BAYESIAN ESTIMATION OF THE SHAPE PARAMETER OF GENERALIZED RAYLEIGH DISTRIBUTION UNDER SYMMETRIC AND ASYMMETRIC LOSS FUNCTIONS

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APRIL, 2016

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B.Sc. (A.B.U. 2011)

M.Sc./SCIE/44695/2012-2013

A DISSERTATION SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF A MASTER DEGREE IN STATISTICS

DEPARTMENT OF MATHEMATICS,

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AHMADU BELLO UNIVERSITY, ZARIA

NIGERIA

APRIL, 2016

Declaration

I declare that the work in this dissertation titled “Bayesian Estimation of the Shape Parameter of Generalized Rayleigh Distribution Under Symmetric and Asymmetric Loss Functions” has been carried out by me in the Department of Mathematics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other institution.

_____ _____

Aliyu Yakubu Date

Certification

This dissertation titled BAYESIAN ESTIMATION OF THE SHAPE PARAMETER OF GENERALIZED RAYLEIGH DISTRIBUTION UNDER SYMMETRIC AND ASYMMETRIC LOSS FUNCTIONS by YAKUBU ALIYU meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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Dedication

This work is dedicated to the Almighty Allah.

Acknowledgments

All praises and thanks be to the Cherisher and Sustainer of the world for giving me the opportunity to complete this work. My sincere appreciation goes to my parents, my late uncle (Maj. J.Y. Aliyu) and his wives for their guidance, prayers and moral support. May the Almighty Allah grant you the best in your final abode, Amen. This work would not have been possible without the professional guidance, comments and suggestions of my major supervisor Dr. A. Yahaya who was able to go through my work despite his tight schedule and workload. I am grateful and may the Almighty Allah continue to reward you abundantly, Amen.

My appreciation goes to my minor supervisor Prof. O.E. Asiribo for his fatherly advice and invaluable contribution towards this research, thank you sir and may God bless you.

My special thanks go to the Head of Department, Prof. B. Sani and Dr. H.G. Dikko for their continous support and encouragement. I will also wish to thank all those that contributed to the success of this research work in one way or the other who are too numerous to mention especially Mr. N.N. Chibuike, Mal. G.Y. Jamilu, all the lecturers in Mathematics Department, my M.Sc. classmates and so on.

Finally, I will like to deeply appreciate the understanding of my wife and daughter in enduring my long absence especially during the weekends in the process of making this dream come to reality. May the Almighty Allah continue to guide and protect you, Amen.

Abstract

In 2001, Surles & Padgett introduced Generalized Rayleigh Distribution (GRD). This skewed distribution can be used quiet effectively in modeling life time data. In this work, Bayesian estimates of the shape parameter of a GRD were determined under the assumption of both informative (gamma) and non-informative (Extended Jeffery's and Uniform) priors. The Bayes estimates were obtained under both symmetric and asymmetric loss functions. The performances of these estimates were compared to the Maximum Likelihood Estimates (MLEs) using Monte Carlo simulation.

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CHAPTER ONE: INTRODUCTION

1.1 Background to the Study

Statistical Inference is the branch of statistics concerned with using probability concept to deal with uncertainty in decision-making. It refers to the process of selecting a sample and using a sample statistic to draw inference about a given population parameter. The field of statistical inference is divided into the theory of estimation and hypothesis testing.

1.1.1 Theory of estimation

Statistical estimation or simply estimation is concerned with the methods by which population characteristics are estimated based on information drawn from a sample. The theory of estimation is further sub-divided into Point and Interval Estimation. A point estimator is a random variable varying from sample to sample and its value is called point estimate i.e. a point estimate is a single value estimate for the parameter. There are several methods of finding a point estimator which can all be broadly classified into the Classical Methods and Non-classical/ Bayesian Methods.

In classical approach, the unknown parameter θ is assumed to be fixed quantity. Inferences in classical approaches are based on a random sample only i.e. if a random sample $X_1, X_2, \cdots, X_n$ is drawn from a population with probability function $f(x;\ \theta)$ and based on the sample knowledge the estimate of θ is obtained. The classical approach is based on the concept of sampling distribution and it does not use any of the prior information available as a result of familiarity with previous studies. There are different methods of point estimation under the classical approach. These include: Maximum Likelihood Estimation (MLE), Method of Moment Estimation, Percentile Estimation, Least Square Estimation, Weighted Least Square Estimates, and so on. On the other hand, in Non-classical/ Bayesian

approach, the parameter θ is assumed to be a random variable which can be described by a probability distribution (known as prior distribution) that is, the unknown parameter θ (being random) follows a prior distribution. Hence, Bayesian approach combines new information that is available with prior information to form basis for inference.

The fundamental difference between the Bayesian and frequentist approaches to statistical inference is characterized in the way they interpret probability, represent the unknown parameters, acknowledge the use of prior information and make the final inferences. The frequentist approach considers probability as a limiting long-run frequency, while the Bayesian approach regards probability as a measure of the degree of personal belief about the value of an unknown parameter θ.

1.1.2 Generalized Rayleigh distribution (GRD)

Twelve different families of cummulative distribution function used in modeling lifetime data were suggested by Burr (1942). Burr Type X distribution is among the most popular distributions that receives the most attention among these families of cummulative distributions.

In 2001, two-parameter Burr Type X distribution was introduced by Surles & Padgett. Kundu and Raqab (2005), Lio, *et.al,* (2011) and Abdel-Hady (2013) prefer to call this distribution GRD which will be adopted in this work. For α>0 and λ>0, the Cumulative Distribution Function (CDF) of the two-parameter GRD is given by:

$$F(x;\ \alpha,\lambda) = \left(1 - e^{-(\lambda x)^2}\right)^{\alpha} \qquad \text{for } x,\ \alpha,\ \lambda > 0 \qquad (1.1)$$

Its probability density function (pdf) is given by:

$$f(x;\ \alpha,\lambda)\ = \begin{cases} 2\alpha\lambda^2 x\left(1-e^{-(\lambda x)^2}\right)^{\alpha-1} e^{-(\lambda x)^2}, & x>0 \\ 0, & elsewhere \end{cases} \qquad (1.2)$$

where α and λ are shape and scale parameters respectively.

Shape and scale parameters are used to determine the shape and location of a distribution. Shape parameter allows a distribution to take on a variety of shapes depending on the value of the shape parameter, while the scale parameter stretches or squeezes the graph of a probability distribution. In general, the larger the scale parameter, the more spread out the distribution and the smaller the parameter, the more compressed the distribution appears to be.

GRD is widely used in modeling events that occur in different fields such as medicine, social and natural sciences. In Physics for instance, the GRD is used in the study of various types of radiation such as light and sound measurements. It is used as a model for wind speed and is often applied to wind driven electrical generation. For details, see Samaila and Cenac, (2006). It is also used in modeling strength and lifetime data (Surles and Padgett (2001), Lio, *et.al* (2011), Kundu and Raqab, (2005)). Hence, the GRD has a survival and hazard functions as shown in equations (1.3) and (1.4) respectively.

Survival function $S(x;\ \alpha,\lambda)\ =\ 1-F(x;\ \alpha,\lambda)\ =\ 1-\left(1-e^{-(\lambda x)^2}\right)^{\alpha}$ (1.3)

Hazard function $h(x;\ \alpha,\lambda)\ =\frac{f(x;\ \alpha,\lambda)}{S(x;\ \alpha,\lambda)}=\frac{2\alpha\lambda^2 x\left(1-e^{-(\lambda x)^2}\right)^{\alpha-1}e^{-(\lambda x)^2}}{1-\left(1-e^{-(\lambda x)^2}\right)^{\alpha}}$ (1.4)

Survival function is the probability that the survival time X takes a value greater than a specific value x ie $S(x)=p(X>x)$, while the hazard function is a measure of how likely

an individual experiences an event as a function of his/her age. These two functions are used to describe the distribution of survival time data.

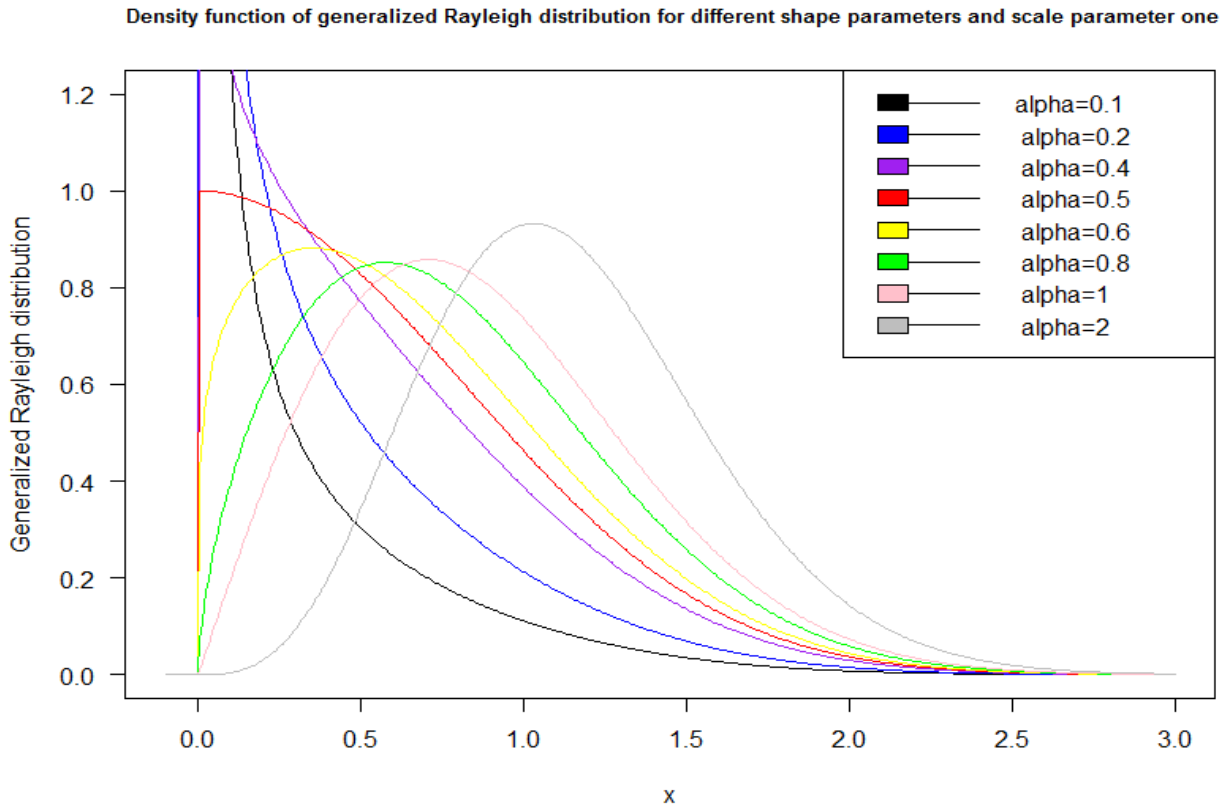


Figure 1. 1: The Graph of Generalized Rayleigh Distribution for different values of shape parameters when the scale parameter takes the value one

The graph of the distribution is shown in Figure 1.1 for different shape parameter values. It is clear from the Figure that the pdf of a GRD is a decreasing function if $\alpha \leq ½$ and it is right skewed uni-modal when $\alpha > ½$. (See also Kundu and Raqab, 2005).

1.2 Statement of the Problem

Bayesian inference requires appropriate choice of priors for parameters. But there is no way to conclude that one prior is better than another. In a situation where one have enough

information about the parameter(s) then using informative prior(s) will be the best practice for choosing a prior, otherwise, a non-informative prior suffices.

1.3 Aim and Objectives of the Study

The aim of this work is to estimate the shape parameter of GRD using Bayesian approach. We wish to achieve the stated aim through the following objectives

i. By estimating the shape parameter (α) when the scale parameter (λ) is known using both informative and non-informative priors under symmetric loss function
ii. By estimating the shape parameter (α) when the scale parameter (λ) is known using both informative and non-informative priors under asymmetric loss functions
iii. To compare the performances of the proposed estimators with that of Maximum Likelihood Estimators in terms of Mean Square Error

1.4 Significance of the Study

In Bayesian approach, the parameter is viewed as random variable behaving according to a subjective (prior) probability distribution that describes our confidence about the actual behavior of the parameter, whereas in classical approach, the parameter is assumed to be fixed but unknown. In Bayesian inference, conclusions are made conditional on the observed data i.e. there is no need to discuss sampling distribution using this method. While in the classical approach one needs not be concerned about any prior knowledge other than the available information observed. Bayesian inference also provides a convenient model for implementing scientific method. The prior distribution is used to state the prior knowledge we have about the parameter of interest, while the posterior distribution reflects

the updated knowledge about the population parameter in line with the new information collected from data.

1.5 Motivation

Statistically, modeling of real life scenario help us to better understand and explain unforeseen eventualities when they take place, thereby enabling us to reproduce such a scenario either on a large and/ or on a simplified scale aimed at describing only critical parts of the phenomenon. These real life phenomena are captured by means of statistical distribution models, which are extracted or learned directly from data gathered about them. Every distribution model has a set of parameters that needs to be estimated. These parameters specify any constant appearing in the model and provide a mechanism for efficient and accurate use of data.

1.6 Limitation

The study will focus only on estimating the shape (α) parameter when the scale (λ) parameter is known under the symmetric and asymmetric loss functions assuming informative and non-informative priors.

1.7 Definition of Terms

1.9.1 Estimator

Let X be a random variable that follows a probability distribution function $f(x;\ \theta)$ indexed by a parameter θ. Let $X_1,\ X_2, \cdots, X_n$ be a random sample from the given population. Any statistic that can be used to estimate the parameter θ is called an estimator of θ. The numerical value of this statistic is called an estimate of θ and is denoted by $\hat{\theta}$.

1.9.2 Prior distribution

A prior distribution is a probability distribution that captures the information about a parameter(s) before data are taken into account. Prior distributions are sub-divided into two classes: Informative and non-informative priors.

Let $X_1, X_2, \cdots X_n$ be a random sample from a distribution with density $f(x/\theta)$, where θ (assumed random) is the unknown parameter to be estimated. The probability distribution of θ is called the prior distribution of θ and is usually denoted as $\pi(\theta)$.

1.9.3 Posterior distribution

Let $X_1, X_2, \cdots X_n$ be a random sample from a distribution with density $f(x/\theta)$, where θ is an unknown parameter to be estimated. The conditional density $\pi(\theta/\, x_1, x_2 \cdots, x_n)$ is called the posterior distribution of θ and is given by

$$\prod(\theta/\, x_1, x_2 \cdots, x_n) = \frac{f\,(x/\theta)\prod(\theta)}{g\,(x)} \tag{1.5}$$

where $g(x)$ is the marginal distribution of X and is given by

$$g(x) = \begin{cases} \sum_\theta f(x/\theta)\pi(\theta) & when\ \theta\ is\ discrete \\ \int_{-\infty}^{\infty} f(x/\theta)\pi(\theta), & when\ \theta\ is\ continous \end{cases} \tag{1.6}$$

where $\pi(\theta)$is the prior distribution of θ.

1.9.4 Loss function

Let $X_1, X_2, \cdots X_n$ be a random sample from a distribution with density $f(x/\theta)$, where θ is an unknown parameter to be estimated. Let $\hat{\theta}$ be an estimator of θ. The function $\mathcal{L}(\hat{\theta}, \theta)$ represents the loss incurred when $\hat{\theta}$ is used in place of θ.

CHAPTER TWO: LITERATURE REVIEW

Kundu and Raqab (2005) considered different estimation procedures (maximum likelihood estimation, modified moment estimator, least square estimator, weighted least square estimator, percentile-based estimator and modified L-moment estimator) to estimate the unknown parameter(s) of a GRD. The study compared the performance of the different estimators using Monte-Carlo simulations mainly with respect to their biases and mean square errors for different sample sizes and different parameter values. (Kundu and Raqab, 2005) showed that, when the sample size is small (say, n=10), the performance of most of the methods are quite bad. In particular, the estimation of the shape parameter (α) becomes very difficult for small sample sizes. The biases of all the methods are quite severe for small and moderate sample sizes ($n \leq 20$). The study also showed that, the least square estimate performs quite well for $n \leq 20$ and for $n \geq 30$, the weighted least square estimates outperforms the least square estimate marginally. If mean square error is considered, the weighted least square performs better than the rest in most of the cases considered.

Different methods of estimating $R = P(Y < X)$ when Y and X both follow GRD with different shape parameters but the same scale parameter were compared by (Raqab and Kundu, 2005). When the scale parameter is unknown, it is observed that the MLEs of the three unknown parameters can be obtained by solving one non-linear equation. An iterative procedure for computing the MLEs of the unknown parameters and the MLE of R was developed. The asymptotic distribution of R was obtained and this was used to compute the asymptotic confidence intervals. It was observed that, even when the sample size is quite small, the asymptotic confidence intervals work quite well. Two bootstrap confidence

intervals were also proposed and their performances were also quite satisfactory. On the other hand, when the scale parameter is known they compare MLE and UMVUE with different Bayes estimators. It was observed that the Bayes estimators with non-informative priors behave quite similarly with the MLEs

In similar studies, Mahdi (2006), estimated the parameters of a Rayleigh distribution using five different estimation techniques namely: maximum likelihood, method of moment, probability weighted moment method, least square method and least absolute deviation method. He proposed the modified maximum likelihood estimation method for the parameters and compares it with the above methods. In his comparison, all the methods performed reasonably well except the method of moments. On the other hand, the modified maximum likelihood method provides better estimates for the parameters when the sample sizes are not small ($n \geq 10$), while in the case of small samples, the probability weighted moment method outperforms the maximum likelihood method for the estimation of the threshold parameter and performs almost as good as the maximum likelihood method for the estimation of the scale parameter. Hence, Mahdi (2006) recommended the use of modified maximum likelihood method for the parameter estimation of the Rayleigh distribution if the sample size is large.

A conventional method for estimating the two parameters of generalized Rayleigh distribution for different sample sizes (small, medium and large) was suggested by (AL-Naqeeb and Hamed, 2009). The estimators were compared by using mean square error (MSE) as a performance measure based on the simulated data used in the study.

Abdel-Hady (2013) extended the Marshall and Olkin's bivariate exponential model to the Generalized Bivariate Rayleigh (GBR) Distribution. The CDF, pdf and the conditional distribution of the GBR distribution were computed. The maximum likelihood estimation procedure for the estimation of the GBR parameters when all parameters are unknown and the observed Fisher Information Matrix were also derived.

Al – Kanani and Abbas, (2014) studied the Bayesian and non-bayesian methods in estimating the parameters of the GRD. The non-bayesian methods considered are the maximum likelihood estimator and the moment estimator while the single stage shrinkage estimator was determined and derived for the Bayesian method. Reliability function and hazard function were also obtained.

Pak, *et.al.*, (2013) study the problem of estimating the scale parameter of Rayleigh distribution under doubly type-II censoring scheme when the lifetime observations are fuzzy and are assumed to be related to underlying crisp realization of a random sample. A new method to determine the maximum likelihood estimate of the scale parameter was proposed and its asymptotic variance was derived using the missing information principle.

Pathak and Chaturvedi (2014) explored and compared the performance of uniformly minimum variance unbiased estimators as well as maximum likelihood estimators of the reliability functions $\mathrm{R(t)} = \mathrm{P}(\mathrm{X} > t)$ and $\mathrm{P} = \mathrm{P}(\mathrm{X} > Y)$ for the two-parameter exponentiated Rayleigh distribution. (Pathak and Chaturvedi, 2014) showed that, the uniformly minimum variance unbiased estimators of *R(t)* and "*P*" are superior to their maximum likelihood counterparts.

Lio, *et.al.* (2011) developed three methods to estimate the parameters of the two-parameter GRD under progress interval censoring. Abbas, *et.al.* (2013) studied the Bayesian estimators for the unknown parameters of Gumbel type II distribution using the idea of Lindley's approximation under the asymmetric (LINEX) loss function and general entropy loss function with gamma prior. Preda *et.al.* (2010) obtained the Bayesian estimators of the shape and scale parameters of modified Weibull distribution using Lindley's approximation under the squared error loss function, LINEX loss function and generalized entropy loss function. The study also compared the performances of these estimates with that of maximum likelihood for different sample sizes. Guure and Ibrahim (2014) used Lindley's approximation to estimate the parameters of the two-parameter Weibull distribution under the squared error loss function and LINEX loss function.

Zaka and Akhter (2014) estimated the scale parameter of Nakagami distribution using Bayesian approach. In their study, the scale parameter was estimated under three prior distributions, namely; Uniform, Inverse Exponential and Levy priors based on three loss functions: Squared Error Loss Function, Quadratic Loss Function and Precautionary Loss Function. The study of Zaka and Akhter (2014) showed that the Precautionary Loss Function produces the least posterior risk when uniform prior is used while Squared Error Loss Function is the best when inverse exponential and Levy Priors are used. It was further shown that, the posterior risk is inversely proportional to the choice of values of the scale parameter using all priors. Sanku and Tanujit (2011) estimated the parameter of Rayleigh distribution via Bayesian technique. The estimates were obtained under Jeffrey's and extension of Jeffrey's priors using a new loss function introduced by Al-bayyati (2002). Monte Carlo simulation was used to compare the estimates in terms of biases and MSE.

Credible and the highest posterior density intervals for the Rayleigh distribution parameter were derived. It was observed that when sample size increases from 10 to 50, the bias and MSE both decreases quite significantly and as far as biases are concerned, the estimate under the extended Jeffrey's prior is better than the estimates under the Jeffrey's prior.

