commutes with x and x = x e, we have 0 = (a + b) x e–(axe–bxe) = {(a+b) e–(ae–be)} x. Thus e is a multiplicative identity in N\N*, becauseN\N* has no nonzero divisors, e is a distributive element of N\N*e. Let a be an arbitrary nonzero element of N\N*

then a = a(f) where f(a) $\in$< a >. Using Result 5.2.3, we can write f (a) = a f``(a) for some f`'(a) $\in$<1 a >. Now a = $a^2$f``(a), gives a f``(a) = 1 i.e. N\\N* is a division near-ring. Thus N\\N* is additively commutative and is a ring. Hence N\\N* is a periodic commutative ring (Bell, 1978).

Lemma 5.2.5

Let N be a near ring satisfying (5.1). Then NN* = N*N = {0}.

ProofFirst, we claim that Nsatisfying condition (5.1) is zero commutative.

Taking x y = 0, then there exist positive integers m` = m`(y, x) $\geq$1 and n' = n`(y, x) $\geq$1, such that $y^{n`}x^{m`}p\,(y, x)^{`} = 0$. Replacing y by x in (5.1), we find that

$0 = x^{n+m}\ g(x)$ for n+ m $\geq$2 and g(x) = p(x, x). (5.2.1)

If u $\in$N, then by repeated use of (5.2.1), we get $u^2 = 0$. By property (5.1), we have

$u^n x^m h(u, x) u^n = 0$, for n= n(u, x) $\geq$1, m = m(u, x) $\geq$1, for any u $\in$ N*.

Zero commutativity in N gives xu = 0, for u $\in$N* and x $\in$N. Hence N*N = NN* = {0}.

Now we are in a position to prove the main theorem.

Proof of Theorem 5.2.1

In view of Result 5.2.4, for each x $\in$R, there exist distinct positive integers $m = m(x)$ and $n = n(x)$ such that $x^m - x^n \in N$. Using Lemma (5.2.3),$x^{m+1} = x^{n+1}$, for each x $\in$ Nand Nis periodic.

Now, we break the proof in two steps.

Step 1 To prove Ais a subring of N: Let x, y $\in$Aand choose integers m = m (x) >1 and n = n(y) >1 such that $x^m = x$ and $y^n = y$. Let s = (m –1) n –(m –2) = (n –1)m–(n–2). Then, clearly x^s= xand y^s = y. Take e = x^{s-1} and f = y^{s-1} which are idempotentswith ex = x and f y = y.In view of Lemmas5.2.4 and 5.2.5, we have C $\subseteq$N $\subseteq$Z(R) and x^2y = xyx, for all, x, y $\in$ A.Clearly, $(xy)^s = x^s y^s = (xy)^s$, thus x y$\in A$, for all x, y $\in$ A. Therefore, N\$N^*$ has$b^p = b$property and we have an integer k>1 such that

$$(x-y)^k = x - y + u, \quad u \in N \qquad (5.2.2)$$

In view of Result 5.2.4, take an idempotent h with he = e, h f = f, Hence h x= x and h y = y.

Multiplying (2) by h, we obtain $(x-y)^k = x - y$ which implies x –y $\in$ A.

Using Result 5.2.1, we get (A, +) is abelian. Hence A is a subring.

Step 2 We claim thatN = A+B.

Proof It is clear that, $A \cap B = \{0\}$. Takingx + a = y + b with x, y $\in$A and a, b $\in$ B,then x – y = b – a $\in A \cap B = \{0\}$.This implies that x = y and a = b. So, N = A+ B.

Remark 5.2.1

Example 2.5(Frohlich, 1958), demonstrates that one cannot get a direct sum decomposition under the hypotheses of Theorem 5.2.1, even in case of distributive near-rings.

Example 5.2.1

Take the non-abelian group (N,+),which is isomorphic to symmetric group S_3 and define the multiplication in *N* as follows:

.	0	a	B	C	u	v
0	0	0	0	0	0	0
A	0	a	A	A	0	0
B	0	a	A	A	0	0
C	0	a	A	A	0	0
U	0	0	0	0	0	0
V	0	0	0	0	0	0

Then (N, +,·) is a commutative near ring satisfying x y = x^2y^2 $x^2$for all x, y∈N. Hence

M = {0, a}is not an ideal of N.

Corollary 5.2.1

Let N be a (d-g) near-ring with 1 and for each x, y in N, let there be n(x, y) >1, such that $(x\,y+y\,x)^n$= x y + y x . Then N is a commutative ring.

5.3 Derivation in Near-rings

The notion of derivation in rings is quite old and plays a significant role in the integrationanalysis, algebraic geometry and algebra. It has got a tremendous

development after Posner (1957) established two very striking results on derivations in prime rings. Also there has been considerable interest in investigating commutativity of rings, more often that of prime ring and semiprime rings admitting suitably constrained derivations.

Derivations in prime and semiprime rings have been studied by Belland Bresar (1988), Vukman (2007) and so on. in several directions. Motivated by the concept of derivation in rings, Bell and Mason (1987) introduced the concept of derivation in near rings as follows.

Definition 5.3.1

A derivation d on N is defined to be an additive endomorphismatisfying the product rule $d(xy) = xd(y) + d(x)y$ for all $x, y \in N$.

Example 5.3.1

Let $N = N_1 \oplus N_2$, where N_1 is a zero symmetric left near-ring andN_2 is a ring having a nonzero derivation δ. Then $d: N \to N$ defined by $d(x, y) = (0, \delta(y))$ for all $x, y \in N$ is a nonzero derivation of the left near ring N.

Example 5.3.2

Let us consider (C,+,*) where `*' is defined as $x * y = |x|y$ for allx, $y \in C$, then it can be easily seen that (C,+,*) is a zero symmetric left near ring

which is not a right near ring. Assume $N = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} | \alpha, \beta \in C \right\}$, then N is a zerosymmetric left near ring which is not a right near ring. Define d: N $\rightarrow$N by $d\begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$. Then d is a nonzero derivation on N.

The notion of derivation in near-rings has been generalized by introducing the notionsof (σ, τ)-derivation and generalized derivation in near-rings by Golbasi and Aydin (2004) respectively.

Definition 5.3.2

An additive mapping d: N $\rightarrow$N is called a (σ, τ)-derivation of N if$\exists$functions σ,τ: N$\rightarrow$N such that the relation d(xy) = d(x)σ(y)+τ(x)d(y)hold $\forall$ x, y $\in$ N.

Example 5.3.3

Let C_1 = (C,+, ·), the ring of complex numbers with regard to usualAddition '+' and multiplication '.' of complex numbers. Next suppose C_2 = (C,+,*),where C is the set of complex numbers, + is the usual addition of complex numbers, '.'is defined as x*y = $|x|y$ for all x, y $\in$C, where '.' is the usual multiplication of complexnumbers and $|x|$ denotes the modulus of the complex number x. Then C_2 is a zerosymmetric left near-ring. Further it can be easily verified that the set B = $C_1 \times C_2$ is azero symmetric left near-ring with regard to component wise addition and multiplication.

Now suppose $N = \left\{ \begin{pmatrix} (a,b) & (c,d) \\ (0,0) & (0,0) \end{pmatrix} \mid (a,b),(c,d) \in B \right\}$Clearly, N is a non-commutative zero symmetric left near-ring with respect tomatrix addition and matrix multiplication. Define d: N →N and , τ : N →Nsuch that

$$d\begin{pmatrix} (x,y) & (u,v) \\ (0,0) & (0,0) \end{pmatrix} = \begin{pmatrix} (0,0) & (\bar{x},0) \\ (0,0) & (0,0) \end{pmatrix},$$

$$\sigma\begin{pmatrix} (x,y) & (u,v) \\ (0,0) & (0,0) \end{pmatrix} = \begin{pmatrix} (x,y) & (-u,-v) \\ (0,0) & (0,0) \end{pmatrix},$$

and $$\tau\begin{pmatrix} (x,y) & (u,v) \\ (0,0) & (0,0) \end{pmatrix} = \begin{pmatrix} (\bar{x},\bar{y}) & (\bar{u},\bar{v}) \\ (0,0) & (0,0) \end{pmatrix},$$

where$\bar{x}, \bar{y}, \bar{u}$ and $\bar{v}$ denote the conjugates of the complex numbers x, y, u and v respectively.

It can be easily seen that d is a (σ, τ)-derivation of N, where σ and $\tau$are automorphisms of N.

