# INVESTIGATION OF MOTION IN THE CIRCULAR RESTRICTED PROBLEMS OF THREE-BODIES UNDER PERTURBING FORCES WITH VARIABLE AND CONSTANT MASSES

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**19 JANUARY, 2016**

**DECLARATION**

I declare that the work in this thesis titled "investigation of motion in the circular restricted problems of three-bodies under perturbing forces with variable and constant masses" has been performed by me in the Department of Mathematics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any institution.

_ONI SAMSON LEKE _____ _____

**Name of student** **Signature** **Date**

## CERTIFICATION

This Thesis titled "**INVESTIGATION OF MOTION IN THE CIRCULAR RESTRICTED PROBLEMS OF THREE-BODIES UNDER PERTURBING FORCES WITH VARIABLE AND CONSTANT MASSES**" by Samson Leke ONI meets the regulation governing the award of the degree of Doctor of Philosophy in Mathematics of Ahmadu Bello University, and is approved for its' contribution to knowledge and literary presentation.

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## DEDICATION

This thesis is dedicated to the memories of my beloved father, HRH James .T. Oni (The Akinrin of Ekinrin II). To my lovely pretty wife Blessing and gorgeous children Denilson, Daniella, Davidson whom the lord blessed me with; my quest for more education brought you into my life, and for that I will be eternally grateful.

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## ABSTRACT

This thesis investigates the motion of a test body around equilibrium points of the circular restricted problems of three-bodies using the model of the general restricted three-body problem (R3BP) and the Robe's R3BP, and taking into account various chacterizations which include circumbinary disc, oblateness, triaxiality, radiation, deflections due to the Coriolis and centrifugal forces and to whether the masses are varying with time or constant. In the R3BP with variable masses, we assume that the primaries vary isotropically in accordance with the unified Mestschersky law (UML) and their motion governed by the Gylden-Mestschersky problem (GMP). We transform the non-autonomous equations of motion to the autonomized forms using the Mestschersky transformation (MT), the UML, the particular integral and solutions of the GMP. The equilibrium points (EPs) are examined and it is seen that due to the mass variations coupled with other perturbing forces; five collinear , two triangular and four out-of-plane EPs are found. The linear stability of the EPs is investigated. For the autonomized systems, the collinear and triangular EPs are stable while the out-of-plane EPs are unstable. We examine the periodic orbits around stable triangular EPs of the autonomized systems and found that the orbits can be elliptical, parabolic or hyperbolic in nature due to mass variations. The stability of the EPs of the non-autonomous equations is tested using the Lyapunov's characteristic number and are seen to be unstable. In the Robe's R3BP with variable masses, we consider effects of the buoyancy force, the gravitational attraction of the fluid and attraction of the second primary. In the first case, we assumed that the primaries vary in accordance with the UML law and their motion described by the GMP. The transformation from the non-autonomous to the autonomized equations is possible only when the first primary contains no fluid. In this regime, an EP at the center of the first primary and a pair of non-collinear EPs exists. When small change in the

Coriolis and centrifugal forces are introduced, every point on the line joining the primaries is an EP and a pair of non-collinear EP also exists. In the second case, the first primary is a fluid in the shape of a sphere with no shell and the masses of the three bodies vary with time. We use the Gelf'gat transformation to cast the non-autonomous system to the autonomized kind. An EP at the center, another near the center, circular points and a pair of non-collinear EP exists. The collinear EPs are stable while the non-collinear and circular EPs are unstable. The study is applied to the motion of a submarine in the Earth-Moon system. In the study of R3BP with constant masses in the presence of a circumbinary disc under different characterizations, it is seen that two additional collinear EPs are found depending on the mass parameter of the system and the mass of the disc. Our numerical evidence reveals that the existence of the additional EPs depend on the system under study. In the range of linearly stable triangular EPs, we examine the periodic orbits around these points and found that the orbits are ellipses. We test their stability using the Poincare Surfaces of Section (PSS) and found them to be stable. Our problem can be used to study the long-term motion of satellites, submarines and planets in binary systems. These bodies are taken to be test particles moving in the field of circular binary system. A simple question of celestial mechanics is, if the EPs of these bodies near a binary star system, loosing mass isotropically or conserving mass, exist, how long can they keep the bodies from escaping? The determination of ranges of semimajor axes may help to know when the body is likely to escape. The obtained EPs may be used in different problems of stellar dynamics, and also in other astrophysical applications.

# TABLE OF CONTENT

TITLE PAGE……i

APPROVAL PAGE...……iii

DECLARATION ..... iii

CERTIFICATION ..... iv

DEDICATION ..... v

ACKNOWLEDGEMENT ..... vi

ABSTRACT ..... vii

TABLE OF CONTENT ..... ix

LIST OF TABLES ..... xvi

LIST OF ACRONYMS ..... xviii

LIST OF SYMBOLS ..... xviii

CHAPTER ONE ..... 1

1.0 INTRODUCTION ..... 1

1.1 General Introduction ..... 1

1.2 Historical Background ..... 2

1.3 Statement of the Problem ..... 5

1.4 Significance/ Justification of the Study ..... 6

1.5 Aim and Objectives of the Study ..... 7

1.6 Thesis Organization ..... 8

1.7 Theoretical Framework ..... 12

1.7.1 Radiation pressure ..... 12

1.7.2. Small perturbations in Coriolis and centrifugal forces ..... 15

1.7.3. Shapes of the bodies ..... 15

1.7.4 Discs ..... 18

1.7.5. Mass variations ..... 20

1.7.6 Binary and trinary systems ..... 22

1.7.7. Relational definitions ..... 25

1.8.1 The two-body problem and the integral of area ..... 26

1.8.2 Circular restricted three-body problem ..... 29

1.8.3. Robe's restricted three-body problem ..... 32

1.8.4 Equation of motion of two-body with variable masses ..... 41

1.8.5. The Mestschersky transformation and unified Mestschersky law ..... 43

1.8.6 Particular solutions of the Gylden-Mestschersky problem ..... 45

1.8.7 Equations of motion of the restricted three-body problem with isotropic mass variations of the primaries ..... 49

1.8.8 Equations of motion of the restricted problem of three bodies with non-isotropic mass variations ..... 59

1.8.9 Stability of dynamical systems ..... 66

LITERATURE REVIEW ..... 74

2.1. Restricted Three-Body Problem ..... 74

2.2. Restricted Three-Body Problem with Variable Masses ..... 84

2.3. Robe Restricted Three-Body Problem ..... 90

2.4 Periodic Orbits in the Restricted Three-Body Problem ..... 95

CHAPTER THREE ..... 98

3.0 RESTRICTED PROBLEMS OF THREE BODIES WITH VARIABLE MASSES ..... 98

3.1 Introduction ..... 98

3.2. Equilibrium Points and Stability in the R3BP with Perturbing Forces, Oblateness and Variable Masses ..... 100

3.2.1. Equations of motion of the third body ..... 100

3.2.2 Locations of equilibrium points ..... 107

3.2.3. Stability of equilibrium points ..... 133

3.3 Combined Effects of Oblateness, Radiation, Perturbation in the Restricted Three-Body Problem with Variable Masses ..... 153

3.3.1 Equations of motion of the third body ..... 153

3.3.2. Locations of equilibrium points ..... 155

3.3.3. Stability of equilibrium points ..... 175

3.4 Out-of-Plane Equilibrium Points in the R3BP with Variable Mass and Disc ..... 202

3.4.1 Equations of motion of the third body ..... 202

3.4.2. Locations of the out-of-plane equilibrium points ..... 205

3.4.3 Stability of out-of-plane equilibrium points ..... 210

3.5 Effects of Mass Variations, Oblateness, Radiation and Perturbations On the Periodic Orbits Around Triangular Equilibrium Points ..... 212

3.5.1. Frequencies and Orientation of the orbits ..... 212

3.5.3 The semi-axes of the orbits ..... 247

3.6. Robe's Restricted Three-Body Problem with Isotropic Mass Variation when the First Primary is Empty ..... 259

3.6.1Equations of motion of the third body ..... 259

3.6.2. Locations of equilibrium points ..... 266

3.6.3 Stability of equilibrium points ..... 272

3.7 Robe's Restricted Problem with Isotropic Mass Variation and Perturbing Forces when the First Primary is Empty ..... 281

3.7.1 Equations of motion of the third body ..... 281

3.7.2 Locations of equilibrium points ..... 282

3.7.3. Stability of equilibrium points ..... 288

3.8 Robe's Restricted Problem with Non-Isotropic Mass Variation of the Bodies ..... 295

3.8.1 Equations of motion of the third body ..... 295

3.8.2. Locations of the equilibrium points ..... 299

3.8.3 Stability of equilibrium points ..... 306

3.9 Conclusion ..... 321

CHAPTER FOUR ..... 323

4.0 RESTRICTED THREE-BODY PROBLEM WITH CONSTANT MASSES AND DISC UNDER DIFFERENT CHARATERIZATIONS ..... 323

4.1. Introduction ..... 323

4.2 Motion in a Modified Restricted Three-Body Problem with Oblateness and Disc ..... 324

4.2.1Equations of motion ..... 324

4.2.2 Locations of equilibrium points ..... 328

4.2.3 Stability of equilibrium points ..... 336

4.2.4 Periodic orbits around triangular equilibrium points ..... 342

4.3. Analytic Treatment of Motion of Dust Grain Particle around Triangular Equilibrium Points with Post-AGB Binary Star and Disc ..... 349

4.3.1 Equations of motion ..... 349

4.3.2 Location of triangular equilibrium points ..... 351

4.3.3. Stability of triangular equilibrium points ..... 353

4.4 A Generalized Restricted Problem of Three-Oblate Bodies with Radiation Pressure and Disc ..... 363

4.4.1Equations of motion ..... 363

4.4.2 Locations of equilibrium points ..... 365

4.4.3 Stability of equilibrium points ..... 375

4.4.4 Periodic orbits around the triangular equilibrium points ..... 380

4.5. Conclusion ..... 389

CHAPTER FIVE ..... 391

5.0 RESULTS AND DISCUSSION ..... 391

5.1 Numerical Treatments of the Restricted Problem of Three-bodies with Variable Masses 391

5.1.1Restricted three-body with isotropic mass variations ..... 391

5.1.2 Robes restricted-three-body problem with variable masses ..... 429

5.1.2 Numerical Treatments of the Restricted Problem of Three-bodies with Constant Masses ..... 446

CHAPTER SIX ..... 481

6.0. SUMMARY, CONCLUSION AND RECOMMENDATIONS ..... 481

6.1. Summary ..... 481

6.2. Conclusion ..... 484

6.3. Recommendations ..... 486

6.4. Contributions of This Thesis To Knowledge ..... 487

REFERENCES ..... 488

**LIST OF FIGURES**

**1.1**. The Pillars of Creation clouds shaped by radiation pressure and stellar winds……..…13

**1.2:** The shape of an oblate spheroid…..……....16

**1.3:** Artist's conception of Haumea, a triaxial dwarf planet, with its two moons……..16

**1.4:** A spheroid showing the role of the semi-axes on the shape……...…17

**1.5:** An artist's impression of the binary star system **HD 98800 B** surrounded by a disc…….19

**1.6:** An artistic impression of mass loss of an extrasolar planet **HD 209458b**.....20

**1.7:** Kepler-16 system, showing the binary star being orbited by Kepler-16b……..24

**1.8:** The circular restricted three-body problem……30

**1.9:** The five Lagrangian points in CR3BP of the Sun-Earth –Moon system……..….32

**1.10.** Set up of the Robe restricted three-body problem……33

**3.1:** Graph of $f\ \xi$ for the out of plane points $L_{6,7}$ ……127

**3.2:** Out of plane Equilibrium points $L_{6,7}$ ……..132

**3.3:** Out of plane Equilibrium points $L_{8,9}$ ……133

**5.1:** Graph of $\kappa$ and the critical mass parameters $\upsilon_{C_\kappa}$ ……..399

**5.2:** Out-of-plane equilibrium points for PSR B1620-26b …..……..404

**5.3:** Out-of-plane equilibrium points for Kepler-16b in the binary system Kepler-16…….…...405

**5.4:** Out-of-plane equilibrium points of PSR B1620-26b and Kepler-16b……..405

**5.5:** Out-of-plane equilibrium points of PSR B1620-26b…….…...406

**5.6:** Lengths of the Semi-axes…….422

**5.7:** Lengths of the Semi-axes…….422

**5.8**: Lengths of the semi axes……...423

**5.9:** Periodic orbits around $L_4$ points……...424

**5.10:** Effects of the mass parameter on the orbits around $L_4$ points……425

**5.11:** Orbits around triangular points $L_4$ ……..……425

**5.12:** Orbits of the third body around triangular points $L_4$ ……..426

**5.13:** Orbits of the third body around triangular points $L_4$ ……..427

**5.14:** Non-collinear equilibrium points..……..430

**5.15:** Non-collinear equilibrium points..……..430

**5.16:** Non-collinear equilibrium points…….431

**5.17:** Non-collinear equilibrium points……438

**5.18:** Non-collinear equilibrium points……439

**5.19:** Non-collinear equilibrium points……439

**5.20:** Non-collinear equilibrium points……439

**5.21:** A floating Submarine……..441

**5.22:** Image of a submerging submarine……..442

**5.23:** Mechanism of how the bouyancy force is controlled……442

**5.24:** Surface plot of $x-$ coordinate of triangular points…….....451

**5.25:** Surface plot of $x-$ coordinate of triangular points……452

**5.26:** Surface plot of $y-$coordinate of triangular points…….452

**5.27:** Surface plot of $y-$coordinate of triangular points……453

**5.28:** Surface plot of $y-$coordinate of triangular points……453

**5.29:** Surface plot of $y-$coordinate of triangular points……453

**5.30:** Variations in the frequencies……..463

**5.31:** Variations of the eccentricities……...463

**5.32:** Effects of the parameters on the semi-axes of the orbits……466

**5.33:** Effects of the parameters on the semi-axes of the orbits……466

**5.34:** Effects of the parameters on the semi-axes of the orbits……....467

**5.35:** Lengths of the semi-axes of long and short period orbits……...467

**5.36:** Lengths of the semi-axes of long and short period orbits……...468

**5.37:** Long periodic orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system…….469

**5.38:** Short and long periodic orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system……...469

**5.39:** Variations in the orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system……470

**5.40:** Variations in the orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system……471

**5.41:** Poincare Surfaces of Section……..472

**5.42:** Poincare Surfaces of Section……..473

**5.43:** Poincare Surfaces of Section……..473

**5.44:** Poincare Surfaces of Section……..474

**5.45:** Poincare Surfaces of Section……..474

**5.46:** Poincare Surfaces of Section……..475

**5.47:** Poincare Surfaces of Section……..475

**5.48:** Surface plot of $L_4$ ……477

**5.49:** Surface plot of $L_5$ ……478

**5.50:** Surface plot of $L_4$ ……478

**5.51:** Surface plot of $L_5$ ……479

**5.52**: Surface plot of $L_{4,5}$ …….....479

## LIST OF TABLES

**1.1:** List of circumbinary planets……..…….25

**5.1:** Numerical computations of the collinear point $L_{2\kappa}$ ……..……....392

**5.2:** Numerical computations of the critical mass parameter $\upsilon_{C_\kappa}$ …….……...395

**5.3:** Numerical computations of the collinear points…….……..395

**5.4:** Numerical computations of the collinear points……..396

**5.5:** Numerical computations of the collinear points……..……..…398

**5.6:** Numerical approximations of out-of-plane points……...401

**5.7** Effects of $\kappa$ on the eccentricities and lengths of the semi-axes……..……..403

**5.8:** Out-of-plane equilibrium points for PSR B1620-26b and Kepler-16b in the binary systems PSR B1620-26……..421

**5.9:** Positions of non-collinear points and the characteristic roots……..431

**5.10:** Positions of non-collinear points and the characteristic roots..……..432

**5.11:** The non-collinear equilibrium points..……...436

**5.12:** The non-collinear equilibrium points..……...437

**5.13:** The non-collinear equilibrium points.……438

**5.14:** The characteristic roots of the non-collinear points..…….443

**5.15:** The characteristic roots of the non-collinear points..…….443

**5.16:** The characteristic roots of the non-collinear points……...444

**5.17:** The characteristic roots of the non-collinear points…..…….444

**5.18:** Numerical computations of the collinear equilibrium points for the Sun–Earth system…448

**5.19:** Numerical computations of the collinear equilibrium points for the Sun–Jupiter system……..449

**5.20:** Numerical computations of the collinear equilibrium points for the Fomalhaut system...450

**5.21:** Critical mass values for the Sun–Earth and Sun–Jupiter systems……..456

**5.22:** Critical mass values for the Sun–Earth and Sun–Jupiter systems……..456

**5.23:** Computations of the periodic orbits and their terms in the Sun-Earth system……...457

**5.24:** Computations of the periodic orbits and their terms in the Sun-Jupiter system……458

**5.25:** Mass functions for the binary IRAS 11472-0800 and G29-38……...459

**5.26:** Coordinates of triangular equilibrium points in IRAS 11472-0800 and G29-38 system……..460

**5.27:** Critical mass function for mass ratios in IRAS 11472-0800 and G29-38 system……..461

**5.28:** Critical mass function for mass ratios in IRAS 11472-0800 Vs G29-38 system……...462

**5.29:** Effects of the mass distributions on the eccentricities in IRAS 11472-0800 Vs G29-38 System…….....464

**5.30:** Effects of the mass of disc on the lengths of the semi-axes in IRAS 11472-0800 Vs G29-38 System……465

## LIST OF ACRONYMS

2BP: Two-Body Problem

3BP: Three-Body Problem

R3BP: Restricted Three-Body Problem

CR3BP: Circular Restricted Three-Body Problem

UML: Unified Mestschersky Law

GMP: Gylden-Mestschersky Problem

PSS: Poincare Surface of Section

LCN: Lyapunov's Characteristic Number

MT: Mestschersky Transformation

EP(s): Equilibrium Point(s)

## LIST OF SYMBOLS

$m_i \quad i=1,2,3$ : Mass of the first and second primary, and the third-body, respectively

$r_i$, $\rho_i \quad i=1,2$ : Distances from the primaries to the third body

$r$ , $\rho_{12}$ : Distance between the primaries.

$r_c$ , $\rho_c$ : Radial distance of the third body

$G$ : Gravitational constant

$\mu\ t$ : Product of the gravitational constant and sum of the masses of the primaries

$\mu_i\ \ i=1,2$ : Product of the masses of the first and second primary and the gravitational constant

$\mu_{i0}\ \ i=1,2$ : Masses of the primaries at initial time

$\mu_0$ : Sum of the masses of the primaries at initial time

$\mu$ , $\upsilon$ : Mass parameter

$\varphi$ : Parameter for the Coriolis force

$\psi$ : Parameter for the centrifugal force

$\in$ , $\in'$ : Parameters representing a small perturbation in the Coriolis and centrifugal forces

$A_i\ \ i=1,2$ , $\alpha_i\ \ i=1,2,3$ : Oblateness parameters of the bodies

$q_i\ \ i=1,2$ : Radiation pressure parameters of the primaries

$\sigma_i\ \ 1,2$ Parameter representing the triaxiality of a primary

$E$ is the intensity of radiation.

$\kappa$ : Kappa (constant of a particular integral of the GMP)

$\omega$ , $\omega_0$ : Angular velocity of $m_2$ around $m_1$

$x, y, z$ , $\xi, \eta, \zeta$ : Coordinates of the third body

$t$ , $\tau$ : Time

$V$ : Potential Energy

$\Omega$ : Force function

$M\ \ t\ ,\ M_d$ : Mass of the disc

$\delta_i\ \ i=1,3$ : Densities

$D_p$ : Density parameter

$R\ \ t\ ,\ \mu_*$ : Function of time

$\rho_E\ ,\ \rho_P$ : Equatorial and polar radii

$O$ -origin or center of mass

$\Re$ : Radius

$M_i\ \ i=1,2,3$ : Centers of the bodies

$C$ : Constant of the Area integral

$J$ : Jacobi Constant

$\theta$ : Angle between the straight line passing through $m_1$ and $m_2$

$x_i\ \ i=1,2$ : Coordinates of the primaries

$<<$ : Strictly less than

. dot: Differentiation w.r.t time

$L_i\ \ i=1,2,..$ : Equilibrium points

$L$ : Lyapunov's Characteristics Number

$\vec{F}_A$ : Gravitational force of the fluid

$\vec{F}_B$ : Buoyancy force

$F_g$ : Gravitational force

$F_p$ : Radiation pressure force

$\Omega^0_{\xi\xi}$ , $U^0_{xx}$ : Partial derivatives to be calculated at equilibrium points

$\mu_c$ , $\upsilon_{c_\kappa}$ : Critical mass parameter

$M_\odot$ : Solar mass

$u, v, w$: Displacements

$\dot{x}, \dot{y}, \dot{z}$ , $\ddot{x}, \ddot{y}, \ddot{z}$ velocity and acceleration components of the third body

$D$: Discriminant

$s_i$ $i = 1, 2$ : Frequencies of the long and short periodic orbits

$I_i$ $(= 1,2,3)$: Principal moments of inertia of a triaxial body and its center of mass

$I$ , $I'$: Moments of inertia about the line joining the centers of primaries and $m_3$, respectively

$e_i$ $i = 1, 2$ : Eccentricity of long and short orbits

$a_i$ $i = 1, 2$ : Lengths of the major axes of long and short periodic orbits

$b_i$ $i = 1, 2$ : Lengths of the minor axes of long and short periodic orbits

$T^*$: Time period

$\bar{T}$ : Parameter which defines the density profile of Disc

$a$ : Flatness parameter of disc

$b$ : Core parameter of disc

<=>: Less than, equal to or greater than

$\vec{R}$ : Radius vector between the origin and center of the third mass

$O$: Origin and mostly the center of mass of the primaries in a Barycentric coordinate system

$n$ : Mean Motion

# CHAPTER ONE

## 1.0 INTRODUCTION

## 1.1 General Introduction

The restricted three-body problem (R3BP) describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The approximate circular motion of the planets around the sun and the small masses of asteroids and the satellites of planets compared to the planet's masses, originally suggested the formulation of the restricted problem. The study of the R3BP is of great theoretical, practical, historical and educational relevance, and in its many variant, has had important implications in several scientific fields, including among others; celestial mechanics, galactic dynamics, chaos theory and molecular physics. The R3BP is still a stimulating and active research field that has been receiving considerable attention of scientists and astronomers because of its applications in dynamics of the solar and stellar systems, lunar theory, and artificial satellites.

There are so many examples of the restricted problem in space dynamics. One of them is the classical three-body problem viz; the Sun-Earth-Moon combination and describing the motion of the moon. The motion of a Trojan asteroid attracted by the Sun and Jupiter is another example, which has a similarity with the restricted problem. One of the foremost in space science is the

creation of artificial bodies, which are required to move in the neighborhood of two natural celestial bodies. This problem is also similar to the restricted problem.

## 1.2 Historical Background

Great minds throughout the ages have been interested in the celestial motions they observed in the night sky. In 1601, Johannes Kepler became the director of the Prague Observatory following the death of Tycho Brahe. Using Brahe's meticulously collected data; Kepler formulated his famous three laws of planetary motion:

1) Every planet moves in an elliptical orbit, with the sun at one focus.

2) The radius vector sweeps out equal areas in equal times.

3) The square of the period is proportional to the cube of the semi-major axis.

Later that century, in 1687, Newton would go on to devise his three laws of motion and put forth his theory of gravitation:

$$\vec{F} = \frac{Gm_1 m_2}{r^2} \tag{1.1}$$

where G is the universal gravitational constant and $r$ is the distance between mass 1 $(m_1)$ and mass 2 $(m_2)$. Armed with his newly invented calculus and laws of gravitation, Newton set out to explain Kepler's laws. He succeeded in mathematically and geometrically describing the laws that govern planetary motion and derived many of the equations that are used today. Newton discovered the laws of motion and universal law of gravitation, and combined them to explain kepler's laws of planetary motion and solved the two-body problem (2BP). In 1744, Leonhard

Euler went on to find the full analytical solution to the two-body equations. He was also one of the first to derive the equations defining the change in osculating elements with time, giving rise to the analytic theory of perturbed motions. In works published in 1761 and 1771, Johann Lambert used a geometrical approach to generalize Euler's formulas to include elliptical and hyperbolic orbits.

The three body problem (3BP) is a classic problem of celestial mechanics, wherein the interest lies in the motion of a third particle in the presence of two massive bodies. One of the earliest contributors to the 3BP was Sir Isaac Newton in the years after he published his theory of gravitation. The problem of ocean navigation required an understanding of the motion of the Moon, which is strongly affected by both the Earth and the Sun and, hence, beyond the scope of the basic two body problem. This problem was of great interest throughout the eighteenth century and was approached in basically two ways: by an infinite series expansion to describe the solution (Clairaut, 1752) or by variation of parameters (Euler, 1744). It was Euler who first used rotating coordinates to frame the problem. Most of the next generation of mathematicians and physicists tried to extend Newton's analytical method to 3BP (i.e. Sun, Earth, Moon), but this problem turned out to be more difficult to solve. After decades of effort, it was eventually realized that three-body problem was impossible to solve, in the sense of obtaining explicit formulae for the motion of three bodies.

However, it was not until 1772 that Lagrange demonstrated a reduction of the problem from its original form, with 18 unknowns, to a problem of order 7 in the rotating frame. Lagrange proposed the assumptions that led to the "restricted" model, allowing closed form solutions and more detailed analyses of celestial motions.

The history of the "Restricted Problem" goes back to Euler and Lagrange (1772). It was followed by Jacobi (1836), Hill (1878), Poincare (1899), Levi-Civita (1903), Birkhoff (1915) and many more modern mathematicians and astronomers. Euler, in his lunar theory, introduced a rotating (or synodic) co-ordinate system, which was used by Jacobi to find the famous Jacobian integral of the equations of motion of the restricted problem. Lagrange showed that the five points of equilibrium, or points of libration (often denoted by $L_i \;\; i = 1, 2...5$ represent stationary solutions of the restricted problem. Lagrange also mentioned the special case of the restricted problem when the Coriolis forces are omitted and a condition of symmetry regarding the masses of the primaries is introduced, the restricted problem may be solved in terms of elliptic functions. Prior to the lunar theory, in 1760, Euler gave the solution of the problem of two fixed centers of force, in which two fixed masses act on the third body according to Newton's law of gravitation. This is a special case of the restricted problem when both the Coriolis and the centrifugal forces are omitted. Euler's solution of the problem of two fixed centers of forces can be used in connection with the artificial satellite problem, and as a reference orbit for the calculation of perturbations in the restricted problem.

The next significant contribution to the R3BP did not appear until Carl Jacobi in 1836. He discovered the only integral of motion in the circular restricted three-body problem (CR3BP), which bears his name as the Jacobi constant. Jacobi's work was further extended by Hill (1878) who used this lone integral to define curves of zero velocity that limit the motion of the third particle. Hill's investigations focused on the Moon and, thus, constrained the Moon's motion to certain regions of space around the two main bodies (the Sun and the Earth).

Another significant step in understanding the 3BP came about due to a mathematical competition to honor the 60th birthday King Oscar II of Sweden and Norway in 1889. Henri Poincare was awarded the prize for his study of the CR3BP. What set his work apart was that he shifted from a quantitative aspect to a more qualitative assessment. While Poincare was interested in finding solutions to the three-body problem, his approach differed from all previous developments, since he was more interested in the nature of the solutions than in the actual solutions themselves. His mathematical work showed that an infinite number of periodic orbits exist in the 3BP. Poincare also introduced an innovative method, the Poincare section (or stroboscopic map) to study the behavior of solutions as time tends to infinity (Barrow-Green, 1997). Poincare is considered by many to be the father of dynamical systems theory.

Poincare's work is regarded as the highest theoretical achievement in his field. He emphasized on the qualitative approach to celestial mechanics rather than the quantitative one. Poincare, in his first volume, even suggested that the topic of periodic orbits is the only opening through which we can gather significant information about the unsolved general 3BP.

Some lunar and planetary theories in celestial mechanics are based on the R3BP. There are numerous applications of the R3BP in cosmogony and stellar dynamics. In fact, the equations of motion of so many dynamical systems are similar to those of the R3BP.

## 1.3 Statement of the Problem

Effects of some perturbing forces due to radiation pressures, deflecting forces, shapes, and the enclosure of the primaries by a disc of dust under mass variations of the primaries remain open problems; and so is the Robe's R3BP when the masses of the bodies in the configuration vary with time. Some studies of the R3BP when the masses do not change with time that could solve

problems in celestial mechanics have also remained open. Hence, the problem in this thesis is to study the Restricted problems of three-bodies taking into account shapes of the bodies, small change in the Coriolis and centrifugal forces, circumbinary disc and radiation pressure force,

1. when masses of the primaries vary isotropically in accordance with the unified Mestschersky law (UML) (1952) and their motion described by the Gylden-Mestschersky problem (GMP) (Gylden, 1884; Mestschersky, 1902) using the model of the general R3BP.

2. when isotropic mass variations of the primaries and non-isotropic mass variations of the three bodies occur, using the set up of the Robe's (1977) CR3BP.

3. when the masses of the participating bodies are kept constant using the formulation of the general R3BP

## 1.4 Significance/ Justification of the Study

The study of the R3BP is of great theoretical, practical, historical and educational relevance, and in its many variant, has been useful in several scientific fields, including among others; celestial mechanics, galactic dynamics, chaos theory and molecular physics (Szebehely 1967a). The R3BP is still a stimulating and active research field that has been receiving attention of scientists and astronomers because, solar system dynamics, lunar theory, motion of space crafts and satellites, stellar dynamics and planetary theories in celestial mechanics are built on it. The Robe's problem (Robe 1977) has been useful in discussing the stability of the Earth's core.

In this thesis, we examine the motion of the third body around the equilibrium points of the R3BP and the Robe's R3BP when the masses of the participating bodies vary with time and when they are kept constant under additional imposed forces of radiation, shapes, disc mass and deflections of the primaries on circular orbits.

1. The study compares and contrast the restricted problems (R3BP and RR3BP) when the masses are constant with when the masses change with time. It clarifies which of these two problems generalize the other.
2. The study is useful in various astronomical problems in which it is necessary to allow for the time dependences of the masses of celestial bodies in the course of evolution.
3. It predicts the long time stability of the Earth's core which is a key factor for habitation on the Earth.
4. It is useful in describing the motion of a submarine in an ocean under the Earth's and Moon's attraction.
5. The thesis finds its applications in the study of the long-term motion of satellites, gyrostat and planets in binary systems. These bodies are taken to be test particles moving in the field of circular binary system.

## 1.5 Aim and Objectives of the Study

The aim of the thesis is to investigate the existence and stability of equilibrium points, and periodic orbits of the restricted problems of three bodies using the models of the CR3BP and the Robe's CR3BP taking into account the forces due to disc of dust around the primaries, radiation

pressure, slight deflections, shapes (oblateness and triaxiality); when the masses in the configurations vary with time and when they are constant.

The objectives of the study are

1. Derivations of the equations of motion for the formulated restricted problems.
2. Investigation of the equilibrium points (EPs) of the problem with numerical applications to realistic models (Stellar or solar systems).
3. Determination of the stability of EPs for all the stated problems.
4. Investigation of the periodic orbits around stable triangular EPs.

## 1.6 Thesis Organization

This thesis has been written in **SIX** chapters and is organized as follows: Chapter **ONE** gives the introduction of the work. Here, the historical background, the statement of the problem; the justification and significance, and, the aim and objectives of the study form the gateway to the dissertation. The theoretical framework and preliminary studies of the thesis have been given in sections **1.7** and **1.8**, respectively. The various characterizations of the participating bodies, binary and trinary systems, and some definitions are discussed under section **1.7**, while the CR3BP, the Robe's R3BP, equation of motion of two-body with variable masses, the GMP and its consequence; equations of motion of the R3BP with isotropic mass flow of the primaries, and those of the restricted problem of three bodies with non-isotropic mass flow, are reviewed in section **1.8**. The stability of dynamical systems and the relationship between the "old" and "new" time are also contained in this section.

Chapter **TWO** is a detailed review of literature. The literature dates back to the emergence of the study of the three-body problem up to recent works in the R3BP. The literature is reviewed based on the characterizations of the general R3BP. We merge studies that considered the masses as constant in section **2.1**, while those studies which took into account mass variations of one, two or all bodies are discussed in section **2.2**. Sections **2.3** and **2.4** comprehensively capture literature on related works of the Robe's problem and periodic orbits, respectively.

In chapter **THREE,** the restricted problems of three-bodies with variable masses using the model of the R3BP and the Robe's R3BP are dealt with. The chapter encapsulates different characterizations of the primaries which are radiation pressure forces, oblateness, perturbations in the Coriolis and centrifugal forces and the inclusion of a circumbinary disc surrounding the primaries. These characterizations under the study of mass variations of the primaries are examined in different sections of this chapter based on the astrophysical model each addresses. Hence, this chapter has been divided into **nine** sections. The first section is the introduction of the chapter. In section **3.2**, we study the effects of small perturbations in the Coriolis and centrifugal forces when the first primary is an oblate spheroid and the second one is a radiation emitter. This section is further divided into four subsections; the first subsection **3.2.1** describes the equations of motion of the third body while the next one deals with the locations of the equilibrium points of the problem. Subsections **3.2.3** and **3.2.4** capture the linear stability analysis of the equilibrium points and the discussion, respectively.

In section **3.3**, we look at the case when both primaries are oblate spheroids and are radiating under the influence of small perturbations in the Coriolis and centrifugal forces coupled with mass variations of the primaries. This section is also divided into four subsections; each of which

discuss the equations of motion, locations of the equilibrium points, linear stability of the equilibrium points, and the discussion, respectively. Section **3.4** examines the restricted problem with variable masses of the primaries when there is a disc present. Here, the study is restricted to the investigations of the existence and stability of the out-of-plane EPs. The section is divided into four parts. The first captures the equations of motion; the second has to do with the existence of the out-of-plane EPs while the third and last dealt with the stability of the EPs and the discussion, respectively. Section **3.5** focus on the study of the periodic orbits around triangular EPs of the R3BP with variable masses. Here, we study the combined effects of mass variations, oblateness, radiation and perturbations on the periodic orbits around the triangular EPs.

A different description of the R3BP in which the third body is located inside the first primary has had major breakthrough ever since its formulation. In the remaining sections of chapter **THREE**, we study the Robe (1977) R3BP problem under mass variations of the bodies. Section **3.6** is devoted to the restricted problem when the rigid spherical shell which is the first primary is empty and the masses of the primaries vary isotropically in obedience to the UML while their motion is governed by the GMP. In section **3.7**, we extend the problem in section **3.6** by introducing small perturbations in the Coriolis and centrifugal forces. The last problem of this chapter is dealt with in section **3.8**. This problem has the same configurations as the previous ones, but here all the masses of the participating bodies are subject to variations and the first primary is assumed to be a fluid with no rigid spherical shell. The sections aforementioned are divided into four subsections. Each of these subsections examine the equations of motion of the problem, the locations of the EPs, the linear stability of the EPs, respectively. We draw the conclusion of the chapter in section **3.9**.

In chapter **FOUR** of the thesis, we investigate the R3BP when the masses of the participating bodies do not vary with time during the mechanical motion, under the assumptions that includes radiation pressure forces, oblateness, triaxiality, perturbations in the Coriolis and centrifugal forces and discs. In this terrain, we classify the problems in three categories, each of which is studied in **five** sections, respectively. Section **4.1** is the introduction of the chapter while section **4.2** captures the dynamical system when the first primary is a radiation emitter, the second primary and third body both have the shape of an oblate spheroid and both primaries are surrounded by a dust of disc. This section is divided into five subsections. The first four subsections deal with derivations of the equations of motion, locations of the EPs, the linear stability of the EPs and the periodic orbits in the range of linearly stable triangular EPs, respectively, while the last subsection is the discussion of the results. Section **4.3** has the same pattern as section **4.2**, with the difference that here, the first primary is a triaxial rigid body while the second one is an oblate spheroid and the third body is a sphere. Small deflections in the Coriolis and centrifugal forces have been considered to be effective in the presence of disc. We however restrict ourselves to the study of the triangular EPs and apply our results in particular to a binary system which consists of a post AGB (Asymptotic Giant Branch star) and a white dwarf. Investigation of the periodic orbits and the stability of the orbits using the Poincare Surfaces of Sections (PSS) sum up this section.

The last problem of this chapter studies the restricted problem with three oblate bodies when the primaries are radiation emitters under the influence of small perturbations in the Coriolis and centrifugal forces together with the effect of the gravitational potential from the disc mass. This problem is a generalized study of the R3BP with constant masses and is discussed in section **4.4**. The section is divided into four subsections, each of which highlights the equations of motion,

the positions of the EPs, their stability and the periodic orbits, respectively. The chapter is concluded in section **4.5**.

The results and discussion of our study is given in chapter **FIVE**, while the summary, conclusion and recommendations which gives directions for future research works are proposed and the contributions of the thesis to knowledge, are stated in chapter **SIX.**

## 1.7 Theoretical Framework

This section captures the theoretical framework upon which the derivations and findings of this research work is built. In the derivations and study of the classical R3BP, some factors and features that may affect the dynamics of the third body were not taken into consideration. Some of these are discussed in this section.

### 1.7.1 Radiation pressure

It is a well known fact that when one or both the primaries are a source of radiation, the classical restricted three-body problem fails to adequately discuss the motion of the third body. This problem, according to Radzievskii (1950) is called the photogravitational problem, which he applied to the Sun-Planet-Particle and Galaxy Kernel-Sun-Particle. Later, Radzievskii (1953) gave a comprehensive treatment of the problem in his work and showed that the inclusion of radiation resulted in the existence of two additional EPs, (coplanar points) denoted by $L_6$ and $L_7$.

Radiation pressure act as an orbital perturbations and affects the orbits and trajectories of small bodies, all spacecrafts and all natural bodies (comets, asteroids, dust grains, gas molecules) and can cause dust grains to either leave the Solar system or spiral into the Sun. Also, in Solar sailing, an experimental method of spacecraft propulsion, uses radiation pressure from the Sun as

a motive force. The idea of interplanetary travel by light was mentioned by **Jules Verne** in ***From the Earth to the Moon***. Further, in cosmic formation, radiation pressure has had a major effect on the development of the cosmos, from the birth of the universe to ongoing formation of stars and shaping of clouds of dust and gases on a wide range of scales (Fig **1.1**).



**Figure 1.1** ***The Pillars of Creation clouds*** **shaped by radiation pressure and stellar winds (Credit: NASA)**

Also, if the effects of the Sun's radiation pressure on the spacecraft of the **Viking** program had been ignored, the spacecraft would have missed Mars orbit by 15,000 kilometers (Eugene Hecht).

Because of the many importance of radiation pressure, several researchers have included radiation pressure force of either one or both primaries in the study of the R3BP. They include among others, Chernikov (1970), Bhatnagar and Chawla (1979), Singh and Ishwar (1999), AbdulRaheem and Singh (2006, 2008), Singh and Leke (2010), and Singh and Haruna (2014a, 2014b).

Now, the radiation factor $q$ of a body with mass M is determined by the value of a resultant force $F$ due to the gravitational force $F_g$ and the radiation pressure force $F_p$ acting on third body:

$$F = F_g - F_p = F_g\left(1-\frac{F_p}{F_g}\right) = qF_g \tag{1.2}$$

where $F_g = -G\frac{Mm}{r^2} < 0$ , $F_p = E\frac{A}{r^2}$

Here, G is the universal gravity constant, $A$ is the transverse section area of the third body and $E$ is the intensity of radiation.

From equation (1.2), $q$ may be written in the form

$$q = 1-\frac{F_p}{F_g} = 1+\frac{E}{GM}.\frac{A}{m}, \tag{1.3}$$

In this thesis, we use $q_i \quad i = 1,2$ to denote the radiation factors of the first and second primary, respectively, and these are given by

$$F_p = F_g \ \ 1-q_i \tag{1.4}$$

where $0 < 1-q_i << 1$

The inclusion of the light/radiation factors in the model of the classical restricted three-body problem yields the photogravitational restricted problem of three bodies. In stellar systems

numerous examples are available where a body is moving under the gravitational field of two radiating bodies.

### 1.7.2. Small perturbations in Coriolis and centrifugal forces

In the formulation of the classical R3BP, the third body of infinitesimal mass is considered to move, only under the mutual gravitational force of the primaries, but in practice, Coriolis and centrifugal forces are effective and small perturbations affect these forces. Examples include: small deviation of disc stars on circular orbits and motion of a close artificial satellite of the Earth perturbed by the atmospheric friction and the oblateness of the Earth. Hence, it is reasonable and justifiable to include these forces in the study of the R3BP. Several interesting studies when Coriolis and centrifugal forces are slightly perturbed have been carried out by Szebehely (1967b), Bhatnagar and Hallan (1978), Singh and Ishwar (1984, 1985), AbdulRaheem and Singh (2006, 2008), Singh and Begha (2011a, 2011b) and Singh and Haruna (2014a, 2014b).

### 1.7.3. Shapes of the bodies

The formulation of the classical R3BP considered all the bodies to be strictly spherical. But in actual sense, most celestial bodies are not perfect spheres. Some have the shape of an oblate spheroid while some are triaxial in nature. For example, the Earth, Jupiter, Saturn and stars (Archerner, Antares and Altair) have the shape of an oblate spheroid (Fig **1.2**) while the Moon and several Post Asymptotic Giant Branch stars (Post AGB), Haumea, (a scalene dwarf planet, Fig **1.3**) are triaxial in shape. The rotation of a star produces an equatorial bulge due to centrifugal force; as a result, the stars are often oblate in shape. Neutron stars and black dwarfs which is a result of the cooling of white dwarfs are also oblate due to their rapid spinning after formation.

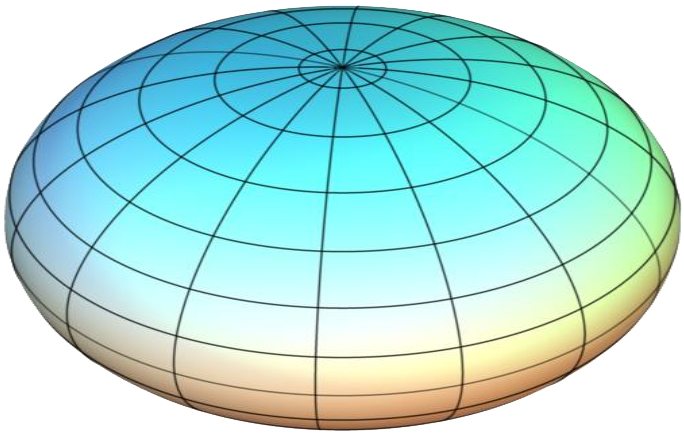


**Fig 1.2: The shape of an oblate spheroid (Credit:** *Wolfram Mathematica***)**



**Fig 1.3:** Artist's conception of Haumea, a triaxial dwarf planet, with its two moons (Credit: NASA, ESA)

An oblate spheroid is a rotationally symmetric ellipsoid having polar axis shorter than the diameter of the equatorial axis. Oblate spheroids are contracted along a line whereas prolate spheroids are elongated like a rugby ball.

The equation of a standard axis-aligned ellipsoid body in an *xyz*-Cartesian coordinate system is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

where *a* and *b* are the equatorial radii (along the *x* and *y* axes) and *c* is the polar radius (along the *z*-axis), all of which are fixed positive real numbers determining the shape of the ellipsoid. If all three radii are equal, the solid body is a sphere; if two radii are equal, the ellipsoid is a spheroid; when the radii are not equal, the body is a triaxial shaped type. The points $a,0,0$ , $0,b,0$ and $0,0,c$ lie on the surface and the line segments from the origin to these points are called the semi-principal axes. These correspond to the semi-major axis and semi-minor axis of the appropriate ellipses. Scalene ellipsoids are frequently called "triaxial ellipsoids", the implication being that all three axes need to be specified to define the shape. Figure **1.4** below shows the assignment of the semi-axes on a spheroid. It is an oblate spheroid when $c < a$ and prolate if $c > a$

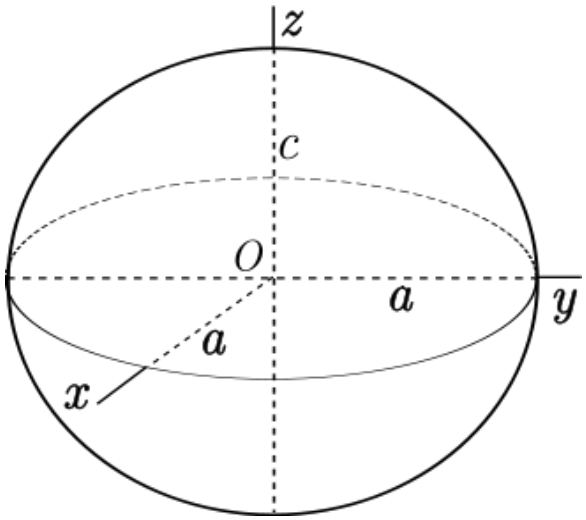



**Fig 1.4:** A spheroid showing the role of the semi-axes on the shape

The lack of sphericity of the planets causes large perturbations from a two-body orbit. Interesting results of studies that have included the oblateness and/or triaxiality of one, two or all

the participating bodies in the R3BP have been produced up to date. Among them are, SubbaRao and Sharma (1975), Sharma (1987), Singh and Ishwar (1999), Khanna and Bhatnagar (1999), Sharma *et al.* (2001), Douskos and Markellos (2006), AbdulRaheem and Singh (2006, 2008), Singh (2009), Singh and Begha (2011a, 2011b), Singh and Laraba (2012), Abouelmagd and El-Shaboury (2012), Singh and Taura (2013) and, Singh and Haruna (2014a, 2014b).

### 1.7.4 Discs

Some studies of our planetary systems have revealed some discs of dust particles, which are regarded as young analogues of the Kuiper Belt in our Solar system. These discs play important roles in the origin of planets' orbital elements. Since the belt of planetesimals often exists within a planetary system and provides the possible mechanism for orbital circularization, it is important to understand the solutions of dynamical systems which show planet-belt interactions. In stellar systems, this phenomenon is also valid. Out of an observed 69 A3-F8 main sequence binary star systems, nearly 60 percent showed dust discs surrounding binary stars. Greaves *et al*. (1998) found a dust ring around a nearby star, e Eridani. *Olivier et al*. (2001) studied dust enshrouded AGB stars in the solar neighborhood. Van Winckel *et al*. (2009, 2012) examined Post-AGB stars with hot circumstellar dust and an extremely depleted pulsating binary post-AGB star IRAS 11472-0800, while Zhao-Geisler, *et al.* (2012) studied dust and molecular shells in asymptotic giant branch stars

Circumbinary discs that may indicate processes of planet formation have been found around several stars, and are in fact common around binaries with separations less than 3 AU (Trilling *et al.* 2007). One notable example is in the HD 98800 system, which comprises two pairs of binary stars separated by around 34 AU (Fig **1.5**). The binary subsystem HD 98800 B, which consists of

two stars of 0.70 and 0.58 solar masses in a highly eccentric orbit with semimajor axis 0.983 AU, is surrounded by a complex dust disc that is being warped by the gravitational effects of the mutually-inclined and eccentric stellar orbits (Akeson *et al*. 2007; Verrier and Evans 2008). The other binary subsystem, HD 98800 A, is not associated with significant amounts of dust (Prato *et al*. 2001).

The importance of the problem in astronomy has been addressed by Jiang and Yeh (2004, 2006), where it was shown that the presence of disc resulted in additional equilibrium points of the system. Other works that took into account the gravitational potential from the belt/disc under different assumptions include Kushvah (2008a), Kishor and Kushvah (2013), Singh and Taura (2013) and Jiang and Yeh (2014).



**Fig 1.5:** An artist's impression of the binary star system **HD 98800 B** surrounded by a disc. (**Credit**: NASA Spitzer Telescope)

### 1.7.5. Mass variations

The classical R3BP assumes that the masses of celestial bodies are constant. However, the phenomenon of isotropic radiation or absorption in stars led scientists to formulate the restricted problem of three bodies with variable mass. During evolution, the masses of celestial bodies change, especially in a double star system were masses change rather intensively. The problem of the motion of astronomical objects with variable mass has many interesting applications in stellar, galactic, and planetary dynamics. As an example, we could mention motion of a satellite around a radiating star surrounded by a cloud and varying its mass due to particles of the cloud, and comets loosing part or all of their mass as a result of roaming around the Sun (or other stars) due to their interaction with the solar wind which blows off particles from their surfaces.

An interesting example of mass loss is the real physical scenario of those transiting exoplanets (Fig **1.6**) whose atmospheres are escaping because of the severe levels of energetic radiations, coming from their very close parent stars, hitting them.

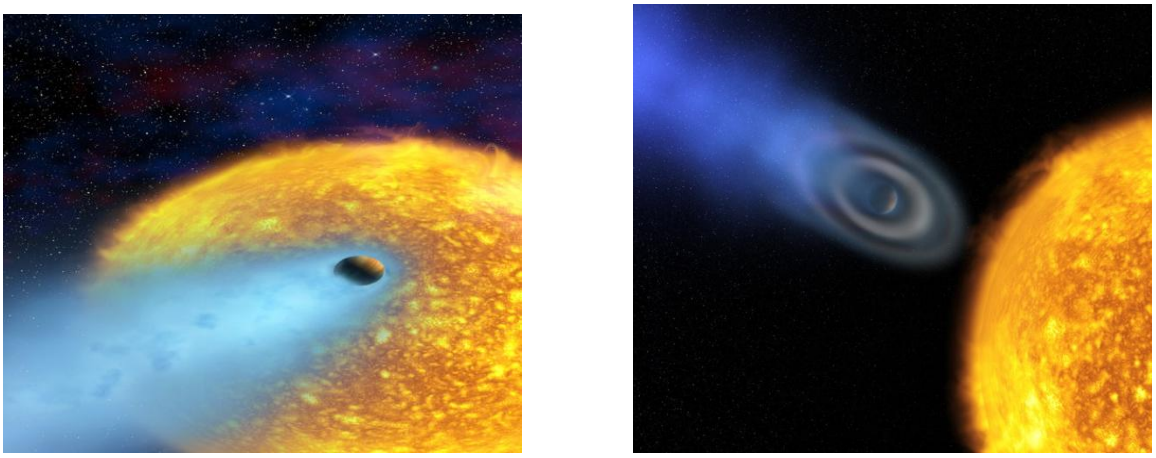


**Fig 1.6:** An artistic impression of mass loss of an extrasolar planet **HD 209458b** (**Credit**: European Astronomical observatory)

This image shows a dramatic scorched extrasolar planet HD 209458b in its orbit around a yellow Sun-like star. The image shows the atmosphere of HD 209458b (shown in blue) evaporating off into space. Much of this planet may eventually disappear, leaving only the core, because the amount of hydrogen gas escaping from it is estimated to be at least 10,000 tonnes per second.

Also, modern concepts of the change in the distance between the Earth and the Moon, and the change in their masses due to, out-gassing, impact of meteorites, asteroids, comets and space dust lead to the necessity of investigating dynamics problem in the Earth-Moon system under these conditions. It is believed that the Earth gains 100,000 kilograms of mass each year from space, one million kilograms of mass every day due to in falling meteors. However, the Earth's mass change appears to be exceedingly tiny, and seemingly, not nearly enough to change the dynamics in any significant way.

Due to the inclusion of mass variations, Bekov (1988) found additional equilibrium points when the masses of the primaries vary with time and called them coplanar equilibrium points.

Hence, one could consider the formulation of the R3BP with variable mass as follows viz.

i. When masses of both primaries vary with time and the mass of the third body is kept constant (Gelf'gat 1973; Bekov 1988; Luk'yanov 1990 and Singh and Leke 2010).

ii. When the mass of the third body is assumed to vary with time and the masses of the primaries is kept constant (Singh and Ishwar 1984, 1985; El-Shaboury 1988; Zhang *et al.* 2012).

iii. When the three masses vary with time (Bekov 1987, Bekov *et al*. 2005; Letelier and Da Silva 2011).

### 1.7.6 Binary and trinary systems

In some segments of this dissertation, where shapes, luminosity, circumbinary disc and mass variations shall be the problem parameters considered. We shall apply this to binary and trinary systems; which includes among others, neutron star-white dwarf systems; Post AGB binaries; Fomalhaut; Kepler 16; PSR B1620-26.

The binary system PSR B1620-26 in the constellation of Scorpius is surrounded by a disc and, composed of a pulsar PSR B1620-26 A with mass $1.35M_{\Theta}$ and a white dwarf star WD B1620-26 having mass $0.34M_{\Theta}$ and host an extrasolar plant; while Kepler-16, is a binary system in the constellation Cygnus. The system consists of a primary Kepler-16A and a companion Kepler-16B. The primary has a mass of $0.6897 \pm 0.0035\, M_{\Theta}$ and the secondary's mass is $0.20255 \pm 0.00065\, M_{\Theta}$. Kepler-16A is a K-type main sequence star, while B is M-type red dwarf. The binary host a planet Kepler-16b whose mass is too small compared to the masses of the host stars.

**Binary system**

Two stars bound by their mutual gravitational attraction in a permanent (or long-term) association constitute a binary system. The nature of the stellar universe is hierarchical. Some stars are single, in binary pairs, in multiple systems, and others still exist in clusters. On large scales, roughly $10^5$ light-years, astronomical objects congregate into galaxies in space. In fact, most stars exist in binary systems. The Sun, with its collection of planets, is an exception. Stars in binaries revolve around their common center of mass (which itself moves through space).

Examples of binary systems include among many, Sirius, Achird, Luyten 726-8, Kruger 60, Alpha Centauri AB, Xi Bootis and Gamma Leporis, e.t.c.

**Circumbinary planet (Trinary system)**

A circumbinary planet is a planet that orbits two stars instead of one. Because of the short orbits of some binary stars, the only way for planets to form is by forming outside the orbit of the two stars. The first confirmed circumbinary extrasolar planet was found orbiting the system PSR B1620-26, which contains a millisecond pulsar and a white dwarf and is located in the globular cluster M4. The existence of the third body was first reported in 1993, and was suggested to be a planet based on 5 years of observational data. In 2003 the planet was characterized as being 2.5 times the mass of Jupiter in a low eccentricity orbit with a semimajor axis of 23 AU.

**Kepler-16**

Kepler-16 is a binary star system in the constellation of Cygnus that was targeted by the Kepler spacecraft. Both stars are smaller than the Sun; the primary, Kepler-16A, is a K-type main-sequence star and the secondary, Kepler-16B, is an M-type red dwarf (Fig 1.7). They are separated by 0.22 AU, and complete an orbit around a common center of mass every 41 days. The system is host to one known extrasolar planet in circumbinary orbit: the Saturn-sized Kepler-16b. Kepler-16b is a gas giant that orbits the two stars in the Kepler-16 system. The planet is a third of Jupiter's mass and slightly smaller than Saturn at 0.7538 Jupiter radii, but is denser. Kepler-16b completes a nearly circular orbit every 228.776 days.

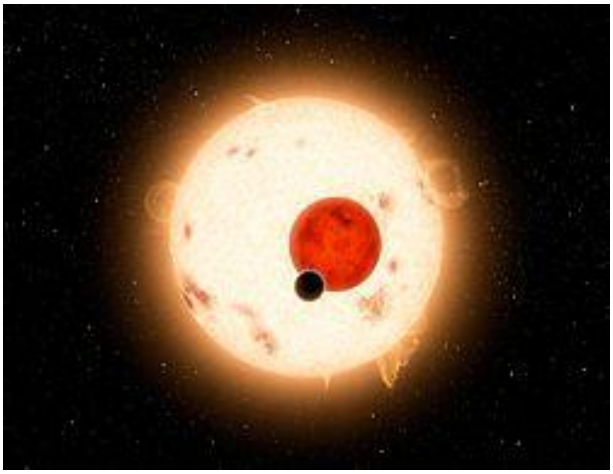


**Fig. 1.7:** An artist's rendering of the Kepler-16 system, showing the binary star being orbited by Kepler-16b (**Credit**: *NASA/JPL-Caltech)*

Table **1.1** below shows the list of confirmed circumbinary systems and the orbiting-planets (Orosz *et al.* 2012) and Qian *et al*. 2012). The mass, semi-major axis, the orbital period and the year of discovery, of the systems are also shown in the table. For more details on the listed circumbinary planets, see Orosz *et al.* (2012), Qian *et al*. (2012).

HW Virginis is an eclipsing binary system comprising a subdwarf B star and a red dwarf. It was announced in 2008 that this binary host a planetary system. In September 2011, astronomers announced the first partial-eclipse-based discovery of a circumbinary planet. The planet called Kepler-16b. All Kepler circumbinary planets are either close to or actually in the habitable zone. None of them are terrestrial planets, but large moons of such planets could be habitable. Because of the stellar binarity, the insolation received by the planet will likely be time-varying in a way, quite unlike the regular sunlight Earth receives.

**Table 1.1:** List of circumbinary planets

| Star system | Planet | Mass ($M_J$) | Semimajor axis AU) | Orbital period (y) | Discovered |
|---|---|---|---|---|---|
| PSR B1620-26 | PSR B1620-26b | 2.5 | 23 | 100 | 2003 |
| DP Leonis | DP Leonisb | 6.28 ± 0.58 | 8.6 | 23.8 | 2009 |
| NN Serpentis | NN Serpentisc | 6.91 ± 0.54 | 5.38 ± 0.20 | 15.50 ± 0.45 | 2010 |
| NN Serpentis | NN Serpentisd | 2.28 ± 0.38 | 3.39 ± 0.10 | 7.75 ± 0.35 | 2010 |
| DT Virginis | DT Virginisc | 8.5 ± 2.5 | 1168 | 33081 | 2010 |
| Kepler-16 | Kepler-16b | 0.333 ± 0.016 | 0.7048 ± 0.0011 | 0.6266 ± 0.0001 | 2011 |
| NY Virginis | NY Virginisb | 2.3 ± 0.3 | 3.3 | 7.9 | 2011 |
| RR Caeli | RR Caelib | 4.2 ± 0.4 | 5.3 ± 0.6 | 11.9 | 2012 |
| Kepler-34 | Kepler-34b | 0.220 ± 0.0011 | 1.0896 ± 0.0009 | 0.7908 ± 0.0002 | 2012 |
| Kepler-35 | Kepler-35b | 0.127 ± 0.02 | 0.603 ± 0.001 | 0.3600 ± 0.1 | 2012 |
| Kepler-38 | Kepler-38b | ≤ 0.38 | 0.4644 ± 0.0082 | 0.289 | 2012 |
| Kepler-47 | Kepler-47b | *Unknown* | 0.2956 ± 0.0047 | 0.136 | 2012 |
| Kepler-47 | Kepler-47c | *Unknown* | 0.989 ± 0.016 | 0.83 | 2012 |
| Kepler 64 | PH1 | 0.11 ± 0.3 | 0.634 ± 0.011 | 0.379 | 2012 |
| ROXs 42B | ROXs 42Bb | 10±4 | 140 | | 2013 |
| FW Tauri | FW Tauri b | 10±4 | 330 | | 2013 |
| Kepler-413 | Kepler-413b | 0.21 | 0.3553 | 0.181 | 2014 |

## 1.7.7. Relational definitions

**(a)Triaxial Rigid Body:** A rigid body which had got three mutually perpendicular symmetrical axes is called triaxial rigid body, such as ellipsoid.

**(b) Oblateness:** The degree of flattening of a celestial object, such as a planet, from a true spherical form, largely as a result of rotation is called the oblateness of the celestial object. It is

numerically equal to the difference between the equatorial and polar diameters of the object divided by the equatorial diameter.

**(c) Primaries:** If an infinitesimal mass is moving under the gravitational field of finite masses, then the finite masses are called the primaries.

**(d) Sidereal Co-Ordinates:** The co-ordinates with respect to an inertial frame are called sidereal co-ordinates.

**(e) Synodic Co-Ordinates:** The co-ordinates with respect to a rotating frame are called synodic co-ordinates.

**(f) Periodic:** A dynamical system is periodic if the same configuration is repeated at regular intervals of time.

## 1.8 Preliminary Ideas

In this section, we discuss some preliminary facts upon which some of our derivations and findings are based on.

### 1.8.1 The two-body problem and the integral of area

The 2BP is the starting point for nearly all reference books in the field of astrodynamics. The basic problem describes the motion of two point-masses in mutual gravitational attraction. Newton's law of gravitation in equation 1.1 leads to closed form solutions to the motions of the bodies with respect to the center of mass. These solutions can be used to analyze orbital properties without the need for cumbersome numerical propagation.

When the formula for gravitation force is applied to the two bodies, the equation relating each body's position with respect to the center of mass of the system may be written:

$$\frac{d\vec{v}}{dt} = -\frac{\mu}{r^2}\frac{\vec{r}}{r} \tag{1.5}$$

where $\vec{v}$ is the velocity, $\mu$ is the product of the gravitational constant and the sum of the masses.

To obtain the integral of area, we take cross product of equation (1.5) with the vector $\vec{r}$, to get

$$\vec{r} \times \frac{d\vec{v}}{dt} = -\frac{\mu}{r^3}\vec{r} \times \vec{r}$$

But $\vec{r} \times \vec{r} = 0$,

And so we have

$$\vec{r} \times \frac{d\vec{v}}{dt} = 0 \tag{1.6}$$

Also, $$\frac{d}{dt}\ \vec{r} \times \vec{v}\ = \frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}$$

From (1.6), we have

$$\frac{d}{dt}\ \vec{r} \times \vec{v}\ = \frac{d\vec{r}}{dt} \times \vec{v} \tag{1.7}$$

But $\frac{d\vec{r}}{dt} = \vec{v}$ and since $\vec{v} \times \vec{v} = 0$,

Equation (1.7) becomes

$$\frac{d}{dt}\ \vec{r}\times\vec{v}\ =0 \tag{1.8}$$

Integrating (1.8), we have

$$\vec{r}\times\vec{v}=\vec{C} \tag{1.9}$$

where $\vec{C}$ is a constant vector called the integral of area, so $\vec{r}\times\vec{v}$ is a constant vector and that $\vec{C}$ is perpendicular to both $\vec{r}$ and $\vec{v}$, which in turns means that $\vec{C}$ is perpendicular to the plane of motion. If in polar coordinates $r$ and $\theta$ are taken in this plane, then the velocity component along and perpendicular to the radius vector joining mass $m_1$ and $m_2$ are $\dot{r}$ and $r\dot{\theta}$, so that

$$\vec{v}=\dot{r}\vec{i}+r\dot{\theta}\vec{j} \tag{1.10}$$

Here, $\vec{i}$ and $\vec{j}$ are unit vectors along and perpendicular to the radius vector, hence by equations (1.9) and (1.10), we have

$$r\vec{i}\times\ \dot{r}\vec{i}+r\dot{\theta}\vec{j}\ =\vec{C}\ .$$

Taking product, yields

$$r^2\dot{\theta}\vec{k}=\vec{C} \tag{1.11}$$

The constant vector $\vec{C}$ can be expressed as

$$\vec{C}=C\vec{k} \tag{1.12}$$

where $\vec{k}$ is a unit vector perpendicular to the plane of the orbit.

Equation (1.11) now takes the form

$$r^2\dot{\theta} = C \tag{1.13}$$

Equation (1.13) is the area integral of the system, $\dot{\theta}$ is the angular velocity of revolution of the bodies of masses $m_1$ and $m_2$, and $C$ is the constant of integration

## 1.8.2 Circular restricted three-body problem

The general 3BP involves three particles of arbitrary masses moving freely under mutual gravitational attraction while, the R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of the two finite masses, called primaries, which move in circular orbits around their common center of mass on account of their mutual attraction and the third body having infinitesimal mass not influencing the motion of the primaries. The motion of the two primaries is governed by the equation of motion of the 2BP with constant masses. The third body even though regarded as having mass infinitesimally small, gives rise to the appearance of chaotic motion and introduces an enormous amount of complexity to the dynamics of the system. We shall throughout this dissertation refer to this body as the third body of infinitesimal mass.

The primaries move in circular orbits irrespective of the third body, while the third body moves under the sole influence of the two primaries. This simplified model is termed as the circular restricted three body problem. Thus the model of the so-called CR3BP is the following

- Two bodies $P_1$ and $P_2$ having masses $m_1 \neq 0$ and $m_2 \neq 0$, respectively, and $m_1 > m_2$ move on circular orbits
- A third body of infinitesimal mass $m_3$ moves in the same plane as the primaries.

Thus, the restricted problem of three bodies is to determine the motion of the third body under the influence of the primaries.

The diagram of the CR3BP is shown in the Fig **1.8** with the first primary shown on the left. An inertial coordinate system $\xi,\eta,\zeta$ and a rotating coordinate system $x,y,z$ are placed with their centers at the center of mass of the two primaries.

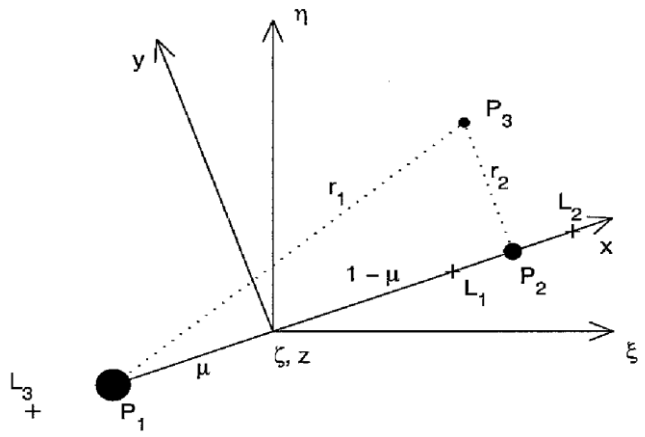



**Fig. 1.8:** The circular restricted three-body problem

The rotating coordinate system rotates with the angular velocity of the primaries so that it's $x-axis$ always points from $P_1$ to $P_2$. The various quantities used in the CR3BP are typically non-dimensionalized so that the mass of the smaller primary is defined to be $\mu$ with the mass of the larger primary given by $1-\mu$. The distance from the origin to $P_1$ and $P_2$ are respectively $\mu$ and $1-\mu$. The non-dimensional time corresponds to the angular distance traveled by the primaries.

The equations of motion of the CR3BP in the barycentric rotating coordinate system $0xyz$, under the gravitational effect of the first and second primary having masses $m_1$ and $m_2$, respectively, have the form:

$$\ddot{x} - 2\dot{y} = \Omega_x$$

$$\ddot{y} + 2\dot{x} = \Omega_y \qquad (1.14)$$

where $\Omega = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$

$r_1^2 = (x-\mu)^2 + y^2$ , $r_2^2 = (x-\mu+1)^2 + y^2$ , $\mu = \frac{m_2}{m_1 + m_2}$

$r_1$ and $r_2$ are distances of the third body from the primaries $m_1$ and $m_2$ positioned at $(\mu, 0, 0)$ and $(1-\mu, 0, 0)$ , respectively; $\mu \in (0, \frac{1}{2}]$ is the mass ratio of the system. The dots denote derivatives w.r.t. time $t$ and subscripts signify partial derivatives.

Equations (1.14) give the equations of motion of the classical R3BP and depend on the mass parameter only. The solutions of this type of problem have been developed over the centuries; from Lagrange (1772), Szebehely (1967a) and to others. There exist five specific solutions called the equilibrium, libration or Lagrangian points. If the third body is placed at any of these points with zero velocity in the coordinate system rotating with the primaries, it will remain at that point in the rotating system. Three of these points, called the collinear equilibrium points, found by Euler (1767) are located along the line joining the primaries and denoted by $L_1, L_2, L_3$ while the other two solutions found by Lagrange (1772) form equilateral triangles with the primaries, one ahead and one behind as they revolve; are called triangular equilibrium points and denoted by $L_4, L_5$. Unlike the collinear points, the triangular points can be stable (Szebehely 1967a; Bhatnagar and Hallan 1978; Singh and Ishwar 1999; Sharma *et al.* 2001; AbdulRaheem and Singh 2006; Kushvah 2008a, 2008b; Singh and Leke 2010; Singh and Begha 2011a; and Singh

and Haruna 2014a) in that a slight displacement of the particle away from the points will not produce unbounded motion but rather an oscillation (libration) about the points. There are asteroids, known as the Trojans, librating about both triangular points in the Sun-Jupiter system.

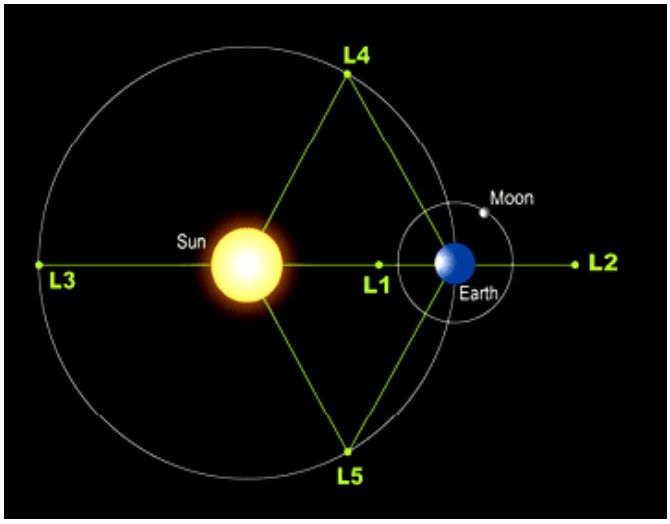



**Fig. 1.9:** The five Lagrangian points in CR3BP (Sun-Earth –Moon system)

Figure **1.9** illustrates the locations of the collinear $L_{1,2,3}$ and triangular $L_{4,5}$ equilibrium points in the Sun-Earth-Moon system. The first collinear point $L_1$ is located between the two primaries, the second collinear point $L_2$ lies to the right of the smaller primary, while the third collinear libration point $L_3$ is located left of the bigger primary.

## 1.8.3. Robe's restricted three-body problem

This restricted problem is a different kind of the R3BP and was formulated by Robe (1977). The problem is modeled in such a way that the first primary is a rigid spherical shell filled with homogenous, incompressible fluid of density $\delta_1$; the second primary is a mass point outside the shell and moving around the first primary, while the third mass is a small sphere of density

$\delta_3$ moving inside the shell and is subject to the attraction of the second primary and the buoyancy force due to the fluid (Fig **1.10**). In estimating the buoyancy force, Robe assumed that the pressure field of the fluid $\delta_1$ has spherical symmetry around the center of the shell.

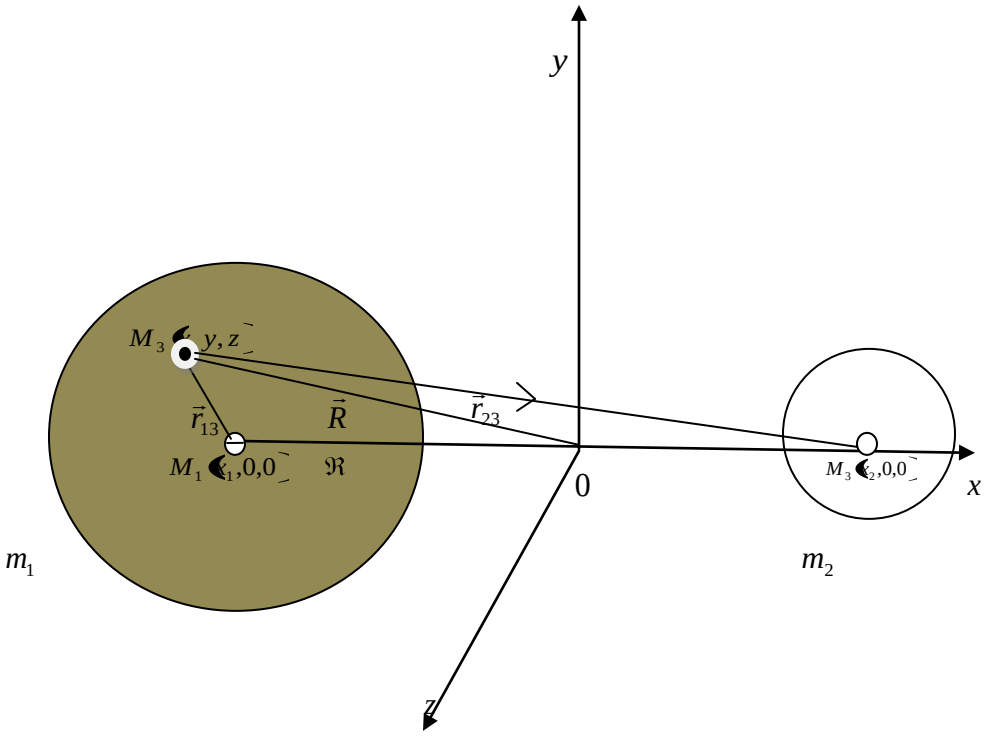



**Fig 1.10:** Set up of the Robe restricted three-body problem

**Equations of motion of the Robe's CR3BP**

Let $m_1$ be the mass of the first primary which is a rigid spherical shell of mass $m_1^*$ filled with a homogenous incompressible fluid of density $\delta_1$. We also let $m_2$ be the mass of the primary outside the shell and $m_3$ the mass of the infinitesimal mass having density $\delta_3$. We suppose that $\Re$ is the radius of the sphere $m_1$, $M_1M_2$ and $M_3$ are the center of $m_1$, $m_2$ and $m_3$ respectively.

Let the positions vector between the center of the mass of the body $m_1$ and $m_2$ be $\vec{r}_{12}$ and that between $m_1$ and $m_3$; and $m_2$ and $m_3$ be $\vec{r}_{13}$ and $\vec{r}_{23}$, respectively; O being the center of mass of the primaries.

Now, by Newton law of gravitation, the force of attraction on a particle of mass $m_1$ due to another particle of mass $m_2$ is given by

$$\vec{F}_1 = \frac{Gm_1m_2\vec{r}_{12}}{r_{12}^3} \tag{1.15}$$

while the force acting on the mass $m_2$ due to $m_1$ is

$$\vec{F}_2 = -\frac{Gm_1m_2\vec{r}_{12}}{r_{12}^3} \tag{1.16}$$

where $G$ is gravitational constant.

The volume $V_L$ of the fluid of density $\delta_1$ inside the first primary is given by

$$V_L = \frac{4\pi}{3}\Re^3 \tag{1.17}$$

Then the mass of the fluid with density $\delta_1$ becomes

$$m_f = \frac{4}{3}\pi^3\Re^3\delta_1 \tag{1.18}$$

Therefore, the mass of the first primary is

$$m_1 = m_1^* + \frac{4}{3}\pi\Re^3\delta_1 \tag{1.19}$$

Using Newton's second law of motion in equations (1.15) and (1.16), we get

$$\ddot{\vec{r}}_{13} = \frac{Gm_2}{r_{12}^3}\vec{r}_{12} \text{ and } \ddot{\vec{r}}_{23} = -\frac{Gm_1}{r_{12}^3}\vec{r}_{12} \tag{1.20}$$

where $\vec{r}_{12} = \vec{r}_{23} - \vec{r}_{13}$

Subtracting the equations of (1.20), we get

$$\ddot{\vec{r}}_{12} = \frac{-G\ (m_1 + m_2)}{r_{12}^3}\vec{r}_{12} \tag{1.21}$$

Equation (1.21) gives the relative motion of the mass $m_2$ about the spherical shell of mass $m_1$.

The forces acting on the third body inside the spherical shell are:

i) The attraction of $m_2$

ii) The gravitational force $\vec{F}_A$ exerted by the fluid

$$\vec{F}_A = -\left(\frac{4}{3}\right)\pi G\delta_1 m_3 \overrightarrow{M_1M_3} \tag{1.22}$$

iii) The buoyancy force which is given by

$$\vec{F}_B = \frac{4}{3}G\pi\delta_1^2\frac{m_3}{\delta_3}\overrightarrow{M_1M_3} \tag{1.23}$$

Let the orbital plane of $m_2$ around $m_1$ be taken as the $XY-$plane, and also let the origin of the coordinate system be at the centre of mass O of the two finite masses. Then the equations of motion of the infinitesimal mass $m_3$ taking into account the forces acting on it is

$$m_3\ddot{\vec{R}} = Gm_2m_3\frac{\vec{r}_{23}}{r_{23}^3} + \vec{F}_A + \vec{F}_B$$

where $\vec{R} = \overrightarrow{0M_3}$ (Fig. **1.10**)

Substituting equations (1.22) and (1.23) and the force of attraction due to the second primary; simplifying, we get

$$\ddot{\vec{R}} = \frac{Gm_2\vec{r}_{23}}{r_{23}^3} - \frac{4}{3}G\pi\delta_1\left(1-\frac{\delta_1}{\delta_3}\right)\overrightarrow{M_1M_3} \tag{1.24}$$

Now, the kinetic energy of the infinitesimal mass revolving with mean motion $n$ is given by

$$T = \frac{1}{2}m_3\left[\ \dot{x}-ny\ ^2 + \ \dot{y}+n\dot{x}\ ^2 + z^2\right]$$

Expanding it, we get

$$T = \frac{1}{2}m_3(\dot{x}^2+\dot{y}^2+\dot{z}^2) + m_3n(x\dot{y}-y\dot{x}) + \frac{1}{2}m_3n^2(x^2+y^2) \tag{1.25}$$

The total potential energy at $m_3$ is given by

$$V^* = -\frac{Gm_2m_3}{r_{23}} + \frac{4}{3}Gm_3\pi\delta_1\left(1-\frac{\delta_1}{\delta_3}\right)r_{13} \tag{1.26}$$

Now let $p_x$, $p_y$ and $p_z$ be the generalized components of momentum then,

$$p_x = \frac{\partial T}{\partial\dot{x}} = m_3(\dot{x}-ny),$$

$$p_y = \frac{\partial T}{\partial \dot{y}} = m_3(\dot{y} + nx), \tag{1.27}$$

$$p_z = \frac{\partial T}{\partial \dot{z}} = m_3\dot{z}$$

This implies

$$\dot{x} = \frac{p_x}{m_3} + ny\,,\ \dot{y} = \frac{p_y}{m_3} - nx \text{ and } \dot{z} = \frac{p_z}{m_3} \tag{1.28}$$

Substituting system (1.28) in (1.25), we get

$$T = \frac{1}{2m_3}(p_x^2 + p_y^2 + p_z^2) \tag{1.29}$$

Now, Hamiltonian H of the system is

$$H = \frac{1}{2m_3}(p_x^2 + p_y^2 + p_z^2) + n(yp_x - xp_y) + V^* \tag{1.30}$$

The Hamiltonian canonical equations are given by

$$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{z} = \frac{\partial H}{\partial p_z}$$

and $$\dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_y = -\frac{\partial H}{\partial y}, \quad \dot{p}_z = -\frac{\partial H}{\partial z}$$

So that $$\dot{p}_x = -(-np_y) - \frac{\partial V^*}{\partial x}$$

$$\dot{p}_y = -np_x - \frac{\partial V^*}{\partial y} \tag{1.31}$$

$$\dot{p}_z = -\frac{\partial V^*}{\partial z}$$

From systems (1.28) and (1.31), we have

$$m_3(\ddot{x} - n\dot{y}) = np_y - \frac{\partial V^*}{\partial x}$$

$$m_3(\ddot{y} + n\dot{x}) = -np_x - \frac{\partial V^*}{\partial y} \tag{1.32}$$

$$m\ddot{z} = -\frac{\partial V^*}{\partial z}$$

Substituting the equations of system (1.27) in those of system (1.32), we have

$$m_3(\ddot{x} - n\dot{y}) = nm_3(\dot{y} + nx) - \frac{\partial V^*}{\partial x}$$

$$m_3(\ddot{y} + n\dot{x}) = -nm_3(\dot{x} - ny) - \frac{\partial V^*}{\partial y}$$

$$m\ddot{z} = -\frac{\partial V^*}{\partial z}$$

Multiplying throughout by $\frac{1}{m_3}$, results in the equations

$$\ddot{x} - 2n\dot{y} = n^2 x - \frac{1}{m_3}\frac{\partial V^*}{\partial x}, \quad \ddot{y} + 2n\dot{x} = n^2 y - \frac{1}{m_3}\frac{\partial V^*}{\partial y}, \quad m\ddot{z} = -\frac{1}{m_3}\frac{\partial V^*}{\partial z} \tag{1.33}$$

We now choose the unit of mass such that the sum of the masses of the primaries is taken as unity, thus we take $m_1 = 1-\mu$, $m_2 = \mu$, $\mu = \frac{m_2}{m_1 + m_2} < 1$. Also, the unit of the distance is taken as the distance between the primaries and is taken to be unity.

Since the masses $m_1 = 1-\mu$ and $m_2 = \mu$ are at $x_1, 0, 0$ and $x_2, 0, 0$, then

$$m_1(x_1) + m_2(x_2) = 0$$

So that $$1-\mu \; x_1 + \mu(x_2) = 0$$

Solving, we get

$$x_1 = -\mu \quad \text{and } x_2 = 1-\mu \tag{1.34}$$

Also from Kepler's third law, we get $G = 1$ and $n = 1$.

Now, from the potential (1.26), we have

$$V^* = -m_3\left[\frac{Gm_2}{r_{32}} - \frac{4\pi G\delta_1}{3}\left(1-\frac{\delta_1}{\delta_3}\right)\frac{r_{13}^2}{2}\right]$$

Then, $$\frac{\partial V^*}{\partial x} = -Gm_3\left[\frac{m_2 \times \frac{1}{2}\left[\,1-\mu-x\,^2+y^2+z^2\,\right]^{-\frac{1}{2}} \times 2\;\,1-\mu-x\quad -1}{1-\mu-x\,^2+y^2+z^2} - D_p\;\;x+\mu\right]$$

or $$\frac{\partial V^*}{\partial x} = -Gm_3\left[\frac{m_2\;\,1-\mu-x}{\left[\,1-\mu-x\,^2+y^2+z^2\,\right]^{\frac{3}{2}}} - D_P\;\;x+\mu\right]$$

Similarly, $$\frac{\partial V^*}{\partial y} = -Gm_3\left[\frac{m_2 y}{\left[1-\mu-x^{\ 2}+y^2+z^2\right]^{\frac{3}{2}}} - D_p y\right]$$

$$\frac{\partial V^*}{\partial z} = -Gm_3\left[\frac{m_2 z}{\left[1-\mu-x^{\ 2}+y^2+z^2\right]^{\frac{3}{2}}} - D_p z\right]$$

Substituting for $\frac{\partial V^*}{\partial x}, \frac{\partial V^*}{\partial y}$ and $\frac{\partial V^*}{\partial z}$ in system (1.33) and simplifying, we get

$$\ddot{x} - 2\dot{y} = x + \frac{m_2\ 1-\mu-x}{\left[1-\mu-x^{\ 2}+y^2+z^2\right]^{\frac{3}{2}}} - D_p\ \ x+\mu$$

$$\ddot{y} + 2\dot{x} = y + \frac{m_2 y}{\left[1-\mu-x^{\ 2}+y^2+z^2\right]^{\frac{3}{2}}} - D_p y \qquad (1.35)$$

$$\ddot{z} = \frac{-m_2 z}{\left[1-\mu-x^{\ 2}+y^2+z^2\right]^{\frac{3}{2}}} - D_p z$$

where $$D_p = \frac{4}{3}\pi\delta_1\left(1-\frac{\delta_1}{\delta_3}\right) \qquad (1.36)$$

These equations are the governing equations of the Robe (1977) CR3BP. This problem was used by Robe (1977) to study the small oscillation of the Earth's core. Since then various studies under different assumptions have been carried out by several researchers e.g., Shrivastava and Garain (1991); Plastino and Plastino (1995), Hallan and Rana (2001a, 2001b); Hallan and

Mangang (2007, 2008); and Kaur and Aggarwal (2012), Singh and Sandah (2012), Singh and Omale (2014), among others.

### 1.8.4 Equation of motion of two-body with variable masses

In this section, we briefly discuss the problem of two bodies with variable masses which also includes the Gylden-Mestschersky problem.

By the problem of two bodies with variable masses, by analogy with the classical problem of two bodies with constant masses, one understands the problem of motion of two primary bodies, the masses $m_1$ and $m_2$ of which vary with time under certain laws and between which only the gravitational force acts. It is usually assumed that the separation of particles from (or their attachment to) the points take place in accordance with Mestschersky's hypothesis, i.e., a contact interaction occurs between the points of variable mass and the separating (or attaching) particles; it is assumed that the masses of the points vary continuously.

The absolute motion of the points is described by the Mestschersky equation for a point of variable mass,

$$\vec{F} = m\dot{\vec{v}} + \ \vec{v} - \vec{u}\ \ \dot{m} \tag{1.37}$$

where $\vec{F}$ is the sum of all the forces acting on the body and $\vec{v}$ is its velocity, both measured in an inertial coordinate system. Also, $\vec{u}$ is the velocity of the center of mass of the absorbed mass immediately before its union with the body (or of the ejected mass immediately after its ejection). The over dot denotes derivation with respect to the time variable.

Gylden represented the relative motion (equation 1.5) of mass $m_2$ about mass $m_1$ under the action of mutual gravitational force, as the sum of the masses of these points as varying with time by a certain law

$$m_1 + m_2 = \mu \ t \tag{1.38}$$

and wrote the differential equation of the problem in the form

$$\ddot{\vec{r}} + \frac{\mu \ t}{r^3}\vec{r} = 0 \tag{1.39}$$

Mestschersky showed that the Gylden problem (1.39) is a particular case of the problem of two bodies with variable mass under the condition that the laws of variation of the two masses are the same.