The shape parameter of Generalized Power Distribution (GPD) was obtained by Almutairi and Heng (2012) via Bayesian approach under the non-informative (uniform) and informative (gamma) priors using the squared error loss function. The Bayesian estimators were found to be dependent upon the parameters of the prior distribution. Monte Carlo simulation was carried out in order to make comparison among the estimates and it was observed that, the shape parameter obtained from GPD function also explored the fact that, the Bayesian method could be used in the process of estimating the shape parameter ranging from small to large samples (n=10000). Sanku (2010) estimated the shape parameter of generalized exponential distribution using extended Jeffrey's prior under the quadratic loss function, squared log error loss function and general entropy loss function. The estimators were compared by obtaining the risk of the estimators under the different loss functions and relative efficiency of the estimators. Monte Carlo simulation was performed in order to compare the estimators. The risks of the estimators were all found to be minimax for the shape parameter of the distribution and it was further found that the risk of general entropy loss function was the least. In addition, the researcher shows that, as the sample size increases, the risks for all the estimators decreases as expected. Hence, (Sanku, 2010) recommended the use of general entropy loss function for the estimation of the shape parameter of generalized exponential distribution using the non-informative prior.

Similarly, (Himanshu and Arun, 2009) estimated the shape parameter of Generalized Pareto Distribution (GPD) using quasi, inverted gamma and uniform prior distributions under the LINEX, precautionary and entropy loss functions. The estimates were compared with that of the Bayes estimators under the squared error loss function. In the work, it was shown that, none of the estimators under quasi prior uniformly dominates any estimator under the various loss functions and hence, they recommended that, the estimator be chosen according to the value of the hyperparameter(d) and the situation at hand. The risks of the inverted gamma prior were further found to be dependent on the population parameter (θ) which is not separable.

The Bayesian estimate of the scale parameter of Nakagami distribution under uniform prior, inverse exponential and levy prior distributions using squared error, quadratic and precautionary loss functions were obtained by Azam and Ahmad (2014). Relative posterior risk and Monte Carlo simulations were used to assess the performance of the estimators. It was discovered that, when uniform prior is used, the precautionary loss function have the least risk while the squared error loss function was discovered to have the least risk when inverse exponential and levy priors are used. Also Wasif and Navid (2015) estimated the scale parameter of Nakagami distribution via Bayesian approach using uniform and gamma priors under the assumption of weighted loss function, balanced weighted loss function, and precautionary loss function. Assessment of the performance of the estimators reveal that, the performance of inverse gamma prior is better than that of uniform prior and weighted balance loss function is better than weighted and precautionary loss functions. Sajid *et.al.* (2012) estimated the scale parameter of Laplace model using different asymmetric loss functions comprising precautionary, weighted squared, modified (quadratic) squared loss

functions. Symmetric loss function was also used. The estimators were obtained under the assumption of inverted gamma, inverted chi-squared, informative levy and Gumbel type II priors. The estimates were compared using the posterior risks under the different loss functions. It was discovered that Gumbel type II prior results are more precise than other priors and modified squared error loss function have the least posterior risk. Hence, it was concluded that Gumbel type II prior is the best prior for the estimation of the scale parameter of laplace model. Also, as discovered by many researchers, the posterior risk decreases as sample size increases. Uniform, Jeffreys, Maxwell, Rayleigh, Chi and normal (considering the location parameter to be zero) priors were used under the squared error, quadratic, entropy and precautionary loss functions by Navid and Muhammad (2012) to derive an estimate of the scale parameter of error function distribution. Bayesian credible interval under each prior was also derived and the estimators were evaluated and compared using simulation study. It was discovered that, the estimators based on entropy loss function under informative priors were the best when estimating the scale parameter of error function distribution.

CHAPTER THREE: METHODOLOGY

3.1 Maximum Likelihood Method

In this section, the definition of Maximum Likelihood Estimator is given and the definition will be use in finding an estimator for the shape parameter of GRD.

Let $x_1, x_2, \cdots, x_n$ be a random sample from a population X with probability density function f(x; θ), where θ is an unknown parameter. The likelihood function, L(θ), is defined to be the joint density of the random variables $x_1, x_2, \cdots, x_n$. That is, $L(\theta) = \prod_{i=1}^{n} f(x_i; \theta)$

The sample statistic that maximizes the likelihood function, L(θ) is called the maximum likelihood estimator of θ and is denoted by $\hat{\theta}$. The GRD(α, λ) has probability density function equation (1.2). If the scale parameter λ is known and without loss of generality let λ=1, then the probability density function of GRD(α, 1) is given by

$$f(x;\ \alpha, 1) = 2\alpha x\left(1 - e^{-x^2}\right)^{\alpha-1} e^{-x^2} \tag{3.1}$$

If $x_1, x_2, \cdots, x_n$ is a random sample from the GRD(α, 1) distribution as in (3.1), then the joint density function is

$$L(x;\alpha,1) = \prod_{i=1}^{n} 2\alpha x_i e^{-x_i^2}\left(1-e^{-x_i^2}\right)^{\alpha-1} = 2^n \alpha^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{\alpha-1} \tag{3.2}$$

and the log likelihood of the joint distribution is

$$\ell nL(x;\alpha,1) = n\ell n2 + n\ell n\alpha - \sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} \ell n x_i + (\alpha-1)\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right) \tag{3.3}$$

By partially differentiating equation (3.3) with respect to α and equating the result to zero, then the estimate of α denoted by $\hat{\alpha}_{MLE}$ is given by

$$\hat{\alpha}_{MLE} = -\frac{n}{\sum_{i=1}^{n} \ln\left(1-e^{-x_i^2}\right)} \tag{3.4}$$

and the variance of the estimator $\hat{\alpha}_{MLE}$ is obtained as

$$var(\hat{\alpha}_{MLE}) = \frac{n^2}{(n-1)^2(n-2)}\alpha_{MLE}^2 \tag{3.5}$$

3.2 Bayes Estimation of the Shape Parameter of GRD

In this section, the Bayesian estimates of the shape parameter of the GRD$(\alpha, 1)$ is obtained using the extended Jeffereys prior, uniform prior and gamma prior under the symmetric (squared error loss function) and asymmetric loss functions involving entropy loss function and precautionary loss function. The Risk function of the estimates were also obtained in this section.

3.2.1 Posterior risk and Bayes estimator

3.2.1.1 Posterior risk

Let $X_1, X_2, \cdots X_n$ be a random sample from a distribution with density$f(x/\theta)$, where θ is an unknown parameter to be estimated. Let $\hat{\theta}$ be an estimator of θ and let $\mathcal{L}(\hat{\theta}, \theta)$ be a given loss function. The expected value of this loss function with respect to the population distribution $f(x/\theta)$, is called the posterior risk function and is given by

$$R_{\mathcal{L}}(\theta) = \int_{-\infty}^{\infty} \mathcal{L}(\hat{\theta}, \theta) f(x/\theta) dx \tag{3.6}$$

The posterior density $\pi(\theta/\ x_1, x_2 \cdots, x_n)$ contains all information about θ, an estimator $\hat{\theta}$ for θ is chosen such that $\pi(\hat{\theta}/\ x_1, x_2 \cdots, x_n)$ is maximum subject to a loss function.

3.2.1.2 Bayes estimator

The risk averaged over the prior distribution of θ is called Bayes risk. The Bayes risk of an estimator $\hat{\theta}$ under a prior distribution π is given by

$$B_\pi(\hat{\theta}) = R(\pi, \hat{\theta}) = \int_{-\infty}^{\infty} R(\theta, \hat{\theta})\pi(\theta)d\theta$$

$$B_\pi(\hat{\theta}) = \iint_{-\infty}^{\infty} L\left(\theta, \hat{\theta}(x)\right) f(x, \theta)dxd\theta \quad (3.7)$$

where $f(x, \theta)$ is the joint density of X and θ. The value of $\hat{\theta}$ that minimizes the Bayes risk is called Bayes estimator.

Theorem 3.1: The Bayes risk $R(\pi, \hat{\theta})$ satisfies

$$R(\pi, \hat{\theta}) = \int_{-\infty}^{\infty} R(\theta, \hat{\theta})\pi(\theta)d\theta$$

Let $\hat{\theta}(x_1, x_2, \cdots x_n)$ be the value of θ that minimizes $R(\theta, \hat{\theta})$, then $\hat{\theta}$ is the Bayes estimator.

Proof

Let $f(x, \theta) = f(x/\theta)\pi(\theta)$ denotes the joint density of X and θ, π be a prior distribution and the observations $X_1, X_2, \cdots, X_n = X_n$. Define the posterior density

$$\pi(\theta/x_n) = \frac{L(x_n/\theta)\pi(\theta)}{\int_{-\infty}^{\infty} L(x_n/\theta)\pi(\theta)d\theta}$$

$$\pi(\theta/x_n)\int_{-\infty}^{\infty} L(x_n/\theta)\pi(\theta)d\theta = L(x_n/\theta)\pi(\theta)$$

also let $m(x_n) = \int_{-\infty}^{\infty} L(x_n/\theta)\pi(\theta)d\theta$ then $L(x_n/\theta)\pi(\theta) = \pi(\theta/x_n)m(x_n)$

rewriting the Bayes risk in the form

$$B_\pi(\hat{\theta}) = R(\pi, \hat{\theta}) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} R(\theta, \hat{\theta})\pi(\theta)dx_n\right]\pi(\theta)d\theta$$

$$= \iint_{-\infty}^{\infty} L\left(\theta, \hat{\theta}(x_n)\right) f(x,\theta) dx_n d\theta$$

$$= \iint_{-\infty}^{\infty} L\left(\theta, \hat{\theta}(x_n)\right) \pi(\theta/x_n) m(x_n) dx_n d\theta$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} L\left(\theta, \hat{\theta}(x_n)\right) \pi(\theta/x_n) d\theta\right] m(x_n) dx_n$$

$$= \int_{-\infty}^{\infty} R\left(\theta, \hat{\theta}(x_n)\right) m(x_n) dx_n \quad (3.8)$$

If $\hat{\theta}(x_n)$ is chosen to be the value of θ that minimizes $R\left(\theta, \hat{\theta}(x_n)\right)$, then the integral at every *x* is minimized and thus the integral equation (3.8) is minimized.

3.2.2 Symmetric loss function

The squared error loss function is defined as

$$\mathcal{L}(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \quad (3.9)$$

The Bayes estimator $\hat{\theta}$ relative to the squared error loss function is obtained as follows:

The Bayes estimator $\hat{\theta}$ minimizes $R(\theta, \hat{\theta}) = \int_{-\infty}^{\infty} (\theta - \hat{\theta})^2 \pi(\theta/x) d\theta$. Taking the derivative under integration of $R(\theta, \hat{\theta})$ with respect to $\hat{\theta}$ and equating to zero, yields

$$2\int_{-\infty}^{\infty} (\theta - \hat{\theta}) \pi(\theta/x) d\theta = 0 \quad (3.10)$$

solving for $\hat{\theta}$, yields $\hat{\theta} = E(\theta/x)$

Hence, the Bayes estimator of θ denoted by $\hat{\theta}_{SL}$ relative to the squared error loss function is given by

$$\hat{\theta}_{SL} = E(\theta) \quad (3.11)$$

The squared error loss function is classified as symmetric loss function, because it assigns equal importance to the losses due to overestimation and underestimation of a parameter. This is impractical in real life situations since overestimation of a parameter can lead to more severe or less severe consequences than under estimation or vice versa as in the estimation of reliability or failure rate function. Hence, the loss function which associate greater importance to overestimation or underestimation and serves as alternative to symmetric loss function is the asymmetric loss function.

3.2.3 Asymmetric loss function

Several asymmetric loss functions have been proposed and studied by many researchers such as Entropy loss function, Precautionary loss function, LINEX loss function, weighted loss function, etc.

3.2.2.1 Entropy loss function

The Entropy loss which was discussed by Dey and Liu (1992) and Dey *et al.* (1987) is defined as

$$\mathcal{L}_{EL}(\hat{\theta},\theta) \propto \left(\frac{\hat{\theta}}{\theta}\right)^{k} - klog\left(\frac{\hat{\theta}}{\theta}\right) - 1 \qquad k \neq 0 \tag{3.12}$$

where $\hat{\theta}$ is an estimator of θ.

The Bayes estimator $\hat{\theta}$ relative to the general entropy loss function is obtained as follows:

The bayes estimator $\hat{\theta}$ minimizes $R(\theta,\hat{\theta}) = \int_{-\infty}^{\infty} w\left[\left(\frac{\hat{\theta}}{\theta}\right)^{k} - klog\left(\frac{\hat{\theta}}{\theta}\right) - 1\right]\pi(\theta/x)d\theta.$

where w is constant of proportionality

Taking the derivative of $R(\theta,\hat{\theta})$ with respect to $\hat{\theta}$ and equating to zero, yields

$$wk\int_{-\infty}^{\infty}\left(\frac{\hat{\theta}^{k-1}}{\theta^{k}}-\frac{1}{\hat{\theta}}\right)\pi(\theta/x)d\theta=0 \tag{3.13}$$

Solving for $\hat{\theta}$, yields $\hat{\theta}=\left(E(\theta^{-k})\right)^{-\frac{1}{k}}$

Hence, the Bayes estimator of θ denoted by $\hat{\theta}_{EL}$ relative to the entropy loss function is given by

$$\hat{\theta}_{EL}=\ [E_{\theta}(\theta^{-k})]^{-\frac{1}{k}} \tag{3.14}$$

provided $E_{\theta}(\theta^{-k})$ exist. If k=-1, the Bayes estimate (3.14) coincides with the Bayes estimate under the squared error loss function and for k=1, the Bayes estimator (3.14) coincides with the Bayes estimator under weighted squared error loss function.

3.2.2.2 Precautionary loss function

Norstrom (1996) introduced an asymmetric loss function called Precautionary loss function which is defined as

$$\mathcal{L}(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^{2}}{\hat{\theta}} \tag{3.15}$$

The Bayes estimator $\hat{\theta}$ relative to the precautionary loss function is obtained as follows:

The bayes estimator $\hat{\theta}$ minimizes $R(\theta,\hat{\theta})=\int_{-\infty}^{\infty}\frac{(\hat{\theta}-\theta)^{2}}{\hat{\theta}}\pi(\theta/x)d\theta$. Taking the derivative under integration of $R(\theta,\hat{\theta})$ with respect to $\hat{\theta}$ and equating to zero, yields

$$\int_{-\infty}^{\infty}\left(1-\frac{\theta^{2}}{\hat{\theta}^{2}}\right)\pi(\theta/x)d\theta=0$$

$$\int_{-\infty}^{\infty} \pi(\theta/x)d\theta = \int_{-\infty}^{\infty} \frac{\theta^2}{\hat{\theta}^2}\pi(\theta/x)d\theta$$

$$1 = \frac{1}{\hat{\theta}^2}\int_{-\infty}^{\infty} \theta^2\pi(\theta/x)d\theta \quad \Rightarrow \hat{\theta}^2 = E(\theta^2)$$

Solving for $\hat{\theta}$, yields $\hat{\theta} = \left(E(\theta^2)\right)^{\frac{1}{2}}$

Hence, the Bayes estimator of θ denoted by $\hat{\theta}_{PL}$ relative to the precautionary loss function is given by

$$\hat{\theta}_{PL} = \left(E(\theta^2)\right)^{\frac{1}{2}} \tag{3.16}$$

3.3 Bayesian Estimates under the Extended Jeffrey's Prior

Jeffrey (1946) proposed a method for generating non-informative priors which is based on the Fisher Information Matrix

$$I(\alpha) = -E\left(\frac{\partial^2 lnf(X;\alpha)}{\partial\alpha_i\partial\alpha_j}\right) \tag{3.17}$$

The Jeffrey's prior is defined by

$$\pi(\alpha) \propto \sqrt{I(\alpha)} \tag{3.18}$$

Al-Kutubi (2005) proposed an extension to Jeffrey's prior which is defined as

$$\pi(\alpha) \propto \left(I(\alpha)\right)^r, \quad r \in \mathbb{R}^+ \tag{3.19}$$

where $I(\alpha)$ is defined as in (3.17)

from equation (3.3)

$$\frac{\partial^2 \ell nL(x;\alpha,1)}{\partial\alpha^2} = -\frac{1}{\alpha^2} \tag{3.20}$$

and hence the Fisher Information $I(\alpha) = \frac{1}{\alpha^2}$

Hence, from equation (3.19), if the constant of proportionality is assumed to be one (1), then it is easy to conclude that the extended Jeffrey's prior is defined as follows;

$$\pi(\alpha)=\frac{1}{\alpha^{r}} \tag{3.21}$$

Let $x_1, x_2, \cdots, x_n$ be a random sample from $GRD(\alpha, 1)$, then the likelihood of the observed data is given in (3.2), the product of the likelihood and the prior is given by

$$Q = L\left(x_1, x_2, \cdots, x_n; \alpha, 1\right)\pi(\alpha) = 2^n \alpha^{n-r} e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{\alpha-1} \tag{3.22}$$

while the posterior distribution of α is given by

$$\pi\left(\alpha / x_1, x_2, \cdots, x_n\right) = \frac{L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha)}{\int_0^{\infty} L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) d\alpha} \tag{3.23}$$

Integrating (3.22) with respect to α, leads to:

$$\int_0^{\infty} Q d\alpha = \int_0^{\infty} 2^n \alpha^{n-r} e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{\alpha-1} d\alpha$$

$$let\, u = \prod_{i=1}^{n}\left(1-e^{-x_i^2}\right)^{\alpha} \Rightarrow \alpha = \frac{-\ell n u}{\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}} \; and \; d\alpha = \frac{du}{-u\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}}$$

$$= \frac{2^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{-1}}{-\left(\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}\right)^{n-r+1}} \int_0^{\infty} \left(-\ell n u\right)^{n-r} du$$

$$let\, q = -\ell n u \Rightarrow du = -e^{-q} dq$$

$$= \frac{2^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{-1}}{\left(\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}\right)^{n-r+1}} \sqrt{n-r+1}$$

$$\int_0^{\infty} L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha)d\alpha = \frac{2^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{-1}}{\left(\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}\right)^{n-r+1}} \lceil n-r+1 \qquad (3.24)$$

Equation (3.22) can be written as follows:

$$L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) = 2^n \alpha^{n-r} e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{\alpha-1}$$

$$let\, u = \prod_{i=1}^{n}\left(1-e^{-x_i^2}\right)^{\alpha} \Rightarrow \ell nu = \alpha \sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right) = -\alpha \sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}$$

taking exponential of both sides, we have

$$u = e^{-\alpha \sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}}$$

then,

$$L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) = 2^n \alpha^{n-r} e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{-1} e^{-\alpha \sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}} \qquad (3.25)$$

and finally using equations (3.24) and (3.25) in equation (3.23), the posterior distribution (3.23) becomes

$$\pi\left(\alpha / x_1, x_2, \cdots, x_n\right) = \frac{\alpha^{\text{n-r}} M^{\,n-r+1} e^{-M\alpha}}{\lceil n-r+1} \qquad (3.26)$$

which is gamma distributed with parameters $\left(n-r+1\right)$ and M.

where $M = \sum_{i=1}^{n} ln(1-e^{-x_i^2})^{-1}$

using equation (3.26) the expectation of α^g where $g \epsilon \mathbb{Z}$ can easily be obtained as follows

$$E\left(\alpha^{g}\right)=\int_{-\infty}^{\infty}\alpha^{g}\prod(\alpha/x)\,d\alpha$$

$$=\int_{0}^{\infty}\alpha^{g}\frac{M^{n-r+1}}{\lceil n-r+1}\alpha^{n-r}e^{-M\alpha}d\alpha$$

$$let\ q=M\alpha\Rightarrow\alpha=\frac{q}{M}\ and\ d\alpha=\frac{dq}{M}$$

Therefore, the expectation of α^{g} is given by (3.27)

$$E\left(\alpha^{g}\right)=\frac{1}{M^{g}\lceil n-r+1}\int_{0}^{\infty}q^{n-r+g}e^{-q}dq=\frac{\lceil n-r+g+1}{M^{g}\lceil n-r+1} \tag{3.27}$$

Let $g=1, g=2\ and\ g=-k,$ then equation (3.27) becomes (3.28), (3.29) and (3.30) respectively

$$E(\alpha)=\frac{\lceil n-r+2}{M\lceil n-r+1}=\frac{n-r+1}{M} \tag{3.28}$$

$$E\left(\alpha^{2}\right)=\frac{\lceil n-r+3}{M^{2}\lceil n-r+1}=\frac{(n-r+2)(n-r+1)}{M^{2}} \tag{3.29}$$

$$E\left(\alpha^{-k}\right)=\frac{\lceil n-r-k+1}{M^{-k}\lceil n-r+1} \tag{3.30}$$

The Bayes estimator of α under the squared error, entropy and precautionary loss functions are given by equations (3.11), (3.14) and (3.16) respectively. Hence, the Bayesian estimators of the parameter α under the aforementioned loss functions denoted by $\hat{\alpha}_{JSL}$, $\hat{\alpha}_{JEL}$ and $\hat{\alpha}_{JPL}$ are respectively given by (3.31), (3.32) and (3.33)

$$\hat{\alpha}_{JSL}=E(\alpha)=\frac{n-r+1}{M} \tag{3.31}$$

$$\hat{\alpha}_{JEL} = \left(E\left(\alpha^{-k}\right)\right)^{-\frac{1}{k}} = \left(\frac{\lceil n-r-k+1}{M^{-k}\lceil n-r+1}\right)^{-\frac{1}{k}} = \frac{1}{M}\left(\frac{\lceil n-r+1}{\lceil n-r-k+1}\right)^{\frac{1}{k}} \tag{3.32}$$

$$\hat{\alpha}_{JPL} = \left(E\left(\alpha^{2}\right)\right)^{\frac{1}{2}} = \sqrt{\frac{(n-r+2)(n-r+1)}{M^{2}}} = \frac{1}{M}\sqrt{(n-r+2)(n-r+1)} \tag{3.33}$$

3.3.1 Transformation of the random variable *M* and its distribution

The need to know the probability distribution of a random variable $Y = \emptyset(x)$ where $\emptyset$ is some known function when the probability distribution of the random variable X is known arise in many statistical applications. Various methods for finding the distribution of a transformed random variable have been developed. These include

i. Distribution method
ii. Transformation method
iii. Convolution method, etc.