Definition 5.3.3

Let N be a near-ring. Then

(i) An additive mapping F: N →N is called a right generalized derivation of N if thereexists a derivation d of N such that F(xy) = F(x)y + d(y)x for all x, y ∈ N.

(ii) An additive mapping F: N →N is called a left generalized derivation of N if thereexists a derivation d of N such that F(xy) = d(x)y + xF(y) for all x,y∈ N.

(iii) An additive mapping F: N →N is called a generalized derivation of N if there exists a derivation d of N such that F(xy) = F(x)y + xd(y) for all x, y ∈ N (F(xy) = d(x)y + xF(y) for all x, y ∈ N).

5.4 Permuting 4-Derivation on Near-rings

In this section, we introduce the notion of a permuting 4- (σ,τ)-derivation in a near-ring and generalize some of the recent results. Ozturk (1999) introduced the concept of permuting 3-derivation. With some properties recent results on properties of prime rings, semi prime rings and near rings with derivations have been investigated in several wayssuch as (Ozturk and Jung, 2004), (Ozturk and Yazarli,2011),(Park, 2009) and (Park and Jung, 2010) introduced the concept of a permuting 3-derivation of a near ring and investigated the conditions for a near ring to be a commutative ring. Motivated by this, it is a natural question that: Could these results can be extended to 4- (σ,τ) derivation? In this context, we give an affirmative answer to this question and introduce the notion of 4- (σ,τ) derivation and give some properties.

An additive mapping $D: N\times N\times N\times N\to N$ is said to be 4-permuting

If $D(x,y,z,t)=D(y,x,z,t)=D(z,y,x,t)=D(t,x,y,z)$ for all $x,y,z,t\in N$.

A mapping: $d: N\to N$ defined by $d(x)=D(x,x,x,x)$ is called the trace of D where $D: N\times N\times N\times N\to N$ is a permuting mapping. It is obvious that, if $D: N\times N\times N\times N\to N$ is a permuting mapping which is also 4 –additive (i.e, additive in all arguments), then the trace of D satisfies the relation $d(x+y)=d(x)+d(y)+4D(x,x,x,y)+6D(x,x,y,y)+4D(x,y,y,y)$ for all $x,y\epsilon N$.A permuting 4- additive mapping $D: N\times N\times N\times N\to N$ is called a permuting 4-derivation if $D(xw,y,z,t)=D(x,y,z,t)w+xD(w,y,z,t)$ is fulfilled for all $x,y,z,t,w\in N$.

Lemma 5.4.1(Bell, 1997)

Let N be a prime ring.

(i) If $z \in Z - \{0\}$, then z is not a zero divisor.

(ii) If $Z - \{0\}$ contains an element z for which $z + z \in Z$, then $(N, +)$ is abelian.

Lemma 5.4.2(Park and Jung, 2010)

Let N be a 3! torsion free near ring,and suppose there exists a permuting fourth additive mapping $D: N \times N \times N \to N$ such that $d(x) = 0$ *for all* $x \in N$, where d is the trace of D. Then we have D=0.

Lemma 5.4.3(Ozturk, 1999)

Let N be a 3! torsion free prime near ring and let $x \in N$. Suppose there exists a non zero permuting fourth- derivation $D: N \times N \times N \to N$ such that $xd(y) = 0$ for all $x, y \in N$, where d is the trace of D. Then we have $x = 0$.

Firstly, we introduce the definition of permuting fourth-(σ, τ)-derivation in a near ring.

Definition 5.2.1

A permuting 4-additive mapping $D: N \times N \times N \times N \to N$ is called a permuting 4-(σ, τ)-derivation if there exist functions

$\sigma, \tau : N \to N \text{ such that } D(xw, y, z, t) = D(x, y, z, t)\sigma(w) + \tau(x)D(w, y, z, t) \text{ for all } x, y, z, t, w \in N.$

Remark 5.4.1

Note that if $\sigma = 1 \text{ and } \tau = 1$ then D is a permuting 4- derivation.

Lemma 5.4.4

Let N be a 4! torsion free prime near ring, D a permuting 4-(σ, τ)-derivation of N and d the trace of D.If $xd(N) = \{0\} \text{ for all } x \in N, \text{ then } x = 0 \text{ or } D = 0$.

ProofWe have $x\ D(w, y, z, t) = 0 \text{ for all } x, y, z, t, w \in N$.

Replacing $z \text{ by } zv, v \in N, \text{to get } x\tau(z)D(w, y, z, t) = 0 \text{ for all } x, y, z, t, w, v \in N.$ Since τ is an automorphism of N, we get $xND(w, y, z, t) = \{0\}$. Again N is a prime near ring, and so we have $x = 0 \text{ or } D = 0.$

Lemma 5.4.5

Let N be a near ring.D is a permuting 4-(σ, τ)-derivation of N if and onlyif $D(xw, y, z, t) = \tau(x)D(w, y, z, t) + D(x, y, z, t)\sigma(w) \text{ for all } x, y.z, t, w \in N.$

Proof Let D be a permuting4-(σ, τ)-derivation of N. Since σ is an automorphism, we get

$D(x(w+w), y, z, t) = D(x, y, z, t)\sigma(w+w) + \tau(x)D(w+w, y, z, t)$

$= D(x, y, z, t)\sigma(w) + D(x, y, z, t)\sigma(w) + \tau(x)D(w, y, z, t) + \tau(x)D(w, y, z, t) \forall x, y, z, t, w \in N.$
This implies that

$$D(x(w+w),y,z,t) = D(xw+xw,y,z,t) = D(xw,y,z,t) + D(xw,y,z,t)$$
$$= D(x,y,z,t)\sigma(w) + \tau(x)D(w,y,z,t) + D(x,y,z,t)\sigma(w) + \tau(x)D(w,y,z,t).$$

Combining the above two equalities, we find that

$$D(x,y,z,t)\sigma(w) + \tau(x)D(w,y,z,t) = \tau(x)D(w,y,z,t) + D(x,y,z,t)\sigma(w).$$

Hence we have

$$D(xw,y,z,t) = \tau(x)D(w,y,z,t) + D(x,y,z,t)\sigma(w) \text{ for all } x,y,z,t,w \in N.$$

Converse can be proved in a similar way.

Lemma 5.4.6

Let N be a near ring, D a permuting 4-(σ,τ)-derivation of N. Then, for all $x,y,z,t,w,v \in N.$ we have

(i) $\left(\left(D(x,y,z,t)\sigma(w) + \tau(x)D(w,y,z,t)\right)v = D(x,y,z,t)\sigma(w)v + \tau(x)D(w,y,z,t)v.\right.$

(ii) $\left[\tau(x)D(w,y,z,t) + D(x,y,z,t)\sigma(w)\right]v = \tau(x)D(w,y,z,t)v + D(x,y,z,t)\sigma(w)v$.

Proof (i) Let D be a permuting 4-(σ,τ)-derivation of N. Since $\sigma \text{ and } \tau$ are automorphisms, we get for all $x,y,z,t,w \epsilon N$,

$$D((xw)t,y,z) = D(xw,y,z)\sigma(t) + \tau(xw)D(t,y,z)$$

$$= \left[D(x,y,z)\sigma(w) + \tau(x)D(w,y,z,t)\right]\sigma(t) + \tau(x)\tau(w)D(t,y,z)$$

This implies that

$$D(x(wt), y.z) = D(x, y, z, t)\sigma(wt) + \tau(x)D(wt, y, z, t)$$

$$= D(x, y, z, t)\sigma(w)\sigma(t) + \tau(x)D(w, y, z, t)\sigma(t) + \tau(x)\tau(w)D(t, y, z).$$

Combining the above two equalities, we find that

$$\left[D(x, y, z, t)\sigma(w) + \tau(x)D(w, y, z, t)\right]\sigma(t) = D(x, y, z, t)\sigma(w)\sigma(t) + \tau(x)D(w, y, z, t)\sigma(t)$$

Therefore, we have

$$\left[D(x, y, z, t)\sigma(w) + \tau(x)D(w, y, z, t)\right]v = D(x, y, z, t)\sigma(w) + \tau(x)D(w, y, z, t)v \quad \forall x, y, z, t, v, w \in N.$$

Similarly (ii) can be proved in the same way.

