SUBSEMIGROUPS GENERATED BY QUASI-IDEMPOTENTS IN CERTAIN FINITE SEMIGROUPS OF MAPPINGS

By

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B.Sc. (ABU, 2009)

MSC/SCIE/283/2010-2011

DEPARTMENT OF MATHEMATICS

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Declaration

I declare that the work in this thesis entitled "SUBSEMIGROUPS GENERATED BY QUASI-IDEMPOTENTS IN CERTAIN FINITE SEMIGROUPS OF MAPPINGS" has been performed by me in the Department of Mathematics under the supervision of Prof. G. U. Garba and Dr. A. M. Ibrahim. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any university.

IMAM, Abdussamad Tanko

Name of student	Signature	Date

Certification

This thesis entitled "SUBSEMIGROUPS GENERATED BY QUASI IDEMPOTENTS IN CERTAIN FINITE SEMIGROUPS OF MAPPINGS" by IMAM, Abdussamad Tanko, meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria, and is approved for its contribution to knowledge and literary presentation.

Prof. G. U. Garba — Date

Chairman Supervisory Committee

Dr. A. M. Ibrahim — Date

Member Supervisory Committee

Dr. B. A. Madu — Date

External Examiner

Dr. B. Sani — Date

Head of Department

Prof. A. A. Joshua — Date

Dean, Postgraduate School

Dedication

To my lovely Mother, Maryam Muhammad.

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Abstract

Let X_n be the finite set $\{1, 2, \ldots, n\}$ and let $\mathcal{T}_n$ and $\mathcal{I}_n$ be the full transformation semigroup and the symmetric inverse semigroup on X_n respectively. In this thesis, we consider quasi-idempotent elements in the two semigroups $\mathcal{T}_n$ and $\mathcal{I}_n$. For the semigroup $\mathcal{T}_n$, we characterise quasi-idempotent elements and establish that the subsemigroup $Sing_n$ of all singular self-maps of $\mathcal{T}_n$ is generated by a set of quasi-idempotents called *quasi-idempotents of type one.* We also obtained the quasi-idempotent rank of $Sing_n$. This number coincides with the rank and idempotent rank of $Sing_n$ which is $\frac{1}{2}n(n-1)$. For the semigroup $\mathcal{I}_n$, we characterise quasi-idempotent elements and prove that the subsemigroup $\mathcal{SI}_n$, of all strictly partial injections of $\mathcal{I}_n$, is generated by quasi-idempotents of type one.

Contents

CHAPTER 1

INTRODUCTION

1.1 Background of the Thesis

The algebraic theory of semigroup has been well-researched during the second half of the twentieth century. Many ideas in the study of semigroup were directly motivated by their analogues in group and ring theory. However certain important and interesting concepts such as the concepts of Green's relations, idempotent rank, nilpotent rank, etc developed largely independently. Consequently this gives semigroup a separate and well deserved field of study.

Just as the set of all permutations of a (finite) set proves to be an important source of examples in group theory, the set of all mappings of a (finite) set into itself provide us with a corresponding object in semigroup theory. The importance of mappings in semigroup theory can be judged by the fact that every semigroup is isomorphic to a semigroup of mappings. In this sense it turns out that the understanding of semigroups of mappings in semigroup theory is of paramount significance. It is observed that the most important contribution related to (some) semigroups of mappings is Ganyushkin and Mazorchuk [4]. Nevertheless, in the recent years, many publications regarding semigroups of mappings are found in significant Journals and conference proceedings (see the reference). In our study, we draw particular attention to Howie [19], Howie [20], Gomes and Howie [14, 15], Howie and McFadden [22], Garba [5, 6, 9, 10] and Madu and Garba [26]. In this thesis we consider two important semigroups of mappings: *full transformation semigroup* and *symmetric inverse semigroup*, and study certain type of elements called *quasi-idempotents*.

1.2 Objectives of the Research

As discussed above, the theory of semigroup of mappings is yet to be fully developed. Therefore the main objective of this thesis is to develop the theory of semigroups of mappings to a greater level. Specifically, we set out the following objectives:

(i) To characterise quasi-idempotents in finite full transformation semigroup;

(ii) To establish that the subsemigroup of singular selfmaps of the full transformation semigroup is quasi-idempotent generated;

(iii) To compute the quasi-idempotent rank of the semigroup of singular selfmaps;

(iv) To characterise quasi-idempotents in finite symmetric inverse semigroup;

(v) To establish that the finite symmetric inverse semigroup is quasi-idempotent generated.

It is hoped that the results obtained by this research will form a part of the development of the theory of semigroups of mappings.

1.3 Research Methodology

The method of research adopted in this thesis is by consulting necessary and relevant papers, in literature, on the theory of transformation semigroups. These papers are thoroughly reviewed to cover a major part of the works done for idempotent elements, nilpotent elements and quasi-idempotent elements in transformation semigroups. The works done on quasi-idempotent elements are then extended to some finite transformation semigroups.

1.4 Outline of the Thesis

This thesis contains four other chapters after this introductory chapter. The outline of the remaining chapters is as follows:

Chapter 2: In this chapter, we present a survey of the necessary and relevant literature for products and rank of idempotents and nilpotents in (some) semigroups of mappings.

Chapter 3: In this chapter, we focus our attention to the full transformation semigroup and give a characterisation of quasi-idempotents. We then proceed to establish that the singular part of the full transformation semigroup is quasi-idempotent generated there by computing its quasi-idempotent rank.

Chapter 4: In this chapter we focus on the symmetric inverse semigroup and prove analogously the same results as in the case of full transformation semigroup.

Chapter 5: In this chapter, we give a summary of the results obtained in the thesis along with some directions for future research.

1.5 Preliminaries

We start by presenting the basic semigroup theory needed for understanding the results of this thesis. All definitions and results are typical and can be found in any introductory text on semigroup theory. The monograph Howie [21] is an excellent introduction to semigroup theory, see also [23].

1.5.1 Semigroups: subsemigroup, ideal and generating set

A *semigroup* is simply a set S which is closed under an associative binary operation, usually denoted by juxtaposition. That is, $(xy)z = x(yz)$ for all $x, y, z \in S$. The semigroup S is called a *monoid* if it has identity, that is, if it contains an element 1 with the property that $x1 = 1x = x$ for all $x \in S$. An element $0 \in S$, with the property that $x0 = 0x = 0$ for all $x \in S$, is called zero element of S and S is called a *semigroup with zero*. Note that if S has no identity or zero, we can adjoin an extra identity or zero to S, so that S becomes a monoid with zero. We write S^1 and S^0 to denote the semigroup with identity or zero adjoined if necessary. Thus,

$$S^1 = \begin{cases} S & \text{if } S \text{ has identity element,} \\ S \cup \{1\} & \text{otherwise,} \end{cases}$$

and

$$S^0 = \begin{cases} S & \text{if } S \text{ has zero element,} \\ S \cup \{0\} & \text{otherwise,} \end{cases}$$

where we define $1s = s1 = s,\ 11 = 1$ and $0s = s0 = 00 = 0$ for all s in S. Note that a semigroup can contain at most one identity element and at most one zero element.

A subset T of a semigroup S is called a *subsemigroup* of S if it is closed under the binary operation of S. For example if S is a semigroup with zero, the set T of all elements of S, in which the product of any two is zero, is a subsemigroup of S.

Let A and B be two non-empty subsets of a semigroup S. We define the product of A and B as

$$AB = \{ab : a \in A, b \in B\}.$$

Therefore, in this sense, $A^2 = \{a_1a_2 : a_1, a_2 \in A\}$ and $Ab = \{ab : a \in A\}$.

A non-empty subset A of S is called a *left ideal* if $SA \subseteq A$, a *right ideal* if $AS \subseteq A$, and an (two-sided) *ideal* if it is both a left and a right ideal. From this definition it is evident that every (right, left, or two-sided) ideal is a subsemigroup, but the converse is not the case. Among the ideals of S are S itself and (if S has a zero element) $\{0\}$. An ideal I such that $\{0\} \subset I \subset S$ (strictly) is called *proper.* A semigroup S is said to be *simple* if it has no proper ideal. A semigroup S with zero is called *0-simple* if $\{0\}$ and S are the only ideals of S. For technical reasons the trivial semigroup and the two-element semigroup are not included among either simple or 0-simple semigroups.

Let $\{T_i : i \in I\}$ be an indexed family of subsemigroups of S. The intersection $\cap T_i$, of all the subsemigroups T_i, is a subsemigroup of S. In particular, for any non-empty subset A of S, the intersection of all subsemigroups of S containing A is a subsemigroup of S containing A. In fact it is the smallest subsemigroup of S containing A and we denote it by $\langle A \rangle$. This subsemigroup consists of all elements of S that can be expressed as finite products of elements in A. It is called the subsemigroup of S generated by A. If $\langle A \rangle = S$, we say that A is a *generating set* for S. In the case $A = \{a\}$, a singleton, we write $\langle a \rangle = \{a, a^2, a^3, \ldots\}$ and refer to $\langle a \rangle$ as the *monogenic subsemigroup* of S generated by a. If $\langle a \rangle = S$, we say that S is a *monogenic* or *cyclic* semigroup [1]. The order of an element $a \in S$ is defined as the *order* (the number of elements) of the subsemigroup $\langle a \rangle$. A semigroup S is called *periodic* if all its elements are of finite order.

An element $e \in S$ is called an *idempotent* if $e^2 = e$. We will use $E(S)$ to denote the set of all idempotents in S. A semigroup S in which $E(S) = S$ is called a *band*. If $\langle E(S) \rangle = S$, then S is called a *semiband*.

Theorem 1.5.1 (Howie [21]) *In a periodic semigroup, every element has a power which makes it an idempotent.*

1.5.2 Regular Semigroups

An element a of a semigroup S is called *regular* if there exists x in S such that $axa = a$. The semigroup S is called *regular* if all its elements are regular. That is, if

$$(\forall\, a \in S)(\exists\, x \in S)\; axa = a. \tag{1.1}$$

A regular semigroup must contain idempotent elements. It immediately follows from (1.1) that both ax and xa are idempotents.

An element $a' \in S$ is an inverse of a if $aa'a = a$ and $a'aa' = a'$. A regular semigroup in which every element has a unique inverse is called an *inverse semigroup*.

Theorem 1.5.2 (Howie [21]) *A semigroup is an inverse semigroup if and only if it is regular and idempotents commute.*

Proof. Suppose that S is regular and that idempotents commute. Let a', a'' be inverses of a. Then

$$\begin{aligned} a' &= a'aa' = a'(aa''a)a' = (a'a)(a''a)a' = a''aa'aa' = a''aa' \\ &= a''(aa''a)a' = a''(aa'')(aa') = a''aa'aa'' = a''(aa'a)a'' = a''aa'' = a''. \end{aligned}$$

Conversely, suppose that inverses are unique, and let e, f be idempotents. Let x be the unique inverse of ef, that is,

$$efxef = ef \quad \text{and} \quad xefx = x.$$

Then fxe is also an inverse of ef, since

$$\begin{aligned} (ef)(fxe)(ef) &= ef^2xe^2f = (ef)x(ef) = ef, \\ (fxe)(ef)(fxe) &= f(xefx)e = fxe. \end{aligned}$$

Also, fxe is an idempotent, since

$$(fxe)^2 = f(xefx)e = fxe.$$

Hence fxe is its own unique inverse, and so $fxe = ef$. In particular, ef is idempotent, and is its own unique inverse. The same holds for fe. But

$$(ef)(fe)(ef) = (ef)^2 = ef \text{ and } (fe)(ef)(fe) = (fe)^2 = fe,$$

and so ef and fe are mutually inverses. Hence $ef = fe$. □

1.5.3 Green's relations

The notion of ideals lead naturally to the consideration of certain equivalence relations on a semigroup. These equivalences, first introduced by Green [17], have played a fundamental role in the development of semigroup theory. Since their introduction, they have become standard tools for investigating the structure of semigroups.