Transformation method and convolution method will be used in this work to find the distribution of *M*

Theorem 3.2: Let X be a continuous random variable with probability density function f(x). Let y=T(x) be an increasing (or decreasing) function. Then the density function of the random variable Y=T(X) is given by

$$g(y) = \left|\frac{dx}{dy}\right| f\left(W(y)\right) \tag{3.34}$$

where $x = W(y)$ is the inverse of $T(x)$

Proof

Suppose $y = T(x)$ is an increasing function. The distribution function $G(y)$ of Y is given by

$$G(y) = P(Y \le y)$$

$$= P(T(x) \le y)$$

$$= P(X \le W(y))$$

$$G(y) = \int_{-\infty}^{w(y)} f(x)\,dx$$

on differentiating, the density function of Y is obtained as

$$g(y) = \frac{dG(y)}{dy} = \frac{d}{dy}\left(\int_{-\infty}^{w(y)} f(x)\,dx \right)$$

$$= f(W(y))\frac{dW(y)}{dy}$$

$$= f(W(y))\frac{dx}{dy} \qquad \text{since } x = W(y) \qquad (3.35)$$

If on the other hand, $y = T(x)$ is a decreasing function, then the distribution of Y is given by

$$G(y) = P(Y \le y)$$

$$P(T(x) \le y)$$

$$P(X \ge W(y)) \qquad \text{since } T(x) \text{ is decreasing}$$

$$G(y) = 1 - \int_{-\infty}^{w(y)} f(x)\,dx$$

on differentiating, the density function of Y is given by

$$g(y) = -f(W(y))\frac{dx}{dy} \qquad (3.36)$$

combining (3.35) and (3.36) the distribution of Y is given by

$$g(y)=\left|\frac{dx}{dy}\right| f\left(W(y)\right) \tag{3.37}$$

Theorem 3.3: let the joint density of the random variables X and Y be $f(x,y)$. The probability density of $X+Y$ is given by

$$h_{X+Y}(V)=\int_{-\infty}^{\infty} f(u,v-u)\,du \tag{3.38}$$

Proof

Let $U=X$ and $V=X+Y$, so that $X=R(U,\ V)=U$ and $Y=S(U,\ V)=V-U$. Hence, the jacobian is given by

$$J=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}=1 \tag{3.39}$$

The joint density function of U and V is given by

$$g(u,v)=|J|f\big(R(u,v),S(u,v)\big)$$
$$f\big(R(u,v),S(u,v)\big)$$
$$f(u,v-u)$$

Hence the marginal density of $V=X+Y$ is given by

$$h_{X+Y}(V)=\int_{-\infty}^{\infty} f(u,v-u)\,du$$

If X and Y are independent and have pdf $f(x)$ and $g(y)$ respectively, then

$$h_{X+Y}(V)=\int_{-\infty}^{\infty} g(y)\,f(z-y)\,dy$$

3.3.2 Convolution

Let *f* and *g* be two real valued functions. The convolution of *f* and *g* is defined as

$$(f*g)(z)=\int_{-\infty}^{\infty} f(z-y)\,g(y)\,dy=\int_{-\infty}^{\infty} f(z-x)\,g(x)\,dx \tag{3.40}$$

Hence, the convolution of *f* and *g* is equal to the convolution of *g* and *f* (i.e. $f*g=g*f$)

Now, if $M=\sum_{i=1}^{n} l\,n\left(1-e^{-x_i^2}\right)^{-1}$, and X ~ $GRD(\alpha,1)$ then, the distribution of M can be obtained as follows:

Let $i=1$, then $M_1=\ell n\left(1-e^{x_1^2}\right)^{-1}$ this implies $x_1^2=-ln(1-e^{-M_1})$ and

$$\frac{dx_1}{dm_1}=-\frac{e^{-m_1}}{2\left(-ln\left(1-e^{-M_1}\right)\right)^{\frac{1}{2}}(1-e^{-m_1})}$$

The jacobian $\left|\frac{dx_1}{dm_1}\right|=\frac{e^{-m_1}}{2\left(-ln\left(1-e^{-M_1}\right)\right)^{\frac{1}{2}}(1-e^{-m_1})}$

Therefore, the pdf of M_1 if X_1 has the pdf (3.1) is given by

$$f(m_1)=f\left(W(x_1)\right)\left|\frac{dx_1}{dm_1}\right|=\alpha e^{-\alpha m_1} \tag{3.41}$$

Following the same technique, for $i=2,3,\cdots,n$ the distribution of $x_2,x_3,\cdots,x_n$ are respectively given by $f(m_2)=\alpha e^{-\alpha m_2}$, $f(m_3)=\alpha e^{-\alpha m_3},\cdots,f(m_n)=\alpha e^{-\alpha m_n}$. But our interest is to find the distribution of $m_1+m_2+\cdots+m_n$ and since $m_1,m_2,\cdots,m_n$ are independent, the convolution method will be used to find the distribution of $m_1+m_2+\cdots+m_n$. First, the density of the random variable $z=m_1+m_2$ is the convolution of m_1 with m_2 that is

$$h(z)=(f*g)(z)=\int_{-\infty}^{\infty} f(z-m_1)\,g(m_1)\,dm_1 \tag{3.42}$$

Note that the sum $z=m_1+m_2$ is between 0 and ∞ and $0<m_1<z$, hence (3.42) becomes

$$h(z)=(f*g)(z)=\int_0^z \alpha e^{-\alpha(z-m_1)}\alpha e^{-\alpha m_1}dm_1=\alpha^2 z e^{-\alpha z} \tag{3.43}$$

This shows that the distribution of $z=m_3+m_4$ following the same technique is $h(m_3+m_4)=h(z)=\alpha^2 z e^{-\alpha z}$. Following the same procedure, the distribution of $z=m_1+m_2+m_3+m_4$ is given by

$$h(z)=\frac{\alpha^4 z^3 e^{-\alpha z}}{6} \tag{3.44}$$

Careful examination of (3.41), (3.43) and (3.44) indicates that

$$f(m_1)=\alpha e^{-\alpha m_1}\sim Gamma(1,\alpha)$$
$$f(z)=\alpha^2 z e^{-\alpha z}\sim Gamma(2,\alpha) \text{ and}$$
$$f(z)=\frac{\alpha^4 z^3 e^{-\alpha z}}{6}\sim Gamma(4,\alpha)$$

Hence, by induction $f(m_1+m_2+\cdots+m_n)$ is given by

$$f(m_1+m_2+\cdots+m_n)=f(M)=\frac{\alpha^n M^{n-1}e^{-\alpha M}}{\lceil n}\sim Gamma(n,\alpha) \tag{3.45}$$

Therefore if $M=\sum_{i=1}^{n}\ell n\left(1-e^{-x_i^2}\right)^{-1}$, and X $\sim GRD(\alpha,1)$ then M $\sim$ Gamma(n,α).

3.3.3 Variance and relative efficiency of the estimates under extended Jeffrey's prior using the various loss functions

From equation (3.45), it is easy to show that

$$E(M^g)=\int_{-\infty}^{\infty} M^g f(M)\,dM$$

$$=\int_0^{\infty} M^g \frac{\alpha^n}{\lceil n} M^{n-1}e^{-\alpha M}dM$$

let $p=\alpha M\Rightarrow M=\frac{p}{\alpha}$ and $dM=\frac{dp}{\alpha}$

then $E\left(M^{g}\right)=\frac{\alpha^{n}}{\lceil n}\int_{0}^{\infty}\left(\frac{p}{\alpha}\right)^{n+g-1}e^{-p}dp=\frac{\alpha^{n}}{\alpha^{n+g}\lceil n}\int_{0}^{\infty}\left(p\right)^{n+g-1}e^{-p}dp$

hence,

$$E\left(M^{g}\right)=\frac{\lceil n+g}{\alpha^{g}\lceil n} \tag{3.46}$$

In order to find the $Var\left(\frac{1}{M}\right)$, we substituting $g=-1\ and\ g=-2$ in equation (3.46), so as to obtain $E\left(\frac{1}{M}\right)$ and $E\left(\frac{1}{M^{2}}\right)$, hence, we have

$$E\left(M^{-1}\right)=E\left(\frac{1}{M}\right)=\frac{\alpha\lceil n-1}{\lceil n}=\frac{\alpha}{n-1} \tag{3.47}$$

$$E\left(M^{-2}\right)=E\left(\frac{1}{M^{2}}\right)=\frac{\alpha^{2}\lceil n-2}{\lceil n}=\frac{\alpha^{2}}{\left(n-1\right)\left(n-2\right)} \tag{3.48}$$

and variance of $\frac{1}{M}$ is obtained as follows:

$$\text{var}\left(\frac{1}{M}\right)=E\left(\frac{1}{M}\right)^{2}-\left(E\left(\frac{1}{M}\right)\right)^{2}=\frac{\alpha^{2}}{\left(n-1\right)^{2}\left(n-2\right)} \tag{3.49}$$

Using (3.49), the variance of the estimates under the extended Jeffrey's prior for the various loss functions given in equations (3.31), (3.32) and (3.33) are obtained as follows:

$$\text{var}\left(\hat{\alpha}_{JSL}\right)=\text{var}\left(\frac{n-r+1}{M}\right)=\left(n-r+1\right)^{2}\text{var}\left(\frac{1}{M}\right) \tag{3.50}$$

substituting equation (3.49), the variance of $\hat{\alpha}_{JSL}$ is obtained by

$$\text{var}\left(\hat{\alpha}_{JSL}\right)=\frac{\left(n-r+1\right)^2}{\left(n-1\right)^2\left(n-2\right)}\alpha^2 \tag{3.51}$$

following the same procedure, the variances of $\hat{\alpha}_{JEL}$ and $\hat{\alpha}_{JPL}$ are respectively given by equations (3.52) and (3.53) respectively

$$\text{var}\left(\hat{\alpha}_{JEL}\right)=\frac{\alpha^2}{\left(n-1\right)^2\left(n-2\right)}\left(\frac{\lceil n-r+1}{\lceil\overline{n-r-k+1}}\right)^{\frac{2}{k}} \tag{3.52}$$

$$Var\left(\hat{\alpha}_{JPL}\right)=\frac{\left(n-r+2\right)\left(n-r+1\right)}{\left(n-1\right)^2\left(n-2\right)}\alpha^2 \tag{3.53}$$

Various statistical properties of point estimators are used to decide which estimator is most appropriate, will give the least risk, the most information at the lowest cost etc. Relative Efficiency (R.E) is one of these desirable properties. The R.E of two estimators $\hat{\theta}_1$ and $\hat{\theta}_2$ is defined as the ratio of their variances, that is

$$R.E=\frac{var(\hat{\theta}_1)}{var(\hat{\theta}_2)} \tag{3.54}$$

If R.E >1, then $\hat{\theta}_2$ is more stable than $\hat{\theta}_1$ otherwise, $\hat{\theta}_1$ is more stable than $\hat{\theta}_2$. The relative efficiency of the estimators under the various loss functions are as follows:

The relative efficiency of $\hat{\alpha}_{JEL}$ with respect to $\hat{\alpha}_{JSL}$ of α is

$$E_{J1}=\frac{Var\left(\hat{\alpha}_{JSL}\right)}{Var\left(\hat{\alpha}_{JEL}\right)}=\left(n-r+1\right)^2\left(\frac{\lceil\overline{n-r-k+1}}{\lceil n-r+1}\right)^{\frac{2}{k}} \tag{3.55}$$

$E_{J1}>1$ for $k\geq 1$, $E_{J1}=1$ for $k=-1$ and $E_{J1}<1$ for $k<-1$ (see fig. A.1&A.2 in appendix I)

The relative efficiency of $\hat{\alpha}_{JEL}$ with respect to $\hat{\alpha}_{JPL}$ of α is

$$E_{J2}=\frac{Var\left(\hat{\alpha}_{JPL}\right)}{Var\left(\hat{\alpha}_{JEL}\right)}=\left(n-r+2\right)\left(n-r+1\right)\left(\frac{\lceil n-r-k+1}{\lceil n-r+1}\right)^{\frac{2}{k}} \tag{3.56}$$

$E_{J2}>1$ for $k\geq -1$, $E_{J2}=1$ for $k=-2$ and $E_{J2}<1$ for $k<-2$ (see fig. A.3&A.4 in appendix I)

The relative efficiency of $\hat{\alpha}_{JSL}$ with respect to $\hat{\alpha}_{JPL}$ of α is

$$E_{J3}=\frac{Var\left(\hat{\alpha}_{JPL}\right)}{Var\left(\hat{\alpha}_{JSL}\right)}=\frac{\left(n-r+2\right)}{\left(n-r+1\right)} \tag{3.57}$$

$E_{J3}>1$ when $n-r\geq 1$ (see fig. A.5 in appendix I)

3.3.4 Posterior Risk

Recall that the posterior risk is defined as the expectation of the loss function with respect to the population distribution $f(x/\theta)$. Hence, the posterior risks for the various loss functions using the extended Jeffrey's prior are as follows:

Posterior risk under squared error loss function

$$R(\hat{\alpha},\alpha)=E\big(L(\hat{\alpha},\alpha)\big)=E(\alpha-\hat{\alpha})^2=E(\hat{\alpha}^2-2\hat{\alpha}\alpha+\alpha^2) \tag{3.58}$$

recall that, the Bayes estimator of $\hat{\alpha}$ under the squared error loss function is given by $\hat{\alpha}=\hat{\alpha}_{SL}=E(\alpha)$ substituting this in equation (3.58) yields

$$E(\alpha^2)-\big(E(\alpha)\big)^2 \tag{3.59}$$

therefore, the posterior risk of $\hat{\alpha}_{JSL}$ is given by

$$\frac{(n-r+2)(n-r+1)}{M^2}-\frac{(n-r+1)^2}{M^2}=\frac{(n-r+1)}{M^2} \tag{3.60}$$

The posterior risk under entropy loss function is obtained as follows:

$$R(\hat{\alpha}_{JEL},\alpha)=E(\hat{\alpha}_{JEL},\alpha)=wE\left[\left(\frac{\hat{\alpha}_{JEL}}{\alpha}\right)^{k}-k\log\left(\frac{\hat{\alpha}_{JEL}}{\alpha}\right)-1\right] \tag{3.61}$$

substituting equation (3.32) yields

$$w\left[\frac{\lceil n-k\rceil n-r+1}{\lceil n\rceil n-r-k+1}-\log\left(\frac{\lceil n-r+1}{\lceil n-r-k+1}\right)+k\log(n-1)-1\right] \tag{3.62}$$

and finally, the posterior risk under the precautionary loss function is obtained as follows

$$R(\hat{\alpha}_{JPL},\alpha)=E(\hat{\alpha}_{PEL},\alpha)=E\left(\frac{(\hat{\alpha}-\alpha)^{2}}{\hat{\alpha}}\right)=E\left(\hat{\alpha}-2\alpha+\frac{\alpha^{2}}{\hat{\alpha}}\right) \tag{3.63}$$

substituting the Bayes estimator under the precautionary loss function in equation (3.63) (that is substituting (3.14) in (3.63)) yields

$$E\left[\left(E(\alpha^{2})\right)^{\frac{1}{2}}-2\alpha+\frac{\alpha^{2}}{\left(E(\alpha^{2})\right)^{\frac{1}{2}}}\right]=2\left(\hat{\alpha}_{PL}-E(\alpha)\right) \tag{3.64}$$

therefore, the posterior risk of $\hat{\alpha}_{JPL}$, is given by

$$R(\hat{\alpha}_{JPL},\alpha)=2\left(\frac{\sqrt{(n-r+2)(n-r+1)}}{M}-\frac{n-r+1}{M}\right) \tag{3.65}$$

3.4 Bayesian Estimates under the Uniform Prior

Let's assume the shape parameter α is a random variable having a uniform prior given by:

$$\pi(\alpha)=\frac{1}{b-a}\ for\ a\le\alpha\le b \tag{3.66}$$

Let $x_1, x_2, \cdots, x_n$ be a random sample from $GRD(\alpha, 1)$, then the likelihood of the observed data is obtained as in (3.2) and following the same procedure as in section 3.3, the product of the likelihood and the prior is given by equation (3.67)

$$\frac{1}{b-a} 2^n \alpha^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{\alpha-1} \tag{3.67}$$

equation (3.67) can be written as follows:

$$L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) = \frac{1}{b-a} 2^n \alpha^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{\alpha-1}$$

$$let\, u = \prod_{i=1}^{n} \left(1-e^{-x_i^2}\right)^{\alpha} \Rightarrow u = e^{-\alpha \sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}}$$

Then,

$$L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) = \frac{1}{b-a} 2^n \alpha^n e^{-\sum_{i=1}^{n} x_i^2} e^{-\alpha \sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{-1} \tag{3.68}$$

Integrating equation (3.68) with respect to α is as follows:

$$\int_0^{\infty} \frac{1}{b-a} 2^n \alpha^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{\alpha-1} d\alpha$$

$$= \frac{2^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{-1}}{\frac{1}{b-a}\left(\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}\right)^{n+1}} \int_0^{\infty} (-\ell nu)^n \, udu$$

$$let\, q = -\ell nu \Rightarrow du = -e^{-q} dq$$

$$= \frac{2^n e^{-\sum_{i=1}^{n} x_i^2} \prod_{i=1}^{n} x_i \left(1-e^{-x_i^2}\right)^{-1}}{\frac{1}{b-a}\left(\sum_{i=1}^{n} \ell n\left(1-e^{-x_i^2}\right)^{-1}\right)^{n+1}} \lceil n+1 \tag{3.69}$$

Finally, substituting equations (3.68) and (3.69) in the posterior distribution (3.70) yields (3.71):

$$\pi\left(x_1, x_2, \cdots, x_n / \alpha\right) = \frac{L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha)}{\int_0^{\infty} L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) d\alpha} \tag{3.70}$$

$$\pi(\alpha / x_1, x_2, \cdots, x_n) = \frac{\alpha^{\mathrm{n}} M^{n+1} e^{-M\alpha}}{\lceil n+1}} \tag{3.71}$$

It is clear that, the posterior distribution (3.71) is gamma with parameters (n+1) and M

where $M = \sum_{i=1}^{n} \ell n(1 - e^{-x_i^2})^{-1}$

Expected value of α^g where $g \in \mathbb{Z}$ can easily be obtained as follows:

$$E\left(\alpha^g\right) = \int_{-\infty}^{\infty} \alpha^g \prod(\alpha / x) d\alpha$$

$$= \int_0^{\infty} \alpha^g \frac{\alpha^n M^{n+1} e^{-M\alpha}}{\lceil n+1} d\alpha$$

$$let\, q = M\alpha \Rightarrow \alpha = \frac{q}{M} \, and \, d\alpha = \frac{dq}{M}$$

$$E\left(\alpha^g\right) = \frac{1}{M^g \lceil n+1} \int_0^{\infty} q^{n+g} e^{-q} dq = \frac{\lceil n+g+1}{M^g \lceil n+1} \tag{3.72}$$

In order to obtain the estimates under the Square error, Entropy and Precautionary loss functions, we let $g = 1, g = 2 \, and \, g = -k$ in equation (3.72). This gives (3.73), (3.74) and (3.75) respectively

$$E\left(\alpha\right) = \frac{\lceil n+2}{M \lceil n+1} = \frac{n+1}{M} \tag{3.73}$$

$$E\left(\alpha^2\right) = \frac{\lceil n+3}{M^2 \lceil n+1} = \frac{(n+2)(n+1)}{M^2} \tag{3.74}$$

$$E\left(\alpha^{-k}\right) = \frac{\lceil n-k+1}{M^{-k} \lceil n+1} \tag{3.75}$$

The Bayes estimator of α under the squared error, entropy and precautionary loss functions are given by equations (3.11), (3.14) and (3.16) respectively. Hence, the Bayesian estimators of the parameter α under the aforementioned loss functions and using the uniform prior denoted by $\hat{\alpha}_{USL}$, $\hat{\alpha}_{UEL}$ and $\hat{\alpha}_{UPL}$ are respectively given by (3.76), (3.77) and (3.78)

$$\hat{\alpha}_{USL} = E(\alpha) = \frac{n+1}{M} \tag{3.76}$$

$$\hat{\alpha}_{UEL} = \left(E\left(\alpha^{-k}\right)\right)^{-\frac{1}{k}} = \left(\frac{\lceil n-k+1}{M^{-k}\lceil n+1}\right)^{-\frac{1}{k}} = \frac{1}{M}\left(\frac{\lceil n+1}{\lceil n-k+1}\right)^{\frac{1}{k}} \tag{3.77}$$

$$\hat{\alpha}_{UPL} = \left(E\left(\alpha^{2}\right)\right)^{\frac{1}{2}} = \sqrt{\frac{(n+2)(n+1)}{M^{2}}} = \frac{1}{M}\sqrt{(n+2)(n+1)} \tag{3.78}$$

The variance of the shape parameter under the aforementioned loss functions are respectively given by (3.80), (3.82) and (3.84)

$$\text{var}\left(\hat{\alpha}_{USL}\right) = (n+1)^{2}\,\text{var}\left(\frac{1}{M}\right) \tag{3.79}$$

substituting equation (3.49) in equation (3.79) yields equation (3.80)

$$\text{var}\left(\hat{\alpha}_{USL}\right) = \frac{(n+1)^{2}}{(n-1)^{2}(n-2)}\alpha^{2} \tag{3.80}$$