Lemma 5.4.7

Let N be a prime near ring. D a non zero permuting 4-(σ, τ)-derivation of N. Then $D(N, N, N, N)x = \{0\} \forall x \in N \mapsto x = 0.$

ProofSuppose $D(y, z, t, w)x = 0 \forall x, y, z, t, w \in N$. Then taking yv instead of y, we have $0 = D(y, z, t, w)\sigma(v)x + \tau(y)D(v, z, t, w)x = D(y, z, t, w)\sigma(v)x$

$$D(x, y, z, t)\sigma(v)\sigma(v) + \tau(x)D(v, y, z, t)\sigma(v) = \sigma(v)D(x, y, z, t)\sigma(v) + \sigma(v)\tau(x)D(v, y, z, t)$$

Since σ is an automorphism, we have

$D(y, z, t, w)Nx = \{0\}$ *for all* $x, y, z, t, w \in N$. Since N is prime near ring and D is non zero, this implies $x = 0.$

Theorem 5.4.1

Let N be a prime near ring and D a non- zero permuting 4-(σ,τ)derivation of N. If $D(N,N,N,N)\subseteq Z,$ then N is a commutative ring.

Proof Since $D(N,N,N,N)\subseteq Z$ and $D$is a non zero permuting 4-(σ,τ)-derivation of N, then there exist non zero elements$x, y, z, t \in N$ such that

$$D(x,y,z,t) \in Z \setminus \{0\}.$$

Replacing x by x + x in the above relation (5.4.7), we find that

$$D(x+x,y,z,t)=D(x,y,z,t)+D(x,y,z,t)\in Z.$$

In view of Lemma 5.4.1, $(N,+)$ is abelian.

$$D(x,y,z,t)\in Z\ \forall\ x,y,z,t\in N$$

This implies that $D(x,y,z,t)\,w = wD(x,y,z,t)$ *for all* $w\in N$.

Replace xv by x to get

$$D(x,y.z,t)\sigma(v)w+\tau(x)D(v,y,z,t)w=wD(x,y,z,t)\sigma(v)+w\tau(x)D(v,y,z,t) \quad (5.4.1)$$

Taking $\sigma(v)$ in place of w, we have

$$D(x,y,z,t)\sigma(v)\sigma(v)+\tau(x)D(v,y,z,t)\sigma(v)=\sigma(v)D(x,y,z,t)\sigma(v)+\sigma(v)\tau(x)D(v,y,z,t)$$

Since $D(N,N,N,N) \subseteq Z,$ we get

$D(v,y,z,t)[\tau(x),\sigma(v)] = 0$ *for all* $x,y,z,t,v \in N.$

Since Z contains no non zero divisors, we observe that for each $v \in N,$

either $D(v,y,z,t) = 0$

or $[\tau(x),\sigma(v)] = 0 \forall\ x,y,z,t \in N.$

Using equation (5.4.1), we find that

$$D(x,y,z,t)[w,\sigma(v)] = 0$$

Since N is a prime near ring and $D \neq 0,$ then

$$[w,\sigma(v)] = 0.$$

Since σ is an automorphism, N is commutative near ring.

If $[\tau(x),\sigma(v)] = 0 \forall\ x,v \in N,$ then N is commutative ring, since σ and τ are automorphisms.

Theorem 5.4.2

Let N be a 4! torsion free prime near ring. D a permuting 4-(σ,τ)-derivation of N and d trace of D. If $d(x)\sigma(y) = \tau(x)d(y) \forall\ x,y \in N$ then $d = 0.$

Proof By hypothesis, we have

$d(x)\sigma(y) = \tau(x)d(y) \forall\ x,y \in N$.

Replacing y by $y+z, z \in N$ in the above relation (5.4.7), we get

$$d(x)\sigma(y+z) = \tau(x)d(y+z)$$

$$\therefore d(x)\sigma(x) + d(x)\sigma(z) = \tau(x)d(y) + 2\tau(x)D(y,y,z,t) + \tau(x)D(y,z,z,t) +$$

$$\tau(x)D(y,y,z,t) + 2\tau(x)D(y,z,z,t) + \tau(x)d(z)$$

This gives

$$2D(y,y,z,t) + D(y,z,z,t) + D(y,y,z,t) + 2D(y,z,z,t) = 0 \qquad (5.4.2)$$

Since τ is an automorphism and N is a prime near ring.Replacing y by $-y$ in equation (5.4.2), we obtain

$$2D(y,y,z,t) - D(y,z,z,t) + D(y,y,z,t) - 2D(y,z,z,t) = 0 \qquad (5.4.3)$$

Replacing y by $z+y, z \in N$ in the hypothesis, we get

$$d(x)\sigma(z+y) = \tau(x)d(z+y)$$

$$d(x)\sigma(z) + d(x)\sigma(y) = \tau(x)d(z) + 2\tau(x)D(z,z,y) + 2D(z,y,y) = 0$$

Since τ is an automorphism and N is a prime near ring.

Comparing (5.4.2) with (5.4.3), we get

$$2D(y,z,z,t) + D(y,y,z,t) + 2D(y,y,z,t) = D(y,y,z,t) - 3D(y,z,z,t)$$

$$+2D(y,y,z,t) \forall y, z \in N \quad (5.4.4)$$

From (5.4.4), we have

$$D(y,y,z,t)-3D(y,z,z,t)+2D(y,y,z,t)=0. \qquad (5.4.5)$$

Replacing y by -y in (5.4.5), we get

$$D(y,y,z,t)+3D(y,z,z,t)+2D(y,y,z,t)=0. \qquad (5.4.6)$$

Comparing (5.4.5) and (5.4.6), we obtain

$$D(y,z,z,t)=0\,\forall\, y,z,t\in N.$$

Replacing z by z+x, we get

$$D(x,y,z,t)=0\,\forall\, x,y,z,t\in N\ .$$

Hence D=0 and then d=0.

Theorem 5.4.3

Let N be a 4! torsion free prime ring near ring, D a permuting 4-(σ,τ) derivation of N, and d the trace of D. If $d(x)$, $d(x)+d(x)\in C(D(N,N,N,N))\,\forall\, x\in N.$, then (N,+) is abelian.

Proof Assume that

$$d(x), d(x)+d(x)\in C(D(y,v,w,t)) \text{ for all } y,v,w,t\in N\ . \qquad (5.2.7)$$

From (5.2.7) we get

$$D(y+t,v,w)(d(x)+d(x))=(d(x)+d(x))D(y+t,v.w)$$

$= (d(x) + d(x))D(y, v, w, s) + (d(x) + d(x))D(t, v, w)$

$= d(x)D(y, v, w, t) + d(x)D(y, v, w, t) + d(x)D(t, v, y, w) + d(x)D(t, v, y, w)$

$= d(x)\left[D(y, v, w, t) + D(y, v, w, t) + D(t, v, w, y) \right]$

$= \left[D(y, v, w, t) + D(y, v, w, t) + D(t, v.w, y) + D(t, v, w, y) \right] d(x)$

and

$D(y + t, v, w, s)(d(x) + d(x)) = D(y + t, v, w, s)d(x) + D(y + t, v, w, s)d(x)$

$= D(y, v, w, t)d(x) + D(t, v, w, y)d(x) + D(y, v, w, t) + D(t, v, w, y)d(x)$

$= \left[D(y, v, w, t) + D(t, v, w, y) + D(y, v, w, t) + D(t, v, w, y) \right] d(x)$ For all $x, y, t, w, s \in N.$

Comparing the last two equations,

$D((y, t), v, w, s)d(x) = 0$ *for* $x, y, , t, v, w, s \in N.$

Replacing y *by* yz *and* t *by* $yt, z \in N$, we get

$0 = D(y(z, t), v, w, s) = D(y, v, w, s)\sigma(z, t) + \tau(y)D((z, t), v, w, s)$

$= D(y, v, w, s)\sigma(z, t)$

Since D is a nonzero permuting 4-(σ, τ) derivation and N is a prime ring, we have $(\sigma, \tau) = 0$. Since σ is an automorphism, $(z, t) = 0 \forall z, t \in N.$ Thus $(N, +)$ is abelian.

5.5 Some Results on Permuting $n-(\sigma,\tau)$Derivation on Near-rings

This work introduces a notion of a permuting $n-(\sigma,\tau)$ derivation in near-rings, Where$n$ is apositive integer, σ and $\tau$are automorphisms on near-rings. In addition,

we prove theirproperties together with a prime near-ring to be a commutative ring.

An additive map *d:* N→ *N* is called a *derivation* if *d*(*x y*) = *d*(*x*) *y* + *x d*(*y*) holds ∀*x, y* ∈ *N*. By a *bi-derivation* we mean a bi-additive map d:*N* ×*N* → *N* (i.e., dis additive in both arguments) which satisfies the relations d(*x y, z*) = d(*x, z*) *y* + *x d* (*y, z*) and

d(*x, y* z) = d(*x, y*)z+ *y d*(*x,* z) for all *x, y,* z ∈ *N*. Let dbe symmetric

(i.e. d(*x, y*) = d(*y, x*) for all *x, y* ∈ N). The mapping δ: *N* → *N* defined by δ(*x*) = d(*x, x*) for all *x* ∈ *N* is said to be the *trace* of d.