There are two special cases of equation (1.37) to be considered. The first one is when the mass is ejected with the same velocity of the body at any moment $\vec{v} = \vec{u}$ , that is, mass ejection does not produce reactive forces. This case can be used to study the motion of a body ejecting mass isotropically (or radiating energy), since the total reactive momentum would be zero.

If $\vec{v} = \vec{u}$ , then equation (1.18) reduces to the form

$$\vec{F} = m\dot{\vec{v}} \tag{1.40}$$

In this case the relative motion of the problem of two bodies with variable masses is described by the equation

$$\ddot{\vec{r}} = -G\frac{m_1 + m_2}{r^3}\vec{r} \tag{1.41}$$

Equation (1.41) is analogous to the equation (1.5) of the classical problem of two bodies with constant masses, with the difference that now; the sum of the masses is a certain function of time. Equality of the velocities of ejected mass and the body at any moment means that isotropic variation of masses (in Mestschersky term) occurs. Equation (1.41) is rightfully called the ***Gylden-Mestschersky problem*** (GMP).

The second case is when mass variation takes place in the presence of reactive forces. In this case the particles are at rest in an inertial coordinate system, that is $\vec{u} = 0$ and equation (1.37) becomes

$$\vec{F} = m\dot{\vec{v}} + \dot{m}\vec{v} = \frac{d}{dt}\ m\vec{v} \tag{1.42}$$

This case can be used to study the orbit of a star moving through a static atmosphere, whose particles attach or detach to the star as it moves.

In this case the relative motion of the problem of two bodies with variable masses exchanging mass with a static atmosphere surrounding them, is described by the equation

$$\frac{d}{dt}\ \mu_*\dot{\vec{r}}\ = -G\frac{\mu_*\ \ m_1 + m_2\ \ \vec{r}}{r^3} \tag{1.43}$$

where $\mu_* = \dfrac{m_1 m_2}{m_1 + m_2}$

## 1.8.5. The Mestschersky transformation and unified Mestschersky law

Mestschersky (1902) showed that the Gylden problem is a particular case of the problem of two bodies with variable mass under the condition that the laws of variation of the two masses are the same , while the relative velocities of the particles separating from them (or attaching to them) equals zero everywhere. He found two laws of variation of the sum of the masses of the points

$$m_1 + m_2 = \mu\ t\ = \frac{\mu_0}{a + \alpha t} \tag{1.44}$$

$$m_1 + m_2 = \mu\ t\ = \frac{\mu_0}{\sqrt{\alpha + \beta t + \gamma t^2}} \tag{1.45}$$

where $\mu_0, a, \alpha, \beta$ and $\gamma$ are constants.

In the work of Mestschersky (1902), he reduced the GMP through the introduction of new variables and "time" to the equations of the classical problem of two bodies with constant masses by a transformation, which was thereafter known as the Mestschersky (1902) transformation and is given as

$$x = \xi R(t),\ y = \eta R(t),\ z = \varsigma R(t),\ \frac{dt}{d\tau} = R^2(t)$$

$$r_i = \rho_i R(t),\ \ (i = 1, 2), r = \rho_{12} R(t) \tag{1.46}$$

where $R\ t\ = \sqrt{\alpha t^2 + 2\beta t + \gamma}$ ; $\xi, \eta, \zeta, \tau$ are the new variables and $\rho_{12}$ is constant.

Later, Mestschersky (1952) came up with a law which considers the masses and their sum to vary in the same proportion in such a way that

$$\mu\ t\ = \frac{\mu_0}{R\ t}\ ,\ \ \mu_1\ t\ = \frac{\mu_{10}}{R\ t}\ ,\ \mu_2\ t\ = \frac{\mu_{20}}{R\ t} \tag{1.47}$$

where $\mu_1\ t\ = Gm_1\ t\ , \mu_2\ t\ = Gm_2\ t\ ,\ \ \mu\ t\ = \mu_1\ t\ + \mu_2\ t\ ,\ \ \mu_{10}$ and $\mu_{20}$ are constants.

The law (1.47) is called the unified Mestschersky (1952) law (UML) and it assures that the centre of the mass of the system moves inertially.

## 1.8.6 Particular solutions of the Gylden-Mestschersky problem.

We let $\mu\ t\ = G\ m_1 + m_2$ in equation (1.41) to get

$$\ddot{\vec{r}} + \frac{\mu}{r^2}\frac{\vec{r}}{r} = 0 \tag{1.48}$$

Now, differentiation of equation (1.13) with respect to time $t$ and multiplying by $\frac{1}{r}$, yields

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0 \tag{1.49}$$

But acceleration in polar coordinates is given by

$$\dot{\vec{v}} = \ \ddot{r} - r\dot{\theta}^2\ \ \vec{i} + \ 2\dot{r}\dot{\theta} + r\ddot{\theta}\ \ \vec{j} \tag{1.50}$$

We substitute equation (1.49) in (1.50), to get

$$\dot{\vec{v}} = \ \ddot{r} - r\dot{\theta}^2\ \ \vec{i} \tag{1.51}$$

Using equation (1.51) in (1.48) yields

$$\ddot{r} - r\dot{\theta}^2 + \frac{\mu}{r^2} = 0 \tag{1.52}$$

Now from equation (1.13), we have

$$\dot{\theta}^2 = \frac{C^2}{r^4} \tag{1.53}$$

Substituting equation (1.53) in (1.52), we get

$$\ddot{r} - \frac{C^2}{r^3} + \frac{\mu}{r^2} = 0, \tag{1.54}$$

where $C = r^2\dot{\theta}$ or $r^2\omega = C$,

$r$ is the distance between the bodies, $\theta$ is the angle between the straight line passing through $m_1$ and $m_2$ and a certain fixed straight line in the plane of motion. $\dot{\theta} = \omega\ t$ is angular velocity of revolution of the bodies and $C$ is the constant of area integral.

Now, from the Mestschersky transformation (1.46), we have

$$r = \rho_{12} R\ t\ , \tag{1.55}$$

Differentiating equation (1.55), yields

$$\dot{r} = \rho_{12}\ \ \alpha t + \beta \quad \alpha t^2 + 2\beta t + \gamma\ ^{-1/2}$$

Differentiating again and simplifying, we get

$$\ddot{r} = \rho_{12} \frac{\alpha\gamma - \beta^2}{R^3\ t} \tag{1.56}$$

From second equation of system (1.54), we have

$$C^2 = r^4\omega^2 \tag{1.57}$$

Substituting equation (1.55), (1.56) and (1.57) in the first equation of system (1.54); multiplying throughout by $\frac{1}{\rho_{12}}$ and simplifying results in the equation

$$\omega\ t\ = \frac{1}{R^2\ t}\left(\alpha\gamma - \beta^2 + \frac{\mu_0}{\rho_{12}^3}\right)^{\frac{1}{2}} \tag{1.58}$$

Now since $\alpha, \beta, \gamma, \mu_0$ and $\rho_{12}$ are constants, we let

$$\left(\alpha\gamma - \beta^2 + \frac{\mu_0}{\rho_{12}^3}\right)^{\frac{1}{2}} = \omega_0,$$

So that equation (1.58) becomes

$$\omega\ t\ = \frac{\omega_0}{R^2\ t} \tag{1.59}$$

Again, substituting equation (1.55) and (1.56) in the first equation of system (1.54) and reducing throughout by $R^3\ t$ , yields

$$\rho_{12}\ \ \alpha\gamma - \beta^2\ \ - \frac{C^2}{\rho_{12}^3} + \frac{\mu_0}{\rho_{12}^2} = 0 \tag{1.60}$$

Applying equation (1.57) in (1.60), we get

$$\alpha\gamma - \beta^2\ \ - \omega_0^2 + \frac{\mu_0}{\rho_{12}^3} = 0 \tag{1.61}$$

Also, substituting equation (1.54) and (1.59) in equation (1.57), and simplifying, gives

$$C = \rho_{12}^2\omega_0 \tag{1.62}$$

Using equation (1.62) in (1.61), we have

$$\frac{\beta^2 - \alpha\gamma + \omega_0^2 \;\; C}{\omega_0} = \frac{\mu_0}{\rho_{12}} \tag{1.63}$$

But $\mu = \frac{\mu_0}{R \;\; t}$,

Substituting for $\mu_0$ in equation (1.63) and multiplying both throughout by $\rho_{12}^2$, yields

$$\frac{\beta^2 - \alpha\gamma + \omega_0^2 \;\; \rho_{12}^2 C}{\omega_0} = \mu\rho_{12} R \;\; t$$

With the help of equations (1.62) and (1.55), we get the equation

$$\frac{\beta^2 - \alpha\gamma + \omega_0^2 \;\; C^2}{\omega_0^2} = r\mu \tag{1.64}$$

Now, if we suppose that

$$\kappa = \frac{\beta^2 - \alpha\gamma + \omega_0^2}{\omega_0^2} \tag{1.65}$$

Therefore, we can express equation (1.64) in the form

$$r\mu = \kappa C^2 \tag{1.66}$$

Equation (1.66) is a particular integral of the Gylden-Mestschersky problem and $\kappa$ is a constant defined by equation (1.65). When $\beta^2 - \alpha\gamma = 0$, we get $\kappa = 1$ and this corresponds to the case when the masses are constant. When $\beta^2 - \alpha\gamma > 0$, this means that $\kappa > 1$ and when $\beta^2 - \alpha\gamma < 0$, this implies that $\kappa < 1$. Since kappa cannot be zero, the range is such that $0 < \kappa < \infty$.

### 1.8.7 Equations of motion of the restricted three-body problem with isotropic mass variations of the primaries

The investigation of the motion and libration points of the infinitesimal mass in the restricted problem, under the condition that the motion of the variable-mass main bodies is determined by the Gylden-Mestschersky problem with isotropic mass variation of the primaries varying in proportion to each other in accordance with the unified Mestschersky law have been studied by Gelf'gat (1973), Bekov (1988), Luk'yanov (1989, 1990) and, Singh and Leke (2010).

Let us consider a rotating frame of reference $Oxyz$, where O is the origin and suppose that $m_1$ and $m_2$ are the masses of the primary bodies and $m_3$ is the mass of the third body. Let the radius vector from $m_3$ to $m_1$ be $r_1$, $m_3$ to $m_2$ be $r_2$ and the distance between the two primaries be $r$ and let $\omega$ be the angular velocity.

Now, the kinetic energy in the rotating frame of reference $0xyz$ is given by

$$T = \frac{1}{2}m_3(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m_3\omega(x\dot{y} - y\dot{x}) + \frac{1}{2}m_3(x^2 + y^2)\omega^2 \tag{1.67}$$

while the potential energy has the form

$$V = Gm_3\left[\frac{m_1\ t}{r_1} + \frac{m_2\ t}{r_2}\right] \tag{1.68}$$

where $r_1^2 = (x - x_1)^2 + y^2 + z^2$, $r_2^2 = (x - x_2)^2 + y^2 + z^2$ and $G$ is the gravitational constant.

We let $p_x$, $p_y$ and $p_z$ be the generalized components of momentum, then

$$\rho_x = m_3\ \dot{x} - \omega y \quad , \quad \rho_y = m_3\ \dot{y} + \omega x \quad , \quad \rho_z = m_3\dot{z} \tag{1.69}$$

Substituting system (1.69) in (1.67), and simplifying yields

$$T = \frac{1}{2m_3}\left(\rho_x^2 + \rho_y^2 + \rho_z^2\right)$$

Now using equations (1.69) in the Hamiltonian H and simplifying, gives

$$H = \frac{1}{2m_3}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega\left(yp_x - xp_y\right) + V \tag{1.70}$$

The Hamiltonian canonical equations are given by

$$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{z} = \frac{\partial H}{\partial p_z}, \quad \dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_y = -\frac{\partial H}{\partial y}, \quad \dot{p}_z = -\frac{\partial H}{\partial z}$$

Therefore,

$$\dot{p}_x = -\left(-\omega p_y\right) - \frac{\partial V}{\partial x}, \quad \dot{p}_y = -\omega p_x - \frac{\partial V}{\partial y}, \quad \dot{p}_z = -\frac{\partial V}{\partial z} \tag{1.71}$$

Now since the primaries move within the framework of the GMP and their masses vary with time in accordance with the UML, then $p_x, p_y, p_z$ and the angular velocity $\omega$ will all be time dependent. Thus, differentiating system $p_x$, $p_y$ and $p_z$ with respect to time, respectively, we get

$$\dot{p}_x = m_3\left(\ddot{x} - \omega\dot{y} - \dot{\omega}y\right), \quad \dot{p}_y = m_3\left(\ddot{y} + \omega\dot{x} + \dot{\omega}x\right), \quad \dot{p}_z = m_3\ddot{z} \tag{1.72}$$

From systems (1.71) and (1.72), we have

$$m_3\left(\ddot{x} - \omega\dot{y} - \dot{\omega}y\right) = \omega p_y - \frac{\partial V}{\partial x}$$

$$m_3\ \ \ddot{y}+\omega\dot{x}+\dot{\omega}x\ \ =-\omega p_x-\frac{\partial V}{\partial y} \tag{1.73}$$

$$m_3\ddot{z}=-\frac{\partial V}{\partial z}$$

Substituting the equations of system (1.69) in those of system (1.73), gives

$$\ddot{x}-2\omega\dot{y}=\omega^2x+\dot{\omega}y-\frac{1}{m_3}\frac{\partial V}{\partial x}$$

$$\ddot{y}+2\omega\dot{x}=\omega^2y-\dot{\omega}x-\frac{1}{m_3}\frac{\partial V}{\partial y} \tag{1.74}$$

$$m_3\ddot{z}=-\frac{1}{m_3}\frac{\partial V}{\partial z}$$

Now, from equation (1.68), we have

$$\frac{\partial V}{\partial x}=Gm_3\left[\frac{m_1\ \ x-x_1}{r_1^3}+\frac{m_2\ \ x-x_2}{r_2^3}\right],\frac{\partial V}{\partial y}=Gm_3\left(\frac{m_1y}{r_1^3}+\frac{m_2y}{r_2^3}\right),\frac{\partial V}{\partial z}=Gm_3\left(\frac{m_1z}{r_1^3}+\frac{m_2z}{r_2^3}\right) \tag{1.75}$$

On substituting equations (1.75) in the respective equations of system (1.74), we get

$$\ddot{x}-2\omega\dot{y}=\omega^2x+\dot{\omega}y-\frac{Gm_1(x-x_1)}{r_1^3}-\frac{Gm_2(x-x_2)}{r_2^3}$$

$$\ddot{y}+2\omega\dot{x}=\omega^2y-\dot{\omega}x-\frac{Gm_1y}{r_1^3}-\frac{Gm_2y}{r_2^3} \tag{1.76}$$

$$\ddot{z}=-\frac{Gm_1z}{r_1^3}-\frac{Gm_2z}{r_2^3}$$

Now, we let

$$Gm_1(t) = \mu_1(t),\ Gm_2(t) = \mu_2(t) \tag{1.77}$$

So that equations (1.76) take the form:

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega} y - \frac{\mu_1(x - x_1)}{r_1^3} - \frac{\mu_2(x - x_2)}{r_2^3}$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega} x - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} \tag{1.78}$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3}$$

where $r_1^2 = (x - x_1)^2 + y^2 + z^2$, $r_2^2 = (x - x_2)^2 + y^2 + z^2$

These equations describe the motion of the third body having infinitesimal mass in the gravitational field of the primary bodies in the barycentric coordinate system $0xyz$, rotating with an angular velocity $\omega\ t$ about the $z-$axis perpendicular to the plane of motion of the primaries, while the $x-$axis always passes through these points with the consideration that both primaries experience mass variations with time.

Now from the property of the center of mass $\mu_1$ at $(x_1, 0, 0)$ and $\mu_2$ at $(x_2, 0, 0)$ with the consideration that the origin is taken as the center of mass, we have

$$\mu_1 x_1 + \mu_2 x_2 = 0 \tag{1.79}$$

From $r^2 = (x_2 - x_1)^2$, we have $x_2 = r + x_1$.

Substituting for $x_2$ in equation (1.79) and solving, we get

$$x_1 = -\frac{\mu_2 r}{\mu_1 + \mu_2} \ , \quad x_2 = \frac{\mu_1 r}{\mu_1 + \mu_2} \tag{1.80}$$

The expressions (1.80) connect the barycentric coordinates $x_1$ and $x_2$ with the mutual distance $r$ .

The equations of motion (1.78) are non-integrable differential equations with variable coefficients, the solutions even for particular steady-state solutions –the EPs are difficult to seek directly from equation (1.78), because these equations contain unknown functions of time. Hence, we use the MT, the UML, the particular integral and solutions of the GMP to reduce the system of equations to the type with constant coefficients.

From the MT (1.46), we have

$$x = \xi R(t) \Rightarrow \quad x = \xi(\alpha t^2 + 2\beta t + \gamma)^{\frac{1}{2}}$$

Differentiating with respect to $t$ and denoting differentiation with respect to $\tau$ by dashes, we have

$$\dot{x} = \xi.\frac{1}{2}(\alpha t^2 + 2\beta t + \gamma)^{-1/2}(2\alpha t + 2\beta) + \frac{\xi'(\alpha t^2 + 2\beta t + \gamma)^{1/2}}{R^2(t)}$$

$$= \frac{\xi}{2}(\alpha t^2 + 2\beta t + \gamma)^{-1/2}.2(\alpha t + \beta) + \frac{\xi'}{R^2(t)}(\alpha t^2 + 2\beta t + \gamma)^{1/2}$$

$$= \frac{\xi(\alpha t + \beta)}{R(t)} + \frac{\xi' R(t)}{R^2(t)}$$

$$\Rightarrow \qquad \dot{x} = \frac{\xi(\alpha t + \beta)}{R(t)} + \frac{\xi'}{R(t)} \tag{1.81}$$

Similarly, $\dot{y} = \frac{\eta(\alpha t + \beta)}{R(t)} + \frac{\eta'}{R(t)}$ , $\dot{z} = \frac{\varsigma(\alpha t + \beta)}{R(t)} + \frac{\varsigma'}{R(t)}$

Now, we find second derivatives of (1.81), so that

$$\ddot{x} = -\frac{\xi}{2}.2(\alpha t + \beta)(\alpha t + \beta)(\alpha t^2 + 2\beta t + \gamma)^{-3/2} + \xi(\alpha t^2 + 2\beta t + \gamma)^{-1/2}\alpha$$

$$+(\alpha t + \beta)(\alpha t^2 + 2\beta t + \gamma)^{-1/2}.\frac{\xi'}{R^2(t)} - \frac{1}{2}\xi'(\alpha t^2 + 2\beta t + \gamma)^{-3/2}.2(\alpha t + \beta)$$

$$+\frac{\xi''}{R^2(t)}(\alpha t^2 + 2\beta t + \gamma)^{-1/2}.$$

$$= -\frac{\xi(\alpha t + \beta)^2}{(\alpha t^2 + 2\beta t + \gamma)^{3/2}} + \frac{\alpha\xi}{(\alpha t^2 + 2\beta t + \gamma)^{1/2}} + \frac{(\alpha t + \beta)\xi'}{(\alpha t^2 + 2\beta t + \gamma)^{1/2}R^2(t)}$$

$$-\frac{(\alpha t + \beta)\xi'}{(\alpha t^2 + 2\beta t + \gamma)^{3/2}} + \frac{\xi''}{R^2(t)(\alpha t^2 + 2\beta t + \gamma)^{1/2}}.$$

Simplifying, we get

$$\ddot{x} = \frac{\xi(\alpha\gamma - \beta^2)}{R^3(t)} + \frac{\xi''}{R^3(t)}$$

Similarly, $\ddot{y} = \frac{\eta(\alpha\gamma - \beta^2)}{R^3(t)} + \frac{\eta''}{R^3(t)}$,

$$\ddot{z} = \frac{\zeta(\alpha\gamma - \beta^2)}{R^2(t)} + \frac{\zeta''}{R^3(t)}$$

Also, from a particular solutions of the GMP, we have

$$\dot{\omega} = -\omega_0(\alpha t^2 + 2\beta t + \gamma)^{-2}.2(\alpha t + \beta)$$

$$= -\frac{2\omega_0(\alpha t + \beta)}{(\alpha t^2 + 2\beta t + \gamma)^2}$$

$$\Rightarrow \qquad \dot{\omega} = -\frac{2\omega_0(\alpha t + \beta)}{R^4(t)}$$

Substituting all the above in system (1.78), we have

$$\frac{\xi\ \ \alpha\gamma - \beta^2}{R^3\ \ t} + \frac{\xi''}{R^3\ \ t} - \frac{2\omega_0}{R^2\ \ t}\left[\frac{\alpha t + \beta\ \ \eta}{R\ \ t} + \frac{\eta'}{R\ \ t}\right] = \frac{\omega_0^2 \xi R\ \ t}{R^4\ \ t} + \left[\frac{-2\omega_0\ \ \alpha t + \beta}{R^4\ \ t}\right]\eta R\ \ t$$
$$-\frac{\mu_{10}}{R\ \ t}\frac{\left[\xi R\ \ t - \xi_1 R\ \ t\right]}{\rho_1^3 R^3\ \ t} - \frac{\mu_{20}}{R\ \ t}\frac{\left[\xi R\ \ t - \xi_2 R\ \ t\right]}{\rho_2^3 R^3\ \ t}$$

or

$$\frac{\xi''}{R^3\ \ t} - \frac{2\omega_0\ \ \alpha t + \beta\ \ \eta}{R^3\ \ t} - \frac{2\omega_0 \eta'}{R^3\ \ t} + \frac{\xi\ \ \alpha\gamma - \beta^2}{R^3\ \ t} = \frac{\omega_0^2 \xi R\ \ t}{R^4\ \ t} - \frac{2\omega_0\ \ \alpha t + \beta\ \ \eta}{R^3\ \ t} - \frac{\mu_{10}}{R\ \ t}\frac{\xi - \xi_1\ \ R\ \ t}{\rho_1^3 R^3\ \ t}$$
$$-\frac{\mu_{20}}{R\ \ t}\frac{\xi - \xi_2\ \ R\ \ t}{\rho_2^3 R^3\ \ t}$$

$$\Rightarrow \frac{\xi''}{R^3\ \ t} - \frac{2\omega_0 \eta'}{R^3\ \ t} = \frac{\omega_0^2 \xi}{R^3\ \ t} - \frac{\xi\ \ \alpha\gamma - \beta^2}{R^3\ \ t} - \frac{\mu_{10}\ \ \xi - \xi_1}{\rho_1^3 R^3\ \ t} - \frac{\mu_{20}\ \ \xi - \xi_2}{\rho_2^3 R^3\ \ t}$$

Multiplying throughout by $R^3\ \ t$ , we get

$$\xi'' - 2\omega_0 \eta' = \omega_0^2 \xi - \xi\ \ \alpha\gamma - \beta^2\ \ - \frac{\mu_{10}\ \ \xi - \xi_1}{\rho_1^3} - \frac{\mu_{20}\ \ \xi - \xi_2}{\rho_2^3} \qquad (1.82)$$

Similarly $\eta'' + 2\omega_0\xi' = \omega_0^2\eta - \eta\ \ \alpha\gamma - \beta^2\ \ - \frac{\mu_{10}\eta}{\rho_1^3} - \frac{\mu_{20}\eta}{\rho_2^3}$

$$\zeta'' = \ \beta^2 - \alpha\gamma\ \ \zeta - \frac{\mu_{10}\zeta}{\rho_1^3} - \frac{\mu_{20}\zeta}{\rho_2^3}$$

Performing same substitution on $r_1^2$ and $r_2^2$, we have

$$\rho_1^2 R^2(t) = \ \xi R(t) - \xi_1 R(t)\ ^2 + \eta^2 R^2(t) + \zeta^2 R^2(t)$$

$$= R^2(t)(\xi - \xi_1)^2 + \eta^2 R^2(t) + \zeta^2 R^2(t)$$

Multiplying throughout by $\frac{1}{R^2(t)}$, we at once have

$$\rho_1^2 = (\xi - \xi_1)^2 + \eta^2 + \zeta^2$$

Similarly, $\rho_2^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2$

Also, we have from equations (1.80), that

$$x_1 = -\frac{\mu_{20}\rho_{12}R(t)}{\mu_0},\ x_2 = \frac{\mu_{10}\rho_{12}R\ \ t}{\mu_0}$$

And can be expressed such that

$$x_1 = \xi_1 R(t), \qquad x_2 = \xi_2 R(t) \tag{1.83}$$

where $\xi_1 = -\frac{\mu_{20}}{\mu_0}\rho_{12}, \quad \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12}$ are constants.

Hence, the system (1.78) of equations of motion with variable coefficients is now reduced to the autonomous form such that

$$\xi'' - 2\omega_0\eta' = \frac{\partial\Omega}{\partial\xi}, \qquad \eta'' + 2\omega_0\xi' = \frac{\partial\Omega}{\partial\eta}, \qquad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{1.84}$$

where $$\Omega = \frac{\omega_0^2 \quad \xi^2 + \eta^2}{2} + \frac{\beta^2 - \alpha\gamma \quad \xi^2 + \eta^2 + \zeta^2}{2} + \frac{\mu_{10}}{\rho_1} + \frac{\mu_{20}}{\rho_2}$$

$$\rho_1^2 = (\xi - \xi_1)^2 + \eta^2 + \zeta^2, \quad \rho_2^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2 \tag{1.85}$$

$$\xi_1 = \frac{-\mu_{20}}{\mu_0}\rho_{12}, \quad \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12}$$

Next, we make choice for units of measurements such that at initial time $t_0$. We choose

$$\mu_0 = G \tag{1.86}$$

For the unit of time and length, we choose them, respectively, such that

$$\omega_0 = 1 \text{ and } \rho_{12} = 1 \tag{1.87}$$

We now introduce the mass parameter $\upsilon$, expressed as

$$\frac{\mu_{10}}{\mu_0} = 1 - \upsilon, \quad \frac{\mu_{20}}{\mu_0} = \upsilon \quad , \quad \text{where } 0 < \upsilon \le \frac{1}{2} \tag{1.88}$$

Substituting (1.86) in (1.88), we have

$$\mu_{10} = G\ 1 - \upsilon \quad , \quad \mu_{20} = G\upsilon \tag{1.89}$$

Now from the particular integral (1.66), we apply equations (1.55) and (1.86) and simplify to get

$$G = \kappa \tag{1.90}$$

If these measurements are substituted in equation (1.65), we get

$$\kappa = \beta^2 - \alpha\gamma + 1 \tag{1.91}$$

where $\kappa$ is the constant of integration of the GMP.

Also, from equation (1.62), we get $C = 1$.

Therefore, substituting the units of measurement in equations of system (1.84) and (1.85), we get the equations of motion of the autonomized system with constant coefficients in the forms

$$\xi'' - 2\eta' = \frac{\partial \Omega}{\partial \xi}, \quad \eta'' + 2\xi' = \frac{\partial \Omega}{\partial \eta}, \quad \zeta'' = \frac{\partial \Omega}{\partial \zeta} \tag{1.92}$$

where, $\Omega = \dfrac{\kappa\ \xi^2 + \eta^2}{2} + \dfrac{\kappa - 1\ \zeta^2}{2} + \dfrac{\kappa\ 1-\upsilon}{\rho_1} + \dfrac{\kappa\upsilon}{\rho_2}$

$$\rho_1 = \sqrt{(\xi + \upsilon)^2 + \eta^2 + \zeta^2}, \qquad \rho_2 = \sqrt{(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2}$$

The equations of system (1.92) possess families of particular solutions-the relative equilibrium states; five of which are analogous to the EPs of the classical R3BP and were established first by Gelf'gat (1973). Later, Bekov (1988) found two additional EPs located outside the plane of motion on the $\xi\zeta$ – plane called coplanar EPs.

The system of equations (1.92) with constant coefficients has seven EPs $L_i\ \ \xi^i, \eta^i, \zeta^i \quad i = 1, 2...7$ ; three collinear, two triangular and two coplanar points. For the

system of equation (1.78) with variable coefficients, these EPs $L_i \;\; i=1,2,...7$ are determined from the MT (1.46) in the form (Luk'yanov 1989):

$$x^i = \xi^i R\ t\ , y^i = \eta^i R\ t\ , z^i = \zeta^i R\ t \tag{1.93}$$

where $\xi^i\ \ \tau\ , \eta^i\ \ \tau$ and $\zeta^i\ \ \tau\ ,\ i=1,2,...7$ are the equilibrium points of the system (1.92) of equations with constant coefficients . Evidently, the equilibrium points of the system with variable coefficients and those of the autonomized systems differ by the function $R\ t$ .

## 1.8.8 Equations of motion of the restricted problem of three bodies with non-isotropic mass variations

Bekov (1987), Bekov *et al*. (2005), Letelier and Da Silva (2011) have all studied the R3BP when the masses of the three bodies change with time. We follow the same derivations for the case when only the primaries vary with the consideration that now, $m_3$ also changes with time. From the generalized components of momentum $p_x$ , $p_y$ and $p_z$ , we have

$$\rho_x = \frac{\partial T}{\partial \dot{x}} = m_3\ \ \dot{x} - \omega y\ \ \ , \ \rho_y = \frac{\partial T}{\partial \dot{y}} = m_3\ \ \dot{y} + \omega x\ \ , \quad \rho_z = \frac{\partial T}{\partial \dot{z}} = m_3 \dot{z}$$

Following the procedure in subsection **1.8.7**, while we note that the mass of the third body also varies with time, we get the equations

$$\ddot{x} - \omega \dot{y} = \dot{\omega} y + \omega\ \ \dot{y} + \omega x\ \ - \frac{1}{m_3}\frac{\partial V}{\partial x} - \frac{\dot{m}_3}{m_3}\ \ \dot{x} - \omega y$$

$$\ddot{y} + \omega \dot{x} = -\dot{\omega} x - \omega\ \ \dot{x} - \omega y\ \ - \frac{1}{m_3}\frac{\partial V}{\partial y} - \frac{\dot{m}_3}{m_3}\ \ \dot{y} + \omega x$$

$$\ddot{z} = -\frac{1}{m_3}\frac{\partial V}{\partial z} - \frac{\dot{m}_3}{m_3}\dot{z}$$

Now, since $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ are same with those in equations **1.75**, we at once have

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega} y - \frac{\mu_1 \ x - x_1}{r_1^3} - \frac{\mu_2 \ x - x_2}{r_2^3} - \frac{\dot{m}_3 \ \dot{x} - \omega y}{m_3}$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega} x - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} - \frac{\dot{m}_3 \ \dot{y} + \omega x}{m_3} \tag{1.94}$$

$$\ddot{z} = -\frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} - \frac{\dot{m}_3 \dot{z}}{m_3}$$

where $r_1^2 = \ x - x_1\ ^2 + y^2 + z^2$, $r_2^2 = \ x - x_2\ ^2 + y^2 + z^2$

These equations govern the motion of the restricted problem of three variable mass bodies. When the mass of the third body is constant, equations (1.94) reduce to those in (1.78).

Now, in order to obtain useful dynamical predictions, we transform $x, y, z, t$ to the autonomous form $\xi, \eta, \zeta, \tau$ . Following Bekov *et al*. (2005), we use the Gelf'gat (1973) transformation:

$$x = \left(\frac{\mu_0}{\mu_*}\right)^3 \xi, \qquad y = \left(\frac{\mu_0}{\mu_*}\right)^3 \eta, \qquad z = \left(\frac{\mu_0}{\mu_*}\right)^3 \zeta, \qquad \frac{dt}{d\tau} = \left(\frac{\mu_0}{\mu_*}\right)^5, \ \omega \ t \ = \left(\frac{\mu_0}{\mu_*}\right)^5,$$

$$\mu_1 = \mu_{01}\frac{\mu_*}{\mu_0}, \qquad \mu_2 = \mu_{02}\frac{\mu_*}{\mu_0}, \qquad m_3 = m_{03}\frac{\mu_*}{\mu_0}, \ r = \rho_{12}\left(\frac{\mu_0}{\mu_*}\right)^3, \ r_1 = \rho_1\left(\frac{\mu_*}{\mu_0}\right)^3 \tag{1.95}$$

$$r_2 = \rho_2 \left( \frac{\mu_*}{\mu_0} \right)^3 : \ \mu_* \ \ t \ = \frac{m_1 m_2}{m_1 + m_2}$$

The particular solutions for the case with variable parameters $\mu_1, \mu_2$ and $m_3$ in the form of the Eddington-Jeans laws with indices $n = 3$ and $n = 6$ is expressed (Bekov *et al.* 2005), with the help of the function:

$$\dot{\mu}_* = \bar{\alpha}_1 \mu_1^n, \quad \dot{\mu}_* = \bar{\alpha}_2 \mu_2^n, \quad \dot{m}_3 = \bar{\alpha}_3 m_3^n \tag{1.96}$$

where $\bar{\alpha}_i \ \ i = 1, 2, 3$ are constants.

The exponent $n = 1$ falls in the stellar range while $n = 2$ and $n = 3$ result, respectively, in the first and second law of Mestschersky (1893, 1902). Equations (1.96) indicate that the laws of variation of the three masses are the same.

Also, the dynamical system has the particular solution (Gelf'gat 1973; Bekov *et al*. 2005):

$$r \mu_* m_3^2 = \kappa C^2 \tag{1.97}$$

where $\kappa > 0$ is a constant, and $C$ is a constant of the area integral.

Now, from equations (1.95) and (1.96), we get the following

$$\dot{x} = \mu_0^3 \left( -3\bar{\alpha}_1 \mu_1^3 \mu_*^{-4} \xi + \frac{\mu_*^2}{\mu_0^5} \xi' \right), \quad \frac{\dot{m}_3}{m_3} \omega y = \frac{\dot{m}_3}{m_3} \frac{\mu_*^2 \omega_0}{\mu_0^2}, \quad \dot{\omega} = 5\bar{\alpha}_1 \frac{\mu_1^3 \mu_*^4 \omega_0}{\mu_0^5}$$

$$\frac{\dot{m}_3}{m_3} \dot{x} = \frac{\dot{m}_3}{m_3} \mu_0^3 \left( -3\bar{\alpha} \mu_1^3 \mu_*^{-4} \xi + \frac{\mu_*^2}{\mu_0^5} \xi' \right), \quad \dot{\omega} y = 5\bar{\alpha}_1 \frac{\mu_1^3 \mu_* \omega_0}{\mu_0^2} \eta \tag{1.98}$$

Taking second derivative $\ddot{x}$ from first equation of (1.98) and using equation (1.97), we substitute the result and equations of system (1.98) in the first equation of (1.94), to get

$$\frac{\mu_*^7}{\mu_0^7}\xi'' + \frac{6\bar{\alpha}_1\mu_*\omega_0\mu_1^3}{\mu_0^2}\eta - 2\omega_0\eta'\frac{\mu_*^7}{\mu_0^7} = \kappa\omega_0^2\xi\frac{\mu_*^7}{\mu_0^7} + \frac{5\bar{\alpha}_1\mu_*\omega_0\mu_1^3}{\mu_0^2}\eta - \mu_{01}\times\frac{\mu_*}{\mu_0}\left(\frac{\mu_0^3}{\mu_*^3}\xi - \frac{\mu_0^3}{\mu_*^3}\xi_1\right)\div\left(\rho_1\frac{\mu_0^3}{\mu_*^3}\right)^3$$

$$-\mu_{02}\times\frac{\mu_*}{\mu_0}\left(\frac{\mu_0^3}{\mu_*^3}\xi - \frac{\mu_0^3}{\mu_*^3}\xi_2\right)\div\left(\rho_2\frac{\mu_0^3}{\mu_*^3}\right)^3 - \frac{\dot{m}_3}{m_3}\mu_0^3\left(-\frac{3\bar{\alpha}\mu_1^3}{\mu_*^4}\xi + \frac{\mu_*^2}{\mu_0^5}\xi'\right) + \frac{\dot{m}_3}{m_3}\mu_*^2\frac{\omega_0\eta}{\mu_0^2}$$

Since $m_3 = m_{03}\frac{\mu_*}{\mu_0}$ and $\frac{\dot{m}_3}{m_3} = \frac{\bar{\alpha}_1\mu_1^3}{\mu_*}$, we get

$$\frac{\mu_*^7}{\mu_0^7}\xi'' - 2\omega_0\eta'\frac{\mu_*^7}{\mu_0^7} = \kappa\omega_0^2\xi\frac{\mu_*^7}{\mu_0^7} - \frac{6\bar{\alpha}_1\mu_*\omega_0\mu_1^3}{\mu_0^2}\eta + \frac{5\bar{\alpha}_1\mu_*\omega_0\mu_1^3}{\mu_0^2}\eta + \frac{\bar{\alpha}_1\mu_1^3}{\mu_*}\times\frac{\mu_*^2\omega_0}{\mu_0^2}\eta - \frac{\bar{\alpha}_1\mu_1^3}{\mu_*}\times\frac{\mu_*^2}{\mu_0^2}\xi'$$

$$+\frac{\bar{\alpha}_1\mu_*\mu_1^3}{\mu_0^2}\xi' - \frac{\mu_{01}\times\frac{\mu_0^2}{\mu_*^2}\ \xi-\xi_1}{\rho_1^3\frac{\mu_0^9}{\mu_*^9}} - \frac{\mu_{02}\times\frac{\mu_0^2}{\mu_*^2}\ \xi-\xi_2}{\rho_2^3\frac{\mu_0^9}{\mu_*^9}}$$

$$\Rightarrow \frac{\mu_*^7}{\mu_0^7}\xi'' - 2\omega_0\eta'\frac{\mu_*^7}{\mu_0^7} = \kappa\omega_0^2\xi\frac{\mu_*^7}{\mu_0^7} - \frac{\mu_{01}\frac{\mu_0^2}{\mu_*^2}\ \xi-\xi_1}{\rho_1^3}\times\frac{\mu_*^9}{\mu_0^9} - \frac{\mu_{02}\frac{\mu_0^2}{\mu_*^2}\ \xi-\xi_2}{\rho_2^3}\times\frac{\mu_*^9}{\mu_0^9}$$

$$\Rightarrow \frac{\mu_*^7}{\mu_0^7}\xi'' - 2\omega_0\eta'\frac{\mu_*^7}{\mu_0^7} = \kappa\omega_0^2\xi\frac{\mu_*^7}{\mu_0^7} - \frac{\mu_{01}\mu_0^2\ \xi-\xi_1}{\mu_*^2\rho_1^3}\frac{\mu_*^9}{\mu_0^9} - \frac{\mu_{02}\mu_0^2\ \xi-\xi_2}{\mu_*^2\rho_2^3}\frac{\mu_*^9}{\mu_0^9}$$

$$\Rightarrow \xi''\frac{\mu_*^7}{\mu_0^7} - 2\omega_0\eta'\frac{\mu_*^7}{\mu_0^7} = \kappa\omega_0^2\xi\frac{\mu_*^7}{\mu_0^7} - \frac{\mu_{01}\ \xi-\xi_1}{\rho_1^3}\frac{\mu_*^7}{\mu_0^7} - \frac{\mu_{02}\ \xi-\xi_2}{\rho_2^3}\frac{\mu_*^7}{\mu_0^7}$$

Multiplying throughout by $\frac{\mu_0^7}{\mu_*^7}$, we get

$$\xi'' - 2\omega_0\eta' = \kappa\omega_0^2\xi - \frac{\mu_{01}\ \xi-\xi_1}{\rho_1^3} - \frac{\mu_{02}\ \xi-\xi_2}{\rho_2^3}$$

Similarly, $\eta'' + 2\omega_0\xi' = \kappa\omega_0^2\eta - \frac{\mu_{01}\eta}{\rho_1^3} - \frac{\mu_{02}\eta}{\rho_2^3}$

$$\zeta'' = \omega_0^2 \ \ \kappa - 1 \ \ \zeta - \frac{\mu_{01}\zeta}{\rho_1^3} - \frac{\mu_{02}\zeta}{\rho_2^3}$$

Hence, we have

$$\xi'' - 2\omega_0\eta' = \frac{\partial\Omega}{\partial\xi}, \ \eta'' + 2\omega_0\xi' = \frac{\partial\Omega}{\partial\eta}, \quad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{1.99}$$

where $\Omega \ \ \xi,\eta,\zeta \ = \frac{\kappa\omega_0^2 \ \ \xi^2 + \eta^2 + \zeta^2}{2} - \frac{\omega_0^2\zeta^2}{2} + \frac{\mu_{01}}{\rho_1} + \frac{\mu_{02}}{\rho_2}$

$$\rho_1^2 = \ \xi - \xi_1 \ ^2 + \eta^2 + \zeta^2, \quad \rho_2^2 = \ \xi - \xi_2 \ ^2 + \eta^2 + \zeta^2, \ \xi_1 = -\frac{\mu_{20}}{\mu_0}\rho_{12}, \ \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12},$$

Now, we choose units as done in subsection **1.8.7** and the system of (1.99), gives

$$\xi'' - 2\eta' = \frac{\partial\Omega}{\partial\xi}, \quad \eta'' + 2\xi' = \frac{\partial\Omega}{\partial\eta}, \quad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{1.100}$$

where $\Omega = \frac{\kappa\left(\xi^2 + \eta^2\right)}{2} + \frac{\left(\kappa - 1\right)\zeta^2}{2} + \frac{\kappa\left(-\upsilon\right)}{\rho_1} + \frac{\kappa\upsilon}{\rho_2}$

$$\rho_1^2 = \left(\xi + \upsilon\right)^2 + \eta^2 + \zeta^2, \quad \rho_2^2 = \left(\xi + \upsilon - 1\right)^2 + \eta^2 + \zeta^2$$

The autonomized system of equations (1.100) is same with that (1.92) in the case of the isotropic mass variations.

**Relationship between the old and new time**

The system of equations with variable coefficients depends on time $t$, which we call the old independent time while the autonomized systems with constant coefficients depend on the new time $\tau$. Next, we establish a relationship connecting them based on **Poincare's** representation of the angular velocity $\omega \ t$ .

By Poincare (1911),

$$\frac{\omega^2\ t}{2\pi G\rho(t)} < 1 \tag{1.101}$$

where $\rho\ t$ is the density and $\omega\ t$ the angular velocity of revolution of the primaries.

From a particular solution of the GMP, knowing that $\omega\ t = \frac{\omega_0}{R^2\ t}$ and applying $\rho\ t = \frac{\rho_0}{R^3\ t}$ in equation (1.101), we get

$$0 < \frac{\omega_0^2}{R^4\ t} < 2\pi G \frac{\rho_0}{R^3\ t}$$

where, $\omega_0, \rho_0$ are constants.

Multiplying throughout by $R^3\ t$ , and using the unit of measurement, i.e. $\omega_0 = 1, G = \kappa$ in the above, yields

$$0 < \frac{1}{R\ t} < 2\pi\kappa\rho_0 : 0 < \kappa < \infty, \tag{1.102}$$

Using the MT, equation (1.102), becomes

$$0 < \frac{d\tau}{dt} < 4\pi^2\kappa^2\rho_0^2$$

Integrating, we have

$$0 < \tau < 2\pi\kappa\rho_0\ ^2 t + A \tag{1.103}$$

At initial time, $\tau = \tau_0, t = t_0$, so equation (1.103) becomes

$$A = \tau_0 - 2\pi\kappa\rho_0^{\ 2} t_0$$

Substituting in (1.103), gives

$$0 < \tau - \tau_0 < 2\pi\kappa\rho_0^{\ 2} \ t - t_0$$

At initial time $t_0 = \tau_0 = 0$, the above expression becomes

$$0 < \frac{\tau}{t} < S^2 \tag{1.104}$$

where $S = 2\pi\kappa\rho_0$ and is finite.

Taking limit as $t$ is approaching $\infty$, we have

$$0 < \lim_{t\to\infty} \tau < S^2 \lim_{t\to\infty} t$$

This implies that $0 < \lim_{t\to\infty} \tau < \infty$

Therefore, as $t$ is approaching $\infty$, $\tau$ is also always approaching a finite value.

Now from (1.104), we have

$$0 < \lim_{t\to\infty} \frac{\tau}{t} < S^2 < \infty \tag{1.105}$$

Equation (1.105) indicates that $\lim_{t\to\infty} \frac{\tau}{t}$ always approaches a finite value.

Now, we let this finite value be $\Gamma$, then

$$\lim_{t\to\infty}\frac{\tau}{t}=\Gamma<S^2 \tag{1.106}$$

where $\Gamma$ is non-negative.

### 1.8.9 Stability of dynamical systems

Stability of dynamical systems is defined in different ways in different context, but the notion of stability must change depending upon the physical system under consideration. Generally speaking, the stability of differential equations refers to the global effects of local perturbations. Lyapunov method introduced in 1899 provides ways of determining the stability of sets of ordinary differential equations. In order to examine the physical stability of systems we are frequently required to determine the physical stability of their discrete approximations. The stability properties of a solution are determined by:

(i) Eigen values (ii) Bifurcation points (iii) Lyapunov Characteristic Numbers.

The stability of constant coefficients linear system of ordinary differential equation is completely determined by the Eigen values and Eigen space structure of the coefficient matrix.

Bifurcation theory is the study of changes in the qualitative structure of the flow that is associated with a system of differential equations, as a result of parameter variations. If the flow does not change substantively for sufficiently small variations of a given parameter, the qualitative nature of the system of equations is said to be structurally stable at the given parameter value. On the other hand, if the system is not structurally stable at this parameter

value, then the parameter has reached a bifurcation value, and the system of equations has evolved to a bifurcation point.

The stability of constant coefficients linear systems of ordinary differential equations is determined completely by the Eigen values of the coefficient matrix. Due to the perturbations induced the position of the body of infinitesimal mass would be displaced a little from the EP. If the resultant motion of the infinitesimal is a rapid departure from the vicinity of the point, we can call such a position of EP an "unstable one", if however, the body merely oscillates about the EP, it is said to be a "stable position" (in the sense of Lyapunov).

Stability of non-autonomous solutions is related to the Lyapunov Characteristic Numbers (LCN) which govern the long-time asymptotic exponential behaviors of solutions.

**Variational equations**

To examine the linear stability of any of the EPs (Szebehely 1967a), we use a small displacement method by shifting the coordinates of the third body and linearize the equations of motion around the EP with coordinates $\xi_0, \eta_0, \zeta_0$ . If we suppose that $u$ , $v$ and $w$ be small displacement in the position of the EP $\xi_0, \eta_0, \zeta_0$ , respectively, then the third body will be displaced to $\xi_0 + u,\ \eta_0 + v,\ \zeta_0 + w$ . That is

$$\xi = \xi_0 + u\ ,\quad \eta = \eta_0 + v\ ,\quad \zeta = \zeta_0 + w \tag{1.107}$$

The force function $\Omega$ of the equations of motion is expanded around the EP, using Taylor's theorem, to get

$$\Omega\ \xi,\eta,\zeta\ =\Omega\ \xi_0+u,\ \eta_0+v,\ \zeta_0+w$$

Using (1.107), we have

$$\Omega=\Omega\ \xi_0,\eta_0,\zeta_0\ +\Omega_\xi\ \xi_0,\eta_0,\zeta_0\ u+\Omega_\eta\ \xi_0,\eta_0,\zeta_0\ v+\Omega_\zeta\ \xi_0,\eta_0,\zeta_0\ w$$

$$+\frac{1}{2!}\Omega_{\xi\xi}\ \xi_0,\eta_0,\zeta_0\ u^2+\Omega_{\xi\eta}\ \xi_0,\eta_0,\zeta_0\ uv+\frac{1}{2!}\Omega_{\eta\eta}\ \xi_0,\eta_0,\zeta_0\ v^2$$

$$+\frac{1}{2!}\Omega_{\zeta\zeta}\ \xi_0,\eta_0,\zeta_0\ w^2+\Omega_{\xi\zeta}\ \xi_0,\eta_0,\zeta_0\ uw+\Omega_{\eta\zeta}\ \xi_0,\eta_0,\zeta_0\ vw+... \qquad (1.108)$$

If we differentiate system (1.107), we have

$$\xi'=u',\quad \eta'=v',\quad \zeta'=w',\quad \xi''=u'',\quad \eta''=v'',\quad \zeta''=w'' \qquad (1.109)$$

We know that at any EP

$$\Omega_\xi\ \xi_0,\eta_0,\zeta_0\ =\Omega_\eta\ \xi_0,\eta_0,\zeta_0\ =\Omega_\zeta\ \xi_0,\eta_0,\zeta_0\ =0$$

Therefore, the partial derivatives in system (1.108) becomes

$$\Omega_\xi=\Omega_{\xi\xi\xi}\ \xi_0,\eta_0,\zeta_0\ u+\Omega_{\xi\eta}\ \xi_0,\eta_0,\zeta_0\ v+\Omega_{\xi\zeta}\ \xi_0,\eta_0,\zeta_0\ w+\Phi\ 2\ +\Phi\ 3$$

$$\Omega_\eta=\Omega_{\xi\eta}\ \xi_0,\eta_0,\zeta_0\ u+\Omega_{\eta\eta}\ \xi_0,\eta_0,\zeta_0\ v+\Omega_{\eta\zeta}\ \xi_0,\eta_0,\xi_0\ w+\Phi\ 2\ +\Phi\ 3$$

$$\Omega_\zeta=\Omega_{\xi\zeta}\ \xi_0,\eta_0,\zeta_0\ u+\Omega_{\eta\zeta}\ \xi_0,\eta_0,\zeta_0\ v+\Omega_{\zeta\zeta}\ \xi_0,\eta_0,\zeta_0\ w+\Phi\ 2\ +\Phi\ 3 \qquad (1.110)$$

where $\Phi\ 2$ and $\Phi\ 3$ stands for second and higher order, and third and higher order terms in $u$, $v$ and $w$, respectively. Ignoring them and evaluating the equations of motion using system (1.109), we have

$$u'' - 2\omega_0 v' = \Omega_{\xi\xi}\ \ \xi_0, \eta_0, \zeta_0\ \ u + \Omega_{\xi\eta}\ \ \xi_0, \eta_0, \zeta_0\ \ v + \Omega_{\xi\zeta}\ \ \xi_0, \eta_0, \zeta_0\ \ w \tag{1.111}$$

$$v'' + 2\omega_0 u' = \Omega_{\eta\xi}\ \ \xi_0, \eta_0, \zeta_0\ \ u + \Omega_{\eta\eta}\ \ \xi_0, \eta_0, \zeta_0\ \ v + \Omega^0_{\eta\zeta}\ \ \xi_0, \eta_0, \zeta_0\ \ w$$

$$w'' = \Omega_{\zeta\xi}\ \ \xi_0, \eta_0, \zeta_0\ \ u + \Omega_{\zeta\eta}\ \ \xi_0, \eta_0, \zeta_0\ \ v + \Omega_{\zeta\zeta}\ \ \xi_0, \eta_0, \zeta_0\ \ w$$

The system (1.111) can be written as

$$u'' - 2\omega_0 v' = \ \Omega^0_{\xi\xi}\ \ u + \ \Omega^0_{\xi\eta}\ \ v + \ \Omega^0_{\xi\zeta}\ \ w$$

$$v'' + 2\omega_0 u' = \ \Omega^0_{\xi\eta}\ \ u + \ \Omega^0_{\eta\eta}\ \ v + \ \Omega^0_{\eta\zeta}\ \ w \tag{1.112}$$

$$w'' = \ \Omega^0_{\zeta\xi}\ \ u + \ \Omega^0_{\zeta\eta}\ \ v + \ \Omega^0_{\zeta\zeta}\ \ w$$

System (1.112) is called the variational equations of motion or precisely the linear variational equations of motion. Here $u$, $v$ and $w$ are the variations or "small displacements". Only linear terms in $u$, $v$ and $w$ have been taken, the second order partial derivatives of $\Omega$ are denoted by subscripts and the superscript $O$ indicates that the derivatives are to be calculated at the equilibrium point under investigation.

**Characteristic equation**

In order to consider the motion in the $\xi\eta - plane$, we let a solution of the system (1.112), be

$$u = A\exp\ \lambda\tau\ ,\ v = B\exp\ \lambda\tau \tag{1.113}$$

where $A, B$ and $\lambda$ are constants.

Then

$$u' = \lambda Ae^{\lambda\tau}, v' = \lambda Be^{\lambda\tau}$$

$$u'' = \lambda^2 Ae^{\lambda\tau}, v'' = \lambda^2 Be^{\lambda\tau}$$

Substituting into the first two equations of system (1.112), we immediately have

$$\lambda^2 Ae^{\lambda\tau} - 2\lambda\omega_0 Be^{\lambda\tau} = \Omega^0_{\xi\xi} Ae^{\lambda\tau} + \Omega^0_{\xi\eta} Be^{\lambda\tau}$$

$$\lambda^2 Be^{\lambda\tau} + 2\lambda\omega_0 Ae^{\lambda\tau} = \Omega^0_{\xi\eta} Ae^{\lambda\tau} + \Omega^0_{\eta\eta} Be^{\lambda\tau}$$

Dividing throughout by $e^{\lambda\tau}$ and expressing it in matrix notation, we get

$$\begin{pmatrix} \lambda^2 - \Omega^0_{\xi\xi} & -\ 2\lambda\omega_0 - \Omega^0_{\xi\eta} \\ 2\lambda\omega_0 - \Omega^0_{\xi\eta} & \lambda^2 - \Omega^0_{\eta\eta} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

It has non-trivial solution if

$$\begin{vmatrix} \lambda^2 - \Omega^0_{\xi\xi} & 2\lambda\omega_0 + \Omega^0_{\xi\eta} \\ 2\lambda\omega_0 - \Omega^0_{\xi\eta} & \lambda^2 - \Omega^0_{\eta\eta} \end{vmatrix} = 0$$

Expanding the determinant, we get

$$\lambda^4 - \left(\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\omega_0^2\right)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 = 0 \qquad (1.114)$$

Equation (1.114) is the characteristic equation corresponding to the variational equations (1.112). If all the four roots of the characteristic equation are distinct imaginary, the EP is stable according to the theorem of Lyapunov, otherwise, it is unstable.

In the case of the EPs on the $\xi\zeta$ – plane (or $xz$ – plane as the case may be), the study of their linear stability is same as when motion takes plane in the $\xi\eta$ – plane. The method requires using the variational equations and computing the partial derivatives.

Now, performing the same routine as done earlier, we suppose that $u = A\exp\ \lambda\tau$ , $v = B\exp\ \lambda\tau$ , $w = C\exp\ \lambda\tau$ ,

where $A, B, C$ and $\lambda$ are constants.

Then, from the variational equations, we get

$$\begin{vmatrix} \lambda^2 - \Omega^0_{\xi\xi} & -2\lambda\omega_0 + \Omega^0_{\xi\eta} & -\Omega^0_{\xi\zeta} \\ 2\omega_0\lambda - \Omega^0_{\xi\eta} & \lambda^2 - \Omega^0_{\eta\eta} & -\Omega^0_{\eta\zeta} \\ -\Omega^0_{\zeta\xi} & -\Omega^0_{\zeta\eta} & \lambda^2 - \Omega^0_{\zeta\zeta} \end{vmatrix} = 0$$

Expanding the determinates, we get

$$\lambda^2-\Omega^0_{\xi\xi}\left[\lambda^2-\Omega^0_{\eta\eta}\quad\lambda^2-\Omega^0_{\zeta\zeta}-\Omega^0_{\zeta\eta}{}^2\right]+2\lambda\omega_0+\Omega^0_{\xi\eta}\left[2\lambda\omega_0-\Omega^0_{\xi\zeta}\quad\lambda^2-\Omega^0_{\zeta\zeta}-\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta}\right]$$
$$-\Omega^0_{\xi\zeta}\left[2\lambda\omega_0+\Omega^0_{\xi\eta}\;\Omega^0_{\zeta\eta}+\Omega^0_{\zeta\xi}\;\lambda^2-\Omega^0_{\eta\eta}-\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta}\right]=0$$

$$\lambda^2-\Omega^0_{\xi\xi}\left[\lambda^4-\lambda^2\Omega^0_{\eta\eta}-\lambda^2\Omega^0_{\zeta\zeta}+\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta}-\Omega^0_{\zeta\eta}{}^2\right]+2\lambda\omega_0+\Omega^0_{\xi\eta}\left[2\omega_0\lambda^2-2\omega_0\lambda\Omega^0_{\zeta\zeta}-\lambda^2\Omega^0_{\eta\xi}\right.$$
$$\left.+\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta}-\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta}\right]-\Omega^0_{\xi\xi}\left[-2\omega_0\lambda\Omega^0_{\zeta\eta}+\Omega^0_{\eta\xi}\Omega^0_{\zeta\eta}+\lambda^2\Omega^0_{\zeta\xi}-\Omega^0_{\zeta\xi}\Omega^0_{\eta\eta}\right]=0$$

$$\lambda^6-\lambda^4\;\Omega^0_{\eta\eta}+\Omega^0_{\zeta\zeta}+\lambda^2\left[\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta}-\Omega^0_{\zeta\eta}{}^2\right]-\lambda^4\Omega^0_{\xi\xi}+\lambda^2\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}-\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta}+\Omega^0_{\xi\xi}\;\Omega^0_{\zeta\eta}{}^2$$
$$+4\omega_0^2\lambda^4-4\lambda^2\omega_0^2\Omega^0_{\zeta\zeta}-2\lambda^3\omega_0\Omega^0_{\xi\eta}+2\lambda\omega_0\;\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta}-\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta}+2\lambda^3\omega_0\Omega^0_{\xi\eta}-2\lambda\omega_0\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta}$$
$$-\lambda^2\Omega^0_{\eta\xi}\Omega^0_{\xi\eta}+\Omega^0_{\xi\eta}\Omega^0_{\eta\xi}\Omega^0_{\zeta\zeta}-\Omega^0_{\xi\eta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta}+2\lambda\omega_0\Omega^0_{\zeta\xi}\Omega^0_{\zeta\eta}-\Omega^0_{\xi\eta}\Omega^0_{\zeta\xi}\Omega^0_{\eta\zeta}-\lambda^2\left(\Omega^0_{\xi\zeta}\right)^2+\left(\Omega^0_{\xi\xi}\right)^2\Omega^0_{\eta\eta}=0$$

Simplifying, we get

$$\lambda^6-\lambda^4\;\Omega^0_{\xi\xi}+\Omega^0_{\eta\eta}+\Omega^0_{\zeta\zeta}-4\omega_0^2+\lambda^2\left[\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}+\Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta}+\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta}-4\omega_0^2\Omega^0_{\zeta\zeta}-\Omega^0_{\zeta\eta}{}^2-\Omega^0_{\xi\xi}{}^2\right.$$
$$\left.-\Omega^0_{\zeta\eta}{}^2\right]-\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta}-2\Omega^0_{\xi\eta}\Omega^0_{\xi\zeta}\Omega^0_{\eta\zeta}+\Omega^0_{\xi\xi}\;\Omega^0_{\zeta\eta}{}^2+\Omega^0_{\xi\eta}{}^2\Omega^0_{\zeta\zeta}+\Omega^0_{\xi\zeta}{}^2\Omega^0_{\eta\eta}=0 \qquad 1.115$$

This gives characteristic equation when motion takes place in the out-of-the orbital plane of motion. The stability of the EP depends on the nature of the characteristic roots. A positive root or a complex root with positive real part of equation (1.115) will induce instability at the EP.

**Lyapunov characteristic numbers**

Lyapunov Characteristic Numbers (LCN) is based on the concept introduced by A.M Lyapunov in 1892. The LCN are a generalization of Lyapunov Type Numbers for ordinary differential equations, and are a tool for discussing the stability of solutions of dynamical systems as well as its orbits, and are a classical method that allows us to distinguish between regular and chaotic dynamical states. The LCN is equivalent to Poincare's characteristic

exponents when orbits are periodic (including librations). They reveal how infinitesimal perturbations on a dynamical system will behave over a long period of time. Especially of great importance is the maximum or largest LCN. The calculations of the LCNs are limited to finding the maximum LCN. This produces an easily computed value that can be used as a metric to give qualitative indication of how stability varies over the solutions. The largest LCN is defined as the limit

$$L\left[ x \;\; t \;\; \right] = \lim_{t \to \infty} \frac{1}{t} \ln \left| x \;\; t \;\; \right|$$

1. When $L > 0$, the solution is unstable.

2. When $L = 0$, this prediction horizon means that solution is stable and corresponds to regular motion in the case of a conservative system of two degrees of freedom.

3. When $L < 0$, the solution is stable.

# CHAPTER TWO

## LITERATURE REVIEW

The R3BP has been treated from many angles by various authors and researchers, some of who have considered the effect of some parameters relating to the shape and nature of orbits of the primaries and of the third body. Others have included perturbing forces of Coriolis, centrifugal, radiation pressure and the effect of the gravitational potential from the disc; while some have looked at the formulation whereby the motion of the third body is confined inside one of the primaries. The investigations of the R3BP with mass variations of one, two or all the masses and studies of the periodic orbits have also received attention.

In this chapter, we review literature of works that have been conducted under the frame of the R3BP. Because of the numerous works we have collected, we divided the chapter into four sections. Section **2.1** is a review of different works on the R3BP with constant masses. The next section highlights contributions of various authors in the investigations of the restricted problem with mass variation. Related studies describing the restricted problem due to Robe (1977), in which the motion of the third body takes place inside the first primary, are given in section **2.3**, while the last section deals with studies of the periodic orbits in the R3BP.

## 2.1. Restricted Three-Body Problem

The R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. Because no general solution in the CR3BP is available, particular solutions are sought to obtain insight into the problem, these particular solution referred to as the equilibrium

or libration points are five for the classical R3BP; two triangular and three collinear. Euler (1767) determined a set of three collinear EPs and Lagrange (1772) found the triangular EPs. Routh (1875) established the condition for linear stability of the triangular EPs. When this condition is satisfied all roots of the characteristic equation are pure imaginary, which leads to pure oscillatory solution. This solution was later referred to as "critical solution" by Lyapunov (1892). The collinear points were found to be unstable in the linear and nonlinear sense, as Routh original result shows, when interpreted in the light of Lyapunov's work.

Poynting (1903) pointed out that small particles approaching or in the vicinity of a luminous celestial body are comparably affected by the gravitational and light radiation forces. The photogravitational R3BP was formulated by Radzievskii (1950). This formulation arises from the classical problem, when one of the interacting masses is an intense emitter of radiation. He discussed it for three specific bodies, the Sun, a planet and a dust particle. He first studied the EPs of the photogravitational problem analogous to the classical case and found that their location strongly depends on the radiation pressure coefficient. First, he considered a planar case (1950) and later, a non-planar one (1953), where he established two additional EPs called coplanar EPs, which he denoted by $L_6$ and $L_7$.

Leontovic (1962) investigated the critical case established by Routh (1875) occurring in the stability of the triangular EPs laying his basis on a theorem by Arnold (1961) and showed that under certain conditions, these equilibrium solutions are stable in the nonlinear sense.

The stability of the five EPs of the classical R3BP was extensively studied by Szebehely (1967a). It was seen that the triangular point can be stable under some conditions while the collinear EPs are unstable. Later, Szebehely (1967b) examined the effect of a small perturbation

of the Coriolis force on the stability of the EPs, keeping the centrifugal force constant. In his study, he maintained that the collinear points remain unstable and obtained in the case of the triangular EPs a relation between the critical value of the mass parameter $\mu_c$ and the change $\in$ in the Coriolis force:

$$\mu_c = \mu_0 + \frac{16 \in}{3\sqrt{69}},$$

This made him conclude that the Coriolis force is a stabilizing force, since the region of stable motion increased.

The work of Radzievskii (1950) was extended by Chernikov (1970). He achieved this by including aberrational deceleration (the Poynting-Robertson effect), and found that despite the absence of a Jacobi integral, the equations of motion admit of particular solutions corresponding to six EPs .

SubbaRao and Sharma (1975) knowing that the participating celestial bodies are not strictly spherical in shape examined the stability of the triangular points when the first primary is an oblate spheroid. They observed that due to the oblateness, the region of stable motion decreases. Later, Sharma and SubbaRao (1976) examined the stationary solutions and their characteristic exponents in the R3BP when the first primary is an oblate spheroid. The first investigation of the stability of the coplanar solutions $L_6$ and $L_7$ in the linear approximation was carried out by Perezhogin (1976) with the help of computer calculation of the roots of the characteristic equation; it was shown that for all possible values of the Sun's system parameter, these libration points are unstable according to Lyapunov.

The work of Szebehely (1967b) was extended by Bhatnagar and Hallan (1978), by putting under consideration the effect of perturbations in the Coriolis and centrifugal forces, and found that collinear points remain unstable; for triangular EPs they obtained the relation

$$\mu_c = \mu_0 + \frac{4\ 36\in -19\in'}{27\sqrt{69}}$$

They inferred that the range of stability increases or decreases depending on whether the points $\in,\in'$ lies in one or the other of the two parts in which the $\in,\in'$ plane is divided by the line $36\in -19\in' = 0$. Schuerman (1980) discussed the restricted problem of three-bodies when the force of radiation pressure and Poynting-Robertson effect (PR-drag) are considered. He found that the PR-drag effect rendered the triangular points $L_4$ and $L_5$ unstable on a time scale long compared to the period of rotation of the two massive bodies. In 1982, Sharma extended the work of SubbaRao and Sharma (1975) by considering the linear stability of triangular EPs of the R3BP when the first primary is a source of radiation and an oblate spheroid as well. He found that the eccentricity of the conditional retrograde elliptical periodic orbits around the triangular points at the critical mass $\mu_c$ increases with the increase in the oblateness coefficient and the radiation force; and becomes unity when $\mu_c$ is zero. Simmons *et al.* (1985) obtained a complete solution of the R3BP. They discussed the existence and linear stability of the EPs for all values of radiation pressures of both luminous bodies and all values of mass ratios, and found that the coplanar points $L_6$ and $L_7$ can be stable for all values of the mass ratio $\upsilon$. The study of the stability of the relative EPs (coplanar points) of the R3BP for a particle in a gas-dust cloud subject to the gravitational field of a binary star system, with allowance for radiation pressure was investigated by Kunitsyn and Tureshbaev (1985), where the existence domain is established

for three-parameter family of coplanar EPs. Elipe and Ferrer (1985) studied the EPs in the circular planar restricted three rigid body-problem. They obtained explicit equations which allow collinear and triangular solutions which in general are not equilateral; also, they established numerically that the influence of non-sphericity of the primaries brings about a significant shift in location of the EPs when compared to the classical case.

A discussion of the results obtained by Simmons *et al.* (1985) on the stability for $\upsilon \neq 0.5$ was initiated by Tureshbaev (1986). He examined the stability of the coplanar libration points in the photogravitational problem for a particle in a gas-dust cloud in the gravitational field and radiation pressure of binary a star whose components have arbitrary mass ratio and obtained domain of the stability of the coplanar point, maintaining that the points can be stable in a special case where just one body radiates as opposed to the contention of Simmons *et al.* (1985) that the points can only be stable if both gravitating bodies are sources of radiation.

El-Shaboury (1990) gave a possibility of nine EPs for small values of oblateness in the photogravitational R3BP when the infinitesimal mass is an axisymmetric body and one of the finite masses is a spherical luminous body, while the other is assumed to be an axisymmetric non-luminous body. Later, El-Shaboury *et al* (1991) established numerically, a shift in the location of EPs due to non-sphericity (triaxiality) and rotation of the primaries, when compared to the classical R3BP. Todoran (1993) claimed that the out-of-plane EPs in the R3BP as concern radiation pressure, do not actually exist. This question was answered by Ragos and Zagouras (1993).

Ragos *et al* (1995) in a further paper investigated the photogravitational CR3BP including the Poynting-Robertson effect to describe the motion of a particle in the vicinity of two massive

radiating bodies. They studied numerically the EPs lying out of the orbital plane of the primaries, and determined the exact number of these points by means of topological degree theory. Kunitsyn and Polyakhova (1995) discussed the modern state of the R3BP when an infinitesimal mass is affected not only by gravitation but also by light pressure from one (or both) of the primaries. Many aspects of the locations and stability of the EPs for all values of radiation pressure and mass ratios were shown and discussed. A retrospective chronological review of the results was also given.

Singh and Ishwar (1999) generalized the R3BP by considering both primaries to be source of radiation and as well oblate spheroid. They observed that the equations of motion and locations of the equilibrium libration points are affected by the radiation pressure forces and oblateness of the primaries. They found that the triangular points are stable for $0 < \mu \le \mu_c$ and unstable for $\mu_c < \mu \le \frac{1}{2}$, and further established that the range of stability depends upon the radiating and oblateness coefficients. In the same year, Khanna and Bhatnagar (1999) investigated the R3BP by modeling the first primary as an oblate spheroid and the second primary as a triaxial rigid body; they maintain the unstable nature of the collinear points and the stable nature of the triangular points for some values of mass parameter; They further observed that the triangular points have long or short periodic elliptical orbits in the same range of the mass parameter. Sharma *et al.* (2001) examined the existence and stability of EPs in the R3BP when the primaries are triaxial rigid bodies and as well source of radiation with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. They established five EPs, two triangular and three collinear. The collinear points are found to be unstable while the triangular points are conditionally stable. The phenomenon of the gravitational capture using the

models of the circular and elliptical R3BP was studied by Mako and Szenkovits (2004). Celletti (2004) studied the frequency analysis of the stability of asteroids in the framework of the R3BP. Erdi (2004) have shown the global regularization of the restricted problem of three bodies.

The R3BP in the presence of a disc was studied by Jiang and Yeh (2004). In their study, they explored possible chaotic and regular orbits for disc-star-planet systems and found that the influence from the disc can change the locations of the EPs and the orbital behavior. Later, Jiang and Yeh (2006), Yeh and Jiang (2006) in these configurations discovered that, in addition to Lagrange points, new EPs can exist under some conditions.

Papadakis (2005) examined the motion around the triangular EPs of the R3BP under angular velocity variation. Ishwar and Kushvah (2006) examined the linear stability of the triangular EPs in the generalized photogravitational R3BP with Poynting-Robertson drag. They found that the triangular points $L_{4,5}$ become unstable due to the Poynting-Robertson effect, but conditionally stable once the P-R drag is ignored.

The combined effect of perturbations, radiation and oblateness on the stability of EPs in the R3BP was studied by AbdulRaheem and Singh (2006). The model was studied under the influence of small perturbations in the Coriolis and centrifugal forces together with the effects of oblateness and radiation pressure of the primaries. They found that the collinear points remain unstable, while the triangular points are stable for $0 < \mu \le \mu_c$ and unstable for $\mu_c < \mu \le \frac{1}{2}$. They observed further that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation and oblateness of the primaries have destabilizing effects; the presence of any one or more of the latter makes weak the stabilizing ability of the former, consequently the overall

effect is that the range of stability of the triangular points decreases. The combined effects of oblateness and radiation of the primaries on the nonlinear stability of the triangular EPs was studied by Singh (2008). It was found that, in the nonlinear sense, the triangular point $L_4$ is stable for all mass ratios in the range of linear stability except for three mass ratios

$$\mu_{c_1} = 0.0242... - 0.1790...A_1 - 0.0368...A_2 - 0.3828... \in +0.0147... \in'$$
$$\mu_{c_2} = 0.0135... - 0.0993...A_1 - 0.093...A_2 - 0.3641... \in +0.0079... \in'$$
$$\mu_{c_3} = 0.0109... - 0.1036...A_1 + 0.5456...A_2 - 0.4748... \in -0.1432... \in'$$

at which the Moser theorem fails.

Kushvah (2008a) studied the linear stability of the EPs in the generalized photogravitational Chermnykh's problem. He considered the first primary as a source of radiation and the second one as an oblate spheroid, and examined the effect of radiation pressure, oblateness and gravitational potential (from the belt) analytically and numerically. The stability of the EPs was investigated using method of Lyapunov, and it was found that collinear points are linearly unstable, while the triangular points are conditionally stable. In another work (Kushvah 2008b) of his, the effect of radiation pressure on the equilibrium points in the generalized photogravitational R3BP was examined. In this work, the existence of EPs and the effect of radiation pressure have been discussed numerically. The problem is generalized by considering the first primary as a source of radiation and second primary as an oblate spheroid. The work also discussed the Poynting- Robertson (P-R) effect which is caused due to radiation pressure, and it was seen that the collinear EPs deviate from the axis joining the two primaries, while the triangular points are not symmetrical due to radiation pressure. They found the collinear points to be linearly unstable while the triangular points are conditionally stable in the sense of Lyapunov

when P-R effect is not considered. Sosnitskii (2008) investigated the stability of triangular lagrangian points in the R3BP. The study explored the stability of triangular EPs in the spatial (three-dimensional) CR3BP. For the analysis of stability, they derived new equations coupling up distances from an infinitesimally small particle to the first and second primaries. The theorem on the instability of triangular EPs and relations between Lyapunov stability and Birkhoff stability, were discussed.

In (2009), Kushvah looked at the Poynting-Robertson effect on the linear stability of equilibrium points in the generalized photogravitational Chermnykh's problem, while the nonlinear stability of the problem studied by AbdulRaheem and Singh (2006) was given by Singh (2009). It was found that in the nonlinear sense, the triangular point $L_4$ is stable for all mass ratios in the range of linear stability except for three mass ratios at which the Moser theorem fails. Singh and Begha (2011a) generalized the restricted problem to include the first primary as a triaxial rigid body while the second one is an oblate spheroid under the influence of small perturbation in the Coriolis and centrifugal forces. They asserted that the position of the triangular and collinear EPs were affected by the non-sphericity of the primaries and the change in the centrifugal force. The stability of the triangular point was investigated and it was seen that triangular points are stable for $0 < \mu \le \mu_c$ and unstable for $\mu_c < \mu \le \frac{1}{2}$, where $\mu_c$ is the critical value of the mass parameter which depends on the joint effect of perturbations, oblateness and triaxiality. The collinear points were found to be unstable in the linear approximation. Ershkov (2012) analyzed the Yarkovsky effect in generalized photogravitational R3BP.

Singh and Taura (2013) investigated the motion of an infinitesimal body in the generalized R3BP. Their problem was generalized in the sense that both primaries are radiating, oblate

bodies, together with the effect of gravitational potential from a belt. The equations of motion were derived and the positions of the EPs were found. It was seen that in addition to the usual five EPs, they found two new collinear points occurring due to the potential from the belt. The linear stability analysis of the EPs show that collinear EPs remain unstable, while the triangular points are stable under some conditions influenced by the oblateness, radiation of the primaries and potential from the belt. A practical application of their model is the study of the motion of a dust particle near the oblate, radiating binary stars systems surrounded by a belt. Kishor and Kushvah (2013) studied the linear stability and resonances in the generalized photogravitational Chermnykh-like problem with a disc. Rambaux (2013) studied the rotational motion of the second largest body "Vesta" of the main asteroid belt, based on its orbital perturbations and its large triaxial shape. The existence and stability of the EPs in the R3BP when one of the primaries is an oblate spheroid, was studied by Javed *et al.* (2013). They observed that there exist three EPs $L_1$, $L_2$ and $L_3$ out of which, $L_1$ is a collinear $L_{2,3}$ are non-collinear EPs. In the stability analysis of the problem, the non-collinear EPs were found to be stable when the mass parameter and the critical value of mass parameter is falls within the range of 0.0267906, while the collinear EPs is unstable in the entire range of the mass parameter. Also, Arif (2013) examined the motion around the EPs in the planar magnetic-binaries problem under the effect of the gravitational forces of the primaries on the small body. The third body was assumed to be a charged particle. In some particular cases, they observed that the orbit around $L_1$ and $L_3$ is an ellipse while around $L_4$, the peculiar looping nature of the particle's path exists. They also noted that the length of the major-axis and orbital period of all the orbits around $L_1$ and $L_3$ decreases as the mass ratio increases. They further studied the motion of the third body in the vicinity of the

second primary, by making some assumptions and transferring the origin of the coordinate system to the second primary. In this case, it was observed that there exist two and three collinear EPs under some respective conditions.

Singh and Umar (2014) have studied the collinear EPs in the elliptic R3BP with a triaxial primary and an oblate secondary. In their work, they examined the motion of a dust grain around a triaxial primary and an oblate companion orbiting each other in elliptic orbits about their common barycenter in the neighborhood of EPs. The positions and stability of these points were found to be affected by the triaxiality and oblateness of the primaries, and by the semi-major axis and eccentricity of their orbits. The stability behaviors of the collinear points remain unchanged; as they are unstable in the Lyapunov sense. Jain *et al*. (2014) performed an analysis of the dynamics of the CR3BP under the effect of the dissipative force, Poynting Robertson drag. They investigated the existence and stability of the EPs and observed that there exist only two non-collinear EPs which are unstable in the linear sense. Finally, Singh and Haruna (2014a) studied the R3BP when the bodies are oblate spheroid and the primaries are radiation sources under effects of small perturbations in the Coriolis and centrifugal forces.

## 2.2. Restricted Three-Body Problem with Variable Masses

The classical R3BP assumes that the masses of celestial bodies are constant. However, the phenomenon of isotropic radiation or absorption in stars led scientists to formulate the restricted problem of three bodies with variable mass. During evolution, the masses of celestial bodies change, especially in a double star system were masses change rather intensively. As an example, we could mention the motion of rockets, black holes formation, motion of a satellite around a radiating star surrounded by a cloud and varying its mass due to particles of the cloud,

and comets loosing part or all of their mass as a result of roaming around the Sun (or other stars) due to their interaction with the solar wind which blows off particles from their surfaces. The problem of the motion of astronomical objects with variable mass has many interesting applications in stellar, galactic, and planetary dynamics.

The problem of two bodies with variable masses came into science practically following the work of Gylden (1884), who for the relative motion of one mass point $m_2$ about the other mass point $m_1$ under the action of mutual gravitational force represented the sum of the masses of these points as varying with time by a certain law $m_1 + m_2 = \mu\ t$ . Later, Dufour (1886) examined the astronomical phenomena of variable mass relating the secular variation of lunar acceleration with the increase of the Earth's mass due to the impact of meteorites. Soon afterwards, Mestschersky (1893, 1902) showed that the Gylden problem is a particular case of the problem of two bodies with variable masses under the condition that the laws of variation of the masses vary isotropically. This problem was rightfully referred to as the Gylden-Mestschersky problem (GMP).

The first investigation of the existence of the EPs for variable masses in the absence of reactive forces was performed by Orlov (1939), in which the plane problem of three bodies with finite variable masses was considered, and the existence of five analogous particular solutions was established. Sersic (1970, 1973) demonstrated the existence of particular solutions in the R3BP, in which the motion of the primary, variable-mass bodies occur along a straight line passing through the center of mass of the system. Gelf'gat (1973) examined the R3BP of variable mass in which the primary bodies move within the framework of the GMP and assumed that the isotropic mass variation of the masses of the primaries occurs in accordance with UML.

He established the existence of three collinear and two triangular EPs analogous to the classical type $L_i$ $i = 1, 2...5$ . Omarov and Mukhametkalieva (1982) studied the evolution of an n-body stellar system with variable mass, describing the behavior of the interacting groups of galaxies by means of numerical methods. It was shown that, if the module of the initial potential energy is much greater than the initial kinetic energy, the evolution of the system will explode and as a result, the velocities of the individual galaxies may increase. The equations of motion of the CR3BP with variable mass under the assumption that the mass of the infinitesimal body vary with respect to time was established by Shrivastava and Ishwar (1983).

Horedt (1984) examined the motion around the triangular EPs $L_4$ and $L_5$ with mass exchange of the primaries within the framework of Brown's theory. The result obtained confirms those of the isotropic mass variation, although the assumed mass variation is considerably less. It was established that the extrema of the elongation with respect to the small mass are not affected by mass exchange and that the equations for the extrema of the Trojan's distance from the sun and for the liberation period are formally the same as in the constant mass problem, but with the understanding that the masses are now time dependent quantities.

Singh and Ishwar (1984, 1985) investigated the effect of small perturbation in the Coriolis and centrifugal forces on the location and stability of EPs in the R3BP with variable mass under the assumption that the third infinitesimal mass is variable and the primaries are spherical with constant masses. The classification of different cases of the problem of two-bodies with variable mass was studied by Razbitnaya (1985). He gave a classification of the physically different cases of the problem on the basis of the DE of relative motion in the problem of two-bodies with variable masses.

Dyakov and Reznikov (1986) studied the motion in the vicinity of triangular EPs when the mass ratio of the components is a variable. They specified by investigating Trojan orbits in a system having components with variable mass ratio within the framework of the CR3BP, and found that when the mass variation over the period of the libration is slight, an increase in the relative mass $\mu$ of the smaller component above a certain limit (lower than the critical value $\mu = 0.0385$ and depends only on the initial deviation from the EPs) result in the cutoff of librations. They investigated the stability and established numerically the regions of relative stability of the motion around the points for different mass ratios. Further they used the results obtained to evaluate the hypothesis that the Trojans originated as former satellites of Jupiter.

The coplanar EPs $L_6$ and $L_7$ for variable mass were first found by Bekov (1988) after he worked on the same problem of Gelf'gat (1973). These EPs are located outside the plane of revolution of the primary bodies, with isotropic mass variation of the masses of the primaries. Later, on the particular solution of same formulation as Gelf'gat (1973) was investigated by Luk'yanov (1989). He found four new coplanar solutions denoted by $L_8$, $L_9$, $L_{10}$ and $L_{11}$, and further sought for the possibility of the existence of infinitely remote solutions $L_{\pm\infty}$. El-Shaboury (1990) investigated the R3BP with variable masses under the assumptions that the infinitesimal body is axisymmetric with constant masses and the primary bodies are spherical with variable masses such that the ratio of their masses remains constant. It was also assumed that the motion of the primaries is determined by the GMP. He found that the collinear $L_{123}$, triangular $L_{4,5}$, and coplanar $L_{6,7}$ EPs do not exist when the infinitesimal body is axisymmetric, but exist when it is spherical.

The stability of the EPs in the R3BP with variable masses of the primaries with respect to time according to the UML, with the consideration that the motion of the variable mass primary body is within the framework of the GMP, was studied by Luk'yanov (1990). He showed that the EPs (collinear, triangular and coplanar) for any parameter are stable with respect to the coordinate introduced by Mestschersky (1952). Minglibaev (1990) investigated the average R3BP with variable mass, while Lu (1990) studied the stability of the triangular points in the R3BP with variable mass, and found that the region of stability of the triangular points does not exist when the restricted problem with constant mass evolves into the problem with variable mass.

The particular solutions in the restricted collinear three-body problem with variable masses were considered by Bekov (1991). In this case, it was also assumed that the motion of the two primary bodies is in accordance with the GMP and the time variation of the masses of the primaries is determined by the UML. He found the collinear solutions $L_{1,2,3}$ and the spatial solution $L_0$ (a Lagrangian ring) for different time dependency of the masses and established the domain of existence of these solutions.

The EPs and Hill surfaces in the same formulation as Gelf'gat (1973) and Bekov (1988) were studied by Bekov (1992) using a Gelf'gat transformation in the autonomization of the equations of motion with variable coefficients, to a system of equations with constant coefficients. He investigated the arising and disappearance of collinear $L_{1,2,3}$, triangular $L_{4,5}$, coplanar $L_{6,7}$, ring $L_0$ and infinitely distant $L_{\pm\infty}$ solutions and the Hill surfaces. The possibility of applying the results obtained to non-stationary double stellar systems was discussed. Bekov (1993a) studied the periodic solutions of the GMP. Further study of the appearance and disappearance of

collinear, triangular, coplanar, ring and infinitely distant solutions in the restricted problem of three variable-mass bodies for different time dependences of the masses of the main bodies and for some additional conditions imposed on the system parameters were considered by Bekov (1993b). An important role in the dynamic evolution of real gravitating systems is their non-stationarity, connected with mass variation of the system and the additional influence of the variable light pressure from the system's components. Debes and Sigurdsson (2002) focused on the stability of multi-planet systems and link stellar mass loss to instability timescales and they demonstrated how multiple planets beyond the reach of the star's expanding envelope became unstable.

Gasanov (2008) investigated the libration points and the general case in the problem of the motion of a star inside a layered inhomogeneous elliptical galaxy with variable mass and established seven EPs of the autonomized equations, located (except for one) outside the gravitating galaxy. He studied the Lyapunov stability of these points using the concept of the LCN. Singh and Leke (2010) investigated the stability of the restricted problem when the primaries are radiation emitters and their masses vary isotropically with time in accordance with the UML with the inclusion that their motion is described by the GMP. Seven EPs were established for both the autonomized and the time-dependent systems. They investigated the stability of the EPs of the time dependent system using the LCNs and found the EPs to be unstable.

The effect of the isotropic variation of the mass of the star in a planetary system and the possible ejection of a planet from the system was studied by Veras *et al*. (2011). Letelier and Da Silva (2011) studied the particular solutions of the R3BP with variable masses. In this study, the particular solutions to the R3BP where the bodies are allowed to either lose or gain mass to or

from a static atmosphere, were found. In the case that all the masses are proportional to the same function of time, they found analogous solution to the five stationary solutions of the usual R3BP of constant masses: the three collinear and the two triangular solutions, however, the relative distance of the bodies change with time at the same rate. Further, they observed that under some restrictions, there are also coplanar, infinitely remote and ring solutions. The triangular EPs in photogravitational R3BP with variable mass, in which both the attracting bodies are radiating as well and the infinitesimal body vary its mass with time according to Jeans' law, was studied by Zhang *et al.* (2012). In this study, they applied the space-time transformation of Mestschersky (1949) and obtained the differential equations of motion of the problem. They obtained the triangular equilibrium points and found that the triangular points are unstable in the linear sense when the problem with constant mass evolves into the problem with decreasing mass.

