DETERMINISTIC INVENTORY MODELS FOR DELAYED DETERIORATING ITEMS WITH INVENTORY LEVEL DEPENDENT DEMAND RATE

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DECLARATION

I declare that the work in this thesis titled, 'Deterministic inventory models for delayed deteriorating items with inventory level dependent demand rate' has been performed by me in the Department of Mathematics under the supervision of Professor B. Sani, Professor A. A. Tijjani and Professor B. K. Jha.

The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other institution.

_____	_____
Yakubu Mamman Baraya	Date

CERTIFICATION

This thesis titled **DETERMINISTIC INVENTORY MODELS FOR DELAYED DETERIORATING ITEMS WITH INVENTORY LEVEL DEPENDENT DEMAND RATE** by Yakubu Mamman BARAYA meets the regulations governing the award of the degree of Doctor of Philosophy (Ph.D) of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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This thesis is dedicated to

my late father, Mal. Mamman Baraya

and

my late mother, Hajia Zainabu Isa.

ABSTRACT

In many real-life situations the consumption rate for certain types of consumer goods (such as fruits, vegetables, doughnuts, cassava, yams and so on) is sometimes influenced by the stock-level. In this research, five deterministic inventory models for delayed deteriorating items with inventory level dependent demand rate have been developed. The models are (1) an economic order quantity (EOQ) model with demand rate as a polynomial function of instantaneous inventory level and constant rate of deterioration, (2) an EOQ model with demand rate as a linear function of instantaneous inventory level and shortages, (3) an economic production quantity (EPQ) model with demand rate as a linear function of instantaneous inventory level with linear-time dependent holding cost, (4) an EPQ model with demand rate as a linear function of instantaneous inventory level and time-dependent deterioration rate, and (5) an EOQ model with demand rate as a polynomial function of instantaneous inventory level and linear time-dependent holding cost. In all the models, theoretical control model results for the optimal replenishment policy of the inventory systems have been determined in order to minimize the inventory system cost or maximize profit per unit time. Analytic formulation and analysis of the inventory problems were developed on the framework of the models assumptions. The proposed inventory systems were represented by systems of differential equations including initial and boundary conditions and typical differential and integral calculus were used to analyse the inventory problems. Newton-Raphson method has been used to determine the optimal solutions (optimal order or production quantity and the optimal cycle length) of the developed cost minimization models for the first four models and the profit maximization model for the fifth model. Some numerical examples to illustrate the applications of the models developed are also

presented. The numerical example presented to demonstrate the fifth model was accompanied by sensitivity analysis. It was shown from the results of the models developed that the impact of stock dependent demand rate, holding cost parameter, deterioration and backlogging rate affect the optimal replenishment policy, and hence should not be left out in developing inventory models for delayed deteriorating items with inventory level dependent demand rate. Finally, the effects of errors in the estimation of various model parameters or even changes in the values of model parameters may occur due to uncertainties surrounding the future of the optimal solutions. In order to examine the implications of these errors or changes, sensitivity analysis was studied based on the given example for the fifth model.

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NOTATION

Q	The Economic Order Quantity which enters into inventory at time t = 0
S_d	The inventory level up to which market demand rate is stock-dependent and depletion is based only on demand
T	The length of optimal ordering/production cycle time
t_d	The length of time for which the product has no deterioration and after this period of time, the inventory starts to depletes due to both deterioration and demand, where t_d is a given constant
θ	This is the rate of deterioration, constant fraction $\theta\ (0 < \theta < 1)$, of the on-hand inventory which deteriorates
$v_1(t)$	The on-hand inventory level at time t for which the product has no deterioration
$v_2(t)$	The on-hand inventory level at time t for which the product has deterioration
C_0	The ordering cost per order for each cycle
c	The cost price per unit item
t_1	The length of time after which the shortage time begins
$B(t)$	The level of backorder at time t for which the product has shortage
μ_1	The unit inventory holding cost per unit time of item, Naira/unit/unit time
μ_2	The unit deterioration cost per unit time of item, Naira/unit/unit time
μ_3	The unit shortage cost for backlogged items per unit time of item, Naira/unit/unit time

μ_4	The unit opportunity cost due to lost sales per unit time of item, Naira/unit/unit time
r	The unit selling price of the item
λ	The constant production rate per unit time
C_s	The set up cost per period
T_1	The production run period
T_2	The time during which there is no production of the product and significant deterioration immediately sets in
$q_1(t)$	The inventory level for the product during the production period
$q_2(t)$	The inventory level for the product during the period when there is no production
V_m	The maximum inventory level
k	The Production unit cost per cycle
P	The production lot-size
TVC	The total inventory cost per unit time

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

This section and the following sections in this chapter will introduce the more commonly used basic inventory terms and concepts (such as definition of inventory, terms connected with inventory systems, some of the features of inventory system and so on) which authors use to develop inventory policy.

Decision-makers are in many instances in charge of assigning widely varied type of resources such as manpower, material, or money. In a relatively uncomplicated form of decision-making problem, the decision-maker can imagine the *best possible course of action* leading to most effective use of the resources. If, however, the problem is complex in nature, the decision-maker cannot estimate the future as clearly as he/she intends by using intuition, common sense or experience. For this reason there is the need to find a procedure which can be effectively used by a decision-maker to identify the *various courses of action available to him/her* and then *recommending* the best course of action. A common analytical technique can be used to find the solution of decision-making problems belonging to the same general category. These are commonly referred to as *Operations Research Techniques*. One of the most prominent Operations research techniques is inventory control. Inventory control (management) has to do with determining *when and how much* of needed resource should be kept as stock which minimizes the total cost or maximizes the profit over a given period of time.

The word *inventory* refers to any kind of stock of resource, such as unused labour (manpower), materials and money having economic value and is being held for future use or sale by an organization. The resources may be *physical resources* such as raw materials, semi-finished goods, finished goods, spare parts, lubricants, and so on, or *human resources* such as unused labour (manpower) and *financial* resources such as working capital.

Maintaining inventory is considered a necessary investment in order to achieve a smooth system of production, distribution and marketing of physical goods. Some of the important reasons for carrying inventory are to: improve customer service, reduce costs, satisfy irregular supply and demand, provide quantity discounts, and avoid stock-outs (shortages).

Inventory management is a common problem in businesses. In a small business, the manager may control his/her inventory and take these decisions by experience, common sense or intuition. However, for large businesses, scientific inventory management is required by way of formulating a mathematical model describing the characteristics of the inventory system and consequently deriving an inventory control model (cost minimization model or profit maximization model equation) for optimal inventory replenishment policy for the model.

In particular, for each order of items, a businessman must incur cost to cover handling and transportation. When large orders (batches) are made infrequently, it leads to low ordering (set-up) cost with high carrying cost. On the other hand, when small orders (batches) are made frequently, it leads to high ordering (set-up) cost with low carrying

cost. There are always conflicting costs to be incurred with having too little or too much inventory on-hand and so there is need to control the inventory by seeking an inventory level that minimizes the sum of these costs, and other costs such as shortage costs, deterioration costs and so on. The costs are conflicting, in the sense that an increase in either ordering cost or holding cost may result in the reduction of the other. The problem reduces to controlling the inventory level by developing an *inventory policy* which answers two questions of *when* and *how much* to order (replenish inventory) so as to minimize the sum of all costs per unit time. The *quantity* to order, so that the sum of ordering cost, carrying cost and other costs such as shortage costs, deterioration cost and so on, are minimized is called Economic Order Quantity (EOQ).

1.2 Classical Square Root EOQ Model

The concept of EOQ was first developed by Harris (1913) for finding the optimum order quantity in order to balance the holding cost and ordering cost. The model developed by the author to determine this order quantity is often called classic square root EOQ model and it is given as follows.

$$\text{EOQ} = \sqrt{\frac{2\left(\text{Demand rate}\right)\left(\text{ Ordering cost per Order}\right)}{\text{Cost of holding a unit for a unit time}}}$$

This is the optimal (economic) ordering quantity which minimizes the total inventory cost comprising of holding cost and ordering cost.

An application of the EOQ concept to production is the Economic Lot Size or Economic Production Quantity (EPQ). In this model, an inventory is produced or manufactured instead of being ordered or purchased in the company itself and subsequently used or

sold. An EPQ model is an inventory control model that determines the amount of product to be produced to minimize the total inventory cost so as to meet a deterministic demand.

1.3 Categorization of Inventory Models

The amount of stock to be maintained will naturally depend upon the consumption or usage of the commodity. It is, therefore, necessary to make a study of the demand pattern of the item under consideration. Mathematical inventory models can be divided into two broad categories; *deterministic models* and *stochastic models.* If the demand in future periods can be predicted with some degree of accuracy, then, this is the case of *known demand* where a deterministic inventory model would be used. This is usually expressed over equal periods (commonly annual) and referred to as demand rate. The demand may vary or remains fixed. However, when demand cannot be predicted well, then this is the case where the demand in any period is a random variable rather than a known constant, and its pattern can be described by a known probability distribution where a stochastic inventory model would be used.

The other categorization refers to whether the current inventory level is being monitored *continuously* or *periodically.* An inventory system may review its inventory continuously or periodically. In *continuous review* an order is placed as soon as the stock level falls below the prescribed reorder point, whereas in the *periodic review* case the inventory is replenished periodically (e.g., ordering at the start of every week or every month). This is usually expressed over equal periods (commonly annual) and referred to as demand rate, the rate at which the product is taken out from inventory in (units / time period). This categorization of demand pattern assumes the availability of reliable data to estimate demand in the future.

One of the ways of analyzing the nature of demand of an item is by using some statistical tools such as evaluating the mean and standard deviation of demand of the item for a given period of time (e.g., monthly).

Letting coefficient of variation as $V = \left(\frac{\text{Standard deviation}}{\text{Mean}}\right)(100\%)$, Taha (2011) suggested that the nature of demand of an item can be estimated using the following guidelines.

1. If the average monthly demand (taken over a number of years) is "approximately" constant and V is reasonably small (< 20%), then the demand may be considered deterministic and constant.

2. If the average monthly demand varies appreciably among the different months but V remains reasonably small for all months, then the demand may be considered deterministic but variable.

3. If in case 1, V is high (> 20%) but approximately constant, then the demand is probabilistic and stationary.

4. When the averages and coefficients of variation vary appreciably month to month, the case is probabilistic and dynamic.

1.4 Components of Inventory Models

The following are some of the basic considerations involved in determining an inventory policy (when and how much to replenish inventory) which are connected with mathematical inventory modeling.

1.4.1 Relevant Inventory costs

It is generally acknowledged that costs have effects on profit of any business organization. There are a number of costs that are associated with inventory policy and some of the basic costs considered in an inventory system are (1) replenishment (production/purchase) cost, (2) holding (carrying) cost, and (3) shortage cost. Other relevant factors include (4) revenue, (5) salvage cost, and (6) discount rate. These six factors are described in turn below.

(i) Ordering or set up cost (in production environment)

This is the cost of placing an order to an outside supplier or releasing a production order to a manufacturing business. An ordering cost is the cost incurred whenever an order is placed. It is independent of the quantity being placed. It is primarily clerical and administrative in nature. Typical elements of this cost include the cost associated with processing, labour, overhead (telephone, postage, e-mails and so on.), and transport (delivery charges). In other environments, the setup cost could be clerical and administrative cost. The cost of purchasing or producing one unit of a material or product is called acquisition cost.

(ii) Product or purchasing cost

This is the actual price paid for the procurement of the item. It is the variable associated with purchasing a single unit. Typically, the unit purchasing cost includes the cost of raw materials associated with purchasing or producing a single unit. This cost may be a constant for all quantities, or it may vary with quantity purchased or produced.

(iii) Holding or carrying cost

This cost is the cost of holding an item in inventory for some given unit of time. It generally includes the cost of capital tied up in the inventory, salary of warehouse employees, insurance cost, pilferage cost and many other costs which are proportional to the amount kept in inventory. It is given in say (₦ / unit time).

(iv) Shortage or stockout cost

When a customer seeks the product and finds the inventory empty, the demand can either go unfulfilled or be satisfied later when the product becomes available. The former case is called a lost sale, and the latter is called a backorder in (₦ /unit/unit time). The shortage cost is the cost incurred when the amount of the inventory demanded exceeds the available stock. This cost depends on whether backordering is allowed or not. For instance, if a customer demands a product and the demand is not met on time, a *stockout*, or *shortage*, is said to occur. If customers will accept delivery at later days, no matter how long it takes them waiting, that demand is said to be backordered. The case in which back-ordering is allowed is often referred to as the *backlogged* demand. If no customer will allow late delivery, then that is a lost-sales case. The process of holding customer orders to be filled later when they cannot be settled immediately because of stockouts is called *backlogging*. The total backorder cost is assumed to be proportional to the number of units backordered and the time the customer must wait.

(v) Backorder cost

This is the cost of handling the backorder (special handling, follow-ups etc) plus whatever loss of goodwill occurs as a result of having to backorder the item.

(vi) Salvage cost

The salvage value of an item is the value of a remaining item beyond which no inventory is needed.

(vii) Discount rate

This is the interest rate one needs to earn on a given amount of money today with a given amount of money in the future. The discount rate considers with other factors such as the time value of money before reaching a decision. When a business ties up its capital in inventory, the business is prevented from using this money elsewhere to earn some kind of return.

1.4.2 Demand for inventory items

The nature of demand (i.e. its rate, size and pattern) for inventory items is one of the most basic features to determine an optimal inventory policy.

(i) Size of demand

This is the number of items required in each period. The size of demand may be either deterministic or probabilistic. This can be fixed or can vary from period to period

(ii) Demand rate

This is the amount of stock that is required for sale, production or other uses. It is expressed as a weekly, monthly, yearly, and so on, rate of demand. The demand may be constant (static) or variable (dynamic).

(iii) Demand pattern

This is the manner in which inventory items are obtained. The demand for a given period of time may be satisfied instantaneously at the beginning of the period, or uniformly during that period.

1.4.3 Order cycle

This is made up of the activities of sensing a need for ordering materials, placing an order, lead time for getting the material delivered, receiving the material and using it. This process is continuously repeated for a material and is thus cyclical. The ordering cycle may be determined in either of the two ways: *continuous review* or *periodic review*.

1.4.4 Lead time

This is the time between ordering an inventory item and actual receipt of the item into stock. The lead-time can be either deterministic (constant or variable) or probabilistic. Constant lead time is when the lead time for each order is a known constant. If the lead time is zero, then we have a special case of instantaneous delivery where there is no need for placing an order in advance.

1.4.5 Inventory replenishment

This simply refers to the actual replenishment of stock at specified intervals which may occur instantaneously or uniformly. Instantaneous replenishment occurs when inventory is received or obtained at one point in time, while uniform replenishment inventory is received over a period of time.

1.4.6 Time horizon

The period over which the inventory level will be controlled is called the time horizon. It can be finite or infinite depending on the nature of demand.

1.4.7 Operating decision rules

There are two types of decisions need to be made in developing inventory models:

(i) How much (size) is to order (or produce) for each replenishment and

(ii) When (timing) it is necessary to place an order (or setup production) to replenish inventory.

1.5 Some Terminologies connected with Inventory Modelling

The following are the basic definitions of the inventory terms commonly used by authors of inventory modelling and used in the thesis.

(i) Inventory level

This refers to the amount of materials on hand in inventory that is ready for use. It is also called the stock level.

(ii) Inventory depletion

This is the removal of inventory items from the stock through demand, deterioration or other factors.

(iii) Buffer stock

This is also called the minimum or safety stock. It is acquired in order to prepare for uncertainties in demand or to correct errors in forecasting the lead time.

(iv) Order Quantity or Lot Size

This is the fixed quantity received in the inventory every inventory replenishment.

(v) Optimal production lot size

It is that particular quantity, which if produced in one production run, will minimize the total cost of setting up and carrying finished goods in inventory.

(vi) Product Cost

This is the unit cost of purchasing the product as part of an order.

(vii) Backlogging

The process of holding customer orders to be filled later when they cannot be settled immediately during stockout period.

(viii) Backorder

This is the item ordered but unable to be delivered on which the customer will wait for delivery without canceling the order.

(ix) Deterministic Inventory Model

This is the inventory model in which demand is assumed to be fixed and known.

1.6 Assumptions in Economic Order Quantity (EOQ) Models

The following are the assumptions made in developing EOQ model.

1. *Repetitive ordering:* This is the system by which orders are placed often at uniform intervals.
2. *Constant demand:* The exact amount of items required during any time period is known at constant rate.
3. *Uniform lead time:* The lead time for each order is known and constant.
4. *Continuous/periodic ordering:* This is a situation in which inventory system is reviewed continuously or periodically.

1.7 The Basic Economic Order Quantity Model

Consider an order of size Q has just arrived at initial time t = 0 and demand occurs at a rate of D per cycle, it will take $\frac{Q}{D}$ time units for inventory to reach zero again. Since demand during any period of length t is Dt, the inventory level over any time interval will decline along a straight line of slope (- D). At which time the inventory is depleted down to zero, an order of amount Q is received instantly, which raises the inventory level back to Q as illustrated pictorially and shown in Figure 1.1.

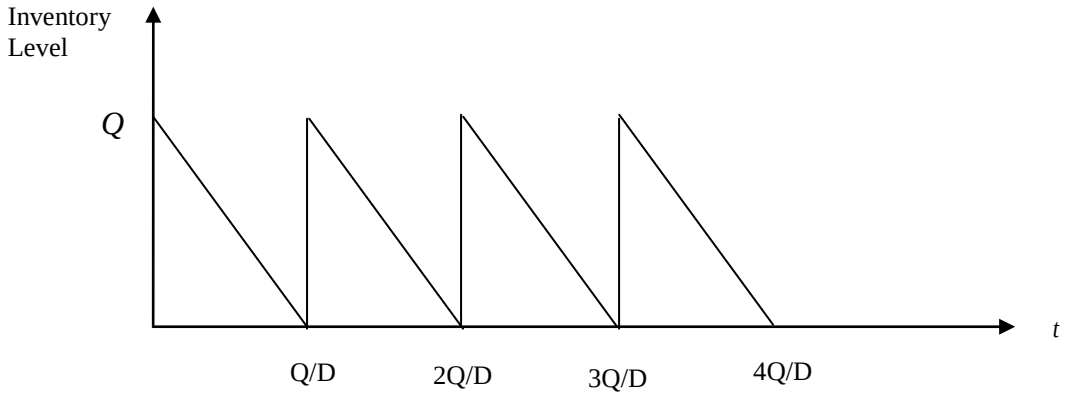


Figure 1.1: Behaviour of Inventory Level I(t) in Basic EOQ Model

1.7.1 Assumptions of basic EOQ model

The basic EOQ model makes the following assumptions.

1. Demand is constant. The exact number of items needed during any time period is known.

2. An ordering or setup cost *K* is fixed and the model does not allow quantity discount.

3. The lead time for each order is zero. Each order arrives in the inventory as soon as it is placed.

4. No shortages are allowed. That is all demands are met on time; a negative inventory is not allowed either.

5. The holding cost per unit per unit time held in inventory is a constant, say *h*.

1.7.2 Derivation of basic EOQ model

The total cost per cycle TC is obtained from the following cost components.

From the above assumption 3, for the reason that the lead time is zero, when an order is placed, it is instantly received in the inventory. If the amount of order is of fixed rate, then there is need for the same quantity *Q*, every time an order is placed.

Production or ordering cost including purchasing cost per cycle is given by

$$PC = K + cQ \tag{1.7.2.1}$$

where c is the unit purchase cost and *Q* is the order quantity that enters the inventory at the beginning of the cycle length.

The average inventory level during a cycle is $\frac{(Q+0)}{2} = \frac{Q}{2}$ units, where *Q* and 0 are regarded as initial and terminal inventory levels respectively. Therefore, the corresponding holding cost is $\frac{hQ}{2}$ per unit time. Since the cycle length is $\frac{Q}{D}$, then holding cost per cycle is given by

$$HC = h\left(\frac{Q}{2}\right)\frac{Q}{D} = \frac{hQ^2}{2D} \tag{1.7.2.2}$$

Then the total cost per cycle for ordering *Q* units is given by

$C_T(Q)$ = Ordering cost/cycle + Holding cost/cycle

$$= K + CQ + \frac{hQ^2}{2D} \tag{1.7.2.3}$$

which leads to the total cost per unit time to be

$$TC(Q) = \frac{C_T}{T} = \frac{K + cQ + \frac{hQ^2}{2D}}{\frac{Q}{D}} = \frac{KD}{Q} + cD + \frac{hQ}{2} \qquad (1.7.2.4)$$

where T is the cycle length.

The necessary condition for finding the value of Q, say Q*, that minimizes cost function TC (*Q*) from equation (1.7.2.4) is when the first derivative with respect to *Q* of this function is zero.

That is

$$\frac{d}{dQ}\left(TC(Q)\right) = -\frac{KD}{Q^2} + \frac{h}{2} = 0, \qquad (1.7.2.5)$$

Solving for *Q* gives the Q^* (the optimal order quantity), which is

$$Q^* = \pm\sqrt{\frac{2KD}{h}}$$

Since we have said from assumption 4 that no shortages are allowed then we do not desire a negative inventory. Hence we get rid of the negative value and take on the positive value as the value that minimizes the total cost function. We now take our EOQ to be,

$$Q^* = \sqrt{\frac{2KD}{h}}, \qquad (1.7.2.6)$$

The second derivative $\frac{d^2}{dQ^2}\left(TC(Q)\right) = \frac{2KD}{Q^3} > 0$ implying Q^* is indeed a minimizer, providing optimal solution to equation (1.7.2.5).

which is the well known basic square root EOQ model. The corresponding cycle time, say t^*, is

$$t^* = \frac{Q^*}{D} = \sqrt{\frac{2K}{hD}}. \qquad (1.7.2.7)$$

We see that Q* is independent of c. It is a function of only k, h and D. This implies that purchasing cost being constant is not included, as is the case in most inventory models, when determining the EOQ.

1.8 The Research Problem

There are several items which do not start deteriorating as soon as they arrive in stock, but they maintain quality or the original condition for some period (fresh product time), before they begin to deteriorate. Consumer goods such as fruits, vegetables, meat, bread, cassava, yams and so on, exhibit this property.

In many real-inventory situations, the demand for these items depends on the amount of inventory items displayed on shelves in a shop. It is usually observed that a large pile of goods displayed on shelves in a shop will influence the customers to buy more and more which will in turn cause sales to rise.

The holding cost and deterioration rate of these types of inventory items increase linearly with time. These inventory systems represent many real-life inventory situations and this is particularly true in the storage of deteriorating and perishable items such as food products. This is because, the longer these deteriorating items are kept in

storage, the higher the holding cost and deterioration cost. These are justified since the cost of holding an item in stock increases with each passing time which eventually will affect total system cost. Deteriorating items will also affect total system cost most especially if the rate of deterioration is significant.

It is assumed that shortages are allowed and the unsatisfied demand is partially backlogged at a fixed fraction of demand rate during the shortage period as in Wee (1995).

It is therefore very desirable to have models that address the problems above. In this thesis, we investigate the inventory policies of delayed deteriorating items with inventory level dependent demand rate which answer some of these problems. Specifically, we attempt to develop models for inventory systems where the demand rate depends on the current stock level and items have constant/variable rate of delayed deterioration and constant/variable holding cost with/without shortages.

1.9 Justification/Significance of the Study

Inventory management is a common problem in business. An inventory consists of usable but idle resources such as men, machines, materials or money. Maintenance of inventory absorbs fluctuations in demand and supply. However, keeping too much of an item on hand increases cost of capital and storage, and keeping too little adversely disrupts production and/or sales. Thus there is a need to seek an inventory level that balances the two extreme situations by minimizing these costs. How to determine "when and how much" of needed resource should be kept as stock that minimizes the total cost or maximizes the net profit can be achieved by modelling the inventory system in question.

We find from the results of the models developed that the effects of stock dependent demand rate, holding cost parameter, deterioration and backlogging rate have impact on the optimal replenishment policy, and hence should not be ignored in developing inventory models. The models will help to determine the inventory policies that minimize the system costs and/or maximize net profit. The proposed models can be used in inventory control of certain delayed deteriorating items such as food items, electronic components, fashionable commodities, vegetables, fruits, yams, potatoes, meat, and so on.

1.10 Aim and Objectives of the Study

The main aim of this research is to develop Deterministic Inventory Models for Delayed Deteriorating Items with Inventory Level Dependent Demand Rate. This aim will be achieved through the following objectives.

(i) To develop an EOQ model for delayed deteriorating items with inventory level dependent demand rate and constant deterioration rate.

(ii) To develop an EOQ model for delayed deteriorating items with inventory level-dependent demand rate and shortages.

(iii) To develop an EPQ model for delayed deteriorating items with stock-dependent and linear time dependent holding cost.

(iv) To develop an EPQ model for delayed deteriorating items with stock-dependent demand rate and time-dependent deterioration rate.

(v) To develop an EOQ model for delayed deteriorating items with inventory level dependent demand rate and linear time-dependent holding cost.

The purpose of the first model (i) is to modify the constant demand rate function of Musa and Sani (2009), by considering the demand rate to be a power-form function of instantaneous inventory level. In the initial stage, inventory depletes down to a certain level of the inventory due to market demand only. In the second stage the inventory level gets depleted due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero at the end of the cycle.

In the second model (ii), we modify the power- form demand rate function in the first model (i) by considering linear-form demand function and allow shortages during stock-out period. The unsatisfied demand is partially backlogged at a rate which is a fixed fraction of demand rate during the shortage period as in Wee (1995).

In the third model (iii), the problem considered has finite replenishment rate rather than infinite replenishment rate as in the case of the first model. Considering same demand rate function as in the second model, we propose an EPQ model for a single item with delayed deterioration, i.e. non- instantaneous deterioration, in which the production rate is constant. The demand rate is linear during and after production run and the holding cost is also a linear function of time. Sugapriya and Jeyarama (2008a) established an inventory model for non-instantaneous deteriorating item with constant demand rate for

both in- production run and out-production run, while in this model we have assumed the demand rate to be inventory level dependent for both in- production run and out-production run.

In the fourth model (iv), we modify the third model (iii) by considering the rate of deterioration to be a linear increasing function of time rather than constant deterioration rate.

In the fifth model (v), we extended the constant holding cost of the first model (i) by considering a linear time dependent holding cost. However, the objective is to determine the optimal replenishment policy by maximizing average net profit per unit time. Unlike in the first four models in which the optimal replenishment policies are obtained by minimizing the average total variable costs per unit time. The reason for using maximization criterion in this case is because of the consideration of stock dependent selling rate r which indicates our inventory system is a sales environment as opposed to non sales environment as in the first four models.

In the literature of inventory models for delayed deteriorating items with stock dependent demand rate, we have not come across any model that studied any of the five models investigated in this thesis.

1.11 Research Methodology

This research being a modelling research has been carried out mainly using literature on (i) Classic square root EOQ model, (ii) Inventory models with constant/variable demand, (iii) Deteriorating inventory models, (iv) delayed (non instantaneous) deteriorating inventory models, (v) Variable holding deteriorating inventory models,

(vi) Production inventory models for deteriorating items, and (vii) Deteriorating inventory models with shortages. Analytic formulation and analysis of the inventory problems are developed on the framework of the model assumptions aimed at determining the inventory policies of the inventory systems. The proposed inventory systems are represented by systems of differential equations including initial and boundary conditions. We then develop a mathematical model for each of the five different inventory systems for non-instantaneous deteriorating items with demand rate depending on the instantaneous inventory level. In all the models, the main objective function to be optimized (minimized) is the function of the total cost for each of the first four models; and the objective function to be optimized (maximized) is the function of net profit per unit time for the fifth model. Differential and integral calculus methods are used to analyze the cost- minimization models and the profit maximization model. We use the approximation method of Newton-Raphson to find the optimal solution (optimal order quantity and the optimal cycle length) of the developed cost minimization model and profit maximization model. Furthermore, we present some numerical examples to illustrate the applications of the models developed. We also carried out a sensitivity analysis on the decision variables with regard to changes in the model parameters for the fifth model. Finally, specific conclusions were given at the end of each of the inventory models developed. However, general conclusions are drawn across the entire research work and future directions for research are presented in the last chapter.