The variance of the estimator under Entropy loss function $Var(\alpha_{UEL})$ is obtain as follows:

$$\text{var}\left(\hat{\alpha}_{UEL}\right) = \left(\frac{\lceil n+1}{\lceil n-k+1}\right)^{\frac{2}{k}}\text{var}\left(\frac{1}{M}\right) \tag{3.81}$$

substituting equation (3.49) in equation (3.81) yields equation (3.82)

$$\operatorname{var}\left(\hat{\alpha}_{UEL}\right)=\frac{\alpha^2}{(n-1)^2(n-2)}\left(\frac{\lceil n+1}{\lceil n-k+1}\right)^{\frac{2}{k}} \tag{3.82}$$

The variance of the estimator under precautionary loss function $Var(\alpha_{UPL})$ is obtain in a similar way as follows:

$$\operatorname{var}\left(\hat{\alpha}_{UPL}\right)=(n+2)(n+1)\operatorname{var}\left(\frac{1}{M}\right) \tag{3.83}$$

substituting equation (3. 49) in equation (3.83) yields equation (3.84)

$$Var\left(\hat{\alpha}_{UPL}\right)=\frac{(n+2)(n+1)}{(n-1)^2(n-2)}\alpha^2 \tag{3.84}$$

The relative efficiency of the estimators under the various loss functions are as follows:

The relative efficiency of $\hat{\alpha}_{UEL}$ with respect to $\hat{\alpha}_{USL}$ of α is

$$E_{U1}=\frac{Var\left(\hat{\alpha}_{USL}\right)}{Var\left(\hat{\alpha}_{UEL}\right)}=(n+1)^2\left(\frac{\lceil n-k+1}{\lceil n+1}\right)^{\frac{2}{k}} \tag{3.85}$$

$E_{U1}>1$ for $k\geq 1$, $E_{U1}=1$ for $k=-1$ and $E_{U1}<1$ for $k<-1$ (see fig. A.6 & A.7 in appendix I)

The relative efficiency of $\hat{\alpha}_{UEL}$ with respect to $\hat{\alpha}_{UPL}$ of α is

$$E_{U2}=\frac{Var\left(\hat{\alpha}_{UPL}\right)}{Var\left(\hat{\alpha}_{UEL}\right)}=(n+2)(n+1)\left(\frac{\lceil n-k+1}{\lceil n+1}\right)^{\frac{2}{k}} \tag{3.86}$$

$E_{U2} > 1$ for $k \geq 1$, $E_{U2} = 1$ for $k = -1, -2$ and $E_{U2} < 1$ for $k < -2$ (see fig. A.8 & A.9 in appendix I)

The relative efficiency of $\hat{\alpha}_{USL}$ with respect to $\hat{\alpha}_{UPL}$ of α is

$$E_{U3} = \frac{Var\left(\hat{\alpha}_{UPL}\right)}{Var\left(\hat{\alpha}_{USL}\right)} = \frac{n+2}{n+1} \tag{3.87}$$

$E_{U3} > 1$ for $n \geq 1$ (see fig. A.10 in appendix I)

The posterior risk is obtained following the same procedure as in section 3.3.4. The posterior risk of the estimators under the aforementioned loss functions using the uniform prior is given by equations (3.88), (3.89) and (3.90) respectively

$$R\left(\hat{\alpha}_{USL}, \alpha\right) = E\left(\alpha^2\right) - \left(E\left(\alpha\right)\right)^2 = \frac{n+1}{M^2} \tag{3.88}$$

$$\begin{aligned} R\left(\hat{\alpha}_{UEL}, \alpha\right) &= E\left(\hat{\alpha}_{UEL}, \alpha\right) = wE\left[\left(\frac{\hat{\alpha}_{UEL}}{\alpha}\right)^k - k\log\left(\frac{\hat{\alpha}_{UEL}}{\alpha}\right) - 1\right] \\ &= w\left[\frac{n}{n-k} - \log\left(\frac{\lceil n+1}{\lceil n-k+1}\right) + k\log\left(n-1\right) - 1\right] \end{aligned} \tag{3.89}$$

$$R\left(\hat{\alpha}_{UPL}, \alpha\right) = 2\left(\frac{\sqrt{\left(n+2\right)\left(n+1\right)}}{M} - \frac{n+1}{M}\right) \tag{3.90}$$

3.5 Bayes Estimates under the Gamma Prior

Let α have the gamma prior distribution given by:

$$\pi(\alpha) = \frac{b^a}{\lceil a}\alpha^{a-1}e^{-b\alpha} \; for \; a, b, \alpha > 0 \tag{3.91}$$

and let $x_1, x_2, \cdots, x_n$ be a random sample from $GRD(\alpha, 1)$, then the likelihood function of the observed data is given in (3.2), the product of the likelihood and the prior is given by

$$L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) = 2^n \alpha^{n+a-1} b^a e^{-\sum_{i=1}^{n} x_i^2} e^{-b\alpha} \prod_{i=1}^{n} x_i \left(1 - e^{-x_i^2}\right)^{\alpha-1} \tag{3.92}$$

while the posterior distribution of α is given by

$$\pi\left(x_1, x_2, \cdots, x_n / \alpha\right) = \frac{L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha)}{\int_0^{\infty} L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) d\alpha} \tag{3.93}$$

equation (3.92) can be written as follows

$$L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) = 2^n \alpha^{n+a-1} b^a e^{-\sum_{i=1}^{n} x_i^2} e^{-b\alpha} \prod_{i=1}^{n} x_i \left(1 - e^{-x_i^2}\right)^{\alpha-1}$$

$$let\, u = \prod_{i=1}^{n} \left(1 - e^{-x_i^2}\right)^{\alpha} \Rightarrow u = e^{-\alpha \sum_{i=1}^{n} \ell n\left(1 - e^{-x_i^2}\right)^{-1}}$$

Then,

$$L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha) = 2^n \alpha^{n+a-1} b^a e^{-\sum_{i=1}^{n} x_i^2} e^{-b\alpha} e^{-\alpha \sum_{i=1}^{n} \ell n\left(1 - e^{-x_i^2}\right)^{-1}} \prod_{i=1}^{n} x_i \left(1 - e^{-x_i^2}\right)^{-1} \tag{3.94}$$

Integrating (3.92) with respect to α, leads to:

$$\int_0^\infty L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha)d\alpha$$

$$= \int_0^\infty 2^n \alpha^{n+a-1} b^a e^{-\sum_{i=1}^n x_i^2} e^{-b\alpha} \prod_{i=1}^n x_i \left(1 - e^{-x_i^2}\right)^{\alpha-1} d\alpha$$

$$let\, u = \prod_{i=1}^n \left(1 - e^{-x_i^2}\right)^\alpha \Rightarrow \alpha = \frac{-\ell nu}{\sum_{i=1}^n \ell n\left(1 - e^{x_i^2}\right)^{-1}} \; and \; d\alpha = \frac{du}{-u\sum_{i=1}^n \ell n\left(1 - e^{x_i^2}\right)^{-1}}$$

$$= \frac{2^n b^a e^{-\sum_{i=1}^n x_i^2} \prod_{i=1}^n x_i \left(1 - e^{-x_i^2}\right)^{-1}}{-\left(\sum_{i=1}^n \ell n\left(1 - e^{-x_i^2}\right)^{-1}\right)^{n+a}} \int_0^\infty (-\ell nu)^{n+a-1} u^{\frac{b}{\sum_{i=1}^n \ell n\left(1 - e^{-x_i^2}\right)^{-1}}} du$$

$$let\, q = -\ell nu \Rightarrow du = -e^{-q} dq$$

$$= \frac{2^n b^a e^{-\sum_{i=1}^n x_i^2} \prod_{i=1}^n x_i \left(1 - e^{-x_i^2}\right)^{-1}}{\left(b + \sum_{i=1}^n \ell n\left(1 - e^{-x_i^2}\right)^{-1}\right)^{n+a}} \lceil n+a$$

$$\int_0^\infty L(x_1, x_2, \cdots, x_n; \alpha, 1)\pi(\alpha)d\alpha = \frac{2^n b^a e^{-\sum_{i=1}^n x_i^2} \prod_{i=1}^n x_i \left(1 - e^{-x_i^2}\right)^{-1}}{\left(b + \sum_{i=1}^n \ell n\left(1 - e^{-x_i^2}\right)^{-1}\right)^{n+a}} \lceil n+a \qquad (3.95)$$

and finally the posterior density (3.93) becomes

$$\pi(\alpha / x_1, x_2, \cdots, x_n) = \frac{\alpha^{n+a-1} H^{n+a} e^{-H\alpha}}{\lceil n+a} \qquad (3.96)$$

From the above distribution, it is clear that the posterior distribution under the gamma prior is gamma distributed with parameters $(n + a)$ and H

where $H = b + \sum_{i=1}^n \ell n(1 - e^{-x_i^2})^{-1} = b + M$

The expectation of α^g where $g \epsilon \mathbb{Z}$, can be obtained as follows:

$$E\left(\alpha^{g}\right)=\int_{-\infty}^{\infty}\alpha^{g}\prod(\alpha/x)\,d\alpha$$

$$=\int_{0}^{\infty}\alpha^{g}\frac{H^{n+a}}{\overline{|n+a}}\alpha^{n+a-1}e^{-H\alpha}d\alpha$$

$$let\ q=H\alpha\Rightarrow\alpha=\frac{q}{H}\ and\ d\alpha=\frac{dq}{H}$$

Therefore

$$E\left(\alpha^{g}\right)=\frac{1}{H^{g}\overline{|n+a}}\int_{0}^{\infty}q^{n+a+g-1}e^{-q}dq=\frac{\overline{|n+a+g}}{H^{g}\overline{|n+a}} \tag{3.97}$$

Let $g=1, g=2\ and\ g=-k,$ then equation (3.97) becomes (3.98), (3.99) and (3.100) respectively

$$E(\alpha)=\frac{\overline{|n+a+1}}{H\overline{|n+a}}=\frac{n+a}{H} \tag{3.98}$$

$$E\left(\alpha^{2}\right)=\frac{\overline{|n+a+2}}{H^{2}\overline{|n+a}}=\frac{(n+a+1)(n+a)}{H^{2}} \tag{3.99}$$

$$E\left(\alpha^{-k}\right)=\frac{\overline{|n+a-k}}{H^{-k}\overline{|n+a}} \tag{3.100}$$

The Bayes estimator of α under the squared error, entropy and precautionary loss functions are given by equations (3.11), (3.14) and (3.16) respectively. Hence, the Bayesian estimators of the parameter α under gamma prior using the aforementioned loss functions denoted by $\hat{\alpha}_{GSL}$, $\hat{\alpha}_{GEL}$ and $\hat{\alpha}_{GPL}$ are respectively given by (3.101), (3.102) and (3.103)

$$\hat{\alpha}_{GSL}=E(\alpha)=\frac{n+a}{H} \tag{3.101}$$

$$\hat{\alpha}_{GEL} = \left(E\left(\alpha^{-k}\right)\right)^{-\frac{1}{k}} = \left(\frac{\lceil n+a-k}{H^{-k}\lceil n+a}\right)^{-\frac{1}{k}} = \frac{1}{H}\left(\frac{\lceil n+a}{\lceil n+a-k}\right)^{\frac{1}{k}} \tag{3.102}$$

$$\hat{\alpha}_{GPL} = \left(E\left(\alpha^{2}\right)\right)^{\frac{1}{2}} = \sqrt{\frac{(n+a+1)(n+a)}{H^{2}}} = \frac{1}{H}\sqrt{(n+a+1)(n+a)} \tag{3.103}$$

3.5.1 The distribution of the random variable H

If $H = b + M$, and M ~ Gamma(n, α) then, the distribution of H is as follows

$$M = H - b \tag{3.104}$$

The jacobian $\frac{dM}{dH} = 1$

Therefore, $f(H) = f\left(w^{-1}(M)\right)$ is given by

$$f(H) = \frac{\alpha^{n}(H-b)^{n-1}e^{-\alpha(H-b)}}{\lceil n} \tag{3.105}$$

$$E\left(H^{-k}\right) = \int_{0}^{\infty} H^{-k} f(H)\, dH = \frac{\alpha^{n}}{\lceil n}\int_{0}^{\infty} H^{-k}(H-b)^{n-1}e^{-\alpha(H-b)}dH \tag{3.106}$$

Let $q = \alpha(H-b) \Rightarrow H = \frac{q}{\alpha} + b \; and \; \frac{dm}{\alpha} = dH$

Therefore,

$$E\left(H^{-k}\right) = \frac{\alpha^{k}}{\lceil n}\int_{0}^{\infty} \frac{q^{n-1}}{(q+\alpha b)^{k}}e^{-q}dq \tag{3.107}$$

The expansion of the denominator of $\frac{q^{n-1}}{(q+\alpha b)^{k}}$ using binomial expansion yields

$$\frac{q^{n-1}}{(q+\alpha b)^k}=\frac{q^{n-1}}{q^k+kq^{k-1}(\alpha b)+\frac{k(k-1)}{2!}q^{k-2}(\alpha b)^2+\cdots+(\alpha b)^k}$$

$$=q^{n-k-1}-k(\alpha b)q^{n-k-2}+\frac{k(k+1)}{2!}(\alpha b)^2q^{n-k-3}-\frac{k(k+1)(k+2)}{3!}(\alpha b)^3q^{n-k-4}+$$

$$\frac{k(k+1)(k+2)(k+3)}{4!}(\alpha b)^4q^{n-k-5}-\frac{k(k+1)(k+2)(k+3)(k+4)}{5!}(\alpha b)^5q^{n-k-6}+\cdots$$

Hence, (3.107) becomes

$$\frac{\alpha^k}{\overline{|n}}\int_0^\infty\left[\begin{array}{l}q^{n-k-1}-k(\alpha b)q^{n-k-2}+\frac{k(k+1)}{2!}(\alpha b)^2q^{n-k-3}-\\ \frac{k(k+1)(k+2)}{3!}(\alpha b)^3q^{n-k-4}+\frac{k(k+1)(k+2)(k+3)}{4!}(\alpha b)^4q^{n-k-5}-\\ \frac{k(k+1)(k+2)(k+3)(k+4)}{5!}(\alpha b)^5q^{n-k-6}+\cdots\end{array}\right]e^{-q}dq \tag{3.108}$$

and finally (3.107) becomes

$$E\left(H^{-k}\right)=\frac{\alpha^k}{\overline{|n}}\left[\begin{array}{l}\overline{|n-k}-k(\alpha b)\overline{|n-k-1}+\frac{k(k+1)}{2!}(\alpha b)^2\overline{|n-k-2}-\\ \frac{k(k+1)(k+2)}{3!}(\alpha b)^3\overline{|n-k-3}+\cdots\end{array}\right] \tag{3.109}$$

Substituting $k=1$ and $k=2$ in equation (3.109) yields equations (3.110) and (3.111) respectively

$$E\left(\frac{1}{H}\right)=\alpha\left[\frac{1}{n-1}-\frac{\alpha b}{(n-1)(n-2)}+\frac{(\alpha b)^2}{(n-1)(n-2)(n-3)}-\frac{(\alpha b)^3}{(n-1)(n-2)(n-3)(n-4)}+\cdots\right] \tag{3.110}$$

$$E\left(\frac{1}{H^2}\right)=\alpha^2\left[\begin{array}{l}\frac{1}{(n-1)(n-2)}-\frac{2\alpha b}{(n-1)(n-2)(n-3)}+\frac{3(\alpha b)^2}{(n-1)(n-2)(n-3)(n-4)}\\ -\frac{4(\alpha b)^3}{(n-1)(n-2)(n-3)(n-4)(n-5)}+\cdots\end{array}\right] \tag{3.111}$$

Taking two terms in $E\left(H^{-1}\right)$ and three terms in the expansion of $E\left(H^{-2}\right)$, then, $Var\left(\frac{1}{H}\right)$

is approximately given by:

$$Var\left(\frac{1}{H}\right)=E\left(\frac{1}{H}\right)^2-\left(E\left(\frac{1}{H}\right)\right)^2=\frac{\alpha^2}{(n-1)(n-2)}-$$

$$\frac{2\alpha^3 b}{(n-1)(n-2)(n-3)}+\frac{3\alpha^4 b^2}{(n-1)(n-2)(n-3)(n-4)}-\left(\frac{\alpha}{n-1}-\frac{\alpha^2 b}{(n-1)(n-2)}\right)^2 \quad (3.112)$$

This becomes

$$Var\left(\frac{1}{H}\right)=\frac{\alpha^2}{(n-1)^2(n-2)}-\frac{4\alpha^3 b}{(n-1)^2(n-2)(n-3)}+\frac{\alpha^4 b^2\left(2n^2-2n-6\right)}{(n-1)^2(n-2)^2(n-3)(n-4)} \quad (3.113)$$

3.5.2 Determination of variance, relative efficiency and posterior risk of the shape parameter under the squares error, entropy and precautionary loss functions

The variances of the estimators under the various loss functions are as follows:

$$\text{var}\left(\hat{\alpha}_{GSL}\right)=(n+a)^2\,\text{var}\left(\frac{1}{H}\right) \quad (3.114)$$

substituting equation (3.113) in equation (3.114) gives equation (3.115)

$$\text{var}\left(\hat{\alpha}_{GSL}\right)=(n+a)^2\left(\frac{\alpha^2}{(n-1)^2(n-2)}-\frac{4\alpha^3 b}{(n-1)^2(n-2)(n-3)}+\frac{\alpha^4 b^2\left(2n^2-2n-6\right)}{(n-1)^2(n-2)^2(n-3)(n-4)}\right) \quad (3.115)$$

The variance of the estimator under gamma prior using Entropy loss function $Var(\alpha_{GEL})$ is obtained as follows:

$$\text{var}\left(\hat{\alpha}_{GEL}\right)=\left(\frac{\overline{|n+a}}{\overline{|n+a-k}}\right)^{\frac{2}{k}}\text{var}\left(\frac{1}{H}\right) \quad (3.116)$$

substituting equation (3.113) in equation (3.116) gives equation (3.117):

$$\operatorname{var}\left(\hat{\alpha}_{GEL}\right)=\left(\frac{\sqrt{n+a}}{\sqrt{n-k+a}}\right)^{\frac{2}{k}}\left(\frac{\alpha^2}{(n-1)^2(n-2)}-\frac{4\alpha^3 b}{(n-1)^2(n-2)(n-3)}+\frac{\alpha^4 b^2\left(2n^2-2n-6\right)}{(n-1)^2(n-2)^2(n-3)(n-4)}\right) \quad (3.117)$$

and finally, the variance of the estimator under gamma prior using precautionary loss function $Var(\alpha_{GPL})$ is obtain in a similar way as follows:

$$\operatorname{var}\left(\hat{\alpha}_{GPL}\right)=(n+a+1)(n+a)\operatorname{var}\left(\frac{1}{H}\right) \quad (3.118)$$

also, substituting equation (3.113) in equation (3.118) gives equation (3.119)

$$Var\left(\hat{\alpha}_{GPL}\right)=(n+a+1)(n+a)\left(\frac{\alpha^2}{(n-1)^2(n-2)}-\frac{4\alpha^3 b}{(n-1)^2(n-2)(n-3)}+\frac{\alpha^4 b^2\left(2n^2-2n-6\right)}{(n-1)^2(n-2)^2(n-3)(n-4)}\right) \quad (3.119)$$

The relative efficiency of the estimators under the various loss functions are as follows:

Relative efficiency of $\hat{\alpha}_{GEL}$ with respect to $\hat{\alpha}_{GSL}$ of α is

$$E_{G1}=\frac{Var\left(\hat{\alpha}_{GSL}\right)}{Var\left(\hat{\alpha}_{GEL}\right)}=(n+a)^2\left(\frac{\sqrt{n+a-k}}{\sqrt{n+a}}\right)^{\frac{2}{k}} \quad (3.120)$$

$E_{G1}>1$ for $k\geq 1$ and $E_{G1}<1$ for $k\leq -2$ (see fig. A.11 & A.12 in appendix I)

Relative efficiency of $\hat{\alpha}_{GEL}$ with respect to $\hat{\alpha}_{GPL}$ of α is

$$E_{G2}=\frac{Var\left(\hat{\alpha}_{GPL}\right)}{Var\left(\hat{\alpha}_{GEL}\right)}=(n+a+1)(n+a)\left(\frac{\sqrt{n+a-k}}{\sqrt{n+a}}\right)^{\frac{2}{k}} \quad (3.121)$$

$E_{G2}>1$ for $k\geq 1$ and $E_{G2}<1$ for $k<-2$ (see fig. A.13 & A.14 in appendix I)

Relative efficiency of $\hat{\alpha}_{GSL}$ with respect to $\hat{\alpha}_{GPL}$ of α is

$$E_{G3} = \frac{Var\left(\hat{\alpha}_{GPL}\right)}{Var\left(\hat{\alpha}_{GSL}\right)} = \frac{n+a+1}{n+a} \tag{3.122}$$

$E_{G3} > 1$ when $n > 0$ (see fig. A.15 in appendix I)

3.5.2.1 Posterior risk

Following the same procedure as in section 3.3.4, the posterior risk of the estimators under the aforementioned loss functions using the gamma prior is given by equations (3.123), (3.125) and (3.126)

$$R\left(\hat{\alpha}_{GSL}, \alpha\right) = E\left(\alpha^2\right) - \left(E\left(\alpha\right)\right)^2 = \frac{n+a}{H^2} \tag{3.123}$$

$$R\left(\hat{\alpha}_{GEL}, \alpha\right) = E\left(\hat{\alpha}_{GEL}, \alpha\right) = wE\left[\left(\frac{\hat{\alpha}_{GEL}}{\alpha}\right)^k - k\log\left(\frac{\hat{\alpha}_{GEL}}{\alpha}\right) - 1\right] \tag{3.124}$$

where w is constant of proportionality

substituting equation (3.102) and taking 2 terms in equation (3.113), the posterior risk of $\hat{\alpha}_{GEL}$ becomes

$$R\left(\hat{\alpha}_{GEL}, \alpha\right) = w\left[\frac{\lceil n+a}{\lceil n+a-k \rceil n}\left(\lceil n-k - k\alpha b \rceil n-k-1\right) - k\log\left(\frac{\lceil n+a}{\lceil n+a-k \rceil n}\left(\lceil n-1 - \alpha b \rceil n-2\right)\right) - 1\right] \tag{3.125}$$

and finally, the posterior risk of the estimator under precautionary loss function is given by:

$$R\left(\hat{\alpha}_{GPL}, \alpha\right) = 2\left(\frac{\sqrt{\left(n+a+1\right)\left(n+a\right)}}{H} - \frac{n+a}{H}\right) \tag{3.126}$$

CHAPTER FOUR: ANALYSIS AND DISCUSSION

In this chapter, results of some numerical experiments were obtained in order to compare the performance of the different estimators proposed in the previous chapter with that of maximum likelihood estimator. Extensive Monte Carlo simulations were carried out to compare the performance of the different estimators with respect to biases and mean-squared errors (MSEs) for different sample sizes (n= 15, 20, 30, 50 and 100) and different shape (α) parameter values of 0.25, 0.5, 1.0, 1.5, 2.0 and 2.5. All results were replicated 10000 times and averaged over.