## 2.3. Robe Restricted Three-Body Problem

A different formulation of the R3BP was worked by Robe (1977). This problem was later called the Robe's R3BP. In this formulation, the first primary of mass $m_1$, is a rigid spherical shell, filled with homogenous, incompressible fluid of density $\rho_1$, with the second mass $m_2$ as a small point outside the shell and moving around the first primary mass in a Keplerian orbit. The third body of infinitesimal mass $m_3$ is taken as a small sphere of density $\rho_3$, moving inside the shell and is subject to the attraction of $m_2$ and the buoyancy force due to the fluid and that the radius of $m_3$ is assumed to be infinitesimal . Robe found an EP at the center of the shell and discussed the linear stability in two cases. In the first case the orbit of $m_2$ around $m_1$ is circular

and in the second case, the orbit is elliptic, but the shell is empty (there is no fluid inside it) or densities of $m_1$ and $m_3$ are equal.

Shrivastava and Garain (1991) investigated the effect of small perturbation in the Coriolis and centrifugal forces on the location of EP in the Robe's CR3BP with consideration that the densities of fluid inside the rigid shell and the third body are equal; in other words, the shell is empty. They observed that the only EP of the problem is $\left(-\mu+\frac{\mu\in'}{1+2\mu},\ 0,\ 0\right)$ and this point depends on the mass parameter and the small perturbation in the centrifugal force. Plastino and Plastino (1995) reanalyzed the Robe's R3BP with a new assumption by considering the first primary as a Roche spheroid in an equilibrium state of a rotational fluid and called the formulation the "Robe-Roche restricted model". They explored its EPs and found that when the density parameter is taken as zero, every point inside the shell is an EP and further analyzed the connection between the effect of the buoyancy forces and a perturbation of the Coriolis force. They also showed that the center of the spheroid is an EP of the Robe-Roche R3BP and it is linearly stable. However, their approximation of the potential due to the second primary was only up to the second degree terms only in the variables involved. Giordano *et al.* (1997) discussed the effect of drag force on the stability of the EPs in the Robe problem (1977). They found that due to drag forces, four regions of stability are changed to instability. The trio also considered the effects on drag forces to the formulation discussed by Plastino and Plastino (1995), and found that in this case all the regions of stability remain unchanged.

The effect of small perturbations $\in$ and $\in'$ in the Coriolis and centrifugal forces respectively on the location and stability of the equilibrium points in the Robe's CR3BP problem as in the case examined by Shrivastava and Garain (1991) was again reconsidered by Hallan and Rana (2001a), when the density of the host primary and that of the third body is equal (i.e., the density parameter being zero). They showed by using a different method, that only one EP which lies on the right or left of the center of the shell on the line joining the center of the shell and the second primary accordingly as $\in'$ is positive or negative and the change $\in'$ in the Coriolis force does not affect the location of the EP. They found that the point is stable in the linear sense for $\mu_C < \mu < 1$ and unstable for $0 < \mu \le \mu_C$ where $\mu_C = \frac{8}{9} + \frac{2}{3}\left[\left(\frac{45}{3}\in' - \frac{10}{3}\in\right)\right]$ and found that the range of stability increases or decreases depending upon whether the point $(\in, \in')$ lies in one or the other of the two regions in which $(\in, \in')$ plane is divided by the line

$$\frac{43}{25}\in' - \frac{10}{3}\in = 0$$

The existence of all the EPs, their location and stability in the Robe's (1977) R3BP was studied by Hallan and Rana (2001b). It was seen that the center of the first primary is always an EP, whatever be the values of the density parameter $D_p$, eccentricity parameter $e$ and the mass parameter $\mu$. The other EP exist only when $D_p \neq 0$ and $e = 0$ that is when the second primary, a mass point moves around the first, a spherical shell filled with fluid, in a circular orbit. They further asserted that when $D_p > 1$, there is one additional EP lying on the line joining the center of first primary and the second primary. When $D_p + \mu = 1$, there are infinite number of EPs in the $x - y$ plane lying on the circle of radius one and center as the second primary, provided the

points are inside the spherical shell. When $D_p < 0$ and $D_p + \mu > 0$, there are two or more EPs lying in the $x-z$ plane forming triangles with the center of the shell and the second primary. In regards to the stability of these EPs, they confirmed the stability result given by Robe (1977) for the point $-\mu, 0, 0$ , whereas triangular and circular EPs are always unstable. The point collinear with the center of the shell and the second primary was found to be stable provided $\mu$ and $D_p$ satisfy the inequality

$$16\mu(D_p - 1)^2 < \left[\mu + \sqrt{\mu\ -4 + 4D_p + \mu}\right]^3$$

Thus, concluding that the Robe's elliptic R3BP has only one EP for all values of the parameters $D_p$ and $\mu$, while the Robe's CR3BP can have two, three or infinite number of EPs.

The locations and linear stability of the EPs in Robe's R3BP under perturbed Coriolis and centrifugal forces was investigated by Shu *et al.* (2005). Their effort was aimed at improving the work of Shrivastava and Garain (1991) as they argued that the assumption that the densities of the first primary and the infinitesimal mass are equal is hard to be fulfilled in practice. Hence, their study regarded the density parameter as $D_p \geq 0$ $\rho_1 \leq \rho_3$ . They found a unique EP point $(x_0, 0, 0) = \left(-\mu + \frac{\mu \in'}{1 + 2\mu - D_p}, 0, 0\right)$ when $\beta - D_p \geq 0$ and estimated the perturbation magnitude of their locations and linear stability. Hallan and Mangang (2007) investigated the existence and stability of EP in the Robe's R3BP by considering the full buoyancy force as in Plastino and Plastino (1995) and assuming the hydrostatic equilibrium figure of the first primary as an oblate spheroid. They derived the pertinent equations of motion and found that there is an

EP near the centre of the first primary. Further they found that there exist another EP on the line joining the centre of the first primary and the second primary, and an infinite number of EPs lying on a circle in the orbital plane of the second primary. The second EP on the line joining the centers of the primaries was found to be conditionally stable while the circular EPs are unstable.

AbdulRaheem (2011) studied the effect of radiation and oblateness of the primaries on the stability of collinear points in the Robe's R3BP and it is observed that the variations of the density parameter $D_p$ with the mass parameter $\mu$ for constant radiation and oblateness factors on the stability of the collinear points $L_1, L_2, L_3$, are unstable for $D_p > 0$ and stable for $D_p < 0$.

Singh and Sandah (2012) studied the existence and linear stability of EPs in the Robe's R3BP with oblateness. They carried out their study under the assumptions that the hydrostatic equilibrium figure of the first primary is an oblate spheroid and the second primary is an oblate body as well. They found EP near the centre of the first primary and one more EPs on the line joining the centers of both primaries. Points on the circle within the first primary are also EPs under certain conditions and they pointed out the existence of two out-of-plane points. They observed that the points near the center of the first primary are conditionally stable, while the circular and out of plane equilibrium points are unstable. Later that year, Singh and Laraba (2012) examined the Robe's CR3BP when the first primary is a fluid in the shape of an oblate spheroid and the second primary is a triaxial rigid body while Kaur and Aggarwal (2012) investigated the Robe's problem of 2+2 bodies and applied it to the study of the motion of two submarines in the Earth-Moon system.

The stability of triangular points of the generalized photogravitational Robes R3BP was studied by AbdulRaheem (2013a). The study examined the linear stability of the triangular points under

the frame of the Robes R3BP when the first primary (rigid shell) is oblate spheroid and the second primary is radiating. The study showed that the critical mass obtained depends on the oblateness of the rigid shell and radiation of the second primary as well as the density parameter of the fluid. Under some conditions, the destabilizing tendencies of the oblateness and radiation factors are enhanced or weakened. In a further paper, AbdulRaheem (2013b) considered the same problem with the assumptions that the primaries have the shape of an oblate spheroid. Singh and Omale (2014) studied the Robe's CR3BP with zonal harmonics.

## 2.4 Periodic Orbits in the Restricted Three-Body Problem

Due to the rotational motion, periodic orbits exist in the R3BP, long and short periodic orbits are found around the EPs. Elements that could be used to describe the motion of test particle relative to the primary and secondary bodies are categorized as orbital and non orbital elements. Angular momentum and total energy are the integrals available to measure the shapes and sizes of the orbits but are not directly observable. Therefore, eccentricities, inclination and semi major axes of the orbits are used to determine the shapes, orientation and sizes of the orbits. Periodic orbits are also used as reference orbits. Enke and Hill have done so, the former using an ellipse and the later the variational orbit as a reference orbit. The study of periodic orbits is a very useful tool for the study of non-integrable dynamical system, because they determine critically the structure of phase space, while the study of periodic orbits, which are close to actual motions, plays an important role in the understanding of the general properties of such a system. Through the study of periodic orbits, one can understand the role of other forces (i.e., drag, tidal etc.) besides gravitational forces. Szebehely's (1967a) famous book gives a fairly good idea about the periodic orbits. Many Mathematicians and astronomers have worked on different types of

periodic orbits in the restricted problem.

Merman (1952) constructed a new class of periodic solutions for the restricted problem and the Hill problem. Meyer and Schmidt (1971) obtained periodic orbits around triangular EPs for mass ratios near the critical mass ratio of Routh. Hadjidemetriou (1975a) discussed the continuation of periodic orbits from the restricted to the general 3BP. In (1975b), he studied the family of periodic orbits in the planar three-body problem. In (1975c) he also developed a method to study the stability of periodic motions of three-body problem in a rotating frame of reference, based on the motion of surface of section. Sharma (1976) established the existence of periodic orbits of the first kind in the R3BP when the first primary is an oblate spheroid.

Gomez and Noguera (1985) gave some numerical results about natural families of periodic orbits, which emanate from limiting orbits around the equilateral EPs of the R3BP when the mass ratio is greater than Routh's critical one. Ragos and Zagouras (1988); studied the periodic solution about the "out of plane" EPs in the Photogravitational R3BP while Zagouras (1991) considered the effect of radiation pressure on the periodic motion of small particle in the vicinity of the triangular EPs of the R3BP and also constructed the second order parametric expansion and determined the family of the periodic orbits numerically for two sets of values of the mass and radiation parameters corresponding to the non-resonant and the resonant case. Elipe and Lara (1997) discussed periodic orbits in the R3BP by taking both the primaries as the source of radiation pressure. By means of some modifications to the method of numerical continuation of natural family of periodic orbits, they found several families of periodic orbits, both in two and three dimensions. Das *et al.* (1999) have shown the existence of periodic orbits of collision in relativistic CR3BP. Poddar (2002 a, b), has shown the existence of periodic orbits, symmetric as well as doubly symmetric when the primaries are triaxial rigid bodies. Corbera and Llibre (2003)

have drawn the periodic orbits of a collinear R3BP. Perdios (2003) studied critical symmetric periodic orbits in the photogravitational R3BP in which the first primary is a source of radiation. This work was later extended by Perdios and Kalantonis (2006) when the first primary is an oblate spheroid. Kalantonis *et.al* (2006) studied the numerical determination of homoclinic and hetroclinic orbits at collinear EPs in the R3BP with oblateness.

The investigations of the combined effects of perturbations, radiation and oblateness, on the periodic orbits in the neighborhood of linearly stable triangular EPs, was carried out by AbdulRaheem and Singh (2008). Later, Singh and Begha (2011b) examined the periodic orbits around the triangular EPs of the R3BP to include the first primary as a triaxial rigid body while the second one is an oblate spheroid under the influence of small perturbation in the Coriolis and centrifugal forces. Singh and Haruna (2014b) generalized the study of the periodic orbits around triangular points in the restricted problem of three oblate bodies. They modeled the bodies as oblate spheroids, under effect of radiation of the main masses and small change in the Coriolis and centrifugal forces. Their study generalized an earlier paper by AbdulRaheem and Singh (2008) with the inclusion that the third body, due to rapid spinning, changes its shape from being a sphere, to an oblate spheroid. They found that the orbits around these points are ellipses with long and short periodic orbits and observed that the consideration of the particle as an oblate spheroid affects the period, orientation, eccentricities, and the semi-major and semi-minor axis of the orbits.

# CHAPTER THREE

## 3.0 RESTRICTED PROBLEMS OF THREE BODIES WITH VARIABLE MASSES

## 3.1 Introduction

In this chapter of the thesis, we study the restricted problem of three-bodies with variable masses using the model of the general R3BP and the Robe's R3BP. This chapter is divided into **Nine** sections. Sections **3.2**, **3.3**, **3.4** and **3.5** investigate the R3BP when the masses of the primaries vary isotropically in accordance with the unified Mestschersky (1952) law and their motion governed by the Gylden-Mestschersky problem (Gylden 1884; Mestschersky 1902) under various characterizations which includes disc, oblateness, radiation, and deflections of the primaries. The first problem in section **3.2**, studies the case when the first primary is taken to be an oblate spheroid while the second one is a radiating body. Small change in the Coriolis and centrifugal forces are considered to be effective. The second problem in section **3.3** is an extension of the first problem and deals with the case whereby both primaries are considered to be oblate spheroids and radiating as well under effects of small perturbations in the Coriolis and centrifugal forces. The periodic orbits around linearly stable triangular points are discussed in section **3.4** while the study of the R3BP with variable masses in the presence of a circumbinary disc forms the third problem and is studied in section **3.5.**

The remaining sections of this chapter discuss the model of the Robe (1977) when the masses are varying with time. Section **3.6** considers the case when the first primary is a rigid spherical shell with no fliud and the third body is located inside it while the second primary orbits the first. The masses of both primaries are assumed to vary with time in accordance with the UML and their

motion determined by the GMP, while the masses of the third body is kept constant. In section **3.7**, we extend the the problem in section **3.6** by considering slight deflections due to the Coriolis and centrifugal forces. The final problem of this chapter examines the restricted problem when the three bodies in the configuration vary their masses at the same rate in a non-isotropic manner. In this problem, the first primary is taken to be a fluid in the shape of a sphere so no need to assume a rigid shell, since in reality most celestial bodies both in our solar and stellar systems do not have rigid coverings. The third body is situated inside the first primary, while the second primary orbits the first. We applied this problem to the dynamics of a submarine under the influence of the Earth and Moon's gravitational influence. We draw the conlusion of the chapter in section **3.9.**

## 3.2. Equilibrium Points and Stability in the R3BP with Perturbing Forces, Oblateness and Variable Masses

In this problem, we extend the model given by Bekov (1988) by assuming that the first primary has the shape of an oblate spheroid and the second one is a source of radiation, under effects of small perturbations in the Coriolis and centrifugal forces. The problem is applied to the binary Cen X-4, which is a low mass X-ray binary consisting of a neutron star and a white dwarf secondary star.

### 3.2.1. Equations of motion of the third body

Let us consider a rotating frame of reference $Oxyz$ , where O is the origin. Let us suppose that $m_1$ and $m_2$ are the masses of the primaries bodies and $m_3$ be the mass of the infinitesimal body. We have taken the line joining $m_1$ and $m_2$ as the $x-$ axis. The coordinates of $m_1$ and $m_2$ are $(x_1,0,0)$ and $(x_2,0,0)$ respectively, and that of the third body of infinitesimal mass is $x,y,z$ . Here we suppose that the masses of the two primaries vary with respect to time with the consideration that the first primary has the shape of an oblate spheroid while the second primary is a source of radiation. Let the radius vector from $m_3$ to $m_1$ be $r_1$, $m_3$ to $m_2$ be $r_2$ and the distance between the two primaries be $r$ and let $\omega$ be the angular velocity.

Now, the kinetic energy in the rotating frame of reference $0xyz$ is given by

$$T=\frac{1}{2}m_3(\dot{x}^2+\dot{y}^2+\dot{z}^2)+m_3\omega(x\dot{y}-y\dot{x})+\frac{1}{2}m_3(x^2+y^2)\omega^2 \tag{3.1}$$

while the potential energy of the third body $m_3$, when the masses and oblateness vary with time is:

$$V = -Gm_3\left[\frac{m_1}{r_1}\left(1+\frac{A_1}{2r_1^2}\right)+\frac{m_2 q_2}{r_2}\right] \tag{3.2}$$

where $r_1^2 = (x-x_1)^2 + y^2 + z^2$, $r_2^2 = (x-x_2)^2 + y^2 + z^2$

$r_1$ and $r_2$ are time dependent distances of the third body from the primaries positioned at $(x_1, 0, 0)$ and $(x_2, 0, 0)$, respectively; $q_2$ is the radiation factor of the second primary; $A_1$ $t$ is the oblateness of the first primary and is also a function of time, while $G$ is gravitational constant.

Now let $p_x$, $p_y$ and $p_z$ be the generalized components of momentum then,

$$p_x = \frac{\partial T}{\partial \dot{x}} = m_3\ \dot{x} - \omega y\ ,\ p_y = \frac{\partial T}{\partial \dot{y}} = m_3\ \dot{y} + \omega x\ ,\ p_z = \frac{\partial T}{\partial \dot{z}} = m_3 \dot{z} \tag{3.3}$$

This implies

$$\dot{x} = \frac{p_x}{m_3} + \omega y\ ,\ \dot{y} = \frac{p_y}{m_3} - \omega x\ , \dot{z} = \frac{p_z}{m_3} \tag{3.4}$$

Substituting equations of system (3.4) in (3.1), we get

$$T = \frac{m_3}{2}\left[\frac{p_x^2}{m_3^2} + \frac{2p_x\omega y}{m_3} + \omega^2 y^2 + \frac{p_y^2}{m_3^2} - \frac{2p_y\omega x}{m_3} + \omega^2 x^2 + \frac{p_z^2}{m_3^2} + \frac{2p_y\omega x}{m_3} - 2\omega^2 x^2 - 2\omega^2 y^2 - \frac{2p_x\omega y}{m_3} + \omega^2 x^2 + \omega^2 y^2\right]$$

Simplifying, we at once have

$$T = \frac{1}{2m_3}\left(p_x^2 + p_y^2 + p_z^2\right)$$

Now, Hamiltonian H is given by

$$H = \frac{1}{2}m_3\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{1}{2}m_3\omega^2\left(x^2 + y^2\right) + V \tag{3.5}$$

Using system (3.4) in equation (3.5), we get

$$H = \frac{1}{2m_3}\left(p_x^2 + p_y^2 + p_z^2\right) + \omega\left(yp_x - xp_y\right) + V \tag{3.6}$$

The Hamiltonian canonical equations are given by

$$\dot{x} = \frac{\partial H}{\partial p_x}, \quad \dot{y} = \frac{\partial H}{\partial p_y}, \quad \dot{z} = \frac{\partial H}{\partial p_z}$$

and $$\dot{p}_x = -\frac{\partial H}{\partial x}, \quad \dot{p}_y = -\frac{\partial H}{\partial y}, \quad \dot{p}_z = -\frac{\partial H}{\partial z}$$

$$\Rightarrow \quad \dot{p}_x = -\left(-\omega p_y - \frac{\partial V}{\partial x}\right), \quad \dot{p}_y = -\omega p_x - \frac{\partial V}{\partial y}, \quad \dot{p}_z = -\frac{\partial V}{\partial z} \tag{3.7}$$

Since the primaries move within the framework of the GMP and their masses vary with time in accordance with the UML, then $p_x, p_y, p_z$ and the angular velocity $\omega$ will all be time dependent. So, we differentiate system (3.3) w.r.t. time and compare with equations (3.7), to get

$$m_3\left(\ddot{x} - \omega\dot{y} - \dot{\omega}y\right) = \omega p_y - \frac{\partial V}{\partial x}, \quad m_3\left(\ddot{y} + \omega\dot{x} + \dot{\omega}x\right) = -\omega p_x - \frac{\partial V}{\partial y}, \quad m_3\ddot{z} = -\frac{\partial V}{\partial z} \tag{3.8}$$

Substituting equations of system (3.4) in those of (3.8), we have

$$m_3\ \ddot{x}-\omega\dot{y}-\dot{\omega}y\ =\omega m_3\ \ \dot{y}+\omega x\ -\frac{\partial V}{\partial x}$$

$$m_3\ \ddot{y}+\omega\dot{x}+\dot{\omega}x\ =-\omega m_3\ \ \dot{x}-\omega y\ -\frac{\partial V}{\partial y}$$

$$m_3\ddot{z}=-\frac{\partial V}{\partial z}$$

If we multiply these equations throughout by $\frac{1}{m_3}$, we get

$$\ddot{x}-2\omega\dot{y}=\omega^2x+\dot{\omega}y-\frac{1}{m_3}\frac{\partial V}{\partial x}$$

$$\ddot{y}+2\omega\dot{x}=\omega^2y-\dot{\omega}x-\frac{1}{m_3}\frac{\partial V}{\partial y} \qquad (3.9)$$

$$m_3\ddot{z}=-\frac{1}{m_3}\frac{\partial V}{\partial z}$$

Now, from equation (3.2), we have

$$\frac{\partial V}{\partial x}=-Gm_3\left[m_1\left(-\frac{1}{r_1^2}\right)\frac{\partial r_1}{\partial x}+m_1A_1\ \ t\ \left(-\frac{3}{2r_1^4}\right)\frac{\partial r_1}{\partial x}+m_2q_2\left(-\frac{1}{r_2^2}\right)\frac{\partial r_2}{\partial x}\right] \qquad (3.10)$$

But $r_1^2=(x-x_1)^2+y^2+z^2$ and $r_2^2=(x-x_2)^2+y^2+z^2$

Taking partial derivatives of these equations, we get respectively

$$\frac{\partial r_1}{\partial x}=\frac{(x-x_1)}{r_1}\ \text{and}\ \ \frac{\partial r_2}{\partial x}=\frac{(x-x_2)}{r_2} \qquad (3.11)$$

Substituting equations (3.11) in equations (3.10), we get

$$\frac{\partial V}{\partial x} = Gm_3\left[\frac{m_1\ \ x-x_1}{r_1^3} + \frac{3A_1\ \ t\ \ m_1\ \ x-x_1}{2r_1^5} + \frac{m_2q_2\ \ x-x_2}{r_2^3}\right]$$

Similarly, for $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$, yield

$$\frac{\partial V}{\partial y} = Gm_3\left[\frac{m_1 y}{r_1^3} + \frac{3A_1\ \ t\ \ m_1 y}{2r_1^5} + \frac{m_2q_2 y}{r_2^3}\right]$$

$$\frac{\partial V}{\partial z} = Gm_3\left[\frac{m_1 z}{r_1^3} + \frac{3A_1\ \ t\ \ m_1 z}{2r_1^5} + \frac{m_2q_2 z}{r_2^3}\right]$$

Substituting for $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ in system (3.9), respectively, we get

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega} y - \frac{Gm_1\ \ x-x_1}{r_1^3} - \frac{Gm_2q_2\ \ x-x_2}{r_2^3} - \frac{3}{2}\frac{A_1\ \ t\ \ Gm_1\ \ x-x_1}{r_1^5}$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega} x - \frac{Gm_1 y}{r_1^3} - \frac{Gm_2q_2 y}{r_2^3} - \frac{3}{2}\frac{A_1\ \ t\ \ Gm_1 y}{r_1^5} \qquad (3.12)$$

$$\ddot{z} = -\frac{Gm_1 z}{r_1^3} - \frac{Gm_2q_2 z}{r_2^3} - \frac{3}{2}\frac{A_1\ \ t\ \ Gm_1 z}{r_1^5}$$

Now, using equations (1.77), the equations of motion takes the form

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega} y - \frac{\mu_1\ \ x-x_1}{r_1^3} - \frac{\mu_2q_2\ \ x-x_2}{r_2^3} - \frac{3}{2}\frac{A_1\mu\ \ t\ \ \ x-x_1}{r_1^5}$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega}x - \frac{Gm_1 y}{r_1^3} - \frac{Gm_2 q_2 y}{r_2^3} - \frac{3}{2}\frac{A_1\ t\ Gm_1 y}{r_1^5} \tag{3.13}$$

$$\ddot{z} = -\frac{Gm_1 z}{r_1^3} - \frac{Gm_2 q_2 z}{r_2^3} - \frac{3}{2}\frac{A_1\ t\ Gm_1 z}{r_1^5}$$

where $r_1^2 = (x - x_1)^2 + y^2 + z^2$, $r_2^2 = (x - x_2)^2 + y^2 + z^2$

These equations describe the motion of the third body in the gravitational field of the primaries in the barycentric coordinate system $0xyz$, rotating with an angular velocity $\omega\ t$ about the $z-$axis when both primaries experience mass variations while the first primary has the shape of an oblate spheroid which also varies with time and the second one is a radiation emitter. Equations (3.13) contain additional terms which do not appear in equations (1.78).

Next, we introduce small perturbations in the Coriolis and centrifugal forces with the help of the parameters $\varphi$ and $\psi$, respectively, and the equations of motion (3.13) take the form

$$\ddot{x} - 2\omega\varphi\dot{y} - \dot{\omega}\varphi y = \omega^2\psi x - \frac{\mu_1(x - x_1)}{r_1^3} - \frac{\mu_2 q_2 (x - x_2)}{r_2^3} - \frac{3\mu_1 A_1 (x - x_1)}{2r_1^5}$$

$$\ddot{y} + 2\omega\varphi\dot{x} + \dot{\omega}\varphi x = \omega^2\psi y - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 q_2 y}{r_2^3} - \frac{3\mu_1 A_1 y}{2r_1^5} \tag{3.14}$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 q_2 z}{r_2^3} - \frac{3\mu_1 A_1 z}{2r_1^5}$$

where $\omega^2(r) = \frac{\mu(r)}{\kappa r^3}\left[1 + \frac{3A_1(r)}{2r^2}\right]$

$\kappa$ is the constant of the integral (1.66) of the GMP and defined in equation (1.65); $\mu$ is defined in equations (1.47) while $\varphi$ and $\psi$ represent small perturbations in the Coriolis and centrifugal forces, respectively and are such that

$$\varphi = 1 + \in,\ \in << 1, \quad \text{and} \quad \psi = 1 + \in' ,\ \in' << 1 \tag{3.15}$$

In order to transform system (3.14), we use the MT (1.46); the UML (1.47); the particular integral (1.66) and solutions of the GMP. Further, in view of the variable oblateness of the first primary, we introduce a transformation

$$A_1\ t\ = \alpha_1 R^2\ t \tag{3.16}$$

where $$\alpha_1 = \frac{\rho_E^2 - \rho_p^2}{5\rho_{12}^2},$$

Here, we have defined $\alpha_1$ as the oblateness coefficient of the first primary following McCuskey (1963) and is a constant such that $\alpha_1 << 1$. $\rho_E$ and $\rho_P$ are the equatorial and polar radii of the first primary, respectively. Consequently, we transform (3.14) from $(x, y, z, t)$ to the autonomized form $(\xi, \eta, \zeta, \tau)$ and choose units of measurements, to get

$$\xi'' - 2\varphi\eta' = \frac{\partial \Omega}{\partial \xi}, \quad \eta'' + 2\varphi\xi' = \frac{\partial \Omega}{\partial \eta}, \quad \zeta'' = \frac{\partial \Omega}{\partial \zeta} \tag{3.17}$$

where $$\Omega = \frac{\omega_0^2(\psi + \kappa - 1)(\xi^2 + \eta^2)}{2} + \frac{\omega_0^2(\kappa - 1)\zeta^2}{2} + \frac{\kappa(1-\upsilon)}{\rho_1} + \frac{q_2\kappa\upsilon}{\rho_2} + \frac{\kappa\alpha_1(1-\upsilon)}{2\rho_1^3}$$

$$\omega_0^2 = 1 + \frac{3}{2}\alpha_1 \ , \quad \rho_1 = \sqrt{(\xi + \upsilon)^2 + \eta^2 + \zeta^2}, \quad \rho_2 = \sqrt{(\xi + \upsilon - 1)^2 + \eta^2 + \zeta^2}$$

### 3.2.2 Locations of equilibrium points

The equilibrium points of the autonomized system are defined by the system of the equations

$$\frac{\partial \Omega}{\partial \xi} = \frac{\partial \Omega}{\partial \eta} = \frac{\partial \Omega}{\partial \zeta} = 0 .$$

That is, they are the solutions of equations (3.17) when the components of velocity and acceleration are zero:

$$\omega_0^2 \ \psi + \kappa - 1 \ \xi - \frac{\kappa \ 1-\upsilon \ \ \xi + \upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon \ \ \xi + \upsilon - 1}{\rho_2^3} - \frac{3}{2} \frac{\alpha_1 \kappa \ 1-\upsilon \ \ \xi + \upsilon}{\rho_1^5} = 0$$

$$\left[ \omega_0^2 \ \psi + \kappa - 1 \ - \frac{\kappa \ 1-\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon}{\rho_2^3} - \frac{3}{2} \frac{\alpha_1 \kappa \ 1-\upsilon}{\rho_1^5} \right] \eta = 0 \qquad (3.18)$$

$$\left[ \omega_0^2 \ \ \kappa - 1 \ - \frac{\kappa \ 1-\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon}{\rho_2^3} - \frac{3}{2} \frac{\alpha_1 \kappa \ 1-\upsilon}{\rho_1^5} \right] \zeta = 0$$

We shall discuss the solutions of (3.18) in the following ways

i. When $\xi \neq 0, \eta = 0 \text{ and } \zeta = 0$

ii. When $\xi \neq 0, \eta \neq 0 \text{ and } \zeta = 0$

iii. When $\xi \neq 0, \eta = 0 \text{ and } \zeta \neq 0$

The first type of solution corresponds to the collinear EPs, while the second and third are the solutions of the triangular and out-of-plane EPs, respectively.

**Collinear equilibrium points**

The positions of the collinear Eps $L_{i\kappa}$ $(=1,2,3)$ are obtained by solving equations of system (3.18) with $\eta = \zeta = 0$, that is

$$\frac{\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa}-\frac{1-\upsilon\quad \xi+\upsilon}{\rho_1^3}-\frac{q_2\upsilon\ \ \xi+\upsilon-1}{\rho_2^3}-\frac{3}{2}\frac{\alpha_1\ 1-\upsilon\quad \xi+\upsilon}{\rho_1^5}=0 \tag{3.19}$$

Each of these points lie in the interval $\upsilon-1,\ \upsilon\ ,\ \upsilon,\ 0$ and $1-\upsilon,\ 2-\upsilon$ respectively and depend on oblateness of the first primary; radiation pressure force of the second primary; a small perturbation given in the centrifugal force, and the arbitrary constant of the GMP. These points lie on the line joining the primaries $(\xi-axis)$. The abscissas are

$$\xi_1=-\upsilon-\varepsilon_1,\ \ \xi_2=1-\upsilon-\varepsilon_2,\quad \xi_3=1-\upsilon+\varepsilon_3, \tag{3.20}$$

where $\varepsilon_i>0,\ i=1,2,3$ are the roots of the equation (3.19) but now, $\rho_1=|\xi+\upsilon|,\quad \rho_2=|\xi+\upsilon-1|$.

To compute the exact positions of the collinear points $L_{2\kappa}$, we substitute $\xi_2=\xi$ and $\psi=1+\in'$, in (3.19), and solve to get an algebraic equation of seventh degree in $\varepsilon_2$:

$$\varepsilon_2^7-a_1\varepsilon_2^6+a_2\varepsilon_2^5-a_3\varepsilon_2^4+a_4\varepsilon_2^3-a_5\varepsilon_2^2+a_6\varepsilon_2-a_7=0 \tag{3.21}$$

where $\quad a_1=\dfrac{5+4\upsilon_2}{1+\upsilon_2},\quad a_2=\dfrac{2\ \ 5+3\upsilon_2}{1+\upsilon_2},\quad a_3=\dfrac{2\ \ 2\upsilon_2+5}{1+\upsilon_2}+\dfrac{\kappa\ \ 1+q_2\upsilon_2}{\omega_0^2\ \ \kappa+\in'\ \ 1+\upsilon_2}$

$$a_4=\frac{\upsilon_2+5}{1+\upsilon_2}+\frac{2\kappa\ \ 1+2q_2\upsilon_2}{\omega_0^2\ \ \kappa+\in'\ \ 1+\upsilon_2}\quad a_5=\frac{1}{1+\upsilon_2}+\frac{\kappa\ \ 2+12q_2\upsilon_2+3\alpha_1}{2\omega_0^2\ \ \kappa+\in'\ \ 1+\upsilon_2}$$

$$a_6 = \frac{4\kappa q_2 \upsilon_2}{\omega_0^2 \ \kappa + \in' \ 1+\upsilon_2} \qquad a_7 = \frac{q_2 \kappa \upsilon_2}{\omega_0^2 \ 1+\upsilon_2 \ \kappa + \in'}, \ \upsilon_2 = \frac{\upsilon}{1-\upsilon} \ .$$

The real roots of the polynomial (3.21) give the collinear EPs. We shall verify numerically using a low mass x-ray binary Cen X-4 which consist of a neutron star and a white dwarf, when we present our results in chapter **FIVE**.

**Triangular equilibrium points**

The triangular EPs of the autonomized system, which lie in the orbital plane $\xi\eta$, denoted by $L_4$ and $L_5$ are the solutions of system (3.18) with, $\eta \neq 0, \ \zeta = 0$. That is, we solve the equations

$$\omega_0^2 \ \psi + \kappa - 1 \ \xi - \frac{\kappa \ 1-\upsilon \ \ \xi+\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon \ \xi+\upsilon-1}{\rho_2^3} - \frac{3\alpha_1 \kappa \ 1-\upsilon \ \ \xi+\upsilon}{2\rho_1^5} = 0$$

$$\omega_0^2 \ \psi + \kappa - 1 \ -\frac{\kappa \ 1-\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon}{\rho_2^3} - \frac{3\alpha_1 \kappa \ 1-\upsilon}{2\rho_1^5} = 0 \qquad (3.22)$$

From the first equation of (3.22), we get

$$\xi + \upsilon \left[ \omega_0^2 \ \psi + \kappa - 1 \ -\frac{\kappa \ 1-\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon}{\rho_2^3} - \frac{3\alpha_1 \kappa \ 1-\upsilon}{2\rho_1^5} \right] = \upsilon \left[ \omega_0^2 \ \psi + \kappa - 1 \ -\frac{q_2 \kappa \upsilon}{\rho_2^3} \right] \qquad (3.23)$$

If we substitute the second equation of (3.22), in equation (3.23), we get

$$\omega_0^2 \ \psi + \kappa - 1 \ -\frac{q_2 \kappa}{\rho_2^3} = 0 \qquad (3.24)$$

Similarly, we have

$$\omega_0^2\ \psi+\kappa-1\ -\frac{\kappa}{\rho_1^3}-\frac{3\alpha_1\kappa}{2\rho_1^5}=0 \tag{3.25}$$

When the shape of the first primary deviates from an oblate spheroid to a sphere and if the second one is not a radiation source, we have $\alpha_1=0$ and $q_2=1$. In this case, the solutions of equations (3.24) and (3.25) are respectively

$$\rho_1=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}} \text{ and } \rho_2=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}}$$

Now, in the presence of these parameters, that is when the first primary is an oblate spheroid and the second one emits radiation, we assume the solutions of (3.24) and (3.25) to be

$$\rho_1=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}}+\varepsilon_1,\qquad \rho_2=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}}+\varepsilon_2 \tag{3.26}$$

where $\varepsilon_i\ \ i=1,2$ are very small quantities.

Now, from the equations of (3.26), we have

$$\rho_i=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}}+\varepsilon_i,\quad i=1,2$$

$$\Rightarrow\quad \rho_i^3=\left[\frac{\kappa^{1/3}}{\psi+\kappa-1^{\ 1/3}}\left\{1+\frac{\psi+\kappa-1^{\ 1/3}\varepsilon_i}{\kappa^{1/3}}\right\}\right]^3$$

Using Binomial expansion and retaining only linear terms in $\varepsilon_i$, yields

$$\rho_i^3 = \frac{\kappa}{\psi+\kappa-1}\left[1+\frac{3\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right]$$

Therefore, we have the following

$$\rho_i = \frac{\kappa^{1/3}}{\psi+\kappa-1^{1/3}}\left[1+\frac{\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right],\ \rho_i^3 = \frac{\kappa}{\psi+\kappa-1}\left[1+\frac{3\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right]$$

$$\rho_i^5 = \frac{\kappa^{5/3}}{\psi+\kappa-1^{5/3}}\left[1+\frac{5\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right],\ \rho_i^7 = \frac{\kappa^{7/3}}{\psi+\kappa-1^{7/3}}\left[1+\frac{7\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right]$$

Consequently,

$$\rho_i^{-3} = \frac{\psi+\kappa-1}{\kappa}\left[1-\frac{3\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right],\ \rho_i^{-5} = \frac{\psi+\kappa-1^{5/3}}{\kappa^{5/3}}\left[1-\frac{5\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right]$$

$$\rho_i^{-7} = \frac{\psi+\kappa-1^{7/3}}{\kappa^{7/3}}\left[1-\frac{7\varepsilon_i\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right] \qquad i=1,2$$

Now, we substitute for $\rho_2^{-3}$ in equation (3.24), to get

$$\omega_0^2\ \psi+\kappa-1\ -\frac{q_2\kappa\ \psi+\kappa-1}{\kappa}\left[1-\frac{3\varepsilon_2\ \psi+\kappa-1^{\frac{1}{3}}}{\kappa^{1/3}}\right]=0$$

Expressing $q_2=\left[1-\ 1-q_2\ \right]$, where $1-q_2\ <<1$, in the above equation results to

$$\omega_0^2\ \psi+\kappa-1\ -\frac{\left[1-\ 1-q_2\ \right]\kappa\ \psi+\kappa-1}{\kappa}\left[1-\frac{3\varepsilon_2\ \psi+\kappa-1^{\frac{1}{3}}}{\kappa^{1/3}}\right]=0$$

Since $1-q_2$ and $\varepsilon_2$ are very small quantities, we ignore their product to get

$$\omega_0^2\ \psi+\kappa-1\ -\ \psi+\kappa-1\ \left[1-\frac{3\varepsilon_2\ \psi+\kappa-1^{\ 1/3}}{\kappa^{1/3}}-\ 1-q_2\ \right]=0$$

$$\Rightarrow\quad \omega_0^2\ \psi+\kappa-1\ -\ \psi+\kappa-1\ +\frac{3\varepsilon_2\ \psi+\kappa-1^{\ 4/3}}{\kappa^{1/3}}+\ \psi+\kappa-1\ \ 1-q_2\ =0$$

$$\Rightarrow\quad \frac{3\varepsilon_2\ \psi+\kappa-1^{\ 4/3}}{\kappa^{1/3}}=-\omega_0^2\ \psi+\kappa-1\ +\ \psi+\kappa-1\ -\ \psi+\kappa-1\ \ 1-q_2$$

$$\Rightarrow\quad \frac{3\varepsilon_2\ \psi+\kappa-1^{\ 4/3}}{\kappa^{1/3}}=\ \psi+\kappa-1\ \left[-\omega_0^2+1-\ 1-q_2\ \right]$$

$$\Rightarrow\quad 3\varepsilon_2=\frac{\kappa^{1/3}\ \psi+\kappa-1}{\psi+\kappa-1^{\ 4/3}}\left[1-\omega_0^2-\ 1-q_2\ \right]$$

$$\Rightarrow\quad \varepsilon_2=\frac{\kappa^{1/3}}{3\ \psi+\kappa-1^{\ \frac{1}{3}}}\left[1-\omega_0^2-\ 1-q_2\ \right]$$

Substituting for $\omega_0^2$, in this equation gives

$$\varepsilon_2=\frac{\kappa^{1/3}}{3\ \psi+\kappa-1^{\ \frac{1}{3}}}\left[-\frac{3}{2}\alpha_1-\ 1-q_2\ \right] \tag{3.27}$$

Similarly, following the same procedure, we substitute $\rho_1^{-3}$ and $\rho_1^{-5}$ in equation (3.25) and solve to get

$$\varepsilon_1 = \frac{\kappa^{1/3}}{3\ \psi+\kappa-1\ ^{\frac{1}{3}}}\left[-\frac{3}{2}\alpha_1+\frac{3}{2}\frac{\alpha_1\ \ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}\right] \tag{3.28}$$

Haven found the small quantities given by equations (3.27) and (3.28). First, we substitute equation (3.28) in the first equation of (3.26), to get

$$\rho_1 = \left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}}\left[1-\frac{1}{2}\alpha_1+\frac{\alpha_1\ \ \psi+\kappa-1\ ^{\frac{2}{3}}}{2\kappa^{2/3}}\right] \tag{3.29}$$

Next, we substitute equation (3.27) in the second equation of (3.26), to get

$$\rho_2 = \left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}}\left[1-\frac{1}{2}\alpha_1-\frac{1}{3}\ 1-q_2\ \right] \tag{3.30}$$

Now, from

$$\rho_1^2 = \ \xi+\upsilon\ ^2+\eta^2 \text{ and } \rho_2^2 = \ \xi+\upsilon-1\ ^2+\eta^2 \tag{3.31}$$

We have $\rho_2^2-\rho_1^2 = \ \xi+\upsilon-1\ ^2-\ \xi+\upsilon\ ^2$

$$\Rightarrow\ \ \rho_2^2-\rho_1^2 = \xi^2+\xi\upsilon-\xi+\upsilon\xi+\upsilon^2-\upsilon-\xi-\upsilon+1-\xi^2-2\upsilon\xi-\upsilon^2$$

$$\Rightarrow\ \ \rho_2^2-\rho_1^2 = 1-2\upsilon-2\xi$$

$$\Rightarrow\ \ \xi = \frac{1}{2}-\upsilon-\frac{\rho_2^2-\rho_1^2}{2} \tag{3.32}$$

Now, we substitute equation (3.32) in the second equation of (3.31), to get

$$\eta^2 = \rho_1^2 - \left[\frac{1}{2} - \frac{\rho_2^2 - \rho_1^2}{2}\right]^2$$

$$\Rightarrow \quad \eta^2 = \rho_1^2 - \left[\frac{1}{4} - \frac{\rho_2^2 - \rho_1^2}{4} - \frac{\rho_2^2 - \rho_1^2}{4} + \frac{\rho_2^2 - \rho_1^{2\ 2}}{4}\right]$$

$$\Rightarrow \quad \eta^2 = \rho_1^2 - \frac{1}{4} + \frac{\rho_2^2 - \rho_1^2}{2} - \frac{\rho_2^2 - \rho_1^{2\ 2}}{4}$$

$$\Rightarrow \quad \eta^2 = -\frac{1}{4} + \frac{\rho_2^2 - \rho_1^2 + 2\rho_1^2}{2} - \frac{\rho_2^2 - \rho_1^{2\ 2}}{4}$$

$$\Rightarrow \quad \eta^2 = \frac{\rho_2^2 + \rho_1^2}{2} - \frac{1}{4} - \frac{\rho_2^2 - \rho_1^{2\ 2}}{4}$$

$$\Rightarrow \quad \eta = \pm\left[\frac{\rho_2^2 + \rho_1^2}{2} - \frac{1}{4} - \frac{\rho_2^2 - \rho_1^{2\ 2}}{4}\right]^{\frac{1}{2}} \tag{3.33}$$

Now,

$$\rho_2^2 - \rho_1^2 = \ \rho_2 - \rho_1 \quad \rho_2 + \rho_1 \tag{3.34}$$

Subtracting equation (3.29) from (3.30), we have

$$\rho_2 - \rho_1 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{\ 1/3}}\left[1 - \frac{1}{2}\alpha_1 - \frac{1}{3}\ 1 - q_2\ \right] - \frac{\kappa^{1/3}}{\psi + \kappa - 1^{\ 1/3}}\left[1 - \frac{1}{2}\alpha_1 + \frac{1}{2}\frac{\alpha_1\ \psi + \kappa - 1^{\ \frac{2}{3}}}{\kappa^{2/3}}\right]$$

$$\Rightarrow \quad \rho_2 - \rho_1 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ 1 - \frac{1}{2}\alpha_1 - \frac{1}{3}\ 1 - q_2\ \ -1 + \frac{1}{2}\alpha_1 - \frac{\alpha_1\ \ \psi + \kappa - 1^{\frac{2}{3}}}{2\kappa^{2/3}} \right]$$

$$\Rightarrow \quad \rho_2 - \rho_1 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ -\frac{1}{3}\ 1 - q_2\ \ - \frac{\alpha_1\ \ \psi + \kappa - 1^{\frac{2}{3}}}{2\kappa^{2/3}} \right] \tag{3.35}$$

Also, summing equations (3.29) and (3.30), we have

$$\rho_2 + \rho_1 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ 1 - \frac{1}{2}\alpha_1 - \frac{1}{3}\ 1 - q_2\ \right] + \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ 1 - \frac{1}{2}\alpha_1 + \frac{1}{2}\frac{\alpha_1\ \ \psi + \kappa - 1^{\frac{2}{3}}}{\kappa^{2/3}} \right]$$

$$\Rightarrow \rho_2 + \rho_1 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ 1 - \frac{1}{2}\alpha_1 - \frac{1}{3}\ 1 - q_2\ \ + 1 - \frac{1}{2}\alpha_1 + \frac{1}{2}\frac{\alpha_1\ \ \psi + \kappa - 1^{\frac{2}{3}}}{\kappa^{2/3}} \right]$$

$$\Rightarrow \rho_2 + \rho_1 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ 2 - \alpha_1 - \frac{1}{3}\ 1 - q_2\ \ + \frac{\alpha_1\ \ \psi + \kappa - 1^{\frac{2}{3}}}{2\kappa^{2/3}} \right] \tag{3.36}$$

With the help of equations (3.35) and (3.36), equation (3.34) becomes

$$\rho_2^2 - \rho_1^2 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ -\frac{1}{3}\ 1 - q_2\ \ - \frac{\alpha_1 + \alpha_2\ \ \psi + \kappa - 1^{\frac{2}{3}}}{2\kappa^{2/3}} \right] \times \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}}$$
$$\times \left[ 2 - \alpha_1 - \frac{1}{3}\ 1 - q_2\ \ + \frac{\psi + \kappa - 1^{\frac{2}{3}}}{2\kappa^{2/3}} \right]$$

$$\Rightarrow \rho_2^2 - \rho_1^2 = \frac{\kappa^{2/3}}{\psi + \kappa - 1^{1/3}} \left[ -\frac{1}{3}\ 1-q_2\ \ -\frac{\alpha_1\ \psi+\kappa-1^{\frac{2}{3}}}{2\kappa^{2/3}} \right] \left[ 2-\alpha_1 -\frac{1}{3}\ 1-q_2\ \ +\frac{\psi+\kappa-1^{\frac{2}{3}}}{2\kappa^{2/3}} \right]$$

Expanding the equation we ignore products of very small quantities and only retain their linear terms. Doing this yields

$$\rho_2^2 - \rho_1^2 = -\frac{2\kappa^{2/3}}{\psi + \kappa - 1^{1/3}} \left[ \frac{1}{3}\ 1-q_2\ \ +\frac{\alpha_1\ \psi+\kappa-1^{2/3}}{2\kappa^{2/3}} \right] \tag{3.37}$$

Now, substituting (3.37) in (3.32), we get

$$\xi = \frac{1}{2} - \upsilon + \frac{\kappa^{2/3}}{\psi + \kappa - 1^{1/3}} \left[ \frac{1}{3}\ 1-q_2\ \ +\frac{\alpha_1\ \psi+\kappa-1^{2/3}}{2\kappa^{2/3}} \right]$$

This gives the location of the triangular points on the $\xi$ – axis. To find the location on the $\eta$ – axis, we must find $\rho_2^2 + \rho_1^2$.

Now, note that

$$\rho_1^2 + \rho_2^2 = \ \rho_1 - \rho_2\ \ \rho_1 - \rho_2\ + 2\rho_1\rho_2 \tag{3.38}$$

But $\rho_1 - \rho_2 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}} \left[ \frac{1}{3}\ 1-q_2\ \ +\frac{\alpha_1\ \psi+\kappa-1^{2/3}}{2\kappa^{2/3}} \right]$

Therefore,

$$\rho_1 - \rho_2\ ^2 = \ \rho_1 - \rho_2\ \ \rho_1 - \rho_2\ = 0 \tag{3.39}$$

If we put (3.39) into (3.38), we get

$$\rho_1^2+\rho_2^2=\frac{2\kappa^{2/3}}{\psi+\kappa-1^{\ 2/3}}\left[1-\alpha_1-\frac{1}{3}\ 1-q_2\ +\frac{\alpha_1\ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right] \tag{3.40}$$

Now, from equation (3.34), we observe that

$$\rho_2^2-\rho_1^{2\ 2}=\ \rho_2^2-\rho_1^2\quad\rho_2^2-\rho_1^2$$

$$\Rightarrow\ \rho_2^2-\rho_1^{2\ 2}=\ \rho_2-\rho_1^{\ 2}\ \rho_2+\rho_1^{\ 2}$$

Using (3.39), we get

$$\rho_2^2-\rho_1^{2\ 2}=0 \tag{3.41}$$

Substituting equations (3.41) and (3.40) in equation (3.33) and simplifying it, we get

$$\eta=\pm\frac{\sqrt{4\kappa^{2/3}-\ \psi+\kappa-1^{\ 2/3}}}{2\ \psi+\kappa-1^{\ 1/3}}\left\{1-\frac{2\kappa^{2/3}}{4\kappa^{2/3}-\ \psi+\kappa-1^{\ 2/3}}\left[\frac{1}{3}\ 1-q_2\ +\alpha_1-\frac{\alpha_1\ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right]\right\}$$

Hence, the coordinates of the triangular equilibrium points $\xi,\pm\eta$ is such that

$$\xi=\frac{1}{2}-\upsilon+\frac{\kappa^{2/3}}{\psi+\kappa-1^{\ 1/3}}\left[\frac{1}{3}\ 1-q_2\ +\frac{\alpha_1\ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right] \tag{3.42}$$

$$\eta=\frac{\sqrt{4\kappa^{2/3}-\ \psi+\kappa-1^{\ 2/3}}}{2\ \psi+\kappa-1^{\ 1/3}}\left\{1-\frac{2\kappa^{2/3}}{4\kappa^{2/3}-\ \psi+\kappa-1^{\ 2/3}}\left[\frac{1}{3}\ 1-q_2\ +\alpha_1-\frac{\alpha_1\ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right]\right\}$$

These points are determined by small perturbation in the centrifugal force, radiation pressure of the second primary, oblateness of the first primary and the constant of a particular integral of the

GMP. The positions of the triangular EPs exist only when $\psi - 1 < \kappa < \infty$, When there is no perturbation in the centrifugal force i.e., $\psi = 1$, these points will exist for $0 < \kappa < \infty$ . The small perturbation in the centrifugal force allows the existence of the parameter kappa; so there can be different solutions for different $\kappa$. When $\alpha_1 = 0$, equations (3.42) are different from the ones obtained by Singh and Leke (2010)

**Out-of-plane equilibrium points**

The out-of-plane EPs which are also referred to as the coplanar libration points, have no analogy in the classical R3BP. These EPs are located outside the plane of motion. Their coordinates are the solutions of (3.18) with $\eta = 0$. That is, we must solve the equations

$$\omega_0^2 \ \psi + \kappa - 1 \ \xi - \frac{\kappa \ 1-\upsilon \ \ \xi+\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon \ \xi+\upsilon-1}{\rho_2^3} - \frac{3\alpha_1 \kappa \ 1-\upsilon \ \ \xi+\upsilon}{2\rho_1^5} = 0 \tag{3.43}$$

$$\omega_0^2 \ \kappa - 1 \ - \frac{\kappa \ 1-\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon}{\rho_2^3} - \frac{3\alpha_1 q_1 \kappa \ 1-\upsilon}{2\rho_1^5} = 0 \tag{3.44}$$

where $\rho_1^2 = \ \xi + \upsilon \ ^2 + \zeta^2$, $\rho_2^2 = \ \xi + \upsilon - 1 \ ^2 + \zeta^2$

Equation (3.43) can be written in the form

$$1-\upsilon \left( \frac{1}{\rho_1^3} + \frac{3\alpha_1}{2\rho_1^5} \right) = \frac{\omega_0^2 \ \psi + \kappa - 1 \ \xi}{\kappa \ \xi + \upsilon} - \frac{q_2 \upsilon \ \xi + \upsilon - 1}{\rho_2^3 \ \xi + \upsilon} \tag{3.45}$$

From equation (3.44), we have

$$1-\upsilon \left( \frac{1}{\rho_1^3} + \frac{3\alpha_1}{2\rho_1^5} \right) = \frac{\omega_0^2 \ \kappa - 1}{\kappa} - \frac{q_2 \upsilon}{\rho_2^3} \tag{3.46}$$

Comparing equations (3.45) and (3.46), we get

$$\frac{\omega_0^2\ \kappa-1}{\kappa}-\frac{\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa\ \xi+\upsilon}+\frac{q_2\upsilon}{\rho_2^3}-\frac{q_2\upsilon}{\rho_2^3\ \xi+\upsilon}-\frac{q_2\upsilon}{\rho_2^3}=0$$

$$\Rightarrow\quad \frac{\omega_0^2\ \kappa-1}{\kappa}-\frac{\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa\ \xi+\upsilon}+\frac{q_2\upsilon}{\rho_2^3}-\frac{q_2\upsilon}{\rho_2^3\ \xi+\upsilon}-\frac{q_2\upsilon}{\rho_2^3}=0$$

$$\Rightarrow\quad \frac{\omega_0^2\ \kappa-1}{\kappa}-\frac{\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa\ \xi+\upsilon}-\frac{q_2\upsilon}{\rho_2^3\ \xi+\upsilon}=0$$

$$\Rightarrow\quad \frac{\omega_0^2\ \kappa-1\ \ \xi+\upsilon\ -\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa}-\frac{q_2\upsilon}{\rho_2^3}=0$$

$$\Rightarrow\quad \frac{\omega_0^2\ \kappa\xi+\kappa\upsilon-\xi-\upsilon\ -\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa}-\frac{q_2\upsilon}{\rho_2^3}=0$$

$$\Rightarrow\quad \frac{\omega_0^2\ \kappa\xi+\kappa\upsilon-\xi-\upsilon-\psi\xi-\kappa\xi+\xi}{\kappa}-\frac{q_2\upsilon}{\rho_2^3}=0$$

$$\Rightarrow\quad \omega_0^2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]=\frac{q_2\kappa\upsilon}{\rho_2^3} \qquad (3.47)$$

Similarly, from equation (3.45), we have

$$1-\upsilon\quad \xi+\upsilon\left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=\frac{\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa}-\frac{q_2\upsilon\ \ \xi+\upsilon-1}{\rho_2^3}$$

$$\Rightarrow\quad \frac{q_2\upsilon\ \ \xi+\upsilon-1}{\rho_2^3}=\frac{\omega_0^2\ \ \psi+\kappa-1\ \xi}{\kappa}-\ 1-\upsilon\quad \xi+\upsilon\left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)$$

$$\Rightarrow \qquad \frac{q_2\upsilon}{\rho_2^3}=\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon-1}-\frac{1-\upsilon\quad \xi+\upsilon}{\xi+\upsilon-1}\left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)$$

$$\Rightarrow \qquad \frac{q_2\upsilon}{\rho_2^3}=\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon-1}-\frac{1-\upsilon\quad \xi+\upsilon}{\xi+\upsilon-1}\left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)$$

Now, if we put this result in equation (3.46), we get

$$\frac{\omega_0^2\ \ \kappa-1}{\kappa}-\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)-\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon-1}+\frac{1-\upsilon\quad \xi+\upsilon}{\xi+\upsilon-1}\left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

Multiplying throughout by $\xi+\upsilon-1$ , yields

$$\frac{\omega_0^2\ \ \kappa-1\quad \xi+\upsilon-1}{\kappa}-\ 1-\upsilon\quad \xi+\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)+\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)-\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa}$$
$$+\ 1-\upsilon\quad \xi+\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

$$\Rightarrow \qquad \frac{\omega_0^2\ \ \kappa-1\quad \xi+\upsilon-1}{\kappa}+\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)-\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa}=0$$

$$\Rightarrow \qquad \frac{\omega_0^2\ \ \kappa\xi+\kappa\upsilon-\kappa-\xi-\upsilon+1\ -\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa}+q_1\ \ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

$$\Rightarrow \qquad \frac{\omega_0^2\ \ \kappa\xi+\kappa\upsilon-\kappa-\xi-\upsilon+1-\psi\xi-\kappa\xi+\xi}{\kappa}+\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

$$\Rightarrow \qquad \frac{\omega_0^2\left[\upsilon\ \ \kappa-1\ -\ \kappa-1\ -\psi\xi\right]}{\kappa}+\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

$$\Rightarrow \qquad \frac{\omega_0^2\left[\ \kappa-1\quad \upsilon-1\ -\psi\xi\right]}{\kappa}+\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

$$\Rightarrow \quad \frac{\omega_0^2\left[-\ \kappa-1\ \ 1-\upsilon\ -\psi\xi\right]}{\kappa}+\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

$$\Rightarrow \quad -\omega_0^2\left[\ \kappa-1\ \ 1-\upsilon\ +\psi\xi\right]+\kappa\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

$$\Rightarrow \quad \omega_0^2\left[\ \kappa-1\ \ 1-\upsilon\ +\psi\xi\right]=\kappa\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right) \tag{3.48}$$

Now, we note that when the first primary is not an oblate spheroid, and the second one does not produce radiation, the solutions of equations (3.47) and (3.48) will respectively, be

$$\rho_1^3=\frac{\kappa\ 1-\upsilon}{\kappa-1\ \ 1-\upsilon\ +\psi\xi} \text{ and } \rho_2^3=\frac{\kappa\upsilon}{\upsilon\ \ \kappa-1\ -\psi\xi}$$

This implies that

$$\rho_1=\left[\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right]^{\frac{1}{3}} \text{ and } \rho_2=\left[\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right]^{\frac{1}{3}}$$

Hence, when the parameters are present, we assume that the solutions of equations (3.47) and (3.48), are

$$\rho_1=\left[\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right]^{\frac{1}{3}}+\varepsilon_1 \text{ and } \rho_2=\left[\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right]^{\frac{1}{3}}+\varepsilon_2 \tag{3.49}$$

where $\varepsilon_i << 1 \quad i=1,2$

Now, from the equations in system (3.49), using Binomial expansion and retaining only linear terms in $\varepsilon_i$, we get

$$\rho_1^3 = \frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\left[1+3\varepsilon_1\left\{\frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon}\right\}^{1/3}\right]$$

$$\rho_1^5 = \frac{\left[\kappa\ 1-\upsilon\ \right]^{5/3}}{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}}\left[1+5\varepsilon_1\left\{\frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon}\right\}^{1/3}\right]$$

$$\rho_1^7 = \frac{\left[\kappa\ 1-\upsilon\ \right]^{7/3}}{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{7/3}}\left[1+7\varepsilon_1\left\{\frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon}\right\}^{1/3}\right]$$

and $\rho_2^3 = \frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\left[1+3\varepsilon_2\left\{\frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\right\}^{1/3}\right]$

$$\rho_2^5 = \left[\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right]^{5/3}\left[1+5\varepsilon_2\left\{\frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\right\}^{1/3}\right]$$

$$\rho_2^7 = \left[\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right]^{7/3}\left[1+7\varepsilon_2\left\{\frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\right\}^{1/3}\right]$$

Consequently, we have

$$\rho_1^{-3} = \frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon}\left[1-3\varepsilon_1\left\{\frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon}\right\}^{1/3}\right]$$

$$\rho_1^{-5} = \frac{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}}{\left[\kappa\ 1-\upsilon\ \right]^{5/3}}\left[1-5\varepsilon_1\left\{\frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon}\right\}^{1/3}\right]$$

$$\rho_1^{-7} = \frac{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{7/3}}{\left[\kappa\ 1-\upsilon\ \right]^{7/3}}\left[1-7\varepsilon_1\left\{\frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon}\right\}^{1/3}\right]$$

and $\rho_2^{-3} = \frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\left[1-3\varepsilon_2\left\{\frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\right\}^{1/3}\right]$

$$\rho_2^{-5} = \left[\frac{-\psi\xi + \upsilon\ \ \kappa - 1}{\kappa\upsilon}\right]^{5/3}\left[1 - 5\varepsilon_2\left\{\frac{-\psi\xi + \upsilon\ \ \kappa - 1}{\kappa\upsilon}\right\}^{1/3}\right]$$

$$\rho_2^{-7} = \left[\frac{-\psi\xi + \upsilon\ \ \kappa - 1}{\kappa\upsilon}\right]^{7/3}\left[1 - 7\varepsilon_2\left\{\frac{-\psi\xi + \upsilon\ \ \kappa - 1}{\kappa\upsilon}\right\}^{1/3}\right]$$

Substituting appropriately in equation (3.48), we have

$$\omega_0^2\left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right] = \kappa\ 1 - \upsilon\ \frac{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}{\kappa\ 1 - \upsilon}\left[1 - 3\varepsilon_1\left\{\frac{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}{\kappa\ 1 - \upsilon}\right\}^{1/3}\right]$$

$$+\frac{3}{2}\alpha_1\kappa\ 1 - \upsilon\ \frac{\left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right]^{5/3}}{\left[\kappa\ 1 - \upsilon\ \right]^{5/3}}\left[1 - 5\varepsilon_1\left\{\frac{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}{\kappa\ 1 - \upsilon}\right\}^{1/3}\right]$$

$$\omega_0^2\left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right] = \left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right]\left[1 - 3\varepsilon_1\left\{\frac{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}{\kappa\ 1 - \upsilon}\right\}^{1/3}\right]$$

$$+\frac{3}{2}\alpha_1\frac{\left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right]^{5/3}}{\left[\kappa\ 1 - \upsilon\ \right]^{2/3}}\left[1 - 5\varepsilon_1\left\{\frac{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}{\kappa\ 1 - \upsilon}\right\}^{1/3}\right]$$

Ignoring product of $\alpha_1$ and $\varepsilon_1$, we get

$$\omega_0^2\left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right] = \left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right]\left[1 - 3\varepsilon_1\left\{\frac{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}{\kappa\ 1 - \upsilon}\right\}^{1/3}\right]$$

$$+\frac{3}{2}\alpha_1\frac{\left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right]^{5/3}}{\left[\kappa\ 1 - \upsilon\ \right]^{2/3}}$$

Multiplying the equation throughout by $\frac{1}{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}$, yields

$$\omega_0^2 = 1 - 3\varepsilon_1\left\{\frac{\psi\xi + \ \kappa - 1\ \ 1 - \upsilon}{\kappa\ 1 - \upsilon}\right\}^{1/3} + \frac{3}{2}\alpha_1\frac{\left[\psi\xi + \ \kappa - 1\ \ 1 - \upsilon\ \right]^{2/3}}{\left[\kappa\ 1 - \upsilon\ \right]^{2/3}}$$

$$\Rightarrow \quad \omega_0^2 = 1 - 3\varepsilon_1 \left\{ \frac{\psi\xi + \ \kappa - 1 \ \ 1 - \upsilon}{\kappa \ 1 - \upsilon} \right\}^{1/3} + \frac{3}{2}\alpha_1 \frac{\left[ \psi\xi + \ \kappa - 1 \ \ 1 - \upsilon \ \right]^{2/3}}{\left[ \kappa \ 1 - \upsilon \ \right]^{2/3}}$$

Substituting for $\omega_0^2$ and simplifying, gives

$$\varepsilon_1 = \frac{1}{3} \left\{ \frac{\kappa \ 1 - \upsilon}{\psi\xi + \ \kappa - 1 \ \ 1 - \upsilon} \right\}^{\frac{1}{3}} \left[ -\frac{3}{2}\alpha_1 + \frac{3\alpha_1 \left[ \psi\xi + \ \kappa - 1 \ \ 1 - \upsilon \ \right]^{\frac{2}{3}}}{2\left[ \kappa \ 1 - \upsilon \ \right]^{2/3}} \right] \tag{3.50}$$

To find $\varepsilon_2$, we substitute for $\rho_2^{-3}$ and $\rho_2^{-5}$ in equation (3.47) to get

$$\omega_0^2 \left[ -\psi\xi + \upsilon \ \ \kappa - 1 \ \right] = q_2 \kappa\upsilon \times \frac{-\psi\xi + \upsilon \ \ \kappa - 1}{\kappa\upsilon} \left[ 1 - 3\varepsilon_2 \left\{ \frac{-\psi\xi + \upsilon \ \ \kappa - 1}{\kappa\upsilon} \right\}^{1/3} \right]$$

$$\omega_0^2 \left[ -\psi\xi + \upsilon \ \ \kappa - 1 \ \right] = q_2 \left[ -\psi\xi + \upsilon \ \ \kappa - 1 \ \right] \left[ 1 - 3\varepsilon_2 \left\{ \frac{-\psi\xi + \upsilon \ \ \kappa - 1}{\kappa\upsilon} \right\}^{1/3} \right]$$

Ignoring product of $\alpha_2$ and $\varepsilon_2$, and multiplying the result throughout by $\frac{1}{-\psi\xi + \upsilon \ \ \kappa - 1}$, yields

$$\omega_0^2 - q_2 = -3q_2\varepsilon_2 \left\{ \frac{-\psi\xi + \upsilon \ \ \kappa - 1}{\kappa\upsilon} \right\}^{1/3}$$

Expressing $q_2 = \left[ 1 - \ 1 - q_2 \ \right]$ and substituting in the above equation, we expand and ignore product of $1 - q_2$ and $\varepsilon_2$, to get

$$\omega_0^2 - 1 + \ 1 - q_2 \ = -3\varepsilon_2 \left[ \frac{-\psi\xi + \upsilon \ \ \kappa - 1}{\kappa\upsilon} \right]^{1/3}$$

Substituting for $\omega_0^2$, at once gives

$$-\frac{3}{2}\alpha_1 - \ 1 - q_2 \ = 3\varepsilon_2 \left\{ \frac{-\psi\xi + \upsilon \ \ \kappa - 1}{\kappa\upsilon} \right\}^{\frac{1}{3}}$$

Therefore,

$$\varepsilon_2 = \frac{1}{3}\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{1/3}\left[-\ 1-q_2\ \ -\frac{3}{2}\alpha_1\right] \tag{3.51}$$

Now, we substitute equations (3.50) and (3.51) in equations (3.49), respectively, to get

$$\rho_1 = \left[\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right]^{\frac{1}{3}} + \frac{1}{3}\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{\frac{1}{3}}\left[-\frac{3}{2}\alpha_1 + \frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{2/3}}{2\left[\kappa\ 1-\upsilon\right]^{2/3}}\right]$$

$$\Rightarrow \quad \rho_1 = \left[\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right]^{1/3}\left[1-\frac{1}{2}\alpha_1 + \frac{\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{2/3}}{2\left[\kappa\ 1-\upsilon\right]^{2/3}}\right]$$

Similarly, $\rho_2 = \left[\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right]^{\frac{1}{3}}\left[1-\frac{1}{3}\ 1-q_2\ \ -\frac{1}{2}\alpha_1\right]$

Now from these equations, using Binomial expansions, we compute that

$$\rho_1^3 = \frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\left[1-\frac{3}{2}\alpha_1 + \frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{2/3}}{2\left[\kappa\ 1-\upsilon\right]^{2/3}}\right]$$

$$\rho_2^3 = \frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\left[1-\ 1-q_2\ \ -\frac{3}{2}\alpha_1\right] \tag{3.52}$$

Knowing that $\rho_1^2 = \ \xi+\upsilon\ ^2+\zeta^2$ and $\rho_2^2 = \ \xi+\upsilon-1\ ^2+\zeta^2$, we can determine the distances of the out-of-plane points on the $\xi-$axis from these relations by substituting $\zeta = 0$ into them, to get

$$\rho_1^3 = \left[\ \xi+\upsilon\ ^2\right]^{3/2} \text{ and } \rho_2^3 = \left[\ \xi+\upsilon-1\ ^2\right]^{3/2} \tag{3.53}$$

Substituting equations (3.53) in (3.52), we have

$$\left[\ \xi+\upsilon\ ^2\right]^{3/2} = \frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\left[1-\frac{3}{2}\alpha_1 + \frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{\frac{2}{3}}}{2\left[\kappa\ 1-\upsilon\right]^{2/3}}\right]$$

$$\Rightarrow \qquad \xi+\upsilon\ ^{2}=\left[\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\left\{1-\frac{3}{2}\alpha_1+\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{2/3}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}\right]^{2/3}$$

So that

$$\xi+\upsilon\ ^{2}=\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{2/3}\left\{1-\frac{3}{2}\alpha_1+\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{2/3}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}^{2/3} \tag{3.54}$$

Similarly, we have

$$\xi+\upsilon-1\ ^{2}=\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{2/3}\left\{1-\ 1-q_2\ \ -\frac{3}{2}\alpha_1\right\}^{2/3} \tag{3.55}$$

Now, if we subtract equation (3.55) from (3.54), we get

$$2\xi+2\upsilon-1=\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{2/3}\left\{1-\frac{3}{2}\alpha_1+\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{2/3}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}^{2/3}$$
$$-\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{2/3}\left\{1-\ 1-q_2\ \ -\frac{3}{2}\alpha_1\right\}^{2/3}$$

Denoting the L.H.S by $f\ \ \xi\ $, gives

$$f\ \ \xi\ =2\xi+2\upsilon-1-\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{\frac{2}{3}}\left\{1-\frac{3}{2}\alpha_1+\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}^{\frac{2}{3}} \tag{3.56}$$
$$+\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{\frac{2}{3}}\left\{1-\ 1-q_2\ \ -\frac{3}{2}\alpha_1\right\}^{\frac{2}{3}}=0$$

Finding the first and second derivatives of (3.56), and observing that $\psi\xi+\left(-\upsilon\right)\left(\kappa-1\right)>0$ ; then, we require that $\upsilon\left(\kappa-1\right)>\psi\xi$. Now, in the interval $-\left(-\upsilon\right)\left(\kappa-1\right)<\psi\xi<\upsilon\left(\kappa-1\right)$, $f\ \ \xi\ $ increases monotonically (Fig **3.1**), so that for any $\upsilon, 0<(\kappa-1)<\infty, q_2$ and $\alpha_1$, there exists a

single root of the equation (3.56), which gives the abscissa of the out-of-plane EPs. The exact abscissa $\xi$ is obtained by using the Newton-Raphson's method.

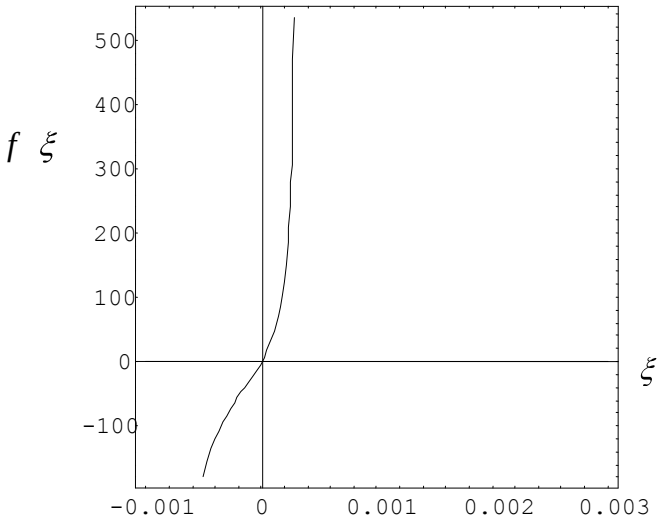



**Fig 3.1:** Graph of $f\ \xi$ : zeros of the function gives the abscissa $\xi$ of the out-of-plane points $L_{6,7}$

Now, taking first derivatives, we get

$$f'\ \xi = 2+\frac{2}{3}\left[\frac{\psi\left[\kappa\ 1-\upsilon\right]^{2/3}}{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{5/3}}-\frac{3\alpha_1\psi\left[\kappa\ 1-\upsilon\right]^{2/3}}{2\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{5/3}}+\frac{\alpha_1\psi\left[\kappa\ 1-\upsilon\right]^{2/3}}{2\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{5/3}}\right]$$
$$+\frac{2}{3}\left[\frac{\psi\ \ \kappa\upsilon\ ^{2/3}}{\left[-\psi\xi+\upsilon\ \ \kappa-1\right]^{5/3}}-\frac{\psi\ \ \kappa\upsilon\ ^{2/3}\ 1-q_2}{\left[-\psi\xi+\upsilon\ \ \kappa-1\right]^{5/3}}-\frac{3\alpha_1\psi\ \ \kappa\upsilon\ ^{2/3}}{2\left[-\psi\xi+\upsilon\ \ \kappa-1\right]^{5/3}}+\frac{\psi\ \ \kappa\upsilon\ ^{2/3}\ 1-q_2}{3\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{5/3}}\right.$$
$$\left.+\frac{\alpha_1\psi\left(\kappa\upsilon\right)^{2/3}}{2\left[-\psi\xi+\upsilon\left(\kappa-1\right)^{5/3}\right]}\right]$$

$$\Rightarrow f'\ \xi = 2+\frac{2}{3}\frac{\psi\left[\kappa\ 1-\upsilon\right]^{2/3}\ 1-\alpha_1}{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{5/3}}+\frac{2}{3}\frac{\psi\ \ \kappa\upsilon\ ^{2/3}}{\left[-\psi\xi+\upsilon\ \ \kappa-1\right]^{5/3}}\left[1-\frac{2\ 1-q_2}{3}-\alpha_1\right] \qquad (3.57)$$

Using Newton-Raphson's method given as

$$\xi = -\frac{f\ 0}{f'\ 0} \tag{3.58}$$

We can find the abscissa by starting with initial value $\xi = 0$ and getting the zeros of $f\ \xi$ and $f'\ \xi$ . Therefore,

$$f\ 0 = 2\upsilon - 1 - \left[\frac{\kappa\ 1-\upsilon}{\kappa-1\ \ 1-\upsilon}\right]^{2/3}\left[1-\frac{3\alpha_1}{2}+\frac{3\alpha_1\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{2/3}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right]^{2/3}+\left[1-\ 1-q_2\ -\frac{3\alpha_1}{2}\right]^{2/3}$$
$$\times\left[\frac{\kappa\upsilon}{\upsilon\ \kappa-1}\right]^{2/3}$$

$$f\ 0 = 2\upsilon - 1 - \left\{\frac{\kappa}{\kappa-1}\right\}^{\frac{2}{3}}\left\{1-\frac{3}{2}\alpha_1+\frac{3\alpha_1\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}^{\frac{2}{3}}+\left\{\frac{\kappa}{\kappa-1}\right\}^{\frac{2}{3}}\left\{1-\ 1-q_2\ -\frac{3}{2}\alpha_1\right\}^{\frac{2}{3}}$$

$$\Rightarrow f\ 0 = 2\upsilon - 1 - \left\{\frac{\kappa}{\kappa-1}\right\}^{\frac{2}{3}}\left\{1-\frac{3}{2}\alpha_1+\frac{3\alpha_1}{2}\left\{\frac{\kappa-1}{\kappa}\right\}^{\frac{2}{3}}\right\}^{\frac{2}{3}}+\left\{\frac{\kappa}{\kappa-1}\right\}^{\frac{2}{3}}\left\{1-\ 1-q_2\ -\frac{3}{2}\alpha_1\right\}^{\frac{2}{3}}$$

Using Binomial expansions, we have

$$f\ 0 = 2\upsilon - 1 - \left\{\frac{\kappa}{\kappa-1}\right\}^{\frac{2}{3}}\left\{1-\alpha_1+\alpha_1\left\{\frac{\kappa-1}{\kappa}\right\}^{\frac{2}{3}}\right\}+\left\{\frac{\kappa}{\kappa-1}\right\}^{\frac{2}{3}}\left\{1-\frac{2}{3}\ 1-q_2\ -\alpha_1\right\}$$

$$\Rightarrow f\ 0 = 2\upsilon - 1 - \left(\frac{\kappa}{\kappa-1}\right)^{\frac{2}{3}}\left[1-\alpha_1+\alpha_1\left(\frac{\kappa-1}{\kappa}\right)^{\frac{2}{3}}-1+\frac{2}{3}\ 1-q_2\ +\alpha_1\right]$$

$$\Rightarrow f\ 0\ = 2\upsilon - 1 - \left(\frac{\kappa}{\kappa-1}\right)^{\frac{2}{3}}\left\{\frac{2}{3}\ 1-q_2\ + \alpha_1\left(\frac{\kappa-1}{\kappa}\right)^{\frac{2}{3}}\right\}$$

$$\Rightarrow f\ 0\ = 2\upsilon - 1 - \frac{2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}} - \alpha_1 \qquad (3.59)$$

Similarly,

$$f'\ 0\ = 2 + \frac{2}{3}\left[\frac{\psi\left[\kappa\ 1-\upsilon\ \right]^{2/3}}{\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}} - \frac{3\alpha_1\psi\left[\kappa\ 1-\upsilon\ \right]^{2/3}}{2\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}} + \frac{\alpha_1\psi\left[\kappa\ 1-\upsilon\ \right]^{2/3}}{2\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}}\right] + \frac{2}{3}\left[\frac{\psi\ \ \kappa\upsilon\ ^{2/3}}{\left[\upsilon\ \ \kappa-1\ \right]^{5/3}}\right.$$
$$\left. - \frac{\psi\ \ \kappa\upsilon\ ^{2/3}\ 1-q_2}{2\left[\upsilon\ \ \kappa-1\ \right]^{5/3}} - \frac{3\alpha_1\psi\ \ \kappa\upsilon\ ^{2/3}}{2\left[\upsilon\ \ \kappa-1\ \right]^{5/3}} + \frac{\psi\ \ \kappa\upsilon\ ^{2/3}\ 1-q_2}{3\left[\upsilon\ \ \kappa-1\ \right]^{5/3}} + \frac{\alpha_1\psi\left(\kappa\upsilon\right)^{2/3}}{2\left[\upsilon\left(\kappa-1\right)\right]^{5/3}}\right]$$

Therefore, we get

$$f'\ 0\ = 2 + \frac{2}{3}\left[\frac{\psi\left[\kappa\ 1-\upsilon\ \right]^{2/3}}{\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}} - \frac{\alpha_1\psi\left[\kappa\ 1-\upsilon\ \right]^{2/3}}{\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}}\right] + \frac{2}{3}\left[\frac{\psi\ \ \kappa\upsilon\ ^{2/3}}{\left[\upsilon\ \ \kappa-1\ \right]^{5/3}} - \frac{2\psi\ \ \kappa\upsilon\ ^{2/3}\ 1-q_2}{3\left[\upsilon\ \ \kappa-1\ \right]^{5/3}}\right.$$
$$\left. - \frac{\alpha_1\psi\ \ \kappa\upsilon\ ^{5/3}}{\left[\upsilon\ \ \kappa-1\ \right]^{5/3}}\right] \qquad (3.60)$$

Substituting equations (3.60) and (3.59) in the Newton-Raphson's formula (3.58) and simplifying we get the abscissa of the out-of-plane EPs such that

$$\xi = \frac{1}{2}\upsilon\left(-\upsilon\right)\left(\kappa-1\right)\left[\frac{3\left(\kappa-1\right)^{2/3}\left(+\alpha_1-2\upsilon\right)+2\kappa^{2/3}\left(-q_2\right)}{3\upsilon\left(-\upsilon\right)\left(\kappa-1\right)^{5/3}+\psi\kappa^{2/3}\left\{1-\alpha_1-\frac{2}{3}\left(-q_2\right)\left(-\upsilon\right)\right\}}\right]$$

From the equation, $\rho_2^2 = \ \xi+\upsilon-1\ ^2+\zeta^2$, we have

$$\zeta^2 = \rho_2^2 - \ \xi+\upsilon-1\ ^2$$

$$\Rightarrow \quad \zeta = \pm\sqrt{\rho_2^2 - \left(\xi + \upsilon - 1\right)^2} \tag{3.61}$$

Substituting for $\rho_2^2$, in (3.61) we get

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\upsilon}{-\psi\xi + \upsilon\ \kappa - 1}\left[1 - \left(1 - q_2\right) - \frac{3}{2}\alpha_1\right]\right\}^{\frac{2}{3}} - \left(\xi + \upsilon - 1\right)^2}$$

$$\Rightarrow \quad \zeta = \pm\sqrt{\left\{\frac{\kappa\upsilon}{-\psi\xi + \upsilon\ \kappa - 1}\right\}^{\frac{2}{3}}\left[1 - \left(1 - q_2\right) - \frac{3}{2}\alpha_1\right]^{\frac{2}{3}} - \left(\xi + \upsilon - 1\right)^2}$$

Using Binomial expansion, we get

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\upsilon}{-\psi\xi + \upsilon\ \kappa - 1}\right\}^{\frac{2}{3}}\left[1 - \frac{2}{3}\left(1 - q_2\right) - \alpha_1\right] - \left(\xi + \upsilon - 1\right)^2}$$

Now, from the equation, $\rho_1^2 = \left(\xi + \upsilon\right)^2 + \zeta^2$, we have

$$\zeta = \pm\sqrt{\rho_1^2 - \left(\xi + \upsilon\right)^2}$$

Substituting for $\rho_1^2$, in the above, we get

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\ 1 - \upsilon}{\psi\xi + \left(\kappa - 1\right)\left(1 - \upsilon\right)}\left[1 - \frac{3}{2}\alpha_1 + \frac{3\alpha_1\left[\psi\xi + \left(\kappa - 1\right)\left(1 - \upsilon\right)\right]^{\frac{2}{3}}}{2\left[\kappa\ 1 - \upsilon\right]^{2/3}}\right]\right\}^{\frac{2}{3}} - \left(\xi + \upsilon\right)^2}$$

$$\Rightarrow \zeta = \pm\sqrt{\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{\frac{2}{3}}\left\{1-\frac{3}{2}\alpha_1+\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{\frac{2}{3}}}{2\left[\kappa\ 1-\upsilon\right]^{2/3}}\right\}^{\frac{2}{3}}-\ \xi+\upsilon^{\ 2}}$$

Therefore, we have

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{\frac{2}{3}}\left\{1-\alpha_1+\frac{\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\right]^{2/3}}\right\}-\ \xi+\upsilon^{\ 2}}$$

Hence, the out of plane EPs are given as

$$\xi = \frac{1}{2}\upsilon\left(-\upsilon\right)\left(\kappa-1\right)\left[\frac{3\left(\kappa-1\right)^{2/3}\left(+\alpha_1-2\upsilon\right)+2\kappa^{2/3}\left(-q_2\right)}{3\upsilon\left(-\upsilon\right)\left(\kappa-1\right)^{5/3}+\psi\kappa^{2/3}\left\{1-\alpha_1-\frac{2}{3}\left(-q_2\right)\left(-\upsilon\right)\right\}}\right]$$

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{\frac{2}{3}}\left[1-\frac{2}{3}\ 1-q_2\ \ -\alpha_1\right]-\ \xi+\upsilon-1^{\ 2}} \qquad (3.62)$$

and

$$\xi = \frac{1}{2}\upsilon\left(-\upsilon\right)\left(\kappa-1\right)\left[\frac{3\left(\kappa-1\right)^{2/3}\left(+\alpha_1-2\upsilon\right)+2\kappa^{2/3}\left(-q_2\right)}{3\upsilon\left(-\upsilon\right)\left(\kappa-1\right)^{5/3}+\psi\kappa^{2/3}\left\{1-\alpha_1-\frac{2}{3}\left(-q_2\right)\left(-\upsilon\right)\right\}}\right]$$

$$\zeta = \pm\sqrt{\left[\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right]^{\frac{2}{3}}\left[1-\alpha_1+\frac{\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\right]^{2/3}}\right]-\ \xi+\upsilon^{\ 2}} \qquad (3.63)$$

Real out-of-plane EP (3.62) and (3.63) exist only for $\kappa > 1$ provided $\psi\xi < \upsilon(\kappa - 1)$ and $\psi\xi + (\kappa - 1)(1 - \upsilon) > 0$, respectively. The coordinate (3.62) is denoted by $L_{6,7}$, while the coordinate (3.63) is denoted by $L_{8,9}$. Hence, there are four out-of-plane EPs. Results of Bekov (1988, 2005), Luk'yanov (1989, 1990) can be deduced from our results when $\alpha_1 = 0,\ q_2 = 1,$ and $\psi = 1$; while the results here are similar to those in Singh and Leke (2010). The positions of $L_{6,7}$ and $L_{8,9}$ are shown in Fig **3.2** and Fig. **3.3**, using equations (3.62) and (3.63), respectively, for $\kappa > 1$, $\upsilon = 0.125$, $\psi = 1.002$, for, $q_2 = 0.9988$ and $\alpha_1 = 0.02$.