If a is an element in a semigroup S, the sets

$$S^1a = Sa \cup \{a\}, \quad aS^1 = aS \cup \{a\} \quad \text{and} \quad S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\},$$

are left, right and two-sided ideals of S respectively. These are respectively the smallest left, right and two-sided ideals of S containing a. We shall call them principal (resp. left, right and two-sided) ideals of S generated by a.

For any two elements $a, b \in S$, we define the equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}$ and $\mathcal{D}$ on S by

$$a\mathcal{L}b \quad \textit{if and only if} \quad S^1a = S^1b, \tag{1.2}$$

$$a\mathcal{R}b \quad \textit{if and only if} \quad aS^1 = bS^1, \tag{1.3}$$

$$a\mathcal{J}b \quad \textit{if and only if} \quad S^1aS^1 = S^1bS^1, \tag{1.4}$$

$$\mathcal{H} \quad = \quad \mathcal{L} \cap \mathcal{R} \tag{1.5}$$

and

$$\mathcal{D} = \mathcal{L} \circ \mathcal{R} \tag{1.6}$$

these five equivalences are known as Green's relations. In many cases the two equivalences $\mathcal{D}$ and $\mathcal{J}$ coincide. This happens when the semigroup is finite, and also in some other circumstances as given in the next theorem.

Theorem 1.5.3 (Howie [21]) *If S is a periodic semigroup, then $\mathcal{D} = \mathcal{J}$.*

For an element $a \in S$, L_a, R_a, H_a, D_a and J_a respectively denote the equivalence classes of a with respect to equivalences $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$, and $\mathcal{J}$. A $\mathcal{D}-class$ is a disjoint union of $\mathcal{R}-classes$, as well as a disjoint union of $\mathcal{L}-classes$. The intersection of an $\mathcal{R}-class$ and an $\mathcal{L}-class$ within the same $\mathcal{D}-class$ is non-empty and is an $\mathcal{H}-class$. So a $\mathcal{D}-class$ can be visualised as an egg-box picture [1], with rows, columns and squares representing, $\mathcal{R}-classes$, $\mathcal{L}-classes$ and $\mathcal{H}-classes$ respectively.

Lemma 1.5.4 (Howie [21], Green's lemma) *Let S be a semigroup and let $a, b \in S$.*

i. If $a\mathcal{R}b$ with $as = b$ and $bt = a$, where $s, t \in S^1$, then the mappings $\lambda_s : x \mapsto xs$ and $\lambda_t : x \mapsto xt$ are mutually inverse $\mathcal{R}-class$ preserving bijections from L_a onto L_b and vice versa.

ii. If $a\mathcal{L}b$ with $sa = b$ and $tb = a$, where $s, t \in S_1$, then the mappings $\rho_s : x \mapsto sx$ and $\rho_t : x \mapsto tx$ are mutually inverse $\mathcal{L}-class$ preserving bijections from R_a onto R_b and vice versa.

Lemma 1.5.5 (Howie [21]) *If $a\mathcal{D}b$ then $|H_a| = |H_b|$*

Theorem 1.5.6 (Howie [21], Green's theorem) *If H is an $\mathcal{H}-class$ of a semigroup S, then either $H^2 \cap H = \emptyset$ or $H^2 \subseteq H$.*

Theorem 1.5.7 (Howie [21]) *If a is a regular element of a semigroup S, then every element in D_a is regular.*

A $\mathcal{D}-class$ whose all elements are regular is called a regular $\mathcal{D}-class$.

Theorem 1.5.8 (Howie [21]) *In a regular $\mathcal{D}-$ cless, each $\mathcal{L}-$ class and each $\mathcal{R}-$ class contains an idempotent.*

Theorem 1.5.9 (Howie [21]) *Let a, b be two elements in a $\mathcal{D}-$ class D. Then $ab \in R_a \cap L_b$ if and only if $L_a \cap R_b$ contains an idempotent.*

1.5.4 Some semigroups of transformations

One of the main motivations for the existence of any abstract algebraic study are interesting and natural examples. For semigroup theory the obvious candidates for such examples are transformation semigroups. Various transformations of different sets appear frequently in mathematics and as the usual composition of transformations is associative, each set of transformations, which is closed with respect to composition, forms a semigroup called transformation semigroup.

For a non-negative integer n, let X_n be the finite set $\{1, 2, \ldots, n\}$. A transformation of X_n is a map $\alpha : X_n \to X_n$. We write α in a tabular form as

$$\alpha = \begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} \tag{1.7}$$

where all $a_i \in X_n$. If $i \in X_n$, the element a_i is called the value of α at i and will be written as $i\alpha$.

The Partial Transformation Semigroup

Let A and B are any two subsets of X_n, a mapping $\alpha : A \to B$ is called a *partial transformation* of X_n. A and B are called domain and range of α respectively, and are denoted by $dom(\alpha)$ and $im(\alpha)$. The set of all partial transformations of X_n is denoted by $\mathcal{P}_n$ and as the composition of partial transformations will be a partial transformation, the set $\mathcal{P}_n$ is a semigroup under composition called the *partial transformation semigroup.* The map $I : X_n \to X_n$ defined by $I(i) = i \quad (\forall i \in X_n)$ acts as the identity of $\mathcal{P}_n$. The zero map $0 : \emptyset \to X_n$ is the zero element of P_n. Thus, the partial transformation semigroup $\mathcal{P}_n$ is a monoid with zero.

If $\alpha \in \mathcal{P}_n$ is a partial transformation with $dom(\alpha) = \{a_1, a_2, \ldots, a_k\} \subseteq X_n$. Then, we write

$$\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_k \\ a_1\alpha & a_2\alpha & \cdots & a_k\alpha \end{pmatrix}. \tag{1.8}$$

It is easy to see, from (1.8), that the semigroup $\mathcal{P}_n$ contains $(n+1)^n$ elements.

For a partial transformation $\alpha \in \mathcal{P}_n$, we refer to the quantity $|im(\alpha)|$ as the *height* of α and the quantity $|dom(\alpha) \setminus im(\alpha)|$ as the *defect* of α.

The Full Transformation Semigroup

The origins of group theory are in the study of permutations and the symmetric group $\mathcal{S}_n$, the group of all permutations of a set, say X_n (a transformation $\alpha : X_n \to X_n$ is called a permutation if it is a bijection). The corresponding object in semigroup theory is the full transformation semigroup.

A map $\alpha : dom(\alpha) \subseteq X_n \to im(\alpha) \subseteq X_n$ is called a *full* (or *total*) *transformation* of X_n if $dom(\alpha) = X_n$. The set $\mathcal{T}_n$, of all full transformations of X_n, forms a semigroup under composition of mappings called the *full transformation semigroup.*

The semigroup $\mathcal{T}_n$ contains n^n elements. The semigroup $\mathcal{T}_n$ has the same universal property as the symmetric group $\mathcal{S}_n$ on X_n:

Theorem 1.5.10 (Howie [21]) *Every (finite) semigroup S is embeddable in a (finite) full transformation semigroup.*

Proof. Let $X = S \cup \{1\}$, where $1 \notin S$. For each s in S, define $\rho_s : X \to X$ by

$$\begin{aligned} x\rho_s &= xs \quad \text{if} \quad x \in S, \\ 1\rho_s &= s. \end{aligned}$$

It is easy to verify that $\rho_s\rho_t = \rho_{st}$, and so the map $s \mapsto \rho_s$ is a homomorphism from S into $\mathcal{T}_X$. It is also one-to-one, since

$$\rho_s = \rho_t \implies (\forall x \in X_n)x\rho_s = x\rho_t \implies 1\rho_s = 1\rho_t \implies s = t.$$

□

Theorem 1.5.11 (Howie [21]) *The semigroup $\mathcal{T}_n$ is regular.*

Proof. Let $\alpha \in \mathcal{T}_n$. Define a mapping $\xi : X_n$ as follows. For each y in $im(\alpha)$, choose an element x in X such that $x\alpha = y$ and let $y\xi = x$; also, for all y not in $im(\alpha)$, choose an arbitrary element z of X_n, and lest $y\xi = z$. Then it is clear that, for all x in X_n. $x\alpha\xi\alpha = x\alpha$. □

Note that since X_n is a subset of itself, each α in $\mathcal{T}_n$ is also a partial transformation. Thus, $\mathcal{T}_n$ is a subsemigroup of $\mathcal{P}_n$.

For $1 \leq r \leq n$, let

$$J_r = \{\alpha \in \mathcal{T}_n : |im(\alpha)| = r\}. \tag{1.9}$$

Then $J_n = S_n$, the symmetric group and J_1 consists of all constant maps of T_n. Let

$$K(n,r) = \{\alpha \in \mathcal{T}_n : |im(\alpha)| \leqslant r\}. \tag{1.10}$$

Then $J_r = K(n,r) \setminus K(n,r-1)$ and since, for all $\alpha, \beta \in \mathcal{T}_n$,

$$|im\alpha\beta| \leqslant min\{|im\alpha|, |im\beta|\}, \tag{1.11}$$

it follows that $K(n,r)$ is an ideal of $\mathcal{T}_n$. Clearly $K(n,n) = \mathcal{T}_n$ and $K(n,n-1) = \mathcal{T}_n \setminus \mathcal{S}_n$. The ideal $K(n,n-1)$ is denoted by $Sing_n$ and is called the *subsemigroup of singular selfmaps* of $\mathcal{T}_n$. The order of $Sing_n$ is $n^n - n!$. The set $P_r = J_r \cup 0$ is a 0-simple semigroup called the principal factor of $K(n,r)$. The product of two elements of P_r is 0 whenever their product in $\mathcal{T}_n$ is of height strictly less then r.

For each $\alpha \in \mathcal{T}_n$ we define an equivalence $ker\alpha$ on X_n by

$$ker\alpha = \{(x,y) \in X_n \times X_n : x\alpha = y\alpha\}. \tag{1.12}$$

For simplicity, we will write each $\alpha \in \mathcal{T}_n$ as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix} \tag{1.13}$$

where $A_1, A_2, \ldots, A_r$ are $ker\alpha - classes$ and $im\alpha = \{a_1, a_2, \ldots, a_r\}$ with $A_i\alpha = a_i$ for each $i = 1, \ldots, r$. The following theorem characterises Green's relations in $\mathcal{T}_n$.

Theorem 1.5.12 (Howie [21]) *Let $\alpha, \beta \in \mathcal{T}_n$. Then*

1. $\alpha\mathcal{L}\beta$ if and only if $im\alpha = im\beta$;

2. $\alpha\mathcal{R}\beta$ if and only if $ker\alpha = ker\beta$;

3. $\alpha\mathcal{J}\beta$ if and only if $|im\alpha| = |im\beta|$;

4. $\mathcal{D} = \mathcal{J}$*.*

As a consequence of this theorem, we see that, the $\mathcal{J}-classes$ in $\mathcal{T}_n$ are J_r and the number of $\mathcal{L}-classes$ is the number of distinct subsets of X_n of cardinality r, that is, the binomial coefficient $\binom{n}{r} = \frac{n!}{r!(n-r)!}$. The number of $\mathcal{R}-classes$ is the number of equivalences on X_n having r classes, that is, the Stirling number of the second kind defined recursively as

$$S(n,r) = S(n-1,r-1) + rS(n-1,r), \tag{1.14}$$

with boundary conditions $S(n,1) = S(n,n) = 1$. Also, $S(n,n-1) = \frac{n(n-1)}{2!}$ and $S(n,2) = 2^{n-1} - 1$.