1.12 Organization of the Thesis

The contents of the thesis have been divided into seven chapters.

Chapter One: Introduction. This chapter starts with the introduction of the research, the basic EOQ model, basic classification of inventory models, components of inventory models and basic terms we use in the work. Also included are the research problem, aim and objectives of the study and research methodology.

Chapter Two: Literature Review. This chapter reviews the published literature related to mathematical modelling of deteriorating items with inventory level dependent demand pattern; and also where deterioration is a function of the on-hand level of inventory. The models have been classified and grouped under specific sections and organized according to: classical square root EOQ Model, inventory models with constant / variable demand, delayed (non-instantaneous) deteriorating inventory models, variable holding deteriorating inventory models, production inventory models for deteriorating items, and deteriorating inventory models with shortages.

Chapter Three: An EOQ Model for Delayed Deteriorating Items with Inventory Level - Dependent Demand Rate and Constant Deterioration Rate . This chapter is concerned with the development of an EOQ model for delayed deteriorating items with stock-dependent demand rate in polynomial functional form, and it is organized as follows. The chapter begins with defining the problem and then building the model assumptions. This is followed by developing the inventory model. Subsequently, the model is analyzed and numerical solution using Newton-Raphson method is presented. Moreover, some numerical examples are presented to illustrate the application of the model developed and, finally concluding remarks are given at the end of the chapter.

Chapter Four: An EOQ Model for Delayed Deteriorating Items with Inventory Level Dependent Demand Rate and Shortages. This chapter presents EOQ model for delayed deteriorating items with stock-dependent demand, allowing shortages and fixed partial backlogging rate, and it is organized as follows. The chapter begins with defining the problem and then building the model assumptions. This is followed by developing the inventory model. Subsequently, the model is analyzed and numerical solution using Newton-Raphson method is presented. Moreover, some numerical examples are presented to illustrate the application of the model developed and, finally concluding remarks are given at the end of the chapter.

Chapter Five: An EPQ Model for Delayed Deteriorating Items with Stock Dependent Demand Rate and Linear Time Dependent Holding Cost with Time Dependent Deterioration Rate. This chapter presents an EPQ model for delayed deteriorating items with stock-dependent demand rate and linear time - dependent holding cost with time - dependent deterioration rate, and it is organized as follows. The chapter presents two models, each having basic introduction to the problems they are related to and each of these problems begins with defining the problem and then building the model assumptions. This is followed by developing the inventory model. Subsequently, the model is analyzed and numerical solution using Newton-Raphson method is reported. Moreover, some numerical examples are presented to illustrate the application of the model developed and, finally concluding remarks are given at the end of the chapter.

Chapter Six: An EOQ Model for Delayed Deteriorating Items with Inventory Level Dependent Demand Rate and Linear Time - Dependent Holding Cost. This chapter is concerned with an inventory replenishment model for delayed deteriorating items

with inventory level dependent demand rate and linear time-dependent holding cost for profit maximization criteria, and it is organized as follows: the chapter begins with defining the problem and then building the model assumptions. This is followed by developing two models: (i) deterioration starts before reorder point occurs and (ii) deterioration sets in after reorder point occurs. Subsequently, the model is analyzed and numerical solution method using Newton-Raphson method is presented. Moreover, some numerical examples are presented to illustrate the application of the model developed and, finally concluding remarks are given at the end of the chapter.

Chapter Seven: Summary, Conclusion and Recommendations. The summary of the entire work, the conclusion with contributions of the research and suggestions on the direction of the related future research is presented in chapter seven.

References and Appendices

The thesis closes with references and appendices in that order.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

In many inventory models a general assumption is that products have indefinitely long lives. However, in general, almost all items deteriorate over time. Often the rate of deterioration is so low that there is little need to consider the deterioration in determining the inventory models. Nevertheless, there are many items in the real life situations that are subject to a significant rate of deterioration. Hence, in such situation, the impact of item deterioration should not be neglected in the inventory decision - making. More specifically, those inventory models where deterioration is a function of the on-hand level of inventory are considered.

There is no clear consensus on the definition of deterioration of items. Raafat (1991) defines deterioration as any process that prevents an item from being used for its intended original use such as: (i) spoilage, as in perishable foodstuffs, fruits, vegetables, periodicals (such as newspapers and magazines); (ii) physical depletion, as in pilferage or highly volatile liquids such as gasoline, alcohol, turpentine, and so on, which undergo physical depletion over time through the process of evaporation. (iii) decay, as in radioactive substances, degradation, as in electronic components, or loss of potential or utility with the passage of time as in photographic films and pharmaceutical drugs. Goyal and Giri (2001) in their review of deteriorating inventory models classified all such models based on obsolescence, deterioration or neither. They reported “items are subject to obsolescence if they lose their value over time because of rapid changes of

technology or the introduction of a new product by a competitor or because they go out of fashion" (Goyal and Giri, 2001, p. 1). Examples of items that are subject to obsolescence are computer chips, mobile phones, fashion and seasonal goods, and so on. Others are style goods such as spare parts for military which become obsolete when a replacement is introduced.

In many real-life situations, for certain types of consumer goods (e.g., fruits, vegetables, meat, bread, beans, cassava, yams and others), the consumption rate is sometimes influenced by stock-level. It is usually observed that a large pile of goods displayed on shelves in a shop will attract customers to buy more and which in turn generate higher sales. The consumption rate may go up or down depending on the on-hand stock level (Gupta and Vrat, 1986). The inventory level as a function of time will decrease rapidly initially, since the quantity demanded will be greater at a high level of inventory. As the inventory is depleted, the quantity demanded will decrease, resulting in the inventory level decreasing more slowly (Baker and Urban, 1988). This type of demand rate which depends on the level of on-hand stock is termed, ''inventory level- dependent demand rate'' (Gupta and Vrat, 1986, p. 19; Baker and Urban, 1988, p. 824). This implies demand rate of this pattern is a function of instantaneous level of inventory.

In the literature of inventory systems, inventory models for deteriorating items assume that deterioration starts as soon as the retailer receives the commodities. However, in real life, many products would have a span of maintaining quality or the original condition for some period. That is during that period there is no deterioration occurring until later, and that phenomenon is termed as "non - instantaneous deterioration" in Wu *et al.* (2006, p. 369) or "delayed deterioration" in Musa and Sani (2012, p. 75).

The aim of this chapter is to review the published literature on mathematical modelling of deteriorating items with inventory level dependent demand pattern which are related to this research. The models have been classified and grouped under specific sections and organized as follows:

(i) Classical square root EOQ Model

(ii) Inventory Models with constant / variable demand

(iii) Delayed (Non-instantaneous) deteriorating inventory models

(iv) Variable holding deteriorating inventory models

(v) Production Inventory Models for Deteriorating Items

(vi) Deteriorating inventory models with shortages

2.2 Classical Square Root EOQ Model

Mathematical modeling of inventory control began with the Harris`s (1913) classic square root EOQ model (as cited in Pentico *et al*., 2009; Baker and Urban, 1988). This model is seen in every basic textbook on inventory management. While this model has been criticized for its unrealistic assumptions, it is, however, widely used (Drake *et al*., 2011; Hsieh and Dye, 2012; Pentico *et al*., 2015). Moreover, it forms the basis for many other models that relax one or more of its assumptions. One of the earliest extensions of this EOQ is the Economic Production Quantity (EPQ) in which the assumption of infinite rate of replenishment is replaced with the finite rate of replenishment. A key assumption of both the classical EOQ and EPQ models is that shortages are not allowed. However, over the years, the classical EOQ and EPQ models, being among the most

popular and useful models in inventory management, have been extended to represent some realistic situations.

There have been a good number of authors who incorporated complete and partial backordering during stock-out periods, into basic EOQ and EPQ models. To mention a few of these authors, we have Pentico and Drake (2009) who developed the deterministic EOQ model with partial backordering; Pentico *et al*. (2009, 2011) who presented a new approach to determine the optimal inventory policy from the three cases viz: to lose all sales, to meet all demand, and to allow stockouts with partial backordering; Drake *et al*. (2011) considered lot-sizing planning for a two-stage system in which the final product is planned using an EPQ model with partial backordering and the production of the components is controlled using basic EPQ models without backordering. Hsieh and Dye (2012) considered again the EPQ models of Pentico et al. (2011) and solved the same model without differential calculus and presented a different decision procedure.

2.3 Inventory Models with Constant / Variable Demand

The basic assumptions of classical Economic Order Quantity (EOQ) model considered demand to be a constant over time. Chung and Lin (2001), Benkherouf *et al*. (2003), Sarker *et al*. (2000), Chang (2004), Chung and Liao (2006), Huang and Liao (2008), Zhang *et al*. (2007a, 2007b) all developed models in which the demand rate is constant. Several authors later, modified the EOQ model to consider variable demand but independent of stock level; these generally fall into three major categories. First, there are those models (Wagner and Whitin, 1958; Silver and Meal, 1969, 1973) in which the demand rates differ for different time periods with a known demand for each time

period. Second, there are those models in which the demand is time dependent such as that of Donaldson (1977) who was the first to solve analytically an EOQ model where the demand was assumed to be a linearly increasing function of time. Giri *et al*. (2000), Teng *et al*. (2002), Wang and Wang (2005), Panda *et al*. (2008), Papachristos and Skouri (2000), Chu and Chen (2002), Khanra and Chaudhuri (2003), Dye *et al*. (2006) all considered time - dependent demand. Some of the authors considered linear positive or negative trend in demand and some considered exponentially increasing or decreasing demand. Ramp type demand being a special type of time-dependent demand was first introduced in inventory study by Hill (1995). Then later, many authors studied this type of demand (Mandal and Pal, 1998; Deng *et al*., 2007). Third, there are also inventory models incorporating economic and marketing policies such as the effect of price, advertising, and so on, on demand rates. Some of these models can be found in (Whitin, 1955; Lee and Rosenblatt, 1986; Peng and Peng, 2004; Teng *et al*., 2005; Qin *et al*. 2006).

Whitin (1957) observed (as cited in Gupta and Vrat, 1988), without empirical evidence that, "An increase in inventories may bring about increased sales of some items". Wolfe (1968) presented empirical evidence of this relationship, noting that the sales of style merchandise, such as women's dresses or sport clothes, are proportional to the amount of inventory displayed. Levin *et al*. (1972) and Silver and Peterson (1985) also observed (as cited in Baker and Vrat, 1988; Mandal and Phaujdar, 1989a; Pal *et al*., 1993) that sales at retail level tend to be proportional to inventory displayed and that a large pile of goods displayed in a supermarket will lead customers to buy more. These observations imply that in real life, demand rate may be influenced by stock levels. As a result, there are many inventory models appearing in the literature dealing with inventory models

using some form of demand rate functions depending on instantaneous inventory level. Different inventory models with stock-dependent demand rate function have used different forms of demand rate function such as the model of Gupta and Vrat (1986) who first developed inventory models depending on the inventory level. They considered the following demand rate functions in their models.

$\lambda = \alpha + \beta Q^{\gamma}$

$\lambda = \alpha - \beta Q^{-\gamma}$

$\lambda = \alpha + \beta e^{Q}$

$\lambda = \alpha - \beta e^{-Q}$

where α, β, γ are positive constants, λ is stock dependent consumption rate and Q is ordering quantity (initial stock level or order level). See Appendix D for notes on α, β and Q.

Baker and Urban (1988) developed an inventory model assuming the demand rate to be in a power-form function depending on the on-hand inventory. That is, the inventory level dependent demand rate was assumed to take the form

$$D\left[i(t)\right] = \alpha\left(i(t)\right)^{\beta}, \quad \alpha > 0, 0 < \beta < 1,$$

where $D\left[i(t)\right]$ is the demand rate of the product at any instant of time t, i(t) is the inventory level at time t, α is the scale parameter and β is the shape parameter which measures the responsiveness of the demand rate to changes in the level of on-hand inventory. Urban and Baker also showed the reasons why this particular functional form was chosen for analysis as are shown in Appendix A. Pal *et al*. (1993) later showed that

the shape parameter β equals the ratio of the percentage change in the quantity demanded to the percentage change in the inventory level as in Appendix C.

Mandal and Phaudjar (1989a) extended the order level dependent demand rate function of Gupta and Vrat (1986) by considering a linearly dependent demand rate *R(q)* on the instantaneous stock level given by

$R(q) = \alpha + \beta q$

where

α and β are nonnegative constants. Explanatory notes on this demand function can be seen in Appendix E.

Datta and Pal (1990a) discussed the model of inventory- level- dependent demand rate in which the demand rate depends upon inventory level down to a certain stock level and then it becomes constant for the rest of the cycle, until the inventory level is zero. They assumed that at the end of each cycle, the inventory level is zero. Urban (1992) relaxed the terminal condition of zero-ending inventory and suggested that, in an inventory system with inventory-level-dependent demand rate, it would be more profitable to utilize higher inventory levels resulting in greater demand. Later on, Goh (1994) discussed the model of Baker and Urban (1988) relaxing the assumption of a constant holding cost. In the model, the author employed the holding cost as (i) a nonlinear function of the length of time the item is held in stock, and (ii) a nonlinear function of the amount of the on-hand inventory. Later on, Alfares (2007) considered the inventory policy for an item with a stock-level-dependent demand rate and a storage-time-dependent holding cost.

2.4 Deteriorating Inventory Models

The inventory problem of deteriorating items was first studied by Whitin (1957) who studied fashion items deteriorating at the end of the storage period. Ghare and Schrader (1963) first modeled negative exponential decaying inventory. They proposed that in a number of inventory situations decay is observed to follow a pattern in which the decay during any given time period is proportional to the magnitude of the inventory at the beginning of the time period. They further assumed that "the ratio of decay during the nth period to the inventory level at the end of the (n-1) period is equal a constant" (Ghare and Schrader, 1963, p. 328). That is to say, the continuously declining inventory function may be represented mathematically by the negative exponential function and expressed symbolically as

$$I_{t+dt} = I_t e^{-\theta dt}$$

where

I_{t+dt} = inventory level at time $(t+dt)$

I_t = inventory level at time t

θ = a fraction representing the rate of decay per time.

They also modeled inventory depletion by demand and decay simultaneously by the differential equation

$$\frac{dI_t}{dt} + \theta I_t = - f_t$$ where f_t is the demand rate at time *t*.

Since then, many different inventory models have been developed which differ in assumptions on type of demand, deteriorating rates, allowing price discounts, shortages and backordering, single or multiple items, one or two warehouses, single or multi-echelon modeling, average cost or discounted cash flows, whether a delay in payment is permissible, finite/infinite rate of replenishment, and instantaneous/non-instantaneous

deterioration of items. In short, considering different combinations of these factors mentioned above, different inventory models were developed.

Mandal and Phaujdar (1989b) presented an Economic Production Quantity (EPQ) model concerning a single item with a variable rate of deterioration and the demand rate which is dependent on instantaneous inventory level. Baraya and Sani (2011b) proposed an EPQ model for delayed deteriorating items with stock-dependent demand rate and linear time dependent holding cost. Padmanabhan and Vrat (1990) developed an inventory model for initial stock dependent consumption rate and exponential decay. Datta and Pal (1990b) extended Mandal and Phaujdar (1989b)'s work for deteriorating items with the assumption that the demand rate is a linear function of the on-hand inventory and allowing shortages, which are completely backlogged for both finite and infinite time horizons. Giri and Chaudhuri (1998) extended Goh (1994) to cover instantaneous deteriorating items where the rate of deterioration is a constant fraction of the on-hand inventory. Later on, inventory models of instantaneous deteriorating items with inventory-level-dependent demand were developed by, Pal *et al*. (1993), Padmanabhan and Vrat (1990, 1995), Giri *et al*. (1996), Chung *et al*. (2000), Dye and Ouyang (2005), Teng *et al*. (2005), Chang *et al.* (2010) and so on. Many researchers have also done quite a lot on inventory models with inventory-level-dependent demand, such as considering, non-linear holding cost (Goh, 1994), perishable items (Padmanabhan and Vrat, 1995; Giri and Chaudhuri, 1996; Chang *et al.*, 2010), perishable items and non – linear holding cost (Giri and Chaudhuri, 1998). Teng *et al*. (2005) extended Datta and Pal's (1990b) EOQ model to allow not only deteriorating items but also non-zero ending inventory. Sana and Chaudhuri (2008) analyzed a kind of EOQ model with current-stock-dependent demand where a supplier gives a retailer

both a credit period and a price discount on the purchase of merchandise. Chang *et al.* (2010) investigated inventory models with stock-and price-dependent demand rate for deteriorating items based on limited shelf space.

As time went on, inventory models of instantaneous deteriorating items with inventory-level-dependent demand were developed by, Pal *et al*. (1993), Padmanabhan and Vrat (1995, 1990), Giri *et al*. (1996), Chung *et al*. (2000), Dye and Ouyang (2005), Teng *et al*. (2005), Yang *et al*. (2011), Chang *et al.* (2010), Lee and Dye (2012), Yadav *et al*. (2012), Roy and Chaudhuri (2012), Glock *et al.* (2014), Pal and Chandra (2014), and so on. In all these models, the holding cost per unit of time was taken as a constant.

2.5 Delayed (Non-instantaneous) Deteriorating Inventory Models

Wu *et al*. (2006) modified the model of Padmanabhan and Vrat (1995) by considering a problem of determining the optimal replenishment policy for non-instantaneous deteriorating items with linearly stock dependent demand and constant unit holding cost. In the model, shortages are allowed and the backlogging rate is a variable and dependent on the waiting time for the next replenishment. Other inventory models using non-instantaneous deteriorating rate with linearly stock-dependent demand rate include those in Jain *et al*. (2008) and Uthayakumar and Geetha (2009) who used linear functional form to represent stock-dependent demand rate. Mahata and Goswami (2009) modified the model of Giri and Chaudhuri (1998) by considering non-instantaneous deteriorating items with fuzzy deteriorating rate and power-form stock-dependent demand rate. Other researchers who considered non-instantaneous deteriorating inventory for items with stock-dependent demand pattern are Chang and Lin (2010), Chang *et al*. (2010) and so on. Baraya and Sani (2011a) developed an inventory model

for delayed deteriorating items with inventory level dependent demand rate. In all of these models, the holding cost per unit of time was taken as a constant.

2.6 Variable Holding Inventory Models

The factor of variability in holding costs has also been studied by many authors. Naddor (1966) gave a detailed derivation of the total inventory cost for a constant demand-rate lot-size-system by considering the holding cost to be $q^m t^n$ where q is the amount of stock held for a time t and m, n are positive integers. Van der Veen (1967) developed an inventory system with constant demand rate, taking the holding cost to be a nonlinear function of inventory. Muhlemann and Valtis-Spanopoulous (1980) studied an EOQ model with constant demand rate and a variable holding cost which is expressed as a percentage of the average value of capital invested in stock. Weiss (1982) studied the model of Van der Veen (1967) by treating holding cost per unit as a nonlinear function of the length of time for which the item was held in stock. Goh (1994) discussed the model of Baker and Urban (1988) relaxing the assumption of a constant holding cost. In the model, Goh employed the holding cost as (i) a nonlinear function of the length of time the item is held in stock, and (ii) a nonlinear function of the amount of the on-hand inventory. Chang (2004) modified the assumption in Giri and Chaudhuri (1998) by changing the objective to maximize the profits instead of minimizing the costs. The author assumed variable holding costs viz: (i) nonlinear time-dependent holding cost and (ii) nonlinear stock-dependent holding cost. Alfares (2007) also presented an inventory model with stock-dependent demand rate and variable holding costs. Other authors who considered holding cost to be a variable include Ferguson *et al*. (2007), Weiss (1982), Goh (1994) and Giri and Chaudhuri (1998).

2.7 Production Inventory Models for Deteriorating Items

Many researchers have studied in the area of production-inventory for deteriorating items. These researchers include Misra (1975) who first studied the EPQ model for deteriorating items with varying and constant rate of deterioration. Others are Goyal and Gunasekaran (1995), Jiang and Du (1998), Chen *et al*. (2002), Gontg and Wang (2005), Maity *et al*. (2007) and so on. With the assumption that the demand rate, production rate and deteriorating rate are all constant, Liao (2007, 2008) established a production inventory model for deteriorating items under the condition of the supplier providing the retailer with retail credit.

In all the models stated above, shortages are not allowed. However, Lin *et al*. (2007) established a production-inventory model with constant production rate, demand rate and deteriorating rate and allowed for shortages. Zhou *et al*. (2003) also considered production-inventory problem in which each cycle of a production-inventory schedule starts with replenishment and ends with a shortage. In the research work of Sana *et al*. (2004) and Zhou and Gu (2007), shortages are allowed and occur at the end of the cycle.

All the deteriorating rates in the models stated above are constant but some authors studied situations with time-dependent deteriorating rates. Such authors include Skouri and Papachristos (2003) and Chen *et al*. (2002) who developed a production-inventory model in which storages are allowed at the beginning of the cycle. In contrast, Manna and Chaudhuri (2006) and Balkhi's (2001), also studied production models in which shortages are also allowed but occur at the end of each cycle. Abad (2003) studied the pricing and lot-sizing problem for deteriorating items under the conditions of finite production, partial backlogging, and lost sale. Teng *et al.* (2007) extended Abad's model

by adding the backlogging cost and the cost of lost goodwill. Tripathy *et al.* (2010) obtained an EPQ model for linear deteriorating item with variable holding cost.

Mandal and Phaujdar (1989b) presented an EPQ model with a variable rate of deterioration and linearly dependent demand rate. Gupta and Agarwal (1995) studied EPQ model in which production rate and demand rate during production remain constant and just as the production stops the demand rate becomes stock dependent. This continued up to a certain level of inventory and then the demand rate becomes constant for the rest of the cycle. Sarker *et al*. (1997) developed an inventory model in which the demand is considered as a composite function consisting of a constant component, and a variable component which is proportional to the inventory level in the periods when there is a positive inventory build-up. The rate of production is considered finite and the decay rate as exponential. Teng and Chang (2005) established an EPQ model for deteriorating items when the demand rate is dependent on both the stock level and the selling price per unit, and in which the deteriorating rate is a constant. Gary *et al.* (2005) obtained production cycle time for an Economic Lot-Size Production (ELSP) with deteriorating items. Note that ELSP is an extended EPQ model in which several products are produced on the same machine leading to determination of both the lot size for each product and when each lot should be produced in order to minimize the total costs.

Sugapriya and Jeyaraman (2008a) proposed an EPQ model for single product subject to non-instantaneous deterioration and in which the holding cost varies with time. Sugapriya and Jeyaraman (2008b) also developed an ELSP model for non-instantaneous deteriorating items using price discount and permissible delay in

payments. Li *et al*. (2010) made a detailed review of deteriorating inventory study including production - inventory study. An economic production quantity (EPQ) model for delayed deteriorating items with stock-dependent demand rate was developed for linear time dependent holding cost (Baraya and Sani 2011b) and time dependent deterioration rate (Baraya and Sani 2012).

2.8 Deteriorating Inventory Models with Shortages

In practice, when shortages occur, some customers are willing to wait for backorder and others are impatient to wait and therefore would turn to buy from other competitors or buy alternative product. For inventory models with stock dependent consumption rate, several inventory researchers such as Papachristos and Skouri (2000), Chung *et al.* (2000), Hou (2006), assumed that shortages are completely backlogged.

Authors such as Park (1982), Hollier and Mak (1983), Wee (1995), Urban (1995) and Uthayakumar and Geetha (2009) considered constant partial backlogging rates during the shortage period.

Padmanabhan and Vrat (1995) considered an EOQ model for perishable items with stock-dependent selling rate for sales environment. Under instantaneous replenishment with zero lead time, they presented three models: (i) without backlogging, (ii) with complete backlogging and (iii) with partial backlogging for which they assumed that the amount of demand backlogged may depend on the amount of orders already backlogged.

Chang and Dye (1999) were among the first to introduce partial backlogging rate in Economic Order Quantity models. They assumed that, the length of the waiting time for the next replenishment becomes main factor for determining the backlogging rate. That is to say, the longer the waiting time is, the smaller the backlogging rate would be. Many researchers considered time-proportional partial backlogging rate in their inventory models. Amongst them are Abad (2001), Papachristos and Skouri (2000, 2003), Wang (2002), Chern *et al*. (2008), Min and Zhou (2009), Dye *et al*. (2007), Hou *et al*. (2011), etc. Abad (1996, 2001) considered the pricing and lot-sizing problem of a perishable good under partial backlogging of demand. The backlogging rate was modeled as a decreasing function of waiting time. Hou (2006) afterwards extended the model of Padmanabhan and Vrat (1995) by incorporating the effects of deterioration rate, inflation and time value of money to develop an inventory model with stock-dependent consumption rate and with complete backlogging for non-sales environment. Dye and Ouyang (2005) developed an inventory model in which the proportion of customers, who would like to accept backlogging, is the reciprocal of a linear function of the waiting time. Wu *et al.* (2006) discussed the problem of determining the optimal replenishment policy for non-instantaneous deteriorating items with stock-dependent demand. In the model, shortages are allowed; the backlogging rate is variable and dependent on the waiting time for the next replenishment.

Wu *et al*. (2006) modified the model of Padmanabhan and Vrat (1995) by considering a problem of determining the optimal replenishment policy for non-instantaneous deteriorating items with linearly stock dependent demand and constant unit holding cost. In the model, shortages are allowed and the backlogging rate is variable and dependent on the waiting time for the next replenishment. Other inventory models using

non-instantaneous deteriorating rate with linearly stock-dependent demand rate include those in Jain *et al*. (2008), Uthayakumar and Geetha (2009) and Mahata and Goswani (2009), Chang and Lin (2010) later proposed a partial backlogging inventory model for non-instantaneous deteriorating items with stock-dependent consumption rate under inflation.

Table 2.1 gives a summary of some related literature of inventory level dependent demand.

Table 2.1: Some related literature of inventory level dependent demand

Author(s) and Year	EOQ / EPQ	Demand Patterns	Zero Ending Inventory	Deterioration Rate	Shortage
Gupta and Vrat(1986)	EOQ	Order quantity	Yes	No	No
Baker and Urban (1988)	EOQ	Polynomial form	No	No	No
Mandal and Phaujdar (1989a)	EPQ	Linear form	Yes	Yes	No
Datta and Pal (1990)	EOQ	Polynomial form	Yes	No	No
Urban(1992)	EOQ	Polynomial form	No	No	No
Pal *et al*. (1993)	EOQ	Polynomial form	No	Yes	No
Padmanabhan and Vrat (1995)	EOQ	Linear form	Yes	Yes	No
Wee (1995)	EOQ	Exponential form	Yes	Yes	Yes
Giri *et al*. (1996)	EOQ	Polynomial form	No	Yes	No
Ray and Chaudhuri (1997)	EOQ	Polynomial form	Yes	No	Yes
Giri and Chaudhuri (1998	EOQ	Polynomial form	Yes	Yes	No
Chung *et al* (2000)	EOQ	Linear form	Yes	Yes	Yes
Kun-Jen *et al* (2000)	EOQ	Linear form	Yes	Yes	Yes
Teng and Chang (2003)	EPQ	Linear form	Yes	No	Yes
Ouyang *et al*. (2003)	EOQ	Linear form	Yes	Yes	Yes
Teng *et al*. (2005)	EOQ	Polynomial form	No	Yes	No
Jhuma and Samanta (2007)	EOQ	Polynomial form	Yes	Yes	No
Jain *et al*.(2008)	EOQ	Linear form	Yes	Yes	Yes
Alfares (2007)	EOQ	Polynomial form	Yes	No	No
Mahata and Goswani (2009)	EOQ	Polynomial form	Yes	No	No
Yang *et al.* (2010)	EOQ	Linear form	Yes	Yes	Yes
Chang *et al* (2010)	EOQ	Linear form	No	Yes	No
Chang and Lin (2010)	EOQ	Linear form	Yes	Yes	Yes

CHAPTER THREE

AN EOQ MODEL FOR DELAYED DETERIORATING ITEM WITH INVENTORY LEVEL - DEPENDENT DEMAND RATE AND CONSTANT DETERIORATION RATE

3.1 Introduction

In many real-life situations, for certain types of consumer goods such as yams, fruits, vegetables, doughnuts, and so on, the demand rate is sometimes influenced by the stock level. It is usually observed that a large pile of goods displayed on shelves in a shop will lead the customer to buy more and then generate higher demand. The consumption rate may go up or down with the on-hand stock level.