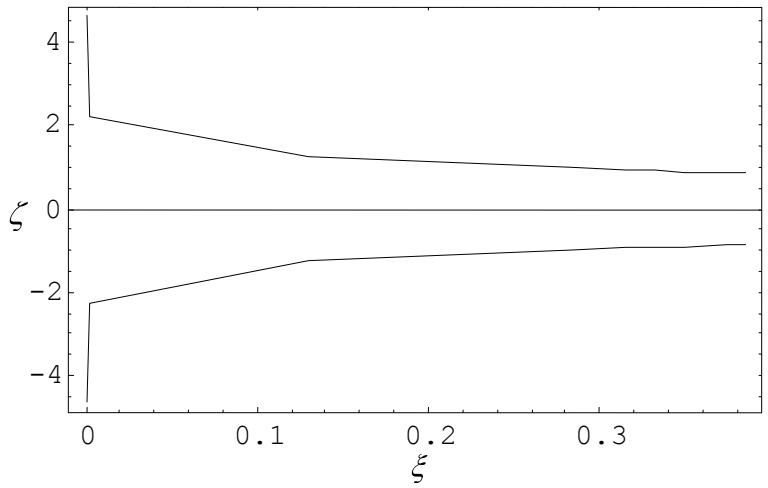



**Fig. 3.2:** Out of plane Equilibrium points $L_{6,7}$

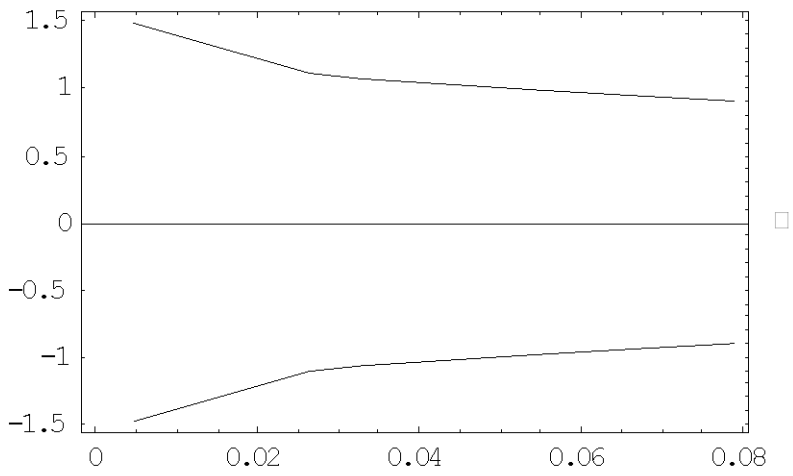



**Fig. 3.3** Out of plane Equilibrium points $L_{8,9}$

The EPs $L_i \;\; i=1,2...9$ of the system of equations of the non-autonomous system are sought using the Mestschersky's transformation (1.46) in the form (Luk'yanov 1990):

$$x^i = \xi^i R\ t\ , \quad y^i = \eta^i R\ t\ , z^i = \zeta^i R\ t\ , \tag{3.64}$$

where, $\xi^i\ \ \tau\ , \eta^i\ \ \tau \quad i=1,2...9$ are the EPs of system autonomized system. All the EPs in this case are function of time $t$, and so the locations of the EPs of equations of motion with variable coefficients always change with time.

### 3.2.3. Stability of equilibrium points

In this section of the dissertation, we study the linear stability of the EPs of the restricted problem. We shall also in the later part of this chapter, investigate the periodic orbits around stable triangular EPs of the autonomized system.

**Stability of the equilibrium points of autonomized system**

To obtain the variational equations for this problem, we follow the same procedure we used in getting the variational equations (1.212), to get

$$\begin{aligned} u'' - 2\phi\omega_0 v' &= \Omega^0_{\xi\xi}\ u + \Omega^0_{\xi\eta}\ v + \Omega^0_{\xi\zeta}\ w, \\ v'' + 2\phi\omega_0 u' &= \Omega^0_{\xi\eta}\ u + \Omega^0_{\eta\eta}\ v + \Omega^0_{\eta\zeta}\ w, \\ w'' &= \Omega^0_{\xi\zeta}\ u + \Omega^0_{\eta\zeta}\ v + \Omega^0_{\zeta\zeta}\ w. \end{aligned} \tag{3.65}$$

These equations differ from those in (1.112) due to the parameter $\varphi$ and the partial derivatives are also evaluated at the equilibrium points

**Collinear equilibrium points**

In order to study the stability of any of the collinear EPs, we compute the partial derivatives and substitute them in the characteristic equation. We consider the point corresponding to $L_{2\kappa}$ with coordinate $1-\upsilon-\varepsilon_{2_\kappa}, 0, 0$ .

The characteristic equation in this premise is also similar and obtained in the same manner as equation (1.114), to get

$$\lambda^4 - \left(\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\varphi^2\omega_0^2\right)\lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 = 0 \tag{3.66}$$

The partial derivatives computed at this point $L_{2\kappa}$ while retaining only linear terms in $1-q_2$, $\alpha_1$ and ignoring products of $\psi - 1$ with $\alpha_1$, are

$$\Omega^0_{\xi\xi} = \psi - 1 + \kappa\left(1 + 2f_{2\kappa}\right) ,$$

$$\Omega^0_{\eta\eta} = \psi - 1 + \kappa(1 - f_{2\kappa}), \tag{3.67}$$

$$\Omega^0_{\xi\eta} = \Omega_{\eta\xi} = 0$$

where $f_{2\kappa} = \frac{1-\upsilon}{\varepsilon_{2\kappa} - 1^{\ 3}} + \frac{\upsilon}{\varepsilon^3_{2\kappa}} + \frac{\upsilon\ 1-q_2}{\varepsilon^3_{2\kappa}} + \frac{3\alpha_1\ 1-\upsilon}{\varepsilon_{2\kappa} - 1^{\ 5}}$

Now, $\varepsilon_{2\kappa} > 0, 0 < \upsilon \le \frac{1}{2}$, $f_{2\kappa} > 0$, consequently, for any $0 < \kappa < \infty$, $\psi - 1 << 1$ and $\alpha_1 << 1$, we always have $\Omega^0_{\xi\xi} > 0$ while $\Omega^0_{\eta\eta}$ can be positive or negative, depending solely on the value of the arbitrary constant $\kappa$ and the small perturbation in the centrifugal force.

Substituting equations (3.67) in the characteristic equation (3.66), we have

$$\lambda^4 + P_2\lambda^2 + Q_2 = 0 \tag{3.68}$$

where $P_2 = 4 - 8\ \phi - 1\ - 2\ \psi - 1\ - \kappa\ 2 + f_{2\kappa}\ + \frac{3}{2}\alpha_1\ 4 - 2\kappa + f_{2\kappa}$

$$Q_2 = \kappa\left[\ \psi - 1\ \ 2 + f_{2\kappa}\ + \kappa\ 1 + f_{2\kappa} - 2f^2_{2\kappa}\ \ 1 + 3\alpha_1\ \right]$$

A careful inspection of the above reveals that the values of $P_2$ and $Q_2$ could be positive or negative depending on the choice of $\kappa$, small perturbations in the Coriolis and centrifugal forces, radiation pressure of the second primary, oblateness of the first and whether $\phi - 1 <=> 0$, $\psi - 1 <=> 0$. If the roots of equation (3.68) are distinct pure imaginary roots, then the EP is stable, otherwise it is unstable.

Now, a condition for pure imaginary roots of (3.68) is that simultaneously, $P_2 > 0$, $Q_2 > 0$, and $P_2 > \sqrt{D_2}$ ; where $D_2 = P_2^2 - 4Q_2$. Consequently, $D_2 > 0$ and the roots in this case can be represented as

$$\lambda_{1,2} = \pm i\sigma_1, \ \lambda_{3,4} = \pm i\sigma_2$$

where $$\sigma_{1,2} = \left( \frac{-P_2}{2} \pm \frac{\sqrt{D_2}}{2} \right)^{1/2}$$

provided $$-\frac{2 + f_{2\kappa} \ \in'}{1 + f_{2\kappa} - 2 f_{2\kappa}^2} < \kappa < \frac{4 - 8 \in -2 \in' + 6\alpha_1}{2 + f_{2\kappa}}$$

The general solutions for pure imaginary roots are given (Szebehely 1967a)

$$u = S_1 \cos\sigma_1\tau + C_1 \sin\sigma_1\tau + S_2 \cos\sigma_2\tau + C_2 \sin\sigma_2\tau$$
$$v = \bar{S}_1 \cos\sigma_1\tau + \bar{C}_1 \sin\sigma_1\tau + \bar{S}_2 \cos\sigma_2\tau + \bar{C}_2 \sin\sigma_2\tau$$

where, $S_i, \bar{S}_i, C_i$ *and* $\bar{C}_i$ $(i = 1, 2)$ are constants.

Hence, the collinear point $L_{2\kappa}$ is stable in this case.

Any case, different from case 1 above, results in a solution of the type

$$u = A_1 e^{\sigma_1\tau} + A_2 e^{-\sigma_1\tau} + A_3 e^{i\sigma_2\tau} + A_4 e^{-i\sigma_2\tau}$$
$$v = B_1 e^{\sigma_1\tau} + B_2 e^{-\sigma_1\tau} + B_3 e^{i\sigma_2\tau} + B_4 e^{-i\sigma_2\tau}$$

Here, $A_{j's}$ and $B_{j's}$ $j = 1, 2, 3, 4,$ are constants and all $\lambda_i, i = 1, 2, 3, 4$ are real.

In this case, the collinear point $L_{2\kappa}$ is unstable due to the presence of exponential functions and a positive root. Therefore, the collinear point is stable or unstable due the a constant of a particular integral of the GMP, the small perturbations in the Coriolis and centrifugal forces, radiation pressure of the second primary, oblateness of the first primary and the mass parameter. The same stability analysis may be followed in case of $L_1$ and $L_3$.

**Triangular equilibrium points**

The characteristic equation in the case of triangular EP is also given by equation (3.66); however, the partial derivatives will be calculated at the triangular EPs. We consider the stability of $L_4$ points; the stability of the point $L_5$ follows in a similar manner.

To compute the partial derivatives, we get the following second order partial derivatives from the force function in equation (3.17):

$$\Omega_{\xi\xi} = \omega_0^2 \ \psi + \kappa - 1 \ - \frac{\kappa \ 1-\upsilon}{\rho_1^3} + \frac{3\kappa \ 1-\upsilon \ \ \xi+\upsilon \ ^2}{\rho_1^5} - \frac{q_2\kappa\upsilon}{\rho_2^3} + \frac{3q_2\kappa\upsilon \ \xi+\upsilon-1 \ ^2}{\rho_2^5} - \frac{3\alpha_1\kappa \ 1-\upsilon}{2\rho_1^5}$$
$$+ \frac{15\alpha_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ ^2}{2\rho_1^7}$$

$$\Omega_{\eta\eta} = \omega_0^2 \ \psi + \kappa - 1 \ - \frac{\kappa \ 1-\upsilon}{\rho_1^3} + \frac{3\kappa \ 1-\upsilon \ \eta^2}{\rho_1^5} - \frac{q_2\kappa\upsilon}{\rho_2^3} + \frac{3q_2\kappa\upsilon\eta^2}{\rho_2^5} - \frac{3\alpha_1\kappa \ 1-\upsilon}{2\rho_1^5} + \frac{15\alpha_1\kappa \ 1-\upsilon \ \eta^2}{2\rho_1^7}$$

$$\Omega_{\xi\eta} = \frac{3\kappa \ 1-\upsilon \ \ \xi+\upsilon \ \eta}{\rho_1^5} + \frac{3q_2\kappa\upsilon \ \xi+\upsilon-1 \ \eta}{\rho_2^5} + \frac{15\alpha_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ \eta}{2\rho_1^7} \qquad (3.69)$$

To get $\Omega_{\xi\xi}^0$, we substitute $\rho_i^{-3}$, $\rho_i^{-5}$, $\rho_i^{-7}$ and $\xi$ in the first equation of (3.69) to get

$$\Omega^0_{\xi\xi} = \omega_0^2 \ \psi+\kappa-1 \ -\kappa \ 1-\upsilon \ \times\frac{\psi+\kappa-1}{\kappa}\left[1+\frac{3}{2}\alpha_1-\frac{3\alpha_1}{2}\frac{\psi+\kappa-1^{\frac{2}{3}}}{\kappa^{2/3}}\right]$$

$$+3\kappa \ 1-\upsilon \ \left[\frac{1}{4}+\frac{\kappa^{2/3}}{\psi+\kappa-1^{\frac{2}{3}}}\left\{\frac{1}{3} \ 1-q_2 \ +\frac{1}{2}\frac{\alpha_1 \ \psi+\kappa-1^{\frac{2}{3}}}{\kappa^{2/3}}\right\}\right]\times\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{5/3}}$$

$$\times\left[1+\frac{5}{2}\alpha_1-\frac{5\alpha_1}{2}\frac{\psi+\kappa-1^{\frac{2}{3}}}{\kappa^{2/3}}\right]-q_2\kappa\upsilon\frac{\psi+\kappa-1}{\kappa}\left[1+\frac{3}{2}\alpha_1+\ 1-q_2 \ \right]$$

$$+3q_2\kappa\upsilon\left[\frac{1}{4}-\frac{\kappa^{2/3}}{\left(\psi+\kappa-1^{\frac{2}{3}}\right)}\left\{\frac{1}{3}\left(1-q_2\right)\frac{1}{2}\frac{\alpha_1\left(\psi+\kappa-1^{\frac{2}{3}}\right)}{\kappa^{2/3}}\right\}\right]\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{5/3}}\left[1+\frac{5}{2}\alpha_1+\frac{5}{3}\left(1-q_2\right)\right.$$

$$-\frac{3}{2}\alpha_1\kappa\left(1-\upsilon\right)\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{5/3}}\times\left[1+\frac{5}{2}\alpha_1-\frac{5\alpha_1}{2}\frac{\left(\psi+\kappa-1^{\frac{2}{3}}\right)}{\kappa^{2/3}}\right]+\frac{15\alpha_1\kappa\left(1-\upsilon\right)\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{2}\frac{}{\kappa^{7/3}}$$

$$\times\left[\frac{1}{4}+\frac{\kappa^{2/3}}{\left(\psi+\kappa-1^{\frac{2}{3}}\right)}\left\{\frac{\left(1-q_2\right)}{3}+\frac{\alpha_1\left(\psi+\kappa-1^{\frac{2}{3}}\right)}{2\kappa^{2/3}}\right\}\right]\left[1+\frac{7}{2}\alpha_1-\frac{7\alpha_1}{2}\frac{\left(\psi+\kappa-1^{\frac{2}{3}}\right)}{\kappa^{2/3}}\right]$$

Expressing $q_2 = 1-\ 1-q_2$ , we have

$$\Omega^0_{\xi\xi}=\omega_0^2\ \ \psi+\kappa-1\ -\ 1-\upsilon\ \ \ \psi+\kappa-1\left[1+\frac{3}{2}\alpha_1-\frac{3\alpha_1}{2}\frac{\psi+\kappa-1^{\frac{2}{3}}}{\kappa^{2/3}}\right]+\left[\frac{1}{4}+\frac{\kappa^{2/3}\ \ 1-q_2}{3\ \psi+\kappa-1^{\frac{2}{3}}}+\frac{1}{2}\alpha_1\right]$$

$$\times 3\ 1-\upsilon\ \left[\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}+\frac{5}{2}\alpha_1\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}-\frac{5}{2}\frac{\alpha_1\ \ \psi+\kappa-1^{\frac{7}{3}}}{\kappa^{4/3}}\right]-\upsilon\left[1-\ 1-q_2\ \right]$$

$$\times\left[\ \psi+\kappa-1\ +\frac{3}{2}\alpha_1\ \ \psi+\kappa-1\ -\frac{3\alpha_1}{2}\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}\right]+3\upsilon\left[1-\left(1-q_2\right)\right]\left[\frac{1}{4}-\frac{\kappa^{2/3}\left(1-q_2\right)}{3\left(\psi+\kappa-1^{\frac{2}{3}}\right)}-\frac{\alpha_1}{2}\right]$$

$$\times\left[\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{5\alpha_1\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{2\kappa^{2/3}}+\frac{5\left(\psi+\kappa-1^{\frac{5}{3}}\right)\left(1-q_2\right)}{3\kappa^{2/3}}\right]-\frac{3}{2}\alpha_1\left(1-\upsilon\right)\left[\frac{\left(\psi+\kappa-1^{5/3}\right)}{\kappa^{2/3}}\right.$$

$$\left.+\frac{5}{2}\alpha_1\frac{\left(\psi+\kappa-1^{5/3}\right)}{\kappa^{2/3}}-\frac{5\alpha_1\left(\psi+\kappa-1^{7/3}\right)}{2\kappa^{4/3}}\right]-\frac{3}{2}\alpha_2\left[1-\left(1-q_2\right)\right]\upsilon$$

$$\times\left[\frac{\left(\psi+\kappa-1^{5/3}\right)}{\kappa^{2/3}}+\frac{5\alpha_1\left(\psi+\kappa-1^{5/3}\right)}{2\kappa^{2/3}}+\frac{5\left(\psi+\kappa-1^{5/3}\right)\left(1-q_2\right)}{3\kappa^{2/3}}\right]+\frac{15\alpha_1}{2}\left(1-\upsilon\right)\left[1-\left(1-q_1\right)\right]$$

$$\times\left[\frac{1}{4}+\frac{\kappa^{2/3}\left(1-q_2\right)}{3\left(\psi+\kappa-1^{2/3}\right)}+\frac{\alpha_1}{2}\right]\left[\frac{\left(\psi+\kappa-1^{7/3}\right)}{\kappa^{4/3}}+\frac{7\alpha_1\left(\psi+\kappa-1^{7/3}\right)}{2\kappa^{4/3}}-\frac{7\alpha_1\left(\psi+\kappa-1^{3}\right)}{2\kappa^{2}}\right]$$

$$\Omega^0_{\xi\xi}=\omega_0^2\ \ \psi+\kappa-1\ -\ 1-\upsilon\ \ \ \psi+\kappa-1\left[1+\frac{3}{2}\alpha_1-\frac{3\alpha_1}{2}\frac{\psi+\kappa-1^{2/3}}{\kappa^{2/3}}\right]+3\ 1-\upsilon$$

$$\times\left[\frac{1}{4}\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}+\frac{5}{8}\alpha_1\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}-\frac{5\alpha_1\ \ \psi+\kappa-1^{\frac{7}{3}}}{8\kappa^{4/3}}+\frac{1-q_2\ \ \psi+\kappa-1}{3}-\frac{1}{2}\alpha_1\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}\right]$$

$$-\upsilon\left[\ \psi+\kappa-1\ +\frac{3}{2}\ \alpha_1+\alpha_2\ \ \ \psi+\kappa-1\ -\frac{3}{2}\frac{\alpha_2\ \ \psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}\right]$$

$$+3\upsilon\left[\frac{1}{4}\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}+\frac{5}{8}\alpha_1\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{5\left(1-q_2\right)\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{12\kappa^{2/3}}+\frac{\left(1-q_2\right)\left(\psi+\kappa-1\right)}{3}\right.$$

$$\left.-\frac{1}{2}\alpha_1\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}-\frac{\left(1-q_2\right)\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{4\kappa^{2/3}}\right]-\frac{3\alpha_1}{2}\left(1-\upsilon\right)\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{15\alpha_1}{2}\left(1-\upsilon\right)\frac{\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{4\kappa^{4/3}}$$

$$\Omega^0_{\xi\xi} = \omega_0^2\ \left(\psi+\kappa-1\right) - \left(\psi+\kappa-1\right) - \frac{3}{2}\alpha_1\ \left(\psi+\kappa-1\right) + \frac{3}{2}\alpha_1\upsilon\ \left(\psi+\kappa-1\right) + \frac{3}{4}\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{2/3}}$$

$$+\frac{3}{8}\alpha_1\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{4/3}}\left[9\kappa^{\frac{2}{3}}-5\ \left(\psi+\kappa-1\right)^{\frac{2}{3}}\right] + \left(1-q_2\right)\ \left(\psi+\kappa-1\right) - \frac{3\alpha_1\upsilon}{8}\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{4/3}}$$

$$\times\left[9\kappa^{2/3}-5\ \left(\psi+\kappa-1\right)^{\frac{2}{3}}\right] - \upsilon\ \left(1-q_2\right)\ \left(\psi+\kappa-1\right) - \frac{3}{2}\alpha_1\upsilon\ \left(\psi+\kappa-1\right)\left[9\kappa^{2/3}-5\ \left(\psi+\kappa-1\right)^{\frac{2}{3}}\right]$$

$$+\frac{3}{8}\alpha_1\upsilon\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{2/3}} + \frac{\upsilon\left(1-q_2\right)\left(\psi+\kappa-1\right)}{4\kappa^{2/3}}\left[5\left(\psi+\kappa-1\right)^{\frac{2}{3}} - 4\kappa^{2/3} - 3\left(\psi+\kappa-1\right)^{\frac{2}{3}}\right]$$

$$+\frac{15\alpha_1\left(\psi+\kappa-1\right)^{\frac{7}{3}}}{8\kappa^{4/3}} - \frac{15\alpha_1\upsilon\left(\psi+\kappa-1\right)^{\frac{7}{3}}}{8\kappa^{4/3}}$$

Simplifying the above equation further and substituting for $\omega_0^2$, we have

$$\Omega^0_{\xi\xi} = \left(\psi+\kappa-1\right)\left(1+\frac{3}{2}\alpha_1-1-\frac{3}{2}\alpha_1\right) + \frac{3}{4}\frac{\left(\psi+\kappa-1\right)^{5/3}}{\kappa^{2/3}} + \frac{3}{8}\alpha_1\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{4/3}}\left[9\kappa^{2/3}-5\ \left(\psi+\kappa-1\right)^{\frac{2}{3}}\right]$$

$$+\ \left(1-q_2\right)\ \left(\psi+\kappa-1\right) - \frac{3}{8}\alpha_1\upsilon\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{4/3}}\left[9\kappa^{2/3}-5\ \left(\psi+\kappa-1\right)^{\frac{2}{3}}\right] - \upsilon\ \left(1-q_2\right)\ \left(\psi+\kappa-1\right)$$

$$+\frac{3\alpha_1\upsilon\ \left(\psi+\kappa-1\right)^{\frac{5}{3}}}{8\kappa^{2/3}} + \frac{\upsilon\ \left(1-q_2\right)\ \left(\psi+\kappa-1\right)}{4\kappa^{2/3}}\left[2\ \left(\psi+\kappa-1\right)^{\frac{2}{3}} - 4\kappa^{\frac{2}{3}}\right] + \frac{15\alpha_1\ \left(\psi+\kappa-1\right)^{\frac{7}{3}}}{8\kappa^{4/3}}$$

$$-\frac{15\alpha_1\upsilon\left(\psi+\kappa-1\right)^{\frac{7}{3}}}{8\kappa^{4/3}}$$

$$\Omega^0_{\xi\xi} = \frac{\left(\psi+\kappa-1\right)^{5/3}}{\kappa^{2/3}}\left\{\frac{3}{4} + \frac{3\alpha_1}{8\kappa^{2/3}}\left[9\kappa^{2/3}-5\ \left(\psi+\kappa-1\right)^{2/3}\right] - \frac{3}{8}\frac{\alpha_1\upsilon\ \left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{4/3}}\left[9\kappa^{2/3}-5\ \left(\psi+\kappa-1\right)^{2/3}\right]\right.$$

$$+\frac{\kappa^{2/3}\ \left(1-q_2\right)}{\left(\psi+\kappa-1\right)^{\frac{2}{3}}} - \frac{\upsilon\kappa^{2/3}\ \left(1-q_2\right)}{\left(\psi+\kappa-1\right)^{2/3}} + \frac{3}{8}\alpha_1\upsilon + \frac{\upsilon\left[\ \left(\psi+\kappa-1\right)\ -2\kappa^{2/3}\right]\left(1-q_2\right)}{2\ \left(\psi+\kappa-1\right)^{2/3}} + \frac{15\alpha_1\ \left(\psi+\kappa-1\right)^{\frac{2}{3}}}{8\kappa^{2/3}}$$

$$\left.-\frac{15\alpha_1\upsilon\ \left(\psi+\kappa-1\right)^{\frac{2}{3}}}{8\kappa^{2/3}}\right\}$$

or

$$\Omega^0_{\xi\xi\xi} = \frac{\psi+\kappa-1\ ^{5/3}}{\kappa^{2/3}}\left\{\frac{3}{4}+\frac{\alpha_1}{\kappa^{2/3}}\left\{\frac{3\left[9\kappa^{2/3}-5\ \psi+\kappa-1\ ^{2/3}\right]}{8}+\frac{5\ \psi+\kappa-1\ ^{2/3}}{8}\right\}+\frac{\kappa^{2/3}\ 1-q_2}{\psi+\kappa-1\ ^{2/3}}\right.$$

$$\left.+\left\{\frac{3}{8}-\frac{3\left[9\kappa^{2/3}-5\ \psi+\kappa-1\ ^{2/3}\right]}{8\kappa^{2/3}}-\frac{15\ \psi+\kappa-1\ ^{2/3}}{8\kappa^{2/3}}\right\}\alpha_1\upsilon+\upsilon\ 1-q_2\ \left[\frac{\psi+\kappa-1\ ^{2/3}-2\kappa^{2/3}}{2\ \psi+\kappa-1\ ^{2/3}}\right.\right.$$

$$\left.\left.-\frac{\kappa^{2/3}}{\psi+\kappa-1\ ^{2/3}}\right]\right\}$$

Simplifying, we have

$$\Omega^0_{\xi\xi\xi} = \frac{\psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}\left\{\frac{3}{4}+\frac{27}{8}\alpha_1+\left(\frac{3}{8}-\frac{27}{8}\right)\alpha_1\upsilon+\frac{\kappa^{2/3}\ 1-q_2}{\psi+\kappa-1\ ^{2/3}}+\frac{\upsilon\ 1-q_2}{2}\left[\frac{\psi+\kappa-1\ ^{2/3}-4\kappa^{2/3}}{\psi+\kappa-1\ ^{2/3}}\right]\right\}$$

$$\Rightarrow \Omega^0_{\xi\xi\xi} = \frac{3\ \psi+\kappa-1\ ^{\frac{5}{3}}}{4\kappa^{2/3}}\left[1+\frac{9}{2}\alpha_1-4\alpha_1\upsilon+\frac{4\kappa^{2/3}\ 1-q_2}{3\ \psi+\kappa-1\ ^{2/3}}+\frac{2\upsilon\ 1-q_2\ N}{3\ \psi+\kappa-1\ ^{2/3}}\right]$$

Similarly,

$$\Omega^0_{\eta\eta} = \frac{3\ \psi+\kappa-1\ ^{\frac{5}{3}}}{4\kappa^{2/3}}\left[\frac{N}{\psi+\kappa-1\ ^{2/3}}+\left\{\frac{12\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}}{2\ \psi+\kappa-1\ ^{2/3}}\right\}\alpha_1-\frac{4\kappa^{2/3}\ 1-q_2}{3}+\frac{2\upsilon\ 1-q_2\ N}{3\ \psi+\kappa-1\ ^{2/3}}\right]$$

$$\Omega^0_{\xi\eta} = \frac{3\sqrt{N}\ \psi+\kappa-1\ ^{4/3}}{8\kappa^{2/3}}\left\{2+\frac{\left[24\kappa^{2/3}-5\ \psi+\kappa-1\ ^{2/3}\right]\alpha_1}{N}+\frac{8\kappa^{2/3}\left[2\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right]}{3N\ \psi+\kappa-1\ ^{2/3}}\ 1-q_2\right.$$

$$\left.-\upsilon\left[4+\frac{\left[32\kappa^{2/3}-6\ \psi+\kappa-1\ ^{2/3}\right]}{N}\alpha_1+\frac{4\ \psi+\kappa-1\ ^{2/3}}{3N}\ 1-q_2\right]\right\}$$

where $N = 4\kappa^{2/3} - \left(\psi+\kappa-1\right)^{2/3}$ (3.70)

Substituting the partial derivatives (3.70) in the characteristic equation (3.66), and ignoring product and higher order terms in $1-q_2, \psi-1$ and $\alpha_1$, we get

$$\lambda^4 + P\lambda^2 + Q = 0, \tag{3.71}$$

where $$\lambda_{1,2}^2 = \frac{-P \pm \sqrt{D}}{2}$$

are the roots of (3.71), and $D = P^2 - 4Q$,

$$P = 4 - 3\kappa\left(1 + \frac{5}{2}\alpha_1 - \upsilon\alpha_1\right) + 6\alpha_1 - 3\ \psi - 1\ + 8\ \phi - 1\ ,$$

$$Q = \frac{3\kappa}{4}\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 2\kappa\ 1 - q_2\ \right]\upsilon\ 1 - \upsilon\ > 0$$

The discriminant $D$ of the polynomial (3.71), is given by

$$\begin{aligned} D = {} & 3\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 2\kappa\ 1 - q_2\ \right]\upsilon^2 \\ & - 3\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 2\kappa\ 1 - q_2\ + 2\ 3\kappa - 2\ \alpha_1\right]\upsilon \\ & + 9\kappa^2\ 1 + 5\alpha_1\ + 6\kappa\left[3\ \psi - 1\ - 8\ \phi - 1\ + 16\alpha_1 - 4\right] + 8\left[8\ \phi - 1\ - 3\ \psi - 1\ + 6\alpha_1 + 2\right] \end{aligned}$$

Now, $D$ is a monotonous function of $\upsilon$ in the interval $(0, ½)$ and has values of opposite signs at endpoints. So there are different values of $\upsilon$, say $\upsilon_{c_\kappa}$ at which the discriminant vanishes. We put $\phi - 1 = \in$ and $\psi - 1 = \in'$ in the discriminant, and solve $D = 0$ for $\upsilon$ to get:

$$\upsilon_{C\kappa} = \upsilon_{0\kappa} + \upsilon_{r\kappa} + \upsilon_{p_1\kappa} + \upsilon_{p_2\kappa} + \upsilon_{\alpha_1\kappa} \tag{3.72}$$

where $\upsilon_{0_\kappa} = \frac{1}{2} - \frac{\mathrm{M}}{18\kappa}$, $\upsilon_{r_\kappa} = \frac{2\ 24\kappa - 9\kappa^2 - 16}{27\kappa M}\ 1 - q_2$ , $\upsilon_{p_1\kappa} = \frac{16\ 4 - 3\kappa\ \in}{3\kappa M}$

$$\upsilon_{p_2\kappa} = \frac{4\left(8\kappa - 9\kappa^2 - 88\right)\in'}{27\kappa^2 M}, \upsilon_{\alpha_{1_\kappa}} = \frac{1}{9\kappa}\left[\ 3\kappa - 4\ + \frac{60\kappa - 9\kappa^2 - 64}{M}\right]\alpha_1,$$

$$\mathrm{M} = \sqrt{3(96\kappa - 9\kappa^2 - 64)}$$

$\upsilon_{C_\kappa}$ are the critical mass parameters which exist for different values of $\kappa$, and describes the joint effect of the involved parameters. Though, $\kappa$ can take values above $\in'$ and below infinity; however, we consider only values in the range $0.71459 < \kappa \leq 9.952135$, for values of $\kappa$ outside this interval do not result to physically meaningful critical mass parameters as they turn out to be either negative or indeterminate quantities.

The critical mass values $\upsilon_{C_\kappa}$, for values of kappa in this interval are:

$$\begin{aligned}
\upsilon_{C_{0.7145319}} &= 0.498958 - 26.6567\delta_2 + 1033.87 \in -798.074 \in' -111.6959\alpha_1 \\
\upsilon_{C_{0.7146}} &= 0.489815 - 2.72606\delta_2 + 105.741 \in -81.6156 \in' -11.5585\alpha_1 \\
\upsilon_{C_1} &= 0.03852 - 0.008917\delta_2 + 0.642058 \in -0.338864 \in' -0.258249\alpha_1
\end{aligned} \tag{3.73}$$

$$\upsilon_{C_{4/3}} = 0 \tag{3.74}$$

$$\begin{aligned}
\upsilon_{C_{1.34}} &= 0.0000082 - 0.0000018\delta_2 - 0.00660061 \in +0.00246018 \in' +0.00594573\alpha_1 \\
\upsilon_{C_2} &= 0.038520 - 0.00891747\delta_2 - 0.321029 \in +0.0713398 \in' +0.818978\alpha_1 \\
\upsilon_{C_5} &= 0.234029 - 0.0748866\delta_2 - 0.490167 \in +0.0190621 \in' +11.9829\alpha_1 \\
\upsilon_{C_6} &= 0.280105 - 0.10189\delta_2 - 0.524006 \in +0.00970381 \in' +19.5514\alpha_1 \\
\upsilon_{C_7} &= 0.322357 - 0.13663\delta_2 - 0.578677 \in +0.0022963 \in' +30.3533\alpha_1
\end{aligned}$$

$$\begin{aligned}
\upsilon_{C_8} &= 0.363917 - 0.189004\delta_2 - 0.680414 \in -0.0047251 \in' +46.8088\alpha_1 \\
\upsilon_{C_{9.9521}} &= 0.499475 - 52.892\delta_2 - 147.284 \in -3.22966 \in' +11533.5\alpha_1 \\
\upsilon_{C_{9.952135}} &= 0.499938 - 451.572\delta_2 - 1257.45 \in -27.5738 \in' +98253.6\alpha_1
\end{aligned} \tag{3.75}$$

where $\delta_2 = 1 - q_2$

Now, since the nature of the characteristic roots depend on the nature of the discriminant, small perturbations, mass ratio, oblateness and the constant $\kappa$, we consider the three regions of the discriminant $D$ coupled with the changes in $P$, which is due to $\kappa$, $\upsilon$, $\alpha_1$ and whether $8 \in -3 \in' <=> 0$.

1. A condition for pure imaginary roots of (3.71) is that simultaneously $0 < \upsilon < \upsilon_{C_\kappa}, D > 0$ and $P > 0$, and are represented as:

$$\lambda_{1,2} = \pm s_1 i,\ \lambda_{3,4} = \pm s_2 i$$

where

$$s_{1,2} = \frac{1}{2}\left[-P \pm \sqrt{D}\right]$$

In this case the triangular EP is stable.

The general solution can be written (Szebehely 1967a) as

$$\begin{aligned} u &= C_1 \cos s_1\tau + S_1 \sin s_1\tau + C_2 \cos s_2\tau + S_2 \sin s_2\tau \\ v &= \bar{C}_1 \cos s_1\tau + \bar{S}_1 \sin s_1\tau + \bar{C}_2 \cos s_2\tau + \bar{S}_2 \sin s_2\tau \end{aligned} \qquad (3.76)$$

Here, $s_1$ and $s_2$ are the frequencies for the long and short periodic orbit respectively, and the coefficients $C_1, S_1, \bar{C}_1$ , $\bar{S}_1$ and $C_2, S_2, \bar{C}_2$ , $\bar{S}_2$ are the long and short periodic terms respectively.

2. For $0 < \upsilon < \upsilon_{C_\kappa}, D > 0$ and $P < 0$. In this case the roots are real and distinct and can be written as

$$\lambda_{1,2} = \pm U_1,\ \lambda_{3,4} = \pm U_2$$

where $U_{1,2} = \sqrt{\frac{1}{2}\ P \pm \sqrt{D}}$

The general solution for real roots with the condition $P < 0,$ is represented as

$$\begin{aligned} u &= P_1 e^{U_1 \tau} + P_2 e^{-U_1 \tau} + P_3 e^{U_2 \tau} + P_4 e^{-U_2 \tau} \\ v &= C_1 P_1 e^{U_1 \tau} + C_1 P_2 e^{-U_1 \tau} + C_2 P_3 e^{U_2 \tau} + C_2 P_4 e^{-U_2 \tau} \end{aligned} \tag{3.77}$$

where $C_1$, $C_2$ , $P_i$ $i = 1, 2..4$ are constants. A positive root induces instability at the triangular point.

3. When $\upsilon_{C_K} < \upsilon \le \frac{1}{2}, D < 0$ and $P < 0$ , $0 < P < 2\sqrt{Q}$ . The real parts of two of the values of $\lambda$ are positive and equal. Therefore, the triangular point is unstable.

4. When $\upsilon = \upsilon_{C_K}$ , $D = 0$ . The following cases are possible.

(i) If $P < 0$ , two roots are real and equal, while the other two are negative and also equal. In this case, the triangular point is unstable.

(ii) If $P = 0$ , here all the roots are zero, and the triangular point is unstable

(iii) If $P > 0$ , all four roots are imaginary, in which two are positive and equal and the other two are negative and equal. In this case we have the resonance case of second order. Here, since the frequencies are equal and of same sign, the equilibrium point is stable (Gozdziewski 2003).

Hence, we conclude that the triangular point of the autonomized system can be stable and unstable in the range $0 < \upsilon \le \upsilon_{c_\kappa}$ depending on $\kappa$, the mass ratio, oblateness coefficient and the small perturbations; but is entirely unstable for $\upsilon_{c_\kappa} < \upsilon \le \frac{1}{2}$.

**Out-of-plane equilibrium points**

We consider the stability of the out-of--plane equilibrium point $L_6$, those of $L_{7,8,9}$ can be analogously obtained. To obtain the characteristic equation in this case, we follow the same procedure used in getting the characteristic equation (1.115). From equations (3.65), we have

$$\begin{vmatrix} \lambda^2 - \Omega^0_{\xi\xi} & -2\varphi\lambda\omega_0 + \Omega^0_{\xi\eta} & -\Omega^0_{\xi\zeta} \\ 2\varphi\omega_0\lambda - \Omega^0_{\xi\eta} & \lambda^2 - \Omega^0_{\eta\eta} & -\Omega^0_{\eta\zeta} \\ -\Omega^0_{\zeta\xi} & -\Omega^0_{\zeta\eta} & \lambda^2 - \Omega^0_{\zeta\zeta} \end{vmatrix} = 0$$

Expanding the determinates and simplifying at once gives

$$\lambda^6 - \lambda^4\ \Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} + \Omega^0_{\zeta\zeta} - 4\varphi^2\omega_0^2\ + \lambda^2\Big[\ \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - 4\varphi^2\omega_0^2\Omega^0_{\zeta\zeta} - \ \Omega^0_{\xi\eta}\ ^2$$
$$- \ \Omega^0_{\xi\zeta}\ ^2 - \ \Omega^0_{\zeta\eta}\ ^2\Big] - \Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - 2\Omega^0_{\xi\eta}\Omega^0_{\xi\zeta}\Omega^0_{\eta\zeta} + \Omega^0_{\xi\xi}\ \ \Omega^0_{\zeta\eta}\ ^2 + \ \Omega^0_{\xi\eta}\ ^2\Omega^0_{\zeta\zeta} + \ \Omega^0_{\xi\zeta}\ ^2\Omega^0_{\eta\eta} = 0 \qquad 3.78$$

where

$$\Omega^0_{\xi\xi} = \psi\left(1 + \frac{3\alpha_1}{2}\right) + 3\ \xi + \upsilon\ ^2\ \frac{\psi\xi + \ 1 - \upsilon\ \ \kappa - 1\ ^{5/3}}{\kappa^{2/3}\ 1 - \upsilon\ ^{2/3}}\left(1 + \frac{5\alpha_1}{2}\right) + 3\ \xi + \upsilon - 1\ ^2\left[1 + \frac{2}{3}\ 1 - q_1\ + \frac{5\alpha_1}{2}\right]$$
$$\times \frac{-\psi\xi + \upsilon\ \ \kappa - 1\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}$$
$$\Omega^0_{\eta\eta} = \psi\left(1 + \frac{3}{2}\alpha_1\right) \quad , \ \Omega^0_{\xi\eta} = \Omega^0_{\eta\zeta} = 0 \qquad (3.79)$$

$$\Omega^0_{\xi\xi} = 3\zeta\left[\frac{\xi+\upsilon\ \ \psi\xi+\ 1-\upsilon\ \ \kappa-1\ ^{\frac{5}{3}}\ 2+5\alpha_1}{2\kappa^{2/3}\ 1-\upsilon\ ^{2/3}} + \frac{\xi+\upsilon-1\ \ -\psi\xi+\upsilon\ \kappa-1\ ^{\frac{5}{3}}}{\kappa\upsilon\ ^{2/3}}\left\{1+\frac{2\ 1-q_2}{3}+\frac{5\alpha_1}{2}\right\}\right]$$

$$\Omega^0_{\zeta\zeta} = 3\zeta^2\left\{\frac{\psi\xi+\ 1-\upsilon\ \ \kappa-1\ ^{5/3}}{\kappa\ 1-\upsilon\ ^{2/3}}\left(1+\frac{5}{2}\alpha_1\right)+\frac{-\psi\xi+\upsilon\ \kappa-1\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}\left(1+\frac{2}{3}\ 1-q_2\ +\frac{5}{2}\alpha_1\right)\right\}$$

These derivatives have been evaluated at the out-of-plane equilibrium points and we have neglected second and higher order terms and product of $\alpha_1$ with $1-q_2$ and $\psi-1=\in'$.

Now, substituting equations of system (3.79) in (3.78), results in

$$\lambda^6 - p\lambda^4 + q\lambda^2 + r = 0 \tag{3.80}$$

where

$$p = \frac{3QY^{5/3}}{A^{2/3}}\left[\ \xi+\upsilon\ ^2+\zeta^2\ \right]+\frac{3RZ^{5/3}}{B^{2/3}}\left[\ \xi+\upsilon-1\ ^2+\zeta^2\ \right]-2-8\in+2\in'$$

$$q = 1+\frac{9\alpha_1}{2}\left[\frac{Y^{5/3}}{A^{2/3}}\ \xi+\upsilon\ ^2+\frac{Z^{5/3}}{B^{2/3}}\ \xi+\upsilon-1\ ^2+\frac{2}{3}-3\zeta^2\left(\frac{Y^{5/3}}{A^{2/3}}+\frac{Z^{5/3}}{B^{2/3}}\right)\right]$$
$$+\varepsilon'\left[2+\frac{3Y^{5/3}}{A^{2/3}}\ \ \xi+\upsilon\ ^2+\zeta^2\ +\frac{3Z^{5/3}}{B^{2/3}}\ \ \xi+\upsilon-1\ ^2+\zeta^2\ \right]-24\varepsilon\left(\frac{Y^{5/3}}{A^{2/3}}+\frac{Z^{5/3}}{B^{2/3}}\right)\zeta^2$$
$$-3\zeta^2\left[\frac{3QY^{5/3}}{A^{2/3}}+\frac{3RZ^{5/3}}{B^{2/3}}+\frac{6V\ YZ\ ^{5/3}\ \xi+\upsilon\ \ \xi+\upsilon-1}{AB\ ^{2/3}}-\frac{Y^{5/3}S}{A^{2/3}}-\frac{Z^{5/3}T}{B^{2/3}}-\frac{3V\ YZ\ ^{5/3}}{AB\ ^{2/3}}\left\{\left(\xi+\upsilon\right)^2+\left(\xi+\upsilon-1\right)^2\right\}\right]$$

$$r = 3\zeta^2\left(1+\frac{3}{2}\alpha_1+\in'\right)\left[3\left\{\frac{Y^{10/3}\ \xi+\upsilon\ ^2U}{A^{4/3}}+\frac{2V\ \ \xi+\upsilon\ \ \xi+\upsilon-1\ \ YZ\ ^{5/3}}{AB\ ^{2/3}}+\frac{Z^{10/3}W\ \ \xi+\upsilon-1\ ^2}{B^{4/3}}\right\}\right.$$
$$\left.-\left\{\frac{Y^{5/3}S}{A^{2/3}}+\frac{Z^{5/3}T}{B^{2/3}}+\frac{3\ \xi+\upsilon\ ^2Y^{10/3}U}{A^{4/3}}+\frac{3\ \xi+\upsilon\ ^2\ YZ\ ^{5/3}V}{AB\ ^{2/3}}+\frac{3\ \xi+\upsilon-1\ ^2\ YZ\ ^{5/3}V}{AB\ ^{2/3}}+\frac{3\ \xi+\upsilon-1\ ^2Z^{10/3}W}{B^{4/3}}\right\}\right]$$

with $Y = 1+\epsilon' \xi + 1-\upsilon \kappa - 1$ , $Z = -1+\epsilon' \xi + \upsilon \kappa - 1$

$$Q = 1+\frac{5}{2}\alpha_1,\ R = 1+\frac{2}{3}(1-q_2) + \frac{5}{2}\alpha_1,\ S = 1+4\alpha_1+\epsilon'$$
$$T = 1+\frac{2}{3}(1-q_2) + 4\alpha_1+\epsilon',\ U = 1+5\alpha_1,\ V = 1+\frac{4}{3}(1-q_2) + 5\alpha_1$$
$$W = 1+\frac{2}{3}(1-q_2) + 5\alpha_1$$

The stability of the out-of-plane EP $L_6$ is determined by the roots of the characteristic equation (3.79). We perform a numerical exploration to compute the out of plane points, the partial derivatives and values of the quantities $p, q$ and $r$ using the software package *Mathematica* (Wolfram 2003) for values of $\kappa > 1$, $q_2 = 0.9855, \phi = 1.001, \psi = 1.002, \alpha_1 = 0.02$ and found that the signs of the quantities $p, q$ and $r$ could be negative or positive depending on the interval where $\kappa$ lies. The following cases have been considered:

i. $p < 0$, $q > 0$ and $r < 0$, there is one change in sign, which implies there is exactly one positive root according to the Descartes rule of sign.

ii. $p > 0$, $q < 0$ and $r < 0$, there is also one change in sign.

iii. $p > 0$, $q < 0$ and $r > 0$, there are two changes in sign indicating there two positive, two negative and two imaginary roots

The stability of the out-of-plane points would have been achieved if a fourth case, that is, $p < 0$, $q > 0$ and $r > 0$, however, this case does not arise. Hence, we conclude that the out-of-plane equilibrium points of the autonomized system are in general an unstable equilibrium points due to at least a positive root.

**Stability of equilibrium points of the non-autonomous system**

The analysis of the stability of the EPs $L_i \;\; i = 1,2...9$ would depend on the methods applied, since these equilibrium points are themselves time dependent. For example, using the definition of a Lyapunov stable solution (Krasnov *et al*. 1983), we have in the triangular case:

$$\lim_{t\to\infty} x^{4,5}\;\; t \;\; = \xi^{4,5} \lim_{t\to\infty} R \;\; t \;\; = \infty \tag{3.81}$$

Equation (3.81) proves the instability of the solutions $x \;\; t$ , and similarly for $y \;\; t$ , according to the Lyapunov's theorem, and is same as the result of Luk'yanov (1990).

The relationship between the old and new independent variables $t$ and $\tau$ are given (Poincare 1911; Singh and Leke 2010) as

$$0 < \lim_{t\to\infty} \tau < \infty \tag{3.82}$$

and $$\omega_0^2 \lim_{t\to\infty} \frac{\tau}{t} = \Gamma < S^* \tag{3.83}$$

where $S^* = 4 \;\; \pi\kappa \;\; ^2$, and finite

Equation (3.82) implies that as $t$ is approaching $\infty$, $\tau$ is also always approaching a finite value, while (3.83) indicates that $\lim_{t\to\infty} \frac{\tau}{t}$ always tends to a positive finite value $\Gamma$. However, this finite values decreases due to oblateness of the first primary.

The system of equations with constant coefficients and the reducible system are regular. The system with variable coefficients is reducible due to the Mestschersky transformation (1.46).

Therefore, we apply Lyapunov's theorem, using the LCN on the stability of the motion around the equilibrium points depending on time of the system. The calculations of the LCNs here are limited to finding the maximum LCN, which gives a computed value that we use as a metric to give a qualitative indication of how stability vary over the solutions. Following Singh and Leke (2010), the LCN of the triangular solutions varying with time with the consideration that as $t \to \infty$, $\tau$ is approaching a finite value, is:

$$L_{4,5}\left[ x\ \ t\ \right] = -\lim_{t\to\infty} \frac{1}{t} \ln\left|\xi R\ \ t\ \right| = 0$$

similarly, $$L_{4,5}\left[ y\ \ t\ \right] = 0 \tag{3.84}$$

Thus, the Lyapunov characteristic number is zero for triangular solutions, therefore the stability or instability of the perturbed motion cannot be determined directly from the triangular solutions.

Using equation (1.107), the equilibrium points (3.64) of the system (3.14) with variable coefficients can be represented (Singh and Leke 2010), with given solutions (3.76) and transformation (1.46) as:

$$x_1 = C_1 \cos s_1 \tau R\ \ t\ \ , \ x_2 = S_1 \sin s_1 \tau R\ \ t\ \ , x_3 = C_2 \cos s_2 \tau R\ \ t\ \ , x_4 = S_2 \sin s_2 \tau R\ \ t$$

$$x_5 = \xi_0 R\ \ t\ \ , y_1 = \bar{C}_1 \cos s_1 \tau R\ \ t\ \ , \ y_2 = \bar{S}_1 \sin s_1 \tau R\ \ t\ \ \ y_3 = \bar{C}_2 \cos s_2 \tau R\ \ t$$

$$y_4 = \bar{S}_2 \sin s_2 \tau R\ \ t\ \ , \ \ y_5 = \eta_0 R\ \ t \tag{3.85}$$

where $\xi_0, \eta_0$ are coordinates of the third body.

These EPs correspond to the region where $0 < \upsilon < \upsilon_{c_\kappa}, P > 0$. Numerically, this holds when $0.002 < \kappa < 1.31169451$ ; when there are no small perturbations in the Coriolis and

centrifugal forces, we have $0<\kappa<1.311$. Further, if the first primary is not an oblate spheroid, then we have $0<\kappa<\frac{4}{3}$, and this agrees with the result in Singh and Leke (2010).

Similarly, the EPs with conditions $0<\upsilon<\upsilon_{c_\kappa}, P<0$ using (1.107), solutions (3.77) and MT (1.46) can be represented as:

$$x_6=e^{\pm U_1\tau}R\ t\ , x_7=e^{\pm U_2\tau}R\ t\ , x_8=\xi_0 R\ t$$

$$y_6=C_1e^{\pm U_1\tau}, y_7=C_2e^{\pm U_2\tau}, y_8=\eta_0 R\ t \tag{3.86}$$

The solutions (3.86) correspond to the region where, $P<0$ and the is determined by the parameter $\kappa$, oblateness of the first primary and whether $8\in -3\in'<=>0$; since in both cases $0<\upsilon<\upsilon_{c_\kappa}$, $D>0$ .

For the solutions (3.85), their LCN's are (Singh and Leke 2010)

$$L\ x_i\ =L\ y_i\ =0,\quad i=1,2...5 \tag{3.87}$$

In view of the EPs (3.86) and using equation (3.83); their LCN's are:

$$L\ x_6\ =\mp U_1\Gamma,\quad L\ x_7\ =\mp U_2\Gamma,\quad L\ x_8\ =0$$

Similarly $\quad L\ y_6\ =\mp U_1\Gamma \quad L\ y_7\ =\mp U_2\Gamma,\quad L\ y_8\ =0$ (3.88)

Therefore, the LCNs are negative for solutions with positive exponents, positive for solutions with negative exponents, and, zero for solutions having imaginary exponents and constant solutions. Hence, when the roots of characteristic equation of the autonomized

equations are positive, then the LCN of the solutions varying with time is negative, and consequently the solutions are unstable according to the Lyapunov theorem. If these roots are pure imaginary quantities, then the LCN of corresponding solution varying with time is zero and this corresponds to regular or periodic motion. Hence, the solution is stable. Finally, when the roots are negative, the LCN of corresponding time-dependent solutions is positive and consequently stable.

The same stability analysis can be done in the case of the collinear and out-of-plane solutions of the non-autonomous system.

## 3.3 Combined Effects of Oblateness, Radiation, Perturbation in the Restricted Three-Body Problem with Variable Masses

In this problem, we extend the R3BP with variable mass of Bekov (1988), by taking both primaries as radiation sources and oblate spheroids under effects of small perturbations in the Coriolis and centrifugal forces when the masses of the primaries vary in accordance with the UML and their motion given by the GMP. This problem is also a modified version of the research problem **3.2** and can be viewed as an extension of the work by AbdulRaheem and Singh (2006), when the masses of the primaries are assumed to vary with time.

### 3.3.1 Equations of motion of the third body

The equations of motion of the third body when both primaries are oblate spheroids and radiating, under effects of small perturbations in Coriolis and centrifugal forces, have the form

$$\ddot{x} - 2\omega\varphi\dot{y} = \omega^2\psi x + \dot{\omega}\varphi y - \frac{\mu_1 q_1 (x - x_1)}{r_1^3} - \frac{\mu_2 q_2 (x - x_2)}{r_2^3} - \frac{3\mu_1 q_1 A_1 (x - x_1)}{2r_1^5} - \frac{3\mu_2 q_2 A_2 (x - x_2)}{r_2^3}$$

$$\ddot{y} + 2\omega\varphi\dot{x} = \omega^2\psi y - \dot{\omega}\varphi x - \frac{\mu_1 q_1 y}{r_1^3} - \frac{\mu_2 q_2 y}{r_2^3} - \frac{3\mu_1 q_1 A_1 y}{2r_1^5} - \frac{3\mu_2 q_2 A_2 y}{r_2^3} \tag{3.89}$$

$$\ddot{z} = -\frac{\mu_1 q_1 z}{r_1^3} - \frac{\mu_2 q_2 z}{r_2^3} - \frac{3\mu_1 q_1 A_1 z}{2r_1^5} - \frac{3\mu_2 q_2 A_2 z}{r_2^3}$$

where $$\omega^2 = \frac{\mu}{\kappa r^3}\left[1 + \frac{3}{2r^2}(A_1 + A_2)\right] \tag{3.90}$$

$A_1$ and $A_2$ is the oblateness of the first and second primary, respectively and $q_i\ (i = 1,2)$ are the radiation pressure forces of the primaries, respectively. The other parameters have been defined earlier.

The autonomized dynamical system with constant coefficients is given by the equations

$$\xi'' - 2\varphi\eta' = \frac{\partial\Omega}{\partial\xi},$$

$$\eta'' + 2\varphi\xi' = \frac{\partial\Omega}{\partial\eta}, \tag{3.91}$$

$$\zeta'' = \frac{\partial\Omega}{\partial\zeta}$$

where

$$\Omega = \frac{\omega_0^2\left(\psi + \kappa - 1\right)\left(\xi^2 + \eta^2\right)}{2} + \frac{\omega_0^2\left(\kappa - 1\right)\zeta^2}{2} + \frac{q_1\kappa\left(1 - \upsilon\right)}{\rho_1} + \frac{q_2\kappa\upsilon}{\rho_2} + \frac{\kappa q_1\alpha_1\left(1 - \upsilon\right)}{2\rho_1^3} + \frac{\kappa q_2\alpha_2\upsilon}{2\rho_2^3}$$

$$\omega_0^2 = 1 + \frac{3}{2}\left(\alpha_1 + \alpha_2\right),$$

$$\rho_1 = \sqrt{\left(\xi + \upsilon\right)^2 + \eta^2 + \zeta^2},$$

$$\rho_2 = \sqrt{\left(\xi + \upsilon - 1\right)^2 + \eta^2 + \zeta^2}$$

where, $\alpha_i$ and $q_i$ $i = 1, 2$ are the oblateness coefficients and radiation pressure force of the primaries, respectively; the other parameters are defined in equations (1.004).

### 3.3.2. Locations of equilibrium points

The EPs are the solutions of the autonomized system when the velocities and components of acceleration are zero. They are defined by the system of the equations

$$\frac{\partial \Omega}{\partial \xi} = \frac{\partial \Omega}{\partial \eta} = \frac{\partial \Omega}{\partial \zeta} = 0 .$$

That is,

$$\omega_0^2 \ \psi + \kappa - 1 \ \xi - \frac{q_1 \kappa \ 1-\upsilon \quad \xi + \upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon \ \xi + \upsilon - 1}{\rho_2^3} - \frac{3}{2}\frac{\alpha_1 q_1 \kappa \ 1-\upsilon \quad \xi + \upsilon}{\rho_1^5} - \frac{3}{2}\frac{q_2 \alpha_2 \kappa \upsilon \ \xi + \upsilon - 1}{\rho_2^5} = 0$$

$$\left[ \omega_0^2 \ \psi + \kappa - 1 \ - \frac{q_1 \kappa \ 1-\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon}{\rho_2^3} - \frac{3}{2}\frac{\alpha_1 q_1 \kappa \ 1-\upsilon}{\rho_1^5} - \frac{3}{2}\frac{q_2 \alpha_2 \kappa \upsilon}{\rho_2^5} = 0 \right] \eta = 0 \qquad (3.92)$$

$$\left[ \omega_0^2 \ \kappa - 1 \ - \frac{q_1 \kappa \ 1-\upsilon}{\rho_1^3} - \frac{q_2 \kappa \upsilon}{\rho_2^3} - \frac{3}{2}\frac{\alpha_1 q_1 \kappa \ 1-\upsilon}{\rho_1^5} - \frac{3}{2}\frac{q_2 \alpha_2 \kappa \upsilon}{\rho_2^5} \right] \zeta = 0$$

When the first equation is solved for $\xi \neq 0$ and $\eta = \zeta = 0$, we get a solution collinear with the line joining the primaries. When the system of equations is solved for $\xi \neq 0, \eta \neq 0$ and $\zeta = 0$, a pair of solution which forms two triangles with the primaries exist. The last type of solution, which is called the out-of-plane or coplanar equilibrium points and occur in pairs, are found when we solve the system of equation (3.92) for $\xi \neq 0, \eta = 0$ and $\zeta \neq 0$. We devote this section to finding the locations of these EPs.

**Collinear equilibrium points**

The positions of the collinear EPs $L_{i\kappa}$ $(=1,2,3)$ are obtained by solving equations of system (3.92) with $\eta = \zeta = 0$, and the abscissas are:

$$\xi_1 = -\upsilon - \varepsilon_1, \quad \xi_2 = 1-\upsilon-\varepsilon_2, \quad \xi_3 = 1-\upsilon+\varepsilon_3, \tag{3.93}$$

where $\varepsilon_i > 0,\ i = 1,2,3$ are the roots of the first equation of (3.92) and,

$$\rho_1 = |\xi + \upsilon|, \quad \rho_2 = |\xi + \upsilon - 1| \tag{3.94}$$

Following problem **3.2**, we compute the collinear points $L_{2\kappa}$, by substituting $\xi_2 = \xi$ and $\psi = 1 + \in'$, in the first equation of (3.92), and solve to get an algebraic equation of ninth degree in $\varepsilon_2$:

$$\varepsilon_2^9 - a_1\varepsilon_2^8 + a_2\varepsilon_2^7 - a_3\varepsilon_2^6 + a_4\varepsilon_2^5 - a_5\varepsilon_2^4 + a_6\varepsilon_2^3 - a_7\varepsilon_2^2 + a_8\varepsilon_2 - a_9 = 0 \tag{3.95}$$

where $a_1 = \dfrac{5+4\upsilon_2}{1+\upsilon_2}$, $a_2 = \dfrac{2\ 5+3\upsilon_2}{1+\upsilon_2}$, $a_3 = \dfrac{2\ 2\upsilon_2+5}{1+\upsilon_2} + \dfrac{\kappa\left[2q_1+2\upsilon_2 q_2-3\ 1+\upsilon_2\quad \alpha_1+\alpha_2\ \right]}{2\ \kappa+\in'\quad 1+\upsilon_2}$

$$a_4 = \frac{\upsilon_2+5}{1+\upsilon_2} + \frac{\kappa\left[\ 2-3\alpha_1-3\alpha_2\quad 1+2\upsilon_2\ -2\ 1-q_1\ -4\upsilon_2\ 1-q_2\ \right]}{\kappa+\in'\quad 1+\upsilon_2}$$

$$a_5 = \frac{1}{1+\upsilon_2} + \frac{\kappa\left[2+3\ \alpha_1-\alpha_2\ -2\ 1-q_1\ +6\upsilon_2\ 2-3\alpha_1-2\alpha_2-2\ 1-q_2\ \right]}{2\ \kappa+\in'\quad 1+\upsilon_2}$$

$$a_6 = \frac{2\kappa\upsilon_2\left[2-3\alpha_1-2\ 1-q_2\ \right]}{\kappa+\in'\quad 1+\upsilon_2} \qquad a_7 = \frac{\kappa\upsilon_2\left[2-3\alpha_1+15\alpha_2-2\ 1-q_2\ \right]}{2\ \kappa+\in'\quad 1+\upsilon_2},$$

$$a_8 = \frac{6\alpha_2\kappa\upsilon_2}{\kappa+\in'\quad 1+\upsilon_2}, \qquad a_9 = \frac{3\alpha_2\kappa\upsilon_2}{2\ \kappa+\in'\quad 1+\upsilon_2}; \quad \upsilon_2 = \frac{\upsilon}{1-\upsilon}\ .$$

The polynomial (3.95) is of degree nine because both primaries have been assumed to have the shape of an oblate spheroids. Notice that the polynomial has degree seven when one of the primaries is an oblate spheroid (Equation 3.21). We can analogously calculate those for the collinear points $L_1$ and $L_3$; in each case, we get an equation of ninth degree, which depend on the mass ratio, small perturbation in the centrifugal force, radiation pressures, oblateness and $\kappa$. The real roots of (3.95) gives the positons of the collinear EPs. Numerically, the binary system known as "peanut stars" is used to present the number of possible real roots of equation (3.95), in chapter **FIVE.**

**Triangular equilibrium points**

The triangular points of the autonomized system, which lie in the orbital plane $\xi\eta$, denoted by $L_4$ and $L_5$ are the solutions of system (3.92) with, $\eta \neq 0,\ \zeta = 0$. Following the same method used in problem **3.2**, we solve the equations:

$$\omega_0^2\ \psi+\kappa-1\ \xi-\frac{q_1\kappa\ 1-\upsilon\ \ \xi+\upsilon}{\rho_1^3}-\frac{q_2\kappa\upsilon\ \ \xi+\upsilon-1}{\rho_2^3}-\frac{3\alpha_1 q_1\kappa\ 1-\upsilon\ \ \xi+\upsilon}{2\rho_1^5}-\frac{3\alpha_2 q_2\kappa\upsilon\ \ \xi+\upsilon-1}{2\rho_2^5}=0$$

$$\omega_0^2\ \psi+\kappa-1\ -\frac{q_1\kappa\ 1-\upsilon}{\rho_1^3}-\frac{q_2\kappa\upsilon}{\rho_2^3}-\frac{3\alpha_1 q_1\kappa\ 1-\upsilon}{2\rho_1^5}-\frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5}=0,\quad \eta\neq 0 \qquad (3.96)$$

From the first equation of (3.96), we get

$$\begin{aligned}\xi+\upsilon&\left[\omega_0^2\ \psi+\kappa-1\ -\frac{q_1\kappa\ 1-\upsilon}{\rho_1^3}-\frac{q_2\kappa\upsilon}{\rho_2^3}-\frac{3\alpha_1 q_1\kappa\ 1-\upsilon}{2\rho_1^5}-\frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5}\right]\\&=\upsilon\left[\omega_0^2\ \psi+\kappa-1\ -\frac{q_2\kappa\upsilon}{\rho_2^3}-\frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5}\right]\end{aligned} \qquad (3.97)$$

Substituting the second equation of (3.96), in (3.97), we get

$$\omega_0^2\ \ \psi+\kappa-1\ -\frac{q_2\kappa}{\rho_2^3}-\frac{3\alpha_2 q_2\kappa}{2\rho_2^5}=0 \tag{3.98}$$

Similarly, we have

$$\omega_0^2\ \ \psi+\kappa-1\ -\frac{q_1\kappa}{\rho_1^3}-\frac{3\alpha_1 q_1\kappa}{2\rho_1^5}=0 \tag{3.99}$$

When the primaries are not radiating and are spherical, we have $q_1=q_2=1$ and $\alpha_1=\alpha_2=0$. In this case, the solutions of equations (3.98) and (3.99) are

$$\rho_1=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{1/3},\ \rho_2=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{1/3}$$

And so, we can assume equation (3.26) to be the solutions when the primaries are oblate spheroids and radiating bodies. That is

$$\rho_1=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{1/3}+\varepsilon_1 \text{ and } \rho_2=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{1/3}+\varepsilon_2 \tag{3.100}$$

where $\varepsilon_i\ \ i=1,2$ are very small quantities.

Substituting for $\rho_2^{-3}$ and $\rho_2^{-5}$ in equation (3.98), we get

$$\omega_0^2\ \ \psi+\kappa-1\ -\frac{q_2\kappa\ \ \psi+\kappa-1}{\kappa}\left[1-\frac{3\varepsilon_2\ \ \psi+\kappa-1\ ^{\frac{1}{3}}}{\kappa^{1/3}}\right]-\frac{3q_2\kappa\alpha_2\ \ \psi+\kappa-1\ ^{\frac{5}{3}}}{2\kappa^{5/3}}\left[1-\frac{5\varepsilon_2\ \ \psi+\kappa-1\ ^{\frac{1}{3}}}{\kappa^{1/3}}\right]=0$$

Expanding and ignoring product of $\alpha_2$ and $\varepsilon_2$, we get

$$\omega_0^2\ \psi+\kappa-1\ -q_2\ \psi+\kappa-1\left[1-\frac{3\varepsilon_2\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right]-\frac{3}{2}\frac{q_2\alpha_2\ \psi+\kappa-1^{5/3}}{\kappa^{2/3}}=0 \tag{3.101}$$

Expressing $q_2=\left[1-\ 1-q_2\ \right]$, where $1-q_2\ <<1$, equation (3.101) becomes

$$\omega_0^2\ \psi+\kappa-1\ -\left[1-\ 1-q_2\ \right]\ \psi+\kappa-1\left[1-\frac{3\varepsilon_2\ \psi+\kappa-1^{1/3}}{\kappa^{1/3}}\right]-\frac{3\left[1-\ 1-q_2\ \right]\alpha_2\ \psi+\kappa-1^{5/3}}{2\kappa^{2/3}}=0$$

Ignoring products of $1-q_2$ with $\alpha_2$ and $\varepsilon_2$, and solving correctly, we get

$$\varepsilon_2=\frac{\kappa^{1/3}}{3\ \psi+\kappa-1^{1/3}}\left[-\frac{3}{2}\ \alpha_1+\alpha_2\ -\ 1-q_2\ +\frac{3}{2}\frac{\alpha_2\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}\right] \tag{3.102}$$

Similarly, if we follow the same procedure and substitute $\rho_1^{-3}$ and $\rho_1^{-5}$ in equation (3.99), and solving, we get

$$\varepsilon_1=\frac{\kappa^{1/3}}{3\ \psi+\kappa-1^{1/3}}\left[-\frac{3}{2}\ \alpha_1+\alpha_2\ -\ 1-q_1\ +\frac{3}{2}\frac{\alpha_1\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}\right] \tag{3.103}$$

Now, substituting equation (3.103) in the first equation of (3.100), yields

$$\rho_1=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{1/3}\left[1-\frac{1}{2}\ \alpha_1+\alpha_2\ -\frac{1}{3}\ 1-q_1\ +\frac{1}{2}\frac{\alpha_1\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}\right] \tag{3.104}$$

Similarly, when we substitute equation (3.102) in the second equation of (3.100), we get

$$\rho_2=\left[\frac{\kappa}{\psi+\kappa-1}\right]^{1/3}\left[1-\frac{1}{2}\ \alpha_1+\alpha_2\ -\frac{1}{3}\ 1-q_2\ +\frac{1}{2}\frac{\alpha_2\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}\right] \tag{3.105}$$

Subtracting equation (3.104) from (3.105), we have

$$\rho_2-\rho_1=\frac{\kappa^{1/3}}{\psi+\kappa-1^{1/3}}\left[1-\frac{1}{2}\ \alpha_1+\alpha_2\ -\frac{1}{3}\ 1-q_2\ +\frac{1}{2}\frac{\alpha_2\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}-1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{1}{3}\ 1-q_1\right.$$
$$\left.-\frac{1}{2}\frac{\alpha_1\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}\right]$$

$$\rho_2-\rho_1=\frac{\kappa^{1/3}}{\psi+\kappa-1^{1/3}}\left[\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ -\frac{1}{2}\frac{\alpha_1\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}+\frac{1}{2}\frac{\alpha_2\ \psi+\kappa-1^{2/3}}{\kappa^{2/3}}\right]$$

$$\rho_2-\rho_1=\frac{\kappa^{1/3}}{\psi+\kappa-1^{1/3}}\left[\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ -\frac{\alpha_1-\alpha_2\ \ \psi+\kappa-1^{2/3}}{2\kappa^{2/3}}\right] \tag{3.106}$$

Summing equations (3.104) and (3.105), gives

$$\rho_2+\rho_1=\frac{\kappa^{1/3}}{\psi+\kappa-1^{1/3}}\left[2-\ \alpha_1+\alpha_2\ -\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ +\frac{\alpha_1+\alpha_2\ \ \psi+\kappa-1^{2/3}}{2\kappa^{2/3}}\right] \tag{3.107}$$

Using equations (3.106) and (3.107) in equation (3.34) and expanding; we ignore products of very small parameters and retain only linear terms, to get

$$\rho_2^2-\rho_1^2=\frac{2\kappa^{2/3}}{\psi+\kappa-1^{1/3}}\left[\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ -\frac{\alpha_1-\alpha_2\ \ \psi+\kappa-1^{2/3}}{2\kappa^{2/3}}\right] \tag{3.108}$$

Substituting (3.108) in (3.32), yields

$$\xi=\frac{1}{2}-\upsilon-\frac{\kappa^{2/3}}{\psi+\kappa-1^{1/3}}\left[\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ -\frac{\alpha_1-\alpha_2\ \ \psi+\kappa-1^{2/3}}{2\kappa^{2/3}}\right] \tag{3.109}$$

To find the $\eta-$coordinate, observe that

$$\rho_1 - \rho_2 = \frac{\kappa^{1/3}}{\psi + \kappa - 1^{1/3}}\left[\frac{1}{3}\ 1-q_2\ \ -\frac{1}{3}\ 1-q_1\ \ + \frac{\alpha_1\ \ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}} - \frac{\alpha_2\ \ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right]$$

Therefore,

$$\rho_1^2 + \rho_2^2 = \frac{2\kappa^{2/3}}{\psi + \kappa - 1^{\ 2/3}}\left[1-\frac{1}{2}\ \alpha_1+\alpha_2\ \ -\frac{1}{3}\ 1-q_2\ \ + \frac{\alpha_2\ \ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}} - \frac{1}{2}\ \alpha_1+\alpha_2\ \ -\frac{1}{3}\ 1-q_1 \right.$$
$$\left. + \frac{\alpha_1\ \ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right] \qquad (3.110)$$

and $\rho_2^2 - \rho_1^{2\ \ 2} = 0$ (3.111)

Substituting equations (3.109) and (3.110) in equation (3.33), we have

$$\eta = \pm\left\{\frac{\kappa^{2/3}}{\psi + \kappa - 1^{\ 2/3}}\left[1-\frac{1}{3}\ 1-q_1\ \ -\frac{1}{3}\ 1-q_2\ \ -\ \alpha_1+\alpha_2\ \ + \frac{\alpha_1+\alpha_2\ \ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right] - \frac{1}{4}\right\}^{1/2}$$

$$\eta = \pm\left\{\frac{\kappa^{2/3}}{\psi + \kappa - 1^{\ 2/3}}\left[1-\frac{1}{3}\ 1-q_1\ \ -\frac{1}{3}\ 1-q_2\ \ -\ \alpha_1+\alpha_2\ \ + \frac{\alpha_1+\alpha_2\ \ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}} - \frac{\psi+\kappa-1^{\ \frac{2}{3}}}{4\kappa^{2/3}}\right]\right\}^{1/2}$$

$$\eta = \pm\left\{\frac{4\kappa^{4/3}\left[1-\frac{1}{3}\ 1-q_1\ \ -\frac{1}{3}\ 1-q_2\ \ -\ \alpha_1+\alpha_2\ \ + \frac{\alpha_1+\alpha_2\ \ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right] - \kappa^{2/3}\ \ \psi+\kappa-1^{\ 2/3}}{4\kappa^{2/3}\ \ \psi+\kappa-1^{\ 2/3}}\right\}^{1/2}$$

$$\eta = \pm\left\{\frac{\kappa^{\frac{2}{3}}}{4\kappa^{2/3}\ \ \psi+\kappa-1\ ^{2/3}}\left\{4\kappa^{2/3}\left[1-\frac{1-q_1}{3}-\frac{1-q_2}{3}-\ \ \alpha_1+\alpha_2\ +\frac{\alpha_1+\alpha_2\quad \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right]\right.\right.$$
$$\left.\left.-\ \ \psi+\kappa-1\ ^{2/3}\right\}\right\}^{1/2}$$

$$\eta = \pm\frac{\kappa^{1/3}}{2\kappa^{1/3}\ \ \psi+\kappa-1\ ^{1/3}}\left\{4\kappa^{2/3}-4\kappa^{2/3}\left[\frac{1-q_1}{3}+\frac{1-q_2}{3}+\ \ \alpha_1+\alpha_2\ -\frac{\alpha_1+\alpha_2\quad \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right]\right.$$
$$\left.-\ \ \psi+\kappa-1\ ^{\frac{2}{3}}\right\}^{1/2}$$

$$\eta = \pm\frac{1}{2\ \ \psi+\kappa-1\ ^{\frac{1}{3}}}\left\{4\kappa^{2/3}-\ \ \psi+\kappa-1\ ^{2/3}+4\kappa^{2/3}\left[-\frac{1-q_1}{3}-\frac{1-q_2}{3}-\ \ \alpha_1+\alpha_2\right.\right.$$
$$\left.\left.+\frac{\alpha_1+\alpha_2\quad \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right]\right\}^{1/2}$$

$$\eta = \pm\frac{1}{2\ \ \psi+\kappa-1\ ^{\frac{1}{3}}}\left\{\left[4\kappa^{2/3}-\ \ \psi+\kappa-1\ ^{2/3}\right]\left[1+\frac{4\kappa^{2/3}}{4\kappa^{\frac{2}{3}}-\ \ \psi+\kappa-1\ ^{2/3}}\left\{-\frac{1-q_1}{3}-\frac{1-q_2}{3}-\ \ \alpha_1+\alpha_2\right.\right.\right.$$
$$\left.\left.\left.+\frac{\alpha_1+\alpha_2\quad \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right\}\right]\right\}^{1/2}$$

$$\eta = \pm\frac{\sqrt{4\kappa^{\frac{2}{3}}-\ \ \psi+\kappa-1\ ^{2/3}}}{2\ \ \psi+\kappa-1\ ^{\frac{1}{3}}}\left[1+\frac{4\kappa^{\frac{2}{3}}}{4\kappa^{\frac{2}{3}}-\ \ \psi+\kappa-1\ ^{2/3}}\left\{-\frac{1-q_1}{3}-\frac{1-q_2}{3}-\ \ \alpha_1+\alpha_2\right.\right.$$
$$\left.\left.\left.+\frac{\alpha_1+\alpha_2\quad \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right\}\right]\right\}^{\frac{1}{2}}$$

Hence, the coordinates of the triangular EPs of the autonomized when the primaries are oblate spheroids and radiation sources under effects of small effective perturbations in the Coriolis and centrifugal forces have the forms

$$\xi = \frac{1}{2} - \upsilon - \frac{\kappa^{2/3}}{\psi + \kappa - 1^{1/3}} \left[ \frac{1}{3}\ 1-q_1\ - \frac{1}{3}\ 1-q_2\ - \frac{\alpha_1 - \alpha_2\quad \psi + \kappa - 1^{\frac{2}{3}}}{2\kappa^{2/3}} \right] \tag{3.112}$$

$$\eta = \pm \frac{\sqrt{4\kappa^{2/3} - \ \psi + \kappa - 1^{\ 2/3}}}{2\ \psi + \kappa - 1^{\ 1/3}} \left\{ 1 - \frac{2\kappa^{2/3}}{4\kappa^{2/3} - \ \psi + \kappa - 1^{\ 2/3}} \left[ \frac{1}{3}\ 1-q_1\ + \frac{1}{3}\ 1-q_2\ +\ \alpha_1 + \alpha_2 \right.\right.$$
$$\left.\left. - \frac{\alpha_1 + \alpha_2\quad \psi + \kappa - 1^{\ 2/3}}{2\kappa^{2/3}} \right] \right\}$$

These solutions exist for various $\kappa\ \ \psi - 1 < \kappa < \infty$ different from the triangular EPs with constant masses. Numerically as $\kappa \to \infty$, there is more than a pair of these points; interestingly, infinite numbers of the points do not exist. When $\kappa = 1$, the solution is similar to those of AbdulRaheem and Singh (2006). However, the triangular points for other values of $\kappa$ have no analogies in any previous works.