Table 5. 1: Average estimates and corresponding MSEs (within parenthesis) for α and n=15

n	method	α= 0.25	α= 0.5	α=1.0	α=1.5	α=2	α =2.5
15	MLE	0.2856 (0.0183)	0.5700 (0.0732)	1.1428 (0.2865)	1.7144 (0.6458)	2.2883 (1.1617)	2.8564 (1.8124)
	JSL, r=1	0.2679 (0.0129)	0.5366 (0.0516)	1.0683 (0.2012)	1.5996 (0.4479)	2.1370 (0.7942)	2.6748 (1.2616)
	JSL, r=2	0.2493 (0.0102)	0.5009 (0.0399)	1.0037 (0.1619)	1.4935 (0.3551)	1.9903 (0.6301)	2.4931 (0.9989)
	JEL, k=1, r=1	0.2494 (0.0100)	0.5022 (0.0407)	1.0038 (0.1613)	1.5060 (0.1892)	1.9934 (0.6324)	2.4928 (0.9936)
	JEL, k=-1, r=1	0.2668 (0.0126)	0.5341 (0.0499)	1.0764 (0.2069)	1.6072 (0.2290)	2.1409 (0.7978)	2.6884 (1.2854)
	JEL, k=1, r=2	0.2328 (0.0084)	0.4668 (0.0339)	0.9288 (0.1307)	1.3939 (0.1735)	1.8605 (0.5245)	2.3164 (0.8291)
	JEL, k=-1, r=2	0.2488 (0.0100)	0.4988 (0.0394)	1.0038 (0.1639)	1.4983 (0.1895)	2.0008 (0.6302)	2.5103 (1.0225)
	JPL, r=1	0.2756 (0.0142)	0.5521 (0.0570)	1.1085 (0.2298)	1.6551 (0.5130)	2.2100 (0.9155)	2.7613 (1.4251)
	JPL, r=2	0.2591 (0.0111)	0.5197 (0.0458)	1.0369 (0.1823)	1.5584 (0.4079)	2.0645 (0.7143)	2.5729 (1.0847)

	USL	0.2852 (0.0164)	0.5714 (0.0660)	1.1446 (0.2614)	1.7179 (0.5977)	2.2857 (1.0525)	2.8548 (1.6267)
	UEL K=1	0.2786 (0.0148)	0.5545 (0.0601)	1.1053 (0.2377)	1.6824 (0.5469)	2.2375 (0.9591)	2.7857 (1.4714)
	UEL K=-1	0.2789 (0.0149)	0.5534 (0.0595)	1.1009 (0.2362)	1.6714 (0.5297)	2.2427 (0.9707)	2.8012 (1.4948)
	UPL	0.2955 (0.0191)	0.5871 (0.0735)	1.1755 (0.2958)	1.7654 (0.6612)	2.3570 (1.1801)	2.9504 (1.8512)
	GSL a=b=.5	0.2742 (0.0135)	0.5408 (0.0498)	1.0625 (0.1850)	1.5659 (0.3902)	2.0624 (0.6984)	2.5196 (1.0563)
	GSL a=b=1	0.2802 (0.0146)	0.5489 (0.0520)	1.0649 (0.1940)	1.5356 (0.4439)	1.9808 (0.8383)	2.4022 (1.4826)
	K=1 GEL a=b=.5	0.2677 (0.0124)	0.5318 (0.0464)	1.0423 (0.1725)	1.5289 (0.3582)	2.0098 (0.6267)	2.4670 (0.9861)
	GEL a=b=1	0.2748 (0.0132)	0.5356 (0.0477)	1.0332 (0.1755)	1.4968 (0.3957)	1.9336 (0.7688)	2.3395 (1.3227)
	K=-1 GEL a=b=.5	0.2666 (0.0121)	0.5288 (0.0465)	1.0400 (0.1674)	1.5342 (0.3623)	2.0121 (0.6447)	2.4723 (0.9968)
	GEL a=b=1	0.2733 (0.0131)	0.5360 (0.0474)	1.0417 (0.1780)	1.5028 (0.4057)	1.9364 (0.7757)	2.3421 (1.3174)
	GPL a=b=.5	0.2831 (0.0157)	0.5610 (0.0584)	1.0971 (0.2100)	1.6182 (0.4480)	2.1172 (0.7628)	2.6025 (1.2036)
	GPL a=b=1	0.2892 (0.0165)	0.5662 (0.0605)	1.0980 (0.2242)	1.5816 (0.5057)	2.0364 (0.9619)	2.4639 (1.6558)

As expected, it is observed that the performances of both the Maximum likelihood and Bayesian estimates become better when sample size increases. Also the MLEs and Bayesian estimates become closer as sample size increases. The MSEs of the Bayesian

Table 5. 2 Average estimates and corresponding MSEs (within parenthesis) for α and n=20

n	method	α= 0.25	α= 0.5	α=1.0	α=1.5	α=2	α =2.5
20	MLE	0.2756 (0.0121)	0.5502 (0.0485)	1.1088 (0.1959)	1.6564 (0.4331)	2.2164 (0.7773)	2.7721 (1.2145)
	JSL, r=1	0.2629 (0.0087)	0.5261 (0.0343)	1.0507 (0.1336)	1.5799 (0.3083)	2.1063 (0.5439)	2.6361 (0.8565)
	JSL, r=2	0.2502 (0.0073)	0.4980 (0.0281)	1.0029 (0.1142)	1.5019 (0.2532)	1.9947 (0.4499)	2.4985 (0.7165)
	JEL, k=1, r=1	0.2493 (0.0072)	0.4993 (0.0285)	0.9997 (0.1136)	1.5022 (0.2595)	2.0000 (0.4576)	2.4953 (0.6997)
	JEL, k=-1, r=1	0.2628 (0.0085)	0.5276 (0.0350)	1.0556 (0.1365)	1.5785 (0.3042)	2.1165 (0.5591)	2.6392 (0.8544)
	JEL, k=1, r=2	0.2357 (0.0063)	0.4735 (0.0246)	0.9476 (0.1023)	1.4218 (0.2236)	1.8889 (0.4018)	2.3657 (0.6278)
	JEL, k=-1, r=2	0.2491 (0.0072)	0.5009 (0.0288)	1.0043 (0.1159)	1.5009 (0.2584)	2.0009 (0.4617)	2.4949 (0.7191)
	JPL, r=1	0.2708 (0.0097)	0.5387 (0.0376)	1.0744 (0.1474)	1.6149 (0.3312)	2.1609 (0.6042)	2.6961 (0.9402)
	JPL, r=2	0.2554 (0.0079)	0.5119 (0.0312)	1.0255 (0.1248)	1.5434 (0.2797)	2.0507 (0.4982)	2.5744 (0.7841)
	USL	0.2765 (0.0106)	0.5535 (0.0423)	1.1046 (0.1646)	1.6563 (0.3745)	2.2104 (0.6615)	2.7694 (1.0597)
	UEL K=1	0.2703 (0.0097)	0.5396 (0.0387)	1.0802 (0.1558)	1.6321 (0.3525)	2.1673 (0.6085)	2.7188 (0.9703)
	UEL K=-1	0.2709 (0.0098)	0.5382 (0.0379)	1.0766 (0.1556)	1.6313 (0.3493)	2.1627 (0.6069)	2.7119 (0.9672)
	UPL	0.2818 (0.0117)	0.5656 (0.0462)	1.1300 (0.1847)	1.7060 (0.4262)	2.2635 (0.7445)	2.8195 (1.1482)
	GSL a=b=.5	0.2676 (0.0091)	0.5310 (0.0344)	1.0516 (0.1283)	1.5554 (0.2763)	2.0409 (0.4661)	2.5230 (0.7369)
	GSL a=b=1	0.2727 (0.0096)	0.5357 (0.0352)	1.0444 (0.1280)	1.5310 (0.2840)	1.9898 (0.5223)	2.4380 (0.8864)

	K=1 GEL a=b=.5	0.2616 (0.0084)	0.5209 (0.0316)	1.0371 (0.1216)	1.5200 (0.2571)	2.0075 (0.4481)	2.4875 (0.6944)
	GEL a=b=1	0.2670 (0.0088)	0.5291 (0.0329)	1.0287 (0.1206)	1.5009 (0.2682)	1.9566 (0.4973)	2.3851 (0.8280)
	K=-1 GEL a=b=.5	0.2621 (0.0084)	0.5225 (0.0325)	1.0303 (0.1220)	1.5258 (0.2596)	2.0023 (0.4468)	2.4724 (0.6750)
	GEL a=b=1	0.2676 (0.0089)	0.5280 (0.0335)	1.0289 (0.1227)	1.5065 (0.2680)	1.9545 (0.4956)	2.3842 (0.8168)
	GPL a=b=.5	0.2738 (0.0101)	0.5449 (0.0385)	1.0732 (0.1390)	1.5939 (0.3041)	2.0950 (0.5187)	2.5831 (0.7995)
	GPL a=b=1	0.2776 (0.0104)	0.5516 (0.0395)	1.0749 (0.1437)	1.5660 (0.3157)	2.0328 (0.5694)	2.4934 (0.9723)

estimate is lower than the MSEs of the MLEs. Hence, the Bayesian estimators are more stable than the MLEs.

The estimates under the extended Jeffrey's prior is more stable than the estimates under the uniform and gamma priors since the MSEs under the Jeffrey's prior is smaller than the estimates under these priors. Also, the estimates under the gamma prior were observed to be more stable than the estimates under the uniform prior. Under the extended Jeffrey's prior, the MSEs when r=2 were found to be smaller than the MSEs when r=1. Hence, the estimates are more stable when r=2 than when r=1. This is also the case for the entropy loss function the estimates are more stable when k=1 compared to when k=-1. Estimates under the entropy and squares error loss functions were further discovered to be more stable than estimates under the precautionary loss function.

Table 5. 3: Average estimates and corresponding MSEs (within parenthesis) for α and n=30

n	method	α= 0.25	α= 0.5	α=1.0	α=1.5	α=2	α =2.5
30	MLE	0.2665 (0.0072)	0.5351 (0.0299)	1.0702 (0.1139)	1.6018 (0.2548)	2.1323 (0.4523)	2.6624 (0.7021)
	JSL, r=1	0.2584 (0.0053)	0.5174 (0.0202)	1.0380 (0.0816)	1.5560 (0.1846)	2.0693 (0.3274)	2.5864 (0.5111)
	JSL, r=2	0.2502 (0.0048)	0.5003 (0.0181)	0.9993 (0.0728)	1.5006 (0.1651)	1.9937 (0.2856)	2.5001 (0.4515)
	JEL, k=1, r=1	0.2488 (0.0047)	0.5002 (0.0182)	0.9998 (0.0738)	1.4995 (0.1635)	1.9991 (0.2916)	2.5086 (0.4520)
	JEL, k=-1, r=1	0.2584 (0.0052)	0.5165 (0.0202)	1.0344 (0.0825)	1.5540 (0.1850)	2.0701 (0.3301)	2.5844 (0.5116)
	JEL, k=1, r=2	0.2408 (0.0043)	0.4835 (0.0166)	0.9681 (0.0671)	1.4454 (0.1476)	1.9269 (0.2675)	2.4215 (0.4107)
	JEL, k=-1, r=2	0.2492 (0.0047)	0.4990 (0.0179)	0.9965 (0.0716)	1.4970 (0.1618)	1.9988 (0.2938)	2.4918 (0.4569)
	JPL, r=1	0.2623 (0.0057)	0.5275 (0.0217)	1.0521 (0.0870)	1.5828 (0.1966)	2.1033 (0.3496)	2.6278 (0.5479)
	JPL, r=2	0.2533 (0.0049)	0.5069 (0.0193)	1.0174 (0.0760)	1.5246 (0.1731)	2.0343 (0.3054)	2.5450 (0.4842)
	USL	0.2668 (0.0060)	0.5325 (0.0230)	1.0698 (0.0951)	1.6047 (0.2110)	2.1379 (0.3839)	2.6787 (0.5915)
	UEL K=1	0.2641 (0.0057)	0.5266 (0.0224)	1.0499 (0.0883)	1.5851 (0.2033)	2.1159 (0.3561)	2.6354 (0.5437)
	UEL K=-1	0.2635 (0.0057)	0.5242 (0.0220)	1.0509 (0.0888)	1.5812 (0.1957)	2.1118 (0.3498)	2.6434 (0.5641)
	UPL	0.2701 (0.0065)	0.5428 (0.0253)	1.0896 (0.1008)	1.6300 (0.2293)	2.1708 (0.4070)	2.7094 (0.6293)
	GSL a=b=.5	0.2616 (0.0055)	0.5229 (0.0211)	1.0331 (0.0789)	1.5368 (0.1707)	2.0303 (0.3011)	2.5233 (0.4599)
	GSL a=b=1	0.2638 (0.0056)	0.5247 (0.0211)	1.0347 (0.0800)	1.5191 (0.1714)	1.9924 (0.3044)	2.4523 (0.4908)

	K=1 GEL a=b=.5	0.2584 (0.0051)	0.5161 (0.0200)	1.0203 (0.0753)	1.5164 (0.1638)	2.0098 (0.2827)	2.4845 (0.4362)
	GEL a=b=1	0.2618 (0.0054)	0.5187 (0.0200)	1.0190 (0.0750)	1.5083 (0.1676)	1.9781 (0.2936)	2.4301 (0.4774)
	K=-1 GEL a=b=.5	0.2572 (0.0052)	0.5150 (0.0196)	1.0205 (0.0766)	1.5192 (0.1671)	1.9983 (0.2813)	2.4854 (0.4446)
	GEL a=b=1	0.2600 (0.0054)	0.5181 (0.0198)	1.0208 (0.0752)	1.5009 (0.1628)	1.9732 (0.2987)	2.4204 (0.4726)
	GPL a=b=.5	0.2654 (0.0058)	0.5306 (0.0224)	1.0536 (0.0844)	1.5616 (0.1827)	2.0644 (0.3158)	2.5569 (0.4845)
	GPL a=b=1	0.2689 (0.0060)	0.5350 (0.0231)	1.0480 (0.0842)	1.5465 (0.1819)	2.0218 (0.3192)	2.4952 (0.5247)

Under the uniform prior, it is observed that the Bayesian estimator under entropy loss function performs better than the Bayesian estimates under the squared error loss function when the sample size is small ($n \leq 15$) and/ moderate ($n \leq 30$), while the performances of the estimates under both the squared error loss function and entropy loss function is almost the same for large sample size. Furthermore, the estimates under the squared error loss function are better than the estimates under the precautionary loss function. When the sample size is less than fifteen, it is observed that the MLE outperforms the estimates under the precautionary loss function marginally.

Table 5. 4: Average estimates and corresponding MSEs (within parenthesis) for α and n=50

n	method	α= 0.25	α= 0.5	α=1.0	α=1.5	α=2	α =2.5
50	MLE	0.2601 (0.0042)	0.5218 (0.0159)	1.0423 (0.0625)	1.5594 (0.1394)	2.0774 (0.2489)	2.6015 (0.3889)
	JSL, r=1	0.2540 (0.0031)	0.5106 (0.0114)	1.0194 (0.0452)	1.5288 (0.1004)	2.0364 (0.1792)	2.5559 (0.2854)
	JSL, r=2	0.2488 (0.0029)	0.5005 (0.0106)	1.0016 (0.0422)	1.5005 (0.0948)	2.0019 (0.1709)	2.4998 (0.2622)
	JEL, k=1, r=1	0.2497 (0.0029)	0.4999 (0.0104)	1.0014 (0.0419)	1.5008 (0.0948)	2.0001 (0.1678)	2.4976 (0.2637)
	JEL, k=-1, r=1	0.2549 (0.0030)	0.5100 (0.0113)	1.0223 (0.0452)	1.5355 (0.1035)	2.0391 (0.1807)	2.5551 (0.2833)
	JEL, k=1, r=2	0.2442 (0.0027)	0.4882 (0.0099)	0.9791 (0.0403)	1.4708 (0.0896)	1.9555 (0.1604)	2.4488 (0.2515)
	JEL, k=-1, r=2	0.2495 (0.0028)	0.5013 (0.0105)	1.0022 (0.0427)	1.4994 (0.0934)	1.9977 (0.1700)	2.4990 (0.2611)
	JPL, r=1	0.2566 (0.0032)	0.5152 (0.0118)	1.0317 (0.0475)	1.5444 (0.1059)	2.0651 (0.1904)	2.5766 (0.2930)
	JPL, r=2	0.2511 (0.0030)	0.5049 (0.0108)	1.0110 (0.0439)	1.5177 (0.0982)	2.0202 (0.1734)	2.5332 (0.2753)
	USL	0.2589 (0.0034)	0.5214 (0.0124)	1.0382 (0.0491)	1.5606 (0.1114)	2.0826 (0.1982)	2.6046 (0.3087)
	UEL K=1	0.2573 (0.0032)	0.5163 (0.0121)	1.0330 (0.0476)	1.5498 (0.1059)	2.0640 (0.1896)	2.5767 (0.2952)
	UEL K=-1	0.2572 (0.0033)	0.5158 (0.0119)	1.0287 (0.0468)	1.5460 (0.1063)	2.0673 (0.1895)	2.5748 (0.2916)
	UPL	0.2613 (0.0036)	0.5255 (0.0129)	1.0509 (0.0518)	1.5751 (0.1159)	2.0994 (0.2075)	2.6224 (0.3212)
	GSL a=b=.5	0.2562 (0.0032)	0.5134 (0.0115)	1.0207 (0.0437)	1.5224 (0.0981)	2.0178 (0.1718)	2.5072 (0.2626)
	GSL a=b=1	0.2584 (0.0032)	0.5148 (0.0115)	1.0173 (0.0436)	1.5126 (0.0969)	1.9977 (0.1715)	2.4682 (0.2660)

	K=1 GEL a=b=.5	0.2540 (0.0030)	0.5077 (0.0109)	1.0144 (0.0432)	1.5101 (0.0945)	2.0023 (0.1654)	2.4872 (0.2527)
	GEL a=b=1	0.2562 (0.0032)	0.5116 (0.0110)	1.0118 (0.0433)	1.5051 (0.0952)	1.9833 (0.1647)	2.4514 (0.2648)
	K=-1 GEL a=b=.5	0.2540 (0.0030)	0.5086 (0.0109)	1.0104 (0.0427)	1.5120 (0.0948)	2.0040 (0.1652)	2.4909 (0.2590)
	GEL a=b=1	0.2562 (0.0031)	0.5111 (0.0111)	1.0129 (0.0429)	1.5040 (0.0950)	1.9816 (0.1667)	2.4543 (0.2652)
	GPL a=b=.5	0.2588 (0.0033)	0.5174 (0.0120)	1.0291 (0.0462)	1.5400 (0.1025)	2.0348 (0.1754)	2.5361 (0.2710)
	GPL a=b=1	0.2604 (0.0035)	0.5197 (0.0120)	1.0312 (0.0462)	1.5286 (0.1003)	2.0198 (0.1793)	2.5048 (0.2811)

Careful examination of the estimates under the gamma prior (conjugate prior) reveal that when a=b=0.5, the estimates are slightly better than the estimates when a=b=1 in terms of MSE. Also as in the case of the extended Jeffrey's prior, it is observed under the gamma prior that the Bayes estimates under the entropy loss function is more stable than the estimates under the squared error loss function and precautionary loss function. On the other hand, the estimates under the squared error loss function were found to be more stable than the estimates under the precautionary loss function. For the entropy loss function, it is also observed that the estimates when k=1 is more stable than the estimates when k=-1.