A map d: $\underbrace{\mathrm{N}\times\mathrm{N}\times\ \ldots\ \times\mathrm{N}}_{n-times}\rightarrow$ Nis said to be *permuting* if the equation

d(x_1, x_2,… , x_n) = d($x_{\pi(1)}$, $x_{\pi(2)}$, . . . , $x_{\pi(n)}$) holds for all x_1, x_2, . . . , x_n ∈ *N* and for every permutation {π(1), π(2), . . . , π(n)}.

A map δ*:* N→ Ndefined by δ(*x*) = d(*x, x,* . . . , *x*) for all *x*∈ N, where

d: $\underbrace{\mathrm{N}\times\mathrm{N}\times\ \ldots\ \times\mathrm{N}}_{n-times}\rightarrow$ Nis a permuting map,which is called the *trace* of d.

A permuting n-additive (i.e., additive in each argument) mapping d: $\underbrace{\mathrm{N}\times\mathrm{N}\times\ \ldots\ \times\mathrm{N}}_{n-times}\rightarrow$ N is called a permuting n-derivation, where n ≥ 1, if, ∀x_1,a_1, x_2,a_2, …, x_n ,a_n ∈N, the relations

d(x_1a_1, x_2, …, x_n) = d(x_1, x_2, …,x_n) a_1 + x_1d(a_1, x_2, …,x_n),

$d(x_1, x_2a_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)\, a_2 + x_2 d(x_1, a_2, \ldots, x_n)$,

. . .

$d(x_1, x_2, \ldots, x_n\, a_n) = d(x_1, x_2, \ldots, x_n)\, a_n + x_n\, d(x_1, x_2, \ldots, a_n)$ are fulfilled.

For an example of a permuting n-derivation, take a fixed positive integer n, and a commutative near-ring N.

Then $M_2(N) = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \| \alpha, \beta, 0 \in N \right\}$ is a non-commutative near-ring with respect to matrix addition and matrix multiplication. Define $d: \underbrace{M_2(N) \times M_2(N) \times \ldots \times M_2(N)}_{n-times} \to M_2(N)$ by

$d\left(\begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & 0 \end{pmatrix} \ldots \begin{pmatrix} \alpha_n & \beta_n \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & \alpha_1\alpha_2 \ldots \alpha_n \\ 0 & 0 \end{pmatrix}$,d is a *permuting n-derivation* of $M_2(N)$.

Next, let d be a permuting n-derivation of a near-ring N. Clearly, $\forall\ 0, x_2, \ldots, x_n \in N$:

$d(0, x_2, \ldots, x_n) = d(0 + 0, x_2, \ldots, x_n) = d(0, x_2, \ldots, x_n) + d(0, x_2, \ldots, x_n)$.

Hence $d(0, x_2, \ldots, x_n) = 0$. One can also see that $d(-x_1, x_2, \ldots, x_n) = -d(x_1, x_2, \ldots, x_n)$.

Before we proceed, we statethe following well known facts.

Lemma5.5.1 (Bell and Mason, 1987)

Let N be a prime near-ring.

(i) If $z \in C \setminus \{0\}$, then z is not a zero divisor.

(ii) If C \ {0} contains an element z for which z + z ∈ C, then (N, +) is abelian.

Lemma5.5.2 Bell and Mason (1987)

If z ∈ C \ {0} and x is an element of a prime near-ring N such that x z ∈ C or z x ∈ C, then x ∈ C.

Lemma5.5.3 (Ashraf and Siddiqee, 2013)

Let N be an n!-torsion free near-ring and let (N,+) be anabelian group, where n is a positive integer. Suppose $x_1, x_2, \ldots, x_n \in N$ satisfy

$ax_1 + a^2x_2 + \cdots + a^n x_n = 0$ for a = 1, 2, . . . ,n. Then $x_i = 0$ for all i.

Lemma5.5.4(Ashraf and Siddiqee, 2013)

Let N be an n!-torsion free near-ring. Suppose that there exists a permuting n-additive mapping d: $\underbrace{N \times N \times \ldots \times N}_{n-times} \rightarrow N$ such that d (N, N, . . . ,N) ⊆Z. If δ(x) = 0 for all x ∈N, where δ is the trace of d, then we have d = 0.

Several authors have proved some results on properties of prime rings, semi-prime rings and near-rings with derivations. There has been a great deal of work concerning derivations, symmetric (permuting) bi-derivations, tri-derivations and n-derivations in near-rings. Golbasi and Aydin (2004) proved that if a prime near-ring *N* admits a nontrivial (*σ,* τ)-derivation *d* for which d(N) ⊆Z, then (N,+) is abelian. Moreover, if Nis 2-torsion free and σ, τcommute with d, then Nis a commutative ring. In addition, this result was extended for symmetric bi-(σ, τ)-derivations by Ceven and Ozturk

(2007)as follows: If N is a 2-torsion free prime near-ring which admits a nonzero symmetric bi-(σ, τ) derivation dsuch that d(N,N) $\subseteq$Z, then N is a commutative ring.

Park and Jung (2010) introduced the concept of a permuting tri-derivation of a near-ring and investigated the conditions for a near ring to be a commutative ring. Further, this result was generalized by Ozturk and Yazarli(2011) in the setting of permuting tri-(σ, τ)-derivation and proved that if Nis a prime near-ring, da nonzero permuting tri- (σ, τ)-derivation of N such that d (N, N, N) $\subseteq$Z, then Nis a commutative ring.

In this line of continuation, a natural question arises: "Does the result of Ozturk and Yazarli(2011) hold for permuting n-(σ, τ) derivation"? Answering this question, we introduce the concept of permuting n-(s, t) derivation and establish some of its properties. Indeed, our results generalize very recent known results of near-rings.

In this sequel, we introduce the notion of permuting n-(s, t) derivation in a near-ring N.

Definition 5.5.1

A permuting n-additive mapping d: $\underbrace{N \times N \times \ldots \times N}_{n-times} \rightarrow N$ is said to be a permuting n-(s ,t)-derivation if there exist automorphisms s, t: N $\rightarrow$N such that

$$d(x_1 a_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)s(a_1) + t(x_1)d(x_1, x_2, \ldots, x_n)$$

For all $x_1, x_2, \ldots, x_n, a_1 \in N$.

Remark 5.5.1

(i) Clearly, a permuting 1-(1, 1) derivation is a derivation, permuting 2-(1, 1) derivation is a permuting bi-derivation, permuting 3-(1, 1) derivation is a permuting tri-derivation, and permuting n-(1, 1) derivation is a permuting n-derivation.

(ii) We give an example ofa nonzero permuting n-(s, t)-derivation on near-ring.

Take a zero symmetric left near-ring B, the set of complex numbers. Let

$$\mathrm{N} = \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \middle| \alpha, \beta \in B \right\}.$$

Clearly, N is a noncommutative zero symmetric left near–ring with respect to matrix addition and matrix multiplication. Define d: N × N × . . . × N$\to$ N by

$$\mathrm{d}\left(\begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & 0 \end{pmatrix} \cdots \begin{pmatrix} \alpha_n & \beta_n \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & \alpha_1^{-}\alpha_2^{-}..\alpha_n^{-} \\ 0 & 0 \end{pmatrix}$$

$$\text{s: N} \to \text{N by s} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ 0 & 0 \end{pmatrix} \text{ and t: N} \to \text{N by t} \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a^{-} & b^{-} \\ 0 & 0 \end{pmatrix},$$

where $\alpha_1^{-}\alpha_2^{-}..\alpha_n^{-}$, a^{-}, b^{-} are the conjugates of the complex numbers $\alpha_1 \alpha_2..\alpha_n$ a and b respectively. One can easily see that d is a permuting n-(s, t) derivation of N, where s and t are automorphisms.

(iii) Next, an example of a permuting n-(s, t) derivation which is not a permuting n-derivation in N.

Define a map d: $\underbrace{\mathrm{N} \times \mathrm{N} \times \ \ldots \times \mathrm{N}}_{n-times} \to$ N by $\mathrm{d}(x_1, x_2, ..., x_n) = \begin{cases} \beta & x_1, x_2 ..., x_n \notin \{0, \alpha\} \\ 0, & Otherwise \end{cases}$

Here(N, +) is the Klein's four group with multiplication defined as given below.