**Out-of-plane equilibrium points**

The out-of-plane EPs are located outside the plane of motion. Their coordinates are the solutions of (3.92) with $\eta = 0$. That is, we must solve the equations

$$\omega_0^2\ \ \psi + \kappa - 1\ \ \xi - \frac{q_1\kappa\ \ 1-\upsilon\quad \xi + \upsilon}{\rho_1^3} - \frac{q_2\kappa\upsilon\ \ \xi + \upsilon - 1}{\rho_2^3} - \frac{3\alpha_1 q_1\kappa\ \ 1-\upsilon\quad \xi + \upsilon}{2\rho_1^5}$$
$$- \frac{3\alpha_2 q_2\kappa\upsilon\ \ \xi + \upsilon - 1}{2\rho_2^5} = 0 \tag{3.113}$$

and

$$\omega_0^2\ \ \kappa-1\ -\frac{q_1\kappa\ \ 1-\upsilon}{\rho_1^3}-\frac{q_2\kappa\upsilon}{\rho_2^3}-\frac{3\alpha_1 q_1\kappa\ \ 1-\upsilon}{2\rho_1^5}-\frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5}=0,\quad \zeta\neq 0 \tag{3.114}$$

where $\rho_1^2=\ \ \xi+\upsilon\ \ ^2+\zeta^2,\ \rho_2^2=\ \ \xi+\upsilon-1\ \ ^2+\zeta^2$

We can express equation (3.113) in the form

$$q_1\ \ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon}-\frac{q_2\upsilon\ \ \xi+\upsilon-1}{\xi+\upsilon}\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right) \tag{3.115}$$

From equation (3.114), we have

$$q_1\ \ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=\frac{\omega_0^2\ \ \kappa-1}{\kappa}-q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right) \tag{3.116}$$

Comparing equations (3.115) and (3.116), results in

$$\frac{\omega_0^2\ \ \kappa-1}{\kappa}-\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon}+\frac{q_2\upsilon\ \ \xi+\upsilon-1}{\xi+\upsilon}\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)-q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=0$$

or

$$\frac{\omega_0^2\ \ \kappa-1}{\kappa}-\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon}+\frac{q_2\upsilon\ \ \xi+\upsilon}{\xi+\upsilon}\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)-\frac{q_2\upsilon}{\xi+\upsilon}\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)-q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=0$$

$$\Rightarrow\frac{\omega_0^2\ \ \kappa-1}{\kappa}-\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon}+q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)-\frac{q_2\upsilon}{\xi+\upsilon}\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)-q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=0$$

$$\Rightarrow\frac{\omega_0^2\ \ \kappa-1}{\kappa}-\frac{\omega_0^2\ \ \psi+\kappa-1\ \ \xi}{\kappa\ \ \xi+\upsilon}-\frac{q_2\upsilon}{\xi+\upsilon}\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=0$$

Multiplying throughout by $\xi+\upsilon$ , yields

$$\frac{\omega_0^2\ \kappa-1\ \ \xi+\upsilon\ -\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa}-q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=0$$

$$\Rightarrow\frac{\omega_0^2\ \ \kappa\xi+\kappa\upsilon-\xi-\upsilon-\psi\xi-\kappa\xi+\xi}{\kappa}-q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=0$$

$$\Rightarrow\frac{\omega_0^2\left[\upsilon\ \kappa-1\ -\psi\xi\right]}{\kappa}-q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=0$$

$$\Rightarrow\omega_0^2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]=q_2\kappa\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right) \qquad (3.117)$$

Next from equation (3.115), we have

$$q_2\upsilon\left(\frac{1}{\rho_2^3}+\frac{3\alpha_2}{2\rho_2^5}\right)=\frac{\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa\ \ \xi+\upsilon-1}-\frac{q_1\ 1-\upsilon\ \ \ \xi+\upsilon}{\xi+\upsilon-1}\left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)$$

On putting this equation in equation (3.114), we have

$$\frac{\omega_0^2\ \kappa-1}{\kappa}-q_1\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)-\frac{\omega_0^2\ \psi+\kappa-1\ \xi}{\kappa\ \ \xi+\upsilon-1}+\frac{q_1\ 1-\upsilon\ \ \ \xi+\upsilon}{\xi+\upsilon-1}\left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right)=0$$

Multiplying throughout by $\xi+\upsilon-1$ and solving we get

$$\omega_0^2\left[\ \kappa-1\ \ 1-\upsilon\ +\psi\xi\right]=q_1\kappa\ 1-\upsilon\ \left(\frac{1}{\rho_1^3}+\frac{3\alpha_1}{2\rho_1^5}\right) \qquad (3.118)$$

Now, we note that when the bodies are spherical and non-luminous, the solutions of equations (3.117) and (3.118) are respectively

$$\rho_1^3=\frac{\kappa\ 1-\upsilon}{\kappa-1\ \ 1-\upsilon\ +\psi\xi}\ \text{and}\ \rho_2^3=\frac{\kappa\upsilon}{\upsilon\ \ \kappa-1\ -\psi\xi}$$

Therefore, we can assume the solution of the equations (3.117) and (3.118) to be

$$\rho_1 = \left[ \frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon} \right]^{\frac{1}{3}} + \varepsilon_1 \text{ and } \rho_2 = \left[ \frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1} \right]^{\frac{1}{3}} + \varepsilon_2 \qquad (3.119)$$

where $\varepsilon_i << 1 \quad i = 1,2$

Now, from these equations in system (3.119), using Binomial expansion and keeping only linear terms in $\varepsilon_i$, we get

$$\rho_1^{-3} = \frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon} \left[ 1-3\varepsilon_1 \left\{ \frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon} \right\}^{1/3} \right]$$

$$\rho_1^{-5} = \frac{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}}{\left[\kappa\ 1-\upsilon\ \right]^{5/3}} \left[ 1-5\varepsilon_1 \left\{ \frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon} \right\}^{1/3} \right]$$

$$\rho_2^{-3} = \frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon} \left[ 1-3\varepsilon_2 \left\{ \frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon} \right\}^{1/3} \right]$$

$$\rho_2^{-5} = \left[ \frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon} \right]^{5/3} \left[ 1-5\varepsilon_2 \left\{ \frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon} \right\}^{1/3} \right]$$

Substituting appropriately in equation (3.118), we have

$$\omega_0^2 \left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right] = q_1 \left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right] \left[ 1-3\varepsilon_1 \left\{ \frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon} \right\}^{1/3} \right]$$

$$+\frac{3}{2}\alpha_1 q_1 \frac{\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{5/3}}{\left[\kappa\ 1-\upsilon\ \right]^{2/3}} \left[ 1-5\varepsilon_1 \left\{ \frac{\psi\xi+\ \kappa-1\ \ 1-\upsilon}{\kappa\ 1-\upsilon} \right\}^{1/3} \right]$$

Ignoring product of $\alpha_1$ and $\varepsilon_1$, and multiplying throughout by $\frac{1}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}$, we get

$$\omega_0^2 - q_1 = -3q_1\varepsilon_1\left\{\frac{\psi\xi + \kappa - 1 \quad 1-\upsilon}{\kappa \quad 1-\upsilon}\right\}^{1/3} + \frac{3}{2}\alpha_1 q_1 \frac{\left[\psi\xi + \kappa - 1 \quad 1-\upsilon\right]^{2/3}}{\left[\kappa \quad 1-\upsilon\right]^{2/3}}$$

Expressing $q_1 = \left[1 - \quad 1 - q_1\right]$, and Ignoring product of $1 - q_1$ with $\varepsilon_1$ and $\alpha_1$; the above equation reduces to

$$\omega_0^2 - 1 + \quad 1 - q_1 = -3\varepsilon_1\left\{\frac{\psi\xi + \kappa - 1 \quad 1-\upsilon}{\kappa \quad 1-\upsilon}\right\}^{1/3} + \frac{3\alpha_1\left[\psi\xi + \kappa - 1 \quad 1-\upsilon\right]^{2/3}}{2\left[\kappa \quad 1-\upsilon\right]^{2/3}}$$

Substituting for $\omega_0^2$, and reducing the equation, we get

$$\varepsilon_1 = \frac{1}{3}\left\{\frac{\kappa \quad 1-\upsilon}{\psi\xi + \kappa - 1 \quad 1-\upsilon}\right\}^{\frac{1}{3}}\left[-\frac{3}{2} \quad \alpha_1 + \alpha_2 \quad - \quad 1 - q_1 \quad + \frac{3\alpha_1\left[\psi\xi + \kappa - 1 \quad 1-\upsilon\right]^{\frac{2}{3}}}{2\left[\kappa \quad 1-\upsilon\right]^{2/3}}\right] \tag{3.120}$$

Similarly, for $\varepsilon_2$, we substitute for $\rho_2^{-3}$ and $\rho_2^{-5}$ in equation (3.117) and solve, to get

$$\varepsilon_2 = \frac{1}{3}\left\{\frac{\kappa\upsilon}{-\psi\xi + \upsilon \quad \kappa - 1}\right\}^{\frac{1}{3}}\left[- \quad 1 - q_2 \quad - \frac{3}{2} \quad \alpha_1 + \alpha_2 \quad + \frac{3\alpha_2\left[-\psi\xi + \upsilon \quad \kappa - 1\right]^{\frac{2}{3}}}{2 \quad \kappa\upsilon^{\quad 2/3}}\right] \tag{3.121}$$

Now, we substitute equations (3.120) and (3.121) in equations (3.119), respectively, to get

$$\rho_1 = \left[\frac{\kappa \quad 1-\upsilon}{\psi\xi + \kappa - 1 \quad 1-\upsilon}\right]^{\frac{1}{3}}\left[1 - \frac{1}{3} \quad 1 - q_1 \quad - \frac{1}{2} \quad \alpha_1 + \alpha_2 \quad + \frac{\alpha_1\left[\psi\xi + \kappa - 1 \quad 1-\upsilon\right]^{\frac{2}{3}}}{2\left[\kappa \quad 1-\upsilon\right]^{2/3}}\right]$$

and

$$\rho_2 = \left[\frac{\kappa\upsilon}{-\psi\xi + \upsilon \quad \kappa - 1}\right]^{\frac{1}{3}}\left[1 - \frac{1}{3} \quad 1 - q_2 \quad - \frac{1}{2} \quad \alpha_1 + \alpha_2 \quad + \frac{\alpha_2\left[-\psi\xi + \upsilon \quad \kappa - 1\right]^{\frac{2}{3}}}{2 \quad \kappa\upsilon^{\quad 2/3}}\right]$$

Now from these equations, using Binomial expansions, we are able compute that

$$\rho_1^3 = \frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\left[1-\ 1-q_1\ -\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right]$$

$$\rho_2^3 = \frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\left[1-\ 1-q_2\ -\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3\alpha_2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{2\ \ \kappa\upsilon\ ^{2/3}}\right] \tag{3.122}$$

Following the pattern used in problem 3.2, we get

$$\xi+\upsilon\ ^2 = \left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{\frac{2}{3}}\left\{1-\ 1-q_1\ -\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}^{2/3} \tag{3.123}$$

and

$$\xi+\upsilon-1\ ^2 = \left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{\frac{2}{3}}\left\{1-\ 1-q_2\ -\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3\alpha_2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{2\ \ \kappa\upsilon\ ^{2/3}}\right\}^{2/3} \tag{3.124}$$

On subtracting equation (3.124) from (3.123), and denoting the LHS by $f\ \ \xi\ $, yields

$$f\ \ \xi\ = 2\xi+2\upsilon-1-\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{\frac{2}{3}}\left\{1-\ 1-q_1\ -\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}{2\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}^{\frac{2}{3}}$$

$$+\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{\frac{2}{3}}\left\{1-\ 1-q_2\ -\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3\alpha_2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{2\ \ \kappa\upsilon\ ^{2/3}}\right\}^{\frac{2}{3}} = 0 \tag{3.125}$$

Finding first and second derivative of (3.125) w.r.t. $\xi$, we get

$$f'(\xi) = 2 + \frac{2}{3}\left[\frac{\psi\left[\kappa(1-\upsilon)\right]^{\frac{2}{3}}}{\left[\psi\xi + \kappa - 1 - (1-\upsilon)\right]^{\frac{5}{3}}} - \frac{\psi\left[\kappa(1-\upsilon)\right]^{\frac{2}{3}}(1-q_1)}{\left[\psi\xi + \kappa - 1 - (1-\upsilon)\right]^{\frac{5}{3}}} - \frac{3\psi(\alpha_1+\alpha_2)\left[\kappa(1-\upsilon)\right]^{\frac{2}{3}}}{2\left[\psi\xi + \kappa - 1 - (1-\upsilon)\right]^{\frac{5}{3}}}\right.$$
$$\left. + \frac{\psi\left[\kappa(1-\upsilon)\right]^{\frac{2}{3}}(1-q_1)}{3\left[\psi\xi + \kappa - 1 - (1-\upsilon)\right]^{\frac{5}{3}}} + \frac{\psi(\alpha_1+\alpha_2)\left[\kappa(1-\upsilon)\right]^{\frac{2}{3}}}{2\left[\psi\xi + \kappa - 1 - (1-\upsilon)\right]^{\frac{5}{3}}}\right] + \frac{2}{3}\left[\frac{\psi(\kappa\upsilon)^{\frac{2}{3}}}{\left[-\psi\xi + \upsilon(\kappa-1)\right]^{\frac{5}{3}}} - \frac{\psi(\kappa\upsilon)^{\frac{2}{3}}(1-q_2)}{\left[-\psi\xi + \upsilon(\kappa-1)\right]^{\frac{5}{3}}}\right.$$
$$\left. - \frac{3\psi(\alpha_1+\alpha_2)(\kappa\upsilon)^{\frac{2}{3}}}{2\left[-\psi\xi + \upsilon(\kappa-1)\right]^{\frac{5}{3}}} + \frac{\psi(\kappa\upsilon)^{\frac{2}{3}}(1-q_2)}{3\left[-\psi\xi + \upsilon(\kappa-1)\right]^{\frac{5}{3}}} + \frac{\psi(\alpha_1+\alpha_2)(\kappa\upsilon)^{\frac{2}{3}}}{2\left[-\psi\xi + \upsilon(\kappa-1)^{\frac{5}{3}}\right]}\right] \quad (3.126)$$

Therefore,

$$f(0) = 2\upsilon - 1 - \left\{\frac{\kappa}{\kappa-1}\right\}^{2/3}\left\{1 - (1-q_1) - \frac{3}{2}(\alpha_1+\alpha_2) + \frac{3\alpha_1}{2}\left\{\frac{\kappa-1}{\kappa}\right\}^{2/3}\right\}^{2/3} + \left\{\frac{\kappa}{\kappa-1}\right\}^{2/3}$$
$$\times\left\{1 - (1-q_2) - \frac{3}{2}(\alpha_1+\alpha_2) + \frac{3\alpha_2}{2}\left\{\frac{\kappa-1}{\kappa}\right\}^{2/3}\right\}^{2/3}$$

Using Binomial expansions, we get

$$f(0) = 2\upsilon - 1 - \left(\frac{\kappa}{\kappa-1}\right)^{\frac{2}{3}}\left\{-\frac{2(1-q_1)}{3} + (\alpha_1-\alpha_2)\left(\frac{\kappa-1}{\kappa}\right)^{\frac{2}{3}} + \frac{2(1-q_2)}{3}\right\}$$

Hence,

$$f(0) = 2\upsilon - 1 + \frac{2\kappa^{2/3}(1-q_1)}{3(\kappa-1)^{2/3}} - \frac{2\kappa^{2/3}(1-q_2)}{3(\kappa-1)^{2/3}} + (\alpha_2-\alpha_1) \quad (3.127)$$

Similarly,

$$f'(0) = 2 + \frac{2}{3}\left[ \frac{\psi\left[\kappa(1-\upsilon)\right]^{2/3}}{\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} - \frac{\psi\left[\kappa(1-\upsilon)\right]^{2/3}(1-q_1)}{\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} - \frac{3\psi(\alpha_1+\alpha_2)\left[\kappa(1-\upsilon)\right]^{2/3}}{2\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} \right.$$
$$\left. + \frac{\psi\left[\kappa(1-\upsilon)\right]^{2/3}(1-q_1)}{3\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} + \frac{\psi(\alpha_1+\alpha_2)\left[\kappa(1-\upsilon)\right]^{2/3}}{2\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} \right] + \frac{2}{3}\left[ \frac{\psi(\kappa\upsilon)^{2/3}}{\left[\upsilon(\kappa-1)\right]^{5/3}} - \frac{\psi(\kappa\upsilon)^{2/3}(1-q_2)}{\left[\upsilon(\kappa-1)\right]^{5/3}} \right.$$
$$\left. - \frac{3\psi(\alpha_1+\alpha_2)(\kappa\upsilon)^{2/3}}{2\left[\upsilon(\kappa-1)\right]^{5/3}} + \frac{\psi(\kappa\upsilon)^{2/3}(1-q_2)}{3\left[\upsilon(\kappa-1)\right]^{5/3}} + \frac{\psi(\alpha_1+\alpha_2)(\kappa\upsilon)^{2/3}}{2\left[\upsilon(\kappa-1)\right]^{5/3}} \right]$$

Therefore, we get

$$f'(0) = 2 + \frac{2}{3}\left[ \frac{\psi\left[\kappa(1-\upsilon)\right]^{2/3}}{\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} - \frac{2\psi\left[\kappa(1-\upsilon)\right]^{2/3}(1-q_1)}{3\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} - \frac{\psi(\alpha_1+\alpha_2)\left[\kappa(1-\upsilon)\right]^{2/3}}{\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} \right]$$
$$+ \frac{2}{3}\left[ \frac{\psi(\kappa\upsilon)^{2/3}}{\left[\upsilon(\kappa-1)\right]^{5/3}} - \frac{2\psi(\kappa\upsilon)^{2/3}(1-q_2)}{3\left[\upsilon(\kappa-1)\right]^{5/3}} - \frac{\psi(\alpha_1+\alpha_2)(\kappa\upsilon)^{2/3}}{\left[\upsilon(\kappa-1)\right]^{5/3}} \right] \quad (3.128)$$

Substituting equations (3.128) and (3.127) in the Newton-Raphson's formula (3.58), we have

$$\xi = -\frac{f(0)}{f'(0)} = 1 - 2\upsilon - \frac{2\kappa^{2/3}(1-q_1)}{3(\kappa-1)^{2/3}} + \frac{2\kappa^{2/3}(1-q_2)}{3(\kappa-1)^{2/3}} + (\alpha_1-\alpha_2) \div \left\{ 2 + \frac{2}{3}\left[ \frac{\psi\left[\kappa(1-\upsilon)\right]^{2/3}}{\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} \right.\right.$$
$$\left. - \frac{2\psi\left[\kappa(1-\upsilon)\right]^{2/3}(1-q_1)}{3\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} - \frac{\psi(\alpha_1+\alpha_2)\left[\kappa(1-\upsilon)\right]^{2/3}}{\left[\kappa-1\ (1-\upsilon)\right]^{5/3}} \right] + \frac{2}{3}\left[ \frac{\psi(\kappa\upsilon)^{2/3}}{\left[\upsilon(\kappa-1)\right]^{5/3}} - \frac{2\psi(\kappa\upsilon)^{2/3}(1-q_2)}{3\left[\upsilon(\kappa-1)\right]^{5/3}} \right.$$
$$\left.\left. - \frac{\psi(\alpha_1+\alpha_2)(\kappa\upsilon)^{2/3}}{\left[\upsilon(\kappa-1)\right]^{5/3}} \right] \right\}$$

$$\xi=\frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2\ +3\ \alpha_1-\alpha_2\ \ \kappa-1^{\frac{2}{3}}}{3\ \kappa-1^{\frac{2}{3}}}\div\left\{2+\frac{2\psi}{3}\left[\frac{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}{\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{5}{3}}}\right.\right.$$

$$\left.-\frac{2\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}\ 1-q_1}{3\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{5}{3}}}-\frac{\alpha_1+\alpha_2\ \left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}{\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{5}{3}}}\right]+\frac{2\psi}{3}\left[\frac{\kappa\upsilon^{\frac{2}{3}}}{\left[\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}-\frac{2\ \kappa\upsilon^{\frac{2}{3}}\ 1-q_2}{3\left[\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}\right.$$

$$\left.\left.-\frac{\alpha_1+\alpha_2\ \ \kappa\upsilon^{\frac{2}{3}}}{\left[\upsilon\left(\kappa-1\right)\right]^{\frac{5}{3}}}\right]\right\}$$

$$\xi=\frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}}\div\left\{2+\frac{2\psi}{3}\left[\frac{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}{\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{5}{3}}}\right.\right.$$

$$\left.\left.+\frac{\kappa\upsilon^{\frac{2}{3}}}{\left[\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}-\frac{2\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}\ 1-q_1}{3\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{5}{3}}}-\frac{2\ \kappa\upsilon^{\frac{2}{3}}\ 1-q_2}{3\left[\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}-\frac{\alpha_1+\alpha_2\ \left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}{\left[\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{5}{3}}}-\frac{\alpha_1+\alpha_2\ \ \kappa\upsilon^{\frac{2}{3}}}{\left[\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}\right]\right\}$$

$$\xi=\frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}}\div\left\{2+\frac{2\psi}{3}\left[\frac{\kappa^{\frac{2}{3}}}{1-\upsilon\ \ \kappa-1^{\frac{5}{3}}}+\frac{\kappa^{\frac{2}{3}}}{\upsilon\ \kappa-1^{\frac{5}{3}}}\right.\right.$$

$$\left.\left.-\frac{2\kappa^{\frac{2}{3}}\ 1-q_1}{3\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}}-\frac{2\kappa^{\frac{2}{3}}\ 1-q_2}{3\upsilon\ \kappa-1^{\frac{5}{3}}}-\frac{\alpha_1+\alpha_2\ \kappa^{\frac{2}{3}}}{1-\upsilon\ \ \kappa-1^{\frac{5}{3}}}-\frac{\alpha_1+\alpha_2\ \kappa^{\frac{2}{3}}}{\upsilon\ \kappa-1^{\frac{5}{3}}}\right]\right\}$$

$$\xi=\frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}}\div\left\{2+\frac{2\psi}{3}\right.$$

$$\left.\times\left[\frac{3\kappa^{\frac{2}{3}}\upsilon+3\kappa^{\frac{2}{3}}-3\kappa^{\frac{2}{3}}\upsilon-2\kappa^{\frac{2}{3}}\upsilon\ 1-q_1\ -2\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ 1-q_2\ -3\kappa^{\frac{2}{3}}\upsilon\ \alpha_1+\alpha_2\ -3\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ \alpha_1+\alpha_2}{3\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}}\right]\right\}$$

$$\xi = \frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}} \div \left\{ 2+\frac{2\psi}{3} \right.$$

$$\left. \times \left[ \frac{3\kappa^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}\upsilon\ 1-q_1\ -2\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ 1-q_2\ -3\alpha_1\kappa^{\frac{2}{3}}\upsilon-3\alpha_2\kappa^{\frac{2}{3}}\upsilon+3\alpha_1\kappa^{\frac{2}{3}}\upsilon+3\alpha_2\kappa^{\frac{2}{3}}\upsilon-3\kappa^{\frac{2}{3}}\ \alpha_1+\alpha_2}{3\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}} \right] \right\}$$

$$\xi = \frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}} \div$$

$$\left\{ 2+\frac{2\psi}{3} \left[ \frac{3\kappa^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}\upsilon\ 1-q_1\ -2\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ 1-q_2\ -3\kappa^{\frac{2}{3}}\ \alpha_1+\alpha_2}{3\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}} \right] \right\}$$

$$\xi = \frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}} \div$$

$$\left\{ \frac{18\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}+2\psi\left[ 3\kappa^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}\upsilon\ 1-q_1\ -2\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ 1-q_2\ -3\kappa^{\frac{2}{3}}\ \alpha_1+\alpha_2 \right]}{9\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}} \right\}$$

$$\xi = \frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\ \kappa-1^{\frac{2}{3}}} \times 9\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}$$

$$\times \frac{1}{18\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}+2\psi\left[ 3\kappa^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}\upsilon\ 1-q_1\ -2\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ 1-q_2\ -3\kappa^{\frac{2}{3}}\ \alpha_1+\alpha_2 \right]}$$

$$\xi = 3\upsilon\ 1-\upsilon\ \ \kappa-1\left[ 3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2 \right]$$

$$\times \frac{1}{18\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}+2\psi\left[ 3\kappa^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}\upsilon\ 1-q_1\ -2\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ 1-q_2\ -3\kappa^{\frac{2}{3}}\ \alpha_1+\alpha_2 \right]}$$

$$\xi = \frac{3\upsilon\ 1-\upsilon\ \ \kappa-1\left[3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2\ \right]}{18\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}+2\psi\left[3\kappa^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}\upsilon\ 1-q_1\ -2\kappa^{\frac{2}{3}}\ 1-\upsilon\ \ 1-q_2\ -3\kappa^{\frac{2}{3}}\ \alpha_1+\alpha_2\ \right]}$$

Hence, we get the abscissa of the out-of-plane equilibrium points:

$$\xi = \frac{1}{2}\upsilon\ 1-\upsilon\ \ \kappa-1\left[\frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\upsilon\ 1-\upsilon\ \ \kappa-1^{\frac{5}{3}}+\psi\kappa^{\frac{2}{3}}\left[1-\frac{2}{3}\upsilon\ 1-q_1\ -\frac{2}{3}\ 1-\upsilon\ \ 1-q_2\ -\ \alpha_1+\alpha_2\ \right]}\right]$$

Now, from the equation, $\rho_2^2 = \ \xi+\upsilon-1\ ^2+\zeta^2$, if we substitute for $\rho_2^2$, in it, we get

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{\frac{2}{3}}\left[1-\ 1-q_2\ -\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3\alpha_2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{2\ \kappa\upsilon\ ^{2/3}}\right]^{\frac{2}{3}}-\ \xi+\upsilon-1\ ^2}$$

Using Binomial expansion, we get

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \ \kappa-1}\right\}^{\frac{2}{3}}\left[1-\frac{2}{3}\ 1-q_2\ -\ \alpha_1+\alpha_2\ +\frac{\alpha_2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{\kappa\upsilon\ ^{2/3}}\right]-\ \xi+\upsilon-1\ ^2}$$

Similarly, from the equation, $\rho_1^2 = \ \xi+\upsilon\ ^2+\zeta^2$, when we substitute for $\rho_1^2$, we get

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right\}^{\frac{2}{3}}\left\{1-\frac{2}{3}\ 1-q_1\ -\ \alpha_1+\alpha_2\ +\frac{\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right\}-\ \xi+\upsilon\ ^2}$$

Hence, the coordinates of the out-of- plane EPs are given as

$$\xi = \frac{1}{2}\upsilon\ 1-\upsilon\quad \kappa-1\left[\frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\upsilon\ 1-\upsilon\quad \kappa-1^{\frac{5}{3}}+\psi\kappa^{\frac{2}{3}}\left[1-\frac{2}{3}\upsilon\ 1-q_1\ -\frac{2}{3}\ 1-\upsilon\ \ 1-q_2\ -\alpha_1-\alpha_2\right]}\right]$$

$$\zeta = \pm\sqrt{\left\{\frac{\kappa\upsilon}{-\psi\xi+\upsilon\ \kappa-1}\right\}^{\frac{2}{3}}\left[1-\frac{2}{3}\ 1-q_2\ -\ \alpha_1+\alpha_2\ +\frac{\alpha_2\left[-\psi\xi+\upsilon\ \kappa-1\ \right]^{\frac{2}{3}}}{\kappa\upsilon^{\ 2/3}}\right]-\ \xi+\upsilon-1^{\ 2}} \qquad 3.129$$

and

$$\xi = \frac{1}{2}\upsilon\ 1-\upsilon\quad \kappa-1\left[\frac{3\ \kappa-1^{\frac{2}{3}}\ 1-2\upsilon+\alpha_1-\alpha_2\ -2\kappa^{\frac{2}{3}}\ 1-q_1\ +2\kappa^{\frac{2}{3}}\ 1-q_2}{3\upsilon\ 1-\upsilon\quad \kappa-1^{\frac{5}{3}}+\psi\kappa^{\frac{2}{3}}\left[1-\frac{2}{3}\upsilon\ 1-q_1\ -\frac{2}{3}\ 1-\upsilon\ \ 1-q_2\ -\alpha_1-\alpha_2\right]}\right]$$

$$\zeta = \pm\sqrt{\left[\frac{\kappa\ 1-\upsilon}{\psi\xi+\ \kappa-1\ \ 1-\upsilon}\right]^{\frac{2}{3}}\left[1-\frac{2\ 1-q_1}{3}-\ \alpha_1+\alpha_2\ +\frac{\alpha_1\left[\psi\xi+\ \kappa-1\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{2/3}}\right]-\ \xi+\upsilon^{\ 2}} \qquad 3.130$$

where $0<\upsilon\le 0.5,\quad -\psi\xi+\upsilon\ \kappa-1\ >0, \psi\xi+\ \kappa-1\ \ 1-\upsilon\ >0,\quad \kappa-1\ >0;\quad \alpha_i<<1$ and $1-q_i\ <<1\ \ i=1,2$ . The coordinate (3.129) is denoted by $L_{6,7}$, while the coordinate (3.130) is denoted by $L_{8,9}$. When $\alpha_2=0$, $q_1=1$, these solutions are same with that of problem **3.2**. Other previous works, such as Singh and Leke (2010) and Bekov (1988) can be confirmed from our results. When $\kappa=1$, the points do not exist, instead, an infinite line of EPs denoted by $L_{\pm\infty}$ exist on the $\zeta-$plane and referred to as infinitely remote solutions.

## 3.3.3. Stability of equilibrium points

We follow the same linear stability analysis used in the previous research problem **3.2.**

**Collinear equilibrium points**

We consider the point corresponding to $L_{1_\kappa}$ with coordinate $-\upsilon-\varepsilon_{1_\kappa},0$ , using $\rho_1=\delta_1<1$, $\rho_2=1+\delta_1>1$, we get the partial derivatives such that

$$\Omega^0_{\xi\xi}=\psi-1+\kappa\ 1+2f_1\ ,\Omega^0_{\eta\eta}=\psi-1+\kappa\ 1-f_1\ ,\Omega^0_{\xi\eta}=\Omega^0_{\eta\xi}=0 \tag{3.131}$$

where $f_1=\dfrac{1-\upsilon}{\delta_1^3}+\dfrac{1-\upsilon\ \ 1-q_1}{\delta_1^3}+\dfrac{\upsilon}{1+\delta_1\ ^3}+\dfrac{\upsilon\ 1-q_2}{1+\delta_1\ ^3}+\dfrac{3\alpha_1\ 1-\upsilon}{\delta_1^5}+\dfrac{3\alpha_2\upsilon}{1+\delta_1\ ^5}$

Now, $\delta_1>0,0<\upsilon\le\dfrac{1}{2}$, $\psi-1<<1$, $\alpha_i<<1,1-q_i<<1$ $f_1>0$; consequently, for any $0<\kappa<\infty$, we always have $\Omega^0_{\xi\xi}>0$ while $\Omega^0_{\eta\eta}$ can be positive or negative, depending solely on the value of the arbitrary constant $\kappa$ and the small perturbation in the centrifugal force.

Now, substituting the derivatives (3.131) in the characteristic equation (3.66) gives

$$\lambda^4+P_1\lambda^2+Q_1=0 \tag{3.132}$$

where $P_1=4-8\ \phi-1\ -2\ \psi-1\ -\kappa\ 2+f_1\ +\dfrac{3}{2}\ \alpha_1+\alpha_2\ \ 4-2\kappa+f_1$

$$Q_2=\kappa\left[\ \psi-1\ \ 2+f_1\ +\kappa\ 1+f_1-2f_1^2\ \ 1+3\alpha_1+3\alpha_2\ \right]$$

Its roots are: $\lambda_{1,2,3,4}=\Lambda_n$ where $\Lambda_n=\sqrt{\dfrac{1}{2}\ -P_1\pm\sqrt{D_1}}\quad n=1,2$ and $D_1$ is the discriminant.

Now, the nature of the roots of the polynomial (3.132) will depend on the discriminant coupled with the changes in $P_1$ and $Q_1$ which depends solely on $\kappa$, while the values are affected by all the involved parameters. The only possibility of a stable collinear point, is when simultaneously $D_1 > 0$, $P_1 > 0$ and $Q_1 > 0$. In this case, all $\lambda_i$ will be distinct pure imaginary roots, consequently, the motion is bounded and the collinear point is stable. The same procedure shows that all the five collinear points are conditionally stable with respect to the involved parameters.

**Triangular equilibrium points**

We first compute the partial derivatives computed at the triangular points. From the force function of the equations of motion (3.91), we obtained the following second order derivatives:

$$\Omega_{\xi\xi} = \omega_0^2 \ \psi + \kappa - 1 \ - \frac{q_1\kappa \ 1-\upsilon}{\rho_1^3} + \frac{3q_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ ^2}{\rho_1^5} - \frac{q_2\kappa\upsilon}{\rho_2^3} + \frac{3q_2\kappa\upsilon \ \xi+\upsilon-1 \ ^2}{\rho_2^5}$$
$$- \frac{3\alpha_1 q_1\kappa \ 1-\upsilon}{2\rho_1^5} + \frac{15\alpha_1 q_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ ^2}{2\rho_1^7} - \frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5} + \frac{15\alpha_2 q_2\kappa\upsilon \ \xi+\upsilon-1 \ ^2}{2\rho_2^7}$$

$$\Omega_{\xi\eta} = \frac{3q_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ \eta}{\rho_1^5} + \frac{3q_2\kappa\upsilon \ \xi+\upsilon-1 \ \eta}{\rho_2^5} + \frac{15\alpha_1 q_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ \eta}{2\rho_1^7} + \frac{15\alpha_2 q_2\kappa\upsilon \ \xi+\upsilon-1 \ \eta}{2\rho_2^7}$$

$$\Omega_{\eta\eta} = \omega_0^2 \ \psi + \kappa - 1 \ - \frac{q_1\kappa \ 1-\upsilon}{\rho_1^3} + \frac{3q_1\kappa \ 1-\upsilon \ \eta^2}{\rho_1^5} - \frac{q_2\kappa\upsilon}{\rho_2^3} + \frac{3q_2\kappa\upsilon\eta^2}{\rho_2^5} - \frac{3\alpha_1 q_1\kappa \ 1-\upsilon}{2\rho_1^5}$$
$$+ \frac{15\alpha_1 q_1\kappa \ 1-\upsilon \ \eta^2}{2\rho_1^7} - \frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5} + \frac{15\alpha_2 q_2\kappa\upsilon\eta^2}{2\rho_2^7} \qquad 3.133$$

$$\Omega_{\xi\zeta} = \frac{3q_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ \zeta}{\rho_1^5} + \frac{3q_2\kappa\upsilon \ \xi+\upsilon-1 \ \zeta}{\rho_2^5} + \frac{15\alpha_1 q_1\kappa \ 1-\upsilon \ \ \xi+\upsilon \ \zeta}{2\rho_1^7} + \frac{15\alpha_2 q_2\kappa\upsilon \ \xi+\upsilon-1 \ \zeta}{2\rho_2^7}$$

$$\Omega_{\zeta\zeta} = \omega_0^2 \ \kappa - 1 \ - \frac{q_1\kappa \ 1-\upsilon}{\rho_1^3} + \frac{3q_1\kappa \ 1-\upsilon \ \zeta^2}{\rho_1^5} - \frac{q_2\kappa\upsilon}{\rho_2^3} + \frac{3q_2\kappa\upsilon\zeta^2}{\rho_2^5} - \frac{3\alpha_1 q_1\kappa \ 1-\upsilon}{2\rho_1^5}$$
$$+ \frac{15\alpha_1 q_1\kappa \ 1-\upsilon \ \zeta^2}{2\rho_1^7} - \frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5} + \frac{15\alpha_2 q_2\kappa\upsilon\zeta^2}{2\rho_2^7}$$

In the case of the triangular EPs, knowing the expressions for $\rho_i \;\; i=1,2$ in equations (3.104) and (3.105), respectively; then for $\Omega^0_{\xi\xi\xi}$, we substitute these in the first equation of (3.133) and get

$$\Omega^0_{\xi\xi\xi} = \omega_0^2 \;\; \psi+\kappa-1 \;\; -q_1\kappa \;\; 1-\upsilon \;\; \times \frac{\psi+\kappa-1}{\kappa}\left[1+\frac{3}{2}\;\; \alpha_1+\alpha_2 \;\; + \;\; 1-q_1 \;\; -\frac{3\alpha_1}{2}\frac{\psi+\kappa-1}{\kappa^{2/3}}^{\frac{2}{3}}\right]$$

$$+3q_1\kappa \;\; 1-\upsilon \;\; \left[\frac{1}{4}+\frac{\kappa^{2/3}}{\psi+\kappa-1}^{\frac{2}{3}}\left\{\frac{1}{3}\;\; 1-q_2 \;\; -\frac{1}{3}\;\; 1-q_1 \;\; +\frac{1}{2}\frac{\alpha_1-\alpha_2 \;\; \psi+\kappa-1}{\kappa^{2/3}}^{\frac{2}{3}}\right\}\right]\times\frac{\psi+\kappa-1}{\kappa^{5/3}}^{\frac{5}{3}}$$

$$\times\left[1+\frac{5}{2}\;\; \alpha_1+\alpha_2 \;\; +\frac{5}{3}\;\; 1-q_1 \;\; -\frac{5\alpha_1}{2}\frac{\psi+\kappa-1}{\kappa^{2/3}}^{\frac{2}{3}}\right]-q_2\kappa\upsilon\frac{\psi+\kappa-1}{\kappa}\left[1+\frac{3}{2}(\alpha_1+\alpha_2)(1-q_2)\right.$$

$$\left.-\frac{3\alpha_2}{2}\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right]+3q_2\kappa\upsilon\left[\frac{1}{4}-\frac{\kappa^{2/3}}{(\psi+\kappa-1)^{\frac{2}{3}}}\left\{\frac{1}{3}(1-q_2)\frac{1}{3}(1-q_1)\frac{1}{2}\frac{(\alpha_1-\alpha_2)(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right\}\right]$$

$$\times\frac{(\psi+\kappa-1)^{\frac{5}{3}}}{\kappa^{5/3}}\left[1+\frac{5}{2}(\alpha_1+\alpha_2)\frac{5}{3}(1-q_2)\frac{5\alpha_2}{2}\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right]-\frac{3}{2}\alpha_1q_1\kappa(1-\upsilon)\frac{(\psi+\kappa-1)^{\frac{5}{3}}}{\kappa^{5/3}}\times$$

$$\left[1+\frac{5}{2}(\alpha_1+\alpha_2)\frac{5}{3}(1-q_1)\frac{5\alpha_1}{2}\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right]-\frac{3}{2}\alpha_2q_2\kappa\upsilon\frac{(\psi+\kappa-1)^{\frac{5}{3}}}{\kappa^{5/3}}\left[1+\frac{5}{2}(\alpha_1+\alpha_2)\frac{5}{3}(1-q_2)\right.$$

$$\left.-\frac{5\alpha_2}{2}\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right]+\frac{15\alpha_1q_1\kappa(1-\upsilon)}{2}\left[\frac{1}{4}+\frac{\kappa^{2/3}}{(\psi+\kappa-1)^{\frac{2}{3}}}\left\{\frac{(1-q_2)}{3}-\frac{(1-q_1)}{3}+\frac{(\alpha_1-\alpha_2)(\psi+\kappa-1)^{\frac{2}{3}}}{2\kappa^{2/3}}\right\}\right]$$

$$\times\frac{(\psi+\kappa-1)^{\frac{7}{3}}}{\kappa^{7/3}}\left[1+\frac{7(\alpha_1+\alpha_2)}{2}+\frac{7(1-q_1)}{3}-\frac{7\alpha_1}{2}\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right]$$

$$+\frac{15}{2}\alpha_2q_2\kappa\upsilon\left[\frac{1}{4}-\frac{\kappa^{2/3}}{(\psi+\kappa-1)^{\frac{2}{3}}}\left\{\frac{1}{3}(1-q_2)\frac{1}{3}(1-q_1)\frac{1}{2}\frac{(\alpha_1-\alpha_2)(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right\}\right]$$

$$\times\frac{(\psi+\kappa-1)^{\frac{7}{3}}}{\kappa^{7/3}}\left[1+\frac{7(\alpha_1+\alpha_2)}{2}+\frac{7(1-q_2)}{3}-\frac{7\alpha_2}{2}\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{\kappa^{2/3}}\right]$$

Simplifying and expressing $q_i = 1-\;\;1-q_i\;\;$, we have

$$\Omega^0_{\xi\xi\xi} = \omega_0^2\ \psi+\kappa-1\ -\left[1-\ 1-q_1\ \right]1-\upsilon\ \ \psi+\kappa-1\left[1+\frac{3}{2}\ \alpha_1+\alpha_2\ +\ 1-q_1\ -\frac{3\alpha_1}{2}\frac{\psi+\kappa-1^{\frac{2}{3}}}{\kappa^{2/3}}\right]$$

$$+3\ 1-\upsilon\ \left[1-\ 1-q_1\ \right]\left[\frac{1}{4}+\frac{\kappa^{2/3}\ 1-q_2}{3\ \psi+\kappa-1^{\frac{2}{3}}}-\frac{\kappa^{2/3}\ 1-q_1}{3\ \psi+\kappa-1^{\frac{2}{3}}}+\frac{1}{2}\ \alpha_1-\alpha_2\ \right]\times$$

$$\left[\frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}+\frac{5}{2}\ \alpha_1+\alpha_2\ \ \frac{\psi+\kappa-1^{\frac{5}{3}}}{\kappa^{2/3}}+\frac{5}{3}\frac{\psi+\kappa-1^{\frac{5}{3}}\ 1-q_1}{\kappa^{2/3}}-\frac{5}{2}\frac{\alpha_1\ \psi+\kappa-1^{\frac{7}{3}}}{\kappa^{4/3}}\right]$$

$$-\upsilon\left[1-\left(1-q_2\right)\right]\left[\left(\psi+\kappa-1\right)\frac{3}{2}\left(\alpha_1+\alpha_2\right)\left(\psi+\kappa-1\right)\left(\psi+\kappa-1\right)\left(1-q_1\right)\frac{3\alpha_1}{2}\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}\right]$$

$$+3\upsilon\left[1-\left(1-q_2\right)\right]\left[\frac{1}{4}-\frac{\kappa^{2/3}\left(1-q_2\right)}{3\left(\psi+\kappa-1^{\frac{2}{3}}\right)}+\frac{\kappa^{2/3}\left(1-q_1\right)}{3\left(\psi+\kappa-1^{\frac{2}{3}}\right)}-\frac{1}{2}\left(\alpha_1-\alpha_2\right)\right]\left[\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{5}{2}\left(\alpha_1+\alpha_2\right)\right.$$

$$\left.\times\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{5}{3}\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)\left(1-q_2\right)}{\kappa^{2/3}}-\frac{5}{2}\frac{\alpha_2\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{\kappa^{4/3}}\right]-\frac{3}{2}\alpha_1\left[1-\left(1-q_1\right)\right]\left(1-\upsilon\right)$$

$$\times\left[\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{5}{2}\left(\alpha_1+\alpha_2\right)\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{5}{3}\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)\left(1-q_1\right)}{\kappa^{2/3}}-\frac{5\alpha_1\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{2\kappa^{4/3}}\right]$$

$$-\frac{3}{2}\alpha_2\left[1-\left(1-q_2\right)\right]\upsilon\left[\frac{\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{\kappa^{2/3}}+\frac{5\left(\alpha_1+\alpha_2\right)\left(\psi+\kappa-1^{\frac{5}{3}}\right)}{2\kappa^{2/3}}+\frac{5\left(\psi+\kappa-1^{\frac{5}{3}}\right)\left(1-q_2\right)}{3\kappa^{2/3}}\right.$$

$$\left.-\frac{5}{2}\frac{\alpha_2\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{\kappa^{4/3}}\right]+\frac{15}{2}\alpha_1\left[1-\left(1-q_1\right)\right]\left(1-\upsilon\right)\left[\frac{1}{4}+\frac{\kappa^{2/3}\left(1-q_2\right)}{3\left(\psi+\kappa-1^{\frac{2}{3}}\right)}-\frac{\kappa^{2/3}\left(1-q_1\right)}{3\left(\psi+\kappa-1^{\frac{2}{3}}\right)}+\frac{1}{2}\left(\alpha_1-\alpha_2\right)\right]$$

$$\times\left[\frac{\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{\kappa^{4/3}}+\frac{7}{2}\left(\alpha_1+\alpha_2\right)\frac{\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{\kappa^{4/3}}+\frac{7}{3}\frac{\left(\psi+\kappa-1^{\frac{7}{3}}\right)\left(1-q_1\right)}{\kappa^{4/3}}-\frac{7}{2}\frac{\alpha_1\left(\psi+\kappa-1^{\frac{3}{3}}\right)}{\kappa^2}\right]$$

$$+\frac{15}{2}\alpha_2\upsilon\left[1-\left(1-q_2\right)\right]\left[\frac{1}{4}-\frac{\kappa^{2/3}\left(1-q_2\right)}{3\left(\psi+\kappa-1^{\frac{2}{3}}\right)}+\frac{\kappa^{2/3}\left(1-q_1\right)}{3\left(\psi+\kappa-1^{\frac{2}{3}}\right)}-\frac{1}{2}\left(\alpha_1-\alpha_2\right)\right]\left[\frac{\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{\kappa^{4/3}}\right.$$

$$\left.+\frac{7}{2}\left(\alpha_1+\alpha_2\right)\frac{\left(\psi+\kappa-1^{\frac{7}{3}}\right)}{\kappa^{4/3}}+\frac{7}{3}\frac{\left(\psi+\kappa-1^{\frac{7}{3}}\right)\left(1-q_2\right)}{\kappa^{4/3}}-\frac{7}{2}\frac{\alpha_2\left(\psi+\kappa-1^{\frac{3}{3}}\right)}{\kappa^2}\right]$$

Neglecting second and higher order terms of $\alpha_i$, $1-q_i$ $i=1,2$ and their product, gives

$$\Omega^0_{\xi\xi}=\omega_0^2\ \left(\psi+\kappa-1\right)-\left(1-\upsilon\right)\left(\psi+\kappa-1\right)\left[1+\frac{3}{2}\left(\alpha_1+\alpha_2\right)+\left(1-q_1\right)-\frac{3\alpha_1}{2}\frac{\left(\psi+\kappa-1\right)^{\frac{2}{3}}}{\kappa^{2/3}}\right]$$

$$+3\left(1-\upsilon\right)\left[\frac{1}{4}\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{2/3}}+\frac{5}{8}\left(\alpha_1+\alpha_2\right)\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{2/3}}+\frac{5\left(1-q_1\right)\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{12\kappa^{2/3}}-\frac{5\alpha_1\left(\psi+\kappa-1\right)^{\frac{7}{3}}}{8\kappa^{4/3}}\right.$$

$$\left.+\frac{\left(1-q_2\right)\left(\psi+\kappa-1\right)}{3}-\frac{\left(1-q_1\right)\left(\psi+\kappa-1\right)}{3}-\frac{1}{2}\left(\alpha_1-\alpha_2\right)\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{2/3}}-\frac{\left(1-q_1\right)\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{4\kappa^{2/3}}\right]$$

$$-\upsilon\left[\left(\psi+\kappa-1\right)\frac{3}{2}\left(\alpha_1+\alpha_2\right)\left(\psi+\kappa-1\right)\frac{3}{2}\frac{\alpha_2\left(\psi+\kappa-1\right)^{5}}{\kappa^{2/3}}\right]$$

$$+3\upsilon\left[\frac{1}{4}\frac{\left(\psi+\kappa-1\right)^{5}}{\kappa^{2/3}}+\frac{5}{8}\left(\alpha_1+\alpha_2\right)\frac{\left(\psi+\kappa-1\right)^{5}}{\kappa^{2/3}}+\frac{5\left(-q_2\right)\left(\psi+\kappa-1\right)^{5}}{12\kappa^{2/3}}-\frac{5\alpha_2\left(\psi+\kappa-1\right)^{7}}{8\kappa^{4/3}}\right.$$

$$\left.+\frac{\left(-q_2\right)\left(\psi+\kappa-1\right)}{3}+\frac{\left(-q_1\right)\left(\psi+\kappa-1\right)}{3}-\frac{1}{2}\left(\alpha_1-\alpha_2\right)\frac{\left(\psi+\kappa-1\right)^{5}}{\kappa^{2/3}}-\frac{\left(-q_2\right)\left(\psi+\kappa-1\right)^{5}}{4\kappa^{2/3}}\right]$$

$$-\frac{3\alpha_1}{2}\left(-\upsilon\right)\frac{\left(\psi+\kappa-1\right)^{5}}{\kappa^{2/3}}-\frac{3\alpha_2\upsilon}{2}\frac{\left(\psi+\kappa-1\right)^{5}}{\kappa^{2/3}}+\frac{15\alpha_1}{2}\left(-\upsilon\right)\frac{\left(\psi+\kappa-1\right)^{7}}{4\kappa^{4/3}}+\frac{15\alpha_2\upsilon}{2}\frac{\left(\psi+\kappa-1\right)^{7}}{4\kappa^{4/3}}$$

Simplifying the above equation and substituting for $\omega_0^2$, we have

$$\Omega^0_{\xi\xi}=\frac{\left(\psi+\kappa-1\right)^{5/3}}{\kappa^{2/3}}\left\{\frac{3}{4}+\frac{3}{8\kappa^{2/3}}\alpha_1\left[9\kappa^{2/3}-5\left(\psi+\kappa-1\right)^{2/3}\right]+\frac{3}{8}\alpha_2+\frac{\left[\left(\psi+\kappa-1\right)^{2/3}-2\kappa^{2/3}\right]\left(1-q_1\right)}{2\left(\psi+\kappa-1\right)^{2/3}}\right.$$

$$+\frac{\kappa^{2/3}\left(1-q_2\right)}{\left(\psi+\kappa-1\right)^{\frac{2}{3}}}-\frac{3}{8}\frac{\alpha_1\upsilon\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{4/3}}\left[9\kappa^{2/3}-5\left(\psi+\kappa-1\right)^{\frac{2}{3}}\right]-\frac{3}{8}\alpha_2\upsilon-\frac{\upsilon\left[\left(\psi+\kappa-1\right)^{\frac{2}{3}}-2\kappa^{2/3}\right]\left(1-q_1\right)}{2\left(\psi+\kappa-1\right)^{\frac{2}{3}}}$$

$$\left.-\frac{\upsilon\kappa^{2/3}\left(1-q_2\right)}{\left(\psi+\kappa-1\right)^{2/3}}+\frac{3}{8\kappa^{2/3}}\alpha_2\upsilon\left[9\kappa^{2/3}-5\left(\psi+\kappa-1\right)^{2/3}\right]+\frac{3}{8}\alpha_1\upsilon+\frac{\upsilon\left[\left(\psi+\kappa-1\right)2\kappa^{2/3}\right]\left(-q_2\right)}{2\left(\psi+\kappa-1\right)^{2/3}}\right.$$

$$-\frac{\upsilon\kappa^{2/3}\ \ 1-q_1\ \ \psi+\kappa-1}{\psi+\kappa-1^{\ 2/3}}+\frac{15\alpha_1\ \ \psi+\kappa-1^{\ \frac{2}{3}}}{8\kappa^{2/3}}-\frac{15\alpha_1\upsilon\ \ \psi+\kappa-1^{\ \frac{2}{3}}}{8\kappa^{2/3}}+\frac{15\alpha_2\upsilon\ \ \psi+\kappa-1^{\ \frac{2}{3}}}{8\kappa^{2/3}}\ \ \Bigg\}$$

or

$$\Omega^0_{\xi\xi\xi}=\frac{\psi+\kappa-1^{\ 5/3}}{\kappa^{2/3}}\left\{\frac{3}{4}+\frac{\alpha_1}{\kappa^{2/3}}\left\{\frac{3\left[9\kappa^{2/3}-5\ \ \psi+\kappa-1^{\ 2/3}\right]}{8}+\frac{5\ \ \psi+\kappa-1^{\ 2/3}}{8}\right\}+\frac{3}{8}\alpha_2+\frac{\kappa^{2/3}\ \ 1-q_2}{\psi+\kappa-1^{\ 2/3}}\right.$$

$$+\frac{1-q_1\ \left[\ \psi+\kappa-1^{\ 2/3}-\kappa^{2/3}\right]}{2\ \ \psi+\kappa-1^{\ 2/3}}+\left\{\frac{3}{8}-\frac{3}{8\kappa^{2/3}}\left[9\kappa^{2/3}-5\ \ \psi+\kappa-1^{\ 2/3}\right]-\frac{15\ \ \psi+\kappa-1^{\ 2/3}}{8\kappa^{2/3}}\right\}\alpha_1\upsilon$$

$$+\left\{\frac{3\left[9\kappa^{2/3}-5\ \ \psi+\kappa-1^{\ 2/3}\right]}{8\kappa^{2/3}}+\frac{15\left(\psi+\kappa-1\right)^{2/3}}{8\kappa^{2/3}}-\frac{3}{8}\right\}\alpha_2\upsilon+\left\{\frac{\kappa^{2/3}}{\left(\psi+\kappa-1\right)^{2/3}}-\frac{\left[\left(\psi+\kappa-1\right)^{2/3}-2\kappa^{2/3}\right]}{2\left(\psi+\kappa-1\right)^{2/3}}\right\}$$

$$\left.\times\upsilon\left(1-q_1\right)\left\{\frac{\left[\left(\psi+\kappa-1\right)^{2/3}-2\kappa^{2/3}\right]}{2\left(\psi+\kappa-1\right)^{2/3}}-\frac{\kappa^{2/3}}{\left(\psi+\kappa-1\right)^{2/3}}\right\}\upsilon\left(1-q_2\right)\right\}$$

or

$$\Omega^0_{\xi\xi\xi}=\frac{\psi+\kappa-1^{\ \frac{2}{3}}}{\kappa^{2/3}}\left\{\frac{3}{4}+\frac{27}{8}\alpha_1+\frac{3}{8}\alpha_2+\left(\frac{3}{8}-\frac{27}{8}\right)\alpha_1\upsilon+\left(\frac{27}{8}-\frac{3}{8}\right)\alpha_2\upsilon+\frac{\psi+\kappa-1^{\ \frac{2}{3}}-2\kappa^{\frac{2}{3}}}{2\ \ \psi+\kappa-1^{\ 2/3}}\ \ 1-q_1\right.$$

$$\left.+\frac{\kappa^{2/3}\ \ 1-q_2}{\psi+\kappa-1^{\ 2/3}}+\left[\frac{4\kappa^{2/3}-\ \ \psi+\kappa-1^{\ 2/3}}{\psi+\kappa-1^{\ 2/3}}\right]\frac{\upsilon\ \ 1-q_1}{2}+\frac{\upsilon\ \ 1-q_2}{2}\left[\frac{\psi+\kappa-1^{\ 2/3}-4\kappa^{2/3}}{\psi+\kappa-1^{\ 2/3}}\right]\right\}$$

Hence, the value of this derivative at the triangular EPs is

$$\Omega^0_{\xi\xi\xi}=\frac{3\ \ \psi+\kappa-1^{\ \frac{5}{3}}}{4\kappa^{2/3}}\left\{1+\frac{9}{2}\alpha_1+\frac{1}{2}\alpha_2-4\alpha_1\upsilon+4\alpha_2\upsilon+\frac{2\ \ 1-q_1}{3}\frac{\psi+\kappa-1^{\ 2/3}-2\kappa^{2/3}}{\psi+\kappa-1^{\ 2/3}}+\frac{4\kappa^{2/3}\ \ 1-q_2}{3\ \ \psi+\kappa-1^{\ 2/3}}\right.$$

$$\left.+\frac{2\upsilon\ \ 1-q_1\ \ N}{3\ \ \psi+\kappa-1^{\ 2/3}}-\frac{2\upsilon\ \ 1-q_2\ \ N}{3\ \ \psi+\kappa-1^{\ 2/3}}\right\}$$

Similarly,

$$\Omega^0_{\eta\eta}=\frac{3\ \psi+\kappa-1\ ^{\frac{5}{3}}}{4\kappa^{2/3}}\left\{\frac{N}{\psi+\kappa-1\ ^{2/3}}+\frac{\left[12\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right]\ \alpha_1+\alpha_2}{2\ \psi+\kappa-1\ ^{2/3}}+\frac{2\ 1-q_1}{3}\right.$$
$$\left.\times\frac{\left[2\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right]}{\psi+\kappa-1\ ^{2/3}}-\frac{4\kappa^{2/3}\ 1-q_2}{3}-\frac{2\upsilon\ 1-q_1\ N}{3\ \psi+\kappa-1\ ^{2/3}}+\frac{2\upsilon\ 1-q_2\ N}{3\ \psi+\kappa-1\ ^{2/3}}\right\} \tag{3.134}$$

and

$$\Omega^0_{\xi\eta}=\frac{3\sqrt{N}\ \psi+\kappa-1\ ^{4/3}}{8\kappa^{2/3}}\left\{2+\frac{\left[24\kappa^{2/3}-5\ \psi+\kappa-1\ ^{2/3}\right]\alpha_1}{N}+\frac{\left[8\kappa^{2/3}-5\ \psi+\kappa-1\ ^{2/3}\right]\alpha_2}{N}\right.$$
$$+\frac{\left[4\ \psi+\kappa-1\ ^{4/3}+4\kappa^{4/3}-4\kappa^{2/3}\ \psi+\kappa-1\ ^{2/3}\right]}{3N\ \psi+\kappa-1\ ^{2/3}}\ 1-q_1\ +\frac{8\kappa^{2/3}\left[2\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right]}{3N\ \psi+\kappa-1\ ^{2/3}}\ 1-q_2$$
$$-\upsilon\left[4+\frac{32\kappa^{2/3}-6\ \psi+\kappa-1\ ^{2/3}}{N}\alpha_1+\frac{\left\{4\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}-16\kappa^{4/3}-10\left(\psi+\kappa-1\right)^{4/3}\right\}}{4\kappa^{2/3}N}\alpha_2\right.$$
$$\left.\left.+\frac{4\ 2\left\{\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right\}}{3N}\left(1-q_1\right)\frac{4\left(\psi+\kappa-1\right)^{2/3}}{3N}\left(1-q_2\right)\right]\right\}$$

where $N$ is defined in equations (3.70)

Factorizing and ignoring product of $\psi-1=\in'$ with $\alpha_i$ and $1-q_i\ \ i=1,2$ and their product, in equations (3.134), yield the respective equations:

$$\Omega^0_{\xi\xi\xi}=\frac{3\kappa}{4}\left\{1+\frac{9}{2}\alpha_1+\frac{1}{2}\alpha_2-4\alpha_1\upsilon+4\alpha_2\upsilon-\frac{2}{3}\ 1-q_1\ +\frac{4}{3}\ 1-q_2\ +2\upsilon\ 1-q_1\ +2\upsilon\ 1-q_2\ +\frac{5\ \psi-1}{3\kappa}\right\}$$
$$\Omega^0_{\eta\eta}=\Omega^0_{\xi\xi\xi}=\frac{3\kappa}{4}\left\{3+\frac{11}{2}\ \alpha_1+\alpha_2\ +\frac{2}{3}\ 1-q_1\ -\frac{4}{3}\ 1-q_2\ -2\upsilon\ 1-q_1\ +2\upsilon\ 1-q_2\ +\frac{7\ \psi-1}{3\kappa}\right\}$$

$$\Omega^0_{\xi\eta} = \frac{3\kappa\sqrt{3}}{8}\left\{2+\frac{19}{3}\alpha_1+\frac{7}{3}\alpha_2-\frac{4}{9}\ 1-q_1\ +\frac{8}{9}\ 1-q_2\ +\frac{7\ \psi-1}{3\kappa}-\upsilon\left[4+\frac{26}{3}\ \alpha_1+\alpha_2\right.\right.$$
$$\left.\left.+\frac{4}{9}\ 1-q_1\ +\frac{4}{9}\ 1-q_2\ +\frac{44\ \psi-1}{3\kappa}\right]\right\} \tag{3.135}$$

The characteristic equation (3.66) after substituting the partial derivatives is

$$\lambda^4+\left\{4-3\kappa\left[1+\frac{5}{2}\alpha_1+\frac{3}{2}\alpha_2-\upsilon\ \alpha_1-\alpha_2\ \right]+6\ \alpha_1+\alpha_2\ -3\in'+8\in\right\}\lambda^2$$
$$+\frac{3\kappa}{4}\left[9\kappa+22\in'+39\kappa\ \alpha_1+\alpha_2\ +2\kappa\ 1-q_1\ +2\kappa\ 1-q_2\ \right]\upsilon\ 1-\upsilon\ =0 \tag{3.136}$$

Its roots are

$$\lambda^2_{1,2}=\frac{3\kappa\left[1+\frac{5}{2}\alpha_1+\frac{3}{2}\alpha_2-\upsilon\ \alpha_1-\alpha_2\ \right]-6\ \alpha_1+\alpha_2\ +3\in'-8\in-4\pm\sqrt{D}}{2}$$

where $D$ is the discriminant given by the equation,

$$D=3\kappa\left[9\kappa+22\in'+39\kappa\ \alpha_1+\alpha_2\ +2\kappa\ 1-q_1\ +2\kappa\ 1-q_2\ \right]\upsilon^2$$
$$-3\kappa\left[9\kappa+22\in'+45\kappa\alpha_1+33\kappa\alpha_2+2\kappa\ 1-q_2\ -8\ \alpha_1-\alpha_2\ \right]\upsilon$$
$$+9\kappa^2\ 1+5\alpha_1+3\alpha_2\ +6\kappa\ 3\in'-8\in-16\alpha_1-12\alpha_2-4\ +8\left[2+8\in-3\in'+6\ \alpha_1+\alpha_2\ \right]$$

**Critical mass parameter**

The critical mass parameter is the value of the mass ratio when the discriminant vanishes. The discriminant is quadratic in $\upsilon$, hence we can solve for $\upsilon$, such that:

$$\upsilon_{c\kappa}=\frac{-b\pm\sqrt{D}}{2a} \tag{3.137}$$

where

$$a=3\kappa\left[9\kappa+22\ \psi-1\ +39\kappa\alpha_1+39\kappa\alpha_2+2\kappa\ 1-q_1\ +2\kappa\ 1-q_2\ \right]$$

$$b = -3\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\ \alpha_1 + \alpha_2\ + 6\kappa\ \alpha_1 - \alpha_2\ + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ - 8\ \alpha_1 - \alpha_2\ \right]$$

$$c = 9\kappa^2 - 24\kappa + 16 + 45\kappa^2\alpha_1 + 27\kappa^2\alpha_2 - 60\kappa\alpha_1 + 36\kappa\alpha_2 - 48\kappa\ \varphi - 1\ + 64\ \varphi - 1\ + 18\kappa\ \psi - 1$$
$$-24\ \psi - 1\ + 48\ \alpha_1 + \alpha_2\ - 36\kappa\ \alpha_1 + \alpha_2$$

Therefore,

$$\upsilon_{c\kappa} = \frac{3\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\ \alpha_1 + \alpha_2\ + 6\kappa\ \alpha_1 - \alpha_2\ + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ - 8\ \alpha_1 - \alpha_2\ \right]}{6\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}$$
$$\pm\frac{\sqrt{9\kappa^2\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]^2 - 4ac}}{6\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}$$

$$\upsilon_{c\kappa} = \frac{3\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\ \alpha_1 + \alpha_2\ + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right] + 3\kappa\left[6\kappa\ \alpha_1 - \alpha_2\ - 8\ \alpha_1 - \alpha_2\ \right]}{6\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}$$
$$\pm\frac{\sqrt{9\kappa^2\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ 6\kappa\ \alpha_1 - \alpha_2\ - 8\ \alpha_1 - \alpha_2\ \right]^2 - 4ac}}{6\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}$$

$$\upsilon_{c\kappa} = \frac{3\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\ \alpha_1 + \alpha_2\ + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}{6\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}$$
$$+\frac{3\kappa\left[6\kappa\ \alpha_1 - \alpha_2\ - 8\ \alpha_1 - \alpha_2\ \right]}{6\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}\pm$$
$$\frac{\sqrt{9\kappa^2\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ 6\kappa\ \alpha_1 - \alpha_2\ - 8\ \alpha_1 - \alpha_2\ \right]^2 - 4ac}}{6\kappa\left[9\kappa + 22\left(\psi - 1\right)39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\left(-q_1\right)2\kappa\left(-q_2\right)\right]}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{\left[6\kappa\ \alpha_1 - \alpha_2\ - 8\ \alpha_1 - \alpha_2\ \right]}{2\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}\pm$$
$$\frac{\sqrt{9\kappa^2\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ + 6\kappa\ \alpha_1 - \alpha_2\ - 8\ \alpha_1 - \alpha_2\ \right]^2 - 4ac}}{6\kappa\left[9\kappa + 22\ \psi - 1\ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa\ 1 - q_1\ + 2\kappa\ 1 - q_2\ \right]}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{\left[ 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2 \ \right]}{2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right]} \pm$$

$$\frac{1}{6}\sqrt{\frac{9\kappa^2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa \ \alpha_1 + \alpha_2 \ + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ + 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2 \ \right]^2 - 4ac}{\kappa^2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right]^2}}$$

For ease, we let

$$l_1 = 9\kappa + 22 \ \psi - 1 \ + 39\kappa \ \alpha_1 + \alpha_2 \ + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ + 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2$$

$$l_2 = 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2$$

Then, we have

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{\left[ 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2 \ \right]}{2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right]} \pm \frac{1}{6}\sqrt{\frac{9\kappa^2 l_1^2 - 4ac}{\kappa^2 l_2^2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{\left[ 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2 \ \right]}{2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right]} \pm \frac{1}{6}\sqrt{\frac{9\kappa^2 l_1^2}{\kappa^2 l_2^2} - \frac{4ac}{\kappa^2 l_2^2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{\left[ 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2 \ \right]}{2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right]} \pm \frac{1}{6}\sqrt{\frac{9l_1^2}{l_2^2} - \frac{4ac}{\kappa^2 l_2^2}}$$

Now,

$$4ac = 12\kappa\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right] \times \left[ 9\kappa^2 - 24\kappa + 16 + 45\kappa^2\alpha_1 \right.$$
$$+27\kappa^2\alpha_2 - 60\kappa\alpha_1 + 36\kappa\alpha_2 - 48\kappa \ \varphi - 1 \ + 64 \ \varphi - 1 \ + 18\kappa \ \psi - 1 \ - 24 \ \psi - 1 \ + 48 \ \alpha_1 + \alpha_2$$
$$\left. -36\kappa \ \alpha_1 + \alpha_2 \ \right]$$

$$\Rightarrow 4ac = 12\kappa l_2 c$$

Therefore, we have

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{\left[ 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2 \ \right]}{2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right]} \pm \frac{1}{6}\sqrt{\frac{9l_1^2}{l_2^2} - \frac{12\kappa l_2 c}{\kappa^2 l_2^2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{\left[ 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2 \ \right]}{2\left[ 9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ \right]} \pm \frac{1}{6}\sqrt{\frac{9l_1^2}{l_2^2} - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{2\left[ 3\kappa \ \alpha_1 - \alpha_2 \ -4 \ \alpha_1 - \alpha_2 \ \right]}{18\kappa\left[ 1 + \frac{22 \ \psi - 1}{9\kappa} + \frac{39}{9}\alpha_1 + \frac{39}{9}\alpha_2 + \frac{2}{9} \ 1 - q_1 \ + \frac{2}{9} \ 1 - q_2 \ \right]} \pm \frac{1}{6}\sqrt{\frac{9l_1^2}{l_2^2} - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{2\left[ 3\kappa \ \alpha_1 - \alpha_2 \ -4 \ \alpha_1 - \alpha_2 \ \right]\left[ 1 - \frac{22 \ \psi - 1}{9\kappa} - \frac{39}{9}\alpha_1 - \frac{39}{9}\alpha_2 - \frac{2}{9} \ 1 - q_1 \ - \frac{2}{9} \ 1 - q_2 \ \right]}{18\kappa}$$
$$\pm \frac{1}{6}\sqrt{9\left( \frac{l_1}{l_2} \right)^2 - \frac{12c}{\kappa l_2}}$$

Ignoring products and second order in very small quantities, we have

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{2\left[ 3\kappa \ \alpha_1 - \alpha_2 \ -4 \ \alpha_1 - \alpha_2 \ \right]}{18\kappa} \pm \frac{1}{6}\sqrt{9\left( \frac{l_1}{l_2} \right)^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \ \ \alpha_1 - \alpha_2}{9\kappa} \pm \frac{1}{6}\sqrt{9\left( \frac{l_1}{l_2} \right)^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \ \ \alpha_1 - \alpha_2}{9\kappa} \pm \frac{1}{6}$$
$$\times\sqrt{9\left[ \frac{9\kappa + 22 \ \psi - 1 \ + 39\kappa \ \alpha_1 + \alpha_2 \ + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2 \ + 6\kappa \ \alpha_1 - \alpha_2 \ -8 \ \alpha_1 - \alpha_2}{9\kappa + 22 \ \psi - 1 \ + 39\kappa\alpha_1 + 39\kappa\alpha_2 + 2\kappa \ 1 - q_1 \ + 2\kappa \ 1 - q_2} \right]^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} \pm \frac{1}{6}\sqrt{9\left[\frac{l_2 + 6\kappa \ \ \alpha_1 - \alpha_2 \ \ -8 \ \ \alpha_1 - \alpha_2}{l_2}\right]^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} \pm \frac{1}{6}\sqrt{9\left[\frac{l_2}{l_2} + \frac{6\kappa \ \ \alpha_1 - \alpha_2 \ \ -8 \ \ \alpha_1 - \alpha_2}{l_2}\right]^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} \pm \frac{1}{6}\sqrt{9\left[1 + \frac{6\kappa - 8 \quad \alpha_1 - \alpha_2}{l_2}\right]^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} \pm \frac{1}{6}\sqrt{9\left[1 + \frac{2 \ \ 3\kappa - 4 \quad \alpha_1 - \alpha_2}{l_2}\right]^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa}$$
$$-\frac{1}{6}\sqrt{9\left[1 + \frac{2 \ \ 3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa\left[1 + \frac{22 \ \ \psi - 1}{9\kappa} + \frac{39}{9}\alpha_1 + \frac{39}{9}\alpha_2 + \frac{2}{9}\ 1 - q_1 \ + \frac{2}{9}\ 1 - q_2\right]}\right]^2 - \frac{12c}{\kappa l_2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa}$$
$$-\frac{1}{6}\sqrt{9\left[1 + \frac{2 \ \ 3\kappa - 4 \quad \alpha_1 - \alpha_2 \ \left[1 - \frac{22 \ \ \psi - 1}{9\kappa} - \frac{13}{3}\alpha_1 - \frac{13}{3}\alpha_2 - \frac{2}{9}\ 1 - q_1 \ - \frac{2}{9}\ 1 - q_2\right]}{9\kappa}\right]^2 - \frac{12c}{\kappa l_2}}$$

Ignoring products and higher order terms, we get

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{6}\sqrt{9\left[1 + \frac{2 \;\; 3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa}\right]^2 - \frac{12c}{\kappa l_2}}$$

Using Binomial expansion and retaining only linear terms, we get

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{6}\sqrt{9\left[1 + \frac{4 \;\; 3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa}\right] - \frac{12c}{9\kappa^2\left[1 + \frac{22 \;\; \psi - 1}{9\kappa} + \frac{39}{9}\alpha_1 + \frac{39}{9}\alpha_2 + \frac{2}{9} \;\; 1 - q_1 \;\; + \frac{2}{9} \;\; 1 - q_2\right]}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{6}\sqrt{9\left[1 + \frac{4 \;\; 3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa}\right] - \frac{4c\left[1 - \frac{22 \;\; \psi - 1}{9\kappa} - \frac{13}{3}\alpha_1 - \frac{13}{3}\alpha_2 - \frac{2}{9} \;\; 1 - q_1 \;\; - \frac{2}{9} \;\; 1 - q_2\right]}{3\kappa^2}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{6}\sqrt{9\left\{1 + \frac{4 \;\; 3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{4c\left[1 - \frac{22 \;\; \psi - 1}{9\kappa} - \frac{13}{3}\alpha_1 - \frac{13}{3}\alpha_2 - \frac{2}{9} \;\; 1 - q_1 \;\; - \frac{2}{9} \;\; 1 - q_2\right]}{27\kappa^2}\right\}}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2}\sqrt{1 + \frac{4 \;\; 3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{4c\left[1 - \frac{22 \;\; \psi - 1}{9\kappa} - \frac{13}{3}\alpha_1 - \frac{13}{3}\alpha_2 - \frac{2}{9} \;\; 1 - q_1 \;\; - \frac{2}{9} \;\; 1 - q_2\right]}{27\kappa^2}} \quad (3.138)$$

Now, $c\left[1-\frac{22\ \psi-1}{9\kappa}-\frac{13}{3}\alpha_1-\frac{13}{3}\alpha_2-\frac{2}{9}\ 1-q_1\ -\frac{2}{9}\ 1-q_2\ \right]=l_3$

This implies

$$l_3=\left[9\kappa^2-24\kappa+16+45\kappa^2\alpha_1+27\kappa^2\alpha_2-60\kappa\alpha_1+36\kappa\alpha_2-48\kappa\ \varphi-1\ +64\ \varphi-1\ +18\kappa\ \psi-1\right.$$

$$\left.-24\ \psi-1\ +48\ \alpha_1+\alpha_2\ -36\kappa\ \alpha_1+\alpha_2\ \right]\left[1-\frac{22\ \psi-1}{9\kappa}-\frac{13}{3}\alpha_1-\frac{13}{3}\alpha_2-\frac{2}{9}\ 1-q_1\ -\frac{2}{9}\ 1-q_2\ \right]$$

For simplicity, we let $\varphi=1+\in$ and $\psi=1+\in'$, so that we get

$$l_3=9\kappa^2-24\kappa+16-48\kappa\in+64\in+18\kappa\in'-24\in'+15\alpha_1\ 3\kappa-4\ +9\alpha_2\ 3\kappa-4\ +48\ \alpha_1+\alpha_2$$

$$-36\kappa\ \alpha_1+\alpha_2\ -\frac{22}{9\kappa}\in'\ 9\kappa^2-24\kappa+16\ -\frac{13}{3}\alpha_1\ 9\kappa^2-24\kappa+16\ -\frac{13}{3}\alpha_2\ 9\kappa^2-24\kappa+16$$

$$-\frac{2\ 9\kappa^2-24\kappa+16\ \ 1-q_1}{9}-\frac{2\ 9\kappa^2-24\kappa+16\ \ 1-q_2}{9}$$

Substitute in equation (3.138) to get

$$\upsilon_{c\kappa}=\frac{1}{2}+\frac{3\kappa-4\ \ \alpha_1-\alpha_2}{9\kappa}-\frac{1}{2}\sqrt{1+\frac{4\ 3\kappa-4\ \ \alpha_1-\alpha_2}{9\kappa}-\frac{4l_3}{27\kappa^2}}$$

$$\upsilon_{c\kappa}=\frac{1}{2}+\frac{3\kappa-4\ \ \alpha_1-\alpha_2}{9\kappa}-\frac{1}{2}\sqrt{1+\frac{12\kappa\ 3\kappa-4\ \ \alpha_1-\alpha_2\ -4l_3}{27\kappa^2}}$$

$$\upsilon_{c\kappa}=\frac{1}{2}+\frac{3\kappa-4\ \ \alpha_1-\alpha_2}{9\kappa}-\frac{1}{2}\sqrt{1+\frac{4}{27\kappa^2}\left[3\kappa\ \ 3\kappa-4\ \ \alpha_1-\alpha_2\ -l_3\right]}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2}\sqrt{\frac{1}{\kappa^2}\left[\kappa^2 + \frac{4 \;\; 4 \;\; 3\kappa - 4 \quad \alpha_1 - \alpha_2 \;\; - l_3}{27}\right]}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\sqrt{\kappa^2 + \frac{4 \;\; 4 \;\; 3\kappa - 4 \quad \alpha_1 - \alpha_2 \;\; - l_3}{27}}$$

Substituting for $l_3$ and simplying, we get

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\sqrt{\kappa^2 + \frac{4l_4}{27}}$$

where

$$l_4 = 9\kappa^2 - 24\kappa + 16 + \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9} + \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - 16 \;\; 4 - 3\kappa \;\; \in$$

$$- \frac{312\kappa - 36\kappa^2 - 352 \;\; \in'}{9\kappa} - \alpha_1\left(-3\kappa^2 + 20\kappa - \frac{64}{3}\right) - \alpha_2\left(-3\kappa^2 + 20\kappa - \frac{64}{3}\right)$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\left\{\kappa^2 + \frac{4 \;\; 9\kappa^2 - 24\kappa + 16}{27} + \frac{4}{27}\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9}\right.\right.$$

$$+ \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \;\; \in'}{9\kappa} - 16 \;\; 4 - 3\kappa \;\; \in - \alpha_1\left(-3\kappa^2 + 20\kappa - \frac{64}{3}\right)$$

$$\left.\left. - \alpha_2\left(-3\kappa^2 + 20\kappa - \frac{64}{3}\right)\right]\right\}^{1/2}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\left\{\frac{27\kappa^2 - 36\kappa^2 + 96\kappa - 64}{27} + \frac{4}{27}\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9}\right.\right.$$

$$+ \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa} - 16 \quad 4 - 3\kappa \quad \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)$$

$$\left.\left.-\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right]\right\}^{1/2}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\left\{\frac{96\kappa - 9\kappa^2 - 64}{27} + \frac{4}{27}\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9}\right.\right.$$

$$+ \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa} - 16 \quad 4 - 3\kappa \quad \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)$$

$$\left.\left.-\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right]\right\}^{1/2}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\left\{\frac{96\kappa - 9\kappa^2 - 64}{27}\left\{1 + \frac{4}{27} \times \frac{27}{96\kappa - 9\kappa^2 - 64}\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9}\right.\right.\right.$$

$$+ \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa} - 16 \quad 4 - 3\kappa \quad \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)$$

$$\left.\left.\left.-\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right] \right\}\right\}^{1/2}$$

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{27}}\left\{1 + \frac{4}{96\kappa - 9\kappa^2 - 64}\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9}\right.\right.$$

$$+ \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa} - 16 \quad 4 - 3\kappa \quad \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)$$

$$\left.\left.-\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right] \right\}^{1/2}$$

Using Binomial expansion, we have

$$\upsilon_{c\kappa} = \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{27}}\left\{1 + \frac{2}{96\kappa - 9\kappa^2 - 64}\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9}\right.\right.$$

$$+ \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa} - 16 \quad 4 - 3\kappa \quad \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)$$

$$\left.\left. -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right]\right\}$$

$$= \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{27}} - \frac{1}{2\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{27}} \times \frac{2}{96\kappa - 9\kappa^2 - 64}$$

$$\times\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9} + \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa}\right.$$

$$\left. -16 \quad 4 - 3\kappa \quad \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right) - \alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right]$$

$$= \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{27}} - \frac{1}{\kappa\sqrt{27}}\frac{1}{\sqrt{96\kappa - 9\kappa^2 - 64}}$$

$$\times\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9} + \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} + \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa}\right.$$

$$\left. +16 \quad 4 - 3\kappa \quad \in +\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right) + \alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right]$$

$$= \frac{1}{2} + \frac{3\kappa - 4 \quad \alpha_1 - \alpha_2}{9\kappa} - \frac{1}{2\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{27}} - \frac{1}{3\kappa\sqrt{3}}\frac{1}{\sqrt{96\kappa - 9\kappa^2 - 64}}$$

$$\times\left[\frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_1}{9} + \frac{18\kappa^2 - 48\kappa + 32 \quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352 \quad \in'}{9\kappa}\right.$$

$$\left. -16 \quad 4 - 3\kappa \quad \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right) - \alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right]$$

$$= \frac{1}{2} + \frac{3\kappa - 4\ \ \alpha_1}{9\kappa} - \frac{3\kappa - 4\ \ \alpha_2}{9\kappa} - \frac{1}{6\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{3}} - \frac{1}{3\kappa\sqrt{3}}\frac{1}{\sqrt{96\kappa - 9\kappa^2 - 64}}$$
$$\times\left[\frac{18\kappa^2 - 48\kappa + 32\quad 1 - q_1}{9} + \frac{18\kappa^2 - 48\kappa + 32\quad 1 - q_2}{9} - \frac{312\kappa - 36\kappa^2 - 352\ \ \in'}{9\kappa}\right.$$
$$\left. -16\ \ 4 - 3\kappa\ \ \in -\alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right) - \alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{3}\right)\right]$$

So that now

$$\upsilon_{c\kappa} = \frac{1}{2} - \frac{1}{6\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{3}} - \frac{2\ \ 9\kappa^2 - 24\kappa + 16\quad 1 - q_1}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}} - \frac{2\ \ 9\kappa^2 - 24\kappa + 16\quad 1 - q_2}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}$$
$$+ \frac{4\ \ 78\kappa - 9\kappa^2 - 88\ \ \in'}{27\kappa^2\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}} + \frac{1}{3\kappa\sqrt{3}}\frac{16\ \ 4 - 3\kappa\ \ \in}{\sqrt{96\kappa - 9\kappa^2 - 64}} + \alpha_1\left(\frac{60\kappa - 9\kappa^2 - 64}{9\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\right)$$
$$+ \alpha_2\left(\frac{60\kappa - 9\kappa^2 - 64}{9\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\right) + \frac{(3\kappa - 4)\alpha_1}{9\kappa} - \frac{(3\kappa - 4)\alpha_2}{9\kappa}$$

$$= \frac{1}{2} - \frac{1}{6\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{3}} - \frac{2\ \ 9\kappa^2 - 24\kappa + 16\quad 1 - q_1}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}} - \frac{2\ \ 9\kappa^2 - 24\kappa + 16\quad 1 - q_2}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}$$
$$+ \frac{4\ \ 78\kappa - 9\kappa^2 - 88\ \ \in'}{27\kappa^2\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}} + \frac{1}{3\kappa\sqrt{3}}\frac{16\ \ 4 - 3\kappa\ \ \in}{\sqrt{96\kappa - 9\kappa^2 - 64}} + \alpha_1\left[\frac{3\kappa - 4}{9\kappa} + \frac{60\kappa - 9\kappa^2 - 64}{9\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\right]$$
$$+ \alpha_2\left[-\frac{3\kappa - 4}{9\kappa} + \frac{60\kappa - 9\kappa^2 - 64}{9\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\right]$$

$$\upsilon_{c\kappa} = \frac{1}{2} - \frac{1}{6\kappa}\sqrt{\frac{96\kappa - 9\kappa^2 - 64}{3}} - \frac{2\ \ 9\kappa^2 - 24\kappa + 16\quad 1 - q_1}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}} - \frac{2\ \ 9\kappa^2 - 24\kappa + 16\quad 1 - q_2}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}} \qquad 3.139$$
$$+ \frac{4\ \ 78\kappa - 9\kappa^2 - 88\ \ \in'}{27\kappa^2\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}} + \frac{1}{3\kappa\sqrt{3}}\frac{16\ \ 4 - 3\kappa\ \ \in}{\sqrt{96\kappa - 9\kappa^2 - 64}} + \frac{1}{9\kappa}\left(3\kappa - 4 + \frac{60\kappa - 9\kappa^2 - 64}{\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\right)\alpha_1$$
$$+ \frac{1}{9\kappa}\left(4 - 3\kappa + \frac{60\kappa - 9\kappa^2 - 64}{\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\right)\alpha_2$$

$\upsilon_{C\kappa}$ exist for different values of $\kappa$, and are a tool in determining the region of stability and also

serve as the technicalities in analyzing the behaviors of these parameters on the motion of the test particle around the triangular equilibrium. Equation (3.139) represents the effect of a constant $\kappa$ of a particular integral of the GMP, oblateness, small perturbation in the Coriolis and the centrifugal forces and radiation pressures of the primaries on the critical mass values $\upsilon_{C\kappa}$.

As analyzed earlier in problem **3.2**, it is seen that, when simultaneously $0<\upsilon<\upsilon_{C_\kappa}$, $D>0$ and coefficient of $\lambda^2$ is positive, the roots of the characteristic equation (3.136) are distinct pure imaginary and the motion is bounded. Therefore the triangular points is stable, otherwise, it is unstable. Our result is different from the case of the R3BP where there exists only one critical mass parameter.