Therefore, a $\mathcal{J}-class$ J_r of $\mathcal{T}_n$ is visualise as an eggbox in which the $\mathcal{L}-classes$ are the columns, the $\mathcal{R}-classes$ are the rows and the $\mathcal{H}-classes$ are the cells. The number of cells is $\binom{n}{r} \times S(n,r)$, and each cell contains $r!$ elements.

A subset $Y = \{a_1, \ldots, a_r\}$, of X_n, is said to be a *transversal* of *(or orthogonal to)* an equivalence ρ, with classes $A_1, A_2, \ldots, A_r$, if each a_i in Y belongs to a unique $\rho-class$ A_j. If Y is a transversal of ρ such that $a_i \in A_i$, for each i, then, the map

$$\epsilon = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ a_1 & a_2 & \cdots & a_r \end{pmatrix}$$

is an idempotent. It is the unique idempotent in the $\mathcal{H}-class$ $H_{Y,\rho}$, in J_r, corresponding to Y and ρ.

Associated with a mapping α in $\mathcal{T}_n$ is a digraph $\Gamma(\alpha)$ whose vertices are labelled $1, 2, \ldots, n$ and there is an edge $i \to j$ if and only if $i\alpha = j$. Let $\alpha \in \mathcal{T}_n$, we define an equivalence relation ω on X_n by

$$\omega = \{(i, j) \in X_n \times X_n : (\exists r, s \geq 0) i\alpha^r = j\alpha^s\}. \tag{1.15}$$

The $\omega-$classes are the connected components of $\Gamma(\alpha)$ called the *orbits* of α. Each orbit Ω has *kernel* $K(\Omega)$, defined by

$$K(\Omega) = \{i \in \Omega : (\exists r > 0) i\alpha^r = i\}. \tag{1.16}$$

To see that $K(\Omega)$ is not empty for each orbit Ω, consider an element i in Ω. The elements

$$i, i\alpha, i\alpha^2, \ldots$$

cannot all be distinct, and so there exist $m \geq 0$ and $r \geq 1$ such that $i\alpha^{m+r} = i\alpha^m$. Thus $i\alpha^m \in K(\Omega)$.

There are four different kinds of orbits that can arise for each α in $\mathcal{T}_n$. An orbit Ω is:

$$\begin{aligned} \textit{standard} \quad &\text{if and only if} \quad 2 \leq K(\Omega) < |\Omega|; \\ \textit{acyclic} \quad &\text{if and only if} \quad 1 = K(\Omega) < |\Omega|; \\ \textit{cyclic} \quad &\text{if and only if} \quad 2 \leq K(\Omega) = |\Omega|; \\ \textit{trivial} \quad &\text{if and only if} \quad 1 = K(\Omega) = |\Omega|. \end{aligned}$$

For example, the map

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 4 & 5 & 6 & 4 & 6 & 9 & 10 & 10 & 12 & 13 & 11 & 14 \end{pmatrix}$$

has orbits $\{1, 2, 3, 4, 5, 6, 7\}$, $\{8, 9, 10\}$, $\{11, 12, 13\}$ and $\{14\}$ which are, respectively, standard, acyclic, cyclic and trivial.

For each $\alpha \in \mathcal{T}_n$ we define the *gravity* of α $g(\alpha)$ by

$$g(\alpha) = n + cycl(\alpha) - fix(\alpha), \tag{1.17}$$

where $cycl(\alpha)$ is the number of cyclic orbits of α and $fix(\alpha)$ is the number of acyclic orbits plus the number of trivial orbits of α.

The Symmetric inverse Semigroup

A map $\alpha \in \mathcal{P}_n$ is called *partial injection* if for all $x, y \in dom(\alpha)$, $x\alpha = y\alpha$ implies $x = y$. Obviously, the composition of two partial injections is a partial injection. Therefore the set of all partial injections, denoted by $\mathcal{I}_n$, is a subsemigroup of $\mathcal{P}_n$ which is an inverse semigroup ([21], Theorem 5.1.5), called the *symmetric inverse semigroup.* The semigroup $\mathcal{I}_n$ is the appropriate analogue in inverse semigroup theory of the symmetric group in group theory and the full transformation semigroup in semigroup theory. The following theorem known as Vagner-Preston Theorem is the analogue of Cayley's Theorem.

Theorem 1.5.13 (Howie [21]) *Every (finite) inverse semigroup is embeddable in a (finite) symmetric inverse semigroup.*

Also in $\mathcal{I}_n$ we have:

Theorem 1.5.14 (Howie [21]) *Let $\alpha, \beta \in \mathcal{I}_n$, then*

1. $\alpha\mathcal{L}\beta$ if and only if $im\alpha = im\beta$;
2. $\alpha\mathcal{R}\beta$ if and only if $ker\alpha = ker\beta$;
3. $\alpha\mathcal{J}\beta$ if and only if $|im\alpha| = |im\beta|$;
4. $\mathcal{D} = \mathcal{J}$.

It is therefore, not hard to see that, the semigroup $\mathcal{I}_n$ contains $n+1$ $\mathcal{J}-classes$

$$J_0, J_1, \dots, J_n.$$

Each $\mathcal{J}-class$ J_r contains $\binom{n}{r}$ $\mathcal{L}-classes$ and $\binom{n}{r}$ $\mathcal{R}-classes$, and each $\mathcal{H}-class$ contains $r!$ elements. Therefore, the number of elements in each J_r is $\binom{n}{r}^2 \times r!$. Thus, since the semigroup $\mathcal{I}_n$ is a union if its $\mathcal{J}-classes$, the number of elements in $\mathcal{I}_n$ is

$$|\mathcal{I}_n| = \sum_{r=0}^{r=n} \binom{n}{r}^2 r!. \tag{1.18}$$

It is well known (see [4]), as for the elements of $\mathcal{T}_n$, that associated to each α in $\mathcal{I}_n$ is a digraph $\Gamma(\alpha)$ in which for each $i, j \in X_n$ there is an edge $i \to j$ if and only if $i \in dom(\alpha)$ and $i\alpha = j$. The connected components of $\Gamma(\alpha)$ are called *orbits* of α. Each of the orbits of α is either a *cycle* or a *chain* of the form

$$a_1 \to a_2 \to \cdots \to a_k,$$

where $a_1, a_2, \dots, a_k \in X_n$. A cycle and a chain of lengths k are written as

$$(a_1, a_2, \dots, a_k) \quad \text{and} \quad [a_1, a_2, \dots, a_k]$$

respectively. Therefore each $\alpha \in \mathcal{I}_n$ can be written in the chain-cycle notation as

$$\alpha = (a_1, \dots, a_k) \cdots (b_1, \dots, b_l)[c_1, \dots, c_p] \cdots [d_1, \dots, d_q]. \tag{1.19}$$

For example, the chain-cycle notation for the element

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 & 10 & 11 & 13 & 14 & 15 \\ 8 & 6 & 4 & 5 & 2 & 7 & 3 & 9 & 11 & 10 & 14 & 15 & 12 \end{pmatrix}$$

is $\alpha = (9)(5, 2, 6, 7, 3, 4)(10, 11)[1, 8][13, 14, 15, 12]$.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

As noted by Sullivan [30], from the 1940's to the early 1960's, the Russian School of semigroups produced some major results on transformations semigorups which concerned generators, morphisms, ideals and congruences. Then in 1966 Howie published his paper [19] on idempotent transformations and throughout the last four decades, many papers have been published on this idea. In this chapter we review some of these works.

2.2 Idempotents: Products and Ranks

In 1966 J. M. Howie published an influential paper [19], in which he proved, among other things, that in any full transformation semigroup the products of idempotent transformations other than the identity are precisely those which are not bijections. Since the appearance of his paper the problem of describing the subsemigroups generated by idempotents in both semigroups of transformations and semigroups of matrices (that is semigroups of endomorphisms of a mathematical structure) have received considerable attention (see, for example,[2, 3, 20, 5, 15]). The analogue of Howie's result was obtained by Erdos [2] for semigroup of matrices. He showed that every singular square matrix can be written as a product of idempotent matrices.

Actually, as noted in [1], p.7, Ex. 10(a), in 1953 Vorobev [33] has shown that if $\alpha \in \mathcal{T}_n$ is of height $r < n$, then $\alpha = \beta\epsilon$ for some $\beta \in \mathcal{T}_n$ of height $r + 1$ and some idempotent $\epsilon \in \mathcal{T}_n$ of height $n - 1$. Consequently, every $\alpha \in \mathcal{T}_n$ of height $n - 1$ can be expressed as a product of a permutation and $n - r$ idempotents of height $n - 1$. In fact, Howie used Vorobev's work to prove the 'easy' part of [19]: namely, the semigroup $Sing_n$ is generated by its idempotnets of height $n - 1$.

Howie was interested in certain combinatorial questions that arose from his discovery. Specifically, given a semigroup S that is generated by idempotents, one ask (among othe questions) the following questions:

1. What is the smallest size of an idempotent generating set for S? This number is called the idempotent rank of S, it is denoted by $idrank(S)$.
2. Can all such minimal number of idempotent generating sets be described?
3. What is the minimal number of idempotents required to generate a particular element of S?
4. What is the minimal number of idempotents required to generate each element of S? This number is called the idempotent depth of S.

The question of finding the minimal number of idempotents required to generate an element of S have been answered by Howie [20] for the singular part of $\mathcal{T}_n$, $Sing_n$. He proved the following theorem via arguments involving some elementary graph theory.

Theorem 2.2.1 (Howie [20]) *If E_1 denote the set of idempotents of height $n-1$ in $Sing_n$. Then, for each $\alpha \in Sing_n$ the least positive integer $k(\alpha)$ for which $\alpha \in E_1^{k(\alpha)}$ is $k(\alpha) = g(\alpha)$, the gravity of α.*

Still in [20] corollary 3.13, Howie proved that the maximum possible value of $g(\alpha)$ for $\alpha \in \mathcal{T}_n$ is $[\frac{3}{2}(n-1)]$ (where $[\ldots]$ denotes the greatest integer function), which is the best possible global lower bound for the number of idempotents required to generate a singular element of $\mathcal{T}_n$. Thus, this number is also the idempotent depth of $Sing_n$ and hence, the question of finding idempotent depth of $Sing_n$ is settled.

Nine years after, the result in Theorem 2.2.1 was generalised by Saito [27] to cover the set of all idempotents in $Sing_n$ (that is not necessarily those of height $n-1$). He also used consept from graph theory in his generalisations which are summarised in the following theorem.

Theorem 2.2.2 (Saito [27]) *Let E denote the set of all idempotents in $Sing_n$. For each $\alpha \in Sing_n$ the unique integer $k(\alpha)$ for which $\alpha \in E^{k(\alpha)}$ and $\alpha \notin E^{k(\alpha)-1}$ is $k(\alpha) = [\frac{g(\alpha)}{d(\alpha)}]$ or $k(\alpha) = [\frac{g(\alpha)}{d(\alpha)}]+1$ and $k(\alpha) = [\frac{g(\alpha)}{d(\alpha)}]$ if $g(\alpha) = 1 mod(d(\alpha))$ (where $[\ldots]$ denote the greatest integer function, $g(\alpha)$ is the gravity of α and $d(\alpha) = n - |im\alpha|$).*

Remark. It is not yet known which $\alpha \in Sing_n$ has $k(\alpha) = [\frac{g(\alpha)}{d(\alpha)}]$ and which has $k(\alpha) = [\frac{g(\alpha)}{d(\alpha)}] + 1$.