If 0, α, β, $\gamma \in$ N, then $0 = 00 = 0\alpha = 0\beta = 0\gamma$; $\alpha 0 = 0 = \alpha\alpha$, $\alpha\beta = \alpha = \alpha\gamma$; $\beta 0 = 0$, $\beta\alpha = \alpha$, $\beta\beta = \beta$, $\beta\gamma = \gamma$; $\gamma 0 = 0$, $\gamma\alpha = \alpha$, $\gamma\beta = \beta$, $\gamma\gamma = \gamma$.

Taking s (0) = s(α) = 0, s(β) = β , s (γ) = γ and $t(x_1, x_2, ..., x_n) = 0$, for all $x_i \in N$, i= 1,2,…,n, then d is a permuting n-(s ,t)derivation on near-ring N. But, since

$d(\beta\alpha, \beta, \ldots, \gamma) = d(\beta, \beta, \ldots, \gamma)\alpha + \beta d(\alpha, \beta, \ldots, \gamma) = \alpha$

and $d(\beta\alpha, \beta, \ldots, \gamma) = d(\alpha, \beta, \ldots, \gamma) = 0$, dis not permuting n-derivation on N.

In remark (iii) aboved is not a permuting $n - (s, t)$ derivation, since t is not an automorphism of N.

Theorem 5.5.1

Let N be a prime near ring and d a nonzero permuting n- (s, t)-derivation of N.

If $d(N, N, \ldots, N) \subseteq Z$, then N is a commutative ring.

In order to prove our Theorem 5.5.1 we first establish the following results.

Lemma5.5.5

Let N be a near ring. d is a permuting n- (s ,t)-derivation of N if and only if

$d(x_1a_1, x_2, x_3, \ldots, x_n) = t(x_1)d(a_1, x_2, x_3, \ldots, x_n) + d(x_1, x_2, x_3, \ldots, x_n)\sigma(a_1)$ for all

$x_1, x_2, x_3, \ldots, x_n, a_1 \in N$.

ProofLet dbe a permuting n- (s, t)derivation of N . If, $\forall x_1, x_2, x_3, \ldots, x_n, a_1 \in N$ and sis an automorphism, then we find that

$d(x_1(a_1 + a_1), x_2, x_3, \ldots, x_n) = d(x_1, x_2, x_3, \ldots, x_n)s(a_1 + a_1) + t(x_1)\, d(a_1 + a_1, x_2, x_3, \ldots, x_n)$

$= d(x_1, x_2, x_3, \ldots, x_n)s(a_1) + d(x_1, x_2, x_3, \ldots, x_n)s(a_1) + t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n)$

$+ t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n)$ (5.5.1)

and

$d(x_1(a_1 + a_1), x_2, x_3, \ldots, x_n) = d(x_1 a_1 + x_1 a_1, x_2, x_3, \ldots, x_n) = d(x_1 a_1, x_2, x_3, \ldots, x_n)$

$+ d(x_1 a_1, x_2, x_3, \ldots, x_n)$

$= d(x_1, x_2, x_3, \ldots, x_n)s(a_1) + t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n) + d(x_1, x_2, x_3, \ldots, x_n)s(a_1)$

$+ t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n)$. (5.5.2)

Combining (5.5.1) and (5.5.2), we obtain

$d(x_1, x_2, x_3, \ldots, x_n)s(a_1) + t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n)$

$= t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n) + d(x_1, x_2, x_3, \ldots, x_n)s(a_1)$.

This implies that

$d(x_1 a_1, x_2, x_3, \ldots, x_n) = t(x_1) d(a_1, x_2, x_3, \ldots, x_n) + d(x_1, x_2, x_3, \ldots, x_n)s(a_1)$ for all

$x_1, x_2, x_3, \ldots, x_n, a_1 \in N$.

Converse part can be proved in a similar way.

Since in a left near-ring N, right distributive law does not hold in general, then we can prove the following partial distributive laws in N.

Lemma5.5.6

Let N be a near- ring, d a permuting n-(s, t) derivation of N and δ the trace of d. Then, for all $x_1, x_2, x_3, \ldots, x_n, a_1, w \in N$,

(i) $[d(x_1, x_2, x_3, \ldots, x_n)s(a_1) + t(x_1)d(a_1, x_2, x_3, \ldots, x_n)]w = d(x_1, x_2, x_3, \ldots, x_n)s(a_1)w$

$+t(x_1)d(a_1, x_2, x_3, \ldots, x_n)w,$

(ii) $[t(x_1)d(a_1, x_2, x_3, \ldots, x_n) + d(x_1, x_2, x_3, \ldots, x_n)s(a_1)]w = t(x_1)d(a_1, x_2, x_3, \ldots, x_n)w$

$+ d(x_1, x_2, x_3, \ldots, x_n)s(a_1)w.$

(iii) $\{\delta(x)x_1 + x\, d(x, x, \ldots, x, x_1)\}w = \delta(x)x_1w + x\, d(x, x, \ldots, x, x_1)w,$

(iv) $\{x\, d(x, x, \ldots, x, x_1) + \delta(x)x_1\}w = x\, d(x, x, \ldots, x, x_1)w + \delta(x)x_1w.$

Proof(i) Let dbe a permuting n-(s, t) derivation of N. If $\forall x_1, x_2, x_3, \ldots, x_n, a_1, b_1, w \in N$, then we have

$d((x_1 a_1)b_1, x_2, x_3, \ldots, x_n) = d(x_1 a_1, x_2, x_3, \ldots, x_n)s(b_1) + t(x_1 a_1)\, d(b_1, x_2, x_3, \ldots, x_n)$

$= [d(x_1, x_2, x_3, \ldots, x_n)s(a_1) + t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n)]\, s(b_1) + t(x_1)\, t(a_1)\, d(b_1, x_2, x_3, \ldots, x_n).$

$\Rightarrow d(x_1(a_1b_1), x_2, x_3, \ldots, x_n) = d(x_1, x_2, x_3, \ldots, x_n)s(a_1b_1) + t(x_1)\, d(a_1b_1, x_2, x_3, \ldots, x_n)$

$= d(x_1, x_2, x_3, \ldots, x_n)\, s(a_1)s(b_1) + t(x_1)d(a_1, x_2, x_3, \ldots, x_n)s(b_1) + t(x_1)t(a_1)\, d(b_1, x_2, x_3, \ldots, x_n).$

Combining the above two equalities, we have

$[d(x_1, x_2, x_3, \ldots, x_n)s(a_1) + t(x_1)\, d(a_1, x_2, x_3, \ldots, x_n)]\, s(b_1)$

$= d(x_1, x_2, x_3, \ldots, x_n)\, s(a_1)s\,(b_1) + t(x_1)d(a_1, x_2, x_3, \ldots, x_n)s(b_1).$

Since s is an automorphism and putting $s\,(b_1) = b_1 = w$ in the above equality, we have

$[d\,(x_1, x_2, x_3, \ldots, x_n)s\,(a_1) + t(x_1)\,d(a_1, x_2, x_3, \ldots, x_n)]\,w$

$= d(x_1, x_2, x_3, \ldots, x_n)\, s(a_1)\, w + t(x_1)d(a_1, x_2, x_3, \ldots, x_n)w.$

(*ii*) Using Lemma 2.2, it can be proved in a similar way as above.

(iii) Substituting $x_1 = x_2 = x_3 = \ldots = x_n = x$ in the proof of (i), we get

$$\{\delta(x)a_1 + x\, d(x, x, \ldots, x, a_1)\}w = \delta(x)a_1w + x\, d(x, x, \ldots, x, a_1)w.$$

Particularly, if $a_1 = x_1$, we obtain

$$\{\delta(x)x_1 + x\, d(x, x, \ldots, x, x_1)\}w = \delta(x)x_1w + x\, d(x, x, \ldots, x, x_1)w.$$

The relation (iv) can be proved in a similar way as above.

Lemma5.5.7

Let N be a prime near-ring, d a non-zero permuting n- (s, t) derivation of N. Then (i) d $(N,N,\ldots,N)x = \{0\}$ for all $x \in N$ which implies $x = 0$

(ii) If $x\, d\,(N, N, \ldots, N) = \{0\}$ where $x \in N$, then $x = 0$.

Proof.(i) For every$x_1, x_2, x_3, \ldots, x_n \in N$, let $d(x_1, x_2, x_3, \ldots, x_n)x = 0$.