Table 5. 5: Average estimates and corresponding MSEs (within parenthesis) for α and n=100

n	method	α= 0.25	α= 0.5	α=1.0	α=1.5	α=2	α =2.5
100	MLE	0.2529 (0.0024)	0.50974 (0.0072)	1.0199 (0.0288)	1.5303 (0.0651)	2.0387 (0.1161)	2.5502 (0.1813)
	JSL, r=1	0.2504 (0.0018)	0.5049 (0.0053)	1.0103 (0.0213)	1.5137 (0.0480)	2.0221 (0.0849)	2.5287 (0.1343)
	JSL, r=2	0.2478 (0.0018)	0.4996 (0.0051)	1.0012 (0.0207)	1.5005 (0.0459)	2.0022 (0.0823)	2.4967 (0.1266)
	JEL, k=1, r=1	0.2484 (0.0017)	0.5005 (0.0052)	0.9990 (0.0205)	1.4999 (0.0460)	2.0012 (0.0823)	2.5005 (0.1286)
	JEL, k=-1, r=1	0.2508 (0.0018)	0.5048 (0.0053)	1.0101 (0.0214)	1.5148 (0.0481)	2.0210 (0.0853)	2.5282 (0.1339)
	JEL, k=1, r=2	0.2453 (0.0018)	0.4947 (0.0050)	0.9902 (0.0201)	1.4821 (0.0449)	1.9806 (0.0804)	2.4705 (0.1261)
	JEL, k=-1, r=2	0.2479 (0.0018)	0.5001 (0.0051)	0.9980 (0.0201)	1.4992 (0.0459)	1.9991 (0.0817)	2.5033 (0.1285)
	JPL, r=1	0.2517 (0.0019)	0.5067 (0.0054)	1.0143 (0.0217)	1.5230 (0.0489)	2.0331 (0.0878)	2.5403 (0.1346)
	JPL, r=2	0.2494 (0.0018)	0.5032 (0.0052)	1.0048 (0.0207)	1.5093 (0.0473)	2.0074 (0.0823)	2.5134 (0.1299)
	USL	0.2532 (0.0018)	0.5093 (0.0055)	1.0194 (0.0220)	1.5256 (0.0487)	2.0425 (0.0894)	2.5555 (0.1408)
	UEL K=1	0.2526 (0.0017)	0.5081 (0.0054)	1.0154 (0.0221)	1.5245 (0.0492)	2.0331 (0.0866)	2.5409 (0.1367)
	UEL K=-1	0.2525 (0.0019)	0.5079 (0.0055)	1.0138 (0.0218)	1.5274 (0.0491)	2.0334 (0.0870)	2.5462 (0.1389)
	UPL	0.2549 (0.0019)	0.5129 (0.0057)	1.0235 (0.0228)	1.5384 (0.0514)	2.0496 (0.0905)	2.5641 (0.1430)
	GSL a=b=.5	0.2513 (0.0019)	0.5065 (0.0054)	1.0097 (0.0211)	1.5117 (0.0471)	2.0071 (0.0823)	2.5022 (0.1284)
	GSL a=b=1	0.2523 (0.0019)	0.5076 (0.0054)	1.0132 (0.0214)	1.5056 (0.0463)	1.9985 (0.0824)	2.4852 (0.1287)

	K=1 GEL a=b=.5	0.2511 (0.0018)	0.5061 (0.0054)	1.0053 (0.0205)	1.5059 (0.0465)	2.0022 (0.0808)	2.4947 (0.1264)
	GEL a=b=1	0.2511 (0.0019)	0.5055 (0.0053)	1.0069 (0.0210)	1.5022 (0.0463)	1.9936 (0.0821)	2.4743 (0.1282)
	K=-1 GEL a=b=.5	0.2505 (0.0018)	0.5050 (0.0053)	1.0050 (0.0206)	1.5051 (0.0457)	2.0002 (0.0812)	2.4933 (0.1263)
	GEL a=b=1	0.2512 (0.0018)	0.5053 (0.0052)	1.0060 (0.0206)	1.5014 (0.0451)	1.9892 (0.0817)	2.4765 (0.1278)
	GPL a=b=.5	0.2524 (0.0020)	0.5085 (0.0054)	1.0152 (0.0216)	1.5207 (0.0484)	2.0216 (0.0840)	2.5207 (0.1293)
	GPL a=b=1	0.2536 (0.0019)	0.5101 (0.0055)	1.0161 (0.0217)	1.5127 (0.0469)	2.0117 (0.0840)	2.5002 (0.1303)

Finally, it is further observed that for the squared error loss function, the estimates under the extended Jeffrey's prior are more stable than the estimates under the uniform and gamma prior. For the entropy and precautionary loss functions, the estimates under the gamma and the extended Jeffrey's priors were found to be more stable than the estimates under the uniform prior even when the hyper-parameters takes the value one for the gamma prior and when r=1 for the extended Jeffrey's prior.

Figures 1-15 in the appendix, show the Relative Efficiency of estimators for different values of n, r and k. The graphs provide an impression on how the efficiencies change with changes in the values of n, r and k. The graphs are plotted using efficiency against sample sizes. It is observed from figures 1, 3, 5, 6, 8, 10, 11, 13 and 15 that as sample size increases, the efficiency decreases while the efficiencies increases as sample size increases in figures 2, 7 and 12 when $k \leq -2$. In figures 4, 9 and 14, the efficiency also increases as sample size increases when $k \leq -3$.

It is further observed that the rate of decrease/ increase in the relative efficiency of the estimators under the Entropy loss function with respect to any other loss function varies for different values of k and as sample size becomes very large, the effect of k on efficiency is negligible.

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

The primary goal of this work as initially stated in the objectives, was to estimate the shape parameter of a GRD when the scale parameter is known using Bayesian approach. The shape parameter was assumed to follow an Extended Jeffrey's prior distribution, Uniform prior distribution and Gamma prior distribution. Symmetric and asymmetric loss functions were considered in estimating the parameter. Squared error loss function which is classified as symmetric loss function because it assigns equal importance to the losses due to both over estimation and under estimation was assumed while in the asymmetric case Entropy and Precautionary loss functions were considered. The Bayes estimators of the aforementioned loss functions were obtained. Relative efficiency and Posterior risk of the estimators under the various loss functions were obtained and finally, Monte Carlo simulation was carried out in order to compare the performance of the estimators using Mean Square Error as a performance measure.

5.2 Conclusion

In this work, MLE and Bayesian estimation of the shape parameter of a generalized Rayleigh distribution is considered. Three prior distributions on the shape parameter is assumed and the Bayes estimators under the symmetric loss function (squared error) and asymmetric (Entropy and precautionary) loss functions were provided. Comparison between the performances of the MLEs and the Bayes estimators were carried out and it was observed that the Non-classical/ Bayesian estimators were more stable than the classical estimators in terms of Mean Square Error. The estimates under the extended

Jeffrey's prior were found to be more stable than the estimates under uniform and gamma priors. It was also observed that under the informative prior, that the estimates under the entropy loss function is better than the estimates under the squared error and precautionary loss functions.

5.3 Recommendations

Based on our observations in the previous chapters, we recommend the following for the estimation of the shape parameter of GRD when the scale parameter is known

i. If you have enough information about the shape parameter that is, it follows gamma, then use entropy loss function in order to estimate the shape parameter.
ii. If on the other hand you have little or no information about the prior distribution of the shape parameter, assumed extended Jeffrey's prior under the assumption of squared error or entropy loss functions.

5.4 Contribution to Knowledge

In this research work, we were able to contribute to the body of knowledge in so many ways, which include the following:

i. We were able to give some formulae for estimating the shape parameter of GRD when the scale parameter is known under three different prior distributions and three different loss functions.
ii. We were also able to give the variances of the various estimators proposed in this work.
iii. Finally, we were able to show that the estimators under extended Jeffery's prior is more stable than estimators under uniform and gamma priors in terms of MSE and

we further show that, when loss functions are considered, the estimators under Entropy loss function is more stable in terms of MSE than the estimators under Square error and precautionary loss functions.

5.5 Areas of Further Research

i. The scale parameter of GRD can be estimated when the shape parameter is known

ii. The shape parameter can be estimated when the scale parameter is unknown and

iii. The scale parameter of GRD can be estimated when the shape parameter is unknown

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Appendix I

The graph of the relative efficiency of the estimators under extended Jeffrey's prior, uniform prior and gamma prior

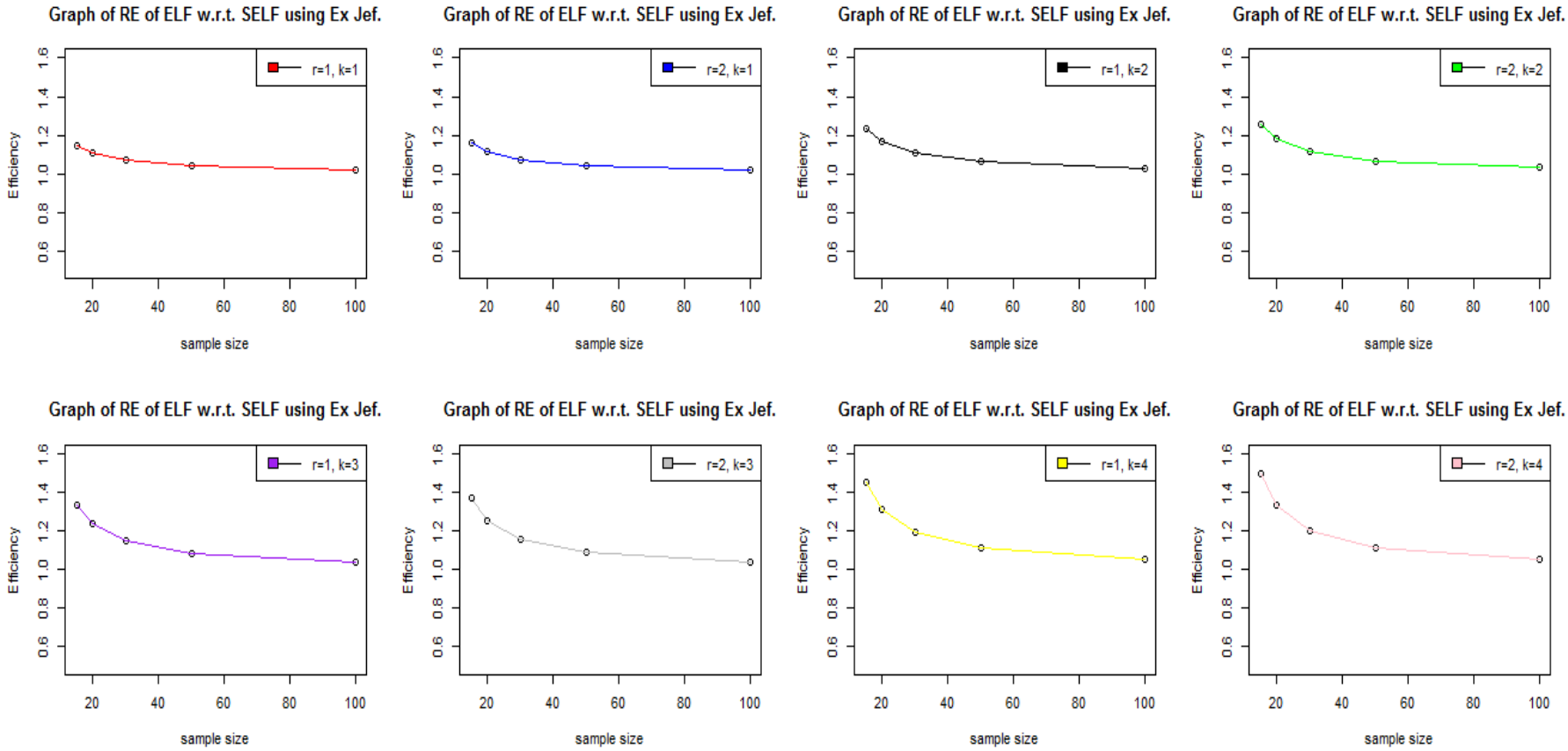


Figure A. 1: Graph of relative efficiency of entropy loss with respect to square error loss functions under extended Jeffrey's prior for different values of k=1, 2, 3, 4 and r=1, 2 when n=15, 20, 30, 50 and 100

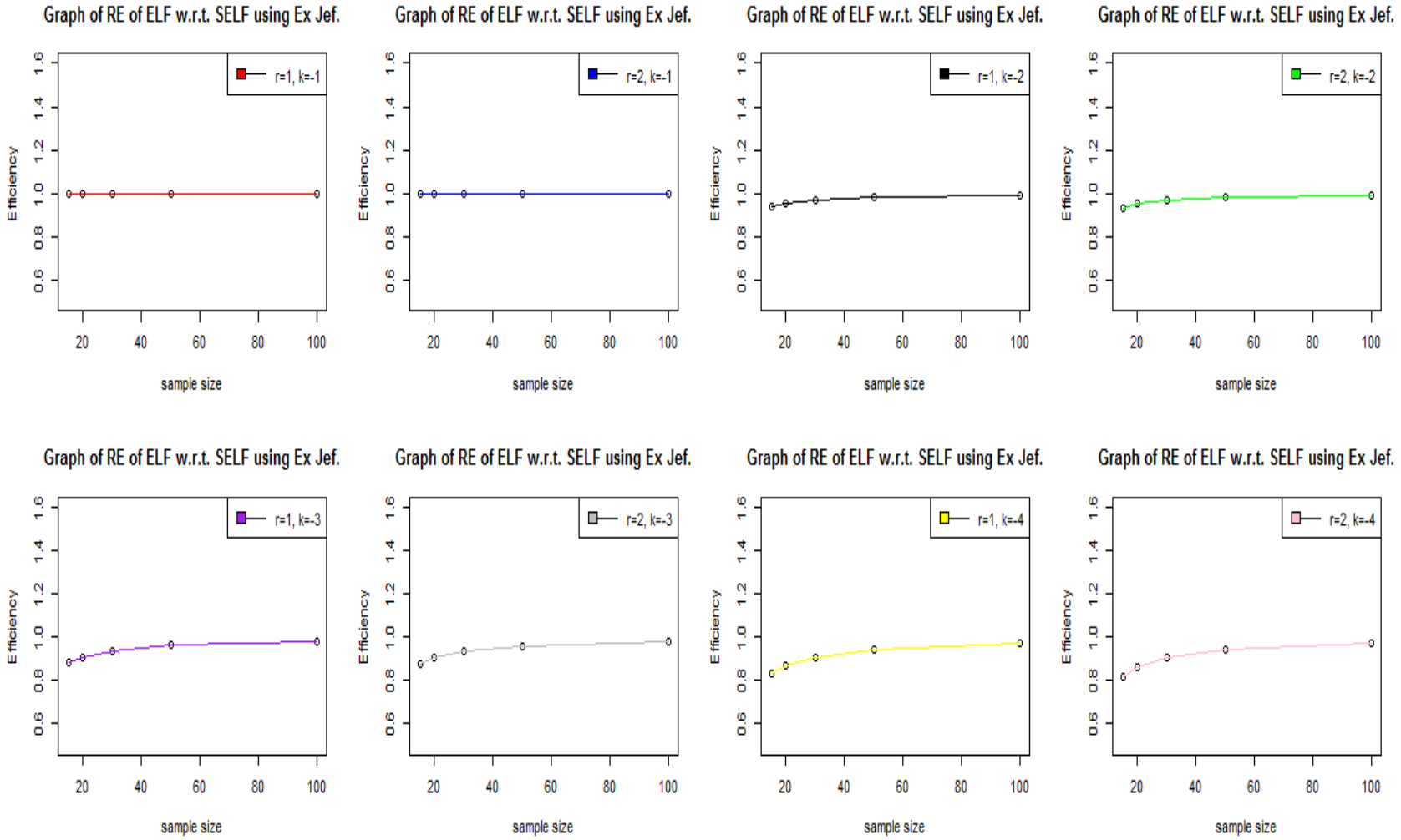


Figure A. 2: Graph of relative efficiency of entropy loss with respect to square error loss functions under extended Jeffrey's prior for different values of k=-1, -2, -3, -4 and r=1, 2 when n=15, 20, 30, 50 and 100

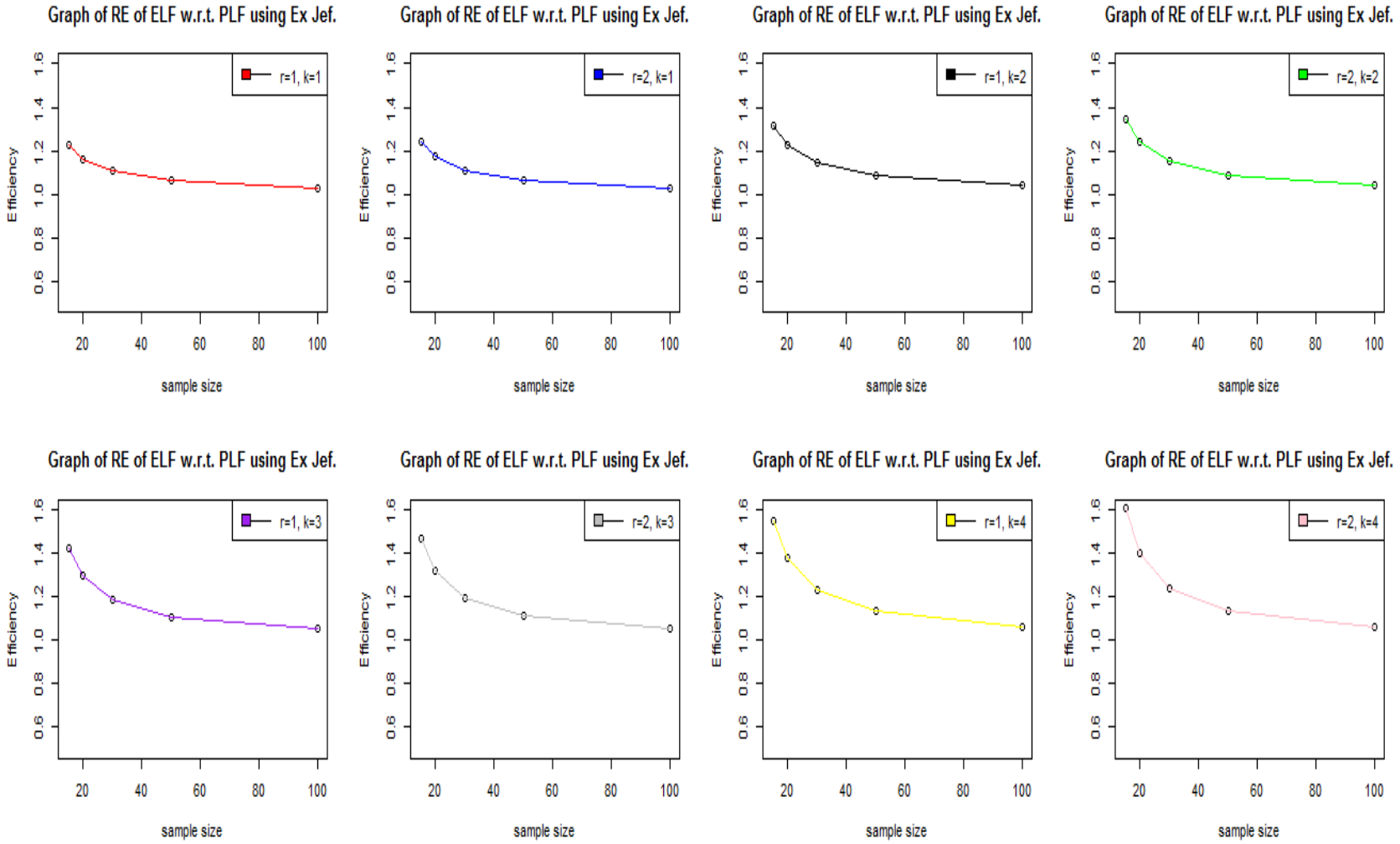


Figure A. 3: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under extended Jeffrey's prior for different values of k=1, 2, 3, 4 and r=1, 2 when n=15, 20, 30, 50 and 100

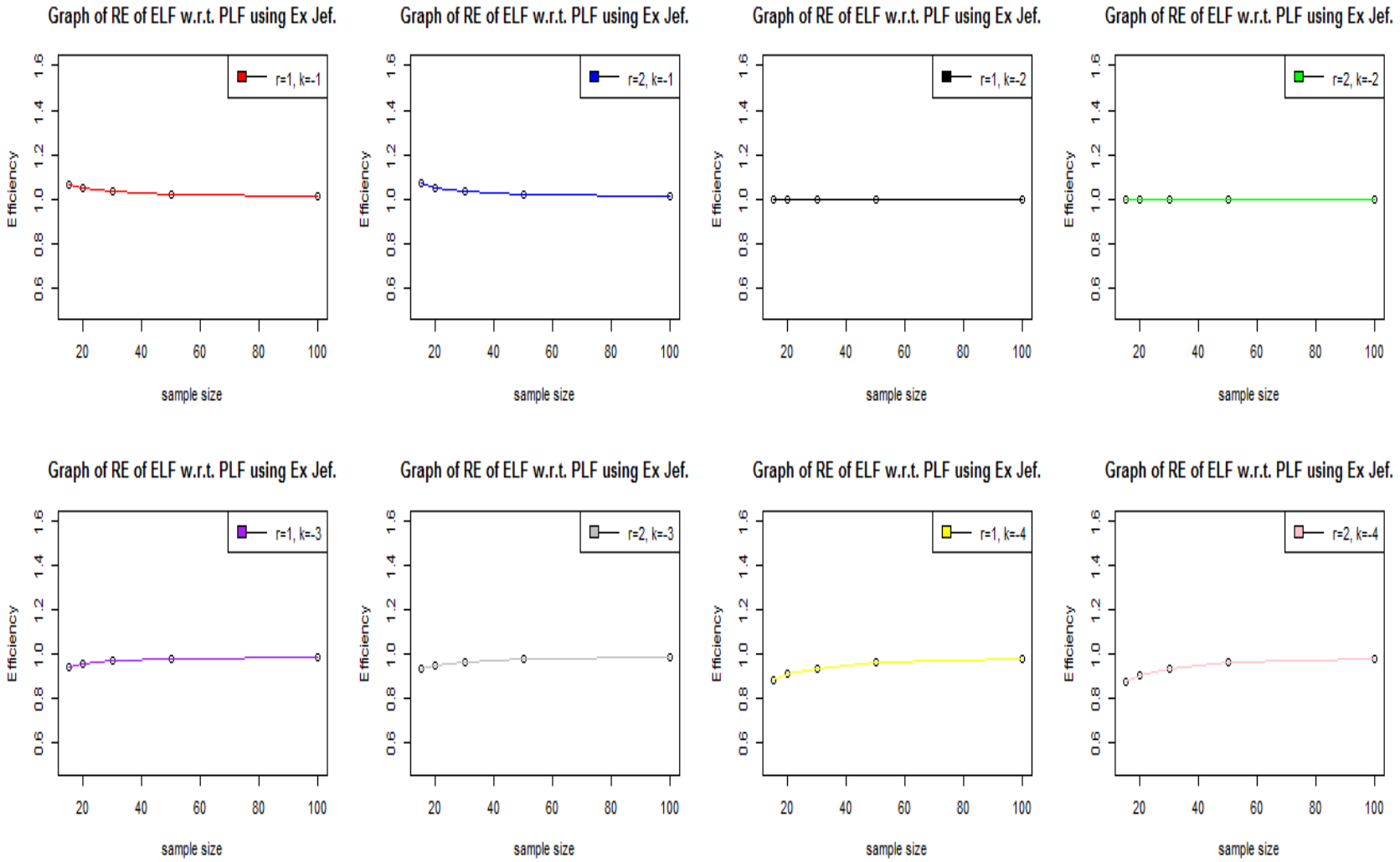


Figure A. 4: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under extended Jeffrey's prior for different values of k=-1, -2, -3,- 4 and r=1, 2 when n=15, 20, 30, 50 and 100