The ideas of Howie [20] and Saito[27] were generalised by Garba [5] to deal with the partial transformation semigroup $\mathcal{P}_n$ by regarding $\mathcal{P}_n$ as a subsemigroup of $\mathcal{T}_{X_n^0}$ where $X_n^0 = \{0, 1, \ldots, n\}$. He discovered that the gravity of a partial transformation α can also be define as $g(\alpha) = n + c(\alpha) - f(\alpha)$ (where $c(\alpha)$ and $f(\alpha)$ are as defined in chapter 1) and that for $\alpha \in \mathcal{P}_n \setminus \mathcal{S}_n$ with $d(\alpha) = n - |im\alpha| = d$. If $n - d$ is even, then $g(\alpha) \leq \frac{1}{2}(3n - d)$. And if $n - d$ is odd, then $g(\alpha) \leq \frac{1}{2}(3n - d - 1)$. He also obtained the result that $\alpha \in E_1^{g(\alpha)}$ and $\alpha \notin E_1^{g(\alpha)-1}$, where E_1 denotes the set of idempotents of height $n - 1$ in $\mathcal{P}_n \setminus \mathcal{S}_n$. Also, $\alpha \in E^{k(\alpha)}$ and $\alpha \notin E^{k(\alpha)-1}$, where $k(\alpha) = [\frac{g(\alpha)}{d(\alpha)}]$ or $k(\alpha) = [\frac{g(\alpha)}{d(\alpha)}] + 1$ and E denotes the set of all idempotents in $\mathcal{P}_n \setminus \mathcal{S}_n$.

Returning to the question of finding the idempotent rank of a transformation semigroup S (that is, the minimal number of idempotents required to generate S), Gomes and Howie [15] have proved (among other things) that

$$idrank(Sing_n) = rank(Sing_n) = \frac{1}{2}n(n-1). \tag{2.1}$$

Howie and McFadden [22] extend this to the two-sided ideal

$$K(n, r) = \{\alpha \in \mathcal{T}_n : |im(\alpha)| \leq r\},$$

for $1 \leq r < n$, and showed that, for $r \geq 2$

$$idrank(K(n, r)) = rank(K(n, r)) = S(n, r). \tag{2.2}$$

where $S(n, r)$ is the stirling number of the second kind (that is, the number of partitions of X_n with exactly r elements) defined by (1.14).

Garba [5] considered the two-sided ideal

$$K^*(n, r) = \{\alpha \in \mathcal{P}_n : |im(\alpha)| \leq r\}, \tag{2.3}$$

for $1 \leq r < n$, of the partial transformation semigroup $\mathcal{P}_n$ and extended the methods of [22] to prove that

$$idrank(K^*(n,r)) = rank(K^*(n,r)) = S(n+1, r+1), \qquad (2.4)$$

the stirling number of the second kind.

Many naturally occurring subsemigroups of both $\mathcal{P}_n$ and $\mathcal{T}_n$ have been studied as idempotent generated semigroups, see, for example [16, 7, 32]. Specifically, the rank and idempotent rank of the subsemigroup $\mathcal{O}_n$ of all order-preserving full transformations (a map $\alpha \in \mathcal{T}_n$ is order-preserving if $x \leq y \implies x\alpha \leq y\alpha$, for all $x, y \in X_n$), were studied by Gomes and Howie [16]. They showed that

$$rank(O_n \setminus \{\iota\}) = n \quad \text{and} \quad idrank(O_n \setminus \{\iota\}) = 2n - 2, \qquad (2.5)$$

where ι denote the identity permutation of X_n. Rank and idempotent rank of certain subsemigroups of $\mathcal{O}_n,\ \mathcal{PO}_n$ *and* $\mathcal{SPO}_n$ were obtained by Garba [7]. Umar [32] also obtained the rank and idempotent rank of the semigroup of all order-decreasing full transformations (a map $\alpha \in \mathcal{T}_n$ is said to be order-decreasing if $x\alpha \leq x$ for all $x \in X_n$).

2.3 Nilpotents: Products and Ranks

If a semigroup S contains zero element 0, then it cantains elements a for which there exists $m \geq 1$ such that $a^m = 0$, called *n*ilpotent. If $a^m = 0$, where $a \neq 0$ and $a^{m-1} \neq 0$, we say that a has index of nilpotency m.

During the mini conference on semigroup theory held at Szeged in 1972, Schwarz [28] suggested that the role of nilpotents in semigroups should be investigated. Motivated by this, Sullivan [29], in 1987, follow the lead from Howie's work in 1966, to initiate the study of subsemigroup of the partial transformation semigroup $\mathcal{P}_X$ of the set X, which is generated by its nilpotents. For the case when X is finite of order n, the answer to the question of nilpotent generated subsemigroup of $\mathcal{P}_n$ depends on whether n is even or odd.

While Sullivan was considering nilpotents in $\mathcal{P}_n$, Gomes and Howie [14] were also considering the same problem in $\mathcal{I}_n$, the symmetric inverse semigroup on X_n. Surprisingly, their answers turn out to be similar to those of Sullivan.

Let $\mathcal{SP}_n$ and $\mathcal{SI}_n$ be the subsemigroup of all strictly partial transformations on X_n and the subsemigroup of all strictly partial one-to-one transformations on X_n respectively. Associated to each $\alpha \in \mathcal{SP}_n$ ($\mathcal{SI}_n$), of height $n-1$, is a permutation $\alpha^* \in \mathcal{S}_n$, the symmetric group on X_n, called the completion of α and defined by

$$i\alpha^* = j \qquad \text{and} \qquad x\alpha^* = x\alpha \text{ (otherwise)},$$

where $dom(\alpha) = X_n \setminus \{i\}$ and $im(\alpha) = X_n \setminus \{j\}$. We summarise the results of [29] and [14], for $\mathcal{SP}_n$ and $\mathcal{SI}_n$, in the following theorems.

Theorem 2.3.1 (Sullivan [29]) *Let N be the set of all the nilpotents in $\mathcal{SP}_n$.*

(i) If n is even, then $\langle N \rangle = \mathcal{SP}_n$;

(ii) If n is odd, then $\langle N \rangle = \mathcal{SP}_n \setminus W_{n-1}$, where W_{n-1} consists of elements of height $n-1$ in $\mathcal{SP}_n$ whose completions are odd permutations.

Theorem 2.3.2 (Gomes and Howie [14]) *Let N_1 be the set of all the nilpotents of height $n-1$ in $\mathcal{SI}_n$.*

(i) If n is even, then $\langle N_1 \rangle = \mathcal{SI}_n$;

(ii) If n is odd, then $\langle N_1 \rangle = \mathcal{SI}_n \backslash W_{n-1}$, where W_{n-1} consists of elements of height $n-1$ in $\mathcal{SI}_n$ whose completions are odd permutations.

It is natural, on establishing that a certain semigroup S is generated by its nilpotents, to adapt the concept of idempotent rank, for idempotent generated semigroups, and ask for the nilpotent rank of S, that is the minimal cardinality of nilpotent generating set for S. This problem was considered by Graba [10] for the semigroups $\mathcal{SI}_n$ and $\mathcal{SP}_n$. He showed that whether n is even or odd the nilpotent rank of the subsemigroups of $\mathcal{SI}_n$ and $\mathcal{SP}_n$ generated by nilpotents are equal to $n+1$ ([10], Propositions 2.6 and 2.7) and $n+2$ ([10], Theorems 3.2 and 3.3) respectively.

Products and rank properties of nilpotents in the semigroups $\mathcal{PO}_n$ and $\mathcal{IO}_n$ of all partial order-preserving transformations and partial one-to-one order-preserving transformations respectively, have been considered by Garba [9, 8]. See also [11, 6] for nilpotents in other subsemigroups of $\mathcal{P}_n$.

2.4 Quasi-idempotents

In any inverse semigroup S, the set $E(S)$ of its idempotents is a semilattice, that is a commutative subsemigroup of S. Thus, $E(S)$ cannot generate S. The symmetric inverse semigroup $\mathcal{I}_n$ is not idempotent generated but its set of nilpotents generates its singular part $\mathcal{SI}_n$ only in the case when n is even. What type of elements will then generate $\mathcal{SI}_n$ for all n? In [8] a description of the elements of the nilpotent generated subsemigroup of $\mathcal{IO}_n$ is given. This subsemigroup is not, in all cases, the whole of $\mathcal{IO}_n$.

Umar [31] studied the subsemigroup $\mathcal{I}_n^-$ of $\mathcal{SI}_n$, of all order-decreasing transformations including the empty map on X_n. Though $\mathcal{I}_n^-$ is not an inverse subsemigroup of $\mathcal{SI}_n$ it is a full subsemigroup of $\mathcal{SI}_n$ in the sense that it contains all the idempotents in $\mathcal{SI}_n$ ([31], Remark 1.2(c)). Thus, $\mathcal{I}_n^-$ is not generated by idempotents. Also, the set of nilpotents in $\mathcal{I}_n^-$ is an ideal ([31], Lemma 1.4) and so generates itself. To find a generating set for $\mathcal{I}_n^-$, Umar [31] introduced the concept of quasi-idempotent elements.

An element a of a semigroup S is called a quasi-idempotent if $a^4 = a^2$, that is, if a^2 is an idempotent. Clearly, all idempotents are quasi-idempotents but not vice-versa. Umar [31] used these elements to generate $\mathcal{I}_n^-$ and obtained its rank via the quasi-idempotent elements, which equals with the minimal cardinality of set of quasi-idempotents generating $\mathcal{I}_n^-$ called the quasi-idempotent rank of $\mathcal{I}_n^-$.

Madu and Garba [26] established that $\mathcal{IO}_n$ is generated by quasi-idempotents. They also proved that the quasi-idempotent rank of $\mathcal{IO}_n$ is $2(n-1)$. In this thesis, we extend the work of Madu [25] by studying quasi-idempotents in the semigroups $\mathcal{I}_n$ and $\mathcal{T}_n$.

CHAPTER 3

QUASI-IDEMPOTENTS IN $\mathcal{T}_n$

In this chapter, we study quasi-idempotent elements in finite full transformation semigroup $\mathcal{T}_n$. In particular we show that the semigroup $Sing_n$, of all singular self maps of X_n, is quasi-idempoten generated. Quasi-idempotent rank of $Sing_n$ is also obtained.

3.1 Introduction

Let X_n be the finite set $\{1, 2, \ldots, n\}$ and $\mathcal{T}_n$ the full transformation semigroup on X_n. We start with the following definition of quasi-idempotent elements.

Definition 3.1.1 A transformation α in $\mathcal{T}_n$ will be called *quasi-idempotent* if α is not an idempotent but α^2 is. That is, if $\alpha \neq \alpha^2 = \alpha^4$.

In [31], quasi-idempotents were defined as elements α for which $\alpha^2 = \alpha^4$, that is elements whose squares are idempotents. Accordingly, all idempotents are quasi-idempotents but not vice-versa. In our definition 3.1.1 of quasi-idempotents as $\alpha \neq \alpha^2 = \alpha^4$, the idempotents are excluded as being quasi-idempotents, therefore throughout what follows quasi-idempotents means non-idempotents elements whose squares are idempotents.

Definition 3.1.2 A transformation α in $\mathcal{T}_n$ will be called a *quasi-idempotent of type one* if α is not an idempotent and $\alpha^2 = \alpha^3$.

It then follows from this definition that every quasi-idempotent of type one is also a quasi-idempotent since, if $\alpha \in \mathcal{T}_n$ is a quasi-idempotent of type one, then $\alpha^4 = \alpha^3\alpha = \alpha^2\alpha = \alpha^3 = \alpha^2$. On the other hand, not all quasi-idempotents are of type one. For example, let $\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5\} & \{6\} \\ 3 & 1 & 2 & 4 \end{pmatrix}$ in $\mathcal{T}_6$ and observe that $\alpha^2 = \begin{pmatrix} \{1,2,6\} & \{3,4,5\} \\ 1 & 3 \end{pmatrix}$ which is an idempotent but $\alpha^3 = \begin{pmatrix} \{1,2,6\} & \{3,4,5\} \\ 3 & 1 \end{pmatrix} \neq \alpha^2$.