Depletion of inventory items may occur for some reasons other than demand, such as by direct spoilage or physical decay or deterioration. Some products such as fruits, vegetables, pharmaceuticals, volatile liquids, and others of this nature deteriorate continuously due to evaporation, obsolescence, spoilage, and so on. Ghare and Schrader (1963) first derived an economic order quantity (EOQ) model by assuming exponential decay.

In reality, not all kinds of inventory items deteriorate as soon as they are received in stock. In the fresh product time, when the product has no deterioration the inventory items retain their original quality. Wu *et al*. (2006) named this phenomenon as "non-instantaneous deterioration", and they established an inventory model for non-instantaneous deteriorating items with permissible delay in payments. Substantial

amount of research has been done on the impact of inventory level on demand for deteriorating inventory items as presented in chapter two. Musa and Sani (2009) proposed an EOQ model for some tropical items that exhibit delay in deterioration where the demand rate is constant.

In this chapter, the work of Musa and Sani (2009) has been extended, by considering a situation where the demand rate is inventory level dependent. In the first stage, inventory depletes down to a certain level of the inventory due to market demand only which is inventory level dependent. In the second stage, the inventory level gets depleted due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero at the end of the cycle. Our aim is to find the optimal ordering quantity and optimal cycle time for the inventory system. We consider the polynomial functional form for inventory level demand rate and establish the theoretical results for the optimal replenishment policy of the inventory system in order to minimize the total average system cost per unit time. Newton-Raphson method is used to find the optimal order quantity and the optimal cycle length. Furthermore, some numerical examples are presented to illustrate the application of the model developed and then use the examples to study the effect of various changes in some possible combinations of model parameters on the decision variables of the system.

3.2 Modelling Assumptions

The inventory system is developed on the basis of the following model assumptions:

1. The unit cost of item and the traditional parameters of replenishment cost and holding cost are known and constant

2. Shortages are not allowed and Lead time is zero.

3. Rate of inflation is constant

4. Replenishment is instantaneous at infinite rate

5. The inventory system involves only one item and one stocking point in each cycle.

6. The time horizon of the inventory system is infinite. Only a typical inventory cycle length is considered, all remaining cycles are identical.

7. A constant fraction θ ($0 < \theta < 1$) representing the rate of deterioration per time period of the on-hand inventory when the item begins to deteriorate.

8. There is no repair or replacement of the deteriorated items.

9. The inventory level dependent demand rate $D\left[v(t)\right]$ of the product at any instant of time t, is assumed to be a polynomial functional form as in (Baker and Urban 1988), where $D\left[v(t)\right] = \alpha\left(v(t)\right)^{\beta}$, $\alpha > 0, 0 < \beta < 1$, α is the scale parameter, β is the measure of responsiveness of the demand rate to changes in the level of on-hand inventory. When $\beta = 0, \text{then } D\left[v(t)\right] = \alpha$, which is a constant demand rate function. On the other hand, when $\beta = 1, \text{then } D\left[v(t)\right] = \alpha v(t)$, which is a linear demand rate function of the inventory level. The reasons why this particular functional form was chosen for analysis are shown by (Baker and Urban, 1988) as in Appendix A. Later on, Pal *et al.* (1993, p.296) have shown as in Appendix C that "The shape parameter β equals the ratio of the percentage change in the quantity demanded to the percentage change in the inventory level (or stock elasticity of demand)". The system parameters α and β can be estimated as in Appendix B.

3.3 Mathematical Analysis

Let Q be the order quantity at time $t = 0$ and $v_1(t)$ be the inventory level at time t $(0 < t < t_d)$ in which the product has no deterioration. The on-hand inventory level gradually falls to the level S_d at time $t = t_d$ due to demand only. The inventory level dependent demand rate $D(v_1(t))$ of the product at any instant of time t, is assumed a polynomial functional form (Baker and Urban 1988), $D(v_1(t)) = \alpha(v_1(t))^{\beta}$, where $\alpha > 0, 0 < \beta < 1$, α is the scale parameter, β is the measure of responsiveness of the demand rate to changes in the level of on-hand inventory. The shape parameter β equals the ratio of the percentage change in the quantity demanded to the percentage change in the inventory level. Since the inventory level is decreasing then the rate of change of inventory level per unit time is negative and this equals the demand rate. Therefore, the inventory level at any time t $(0 \le t \le t_d)$ can be represented by the following differential equation:

$$\frac{dv_1(t)}{dt} = -\alpha(v_1(t))^{\beta}, \quad 0 \le t \le t_d \tag{3.3.1}$$

Let $v_2(t)$ be the inventory level at time t $(t_d \le t \le T)$ in which the product depletes by both effects of market demand and deterioration. Given that θ is a constant fraction of the on-hand inventory that gets deteriorated per unit time, the differential equation governing the instantaneous state of $v_2(t)$ over the time period in $[t_d, T]$ is given by

$$\frac{dv_2(t)}{dt} = -\left(\theta v_2(t) + \alpha(v_2(t))^{\beta}\right), \quad t_d \le t \le T \tag{3.3.2}$$

with conditions $v_1(0) = Q$, $v_1(t_d) = v_2(t_d) = S_d$ and $v_2(T) = 0$.

The inventory system at any time t can be represented by the two differential equations (3.3.1) and (3.3.2) above.

The pictorial representation of the model is given in Figure 3.1.

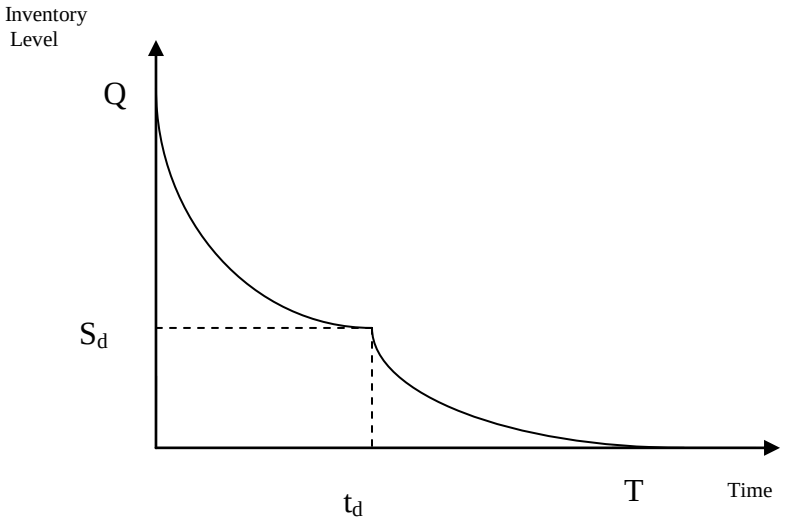


Figure 3.1: The inventory system represented by equations (3.3.1) and (3.3.2)

The solution of the differential equation (3.3.1) using the initial condition $v_1(0) = Q$ will be given by

$$v_1 = \left[Q^{1-\beta} - \alpha(1-\beta)\, t \right]^{\frac{1}{1-\beta}}, \quad 0 \le t \le t_d \tag{3.3.3}$$

where on applying the condition $v_1(t_d) = S_d$ on (3.3.3), we get

$$S_d^{\,1-\beta} = Q^{1-\beta} - \alpha(1-\beta)\, t_d \tag{3.3.4}$$

The nonlinear first order differential equation (3.3.2), may be reduced to the linear form and solved to give

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(S_d^{1-\beta} + \frac{\alpha}{\theta} \right) e^{\theta(1-\beta)(t_d - t)} \tag{3.3.5}$$

Substituting the value of $S_d^{1-\beta}$ from (3.3.4) in (3.3.5), we have

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d + \frac{\alpha}{\theta} \right) e^{\theta(1-\beta)(t_d - t)} \tag{3.3.6}$$

Using the Taylor series expansion for exponential function, we have

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d + \frac{\alpha}{\theta} \right) \left((1 + \frac{\theta(1-\beta)(\, t_d - t\,)}{1!} + \frac{(\theta(1-\beta)(\, t_d - t\,))^2}{2!} + \ldots) \right)$$

Neglecting the higher powers of θ (since $0 < \theta < 1$) and allowing only the first two terms in the expansion, we get

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d + \frac{\alpha}{\theta} \right) \left(\, 1 + \theta(1-\beta)(\, t_d - t\,) \, \right)$$

$$= -\frac{\alpha}{\theta} + \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right) \left(1 + \theta(1-\beta)\,(t_d - t) \, \right) + \frac{\alpha}{\theta} \left(1 + \theta(1-\beta)\,(t_d - t) \, \right)$$

$$= \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right) \left(1 + \theta(1-\beta)\,(t_d - t) \, \right) + \alpha(1-\beta)\,(t_d - t)$$

$$= \quad Q^{1-\beta} - \alpha(1-\beta)t_d + \left[\ \theta\left(\ Q^{1-\beta} - \alpha(1-\beta)t_d\right) + \alpha\ \right](1-\beta)(\ t_d - t)$$

and so

$$v_2 \ = \ \left\{\ Q^{1-\beta} - \alpha(1-\beta)t_d + \left[\ \theta\left(\ Q^{1-\beta} - \alpha(1-\beta)t_d\right) + \alpha\ \right](1-\beta)(\ t_d - t)\right\}^{\frac{1}{1-\beta}} \quad (3.3.7)$$

Using the condition $v_2(T) = 0$ in (3.3.7), we have using equation (3.3.4),

$$t_d - T = \quad \frac{-S_d^{\,1-\beta}}{(1-\beta)\left(\ \theta S_d^{\,1-\beta} + \alpha\right)} \quad (3.3.8)$$

Total inventory during the complete cycle time T is given by

$$M \ = \ \int_0^T v(t)\,dt$$

$$= \int_0^{t_d} v_1(t)\,dt \ + \ \int_{t_d}^{T} v_2(t)\,dt$$

$$= \ \frac{1}{\alpha(\ 2-\beta\ )}\left[\ Q^{\,2-\beta} - \left(\ Q^{\,1-\beta} - \alpha(1-\beta)\ t_d\ \right)^{\frac{2-\beta}{1-\beta}}\right]$$

$$+ \frac{\left\{\ Q^{1-\beta} - \alpha(1-\beta)t_d + \left[\ \theta\left(\ Q^{1-\beta} - \alpha(1-\beta)t_d\right) + \alpha\ \right](1-\beta)(\ t_d - T)\right\}^{\frac{2-\beta}{1-\beta}}}{(\ \beta-2)\left[\ \theta\left(\ Q^{\,1-\beta} - \alpha(1-\beta)\ t_d\ \right) + \alpha\ \right]}$$

$$- \ \frac{1}{(\ \beta-2)\left[\ \theta\left(\ Q^{\,1-\beta} - \alpha(1-\beta)\ t_d\ \right) + \alpha\ \right]}\left[\ Q^{1-\beta} - \alpha(1-\beta)t_d\ \right]^{\frac{2-\beta}{1-\beta}} \quad (3.3.9)$$

Using equations (3.3.4) and (3.3.8) in equation (3.3.9), we have

$$M = \frac{Q^{2-\beta} - S_d^{2-\beta}}{\alpha(2-\beta)} + \frac{S_d^{2-\beta}}{(2-\beta)\left(\theta S_d^{1-\beta} + \alpha\right)} \tag{3.3.10}$$

Total demand during the cycle time T is given by

$$W = \int_0^{t_d} \text{demand rate in } [0, t_d] + \int_{t_d}^{T} \text{demand rate in } [t_d, T]$$

$$= \int_0^{t_d} \alpha\left(\nu_1(t)\right)^{\beta} dt + \int_{t_d}^{T} \alpha\left(\nu_2(t)\right)^{\beta} dt$$

$$= \left[Q - \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right)^{\frac{1}{1-\beta}} \right] + \alpha \frac{\left\{ Q^{1-\beta} - \alpha(1-\beta) t_d \right\}^{\frac{1}{1-\beta}}}{\left[\theta\left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right) + \alpha \right]}$$

$$- \alpha \frac{\left\{ Q^{1-\beta} - \alpha(1-\beta) t_d + \left[\theta\left(Q^{1-\beta} - \alpha(1-\beta) t_d \right) + \alpha \right](1-\beta)(t_d - T) \right\}^{\frac{1}{1-\beta}}}{\left[\theta\left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right) + \alpha \right]}$$

(3.3.11)

Again using equations (3.3.4) and (3.3.8) in equation (3.3.11), we obtain

$$W = Q - S_d + \frac{\alpha S_d}{\left(\theta S_d^{1-\beta} + \alpha\right)} \tag{3.3.12}$$

Total amount of deteriorated inventory items during the cycle time T is given by

I_d = Initial inventory - Total demand in the time period T

= Q - W

Total relevant cost per unit time is given by

$$TVC = \frac{1}{T}\left[\text{ ordering costs } + \text{holding costs } + \text{dete rioration costs } \right]$$

$$= \frac{1}{T}\left[\ C_0 + cQ + \mu_1 cM\ - cW\ )\right], \tag{3.3.13}$$

where μ_1 here is redefined as the inventory carrying charge expressed as a fraction of unit cost/unit/cycle.

T can be obtained by combining equations (3.3.4) and (3.3.8) to have

$$T\ =\ \frac{Q^{1-\beta} - S_d^{\,1-\beta}}{\alpha(1-\beta)}\ +\ \frac{S_d^{\,1-\beta}}{(1-\beta)\left(\ \theta S_d^{\,1-\beta} + \alpha\right)}$$

$$=\ \frac{\left(\ \alpha + \theta S_d^{\,1-\beta}\right)Q^{1-\beta} - \theta S_d^{\,2(1-\beta)}}{\alpha(1-\beta)\left(\ \theta S_d^{\,1-\beta} + \alpha\right)} \tag{3.3.14}$$

Substituting the expressions for M and W from (3.3.10) and (3.3.12) respectively in (3.3.13), we have

$$TVC(Q) = \frac{1}{T}\left\{ \begin{array}{l} C_0 + cQ + \mu_1 c\left[\dfrac{Q^{2-\beta} - S_d^{\,2-\beta}}{\alpha(2-\beta)} + \dfrac{S_d^{\,2-\beta}}{(2-\beta)\left(\theta S_d^{\,1-\beta} + \alpha\right)} \right] \\ -c\left[Q - S_d + \dfrac{\alpha S_d}{\left(\theta S_d^{\,1-\beta} + \alpha\right)} \right] \end{array} \right\}$$

Our problem is to determine the optimal order quantity Q^* which minimizes the total variable cost function per unit time $TVC(Q)$. In this problem, we differentiate $TVC(Q)$ with respect to Q and set the result to zero. A positive value of Q for which the sufficient condition $\dfrac{d^2\left(TVC(Q)\right)}{dQ^2} > 0$ is satisfied gives a minimum for the total average cost function per unit time $TVC(Q)$.

Assuming that $Q > S_d$, in order for the model to represent delayed deterioration situation, the optimal order quantity Q^* can now be obtained as follows:

$$\therefore \frac{d(TVC)}{dQ} = C_0 \frac{d}{dQ}\left(\frac{1}{T}\right) + c\frac{d}{dQ}\left(\frac{Q}{T}\right) + \mu_1 c\frac{d}{dQ}\left(\frac{M}{T}\right) - c\frac{d}{dQ}\left(\frac{W}{T}\right)$$

$$= -\left\{ C_0\alpha(1-\beta)^2\left(\theta S_d^{\,1-\beta} + \alpha\right)^2 Q^{-\beta} \right\} / V^2$$

$$+ c\left\{ \alpha\beta(1-\beta)\left(\theta S_d^{\,1-\beta} + \alpha\right)^2 Q^{1-\beta} - \alpha\theta(1-\beta)\left(\theta S_d^{\,1-\beta} + \alpha\right) S_d^{\,2(1-\beta)} \right\} / V^2$$

$$+ \mu_1 c\frac{1-\beta}{2-\beta}\left\{ \left(\theta S_d^{\,1-\beta} + \alpha\right)^2 Q^{2(1-\beta)} - \theta(2-\beta) S_d^{\,2(1-\beta)}\left(\theta S_d^{\,1-\beta} + \alpha\right) Q^{1-\beta} \right\} / V^2$$

$$+\mu_1 c\theta\frac{1-\beta}{2-\beta}\left\{(1-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{3-2\beta}Q^{-\beta}\right\}/V^2$$

$$-c\left\{\alpha\beta(1-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)Q^{1-\beta}+\alpha\theta(1-\beta)^2\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2-\beta}Q^{-\beta}\right\}/V^2$$

$$+c\alpha\theta(1-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2(1-\beta)}/V^2$$

where,

$$V=\left[\left(\theta S_d^{1-\beta}+\alpha\right)Q^{1-\beta}-\theta S_d^{2(1-\beta)}\right]$$

Setting the derivative to zero and simplifying the terms in powers of Q, we obtain

$$\mu_1 c(1-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2 Q^{2(1-\beta)}$$

$$+\left\{\begin{array}{l} c\alpha\beta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2-\alpha\beta c(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right) \\ -\mu_1 c\theta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2(1-\beta)} \end{array}\right\}Q^{1-\beta}$$

$$+\left\{\begin{array}{l} \mu_1 c\theta(1-\beta)^2\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{3-2\beta} \\ -\alpha C_0(1-\beta)^2(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2-\alpha c\theta(1-\beta)^2(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2-\beta} \end{array}\right\}Q^{-\beta}$$

$$+\alpha c\theta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2(1-\beta)}-c\alpha\theta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2(1-\beta)}=0$$

or

$$\mu_1 c(1-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2 Q^{2-\beta}$$

$$+\left\{\begin{array}{l} c\alpha\beta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2-\alpha\beta c(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right) \\ -\mu_1 c\theta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2(1-\beta)} \end{array}\right\} Q$$

$$+\left\{\begin{array}{l} \mu_1 c\theta(1-\beta)^2\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{3-2\beta} \\ -\alpha C_0(1-\beta)^2(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2 \\ -\alpha c\theta(1-\beta)^2(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2-\beta} \end{array}\right\} = 0$$

Thus,

$$E_1 Q^{2-\beta}+E_2 Q+E_3 = 0 \qquad (3.3.15)$$

where

$$E_1 = \mu_1 c(1-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2$$

$$E_2 = \left\{\begin{array}{l} c\alpha\beta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2 \\ -\alpha\beta c(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right) \\ -\mu_1 c\theta(1-\beta)(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)S_d^{2(1-\beta)} \end{array}\right\}$$

$$E_3 = \left\{ \begin{array}{l} \mu_1 c\theta(1-\beta)^2 \left(\theta S_d^{1-\beta} + \alpha \right) S_d^{3-2\beta} \\ -\alpha C_0 (1-\beta)^2 (2-\beta) \left(\theta S_d^{1-\beta} + \alpha \right)^2 \\ -\alpha c\theta(1-\beta)^2 (2-\beta) \left(\theta S_d^{1-\beta} + \alpha \right) S_d^{2-\beta} \end{array} \right\}$$

Equation (3.3.15) is highly nonlinear and therefore difficult to solve by any analytic method. We may therefore evaluate the roots of the equation in the range $Q > S_d$ by using any suitable numerical approximation method. In our case, we used Newton – Raphson method. However, in our examples we used *Direct Search Method* to show the convexity of the cost function in those examples in which the sufficient condition $\left(\frac{d^2(TVC)}{dQ^2} \right)_{Q=Q^*} > 0$ is satisfied, implying that the Q* so determined is indeed the unique optimal solution in the considered region.

3.4 Numerical Examples

To illustrate the minimization control model developed from equation (3.3.15), the following numerical examples in Table 3.1 using different values of parameters have been prescribed arbitrarily to obtain optimal ordering quantity Q^* and optimal cycle length T*. Using the approximation method of Newton-Raphson, Q^* and T* were determined from equations (3.3.15) and (3.3.14) respectively for each of the eight examples.

Table 3.1: Some examples to illustrate the model developed

S/No.	α	β	θ	μ_1	C ₦/unit	C_0 ₦/Order	S_d Units	Q* Units	T* Cycle Time
1	1000	0.04	0.3	0.03	80	60	120	98.32	0.08
2	900	0.05	0.7	0.05	70	150	110	162.77	2.53
3	800	0.06	0.4	0.06	70	40	100	64.40	0.65
4	600	0.05	0.5	0.04	60	30	90	101.29	1.35
5	500	0.10	0.8	0.02	20	30	200	294.02	5.01
6	400	0.03	0.6	0.02	50	200	80	311.24	5.34
7	100	0.02	0.7	0.01	40	300	70	628.62	11.31
8	50	0.01	0.8	0.01	30	100	60	558.33	9.99

In case $Q^* < S_d$ as can be seen in examples 1 and 3 in the above table, we need not use Q^* from the model, since in that case, the problem is infeasible. It follows that, the model represented by equation (3.3.15) in that case, no longer represents delayed deterioration model for item having this type of stock dependent demand rate function. We may then change the existing assumptions and apply the conventional models where there is no deterioration.

3.5 Concluding Remarks

In this chapter, a model has been presented of an inventory system for delayed deteriorating items with stock-dependent demand rate which is in polynomial functional form. There is terminal condition of zero inventory at the end of the cycle time period. This model helps to determine the optimum ordering quantity and optimum ordering cycle for situations where shortages are not permitted and the replenishment rate is

infinite. Some numerical examples are presented to illustrate the application of the model. The model so developed is highly non-linear and therefore, approximation solution method of Newton- Raphson is used to evaluate the optimal solution for different values of parameters. As expected, by the results shown in Table 3.1, indicate that deterioration rate θ and stock-dependent parameter β have impact on the optimal solutions and hence, should not be neglected in the modelling of deteriorating inventory models with inventory dependent demand rate of this nature.

CHAPTER FOUR

AN EOQ MODEL FOR DELAYED DETERIORATING ITEMS WITH INVENTORY LEVEL DEPENDENT DEMAND RATE AND SHORTAGES

4.1 Introduction

In practice, when shortages occur, some customers are willing to wait for backorder while others are impatient to wait and therefore would turn to buy from other competitors or buy alternative item. For inventory models with stock dependent consumption rate, some authors assumed shortages to be completely backlogged while others assume the shortages to be partially backlogged.

In this chapter, we amend the model in chapter three where shortages are not allowed and the demand rate function is a power function of inventory level by proposing an EOQ model for delayed deteriorating items with linear inventory level dependent demand rate function and partial backlogging rate. We assume a fixed fraction of demand rate during the shortage period to be the backlogging rate as in Wee (1995). In the initial stage, inventory depletes down to a certain level of the inventory due to market demand only but dependent on stock level. In the second stage the inventory level gets depleted due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero. In the final phase of the cycle, shortages are allowed and the unsatisfied demand is partially backlogged at a rate which is a fixed fraction of demand rate during the shortage period. Theoretical results for the optimal replenishment policy of the inventory system in order to minimize the total average system cost per unit time are established. Furthermore, some numerical

examples are presented to illustrate the application of the model developed and then use the examples to study the effects of various changes in some model parameters on the decision variables.

4.2 Modelling Assumptions

The inventory system in this case is based on the same assumptions as in model assumptions in section 3.2 except that the demand rate function is a linear stock dependent demand rate function and shortages are allowed. These exceptional assumptions are made as follows:

1. The inventory level dependent demand rate $D\left[v(t)\right]$ of the product at any instant of time t, is assumed a linearly dependent demand rate function (Mandal and Phaudjar 1989a) such that $D\left[v(t)\right] = \alpha + \beta v(t)$, where α is a positive constant, β is the stock dependent demand rate parameter, $0 < \beta < 1$, and $v(t)$ is the inventory level at time *t*.

2. Shortages are allowed and partially backlogged. We adopt the concept used in Wee (1995), where the unsatisfied demand is backlogged, and the partial backlogging rate parameter is $\delta\,(0 < \delta < 1)$. The extreme cases $\delta = 0$ and $\delta = 1$ represent the scenarios of no shortage allowed and complete backlogging, respectively.

4.3 Mathematical Model and Analysis

Using the above assumptions, a typical inventory cycle for the variation of inventory level with time where the shortages are allowed is shown in figure 4.1.

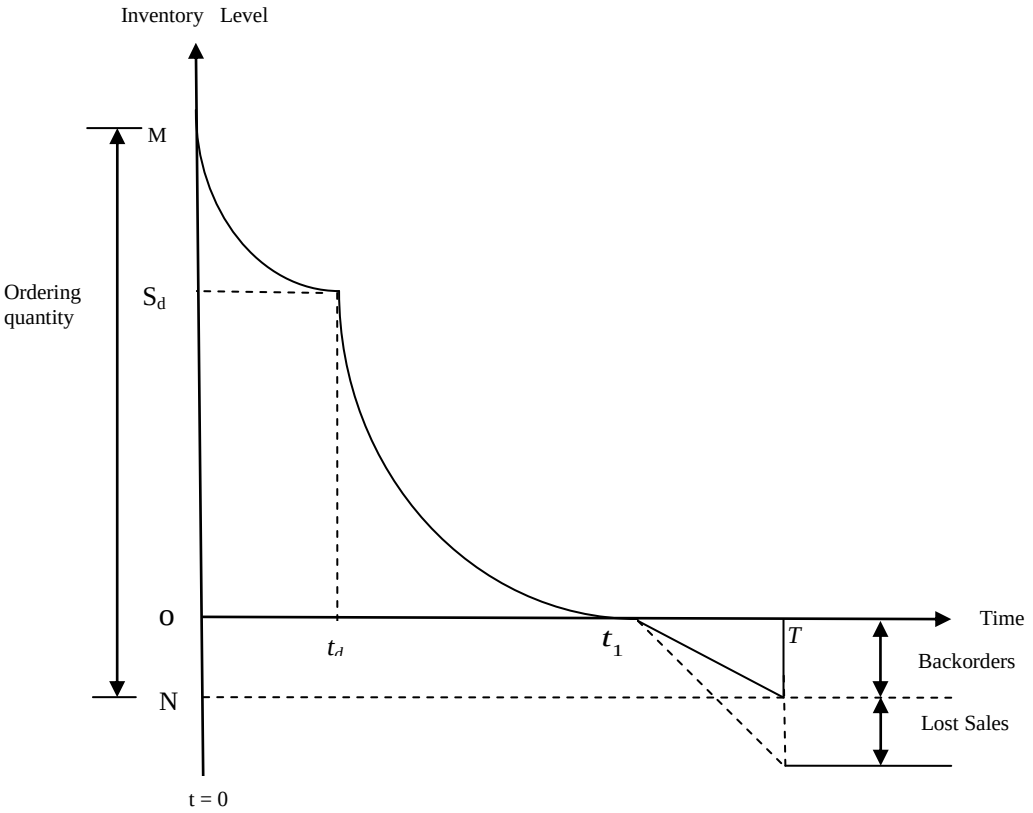


Figure 4.1: The graph of the inventory system represented by the equations (4.3.1), (4.3.3) and (4.3.5)

The inventory level dependent demand rate $D\left[v(t)\right]$ of the product at any instant of time t is a linear function of $v(t)$ and is given by $D\left[v(t)\right] = \alpha + \beta v(t)$. An order of size M units of item is received in the inventory at the beginning of each cycle. During the time interval $[0, t_d]$, the on-hand inventory level gradually depletes from M to the level S_d at time $t = t_d$, due to market demand only. The inventory level further depletes from S_d to zero due to market demand and deterioration during the time interval $[t_d, t_1]$. Then during the time interval $[t_1, T]$ shortages set in and unsatisfied demand is backlogged at a rate δ of the demand rate. The whole process is then repeated.