Replacing x_1 by $w\, x_1$, we have

$d(w, x_2, x_3, \ldots, x_n)\, s(x_1)x + \tau\,(w)d(x_1, x_2, x_3, \ldots, x_n)x = 0,$

$= d\,(w, x_2, x_3, \ldots, x_n)s(x_1)x$

Since sis an automorphism, we have

$d(w, x_2, x_3, \ldots, x_n)N\ x = \{0\}$ for all $x_1, x_2, x_3, \ldots, x_n, w \in N$.

or

$d(N,N,\ldots,N)\ N\ x = \{0\}$. Since N is a prime near-ring and d nonzero, which implies x = 0.

(ii) It can be proved in a similar way.

Lemma5.5.8

Let d be a nonzero permuting n-(s, t) derivation of a prime near-ring

N. Then $d(C, C, \ldots, C) \neq \{0\}$ where $C \neq \{0\}$.

Proof. On the contrary, assume $d(C, C, \ldots, C) = \{0\}$, then $d(c_1, c_2, \ldots, c_n) = 0$

for all $c_1, c_2, \ldots, c_n \in C$. For all $r_1 \in N$ and $c_1 \in C$ we get $r_1c_1 \in C$. Also

$d(r_1c_1, c_2, \ldots, c_n) = 0$ implies $t(r_1)\, d(c_1, c_2, \ldots, c_n) + d(r_1, c_2, \ldots, c_n)s(c_1) = 0$.

Thus we get

$$d(r_1, c_2, \ldots, c_n)\, s(c_1) = 0. \qquad (5.5.3)$$

Replacing c_1 by xc_1 in equation (5.5.3) where $x \in N$ and s is an automorphism, we find that

$d(r_1, c_2, \ldots, c_n)Nc_1 = \{0\}$. Primeness of N gives,

$$d(r_1, c_2, \ldots, c_n) = 0. \qquad (5.5.4)$$

Next, putting $r_2c_2 \in C$ in place of c_2 where $r_2 \in N$ in equation (5.5.4), and proceeding as above we have $d(r_1, r_2, \ldots, c_n) = 0$. Continuing this process inductively,we find that $d(r_1, r_2, \ldots, r_n) = 0$, for all $r_1, r_2, \ldots, r_n \in N$, a contradiction.

Lemma5.5.9

Let N be an n!-torsion free prime near-ring and let $x \in N$. Suppose there exists a nonzero permuting n-(σ, τ)derivation $d: \underbrace{N \times N \times \ldots \times N}_{n-times} \to N$ such that d(N,N, . . . ,N) $\subseteq Z$. Let $x\delta(y) = 0$ for all $x, y \in N$, where δ is the trace of d. Then x = 0 or d = 0.

Proof Using same method as in the proof of Result 5.5.3, we have $d(x_1, x_2, \ldots, x_n) = 0$.

If, $\forall x_1, x_2, \ldots, x_n, x \in N$, then we can write

$$xd(x_1, x_2, \ldots, x_n) = 0.$$

Replacing $x_1$by $x_1 v$, $v \in N$ in the above relation, we get

$$xt(x_1)\, d(v, x_2, \ldots, x_n) = 0 \ \forall x_1, x_2, \ldots, x_n, v \in N.$$

Since tis an automorphism of *N*, then we get

$$x\, N\, d(v, x_2, \ldots, x_n) = \{0\}.$$

Again *N* is prime near-ring and so we have x= 0 or d= 0.

Proof of Theorem 5.5.1

Since d(N, N, …,N)$\subseteq$Zand dis a nonzero permuting n-(s ,t)-derivation of *N* , then there exist nonzero elements $x_1, x_2, \ldots, x_n \in N$ such that $d(x_1, x_2, \ldots, x_n) \in Z - \{0\}$. Then

$d(x_1+ x_1, x_2, \ldots, x_n) = d(x_1, x_2, \ldots, x_n)+ d(x_1, x_2, \ldots, x_n) \in Z$ and in view of Lemma 5.3.2, $(N,+)$ is abelian. With this condition, we have $d(x_1, x_2,\ldots, x_n) \in Z$, for $x_1, x_2, \ldots, x_n \in N$, which implies $d(x_1, x_2, \ldots, x_n)w = w\, d(x_1, x_2, \ldots, x_n)$ for all $w \in N$.

Replacing x_1 in the above relation for x_1v and using d a permuting n- (s, t) derivation,we get

$$\{d(x_1, x_2, \ldots, x_n)s(v) + t(x_1)d(v, x_1, x_2, \ldots, x_n)\}w$$

$$= w\, \{d(x_1, x_2, \ldots, x_n)\, s(v)+ t(x_1)\, d(v, x_1, x_2, \ldots, x_n)\} \forall\ v, w, x_1, x_2, \ldots, x_n \in N \quad (5.5.5)$$

Replacing $s(v)$ by w in the relation (5.3.5) and using Lemmm5.3.4(i), we have $d(x_1, x_2, \ldots, x_n)s(v)\, s(v) + t(x_1)d(v, x_1, x_2, \ldots, x_n)\, s(v) = s(v)d(x_1, x_2, \ldots, x_n)\, s(v)+ s(v)\, t(x_1)\, d(v, x_1, x_2, \ldots, x_n)$.

In view of the hypothesis $(d(N, N, \ldots, N) \subseteq Z)$, the above relation reduces to

$$d(v, x_2, \ldots, x_n)\, N[s(v), t(x_1)] = \{0\} \forall\ v, x_1, x_2, \ldots, x_n \in N.$$

One can observe that for each $x_1 \in N$, either $d(v, x_2, \ldots, x_n) = 0$ or $[s(v), t(x_1)] = 0$, because N is a prime near-ring.

Let $d(v, x_2, \ldots, x_n) = 0$. Then by hypothesis, relation (5.3.5) becomes $d(x_1, x_2, \ldots, x_n)N[w, s(v)] = \{0\}$ for all $w \in N$. Since N is a prime near-ring and $d \neq 0$, the above relation yields

$[w, s(v)] = 0$ for all w and $v \in N$.

Since s is an automorphism, [w, v] = 0 and v $\in$Z, N is commutative near ring.If [s(v), t(x_1)] = 0 for all v and $x_1 \in$N, then since s and t are automorphisms, [v,x_1] = 0, we have $x_1 \in$Z, and so we get N as a commutative near-ring.

Remark 5.3.7

The following example shows that for N to be prime near-ring is essential in the hypothesis of Theorem 5.3.1.

Example 5.5.14

Taking N = P + Q, where P is the polynomial ring in z over the field of complex numbers C and Q is the zero symmetric noncommutative prime left near-ring, then N forms a zero symmetric left near-ring with respect to component wise addition and multiplication. Define d: $\underbrace{\mathrm{N} \times \mathrm{N} \times \ \ldots \times \mathrm{N}}_{n-times} \rightarrow$ N by

$$d((p_1(z), x_1), (p_2(z), x_2), \ldots, (p_n(z), x_n)) = ((g(p_1(z)), (g(p_2(z)), \ldots, (g(p_n(z)), 0),$$

where $g(p_k(z))$, $1 \leq$ k $\leq$ n is the usual differentiation of $p_k(z) \in C[z]$, $1 \leq$ k $\leq$ n. One can easily check that d is a nonzero permuting n- (s, t) derivation of N, where s and t are identity automorphisms of N. Also, N satisfies the property d (N, N, …,N) $\subseteq$Z but N isnot a prime near-ring. Thus, N is not a commutative ring.Now, we establish the following:

Theorem 5.5.2

Let N be an n!-torsion free prime near-ring, d a permuting n- (s, t)-derivation of N and δ the trace of d. If δ(x) s (y) = t(x) δ(y) for all x, y$\in$N, then δ= 0.

ProofBy hypothesis, we have

$\delta(x)s(y) = t(x)\delta(y)$ for all $x, y \in N$. (5.5.6)

Replacing y by $y + z$, $z \in N$ in the relation (5.3.6), we get $\delta(x)\, s(y + z) = t(x)\, \delta(y + z)$.