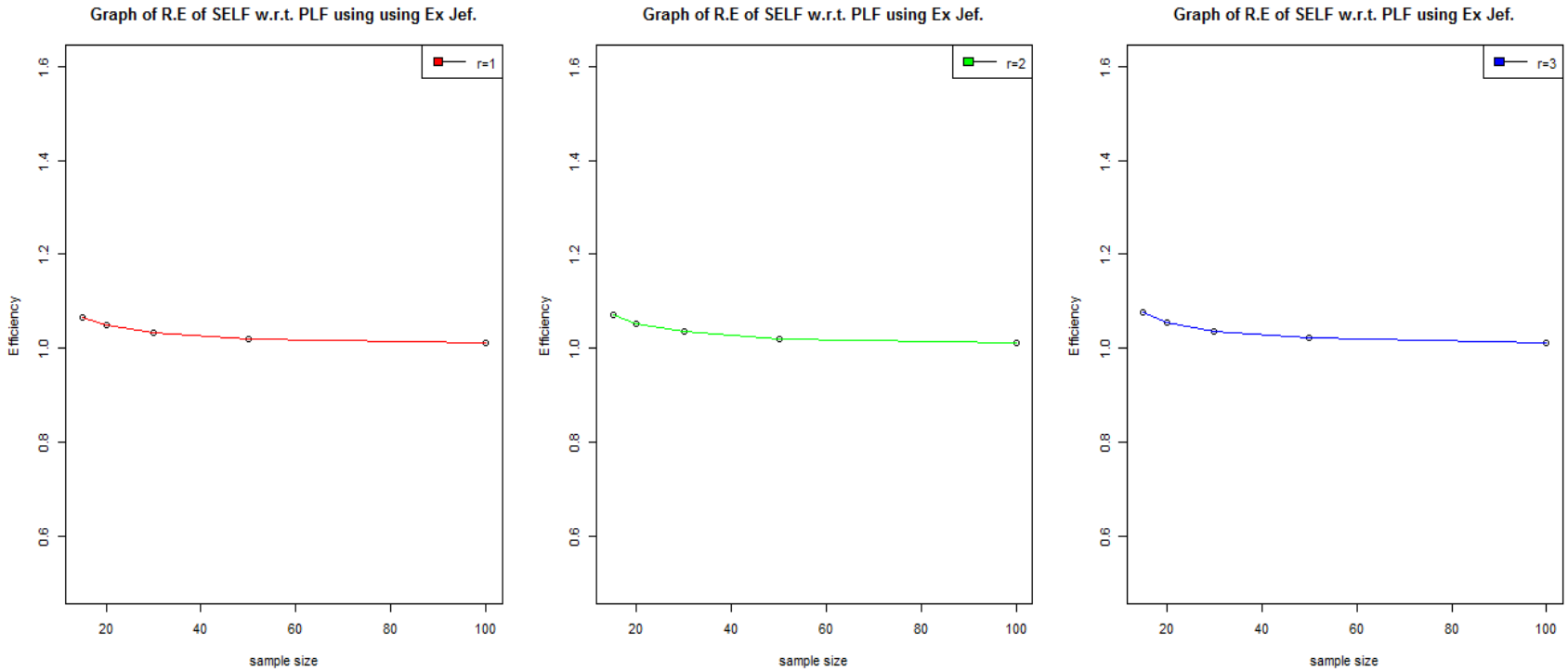


Figure A. 5: Graph of relative efficiency of squared error with respect to precautionary error loss functions under extended Jeffrey's prior for different values of r=1, 2 and 3 when n=15, 20, 30, 50 and 100

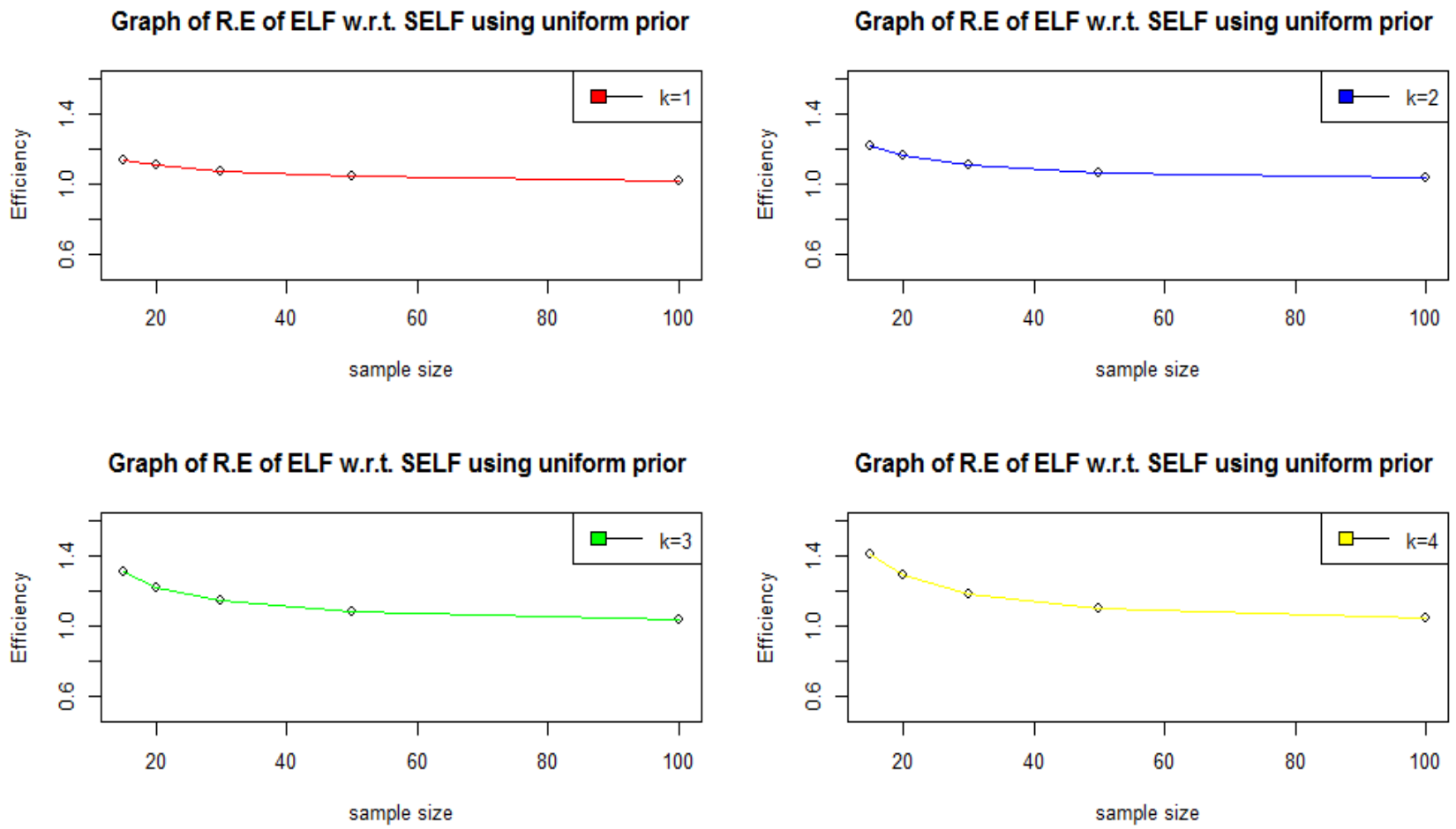


Figure A. 6: Graph of relative efficiency of entropy loss with respect to squared error loss functions under uniform prior for different values of k=1, 2, 3, 4 when n=15, 20, 30, 50 and 100

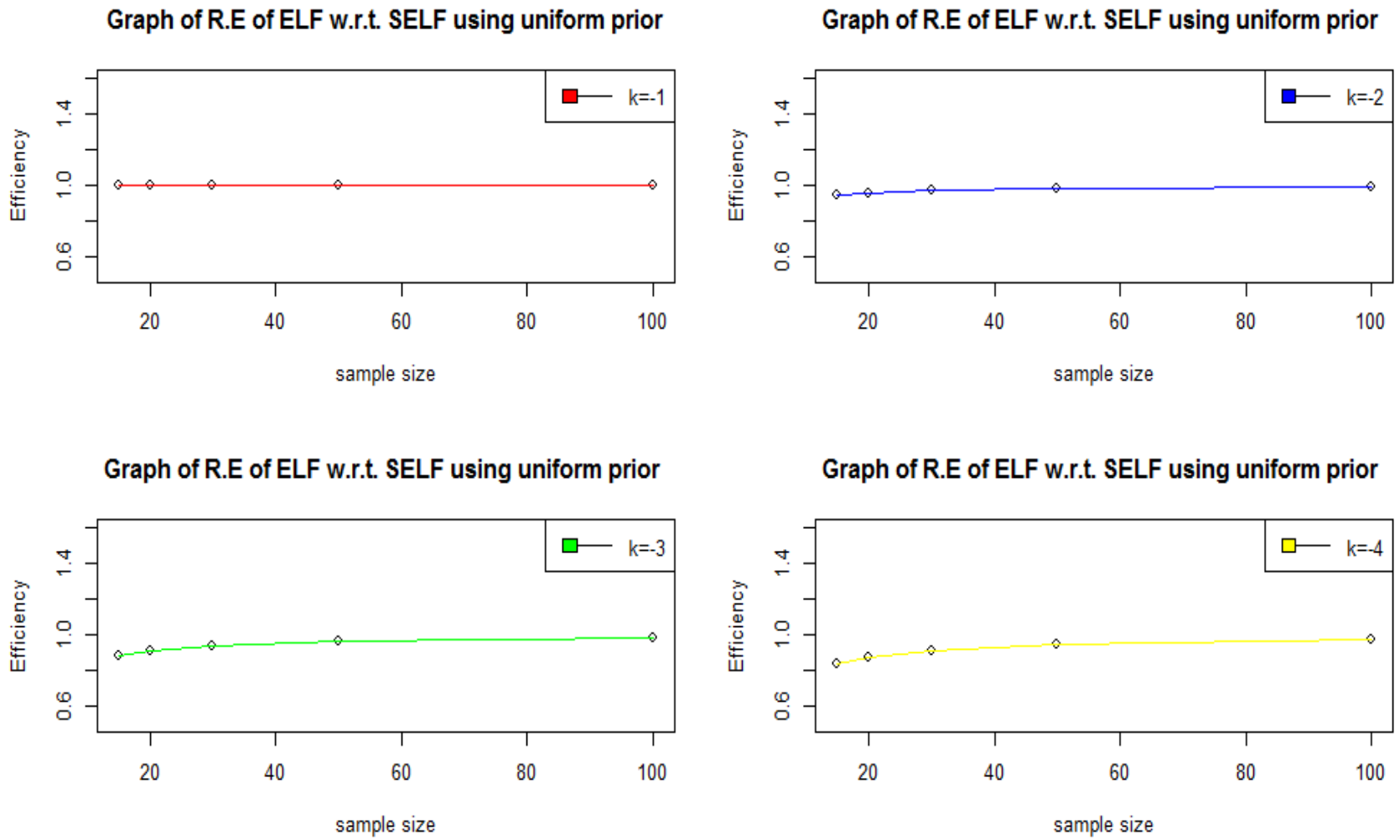


Figure A. 7: Graph of relative efficiency of entropy loss with respect to squared error loss functions under uniform prior for different values of k=-1, -2, -3, -4 when n=15, 20, 30, 50 and 100

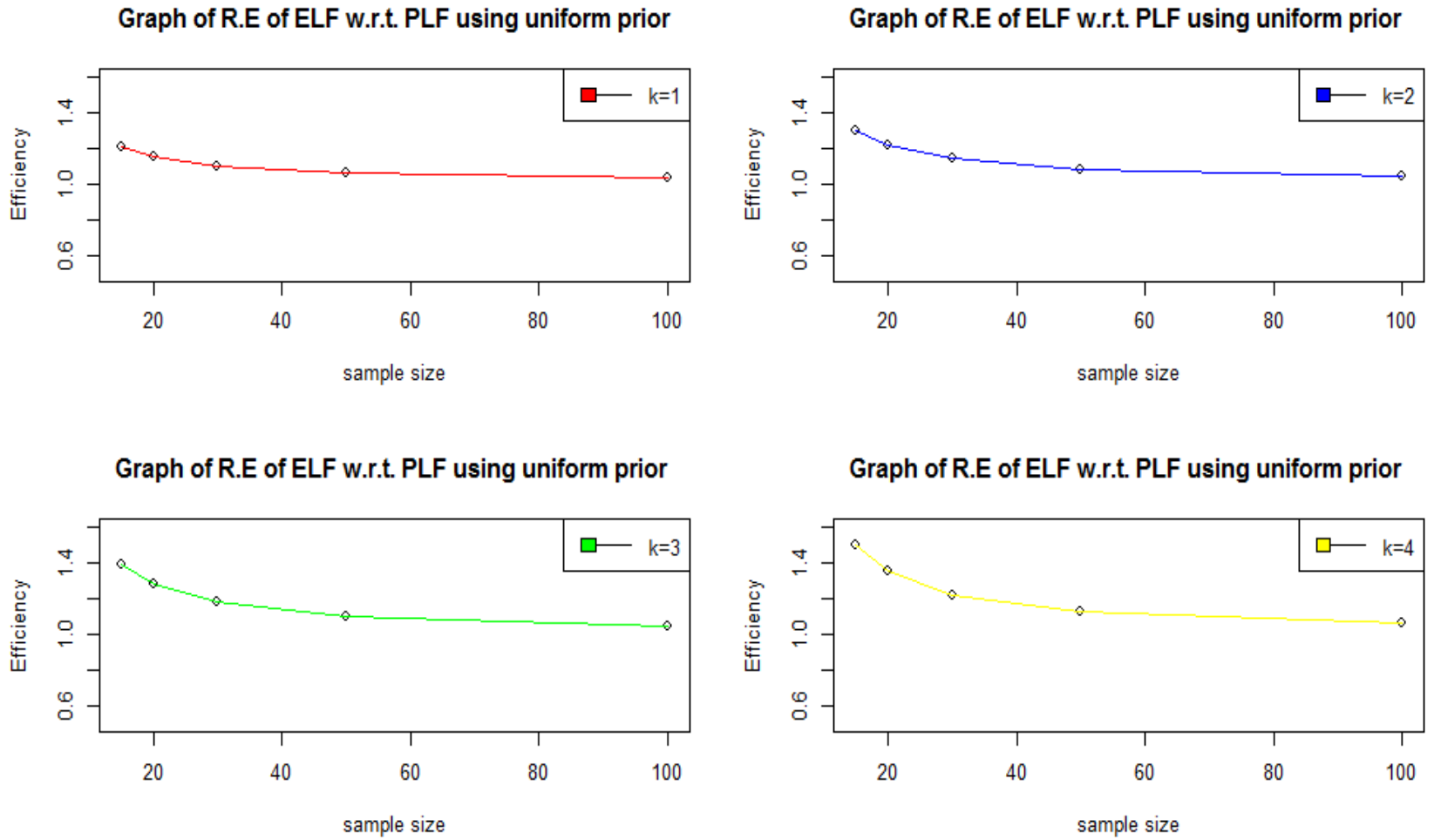


Figure A. 8: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under uniform prior for different values of k=1, 2, 3, 4 when n=15, 20, 30, 50 and 100

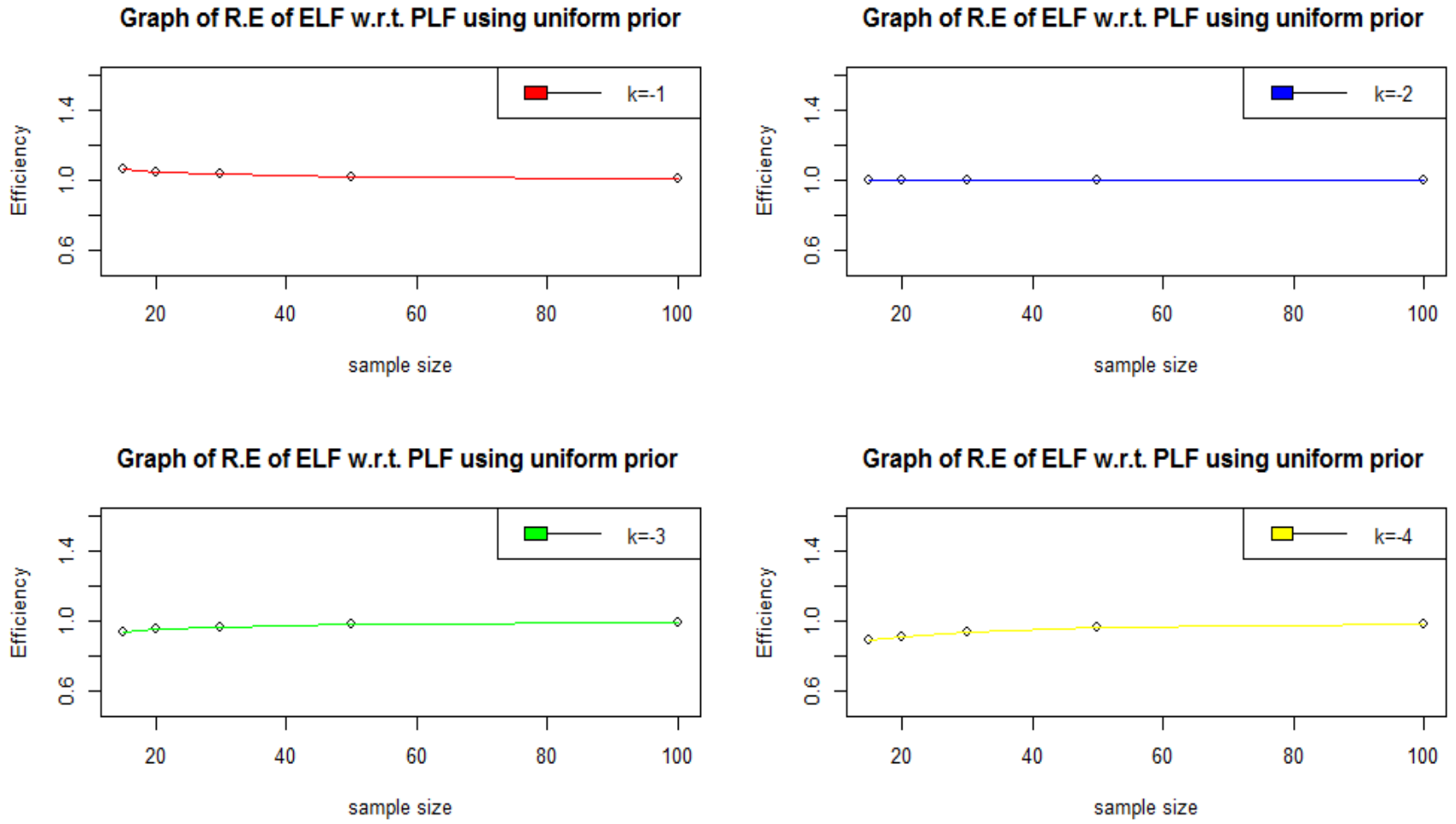


Figure A. 9: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under uniform prior for different values of k=-1, -2, -3, -4 when n=15, 20, 30, 50 and 100