The inventory level at any time t in the interval $[0, t_d]$ can be represented by the following differential equation:

$$\frac{dv_1(t)}{dt} = -\left(\alpha + \beta v_1(t)\right), \quad 0 \le t \le t_d, \tag{4.3.1}$$

with the initial condition $v_1(0) = \mathrm{M}$.

Solution of equation (4.3.1) is obtained as follows:

$\frac{dv_1(t)}{dt} + \beta v_1(t) = -\alpha$ is a linear differential equation in $v_1(t)$ whose integrating factor is $e^{\beta t}$

so that

$v_1(t)e^{\beta t} = -\frac{\alpha}{\beta}e^{\beta t} + k_1$ where, k_1 is a constant.

With the initial condition $v_1(0) = M$, we have $k_1 = M + \frac{\alpha}{\beta}$

and

$$v_1(t) = Me^{-\beta t} - \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right) \tag{4.3.2}$$

During the time interval $[t_d, t_1]$ the product depletes owing to the effect of both market demand and deterioration. Given that θ is a constant fraction of the on-hand inventory that gets deteriorated per unit time, the differential equation governing the instantaneous state of $v_2(t)$ over the time period in $[t_d, t_1]$ is given by

$$\frac{dv_2(t)}{dt} + \theta v_2(t) = -(\alpha + \beta v_2(t)), \quad t_d \le t \le t_1 \tag{4.3.3}$$

with boundary condition $v_2(t_1) = 0$.

Solution of equation (4.3.3) is obtained as follows:

$\frac{dv_2(t)}{dt} + (\beta + \theta) v_2(t) = -\alpha$ is also a linear differential equation in $v_2(t)$ whose integrating factor is $e^{(\beta+\theta)t}$

so that

$v_2(t) e^{(\beta+\theta)t} = -\frac{\alpha}{\beta+\theta} e^{(\beta+\theta)t} + k_2$ where, k_2 is a constant of integration.

With the boundary condition $v_2(t_1) = 0$, we have $k_2 = \frac{\alpha}{\beta+\theta} e^{(\beta+\theta)t_1}$

and

$$v_2(t) = \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t)} - 1\right) \tag{4.3.4}$$

During the shortage interval $[t_1, T]$, the demand at time t is partially backlogged at a fixed fraction $\delta\,(0<\delta<1)$ of the demand rate. Thus, the backorder level is governed by the differential equation:

$\frac{dB(t)}{dt} = -\alpha\delta,$ since there is no stock on which demand depends upon after t_1,

and so

$$\frac{dB(t)}{dt} = -\alpha\delta,\ t_1 \le t \le T \tag{4.3.5}$$

Solution of equation (4.3.5) is obtained as follows:

$B(t) = -\alpha\delta t + k_3$, where k_3 is a constant.

With the boundary condition $B(t_1) = 0$, we have

$$k_3 = \alpha\delta t_1$$

$$\therefore \quad B(t) = -\alpha\delta(t - t_1) \tag{4.3.6}$$

Since $v(t)$ must be continuous at $t = t_d$, then from equations (4.3.2) and (4.3.4), we get

$$Me^{-\beta t_d} - \frac{\alpha}{\beta}\left(1 - e^{-\beta t_d}\right) = \frac{\alpha}{\beta + \theta}\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)$$

or

$$M = \frac{\alpha}{\beta + \theta}\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)e^{\beta t_d} + \frac{\alpha}{\beta}\left(e^{\beta t_d} - 1\right) \tag{4.3.7}$$

Substituting equation (4.3.7) into equation (4.3.2), we obtain

$$v_1(t) = \left[\frac{\alpha}{\beta + \theta}\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)e^{\beta t_d} + \frac{\alpha}{\beta}\left(e^{\beta t_d} - 1\right)\right]e^{-\beta t} - \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

$$= \left[\frac{\alpha}{\beta + \theta}\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)e^{\beta(t_d - t)} + \frac{\alpha}{\beta}\left(e^{\beta(t_d - t)} - e^{-\beta t}\right)\right] - \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

$$= \left[\frac{\alpha}{\beta + \theta}\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)e^{-\beta(t - t_d)} + \frac{\alpha}{\beta}\left(e^{-\beta(t - t_d)} - e^{-\beta t}\right)\right] - \frac{\alpha}{\beta}\left(1 - e^{-\beta t}\right)$$

(4.3.8)

Substituting $t = T$ in equation (4.3.6), we obtain the maximum amount of demand backlogged per cycle which is given by

$$N = -B(T)$$

$$= \alpha\delta(T - t_1) \tag{4.3.9}$$

Combining equations (4.3.7) and (4.3.9), we obtain the order quantity, Q, as

$$Q = M + N$$

$$= \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)e^{\beta t_d} + \frac{\alpha}{\beta}\left(e^{\beta t_d}-1\right) + \alpha\delta\left(T-t_1\right) \qquad (4.3.10)$$

4.3.1 Total inventory during the complete cycle time T

This is given by

$$C_h = \int_0^{t_d} v_1(t)\,dt + \int_{t_d}^{t_1} v_2(t)\,dt$$

$$= \int_0^{t_d}\left(\frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)e^{-\beta(t-t_d)} + \frac{\alpha}{\beta}\left(e^{-\beta(t-t_d)}-e^{-\beta t}\right) - \frac{\alpha}{\beta}\left(1-e^{-\beta t}\right)\right)dt$$

$$+ \frac{\alpha}{\beta+\theta}\int_{t_d}^{t_1}\left(e^{(\beta+\theta)(t_1-t)}-1\right)dt$$

$$= \left[\frac{\alpha}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)e^{-\beta(t-t_d)} - \frac{\alpha}{\beta^2}\left(e^{-\beta(t-t_d)}-e^{-\beta t}\right) - \frac{\alpha}{\beta}\left(t+\frac{e^{-\beta t}}{\beta}\right)\right]_0^{t_d}$$

$$- \frac{\alpha}{\beta+\theta}\left[t+\frac{e^{(\beta+\theta)(t_1-t)}}{\beta+\theta}\right]_{t_d}^{t_1}$$

$$= \left[\frac{\alpha}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)-\frac{\alpha}{\beta^2}-\frac{\alpha t_d}{\beta}\right] - \left[\frac{\alpha e^{\beta t_d}}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)-\frac{\alpha e^{\beta t_d}}{\beta^2}\right]$$

$$-\frac{\alpha}{\beta+\theta}\left[\left(t_1+\frac{1}{\beta+\theta}\right)-\left(t_d+\frac{e^{(\beta+\theta)(t_1-t_d)}}{\beta+\theta}\right)\right]$$

$$= \frac{\alpha}{\beta(\beta+\theta)}\left[1-e^{(\beta+\theta)(t_1-t_d)}-e^{\beta t_d}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)\right]-\left[\frac{\alpha}{\beta^2}+\frac{\alpha t_d}{\beta}-\frac{\alpha e^{\beta t_d}}{\beta^2}\right]$$

$$-\frac{\alpha}{\beta+\theta}\left[\left(t_1+\frac{1}{\beta+\theta}\right)-\left(t_d+\frac{e^{(\beta+\theta)(t_1-t_d)}}{\beta+\theta}\right)\right]$$

$$= \frac{\alpha}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)\left(1-e^{\beta t_d}\right)+\frac{\alpha}{\beta^2}\left(e^{\beta t_d}-\beta t_d-1\right)$$

$$+\frac{\alpha}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right) \qquad (4.3.11)$$

4.3.2 Total amount of items which deteriorate during the cycle time

This is given by

$$C_{\text{det}} = v_2(t_d) - \text{demand during the time interval } [t_d,\ t_1]$$

$$= \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-\left(\int_{t_d}^{t_1}\alpha\, dt + \frac{\alpha\beta}{\beta+\theta}\int_{t_d}^{t_1}\left(e^{(\beta+\theta)(t_1-t)}-1\right)dt\right)$$

$$= \frac{\alpha}{\beta+\theta}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-\alpha(t_1-t_d)-\frac{\alpha\beta}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right)$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{(\beta+\theta)\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-(\beta+\theta)^2(t_1-t_d)-\beta\left(e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right)\right\}$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{\theta e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)-(\beta+\theta)^2(t_1-t_d)+\beta(\beta+\theta)(t_1-t_d)+\beta\right\}$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{\theta e^{(\beta+\theta)(t_1-t_d)}-\theta-(\beta+\theta)(t_1-t_d)\left((\beta+\theta)-\beta\right)\right\}$$

$$= \frac{\alpha}{(\beta+\theta)^2}\left\{\theta e^{(\beta+\theta)(t_1-t_d)}-\theta-\theta(\beta+\theta)(t_1-t_d)\right\}$$

$$= \frac{\alpha\theta}{(\beta+\theta)^2}\left\{e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right\} \qquad (4.3.12)$$

4.3.3 Total amount of backordered items during the cycle time

This is given as

$$C_m = \int_{t_1}^{T}\left(-B(t)\right)dt$$

$$= \alpha\delta\int_{t_1}^{T}(t-t_1)\,dt$$

$$= \alpha\delta\left\{\left(\frac{T^2}{2}-Tt_1\right)-\left(\frac{t_1^2}{2}-t_1^2\right)\right\}$$

$$= \frac{\alpha\delta}{2}(T-t_1)^2 \qquad (4.3.13)$$

4.3.4 Total amount of demand items unsatisfied during the cycle time

This is given by

$$C_{lost} = \int_{t_1}^{T} \alpha(1-\delta)\, dt$$

$$= \alpha(1-\delta)(T - t_1) \tag{4.3.14}$$

4.3.5 Total relevant inventory cost per unit time

This is obtained as:

$$\text{TVC}\ (t_1, T) = \frac{1}{T}\left\{\begin{array}{l} \text{ordering cost per cycle} \\ + \text{inventory holding cost per cycle} \\ + \text{the deterioration cost per cycle} \\ + \text{shortage cost per cycle due to backlog} \\ + \text{opportunity cost per cycle due to lost sales} \end{array}\right\}$$

$$= \frac{1}{T}\left\{C_0 + \mu_1 C_h + \mu_2 C_{\text{det}} + \mu_3 C_m + \mu_4 C_{lost}\right\}$$

$$= \frac{1}{T} C_0 + \frac{1}{T}\mu_1 C_h + \frac{1}{T}\mu_2 C_{\text{det}} + \frac{1}{T}\mu_3 C_m + \frac{1}{T}\mu_4 C_{lost}$$

$$= \frac{C_0}{T} + \frac{\mu_1}{T}\left\{\frac{\alpha}{\beta(\beta+\theta)}\left(1 - e^{(\beta+\theta)(t_1 - t_d)}\right)\left(1 - e^{\beta t_d}\right)\right\}$$

$$+ \frac{\mu_1}{T}\left\{\frac{\alpha}{\beta^2}\left(e^{\beta t_d} - \beta t_d - 1\right)\right\}$$

$$+ \frac{\mu_1}{T}\left\{\frac{\alpha}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1 - t_d)} - (\beta+\theta)(t_1 - t_d) - 1\right)\right\}$$

$$+\frac{\mu_2}{T}\left\{\frac{\alpha\,\theta}{(\beta+\theta)^2}\left(e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right)\right\}$$

$$+\frac{\mu_3}{T}\left\{\frac{\alpha\delta}{2}(T-t_1)^2\right\}+\frac{\mu_4}{T}\left\{\alpha(1-\delta)(T-t_1)\right\}$$

$$=\frac{C_0}{T}+\frac{\mu_1}{T}\left\{\frac{\alpha}{\beta(\beta+\theta)}\left(1-e^{(\beta+\theta)(t_1-t_d)}\right)\left(1-e^{\beta t_d}\right)\right\}$$

$$+\frac{\mu_1}{T}\left\{\frac{\alpha}{\beta^2}\left(e^{\beta t_d}-\beta t_d-1\right)\right\}$$

$$+\frac{\alpha(\mu_1+\theta\mu_2)}{(\beta+\theta)^2 T}\left\{e^{(\beta+\theta)(t_1-t_d)}-(\beta+\theta)(t_1-t_d)-1\right\}$$

$$+\frac{\mu_3}{T}\left\{\frac{\alpha\delta}{2}(T-t_1)^2\right\}+\frac{\mu_4}{T}\left\{\alpha(1-\delta)(T-t_1)\right\} \qquad (4.3.15)$$

4.4 Solution Procedure

Based on the non-linear objective function (4.3.15), *TVC* (t_1, T) is a function of t_1 and T. The necessary conditions for the total system cost per unit time in equation (4.3.15) to be minimized are

$$\frac{\partial}{\partial t_1}\left(TVC(T,t_1)\right)=0 \text{ and } \frac{\partial}{\partial T}\left(TVC(T,t_1)\right)=0 \qquad (4.3.16)$$

The differential equation $\frac{\partial}{\partial t_1}\left(TVC(T,t_1)\right) = 0$ is obtained as follows:

$$\frac{\partial}{\partial t_1}\left(TVC(T,t_1)\right) = -\frac{\alpha\mu_1}{\beta T}\left(1-e^{\beta t_d}\right)e^{(\beta+\theta)(t_1-t_d)} + \frac{\alpha\mu_1}{(\beta+\theta)T}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)$$

$$+\frac{\alpha\mu_2\theta}{(\beta+\theta)T}\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-\frac{\mu_3\alpha\delta}{T}\left(T-t_1\right)-\frac{\mu_4\alpha}{T}\left(1-\delta\right) \quad = 0$$

(4.3.17)

Multiplying both sides of equation (4.3.17) by $T\beta(\beta+\theta)$, simplifying and rearranging the terms, yields,

$$-\alpha\mu_1(\beta+\theta)\left(1-e^{\beta t_d}\right)e^{(\beta+\theta)(t_1-t_d)} + \alpha\mu_1\beta\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)$$

$$+\alpha\mu_2\theta\beta\left(e^{(\beta+\theta)(t_1-t_d)}-1\right)-\mu_3\alpha\delta\beta(\beta+\theta)(T-t_1)-\mu_4\alpha\beta(\beta+\theta)(1-\delta) = 0$$

(4.3.18)

Grouping the terms in (4.3.18), we have

$$\left\{\alpha\beta(\mu_1+\theta\mu_2)-\alpha\mu_1(\beta+\theta)\left(1-e^{\beta t_d}\right)\right\}e^{(\beta+\theta)(t_1-t_d)} - \mu_3\alpha\delta\beta(\beta+\theta)T + \mu_3\alpha\delta\beta(\beta+\theta)t_1$$

$$-\alpha\beta\left\{(\mu_1+\mu_2\theta)+\mu_4(\beta+\theta)(1-\delta)\right\} = 0$$

which can be written as

$$\omega_1 e^{(\beta+\theta)(t_1-t_d)} + \omega_2 T + \omega_3 t_1 + \omega_4 = 0 \qquad (4.3.19)$$

where

$$\omega_1 = \alpha\beta(\mu_1 + \theta\mu_2) - \alpha\mu_1(\beta + \theta)\left(1 - e^{\beta t_d}\right),$$

$$\omega_2 = -\mu_3\alpha\delta\beta(\beta + \theta),$$

$$\omega_3 = \mu_3\alpha\delta\beta(\beta + \theta),$$

$$\omega_4 = -\alpha\beta\{(\mu_1 + \mu_2\theta) + \mu_4(\beta + \theta)(1 - \delta)\}$$

Similarly, $\frac{\partial}{\partial T}(TVC(T, t_1)) = 0$ is obtained as follows:

$$\frac{\partial}{\partial T}(TVC(T, t_1)) = -\frac{C_0}{T^2} - \frac{\mu_1}{T^2}\left\{\frac{\alpha}{\beta(\beta + \theta)}\left(1 - e^{(\beta+\theta)(t_1 - t_d)}\right)\left(1 - e^{\beta t_d}\right) + \frac{\alpha}{\beta^2}\left(e^{\beta t_d} - \beta t_d - 1\right)\right\}$$

$$- \frac{\alpha(\mu_1 + \theta\mu_2)}{(\beta + \theta)^2 T^2}\left\{e^{(\beta+\theta)(t_1 - t_d)} - (\beta + \theta)(t_1 - t_d) - 1\right\}$$

$$+ \frac{\mu_3}{T^2}\left\{\frac{\alpha\delta}{2}\left(T^2 - t_1^2\right)\right\} + \frac{\mu_4}{T^2}\left\{\alpha(1 - \delta)t_1\right\} = 0 \qquad (4.3.20)$$

Multiplying both sides of equation (4.3.20) by $T^2(\beta + \theta)^2\beta^2$, we have,

$$-C_0(\beta + \theta)^2\beta^2 - \mu_1\left\{\alpha\beta(\beta + \theta)\left(e^{(\beta+\theta)(t_1 - t_d)} - 1\right)\left(e^{\beta t_d} - 1\right) + \alpha(\beta + \theta)^2\left(e^{\beta t_d} - \beta t_d - 1\right)\right\}$$

$$- \alpha\beta^2(\mu_1 + \theta\mu_2)\left\{e^{(\beta+\theta)(t_1-t_d)} - (\beta+\theta)(t_1-t_d) - 1\right\}$$

$$+\mu_3\beta^2(\beta+\theta)^2\frac{\alpha\delta}{2}(T^2 - t_1^2) + \mu_4\beta^2(\beta+\theta)^2\alpha(1-\delta)t_1 = 0 \qquad (4.3.21)$$

Simplifying and rearranging the terms in (4.3.21), we get

$$\left\{\alpha\mu_1\beta(\beta+\theta)(1-e^{\beta t_d}) - \alpha\beta^2(\mu_1+\theta\mu_2)\right\}e^{(\beta+\theta)(t_1-t_d)}$$

$$+\mu_3\beta^2(\beta+\theta)^2\frac{\alpha\delta}{2}T^2 - \mu_3\beta^2(\beta+\theta)^2\frac{\alpha\delta}{2}t_1^2$$

$$+\left\{\alpha\beta^2(\beta+\theta)(\mu_1+\theta\mu_2) + \mu_4\beta^2(\beta+\theta)^2\alpha(1-\delta)\right\}t_1$$

$$-\mu_1\left\{\alpha\beta(\beta+\theta)(1 - e^{\beta t_d}) + \alpha(\beta+\theta)^2(e^{\beta t_d} - \beta t_d - 1)\right\}$$

$$- \alpha\beta^2(\mu_1+\theta\mu_2)\{(\beta+\theta)t_d - 1\} - C_0(\beta+\theta)^2\beta^2 = 0 \qquad (4.3.22)$$

which also reduces to

$$\Upsilon_1 e^{(\beta+\theta)(t_1-t_d)} + \Upsilon_2 T^2 + \Upsilon_3 t_1^2 + \Upsilon_4 t_1 + \Upsilon_5 = 0 \qquad (4.3.23)$$

where

$$\Upsilon_1 = \alpha\mu_1\beta(\beta+\theta)\left(1-e^{\beta t_d}\right) - \alpha\beta^2(\mu_1+\theta\mu_2),$$

$$\Upsilon_2 = \mu_3\beta^2(\beta+\theta)^2\frac{\alpha\delta}{2},$$

$$\Upsilon_3 = -\mu_3\beta^2(\beta+\theta)^2\frac{\alpha\delta}{2}$$

$$\Upsilon_4 = \alpha\beta^2(\beta+\theta)(\mu_1+\theta\mu_2) + \mu_4\beta^2(\beta+\theta)^2\alpha(1-\delta),$$

$$\Upsilon_5 = -\mu_1\left\{\alpha\beta(\beta+\theta)\left(1 - e^{\beta t_d}\right) + \alpha(\beta+\theta)^2\left(e^{\beta t_d} - \beta t_d - 1\right)\right\}$$

$$- \alpha\beta^2(\mu_1+\theta\mu_2)\left\{(\beta+\theta)t_d - 1\right\} - C_0(\beta+\theta)^2\beta^2$$

Solving the two non-linear equations (4.3.19) and (4.3.23) simultaneously will give the optimum values t_1* of t and T* of T provided these values t_1* and T* satisfy the conditions

$$\frac{\partial^2 TVC(T,t_1)}{\partial T^2} > 0 \qquad (4.3.24)$$

and

$$\det\left[H_m(T,t_1)\right] > 0, \qquad (4.3.25)$$

where

$\det\left[H_m(T,t_1)\right]$, is the determinant of the Hessian matrix given by

$$H_m(T,t_1) = \begin{bmatrix} \frac{\partial^2 TVC(T,t_1)}{\partial T^2} & \frac{\partial^2 TVC(T,t_1)}{\partial T \partial t_1} \\ \frac{\partial^2 TVC(T,t_1)}{\partial t_1 \partial T} & \frac{\partial^2 TVC(T,t_1)}{\partial t_1^2} \end{bmatrix}$$

It is not however easy to get the solution of equations (4.3.19) and (4.3.23) by an analytic method. Hence, there is need for an approximation solution method.

From equation (4.3.19), we have

$$T = -\frac{1}{\omega_2}\left(\omega_1 e^{(\beta+\theta)(t_1-t_d)} + \omega_3 t_1 + \omega_4\right) \quad (4.3.26)$$

Substituting this value into (3.3.23), we get an equation in a single variable t_1 as,

$$\Upsilon_1 e^{(\beta+\theta)(t_1-t_d)} + \frac{\Upsilon_2}{(\omega_2)^2}\left(\omega_1 e^{(\beta+\theta)(t_1-t_d)} + \omega_3 t_1 + \omega_4\right)^2 + \Upsilon_3 t_1^2 + \Upsilon_4 t_1 + \Upsilon_5 = 0 \quad (4.3.27)$$

We can now use equation (4.3.27) to solve for t_1 using Newton-Raphson method and then substitute the solution obtained from (4.3.27) into (4.3.26) to compute T. These solutions T* and t_1* will now jointly make the optimal solution of (4.3.19) and (4.3.23) provided equations (4.3.24) and (4.3.25) are satisfied. By using the optimal values of t_1* and T* in (4.3.15) and (4.3.10), optimal minimum system cost per unit time $TVC\left(t_1^*, T*\right)$ and optimal order quantity Q* respectively are obtained.

4.5 Numerical Examples

Four tables (Table 4.1, Table 4.2, Table 4.3 and Table 4.4) of examples are presented to illustrate the model developed and the effects of changes of some model parameters on the decision variables. From Table 4.1, an inventory system in the first example has the following values of parameters: $\alpha = 180$, $\beta = 0.7$, $\theta = 0.2$, $\delta = 0.2$, C_o = ₦ 80 per order, μ_1 = ₦ 0.5 per unit, μ_2 = ₦ 2 per unit, μ_3 = ₦ 2.5 per unit, μ_4 = ₦ 1.2 per unit, $t_d = 0.1$ time units. By using Newton-Raphson method, the optimal length of time with positive inventory t_1* is obtained using expression (4.3.27) as $t^* = 0.724$ units, the optimum cycle length of time T* is determined using expression (4.3.26) as $T^* = 1.028$ units and corresponding minimum total system cost per unit time is determined using expression (4.3.15) as TVC* = ₦153.04 while the optimal order quantity per cycle is determined using expression (4.3.10) as Q* = 224.12 units. For other values of the parameters as reflected in the table, the optimal length of time with positive inventory t_1*, the optimum cycle length of time T*, the corresponding minimum total system cost per unit time TVC*, the optimal order quantity per cycle Q* are similarly determined. The results are tabulated in Table 4.1. Also, Table 4.2, Table 4.3 and Table 4.4 are similarly drawn up, with changes as indicated in the heading of the table.

Note that in all examples given, the sufficient conditions expressed by equations (4.3.24) and (4.3.25) are satisfied using *direct search method* implying that t_1* and T* so determined are indeed the minimum values in the considered region.

Table 4.1: Effect of changing the constant demand rate α on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	T*	Q*	TVC*
1	180	0.7	0.2	0.2	0.5	2	2.5	1.2	80	0.1	0.724	1.028	224.12	153.04
2	185	0.7	0.2	0.2	0.5	2	2.5	1.2	80	0.1	0.718	1.015	227.01	154.67
3	190	0.7	0.2	0.2	0.5	2	2.5	1.2	80	0.1	0.711	1.002	229.86	156.28

Table 4.2: Effect of changing the stock-dependent demand rate β on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	T*	Q*	TVC*
1	150	0.3	0.7	0.15	0.5	2	2.5	1.2	120	0.08	1.083	1.501	251.73	182.94
2	150	0.4	0.7	0.15	0.5	2	2.5	1.2	120	0.06	1.047	1.472	255.40	184.65
3	150	0.5	0.7	0.15	0.5	2	2.5	1.2	120	0.06	1.013	1.446	258.96	186.30

Table 4.3: Effect of changing the deterioration rate θ on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	T*	Q*	TVC*
1	160	0.8	0.9	0.15	0.7	2	2.5	1.2	80	0.08	0.680	1.040	203.67	150.34
2	160	0.8	0.9	0.20	0.7	2	2.5	1.2	80	0.08	0.643	1.013	196.16	153.26
3	160	0.8	0.9	0.25	0.7	2	2.5	1.2	80	0.08	0.609	0.990	189.69	155.81

Table 4.4: Effect of changing the backlogging rate δ on the decision variables

S/No	α	β	δ	θ	μ_1	μ_2	μ_3	μ_4	C_o	t_d	t_1*	T*	Q*	TVC*
1	180	0.8	0.5	0.1	0.5	1.5	2	1.7	80	0.08	0.913	0.958	242.92	152.06
2	180	0.8	0.55	0.1	0.5	1.5	2	1.7	80	0.08	0.909	1.021	247.89	151.02
3	180	0.8	0.6	0.1	0.5	1.5	2	1.7	80	0.08	0.902	1.066	251.49	149.57

4.6 Results and Discussion

A careful study of the computational results as shown in Tables 4.1 – 4.4, and within the range of values of the chosen parameters, reveals the following observations:

(1) From Table 4.1, a higher value of α results in higher values of Q* and TVC*, but lower values of t_1* and T*. This implies that increase in the demand rate will result in the increase of the optimal order quantity per cycle Q* and the minimum total system cost per unit time TVC*, but decrease in the length of time with positive inventory t_1* and the optimum cycle length of time T*. This is expected since if demand rate is higher, stock will take less time to finish and so t_1* and T* decrease. Increasing the total demand will, also in turn, increase the order quantity and the overall system cost per unit time.

(2) From Table 4.2, a higher value of β results in higher values of Q* and TVC*, but lower values of t_1* and T*. This implies that increase in the stock-dependent demand rate will result in the increase of the optimal order quantity per cycle and the minimum total system cost per unit time, but decrease in the length of time with positive inventory and the optimum cycle length of time. This is expected since if stock dependent demand rate is higher, stock will finish earlier and so T* will decrease. On the other hand, increasing the stock- dependent demand parameter β will effectively increase the total demand which, in turn, will increase the overall system cost per unit time.

(3) From Table 4.3, higher values of θ result in higher values of TVC*, but lower values of t_1*, T* and Q*. In this case, it implies that increase in deterioration rate will result in increase of the minimum total system cost per unit time, but decrease in the

optimal length of time with positive inventory, optimum cycle length of time and optimal order quantity per cycle. The total variable cost per unit time, however, increases in this case, which is also expected since when deterioration cost increases, the total variable cost per unit time will also increase. With deterioration rate higher however, stock will finish earlier resulting in lower T*.

(4) From Table 4.4, higher values of δ result in higher values T* and Q*, but lower values of t_1* and TVC*. In this case, it implies that increase in backlogging rate will result in increase of the optimum cycle length of time and the optimal order quantity per cycle and , but decrease in the values of the optimal length of time with positive inventory and the optimal minimum total system cost per unit time. The total variable cost per unit time in this case is certainly expected to decrease due to zero stocking cost. It is also expected that increasing the backlogging rate δ increases the order quantity Q.