**Out-of-plane equilibrium points**

We consider the stability of the out-of-plane EP $L_6$. To compute the partial derivatives using this equilibrium point, we first substitute $\eta=0$ in the second equation of (3.133) to get

$$\Omega^0_{\xi\eta}=0 \tag{3.140}$$

Similarly, when we perform same substitution in the third equation of (3.133), we at once have

$$\Omega^0_{\eta\eta}=\omega_0^2\ \ \psi+\kappa-1\ -\frac{q_1\kappa\ \ 1-\upsilon}{\rho_1^3}-\frac{q_2\kappa\upsilon}{\rho_2^3}-\frac{3\alpha_1 q_1\kappa\ \ 1-\upsilon}{2\rho_1^5}-\frac{3\alpha_2 q_2\kappa\upsilon}{2\rho_2^5} \tag{3.141}$$

But here, we know that

$$\rho_1^{-3}=\frac{\psi\xi+\ \ 1-\upsilon\ \ \ \kappa-1}{\kappa\ \ 1-\upsilon}\left[1+\ \ 1-q_1\ \ +\frac{3}{2}\left\{1-\frac{\left[\psi\xi+\ \ 1-\upsilon\ \ \ \kappa-1\ \right]^{\frac{2}{3}}}{\left[\kappa\ \ 1-\upsilon\ \right]^{\frac{2}{3}}}\right\}\alpha_1+\frac{3}{2}\alpha_2\right]$$

$$\rho_2^{-3}=\frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\left[1+\ \ 1-q_2\ \ +\frac{3}{2}\alpha_1+\frac{3}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{\kappa\upsilon\ ^{\frac{2}{3}}}\right\}\alpha_2\right]$$

$$\rho_1^{-5}=\left[\frac{\psi\xi+\ 1-\upsilon\quad\kappa-1}{\kappa\ 1-\upsilon}\right]^{\frac{5}{3}}\left[1+\frac{5}{3}\ 1-q_1\ +\frac{5}{2}\left\{1-\frac{\left[\psi\xi+\ 1-\upsilon\quad\kappa-1\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\right\}\alpha_1+\frac{5}{2}\alpha_2\right] \tag{3.142}$$

$$\rho_2^{-5}=\left[\frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\right]^{\frac{5}{3}}\left[1+\frac{5}{3}\ 1-q_2\ +\frac{5}{2}\alpha_1+\frac{5}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{\kappa\upsilon^{\ \frac{2}{3}}}\right\}\alpha_2\right]$$

$$\rho_1^{-7}=\left[\frac{\psi\xi+\ 1-\upsilon\quad\kappa-1}{\kappa\ 1-\upsilon}\right]^{\frac{7}{3}}\left[1+\frac{7}{3}\ 1-q_1\ +\frac{7}{2}\left\{1-\frac{\left[\psi\xi+\ 1-\upsilon\quad\kappa-1\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\right\}\alpha_1+\frac{7}{2}\alpha_2\right]$$

$$\rho_2^{-7}=\left[\frac{-\psi\xi+\upsilon\ \ \kappa-1}{\kappa\upsilon}\right]^{\frac{7}{3}}\left[1+\frac{7}{3}\ 1-q_2\ +\frac{7}{2}\alpha_1+\frac{7}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{\kappa\upsilon^{\ \frac{2}{3}}}\right\}\alpha_2\right]$$

Substituting the first four equations of (3.142) in equation (3.141), we have

$$\Omega_{\eta\eta}^0=\omega_0^2\ \ \psi+\kappa-1\ -q_1\left[\psi\xi+\ 1-\upsilon\quad\kappa-1\ \right]\left[1+\ 1-q_1\ +\frac{3\alpha_1}{2}\left\{1-\frac{\left[\psi\xi+\ 1-\upsilon\quad\kappa-1\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\right\}+\frac{3\alpha_2}{2}\right]$$

$$-q_2\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]\left[1+\ 1-q_2\ +\frac{3}{2}\alpha_1+\frac{3}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{\kappa\upsilon^{\ \frac{2}{3}}}\right\}\alpha_2\right]-\frac{3\alpha_1q_1\kappa\ 1-\upsilon}{2}$$

$$\times\left[\frac{\psi\xi+\ 1-\upsilon\quad\kappa-1}{\kappa\ 1-\upsilon}\right]^{\frac{5}{3}}\left[1+\frac{5}{3}\left(1-q_1\right)\frac{5}{2}\left\{1-\frac{\left[\psi\xi+\left(1-\upsilon\right)\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left[\kappa\left(1-\upsilon\right)\right]^{\frac{2}{3}}}\right\}\alpha_1+\frac{5}{2}\alpha_2\right]$$

$$-\frac{3\alpha_2q_2\kappa\upsilon}{2}\left[\frac{-\psi\xi+\upsilon\left(\kappa-1\right)}{\kappa\upsilon}\right]^{\frac{5}{3}}\left[1+\frac{5}{3}\left(1-q_2\right)\frac{5}{2}\alpha_1+\frac{5}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left(\kappa\upsilon\right)^{\frac{2}{3}}}\right\}\alpha_2\right]$$

Expressing $q_i=\left[1-\ 1-q_i\ \right],\quad i=1,2$ where $1-q_i\ <<1$; simplifying the above while ignoring products of $1-q_i\ <<1$ with $\alpha_i$ and retaining only linear terms in $1-q_i$ , we get

$$\Omega_{\eta\eta}^{0}=\omega_{0}^{2}\ \left[\psi+\kappa-1\right]-\psi\xi-\left[1-\upsilon\right]\left[\kappa-1\right]-\frac{3}{2}\alpha_{1}\left[\psi\xi+\left[1-\upsilon\right]\left[\kappa-1\right]\right]+\frac{3\alpha_{1}}{2}\frac{\left[\psi\xi+\left[1-\upsilon\right]\left[\kappa-1\right]\right]^{\frac{5}{3}}}{\left[\kappa\left[1-\upsilon\right]\right]^{\frac{2}{3}}}$$

$$-\frac{3\alpha_{2}}{2}\left[\psi\xi+\left[1-\upsilon\right]\left[\kappa-1\right]\right]+\psi\xi-\upsilon\left[\kappa-1\right]-\frac{3}{2}\alpha_{1}\left[-\psi\xi+\upsilon\left[\kappa-1\right]\right]+\frac{3\alpha_{2}}{2}\frac{\left[-\psi\xi+\upsilon\left[\kappa-1\right]\right]^{\frac{5}{3}}}{\kappa\upsilon^{\frac{2}{3}}}$$

$$-\frac{3\alpha_{2}}{2}\left[-\psi\xi+\upsilon\left[\kappa-1\right]\right]-\frac{3\alpha_{1}}{2}\frac{\left[\psi\xi+\left[1-\upsilon\right]\left[\kappa-1\right]\right]^{\frac{5}{3}}}{\left[\kappa\left[1-\upsilon\right]\right]^{\frac{2}{3}}}-\frac{3\alpha_{2}}{2}\frac{\left[-\psi\xi+\upsilon\left[\kappa-1\right]\right]^{\frac{5}{3}}}{\kappa\upsilon^{\frac{2}{3}}}$$

Substituting $\omega_{0}^{2}=1+\frac{3}{2}\left[\alpha_{1}+\alpha_{2}\right]$ , and simplifying, we get

$$\Omega_{\eta\eta}^{0}=\psi+\kappa-1+\frac{3}{2}\left[\alpha_{1}+\alpha_{2}\right]\left[\psi+\kappa-1\right]-\kappa+1+\kappa\upsilon-\upsilon-\kappa\upsilon+\upsilon-\frac{3}{2}\left[\kappa-1-\kappa\upsilon+\upsilon+\kappa\upsilon-\upsilon\right]\alpha_{1}$$

$$-\frac{3}{2}\left[\kappa-1-\kappa\upsilon+\upsilon+\kappa\upsilon-\upsilon\right]\alpha_{2}$$

$$\Rightarrow\Omega_{\eta\eta}^{0}=\psi+\frac{3}{2}\left[\alpha_{1}+\alpha_{2}\right]\left[\psi+\kappa-1\right]-\frac{3}{2}\left[\kappa-1\right]\alpha_{1}-\frac{3}{2}\left[\kappa-1\right]\alpha_{2}$$

$$\Rightarrow\Omega_{\eta\eta}^{0}=\psi+\frac{3}{2}\alpha_{1}\left[\psi+\kappa-1\right]+\frac{3}{2}\alpha_{2}\left[\psi+\kappa-1\right]-\frac{3}{2}\left[\kappa-1\right]\alpha_{1}-\frac{3}{2}\left[\kappa-1\right]\alpha_{2}$$

$$\Rightarrow\Omega_{\eta\eta}^{0}=\psi+\frac{3}{2}\alpha_{1}\left[\psi+\kappa-1-\kappa+1\right]+\frac{3}{2}\alpha_{2}\left[\psi+\kappa-1-\kappa+1\right]$$

$$\Rightarrow\Omega_{\eta\eta}^{0}=\psi+\frac{3}{2}\alpha_{1}\psi+\frac{3}{2}\alpha_{2}\psi$$

Hence, we get

$$\Omega_{\eta\eta}^{0}=\psi\left(1+\frac{3}{2}\alpha_{1}+\frac{3}{2}\alpha_{2}\right)$$

For the partial derivative $\Omega_{\xi\xi}^{0}$, we have

$$\Omega^{0}_{\xi\xi\xi}=-q_1\kappa\ 1-\upsilon\ \frac{\psi\xi+\ 1-\upsilon\quad\kappa-1}{\kappa\ 1-\upsilon}\left[1+\ 1-q_1\ +\frac{3}{2}\left\{1-\frac{\left[\psi\xi+\ 1-\upsilon\quad\kappa-1\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\right\}\alpha_1+\frac{3}{2}\alpha_2\right]$$

$$+\frac{3q_1\kappa\ 1-\upsilon\quad\xi+\upsilon\ ^2\left[\psi\xi+\ 1-\upsilon\quad\kappa-1\ \right]^{\frac{5}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{5}{3}}}\left[1+\frac{5\ 1-q_1}{3}+\frac{5\alpha_1}{2}\left\{1-\frac{\left[\psi\xi+\ 1-\upsilon\quad\kappa-1\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\right\}+\frac{5\alpha_2}{2}\right]$$

$$-q_2\kappa\upsilon\times\frac{-\psi\xi+\upsilon\left(\kappa-1\right)}{\kappa\upsilon}\left[1+\left(-q_2\right)\frac{3}{2}\alpha_1+\frac{3}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left(\kappa\upsilon\right)^{\frac{2}{3}}}\right\}\alpha_2\right]+\omega_0^2\left(\psi+\kappa-1\right)$$

$$+3q_2\kappa\upsilon\left(\xi+\upsilon-1\right)^2\left[\frac{-\psi\xi+\upsilon\left(\kappa-1\right)}{\kappa\upsilon}\right]^{\frac{5}{3}}\left[1+\frac{5}{3}\left(-q_2\right)+\frac{5}{2}\alpha_1+\frac{5}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left(\kappa\upsilon\right)^{\frac{2}{3}}}\right\}\alpha_2\right]$$

$$-\frac{3\alpha_1q_1\kappa\left(1-\upsilon\right)}{2}\left[\frac{\psi\xi+\left(1-\upsilon\right)\left(\kappa-1\right)}{\kappa\left(1-\upsilon\right)}\right]^{\frac{5}{3}}\left[1+\frac{5}{3}\left(-q_1\right)+\frac{5}{2}\left\{1-\frac{\left[\psi\xi+\left(1-\upsilon\right)\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left[\kappa\left(1-\upsilon\right)\right]^{\frac{2}{3}}}\right\}\alpha_1+\frac{5}{2}\alpha_2\right]$$

$$+\frac{15\alpha_1q_1\kappa\left(1-\upsilon\right)\left(\xi+\upsilon\right)^2\left[\psi\xi+\left(1-\upsilon\right)\left(\kappa-1\right)\right]^{\frac{7}{3}}}{2\left[\kappa\left(1-\upsilon\right)\right]^{\frac{7}{3}}}\left[1+\frac{7\left(-q_1\right)}{3}+\frac{7\alpha_1}{2}\left\{1-\frac{\left[\psi\xi+\left(1-\upsilon\right)\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left[\kappa\left(1-\upsilon\right)\right]^{\frac{2}{3}}}\right\}\right.$$

$$\left.+\frac{7\alpha_2}{2}\right]-\frac{3\alpha_2q_2\kappa\upsilon}{2}\left[\frac{-\psi\xi+\upsilon\left(\kappa-1\right)}{\kappa\upsilon}\right]^{\frac{5}{3}}\left[1+\frac{5}{3}\left(-q_2\right)+\frac{5}{2}\alpha_1+\frac{5}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left(\kappa\upsilon\right)^{\frac{2}{3}}}\right\}\alpha_2\right]$$

$$+\frac{15\alpha_2q_2\kappa\upsilon\left(\xi+\upsilon-1\right)^2}{2}\left[\frac{-\psi\xi+\upsilon\left(\kappa-1\right)}{\kappa\upsilon}\right]^{\frac{7}{3}}\left[1+\frac{7}{3}\left(-q_2\right)+\frac{7}{2}\alpha_1+\frac{7}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left(\kappa\upsilon\right)^{\frac{2}{3}}}\right\}\alpha_2\right]$$

Expressing the radiation factors of the primaries and following the same previous procedures, we simplify to get

$$\Omega^0_{\xi\xi} = \psi\left(1+\frac{3\alpha_1}{2}+\frac{3\alpha_2}{2}\right)+\frac{3\ \xi+\upsilon\ ^2\left[\psi\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{\frac{5}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\left[1+\frac{2\ 1-q_1}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]$$

$$-\frac{15\alpha_1\ \ \xi+\upsilon\ ^2}{2}\frac{\left[\psi\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{\frac{7}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{4}{3}}}+\frac{3\ \xi+\upsilon-1\ ^2\left[-\psi\xi+\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}{\kappa\upsilon^{\frac{2}{3}}}\left[1+\frac{2\ 1-q_2}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]$$

$$-\frac{15\alpha_2\ \ \xi+\upsilon-1\ ^2}{2}\frac{\left[-\psi\xi+\upsilon\ \kappa-1\ \right]^{\frac{7}{3}}}{\kappa\upsilon^{\frac{4}{3}}}+\frac{15\alpha_1\left(\xi+\upsilon\right)^2}{2}\frac{\left[\psi\xi+\left(-\upsilon\right)\left(\kappa-1\right)\right]^{\frac{7}{3}}}{\left[\kappa\left(-\upsilon\right)\right]^{\frac{4}{3}}}+\frac{15\alpha_2\left(\xi+\upsilon-1\right)^2}{2}$$

$$\times\frac{\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{7}{3}}}{\left(\kappa\upsilon^{\frac{4}{3}}\right)}$$

Hence,

$$\Omega^0_{\xi\xi} = \psi\left(1+\frac{3\alpha_1}{2}+\frac{3\alpha_2}{2}\right)+\frac{3\ \xi+\upsilon\ ^2\left[\psi\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{\frac{5}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\left[1+\frac{2\ 1-q_1}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]$$

$$+\frac{3\ \xi+\upsilon-1\ ^2\left[-\psi\xi+\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}{\kappa\upsilon^{\frac{2}{3}}}\left[1+\frac{2\ 1-q_2}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]$$

Similarly, for $\Omega^0_{\xi\zeta}$, we get

$$\Omega^0_{\xi\zeta} = \frac{3\ \xi+\upsilon\ \ \zeta\left[\psi\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{\frac{5}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\left[1+\frac{2\ 1-q_1}{3}+\frac{5\alpha_1}{2}\left\{1-\frac{\left[\psi\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{\frac{2}{3}}}{\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}}\right\}+\frac{5\alpha_2}{2}\right]$$

$$+\frac{3\ \xi+\upsilon-1\ \zeta\left[-\psi\xi+\upsilon\ \kappa-1\ \right]^{\frac{5}{3}}}{\kappa\upsilon^{\frac{2}{3}}}\left[1+\frac{2}{3}\ 1-q_2\ +\frac{5}{2}\alpha_1+\frac{5}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\ \kappa-1\ \right]^{\frac{2}{3}}}{\kappa\upsilon^{\frac{2}{3}}}\right\}\alpha_2\right]$$

$$+\frac{15\alpha_1\ \ \xi+\upsilon\ }{2}\frac{\zeta\left[\psi\xi+\ 1-\upsilon\ \left(\kappa-1\right)\right]^{\frac{7}{3}}}{\left[\kappa\left(-\upsilon\right)\right]^{\frac{4}{3}}}\left[1+\frac{4\left(-q_1\right)}{3}+\frac{7\alpha_1}{2}\left\{1-\frac{\left[\psi\xi+\left(-\upsilon\right)\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left[\kappa\left(-\upsilon\right)\right]^{\frac{2}{3}}}\right\}+\frac{7\alpha_2}{2}\right]$$

$$+\frac{15\alpha_2\left(\xi+\upsilon-1\right)}{2}\frac{\zeta\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{7}{3}}}{\left(\kappa\upsilon^{\frac{4}{3}}\right)}\left[1+\frac{7}{3}\left(-q_2\right)+\frac{7}{2}\alpha_1+\frac{7}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{2}{3}}}{\left(\kappa\upsilon^{\frac{2}{3}}\right)}\right\}\alpha_2\right]$$

Retaining only linear terms in $\alpha_i \;\; i=1,2$ and ignoring its product with $1-q_i$, we have

$$\Omega^0_{\xi\xi} = \frac{3\;\xi+\upsilon\;\zeta\left[\psi\xi+\;1-\upsilon\quad\kappa-1\;\right]^{\frac{5}{3}}}{\left[\kappa\;1-\upsilon\;\right]^{\frac{2}{3}}}\left[1+\frac{2\;1-q_1}{3}+\frac{5\alpha_1}{2}\left\{1-\frac{\left[\psi\xi+\;1-\upsilon\quad\kappa-1\;\right]^{\frac{2}{3}}}{\left[\kappa\;1-\upsilon\;\right]^{\frac{2}{3}}}\right\}+\frac{5\alpha_2}{2}\right]$$
$$+\frac{3\;\xi+\upsilon-1\;\zeta\left[-\psi\xi+\upsilon\;\kappa-1\;\right]^{\frac{5}{3}}}{\kappa\upsilon^{\;\frac{2}{3}}}\left[1+\frac{2}{3}\;1-q_2\;+\frac{5}{2}\alpha_1+\frac{5}{2}\left\{1-\frac{\left[-\psi\xi+\upsilon\;\kappa-1\;\right]^{\frac{2}{3}}}{\kappa\upsilon^{\;\frac{2}{3}}}\right\}\alpha_2\right]$$
$$+\frac{15\alpha_1\;\xi+\upsilon}{2}\frac{\zeta\left[\psi\xi+\;1-\upsilon\;\left(\kappa-1\right)\right]^{\frac{7}{3}}}{\left[\kappa\left(1-\upsilon\right)\right]^{\frac{4}{3}}}+\frac{15\alpha_2\left(\xi+\upsilon-1\right)\zeta\left[-\psi\xi+\upsilon\left(\kappa-1\right)\right]^{\frac{7}{3}}}{\left(\kappa\upsilon\right)^{\frac{4}{3}}}$$

Simplifying, we get

$$\Omega^0_{\xi\zeta} = 3\zeta\left\{\frac{\xi+\upsilon\;\left[\psi\xi+\;1-\upsilon\quad\kappa-1\;\right]^{\frac{5}{3}}}{\left[\kappa\;1-\upsilon\;\right]^{\frac{2}{3}}}\left[1+\frac{2\;1-q_1}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]+\frac{\xi+\upsilon-1\;\left[-\psi\xi+\upsilon\;\kappa-1\;\right]^{\frac{5}{3}}}{\kappa\upsilon^{\;\frac{2}{3}}}\right.$$
$$\left.\times\left[1+\frac{2\;1-q_2}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]\right\}$$

Lastly, for $\Omega^0_{\zeta\zeta}$, we get

$$\Omega^0_{\zeta\zeta} = 3\zeta^2\left\{\frac{\left[\psi\xi+\;1-\upsilon\quad\kappa-1\;\right]^{\frac{5}{3}}}{\left[\kappa\;1-\upsilon\;\right]^{\frac{2}{3}}}\left[1+\frac{2\;1-q_1}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]+\frac{\left[-\psi\xi+\upsilon\;\kappa-1\;\right]^{\frac{5}{3}}}{\kappa\upsilon^{\;\frac{2}{3}}}\right.$$
$$\left.\times\left[1+\frac{2\;1-q_2}{3}+\frac{5\alpha_1}{2}+\frac{5\alpha_2}{2}\right]\right\}$$

Therefore, the partial derivatives computed at the out-of-plane equilibrium point $L_6$ are

$$\Omega^0_{\xi\xi\xi} = \psi\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)+3\;\xi+\upsilon\;^2\frac{Y^{5/3}}{a^{2/3}}\left(1+\frac{2}{3}\;1-q_1\;+\frac{5}{2}\alpha_1+\frac{5}{2}\alpha_2\right)$$
$$+3\;\xi+\upsilon-1\;^2\frac{Z^{5/3}}{b^{2/3}}\left(1+\frac{2}{3}\;1-q_2\;+\frac{5}{2}\alpha_1+\frac{5}{2}\alpha_2\right) \qquad (3.143)$$

$$\Omega^0_{\eta\eta} = \psi\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right) \quad , \ \Omega^0_{\xi\eta} = \Omega^0_{\eta\zeta} = 0$$

$$\Omega^0_{\xi\xi} = 3\zeta\left\{\frac{Y^{5/3}\ \xi+\upsilon}{a^{2/3}}\left(1+\frac{2}{3}\ 1-q_1\ +\frac{5}{2}\alpha_1+\frac{5}{2}\alpha_2\right)+\frac{Z^{5/3}\ \xi+\upsilon-1}{b^{2/3}}\left(1+\frac{2}{3}\ 1-q_2\ +\frac{5}{2}\alpha_1+\frac{5}{2}\alpha_2\right)\right\}$$

$$\Omega^0_{\zeta\zeta} = 3\zeta^2\left\{\frac{Y^{5/3}}{a^{2/3}}\left(1+\frac{2}{3}\ 1-q_1\ +\frac{5}{2}\alpha_1+\frac{5}{2}\alpha_2\right)+\frac{Z^{5/3}}{b^{2/3}}\left(1+\frac{2}{3}\ 1-q_2\ +\frac{5}{2}\alpha_1+\frac{5}{2}\alpha_2\right)\right\}$$

where $a=\kappa\ 1-\upsilon\ , b=\kappa\upsilon\ , Y=\psi\xi+\ 1-\upsilon\ \ \kappa-1\ , Z=-\psi\xi+\upsilon\ \kappa-1$

Substituting the derivatives (3.143) in the characteristic equation (3.78), gives

$$\lambda^6 - d_1\lambda^4 + d_2\lambda^2 + d_3 = 0 \tag{3.144}$$

Its roots are found using M*athematica* software as

$$\lambda_{1,2} \to \pm\left[\frac{d_1}{3}+\frac{d_1^2-d_2}{3}\left(\frac{2}{\bar{n}}\right)^{1/3}+\frac{\bar{n}^{1/3}}{32^{1/3}}\right]^{1/2}$$

$$\lambda_{3,4} \to \pm\left[\frac{d_1}{3}-\frac{d_1^2}{32^{2/3}\bar{n}^{1/3}}+\frac{id_1^2}{2^{2/3}\sqrt{3}\bar{n}^{1/3}}+\frac{d_2}{2^{2/3}\bar{n}^{1/3}}-\frac{id_2\sqrt{3}}{2^{2/3}\bar{n}^{1/3}}+\frac{\bar{n}^{1/3}}{22^{1/3}\sqrt{3}}+\frac{\bar{n}^{1/3}}{62^{1/3}}\right]^{1/2}$$

$$\lambda_{5,6} \to \pm\left[\frac{d_1}{3}-\frac{d_1^2}{32^{2/3}\bar{n}^{1/3}}-\frac{id_1^2}{2^{2/3}\sqrt{3}\bar{n}^{1/3}}+\frac{d_2}{2^{2/3}\bar{n}^{1/3}}+\frac{i\sqrt{3}d_2}{2^{2/3}\bar{n}^{1/3}}+\frac{\bar{n}^{1/3}}{22^{1/3}\sqrt{3}}-\frac{\bar{n}^{1/3}}{62^{1/3}}\right]^{1/2}$$

where $d_1=\frac{3Y^{5/3}}{a^{2/3}}\left[\ \xi+\upsilon\ ^2+\zeta^2\right]\bar{Q}+\frac{3Z^{5/3}}{b^{2/3}}\left[\ \xi+\upsilon-1\ ^2+\zeta^2\right]\bar{R}-2-8\in+2\in'$

$$d_2 = 1+\frac{9}{2}\left[\frac{2}{3}+\frac{Y^{5/3}}{a^{2/3}}\ \xi+\upsilon\ ^2+\frac{Z^{5/3}}{b^{2/3}}\ \xi+\upsilon-1\ ^2-\zeta^2\left(\frac{Y^{5/3}}{a^{2/3}}+\frac{Z^{5/3}}{b^{2/3}}\right)\right]\ \alpha_1+\alpha_2\ -24\zeta^2\left(\frac{Y^{5/3}}{a^{2/3}}+\frac{Z^{5/3}}{b^{2/3}}\right)\in$$
$$+\left[2+\frac{3Y^{5/3}}{a^{2/3}}\ \xi+\upsilon\ ^2+\zeta^2\ \bar{Q}+\frac{3Z^{5/3}}{b^{2/3}}\ \xi+\upsilon-1\ ^2+\zeta^2\ \bar{R}\right]\in'-3\zeta^2\left[\frac{3Y^{5/3}}{a^{2/3}}\bar{Q}+\frac{3Z^{5/3}}{b^{2/3}}\bar{R}-\frac{Y^{5/3}}{a^{2/3}}\bar{S}\right.$$
$$\left.-\frac{Z^{5/3}}{b^{2/3}}\bar{T}+\frac{6\ (YZ)^{-5/3}}{(ab)^{2/3}}\{(\xi+\upsilon)(\xi+\upsilon-1)\bar{V}-\frac{3\ (YZ)^{-5/3}}{(ab)^{2/3}}\{(\xi+\upsilon)^2+(\xi+\upsilon-1)^2\}\bar{W}\right]$$

$$d_3 = 3\zeta^2\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2+\in'\right)\left[\frac{3Y^{10/3}\ \xi+\upsilon\ ^2}{a^{4/3}}\bar{U}+\frac{6\ YZ\ ^{5/3}\ \xi+\upsilon\ \ \xi+\upsilon-1}{ab\ ^{2/3}}\bar{V}+\frac{3Z^{10/3}\ \xi+\upsilon-1\ ^2}{b^{4/3}}\bar{W}\right]$$
$$\left.-\frac{Y^{5/3}}{a^{2/3}}\bar{S}-\frac{Z^{5/3}}{b^{2/3}}\bar{T}-\frac{3\ \xi+\upsilon\ ^2Y^{10/3}}{a^{4/3}}\bar{U}-\frac{3\ YZ\ ^{5/3}\ \ \xi+\upsilon\ ^2+\ \xi+\upsilon-1\ ^2}{ab\ ^{2/3}}\bar{V}-\frac{3Z^{10/3}\ \xi+\upsilon-1\ ^2}{b^{4/3}}\bar{W}\right]$$

$$\bar{n} = 2d_1^3-9d_1d_2+27d_3+\sqrt{4\ 3d_2-d_1^2\ ^3+\ 2d_1^3-9d_1d_2+27d_3\ ^2}$$

$$\bar{Q} = 1+\frac{2}{3}\ 1-q_1\ +\frac{5}{2}\ \alpha_1+\alpha_2\ ,\qquad \bar{R} = 1+\frac{2}{3}\ 1-q_2\ +\frac{5}{2}\ \alpha_1+\alpha_2$$

$$\bar{S} = 1+\frac{2}{3}\ 1-q_1\ +4\ \alpha_1+\alpha_2\ +\in',\quad \bar{T} = 1+\frac{2}{3}\ 1-q_2\ +4\ \alpha_1+\alpha_2\ +\in'$$

$$\bar{U} = 1+\frac{4}{3}\ 1-q_1\ +5\ \alpha_1+\alpha_2\ ,\qquad \bar{V} = 1+\frac{4}{3}\ 1-q_2\ +5\ \alpha_1+\alpha_2$$

$$\bar{W} = 1+\frac{2}{3}\ 1-q_1\ +\frac{2}{3}\ 1-q_2\ +5\ \alpha_1+\alpha_2$$

The roots of (3.144) shows that there exist at least one positive root and a complex root with positive real part. Hence, we conclude that the out-of-plane point $L_6$ is unstable due to a positive roots or a positive real part of the complex roots. The same stability process can be done for the out-of-plane points $L_7, L_8$ and $L_9$.

The EPs of non-autonomous solutions on the basis of Lyapunov's definition are generally unstable because the solutions do not converge with $t$ tending to infinity. Also, because when $t$ is tending to infinity $\tau$ is approaching a finite value; the LCNs are positive for solutions with negative exponents, negative for solutions with positive exponents and zero for solutions with oscillatory and constant solutions. Consequently, the solutions with positive exponents which are unstable induce instability. Hence, the equilibrium points of the non-autonomous system are unstable.

## 3.4 Out-of-Plane Equilibrium Points in the R3BP with Variable Mass and Disc

The coplanar or out-of-plane points for variable mass were first found by Bekov (1988). Here what permits the existence of these points is neither the inclusion of radiation pressures of the primaries as in Radzievskii (1953), nor oblateness as in Douskos and Markellos (2006); but a constant (greater than unity) of a particular integral of the GMP.

In this problem, using the equations of the three-dimensional problem when the masses of the primaries vary with time at the same rate according to the UML and their motion governed by the GMP with further assumptions that they are enclosed by a circumbinary disc; we report the existence of the out of plane non-planar equilibrium points of the non-autonomous and autonomized systems. We obtain their positions both numerically and by analytical approximations in the form of power series in the disc mass coefficient.

### 3.4.1 Equations of motion of the third body

We consider that the configuration has a disc to interact with and we write the potential due to the mass of the disc as proposed by Miyamoto and Nagai (1975) which describes the gravitational potential within a stellar system (or subsystem); however, we assume that the parameters vary with time:

$$V\ r,z\ \ t\ = -\frac{GM\ \ t}{\sqrt{r^2+\ \ a+\sqrt{z^2+b^2}^{\ 2}}}$$

where $M$ is the mass of the disc; $a$ and $b$ are parameters which determine the density profile of the disc. $a$ is the flatness parameter and controls the flatness of the profile while $b$ is the core parameter that controls the size of the core of the density profile, and $G$ is the gravitational

constant. The behavior of the two parameters having dimension of distance, *a* and *b*, determines the two limiting cases. If we have $a = 0$, then the potential becomes the case known as the Plummer or the Schuster one, in which the axial symmetry is reduced to its special case (the spherical one). On the other hand, if $b = 0$, it becomes Kuzmin's case (or Kuzmin-Toomre case) for which the system has a collapse.

The equations of motion of a passively gravitating third body of mass $m_3$ in the barycentric rotating coordinate system $Oxyz$, the $xy$ plane of which coincides with the plane of motion of the primaries while the $x-$axis always passes through these points, when the primaries are surrounded by circumbinary disc and their masses $m_1$ and $m_2$ are time dependent, have the form

$$\ddot{x} - 2\omega\dot{y} - \dot{\omega}y = \omega^2 x - \frac{\mu_1 \ x - x_1}{r_1^3} - \frac{\mu_2 \ x - x_2}{r_2^3} - \frac{xGM \ t}{\left(r^2 + \ a + \sqrt{z^2 + b^2}^{\ 2}\right)^{3/2}}$$

$$\ddot{y} + 2\omega\dot{x} + \dot{\omega}x = \omega^2 y - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} - \frac{yGM \ t}{\left(r^2 + \ a + \sqrt{z^2 + b^2}^{\ 2}\right)^{3/2}} \qquad (3.145)$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} - \frac{zGM \ t}{\left(r^2 + \ a + \sqrt{z^2 + b^2}^{\ 2}\right)^{3/2}}\left[1 + \frac{a + \sqrt{z^2 + b^2}}{\sqrt{z^2 + b^2}}\right]$$

where $$\omega^2 \ t = \frac{\mu \ t}{\kappa}\left[\frac{1}{r^3} + \frac{2GM \ t \ r_c}{r_c^2 + \ a + \sqrt{z^2 + b^2}^{\ 3/2}}\right]$$

$\mu \ t$ is the product of the gravitational constant and the sum of the masses; $r_c$ is the radial distance of the third body in the R3BP with variable mass.

Now, using the MT (1.46), the UML (1.47), the particular integral and solutions of the GMP and in addition, we assume that the mass of the disc varies at the same rate as the masses of the primaries, while the parameters $a$ and $b$ also vary as the distsnces, such that

$$M(t) = \frac{M_d}{R(t)}, \quad a = a_0 R(t), \; b = b_0 R(t) \tag{3.146}$$

where $M_d$, $a_0$ and $b_0$ are constants.

The autonomized forms of (3.145) with constant coefficients are:

$$\xi'' - 2\eta' = \frac{\partial \Omega}{\partial \xi}, \; \eta'' + 2\xi' = \frac{\partial \Omega}{\partial \eta}, \; \zeta'' = \frac{\partial \Omega}{\partial \zeta} \tag{3.147}$$

where $\Omega = \frac{\kappa\omega_0^2}{2}\left(\xi^2 + \eta^2\right) + \frac{\omega_0^2\left(\kappa - 1\right)\zeta^2}{2} + \frac{\kappa\left(1-\upsilon\right)}{\rho_1} + \frac{\kappa\upsilon}{\rho_2} + \frac{\kappa M_d}{\left(\rho^2 + \left(a_0 + \sqrt{\zeta^2 + b_0^2}\right)^2\right)^{1/2}}$

$$\omega_0^2 = 1 + \frac{2\kappa M_d \rho_c}{\left(\rho_c^2 + \left(a_0 + \sqrt{\zeta^2 + b_0^2}\right)^2\right)^{3/2}} \; ; \qquad \rho_c^2 = \kappa\left(-\upsilon + \upsilon^2\right), \; \rho^2 = \xi^2 + \eta^2 + \zeta^2$$

$$\rho_1 = \sqrt{\left(\xi + \upsilon\right)^2 + \eta^2 + \zeta^2}, \; \rho_2 = \sqrt{\left(\xi + \upsilon - 1\right)^2 + \eta^2 + \zeta^2},$$

$\rho_c$ is the radial distance of the infinitesimal body in the restricted three-body problem with variable mass. When $\kappa = 1$, the radial distance $\rho_c$ reduces to one of the classical restricted three-body problem.

We restrict ourselves to the case of a point mass when $a_0 = b_0$. In this premise, the system of equations (3.147) becomes

$$\xi'' - 2\eta' = \frac{\partial \Omega}{\partial \xi},\ \eta'' + 2\xi' = \frac{\partial \Omega}{\partial \eta},\ \zeta'' = \frac{\partial \Omega}{\partial \zeta} \tag{3.148}$$

where $\Omega = \frac{\kappa\omega_0^2}{2}\left(\xi^2 + \eta^2\right) + \frac{\omega_0^2\left(\kappa - 1\right)\zeta^2}{2} + \frac{\kappa\left(1-\upsilon\right)}{\rho_1} + \frac{\kappa\upsilon}{\rho_2} + \frac{\kappa M_d}{\left(\rho^2 + \zeta^2\right)^{1/2}}$

$$\omega_0^2 = 1 + \frac{2\kappa M_d \rho_c}{\left(\rho_c^2 + \zeta^2\right)^{3/2}} \tag{3.149}$$

$$\rho_c^2 = \kappa\left(-\upsilon + \upsilon^2\right) \tag{3.150}$$

$$\rho_1 = \sqrt{\left(\xi + \upsilon\right)^2 + \eta^2 + \zeta^2},\ \rho_2 = \sqrt{\left(\xi + \upsilon - 1\right)^2 + \eta^2 + \zeta^2},\ \rho^2 = \xi^2 + \eta^2 + \zeta^2 \tag{3.151}$$

### 3.4.2. Locations of the out-of-plane equilibrium points

The positions of the out-of-plane EPs can be found from the equations of motion (3.148) by setting all velocity and acceleration components equal to zero and solving the resulting system, $\frac{\partial \Omega}{\partial \xi} = \frac{\partial \Omega}{\partial \eta} = \frac{\partial \Omega}{\partial \zeta} = 0$ for $\xi, \eta, \zeta$ .The second equation is satisfied for $\eta = 0$,so we must solve the remaining two equations for $\xi \neq 0$ and $\zeta \neq 0$ .That is,

$$\omega_0^2 \xi - \frac{\left(-\upsilon\right)\left(\xi + \upsilon\right)}{\rho_1^3} - \frac{\upsilon\left(\xi + \upsilon - 1\right)}{\rho_2^3} - \frac{M_d \xi}{\left(\rho_c^2 + \zeta^2\right)^{3/2}} = 0$$

and $\frac{\omega_0^2\left(\kappa - 1\right)}{\kappa} - \frac{\left(-\upsilon\right)}{\rho_1^3} - \frac{\upsilon}{\rho_2^3} - \frac{2M_d}{\left(\rho_c^2 + \zeta^2\right)^{3/2}} = 0$ , $\zeta \neq 0$ (3.152)

where $\rho_1^2 = \left(\xi + \upsilon\right)^2 + \zeta^2, \rho_2^2 = \left(\xi + \upsilon - 1\right)^2 + \zeta^2$ . (3.153)

Now, the first equation of (3.152) can be expressed in two forms as

$$\frac{\upsilon}{\rho_2^3} = \frac{\omega_0^2 \xi}{(\xi + \upsilon - 1)} - \frac{M_d \xi}{(\xi + \upsilon - 1)(\zeta_c^2 + \zeta^2)^{3/2}} - \frac{(1-\upsilon)}{\rho_1^3}\left(\frac{\xi + \upsilon}{\xi + \upsilon - 1}\right)$$

and $$\frac{(1-\upsilon)}{\rho_1^3} = \frac{\omega_0^2 \xi}{(\xi + \upsilon)} - \frac{M_d \xi}{(\xi + \upsilon)(\zeta_c^2 + \zeta^2)^{3/2}} - \frac{\upsilon(\xi + \upsilon - 1)}{(\xi + \upsilon)\rho_2^3}$$

When we substitute these equations, respectively, into the second equation of (3.153) and simplify, we obtain the respective equations:

$$\frac{\kappa(1-\upsilon)}{\rho_1^3} = \omega_0^2 [\xi + (1-\upsilon)(\kappa - 1)] - \frac{M_d \kappa [\xi + 2(1-\upsilon)]}{(\zeta_c^2 + \zeta^2)^{3/2}}$$

and $$\frac{\kappa\upsilon}{\rho_2^3} = \omega_0^2 [\xi + \upsilon(\kappa - 1)] - \frac{M_d \kappa (\xi + 2\upsilon)}{(\zeta_c^2 + \zeta^2)^{3/2}} \qquad (3.154)$$

To solve equations (3.154) analytically, we use perturbation method, by first obtaining the solutions when the parameter representing the potential from the disc is neglected. In this case, $M_d = 0$ and the solutions of equations (3.154) are:

$$\rho_1 = \left[\frac{\kappa(1-\upsilon)}{[\xi + (1-\upsilon)(\kappa - 1)]}\right]^{1/3}, \rho_2 = \left[\frac{\kappa\upsilon}{[\upsilon(\kappa - 1) - \xi]}\right]^{1/3} \qquad (3.155)$$

Next, we treat the problem when the disc is present, as a perturbation. In this case $M_d \neq 0$ and with the help of (3.155), we assume the solutions now to be

$$\rho_1 = \left[\frac{\kappa(1-\upsilon)}{[\xi + (1-\upsilon)(\kappa - 1)]}\right]^{1/3} + \pi_1, \quad \rho_2 = \left[\frac{\kappa\upsilon}{[\upsilon(\kappa - 1) - \xi]}\right]^{1/3} + \pi_2; \; \pi_i << 1 \; (i = 1,2) \qquad (3.156)$$

Expanding equations (3.156) using Binomial, we substitute them together with equation (3.149) into equations (3.154) and simplify to get the respective expressions:

$$\pi_1 = -\frac{1}{3}\frac{M_d\kappa}{(\rho_c^2+\zeta^2)^{3/2}}\left[\frac{\kappa(1-\upsilon)}{[\xi+(1-\upsilon)(\kappa-1)]}\right]^{1/3}\left\{2\rho_c - \frac{(2-2\upsilon-\xi)}{[\xi+(1-\upsilon)(\kappa-1)]}\right\}$$

$$\pi_2 = -\frac{1}{3}\frac{M_d\kappa}{(\rho_c^2+\zeta^2)^{3/2}}\left[\frac{\kappa\upsilon}{[(\kappa-1)-\xi]}\right]^{1/3}\left\{2\rho_c - \frac{(\xi+2\upsilon)}{[(\kappa-1)-\xi]}\right\}$$

Substituting these equations into equations (3.156) and using Binomial expansion, we get

$$\rho_1 = \left[\frac{\kappa(1-\upsilon)}{[\xi+(1-\upsilon)(\kappa-1)]}\right]^{1/3}\left[1-\frac{M_d\kappa}{3(\rho_c^2+\zeta^2)^{3/2}}\left\{2\rho_c - \frac{(2-2\upsilon-\xi)}{[\xi+(1-\upsilon)(\kappa-1)]}\right\}\right]$$

and (3.157)

$$\rho_2 = \left[\frac{\kappa\upsilon}{[(\kappa-1)-\xi]}\right]^{1/3}\left[1-\frac{M_d\kappa}{3(\rho_c^2+\zeta^2)^{3/2}}\left\{2\rho_c - \frac{(\xi+2\upsilon)}{[(\kappa-1)-\xi]}\right\}\right]$$

The distances of the coplanar points on the $\xi-$axis are obtained by putting $\zeta=0$ into (3.153). Substituting equations (3.157), respectively in equations (3.153), we subtract the resulting two equations and then denote L.H.S by $f(\xi)$, to get

$$f(\xi) = 2(\xi+\upsilon)-1-\left[\frac{\kappa(1-\upsilon)}{[\xi+(1-\upsilon)(\kappa-1)]}-\frac{M_d\kappa^2(1-\upsilon)}{\rho_{c\kappa}^3[\xi+(1-\upsilon)(\kappa-1)]}\left\{2\xi-\frac{2-2\upsilon-\xi}{[\xi+(1-\upsilon)(\kappa-1)]}\right\}\right]^{2/3}$$

$$+\left[\frac{\kappa\upsilon}{[-\xi+\upsilon(\kappa-1)]}-\frac{M_d\kappa^2\upsilon}{\rho_{c\kappa}^3[-\xi+\upsilon(\kappa-1)]}\left\{2\xi-\frac{\xi+2\upsilon}{[-\xi+\upsilon(\kappa-1)]}\right\}\right]^{2/3} \quad (3.158)$$

Finding first and second derivative of (3.158) w.r.t.$\xi$, we see that in the

interval $(-\upsilon)(\kappa-1) \le \xi < \upsilon(\kappa-1)$, the function increases monotonically; so that for any $M_d \ge 0$, $0 < \upsilon \le \frac{1}{2}$ and $\kappa > 1$, there exists a single root of the equation (3.158), which gives the abscissas of the out of-plane EP $L_6$. Using Newton-Raphson's method and starting with initial value $\xi = 0$ and knowing that in the present problem $M_d > 0$, we get

$$\xi = \frac{9\rho_{c\kappa}^3 \upsilon\ 1-\upsilon\ \ 1-2\upsilon\ \ \kappa-1\ ^{8/3}}{6\ \kappa-1\ \rho_{c\kappa}^3\left[3\upsilon\ 1-\upsilon\ \ \kappa-1\ ^{5/3}+\kappa^{2/3}\right]-2\upsilon\kappa^{5/3}M_d\left[4\kappa\ 1-\upsilon\ +3\ 2\ 1-\upsilon\ \ \kappa-1\ ^2+3+\kappa\right]}$$

Making $\zeta$ subject of formula in both equations of system (3.153), and substituting the equations of (3.157), in them, respectively, we get

$$\zeta = \pm\sqrt{\frac{\left[\kappa\ 1-\upsilon\ \right]^{2/3}}{\left[\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{2/3}}\left[1-\frac{2M_d\kappa}{3\ \rho_c^2+\zeta^2\ ^{3/2}}\left\{2\rho_c-\frac{\xi+2\upsilon}{\left[\xi+\ 1-\upsilon\ \ \kappa-1\ \right]}\right\}\right]-\ \xi+\upsilon\ ^2}$$

and

$$\zeta = \pm\sqrt{\frac{(\kappa\upsilon)^{2/3}}{\left[(\kappa-1)+\xi\right]^{2/3}}\left[1-\frac{2M_d\kappa}{3(\rho_c^2+\zeta^2)^{3/2}}\left\{2\rho_c-\frac{(\xi+2\upsilon)}{\left[(\kappa-1)+\xi\right]}\right\}\right]-(\xi+\upsilon-1)^2} \qquad (3.159)$$

Starting with initial value, we select $\zeta = \sqrt{2\kappa M_d}$ and knowing that $\rho_c^2 = \kappa(-\upsilon+\upsilon^2)$. We use series expansion in equations (3.159), respectively, to get

$$\zeta = \pm\left[-\upsilon^2+\frac{\left[\kappa\ 1-\upsilon\ \right]^{2/3}}{\left[\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{2/3}}-2\upsilon\xi-\xi^2+\frac{2\kappa M_d\left[\kappa\ 1-\upsilon\ \right]^{\frac{2}{3}}\left[\upsilon\ 2+2\rho_c-2\kappa\rho_c\ +\xi+2\xi\rho_c\right]}{3\rho_c^3\left[\xi+\ 1-\upsilon\ \ \kappa-1\ \right]^{5/3}}\right]^{1/2}$$

and

$$\zeta = \pm\left[-1-\upsilon^2+\frac{(\kappa\upsilon)^{2/3}}{[(\kappa-1)-\xi]^{2/3}}-2\upsilon(\xi-1)+2\xi-\xi^2+\frac{2\kappa M_d(\kappa\upsilon)^{2/3}[(\kappa+2\rho_c-2\kappa\rho_c)+\xi+2\xi\rho_c]}{3\rho_c^3[(\kappa-1)-\xi]^{5/3}}\right]^{1/2}$$

Hence, we have the solutions:

$$\xi = \frac{9\rho_{c\kappa}^3\upsilon\ 1-\upsilon\ \ 1-2\upsilon\ \ \kappa-1^{8/3}}{6\ \kappa-1\ \rho_{c\kappa}^3\left[3\upsilon\ 1-\upsilon\ \ \kappa-1^{5/3}+\kappa^{2/3}\right]-2\upsilon\kappa^{5/3}M_d\left[4\kappa\ 1-\upsilon\ +3\ 2\ 1-\upsilon\ \ \kappa-1^{\ 2}+3+\kappa\right]} \quad (3.160)$$

$$\zeta = \pm\left[-\upsilon^2+\frac{[\kappa\ 1-\upsilon\ ]^{2/3}}{[\xi+\ 1-\upsilon\ \ \kappa-1\ ]^{2/3}}-2\upsilon\xi-\xi^2+\frac{2\kappa M_d[\kappa\ 1-\upsilon\ ]^{\frac{2}{3}}[\upsilon\ 2+2\rho_c-2\kappa\rho_c\ +\xi+2\xi\rho_c]}{3\rho_c^3[\xi+\ 1-\upsilon\ \ \kappa-1\ ]^{5/3}}\right]^{1/2}$$

and

$$\xi = \frac{9\rho_{c\kappa}^3\upsilon\ 1-\upsilon\ \ 1-2\upsilon\ \ \kappa-1^{8/3}}{6\ \kappa-1\ \rho_{c\kappa}^3\left[3\upsilon\ 1-\upsilon\ \ \kappa-1^{5/3}+\kappa^{2/3}\right]-2\upsilon\kappa^{5/3}M_d\left[4\kappa\ 1-\upsilon\ +3\ 2\ 1-\upsilon\ \ \kappa-1^{\ 2}+3+\kappa\right]} \quad (3.161)$$

$$\zeta = \pm\left[-1-\upsilon^2+\frac{(\kappa\upsilon)^{2/3}}{[(\kappa-1)-\xi]^{2/3}}-2\upsilon(\xi-1)+2\xi-\xi^2+\frac{2\kappa M_d(\kappa\upsilon)^{2/3}[(\kappa+2\rho_c-2\kappa\rho_c)+\xi+2\xi\rho_c]}{3\rho_c^3[(\kappa-1)-\xi]^{5/3}}\right]^{1/2}$$

These points give the coordinates of the relative equilibrium positions (coplanar points), which exist in the plane passing through the primaries and to the plane of their orbits. The first pair (equation 3.160) is denoted by $L_{6,7}$ and exist for $\kappa>1$ while the second pair (equation 3.161) is denoted by $L_{8,9}$ and exist simultaneously, when $\kappa>1$ and $\xi<\upsilon\ \kappa-1$ . We note the equations are unique, in the sense that the same solutions are obtained when the initial condition was selected

as $\zeta = \sqrt{2M_d}$ . Evidently, the out of plane solutions are affected by the mass parameter, the mass of disc $M_d$ and the arbitrary constant $\kappa$ of the GMP. When $M_d = 0$, the coordinates are fully analogous with those obtained by Singh and Leke (2010).

### 3.4.3 Stability of out-of-plane equilibrium points

The partials derivatives computed at the out-of-plane points in the presence of a disc are

$$\Omega^0_{\xi\xi} = \kappa\left[1 - \frac{1-\upsilon}{\rho_{10}^3} - \frac{\upsilon}{\rho_{20}^3} + \frac{3\ 1-\upsilon\ \ \xi+\upsilon\ ^2}{\rho_{10}^5} + \frac{3\upsilon\ \ \xi+\upsilon-1\ ^2}{\rho_{20}^5} + \frac{M_d}{\rho_c^2+\zeta_0^2\ ^{3/2}}\left\{2\rho_c\kappa - 1 + \frac{3\xi_0^2}{\rho_c^2+\zeta_0^2}\right\}\right]$$

$$\Omega^0_{\eta\eta} = \kappa\left[1 - \frac{1-\upsilon}{\rho_{10}^3} - \frac{\upsilon}{\rho_{20}^3} + \frac{M_d\ \ 2\rho_c\kappa - 1}{\rho_c^2+\zeta_0^2\ ^{3/2}}\right], \Omega^0_{\xi\eta} = \Omega^0_{\eta\zeta} = 0$$

$$\Omega^0_{\xi\zeta} = 3\kappa\zeta_0\left[\frac{1-\upsilon\ \ \xi+\upsilon}{\rho_{10}^5} + \frac{\upsilon\ \ \xi+\upsilon-1}{\rho_{20}^5} + \frac{2M_d\xi_0}{\rho_c^2+\zeta_0^2\ ^{5/2}}\right] \tag{3.162}$$

$$\Omega^0_{\zeta\zeta} = -1 + \kappa\left[1 - \frac{1-\upsilon}{\rho_{10}^3} - \frac{\upsilon}{\rho_{20}^3} + \frac{3\ 1-\upsilon\ \ \zeta_0^2}{\rho_{10}^5} + \frac{3\upsilon\zeta_0^2}{\rho_{20}^5} + \frac{2M_d}{\rho_c^2+\zeta_0^2\ ^{3/2}}\left\{\rho_c\ \ \kappa - 1\ \ -1 + \frac{6\zeta_0^2}{\rho_c^2+\zeta_0^2}\right\}\right]$$

Substituting equations (3.162) in the characteristic equation (1.115), we get

$$\lambda^6 - p\lambda^4 + q\lambda^2 - r = 0 \tag{3.163}$$

where

$$p = \Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} + \Omega^0_{\zeta\zeta}$$

$$q = \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - 4\omega_0^2\Omega^0_{\zeta\zeta} - \ \Omega^0_{\xi\zeta}\ ^2$$

$$r = \Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} + \Omega^0_{\eta\eta}\ \ \Omega^0_{\xi\zeta}\ ^2, \omega_0^2 = 1 + \frac{2M_d\kappa\rho_c}{\left(\rho_c^2 + \zeta_0^2\right)^{3/2}},$$

We use the *Mathematica* soft ware (Wolfram 2003) package to determine the nature of the roots of equation (3.163) and found that the roots contain a positive and negative real root; and four complex roots; two of which have positive real parts. The existence of a positive root and a complex root with positive real part, render the motion around the out-of-plane equilibrium point unstable. This result agrees with Ragos and Zafiropoulos (1995), Douskos and Markellos (2006), Singh and Leke (2010), and Singh (2013)

In view of the system of equations with variable coefficients, the out-of-plane equilibrium points are determined (Luk'yanov, 1990) using the Mestschersky transformation (1.46), so that the equilibrium points of the non-autonomous system differ from the equilibrium points of the autonomized system only by the function $R(\quad)$. The stability is tested using the definition of a Lyapunov stable solution (Krasnov *et al.* 1983), we have

$$\lim_{t\to\infty} x(\quad) = \xi \lim_{t\to\infty} R(\quad) = \infty \quad \text{and} \quad \lim_{t\to\infty} z(\quad) = \pm\zeta \lim_{t\to\infty} R(\quad) = \pm\infty \tag{3.164}$$

This proves that the out-of-plane point lying on the $xz\ \ t$ plane of motion is unstable according to the Lyapunov theorem.

## 3.5 Effects of Mass Variations, Oblateness, Radiation and Perturbations On the Periodic Orbits Around Triangular Equilibrium Points

The effects of mass variation of the primaries on the orbits of the third body around triangular EPs in the frame of the R3BP, is studied when the motion of the primaries is described by the GMP while their mass variation occurs in accordance with the UML under oblateness and radiation of the primaries coupled with the influence of small perturbations in the Coriolis and centrifugal forces.

### 3.5.1. Frequencies and Orientation of the orbits

The differential equations of motion of the R3BP with variable masses are described by the equations (3.89) and (3.91), while the triangular points and the partial derivatives are presented in equations (3.112) and (3.134), respectively. Substituting the partial derivatives (3.134) in the characteristic equation (3.66), we get

$$\lambda^4+\left[4\varphi^2-3\ \psi+\kappa-1\ -\left(\frac{9\kappa^{2/3}\ \psi+\kappa-1\ +6\ \psi+\kappa-1\ ^{5/3}}{2\kappa^{2/3}}\right)\alpha_1-\frac{9\ \psi+\kappa-1}{2}\alpha_2+6\varphi^2\ \ \alpha_1+\alpha_2\right.$$

$$\left.+\frac{3\ \alpha_1-\alpha_2\ \upsilon\ \psi+\kappa-1\ ^{5/3}}{\kappa^{2/3}}\right]\lambda^2+\frac{3\ \psi+\kappa-1\ ^{8/3}}{4\kappa^{4/3}}\left[3\ 16\kappa^{2/3}-3\ \psi+\kappa-1\ ^{2/3}\ \upsilon\ 1-\upsilon\ \ \alpha_1+\alpha_2\right.$$

$$+3N\upsilon\ 1-\upsilon\ +\frac{2\upsilon\ 5\kappa^{4/3}-3\kappa^{2/3}\ \psi+\kappa-1\ ^{2/3}-\ \psi+\kappa-1\ ^{4/3}}{\psi+\kappa-1\ ^{2/3}}\ 1-q_1\ +2\kappa^{2/3}\upsilon\left(-q_2\right)\qquad(3.165)$$

$$\left.-2\upsilon^2\left(2\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right)\left(-q_1\right)+2\upsilon^2\left(\psi+\kappa-1\right)^{2/3}\left(-q_2\right)\right]=0$$

The roots of equation (3.165) are

$$\lambda_{1,2}=\pm s_1 i\ ,\ \lambda_{3,4}=\pm s_2 i$$

where

$$s_{1,2}=\frac{1}{2}\ \ P\pm\sqrt{D}$$

$$P=\left[4\varphi^2-3\ \psi+\kappa-1\ \right]\left[1-\left\{\frac{9\kappa^{2/3}\ \psi+\kappa-1\ +6\ \psi+\kappa-1\ ^{5/3}}{2\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\right\}\alpha_1-\frac{9\ \psi+\kappa-1}{2\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\alpha_2\right.$$
$$\left.+\frac{3\ \psi+\kappa-1\ ^{5/3}\upsilon\ \alpha_1-\alpha_2}{\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}+\frac{6\varphi^2\ \alpha_1+\alpha_2}{\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\right] \quad (3.166)$$

Using binomial expansion and retaining only linear terms in $\alpha_i\ (=1,2)$, we get

$$P^2=\left[4\varphi^2-3\ \psi+\kappa-1\ \right]^2\left[1-\left(\frac{9\kappa^{2/3}\ \psi+\kappa-1\ +6\ \psi+\kappa-1\ ^{5/3}}{\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\right)\alpha_1-\frac{9\ \psi+\kappa-1\ \alpha_2}{\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\right.$$
$$\left.+\frac{6\ \psi+\kappa-1\ ^{5/3}\upsilon\ \alpha_1-\alpha_2}{\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}+\frac{12\varphi^2\ \alpha_1+\alpha_2}{\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\right]$$

Therefore, the discriminant is

$$D=\left[4\varphi^2-3\ \psi+\kappa-1\ \right]^2\left[1-\left(\frac{9\kappa^{2/3}\ \psi+\kappa-1\ +6\ \psi+\kappa-1\ ^{5/3}}{\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\right)\alpha_1-\frac{9\ \psi+\kappa-1}{\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\alpha_2\right.$$
$$\left.+\frac{6\ \psi+\kappa-1\ ^{5/3}\upsilon\ \alpha_1-\alpha_2}{\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}+\frac{12\alpha^2\ \alpha_1+\alpha_2}{\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\right]-\frac{3\ \psi+\kappa-1\ ^{8/3}}{\kappa^{4/3}}\left[3N\upsilon\ 1-\upsilon\ +3\upsilon\ 1-\upsilon\right.$$
$$\times\left\{6\kappa^{2/3}-3\ \psi+\kappa-1\ ^{2/3}\right\}\left(\alpha_1+\alpha_2\right)\frac{2\upsilon\left\{3\kappa^{4/3}-3\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}-\left(\psi+\kappa-1\right)^{4/3}\right\}\left(-q_1\right)}{\left(\psi+\kappa-1\right)^{2/3}}$$
$$\left.-2\upsilon^2\left\{2\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right\}\left(-q_1\right)+2\kappa^{2/3}\upsilon\left(-q_2\right)+2\upsilon^2\left(\psi+\kappa-1\right)^{2/3}\left(-q_2\right)\right]$$

Expanding, we get

$$D=\left[4\varphi^2-3\ \beta+\kappa-1\ \right]^2-\frac{1}{\kappa^{2/3}}\ 9\kappa^{2/3}\ \psi+\kappa-1\ +6\ \psi+\kappa-1\ ^{5/3}\ \left[4\varphi^2-3\ \psi+\kappa-1\ \right]\alpha_1$$

$$-9\ \psi+\kappa-1\ \left[4\varphi^2-3\ \psi+\kappa-1\ \right]\alpha_2+\frac{6\ \psi+\kappa-1\ ^{5/3}\upsilon\ \alpha_1-\alpha_2}{\kappa^{2/3}}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]$$

$$+12\varphi^2\ \alpha_1+\alpha_2\ \left[4\varphi^2-3\ \psi+\kappa-1\ \right]-\frac{3\ \psi+\kappa-1\ ^{8/3}}{\kappa^{4/3}}\left[3N\upsilon\ 1-\upsilon\ +3\ 16\kappa^{2/3}-3\ \psi+\kappa-1\ ^{2/3}\right.$$

$$\times\upsilon\ 1-\upsilon\ \alpha_1+\alpha_2\ -2\upsilon^2\ 2\left(\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right)\left(-q_1\right)\frac{2\upsilon}{\left(\psi+\kappa-1\right)^{2/3}}\left(5\kappa^{4/3}-3\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}\right.$$

$$\left.\left.-\left(\psi+\kappa-1\right)^{4/3}\right)\left(-q_1\right)+2\kappa^{2/3}\upsilon\left(-q_2\right)+2\upsilon^2\left(\psi+\kappa-1\right)^{2/3}\left(-q_2\right)\right]$$

or

$$D=\left[4\varphi^2-3\ \psi+\kappa-1\ \right]^2-\left(\frac{9\kappa^{2/3}\ \psi+\kappa-1\ +6\ \psi+\kappa-1\ ^{5/3}}{\kappa^{2/3}}\right)\left[4\varphi^2-3\ \psi+\kappa-1\ \right]\alpha_1$$

$$-9\ \psi+\kappa-1\ \left[4\varphi^2-3\ \psi+\kappa-1\ \right]\alpha_2+\frac{6\ \psi+\kappa-1\ ^{5/3}\upsilon\ \alpha_1-\alpha_2}{\kappa^{2/3}}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]$$

$$+12\varphi^2\ \alpha_1+\alpha_2\ \left[4\varphi^2-3\ \psi+\kappa-1\ \right]-\frac{3\upsilon\ 1-\upsilon\ \ \psi+\kappa-1\ ^{8/3}}{\kappa^{4/3}}\left[3N+3\ 16\kappa^{2/3}-3\ \psi+\kappa-1\ ^{2/3}\ \ \alpha_1+\alpha_2\ \right]$$

$$-\frac{3\upsilon\left(\psi+\kappa-1\right)^{8/3}}{\kappa^{4/3}}\left[\frac{\left(10\kappa^{4/3}-6\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}-2\left(\psi+\kappa-1\right)^{4/3}\right)}{\left(\psi+\kappa-1\right)^{2/3}}\left(-q_1\right)2\kappa^{2/3}\left(-q_2\right)\right]$$

$$-\frac{3\upsilon^2\left(\psi+\kappa-1\right)^{8/3}}{\kappa^{4/3}}\left[-2\left(2\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right)\left(-q_1\right)2\left(\psi+\kappa-1\right)^{2/3}\left(-q_2\right)\right]$$

Factorizing $\upsilon$ and $\upsilon^2$, we have

$$D = \left(4\varphi^2 - 3\left(\psi+\kappa-1\right)\right)^2 - \frac{9\kappa^{2/3}\left(\psi+\kappa-1\right) + 6\left(\psi+\kappa-1\right)^{5/3}}{\kappa^{2/3}}\left(4\varphi^2 - 3\left(\psi+\kappa-1\right)\right)\alpha_1$$

$$-9\left(\psi+\kappa-1\right)\left(4\varphi^2 - 3\left(\psi+\kappa-1\right)\right)\alpha_2 + 12\varphi^2\left(\alpha_1+\alpha_2\right)\left(4\varphi^2 - 3\left(\psi+\kappa-1\right)\right)$$

$$+\frac{3\upsilon^2\left(\psi+\kappa-1\right)^{8/3}}{\kappa^{4/3}}\left[3N + 3\left(16\kappa^{2/3} - 3\left(\psi+\kappa-1\right)^{2/3}\right)\left(\alpha_1+\alpha_2\right)\right.$$

$$\left. + 2\left(2\kappa^{2/3} - \left(\psi+\kappa-1\right)^{2/3}\right)\left(1-q_1\right) + 2\left(\psi+\kappa-1\right)^{2/3}\left(1-q_2\right)\right]$$

$$-\frac{3\upsilon\left(\psi+\kappa-1\right)^{8/3}}{\kappa^{4/3}}\left[3N + 3\left(16\kappa^{2/3} - 3\left(\psi+\kappa-1\right)^{2/3}\right)\left(\alpha_1+\alpha_2\right) - \frac{2\kappa^{2/3}\left(\alpha_1-\alpha_2\right)}{\left(\psi+\kappa-1\right)}\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}\right.$$

$$\left. + \frac{2\left\{5\kappa^{4/3} - 3\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} - \left(\psi+\kappa-1\right)^{4/3}\right\}}{\left(\psi+\kappa-1\right)^{2/3}}\left(1-q_1\right) - 2\kappa^{2/3}\left(1-q_2\right)\right]$$

So that

$$D = \frac{3\upsilon^2\left(\psi+\kappa-1\right)^{8/3}}{\kappa^{4/3}}\left[3N + \left(48\kappa^{2/3} - 9\left(\psi+\kappa-1\right)^{2/3}\right)\left(\alpha_1+\alpha_2\right) + \left(4\kappa^{2/3} - 2\left(\psi+\kappa-1\right)^{2/3}\right)\left(1-q_1\right)\right.$$

$$\left. + 2\left(\psi+\kappa-1\right)^{\frac{2}{3}}\left(1-q_2\right)\right] - \frac{3\upsilon\left(\psi+\kappa-1\right)^{8/3}}{\kappa^{4/3}}\left[3N + \left(48\kappa^{2/3} - 9\left(\psi+\kappa-1\right)^{2/3}\right)\left(\alpha_1+\alpha_2\right)\right.$$

$$\left. - \frac{2\kappa^{2/3}\left(4\varphi^2 - 3\left(\psi+\kappa-1\right)\right)\left(\alpha_1-\alpha_2\right)}{\psi+\kappa-1} + \frac{2\left(5\kappa^{4/3} - 3\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} - \left(\psi+\kappa-1\right)^{4/3}\right)\left(1-q_1\right)}{\left(\psi+\kappa-1\right)^{2/3}}\right.$$

$$\left. + 2\kappa^{\frac{2}{3}}\left(1-q_2\right)\right] + \left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}^2\frac{\left\{9\kappa^{2/3}\left(\psi+\kappa-1\right) + 6\left(\psi+\kappa-1\right)^{5/3}\right\}\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}\alpha_1}{\kappa^{2/3}}$$

$$-9\left(\beta+\kappa-1\right)\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}\alpha_2 + 12\varphi^2\left(\alpha_1+\alpha_2\right)\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\} \qquad (3.167)$$

But

$$s_{1,2} = \left[\frac{1}{2}\left(P \pm \sqrt{D}\right)\right]^{1/2}$$

Using equations (3.166) and (3.167) in the above, we have

$$s_1 = \left[\frac{1}{2}\left[4\varphi^2 - 3\ \psi+\kappa-1\ - \left\{\frac{9\kappa^{2/3}\ \psi+\kappa-1\ + 6\ \psi+\kappa-1\ ^{5/3}}{2\kappa^{2/3}}\right\}\alpha_1 - \frac{9\ \psi+\kappa-1\ }{2}\alpha_2 + 6\varphi^2\right.\right.$$

$$\times\ \alpha_1+\alpha_2\ + \frac{3\ \psi+\kappa-1\ ^{5/3}\upsilon\ \ \alpha_1-\alpha_2\ }{\kappa^{2/3}} - \left\{\frac{3\ \psi+\kappa-1\ ^{8/3}}{\kappa^{4/3}}\left[3N + \ 48\kappa^{2/3} - 9\ \psi+\kappa-1\ ^{2/3}\ \ \alpha_1+\alpha_2\right.\right.$$

$$\left.\left.+2\ 1-q_1\ \ 2\kappa^{2/3} - \ \psi+\kappa-1\ ^{2/3}\ + 2\ 1-q_2\ \ \psi+\kappa-1\ ^{2/3}\ \right]\upsilon^2\right.$$

$$-\frac{3\ \psi+\kappa-1\ ^{8/3}}{\kappa^{4/3}}\left[3N + \left\{48\kappa^{2/3} - 9\left(\psi+\kappa-1\right)^{2/3}\right\}\left(\alpha_1+\alpha_2\right) - \frac{2\kappa^{2/3}\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}\left(\alpha_1-\alpha_2\right)}{\left(\psi+\kappa-1\right)}\right.$$

$$\left.+\frac{2\left(1-q_1\right)\left\{\kappa^{4/3} - 3\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} - \left(\psi+\kappa-1\right)^{4/3}\right\}}{\left(\psi+\kappa-1\right)^{2/3}} + 2\kappa^{2/3}\left(1-q_2\right)\right]\upsilon$$

$$+\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)^2\right\} - \frac{\left\{9\kappa^{2/3}\left(\psi+\kappa-1\right) - 6\left(\psi+\kappa-1\right)^{5/3}\right\}}{\kappa^{2/3}}\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}\alpha_1$$

$$\left.\left.-9\left(\psi+\kappa-1\right)\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}\alpha_2 + 12\varphi^2\left(\alpha_1+\alpha_2\right)\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}\right\}\right]^{1/2}$$

Using binomial expansion and still retaining only linear terms in the parameters $\alpha_i$ and $(-q_i)$

where $i = 1,2$, and simplifying, we get

On simplifying, we get

$$s_1 = \left[\frac{1}{2}\left[\frac{9N\ \ \psi+\kappa-1\ ^{8/3}\upsilon\ \ 1-\upsilon}{2\kappa^{4/3}\ \ 4\varphi^2 - 3\ \psi+\kappa-1} + \frac{144\kappa^{2/3}\ \ \psi+\kappa-1\ ^{8/3} - 27\ \ \psi+\kappa-1\ ^{10/3}\ \ \upsilon\ \ 1-\upsilon\ \ \ \alpha_1+\alpha_2}{2\kappa^{4/3}\ \ 4\varphi^2 - 3\ \psi+\kappa-1}\right.\right.$$

$$-\frac{3\upsilon^2\ \ 2\kappa^{2/3}\ \ \psi+\kappa-1\ ^{8/3} - \ \psi+\kappa-1\ ^{10/3}\ \ \ 1-q_1}{\kappa^{4/3}\ \ 4\varphi^2 - 3\ \psi+\kappa-1} + \frac{3\upsilon\ \ \psi+\kappa-1\ ^{8/3}\ \ 1-q_2}{\kappa^{2/3}\ \ 4\varphi^2 - 3\ \psi+\kappa-1}$$

$$\left.\left.+\frac{3\upsilon\ \left\{\kappa^{4/3}\ \ \psi+\kappa-1\ ^2 - 3\kappa^{2/3}\ \ \psi+\kappa-1\ ^{8/3} - \left(\psi+\kappa-1\right)^{10/3}\right\}\left(1-q_1\right)}{\kappa^{4/3}\ \ 4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}} - \frac{3\upsilon^2\left(\psi+\kappa-1\right)^{10/3}\left(1-q_2\right)}{\kappa^{4/3}\ \ 4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}}\right]^{1/2}$$

or

$$s_1 = \left[ \frac{9N\ (\psi+\kappa-1)^{8/3}\ \upsilon\ (1-\upsilon)}{4\kappa^{4/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)} + \frac{144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 27\ (\psi+\kappa-1)^{10/3}\ \upsilon\ (1-\upsilon)\ (\alpha_1+\alpha_2)}{4\kappa^{4/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)} \right.$$

$$- \frac{3\upsilon^2\ (2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - (\psi+\kappa-1)^{10/3})\ (1-q_1)}{2\kappa^{4/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)} + \frac{3\upsilon\ (\psi+\kappa-1)^{8/3}\ (1-q_2)}{2\kappa^{2/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)}$$

$$\left. + \frac{3\upsilon\ \{5\kappa^{4/3}\ (\psi+\kappa-1)^2 - 3\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - (\psi+\kappa-1)^{10/3}\}\ (1-q_1)}{2\kappa^{4/3}\ \{4\varphi^2-3(\psi+\kappa-1)\}} - \frac{3\upsilon^2\ (\psi+\kappa-1)^{10/3}\ (1-q_2)}{2\kappa^{4/3}\ \{4\varphi^2-3(\psi+\kappa-1)\}} \right]^{1/2}$$

Factorizing the above yields

$$s_1 = \left[ \frac{\upsilon\ (1-\upsilon)}{4\varphi^2-3\ (\psi+\kappa-1)} \left\{ \frac{9N\ (\psi+\kappa-1)^{8/3}}{4\kappa^{4/3}} + \frac{144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 27\ (\psi+\kappa-1)^{10/3}\ (\alpha_1+\alpha_2)}{4\kappa^{4/3}} \right.\right.$$

$$- \frac{3\upsilon\ (2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - (\psi+\kappa-1)^{10/3})\ (1-q_1)}{2\kappa^{4/3}\ (1-\upsilon)} + \frac{3\ (\psi+\kappa-1)^{8/3}\ (1-q_2)}{2\kappa^{2/3}\ (1-\upsilon)}$$

$$\left.\left. + \frac{3\ \{5\kappa^{4/3}\ (\psi+\kappa-1)^2 - 3\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - (\psi+\kappa-1)^{10/3}\}\ (1-q_1)}{2\kappa^{4/3}\ (1-\upsilon)} - \frac{3\upsilon\ (\psi+\kappa-1)^{10/3}\ (1-q_2)}{2\kappa^{4/3}\ (1-\upsilon)} \right\} \right]^{1/2}$$

Hence,

$$s_1 = \left[ \frac{\upsilon\ (1-\upsilon)}{4\varphi^2-3\ (\psi+\kappa-1)} \left\{ \frac{9N\ (\psi+\kappa-1)^{8/3}}{4\kappa^{4/3}} + \frac{\left[144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 27\ (\psi+\kappa-1)^{10/3}\right]\ (\alpha_1+\alpha_2)}{4\kappa^{4/3}} \right.\right.$$

$$+ \frac{3\upsilon\ (\psi+\kappa-1)^2\ (1-q_1)}{2\kappa^{4/3}\ (1-\upsilon)} \left[ 5\kappa^{4/3} - 3\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} - (\psi+\kappa-1)^{4/3} - \upsilon\ (2\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} \right.$$

$$\left.\left. - (\psi+\kappa-1)^{4/3} \right] + \frac{3\ (\psi+\kappa-1)^{8/3}\ (1-q_2)}{2\kappa^{2/3}\ (1-\upsilon)} \left[ 1 - \frac{\upsilon\ (\psi+\kappa-1)^{2/3}}{\kappa^{2/3}} \right] \right\} \right]^{1/2} \tag{3.168}$$

Similarly, using binomial expansion and retaining only linear terms in the parameters $\alpha_i$ and $(1-q_i): i=1,2$, we get

$$s_2 = \left[ \frac{1}{2} \left[ 4\varphi^2 - 3\ (\psi + \kappa - 1\ - \left\{ \frac{9\kappa^{2/3}\ (\psi + \kappa - 1\ + 6\ (\psi + \kappa - 1\ ^{5/3}}{2\kappa^{2/3}} - 6\varphi^2 \right\} \alpha_1 - \left\{ \frac{9\ (\psi + \kappa - 1}{2} - 6\varphi^2 \right\} \alpha_2 \right.\right.$$

$$+ \frac{3\upsilon\ (\psi + \kappa - 1\ ^{5/3}\ (\alpha_1 - \alpha_2}{\kappa^{2/3}} + \ 4\varphi^2 - 3\ (\psi + \kappa - 1 \left[ 1 - \frac{9\kappa^{2/3}\ (\psi + \kappa - 1\ + 6\ (\psi + \kappa - 1\ ^{5/3}}{2\kappa^{2/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1} \alpha_1 \right.$$

$$- \frac{9\ (\psi + \kappa - 1\ \alpha_2}{2\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1} + \frac{6\varphi^2\ \ \alpha_1 + \alpha_2}{4\varphi^2 - 3\ (\psi + \kappa - 1} - \frac{9N\left((\psi + \kappa - 1\right)^{8/3} \upsilon\left(1 - \upsilon\right)}{2\kappa^{4/3}\ \left\{4\varphi^2 - 3\left((\psi + \kappa - 1\right)^2\right\}}$$

$$- \frac{144\kappa^{2/3}\ (\psi + \kappa - 1\ ^{8/3} - 27\ (\psi + \kappa - 1\ ^{10/3}\ \upsilon\ (1 - \upsilon\ \ \alpha_1 + \alpha_2}{2\kappa^{4/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1\ ^2} + \frac{3\upsilon\ (\psi + \kappa - 1\ ^{5/3}\ (\alpha_1 - \alpha_2}{\kappa^{2/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1}$$

$$+ \frac{3\upsilon^2\ \ 2\kappa^{2/3}\ (\psi + \kappa - 1\ ^{8/3} - \ (\psi + \kappa - 1\ ^{10/3}\ \ (1 - q_1}{\kappa^{4/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1\ ^2} + \frac{3\upsilon^2\ \ (\psi + \kappa - 1\ ^{10/3}\ \ (1 - q_2}{\kappa^{4/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1\ ^2}$$

$$\left. \left. - \frac{3\upsilon\ \ 5\kappa^{4/3}\ (\psi + \kappa - 1\ ^2 - 3\kappa^{2/3}\ (\psi + \kappa - 1\ ^{8/3} - \ (\psi + \kappa - 1\ ^{10/3}\ \left(-q_1\right)}{\kappa^{4/3}\ \left\{4\varphi^2 - 3\left((\psi + \kappa - 1\right)^2\right\}} - \frac{3\upsilon\left((\psi + \kappa - 1\right)^{8/3}\left(-q_2\right)}{\kappa^{2/3}\ \left\{4\varphi^2 - 3\left((\psi + \kappa - 1\right)^2\right\}} \right] \right] \right]^{1/2}$$

On simplifying, we get

$$s_2 = \left[ \ \ 4\varphi^2 - 3\ (\psi + \kappa - 1\ \ - \left[ \frac{9\kappa^{2/3}\ \ (\psi + \kappa - 1\ + 6\ (\psi + \kappa - 1\ ^{5/3} - 12\kappa^{2/3}\varphi^2}{2\kappa^{2/3}} \right.\right.$$

$$+ \left\{ \frac{144\kappa^{2/3}\ \ (\psi + \kappa - 1\ ^{8/3} - 27\ \ (\psi + \kappa - 1\ ^{10/3} - 12\kappa^{2/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1\ \ \ (\psi + \kappa - 1\ ^{5/3}}{4\kappa^{4/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1} \right\} \upsilon$$

$$\left. - \left\{ \frac{144\kappa^{2/3}\ \ (\psi + \kappa - 1\ ^{8/3} - 27\ \ (\psi + \kappa - 1\ ^{10/3}}{4\kappa^{4/3}\ \ 4\varphi^2 - 3\ (\psi + \kappa - 1} \right\} \upsilon^2 \right] \alpha_1 - \left[ \frac{9\left((\psi + \kappa - 1\right)}{2} - 6\varphi^2 \right.$$

$$+ \left\{ \frac{144\kappa^{2/3}\left((\psi + \kappa - 1\right)^{8/3} - 27\left((\psi + \kappa - 1\right)^{10/3} - 12\kappa^{2/3}\ \left\{4\varphi^2 - 3\left((\psi + \kappa - 1\right)\right\}\left((\psi + \kappa - 1\right)^{5/3}}{4\kappa^{4/3}\ \left\{4\varphi^2 - 3\left((\psi + \kappa - 1\right)\right\}} \right\} \upsilon$$

$$-\left\{\frac{144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-27\ (\psi+\kappa-1)^{10/3}}{4\kappa^{4/3}\ (4\varphi^2-3\ (\psi+\kappa-1)}\right\}\upsilon^2\Bigg]\alpha_2-\frac{\upsilon\ (1-\upsilon)}{4\varphi^2-3\ (\psi+\kappa-1)}\left\{\frac{9N\ (\psi+\kappa-1)^{8/3}}{4\kappa^{4/3}}\right.$$

$$+\frac{3\ (1-q_1)}{2\kappa^{4/3}\ (1-\upsilon)}\Big[5\kappa^{4/3}\ (\psi+\kappa-1)^{2}-3\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-(\psi+\kappa-1)^{10/3}-\upsilon\ (2\kappa^{4/3}\ (\psi+\kappa-1)^{8/3}$$

$$-(\psi+\kappa-1)^{10/3}\Big]+\frac{3\ (1-q_2}{\kappa^{2/3}\ (1-\upsilon)}\left[(\psi+\kappa-1)^{8/3}-\frac{\upsilon\ (\psi+\kappa-1)^{10/3}}{\kappa^{2/3}}\right]\Bigg\}\Bigg]^{1/2}$$

$$s_2=\Bigg[4\varphi^2-3\ (\psi+\kappa-1)-\Bigg[\frac{9\kappa^{2/3}\ (\psi+\kappa-1)+6\ (\psi+\kappa-1)^{5/3}-12\kappa^{2/3}\varphi^2}{2\kappa^{2/3}}$$

$$+\left\{\frac{144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-27\ (\psi+\kappa-1)^{10/3}+36\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-48\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}}{4\kappa^{4/3}\ (4\varphi^2-3\ (\psi+\kappa-1)}\right\}\upsilon$$

$$-\left\{\frac{144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-27\ (\psi+\kappa-1)^{10/3}}{4\kappa^{4/3}\ (4\varphi^2-3\ (\psi+\kappa-1)}\right\}\upsilon^2\Bigg]\alpha_1-\Bigg[\frac{9\ (\psi+\kappa-1)}{2}-6\varphi^2$$

$$+\left\{\frac{144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-27\ (\psi+\kappa-1)^{10/3}-36\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}+48\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}}{4\kappa^{4/3}\ (4\varphi^2-3\ (\psi+\kappa-1)}\right\}\upsilon$$

$$-\left\{\frac{(144\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-27\ (\psi+\kappa-1)^{10/3})}{4\kappa^{4/3}\ (4\varphi^2-3\ (\psi+\kappa-1)}\right\}\upsilon^2\Bigg]\alpha_2-\frac{\upsilon\ (1-\upsilon)}{(4\varphi^2-3\ (\psi+\kappa-1)}\left\{\frac{9N\ (\psi+\kappa-1)^{8/3}}{4\kappa^{4/3}}\right.$$

$$+\frac{3}{2\kappa^{4/3}\ (1-\upsilon)}\Big[5\kappa^{4/3}\ (\psi+\kappa-1)^{2}-3\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-(\psi+\kappa-1)^{10/3}-\upsilon\ (2\kappa^{4/3}\ (\psi+\kappa-1)^{8/3}$$

$$-(\psi+\kappa-1)^{10/3}\Big]\ (1-q_1)\ \frac{3}{\kappa^{2/3}\ (1-\upsilon)}\left[(\psi+\kappa-1)^{8/3}-\frac{\upsilon\ (\psi+\kappa-1)^{10/3}}{\kappa^{2/3}}\right]\ (1-q_2)\Bigg\}\Bigg]^{1/2}$$

Hence, we have

$$s_2 = \left[ 4\varphi^2 - 3\ \psi+\kappa-1\ - \left[ \frac{9\kappa^{2/3}\ \psi+\kappa-1\ + 6\ \psi+\kappa-1\ ^{5/3} - 12\kappa^{2/3}\varphi^2}{2\kappa^{2/3}} \right.\right.$$

$$+ \left\{ \frac{180\kappa^{2/3}\ \psi+\kappa-1\ ^{8/3} - 27\ \psi+\kappa-1\ ^{10/3} - 48\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ ^{5/3}}{4\kappa^{4/3}\ \ 4\varphi^2 - 3\ \psi+\kappa-1} \right\} \upsilon$$

$$\left. \left. - \left\{ \frac{144\kappa^{2/3}\ \psi+\kappa-1\ ^{8/3} - 27\ \psi+\kappa-1\ ^{10/3}}{4\kappa^{4/3}\ \ 4\varphi^2 - 3\ \psi+\kappa-1} \right\} \upsilon^2 \right] \alpha_1 - \left[ \frac{9\ \psi+\kappa-1}{2} - 6\varphi^2 \right.\right.$$

$$+ \left\{ \frac{108\kappa^{2/3}\left(\psi+\kappa-1\right)^{8/3} - 27\left(\psi+\kappa-1\right)^{10/3} + 48\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3}}{4\kappa^{4/3}\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}} \right\} \upsilon$$

$$\left. \left. - \left\{ \frac{\left(144\kappa^{2/3}\left(\psi+\kappa-1\right)^{8/3} - 27\left(\psi+\kappa-1\right)^{10/3}\right)}{4\kappa^{4/3}\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}} \right\} \upsilon^2 \right] \alpha_2 - \frac{\upsilon\left(1-\upsilon\right)}{\left\{4\varphi^2 - 3\left(\psi+\kappa-1\right)\right\}} \left\{ \frac{9N\left(\psi+\kappa-1\right)^{8/3}}{4\kappa^{4/3}} \right.\right.$$

$$+ \frac{3}{2\kappa^{4/3}\left(1-\upsilon\right)} \left[ 5\kappa^{4/3}\left(\psi+\kappa-1\right)^2 - 3\kappa^{2/3}\left(\psi+\kappa-1\right)^{8/3} - \left(\psi+\kappa-1\right)^{10/3} - \upsilon\left\{2\kappa^{4/3}\left(\psi+\kappa-1\right)^{8/3} \right.\right.$$

$$\left. \left. \left. - \left(\psi+\kappa-1\right)^{10/3} \right\} \right] \left(1-q_1\right) \frac{3}{\kappa^{2/3}\left(1-\upsilon\right)} \left[ \left(\psi+\kappa-1\right)^{8/3} - \frac{\upsilon\left(\psi+\kappa-1\right)^{10/3}}{\kappa^{2/3}} \right] \left(1-q_2\right) \right\} \right]^{1/2}$$

$$s_2=\left[4\varphi^2-3\ \psi+\kappa-1\ -\left[\frac{9\kappa^{2/3}\ \psi+\kappa-1\ +6\ \psi+\kappa-1\ ^{5/3}-12\kappa^{2/3}\varphi^2}{2\kappa^{2/3}}\right.\right.$$

$$+\left\{\frac{180\kappa^{2/3}\ \psi+\kappa-1\ ^{8/3}-27\ \psi+\kappa-1\ ^{10/3}-48\alpha^2\kappa^{2/3}\ \psi+\kappa-1\ ^{5/3}}{4\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\right\}\upsilon$$

$$\left.\left.-\left\{\frac{144\kappa^{2/3}\ \psi+\kappa-1\ ^{8/3}-27\ \psi+\kappa-1\ ^{10/3}}{4\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\right\}\upsilon^2\right]\alpha_1-\left[\frac{9\ \psi+\kappa-1}{2}-6\varphi^2\right.\right.$$

$$+\left\{\frac{108\kappa^{2/3}\left(\psi+\kappa-1\right)^{8/3}-27\left(\psi+\kappa-1\right)^{10/3}+48\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3}}{4\kappa^{4/3}\left\{4\varphi^2-3\left(\psi+\kappa-1\right)\right\}}\right\}\upsilon$$

$$\left.\left.-\left\{\frac{\left(144\kappa^{2/3}\left(\psi+\kappa-1\right)^{8/3}-27\left(\psi+\kappa-1\right)^{10/3}\right)}{4\kappa^{4/3}\left\{4\varphi^2-3\left(\psi+\kappa-1\right)\right\}}\right\}\upsilon^2\right]\alpha_2-\frac{\upsilon\left(1-\upsilon\right)}{\left\{4\varphi^2-3\left(\psi+\kappa-1\right)\right\}}\times$$

$$\left\{\frac{9N\left(\psi+\kappa-1\right)^{8/3}}{4\kappa^{4/3}}+\frac{3\left(\psi+\kappa-1\right)^{2}\left(1-q_1\right)}{2\kappa^{4/3}\left(1-\upsilon\right)}\left[5\kappa^{4/3}-3\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}-\left(\psi+\kappa-1\right)^{4/3}\right.\right.$$

$$\left.\left.-\upsilon\left\{2\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}-\left(\psi+\kappa-1\right)^{4/3}\right\}\right]+\frac{3\left(\psi+\kappa-1\right)^{8/3}\left(1-q_2\right)}{\kappa^{2/3}\left(1-\upsilon\right)}\left[1-\frac{\upsilon\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{2/3}}\right]\right\}\Bigg]^{1/2} \quad (3.169)$$

Now, for simplicity, we let $\varphi=1+\in$ and $\psi=1+\in'$. Expanding and ignoring products and higher terms in $\in,\in',\alpha_i$ and $1-q_i\ \ i=1,2$ , and simplifying; equation (3.168) gives

$$s_1=\left\{\ \frac{1}{2}\left[-\frac{45\kappa^2-96\kappa+48\ \alpha_1}{2\ 4-3\kappa}-\left(\frac{15}{2}\kappa-6\right)\alpha_1-\frac{27\kappa^2-72\kappa+48\ \alpha_2}{2\ 4-3\kappa}-\left(\frac{9}{2}\kappa-6\right)\alpha_2\right.\right.$$

$$+\frac{24-18\kappa\ \in'}{2\ 4-3\kappa}-3\in'+8\in-\frac{64-48\kappa\ \in}{2\ 4-3\kappa}+3\kappa\upsilon\ \alpha_1-\alpha_2\ -\frac{24\kappa\upsilon\ \alpha_1-\alpha_2}{2\ 4-3\kappa}-\frac{1}{2\ 4-3\kappa}\times$$

$$\left.\left\{-27\kappa^2\left(1+\frac{22\in'}{9\kappa}\right)\upsilon\ 1-\upsilon\ -6\kappa^2\ \in_1+\in_2\ \upsilon\ 1-\upsilon\ -117\kappa^2\ \alpha_1+\alpha_2\ \upsilon\ 1-\upsilon\ -18\kappa^2\upsilon\left(\alpha_1-\alpha_2\right)\right\}\right]\Bigg\}^{1/2}$$

$$s_1=\left\{\frac{1}{2}\left[-\left\{\frac{45\kappa^2-96\kappa+48}{2\ \ 4-3\kappa}-6+\frac{15\kappa}{2}\right\}\alpha_1-\left\{\frac{27\kappa^2-72\kappa+48}{2\ \ 4-3\kappa}-6+\frac{9\kappa}{2}\right\}\alpha_2+\left\{\frac{24-18\kappa}{2\ \ 4-3\kappa}-3\right\}\in'\right.\right.$$

$$+\left\{8-\frac{64-48\kappa}{2\ \ 4-3\kappa}\right\}\in+\left\{3-\frac{24}{2\ \ 4-3\kappa}\right\}\kappa\upsilon\ \ \alpha_1-\alpha_2\ \ +\frac{1}{2\ \ 4-3\kappa}\left\{27\kappa^2\left(1+\frac{22}{9\kappa}\in'\right)\upsilon\ \ 1-\upsilon\right.$$

$$\left.\left.+6\kappa^2\left(\in_1+\in_2\right)\upsilon\left(1-\upsilon\right)117\kappa^2\left(\alpha_1+\alpha_2\right)\upsilon\left(1-\upsilon\right)18\kappa^2\upsilon\left(\alpha_1-\alpha_2\right)\right\}\ \ \right]\right\}^{1/2}$$

$$s_1=\left\{\frac{1}{2}\left[-\left\{\frac{45\kappa^2-96\kappa+48-12\ \ 4-3\kappa\ \ +15\kappa\ \ 4-3\kappa}{2\ \ 4-3\kappa}\right\}\alpha_1+\left\{\frac{24-18\kappa\ \ -6\ \ 4-3\kappa}{2\ \ 4-3\kappa}\right\}\in'\right.\right.$$

$$-\left\{\frac{27\kappa^2-72\kappa+48-12\ \ 4-3\kappa\ \ +9\kappa\ \ 4-3\kappa}{2\ \ 4-3\kappa}\right\}\alpha_2+\left\{\frac{16\ \ 4-3\kappa\ \ -\ \ 64-48\kappa}{2\ \ 4-3\kappa}\right\}\in$$

$$+\left\{\frac{6\ \ 4-3\kappa\ \ -24}{2\ \ 4-3\kappa}\right\}\kappa\upsilon\ \ \alpha_1-\alpha_2\ \ +\frac{\upsilon\ \ 1-\upsilon}{2\left(4-3\kappa\right)}\left[27\kappa^2\left(1+\frac{22}{9\kappa}\in'\right)+6\kappa^2\left(\in_1+\in_2\right)117\kappa^2\left(\alpha_1+\alpha_2\right)\right.$$

$$\left.\left.+\frac{18\kappa^2\upsilon\left(\alpha_1-\alpha_2\right)}{2\left(4-3\kappa\right)}\right]\right\}^{1/2}$$

$$s_1=\left\{\frac{1}{2}\left[-\left\{\frac{45\kappa^2-96\kappa+48-48+36\kappa+60\kappa-45\kappa^2}{2\ \ 4-3\kappa}\right\}\alpha_1+\left\{\frac{24-18\kappa-24+18\kappa}{2\ \ 4-3\kappa}\right\}\in'\right.\right.$$

$$-\left\{\frac{27\kappa^2-72\kappa+48-48+36\kappa+36\kappa-9\kappa^2}{2\ \ 4-3\kappa}\right\}\alpha_2+\left\{\frac{64-48\kappa-64+48\kappa}{2\ \ 4-3\kappa}\right\}\in+\frac{9\kappa^2\upsilon\ \ \alpha_1-\alpha_2}{4-3\kappa}$$

$$\left.\left.+\left\{\frac{24-18\kappa-24}{2\ \ 4-3\kappa}\right\}\kappa\upsilon\left(\alpha_1-\alpha_2\right)\frac{\upsilon\left(1-\upsilon\right)}{2\left(4-3\kappa\right)}\left[27\kappa^2\left(1+\frac{22}{9\kappa}\in'\right)+6\kappa^2\left(\in_1+\in_2\right)117\kappa^2\left(\alpha_1+\alpha_2\right)\right]\right]\right\}^{\frac{1}{2}}$$

Cancelling, we get

$$s_1=\left[\frac{1}{2}\left\{\frac{9\kappa^2\upsilon\ \ \alpha_1-\alpha_2}{4-3\kappa}-\left(\frac{9\kappa}{4-3\kappa}\right)\kappa\upsilon\ \ \alpha_1-\alpha_2\ \ +\frac{\upsilon\ \ 1-\upsilon}{2\ \ 4-3\kappa}\left[27\kappa^2\left(1+\frac{22}{9\kappa}\in'\right)+6\kappa^2\ \ \in_1+\in_2\right.\right.\right.$$

$$\left.\left.\left.+117\kappa^2\ \ \alpha_1+\alpha_2\ \ \right]\right\}\right]^{\frac{1}{2}}$$

$$\Rightarrow s_1 = \left[\frac{1}{2}\left\{\frac{\upsilon\ 1-\upsilon}{2\ \ 4-3\kappa}\left[27\kappa^2\left(1+\frac{22}{9\kappa}\in'\right)+6\kappa^2\ \ 1-q_1\ \ +6\kappa^2\ \ 1-q_2\ \ +117\kappa^2\ \ \alpha_1+\alpha_2\ \right]\right\}\right]^{\frac{1}{2}}$$

$$\Rightarrow s_1 = \left\{\frac{\upsilon\ 1-\upsilon}{4\ \ 4-3\kappa}\left[27\kappa^2\left(1+\frac{22}{9\kappa}\in'\right)+6\kappa^2\ \ 1-q_1\ \ +6\kappa^2\ \ 1-q_2\ \ +117\kappa^2\ \ \alpha_1+\alpha_2\ \right]\right\}^{\frac{1}{2}}$$

$$\Rightarrow s_1 = \left\{\frac{\upsilon\ 1-\upsilon}{4-3\kappa}\left[\frac{27\kappa^2}{4}\left(1+\frac{22}{9\kappa}\in'\right)+\frac{3\kappa^2}{2}\ \ 1-q_1\ \ +\frac{3\kappa^2}{2}\ \ 1-q_2\ \ +\frac{117\kappa^2}{4}\ \ \alpha_1+\alpha_2\ \right]\right\}^{\frac{1}{2}}$$

$$\Rightarrow s_1 = \left\{\frac{\kappa^2\upsilon\ 1-\upsilon}{4-3\kappa}\left[\frac{27}{4}\left(1+\frac{22}{9\kappa}\in'\right)+\frac{3}{2}\ 1-q_1\ \ +\frac{3}{2}\ 1-q_2\ \ +\frac{117}{4}\alpha_1+\frac{117}{4}\alpha_2\ \right]\right\}^{\frac{1}{2}}$$

$$\Rightarrow s_1 = \left\{\frac{\kappa^2\upsilon\ 1-\upsilon}{4-3\kappa}\left[\frac{27}{4}+\frac{33}{2\kappa}\in'+\frac{3}{2}\ 1-q_1\ \ +\frac{3}{2}\ 1-q_2\ \ +\frac{117}{4}\alpha_1+\frac{117}{4}\alpha_2\ \right]\right\}^{\frac{1}{2}} \qquad (3.170)$$

Similarly, from equation (3.169), we get

$$s_2 = \left[4-3\kappa-3\in'+8\in+\frac{\alpha_1}{4\ \ 4-3\kappa}\ \ 90\kappa^2-192\kappa+96-\ 153\kappa-48\ \ \kappa\upsilon+117\kappa^2\upsilon^2\ \ +\frac{\alpha_2}{4\ \ 4-3\kappa}\right. \qquad 3.171$$

$$\left.\times\ 54\kappa^2-144\kappa+96-\ 81\kappa+48\ \ \kappa\upsilon+117\kappa^2\upsilon^2\ \ -\frac{\kappa^2\upsilon\ 1-\upsilon}{4-3\kappa}\left\{\frac{27}{4}+\frac{33\in'}{2\kappa}+\frac{3\ 1-q_1}{2}+\frac{3\ 1-q_2}{2}\right\}\right]^{\frac{1}{2}}$$

Now, we expand the potential around the triangular points to get

$$\Omega = \Omega^0+\frac{1}{2}\Omega^0_{\xi\xi}u^2+\Omega^0_{\xi\eta}uv+\frac{1}{2}\Omega^0_{\eta\eta}v^2+O\ \ 3 \qquad (3.172)$$

where $O\ \ 3$ represents the third and higher terms in $u$ and $v$.