3.2 Characterisation of quasi-idempotents in $\mathcal{T}_n$

In this section we present a set theoretic characterisation of quasi-idempotent in $\mathcal{T}_n$, this will enable us to identify them among other elements of $\mathcal{T}_n$. To begin with, we have the following definition.

Definition 3.2.1 Let α be a map in $\mathcal{T}_n$. We define a *matching pair* of α to be a pair (A_i, A_j) of blocks of α for which $A_i\alpha \in A_j$ and $A_j\alpha \in A_i$. The blocks A_i and A_j will be referred to as *matching blocks* of α. If $i = j$ then we will say that (A_i, A_i) is a *self-matching pair* and A_i is a *stationary block* of α.

The next three lemmas lead to our characterisation of quasi-idempotents in terms of matching blocks.

Lemma 3.2.1 *If A_i is a matching block of $\alpha \in \mathcal{T}_n$, then A_i is a stationary block of α^2.*

Proof. Suppose (A_i, A_j) is a matching pair of blocks of α. Then $A_i\alpha^2 = (A_i\alpha)\alpha \in A_j\alpha \in A_i$. Showing that A_i is a stationary block of α^2. □

Lemma 3.2.2 *If $\alpha \in \mathcal{T}_n$ consists only of matching blocks (not all stationary), then α is a quasi-idempotent.*

Proof. Suppose $\alpha \in \mathcal{T}_n$ is a map containing only matching pair of blocks and let A_i be an arbitrary block of α; A_i is a matching block α and by lemma 3.2.1 A_i is a stationary block of α^2. Thus α^2 is an idempotent. □

Lemma 3.2.3 *Every quasi-idempotent $\alpha \in \mathcal{T}_n$ must contain at least one matching pair of blocks (either self-matching or non-stationary).*

Proof. Suppose $\alpha \in \mathcal{T}_n$ is a quasi-idempotent containing no matching pair of blocks and let A_i be any of its blocks. Then there exists a block A_j $(j \neq i)$ of α for which $A_i\alpha \in A_j$ and $A_j\alpha \notin A_i$. Therefore, $A_i\alpha^2 = (A_i\alpha)\alpha \in A_j\alpha \notin A_i$ implying that A_i is not fixed by α^2, and so, α^2 is not an idempotent, contradicting the choice of α as a quasi-idempotent. Hence α must contain at least one matching pair of blocks. □

Now we are at a level to present our characterisation of quasi-idempotents in $\mathcal{T}_n$. Thus, we have

Theorem 3.2.4 *Let $\alpha \in \mathcal{T}_n$. Then α is a quasi-idempotent if and only if the image of each non-matching block of α is contained in a matching block of α.*

Proof. If $\alpha \in \mathcal{T}_n$ contains no non-matching blocks, then the argument is trivial by lemma 3.2.2. Now suppose $\alpha \in \mathcal{T}_n$ is a map consisting of both matching and non-matching blocks in which every non-matching block is mapped into a matching block. We shall show that α is a quasi-idempotent. Note that α can be depicted (without loss of generality) as follows

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_r & A_{r+1} & A_{r+2} & \cdots & A_{r+s} & B_1 & \cdots & B_t \\ a_1 & \cdots & a_r & a_{r+2} & a_{r+1} & \cdots & a_{r+s-1} & b_1 & \cdots & b_t \end{pmatrix}$$

where $A_1, \cdots, A_r$ are stationary blocks of α, $A_{r+1}, \cdots, A_{r+s}$ are matching blocks of α, and $B_1, \cdots, B_t$ are non-matching blocks of α. Also $a_i \in A_i$ $(1 \leq i \leq r+s)$ and each b_j $(1 \leq j \leq t)$ belongs to one of $A_i's$. Observe that, by lemma 2.1, $A_i\alpha^2 = a_i$ for each $i = 1, \ldots, r+s$, and if $b_j \in A_i$, then $(A_i \cup B_j)\alpha^2 = a_i$. Therefore α^2 can be depicted (without loss of generality) as

$$\alpha^2 = \begin{pmatrix} C_1 & \cdots & C_u & A_{u+1} & \cdots & A_r & C_{r+1} & \cdots & C_{r+v} & A_{r+v+1} & \cdots & A_{r+s} \\ a_1 & \cdots & a_u & a_{u+1} & \cdots & a_r & a_{r+1} & \cdots & a_{r+v} & a_{r+v+1} & \cdots & a_{r+s} \end{pmatrix}$$

with the assumption that $A_1, \cdots, A_u$ and $A_{r+1}, \cdots, A_{r+v}$ are respectively the stationary and matching blocks of α containing $b_j's$ $(j = 1, \cdots, t)$, and for each $i = 1, \cdots, u, r+1, \cdots, r+v$, $C_i = A_i \cup (\cup_j B_j)$, where the second union runs over each j for which $b_j \in A_i$. Then it is clear that α^2 is an idempotent and α is a quasi-idempotent.

Conversely, suppose α is a quasi-idempotent. Then by lemma 3.2.3 α must contain at least one matching pair of blocks. If all its blocks are matching, there is nothing to prove. Therefore let α contains A_i $(1 \leq i \leq s)$ and B_j $(1 \leq j \leq t)$ matching and non-matching blocks respectively. Let us further suppose, by way of contradiction, that for some j $B_j\alpha \notin A_i$ for all i. Then either $B_j\alpha^2 \in A_i$ for some i or $B_j\alpha^2 \notin A_i$ for all i. In the former, since $B_j\alpha \notin A_i$, B_j is not fixed by α^2. In the latter, $B_j\alpha^2 \in B_k$ for some $k \neq j$ and again B_j is not fixed by α^2. Thus in both cases α^2 is not an idempotent contradicting the choice of α as a quasi-idempotent. $\square$

Example 3.2.1 Let $\alpha = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} & \{9,10\} \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$ in $\mathcal{T}_{10}$. Then α has

$$\begin{aligned} matching\ blocks &: \{1,2\}, \{3,4\}; \\ self - matching\ block &: \{5,6\}; \\ non - matching\ blocks &: \{7,8\}, \{9,10\}. \end{aligned}$$

observe that the non-matching blocks $\{7,8\}$, $\{9.10\}$ are mapped into the matching blocks $\{1,2\}$ and $\{3,4\}$, and

$$\begin{aligned} \{5,6\}\alpha^2 &= 5\alpha = 5, \\ \{1,2\}\alpha^2 &= 3\alpha = 1, \\ \{3,4\}\alpha^2 &= 1\alpha = 3, \\ \{7,8\}\alpha^2 &= 2\alpha = 3, \\ \{9,10\}\alpha^2 &= 4\alpha = 1. \end{aligned}$$

Thus, $\{5,6\}\alpha^2 = 5$, $\{1,2,9,10\}\alpha^2 = 1$ and $\{3,4,7,8\}\alpha^2 = 3$ and so

$$\alpha^2 = \begin{pmatrix} \{1,2,9,10\} & \{3,4,7,8\} & \{5,6\} \\ 1 & 3 & 5 \end{pmatrix}$$

which is an idempotent.

Also, consider $\beta = \begin{pmatrix} \{1,2\} & \{3,4\} & \{5,6\} & \{7,8\} & \{9,10\} \\ 3 & 1 & 2 & 4 & 7 \end{pmatrix}$ in $\mathcal{T}_{10}$. Then, the non-matching blocks of β are $\{5,6\}$, $\{7,8\}$, $\{9,10\}$. Observe that the non-matching block $\{9,10\}$ is not mapped into a matching block and $\{9,10\}\alpha^2 = 7\alpha = 4$. Thus, $\{9,10\}$ is not fixed by α^2 implying that β cannot be a quasi-idempotent.

Next we characterise quasi-idempotents of type one

Theorem 3.2.5 *Let $\alpha \in \mathcal{T}_n$ be a quasi-idempotent. Then α is of type one if and only if all matching blocks of α are self-matching.*

Proof. Suppose $\alpha \in \mathcal{T}_n$ is a quasi-idempotent containing no non-stationary matching blocks. Then by lemma 3.2.3 α contains two type of blocks, namely, stationary and non-matching blocks. And by theorem 3.2.4 every non-matching block of α must be mapped into a stationary block of α. Therefore, α can be written (without loss of generality) as

$$\alpha = \begin{pmatrix} A_1 & \cdots & A_s & B_1 & \cdots & B_t \\ a_1 & \cdots & a_s & b_1 & \cdots & b_t \end{pmatrix},$$

where, as in the proof of theorem 2.4, $a_i \in A_i$ $(1 \le i \le s)$ and each b_j $(1 \le j \le t)$ belong to one of $A_i's$. If $b_j \in A_i$, then $(A_i \cap A_j)\alpha^2 = a_i$, so that

$$\alpha = \begin{pmatrix} C_1 & \cdots & C_r & A_{r+1} & \cdots & A_s \\ a_1 & \cdots & a_r & a_{r+1} & \cdots & a_s \end{pmatrix},$$

with the assumption that $A_1, \cdots, A_r$ are the stationary blocks of α containing b_j $(1 \le j \le t)$, and for each $k = 1, \cdots, r$, $C_k = A_k \cup (\cup_j B_j)$, where the second union runs over all j for which $b_j \in A_k$. Then it can be easily verified that $\alpha^2 = \alpha^3$.

Conversely, suppose $\alpha \in \mathcal{T}_n$ is a quasi-idempotent of type one. We shall show that α contains no non-stationary matching block. Let us assume contrary, that is, α contains a non-stationary matching block say A_i. Then there is a unique block say A_j such that $A_i\alpha \in A_j$ and $A_j\alpha \in A_i$. But then, $A_i\alpha^2 \in A_i$ and $A_i\alpha^3 \in A_j \neq A_i$, and so $\alpha^2 \neq \alpha^3$. □

Corollary 3.2.6 *Every quasi-idempotent $\alpha \in \mathcal{T}_n$ of type one must contain both stationary and non-matching blocks.*

Proof. By theorem 3.2.5 α cannot contain non-stationary matching blocks. Now if all its blocks are stationary, it is an idempotent. On the other hand, if all its blocks are non-matching, by lemma 3.2.3, it cannot be a quasi-idempotent. $\square$

Corollary 3.2.7 *A map $\alpha \in \mathcal{T}_n$ is a quasi-idempotent of type one if and only if every non-matching block of α is mapped to an element in the stationary blocks of α.*

Proof. This is a consequence of theorem 3.2.4, theorem 3.2.5 and corollary 3.2.6.

$\square$

Remark. It is evidently clear from the previous observations that if $\alpha \in \mathcal{T}_n$ is a quasi-idempotent, then α cannot be a permutation. And so all quasi-idempotents of $\mathcal{T}_n$ lie within the subsemigroup $Sing_n$ of all singular maps of $\mathcal{T}_n$. Therefore henceforth we will be discussing quasi-idempotents in $Sing_n$ rather than in $\mathcal{T}_n$.

3.3 Products of quasi-idempotents in $Sing_n$

Our main objective in this section is to show that the semigroup $Sing_n$ is generated by set of quasi-idempotents of type one. We record the following lemma from [21].