This implies that

$$\delta(x)s(y) + \delta(x)s(z) = t(x)\,\delta(y) + t(x)\sum_{k=1}^{n-1}\binom{n}{k} d\left(\underbrace{y, y,..., y}_{n-k-times}, \underbrace{z, z,..., z}_{k-times}\right) + t(x)\delta(z)$$

$\forall x, y, z \in N$

Again, in view of (5.5.6), we find that

$$t(x)\sum_{k=1}^{n-1}\binom{n}{k} d\left(\underbrace{y, y,..., y}_{n-k-times}, \underbrace{z, z,..., z}_{k-times}\right) = n!\, t(x)\, d(y, y,\ldots, y, z) = 0. \quad (5.5.7)$$

Since t is an automorphism and N is an n!-torsion free prime near-ring, then (5.3.7) becomes $d(y, y,\ldots, y, z) = 0$. (5.5.8)

Replacing y by zin (5.3.8), we get $d(y, y,\ldots, y, y) = 0$, which implies $d = 0$ and an application of Lemma 5.3.4 yields $\delta = 0$.

Remark 5.5.3

Substituting $y + y$ for y in (5.3.6) and applying (5.3.6)gives $(2^n - 2)\,\delta(y) = 0$, so Theorem 5.3.1 holds if one can assume the hypothesis that N is $2^n - 2$ torsion-free.

Corollary 5.5.2(Ozturk and Yazarli (2011)

Let N be a 3!-torsion free prime near-ring, d a permuting tri- (s, t)-derivation of N and δ the trace of d. If δ (x) s (y) = t (x) δ (y) for all x, y∈N, then δ = 0.

Theorem 5.5.3

Let N be an!-torsion free prime near-ring, d a permuting n-(s, t) derivation of N and δ the trace of d. If δ(x), δ(x) + δ(x) ∈C (d (N, N,…, N))∀ x ∈N, then (N,+) is abelian.

Proof By hypothesis, we assume that

$$\delta(x), \delta(x) + \delta(x) \in C(d(x_1, x_2, \ldots, x_n)) \forall\ x, x_1, x_2, \ldots, x_n \in N. \qquad (5.5.9)$$

From the relation (5.5.9), if ∀ x, $x_1, x_2, \ldots, x_n$, w∈N, then we find that

$$d(x_1+w, x_2, \ldots, x_n)(\delta(x) + \delta(x)) = (\delta(x) + \delta(x))\, d(x_1+w, x_2, \ldots, x_n)$$

$$= \delta(x)\, d(x_1, x_2, \ldots, x_n) + \delta(x)\, d(x_1, x_2, \ldots, x_n) + \delta(x) d(w, x_2, \ldots, x_n) + \delta(x)\, d(w, x_2, \ldots, x_n)$$

$$= \delta(x)[d(x_1, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n) + d(w, x_2, \ldots, x_n) + d(w, x_2, \ldots, x_n)]$$

$$= [d(x_1, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n) + d(w, x_2, \ldots, x_n) + d(w, x_2, \ldots, x_n)]\, \delta(x)$$

and

$$d(x_1+w, x_2, \ldots, x_n)(\delta(x) + \delta(x)) = d(x_1, x_2, \ldots, x_n)\, \delta(x) + d(w, x_2, \ldots, x_n)\, \delta(x)$$

$$+ d(x_1, x_2, \ldots, x_n)\, \delta(x) + d(w, x_2, \ldots, x_n)\, \delta(x)$$

$$= [d(x_1, x_2, \ldots, x_n) + d(w, x_2, \ldots, x_n) + d(x_1, x_2, \ldots, x_n) + d(w, x_2, \ldots, x_n)]\, \delta(x).$$

From the above two relations, we find that

$[d(x_1, x_2, \ldots, x_n)+ d(w, x_2, \ldots, x_n) - d(x_1, x_2, \ldots, x_n)- d(w, x_2, \ldots, x_n)]\ \delta(x) = 0.$

Using Result 5.5.6 in the above relation, we find that

$d((x_1,w), x_2, \ldots, x_n) = 0\ \forall\ x, x_1, x_2, \ldots, x_n, w \in N.$ (5.5.10)

Substituting $x_1$by $x_1$vand wby x_1w in the relation (5.5.10), v ∈N and using (5.5.10), we getd $(x_1(v, w)), x_2, \ldots, x_n) = 0$. But dis a permuting n-(s, t)-derivation on N, we haved $(x_1, x_2, \ldots, x_n)\ s(v, w) + t(x_1)\ d((v, w), x_2, \ldots, x_n) = 0.$

This implies that

$d(x_1, x_2, \ldots, x_n)\ s(v, w) = 0\ \forall\ x, x_1, x_2, \ldots, x_n, w \in N.$ (5.5.11)

Replacing z v in the relation (5.5.11), where z∈ N for v and using it again we find thatd $(x_1, x_2, \ldots, x_n)s(z)\ s(v, w) = 0.$

or

$d(x_1, x_2, \ldots, x_n)N\ s(v, w) = \{0\}$ for all $x_1, x_2, \ldots, x_n,$ z, v, w∈N.

Next, since dis a nonzero permuting n-(s, t)-derivation and *N* is a prime near-ring, the last relation gives s (v, w) = 0. Since sis an automorphism, (v, w) = 0 for all v, w∈*N*. Hence (N, +) is abelian.

Corollary 5.5.3 (Ozturk and Yazarli, 2011)

Let N be a 3!-torsion free prime near ring, d a permuting tri- (σ, τ)-derivation of N and δ the trace of d . If δ (x), δ(x) + δ (x) ∈C(d (N, N, N)) for all x∈N , then (N, +) is abelian.

Now, we establish some results related to permuting n-(s, t) derivation in a prime near-ring.

We find the following results without torsion restrictions.

Theorem 5.5.4

Suppose d_1 and d_2 are two non zero permuting n(s, t)-derivation of prime near-ring N. Given $[d_1(N, N, \ldots, N), d_2(N, N, \ldots, N)] = \{0\}$, then (N,+) is an abelian group.

Proof By hypothesis, we have

$d_1(N, N, \ldots, N)\, d_2(N, N, \ldots, N) = d_2(N, N, \ldots, N)\, d_1(N, N, \ldots, N)]$

Let z and z + z commute element wise with $d_2(N,N, \ldots, N)$, then, using same

technique as in (Ashraf and Siddeeque.,2013) we arrive at $z\, d_2(x_1 + a_1 - x_1 - a_1, x_2, \ldots, x_n) = 0$.

or

$z\, d_2((x_1, a_1), x_2, \ldots, x_n) = 0$. Replacing $z = d_1(y_1, y_2, \ldots, y_n)$ we get $d_1(y_1, y_2, \ldots, y_n)\, d_2((x_1, a_1), x_2, \ldots, x_n) = 0$.

Using Lemma5.5.4 (i) we conclude that

$d_2((x_1, a_1), x_2, \ldots, x_n) = 0.$ (5.5.12)

Substituting $m(x_1, a_1)$ in place of additive commutator (x_1, a_1) where $m \in N$ we have

$d_2(m(x_1, a_1), x_2, \ldots, x_n) = 0.$

Since d_2 is a Permuting n-(s, t) derivation, then

$d_2(m, x_2, \ldots, x_n)\, s(x_1, a_1) + t(m)d_2((x_1, a_1), x_2, \ldots, x_n) = 0.$

Using the relation (5.5.12) in the above equality, we get

$d_2(m, x_2, \ldots, x_n)\, s(x_1, a_1) = 0.$

In view of Result 5.5.4 (i) and s is an automorphism, we find that

$(x_1, a_1) = 0$. Hence (N, +) is an abelian group.

Corollary 5.5.4(Ashraf and Siddeeque,2013)

Let N be a prime near-ring and d_1 and d_2 be any two non zero permuting n-derivations of N. If $[d_1(N, N, \ldots, N), d_2(N, N, \ldots, N)] = \{0\}$, then (N,+) is an abelian group.

Theorem 5.5.5

Suppose N is a prime near-ring with non zero permuting n-(s, t) derivationsd_1 and d_2 such that $d_1(x_1, x_2, \ldots, x_n)\, d_2(y_1, y_2, \ldots, y_n) + d_2(x_1, x_2, \ldots, x_n)d_1(y_1, y_2, \ldots, y_n) = 0$, for all $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n \in N$. Then (N, +) is an abelian group.