A symmetric form of the potential in equations (3.91) with $\zeta = 0$ is given by the equation

$$\Omega^0 = \frac{\omega_0^2\ \psi+\kappa-1}{2}\left[\ 1-\upsilon\ \ \rho_1^2+\upsilon\rho_2^2\ \right]+\frac{q_1\kappa\ 1-\upsilon}{\rho_1}+\frac{q_1\alpha_1\kappa\ 1-\upsilon}{2\rho_1^3}+\frac{q_2\kappa\upsilon}{\rho_2}+\frac{q_2\alpha_2\kappa\upsilon}{2\rho_2^3} \tag{3.173}$$

Now, from equations (3.104) and (3.105), we get

$$\rho_i = \left[\frac{\kappa}{\psi+\kappa-1}\right]^{\frac{1}{3}}\left[1-\frac{1}{2}\ \alpha_1+\alpha_2\ -\frac{1}{3}\ 1-q_i\ +\frac{1}{2}\frac{\alpha_i\ \psi+\kappa-1^{\frac{2}{3}}}{\kappa^{2/3}}\right]$$

$$\rho_i^{-3} = \frac{\psi+\kappa-1}{\kappa}\left[1+\frac{3}{2}\ \alpha_1+\alpha_2\ +\ 1-q_i\ -\frac{3}{2}\frac{\alpha_i\ \psi+\kappa-1^{\frac{2}{3}}}{\kappa^{2/3}}\right] \qquad i=1,2 \tag{3.174}$$

Substituting equations (3.174) in equation of (3.173), we have

$$\Omega^0 = \frac{\omega_0^2\ \psi+\kappa-1}{2}\left\{\ 1-\upsilon\ \left[\frac{\kappa}{\psi+\kappa-1}\right]^{2/3}\left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1^{\ 2/3}}{\kappa^{2/3}}\right]\right.$$

$$\left.+\upsilon\left[\frac{\kappa}{\psi+\kappa-1}\right]^{2/3}\left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_2\ +\frac{\alpha_2\ \psi+\kappa-1^{\ 2/3}}{\kappa^{2/3}}\right]\right\}+q_1\kappa\ 1-\upsilon\ \left[\frac{\psi+\kappa-1}{\kappa}\right]^{1/3}$$

$$\times\left[1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{1}{3}\ 1-q_1\ -\frac{\alpha_1\ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right]+\frac{q_1\alpha_1\kappa\left(-\upsilon\right)\left(\psi+\kappa-1\right)}{2\ \kappa}\left[1+\frac{3}{2}\left(\alpha_1+\alpha_2\right)\left(-q_1\right)\right.$$

$$\left.-\frac{3}{2}\frac{\alpha_1\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{2/3}}\right]+q_2\kappa\upsilon\left[\frac{\left(\psi+\kappa-1\right)}{\kappa}\right]^{1/3}\left[1+\frac{1}{2}\left(\alpha_1+\alpha_2\right)\frac{1}{3}\left(-q_2\right)\frac{\alpha_2\left(\psi+\kappa-1\right)^{2/3}}{2\kappa^{2/3}}\right]$$

$$+\frac{q_2\alpha_2\kappa\upsilon}{2}\frac{\left(\psi+\kappa-1\right)}{\kappa}\left[1+\frac{3}{2}\left(\alpha_1+\alpha_2\right)\left(-q_2\right)\frac{3}{2}\frac{\alpha_2\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{2/3}}\right]$$

Expressing $q_i = 1-\ 1-q_i\ $, we have

$$\Omega^0=\frac{\kappa^{2/3}\omega_0^2\ \psi+\kappa-1\ ^{1/3}}{2}\left\{1-\upsilon\ \left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}}\right]+\upsilon\left[1-\ \alpha_1+\alpha_2\right.\right.$$
$$\left.\left.-\frac{2}{3}\ 1-q_2\ +\frac{\alpha_2\ \psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}}\right]\right\}+\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}\ 1-\upsilon\ \left[1-\ 1-q_1\ \right]\left[1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{1}{3}\ 1-q_1\right.$$
$$\left.+\frac{\alpha_1\ \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right]+\kappa^{2/3}\ \psi+\kappa-1\ ^{\frac{1}{3}}\upsilon\left[1-\ 1-q_2\ \right]\left[1+\frac{1}{2}\left(\alpha_1+\alpha_2\right)\frac{1}{3}\left(1-q_2\right)\frac{\alpha_2\left(\psi+\kappa-1\right)^{2/3}}{2\kappa^{2/3}}\right]$$
$$+\frac{\left(\psi+\kappa-1\right)\left(1-\upsilon\right)\alpha_1}{2}\left[1-\left(1-q_1\right)\right]\times\left[1+\frac{3}{2}\left(\alpha_1+\alpha_2\right)\left(1-q_1\right)\frac{3}{2}\frac{\alpha_1\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{2/3}}\right]$$
$$+\frac{\left(\psi+\kappa-1\right)\upsilon\alpha_2}{2}\left[1-\left(1-q_2\right)\right]\times\left[1+\frac{3}{2}\left(\alpha_1+\alpha_2\right)\left(1-q_2\right)\frac{3}{2}\frac{\alpha_2\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{2/3}}\right]$$

Ignoring products of $1-q_i$ and $\alpha_i$, we get

$$\Omega^0=\frac{\kappa^{2/3}\omega_0^2\ \psi+\kappa-1\ ^{1/3}}{2}\left\{1-\upsilon\ \left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}\right]\right.$$
$$\left.+\upsilon\left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_2\ +\frac{\alpha_2\ \psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}}\right]\right\}+\frac{\psi+\kappa-1\ \ 1-\upsilon}{2}\alpha_1+\frac{\psi+\kappa-1\ \upsilon}{2}\alpha_2$$
$$+\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}\ 1-\upsilon\ \left[1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\ 1-q_1\ -\frac{\alpha_1\ \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right]+\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}\upsilon$$
$$\times\left[1+\frac{1}{2}\left(\alpha_1+\alpha_2\right)\frac{4}{3}\left(1-q_2\right)\frac{\alpha_2\left(\psi+\kappa-1\right)^{2/3}}{2\kappa^{2/3}}\right]$$

$$\Omega^0=\frac{\omega_0^2\kappa^{2/3}\ \psi+\kappa-1^{\ 1/3}}{2}\left\{1-\upsilon\left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1^{\ \frac{2}{3}}}{\kappa^{2/3}}\right.\right.$$

$$\left.+\frac{2}{\omega_0^2}\left\{1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\frac{1-q_1}{}-\frac{\alpha_1\ \psi+\kappa-1^{\ \frac{2}{3}}}{2\kappa^{2/3}}\right\}\right]+\upsilon\left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\frac{1-q_2}{}+\frac{\alpha_2\ \psi+\kappa-1^{\ \frac{2}{3}}}{\kappa^{2/3}}\right.$$

$$\left.\left.\left.+\frac{2}{\omega_0^2}\left\{1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\ 1-q_2\ -\frac{\alpha_2\ \psi+\kappa-1^{\ 2/3}}{2\kappa^{2/3}}\right\}\right]\right\}+\frac{\psi+\kappa-1\ 1-\upsilon\ \alpha_1}{2}+\frac{\psi+\kappa-1\ \upsilon\alpha_2}{2}$$

Substituting for $\omega_0^2$,

$$\Omega^0=\frac{\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)\kappa^{2/3}\ \psi+\kappa-1^{\ 1/3}}{2}\left\{1-\upsilon\left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1^{\ \frac{2}{3}}}{\kappa^{2/3}}\right.\right.$$

$$\left.+\frac{2}{\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)}\left\{1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\frac{1-q_1}{}-\frac{\alpha_1\ \psi+\kappa-1^{\ \frac{2}{3}}}{2\kappa^{2/3}}\right\}\right]+\upsilon\left[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\frac{1-q_2}{}\right.$$

$$\left.\left.+\frac{\alpha_2\ \psi+\kappa-1^{\ \frac{2}{3}}}{\kappa^{2/3}}+\frac{2}{\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)}\left\{1+\frac{1}{2}\left(\alpha_1+\alpha_2\right)+\frac{4}{3}\left(1-q_2\right)+\frac{\alpha_2\left(\psi+\kappa-1\right)^{2/3}}{2\kappa^{2/3}}\right\}\right]\right\}$$

$$+\frac{\psi+\kappa-1\ 1-\upsilon\ \alpha_1}{2}+\frac{\psi+\kappa-1\ \upsilon\alpha_2}{2}$$

Multiplying and retaining only linear terms in $\alpha_i$ and ignoring products with $1-q_i$

$$\Omega^0=\frac{\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)\kappa^{2/3}\ \ \psi+\kappa-1\ ^{1/3}}{2}\Bigg\{\ 1-\upsilon\ \Bigg[1-\ \alpha_1+\alpha_2\ \ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}$$

$$+2\left\{1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\frac{1-q_1}{}-\frac{\alpha_1\ \psi+\kappa-1\ ^{\frac{2}{3}}}{2\kappa^{2/3}}-\frac{3}{2}\alpha_1-\frac{3}{2}\alpha_2\right\}\Bigg]+\upsilon\Bigg[1-\ \alpha_1+\alpha_2\ \ -\frac{2}{3}\frac{1-q_2}{}$$

$$+\frac{\alpha_2\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}+2\left\{1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\ 1-q_2\ \ -\frac{\alpha_2\ \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}-\frac{3}{2}\alpha_1-\frac{3}{2}\alpha_2\right\}\Bigg]\Bigg\}$$

$$+\frac{\left(\psi+\kappa-1\right)\alpha_1}{2}-\frac{\left(\psi+\kappa-1\right)\upsilon\alpha_1}{2}+\frac{\left(\psi+\kappa-1\right)\upsilon\alpha_2}{2}$$

$$\Omega^0=\frac{\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)\kappa^{2/3}\ \ \psi+\kappa-1\ ^{1/3}}{2}\Bigg\{\ 1-\upsilon\ \Bigg[1-\ \alpha_1+\alpha_2\ \ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}$$

$$+2\left\{1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\frac{1-q_1}{}-\frac{\alpha_1\ \psi+\kappa-1\ ^{\frac{2}{3}}}{2\kappa^{2/3}}-\frac{3}{2}\alpha_1-\frac{3}{2}\alpha_2\right\}\Bigg]+\upsilon\Bigg[1-\ \alpha_1+\alpha_2\ \ -\frac{2}{3}\frac{1-q_2}{}$$

$$+\frac{\alpha_2\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}+2\left\{1+\frac{1}{2}\ \alpha_1+\alpha_2\ +\frac{4}{3}\ 1-q_2\ \ -\frac{\alpha_2\ \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}-\frac{3}{2}\alpha_1-\frac{3}{2}\alpha_2\right\}\Bigg]\Bigg\}$$

$$+\frac{\left(\psi+\kappa-1\right)\alpha_1}{2}-\frac{\upsilon\left(\psi+\kappa-1\right)\left(\alpha_1-\alpha_2\right)}{2}$$

$$\Omega^0=\frac{\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)\kappa^{2/3}\ \ \psi+\kappa-1\ ^{1/3}}{2}\Bigg\{\ 1-\upsilon\ \Bigg[1-\ \alpha_1+\alpha_2\ -\frac{2}{3}\ 1-q_1\ +\frac{\alpha_1\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}$$

$$+2+\ \alpha_1+\alpha_2\ +\frac{8\ 1-q_1}{3}-\frac{\alpha_1\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}-3\ \alpha_1+\alpha_2\ \Bigg]+\upsilon\Bigg[1-\ \alpha_1+\alpha_2\ -\frac{2\ 1-q_2}{3}$$

$$+\frac{\alpha_2\ \psi+\kappa-1\ ^{\frac{2}{3}}}{\kappa^{2/3}}+2+\ \alpha_1+\alpha_2\ +\frac{8}{3}\ 1-q_2\ -\frac{\alpha_2\ \psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}}-3\ \alpha_1+\alpha_2\ \Bigg]\Bigg\}$$

$$+\frac{\left(\psi+\kappa-1\right)\alpha_1}{2}-\frac{\upsilon\left(\psi+\kappa-1\right)\left(\alpha_1-\alpha_2\right)}{2}$$

Simplifying, we get

$$\Omega^0=\frac{\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)\kappa^{2/3}\ \ \psi+\kappa-1\ ^{1/3}}{2}\Bigg\{\ 1-\upsilon\ \Bigg[3+2\ 1-q_1\ -3\ \alpha_1+\alpha_2\ +\frac{\psi+\kappa-1\ ^{2/3}\alpha_1}{\kappa^{2/3}\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)}\Bigg]$$

$$+\upsilon\Bigg[3+2\ 1-q_2\ -3\ \alpha_1+\alpha_2\ +\frac{\psi+\kappa-1\ ^{2/3}\alpha_2}{\kappa^{2/3}\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2\right)}\Bigg]\Bigg\}$$

$$\Omega^0=\frac{\omega_0^2\kappa^{2/3}\ \ \psi+\kappa-1\ ^{1/3}}{2}\Bigg\{\ 1-\upsilon\ \Bigg[3+2\ 1-q_1\ -3\ \alpha_1+\alpha_2\ +\frac{\left(1-\frac{3}{2}\alpha_1-\frac{3}{2}\alpha_2\right)\ \psi+\kappa-1\ ^{2/3}\alpha_1}{\kappa^{2/3}}\Bigg]$$

$$+\upsilon\Bigg[3+2\ 1-q_2\ -3\ \alpha_1+\alpha_2\ +\frac{\psi+\kappa-1\ ^{2/3}\alpha_2\left(1-\frac{3}{2}\alpha_1-\frac{3}{2}\alpha_2\right)}{\kappa^{2/3}}\Bigg]\Bigg\}$$

$$\Omega^0 = \frac{\omega_0^2 \kappa^{2/3} \left(\psi + \kappa - 1\right)^{1/3}}{2} \left\{ 1 - \upsilon \left[ 3 + 2\left(1 - q_1\right) - 3\left(\alpha_1 + \alpha_2\right) + \frac{\left(\psi + \kappa - 1\right)^{2/3} \alpha_1}{\kappa^{2/3}} \right] \right.$$

$$\left. + \upsilon \left[ 3 + 2\left(1 - q_2\right) - 3\left(\alpha_1 + \alpha_2\right) + \frac{\left(\psi + \kappa - 1\right)^{2/3} \alpha_2}{\kappa^{2/3}} \right] \right\}$$

$$\Omega^0 = \frac{\left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2\right) \kappa^{2/3} \left(\psi + \kappa - 1\right)^{1/3}}{2} \left\{ 3 + 2\left(1 - q_1\right) - 3\left(\alpha_1 + \alpha_2\right) + \frac{\left(\psi + \kappa - 1\right)^{2/3} \alpha_1}{\kappa^{2/3}} \right.$$

$$\left. - \upsilon \left[ 3 + 2\left(1 - q_1\right) - 3\left(\alpha_1 + \alpha_2\right) + \frac{\left(\psi + \kappa - 1\right)^{2/3} \alpha_1}{\kappa^{2/3}} \right] + \upsilon \left[ 3 + 2\left(1 - q_2\right) - 3\left(\alpha_1 + \alpha_2\right) + \frac{\left(\psi + \kappa - 1\right)^{2/3} \alpha_2}{\kappa^{2/3}} \right] \right\}$$

$$\Omega^0 = \frac{\left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2\right) \kappa^{2/3} \left(\psi + \kappa - 1\right)^{1/3}}{2} \left\{ 3 + 2\left(1 - q_1\right) - 3\left(\alpha_1 + \alpha_2\right) + \frac{\left(\psi + \kappa - 1\right)^{2/3} \alpha_1}{\kappa^{2/3}} - 3\upsilon \right.$$

$$\left. - 2\upsilon\left(1 - q_1\right) + 3\upsilon\left(\alpha_1 + \alpha_2\right) - \frac{\upsilon\left(\psi + \kappa - 1\right)^{2/3} \alpha_1}{\kappa^{2/3}} + 3\upsilon + 2\upsilon\left(1 - q_2\right) - 3\upsilon\left(\alpha_1 + \alpha_2\right) + \frac{\upsilon\left(\psi + \kappa - 1\right)^{2/3} \alpha_2}{\kappa^{2/3}} \right\}$$

$$\Omega^0 = \frac{\left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2\right) \kappa^{2/3} \left(\psi + \kappa - 1\right)^{1/3}}{2} \left\{ 3 + 2\left(1 - q_1\right) - 3\left(\alpha_1 + \alpha_2\right) + \frac{\left(\psi + \kappa - 1\right)^{2/3} \alpha_1}{\kappa^{2/3}} \right.$$

$$\left. - 2\upsilon\left(1 - q_1\right) + \frac{\upsilon\left(\psi + \kappa - 1\right)^{2/3} \left(\alpha_2 - \alpha_1\right)}{\kappa^{2/3}} + 2\upsilon\left(1 - q_2\right) \right\}$$

$$\Omega^0 = \frac{\left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2\right) \kappa^{2/3} \left(\psi + \kappa - 1\right)^{1/3}}{2} \left\{ 3 + 2\left(1 - q_1\right) - \left[ 3 - \frac{\left(\psi + \kappa - 1\right)^{2/3}}{\kappa^{2/3}} \right] \alpha_1 - 3\alpha_2 \right.$$

$$\left. + \upsilon \left[ 2\left(1 - q_2\right) - 2\left(1 - q_1\right) + \frac{\upsilon\left(\psi + \kappa - 1\right)^{2/3} \left(\alpha_2 - \alpha_1\right)}{\kappa^{2/3}} \right] \right\}$$

$$\Omega^0 = \frac{\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}}{2}\left\{3+2\ 1-q_1\ -\left[3-\frac{\psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}}\right]\alpha_1-3\alpha_2+\upsilon\left[\frac{\upsilon\ \psi+\kappa-1\ ^{2/3}\ \alpha_2-\alpha_1}{\kappa^{2/3}}\right.\right.$$
$$\left.\left.+2\ 1-q_2\ -2\ 1-q_1\ \right]+\frac{9}{2}\ \alpha_1+\alpha_2\ \right\}$$

$$\Omega^0 = \frac{\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}}{2}\left\{3+2\ 1-q_1\ -\left[3-\frac{\psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}}-\frac{9}{2}\right]\alpha_1-\left(3-\frac{9}{2}\right)\alpha_2\right.$$
$$\left.+\upsilon\left[2\ 1-q_2\ -2\ 1-q_1\ +\frac{\upsilon\ \psi+\kappa-1\ ^{2/3}\ \alpha_2-\alpha_1}{\kappa^{2/3}}\right]\right\}$$

$$\Omega^0 = \frac{\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}}{2}\left\{3+2\ 1-q_1\ +\left[\frac{3}{2}+\frac{\psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}}\right]\alpha_1+\frac{3}{2}\alpha_2+\upsilon\left[2\ 1-q_2\ -2\ 1-q_1\right.\right.$$
$$\left.\left.+\frac{\psi+\kappa-1\ ^{2/3}\ \alpha_2-\alpha_1}{\kappa^{2/3}}\right]\right\}$$

$$\Omega^0 = \frac{3\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}}{2}-\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}\ 1-\upsilon\ \ 1-q_1\ -\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}\upsilon\ 1-q_2 \qquad 3.175$$
$$+\kappa^{\frac{2}{3}}\ \psi+\kappa-1\ ^{\frac{1}{3}}\left\{\frac{3}{4}+\frac{\psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}-\frac{\upsilon\ \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right\}\alpha_1+\kappa^{\frac{2}{3}}\ \psi+\kappa-1\ ^{\frac{1}{3}}\left\{\frac{3}{4}+\frac{\upsilon\ \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right\}\alpha_2$$

Substituting equation (3.175) and the partial derivatives (3.134) in equation (3.172), we have

$$\Omega = \frac{3\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}}{2}-\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}\ 1-\upsilon\ \ 1-q_1\ -\kappa^{2/3}\ \psi+\kappa-1\ ^{1/3}\upsilon\ 1-q_2$$
$$+\kappa^{\frac{2}{3}}\ \psi+\kappa-1\ ^{\frac{1}{3}}\left\{\frac{3}{4}+\frac{\psi+\kappa-1\ ^{\frac{2}{3}}}{2\kappa^{2/3}}-\frac{\upsilon\ \psi+\kappa-1\ ^{\frac{2}{3}}}{2\kappa^{2/3}}\right\}\alpha_1+\kappa^{\frac{2}{3}}\ \psi+\kappa-1\ ^{\frac{1}{3}}\left\{\frac{3}{4}+\frac{\upsilon\ \psi+\kappa-1\ ^{2/3}}{2\kappa^{2/3}}\right\}\alpha_2$$
$$+\frac{\psi+\kappa-1\ ^{\frac{5}{3}}}{\kappa^{2/3}}\left\{\frac{3}{8}+\left(\frac{27}{16}-\frac{3\upsilon}{2}\right)\alpha_1+\left(\frac{3}{16}+\frac{3\upsilon}{2}\right)\alpha_2+\left[\frac{\psi+\kappa-1\ ^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}}{4\left(\psi+\kappa-1\right)^{2/3}}+\frac{N\upsilon}{4\left(\psi+\kappa-1\right)^{2/3}}\right]\left(1-q_1\right)\right.$$

$$+\left[\frac{\kappa^{2/3}}{2\ (\psi+\kappa-1)^{2/3}}-\frac{N\upsilon}{4\ (\psi+\kappa-1)^{2/3}}\right](1-q_2)\Bigg\}u^2+\frac{(\psi+\kappa-1)^{\frac{4}{3}}\sqrt{N}}{\kappa^{2/3}}\Bigg[\left(\frac{3}{4}-\frac{3\upsilon}{2}\right)$$

$$+\left\{\frac{72\kappa^{2/3}-15\ (\psi+\kappa-1)^{2/3}}{8N}-\frac{\left[96\kappa^{2/3}-18\ (\psi+\kappa-1)^{2/3}\right]\upsilon}{8N}\right\}\alpha_1+\left\{\frac{24\kappa^{2/3}-3\ (\psi+\kappa-1)^{2/3}}{8N}\right.$$

$$\left.-\frac{\left[96\kappa^{2/3}-18\ (\psi+\kappa-1)^{2/3}\right]\upsilon}{8N}\right\}\alpha_2+\left\{\frac{-4\ (\psi+\kappa-1)^{4/3}+4\kappa^{4/3}-4\kappa^{2/3}(\psi+\kappa-1)^{2/3}}{8N(\psi+\kappa-1)^{2/3}}\right.$$

$$\left.-\frac{\upsilon\left[2\kappa^{2/3}-(\psi+\kappa-1)^{2/3}\right]}{2N}\right\}(1-q_1)\left\{\frac{\kappa^{2/3}\left[2\kappa^{2/3}-(\psi+\kappa-1)^{2/3}\right]}{N(\psi+\kappa-1)^{2/3}}-\frac{\upsilon(\psi+\kappa-1)^{2/3}}{2N}\right\}(1-q_2)\Bigg]uv$$

$$+\frac{(\psi+\kappa-1)^{\frac{5}{3}}}{\kappa^{2/3}}\Bigg[\frac{3N}{8(\psi+\kappa-1)^{2/3}}+\frac{\left[36\kappa^{2/3}-3(\psi+\kappa-1)^{2/3}\right](\alpha_1+\alpha_2)}{16(\psi+\kappa-1)^{2/3}}$$

$$+\left\{\frac{\left[2\kappa^{2/3}-(\psi+\kappa-1)^{2/3}\right]}{4(\psi+\kappa-1)^{2/3}}-\frac{\upsilon N}{4(\psi+\kappa-1)^{\frac{2}{3}}}\right\}(1-q_1)\left\{\frac{-\kappa^{2/3}}{2(\psi+\kappa-1)^{\frac{2}{3}}}+\frac{\upsilon N}{4(\psi+\kappa-1)^{\frac{2}{3}}}\right\}(1-q_2)\Bigg]v^2$$

$$+O(3)$$

$$\Omega=\kappa^{2/3}\ (\psi+\kappa-1)^{1/3}\Bigg[\frac{3}{2}-(1-\upsilon)(1-q_1)-\upsilon(1-q_2)\left\{\frac{3}{4}+\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{2\kappa^{2/3}}-\frac{\upsilon(\psi+\kappa-1)^{\frac{2}{3}}}{2\kappa^{2/3}}\right\}\alpha_1$$

$$+\left\{\frac{3}{4}+\frac{\upsilon\ (\psi+\kappa-1)^{2/3}}{2\kappa^{2/3}}\right\}\alpha_2\Bigg]+\frac{(\psi+\kappa-1)^{\frac{5}{3}}}{\kappa^{2/3}}\left\{\frac{3}{8}+\left(\frac{27}{16}-\frac{3\upsilon}{2}\right)\alpha_1+\left(\frac{3}{16}+\frac{3\upsilon}{2}\right)\alpha_2\right.$$

$$+\left[\frac{(\psi+\kappa-1)^{\frac{2}{3}}-2\kappa^{\frac{2}{3}}+\upsilon\left[4\kappa^{\frac{2}{3}}-(\psi+\kappa-1)^{\frac{2}{3}}\right]}{4(\psi+\kappa-1)^{2/3}}\right](1-q_1)\left[\frac{2\kappa^{2/3}+\upsilon\left[(\psi+\kappa-1)^{2/3}-4\kappa^{\frac{2}{3}}\right]}{4(\psi+\kappa-1)^{2/3}}\right](1-q_2)\Bigg\}u^2$$

$$+\frac{(\psi+\kappa-1)^{\frac{4}{3}}\sqrt{N}}{\kappa^{2/3}}\Bigg[\left(\frac{3}{4}-\frac{3\upsilon}{2}\right)-\left\{\frac{72\kappa^{2/3}-15(\psi+\kappa-1)^{2/3}-\upsilon\left[96\kappa^{2/3}-18(\psi+\kappa-1)^{2/3}\right]}{8N}\right\}\alpha_1$$

$$-\left\{\frac{24\kappa^{2/3}-3\ \psi+\kappa-1\ ^{2/3}-\upsilon\left[96\kappa^{2/3}-18\ \psi+\kappa-1\ ^{2/3}\right]}{8N}\right\}\alpha_2$$

$$-\left\{\frac{4\kappa^{4/3}-4\ \psi+\kappa-1\ ^{4/3}-4\kappa^{2/3}\ \psi+\kappa-1\ ^{2/3}-\upsilon\ \psi+\kappa-1\ ^{2/3}\left[8\kappa^{2/3}-4\ \psi+\kappa-1\ ^{2/3}\right]}{8N\ \psi+\kappa-1\ ^{2/3}}\right\}1-q_1$$

$$-\left\{\frac{2\kappa^{2/3}\left[2\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right]-\upsilon\ \psi+\kappa-1\ ^{4/3}}{2N\ \psi+\kappa-1\ ^{2/3}}\right\}1-q_2\Bigg]uv+\frac{\left(\psi+\kappa-1\right)^{\frac{5}{3}}}{\kappa^{2/3}}$$

$$\times\left[\frac{3N}{8\left(\psi+\kappa-1\right)^{2/3}}+\frac{\left[36\kappa^{2/3}-3\left(\psi+\kappa-1\right)^{2/3}\right]\left(\alpha_1+\alpha_2\right)}{16\left(\psi+\kappa-1\right)^{2/3}}+\left\{\frac{\left[2\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right]-\upsilon N}{4\left(\psi+\kappa-1\right)^{2/3}}\right\}\left(1-q_1\right)\right.$$

$$\left.+\left\{\frac{-2\kappa^{2/3}+\upsilon N}{4\left(\psi+\kappa-1\right)^{2}}\right\}\left(1-q_2\right)\right]v^2 \qquad (3.176)$$

Now, we introduce the variables $\bar{u}$ and $\bar{v}$ by the transformation

$$u=\bar{u}\cos\alpha-\bar{v}\sin\alpha,\ v=\bar{u}\sin\alpha+\bar{v}\cos\alpha \qquad (3.177)$$

which is equivalent to rotating the coordinate system through an angle $\alpha$ .

For simplicity, we put $\psi=1+\in'$; factorizing and ignoring higher order terms in $\in'$, $1-q_i$ and $\alpha_i$; we get the new quadratic form of (3.176) , such that

$$\bar{\Omega} = \kappa^{2/3}\ \left(\kappa + \epsilon'\right)^{1/3}\left[\frac{3}{2} - \left(1-\upsilon\right)\left(1-q_1\right) - \upsilon\left(1-q_2\right) + \left\{\frac{3}{4} + \frac{\left(\kappa+\epsilon'\right)^{2/3}}{2\kappa^{2/3}} - \frac{\upsilon\left(\kappa+\epsilon'\right)^{2/3}}{2\kappa^{2/3}}\right\}\alpha_1\right.$$

$$\left. + \left\{\frac{3}{4} + \frac{\upsilon\left(\kappa+\epsilon'\right)^{2/3}}{2\kappa^{2/3}}\right\}\alpha_2\right] + \frac{\left(\kappa+\epsilon'\right)^{5/3}}{\kappa^{2/3}}\left[\frac{3}{8} + \left(\frac{27}{16} - \frac{3\upsilon}{2}\right)\alpha_1 + \left(\frac{3}{16} + \frac{3\upsilon}{2}\right)\alpha_2\right.$$

$$\left. + \left\{\frac{\left(\kappa+\epsilon'\right)^{2/3} - 2\kappa^{2/3} + \upsilon\left[4\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3}\right]}{4\left(\kappa+\epsilon'\right)^{2/3}}\right\}\left(1-q_1\right) + \left\{\frac{2\kappa^{2/3} + \upsilon\left[\left(\kappa+\epsilon'\right)^{2/3} - 4\kappa^{2/3}\right]}{4\left(\kappa+\epsilon'\right)^{2/3}}\right\}\left(1-q_2\right)\right]$$

$$\times\left[\bar{u}^2\cos^2\alpha + \bar{v}^2\sin^2\alpha - 2\bar{u}\bar{v}\sin\alpha\cos\alpha\right] + \frac{\left(\kappa+\epsilon'\right)^{5/3}\sqrt{4\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3}}}{\kappa^{2/3}}\left[\left(\frac{3}{4} - \frac{3\upsilon}{2}\right)\right.$$

$$\left. - \left\{\frac{4\kappa^{4/3} - 4\left(\kappa+\epsilon'\right)^{4/3} - 4\kappa^{2/3}\left(\kappa+\epsilon'\right)^{2/3} - \upsilon\left(\kappa+\epsilon'\right)^{2/3}\left[8\kappa^{2/3} - 4\left(\kappa+\epsilon'\right)^{2/3}\right]}{8\left(\kappa+\epsilon'\right)^{2/3}\left[4\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3}\right]}\right\}\left(1-q_1\right)\right.$$

$$\left. - \left\{\frac{2\kappa^{2/3}\left[2\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3}\right] - \upsilon\left(\kappa+\epsilon'\right)^{4/3}}{2\left(\kappa+\epsilon'\right)^{2/3}\left[4\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3}\right]}\right\}\left(1-q_2\right)\right]\left[\left(\bar{u}^2 - \bar{v}^2\right)\sin\alpha\cos\alpha + \bar{u}\bar{v}\left(\cos^2\alpha - \sin^2\alpha\right)\right]$$

$$+ \frac{\left(\kappa+\epsilon'\right)^{5/3}}{\kappa^{2/3}}\left[\frac{\left[12\kappa^{2/3} - 3\left(\kappa+\epsilon'\right)^{2/3}\right]}{8\left(\kappa+\epsilon'\right)^{2/3}} + \frac{\left[36\kappa^{2/3} - 3\left(\kappa+\epsilon'\right)^{2/3}\right]}{16\left(\kappa+\epsilon'\right)^{2/3}}\left(\alpha_1 + \alpha_2\right)\right.$$

$$\left. + \left\{\frac{2\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3} - \upsilon\left[4\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3}\right]}{4\left(\kappa+\epsilon'\right)^{2/3}}\right\}\left(1-q_1\right) + \left\{\frac{-2\kappa^{2/3}\left(\kappa+\epsilon'\right)^{2/3} + \upsilon\left[4\kappa^{2/3} - \left(\kappa+\epsilon'\right)^{2/3}\right]}{4\left(\kappa+\epsilon'\right)^{2/3}}\right\}\right.$$

$$\left.\times\left(1-q_2\right)\left[\bar{u}^2\sin^2\alpha + \bar{v}^2\cos^2\alpha + 2\bar{u}\bar{v}\sin\alpha\cos\alpha\right]\right.$$

Now, we select $\alpha$ such that the terms $\bar{u}\bar{v}$ is set zero; we then use Binomial expansion and simplify, to get

$$\bar{\Omega} = \kappa\left[\frac{3}{2}+\frac{1}{2\kappa}\in' - \ 1-\upsilon \ \ 1-q_1 \ -\upsilon \ 1-q_2 \ +\left(\frac{5}{4}-\frac{\upsilon}{2}\right)\alpha_1+\left(\frac{3}{4}+\frac{\upsilon}{2}\right)\alpha_2\right]$$

$$+\kappa\left[\frac{3}{8}+\frac{5}{8\kappa}\in'+\left(\frac{27}{16}-\frac{3\upsilon}{2}\right)\alpha_1+\left(\frac{3}{16}+\frac{3\upsilon}{2}\right)\alpha_2+\left(-\frac{1}{4}+\frac{3\upsilon}{4}\right)\ 1-q_1 \ +\left(\frac{1}{2}-\frac{3\upsilon}{4}\right)\ 1-q_2 \ \right]$$

$$\times\left[\bar{u}^2\cos^2\alpha+\bar{v}^2\sin^2\alpha\right]+\kappa\sqrt{3}\left(1+\frac{4}{3\kappa}\in'\right)\left(1-\frac{1}{9\kappa}\in'\right)\left[\frac{3}{4}-\frac{3\upsilon}{2}+\frac{57-78\upsilon}{24\left(1-\frac{2}{9\kappa}\in'\right)}\right]\alpha_1$$

$$+\left\{\frac{21-78\upsilon}{24\left(1-\frac{2}{9\kappa}\in'\right)}\right\}\alpha_2+\left\{\frac{-4+4\upsilon}{24\left(1-\frac{2}{9\kappa}\in'\right)}\right\}(1-q_1)\left\{\frac{2-\upsilon}{6\left(1-\frac{2}{9\kappa}\in'\right)}\right\}(1-q_2)\left[(\bar{u}^2-\bar{v}^2)\sin\alpha\cos\alpha\right]$$

$$+\left(\kappa+\frac{5}{3}\in'\right)\left[\left(\frac{12-3}{8}-\frac{2}{\kappa}\in'\right)\left(1-\frac{2}{3\kappa}\in'\right)+\frac{33}{16}(\alpha_1+\alpha_2)\left(\frac{1}{4}-\frac{3\upsilon}{4}\right)(1-q_1)\left(-\frac{1}{2}+\frac{3\upsilon}{4}\right)(1-q_2)\right]$$

$$\times\left[\bar{u}^2\sin^2\alpha+\bar{v}^2\cos^2\alpha\right]$$

$$\bar{\Omega} = \kappa\left[\frac{3}{2}+\frac{1}{2\kappa}\in' - \ 1-\upsilon \ \ 1-q_1 \ -\upsilon \ 1-q_2 \ +\left(\frac{5}{4}-\frac{\upsilon}{2}\right)\alpha_1+\left(\frac{3}{4}+\frac{\upsilon}{2}\right)\alpha_2\right]$$

$$+\kappa\left[\frac{3}{8}+\frac{5}{8\kappa}\in'+\left(\frac{27}{16}-\frac{3\upsilon}{2}\right)\alpha_1+\left(\frac{3}{16}+\frac{3\upsilon}{2}\right)\alpha_2+\left(-\frac{1}{4}+\frac{3\upsilon}{4}\right)\ 1-q_1 \ +\left(\frac{1}{2}-\frac{3\upsilon}{4}\right)\ 1-q_2 \ \right]$$

$$\times\left[\bar{u}^2\cos^2\alpha+\bar{v}^2\sin^2\alpha\right]+\kappa\sqrt{3}\left(1+\frac{11}{9\kappa}\in'\right)\left[\frac{3}{4}-\frac{3\upsilon}{2}+\frac{57-78\upsilon\ \left(1-\frac{2}{9\kappa}\in'\right)}{24}\right]\alpha_1$$

$$+\left\{\frac{(21-78\upsilon)\left(1-\frac{2}{9\kappa}\in'\right)}{24}\right\}\alpha_2+\left\{\frac{(-4-4\upsilon)\left(1-\frac{2}{9\kappa}\in'\right)}{24}\right\}(1-q_1)\left\{\frac{(2-\upsilon)\left(1-\frac{2}{9\kappa}\in'\right)}{6}\right\}(1-q_2)$$

$$\times\left[(\bar{u}^2-\bar{v}^2)\sin\alpha\cos\alpha\right]+\left[\bar{u}^2\sin^2\alpha+\bar{v}^2\cos^2\alpha\right]\left(\kappa+\frac{5}{3}\in'\right)$$

$$\times\left[\left(\frac{12-3}{8}-\frac{2}{\kappa}\in'\right)\left(1-\frac{2}{3\kappa}\in'\right)+\frac{33}{16}(\alpha_1+\alpha_2)\left(\frac{1}{4}-\frac{3\upsilon}{4}\right)(1-q_1)\left(-\frac{1}{2}+\frac{3\upsilon}{4}\right)(1-q_2)\right]$$

Ignoring product of $\in'$, $1-q_i$ and $\alpha_i$; we get

$$\bar{\Omega} = \kappa\left[\frac{3}{2}+\frac{1}{2\kappa}\in' - (1-\upsilon)(1-q_1) - \upsilon(1-q_2) + \left(\frac{5}{4}-\frac{\upsilon}{2}\right)\alpha_1 + \left(\frac{3}{4}+\frac{\upsilon}{2}\right)\alpha_2\right]$$

$$+\kappa\left[\frac{3}{8}+\frac{5}{8\kappa}\in' + \left(\frac{27}{16}-\frac{3\upsilon}{2}\right)\alpha_1 + \left(\frac{3}{16}+\frac{3\upsilon}{2}\right)\alpha_2 + \left(-\frac{1}{4}+\frac{3\upsilon}{4}\right)(1-q_1) + \left(\frac{1}{2}-\frac{3\upsilon}{4}\right)(1-q_2)\right]$$

$$\times\left[\bar{u}^2\cos^2\alpha + \bar{v}^2\sin^2\alpha\right] + \kappa\sqrt{3}\left[\frac{3}{4}-\frac{3\upsilon}{2}+\frac{11}{12\kappa}\in' - \frac{11}{6\kappa}\upsilon\in' + \frac{19-26\upsilon}{8}\right]\alpha_1$$

$$+\frac{(7-26\upsilon)}{8}\alpha_2 - \frac{(1+\upsilon)}{6}(1-q_1) - \frac{(2-\upsilon)}{6}(1-q_2)\Big]\left[(\bar{u}^2-\bar{v}^2)\sin\alpha\cos\alpha\right]$$

$$+\kappa\left[\frac{9}{8}+\frac{7}{8\kappa}\in' + \frac{33}{16}(\alpha_1+\alpha_2) + \left(\frac{1}{4}-\frac{3\upsilon}{4}\right)(1-q_1) + \left(-\frac{1}{2}+\frac{3\upsilon}{4}\right)(1-q_2)\Big]\left[\bar{u}^2\sin^2\alpha + \bar{v}^2\cos^2\alpha\right]$$

Expanding, we get

$$\bar{\Omega} = \bar{u}^2\bar{r} + \bar{v}^2\bar{s} + \bar{t}$$

where

$$\bar{t} = \kappa\left[\frac{3}{2}+\frac{1}{2\kappa}\in' - (1-\upsilon)(1-q_1) - \upsilon(1-q_2) + \left(\frac{5}{4}-\frac{\upsilon}{2}\right)\alpha_1 + \left(\frac{3}{4}+\frac{\upsilon}{2}\right)\alpha_2\right]$$

$$\bar{r} = \kappa\left[\frac{3}{8}+\frac{3}{4}\sin^2\alpha - \frac{3\sqrt{3}}{8}(1-2\upsilon)\sin 2\alpha + \left\{\frac{5}{8\kappa}+\frac{1}{4\kappa}\cos^2\alpha + \frac{11}{24\kappa}\sqrt{3}(1-2\upsilon)\sin 2\alpha\right\}\in'\right.$$

$$+\left\{\left(-\frac{1}{4}+\frac{3\upsilon}{4}\right)\cos 2\alpha + \frac{\sqrt{3}}{12}(1+\upsilon)\sin 2\alpha\right\}(1-q_1) + \left\{\left(\frac{1}{2}-\frac{3\upsilon}{4}\right)\cos 2\alpha + \frac{\sqrt{3}}{12}(-2+\upsilon)\sin 2\alpha\right\}(1-q_2)$$

$$+\left\{\frac{27}{16}-\frac{3\upsilon}{2}+\left(\frac{3}{8}+\frac{3\upsilon}{2}\right)\sin^2\alpha - \frac{\sqrt{3}(19-26\upsilon)\sin 2\alpha}{16}\right\}\alpha_1 + \left\{\frac{3}{16}+\frac{3\upsilon}{2}+\left(\frac{15}{8}-\frac{3\upsilon}{2}\right)\sin^2\alpha\right.$$

$$\left.\left.-\frac{\sqrt{3}(19-26\upsilon)\sin 2\alpha}{16}\right\}\alpha_2\right]$$

$$\bar{s} = \kappa\left[\frac{3}{8}+\frac{3}{4}\cos^2\alpha+\frac{3\sqrt{3}}{8}\ 1-2\upsilon\ \sin 2\alpha+\left\{\frac{5}{8\kappa}+\frac{1}{4\kappa}\cos^2\alpha+\frac{11}{24\kappa}\sqrt{3}\ 1-2\upsilon\ \sin 2\alpha\right\}\in'\right.$$
$$+\left\{\left(\frac{1}{4}-\frac{3\upsilon}{4}\right)\cos 2\alpha-\frac{\sqrt{3}}{12}\ 1+\upsilon\ \sin 2\alpha\right\}\ 1-q_1\ +\left\{\left(-\frac{1}{2}+\frac{3\upsilon}{4}\right)\cos 2\alpha-\frac{\sqrt{3}}{12}\ -2+\upsilon\ \sin 2\alpha\right\}\ 1-q_2$$
$$+\left\{\frac{27}{16}-\frac{3\upsilon}{2}+\left(\frac{3}{8}+\frac{3\upsilon}{4}\right)\cos^2\alpha+\frac{\sqrt{3}\ 19-26\upsilon\ \sin 2\alpha}{16}\right\}\alpha_1+\left\{\frac{3}{16}+\frac{3\upsilon}{2}+\left(\frac{15}{8}-\frac{3\upsilon}{2}\right)\cos^2\alpha\right.$$
$$\left.-\frac{\sqrt{3}\left(19-26\upsilon\right)\sin 2\alpha}{16}\right\}\alpha_2\Bigg]$$

Now, setting the terms containing $uv$ to zero in equation (3.176), we subtract terms containing $u^2$ from those $v^2$ and add it to terms of $uv$ to get

$$\tan 2\alpha = \frac{4\ \psi+\kappa-1^{\ 1/3}\sqrt{N}}{3\left[2\kappa^{2/3}-\ \psi+\kappa-1^{\ 2/3}\right]}\left[\frac{3}{4}-\frac{3}{2}\upsilon+\left\{\frac{9\kappa^{4/3}-3\kappa^{2/3}\ \psi+\kappa-1^{\ \frac{2}{3}}}{B}+\frac{3\upsilon^2\ \psi+\kappa-1^{\ 2/3}}{\left[2\kappa^{2/3}-\ \psi+\kappa-1^{\ 2/3}\right]}\right.\right.$$
$$\left.-\frac{\upsilon\left[6\kappa^{4/3}+9\kappa^{2/3}\ \psi+\kappa-1^{\ 2/3}-3\ \psi+\kappa-1^{\ 4/3}\right]}{B}\right\}\alpha_1+\left\{\frac{-3\kappa^{4/3}}{B}-\frac{3\upsilon^2\ \psi+\kappa-1^{\ 2/3}}{\left[2\kappa^{2/3}-\ \psi+\kappa-1^{\ 2/3}\right]}\right.$$
$$\left.+\frac{\upsilon\left[-6\kappa^{4/3}+15\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}-3\left(\psi+\kappa-1\right)^{4/3}\right]}{B}\right\}\alpha_2+\left\{\frac{\left[2\kappa^2-11\kappa^{4/3}+5\kappa^{2/3}\left(\psi+\kappa-1\right)^{4/3}\right]}{2B\left(\psi+\kappa-1\right)^{2/3}}\right. \quad (3.178)$$
$$\left.+\frac{\upsilon\left[28\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}-16\kappa^{2/3}\left(\psi+\kappa-1\right)^{4/3}+2\left(\psi+\kappa-1\right)^{2}\right]}{2B\left(\psi+\kappa-1\right)^{2/3}}-\frac{\upsilon^2\left[4\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right]}{\left[2\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right]}\right\}\left(1-q_1\right)$$
$$+\left\{\frac{\left[8\kappa^2-\left(\psi+\kappa-1\right)^{2}-4\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}+5\kappa^{2/3}\left(\psi+\kappa-1\right)^{4/3}\right]}{2B\left(\psi+\kappa-1\right)^{2/3}}-\frac{\upsilon^2\left[4\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right]}{\left[2\kappa^{2/3}-\left(\psi+\kappa-1\right)^{2/3}\right]}\right.$$
$$\left.\left.+\frac{\upsilon\left[24\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}-2\left(\psi+\kappa-1\right)^{2}\right]}{2B\left(\psi+\kappa-1\right)^{2/3}}\right\}\left(1-q_2\right)\right]$$

where $B = 8\kappa^{4/3}-6\kappa^{2/3}\ \psi+\kappa-1^{\ 2/3}+\ \psi+\kappa-1^{\ 4/3}$

If we substitute $\psi = 1 + \in'$; factorizing and ignoring higher order terms in $\in'$, $1-q_i$ and $\alpha_i$, and their products, we get

$$\tan 2\alpha = \frac{4\sqrt{3}}{3}\left[\frac{3}{4} - \frac{3\upsilon}{2} + \frac{2\in'}{3\kappa} - \frac{4\upsilon\in'}{3\kappa} + \left(2 - 4\upsilon + 3\upsilon^2\right)\alpha_1 - \left(1 - 2\upsilon + 3\upsilon^2\right)\alpha_2 - \left(+\frac{2}{3} - \frac{7\upsilon}{3} + 3\upsilon^2\right)1 - q_1 + \left(\frac{4}{3} - \frac{11}{3}\upsilon - 3\upsilon^2\right)1 - q_2\right] \quad (3.179)$$

## 3.5.2 Eccentricities of the orbits

The associated characteristic equation when the potential in equation (3.176) admit the Jacobi integral $J = 2\Omega$, is

$$\lambda^2 - \left(\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta}\right)\lambda + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 = 0 \quad (3.180)$$

where

$$\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} = \frac{3\left(\psi + \kappa - 1\right)^{\frac{5}{3}}}{\kappa^{2/3}}\left[\frac{\kappa^{2/3}}{\left(\psi + \kappa - 1\right)^{\frac{2}{3}}} + \frac{3\kappa^{\frac{2}{3}} + 2\left(\psi + \kappa - 1\right)^{\frac{2}{3}}}{2\left(\psi + \kappa - 1\right)^{2/3}}\alpha_1 + \frac{3\kappa^{2/3}\alpha_2}{2\left(\psi + \kappa - 1\right)^{\frac{2}{3}}} - \upsilon\left(\alpha_1 - \alpha_2\right)\right]$$

$$\Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \left(\Omega^0_{\xi\eta}\right)^2 = \frac{9\left(\psi + \kappa - 1\right)^{\frac{8}{3}}}{16\kappa^{4/3}}\left[4N\upsilon\left(1 - \upsilon\right) + \frac{8\upsilon\left(1 - q_1\right)}{3\left(\psi + \kappa - 1\right)^{2/3}}\left(5\kappa^{4/3} - 3\kappa^{2/3}\left(\psi + \kappa - 1\right)^{2/3}\right.\right.$$
$$\left. - \left(\psi + \kappa - 1\right)^{4/3}\right) - \frac{8\upsilon^2\left(1 - q_1\right)}{3}\left(2\kappa^{2/3} - \left(\psi + \kappa - 1\right)^{2/3}\right) + \frac{8\kappa^{2/3}\upsilon\left(1 - q_2\right)}{3} - \frac{8\upsilon^2\left(\psi + \kappa - 1\right)^{2/3}\left(1 - q_2\right)}{3}$$
$$\left. + 4\upsilon\left(1 - \upsilon\right)\left(16\kappa^{2/3} - 3\left(\psi + \kappa - 1\right)^{2/3}\right)\left(\alpha_1 + \alpha_2\right)\right]$$

Substituting these in the characteristic equation (3.180), and simplifying, we get

$$\lambda^2-\left[3\ \psi+\kappa-1\ +\frac{\psi+\kappa-1\ \ 9\kappa^{2/3}+\ \psi+\kappa-1\ ^{2/3}\ \alpha_1}{2\kappa^{2/3}}+\frac{9\ \psi+\kappa-1\ \alpha_2}{2}-3\upsilon\ \alpha_1-\alpha_2\right.$$

$$\left.\times\frac{\psi+\kappa-1\ ^{\frac{5}{3}}}{2\kappa^{2/3}}\right]\lambda+\frac{9\ \psi+\kappa-1\ ^{8/3}}{16\kappa^{4/3}}\left[4N\upsilon\ 1-\upsilon\ +4\upsilon\ 1-\upsilon\ \ 16\kappa^{2/3}-3\ \psi+\kappa-1\ ^{2/3}\ \ \alpha_1+\alpha_2\right.$$

$$+\frac{8\upsilon}{3}\left\{\frac{5\kappa^{4/3}-3\kappa^{2/3}\ \ \psi+\kappa-1\ ^{2/3}-\ \psi+\kappa-1\ ^{4/3}}{\psi+\kappa-1\ ^{2/3}}-\upsilon\ \ 2\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right\}(1-q_1)$$

$$\left.+\frac{8\upsilon}{3}\left\{\kappa^{2/3}-\upsilon(\psi+\kappa-1)^{2/3}\right\}(1-q_2)\right]=0 \qquad (3.181)$$

Its roots are

$$\lambda_1=3\ \psi+\kappa-1\ -\frac{3N\ \ \psi+\kappa-1\ ^{5/3}\upsilon\ 1-\upsilon}{4\kappa^{4/3}}+\left\{\frac{18\kappa^{4/3}\ \ \psi+\kappa-1\ +12\kappa^{2/3}\ \ \psi+\kappa-1\ ^{5/3}}{4\kappa^{4/3}}\right.$$

$$\left.-\frac{\upsilon\left[66\kappa^{2/3}\ \ \psi+\kappa-1\ ^{5/3}-9\ \ \psi+\kappa-1\ ^{7/3}\right]}{4\kappa^{4/3}}+\frac{\upsilon^2\left[48\kappa^{2/3}\ \ \psi+\kappa-1\ ^{5/3}-9\ \ \psi+\kappa-1\ ^{7/3}\right]}{4\kappa^{4/3}}\right\}\alpha_1$$

$$+\left\{\frac{9\ \psi+\kappa-1}{2}-\frac{\upsilon\left[36\kappa^{2/3}\ \ \psi+\kappa-1\ ^{5/3}-9\ \ \psi+\kappa-1\ ^{7/3}\right]}{4\kappa^{4/3}}+\left[48\kappa^{2/3}(\psi+\kappa-1)^{5/3}-9(\psi+\kappa-1)^{7/3}\right]\right.$$

$$\left.\times\frac{\upsilon^2}{4\kappa^{4/3}}\right\}\alpha_2-\frac{(\psi+\kappa-1)\upsilon(1-q_1)}{2\kappa^{4/3}}\left\{5\kappa^{4/3}-3\kappa^{2/3}(\psi+\kappa-1)^{2/3}-(\psi+\kappa-1)^{4/3}-\upsilon\left[2\kappa^{2/3}(\psi+\kappa-1)^{2/3}\right.\right.$$

$$\left.\left.-(\psi+\kappa-1)^{4/3}\right]\right\}\frac{(\psi+\kappa-1)^{5/3}\upsilon(1-q_2)}{2\kappa^{4/3}}\left\{\kappa^{2/3}-\upsilon(\psi+\kappa-1)^{2/3}\right\} \qquad (3.182)$$

$$\lambda_2=\upsilon\ 1-\upsilon\ \left\{\frac{3N\ \ \psi+\kappa-1\ ^{5/3}}{4\kappa^{4/3}}+\frac{\left[48\kappa^{2/3}\ \ \psi+\kappa-1\ ^{5/3}-9\ \ \psi+\kappa-1\ ^{7/3}\right]\ \alpha_1+\alpha_2}{4\kappa^{4/3}}\right\}+\frac{\psi+\kappa-1}{2\kappa^{4/3}}$$

$$\times\left\{5\kappa^{\frac{4}{3}}-3\kappa^{2/3}\ \ \psi+\kappa-1\ ^{2/3}-\ \psi+\kappa-1\ ^{4/3}-\upsilon\left[2\kappa^{\frac{2}{3}}\ \ \psi+\kappa-1\ ^{2/3}-\ \psi+\kappa-1\ ^{\frac{4}{3}}\right]\right\}1-q_1$$

$$+\frac{\psi+\kappa-1\ ^{5/3}\ \ \kappa^{2/3}-\upsilon\ \ \psi+\kappa-1\ ^{2/3}\ \ \upsilon}{2\kappa^{4/3}}\ 1-q_2 \qquad (3.183)$$

With the help of the first root above, the eccentricities of the orbits can then be obtained following Szebehely (1967a):

$$e_i = \sqrt{1-\gamma_i^2} \quad i = 1,2 \tag{3.184}$$

where $\gamma_i^2 = 4s_i^2 \left(s_i^2 + \lambda_i\right)^{-2}$

Here, $s_i \ \ i = 1,2$ are given in equations (3.168) and (3.169), respectively, while $\lambda_i$ are the roots of the characteristic equation (3.180). Now, going through some rigorous calculations, we get

$$\begin{aligned}
s_1^2 + \lambda_1^{-2} &= \frac{1}{9\left(\psi+\kappa-1\right)^2}\left\{1 - \frac{N\upsilon\left(1-\upsilon\right)\left[3\left(\psi+\kappa-1\right)-2\varphi^2\right]\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]} - \left[3 + \frac{2\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{2/3}}\right.\right. \\
&+ \frac{\upsilon\left[54\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3} - 9\left(\psi+\kappa-1\right)^{7/3} - 40\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} + 6\varphi^2\left(\psi+\kappa-1\right)^{4/3}\right]}{\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]} \\
&\left.\left. + \frac{\upsilon^2\left[9\left(\psi+\kappa-1\right)^{7/3} - 48\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3} + 32\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} - 6\varphi^2\left(\psi+\kappa-1\right)^{4/3}\right]}{\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]}\right]\alpha_1\right. \\
&- \left[3 + \frac{\upsilon\left[42\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3} - 9\left(\psi+\kappa-1\right)^{7/3} - 24\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} + 6\varphi^2\left(\psi+\kappa-1\right)^{4/3}\right]}{\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]}\right. \\
&\left. + \frac{\upsilon^2\left[9\left(\psi+\kappa-1\right)^{7/3} - 48\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3} + 32\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} - 6\varphi^2\left(\psi+\kappa-1\right)^{4/3}\right]}{4\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]}\right]\alpha_2 \\
&- \frac{2\upsilon\left(1-q_1\right)}{3\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]}\left[15\kappa^{4/3}\left(\psi+\kappa-1\right) - 9\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3} - 3\left(\psi+\kappa-1\right)^{7/3}\right. \\
&+ 2\varphi^2\left(\psi+\kappa-1\right)^{4/3} + 6\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} - 10\varphi^2\kappa^{4/3} - \upsilon\left\{8\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3} - 3\left(\psi+\kappa-1\right)^{7/3}\right. \\
&\left.\left. - 4\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} + 2\varphi^2\left(\psi+\kappa-1\right)^{4/3}\right\} - \frac{2\upsilon\left(1-q_2\right)}{3\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]}\left[3\kappa^{2/3}\left(\psi+\kappa-1\right)\right. \\
&\left.\left. - 2\varphi^2\kappa^{2/3} - \upsilon\left\{\left(\psi+\kappa-1\right)^{2/3}\left(\psi+\kappa-1\right) - 2\varphi^2\right\}\right]\right\}
\end{aligned} \tag{3.185}$$

$$s_2^2+\lambda_1^{-2}=\frac{1}{16\varphi^4}\left\{1+\frac{3N\upsilon\ 1-\upsilon\ \ \psi+\kappa-1^{5/3}}{2\kappa^{4/3}}-\left[3-\frac{\upsilon\left[48\kappa^{2/3}\ \psi+\kappa-1^{5/3}-9\ \psi+\kappa-1^{7/3}\right]}{2\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]}\right.\right.$$

$$\left.\left.+\frac{\upsilon^2\left[48\kappa^{2/3}\ \psi+\kappa-1^{5/3}-9\ \psi+\kappa-1^{7/3}\right]}{2\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]}\right]\alpha_1+\alpha_2\ +\frac{\upsilon\ \psi+\kappa-1\ \ 1-q_1}{\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]}\left[5\kappa^{4/3}\ \psi+\kappa-1\right.\right.$$

$$\left.\left.-3\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}-\left(\psi+\kappa-1\right)^{4/3}-\upsilon\left\{2\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}-\left(\psi+\kappa-1\right)^{4/3}\right\}\right]\right.$$

$$\left.+\frac{\upsilon\left(\psi+\kappa-1\right)^{5/3}\left(1-q_2\right)}{\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]}\left[\kappa^{2/3}-\upsilon\left(\psi+\kappa-1\right)^{2/3}\right]\right\} \qquad (3.186)$$

Substituting equations (3.168) and (3.185) in (3.184) and simplifying, we get

$$\gamma_1^2=\frac{N\upsilon\ \psi+\kappa-1^{2/3}}{\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}-\frac{N\upsilon^2\ \psi+\kappa-1^{2/3}}{\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\left[1+\frac{N\ \psi+\kappa-1^{2/3}\left[3\ \psi+\kappa-1\ -2\varphi^2\right]}{\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\right]$$

$$-\left[\frac{\left[3N\kappa^{2/3}\ \psi+\kappa-1^{2/3}+2N\ \psi+\kappa-1^{4/3}-16\kappa^{4/3}\ \psi+\kappa-1^{2/3}+3\kappa^{2/3}\ \psi+\kappa-1^{4/3}\right]\upsilon}{4\kappa^2\ \ 4\varphi^2-3\ \psi+\kappa-1}\right.$$

$$+\frac{\upsilon^2}{\kappa^{8/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\left\{108N\kappa^{2/3}\left(\psi+\kappa-1\right)^{7/3}-18N\left(\psi+\kappa-1\right)^3-80N\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{4/3}\right.$$

$$+12N\varphi^2\left(\psi+\kappa-1\right)^2-12N\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}+9N\kappa^{4/3}\left(\psi+\kappa-1\right)^{5/3}+64\varphi^2\kappa^2\left(\psi+\kappa-1\right)^{2/3}$$

$$\left.\left.-48\kappa^2\left(\psi+\kappa-1\right)^{5/3}-12\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{4/3}+9\kappa^{4/3}\left(\psi+\kappa-1\right)^{7/3}\right\}\right]\alpha_1$$

$$-\left[\frac{\left[3N\left(\psi+\kappa-1\right)^{2/3}-16\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}+3\left(\psi+\kappa-1\right)^{4/3}\right]\upsilon}{4\kappa^{4/3}\ \ 4\varphi^2-3\left(\psi+\kappa-1\right)}+\frac{\upsilon^2}{\kappa^{8/3}\ \ 4\varphi^2-3\left(\psi+\kappa-1\right)}\right.$$

$$\times\left\{90N\kappa^{2/3}\left(\psi+\kappa-1\right)^{7/3}-18N\left(\psi+\kappa-1\right)^3-56N\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{4/3}+12N\varphi^2\left(\psi+\kappa-1\right)^2\right.$$

$$-12N\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}+9N\kappa^{4/3}\left(\psi+\kappa-1\right)^{5/3}+64\varphi^2\kappa^2\left(\psi+\kappa-1\right)^{2/3}-48\kappa^2\left(\psi+\kappa-1\right)^{5/3}$$

$$\left.\left.-12\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{4/3}+9\kappa^{4/3}\left(\psi+\kappa-1\right)^{7/3}\right\}\right]\alpha_2$$

$$+\frac{2\ 1-q_1}{3\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\left\{\left[5\kappa^{4/3}-3\kappa^{2/3}\ \ \psi+\kappa-1\ ^{2/3}-\ \psi+\kappa-1\ ^{4/3}\right]\upsilon-\upsilon^2\left[2\kappa^{2/3}\ \ \psi+\kappa-1\ ^{2/3}\right.\right.$$

$$\left.-\ \psi+\kappa-1\ ^{4/3}+\frac{N}{\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\ 15\kappa^{4/3}\ \ \psi+\kappa-1\ -10\varphi^2\kappa^{4/3}-9\kappa^{2/3}\ \ \psi+\kappa-1\ ^{5/3}\right.$$

$$+6\varphi^2\kappa^{2/3}\ \ \psi+\kappa-1\ ^{2/3}-3\ \psi+\kappa-1\ ^{7/3}+2\varphi^2\ \ \psi+\kappa-1\ ^{2}+6\varphi^2\kappa^{2/3}\ \ \psi+\kappa-1\ ^{4/3}-3\ \psi+\kappa-1\ ^{3}$$

$$\left.\left.-10\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}+15\kappa^{4/3}\left(\psi+\kappa-1\right)^{5/3}+9\kappa^{2/3}\left(\psi+\kappa-1\right)^{7/3}\right\}\right]\right\}+\frac{2\upsilon\left(\psi+\kappa-1\right)^{2/3}\left(-q_2\right)}{3\kappa^{2/3}\ \left\{4\varphi^2-3\left(\psi+\kappa-1\right)\right\}}$$

$$\times\left[1-\upsilon\left\{\frac{\left(\psi+\kappa-1\right)^{2/3}}{\kappa^{2/3}}+\frac{2N\left(\psi+\kappa-1\right)^{2/3}\left[3\left(\psi+\kappa-1\right)-2\varphi^2\right]}{\kappa^{4/3}\ \left\{4\varphi^2-3\left(\psi+\kappa-1\right)\right\}}\right\}\right] \qquad (3.187)$$

Now, substituting equation (3.187) in equation (3.184) for $i=1$ and simplifying, we get

$$e_1=1-\frac{N\upsilon\ \ \psi+\kappa-1\ ^{2/3}}{2\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}+\frac{N\upsilon^2\ \ \psi+\kappa-1\ ^{2/3}}{2\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\left[1+\frac{N\ \ \psi+\kappa-1\ ^{2/3}\left[3\ \psi+\kappa-1\ -2\varphi^2\right]}{\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\right]$$

$$+\ A_{11}\upsilon+A_{12}\upsilon^2\ \ \alpha_1+\ B_{11}\upsilon+B_{12}\upsilon^2\ \ \alpha_2+\ C_{11}\upsilon-C_{12}\upsilon^2\ \ 1-q_1\ -\ D_{11}\upsilon+D_{12}\upsilon^2\ \ 1-q_2 \qquad (3.188)$$

where

$$A_{11}=\frac{\psi+\kappa-1\ ^{2/3}}{\kappa^2\ \ 4\varphi^2-3\ \psi+\kappa-1}\left[4\kappa^{2/3}\ \ \psi+\kappa-1\ ^{2/3}-2\kappa^{4/3}-\ \psi+\kappa-1\ ^{4/3}\right]$$

$$A_{12}=\frac{\psi+\kappa-1\ ^{2/3}}{\kappa^{8/3}\ \ 4\varphi^2-3\ \psi+\kappa-1\ ^{2}}\left[216\kappa^{4/3}\ \ \psi+\kappa-1\ ^{5/3}+9\ \psi+\kappa-1\ ^{3}-90\kappa^{2/3}\ \ \psi+\kappa-1\ ^{7/3}\right.$$

$$\left.-160\varphi^2\kappa^{4/3}\ \ \psi+\kappa-1\ ^{2/3}+64\varphi^2\kappa^{2/3}\ \ \psi+\kappa-1\ ^{4/3}-6\varphi^2\ \ \psi+\kappa-1\ ^{2}+8\kappa^2\varphi^2-6\kappa^2\ \ \psi+\kappa-1\ \right]$$

$$B_{11}=-\frac{2\ \psi+\kappa-1\ ^{2/3}}{\kappa^{2/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}$$

$$B_{12}=\frac{\psi+\kappa-1^{\,2/3}}{\kappa^{8/3}\ \ 4\varphi^2-3\ \ \psi+\kappa-1^{\,2}}\Big[180\kappa^{4/3}\ \ \psi+\kappa-1^{\,5/3}-81\kappa^{2/3}\ \ \psi+\kappa-1^{\,7/3}+9\ \ \psi+\kappa-1^{\,3}$$

$$-112\kappa^{4/3}\ \ \psi+\kappa-1^{\,2/3}+52\varphi^2\kappa^{2/3}\ \ \psi+\kappa-1^{\,4/3}-6\varphi^2\ \ \psi+\kappa-1^{\,2}+8\kappa^2\varphi^2-6\kappa^2\ \ \psi+\kappa-1\ \Big]$$

$$C_{11}=\frac{5\kappa^{4/3}-3\kappa^{2/3}\ \ \psi+\kappa-1^{\,2/3}-\ \ \psi+\kappa-1^{\,4/3}}{3\kappa^{4/3}\ \ 4\varphi^2-3\ \ \psi+\kappa-1}$$

$$C_{12}=\frac{1}{3\kappa^{8/3}\ \ 4\varphi^2-3\ \ \psi+\kappa-1}^{\,2}\Big[2\kappa^{2/3}\ \ \psi+\kappa-1^{\,2/3}-\ \ \psi+\kappa-1^{\,4/3}+60\kappa^2\ \ \psi+\kappa-1\ -40\kappa^2\varphi^2$$

$$-51\kappa^{4/3}\ \ \psi+\kappa-1^{\,5/3}+34\varphi^2\kappa^{4/3}\ \ \psi+\kappa-1^{\,2/3}-3\kappa^{2/3}\ \ \psi+\kappa-1^{\,7/3}+2\varphi^2\kappa^{2/3}\ \ \psi+\kappa-1^{\,4/3}$$

$$+2\varphi^2\kappa^{2/3}\ \ \psi+\kappa-1^{\,2}+34\varphi^2\kappa^{4/3}\ \ \psi+\kappa-1^{\,4/3}-40\varphi^2\kappa^2\ \ \psi+\kappa-1^{\,2/3}-3\kappa^{2/3}\ \ \psi+\kappa-1^{\,3}$$

$$+60\kappa^2\ \ \psi+\kappa-1^{\,5/3}-51\kappa^{4/3}\left(\psi+\kappa-1\right)^{7/3}+3\left(\psi+\kappa-1\right)^{3}-2\varphi^2\left(\psi+\kappa-1\right)^{2}-2\varphi^2\left(\psi+\kappa-1\right)^{8/3}$$

$$+3\left(\psi+\kappa-1\right)^{11/3}\Big]$$

$$D_{11}=\frac{\psi+\kappa-1^{\,2/3}}{3\kappa^{2/3}\ \ 4\varphi^2-3\ \ \psi+\kappa-1}$$

$$D_{12}=\frac{\psi+\kappa-1^{\,4/3}}{3\kappa^{2}\ \ 4\varphi^2-3\ \ \psi+\kappa-1}^{\,2}\Big[21\kappa^{2/3}\ \ \psi+\kappa-1\ -12\varphi^2\kappa^{2/3}-6\ \ \psi+\kappa-1^{\,5/3}+4\varphi^2\ \ \psi+\kappa-1^{\,2/3}\Big]$$

Similarly to get the eccentricity of the short period orbits, we get

$$\gamma_2^2 = \frac{1}{4\varphi^4}\left[ 4\varphi^2 - 3\ (\psi+\kappa-1\ - \frac{3N\upsilon\ (\psi+\kappa-1)^{5/3}}{4\kappa^{4/3}}\left\{ \frac{3\ (\psi+\kappa-1\ - 2\left[4\varphi^2 - 3\ (\psi+\kappa-1\right]^2}{4\varphi^2 - 3\ (\psi+\kappa-1} \right\} \right. \quad 3.189$$

$$+ \frac{3N\upsilon^2\ (\psi+\kappa-1)^{5/3}}{8\kappa^{8/3}\left[4\varphi^2 - 3\ (\psi+\kappa-1\right]}\ 6\kappa^{4/3}\ (\psi+\kappa-1\ - 4\kappa^{4/3}\left[4\varphi^2 - 3\ (\psi+\kappa-1\right]^2 - 9N\ (\psi+\kappa-1$$

$$+\alpha_1\left\{ \frac{-12\varphi^2\kappa^{2/3} + 9\kappa^{2/3}\ (\psi+\kappa-1\ - 6\ (\psi+\kappa-1)^{5/3}}{2\kappa^{2/3}} + A_{21}\upsilon + A_{22}\upsilon^2 \right\} + \alpha_2\left\{ \frac{-12\varphi^2 + 9\left(\psi+\kappa-1\right)}{2} \right.$$

$$\left. +B_{21}\upsilon + B_{22}\upsilon^2\ \right\} + \left(C_{21}\upsilon + C_{22}\upsilon^2\ \right)\left(-q_1\right) + \left(D_{21}\upsilon + D_{22}\upsilon^2\ \right)\left(-q_2\right)$$

where

$$A_{21} = \frac{9\ (\psi+\kappa-1)^{2/3}}{4\kappa^2\ \ 4\varphi^2 - 3\ (\psi+\kappa-1}\left[ 9\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} - 52\kappa^{4/3}\ (\psi+\kappa-1\ - 8\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} \right.$$

$$+48\kappa^{4/3}\varphi^2 + 3N\kappa^{2/3}\ (\psi+\kappa-1\ + 16\varphi^4 N\kappa^{2/3} - 8N\varphi^2\ (\psi+\kappa-1)^{5/3} + 9N\kappa^{2/3}\ (\psi+\kappa-1)^2$$

$$\left. +6N\ (\psi+\kappa-1)^{8/3} - 24N\varphi^2\kappa^{2/3}\ (\psi+\kappa-1\right]$$

$$A_{22} = \frac{3\ (\psi+\kappa-1)^{5/3}}{4\kappa^{8/3}\ \ 4\varphi^2 - 3\ (\psi+\kappa-1}\left[ 27N\ (\psi+\kappa-1)^{10/3} + 48N\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} + 144\kappa^2\ (\psi+\kappa-1\right.$$

$$-9N\kappa^{4/3}\ (\psi+\kappa-1\ - 180N\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 27\kappa^{4/3}\ (\psi+\kappa-1)^{5/3} + 72N\varphi^2\kappa^{4/3}\ (\psi+\kappa-1$$

$$\left. -48N\kappa^{4/3}\varphi^4 - 128\varphi^2\kappa^2 + 24\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{2/3} - 27N\kappa^{4/3}\ (\psi+\kappa-1)^2\right]$$

$$B_{21} = \frac{3\ (\psi+\kappa-1)^{2/3}}{4\kappa^{4/3}\ \ 4\varphi^2 - 3\ (\psi+\kappa-1}\left[ 27\ (\psi+\kappa-1)^{8/3} + 27N\ (\psi+\kappa-1)^3 + 112\varphi^2\kappa^{2/3}\ (\psi+\kappa-1\right.$$

$$-36\kappa^{2/3} + 9N\ (\psi+\kappa-1)^2 + 48\varphi^2 N\ (\psi+\kappa-1\ - 72\varphi^2 N\ (\psi+\kappa-1)^2 - 24\varphi^2\ (\psi+\kappa-1)^{5/3}$$

$$\left. -96\kappa^{2/3}\ (\psi+\kappa-1)^2\right]$$

$$B_{22}=\frac{3\ (\psi+\kappa-1)^{5/3}}{8\kappa^{8/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)}\Big[54N\ (\psi+\kappa-1)^{10/3}-252N\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}+288\kappa^2\ (\psi+\kappa-1)$$
$$-48N\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}-54\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}-18N\kappa^{2/3}\ (\psi+\kappa-1)\ -96N\varphi^4\kappa^{4/3}-256\varphi^2\kappa^2$$
$$+144N\kappa^{4/3}\varphi^2\ (\psi+\kappa-1)\ +48\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}-54N\kappa^{4/3}\ (\psi+\kappa-1)^2\Big]$$

$$C_{21}=\frac{\psi+\kappa-1}{2\kappa^{4/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)}\Big[27\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}+9\ (\psi+\kappa-1)^{7/3}-45\kappa^{4/3}\ (\psi+\kappa-1)$$
$$+40\varphi^2\kappa^{4/3}-24\kappa^{2/3}\varphi^2\ (\psi+\kappa-1)^{2/3}-8\varphi^2\ (\psi+\kappa-1)^{4/3}\Big]$$

$$C_{22}=\frac{1}{4\kappa^{8/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)^2}\Big[27N\kappa^{2/3}\ (\psi+\kappa-1)^{13/3}+9N\ (\psi+\kappa-1)^5-45N\kappa^{4/3}\ (\psi+\kappa-1)^{11/3}$$
$$-128\kappa^2\varphi^4\ (\psi+\kappa-1)^{5/3}+64\kappa^{4/3}\varphi^4\ (\psi+\kappa-1)^{7/3}+240\kappa^2\varphi^2\ (\psi+\kappa-1)^{8/3}-108\kappa^2\ (\psi+\kappa-1)^{11/3}$$
$$-120\kappa^{4/2}\varphi^2\ (\psi+\kappa-1)^{10/3}+54\kappa^{4/3}\ (\psi+\kappa-1)^{13/3}\Big]$$

$$D_{21}=\frac{(\psi+\kappa-1)^{5/3}\ (8\varphi^2-9)\ (\psi+\kappa-1)}{2\kappa^{2/3}\ (4\varphi^2-3)\ (\psi+\kappa-1)}$$

$$D_{22}=\frac{(\psi+\kappa-1)^{7/3}}{4\kappa^2\ (4\varphi^2-3)\ (\psi+\kappa-1)^2}\Big[120\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)\ -64\varphi^4\kappa^{2/3}-9N\ (\psi+\kappa-1)^2$$
$$-54\kappa^{2/3}\ (\psi+\kappa-1)^2\Big]$$

Therefore,

$$1-\gamma_2^2 = 1-\frac{4\varphi^2-3\ \psi+\kappa-1}{4\varphi^4}+\frac{3N\upsilon\ \psi+\kappa-1^{\ 5/3}}{16\varphi^4\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]}\ 3\ \psi+\kappa-1\ +48\varphi^2\ \psi+\kappa-1$$

$$-32\varphi^4-18\ \psi+\kappa-1^{\ 2}\ -\frac{3N\upsilon^2\ \psi+\kappa-1^{\ 5/3}}{32\varphi^4\kappa^{8/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]}\ 6\kappa^{4/3}\ \psi+\kappa-1\ -36\kappa^{4/3}\ \psi+\kappa-1^{\ 2}$$

$$-64\varphi^4\kappa^{\frac{4}{3}}+96\varphi^2\kappa^{\frac{4}{3}}\ \psi+\kappa-1\ \Big\}-\left[\frac{9\kappa^{2/3}\ \psi+\kappa-1\ -12\varphi^2\kappa^{2/3}-6\ \psi+\kappa-1^{\ 5/3}}{8\varphi^4\kappa^{2/3}}+\frac{A_{21}\upsilon}{4\varphi^4}+\frac{A_{22}\upsilon^2}{4\varphi^4}\right]\alpha_1$$

$$+\left[\frac{9\left(\psi+\kappa-1\right)12\varphi^2}{8\varphi^4}+\frac{B_{21}\upsilon}{4\varphi^4}+\frac{B_{22}\upsilon^2}{4\varphi^4}\right]\alpha_2-\frac{\left(C_{21}\upsilon+C_{22}\upsilon^2\right)\left(1-q_1\right)}{4\varphi^4}-\frac{\left(D_{21}\upsilon+D_{22}\upsilon^2\right)\left(1-q_2\right)}{4\varphi^4} \quad (3.190)$$

Substituting $N=4\kappa^{4/3}-\ \psi+\kappa-1^{\ 2/3}$ in equation (3.190) and using the result in equation (3.184) for $i=2$, we simplify to get the eccentricity of the short period orbits:

$$e_2=\frac{\sqrt{4\varphi^2\ \varphi^2-1\ +3\ \psi+\kappa-1}}{2\varphi^2}\left[1+\frac{1}{\left[4\varphi^2\ \varphi^2-1\ +3\ \psi+\kappa-1\right]}\left\{K_1\upsilon-K_2\upsilon^2-\left[K_3+\frac{A_{21}\upsilon}{2}\right.\right.\right.$$

$$\left.\left.\left.+\frac{A_{22}\upsilon}{2}\right]\alpha_1-\left[K_4+\frac{B_{21}\upsilon}{2}+\frac{B_{22}\upsilon^2}{2}\right]\alpha_2-\frac{1}{2}\ C_{21}\upsilon+C_{22}\upsilon^2\ \ 1-q_1\ -\frac{1}{2}\ D_{21}\upsilon+D_{22}\upsilon^2\ \ 1-q_2\ \right\}\right] \quad 3.191$$

where

$$K_1=\frac{3\ \psi+\kappa-1^{\ 5/3}}{8\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\Big[12\kappa^{2/3}\ \psi+\kappa-1\ +192\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ -72\kappa^{2/3}\ \psi+\kappa-1^{\ 2}$$

$$-128\varphi^4\kappa^{2/3}-3\ \psi+\kappa-1^{\ 5/3}+32\varphi^4\ \psi+\kappa-1^{\ 2/3}-48\varphi^2\ \psi+\kappa-1^{\ 5/3}+18\ \psi+\kappa-1^{\ 8/3}\Big]$$

$$K_2=\frac{3\ \psi+\kappa-1^{\ 5/3}}{16\kappa^{4/3}\ \ 4\varphi^2-3\ \psi+\kappa-1}\Big[24\kappa^{2/3}\ \psi+\kappa-1\ -144\kappa^{2/3}\ \psi+\kappa-1^{\ 2}+384\varphi^2\kappa^{2/3}\ \psi+\kappa-1$$

$$-256\varphi^4\kappa^{2/3}-144\ \psi+\kappa-1\ +72\kappa^{-2/3}\ \psi+\kappa-1^{\ 5/3}-6\ \psi+\kappa-1^{\ 5/3}+36\ \psi+\kappa-1^{\ 8/3}$$

$$+64\varphi^4\ \psi+\kappa-1^{\ 2/3}-96\varphi^2\ \psi+\kappa-1^{\ 5/3}-9\kappa^{-4/3}\ \psi+\kappa-1^{\ 7/3}\Big]$$

$$A_{21}=\frac{9\ (\psi+\kappa-1)^{5/3}}{4\kappa^{2}\ (4\varphi^{2}-3)\ (\psi+\kappa-1)}\Big[64\varphi^{4}\kappa^{4/3}+48\varphi^{2}\kappa^{4/3}+6\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}-40\kappa^{4/3}\ (\psi+\kappa-1)$$
$$-8\varphi^{2}\kappa^{2}\ (\psi+\kappa-1)^{2/3}-8\varphi^{2}\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}+36\kappa^{4/3}\ (\psi+\kappa-1)^{2}+15\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}$$
$$-96\varphi^{2}\kappa^{4/3}\ (\psi+\kappa-1)-16\varphi^{4}\kappa^{2/3}\ (\psi+\kappa-1)^{2/3}+8\varphi^{2}\ (\psi+\kappa-1)^{7/3}-6\ (\psi+\kappa-1)^{10/3}\Big]$$

$$A_{22}=\frac{3\ (\psi+\kappa-1)^{2/3}}{4\kappa^{8/3}\ (4\varphi^{2}-3)\ (\psi+\kappa-1)}\Big[288\kappa^{2/3}\ (\psi+\kappa-1)^{10/3}+120\varphi^{2}\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}-27\ (\psi+\kappa-1)^{4}$$
$$-192\varphi^{4}\kappa^{2}-693\kappa^{4/3}\ (\psi+\kappa-1)^{8/3}+288\varphi^{2}\kappa^{2}\ (\psi+\kappa-1)-108\kappa^{2}\ (\psi+\kappa-1)^{2}-18\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}$$
$$+108\kappa^{2}\ (\psi+\kappa-1)-48\varphi^{2}\kappa^{2/3}\ (\psi+\kappa-1)^{7/3}+48\varphi^{4}\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}+24\varphi^{2}\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}$$
$$-128\kappa^{2}\varphi^{2}\Big]$$

$$B_{21}=\frac{3\ (\psi+\kappa-1)^{5/3}}{4\kappa^{4/3}\ (4\varphi^{2}-3)\ (\psi+\kappa-1)}\Big[90\ (\psi+\kappa-1)^{8/3}-348\kappa^{2/3}\ (\psi+\kappa-1)^{2}-72\varphi^{2}\ (\psi+\kappa-1)^{5/3}$$
$$-36\kappa^{2/3}+304\varphi^{2}\kappa^{2/3}\ (\psi+\kappa-1)+108\kappa^{2/3}\ (\psi+\kappa-1)^{3}-27\ (\psi+\kappa-1)^{11/3}\Big]$$

$$B_{22}=\frac{3\ (\psi+\kappa-1)^{5/3}}{8\kappa^{8/3}\ (4\varphi^{2}-3)\ (\psi+\kappa-1)}\Big[48\varphi^{2}\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}-256\varphi^{2}\kappa^{2}-36\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}$$
$$+216\kappa^{2}\ (\psi+\kappa-1)+468\kappa^{2/3}\ (\psi+\kappa-1)^{10/3}-954\kappa^{4/3}\ (\psi+\kappa-1)^{8/3}-336\varphi^{2}\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}$$
$$-384\varphi^{4}\kappa^{2}+576\varphi^{2}\kappa^{2}\ (\psi+\kappa-1)-216\kappa^{2}\ (\psi+\kappa-1)^{2}-54\ (\psi+\kappa-1)^{4}+48\varphi^{2}\kappa^{2/3}\ (\psi+\kappa-1)^{7/3}$$
$$+96\varphi^{4}\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}\Big]$$

$$C_{21}=\frac{\psi+\kappa-1}{2\kappa^{4/3}\ (4\varphi^{2}-3)\ (\psi+\kappa-1)}\Big[27\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}+9\ (\psi+\kappa-1)^{7/3}-45\kappa^{4/3}\ (\psi+\kappa-1)$$
$$+40\varphi^{2}\kappa^{4/3}-24\varphi^{2}\kappa^{2/3}\ (\psi+\kappa-1)^{2/3}-8\varphi^{2}\ (\psi+\kappa-1)^{4/3}\Big]$$

$$C_{22}=\frac{1}{4\kappa^{8/3}\ \left(4\varphi^2-3\right)\ \psi+\kappa-1\ ^2}\Big[207\kappa^{2/3}\ \psi+\kappa-1\ ^{13/3}+9\kappa^{2/3}\ \psi+\kappa-1\ ^5-288\kappa^2\ \psi+\kappa-1\ ^{11/3}$$
$$+64\varphi^4\kappa^{4/3}\ \psi+\kappa-1\ ^{7/3}-128\varphi^4\kappa^2\ \psi+\kappa-1\ ^{5/3}+240\varphi^2\kappa^2\ \psi+\kappa-1\ ^{8/3}-120\varphi^2\kappa^{4/3}\ \psi+\kappa-1\ ^{10/3}$$
$$-9\ \psi+\kappa-1\ ^{17/3}\Big]$$

$$D_{21}=\frac{\psi+\kappa-1\ ^{5/3}\left[8\varphi^2-9\ \psi+\kappa-1\ \right]}{2\kappa^{2/3}\ \left(4\varphi^2-3\right)\ \psi+\kappa-1}$$

$$D_{22}=\frac{\psi+\kappa-1\ ^{4/3}}{4\kappa^2\ \left(4\varphi^2-3\right)\ \psi+\kappa-1\ ^2}\Big[120\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ -54\kappa^{2/3}\ \psi+\kappa-1\ ^2-36\kappa^{2/3}\ \psi+\kappa-1\ ^{5/3}$$
$$-64\varphi^4\kappa^{2/3}+9\ \psi+\kappa-1\ ^5\Big]$$

Equations (3.188) and (3.191) are the eccentricities of the long and short period orbits, respectively. These equations are defined by the mass parameter, the parameters representing the perturbations in the Coriolis and centrifugal forces, oblateness, radiation pressure of the primaries and the constant (kappa) of a particular integral of the GMP.