Lemma 3.3.1 (Howie [21]) *Let $\alpha \in J_r$, where $1 \leq r \leq n-1$. Then there exist an idempotent $\epsilon \in J_{n-1}$ and β in J_{r+1} such that $\alpha = \epsilon\beta$.*

Proof. Write $\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$. Since not all the sets A_i are singleton, we may assume without loss of generality that $A_1 = \{a_1, a_1', \ldots\}$ has at least two elements. Then let

$$\epsilon = \begin{pmatrix} a_1 \\ a_1' \end{pmatrix}, \quad \beta = \begin{pmatrix} A_1 \setminus \{a_1\} & A_2 & \cdots & A_r & \{a_1\} \\ b_1 & b_2 & \cdots & b_r & b_{r+1} \end{pmatrix},$$

where $b_{r+1} \notin im(\alpha)$, and verify that $\alpha = \epsilon\beta$ $\square$

As a consequence of this lemma we easily deduce that $Sing_n$ is generated by J_{n-1}. Next we have

Lemma 3.3.2 *Every idempotent $\epsilon \in Sing_n$ is expressible as a product of two quasi-idempotents of type one $\alpha, \beta \in Sing_n$ each having only one non-stationary block and of the same height as ϵ.*

Proof. Let $\epsilon = \begin{pmatrix} A_1 & \cdots & A_i & \cdots & A_r \\ a_1 & \cdots & a_i & \cdots & a_r \end{pmatrix}$ be an idempotent of height r in $Sing_n$. Without loss of generality, we assume that $A_1 = \{a_1,\ a_1', \ldots\}$, and choose $\alpha, \beta \in Sing_n$ as follows:

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_{i-1} & A_i & A_{i+1} & \cdots & A_r \\ a_1' & a_2 & \cdots & a_{i-1} & a_1 & a_{i+1} & \cdots & a_r \end{pmatrix},$$

$$\beta = \begin{pmatrix} Z_1 & A_2 & \cdots & A_{i-1} & Z_2 & A_{i+1} & \cdots & A_r \\ a_i & a_2 & \cdots & a_{i-1} & a_1 & a_{i+1} & \cdots & a_r \end{pmatrix},$$

where $Z_1 = (A_1 \cup \{a_i\}) \setminus \{a_1'\}$, $Z_2 = (A_i \cup \{a_1'\}) \setminus \{a_i\}$. □

Example 3.3.1 Let $\epsilon = \begin{pmatrix} \{1,2\} & \{3\} & \{4\} & \{5\} \\ 1 & 3 & 4 & 5 \end{pmatrix}$ be an idempotent in $Sing_5$. We may choose $\alpha = \begin{pmatrix} \{1,2\} & \{3\} & \{4\} & \{5\} \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} \{1,5\} & \{3\} & \{4\} & \{2\} \\ 5 & 3 & 4 & 1 \end{pmatrix}$. Then it is straight forward to check (using corollary 3.2.7) that α, β are both quasi-idempotents and that $\epsilon = \alpha\beta$.

Let us denote the set of all quasi-idempotents of type one and of height $n-1$ by QI_{n-1}^*. A typical element of QI_{n-1}^* will be of the form

$$\xi = \begin{pmatrix} 1 & \cdots & j & \cdots & k & \cdots & n \\ 1 & \cdots & i & \cdots & j & \cdots & n \end{pmatrix}.$$

We denote this map by $\begin{pmatrix} j & k \\ i & j \end{pmatrix}$, where i, j and k are distinct. With this we present the main result of this section

Theorem 3.3.3 *The semigroup $Sing_n$ is generated by QI_{n-1}^*.*

Proof. Howie [19] showed that every element $\alpha \in Sing_n$ is a product of idempotents of height $n-1$. Now if $\epsilon = \binom{i}{j}$ is an idempotent of height $n-1$. Then by lemma 3.3.2, $\epsilon = \begin{pmatrix} j & k \\ i & j \end{pmatrix}\begin{pmatrix} j & i \\ k & j \end{pmatrix}$, where $k(\neq i, j)$ is any element of X_n fixed by ϵ. Hence α is a product of elements in QI^*_{n-1}. $\square$

3.4 Quasi-idempotent rank of $Sing_n$

Let S be a semigroup generated by a set of quasi-idempotents. We define the *quasi-idempotent rank* of S as

$$qidrank(S) = min\{|QI| : QI \text{ is a set of quasi} - \text{idempotents in } S \text{ and } \langle QI \rangle = S\}. \tag{3.1}$$

In this section we investigate the $qidrank(Sing_n)$. For convenience we record the following definition, which may be found in [24].

Definition 3.4.1 A partition π of weight r and an $r-subset$ (a subset of cardinality r) A of X_n are said to be orthogonal if every class of π contains exactly one element of A. An orthogonally labelled list $A_1\pi_1 \ldots A_m\pi_m$, $m = \binom{n}{r}$, is an alternating sequence of distinct partitions $\pi_1, \ldots, \pi_m$ of weight r of X_n and distinct $r - subsets$ $A_1, \ldots, A_m$ of X_n, such that for $i = 2, \ldots, m$, π_i is orthogonal to A_{i-1} and A_i and π_1 is orthogonal to A_1 and A_m.

The next lemma establishes the existence of orthogonally labelled list for any integer $n \geq 3$.

Lemma 3.4.1 *For each positive integer $n \geq 3$, there exists an orthogonally labelled list $A_1\pi_1 \ldots A_n\pi_n$ of X_n in which π_i is not orthogonal to A_{i+1} for $i = 1, \ldots, n-1$ and π_n is not orthogonal to A_1.*

Proof. Consider the list of subsets A_i and partitions π_i in [22], lemma 7, as: $A_i = X_n \setminus \{n-i+1\}$ for each i and for $i = 2, \ldots, n$, π_i is the partition with a unique non-singleton class $\{n-i+1, n-i+2\}$ and all other classes being singleton, while π_1 have a unique non-singleton class $\{1, n\}$ with all other classes singleton. Then these subset and partitions satisfy the lemma. □

Lemma 3.4.2 (Howie [21]) *Let x, y be elements in the same $\mathcal{D}-$ class of a semigroup S. Then $xy \in R_x \cap L_y$ if and only if $L_x \cap R_y$ contains an idempotent.*

Lemma 3.4.3 *Let $n \geq 3$ and $1 < r \leq n-1$. Then every non-group $\mathcal{H}-$class of J_r contains a quasi-idempotent of type one.*

Proof. Let $H_{\pi,A}$ be a non-group $\mathcal{H}-$class in J_r corresponding to an $r-$subset A and an $r-$partition π of X_n. Then A is not orthogonal to π and so, there exists at least one block of π containing more than one element of A. Let $A = \{a_1, \ldots, a_r\}$, we may list the blocks of π as $A_1, \ldots, A_i, \ldots, A_r$ such that $A \subset \cup_{j=1}^{j=i} A_j$ and $A \cap (\cup_{j=i+1}^{j=r} A_j) = \emptyset$. Choose $a_1, \ldots, a_i \in A$ such that $a_j \in A_j$ for each $j1, \ldots, i$. Then the map $\begin{pmatrix} A_1 & \cdots & A_i & A_{i+1} & \cdots & A_r \\ a_1 & \cdots & a_i & a_{i+1} & \cdots & a_r \end{pmatrix}$ is a quasi-idempotent of type one in $H_{\pi,A}$. □

Now the main result of the section is

Theorem 3.4.4 *For $n \geq 3, qidrank(Sing_n) = \frac{n(n-1)}{2}$.*

Proof. Let $A_1\pi_1 \ldots A_n\pi_n$ be an orthogonally labelled list, of $(n-1)$-subsets and partitions of weight $(n-1)$ of X_n, satisfying lemma 3.4.1. In [22], each $\mathcal{H}-class$ (π_i, A_i), for $i = 1, \ldots, n$, contains an idempotent ϵ_i and there exist idempotents $\epsilon_{n+1}, \ldots, \epsilon_{\frac{n(n-1)}{2}}$ (chosen in a way that the set $\{\epsilon_i : i = 1, \ldots, \frac{n(n-1)}{2}\}$ covers the $\mathcal{R}-$classes of J_{n-1}) such that $\{\epsilon_i : i, \ldots, \frac{n(n-1)}{2}\}$ is a generating set for $Sing_n$. Also, lemma 3.4.1 ensures that each non-group $\mathcal{H}-$class (π_i, A_{i+1}), for $i = 1, \ldots, n-1$, and (π_n, A_1) contains a quasi-idempotent $\xi_{i,i+1}$ and $\xi_{n,1}$ respectively. Choose quasi-idempotents $\xi_{n+1}, \ldots, \xi_{\frac{n(n-1)}{2}}$ such that for each $j = n+1, \ldots, \frac{n(n-1)}{2}$, $\epsilon_j \mathcal{R} \xi_j$. Then the set

$$\{\xi_{1,2}, \ldots, \xi_{n-1,n}, \xi_{n,1}, \xi_{n+1}, \ldots, \xi_{\frac{n(n-1)}{2}}\}$$

covers the $\mathcal{R}-$classes of J_{n-1}. We aim at showing that, for each $i = 1, \ldots, \frac{n(n-1)}{2}$, ϵ_i is an element of $\left\langle \xi_{1,2}, \ldots, \xi_{n-1,n}, \xi_{n,1}, \xi_{n+1}, \ldots, \xi_{\frac{n(n-1)}{2}} \right\rangle$. Notice first that the product $\xi_{i,i+1}\xi_{i+1,i+2}$ $(i = 1, \ldots, n-2)$ is of height $n-1$ and, by lemma 4.1, is in the $\mathcal{H}-$class $L_{\xi_{i+1,i+2}} \cap R_{\xi_{i,i+1}}$. Also for the same reasons the products $\xi_{n-1,n}\xi_{n,1}$ and $\xi_{n,1}\xi_{1,2}$ are of height $n-1$. Now, for $i = 1, \ldots, n-1$, the element $\xi_{i,i+1} \ldots \xi_{n-1,n}\xi_{n,1}\xi_{1,2} \ldots \xi_{i-1,i}$ is of height $n-1$ and belong to the same $\mathcal{H}-$class as ϵ_i. Hence for some $q \geq 1$, $\epsilon_i = (\xi_{i,i+1} \ldots \xi_{n-1,n}\xi_{n,1}\xi_{1,2} \ldots \xi_{i-1,i})^q$. Similarly, the element $\xi_{n,1}\xi_{1,2} \ldots \xi_{n-1,n}$ is of height $n-1$ and belong the same group $\mathcal{H}-$class as ϵ_n, and so for some $q \geq 1$, $\epsilon_n = (\xi_{n,1}\xi_{1,2} \ldots \xi_{n-1,n})^q$.

For each $j \in \{n+1, \ldots, \frac{n(n-1)}{2}\}$, $\epsilon_j \mathcal{R} \xi_j$ and $\xi_{n,1} \mathcal{L} \epsilon_j$ or there exists a unique $i \in \{1, \ldots, n-1\}$ such that $\xi_{i,i+1} \mathcal{L} \epsilon_j$. Also, either $\xi_{n,1} \mathcal{L} \xi_j$ or there exists a unique $k \in \{1, \ldots, n-1\}$ such that $\xi_{k,k+1} \mathcal{L} \xi_j$. Now, if $\xi_{n,1} \mathcal{L} \xi_j$ and $k < i$, the product $\xi_j \xi_{k+1,k+2} \ldots \xi_{i,i+1}$ is of height $n-1$ and belongs to the same group $\mathcal{H}-$class as ϵ_j. Hence for some $q \geq 1$, $\epsilon_j = (\xi_j \xi_{k+1,k+2} \ldots \xi_{i,i+1})^q$. If $k > i$, then the product

$$\xi_j \xi_{k+1,k+2} \cdots \xi_{n-1,n}\xi_{n,1}\xi_{1,2} \cdots \xi_{i,i+1}$$

is of height $n-1$ and belongs to the same group $\mathcal{H}-$class as ϵ_j. Hence, for some $q \geq 1$, $\epsilon_j = (\xi_j \xi_{k+1,k+2} \cdots \xi_{n-1,n}\xi_{n,1}\xi_{1,2} \cdots \xi_{i,i+1})^q$.