Proof Substituting y_1 by $y_1 + b_1$ in the given hypothesis, we find that

$d_1(x_1, x_2, \ldots, x_n)\, d_2(y_1 + b_1, y_2, \ldots, y_n) + d_2(x_1, x_2, \ldots, x_n)d_1(y_1 + b_1, y_2, \ldots, y_n) = 0$. Again, using our hypothesis we obtain,

$d_1(x_1, x_2, \ldots, x_n)\, d_2(y_1, y_2, \ldots, y_n) + d_1(x_1, x_2, \ldots, x_n)\, d_2(b_1, y_2, \ldots, y_n)$

$+ d_1(x_1, x_2, \ldots, x_n)\, d_2(-y_1, y_2, \ldots, y_n) + d_1(x_1, x_2, \ldots, x_n)\, d_2(-b_1, y_2, \ldots, y_n) = 0.$

This implies that

$$d_1(x_1, x_2, \ldots, x_n)\, d_2((y_1, b_1), y_2, \ldots, y_n) = 0.$$

Using Result 5.5.4(i), we find that

$$d_2((y_1, b_1), y_2, \ldots, y_n) = 0. \qquad (5.5.13)$$

Substituting $m(y_1, b_1)$ for (y_1, b_1) where $m \in N$ in the equality (5.5.13) and using Result 5.5.4(i), and same method as used in the Theorem 5.5.4, we obtain (N,+) is an abelian group.

Corollary 5.5.5 (Beidar, 1996).

Let N be a prime near-ring with non zero derivations d_1 and d_2 such that $d_1(x)d_2(y) + d_2(x)d_1(y)$ for all $x, y \in N$. Then (N,+) is an abelian group.

Theorem 5.5.6

Assume that d is a non zero permuting n -(s, t) derivation of prime near-ring N. Let $d(C, N, N, \ldots, N) = \{0\}$. Then (N, +) is an abelian group.

Proof By hypothesis we have

$$d(c, r_2, \ldots, r_n) = 0 \text{ for all } c \in C \text{ and for all } r_2, \ldots, r_n \in N. \qquad (5.5.14)$$

Replacing c by mc, $m \in N$, in the relation (5.5.14), we get

$$d(mc, r_2, \ldots, r_n) = 0 \text{ for all } c \in C \text{ and for all } r_2, \ldots, r_n \in N.$$

Using the fact that d is a non zero permuting n-(s, t) derivation of N in the above equality, we get

$$d(m, r_2, \ldots, r_n)\, s(c) + t(m)\, d(c, r_2, \ldots, r_n) = 0.$$

Using the relation (5.5.14) and s as an automorphism in the above equality, we get

$d(m, r_2, \ldots, r_n)c = 0$. But $d \neq 0$, and by Result 5.4.3(i), we get $c = 0$.

Thus (N,+) is an abelian group.

Corollary 5.5.6(Ashraf and Siddeeque, 2013)

Let d be a non zero permuting n-derivation of prime near- ring N. If $d(C, N, N, \ldots, N) = \{0\}$, then (N,+) is an abelian group.

Theorem 5.5.7

Let N be a semi prime near-ring and d a permuting n-(s, t) derivation of N. Given $d(x_1, x_2, \ldots, x_n)\, s(y_1) = t(x_1)d(y_1, y_2, \ldots, y_n)\ \forall\ x_i, y_i \in N$, $i = 1, 2, \ldots, n$, then $d = 0$.

Proof Replacing y_1 by y_1b_1, where $b_1 \in N$ in the given hypothesis, we get

$d(x_1, x_2, \ldots, x_n)\, s(y_1b_1) = t(x_1)\, d(y_1b_1, y_2, \ldots, y_n)$

$= t(x_1)d\,(y_1, y_2, \ldots, y_n)\, s(b_1) + t(x_1)\, t(y_1)\, d(b_1, y_2, \ldots, y_n)$.

Using the given hypothesis, we get

$d(x_1, x_2, \ldots, x_n)s(y_1)\, s(b_1) = d\,(x_1, x_2, \ldots, x_n)s(y_1)\, s(b_1) + t(x_1)\, t(y_1)\, d\,(b_1, y_2, \ldots, y_n)$.

This implies that $t(x_1)\, t(y_1)\, d(b_1, y_2, \ldots, y_n) = 0$. Since t is an automorphism of N in the above relation, we obtain

$x_1y_1\, d(b_1, y_2, \ldots, y_n) = 0$, for all $x_1, y_1 \in N$.

Replacing x_1 by $d\,(b_1, y_2, \ldots, y_n)$ in the above relation, we get

$d\,(b_1, y_2, \ldots, y_n)N\, d\,(b_1, y_2, \ldots, y_n) = \{0\}$.

In view of N being a semi prime near-ring, we find that d = 0.

Corollary 5.5.7(Ashraf and Siddeeque, 2013)

Let N be a semi prime near-ring and d a permuting n- derivation of N. If d (x_1, x_2, . . . , x_n)y_1 = x_1 d(y_1, y_2, . . ., y_n) $\forall$ x_1, x_2, . . .,x_n, y_1, y_2, . $y_n \in N$, then d = 0.

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Summary

This thesis presents some contributions to commutativity theorems for rings, near-rings, and applications in permuting derivation in near-rings.

We first discuss the general introduction of the thesis, which includes the historical background of the study, motivation and justification, and the aim and objectives of the study. Then a comprehensive and critical literature survey of the fundamentals of commutativity of rings and near-rings is presented.

We then establish the extension of commutativity condition, and then prove that certain classes of rings such as semi prime rings, when the commutators $[x, y] = x y - y x$ are central, with condition given below:

$[(x^n o\ y^{m)k} \pm (x^m o\ y), x] = 0$ or $[(x^n o\ y^m)^k \pm (x^m o\ y), y] = 0$

for all ring elements x, y and fixed natural numbers $\mathrm{m, n, k}$.

Next a ring theoretic analogue of group theoretic results which asserts that a group G is commutative if and only if $(xy)^2 = xy^2x$ for all x, y in G is presented.We then studyother polynomial identities and investigate the classes of commutative rings satisfyinggeneralized forms of these identities. We obtain a result which states that if R is a ring with 1 satisfying $x^p[x^n, \mathrm{y}]\ y^q = \mathrm{f(y)}\ [\mathrm{x},\ y^t \mathrm{g(y)}]^k \mathrm{h(y)},$ where for every $y \in R,$ there exist polynomials $f(X), g(X), h(X) \in \boldsymbol{Z}[X]$ and $\mathrm{p \geq 0, q \geq 0, t \geq 0, n > 1, k > 1}$ are fixed positiveintegers.Also $Q(n); \forall\ \mathrm{x, y} \in \mathrm{R},$with

positive integer n and $\mathrm{n}[\mathrm{x}, \mathrm{y}] = 0$ implies $[\mathrm{x}, \mathrm{y}] = 0$, together with at least one of p, q is zero, then R is a commutative ring. Also some results onrings with unity and s-unital rings are presented.

We then present our results which were derived after relaxation of Khan (2001), Putcha and Yaqub (1973) that a ring satisfying a polynomial identity of the form xy = w(x,y), where w (x, y) $\in W$(X, Y) is a word different from XY in non commuting in determinates X and Y, must have a commutator ideal.

Furthermore, we introduce the new notion of permuting n -(s, t) derivations on prime near rings and establishes related results.

Finally we prove the Posner's first theorem in to σ -prime ring Jordan left σ ring (respectively Jordan right σ ring).The generalizations of some ofthe results presented in this thesis may be possible, but the choice of our examples show that they cannot be extended arbitrarily.

6.2 Conclusion

In this thesis, some results on commutativity of semi prime rings, ring with unity, s-unital rings and permuting $4-(\sigma,\tau)$ derivation as well as permuting $n-(\sigma,\tau)$ derivations on prime near-rings studied. We obtained some related results for one sided s-unital rings and n-torsion free rings, established the results of Jordan right σ −derivation and generalized Jordan right σ −derivation on $\sigma-$ rings and showed that additive commutativity of a near ring N satisfied certain identities involving permuting $n-(\sigma,\tau)$ derivations of a prime near ring. We extended Posner's first theorem to σ −prime rings of characteristic different from two and examined

polynomial identities with constraints such as commutators, torsion free conditions on rings and near-rings thereby establish commutativity theorems.

6.3 Recommendations

The notion provided in subsection 5.2.1 and 5.2.2 may be found useful in addressing some real life problems which require symmetry of abstraction. The concept of semiprime rings, rings with identity and prime near-rings can be very useful in many areas like coding theory, cryptography and decision making, etc. In view of the derivation application of rings and near-rings,it needs to be investigated whether or not, similar to the result that every semiprime rings can be extended into arbitrary rings found in (Khan *et al.*, 2013) also holds for s-unital rings as well. One can investigate the possibility of the commutativity of addition and multiplication of near-rings satisfying some algebraic or differential identities involving one of the properties, permuting n -derivations, permuting n generalised derivations, permuting $n-(s,t)$ derivation andpermuting$n-(s,t)$ generalised derivationon semi group ideals of near-rings.

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