4.7 Concluding Remarks

In this chapter, an EOQ model for delayed deteriorating items with stock-dependent demand, allowing shortages and fixed backlogging rate is presented. The impact of stock dependent demand rate, constant rate of deterioration and partial backlogging parameters on order quantity, maximum inventory level and total system cost per unit time are reported. We present some numerical examples to illustrate the application of the model developed and use the examples to study the effect of various changes in some possible combinations of model parameters on the decision variables of the system. We find from the above results that the effects of stock dependent demand rate, deterioration and backlogging rate on the optimal replenishment policy are significant, and hence should not be ignored in developing inventory models. The proposed model

can be used in inventory control of certain delayed deteriorating items such as food items, electronic components, fashionable commodities, vegetables, fruits, yams, potatoes, and so on. The model we have presented in this study provides a basis for several possible extensions. For future research, the model can be enriched by incorporating inflation, time value of money, variable holding cost, and so on.

CHAPTER FIVE

AN EPQ MODEL FOR DELAYED DETERIORATING ITEMS WITH STOCK-DEPENDENT DEMAND RATE AND LINEAR TIME - DEPENDENT HOLDING COST WITH TIME- DEPENDENT DETERIORATION RATE

5.1 Introduction

In this chapter, two economic production quantity (EPQ) models are presented. In both cases, there is a delay in deterioration and the production rate is constant, demand rate is inventory level dependent in a linear functional form, both before and after production. In the first model, the holding cost is a linear function of time and the deterioration rate is a constant while in the second model, the holding cost is a constant but the deterioration rate is time dependent.

5.2 Time Dependent Holding Cost Model

In this section, we present the first model which is an economic production quantity (EPQ) model for single product with delayed deterioration and in which the production rate is constant, demand rate is inventory level dependent in a linear functional form, both before and after production. The holding cost is a linear function of time while the deterioration rate is a constant. The main objective is to determine the optimal replenishment cycle time such that the total variable cost is minimised. Numerical examples are provided to illustrate the theoretical results.

5.2.1 Modelling assumptions

This model is also based on the same assumptions as in model of chapter three except that of instantaneous replenishment, polynomial functional form of demand and constant holding cost.

(1) In many real inventory situations, the amount ordered is not delivered all at once, but available at a finite rate per unit of time. Thus order is either produced internally or will be supplied at some constant rate.

(2) The linear demand rate function $D\left[q(t)\right]$ given by $D\left[q(t)\right] = \alpha + \beta q(t)$ is considered rather than power form function $D\left[q(t)\right]$ given by $D\left[q(t)\right] = \alpha\left(q(t)\right)^{\beta}$.

(3) Inventory holding cost per unit time increases linearly with time. This inventory holding cost function per period is given by $C_h(t) = \delta_1 + \delta_2 t$, where δ_1 and δ_2 are positive constants.

The additional assumptions made are as follows:

(4) Production rate λ is known and constant and is greater than the demand rate.

(5) Once the product starts deteriorating, the production is terminated, thus delay in deterioration during production period.

5.2.2 Model development

The objective of the inventory problem here is to determine the optimal production quantity so as to keep the total relevant cost as low as possible.

At start t = 0, the inventory level is zero. The production and supply start simultaneously and the production stops at $t = T_1$ during which the maximum inventory V_m is reached. The demand rate during production is assumed to be $\alpha + \beta q_1(t), \text{where}, 0 < \beta < 1,$ and

so the inventory built up is at the rate $\left(\lambda - \left(\alpha + \beta q_1(t)\right)\right) > 0$ in the interval [0, T_1], and in that interval there is no deterioration. After the time T_1, the produced units start to deteriorate significantly. There is no fall in demand, when the inventory reduces to zero level and production run begins. The inventory level of the product at time t over period [0, *T*] can be represented by the following systems of differential equations:

$$\frac{dq_1(t)}{dt} = \lambda - \left(\ \alpha + \beta q_1(t)\right),\ for\ 0 \le t \le T_1 \qquad (5.2.1)$$

$$\frac{dq_2(t)}{dt} + \theta q_2(t)\ =\ -\left(\ \alpha + \beta q_2(t)\right),\ for\ T_1 \le t \le T \Leftrightarrow 0 \le t \le T_2 \qquad (5.2.2)$$

That is the demand rate after production is $\alpha + \beta q_2(t)$ and the product has then started deteriorating at the rate of θ.

The pictorial representation of the model is given in Figure 5.1

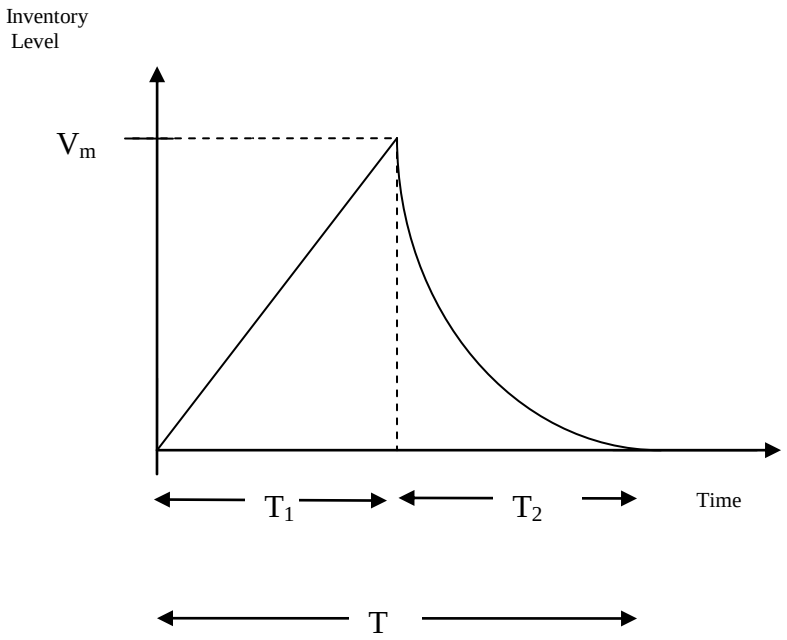


Figure 5.1: The graph of inventory system represented by the equations (5.2.1) and (5.2.2)

On integrating (5.2.1) with respect to t and using the initial condition $q_1(0) = 0$, we have

$$q_1 e^{\beta t} = \frac{(\lambda - \alpha) e^{\beta t}}{\beta} - \frac{(\lambda - \alpha)}{\beta} \quad , \ 0 \le t \le T_1$$

and from which

$$q_1(t) = \frac{(\lambda - \alpha)}{\beta}\left(1 - e^{-\beta t}\right) \tag{5.2.3}$$

Using the Taylor series expansion for exponential function on equation (5.2.3), we have

$$q_1(t) = \left(\frac{\lambda - \alpha}{\beta} \right) \left\{ 1 - \left(1 - \beta t + \frac{(\beta t)^2}{2} ..... \right) \right\}$$

Approximating the value by neglecting those terms in β t (since $0 < \beta < 1$) of degree greater than 2, we have

$$q_1(t) = (\lambda - \alpha) \left(t - \frac{\beta t^2}{2} \right), \qquad 0 \le t \le T_1 \tag{5.2.4}$$

From (5.2.2) we have

$$\frac{dq_2(t)}{dt} + (\theta + \beta) q_2(t) = -\alpha$$

which on integration with respect to t gives

$$q_2(t) = \frac{\alpha}{\theta + \beta} \left[e^{(\theta + \beta)(T_2 - t)} - 1 \right], \quad 0 \le t \le T_2 \quad , \tag{5.2.5}$$

Using the Taylor series expansion for exponential function, we have

$$q_2(t) = \frac{\alpha}{\theta + \beta} \left\{ \left(1 + (\theta + \beta)(T_2 - t) + \frac{\left((\theta + \beta)(T_2 - t) \right)^2}{2} ..... \right) - 1 \right\} ,$$

Approximating the value by neglecting those terms in β t (since $0 < \beta < 1$) of degree greater than 2, we have

$$q_2(t) = \alpha\left\{ (T_2 - t) + \frac{(\theta+\beta)(T_2-t)^2}{2} \right\}, 0 \le t \le T_2 \qquad (5.2.6)$$

5.2.2.1 The holding cost per period

This is given by

$$C_p = \int_0^{T_1} C_h(t)\, q_1(t)dt + \int_0^{T_2} C_h(t)\, q_2(t)dt$$

$$= \int_0^{T_1} (\delta_1 + \delta_2 t)(\lambda - \alpha)\left(t - \frac{\beta t^2}{2} \right) dt + \int_0^{T_2} (\delta_1 + \delta_2 t)\alpha\left\{ (T_2 - t) + \frac{(\theta+\beta)(T_2-t)^2}{2} \right\} dt$$

$$= (\lambda - \alpha)\int_0^{T_1} (\delta_1 + \delta_2 t)\left(t - \frac{\beta t^2}{2} \right) dt + \alpha\int_0^{T_2} (\delta_1 + \delta_2 t)(T_2 - t)dt + \frac{\alpha(\theta+\beta)}{2}\int_0^{T_2} (\delta_1 + \delta_2 t)(T_2 - t)^2\, dt$$

$$= (\lambda - \alpha)\int_0^{T_1}\left(\delta_1 t - \frac{\delta_1\beta t^2}{2} + \delta_2 t^2 - \frac{\delta_2\beta t^3}{2} \right) dt + \alpha\int_0^{T_2}\left(\delta_1 T_2 - \delta_1 t + \delta_2 T_2 t - \delta_2 t^2 \right) dt$$

$$+ \frac{\alpha(\theta+\beta)}{2}\int_0^{T_2}\left(\delta_1 T_2^{\,2} - 2\delta_1 T_2 t + \delta_1 t^2 + \delta_2 T_2^{\,2} t - 2\delta_2 T_2 t^2 + \delta_2 t^3 \right) dt$$

$$= (\lambda - \alpha)\left(\frac{\delta_1 T_1^{\,2}}{2} - \frac{\delta_1\beta T_1^{\,3}}{6} + \frac{\delta_2 T_1^{\,3}}{3} - \frac{\delta_2\beta T_1^{\,4}}{8} \right) + \alpha\left(\frac{\delta_1 T_2^{\,2}}{2} + \frac{\delta_2 T_2^{\,3}}{6} \right)$$

$$+ \frac{\alpha(\theta+\beta)}{2}\left(\frac{\delta_1 T_2^{\,3}}{3} + \frac{\delta_2 T_2^{\,4}}{12} \right)$$

$$= (\lambda - \alpha)\left(\frac{\delta_1 T_1^2}{2} - \frac{\delta_1 \beta T_1^3}{6} + \frac{\delta_2 T_1^3}{3} - \frac{\delta_2 \beta T_1^4}{8} \right) + \alpha\left(\frac{\delta_1 (T - T_1)^2}{2} + \frac{\delta_2 (T - T_1)^3}{6} \right)$$

$$+ \frac{\alpha(\theta + \beta)}{2}\left(\frac{\delta_1 (T - T_1)^3}{3} + \frac{\delta_2 (T - T_1)^4}{12} \right) \tag{5.2.7}$$

5.2.2.2 The demand in the time period T_2

This is given by

$$M_2 = \int_0^{T_2} (\alpha + \beta t)\, dt$$

$$= \alpha T_2 + \frac{\beta T_2^2}{2} \tag{5.2.8}$$

5.2.2.3 The number of Deteriorated items per cycle

This is given by

$$I_d = q_2(0) - M_2$$

$$= \alpha\left\{ T_2 + \frac{(\theta + \beta) T_2^2}{2} \right\} - \left\{ \alpha T_2 + \frac{\beta T_2^2}{2} \right\}$$

$$= \frac{1}{2}(\alpha\theta + \alpha\beta - \beta) T_2^2 \tag{5.2.9}$$

5.2.2.4 *The average total variable inventory cost per unit time, TVC (T),* is given by

TVC(T) = $\frac{1}{T}$ {set up cost (Cs) + cost of deteriorated items (kId) +holding cost(Cp)}

$$= \frac{1}{T}\left(C_s + k\, I_d + C_p \right)$$

$$= \frac{C_s}{T} + \frac{k}{T}(\alpha\theta + \alpha\beta - \beta)\frac{(T - T_1)^2}{2} + \frac{1}{T}(\lambda - \alpha)\left(\frac{\delta_1 T_1^2}{2} - \frac{\delta_1 \beta T_1^3}{6} + \frac{\delta_2 T_1^3}{3} - \frac{\delta_2 \beta T_1^4}{8}\right)$$

$$+ \frac{\alpha}{T}\left(\frac{\delta_1 (T - T_1)^2}{2} + \frac{\delta_2 (T - T_1)^3}{6}\right) + \frac{\alpha(\theta + \beta)}{2T}\left(\frac{\delta_1 (T - T_1)^3}{3} + \frac{\delta_2 (T - T_1)^4}{12}\right) \quad (5.2.10)$$

To minimize the total variable cost per unit time TVC (T), we differentiate it with respect to T and set the result to zero. A positive value of T for which the sufficient condition $\frac{d^2(TVC(T))}{dT^2} > 0$ gives a minimum for the total average cost function per unit time *TVC(T)*.

$$\frac{d}{dT}(TVC(T)) = -C_s\left(\frac{1}{T^2}\right) + k(\alpha\theta + \alpha\beta - \beta)\left(\frac{T^2 - T_1^2}{2T^2}\right)$$

$$- (\lambda - \alpha)\left(\frac{\delta_1 T_1^2}{2} - \frac{\delta_1 \beta T_1^3}{6} + \frac{\delta_2 T_1^3}{3} - \frac{\delta_2 \beta T_1^4}{8}\right)\left(\frac{1}{T^2}\right)$$

$$+ \frac{\alpha\delta_1}{2}\left(1 - \frac{T_1^2}{T^2}\right) + \frac{\alpha\delta_2}{6}\left(2T - 3T_1 + \frac{T_1^3}{T^2}\right) + \left(\frac{\alpha\delta_1(\theta + \beta)}{6}\left(2T - 3T_1 + \frac{T_1^3}{T^2}\right)\right)$$

$$+ \left(\frac{\alpha(\theta + \beta)\delta_2}{24}\left(3T^2 - 8TT_1 + 6T_1^2 - \frac{T_1^4}{T^2}\right)\right)$$

$$= -C_s\left(\frac{1}{T^2}\right) + k(\alpha\theta + \alpha\beta - \beta)\left(\frac{T^2 - T_1^2}{2T^2}\right)$$

$$- (\lambda - \alpha)\left(\frac{\delta_1 T_1^2}{2} - \frac{\delta_1 \beta T_1^3}{6} + \frac{\delta_2 T_1^3}{3} - \frac{\delta_2 \beta T_1^4}{8} \right)\left(\frac{1}{T^2} \right)$$

$$+ \frac{\alpha \delta_1}{2T^2}\left(T^2 - T_1^2 \right) + \frac{\alpha \delta_2}{6T^2}\left(2T^3 - 3T^2 T_1 + T_1^3 \right)$$

$$+ \left(\frac{\alpha \delta_1 (\theta + \beta)}{6T^2}\left(2T^3 - 3T^2 T_1 + T_1^3 \right) \right)$$

$$+ \left(\frac{\alpha (\theta + \beta) \delta_2}{24T^2}\left(3T^4 - 8T^3 T_1 + 6T^2 T_1^2 - T_1^4 \right) \right)$$

Setting it to zero and simplifying, we have

$$\left\{ \begin{array}{l} -C_s\left(\frac{1}{T^2} \right) + k\left(\alpha\theta + \alpha\beta - \beta \right)\left(\frac{T^2 - T_1^2}{2T^2} \right) \\ - (\lambda - \alpha)\left(\frac{\delta_1 T_1^2}{2} - \frac{\delta_1 \beta T_1^3}{6} + \frac{\delta_2 T_1^3}{3} - \frac{\delta_2 \beta T_1^4}{8} \right)\left(\frac{1}{T^2} \right) \\ + \frac{\alpha \delta_1}{2T^2}\left(T^2 - T_1^2 \right) + \frac{\alpha \delta_2}{6T^2}\left(2T^3 - 3T^2 T_1 + T_1^3 \right) \\ + \left(\frac{\alpha \delta_1 (\theta + \beta)}{6T^2}\left(2T^3 - 3T^2 T_1 + T_1^3 \right) \right) \\ + \left(\frac{\alpha (\theta + \beta) \delta_2}{24T^2}\left(3T^4 - 8T^3 T_1 + 6T^2 T_1^2 - T_1^4 \right) \right) \end{array} \right\} = 0$$

or

$$\left\{\begin{array}{l} -24C_s + 12k\left(\alpha\theta + \alpha\beta - \beta\right)\left(T^2 - T_1^2\right) \\ -\left(\lambda - \alpha\right)\left(12\delta_1 T_1^2 - 4\delta_1\beta T_1^3 + 8\delta_2 T_1^3 - 3\delta_2\beta T_1^4\right) \\ +\ 12\alpha\delta_1\left(T^2 - T_1^2\right) + \ 4\alpha\delta_2\left(2T^3 - 3T^2 T_1 + T_1^3\right) \\ +\left(4\alpha\delta_1(\theta + \beta)\left(2T^3 - 3T^2 T_1 + T_1^3\right)\right) \\ +\ \alpha(\theta + \beta)\delta_2\left(3T^4 - 8T^3 T_1 + 6T^2 T_1^2 - T_1^4\right) \end{array}\right\} = 0$$

This simplifies to

$$\left\{\begin{array}{l} 3\alpha\delta_2(\theta + \beta)T^4 + \left\{8\alpha\delta_2 + 8\alpha\delta_1(\theta + \beta) - 8\alpha\delta_2 T_1(\theta + \beta)\right\}T^3 \\ +\left\{12k(\alpha\theta + \alpha\beta - \beta) + 12\alpha\delta_1 - 12\alpha\delta_2 T_1 - 12\alpha\delta_1 T_1(\theta + \beta) + 6\alpha T_1^2\delta_2(\theta + \beta)\right\}T^2 \\ +\left\{-24C_s - 12T_1^2\, k(\alpha\theta + \alpha\beta - \beta) - \left(\lambda - \alpha\right)\left(12\delta_1 T_1^2 - 4\delta_1\beta T_1^3 + 8\delta_2 T_1^3 - 3\delta_2\beta T_1^4\right)\right. \\ \left. -12\alpha\delta_1 T_1^2 + 4\alpha\delta_2 T_1^3 + 4\alpha\delta_1(\theta + \beta)T_1^3 - \alpha\delta_2(\theta + \beta)T_1^4\right\} \end{array}\right\} = 0$$

or

$$\psi_1 T^4 + \ \psi_2 T^3 + \ \psi_3 T^2 + \psi_4 \ = 0 \qquad (5.2.11)$$

where

$$\psi_1 = 3\alpha\delta_2(\theta+\beta)$$

$$\psi_2 = 8\alpha\delta_2 + 8\alpha\delta_1(\theta+\beta) - 8\alpha\delta_2 T_1(\theta+\beta)$$

$$\psi_3 = 12k(\alpha\theta+\alpha\beta-\beta) + 12\alpha\delta_1 - 12\alpha\delta_2 T_1 - 12\alpha\delta_1 T_1(\theta+\beta) + 6\alpha T_1^2\delta_2(\theta+\beta)$$

$$\psi_4 = \Big\{ -24C_s - 12T_1^2\, k(\alpha\theta+\alpha\beta-\beta) - (\lambda-\alpha)\Big(12\delta_1 T_1^2 - 4\delta_1\beta T_1^3 + 8\delta_2 T_1^3 - 3\delta_2\beta T_1^4 \Big)$$

$$-12\alpha\delta_1 T_1^2 + 4\alpha\delta_2 T_1^3 + 4\alpha\delta_1(\theta+\beta)T_1^3 - \alpha\delta_2(\theta+\beta)T_1^4 \Big\}$$

The cost minimization model equation (5.2.11) can be solved numerically by using any suitable numerical method. In our case, we used Newton – Raphson method. However, in our examples we used *Direct Search Method* to show the convexity of the cost function in those examples.

5.2.3 Numerical examples

To illustrate the control model developed by equation (5.2.11), the following five numerical examples using different values of parameters have been considered.

In Table 5.1, an inventory system represented in the first example has the following values of parameters: $\alpha = 75$, $\beta = 0.70$, $\theta = 0.05$, $\delta_1 = 30$, $\delta_2 = 40$, $\lambda = 200$, C_s = ₦ 1200 per order, k = ₦ 15 per unit, $T_1 = 29.20$ time units, the optimum cycle length of time T* is determined by Newton-Raphson method using equation (5.2.11) as $T^* = 81.03$ units and corresponding minimum total system cost per unit time is determined using the equation (5.2.10) as TVC* = ₦5649.81 and optimal production

quantity determined using equation (5.2.4) at $t = T_1$ as $V_m^* = 2.38$. The remaining four examples in the table are similarly drawn up with changes as indicated in the heading of the table. Note that in all examples given, the sufficient condition $\frac{d^2}{dT^2}\left(TVC(T)\right) > 0$ is satisfied implying that T* so determined is indeed the minimum in the considered region.

Table 5.1: Some numerical examples to illustrate the theoretical results

S/N	α	β	θ	δ_1	δ_2	C_s	T_1(days)	λ	k	T (days)	TVC	EPQ (V_m)
1	75	0.70	0.1	30	40	1200	29.20	200	15	81.03	5649.81	2.38
2	50	0.60	0.2	40	50	1100	32.85	180	20	75.56	4659.84	13.65
3	40	0.50	0.05	30	40	1000	36.50	160	25	80.67	4652.28	11.70
4	90	0.04	0.08	60	75	1400	54.75	250	10	75.92	7357.61	20.47
5	80	0.10	0.02	75	95	1300	73.00	220	5	100.01	5690.19	27.72

5.2.4 Concluding Remarks

In this part of the chapter, an economic production quantity (EPQ) model is presented for single product with delayed deterioration in which the production rate is constant, demand rate is inventory level dependent in a linear functional form before and after production and the holding cost is a linear function of time. An analytic formulation of the problem on the framework described above has been provided and also an optimal solution procedure with the help of Newton-Raphson method was used to find the optimal replenishment policy. Some numerical examples are also provided to illustrate the theoretical results. The results as tabulated in Table 5.1 are for various values of inventory level dependent demand rate parameter $(\beta = 0.70, 0.60, 0.50, 0.04, 0.10)$

and deterioration rate $\left(\theta = 0.10, 0.20, 0.05, 0.08, 0.02 \right)$. The optimum values of production quantity and average total cost function per unit time were determined using the expressions for (5.2.4) and (5.2.10) respectively.

5.3 Linear Time Dependent Deterioration Rate Model

In this section, we present the second model which is an EPQ model for delayed deteriorating items with linear inventory level dependent demand rate and the deteriorating rate is a linear increasing function of time. The main difference between the first model under Section 5.2 and this model is that the former considers constant rate of deterioration while the later considers variable rate of deterioration and constant inventory holding cost per cycle. The main objective is to determine the optimal replenishment policy such that the total variable cost is minimised.

5.3.1 Modelling assumptions

The inventory system in this case is based on the same assumptions as in model 5.2.1 except that:

1. The deterioration rate of the item is time dependent and it is given by $\theta(t) = \omega t, \quad 0 < \theta(t), \omega < 1$.
2. Constant inventory holding cost per cycle.

5.3.2 Model development

The objective of the inventory problem here is to determine the optimal production quantity so as to keep the total relevant cost as low as possible. At the start, t = 0, the inventory level is zero. The production and supply start simultaneously and the production stops at t = T_1 during which the maximum inventory V_m is reached. The demand rate during production is assumed to be $\alpha + \beta q_1(t)$, where, $0 < \beta < 1$ and so the inventory built up is at the rate $\lambda - (\alpha + \beta q_1(t))$ in the interval [0, T_1], and in that interval there is no deterioration. After the time T_1, the produced units start deteriorating. There is no fall in demand, when the inventory reduces to zero level and production run begins. The inventory level of the product at time t over period [0, *T*] can be represented by the following differential equations:

$$\frac{d}{dt}\left(q_1(t)\right) = \lambda - (\alpha + \beta q_1(t)), \text{ for } 0 \le t \le T_1 \qquad (5.3.1)$$

$$\frac{d}{dt}\left(q_2(t)\right) + \theta(t) q_2(t) = -(\alpha + \beta q_2(t)), \qquad (5.3.2)$$

where $\theta(t) = \omega t$ with $0 < \omega < 1$ for $T_1 \le t \le T \Leftrightarrow 0 \le t \le T_2$

That is, the demand rate after production is $\alpha + \beta q_2(t)$ and the product has then started deteriorating at the rate of θ(t), i.e. the rate of deterioration increases linearly.

The pictorial representation of the model is given in figure 5.2.

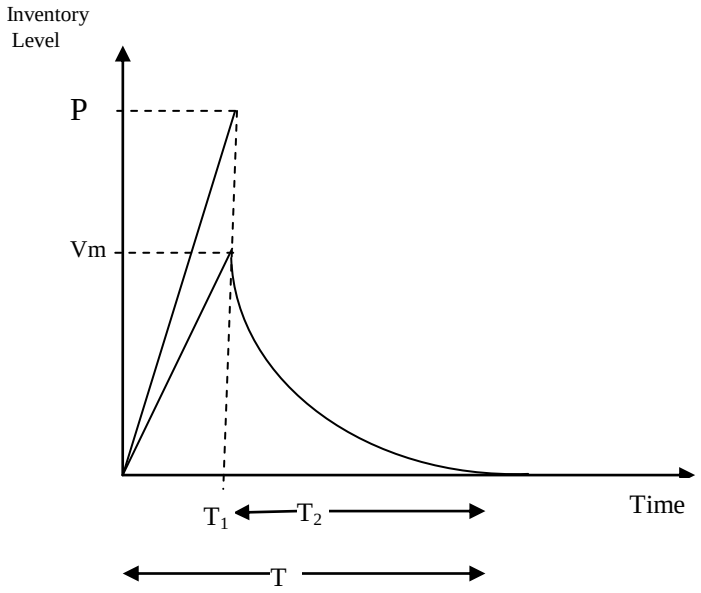


Figure 5.2: The graph of the inventory system represented by the equations (5.3.1) and (5.3.2).

λ = constant production rate,

demand rate during production period = $\alpha + \beta q_1(t)$

rate of inventory built up = $\lambda - (\alpha + \beta q_1(t)) > 0$

On integrating (5.3.1) with respect to t, we have

$$q_1 e^{\beta t} = \frac{(\lambda - \alpha) e^{\beta t}}{\beta} + k_1 \qquad (5.3.3)$$

where k_1 is a constant of integration.

Using the boundary condition $q_1(0) = 0$ in equation (5.3.3), we have

$$q_1 e^{\beta t} = \frac{(\lambda - \alpha)\, e^{\beta t}}{\beta} - \frac{(\lambda - \alpha)}{\beta} \quad , \; 0 \le t \le T_1$$

and from which

$$q_1(t) = \frac{(\lambda - \alpha)}{\beta}\left(1 - e^{-\beta t}\right)$$

Using the Taylor series expansion for exponential function, we have

$$q_1(t) = \left(\frac{\lambda - \alpha}{\beta}\right)\left\{1 - \left(1 - \beta t + \frac{(\beta t)^2}{2} - \cdots\right)\right\}$$

Approximating the value by neglecting those terms in β t $(\text{since } 0 < \beta < 1)$ of degree greater than 2, we have

$$q_1(t) = (\lambda - \alpha)\left(t - \frac{\beta t^2}{2}\right), \qquad 0 \le t \le T_1 \tag{5.3.4}$$

From (5.3.2) we have

$$\frac{d}{dt}(q_2(t)) + (\omega t + \beta) q_2(t) = -\alpha$$

Also integrating it with respect to t gives

$$q_2(t)\exp\left(\frac{\omega t^{\,2}}{2}+\beta t\right) = -\alpha\int\exp\left(\frac{\omega t^{\,2}}{2}+\beta t\right)$$

$$= -\alpha\int\left\{1+\left(\frac{\omega t^{\,2}}{2}+\beta t\right)+\frac{1}{2!}\left(\frac{\omega t^{\,2}}{2}+\beta t\right)^2+\ldots\right\}dt$$

Since $0<\beta<1$ and $0<\omega<1$, then neglecting all terms of $\beta\omega$ of degree greater than or equal to 1 in the Taylor`s series expansion, we have

$$q_2(t)\exp\left(\frac{\omega t^{\,2}}{2}+\beta t\right) = -\alpha\int\left\{1+\beta t+\frac{\omega t^{\,2}}{2}\right\}dt$$

$$= -\alpha\left\{t+\frac{\beta t^2}{2}+\frac{\omega t^{\,3}}{6}\right\}+k_2 \qquad (5.3.5)$$

where k_2 is a constant of integration.