We have shown that every idempotent in $\{\epsilon_i : i = 1, \ldots, \frac{n(n-1)}{2}\}$ is a product of $\frac{n(n-1)}{2}$ quasi-idempotents $\xi_{1,2}, \ldots, \xi_{n-1,n}, \xi_{n,1}, \xi_{n+1}, \ldots, \xi_{\frac{n(n-1)}{2}}$. Then by [22] it follows that $\left\langle \xi_{1,2}, \ldots, \xi_{n-1,n}, \xi_{n,1}, \xi_{n+1}, \ldots, \xi_{\frac{n(n-1)}{2}} \right\rangle = Sing_n$. Since it is clear that no lesser number of quasi-idempotents can generate $Sing_n$, we have $qidrank(Sing_n) = \frac{n(n-1)}{2}$.

□

To exemplify the processes in the proof of this theorem. Let $n = 5$ and consider the $\mathcal{J}-$class $J_4 = \{\alpha \in Sing_5 : |im(\alpha)| = 4\}$. J_4 contains 5 $\mathcal{L}-$classes and 10 $\mathcal{R}-$classes. Let

$$A_1 = \{1,2,3,4\}, A_2 = \{1,2,3,5\}, A_3 = \{1,2,4,5\}, A_4 = \{1,3,4,5\}, A_5 = \{2,3,4,5\},$$

and

$$\pi_1 = \{1,5\},\{2\},\{3\},\{4\}, \quad \pi_2 = \{1\},\{2\},\{3\},\{4,5\},$$

$$\pi_3 = \{1\},\{2\},\{3,4\},\{5\}, \quad \pi_4 = \{1\},\{2,3\},\{4\},\{5\},$$

$$\pi_5 = \{1,2\},\{3\},\{4\},\{5\}, \quad \pi_6 : \{1,3\},\{2\},\{4\},\{5\},$$

$$\pi_7 = \{1,4\},\{2\},\{3\},\{5\}, \quad \pi_8 = \{1\},\{2\},\{4\},\{3,5\},$$

$$\pi_9 = \{1\},\{2,5\},\{3\},\{4\}, \quad \pi_{10} = \{1\},\{2,4\},\{3\},\{5\}.$$

Then these $4-$subsets and $4-$partitions form an orthogonally labelled list satisfying lemma 3.4.1. Thus, we have the following egg-box picture for J_4.

Table 3.1: *Egg-box picture of $J_{5,4}$ showing the elements of QI_4^* that generate $Sing_5$.*

J_4	$A_1 = \{1,2,3,4\}$	$A_2 = \{1,2,3,5\}$	$A_3 = \{1,2,4,5\}$	$A_4 = \{1,3,4,5\}$	$A_5 = \{2,3,4,5\}$
π_1	ϵ_1	ξ_1			
π_2		ϵ_2	ξ_2		
π_3			ϵ_3	ξ_3	
π_4				ϵ_4	ξ_4
π_5	ξ_5				ϵ_5
π_6		ξ_6	ϵ_6		
π_7		ϵ_7	ξ_7		
π_8			ϵ_8	ξ_8	
π_9				ϵ_9	ξ_9
π_{10}	ξ_{10}	ϵ_{10}			

By the results of theorem 3.4.4 the quasi-idempotents

$$\xi_1 = \begin{pmatrix} \{1,5\} & \{2\} & \{3\} & \{4\} \\ 1 & 2 & 3 & 5 \end{pmatrix}, \qquad \xi_2 = \begin{pmatrix} \{1\} & \{2\} & \{3\} & \{4,5\} \\ 1 & 2 & 4 & 5 \end{pmatrix},$$
$$\xi_3 = \begin{pmatrix} \{1\} & \{2\} & \{3,4\} & \{5\} \\ 1 & 3 & 4 & 5 \end{pmatrix}, \qquad \xi_4 = \begin{pmatrix} \{1\} & \{2,3\} & \{4\} & \{5\} \\ 2 & 3 & 4 & 5 \end{pmatrix},$$
$$\xi_5 = \begin{pmatrix} \{1,2\} & \{3\} & \{4\} & \{5\} \\ 1 & 2 & 3 & 4 \end{pmatrix}, \qquad \xi_6 = \begin{pmatrix} \{1,3\} & \{2\} & \{4\} & \{5\} \\ 1 & 2 & 3 & 5 \end{pmatrix},$$
$$\xi_7 = \begin{pmatrix} \{1,4\} & \{2\} & \{3\} & \{5\} \\ 1 & 2 & 4 & 5 \end{pmatrix}, \qquad \xi_8 = \begin{pmatrix} \{1\} & \{2\} & \{4\} & \{3,5\} \\ 1 & 3 & 4 & 5 \end{pmatrix},$$
$$\xi_9 = \begin{pmatrix} \{1\} & \{2,5\} & \{3\} & \{4\} \\ 5 & 2 & 3 & 4 \end{pmatrix}, \qquad \xi_{10} = \begin{pmatrix} \{1\} & \{2,4\} & \{3\} & \{5\} \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

generate $Sing_5$ and $qidrank(Sing_5) = 10$.

The result of theorem 3.4.4 can be generalised to cover the ideal $K(n, r)$ of $Sing_n$.

Theorem 3.4.5 *Let* $n \geq 3$ *and* $1 < r \leq n-1$. *Then* $qidrank(K(n,r)) = S(n,r)$, *the stirling number of the second kind.*

Proof. In [12], Lemma 5, Garba and Madu proved that there is a way of listing the $r-$subsets of X_n as $A_1, \ldots, A_{\binom{n}{r}}$ (with $A_1 = \{1, \ldots, r\}, A_{\binom{n}{r}} = \{n-r+1, \ldots, n\}$) so that there exist $r-$partitions $\pi_1, \ldots, \pi_{\binom{n}{r}}$ of X_n such that A_{i-1}, A_i are orthogonal to π_i $(i = 2, \ldots, \binom{n}{r})$; $A_1, A_{\binom{n}{r}}$ are orthogonal to π_1; A_{i+1} is not orthogonal to π_i $(i = 1, \ldots, \binom{n}{r} - 1)$; and A_1 is not orthogonal to $\pi_{\binom{n}{r}}$. Now if we choose $A_1, \ldots, A_{\binom{n}{r}}$ and $\pi_1, \ldots, \pi_{\binom{n}{r}}$ as a list of $r-$subsets and $r-$partitions of X_n satisfying the above properties, then the remainder of the proof follows exactly as the proof of theorem 4 in [12] with all the occurrences of quasi-nilpotent(s) replaced by quasi-idempotent(s).

$\square$

CHAPTER 4

QUASI-IDEMPOTENTS IN $\mathcal{I}_n$

In this chapter, we study quasi-idempotent elements in finite symmetric inverse semigroup $\mathcal{I}_n$. In particular we characterise quasi-idempotents in $\mathcal{I}_n$ and show that the semigroup $\mathcal{SI}_n$, of all strictly partial injections on X_n, is quasi-idempotent generated.

4.1 Characterisation of quasi-idempotents in $\mathcal{I}_n$

In this section we give a set-theoritic characterisation of quasi-idempotents in the symmetric inverse semigroup $\mathcal{I}_n$.

For a map α in $\mathcal{I}_n$ we will write $dom(\alpha)$ and $im(\alpha)$ for the domain and range of α respectively. We shall refer to $|im(\alpha)|$ as the *height* of α. Let $i \in X_n$ be such that $i \in dom(\alpha)$, then we shall say that i is an *invariant* point of α if $i\alpha \in dom(\alpha)$. The point $i \in dom(\alpha)$ will be called a *variant* point of α if $i\alpha \notin dom(\alpha)$. An invariant point i of α will be called twin if there exists $j \in dom(\alpha)$ such that $i\alpha = j$ and $j\alpha = i$. If $i\alpha = i$, i is a fixed point of α. Note that every fixed point of α is also a twin point. We shall denote by $Invar(\alpha), Var(\alpha), Twin(\alpha)$ and $Fix(\alpha)$ the sets of all invariant, variant, twin and fixed points of α respectively.

Lemma 4.1.1 *Let $\alpha \in \mathcal{I}_n$ be a quasi-idempotent. If $i \in dom(\alpha)$, then either $i \in Twin(\alpha)$ or $i \in Var(\alpha)$.*

Proof. Let $\alpha \in \mathcal{I}_n$ be a quasi-idempotent and suppose $i \in dom(\alpha)$ with $i \notin Twin(\alpha)$. We shall show that $i\alpha \notin dom(\alpha)$. Let us assume otherwise, that is $i\alpha \in dom(\alpha)$, since $i \notin Twin(\alpha)$ we must have $i\alpha^2 \neq i$ which is a contradiction to the choice of α as a quasi-idempotent. Therefore $i\alpha \notin dom(\alpha)$ so that $i \in Var(\alpha)$. On the other hand if $i \notin Var(\alpha)$ then $i\alpha \in dom(\alpha)$ and, since α is a quasi-idempotent, $i\alpha^2 = (i\alpha)\alpha = i$ which implies that $i \in Twin(\alpha)$. □

Corollary 4.1.2 *If $\alpha \in \mathcal{I}_n$ is a quasi-idempotent then every invariant point of α is twin.*

Proof. If $i \in Invar(\alpha)$ then, by lemma 2.1, i must be in $Twin(\alpha)$. □

Theorem 4.1.3 *A map $\alpha \in \mathcal{I}_n$ is a quasi-idempotent if and only if*

$$dom(\alpha) \cap im(\alpha) = Twin(\alpha).$$

Proof. Suppose that $dom(\alpha) \cap im(\alpha) = Twin(\alpha)$. Then $dom(\alpha^2) = (im(\alpha) \cap dom(\alpha))\alpha^{-1} = (Twin(\alpha))\alpha^{-1} = Twin(\alpha)$. Thus $dom(\alpha^2)\alpha^2 = (Twin(\alpha))\alpha^2 = Fix(\alpha^2)$. Hence α^2 is an idempotent and α is a quasi-idempotent. Conversely, suppose that $(dom(\alpha) \cap im(\alpha)) \setminus Twin(\alpha) \neq \emptyset$ and let $x \in (dom(\alpha) \cap im(\alpha)) \setminus Twin(\alpha)$. Then there exist $y \in dom(\alpha)$ and $z \in im(\alpha)$ such that $y\alpha = x$ and $x\alpha = z$. Therefore $y\alpha^2 = z$. Since $x \notin Twin(\alpha)$, we must have $y \neq z$ so that α^2 is not an idempotent and α is not a quasi-idempotent. □

Theorem 4.1.4 *Let $\alpha \in \mathcal{I}_n$ be a quasi-idempotent. Then α is of type one if and only if $Twin(\alpha) = Fix(\alpha)$.*

Proof. Suppose $\alpha \in \mathcal{I}_n$ is a quasi-idempotent of type one and let $i \in dom(\alpha^2)$. Then $i\alpha = (i\alpha^2)\alpha = i\alpha^3 = i$. Therefore $i \in Fix(\alpha)$ implying that $dom(\alpha^2) \subseteq Fix(\alpha)$. If $j \in Fix(\alpha)$, then $j\alpha^2 = j$, therefore $j \in dom(\alpha^2)$ implying that $Fix(\alpha) \subseteq dom(\alpha^2)$. Hence $dom(\alpha^2) = Fix(\alpha)$, but by theorem 2.3 $dom(\alpha^2) = (im(\alpha) \cap dom(\alpha))\alpha^{-1} = (Twin(\alpha))\alpha^{-1} = Twin(\alpha)$. Therefore $Twin(\alpha) = dom(\alpha^2) = Fix(\alpha)$. Conversely, let $\alpha \in \mathcal{I}_n$ be a quasi-idempotent and suppose $Twin(\alpha) = Fix(\alpha)$. Then by theorem 2.3 $dom(\alpha^2) = Fix(\alpha)$ and so $dom(\alpha^3) = (im(\alpha) \cap dom(\alpha^2))\alpha^{-1} = (im(\alpha) \cap Fix(\alpha))\alpha^{-1} = (Fix(\alpha))\alpha^{-1} = Fix(\alpha) = dom(\alpha^2)$. Therefore since each point of $Fix(\alpha)$ is also fixed α^3 we have $\alpha^2 = \alpha^3$. And so α is of type one. $\square$