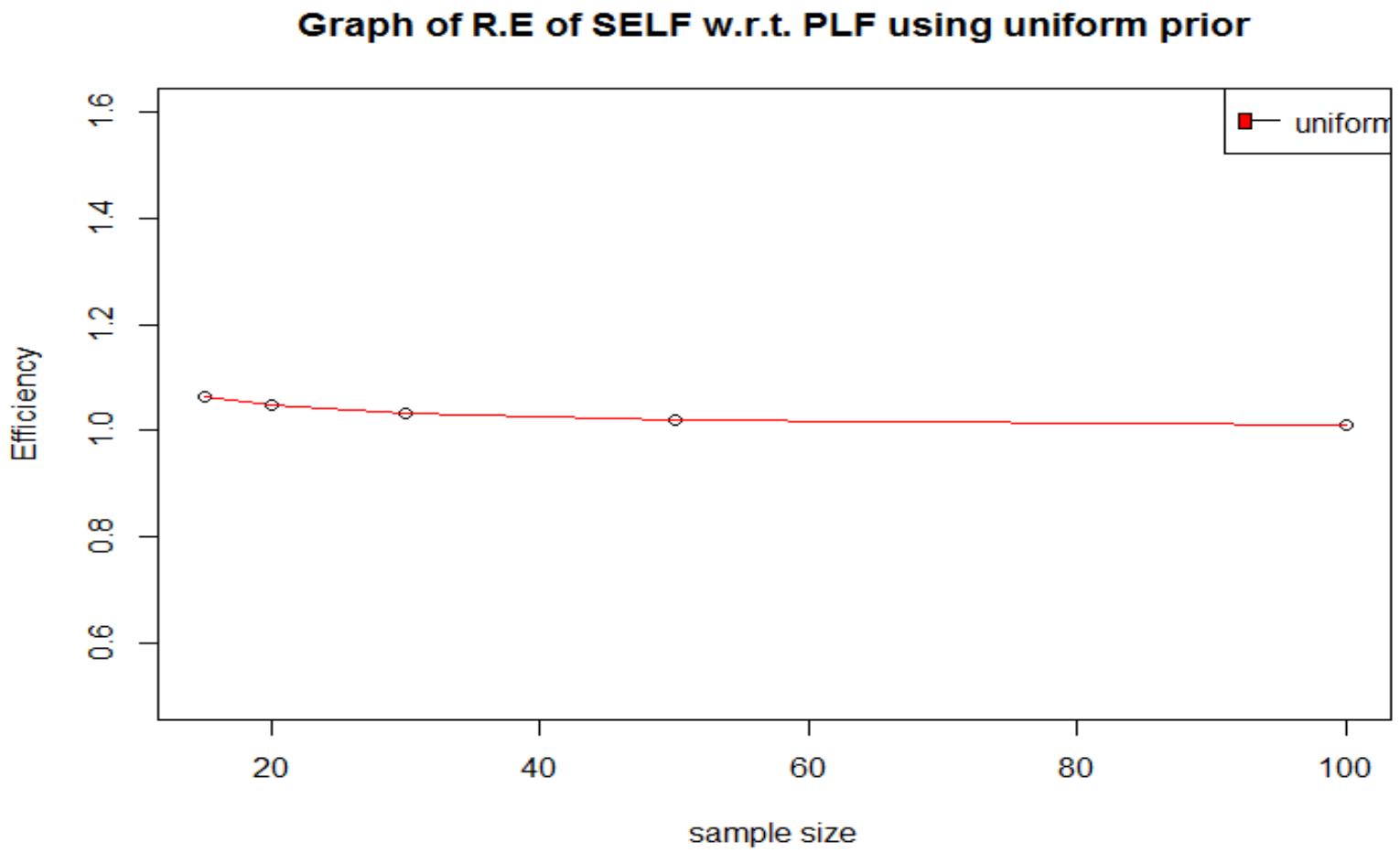


Figure A. 10: Graph of relative efficiency of squared error loss with respect to precautionary error loss functions under uniform prior for different values n=15, 20, 30, 50 and 100

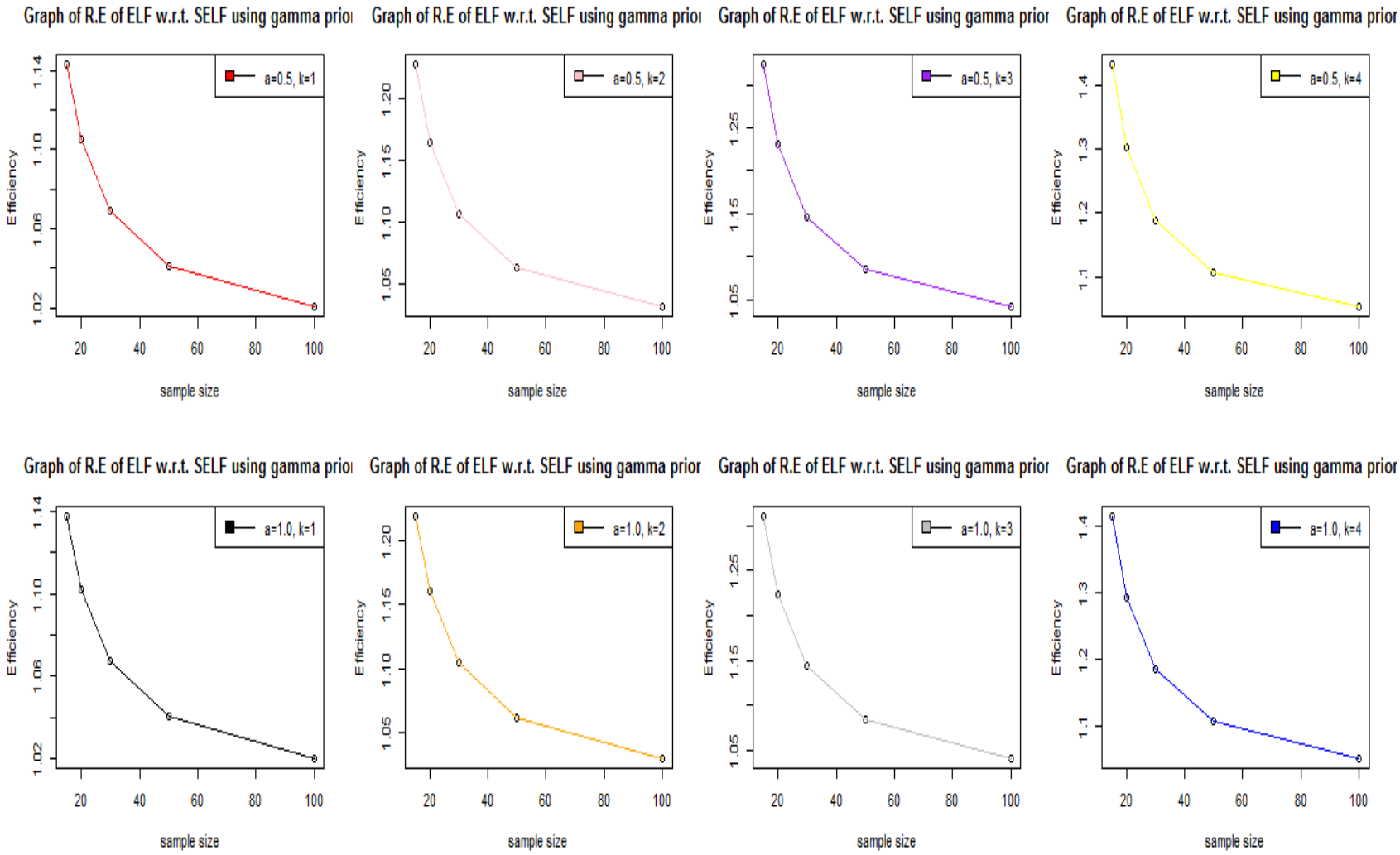


Figure A. 11: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under gamma prior for different values of k=1, 2, 3, 4 and a=0.5, 1.0 when n=15, 20, 30, 50 and 100

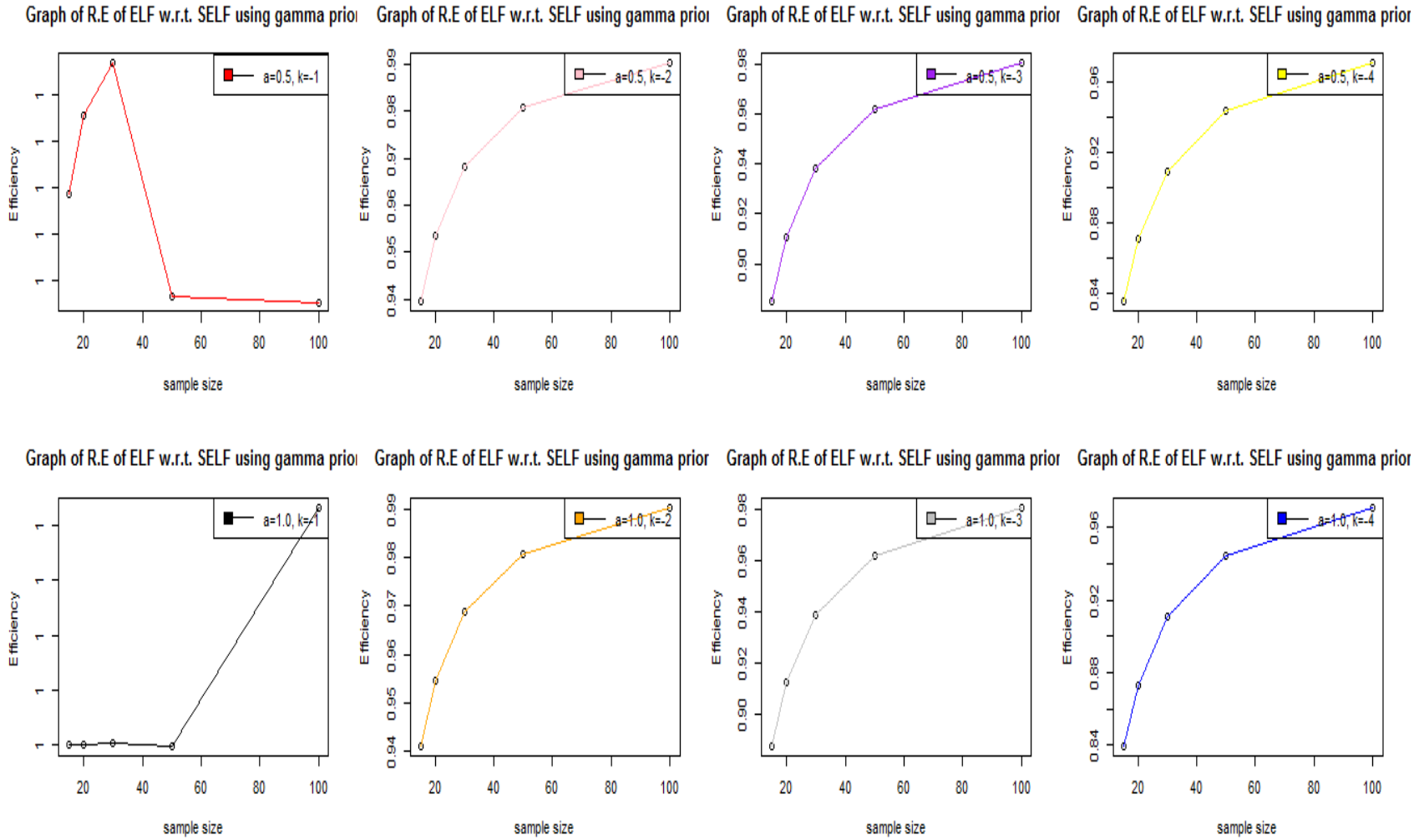


Figure A. 12: Graph of relative efficiency of entropy loss with respect to squared error loss functions under gamma prior for different values of k=-1, -2, -3, -4 and a=0.5, 1.0 when n=15, 20, 30, 50 and 100

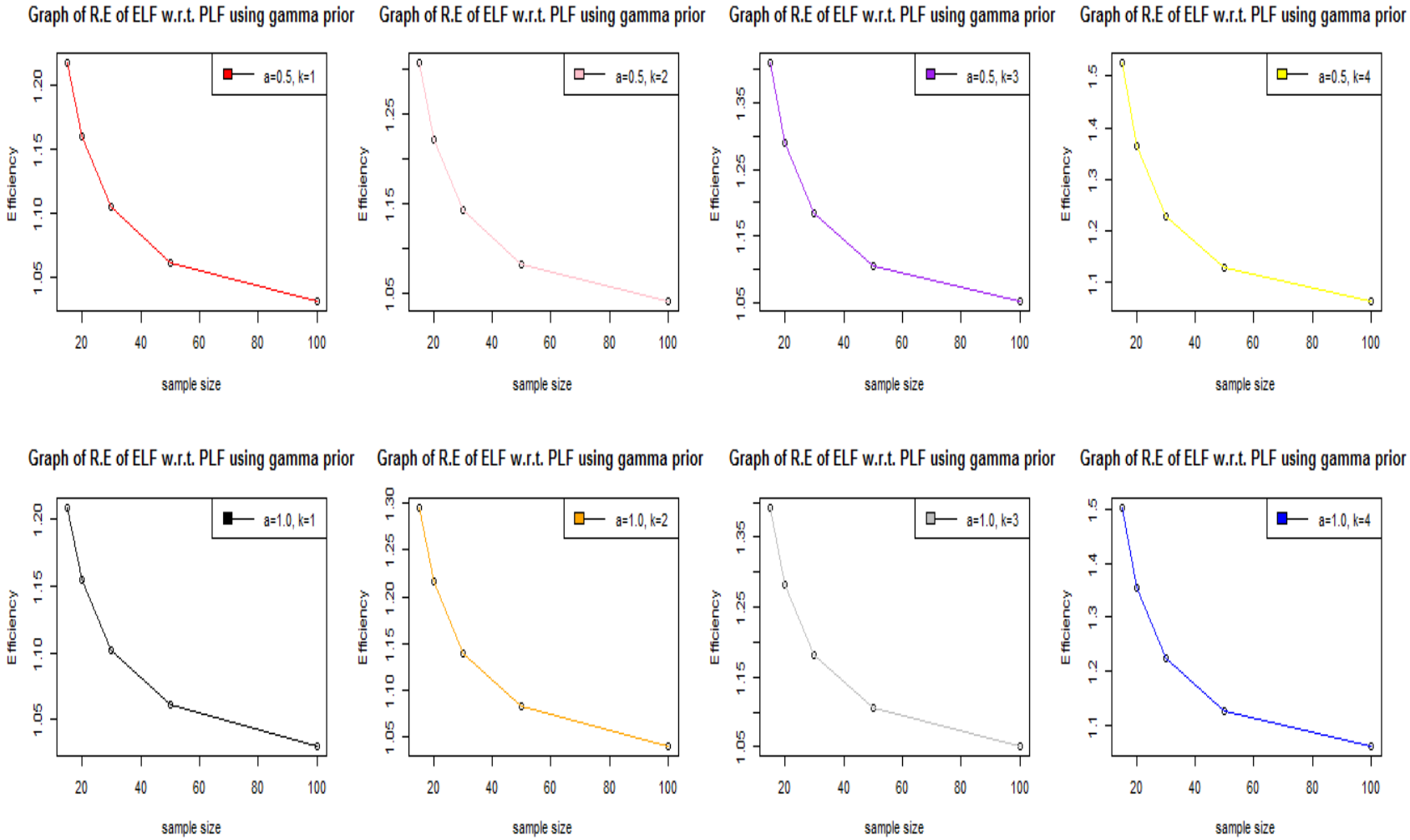


Figure A. 13: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under gamma prior for different values of k=1, 2, 3, 4 and a=0.5, 1.0 when n=15, 20, 30, 50 and 100

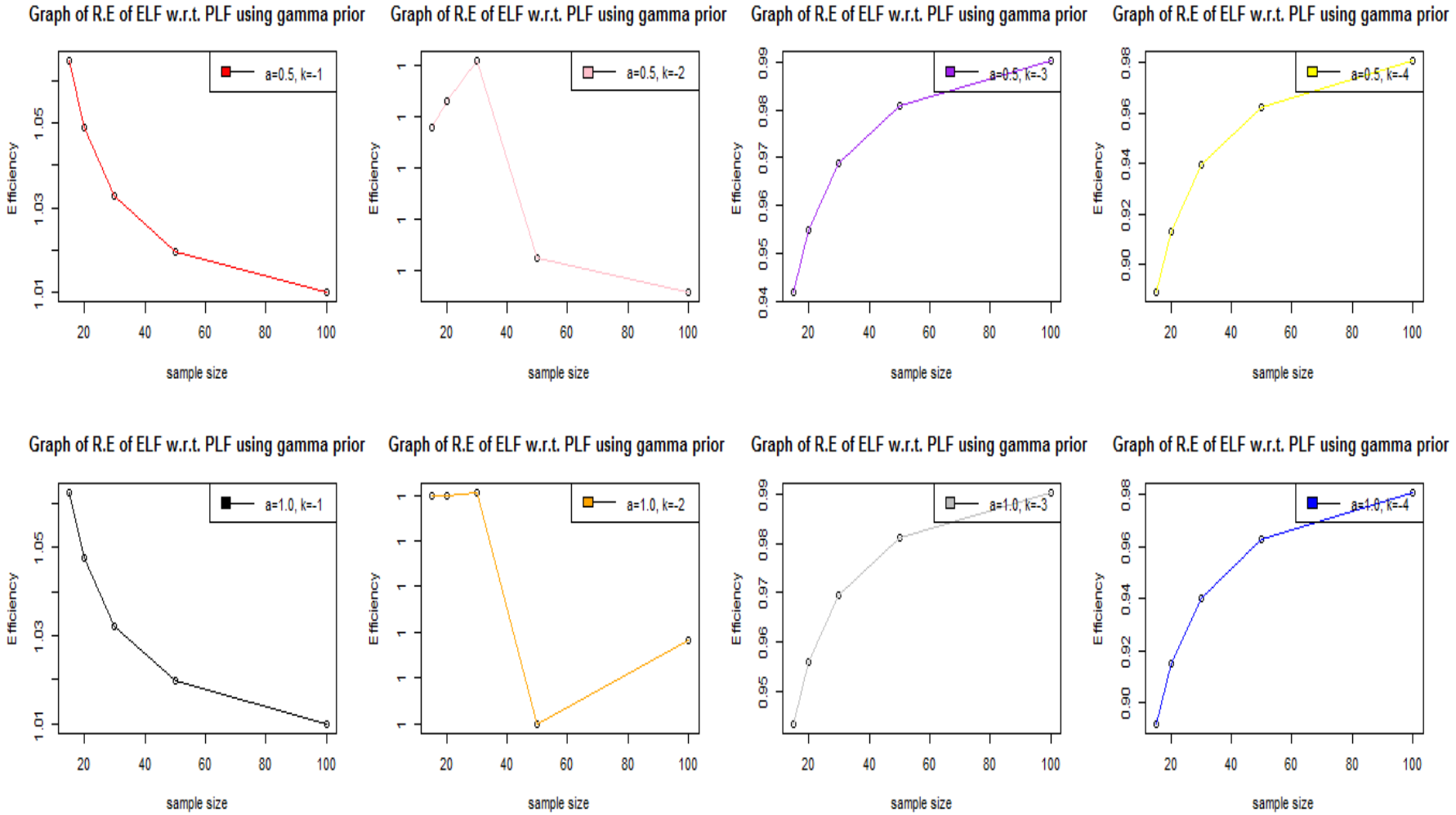


Figure A. 14: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under gamma prior for different values of k=-1, -2, -3, -4 and a=0.5, 1.0 when n=15, 20, 30, 50 and 100

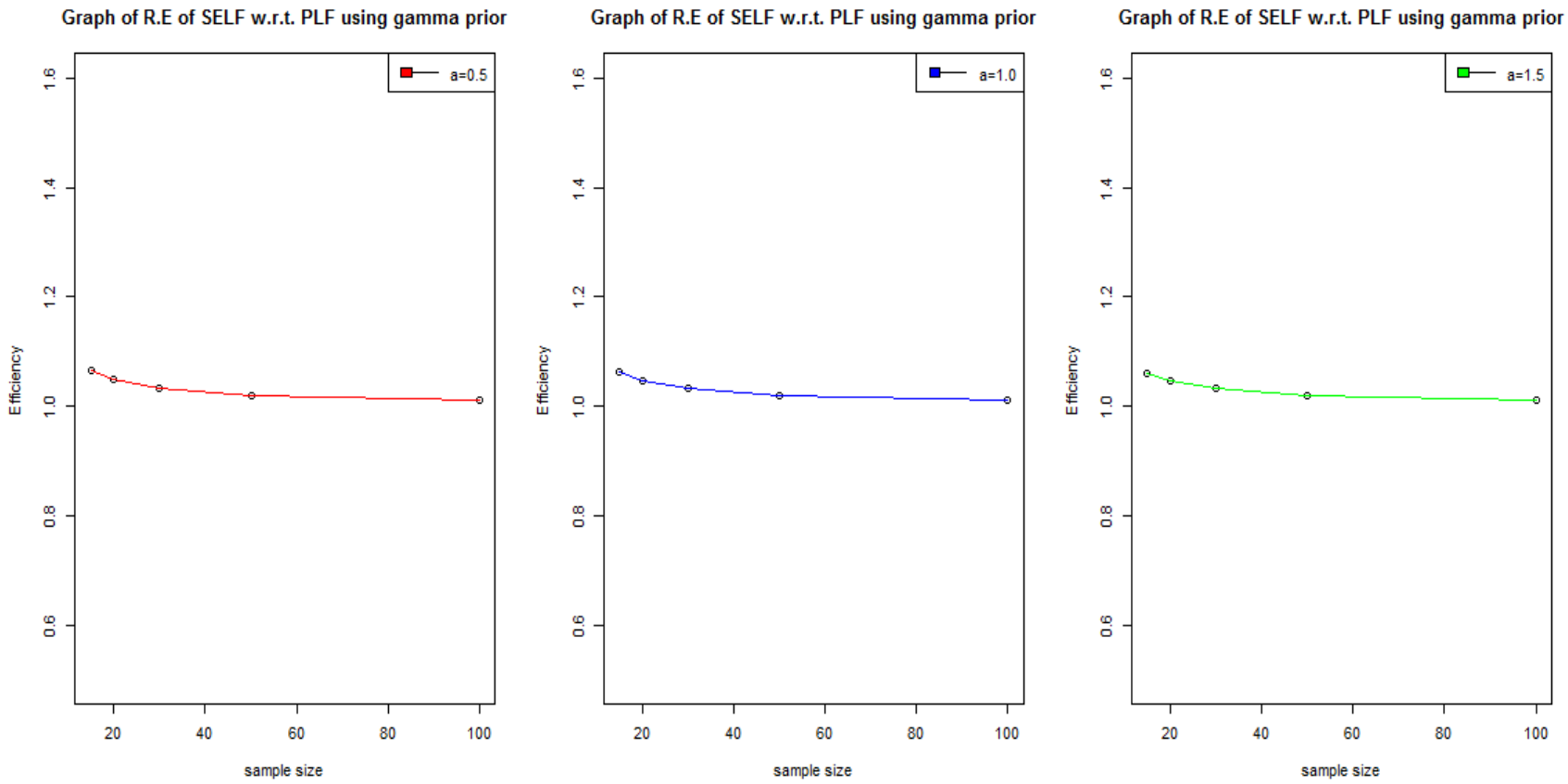


Figure A. 15: Graph of relative efficiency of entropy loss with respect to precautionary error loss functions under gamma prior for different values of a=0.5, 1.0, and 1.5 when n=15, 20, 30, 50 and 100