And using the condition $q_2(T_2)=0$ in (5.3.5) gives

$$k_2=\alpha\left(T_2+\frac{\beta T_2^{\,2}}{2}+\frac{\omega T_2^{\,3}}{6}\right)$$

and

$$q_2(t)\exp\left(\frac{\omega t^2}{2}+\beta t\right) = \alpha\left(T_2+\frac{\beta T_2^{\,2}}{2}+\frac{\omega T_2^{\,3}}{6}-t-\frac{\beta t^2}{2}-\frac{\omega t^3}{6}\right)$$

Thus

$$q_2(t) = \alpha\left(T_2 + \frac{\beta T_2^2}{2} + \frac{\omega T_2^3}{6} - t - \frac{\beta t^2}{2} - \frac{\omega t^3}{6} \right)\exp\left(-\left(\frac{\omega t^2}{2} + \beta t \right)\right) \tag{5.3.6}$$

5.*32.1 The production cost per period*

This is given by

$$\text{PC} = k\lambda T_1 \tag{5.3.7}$$

5.3.2.2 The holding cost per period

This is given by

$$\text{HC} = \mu_1 \int_0^{T_1} q_1(t)\, dt + \mu_1 \int_0^{T_2} q_2(t)\, dt$$

$$= \mu_1 \int_0^{T_1} (\lambda - \alpha)\left(t - \frac{\beta t^2}{2} \right) dt$$

$$+ \alpha\mu_1 \int_0^{T_2} \left(T_2 + \frac{\beta T_2^2}{2} + \frac{\omega T_2^3}{6} - t - \frac{\beta t^2}{2} - \frac{\omega t^3}{6} \right)\exp\left(-\left(\frac{\omega t^2}{2} + \beta t \right)\right) dt$$

We take the first two terms from the Taylor`s series expansion of the exponential function, since $0 < \beta, \omega < 1$, to have

$$HC = \mu_1 \int_0^{T_1} (\lambda - \alpha) \left(t - \frac{\beta t^2}{2} \right) dt$$

$$+ \alpha\mu_1 \int_0^{T_2} \left(T_2 + \frac{\beta T_2^{\,2}}{2} + \frac{\omega T_2^{\,3}}{6} - t - \frac{\beta t^2}{2} - \frac{\omega t^3}{6} \right) \left(1 - \beta t - \frac{\omega t^2}{2} \right) dt$$

$$= \mu_1 \int_0^{T_1} (\lambda - \alpha) \left(t - \frac{\beta t^2}{2} \right) dt + \alpha\mu_1 \int_0^{T_2} \left(\xi_1(t) - \beta\xi_2(t) - \frac{\omega}{2} \xi_3(t) \right) dt \qquad (5.3.8)$$

where

$$\xi_1(t) = \left(T_2 + \frac{\beta T_2^{\,2}}{2} + \frac{\omega T_2^{\,3}}{6} - t - \frac{\beta t^2}{2} - \frac{\omega t^3}{6} \right)$$

$$\xi_2(t) = \left(T_2 t + \frac{\beta T_2^{\,2} t}{2} + \frac{\omega T_2^{\,3} t}{6} - t^2 - \frac{\beta t^3}{2} - \frac{\omega t^4}{6} \right)$$

$$\xi_3(t) = \left(T_2\, t^2 + \frac{\beta T_2^{\,2} t^2}{2} + \frac{\omega T_2^{\,3}}{6} t^2 - t^3 - \frac{\beta t^4}{2} - \frac{\omega t^5}{6} \right)$$

Evaluating the integral (5.3.8) with respect to t, we have

$$HC = \mu_1 (\lambda - \alpha) \left(\frac{T_1^{\,2}}{2} - \frac{\beta T_1^{\,3}}{6} \right) + \alpha\mu_1 \left\{ \left(\frac{T_2^{\,2}}{2} + \frac{\beta T_2^{\,3}}{3} + \frac{\omega T_2^{\,4}}{8} \right) - \beta \left(\frac{T_2^{\,3}}{6} + \frac{\beta T_2^{\,4}}{8} + \frac{\omega T_2^{\,5}}{20} \right) \right\}$$

$$-\frac{\alpha\mu_1\omega}{2}\left(\frac{T_2^{\,4}}{12}+\frac{\beta T_2^{\,5}}{15}+\frac{\omega T_2^{\,6}}{36}\right) \tag{5.3.9}$$

5.3.2.3 Total amount of demand in the time period T_2

This is given by

$$MD=\int_0^{T_2}(\alpha+\beta t)\,dt \;=\; \alpha T_2+\frac{\beta\,T_2^{\,2}}{2} \tag{5.3.10}$$

5.3.2.4 The number of deteriorated items per cycle

This is given by

$$DP \;=\; q_2(0)-MD$$

$$=\;\alpha\left(\;T_2+\frac{\beta\,T_2^{\,2}}{2}+\frac{\omega\,T_2^{\,3}}{6}\;\right)-\left(\;\alpha T_2+\frac{\beta T_2^{\,2}}{2}\right)$$

$$=\;\frac{\alpha\omega\,T_2^{\,3}}{6}+\frac{\alpha\beta T_2^{\,2}}{2}-\frac{\beta T_2^{\,2}}{2} \tag{5.3.11}$$

5.3.2.5 Total average variable inventory cost per unit time

This is given by

$$TVC(T) \;=\; \frac{1}{T}\left(C_s+PC+kDP+HC\right)$$

$$= \frac{C_s}{T} + \frac{1}{T}(k\lambda T_1) + \frac{k}{T}\left(\frac{\alpha\omega T_2^{\,3}}{6} + \frac{\alpha\beta T_2^{\,2}}{2} - \frac{\beta T_2^{\,2}}{2} \right) + \frac{1}{T}\left(\mu_1(\lambda-\alpha)\left(\frac{T_1^2}{2} - \frac{\beta T_1^3}{6} \right) \right)$$

$$+ \frac{\alpha\mu_1}{T}\left\{ \left(\frac{T_2^{\,2}}{2} + \frac{\beta T_2^{\,3}}{3} + \frac{\omega T_2^{\,4}}{8} \right) - \beta\left(\frac{T_2^{\,3}}{6} + \frac{\beta T_2^{\,4}}{8} + \frac{\omega T_2^{\,5}}{20} \right) - \frac{\omega}{2}\left(\frac{T_2^{\,4}}{12} + \frac{\beta T_2^{\,5}}{15} + \frac{\omega T_2^{\,6}}{36} \right) \right\}$$

(5.3.12)

To minimize the total variable cost per unit time TVC (T), we differentiate TVC (T) with respect to T and set the result to zero with $T_2 = T - T_1$ as follows:

$$\frac{d\left(TVC(T)\right)}{dT} = -\frac{C_s}{T^2} - \frac{k\lambda T_1}{T^2} + \frac{k}{T^2}\left\{ \frac{\alpha\omega}{6}\left(2T^3 - 3T^2T_1 + T_1^3\right) + \frac{\alpha\beta}{2}\left(T^2 - T_1^2\right) - \frac{\beta}{2}\left(T^2 - T_1^2\right) \right\}$$

$$- \frac{\mu_1(\lambda-\alpha)}{T^2}\left(\frac{T_1^2}{2} - \frac{\beta T_1^3}{6} \right)$$

$$+ \frac{\alpha\mu_1}{T^2}\left\{ \frac{T^2 - T_1^2}{2} + \frac{\beta}{3}\left(2T^3 - 3T^2T_1 + T_1^3\right) + \frac{\omega}{8}\left(3T^4 - 8T^3T_1 + 6T^2T_1^2 - T_1^4\right) \right\}$$

$$- \frac{\alpha\mu_1\beta}{T^2}\left\{ \frac{1}{6}\left(2T^3 - 3T^2T_1 + T_1^3\right) + \frac{\beta}{8}\left(3T^4 - 8T^3T_1 + 6T^2T_1^2 - T_1^4\right) \right\}$$

$$- \frac{\alpha\mu_1\beta\omega}{12T^2}\left(4T^5 - 15T^4T_1 + 20T^3T_1^2 - 10T^2T_1^3 + T_1^5\right)$$

$$-\frac{\alpha\mu_1\omega}{24T^2}\left(3T^4-8T^3T_1+6T^2T_1^2-T_1^4\right)$$

$$-\frac{\alpha\mu_1\omega^2}{72T^2}\left(5T^6-24T^5T_1+45T^4T_1^2-40T^3T_1^3+15T^2T_1^4-T_1^6\right) \qquad (5.3.13)$$

Setting equation (5.3.13) to zero and multiplying both sides by $360T^2$, simplifying and rearranging the terms, yields

$$-360C_s-360k\lambda T_1+k\left\{60\alpha\omega\left(2T^3-3T^2T_1+T_1^3\right)+180\alpha\beta\left(T^2-T_1^2\right)-180\beta\left(T^2-T_1^2\right)\right\}$$

$$-\ \mu_1(\lambda-\alpha)\left(180T_1^2-60\beta T_1^3\right).$$

$$+\alpha\mu_1\left\{180\left(T^2-T_1^2\right)+120\beta\left(2T^3-3T^2T_1+T_1^3\right)+45\omega\left(3T^4-8T^3T_1+6T^2T_1^2-T_1^4\right)\right\}$$

$$-\alpha\mu_1\beta\left\{60\left(2T^3-3T^2T_1+T_1^3\right)+45\beta\left(3T^4-8T^3T_1+6T^2T_1^2-T_1^4\right)\right\}$$

$$-30\alpha\mu_1\beta\omega\left(4T^5-15T^4T_1+20T^3T_1^2-10T^2T_1^3+T_1^5\right)$$

$$-15\alpha\mu_1\omega\left(3T^4-8T^3T_1+6T^2T_1^2-T_1^4\right)$$

$$-5\alpha\mu_1\omega^2\left(5T^6-24T^5T_1+45T^4T_1^2-40T^3T_1^3+15T^2T_1^4-T_1^6\right) \quad = \quad 0$$

This further simplifies to

$$\psi_1 T^6 + \psi_2 T^5 + \psi_3 T^4 + \psi_4 T^3 + \psi_5 T^2 + \psi_6 = 0 \qquad (5.3.14)$$

where

$$\psi_1 = -25\alpha\mu_1\omega^2$$

$$\psi_2 = 120\alpha\mu_1\omega^2 T_1 - 120\alpha\mu_1\beta\omega$$

$$\psi_3 = -225\alpha\mu_1\omega^2 T_1^2 + 450\alpha\mu_1\beta\omega T_1 - 135\alpha\mu_1\beta^2 + 90\alpha\mu_1\omega$$

$$\psi_4 = 200\alpha\mu_1\omega^2 T_1^3 - 600\alpha\mu_1\beta\omega T_1^2 - 240\alpha\mu_1\omega T_1 + 120\alpha\mu_1\beta + 360\alpha\mu_1\beta^2 T_1 + 120k\alpha\omega$$

$$\psi_5 = -75\alpha\mu_1\omega^2 T_1^4 + 300\alpha\mu_1\omega\beta T_1^3 + 180\alpha\mu_1\omega T_1^2{}_1 - 270\alpha\mu_1\beta^2 T_1^2 - 180\alpha\mu_1\beta T_1$$

$$+180\alpha\mu_1 - 180k\alpha\omega T_1 + 180k\alpha\beta - 180k\beta$$

$$\psi_6 = 5\alpha\mu_1\omega^2 T_1^6 - 30\alpha\mu_1\beta\omega T_1^5 - 30\alpha\mu_1\omega T_1^4 + 45\alpha\mu_1\beta^2 T_1^4 - 180\alpha\mu_1 T_1^2 + 60\alpha\mu_1\beta T_1^3 - 360C_s$$

$$-360k\lambda T_1 + 60k\alpha\omega T_1^3 - 180k\alpha\beta T_1^2 + 180k\beta T_1^2 - \mu_1(\lambda - \alpha)\left(180T_1^2 - 60\beta T_1^3\right)$$

The roots of equation (5.3.14) can be solved by any suitable numerical approximation method such as Newton-Raphson method which we used in our case when the values of the different parameters are prescribed. Such a positive root T* of equation (5.3.14) for

which $\frac{d^2\left(TVC(T)\right)}{dT^2} > 0$ gives a minimum for the total system cost. This is found to be so for all our examples under Section 5.3.3.

5.3.3 Numerical examples

Four tables (Table 5.2, Table 5.3, Table 5.4 and Table 5.5) of examples are presented to illustrate the model developed and the effects of changes of some model parameters on the decision variables. From Table 5.2, an inventory system in the first example has the following values of parameters: $\alpha = 90$, $\beta = 0.7$, $\omega = 0.05$, $\delta = 2$, $\lambda = 110$, C_s = ₦ 120 per order, k = ₦ 10 per unit, $T_1 = 0.1$ time units, the optimum cycle length of time T* is determined by Newton-Raphson method using expression (5.3.14) as $T^* = 0.74$ units and corresponding minimum total system cost per unit time is determined using the expression (5.3.12) as TVC* = ₦548.70. For other values of the parameters as reflected in the table, the optimum cycle length of time T* and the corresponding minimum total system cost per unit time TVC* are similarly determined. The results are tabulated in Table 5.2. Also, Table 5.3, Table 5.4 and Table 5.5 are similarly drawn up, with changes as indicated in the heading of the table.

Note that in all examples given, the sufficient condition $\frac{d^2}{dT^2}\left(TVC(T)\right) > 0$ is satisfied implying that T* so determined is indeed the minimum in the considered region.

Table 5.2: Effects of changing the stock-dependent demand rate β on the decision variables

S/No	α	β	ω	μ_1	λ	C_s	T_1	k	T*	TVC*
1	90	0.7	0.05	2	110	120	0.1	10	0.74	548.70
2	90	0.5	0.05	2	110	120	0.1	10	0.83	492.69
3	90	0.3	0.05	2	110	120	0.1	10	0.97	425.65

Table 5.3: Effects of changing the deterioration rate ω on the decision variables

S/No	α	β	ω	μ_1	λ	C_s	T_1	k	T*	TVC*
1	100	0.5	0.1	5	130	150	0.05	11	0.62	655.35
2	100	0.5	0.3	5	130	150	0.05	11	0.60	666.51
3	100	0.5	0.5	5	130	150	0.05	11	0.58	676.95

Table 5.4: Effects of changing the set up cost C_s on the decision variables

S/No	α	β	ω	μ_1	λ	C_s	T_1	k	T*	TVC*
1	80	0.7	0.1	4	90	120	0.01	8	0.55	453.75
2	80	0.7	0.1	4	90	100	0.01	8	0.51	413.92
3	80	0.7	0.1	4	90	80	0.01	8	0.46	370.65

Table 5.5: Effects of changing the constant production rate λ on the decision variables

S/No	α	β	ω	μ_1	λ	C_s	T_1	k	T*	TVC*
1	110	0.5	0.2	3	160	130	0.08	7	0.73	533.86
2	110	0.5	0.2	3	200	130	0.08	7	0.77	565.53
3	110	0.5	0.2	3	500	130	0.08	7	0.99	771.89

5.3.4 Results and discussion

A careful study of the computational results as shown in Tables 5.2 – 5.5, and within the range of values of the chosen parameters, reveal the following observations:

(1) From Table 5.2, a higher value of β results in lower values of T*, but higher values of TVC*. This implies that increase in the stock-dependent demand rate will result in the decrease of optimal cycle, but higher values of optimal total system cost per unit time. This is expected since if stock dependent demand rate is higher, stock will finish earlier and so T* will decrease. On the other hand, increasing the stock-dependent demand parameter β will effectively increase the total demand which, in turn, will increase the overall system cost per unit time.

(2) From Table 5.3, a higher value of ω results in lower values of T*, but higher values of TVC*. This implies that increase in deterioration rate will result in decrease in the optimal length. The total variable cost per unit time, however, increases in this case, which is also expected since when deterioration cost increases, the total variable cost per unit time will also increase. With deterioration rate higher however, stock will finish earlier resulting in lower T*.

(3) From Table 5.4, higher values of C_s result in higher values of T*and TVC*. In this case, it implies that increase in setup cost will result in the increase of optimal cycle length and the total variable cost per unit time. This is clearly expected since excess stocking is encouraged as a result of higher set up cost. The total variable cost per unit time is therefore expected to increase due to increase in stocking cost, and to avoid more frequency of ordering, T* increases.

(4) From Table 5.5, higher values of λ result in higher values of T*and TVC*. In this case, it implies that increase in production rate will result in the increase of optimal cycle length of time and the total variable cost per unit time. The total variable cost per unit time is also expected to increase due to increase in stocking cost and higher production cost. Clearly also T* should increase since much has been produced and so takes longer time to finish.

5.3.5 Concluding Remarks

In this part, an Economic Production Quantity (EPQ) model is presented for single product with delayed deterioration in which the production rate is constant, demand rate is inventory level dependent in a linear functional form before and after production and deteriorating rate is a linear increasing function of time. The rate of deterioration is a linear increasing function of time. An analytic formulation of the problem on the framework described above has been provided and also an optimal solution procedure. We present some numerical examples to illustrate the application of the model developed and use the examples to study the effect of various changes in some possible combinations of model parameters on the decision variables of the system. We find from the above results that the effects of changing the model parameters $\beta, \omega, C_{s.}$ and λ on the optimal replenishment policy reveal the following:

(1) When the stock-dependent consumption rate β is increasing, the optimal cost is increasing.

(2) When the deterioration rate ω is increasing, the optimal cost is increasing.

(3) When the set up cost C_s is increasing, the optimal cost is increasing.

(4) When the production rate λ is increasing, the optimal cost is increasing.

The model we have presented in this section provides a basis for several possible extensions. For future research, the model can be enriched by incorporating planned shortages, inflation and time value of money, variable holding cost, other forms of stock dependent demand rate function, and so on.

CHAPTER SIX

AN EOQ MODEL FOR DELAYED DETERIORATING ITEMS WITH INVENTORY LEVEL DEPENDENT DEMAND RATE AND LINEAR TIME-DEPENDENT HOLDING COST

6.1 Introduction

To keep sales high, inventory level needs to remain high. Of course, this would also result in high holding or procurement costs (Baker and Urban, 1988). Larson and DeMarais (1990) suggested that only those items with high direct product profitability (DPP) and high sales volume need to be considered for positive ending- inventory. An inventory system which possesses inventory -level-dependent demand rate, and in which the inventory level falls to zero at the end of the order cycle, may not provide the optimal solution (Urban, 1992 and chang, 2004). However, Urban (1992) stated that having inventory remaining at the end of the order cycle is not a general requirement but a possibility. Maximising profit rather than minimising cost is justified in this model (see Urban, 1992 and chang, 2004 for instance) since the cost of holding an item in stock increases with each passing time and the aim in this environment is to hold more stock so as to generate more profit.

In this chapter, we develop an optimal replenishment policy for delayed deteriorating items with power-form inventory level dependent demand rate. The holding cost is assumed to be linear while the deterioration rate is a constant. In the initial stage, inventory depletes down to a certain level of the inventory due to market demand only and dependent on inventory level. In the second stage the inventory level gets depleted

due to the effect of both market demand and deterioration but still dependent on inventory level until the inventory level falls to zero at the end of the cycle. The difference between this model and that in chapter three is that unlike in chapter three where the holding cost is a constant. The holding cost in this model is linear time dependent. Moreover, the aim in chapter three is to minimize cost while the aim in this model is to maximize profit. This structure of linear time dependent holding cost is representative of many real-life situations and this is particularly true in the storage of deteriorating and perishable items such as food products. Hence, the longer these deteriorating items are kept in storage, the higher the holding cost. The linear holding cost is justified for that inventory system since the cost of holding an item in stock increases with each passing time for such items. We consider two cases: (i) when replenishment occurs after deterioration sets in $\left(0 < \mathrm{t_d} < \mathrm{T}\right)$ and (ii) when replenishment occurs before deterioration sets in $\left(\mathrm{t_d} \geq \mathrm{T} \text{ or } 0 < T \leq t_d\right)$. The inventory system in each case is zero-ending. The objective of this model is to determine the optimal replenishment policy to maximize average net profit per unit time. In case (i), if $t_d = 0$ and the holding cost were to be constant, our model would reduce to Pal *et al*. (1993) for zero-ending inventory. In case (ii), if the holding cost per unit were to be a constant, then our model would reduce to the model of Baker and Urban (1988). We obtain in each case maximization models (6.3.19) and (6.3.28) respectively for the net average profit per unit time for the inventory system. The results are illustrated with the help of numerical examples and we also carried out a sensitivity analysis on the decision variables with regard to changes in the model parameters

6. 2 Modelling Assumptions

This model is based on the same assumptions as in model assumptions of chapter three except that:

(1) The traditional holding cost is a variable holding cost which is given by $C_h(t) = Rt$. This is the cumulative holding cost for one unit that has been stored during t units of time, where the constant *R* is the cost to hold one unit for one time period.

(2) The optimal replenishment policy is determined by maximizing the profit function per unit time rather than minimizing the total cost function.

6.3 Mathematical Model and Theoretical Results

We consider two cases: (i) when $0 < t_d < T$ and (ii) when $t_d \geq T$.

6.3.1 Analysis for the case $0 < t_d < T$

At the beginning of each cycle, the on-hand inventory level gradually falls from *Q* to the level S_d at time $t = t_d$, due to demand only. After time $t = t_d$, the inventory level further depletes due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero at the end of the cycle.

The pictorial representation of the model is given in figure 6.1.

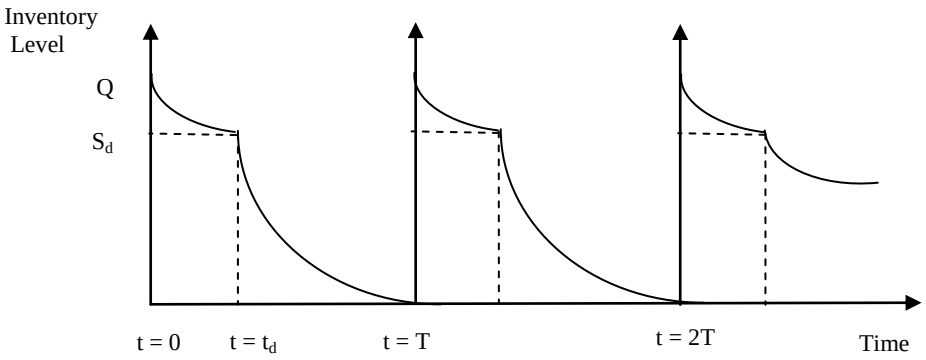


Figure 6.1: The inventory system of the situation for $0 < t_d < T$

The inventory level $v_1(t)$ at any time t $(0 \le t \le t_d)$ can be represented by the differential equation

$$\frac{dv_1(t)}{dt} = -\alpha\left(v_1(t)\right)^{\beta}, \quad 0 \le t \le t_d \tag{6.3.1}$$

with conditions $v_1(0) = Q$ and $v_1(t_d) = S_d$

The differential equation governing the instantaneous state of $v_2(t)$ over the time period in $[t_d, T]$ is given by

$$\frac{dv_2(t)}{dt} = -\left(\theta v_2(t) + \alpha(v_2(t))^{\beta}\right), t_d \le t \le T \tag{6.3.2}$$

with $v_2(t_d) = S_d$ and $v_2(T) = 0.$

The inventory system at any time t $(0 \le t \le T)$ can be represented by the two differential equations (6.3.1) and (6.3.2) above.

The solution of the differential equation (6.3.1) using the initial condition $v_1(0) = Q$ is given by

$$v_1 = \left[Q^{1-\beta} - \alpha(1-\beta)t\right]^{1/1-\beta}, \quad 0 \le t \le t_d \tag{6.3.3}$$

and from which

$$t = \frac{\left(Q^{1-\beta} - v_1^{1-\beta}\right)}{\alpha(1-\beta)} \tag{6.3.4}$$

Applying the condition $v_1(t_d) = S_d$ in (6.3.3), we get

$$S_d = \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d\right)^{1/1-\beta} \tag{6.3.5}$$

The solution of the differential equation (6.3.2), using the boundary condition $v_2(t_d) = S_d$ is given by

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(S_d^{1-\beta} + \frac{\alpha}{\theta}\right) e^{\theta(1-\beta)(t_d - t)} \tag{6.3.6}$$

Substituting the value of $S_d^{1-\beta}$ from (6.3.5) in (6.3.6), we have

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d + \frac{\alpha}{\theta}\right) e^{\theta(1-\beta)(t_d - t)} \tag{6.3.7}$$

Using Taylor series expansion for exponential function, we have

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d + \frac{\alpha}{\theta}\right) \left(\left(1 + \frac{\theta(1-\beta)(t_d - t)}{1!} + \frac{(\theta(1-\beta)(t_d - t))^2}{2!} + \ldots\right)\right)$$

Neglecting the higher powers of $\theta \left(\text{since } 0 < \theta < 1\right)$ and allowing only the first two terms in the expansion, we get

$$v_2^{1-\beta} = -\frac{\alpha}{\theta} + \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d + \frac{\alpha}{\theta} \right) \left(1 + \theta(1-\beta)(t_d - t) \right)$$

$$= \quad Q^{1-\beta} - \alpha(1-\beta)t_d + \left[\theta \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right) + \alpha \right] (1-\beta)(t_d - t)$$

and so

$$v_2 = \left\{ Q^{1-\beta} - \alpha(1-\beta)\, t_d + \left[\theta \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right) + \alpha \right] (1-\beta)(t_d - t) \right\}^{1/1-\beta} \qquad (6.3.8)$$

Using the condition $v_2(T) = 0$ in (6.3.8), we have using equation (6.3.5),

$$T = t_d + \frac{S_d^{1-\beta}}{(1-\beta)\left(\theta S_d^{1-\beta} + \alpha \right)} \qquad (6.3.9)$$

6.3.1.1 Total inventory holding cost during the complete cycle time T

This is given by

$$HC = \int_0^{t_d} (Rt) v_1 dt + \int_{t_d}^{T} (Rt) v_2 dt$$

$$= R \int_0^{t_d} t \left(Q^{1-\beta} - \alpha(1-\beta)\, t \right)^{\frac{1}{1-\beta}} dt$$

$$+ R \int_{t_d}^{T} t \left\{ Q^{1-\beta} - \alpha(1-\beta)\, t_d + \left[\theta \left(Q^{1-\beta} - \alpha(1-\beta)t_d \right) + \alpha \right] (1-\beta)(t_d - t) \right\}^{\frac{1}{1-\beta}} dt$$

Using the method of integration by parts to evaluate the definite integral and using the expression for $S_d^{1-\beta}$ from (6.3.5), we obtain

$$HC = -\frac{R\,t_d}{\alpha(2-\beta)}\left\{Q^{1-\beta} - \alpha(1-\beta)t_d\right\}^{(2-\beta)/(1-\beta)} - \frac{R\left\{Q^{1-\beta} - \alpha(1-\beta)t_d\right\}^{(3-2\beta)/(1-\beta)}}{\alpha^2(2-\beta)(3-2\beta)}$$

$$+\frac{RQ^{3-2\beta}}{\alpha^2(2-\beta)(3-2\beta)} + \frac{Rt_d S_d^{2-\beta}}{(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)} + \frac{RS_d^{3-2\beta}}{(2-\beta)(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2}$$

$$= \frac{RQ^{3-2\beta}}{\alpha^2(2-\beta)(3-2\beta)} - \frac{Rt_d\,S_d^{2-\beta}}{\alpha(2-\beta)} - \frac{RS_d^{3-2\beta}}{\alpha^2(2-\beta)(3-2\beta)}$$

$$+\frac{Rt_d S_d^{2-\beta}}{(2-\beta)\left(\theta S_d^{1-\beta}+\alpha\right)} + \frac{RS_d^{3-2\beta}}{(2-\beta)(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2} \qquad (6.3.10)$$

6.3.1.2 Total demand during the cycle time T

This is given by

$$DM = \int_0^{t_d} \alpha\left(\nu_1(t)\right)^{\beta} dt + \int_{t_d}^{T} \alpha\left(\nu_2(t)\right)^{\beta} dt$$

$$= \alpha\int_0^{t_d}\left(Q^{1-\beta} - \alpha(1-\beta)\,t\right)^{\beta/(1-\beta)} dt$$

$$+\alpha\int_{t_d}^{T}\left\{Q^{1-\beta} - \alpha(1-\beta)\,t_d + \left[\theta\left(Q^{1-\beta} - \alpha(1-\beta)\,t_d\right)+\alpha\right](1-\beta)(t_d - t)\right\}^{\beta/(1-\beta)} dt$$

$$= Q - \left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right)^{1/(1-\beta)}$$

$$- \frac{\alpha\left\{ Q^{1-\beta} - \alpha(1-\beta)t_d + \left[\theta\left(Q^{1-\beta} - \alpha(1-\beta)t_d \right) + \alpha \right](1-\beta)(t_d - T)\right\}^{1/(1-\beta)}}{\alpha\left\{ Q^{1-\beta} - \alpha(1-\beta)t_d \right\}^{1/(1-\beta)}}$$

$$+ \frac{\alpha\left\{ Q^{1-\beta} - \alpha(1-\beta)t_d \right\}^{1/(1-\beta)}}{\left(\theta\left(Q^{1-\beta} - \alpha(1-\beta)\, t_d \right) + \alpha \right)}$$

Again using equations (6.2.5) and (6.2.9) in this expression, we obtain

$$DM = Q - S_d + \frac{\alpha S_d}{\left(\theta S_d^{\,1-\beta} + \alpha \right)} \tag{6.3.11}$$

6.3.1.3 Total variable cost per cycle

This is given by

$$TVC = \left(\text{purchase cost+ordering cost+holding cost}\right)$$

$$= \left(c\,Q + C_o + HC \right) \tag{6.3.12}$$

6.3.1.4 The net profit per unit time

This is given by

$$Z = \frac{1}{\mathrm{T}}\left\{\text{Sales revenue per cycle -Total cost per cycle}\right\}$$

$$= \frac{rDM}{T} - \frac{TVC}{T}$$

$$= \frac{1}{T}\left\{ \left(r - c \right) Q + r\left(\frac{\alpha S_d}{\left(\theta S_d^{1-\beta} + \alpha \right)} - S_d \right) \right\} - \frac{1}{T}\left(C_0 + HC \right), \qquad (6.3.13)$$

where

Z, T, HC and S_d are functions of Q.