The following corollary follows from theorems 2.3 and 2.4

Corollary 4.1.5 *Let $\alpha \in \mathcal{I}_n$. Then α is a quasi-idempotent of type one if and only if $dom(\alpha) \cap im(\alpha) = Fix(\alpha)$.*

4.2 Products of quasi-idempotents in $\mathcal{SI}_n$

Recall that (see [4]) the semigroup $\mathcal{I}_n$ has $n+1$ $J-$classes $J_0, \ldots, J_n$ where $J_r = \{\alpha \in \mathcal{I}_n : |im(\alpha)| = r\}$ $(0 \leq r \leq n)$. Let $L(n,r) = \{\alpha \in \mathcal{I}_n : |im(\alpha)| \leq r\}$ $(1 \leq r \leq n-1)$. This is an ideal of $\mathcal{I}_n$, its principal factor $J_r^* = J_r \cup \{0\}$ is a 0-simple semigroup with multiplication

$$\alpha * \beta = \begin{cases} \alpha\beta & if\ |im(\alpha\beta)| = r, \\ 0 & if\ |im(\alpha\beta)| \leq r. \end{cases}$$

Lemma 4.2.1 *Let α be an element of $\mathcal{I}_n$ of height r, where $1 \leq r \leq n-2$. Then there exist β, γ of height $r+1$ such that $\alpha = \beta\gamma$.*

Proof. Let $\alpha = \binom{a_1 \ a_2 \ \cdots \ a_r}{b_1 \ b_2 \ \cdots \ b_r}$. Since $r \leq n-2$, we can choose $a_{r+1} \in X_n \setminus \{a_1, \ldots, a_r\}$ and $b_{r+1} \in X_n \setminus \{b_1, \ldots, b_r\}$ such that $a_{r+1} \neq b_{r+1}$. Then there exists $c_1, \ldots, c_r \in X_n$ such that $c_i \notin \{a_{r+1}, b_{r+1}\}$ $(1 \leq i \leq r)$. Define $\beta = \binom{a_1 \ a_2 \ \cdots \ a_r \ a_{r+1}}{c_1 \ c_2 \ \cdots \ c_r \ b_{r+1}}$ and $\gamma = \binom{c_1 \ c_2 \ \cdots \ c_r \ a_{r+1}}{b_1 \ b_2 \ \cdots \ b_r \ b_{r+1}}$. Then it is easy to verify that β, γ are of height $r+1$ and $\beta\gamma = \alpha$. □

As a consequence of this lemma the elements of J_r^* generate the elements of $L(n, r)$ and a set of elements of height r generates J_r^* if and only if it generates $L(n, r)$.

Let QE'_{n-1} denote the set of quasi-idempotents of type one in J_{n-1}. Then, by theorem 2.5, each $\xi \in QE'_{n-1}$ must be of the form

$$\xi = \begin{pmatrix} 1 & \cdots & i-1 & i & i+1 & \cdots & j-1 & j+1 & \cdots & n \\ 1 & \cdots & i-1 & j & i+1 & \cdots & j-1 & j+1 & \cdots & n \end{pmatrix}$$

where $dom(\xi) = X_n \setminus \{j\}$, $im(\xi) = X_n \setminus \{i\}$ and ξ maps i to j and every other point of $dom(\xi)$ identically. The map ξ, in the chain-cycle notation, can be written as

$$\xi = (1) \cdots (j-1)(j+1) \cdots (n)[ij].$$

We shall omit all the fixed points of ξ and write ξ simply as $\xi = \binom{i}{j}$. We shall refer to i as the *upper entry* of ξ and to j as the *lower entry* of ξ. It is then clear that there are two quasi-idempotents of type one corresponding to any two distinct points of X_n. Thus we have

Lemma 4.2.2 $|QE'_{n-1}| = 2 \times \binom{n}{2} = n(n-1)$.

Theorem 4.2.3 *Let* $J_{n-1} = \{\alpha \in \mathcal{SI}_n : |im(\alpha)| = n-1\}$ *and let* QE'_{n-1} *denote the set of all quasi-idempotents of type one in* J_{n-1}*. Then each element of* J_{n-1} *is expressible as a product of elements in* QE'_{n-1}*.*

Proof. Let $\alpha \in J_{n-1}$. Then, since $|X_n \backslash im(\alpha)| = 1$, α is either a chain or has exactly one chain orbit. If α is a chain, we can write it as $\alpha = [a_1, a_2, \ldots, a_n]$ ($a_n \notin dom(\alpha)$). And so

$$\alpha = \begin{pmatrix} a_{n-1} \\ a_n \end{pmatrix}\begin{pmatrix} a_{n-2} \\ a_{n-1} \end{pmatrix}\cdots\begin{pmatrix} a_1 \\ a_2 \end{pmatrix},$$

is the required decomposition of α as a product of quasi-idempotents in QE'_{n-1}. Suppose α has $m \geq 1$ cyclic orbits $\Omega_1, \ldots, \Omega_m$ and a single chain $[c_1, \ldots, c_p]$. Consider a typical cyclic orbit $\Omega = (a_1, \ldots, a_k)$ of α and let

$$\beta = \begin{pmatrix} a_k \\ c_p \end{pmatrix}\begin{pmatrix} a_{k-1} \\ a_k \end{pmatrix}\cdots\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\begin{pmatrix} c_p \\ a_1 \end{pmatrix}.$$

Observe that each element of Ω appears exactly once as an upper entry in the product β and, with the sole exception of c_p, an element appearing as a lower entry never subsequently reappears as an upper entry. Hence each $x(\neq a_k) \in \Omega$ is moved by exactly one the quasi-idempotents, in the product β, to $x\alpha$. The point a_k is moved to c_p by the first quasi-idempotent and then to a_1 $(= a_k\alpha)$ by the last quasi-idempotent in the product β. Thus, $x\beta = x\alpha$ for all x in Ω, while $x\beta = x$ for each $x(\neq c_p) \notin \Omega$ and $c_p\beta = \emptyset$ (since c_p in not in the domain of the first quasi-idempotent in β). This argument applies to each of the cyclic orbits.

Now, let $\Omega = [c_1, c_2, \ldots, c_p]$ and let

$$\gamma = \begin{pmatrix} c_{p-1} \\ c_p \end{pmatrix}\begin{pmatrix} c_{p-2} \\ c_{p-1} \end{pmatrix}\cdots\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Then it can be observed as above that for each $x(\neq c_p) \in \Omega$ $x\gamma = x\alpha$ while $x\gamma = x$ for $x(\neq c_p) \notin \Omega$ and $c_p\gamma = \emptyset$. Therefore it follows that, for all $x \in dom(\alpha)$, $x\beta_1 \cdots \beta_m\gamma = x\alpha$ and, since the orbits of α are disjoint, $\alpha = \beta_1 \cdots \beta_m\gamma$, where β_i $(1 \leq i \leq m)$ correspond to the decomposition of the cyclic orbit Ω_i of α. $\square$

Example 4.2.1 Let $n = 9$ and consider the element

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 9 \\ 8 & 6 & 4 & 5 & 2 & 7 & 3 & 9 \end{pmatrix}.$$

First, we write α in the chain-cycle form $\alpha = (9)(5,2,6,7,3,4)[1,8]$ then, by theorem 3.2, $\alpha = \beta_1\beta_2\gamma$ where

$$\begin{aligned} \beta_1 &= \begin{pmatrix} 9 \\ 8 \end{pmatrix}\begin{pmatrix} 8 \\ 9 \end{pmatrix}, \\ \beta_2 &= \begin{pmatrix} 4 \\ 8 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix}\begin{pmatrix} 7 \\ 3 \end{pmatrix}\begin{pmatrix} 6 \\ 7 \end{pmatrix}\begin{pmatrix} 2 \\ 6 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}\begin{pmatrix} 8 \\ 5 \end{pmatrix}, \\ \gamma &= \begin{pmatrix} 1 \\ 8 \end{pmatrix}. \end{aligned}$$

Theorem 4.2.4 *The inverse semigroup $\mathcal{SI}_n = \mathcal{I}_n \setminus S_n$ is quasi-idempotent generated.*

Proof. This follows from theorem 3.2 and lemma 3.1. $\square$

CHAPTER 5

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In this thesis, we have considered two important semigroups of transformations, namely, the full transformation semigroup $\mathcal{T}_n$ and the symmetric inverse semigroup $\mathcal{I}_n$. The main objects of study in these semigroups are special type of elements called quasi-idempotents, that is, elements $\alpha \in \mathcal{T}_n$ with the property that $\alpha \neq \alpha^2 = \alpha^4$. Of more interest are quasi-idempotents α satisfying $\alpha^2 = \alpha^3$ called quasi-idempotents of type one.

In chapter three, theorems 3.2.4 and 3.2.5, we characterised quasi-idempotents and quasi-idempotents of type one respectively in $\mathcal{T}_n$. We then proved, in theorem 3.3.3, that the singular part of $\mathcal{T}_n$ is generated by quasi-idempotents of type one. Theorem 3.4.4 established that the quasi-idempotent rank of $Sing_n$ is $\frac{1}{2}n(n-1)$.

In chapter four, theorems 4.1.3 and 4.1.4, we characterised quasi-idempotents and quasi-idempotents of type one respectively in $\mathcal{I}_n$. We then proved, in theorem 4.2.4, that the semigroup $\mathcal{SI}_n$ is generated by quasi-idempotents of type one.

5.2 Conclusion

In this study, quasi-idempotent elements have been used to generate the subsemigroup $Sing_n$ of $\mathcal{T}_n$ (which was shown by Howie [19] to be generated by idempotents). Quasi-idempotent rank of $Sing_n$ is also found to be equal to its idempotent rank.

The symmetric inverse semigroup $\mathcal{I}_n$ (being an inverse semigroup) is not generated by idempotent elements. Gomes and Howie [14] showed that, when n is even, the subsemigroup $\mathcal{SI}_n$ of $\mathcal{I}_n$ is generated by set of nilpotents, and that the nilpotents only generate a proper subsemigroup of $\mathcal{SI}_n$ when n is odd. In this study, the entire semigroup $\mathcal{SI}_n$ has been generated by a set of quasi-idempotents, regardless of whether n is even or odd. Thus, quasi-idempotent elements have generated an idempotent generated semigroup and a semigroup which is neither generated by idempotent nor by nilpotent elements, namely, $Sing_n$ and $\mathcal{SI}_n$ respectively. Therefore, we conclude that quasi-idempotent elements are more promising than the idempotent elements and that the quasi-idempotent elements may replace the idempotents in the theory of transformation semigroups.

5.3 Recommendations

Various classes of transformations of X_n have been proved to form a semigroup under composition of mappings. Some of these semigroups have not been (or may not be) generated by idempotents and/or nilpotents. We therefore recommend that quasi-idempotent elements should be investigated in these semigroups and that the role of quasi-idempotents should be investigated further in the algebraic theory of semigroups. This may be achieved via exploring the quasi-idempotent rank properties of a quasi-idempotent generated semigroup and on comparison with corresponding properties for idempotents and/or nilpotents.

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