Note: S_d is only a symbol representing the amount of inventory left when deterioration begins and it is useful in the analysis of the problem.

The optimum value Q^* of Q giving the optimal net profit per unit time is the solution of

$\frac{d}{dQ}\left(Z(Q) \right) = 0$ provided the sufficient condition $\left(\frac{d^2 Z}{dQ^2} \right)_{Q = Q^*} < 0$ is satisfied.

The necessary condition for optimal order quantity Q^* is given by

$$\frac{d}{dQ} Z\left(Q \right) = \frac{d}{dQ}\left\{ \frac{\left(r - c \right) Q}{T} + \frac{r}{T}\left(\frac{\alpha S_d}{\theta S_d^{1-\beta} + \alpha} - S_d \right) \right\} - \frac{d}{dQ}\left\{ \frac{C_0}{T} + \frac{HC}{T} \right\} = 0 \qquad (6.3.14)$$

Letting $\pi_1 = \frac{\left(r - c \right) Q}{T} + \frac{r}{T}\left(\frac{\alpha S_d}{\theta S_d^{1-\beta} + \alpha} - S_d \right)$ and $\pi_2 = \left\{ \frac{C_0}{T} + \frac{HC}{T} \right\}$,

we have

$$\frac{d\pi_1}{dQ} = \frac{\left(r - c \right)\left(T - T' Q \right)}{T^2} + \frac{r\, \alpha S_d' T\left(\theta S_d^{1-\beta} + \alpha \right)}{\left(T\left(\theta S_d^{1-\beta} + \alpha \right) \right)^2}$$

$$-\frac{r\,\alpha S_d\left(T'\left(\theta S_d^{1-\beta}+\alpha\right)+\left(\theta S_d^{1-\beta}+\alpha\right)'T\right)}{\left(T\left(\theta S_d^{1-\beta}+\alpha\right)\right)^2}-\frac{r\left(S_d'T-T'S_d\right)}{T^2} \tag{6.3.15}$$

and

$$\frac{d\pi_2}{dQ}=-\frac{C_o T'}{T^2}+R\left\{\frac{(3-2\beta)Q^{2(1-\beta)}T-T'Q^{3-2\beta}}{\alpha^2(2-\beta)(3-2\beta)T^2}\right\}-\frac{Rt_d}{\alpha(2-\beta)}\left\{\frac{\left(S_d^{2-\beta}\right)'T-T'S_d^{2-\beta}}{T^2}\right\}$$

$$+\frac{Rt_d}{(2-\beta)}\left\{\frac{\left(S_d^{2-\beta}\right)'\left(\theta S_d^{1-\beta}+\alpha\right)T-S_d^{2-\beta}\left(\theta S_d^{1-\beta}+\alpha\right)'T-T'S_d^{2-\beta}\left(\theta S_d^{1-\beta}+\alpha\right)}{\left(\theta S_d^{1-\beta}+\alpha\right)^2T^2}\right\}$$

$$+R\left\{\frac{\left(S_d^{3-2\beta}\right)'\left(\theta S_d^{1-\beta}+\alpha\right)^2T-S_d^{3-2\beta}\left(\left(\theta S_d^{1-\beta}+\alpha\right)^2\right)'T-T'S_d^{3-2\beta}\left(\theta S_d^{1-\beta}+\alpha\right)^2}{(2-\beta)(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^4T^2}\right\}$$

$$-R\left\{\frac{\left(S_d^{3-2\beta}\right)'T-T'S_d^{3-2\beta}}{\alpha^2(2-\beta)(3-2\beta)T^2}\right\} \tag{6.3.16}$$

where $(\ )'$ denotes the derivative of the function in the bracket with respect to Q. The derivatives of T and S_d which are functions of Q were obtained using equations (6.3.5) and (6.3.9). See Appendix F for those derivatives.

Substituting the values of $\frac{d\pi_1}{dQ}$ and $\frac{d\pi_2}{dQ}$ from equations (6.3.15) and (6.3.16) respectively into equation (6.3.14), we have

$$\frac{d\pi_1}{dQ}-\frac{d\pi_2}{dQ}=0 \tag{6.3.17}$$

Multiplying both sides of equation (6.3.17) by the expression,

$\alpha^2(2-\beta)(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^4 T^2$, we obtain

$$(r-c)(T-T'Q)\alpha^2(2-\beta)(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^4 + r\alpha^3 S_d'(2-\beta)(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^3 T$$

$$- r S_d\alpha^3(2-\beta)(3-2\beta)\left\{T'\left(\theta S_d^{1-\beta}+\alpha\right)+\theta(1-\beta)Q^{-\beta}T\right\}\left(\theta S_d^{1-\beta}+\alpha\right)^2$$

$$- r\alpha^2(2-\beta)(3-2\beta)\left(S_d'T-T'S_d\right)\left(\theta S_d^{1-\beta}+\alpha\right)^4$$

$$- C_o\alpha^2(2-\beta)(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^4 T'$$

$$+ R\left(\theta S_d^{1-\beta}+\alpha\right)^4\left\{(3-2\beta)Q^{2(1-\beta)}T-T'Q^{3-2\beta}\right\}$$

$$- Rt_d\alpha(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^4\left\{\left(S_d^{2-\beta}\right)'T-T'S_d^{2-\beta}\right\}$$

$$+Rt_d\alpha^2(3-2\beta)\left(\theta S_d^{1-\beta}+\alpha\right)^2\left\{\left(S_d^{2-\beta}\right)'\left(\theta S_d^{1-\beta}+\alpha\right)T-S_d^{2-\beta}\left(\theta S_d^{1-\beta}+\alpha\right)'T-T'S_d^{2-\beta}\left(\theta S_d^{1-\beta}+\alpha\right)\right\}$$

$$+R\alpha^2\left\{\left(S_d^{3-2\beta}\right)'\left(\theta S_d^{1-\beta}+\alpha\right)^2 T-S_d^{3-2\beta}\left(\left(\theta S_d^{1-\beta}+\alpha\right)^2\right)'T-T'S_d^{3-2\beta}\left(\theta S_d^{1-\beta}+\alpha\right)^2\right\}$$

$$- R\left(\theta S_d^{1-\beta}+\alpha\right)^4\left\{\left(S_d^{3-2\beta}\right)'T-T'S_d^{3-2\beta}\right\} = 0 \qquad (6.3.18)$$

Simplifying and rearranging (6.3.18) in terms of the powers of Q by multiplying both sides with Q^{β}

and letting $N = \theta S_d^{1-\beta} + \alpha$ *and* $P = t_d(1-\beta)\left(\theta S_d^{1-\beta} + \alpha\right) + S_d^{1-\beta} = t_d(1-\beta)N + S_d^{1-\beta}$,

we get,

$$\left(R\alpha N^2\right)Q^{3-2\beta} - \left(\frac{R(3-2\beta)N^3P}{(1-\beta)}\right)Q^{2-\beta} + \left(\frac{(r-c)\alpha^2(2-\beta)(3-2\beta)N^3P}{(1-\beta)}\right)Q^{\beta}$$

$$-(r-c)\alpha^3(2-\beta)(3-2\beta)N^2Q \qquad + \frac{R\alpha t_d N^3(2-\beta)(3-2\beta)S_dP}{(1-\beta)}$$

$$+ \frac{r\alpha^3 S_d^{\beta}(2-\beta)(3-2\beta)N^2P}{(1-\beta)} \quad - \frac{R\alpha^2 N(3-2\beta)S_d^{2-\beta}P}{(1-\beta)} \quad + \frac{2R\theta\alpha^2 S_d^{3-2\beta}P}{(1-\beta)}$$

$$+ \frac{RN^3(3-2\beta)S_d^{2-\beta}P}{(1-\beta)} \quad - \frac{Rt_d\alpha^2N^2(2-\beta)(3-2\beta)S_dP}{(1-\beta)} \quad - \frac{rS_d^{\beta}\alpha^2(2-\beta)(3-2\beta)N^3P}{(1-\beta)}$$

$$+ R\theta t_d\alpha^2N(3-2\beta)S_d^{2-\beta}P \quad - r\alpha^3S_d\theta(2-\beta)(3-2\beta)NP$$

$$+ C_o\alpha^3N^2(2-\beta)(3-2\beta) \quad - r\alpha^4S_d(2-\beta)(3-2\beta)N \quad - Rt_d\alpha^2N^2(3-2\beta)S_d^{2-\beta}$$

$$+ r\alpha^3S_d(2-\beta)(3-2\beta)N^2 - R\alpha N^2S_d^{3-2\beta} \quad + Rt_d\alpha^3N(3-2\beta)S_d^{2-\beta} \quad + R\alpha^3S_d^{3-2\beta} \quad = 0$$

or

$$\omega_1 Q^{3-2\beta} + \omega_2 Q^{2-\beta} + \omega_3 Q^{\beta} + \omega_4 Q + \omega_5 = 0 \qquad (6.3.19)$$

where

$$\omega_1 = R\alpha N^2$$

$$\omega_2 = -\frac{R(3-2\beta)N^3 P}{(1-\beta)}$$

$$\omega_3 = \frac{(r-c)\alpha^2(2-\beta)(3-2\beta)N^3 P}{(1-\beta)}$$

$$\omega_4 = -(r-c)\alpha^3(2-\beta)(3-2\beta)N^2$$

$$\omega_5 = \frac{R\alpha t_d N^3(2-\beta)(3-2\beta)S_d P}{(1-\beta)} + \frac{r\alpha^3 S_d^{\beta}(2-\beta)(3-2\beta)N^2 P}{(1-\beta)}$$

$$+\frac{RN^3(3-2\beta)S_d^{2-\beta}P}{(1-\beta)} - \frac{Rt_d\alpha^2 N^2(2-\beta)(3-2\beta)S_d P}{(1-\beta)} - \frac{rS_d^{\beta}\alpha^2(2-\beta)(3-2\beta)N^3 P}{(1-\beta)}$$

$$-\frac{R\alpha^2 N(3-2\beta)S_d^{2-\beta}P}{(1-\beta)} + R\theta t_d\alpha^2 N(3-2\beta)S_d^{2-\beta}P + \frac{2R\theta\alpha^2 S_d^{3-2\beta}P}{(1-\beta)}$$

$$-r\alpha^3 S_d\theta(2-\beta)(3-2\beta)NP$$

$$+C_o\alpha^3N^2(2-\beta)(3-2\beta) \quad -r\alpha^4S_d(2-\beta)(3-2\beta)N - Rt_d\alpha^2N^2(3-2\beta)S_d^{2-\beta}$$

$$+r\alpha^3S_d(2-\beta)(3-2\beta)N^2 - R\alpha N^2S_d^{3-2\beta} \quad + Rt_d\alpha^3N(3-2\beta)S_d^{2-\beta} \quad + R\alpha^{3}S_d^{3-2\beta}$$

Equation (6.3.19) is highly nonlinear and therefore difficult to solve by an analytic method. We may therefore evaluate the roots (satisfying the sufficient condition $\frac{d^2Z}{dQ^2} < 0$) of the equation using any suitable numerical method. In our case, we used Newton-Raphson method. Since the problem is complex, it is also difficult to show analytically the existence and uniqueness of the optimal solution. However, in our examples we used *Direct Search Method* to show the concavity of the profit function in those examples. That is to say, the sufficient condition $\left(\frac{d^2Z}{dQ^2}\right)_{Q=Q^*} < 0$ is satisfied, implying that the Q* so obtained is indeed the unique optimal solution in the considered region.

6.3.2 Analysis for the case when $t_d \geq T$ or $0 < T \leq t_d$

In this case, $t_d \geq T$ or $0 < T \leq t_d$ the reorder point occurs before deterioration takes place and the inventory system has zero-ending inventory. If the holding cost per unit were to be a constant, then our model would reduce to the model of Baker and Urban (1988).

Let $v(t)$ be the inventory level at any time t $(0 \leq t \leq T)$. The differential equation governing the instantaneous state of $v(t)$ over the time period in $[0, T]$ is given by

$$\frac{dv(t)}{dt} = -\alpha(v(t))^{\beta}, 0 \leq t \leq T \qquad (6.3.20)$$

with initial value $v(0) = Q_1$.

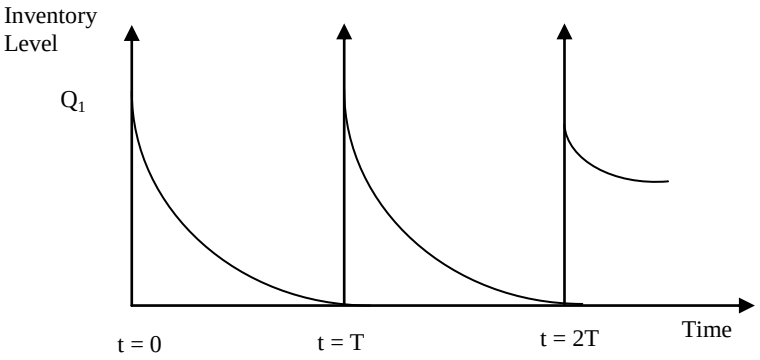


Figure 6.2: The inventory system of the situation for $t_d \geq T$

The solution of equation (6.3.20) is given by

$$v(t) = \left[Q_1^{1-\beta} - \alpha(1-\beta)\, t \right]^{\frac{1}{1-\beta}}, \quad 0 \leq t \leq T \tag{6.3.21}$$

Using the condition $v(T) = 0$ in (6.3.21), we have

$$T = T_1 = \frac{Q_1^{1-\beta}}{\alpha(1-\beta)} \tag{6.3.22}$$

which is the same as equation (6.3.9) when $t_d = 0, \theta = 0$ and $S_a = Q_1 = Q$

6.3.2.1 The inventory holding cost per cycle

This is given by

$$H_1 C_1 = \int_0^T (Rt)\, v(t)\, dt$$

$$= \int_0^T (Rt)\left\{Q_1^{1-\beta} - \alpha(1-\beta)\,t\right\}^{\frac{1}{1-\beta}} dt$$

$$= -\frac{RT\left\{Q_1^{1-\beta} - \alpha(1-\beta)T\right\}^{\frac{2-\beta}{1-\beta}}}{\alpha(2-\beta)} - \frac{R\left\{Q_1^{1-\beta} - \alpha(1-\beta)T\right\}^{\frac{3-2\beta}{1-\beta}}}{\alpha(2-\beta)(3-2\beta)} + \frac{RQ_1^{3-2\beta}}{\alpha^2(2-\beta)(3-2\beta)}$$

Using equation (6.3.22) in this expression, we have

$$H_1C_1 = \frac{RQ_1^{3-2\beta}}{\alpha^2(2-\beta)(3-2\beta)} \tag{6.3.23}$$

6.3.2.2 Total demand during the cycle time T

This is given by

$$D_1M_1 = \int_0^T \alpha\left(\nu(t)\right)^{\beta} dt = Q_1 \tag{6.3.24}$$

which is the same as equation (6.3.10) when $t_d = 0, \theta = 0$ and $S_d = Q_1 = Q$

6.3.2.3 Total cost per cycle

This is given by

$T_1C_1 = (\text{purchase cost+ordering cost+holding cost})$

$$= \left(c\,Q + C_o + H_1C_1\right) \tag{6.3.25}$$

which is the same as equation (6.3.11) when $t_d = 0, \theta = 0$ and $S_d = Q_1 = Q$

6.3.2.4 The net profit per unit time

This is given by

$$Z_1 = \frac{1}{\mathrm{T}}\{\text{Sales revenue per cycle -Total cost per cycle}\}$$

$$= \frac{1}{T}(r - c)Q_1 - \frac{1}{T}(\ C_0 + H_1 C_1),$$

$$= (r - c)\alpha(1 - \beta)Q_1^{\beta} \ - \ C_o\alpha(1 - \beta)Q_1^{-(1-\beta)} \ - \ \frac{(1 - \beta)RQ_1^{(2-\beta)}}{\alpha(2 - \beta)(3 - 2\beta)} \qquad (6.3.26)$$

which is the same as equation (6.3.13) when $t_d = 0, \theta = 0, \ S_d = Q$ and $HC = H_1 C_1$

The optimum value Q^* of Q giving the optimal profit per unit time is the solution of necessary condition $\frac{d}{dQ_1}(Z_1(Q_1)) = 0$ provided the sufficient condition $\left(\frac{d^2 Z_1}{dQ_1^2}\right)_{Q_1 = Q_1^*} < 0$ is satisfied.

Theorem 6.1: $Z_1(Q_1)$ *is concave with respect to* Q_1.

Proof:

The present net profit per unit time is given by

$$Z_1(Q_1) = (r - c)\alpha(1 - \beta)Q_1^{\beta} \ - \ C_o\alpha(1 - \beta)Q_1^{-(1-\beta)} \ - \ \frac{(1 - \beta)RQ_1^{(2-\beta)}}{\alpha(2 - \beta)(3 - 2\beta)}$$

From which, we have

$$\frac{dZ_1}{dQ_1} = \alpha\beta(1-\beta)(r-c)Q_1^{-(1-\beta)} + C_0\alpha(1-\beta)^2 Q_1^{-2(1-\beta)} - \frac{R(1-\beta)Q_1^{(1-\beta)}}{\alpha(3-2\beta)}$$

and

$$\frac{d^2Z_1}{dQ_1^2} = -\alpha\beta(1-\beta)^2(r-c)Q_1^{-(2-\beta)} - (2-\beta)C_0\alpha(1-\beta)^2 Q_1^{-(3-\beta)} - \frac{R(1-\beta)^2 Q_1^{-\beta}}{\alpha(3-2\beta)}$$

$$= -\left(\alpha\beta(1-\beta)(r-c)Q_1^{-(2-\beta)} + C_0\alpha(2-\beta)(1-\beta)^2 Q_1^{-(3-\beta)} + \frac{R(1-\beta)^2 Q_1^{-\beta}}{\alpha(3-2\beta)}\right) < 0$$

Consequently, $Z_1(Q_1)$ is concave with respect to Q_1. This completes the proof.

Theorem 6.1 shows that $\frac{d^2Z_1}{dQ_1^2}$ is negative for all positive values of the parameters. So the optimal value Q_1* of Q_1 can be obtained from equation (6.3.27) by using any numerical methods such as Newton-Raphson method. Using the optimal solution we obtained from equation (6.3.28), we can find the optimal cycle length T_1* from equation (6.3.22) and corresponding optimal profit Z_1* from equation (6.3.26).

$$\frac{dZ_1}{dQ_1} = \alpha\beta(1-\beta)(r-c)Q_1^{-(1-\beta)} + C_0\alpha(1-\beta)^2 Q_1^{-2(1-\beta)} - \frac{R(1-\beta)Q_1^{(1-\beta)}}{\alpha(3-2\beta)} = 0 \qquad (6.3.27)$$

which leads to

$$\omega_{11} Q_1^{3(1-\beta)} + \omega_{12} Q_1^{(1-\beta)} + \omega_{13} = 0 \qquad (6.3.28)$$

where

$$\omega_{11} = - R$$

$$\omega_{12} = \alpha^2 \beta(3-2\beta)(r-c)$$

$$\omega_{13} = C_0\alpha^2(1-\beta)(3-2\beta)$$

6.4 Selection Criteria

To select the most appropriate Q and T for each problem, we follow the following steps:

I Determine Q and Q_1 from equations (6.3.19) and (6.3.28) respectively

II Determine the corresponding T and T_1 from equations (6.3.9) and (6.3.22) respectively

III If $T_1 < t_d$ and $T \geq t_d$ $\left(\text{provided } \frac{d^2Z}{dQ^2} < 0 \text{ or } \frac{d^2Z_1}{dQ_1^2} < 0\right)$ then compare the profits using equations (6.3.26) and (6.3.13) respectively, in order to select T or T_1 corresponding with the highest profit. Hence select the corresponding Q or Q_1

IV If $T_1 < t_d$ but $T < t_d$ then select T_1 and Q_1

V If $T_1 \geq t_d$ but $T \geq t_d$ then select T and Q $\left(\text{provided } \frac{d^2Z}{dQ^2} < 0\right)$

VI If both $T_1 \geq t_d$ and $T < t_d$ then the solution is infeasible.

6.5 Numerical Examples

To illustrate the proposed models developed, we consider the following two examples:

Example 6.1**:** We take the values of the parameters of the inventory system as $\alpha = 25$ units per time period, $R = 28$ per unit per time period, $t_d = 0.08$, C_o = ₦ 80 per order, c = ₦ 0.5 per unit, r = ₦ 3.5 per unit. For $\beta = 0.5$ and $\theta = 0.05$, the optimum ordering quantity Q^* is determined by Newton-Raphson method using expression (6.3.19) as $Q^* = 88.966$ units and corresponding optimal profit per unit time Z^* is determined using the equation (6.3.13) as Z^* = ₦ 90.211 and optimal cycle time T^* from equation (6.3.9) as T^* = 0.743.

Example 6.2**:** We take the same values of the parameters of the inventory system as Example 6.1 except the parametric values of t_d and θ. They are $\alpha = 25$ units per time period, $R = 28$ per unit per time period, C_o = ₦ 80 per order, c = ₦ 0.5 per unit, r = ₦ 3.5 per unit and $\beta = 0.5$. Having established the sufficient condition, $\frac{d^2 Z_1}{dQ_1^2} < 0$ for optimality of Z_1, then the optimum ordering quantity Q_1^* is determined by Newton-Raphson method using expression (6.3.28) as $Q_1^* = 87.551$ units and corresponding optimal profit per unit time Z_1^* is determined using the expression (6.3.26) as Z_1^* = ₦ 91.092 and optimal cycle time T_1^* from (6.3.22) as T_1^* = 0.748.

The results of Examples 6.1 and 6.2 comparatively show that:

(1) Optimal profit per unit time Z^*= 90.211 of Example 6.1 is smaller than the optimal profit per unit time Z_1^* = 91.092 of Example 6.2.

(2) Optimal ordering quantity Q^*= 88.966 of Example 6.1 is larger than the optimum ordering quantity Q_1^* = 87.551 of Example 6.2.

(3) Optimal cycle time T^*= 0.743 of Example 6.1 is smaller than the optimal cycle time T_1^* = 0.748 of Example 6.2.

The comparative conclusions are obviously expected since when there is no deterioration, the optimal ordering quantity, however decreases and the cycle time will get prolonged, which is also expected, resulting in higher profits.

Table 6.1a: Sensitivity analysis of decision variables Q^*, T^* and Z^* (%) of Example 6.1 with respect to various parameters using percentage estimation from -15 to 15

		Percentage of under estimation and over estimation of parameters						
Variation Parameter	Decision variables	-15	-10	-5		5	10	15
C_o	Q*	-2.91	-1.92	-0.96	0	0.94	1.87	2.78
	T*	-1.39	-0.90	-0.42	0	0.51	0.97	1.41
	Z*	18.11	12.04	6.01	0	-5.98	-11.93	-17.86
C	Q*	1.65	1.10	0.55	0	-0.55	-1.09	-1.61
	T*	0.86	0.59	0.32	0	-0.22	-0.49	-0.76
	Z *	10.08	6.71	3.35	0	-3.34	-6.67	-9.99
α	Q*	-22.76	-15.51	-7.93	0	8.28	16.90	25.89
	T*	3.38	2.14	1.04	0	-0.83	-1.62	-2.34
	Z *	-60.45	-41.54	-21.38	0	22.59	46.39	71.37
t_d	Q*	0.12	0.08	0.04	0	-0.04	-0.08	-0.11
	T*	0.06	0.05	0.05	0	0.05	0.05	0.05
	Z *	-0.07	-0.04	-0.02	0	0.02	0.04	0.06
β	Q*	-36.91	-27.76	-15.87	0	21.77	52.62	97.93
	T*	-6-60	-4.86	-2.66	0	3.39	7.46	12.40
	Z *	-82.39	-60.24	-33.31	0	42.07	96.57	169.29
θ	Q*	-0.27	-0.18	-0.09	0	0.09	0.18	0.27
	T*	0.14	0.11	0.08	0	0.02	-0.01	-0.04
	Z *	0.15	0.10	0.05	0	-0.05	-1.10	-0.15
R	Q*	14.67	9.12	4.33	0	-3.92	-7.50	-10.78
	T*	6.91	4.43	2.15	0	-1.89	-3.71	-5.40
	Z *	28.00	18.03	8.73	0	-8.21	-15.96	-23.30

Table 6.1b: Sensitivity analysis of decision variables Q^*, T^* and Z^* (%) of Example 6.1 with respect to various parameters using percentage estimation from -5 to 5

		Percentage of under estimation and over estimation of parameters									
Variation Parameter	Decision Variables	-5	-4	-3	-2	-1	1	2	3	4	5
C_o	Q*	-0.96	-0.76	-0.57	-0.38	-0.19	0.19	0.38	0.57	0.76	0.94
	T*	-0.42	-0.32	-0.23	-0.14	-0.04	0.15	0.24	0.33	0.42	0.51
	Z*	6.01	4.80	3.60	2.40	1.20	-1.20	-2.40	-3.60	-4.79	-5.98
C	Q*	0.55	0.44	0.33	0.22	0.11	-0.11	-0.22	-0.33	-0.44	-0.55
	T*	0.32	0.26	0.21	0.16	0.11	-0.02	-0.06	-0.11	-0.16	-0.22
	Z *	3.35	2.68	2.80	1.34	0.67	-0.67	-1.34	-2.00	-2.67	-3.34
α	Q*	-7.93	-6.37	-4.80	-3.21	-1.61	1.63	3.27	4.93	6.60	8.28
	T*	1.04	0.83	0.63	0.43	0.24	-0.13	-0.31	-0.49	-0.66	-0.83
	Z *	-21.38	-17.20	-12.97	-8.70	-4.37	4.42	8.89	13.41	17.99	22.59
t_d	Q*	0.04	0.03	0.02	0.02	0.01	-0.01	-0.02	-0.02	-0.03	-0.04
	T*	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05
	Z *	-0.02	-0.03	-0.01	-0.01	-0.00	0.01	0.01	0.01	0.02	0.02
β	Q*	-15.87	-13.06	-10.09	-6.93	-3.57	3.80	7.86	12.19	16.82	21.77
	T*	-2.66	-2.17	-1.65	-1.10	-0.54	0.67	1.31	1.97	2.67	3.90
	Z *	-33.31	-27.22	-20.86	-14.22	-7.27	7.62	15.61	23.99	32.81	42.08
θ	Q*	-0.09	-0.07	-0.05	-0.04	-0.02	0.02	0.04	0.05	0.07	0.09
	T*	0.08	-0.08	-0.07	-0.06	-0.06	0.05	0.04	0.03	0.03	0.02
	Z *	0.05	0.04	0.03	0.02	0.01	-0.01	-0.02	-0.03	-0.04	-0.05
R	Q*	4.33	3.43	2.54	1.68	0.83	-0.82	-1.61	-2.40	-3.17	-3.92
	T*	2.15	1.72	1.30	0.87	0.46	-0.35	-0.74	-1.13	-1.52	-1.89
	Z *	8.72	6.94	5.17	3.43	1.70	-1.68	-3.34	-4.98	-6.61	-8.21

6.6 Results and Discussion

Based on computational results of Example 6.1 as shown in Table 6.1a and Table 6.1b, the observations can be made as follows:

(1) A higher value of β results in higher values of Q^*, T^* and Z^*. This implies that increase in the stock-dependent demand rate will result in the increase of optimal ordering quantity, optimal length of cycle time and optimal average profit per unit time. This is expected since the deterioration rate remains constant.

(2) A higher values of θ result in lower values of Q^*, T^* and Z^*. This implies that increase in deterioration rate will result in the decrease of optimal ordering quantity, optimal length of cycle time and optimal average profit per unit time. This is also clearly expected since deterioration of the item discourages its stocking, and so every one of the three parameters decreases.

(3) A higher value of t_d results in higher values of T^* and Z^*, but lower values of Q^*. This implies increasing the fresh product time increases the length of cycle time and the optimal average profit per unit time. The optimal ordering quantity, however decrease in this case, which is also expected since when there is delay in deterioration, the cycle time will get prolonged resulting in higher profits since there is no much deterioration cost.

(4) A higher values of R result in lower values of Q^*, T^* and Z^*. This implies that increase in unit holding cost per unit time will result in the decrease of optimal ordering quantity, optimal length of cycle time and the optimal average profit per unit time.

This is clearly expected since increase in the holding cost of the item discourages its stocking, and so reduces the profit.

6.7 Sensitivity Analysis

Generally, models usually are formulated in order to select some future course of action. Therefore, the parameters used would be based on a prediction of future conditions, which inevitably introduces some degree of uncertainty. For this reason it usually is important to conduct a sensitivity analysis after finding the models solution with the assumed parameter values. The general purpose is to identify those parameters that cannot be changed much without changing the optimal solution, and then to select a solution which remains a good one over the ranges of likely values of the sensitive parameters. Using the data of Example 6.1 for a combination of $\beta = 0.5$ and $\theta = 0.05$, the sensitivity analysis of the decision variables Q^*, T* and Z* has been carried out by changing the value of each of the parameters by -15%, -10%, -5%, 5%, 10% and 15% (Table 6.1a) and also by -5%, -4%, -3%, -2%, -1%, 1%, 2%, 3%, 4% and 5% (Table 6.1b) , taking one parameter at a time and keeping the remaining parameters unchanged. The results are tabulated in table 6.1a and Table 6.1b. The main observations of the result of the sensitivity analysis with respect to the parameters are as follows:

(i) Case for Table 6.1a

(1a) A 15% over estimation in the value of the ordering cost C_o results in increases of Q^* and T^* by 3% and 1% respectively but decrease in Z^* by 18%. A 15% under estimation in C_o results in decreases of Q^* and T^* by 3% and 1% respectively but increases in Z^* by 18%. Thus $Q^*$and T^* are less sensitive and Z^* is highly sensitive to changes in the values of the parameter C_o.

(2a) A 15% over estimation in the value of the scale parameter α results in increases of Q^* and Z^* by 26% and 71% respectively while T^* decreases by 2%. For a 15% under estimation in α results in decreases of Q^* and Z^* by 23% and 60% respectively while T^* increases by 3%. Thus $Q^*$and T^* are highly sensitive and Z^* is less sensitive to changes in the values of the parameter α .

(3a) A 15% over estimation in the value of the scale parameter β results in increases of Q^*, T^* and Z^* by 97%, 13% and 169% respectively. On the other hand, 15% under estimation in β results in decreases of Q^*, T^* and Z^* by 37%, 7% and 82% respectively. Thus Q^* and Z^* are highly sensitive and T^* is moderately sensitive to over estimation and insensitive to under estimation to changes in the values of the parameter β .

(4a) A 15% over estimation in the value of the unit holding cost R results in decreases of Q^*, T^* and Z^* by 11%, 5% and 23% respectively. On the other hand, 15% under estimation in R results in increases of Q^*, T^* and Z^* by 14%, 7% and 28% respectively. Thus Q^* is moderately sensitive, T^* is fairly sensitive and Z^* is highly sensitive to changes in the values of the parameter R.

(ii) Case for Table 6.1b

(1b) A 5% over estimation in the value of the ordering cost C_o results in increases of Q^* and T^* by 0.94% and 0.51% respectively but decrease in Z^* by 5.98%. A 5% under estimation in C_o results in decreases of Q^* and T^* by 0.96% and 0.42% respectively but

increases in Z^* by 6.01%. Thus $Q^*$and T^* are less sensitive and Z^* is highly sensitive to changes in the values of the parameter C_o.

(2b) A 5% over estimation in the value of the scale parameter α results in increases of Q^* and Z^* by 8.28% and 22.59% respectively while T^* decreases by 0.83%. For a 5% under estimation in α results in decreases of Q^* and Z^* by 7.93% and 21.38% respectively while T^* increases by 1.04%. Thus $Q^*$and T^* are highly sensitive and Z^* is less sensitive to changes in the values of the parameter α .

(3b) A 5% over estimation in the value of the shape parameter β results in increases of Q^*, T^* and Z^* by 21.77%, 3.90% and 42.08% respectively. On the other hand, 5% under estimation in β results in decreases of Q^*, T^* and Z^* by 15.87%, 2.66% and 33.31% respectively. Thus Q^* and Z^* are highly sensitive and T^* is moderately sensitive to over estimation and insensitive to under estimation to changes in the values of the parameter β .

(4b) A 5% over estimation in the value of the unit holding cost R results in decreases of Q^*, T^* and Z^* by 3.92%, 1.89% and 8.21% respectively. On the other hand, 5% under estimation in R results in increases of Q^*, T^* and Z^* by 4.33%, 2.15% and 8.72% respectively. Thus Q^* is moderately sensitive, T^* is fairly sensitive and Z^* is highly sensitive to changes in the values of the parameter R.

Using the same argument as above for both Table 6.1a and Table 6.1b, we observe that all the decision variables Q^*, T^* and Z^* are insensitive to both over estimation and

under estimation of the parameters t_d, c and θ, except Z^* which is moderately sensitive to changes in the values of c.

6.8 Concluding Remarks

In this chapter, an EOQ model for delayed deteriorating items with inventory level dependent demand rate was developed. The model considers linear time dependent holding cost but taking a constant deterioration rate. The demand rate is in polynomial functional form (Urban and Baker, 1988) and dependent on the instantaneous inventory level. In the initial stage, inventory depletes down to a certain level of the inventory due to market demand only and dependent on inventory level. In the second stage the inventory level gets depleted due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero at the end of the cycle. Several items (especially in tropical countries) such as cassava, yams, potatoes, some fruits and vegetables, and so on exhibit this property. This structure of linear time dependent holding cost is representative of many real-life situations and this is particularly true in the storage of some deteriorating and perishable items such as food products. We considered two cases: (i) when reorder point occurs after deterioration takes $(0 < t_d < T)$ and (ii) when reorder point occurs before deterioration takes $(t_d \geq T \text{ or } 0 < T \leq t_d)$; and the inventory system in each case has zero-ending inventory.

The objective of the models in each case is to determine the optimal replenishment policy to maximize average net profit per unit time. Our findings show that if the holding cost per unit were to be a constant, then (i) our model of section 6.3.2 would reduce to the model of Baker and Urban (1988) and (ii) if $t_d = 0$, our model of section 6.3.1 would reduce to zero - ending inventory of Pal *et al*. (1993). We obtain an

expression for the net average profit per unit time for the inventory system in each case in order to find the optimal order quantity and optimal replenishment cycle time. The results are illustrated with the help of numerical examples and a sensitivity analysis (only for model of section 6.3.1) of the decision variables with regard to changes in the model parameters.

Our results indicate that the effect of stock dependent demand parameter β and holding cost parameter R on the profit is significant. On examining the effect of system parameters on the decision variables, we find that Q^* and Z^* are sensitive to the over estimation and under estimation of the parameters C_o, α , β and R while they are more or less insensitive to the parameters t_d, c and θ. The proposed model can further be extended by incorporating inflation, time-value of money, quantity discounts, trade credits, relax the restriction of constant deterioration rate, allow shortage and so on.

CHAPTER SEVEN

SUMMARY, CONCLUSION AND RECOMMENDATIONS

7.1 Summary

The main aim of this study is to develop deterministic inventory models for delayed deteriorating items with inventory level dependent demand rate. The aim has been achieved through five main objectives. These objectives (as already stated in section 1.10) were accomplished by way of developing five models of Economic Order Quantity (EOQ) and Economic Production Quantity (EPQ) for delayed deteriorating items with inventory -level- dependent demand rate.

The first EPQ model presented is that of an inventory system for delayed deteriorating items with stock-dependent demand rate which is in polynomial functional form. In the first stage, inventory depletes down to a certain level of the inventory due to market demand only which is inventory level dependent. In the second stage, the inventory level gets depleted due to the effect of both market demand and deterioration but still dependent on stock until the inventory level falls to zero at the end of the cycle. The aim is to find the optimal ordering quantity and optimal cycle time for the inventory system, in order to minimize the total average system cost per unit time. Newton-Raphson method has been used to find the optimal order quantity and the optimal cycle length. Furthermore, some numerical examples have been presented to illustrate the application of the model developed and use the examples to study the effect of various changes in some possible combinations of model parameters on the decision variables.

The second model is an EOQ model for delayed deteriorating items with stock-dependent demand, allowing shortages and with fixed backlogging rate. The model is an amendment of the first model by allowing shortages. We assume a fixed fraction of demand rate to be backlogged during the shortage period as in Wee (1995).

The next two models presented are economic production quantity (EPQ) models. In both models, there is a delay in deterioration and the production rate is constant, demand rate is inventory level dependent in a linear functional form, both before and after production. In the first model, the holding cost is a linear function of time and the deterioration rate is a constant while in the second model, the holding cost is a constant but the deterioration rate is time dependent. Deterioration starts immediately production stops and so the delay in deterioration is that period when production is in progress. The main objective in both cases is to determine the optimal replenishment cycle time such that the total variable cost is minimised. Some numerical examples are presented to illustrate the application of the models developed and the examples are used to study the effect of various changes in some possible combinations of model parameters on the decision variables of the system.

The last model is for an optimal replenishment policy for delayed deteriorating items with power-form inventory level dependent demand rate. The holding cost is assumed to be linear while the deterioration rate is a constant. The difference between this model and the first model is that the holding cost in this model is linear time dependent and not a constant as in the first model. Moreover, the aim in this model is to maximize profit while cost is minimized in the first model. We consider two cases in this model: (i) when replenishment occurs after deterioration sets in $\left(t_d < T\right)$ and (ii) when

replenishment occurs before deterioration sets in $\left(\mathrm{t_d} \geq \mathrm{T} \text{ or } 0 < T \leq t_d\right)$. The inventory system in each case is zero-ending and the objective is to determine the optimal replenishment policy to maximize average net profit per unit time. If the holding cost per unit were to be a constant, then (i) our model of section 6.3.2 would reduce to the model of Baker and Urban (1988), (ii) if $t_d = 0$, our model of section 6.3.1 would reduce to Pal *et al*. (1993), provided its terminal inventory $i_T = 0$. The application of the model is illustrated with the help of numerical examples and a sensitivity analysis is carried out on the decision variables to see the effect of changes in the model parameters.

7.2 Conclusion

Different model parameters and assumptions govern the different inventory systems investigated in this thesis. Specific conclusions were given at the end of each of the inventory models investigated. However, general conclusions are drawn across the entire research work which is summarized as follows:

1. A model has been presented of an inventory system for delayed deteriorating items with stock-dependent demand rate which is in polynomial functional form. As expected by the result shown in Table 3.1, the deterioration rate θ and the stock-dependent parameter β have impact on the optimal solutions and hence, should not be neglected in modelling deteriorating inventory models with inventory dependent demand. This model helps to determine the optimum ordering quantity and optimum ordering cycle for situations where shortages are not permitted and the replenishment rate is infinite. For that reason, any inventory system with the kind of characteristics of

the first model assumptions in Section 1.10 can use the model to determine its inventory policy.

2. An EOQ model is presented for delayed deteriorating items with stock-dependent demand, allowing shortages and fixed backlogging rate. The impact of stock dependent demand rate, constant rate of deterioration and partial backlogging parameters on order quantity, maximum inventory level and total system cost per unit time were reported. We find from the results (Tables 4.1 - 4.4) that the effects of stock dependent demand rate, holding cost parameter, deterioration and backlogging rate on the optimal replenishment policy are significant, and hence should not be ignored in developing such inventory models. The proposed model can be used in controlling the inventory of certain delayed deteriorating items such as food items, electronic components, fashionable commodities, vegetables, fruits, yams, potatoes, and so on.

3. The first EPQ model presented under Section 5.2 in chapter five is for single product with delayed deterioration in which the production rate is constant, demand rate is inventory level dependent and in a linear functional form before and after production. The holding cost is a linear function of time. The results as tabulated in Table 5.1 reveal that for various values of inventory level dependent demand rate parameter $(\beta = 0.70, 0.60, 0.50, 0.04, 0.10)$ and deterioration rate $(\theta = 0.10, 0.20, 0.05, 0.08, 0.02)$, the optimum values of production quantity and average total cost function per unit time were determined using the expressions for (4.2.4) and (4.2.10) respectively. This structure of linear time dependent holding cost is representative of many real-life

situations and this is particularly true in the storage of some deteriorating and perishable items such as food products.

4. The second EPQ model under Section 5.3 in chapter five, is for a single product with delayed deterioration in which the production rate is constant, demand rate is inventory level dependent and in a linear functional form before and after production. The deteriorating rate is a linear function of time. We find from the results (Tables 5.2 - 5.5) that the effects of changing the model parameters $\beta, \omega, C_{s.}$ and λ on the optimal replenishment policy reveal the following:

(i) When the stock-dependent consumption rate β is increasing, the optimal cost is increasing.

(ii) When the deterioration rate ω is increasing, the optimal cost is increasing.

(iii) When the set up cost C_s is increasing, the optimal cost is increasing.

(iv) When the production rate λ is increasing, the optimal cost is increasing.

The model helps in determining the optimum ordering quantity for stock dependent demand rate items under linear time dependent holding cost and delayed deterioration.

5. In chapter six, an EOQ model for delayed deteriorating items with inventory level dependent demand rate is considered. The model considers linear time dependent holding cost but taking a constant deterioration rate. This structure of linear time dependent holding cost is representative of many real-life situations and this is particularly true in the storage of some deteriorating and perishable items such as food products. Our results indicate that the effect of stock dependent demand parameter β and holding cost parameter R on the profit is significant.

The results of sensitivity analysis of the decision variables with regard to changes in the model parameters indicate that Q^* and Z^* are sensitive to over estimation and under estimation of the parameters C_o, α, β and R while they are more or less insensitive to the parameters t_d, c and θ. This is equivalent to saying that the optimum ordering quantity and net profit per unit time are sensitive to over estimation and under estimation of ordering cost, stock dependent demand parameter and instantaneous rate of holding cost while they are more or less insensitive to the parameters of fresh product time, unit purchase cost and rate of deterioration.

6. Our findings show that if the holding cost per unit were to be a constant, then (i) our model of section 6.3.2 would reduce to the model of Baker and Urban (1988) and (ii) if $t_d = 0$, our model of section 6.3.1 would reduce to zero - ending inventory of Pal *et al*. (1993).

The contributions of this research include the following:

(1) A model has been developed of an inventory system for delayed deteriorating items with stock-dependent demand rate by extending the work of Musa and Sani (2009) where the demand rate is a constant by considering a situation where the demand rate is a power-form function as in Urban and Baker (1988). As expected by the result of the model developed it has been shown that, the deterioration rate θ and the stock-dependent parameter β have impact on the optimal solutions and hence, should not be left out in modelling delayed deteriorating items which are inventory level dependent demand rate.

(2) A model has been presented of an inventory system for delayed deteriorating items with inventory level dependent demand rate as a linear function of inventory level and assumed a fixed fraction of demand rate to be the backlogging rate during the shortage period as in Wee (1995). This is an extension of the first contribution where shortages are not allowed and the demand rate function is a power-form function.

(3) An attempt has been made to extend the model of Sugapriya and Jeyarama (2008a) who established an inventory model for non-instantaneous deteriorating item with constant demand rate for both in- production run and out-production run, while we have developed two EPQ models for delayed deteriorating item in which the production rate is constant, demand rate is inventory level dependent and (1) the holding cost is a linear function of time to reflect the fact that holding cost increases linearly with time, (2) the deteriorating rate is a linear function of time to also reflect the fact that deterioration increases with each passing time.

(4) Some authors including Urban (1992) and Chang (2004) have suggested that an inventory system that possesses an inventory -level-dependent demand rate, in which the inventory level falls to zero at the end of the order cycle may not provide the optimal solution. However, we find from the results of the models developed that the terminal condition of zero-ending inventory provide optimal solution in which inventory item is the stock dependent and the holding cost is linear time-dependent. In addition, our findings show that if the holding cost per unit were to be a constant, then (i) our model of section 6.3.2 would reduce to the model of Baker and Urban (1988) and (ii) if $t_d = 0$, our model of section 6.3.1 would reduce to zero - ending inventory of Pal *et al*. (1993).

7.4 Recommendations

The models presented in this research provide basis for several possible extensions including incorporating inflation, time - value of money, discount rates, salvage cost, quantity discounts, trade credits; other variable forms of deterioration rate, variable partial backlogging rate and variable holding cost functions, and so on.

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Appendix A

Explanation of Polynomial demand rate function of Baker and Urban (1988)

The equation $D\left[i(t)\right] = \alpha\left(i(t)\right)^{\beta}$, where $D\left[i(t)\right]$ is the demand rate of the product at any instant of time t, i(t) is the inventory level, $\alpha > 0$ is the scale parameter, and $\beta\left(0 < \beta < 1\right)$ is the shape parameter that measures the responsiveness of the demand rate to changes in the level of on-hand inventory are characterized by:

(1) Diminishing returns: By constraining the shape parameter, β, to remain between zero and one, the marginal increase in demand rate will decrease for larger values of the inventory level. Although there is no maximum limit on the level of demand with this functional form (that is, the curve does not approach a constant limiting value as the inventory level approaches infinity), it can be shown that the slope of the curve approaches zero. We would not realistically expect a situation in which the demand rate continued to increase with an 'infinitely' large level of inventory.

(2) Inventory-level elasticity: The shape parameter of the demand rate function represents the ratio of the percentage change in the quantity demanded to the percentage change in the inventory level. It is a measure of responsiveness of the demand rate to changes in the level of all other things being equal.

(3) Richness: Unless the actual distribution is quite irregular, we can expect this function parameters (varying values of the shape and scale parameters) to provide a good approximation of the situations. As β approaches zero, the function will approach a horizontal line such that the demand rate function will equal a constant, α. On the

other extreme, as β approaches one, the demand rate function will approach a straight line with an intercept of zero and a positive slope (αi).

(4) Intrinsic linearity: Another reason this function is desirable is its simplicity and linearity. This function is intrinsically linear; a logarithmic transformation can be used to transform the demand function into a straight line. The functional form of *D(i)* possesses intrinsic linearity ln(D(i)) and is linearly dependent on ln(i(t)). This helps in estimation of the parameters by using simple linear regression, assuming that the error terms are also assumed to be multiplicative.

Appendix B

Estimation of the parameters α and β

Finding appropriate values of α and β in $D(v) = \alpha\left(v(t)\right)^{\beta}$ is an important task for practical users of the proposed model. The system parameters can be estimated by applying the least square regression to a set of observed data $\left(v, D(v)\right)$, where v is the starting inventory level and D(v) the quantity demanded during a relatively short length of time. Taking the logarithm on both sides of the demand function, we have $\log D(v) = \log\alpha + \beta \log v$. Setting $Y = \log D(v), A = \log\alpha, B = \beta$ and $X = \log v$, the above transformed equation be rewritten as Y = A +BX. The slope B and the intercept A of a straight line are estimated from the application of regression to the observed values of (X, Y), from which the corresponding values of α and β are found.

The parameter θ can be estimated by keeping records about physical decay of the item over time.

Appendix C

Proof of shape parameter β as the ratio of the percentage change in the quantity demanded to the percentage change in the inventory level by (Pal *et al*. 1993)

The inventory-level-dependent demand rate function is taken in the form (Baker and Urban, 1988).

$R(i) = \alpha i^{\beta}$, $\alpha > 0, 0 < \beta < 1$.

The marginal increase in demand rate is

$$MD = \frac{dR}{di} = \frac{\alpha\beta}{i^{1-\beta}} > 0.$$

This shows that the demand rate R(i) increases as the inventory level i(t) increases.

Again,

$$\frac{d^2R}{di^2} = \frac{d(MD)}{di} = \frac{\alpha\beta(\beta-1)}{i^{2-\beta}} < 0 \text{ since } 0 < \beta < 1$$

Hence, the marginal demand (MD) is a decreasing function of the inventory level i(t). Thus, the demand rate increases at a decreasing rate as the inventory level increases. Although $R(i) \to \infty$ as $i(t) \to \infty$, we find that $MD \to 0$ as $i(t) \to \infty$ since $1-\beta > 0$. This means that, even if there is no upper bound for the level of demand with the present functional form for R(i), the slope of the demand curve tends to zero as the inventory level decreases.

It is noted further that

$$\frac{dR}{di} = \alpha\beta i^{\beta-1} = \frac{\beta}{i}R,$$

so that $\left[100(dR/R)\right] / \left[100(di/i)\right] = \beta$

Thus, the shape parameter β equals the ratio of the percentage change in the quantity demanded to the percentage change in the inventory level.

Also,

$$\beta = \frac{dR}{di} \cdot \frac{i}{R}$$

is the elasticity of the demand rate with respect to the inventory level. Thus, β measures the response of the demand rate to changes in the inventory level.

As $\beta \to 0,\ R(i) \to \alpha$ so that the demand rate becomes constant. On the other hand, $R(i) \to \alpha i$ as $\beta \to 1$; this indicates linear dependence of the consumption rate on the inventory level.

Another advantage of the present functional form of R(i) is its intrinsic linearity in that lnR(i) is linearly dependent on ln i(t). This helps in estimation of the parameters by using simple linear regression assuming that the error terms are also multiplicative.

Appendix D

Linear Demand rate function of Gupta and Vrat (1986)

Gupta and Vrat (1986) have adopted the following forms of demand rate function:

$$R(Q) = \alpha + \beta Q^{\gamma} \quad \text{(i)}$$

$$R(Q) = \alpha + \beta e^{Q} \quad \text{(ii)}$$

where Q denotes the order quantity and α, β, γ are all positive constants. In both of these forms, *R(Q)* reduces to a constant when $Q = 0$. Moreover, if $\gamma > 1$ in equation (i), *R(Q)* as well its slope MD goes on increasing without limit as Q increases. If $\gamma > 2$, *R(Q)* increases at an increasing rate Q increases. These features of *R(Q)* are obviously unrealistic.

In the form (ii), we find that R(Q), MD and the rate of change of MD are all increasing exponentially with Q. This sort of situation is not observed in any practical inventory system.

Appendix E

Explanation of demand rate function of Mandal and Phaudjar (1989a)

Mandal and Phaudjar (1989a) have used the form:

$$R(i) = \alpha + \beta i, \quad \alpha \geq 0, \ \beta \geq 0 \qquad \text{(C1)}$$

Here $R(i)$ increases linearly with i(t). Equation C1 is a particular case of R(i) in Baker and Urban (1988) for $\alpha = 0$. For $\alpha > 0$, this form implies constant demand rate when the inventory level falls to zero. At a first glance, this may appear to be queer in the inventory level dependent demand rate; one may argue that R(i) should be zero when i(t) = 0. However, there is no reason to specify that the demand rate cannot be constant (positive) with zero inventory for all items. Some products that have inventory – level - dependent demand rate may be such that some demand is realized even if there is no stock, although considerably less than that with a positive inventory level

Appendix F

Simplification of derivatives of S_d , T and S_d in powers from equations (6.3.5) and (6.3.9) of section 6.3

From equation (6.3.9)

$$T = \frac{t_d(1-\beta)\left(\theta S_d^{1-\beta} + \alpha\right) + S_d^{1-\beta}}{(1-\beta)\left(\theta S_d^{1-\beta} + \alpha\right)} = \frac{P}{(1-\beta)N}$$

where

$$N = \theta S_d^{1-\beta} + \alpha \text{ and } P = t_d(1-\beta)\left(\theta S_d^{1-\beta} + \alpha\right) + S_d^{1-\beta} = t_d(1-\beta)N + S_d^{1-\beta}$$

$$T' = \frac{\alpha Q^{-\beta}}{\left(\theta S_d^{1-\beta} + \alpha\right)^2} = \frac{\alpha Q^{-\beta}}{N^2}$$

$$N' = \left(\theta S_d^{1-\beta} + \alpha\right)' = \theta\left(S_d^{1-\beta}\right)' = \theta(1-\beta)Q^{-\beta}$$

Using equation (6.3.5), we obtain

$$\left(S_d\right)' = \left(Q^{1-\beta} - \alpha(1-\beta)\,t_d\right)^{\frac{\beta}{1-\beta}} Q^{-\beta} = S_d^{\beta} Q^{-\beta}$$

$$\left(S_d^{\beta}\right)' = \beta S_d^{2\beta-1} Q^{-\beta}, \qquad \left(S_d^{1-\beta}\right)' = (1-\beta)Q^{-\beta}$$

$$\left(S_d^{2(1-\beta)}\right)' = 2(1-\beta)S_d^{1-\beta}Q^{-\beta}, \qquad \left(S_d^{2-\beta}\right)' = (2-\beta)S_d Q^{-\beta}$$

$$\left(S_d^{3-2\beta}\right)' = (3-2\beta)S_d^{2-\beta}Q^{-\beta}, \qquad \left(S_d^{-2\beta}\right)' = -2\beta S_d^{-(\beta+1)}Q^{-\beta}$$