### 3.5.3 The semi-axes of the orbits

The lengths of the semi-major axis, $a_i$ and the semi-minor axis, $b_i$ of the long and short period orbits are obtained from the respective equations below:

$$a_i=\frac{1}{\delta_j}\ \xi^2\delta_i^2+\eta^2\ ^{1/2},b_j=\ \xi^2\delta_j^2+\eta^2\ ^{1/2}\quad i=1,2 \tag{3.192}$$

where $\xi$ and $\eta$ are the triangular equilibrium points.

Solving, we get

$$a_1 = \frac{\sqrt{L_1}}{2\ (\psi+\kappa-1)^{2/3}}\left[1-\frac{2\upsilon\ (\psi+\kappa-1)^{4/3}}{L_1}+\frac{2\upsilon^2\ (\psi+\kappa-1)^{4/3}}{L_1}+\frac{\kappa^{4/3}\left[4\varphi^2-3\ (\psi+\kappa-1)\right]}{2\upsilon L_1}\right. \tag{3.193}$$

$$\left.+\frac{2\alpha_1 L_2\ (\psi+\kappa-1)^{4/3}}{L_1}+\frac{2\alpha_2 L_3\ (\psi+\kappa-1)^{4/3}}{L_1}+\frac{2L_4\ (\psi+\kappa-1)^{4/3}\ (1-q_1)}{L_1}+\frac{2L_5\ (\psi+\kappa-1)^{4/3}\ (1-q_2)}{L_1}\right]$$

where

$$L_1 = 4\varphi^2\kappa^{4/3}-3\kappa^{4/3}\ (\psi+\kappa-1)+12\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}-8\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{2/3}+\ (\psi+\kappa-1)^{4/3}$$
$$-3\ (\psi+\kappa-1)^{7/3}+2\varphi^2\ (\psi+\kappa-1)^{4/3}$$

$$L_2 = \frac{1}{2N\kappa^{4/3}\ (\psi+\kappa-1)^{4/3}}\left[216\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}+9\ (\psi+\kappa-1)^{3}-90\kappa^{2/3}\ (\psi+\kappa-1)^{7/3}+8\varphi^2\kappa^2\right.$$
$$-6\kappa^2\ (\psi+\kappa-1)-3\kappa^{8/3}\ (\psi+\kappa-1)^{5/3}+18\kappa^{4/3}\ (\psi+\kappa-1)^{7/3}+6\kappa^{10/3}\ (\psi+\kappa-1)^{7/3}-6\varphi^2\ (\psi+\kappa-1)^2$$
$$-3\kappa^{2/3}\ (\psi+\kappa-1)^3-24\kappa^2\ (\psi+\kappa-1)^{5/3}+64\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{4/3}-160\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}$$
$$+4\varphi^2\kappa^{8/3}\ (\psi+\kappa-1)^{4/3}-8\varphi^2\kappa^{10/3}(\psi+\kappa-1)^{4/3}-12\varphi^2\kappa^{4/3}(\psi+\kappa-1)^{4/3}+16\varphi^2\kappa^2(\psi+\kappa-1)^{2/3}$$
$$\left.+2\varphi^2\kappa^{2/3}(\psi+\kappa-1)^2+4\kappa^2(\psi+\kappa-1)^{4/3}-\kappa^{4/3}(\psi+\kappa-1)^2\right]-\upsilon+\frac{\kappa^{2/3}\left[4\varphi^2-3(\psi+\kappa-1)\right]}{2N\upsilon(\psi+\kappa-1)^{4/3}}$$
$$\times\left[5\kappa^{2/3}(\psi+\kappa-1)^{2/3}-4\kappa^{2/3}-(\psi+\kappa-1)^{4/3}\right]$$

$$L_3 = \frac{1}{2N\ (\psi+\kappa-1)^{4/3}}\left[159\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}-63\kappa^{2/3}\ (\psi+\kappa-1)^{7/3}-112\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}+8\varphi^2\kappa^2\right.$$
$$+6\ (\psi+\kappa-1)^3+40\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{4/3}-6\kappa^2\ (\psi+\kappa-1)-4\varphi^2\ (\psi+\kappa-1)^2-4\kappa^{2/3}\ (\psi+\kappa-1)^{4/3}$$
$$\left.+12\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}+\ (\psi+\kappa-1)^2\right]+\upsilon-\frac{\kappa^{4/3}\left[4\varphi^2-3\ (\psi+\kappa-1)\right]}{2\upsilon\ (\psi+\kappa-1)^{4/3}}$$

$$L_4 = \frac{1}{6N\ (\psi+\kappa-1)^{4/3}}\Big[2\kappa^{2/3} - (\psi+\kappa-1)^{2/3} + 60\kappa^2\ (\psi+\kappa-1)^{1/3} - 51\kappa^{4/3}\ (\psi+\kappa-1) - 48\varphi^2\kappa^2$$
$$+3\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} - 3\kappa^{2/3}\ (\psi+\kappa-1)^{7/3} + 66\kappa^2\ (\psi+\kappa-1) - 75\kappa^{4/3}\ (\psi+\kappa-1)^{5/3} + 3\ (\psi+\kappa-1)^{7/3}$$
$$+3\ (\psi+\kappa-1)^3 - 8\kappa^{4/3}\ (\psi+\kappa-1)^{2/3} + 2\kappa^{2/3}\ (\psi+\kappa-1)^{4/3} + 34\varphi^2\kappa^{4/3} - 40\varphi^2\kappa^2\ (\psi+\kappa-1)^{-2/3}$$
$$+2\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} - 2\varphi^2\left(\psi+\kappa-1\right)^{4/3} - 2\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{4/3} + 50\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}$$
$$-2\varphi^2\left(\psi+\kappa-1\right)^2\Big] + \frac{2\upsilon\kappa^{2/3}}{3\left(\psi+\kappa-1\right)^{2/3}} - \frac{\kappa^{4/3}\left[4\varphi^2-3\left(\psi+\kappa-1\right)\right]}{6N\upsilon\left(\psi+\kappa-1\right)^2}\Big[5\kappa^{4/3} - \kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}$$
$$-\left(\psi+\kappa-1\right)^{4/3}\Big]$$

$$L_5 = \frac{\kappa^{2/3}}{6N\ (\psi+\kappa-1)^{4/3}}\Big[6\kappa^{4/3}\ (\psi+\kappa-1) - 3\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} + 8\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} - 2\ (\psi+\kappa-1)^{4/3}$$
$$+4\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} - 8\varphi^2\kappa^{4/3}\Big] - \frac{2\kappa^{2/3}\upsilon}{3\ (\psi+\kappa-1)^{2/3}} - \frac{\kappa^2\left[4\varphi^2-3\ (\psi+\kappa-1)\right]}{2N\upsilon\ (\psi+\kappa-1)^{4/3}}$$

Equation (3.193) gives the length of the semi-major axes of the long period orbits.

Similarly, the length of the semi-minor axis of the long period orbit gives

$$b_1 = \frac{\sqrt{N}}{2\ (\psi+\kappa-1)^{\frac{2}{3}}}\Bigg[1 + \frac{\upsilon\ (\psi+\kappa-1)^{4/3}}{2\kappa^{2/3}\left[4\varphi^2-3\ (\psi+\kappa-1)\right]} - M_1\upsilon^2 + M_2\upsilon + M_3\upsilon^2 + M_4\ \alpha_1$$
$$+ M_5\upsilon + M_6\upsilon^2 + M_7\ \alpha_2 + M_8Y_6\upsilon + M_9Y_7\upsilon^2 - M_{10}\ (1-q_1) + M_{11}\upsilon + M_{12}\upsilon^2 - M_{13}\ (1-q_2)\Bigg] \qquad (3.194)$$

where

$$M_1 = \frac{(\psi+\kappa-1)^{4/3}}{2\kappa^2\left[4\varphi^2-3\ (\psi+\kappa-1)\right]^2}\Big[-8\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} + 2\varphi^2\ (\psi+\kappa-1)^{4/3} - 3\ (\psi+\kappa-1)^{8/3}$$
$$+12\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} - 15\kappa^{4/3}\ (\psi+\kappa-1) + 20\varphi^2\kappa^{4/3}\Big]$$

$$M_2 = \frac{\psi+\kappa-1\ ^{2/3}}{N\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]}\left[\kappa^{2/3}\ \psi+\kappa-1\ ^{2/3}-5\kappa^{2/3}\ \psi+\kappa-1\ ^{4/3}+\ \psi+\kappa-1\ ^{2}\right]$$

$$M_3 = \frac{\psi+\kappa-1\ ^{4/3}}{2\kappa^2\left[4\varphi^2-3\ \psi+\kappa-1\right]}\left[132\kappa^2\ \psi+\kappa-1\ -642\kappa^{4/3}\ \psi+\kappa-1\ ^{5/3}+536\varphi^2\kappa^{4/3}\ \psi+\kappa-1\ ^{2/3}\right.$$
$$\left.-176\varphi^2\kappa^2+24\kappa^{2/3}\ \psi+\kappa-1\ ^{8/3}-192\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ ^{4/3}+288\kappa^{2/3}\ \psi+\kappa-1\ ^{7/3}-6\ \psi+\kappa-1\ ^{10/3}\right.$$
$$\left.+16\varphi^2\ \psi+\kappa-1\ ^{2}-18\ \psi+\kappa-1\ ^{3}\right]$$

$$M_4 = M_7 = \frac{\psi+\kappa-1\ ^{2/3}-2\kappa^{2/3}}{N}$$

$$M_5 = -\frac{\psi+\kappa-1\ ^{4/3}\left[2\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right]}{N\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}$$

$$M_6 = \frac{\psi+\kappa-1\ ^{4/3}}{2N\kappa^2\left[4\varphi^2-3\ \psi+\kappa-1\ \right]^2}\left[-12\kappa^2\ \psi+\kappa-1\ -88\varphi^2\kappa^{4/3}\ \psi+\kappa-1\ ^{2/3}+8\varphi^2\ \psi+\kappa-1\ ^{2}\right.$$
$$\left.-246\kappa^{4/3}\ \psi+\kappa-1\ ^{2/3}-16\varphi^2\kappa^2+114\kappa^{2/3}\ \psi+\kappa-1\ ^{7/3}-12\ \psi+\kappa-1\ ^{3}-72\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ ^{4/3}\right.$$
$$\left.+224\kappa^{4/3}\ \psi+\kappa-1\ ^{2/3}\right]$$

$$M_8 = -\frac{\psi+\kappa-1\ ^{4/3}}{3N\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}\left[3\kappa^{4/3}+\kappa^{2/3}\ \psi+\kappa-1\ ^{2/3}+\ \psi+\kappa-1\ ^{4/3}\right]$$

$$M_9 = \frac{\psi+\kappa-1\ ^{2/3}}{3N\kappa^2\left[4\varphi^2-3\ \psi+\kappa-1\right]^2}\Big[-2\kappa^{2/3}\ \psi+\kappa-1\ ^{4/3}+\ \psi+\kappa-1\ ^2+18\kappa^2\ \psi+\kappa-1\ ^{5/3}$$
$$-9\kappa^{4/3}\ \psi+\kappa-1\ ^{7/3}+14\varphi^2\kappa^{4/3}\ \psi+\kappa-1\ ^{4/3}+9\kappa^{2/3}\ \psi+\kappa-1\ ^3-6\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ ^2$$
$$-2\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ ^{8/3}-34\varphi^2\kappa^{4/3}\ \psi+\kappa-1\ ^2+40\varphi^2\kappa^2\ \psi+\kappa-1\ ^{4/3}+3\kappa^{2/3}\ \psi+\kappa-1\ ^{11/3}$$
$$-60\kappa^2\ \psi+\kappa-1\ ^{7/3}+51\kappa^{4/3}\left(\psi+\kappa-1\right)^3-3\left(\psi+\kappa-1\right)^{11/3}+2\varphi^2\left(\psi+\kappa-1\right)^{8/3}+2\varphi^2\left(\psi+\kappa-1\right)^{10/3}$$
$$-3\left(\psi+\kappa-1\right)^{13/3}+16\varphi^2\kappa^{2/3}-12\kappa^{8/3}\left(\psi+\kappa-1\right)\Big]$$

$$M_{10} = M_{13} = \frac{-2\kappa^{2/3}}{3N}$$

$$M_{11} = \frac{\psi+\kappa-1\ ^{2/3}\left[8\kappa^{2/3}-\ \psi+\kappa-1\ ^{2/3}\right]}{3N\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}$$

$$M_{12} = \frac{\psi+\kappa-1\ ^{\frac{2}{3}}}{6N\kappa^{\frac{4}{3}}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]^2}\Big[54\kappa^{\frac{2}{3}}\ \psi+\kappa-1\ ^{\frac{7}{3}}+144\kappa^2\ \psi+\kappa-1\ -204\kappa^{4/3}\ \psi+\kappa-1\ ^{5/3}$$
$$-40\varphi^2\kappa^{2/3}\ \psi+\kappa-1\ ^{4/3}+144\varphi^2\kappa^{\frac{4}{3}}\ \psi+\kappa-1\ ^{\frac{2}{3}}-192\varphi^2\kappa^2\Big]$$

Similarly, the length of the semi-major axes for the short period orbits are

$$a_2 = \frac{\sqrt{4N\varphi^4+\ \psi+\kappa-1\ ^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]}}{2\ \psi+\kappa-1\ ^{2/3}\sqrt{4\varphi^2-3\ \psi+\kappa-1}}\Big[1+\frac{J_1}{2J_2}\upsilon+\frac{1}{2}J_3\upsilon^2+\frac{1}{2}\ J_4+J_5\upsilon+J_6\upsilon^2\ \alpha_1$$
$$+\frac{1}{2}\ J_7+J_8\upsilon+J_9\upsilon^2\ \alpha_2+\frac{1}{2}\ J_{10}+J_{11}\upsilon+J_{12}\upsilon^2\ 1-q_1\ +\frac{1}{2}W\ J_{13}+J_{14}\upsilon+J_{15}\upsilon^2\ 1-q_2\ \Big] \qquad 3.195$$

where

$$J_1 = 144\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^{8/3} - 72\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{10/3} + 9\varphi^4\ (\psi+\kappa-1)^4 - 960\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}$$
$$+768\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{7/3} - 96\varphi^4\ (\psi+\kappa-1)^3 + 2304\varphi^6\kappa^{4/3}\ (\psi+\kappa-1)^{8/3} - 54\varphi^4\ (\psi+\kappa-1)^5$$
$$-1152\varphi^6\kappa^{2/3}\ (\psi+\kappa-1)^{10/3} + 144\varphi^6\ (\psi+\kappa-1)^4 - 864\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^{11/3} + 108\kappa^{4/3}\ (\psi+\kappa-1)^{11/3}$$
$$+432\varphi^4\kappa^{2/3}(\psi+\kappa-1)^{13/3} - 256\varphi^6\kappa^{4/3}(\psi+\kappa-1)^{2/3} - 432\varphi^2\kappa^{4/3}(\psi+\kappa-1)^{8/3}$$

$$J_2 = -48\varphi^4\ (\psi+\kappa-1)^{5/3} - 384\varphi^4\kappa^{2/3}\ (\psi+\kappa-1) + 144\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^2 + 72\varphi^2\ (\psi+\kappa-1)^{8/3}$$
$$-27\ (\psi+\kappa-1)^{11/3} + 256\varphi^6\kappa^{2/3}$$

$$J_3 = \frac{(\psi+\kappa-1)^{2/3}}{2\kappa^{8/3}\left[16\varphi^4 - 24\varphi^2\ (\psi+\kappa-1) + 9\ (\psi+\kappa-1)^2\right]\left[16\varphi^2\kappa^{2/3} - 3\ (\psi+\kappa-1)^{5/3}\right]}\Big[512\varphi^6\kappa^{8/3}$$
$$-1440\varphi^4\kappa^{8/3}\ (\psi+\kappa-1) + 864\varphi^2\kappa^{8/3}\ (\psi+\kappa-1)^2 - 216\kappa^{8/3}\ (\psi+\kappa-1)^3 + 3072\varphi^8\kappa^2\ (\psi+\kappa-1)$$
$$-18\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^{10/3} + 1728\varphi^4\kappa^{8/3}\ (\psi+\kappa-1)^3 - 27\varphi^4\ (\psi+\kappa-1)^{17/3} + 32\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^5$$
$$-1296\varphi^4\kappa^{4/3}(\psi+\kappa-1)^{13/3} + 144\varphi^4\kappa^2(\psi+\kappa-1)^{8/3} - 1536\varphi^8\kappa^{4/3}(\psi+\kappa-1)^{5/3} + 2304\varphi^6\kappa^2$$
$$\times(\psi+\kappa-1)^{8/3} + 192\varphi^8\kappa^{2/3}(\psi+\kappa-1)^{7/3} - 4608\varphi^6\kappa^{8/3}(\psi+\kappa-1)^2 - 288\varphi^6\kappa^{4/3}(\psi+\kappa-1)^{10/3}$$
$$+108\varphi^4\kappa^{4/3}(\psi+\kappa-1)^2 + 864\varphi^4\kappa^2(\psi+\kappa-1)^{11/3}\Big]$$

$$J_4 = \frac{2}{\kappa^{2/3}\left[4\varphi^2 - 3\ (\psi+\kappa-1)\right]\left[4N\varphi^4 + (\psi+\kappa-1)^{2/3}\ 4\varphi^2 - 3\ (\psi+\kappa-1)\right]}\Big[4\varphi^6\kappa^{2/3}\ (\psi+\kappa-1)^{2/3}$$
$$+21\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} + 16\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} + 9\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 24\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}$$
$$-6\varphi^4\ (\psi+\kappa-1)^{7/3} - 12\varphi^4\kappa^{4/3}\ (\psi+\kappa-1) + 16\varphi^4\kappa^{4/3}\Big]$$

$$J_5=\frac{6}{\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]^2\left[4N\varphi^4+\ \psi+\kappa-1\ ^{2/3}\ 4\varphi^2-3\ \psi+\kappa-1\ \right]}\Big[32\varphi^8\ \ \psi+\kappa-1\ ^3$$
$$+18\varphi^4\kappa^{2/3}\ \ \psi+\kappa-1\ ^{10/3}-192\varphi^8\kappa^{2/3}\ \ \psi+\kappa-1\ ^{7/3}+288\varphi^6\kappa^{2/3}\ \ \psi+\kappa-1\ ^{10/3}-3\varphi^4\ \ \psi+\kappa-1\ ^4$$
$$-108\varphi^4\kappa^{2/3}\ \ \psi+\kappa-1\ ^{13/3}-24\varphi^4\kappa^{4/3}\ \ \psi+\kappa-1\ ^{8/3}+256\varphi^8\kappa^{4/3}\ \ \psi+\kappa-1\ ^{5/3}-48\varphi^6\ \ \psi+\kappa-1\ ^4$$
$$-384\varphi^6\kappa^{4/3}\left(\psi+\kappa-1\right)^{8/3}+144\varphi^4\kappa^{4/3}\left(\psi+\kappa-1\right)^{11/3}+18\varphi^4\left(\psi+\kappa-1\right)^5+\frac{1}{3}\Big\{288\varphi^4\kappa^{4/3}\left(\psi+\kappa-1\right)$$
$$+128\varphi^6\kappa^{4/3}-216\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^2+54\kappa^{4/3}\left(\psi+\kappa-1\right)^3\Big\}\frac{2}{3}A_{21}\varphi^4\kappa^{4/3}\ \Big\{16\varphi^2\kappa^{2/3}+3\left(\psi+\kappa-1\right)^{5/3}$$
$$-12\kappa^{\ 2/3}\left(\psi+\kappa-1\right)-4\varphi^2\left(\psi+\kappa-1\right)^{2/3}\Big\}\Big]$$

$$J_6=\frac{3N\varphi^4\ \ \psi+\kappa-1\ ^{5/3}}{\kappa^{8/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]^2\left[4N\varphi^4+\ \psi+\kappa-1\ ^{2/3}\ 4\varphi^2-3\ \psi+\kappa-1\ \right]}\Big[12\kappa^2\ \ \psi+\kappa-1$$
$$-72\kappa^2\ \ \psi+\kappa-1\ ^2-36\kappa^{4/3}\ \ \psi+\kappa-1\ ^{8/3}+54\kappa^{2/3}\ \ \psi+\kappa-1\ ^{10/3}-9\ \ \psi+\kappa-1\ ^4+192\varphi^4\kappa^2$$
$$-6\kappa^{4/3}\ \ \psi+\kappa-1\ ^{5/3}+64\varphi^4\kappa^{4/3}\ \ \psi+\kappa-1\ ^{2/3}-96\varphi^2\kappa^{4/3}\ \ \psi+\kappa-1\ ^{5/3}-288\kappa^{2/3}\ \ \psi+\kappa-1\ ^{10/3}$$
$$-120\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{5/3}+693\kappa^{4/3}\left(\psi+\kappa-1\right)^{8/3}-108\kappa^2\left(\psi+\kappa-1\right)288\varphi^2\kappa^2\left(\psi+\kappa-1\right)$$
$$+108\kappa^2\left(\psi+\kappa-1\right)^2+27\left(\psi+\kappa-1\right)^4+48\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{7/3}+18\kappa^{4/3}\left(\psi+\kappa-1\right)^{5/3}+128\varphi^2\kappa^2$$
$$-48\varphi^4\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}-24\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}\Big]$$

$$J_7=\frac{2}{\kappa^{2/3}\left[4\varphi^2-3\ \psi+\kappa-1\ \right]\left[4N\varphi^4+\ \psi+\kappa-1\ ^{2/3}\ 4\varphi^2-3\ \psi+\kappa-1\ \right]}\Big[4\varphi^6\ \ \psi+\kappa-1\ ^{2/3}$$
$$-3\varphi^4\ \ \psi+\kappa-1\ ^{5/3}-12\varphi^4\kappa^{2/3}\ \ \psi+\kappa-1\ -16\varphi^4\ \ \psi+\kappa-1\ ^{2/3}+24\varphi^2\ \ \psi+\kappa-1\ ^{5/3}+16\varphi^4\kappa^{2/3}$$
$$-9\ \ \psi+\kappa-1\ ^{8/3}\Big]$$

$$J_8=\frac{\psi+\kappa-1^{5/3}}{\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]^2\left[4N\varphi^4+\ \psi+\kappa-1^{2/3}\ 4\varphi^2-3\ \psi+\kappa-1\right]}\Big[688\varphi^4\kappa^{4/3}$$
$$+1728\varphi^6\kappa^{2/3}\ \psi+\kappa-1^{8/3}-2016\varphi^4\kappa^{2/3}\ \psi+\kappa-1^{8/3}-1152\varphi^8\kappa^{2/3}\ \psi+\kappa-1^{5/3}$$
$$+4032\varphi^4\kappa^{4/3}\ \psi+\kappa-1^{2}-576\varphi^4\kappa^{4/3}\ \psi+\kappa-1\ -432\varphi^4\kappa^{4/3}\ \psi+\kappa-1^{3}+1536\varphi^8\kappa^{4/3}\ \psi+\kappa-1$$
$$-2304\varphi^6\kappa^{4/3}\left(\psi+\kappa-1\right)^2-3648\varphi^6\kappa^{4/3}\left(\psi+\kappa-1\right)-1776\varphi^6\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3}-288\varphi^6\left(\psi+\kappa-1\right)^{10/3}$$
$$+252\varphi^4\left(\psi+\kappa-1\right)^{10/3}+192\varphi^8\left(\psi+\kappa-1\right)^{7/3}+27\varphi^4\left(\psi+\kappa-1\right)^{13/3}-108\varphi^4\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3}$$
$$-216\varphi^6\left(\psi+\kappa-1\right)^{7/3}+432\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)^2-108\kappa^{4/3}\left(\psi+\kappa-1\right)^3\Big]$$

$$J_9=\frac{-3N\varphi^4\ \psi+\kappa-1^{5/3}}{2\kappa^{8/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]^2\left[4N\varphi^4+\ \psi+\kappa-1^{2/3}\ 4\varphi^2-3\ \psi+\kappa-1\right]}\Big[192\kappa^2\ \psi+\kappa-1$$
$$+48\varphi^2\kappa^{4/3}\ \psi+\kappa-1^{2/3}+48\varphi^2\kappa^{2/3}\ \psi+\kappa-1^{7/3}-144\varphi^2\kappa^{4/3}\ \psi+\kappa-1^{5/3}-72\kappa^2\ \psi+\kappa-1^{2}$$
$$+360\kappa^{2/3}\ \psi+\kappa-1^{10/3}-882\kappa^{4/3}\ \psi+\kappa-1^{8/3}+192\varphi^2\kappa^2\ \psi+\kappa-1\ -24\kappa^{4/3}\ \psi+\kappa-1^{5/3}$$
$$-36\left(\psi+\kappa-1\right)^4-32\varphi^4\kappa^{4/3}\left(\psi+\kappa-1\right)^{2/3}-256\varphi^2\kappa^2\Big]$$

$$J_{10}=-\frac{4\kappa^{2/3}\left[4\varphi^4+4\varphi^2-3\ \psi+\kappa-1\right]}{3\left[4N\varphi^4+\ \psi+\kappa-1^{2/3}\ 4\varphi^2-3\ \psi+\kappa-1\right]}$$

$$J_{11}=\frac{2}{3\kappa^{4/3}\left[4\varphi^2-3\ \psi+\kappa-1\right]^2\left[4N\varphi^4+\ \psi+\kappa-1^{2/3}\ 4\varphi^2-3\ \psi+\kappa-1\right]}\Big[27\varphi^4\ \psi+\kappa-1^{4}$$
$$+408\varphi^6\kappa^{4/3}\ \psi+\kappa-1^{5/3}+540\varphi^4\kappa^2\ \psi+\kappa-1^{2}+432\varphi^2\kappa^2\ \psi+\kappa-1^{2}-576\varphi^4\kappa^2\ \psi+\kappa-1$$
$$+531\varphi^4\kappa^{4/3}\ \psi+\kappa-1^{8/3}-9\varphi^4\kappa^{2/3}\ \psi+\kappa-1^{10/3}+96\varphi^6\kappa^{2/3}\ \psi+\kappa-1^{7/3}+432\varphi^4\ \psi+\kappa-1^{11/3}$$
$$-1152\varphi^6\kappa^{4/3}\left(\psi+\kappa-1\right)^{8/3}+7688\varphi^8\kappa^{4/3}\left(\psi+\kappa-1\right)^{5/3}-72\varphi^6\kappa^{2/3}\left(\psi+\kappa-1\right)^{7/3}-24\varphi^6\left(\psi+\kappa-1\right)^3$$
$$+288\varphi^6\kappa^{2/3}\left(\psi+\kappa-1\right)^{10/3}-192\varphi^8\kappa^{2/3}\left(\psi+\kappa-1\right)^{7/3}-108\varphi^4\kappa^{2/3}\left(\psi+\kappa-1\right)^{13/3}-108\kappa^2\left(\psi+\kappa-1\right)^3$$
$$-224\varphi^6\kappa^2\Big]$$

$$J_{12}=\frac{N\varphi^4\ (\psi+\kappa-1)^{5/3}}{\kappa^{8/3}\left[4\varphi^2-3\ (\psi+\kappa-1)\right]^3\left[4N\varphi^4+\ (\psi+\kappa-1)^{2/3}\ (4\varphi^2-3\ (\psi+\kappa-1)\right]}\Big[252\kappa^2\ (\psi+\kappa-1)^2$$
$$-207\kappa^{4/3}\ (\psi+\kappa-1)^{8/3}-512\varphi^6\kappa^2+128\varphi^4\kappa^2-64\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^{2/3}+120\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{5/3}$$
$$+216\kappa^2\ (\psi+\kappa-1)^3-288\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{8/3}+72\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{10/3}-192\varphi^2\kappa^2(\psi+\kappa-1)$$
$$+1152\varphi^4\kappa^2(\psi+\kappa-1)-864\varphi^2\kappa^2(\psi+\kappa-1)^2+216\kappa^{4/3}(\psi+\kappa-1)^{11/3}-9\kappa^{2/3}(\psi+\kappa-1)^{10/3}$$
$$-54\kappa^{2/3}(\psi+\kappa-1)^{13/3}+9(\psi+\kappa-1)^4\Big]$$

$$J_{13}=\frac{4\kappa^{2/3}\left[4\varphi^2\ (1-\varphi^2\ -3\ (\psi+\kappa-1)\right]}{3\left[4N\varphi^4+\ (\psi+\kappa-1)^{2/3}\ (4\varphi^2-3\ (\psi+\kappa-1)\right]}$$

$$J_{14}=\frac{2}{3\kappa^{2/3}\left[4\varphi^2-3\ (\psi+\kappa-1)\right]^2\left[4N\varphi^4+\ (\psi+\kappa-1)^{2/3}\ (4\varphi^2-3\ (\psi+\kappa-1)\right]}\Big[24\varphi^6\ (\psi+\kappa-1)^{7/3}$$
$$-96\varphi^6\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}+36\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}+768\varphi^8\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}+288\varphi^6\ (\psi+\kappa-1)^{10/3}$$
$$-1152\varphi^6\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}+432\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{11/3}-9\varphi^4\ (\psi+\kappa-1)^{10/3}-192\varphi^8\ (\psi+\kappa-1)^{7/3}$$
$$-108\varphi^4(\psi+\kappa-1)^{11/3}+576\varphi^4\kappa^{4/3}(\psi+\kappa-1)-432\varphi^2\kappa^{4/3}(\psi+\kappa-1)^3+108\kappa^{4/3}(\psi+\kappa-1)^3$$
$$-256\varphi^6\kappa^{4/3}\Big]$$

$$J_{15}=\frac{N\varphi^4\ (\psi+\kappa-1)^{5/3}}{\kappa^2\left[4\varphi^2-3\ (\psi+\kappa-1)\right]^3\left[4N\varphi^4+\ (\psi+\kappa-1)^{2/3}\ (4\varphi^2-3\ (\psi+\kappa-1)\right]}\Big[192\varphi^4\kappa^{2/3}\ (\psi+\kappa-1$$
$$-2048\varphi^8\kappa^{4/3}+6144\varphi^6\kappa^{4/3}\ (\psi+\kappa-1)\ -1152\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}+288\varphi^4\ (\psi+\kappa-1)^{10/3}$$
$$-288\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^2-6912\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^2+3456\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^3+108\kappa^{4/3}\ (\psi+\kappa-1)^3$$
$$+1728\varphi^2\kappa^{2/3}(\psi+\kappa-1)^{11/3}-432\varphi^2(\psi+\kappa-1)^{13/3}-648\kappa^{4/3}(\psi+\kappa-1)^4-648\kappa^{2/3}(\psi+\kappa-1)^{14/3}$$
$$+162(\psi+\kappa-1)^{16/3}-120\varphi^2\kappa^{2/3}(\psi+\kappa-1)^{5/3}+54\kappa^{2/3}(\psi+\kappa-1)^{8/3}+64\varphi^4\kappa^{2/3}(\psi+\kappa-1)^{2/3}$$
$$+36\kappa^{2/3}(\psi+\kappa-1)^5-9(\psi+\kappa-1)^{17/3}\Big]$$

Lastly, for the length of the semi-minor axes of the short period orbits, we get

$$b_2=\frac{\sqrt{A^*}}{4\varphi^2\ (\psi+\kappa-1)^{1/3}}\Bigg[1-\frac{N_1\ (\psi+\kappa-1)^{2/3}\upsilon}{8\kappa^{4/3}A^*\left[4\varphi^2-3\ (\psi+\kappa-1)\right]}+\frac{N_2\ (\psi+\kappa-1)^{2/3}\upsilon^2}{16\kappa^{8/3}A^*\left[4\varphi^2-3\ (\psi+\kappa-1)\right]}$$

$$+\frac{(\psi+\kappa-1)^{2/3}}{4\kappa^{2/3}A^*}\Bigg\{(\psi+\kappa-1)\left[4\varphi^2\kappa^{2/3}-6\ (\psi+\kappa-1)^{2/3}\right]+\frac{N_3\upsilon}{\left[4\varphi^2-3\ (\psi+\kappa-1)\right]}$$

$$+\frac{N_4\upsilon^2}{2\kappa^2\left[4\varphi^2-3\ (\psi+\kappa-1)\right]}\Bigg\}\alpha_1+\frac{(\psi+\kappa-1)^{\frac{2}{3}}}{4A^*}\Bigg\{9\left[3(\psi+\kappa-1)-4\varphi^2\right]+\frac{16\varphi^4\left[(\psi+\kappa-1)^{2/3}-\kappa^{2/3}\right]}{(\psi+\kappa-1)^{2/3}}$$

$$+\frac{N_5\upsilon}{\kappa^{\frac{4}{3}}\left[4\varphi^2-3(\psi+\kappa-1)\right]}+\frac{N_4\upsilon^2}{2\kappa^{\frac{8}{3}}\left[4\varphi^2-3(\psi+\kappa-1)\right]}\Bigg\}\alpha_2+\frac{1}{3(\psi+\kappa-1)^{\frac{2}{3}}A^*}\Bigg\{2\kappa^{\frac{2}{3}}\left[3(\psi+\kappa-1)\right.$$

$$\left.-4\varphi^2-4\varphi^4(\psi+\kappa-1)^{2/3}\right]+\frac{N_6\upsilon}{2\kappa^{2/3}\left[4\varphi^2-3(\psi+\kappa-1)\right]}+\frac{3(\psi+\kappa-1)^{2/3}N_7\upsilon^2}{4\kappa^2\left[4\varphi^2-3(\psi+\kappa-1)\right]}\Bigg\}(-q_1)$$

$$+\Bigg\{2\kappa^{2/3}\left[4\varphi^2-3(\psi+\kappa-1)\right]4\varphi^4(\psi+\kappa-1)^{2/3}\Big]+\frac{N_8\upsilon}{2\kappa^{2/3}\left[4\varphi^2-3(\psi+\kappa-1)\right]} \qquad (3.196)$$

$$+\frac{3(\psi+\kappa-1)^{2/3}N_9\upsilon^2}{4\kappa^2\left[4\varphi^2-3(\psi+\kappa-1)\right]}\Bigg\}(-q_2)\Bigg]$$

where

$$N_1=36\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}-384\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{5/3}+576\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}+256\varphi^4\kappa^{4/3}$$
$$-216\kappa^{2/3}\ (\psi+\kappa-1)^{11/3}-9\ (\psi+\kappa-1)^{10/3}+96\kappa^{2/3}\varphi^4\ (\psi+\kappa-1)^{7/3}-144\varphi^2\ (\psi+\kappa-1)^{10/3}$$
$$+54\ (\psi+\kappa-1)^{13/3}-384\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)\ +144\kappa^{4/3}\ (\psi+\kappa-1)^2$$

$$N_2=360\kappa^2\ (\psi+\kappa-1)^{8/3}-3840\varphi^4\kappa^2\ (\psi+\kappa-1)^{5/3}+5760\varphi^2\kappa^2\ (\psi+\kappa-1)^{8/3}+512\varphi^4\kappa^{8/3}$$
$$-2160\kappa^2\ (\psi+\kappa-1)^{11/3}+108\kappa^{4/3}\ (\psi+\kappa-1)^{13/3}+216\kappa^{2/3}\ (\psi+\kappa-1)^5-90\kappa^{4/3}\ (\psi+\kappa-1)^{10/3}$$
$$+960\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^{7/3}-1440\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{10/3}-27\ (\psi+\kappa-1)^{17/3}-768\varphi^2\kappa^{8/3}\ (\psi+\kappa-1)$$
$$+288\kappa^{8/3}(\psi+\kappa-1)^2$$

$$N_3 = 672\kappa^{2/3}\varphi^4\ (\psi+\kappa-1)^{5/3} - 288\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 1008\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} + 64\varphi^4\kappa^{4/3}$$
$$+378\kappa^{2/3}\ (\psi+\kappa-1)^{11/3} - 168\varphi^4\ (\psi+\kappa-1)^{7/3} + 36\ (\psi+\kappa-1)^{10/3} + 108\varphi^2\ (\psi+\kappa-1)^{10/3}$$
$$+\frac{27}{2}\ (\psi+\kappa-1)^{13/3} - 96\kappa^{2/3}\varphi^2\ (\psi+\kappa-1) + 312\kappa^{2/3}\varphi^2\ (\psi+\kappa-1)^{5/3} + 36\kappa^{4/3}\ (\psi+\kappa-1)$$
$$-36\kappa^{4/3}\varphi^4\ (\psi+\kappa-1)^{7/3} + 36\kappa^{-2/3}\varphi^2\left(\psi+\kappa-1\right)^4 - 27\kappa^{-2/3}\left(\psi+\kappa-1\right)^5$$

$$N_4 = 936\varphi^2\ (\psi+\kappa-1)^{7/3} - 4032\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} + 1728\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} - 384\varphi^4\kappa^{4/3}$$
$$-1080\ (\psi+\kappa-1)^{10/3} + 4284\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} + 486\kappa^{-2/3}\ (\psi+\kappa-1)^5 - 2592\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}$$
$$+360\varphi^2\ (\psi+\kappa-1)^{10/3} - 342\kappa^{2/3}\ (\psi+\kappa-1)^{11/3} - 54\kappa^{-4/3}\ (\psi+\kappa-1)^{17/3} - 999\ (\psi+\kappa-1)^{13/3}$$
$$+72\varphi^2\kappa^{-2/3}\left(\psi+\kappa-1\right)^4 - 432\varphi^4\left(\psi+\kappa-1\right)^{7/3} + 432\kappa^{2/3}\left(\psi+\kappa-1\right)^{2/3} + 576\varphi^2\kappa^{4/3}\left(\psi+\kappa-1\right)$$
$$-216\kappa^{4/3}\left(\psi+\kappa-1\right)^2$$

$$N_5 = 320\varphi^4\kappa^{4/3} - 486\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 384\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} + 576\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3}$$
$$-54\kappa^{2/3}\ (\psi+\kappa-1)^{11/3} + 126\ (\psi+\kappa-1)^{10/3} + 96\varphi^4\ (\psi+\kappa-1)^{7/3} - 108\varphi^2\ (\psi+\kappa-1)^{7/3}$$
$$144\varphi^2\ (\psi+\kappa-1)^{10/3} + \frac{27}{2}\ (\psi+\kappa-1)^{13/3} - 480\varphi^2\kappa^{4/3}\ (\psi+\kappa-1) + 180\kappa^{4/3}\ (\psi+\kappa-1)^2$$
$$-54\kappa^{2/3}\ (\psi+\kappa-1)^{2/3} + 456\varphi^2\kappa^{2/3}\left(\psi+\kappa-1\right)^{5/3}$$

$$N_6 = \frac{63}{2}\ (\psi+\kappa-1)^{10/3} - \frac{63}{2}\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} + 60\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} + 96\varphi^4\ (\psi+\kappa-1)^{7/3}$$
$$-36\varphi^2\ (\psi+\kappa-1)^{7/3} - 192\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{8/3} + 576\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - 144\varphi^2\ (\psi+\kappa-1)^{10/3}$$
$$-384\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} - 216\kappa^{2/3}\ (\psi+\kappa-1)^{11/3} + 54\ (\psi+\kappa-1)^{13/3} + \frac{27}{2}\kappa^{-2/3}\ (\psi+\kappa-1)^4$$
$$-12\varphi^2\kappa^{-2/3}\left(\psi+\kappa-1\right)^3 + 72\kappa^{4/3}\left(\psi+\kappa-1\right)^2 + 128\varphi^4\kappa^{4/3}$$

$$N_7 = \frac{1}{2\kappa^2\left[4\varphi^2 - 3\ (\psi+\kappa-1)\right]}\Big[207\kappa^{4/3}\ (\psi+\kappa-1)^{13/3} + 9\kappa^{2/3}\ (\psi+\kappa-1)^{5} - 288\kappa^{2}\ (\psi+\kappa-1)^{11/3}$$
$$+64\varphi^4\kappa^{4/3}\ (\psi+\kappa-1)^{7/3} - 128\varphi^4\kappa^{2}\ (\psi+\kappa-1)^{5/3} + 240\varphi^2\kappa^{2}\ (\psi+\kappa-1)^{8/3} - 9\ (\psi+\kappa-1)^{17/3}$$
$$-120\varphi^2\kappa^{4/3}\ (\psi+\kappa-1)^{10/3}\Big] - 90\ (\psi+\kappa-1)^{8/3} + 108\kappa^{2/3}\ (\psi+\kappa-1)^{2} - 160\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)$$
$$+96\varphi^2\ (\psi+\kappa-1)^{5/3} + 768\varphi^4\kappa^{2/3}(\psi+\kappa-1) - 1152\varphi^2\kappa^{2/3}(\psi+\kappa-1)^{2} + 432\kappa^{2/3}(\psi+\kappa-1)^{3}$$
$$-192\varphi^4(\psi+\kappa-1)^{5/3} + 288\varphi^2(\psi+\kappa-1)^{8/3} + 36(\psi+\kappa-1)^{11/3} - 72\kappa^{-2/3}(\psi+\kappa-1)^{13/3}$$
$$+9\kappa^{-4/3}(\psi+\kappa-1)^{5} - 36\kappa^{-2/3}(\psi+\kappa-1)^{10/3} + 32\varphi^2\kappa^{-2/3}(\psi+\kappa-1)^{7/3}$$

$$N_8 = 12\varphi^2\ (\psi+\kappa-1)^{7/3} - 96\varphi^4\ (\psi+\kappa-1)^{7/3} - 54\ (\psi+\kappa-1)^{13/3} + 144\varphi^2\ (\psi+\kappa-1)^{10/3}$$
$$+192\varphi^2\kappa^{4/3}\ (\psi+\kappa-1) - 72\kappa^{4/3}\ (\psi+\kappa-1)^{2} - 36\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} - \frac{9}{2}\ (\psi+\kappa-1)^{10/3}$$
$$+384\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)^{5/3} - 576\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{8/3} + 216\kappa^{2/3}\ (\psi+\kappa-1)^{11/3} - 128\varphi^4\kappa^{4/3}$$

$$N_9 = \frac{(\psi+\kappa-1)^{7/3}}{2\kappa^{4/3}\left[4\varphi^2 - 3\ (\psi+\kappa-1)\right]}\Big[120\varphi^2\kappa^{2/3}\ (\psi+\kappa-1) - 54\kappa^{2/3}\ (\psi+\kappa-1)^{2} + 9\ (\psi+\kappa-1)^{5}$$
$$-64\varphi^4\kappa^{2/3} - 36\kappa^{2/3}\ (\psi+\kappa-1)^{13/3}\Big] + 18\ (\psi+\kappa-1)^{8/3} - 32\varphi^2\ (\psi+\kappa-1)^{5/3} - 768\varphi^4\kappa^{2/3}\ (\psi+\kappa-1)$$
$$+192\varphi^4\ (\psi+\kappa-1)^{5/3} + 72\kappa^{2/3}\ (\psi+\kappa-1)^{2} - 432\kappa^{2/3}\ (\psi+\kappa-1)^{3} + 1152\varphi^2\kappa^{2/3}\ (\psi+\kappa-1)^{2}$$
$$+72\kappa^{-2/3}(\psi+\kappa-1)^{13/3} - 288\varphi^2(\psi+\kappa-1)^{8/3} - 36(\psi+\kappa-1)^{11/3} - 9\kappa^{-4/3}(\psi+\kappa-1)^{5}\Big]$$

$$A^{*} = 4N\varphi^4 + \ (\psi+\kappa-1)^{2/3}\ \ 4\varphi^2 - 3\ (\psi+\kappa-1)$$

We have generalized the study of the periodic orbits around linearly stable triangular EPs when the masses of the primaries are changing with time under the dictates of the UML and their motion described by the GMP. The primaries have been characterized as radiating bodies and having the shape of an oblate spheroid under small perturbations in the Coriolis and centrifugal forces.The orbits about EPs are important for scientific investigations. Since, a number of space missions have been completed and some are being proposed by various space agencies.

## 3.6. Robe's Restricted Three-Body Problem with Isotropic Mass Variation when the First Primary is Empty

In this section, we discuss the motion of the third body in the restricted problem described by Robe (1977), when the masses of the primaries vary isotropically with time at the same rate according to the UML and their motion described by the GMP. The first primary is a spherical shell filled with a fluid and contains the third body. In this study, we consider the case when the shell is empty and contains no fluid. The equations of motion are given and the positions and stability of the EPs are investigated. This study is a direct investigation of the Robe (1977) problem with further assumptions that the primaries have variable masses.

### 3.6.1Equations of motion of the third body

Here, we extend the formulation of Robe (1977) by assuming that the masses of the primaries change with time according to the UML and their motion determined by the GMP.

Let $m_1$ be the mass of the first primary which is a rigid spherical shell of constant radius $\Re$ with center at $M_1$ , and filled with a homogenous incompressible fluid of constant mass $m_f$ , density $\delta_1$ and volume $V_f$ . Also, let $m_2$ be the mass of the second primary with center at $M_2$ . Both masses are assumed to vary with time due to the attachment or separation of particles to or from them. We assume that the variation in mass of the first primary occurs isotropically and does not affect the fluid inside it since there is a rigid shell. Further, we let $m_3$ be the mass of the third body with center at $M_3$, having density $\delta_3$ and its motion is confined inside the spherical shell (Fig. **1.10**). Consequently, following the Robe (1977) and knowing that the distances between the centers of the primaries vary with time; then, the forces acting on $m_3$ are

i. The force of attraction $m_2$ of which is given by

$$\vec{F}_{m_2} = -\frac{Gm_2 m_3 \vec{r}_{23}}{r_{23}^3}$$

ii. The gravitational force $\vec{F}_A$ exerted by the fluid (3.197)

$$\vec{F}_A = -\frac{4\pi G\Re^3 m_3 \delta_1 \vec{r}_{13}}{3r_{13}^3}$$

iii) The buoyancy force which is given by

$$\vec{F}_B = \frac{4G\pi\Re^3 m_3}{3r_{13}^3}\frac{\delta_1^2 \vec{r}_{13}}{\delta_3}$$

where $\vec{r}_{13} = \overrightarrow{M_1M_3}$ and $\vec{r}_{23} = \overrightarrow{M_2M_3}$ are the position vectors of the line joining the centers of the first and second primary and the center of the third body, respectively. $G$ is the gravitational constant.

Now, the equation of motion of the third body in the inertial system, taking into account the combined forces in system (3.197) acting on it, is

$$\ddot{\vec{R}} = \frac{Gm_2\vec{r}_{23}}{r_{23}^3} - \frac{4\pi\Re^3}{3}\frac{G\delta_1}{r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right)\vec{r}_{13} \qquad (3.198)$$

where $\vec{R} = \overrightarrow{OM_3}$.

In a synodic coordinates system $Oxyz$ rotating with angular velocity $\vec{\omega}$ and origin at the center of mass, O, of the primaries; the equation of motion (3.198) is

$$\frac{\partial^2 \vec{R}}{\partial t^2} + 2\vec{\omega} \times \frac{\partial \vec{r}}{\partial t} + \vec{R} \times \frac{\partial \vec{\omega}}{\partial t} + \vec{\omega} \times \left( \vec{\omega} \times \vec{R} \right) = -\frac{4\pi \Re^3}{3} \frac{G\delta_1}{r_{13}^3} \left( 1 - \frac{\delta_1}{\delta_3} \right) \left( x - x_1 \right) - \frac{Gm_2 \left( x - x_2 \right)}{r_{23}^3}$$

where $\vec{R}$ is the position vector of the third body relative to the center of mass O, and $\vec{\omega} = \omega \hat{k} : \hat{k}$ is a unit vector.

Therefore, the equations of motion of the third body under the set up of the Robe's CR3BP with variable masses in a Cartesian coordinate system, takes the form

$$\ddot{x} - 2\omega \dot{y} = \omega^2 x + \dot{\omega} y - \frac{4\pi G \Re^3 \delta_1}{3 r_{13}^3} \left( 1 - \frac{\delta_1}{\delta_3} \right) \left( x - x_1 \right) - \frac{\mu_2 \left( x - x_2 \right)}{r_{23}^3}$$

$$\ddot{y} + 2\omega \dot{x} = \omega^2 y - \dot{\omega} x - \frac{4\pi G \Re^3 y \delta_1}{3 r_{13}^3} \left( 1 - \frac{\delta_1}{\delta_3} \right) - \frac{\mu_2 y}{r_{23}^3} \tag{3.199}$$

$$\ddot{z} = -\frac{4\pi G \Re^3 z \delta_1}{3 r_{13}^3} \left( 1 - \frac{\delta_1}{\delta_3} \right) - \frac{\mu_2 z}{r_{23}^3}$$

where $\mu = \mu_1 + \mu_2 : \mu_1 = Gm_1, \mu_2 \ t = Gm_2 \ t$

The over-dot denotes differentiation with respect to time $t$ while the coordinates $x, y, z$ of the third body are connected with the distances between the center of the third body and centers of the primaries by the relation

$r_{13}^2 = \left( x - x_1 \right)^2 + y^2 + z^2$ , $r_{23}^2 = \left( x - x_2 \right)^2 + y^2 + z^2$ and $\omega$ is the angular velocity of revolution of the primaries.

Now, system (3.199) is a non-autonomous system of equations and we have to transform it to the autonomized forms. However, it is impossible to carry out a complete transformation using the MT, the UML, the particular solutions and integral of the GMP when the density parameter is not zero. In this light, we restrict ourselves to the case when the densities of the first and third body are equal or the shell is empty $\delta_1 = 0$. Hence, the system of equations (3.199) when the shell is empty $\delta_1 = 0 \Rightarrow m_f = 0$ reduces to

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega} y - \frac{\mu_2 \ x - x_2}{r_{23}^3}$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega} x - \frac{\mu_2 y}{r_{23}^3} \tag{3.200}$$

$$\ddot{z} = -\frac{\mu_2 z}{r_{23}^3}$$

$$r_{23}^2 = \ x - x_2 \ ^2 + y^2 + z^2$$

Now, from the principle of the center of mass, we have

$$x_1 = -\frac{\mu_2 r_{12}}{\mu \ t}, \ x_2 = \frac{\mu_1 r_{12}}{\mu \ t} \tag{3.201}$$

Now, From the MT (1.46), we have

$$\dot{x} = \frac{\xi \ \alpha t + \beta}{R \ t} + \frac{\xi'}{R \ t}, \dot{y} = \frac{\eta \ \alpha t + \beta}{R \ t} + \frac{\eta'}{R \ t} \text{ and } \dot{z} = \frac{\zeta \ \alpha t + \beta}{R \ t} + \frac{\zeta'}{R \ t} \tag{3.202}$$

Similarly, we get

$$\dot{x} = \frac{\xi \ \alpha\gamma - \beta^2}{R^3 \ t} + \frac{\xi''}{R^3 \ t}, \ddot{y} = \frac{\eta \ \alpha\gamma - \beta^2}{R^3 \ t} + \frac{\eta''}{R^3 \ t}, \ \ddot{z} = \frac{\zeta \ \alpha\gamma - \beta^2}{R^3 \ t} + \frac{\zeta''}{R^3 \ t}$$

and $$\dot{\omega} = -\frac{2\omega_0 \ \alpha t + \beta}{R^4 \ t} \tag{3.203}$$

Now, using equations (3.203), the UML (1.47) and the particular solutions of the GMP, in the system of equations (4.6), we have

$$\frac{\xi\ \ \alpha\gamma-\beta^2}{R^3\ \ t}+\frac{\xi''}{R^3\ \ t}-\frac{2\omega_0}{R^2\ \ t}\left[\frac{\eta\ \ \alpha t+\beta}{R\ \ t}+\frac{\eta'}{R\ \ t}\right]=\frac{\omega_0^2\xi R\ \ t}{R^4\ \ t}+\left[\frac{-2\omega_0\ \ \alpha t+\beta}{R^4\ \ t}\right]\eta R\ \ t$$
$$-\frac{\mu_{20}}{R\ \ t}\left[\frac{\xi R\ \ t\ -\xi_2 R\ \ t}{\rho_{23}^3 R^3\ \ t}\right]$$

Simplifying, we get

$$\frac{\xi''}{R^3\ \ t}-\frac{2\omega_0\eta'}{R^3\ \ t}=\frac{\omega_0^2\xi}{R^3\ \ t}-\frac{\xi\ \ \alpha\gamma-\beta^2}{R^3\ \ t}-\frac{\mu_{20}\ \ \xi-\xi_2}{\rho_{23}^3 R^3\ \ t}$$

Multiplying throughout by $R^3\ \ t$ , we at once have

$$\xi''-2\omega_0\eta'=\omega_0^2\xi-\xi\ \ \alpha\gamma-\beta^2\ \ -\frac{\mu_{20}\ \ \xi-\xi_2}{\rho_{23}^3}$$

Similarly, for the second and third equations of system (3.200), we respectively get

$$\eta''+2\omega_0\xi'=\omega_0^2\eta-\eta\ \ \alpha\gamma-\beta^2\ \ -\frac{\mu_{20}\eta}{\rho_{23}^3}$$

$$\zeta''=-\zeta\ \ \alpha\gamma-\beta^2\ \ -\frac{\mu_{20}\zeta}{\rho_{23}^3}$$

Therefore, we have the equations

$$\xi''-2\omega_0\eta'=\ \ \omega_0^2+\beta^2-\alpha\gamma\ \ \xi-\frac{\mu_{20}\ \ \xi-\xi_2}{\rho_{23}^3}$$

$$\eta''+2\omega_0\xi'=\ \ \omega_0^2+\beta^2-\alpha\gamma\ \ \eta-\frac{\mu_{20}\eta}{\rho_{23}^3}$$

$$\zeta''=\ \ \beta^2-\alpha\gamma\ \ \zeta-\frac{\mu_{20}\zeta}{\rho_{23}^3}$$

where $\rho_{23}^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2$

These equations can be expressed as

$$\xi'' - 2\omega_0\eta' = \Omega_\xi$$

$$\eta'' + 2\omega_0\xi' = \Omega_\eta \tag{3.204}$$

$$\zeta'' = \Omega_\zeta$$

where $$\Omega = \frac{\omega_0^2}{2}(\xi^2 + \eta^2) + \frac{\beta^2 - \alpha\gamma}{2}(\xi^2 + \eta^2 + \zeta^2) - \frac{\mu_{20}}{\rho_{23}}$$

$$\rho_{23}^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2 \tag{3.205}$$

Now, from the principle of the center of mass given by equations (3.201), we get

$$\xi_1 = -\frac{\mu_{20}}{\mu_0}\rho_{12},\ \xi_2 = \frac{\mu_{10}}{\mu_0}\rho_{12} \tag{3.206}$$

Next, we introduce the mass parameter $\upsilon$ defined as

$$\frac{\mu_{10}}{\mu_0} = 1 - \upsilon,\ \frac{\mu_{20}}{\mu_0} = \upsilon:\ 0 < \upsilon < 1 \tag{3.207}$$

Using equations (3.207) in (3.206), we get

$$\xi_1 = -\upsilon\rho_{12},\ \xi_2 = (1 - \upsilon)\rho_{12} \tag{3.208}$$

Now, we choose units of measurements for the autonomized system. For the unit of mass, we choose

$$\mu_0 = G \tag{3.209}$$

For unit of length and time, we choose respectively

$$\omega_0 = 1 \text{ and } \rho_{12} = 1 \tag{3.210}$$

Applying equations (3.210) and (3.209) in equations (3.208), we get

$$\xi_1 = -\upsilon,\ \xi_2 = 1-\upsilon \tag{3.211}$$

From the particular integral of the GMP, that is $r\mu = \kappa C^2$, we get

$$G = \kappa \tag{3.212}$$

Here, the gravitational constant is not one as in the Robe (1977) problem but adopts an arbitrary values since $0 < \kappa < \infty$ .

From equations (3.212) and (3.209), we get

$$\mu_0 = G = \kappa \tag{3.213}$$

Hence, equations (3.207) result in

$$\mu_{10} = \kappa\ 1-\upsilon \text{ and } \mu_{20} = \kappa\upsilon \tag{3.214}$$

Substituting all these in the autonomized equations of motion (3.204), we obtain a system of equation with constant coefficients, given as

$$\xi'' - 2\eta' = \frac{\partial\Omega}{\partial\xi},\quad \eta'' + 2\xi' = \frac{\partial\Omega}{\partial\eta},\quad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{3.215}$$

where

$$\Omega = \frac{\kappa\left(\xi^2 + \eta^2\right)}{2} + \frac{\left(\kappa - 1\right)\zeta^2}{2} + \frac{\kappa\upsilon}{\rho_{32}} \tag{3.216}$$

$$\rho_{23}^2 = \left(\xi + \upsilon - 1\right)^2 + \eta^2 + \zeta^2 \tag{3.217}$$

where $\upsilon$ is the mass parameter and is such that $0 < \upsilon < 1$, while $\kappa$ is the constant of the particular integral of the GMP.

## 3.6.2. Locations of equilibrium points

The positions of the EPs are found by solving the equations $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$

That is

$$\kappa\xi - \frac{\kappa\upsilon\ (\xi+\upsilon-1)}{\left[\ (\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} = 0$$

$$\kappa\eta - \frac{\kappa\upsilon\eta}{\left[\ (\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} = 0$$

$$\kappa-1\ \zeta - \frac{\kappa\upsilon\zeta}{\left[\ (\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} = 0$$

These equations can be expressed as

$$\kappa\left[\xi - \frac{\upsilon\ (\xi+\upsilon-1)}{\left[\ (\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}}\right] = 0$$

$$\left[1 - \frac{\upsilon}{\left[\ (\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}}\right]\kappa\eta = 0 \qquad (3.218)$$

$$\left[\kappa-1\ - \frac{\kappa\upsilon}{\left[\ (\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}}\right]\zeta = 0$$

Clearly, from the last equation in (3.218), if $\kappa = 1$, we'll have

$$\frac{\upsilon\zeta}{\left[\ (\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} = 0$$

If $\zeta \neq 0$, then

$$\frac{\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}+\zeta^{2}\right]^{\frac{3}{2}}}=0,$$

which is not possible, so we must have $\zeta = 0$ when $\kappa = 1$.

Now, if in equations (3.218), we suppose that $\zeta = 0$, we have

$$\xi-\frac{\upsilon\ \ \xi+\upsilon-1}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\right]^{\frac{3}{2}}}=0,\quad \kappa\neq 0$$

and $$1-\frac{\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\right]^{\frac{3}{2}}}=0,\quad \kappa\eta\neq 0 \tag{3.219}$$

From the first equation of (3.219), we get

$$\left[1-\frac{\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\right]^{\frac{3}{2}}}\right]\xi+\frac{\upsilon\ 1-\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\right]^{\frac{3}{2}}}=0 \tag{3.220}$$

Using the second equation of (3.219) in (3.220), we get

$$\frac{\upsilon\ 1-\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\right]^{\frac{3}{2}}}=0 \tag{3.221}$$

But $\upsilon\ 1-\upsilon\ \neq 0$, since $0<\upsilon<1$, hence, a contradiction and so we must have $\eta = 0$.

Hence, there is not solution on the $\xi\eta-$ plane and so we search for the solutions on the $\xi\zeta-$ plane only.

**Equilibrium point at the center of the shell**

The EP at the center of the shell is found by solving first equation of (3.218) with $\kappa \neq 0$, $\eta = \zeta = 0$, to get

$$\xi + \frac{\upsilon}{1-\upsilon-\xi^{2}} = 0$$

Now, we denote this equation by $f(\xi)$, that is

$$f(\xi) = \xi + \frac{\upsilon}{\xi+\upsilon-1^{2}} = 0 \tag{3.222}$$

Now, $f'(\xi) > 0$ for $\xi < 1-\upsilon$; therefore, $f(\xi)$ is a strictly increasing function in the interval $(-\infty, 1-\upsilon)$. Also, as $\xi \to -\infty$, $f(\xi) \to -\infty$ and as $\xi \to 1-\upsilon$, $f(\xi) \to +\infty$. Consequently, $f(\xi)$ vanishes only once in the interval $(-\infty, 1-\upsilon)$. Hence, equation (3.222) has only one root in this interval. Solving it, we have

$$\xi(1-2\upsilon-2\xi+\upsilon^{2}+2\upsilon\xi+\xi^{2}+\upsilon) = 0$$

Expanding, we get

$$\xi^{3} - 2(1-\upsilon)\xi^{2} + (1-2\upsilon+\upsilon^{2})\xi + \upsilon = 0$$

$$\Rightarrow \xi^{2}(\xi+\upsilon) - 2\xi(2-\upsilon)(\xi+\upsilon) + (\xi+\upsilon) = 0$$

$$\Rightarrow (\xi+\upsilon)\left[\xi^{2} - 2\xi(2-\upsilon) + 1\right] = 0$$

$$\Rightarrow (\xi+\upsilon) = 0 \text{ or } \xi^{2} - 2\xi(2-\upsilon) + 1 = 0$$

Hence,

$$\xi = -\upsilon \tag{3.223}$$

Now, if $\xi^2 - 2\xi\ \ 2-\upsilon\ \ +1=0$, the two roots are $\xi_{2,3} = \dfrac{\upsilon - 2 \pm \sqrt{\upsilon\ \ \upsilon - 4}}{-2}$

Since $\upsilon - 4 < 0$, we must have $\upsilon\ \ \upsilon - 4\ \ < 0$.

Hence, the two roots are complex and so do not exist.

Equation (3.223) gives the EP located at the center of the spherical shell and is fully analogous to that obtained by Robe (1977).

**Non-collinear equilibrium points**

The existence of the non-collinear EPs was not pointed out in the Robe (1977) problem. The non-collinear EPs of the autonomized systems are the solutions of system (3.218), when $\eta = 0$, $\zeta \neq 0$. That is, they are obtained by solving the first and third equations of system (3.218), provided $\kappa - 1 \neq 0$ and $\zeta \neq 0$. These equations are

$$\kappa\xi - \frac{\kappa\upsilon\ \ \xi + \upsilon - 1}{\left[\ \ \xi + \upsilon - 1\ \ ^2 + \zeta^2\right]^{\frac{3}{2}}} = 0$$

$$\kappa - 1 - \frac{\kappa\upsilon}{\left[\ \ \xi + \upsilon - 1\ \ ^2 + \zeta^2\right]^{\frac{3}{2}}} = 0 \tag{3.224}$$

$$\rho_{23}^2 = \ \ \xi + \upsilon - 1\ \ ^2 + \zeta^2 \tag{3.225}$$

From the second equation of (3.224), we have

$$\kappa - 1 - \frac{\kappa\upsilon}{\rho_{23}^3} = 0$$

$$\Rightarrow \rho_{23} = \left(\frac{\kappa\upsilon}{\kappa - 1}\right)^{\frac{1}{3}} \tag{3.226}$$

Substituting equation (3.226) in the first equation of (3.224), we get

$$\kappa\xi - \frac{\kappa\upsilon\ \ \xi+\upsilon-1\ \ \kappa-1}{\kappa\upsilon} = 0$$

$$\Rightarrow \kappa\xi - \ \xi+\upsilon-1\ \ \kappa-1\ = 0$$

$$\Rightarrow \kappa\xi - \ \kappa-1\ \xi+\ 1-\upsilon\ \ \kappa-1\ = 0$$

$$\Rightarrow\ \kappa-\kappa+1\ \xi+\ 1-\upsilon\ \ \kappa-1\ = 0$$

$$\Rightarrow \xi+\ 1-\upsilon\ \ \kappa-1\ = 0$$

$$\Rightarrow \xi = -\ \kappa-1\ \ 1-\upsilon \qquad (3.227)$$

This gives the abscissa of the non-collinear EPs.

Next, we substitute equation (3.227) and (3.226) in equation (3.225), to get

$$\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} = \ \xi+\upsilon-1\ ^2+\zeta^2$$

Using equation (3.227) yields

$$\Rightarrow\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} = \left[-\ \kappa-1\ \ 1-\upsilon\ +\upsilon-1\right]^2+\zeta^2$$

$$\Rightarrow\zeta^2 = \left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \left[-\ \kappa-1\ \ 1-\upsilon\ +\upsilon-1\right]^2$$

$$\Rightarrow\zeta^2 = \left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \left[-\ 1-\upsilon\ \ \kappa-1\ +\ \upsilon-1\ \right]^2$$

$$\Rightarrow\zeta^2 = \left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \left[\ 1-\upsilon\ \ \kappa-1\ +\ 1-\upsilon\ \right]^2$$

Therefore, we have $\zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \kappa^2\ \ 1-\upsilon\ ^2}$

And so we have the coordinates of the non-collinear EPs is represented by the equations

$$\xi = -\ \kappa-1\ \ 1-\upsilon\ , \qquad \zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \kappa^2\ \ 1-\upsilon\ ^2} \tag{3.228}$$

Equations (3.228) give the position of a pair of EPs $\xi, 0, \zeta$ which exist for $\kappa > 1$ and lies in the $\xi\zeta-$plane. We call these points, non-collinear Eps and they depend on the mass ratio and a constant of a particular integral (1.66) of the GMP. These EPs do not have an analogy in the Robe (1977) problem when the spherical shell is empty. They are also different from those of the R3BP of Bekov (1988), Luk'yanov (1989, 1990) and, Singh and Leke (2010).

When $\kappa = 1$, we have $\xi = 0$ which lies outside the shell and since $\zeta = \pm\infty$, it is seen that infinite remote EPs $L \pm \infty$ do not exist for any value of $0 < \upsilon < 1$.

For the non-autonomous system, the equilibrium point at the center of the shell and the non-collinear points are sought using the MT. These points differ from those of the autonomized system with constant coefficient only by the function $R\ t$ . We express them as

$x^{1}\ \ t\ = -\upsilon R\ t\ , x^{2}\ \ t\ = \xi^{2} R\ t\ ,\ z^{2,3}\ \ t\ = \zeta^{2,3} R\ t$

where $\xi^{2}$ and $\zeta^{2,3}$ are the non-collinear EPs of the autonomized system.

### 3.6.3 Stability of equilibrium points

The stability analysis of the EPs is same with that of the general R3BP. We apply small displacement $u$, $v$ and $w$ to the coordinates $\xi_0, \eta_0, \zeta_0$ of the third body, to the positions, $\xi = \xi_0 + u$, $\eta = \eta_0 + v$ and $\zeta = \zeta_0 + w$. If its motion departs from the vicinity of the point, such a position is an unstable one. However, if it oscillates about the point, then it is a stable position.

We obtain the variational equations in this case from (1.112) by substituting $\omega_0 = 1$ in it, to get

$$\begin{aligned} u'' - 2v' &= \Omega^0_{\xi\xi}\ u + \Omega^0_{\xi\eta}\ v + \Omega^0_{\xi\zeta}\ w, \\ v'' + 2u' &= \Omega^0_{\xi\eta}\ u + \Omega^0_{\eta\eta}\ v + \Omega^0_{\eta\zeta}\ w, \\ w'' &= \Omega^0_{\xi\zeta}\ u + \Omega^0_{\eta\zeta}\ v + \Omega^0_{\zeta\zeta}\ w. \end{aligned} \tag{3.229}$$

The superscript 0 depicts that the partial derivatives are to be evaluated at the equilibrium points. Now, from the potential (3.216), we calculate the second partial derivatives, to get

$$\Omega_{\xi\xi} = \kappa\left[1 - \frac{\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} + \frac{3\upsilon\ \ \xi+\upsilon-1\ ^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{\frac{5}{2}}}\right]$$

$$\Omega_{\eta\eta} = \kappa\left[1 - \frac{\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} + \frac{3\upsilon\eta^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{\frac{5}{2}}}\right]$$

$$\Omega_{\zeta\zeta} = \kappa - 1 - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} + \frac{3\kappa\upsilon\zeta^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{\frac{5}{2}}} \tag{3.230}$$

$$\Omega_{\xi\eta} = \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ \ \eta}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{5/2}}\ ,\ \Omega_{\eta\zeta} = \frac{3\kappa\upsilon\eta\zeta}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\right]^{5/2}}\ ,$$

$$\Omega_{\xi\zeta} = \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ \ \zeta}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}}$$

We now study the stability of the equilibrium points in the subsections below.

**Equilibrium point at the center of the shell**

To study the stability of the equilibrium point located at the center of the empty spherical shell, we find the characteristic equation which corresponds to the first two equations of the variational equations (3.229). In this case, the characteristic equation is obtained by substituting $\omega_0 = 1$ in the characteristic equation (1.114), to get

$$\lambda^4 - \ \Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} - 4\ \ \lambda^2 + \Omega^0_{\xi\xi}\Omega^0_{\eta\eta} - \ \Omega^0_{\xi\eta}\ ^2 = 0 \tag{3.231}$$

Equation (3.231) is the characteristic equation corresponding to the variational equations (3.229) when motion is considered in the $\xi\eta$-plane. The superscript $0$ indicates that the second order partial derivatives will be computed at the EP $-\upsilon, 0, 0$ .

To find the partial derivatives at the point $-\upsilon, 0, 0$ , we substitute $\eta = \zeta = 0$ and $\xi = -\upsilon$ in the equations (3.230), to get

$$\Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon}{\left[\ -\upsilon+\upsilon-1\ ^2\right]^{\frac{3}{2}}} + \frac{3\kappa\upsilon\ \ -\upsilon+\upsilon-1\ ^2}{\left[\ -\upsilon+\upsilon-1\ ^2\right]^{\frac{5}{2}}}$$

$$\Rightarrow \qquad \Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon}{\left|1+\upsilon-\upsilon\right|^3} + \frac{3\kappa\upsilon}{\left|1+\upsilon-\upsilon\right|^3}$$

$$\Rightarrow \qquad \Omega^0_{\xi\xi\xi} = \kappa + \frac{2\kappa\upsilon}{\left|1+\upsilon-\upsilon\right|^3}$$

$$\Rightarrow \qquad \Omega^0_{\xi\xi\xi} = \kappa\ \ 1+2\upsilon$$

Similarly, from second equation of system (3.230), we get

$$\Omega^0_{\eta\eta} = \kappa - \kappa\upsilon$$

The third equation of (4.36), at the EP located at the center of the shell is

$$\Omega^0_{\zeta\zeta} = \kappa - 1 - \frac{\kappa\upsilon}{\left[ -\upsilon + \upsilon - 1^{\ 2} \right]^{\frac{3}{2}}}$$

$$\Rightarrow \qquad \Omega^0_{\zeta\zeta} = \kappa - 1 - \frac{\kappa\upsilon}{\left|1 + \upsilon - \upsilon\right|^3}$$

$$\Rightarrow \qquad \Omega^0_{\zeta\zeta} = \kappa - 1 - \kappa\upsilon$$

$$\Rightarrow \qquad \Omega^0_{\zeta\zeta} = \kappa\ 1 - \upsilon\ - 1$$

Lastly, the last three equations of system (3.230) at the EP located at the center are

$$\Omega^0_{\xi\xi} = 0\,,\ \Omega^0_{\eta\zeta} = 0\,, \Omega^0_{\xi\eta} = 0$$

Hence, the partial derivatives computed at the point near the center of the shell under small effect of perturbations have the forms:

$$\Omega^0_{\xi\xi} = \kappa\ 1 + 2\upsilon\ \,, \Omega^0_{\eta\eta} = \kappa - \kappa\upsilon\,,\ \Omega^0_{\zeta\zeta} = \kappa\ 1 - \upsilon\ - 1 \qquad (3.232)$$

$$\Omega^0_{\xi\zeta} = 0\,,\ \Omega^0_{\eta\zeta} = 0\,, \Omega^0_{\xi\eta} = 0$$

When $\kappa = 1$, these equations reduce to those in Robe (1977).

Next, we substitute these derivatives in the variational equations (3.229), to get

$$u'' - 2v' = \kappa\ 1 + 2\upsilon\ u$$

$$v'' + 2u' = \kappa\ 1 - \upsilon\ v \qquad (3.233)$$

$$w'' = \ \kappa - 1 - \kappa\upsilon\ w \qquad (3.234)$$

Equation of (3.234) is independent of the first two equations (3.233) and depicts that the motion parallel to the $\zeta$ – axis is stable when $1 \le \kappa < \frac{1}{1-\upsilon}$ , otherwise it is unstable.

Now, substituting the partial derivatives in the characteristic equation (3.231) and simplifying gives

$$\lambda^4 + \ 4-2\kappa-\kappa\upsilon \ \lambda^2+\kappa^2 \ 1-\upsilon \ \ 1+2\upsilon \ =0 \tag{3.235}$$

Clearly, $\kappa^2 \ 1-\upsilon \ \ 1+2\upsilon \ >0$ since $0<\upsilon<1 \text{and} \kappa>0$; but the coefficient of $\lambda^2$ may be negative, zero or positive, depending on whether $\kappa <=> \frac{4}{2+\upsilon}$, respectively.

The roots of equation (3.235) are

$$\lambda_{1,2}^2 = \frac{\kappa \ 2+\upsilon \ -4 \pm \sqrt{D}}{2}$$

where

$$D=-16 \ \kappa-1 \ +\kappa\upsilon \ 9\kappa\upsilon-8 \tag{3.236}$$

Now when $\kappa=1$, $D$ is only zero when $\upsilon=\frac{8}{9}$; When $\kappa \neq 1$, the discriminant vanishes when

$$\upsilon_{c\kappa} = \frac{4}{9\kappa} \ 1+\sqrt{9\kappa-8} \tag{3.237}$$

where $\upsilon_{c\kappa}$ are the values of $\upsilon$ at which the discriminant is zero. These values are called the critical mass values which exist for $\kappa \geq \frac{8}{9}$ and different from that found by Robe (1977).

Now, from equation (3.236), we have

$$\frac{dD}{d\upsilon} = 2\kappa\ \ 9\kappa\upsilon - 4 \tag{3.238}$$

Also, $\frac{dD}{d\upsilon} = 0$ when $\upsilon = \frac{4}{9\kappa}$.

Therefore, equation (3.238) is positive when $\upsilon > \frac{4}{9\kappa}$ and negative when $\upsilon < \frac{4}{9\kappa}$.

This means that $D$ is strictly increasing when the former holds and strictly decreasing when the later occurs.

Now when $\upsilon = 0$ in Equation (3.236), we have

$$D = -16\ \ \kappa - 1 \tag{3.239}$$

However, when $\upsilon = \frac{4}{9\kappa}$, equation (3.236) reduces to

$$D = -\frac{16\ \ 9\kappa - 10}{9} \tag{3.239}$$

We observe that equation (3.239) is negative when $\kappa > \frac{10}{9}$. Further, as $\upsilon$ increases to $\upsilon_{c\kappa}$, $D$ increases from the value in (3.239) to zero. Finally, $\upsilon_{c\kappa}$ increases to 1, when $D$ increases from 0 to

$$D = 9\kappa^2 - 24\kappa + 16 \tag{3.240}$$

An inspection of equation (3.240) reveals that for $0 < \kappa < \infty$, we have $D > 0$.

Now, since the natures of the roots depend on the discriminant, the mass ratio, and kappa; we consider the three regions of equation (3.236) coupled with the changes in the coefficient of $\lambda^2$.

1. When simultaneously, $\kappa > \frac{4}{2+\upsilon}$ or $\kappa < \frac{4}{2+\upsilon}$ and $0 < \upsilon < \upsilon_{c\kappa}$, $D < 0$; the roots are respectively

   $$\lambda_{1,2} = \pm\frac{1}{\sqrt{2}}\left[\kappa\ 2+\upsilon\ -4+i\beta_1\right]^{1/2} \quad , \quad \lambda_{3,4} = \pm\frac{1}{\sqrt{2}}\left[\kappa\ 2+\upsilon\ -4-i\beta_1\right]^{1/2}$$

   and

   $$\lambda_{1,2} = \pm\frac{1}{\sqrt{2}}\left[4-\kappa\ 2+\upsilon\ -i\beta_1\right]^{1/2}, \lambda_{3,4} = \pm\frac{1}{\sqrt{2}}\left[4-\kappa\ 2+\upsilon\ -i\beta_1\right]^{1/2}$$

   where $i\beta_1 = \sqrt{D}$

   The real parts of two roots are positive and equal in both cases. Therefore, the equilibrium point is unstable

2. When $\upsilon = \upsilon_{c\kappa}$, $D = 0$; the following cases are possible

   a. If $\kappa < \frac{4}{2+\upsilon}$, two roots are real and equal while the remaining two are negative and equal as well. The EP is unstable.

b. If $\kappa = \frac{4}{2+\upsilon}$, here all the roots are zero and the EP is unstable.

c. If $\kappa > \frac{4}{2+\upsilon}$, all the four roots are imaginary with two positive and equal and the other two are negative and equal. In this case, we have positive stable resonance.

3. When $\upsilon_{c\kappa} < \upsilon < 1$, $D$ is positive; here when $\kappa > \frac{4}{2+\upsilon}$, the characteristic roots are all distinct pure imaginary. In this case, the EP is stable. However, when $\kappa = \frac{4}{2+\upsilon}$, there are two positive and negative equal roots; while for $\kappa < \frac{4}{2+\upsilon}$, the roots are real and distinct. The EP is unstable in both cases due to a positive root.

Hence, we conclude that the EP is stable for $\upsilon_{c\kappa} < \upsilon < 1$ provided $\kappa > \frac{4}{2+\upsilon}$; otherwise it is unstable.

**Non-collinear equilibrium points**

The characteristic equation in this is obtained from the variational equations (1.115) by substituting $\omega_0 = 1$, to get

$$\lambda^6 - \lambda^4\left(\Omega^0_{\xi\xi} + \Omega^0_{\eta\eta} + \Omega^0_{\zeta\zeta} - 4\right) + \lambda^2\left[\Omega^0_{\xi\xi}\Omega^0_{\eta\eta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - 4\Omega^0_{\zeta\zeta} - \left(\Omega^0_{\xi\eta}\right)^2 - \left(\Omega^0_{\xi\zeta}\right)^2 - \left(\Omega^0_{\zeta\eta}\right)^2\right]$$
$$-\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}\Omega^0_{\zeta\zeta} - 2\Omega^0_{\xi\eta}\Omega^0_{\xi\zeta}\Omega^0_{\eta\zeta} + \Omega^0_{\xi\xi}\left(\Omega^0_{\eta\zeta}\right)^2 + \Omega^0_{\zeta\zeta}\left(\Omega^0_{\xi\eta}\right)^2 + \Omega^0_{\eta\eta}\left(\Omega^0_{\xi\zeta}\right)^2 = 0 \qquad 3.241$$

The partial derivatives computed at these EPs are obtained by substituting the coordinate (3.228) and $\eta = 0$ in equations (3.230), to get

$$\Omega^0_{\xi\xi} = 1 + \frac{3\kappa^{4/3}\ (1-\upsilon)^2\ (\kappa-1)^{5/3}}{\upsilon^{2/3}},\quad \Omega^0_{\zeta\zeta} = 3\ (\kappa-1)\left[1 - \frac{3\kappa^{4/3}\ (1-\upsilon)^2\ (\kappa-1)^{2/3}}{\upsilon^{2/3}}\right] \tag{3.242}$$

$$\Omega^0_{\eta\eta} = 1,\ \Omega^0_{\xi\zeta} = -\frac{3\zeta\kappa^{1/3}\ (1-\upsilon)\ (\kappa-1)^{5/3}}{\upsilon^{2/3}},\ \Omega^0_{\eta\zeta} = \Omega^0_{\xi\eta} = 0$$

If we substitute $\Omega^0_{\eta\zeta} = \Omega^0_{\xi\eta} = 0$ and $\Omega^0_{\eta\eta} = 1$ in the characteristic equation (3.241), it reduces to

$$\lambda^6 - \lambda^4\ (\Omega^0_{\xi\xi} + \Omega^0_{\zeta\zeta} - 3) + \lambda^2\left[\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + \Omega^0_{\xi\xi} - 4\Omega^0_{\zeta\zeta} - (\Omega^0_{\xi\zeta})^2\right] - \Omega^0_{\xi\xi}\Omega^0_{\zeta\zeta} + (\Omega^0_{\xi\zeta})^2 = 0$$

Now, we substitute the values of the other partial derivatives and simplify to get

$$\lambda^6 - p_1\lambda^4 + p_2\lambda^2 - p_3 = 0 \tag{3.243}$$

where

$$p_1 = 3\kappa - 5$$

$$p_2 = 7 - 6\kappa - \frac{3\kappa^{4/3}\ (\kappa-1)^{5/3}}{\upsilon^{2/3}}\left[\frac{3\zeta^2\ (\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}} - (1-\upsilon)^2\ (5-3\kappa) + \frac{3\kappa^{4/3}\ (1-\upsilon)^4\ (\kappa-1)^{5/3}}{\upsilon^{2/3}}\right]$$

$$p_3 = 1 + \frac{3\kappa^2\ (1-\upsilon)^2\ (\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}}\left[\frac{1}{(\kappa\upsilon)^{2/3}} - 6\ (\kappa-1) + \frac{6\kappa^2\ (1-\upsilon)^2\ (\kappa-1)^{5/3}}{(\kappa\upsilon)^{2/3}} - \frac{6\kappa\ (1-\upsilon)^2\ (\kappa-1)^{8/3}}{(\kappa\upsilon)^{2/3}}\right]$$

The stability of the non-collinear EPs is completely determined by the roots of the characteristic equation in (3.243). Using the software package *Mathematica* (Wolfram 2003), the roots are:

$$\lambda_{1,2} \to \pm\left[\frac{p_1}{3} + \frac{p_1^2 - p_2}{3}\left(\frac{2}{N}\right)^{1/3} + \frac{N^{1/3}}{32^{1/3}}\right]^{1/2}$$

$$\lambda_{3,4} \to \pm\left[\frac{p_1}{3} - \frac{p_1^2}{32^{2/3}N^{1/3}} + \frac{ip_1^2}{2^{2/3}\sqrt{3}N^{1/3}} + \frac{p_2}{2^{2/3}N^{1/3}} - \frac{ip_2\sqrt{3}}{2^{2/3}N^{1/3}} + \frac{N^{1/3}}{22^{1/3}\sqrt{3}} + \frac{N^{1/3}}{62^{1/3}}\right]^{1/2}$$

$$\lambda_{5,6} \to \pm\left[\frac{p_1}{3} - \frac{p_1^2}{32^{2/3}N^{1/3}} - \frac{ip_1^2}{2^{2/3}\sqrt{3}N^{1/3}} + \frac{p_2}{2^{2/3}N^{1/3}} + \frac{i\sqrt{3}p_2}{2^{2/3}N^{1/3}} + \frac{N^{1/3}}{22^{1/3}\sqrt{3}} - \frac{N^{1/3}}{62^{1/3}}\right]^{1/2}$$

where $N = 2p_1^3 - 9p_1p_2 + 27p_3 + \sqrt{4\ 3p_2 - p_1^2\ ^3 + \ 2p_1^3 - 9p_1p_2 + 27p_3\ ^2}$

A positive root and positive real part of the complex roots induce instability at the non-collinear EPs. Hence the motion of the third body around the non-collinear EP is unstable.

In view of the EPs of the non-autonomous system, because the solutions contain the function $R\ t$ , they diverge with time and consequently motion in the vicinity of these points departs away rapidly. Hence, the EPs of the non-autonomous system are unstable according to the Lyapunov's theorem.

## 3.7 Robe's Restricted Problem with Isotropic Mass Variation and Perturbing Forces when the First Primary is Empty

In this problem, we modify problem **3.6** by introducing small perturbations in the Coriolis and centrifugal forces. The motivation for this study is drawn from the works conducted by Shrivastava and Garain (1991) and Hallan and Rana (2001a).

### 3.7.1 Equations of motion of the third body

The equations of motion of the third body in a rotating coordinate system under effect of small perturbations in the Coriolis and centrifugal forces have the form:

$$\ddot{x} - 2\omega\varphi\dot{y} = \omega^2\psi x + \dot{\omega}y - \frac{D_p\ (x-x_1)}{r_{13}^3} - \frac{\mu_2\ (x-x_2)}{r_{23}^3}$$

$$\ddot{y} + 2\omega\varphi\dot{x} = \omega^2\psi y - \dot{\omega}x - \frac{D_p y}{r_{13}^3} - \frac{\mu_2 y}{r_{23}^3} \qquad (3.244)$$

$$\ddot{z} = -\frac{D_p z}{r_{13}^3} - \frac{\mu_2 z}{r_{23}^3}$$

where $D_p = \dfrac{4\pi G\Re^3\delta_1}{3r_{13}^3}\left(1-\dfrac{\delta_1}{\delta_3}\right)$, $r_{13}^2 = (x-x_1)^2 + y^2 + z^2$ , $r_{23}^2 = (x-x_2)^2 + y^2 + z^2$

$\varphi = 1+\in$, $\in << 1$, and $\psi = 1+\in'$ , $\in' << 1$; are the perturbations in the Coriolis and centrifugal forces, respectively. The other parameters remained as defined in problem **3.6**.

Now, because the system of equations (3.244) cannot be transformed to the autonomized form using the MT, the UML, the particular solutions and integral of the GMP when the density parameter is non-zero; we restrict ourselves to the case when the there is no fluid in the shell. Hence, we substitute $D_p = 0$ in equations (3.244) and then we can completely transform it.

Now, following same procedure used in problem **3.6**, the autonomized equations of motion with constant coefficients are

$$\xi'' - 2\varphi\eta' = \frac{\partial\Omega}{\partial\xi}, \quad \eta'' + 2\varphi\xi' = \frac{\partial\Omega}{\partial\eta}, \qquad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{3.245}$$

where $\Omega = \dfrac{(\psi+\kappa-1)(\xi^2+\eta^2)}{2} + \dfrac{(\kappa-1)\zeta^2}{2} + \dfrac{\kappa\upsilon}{\rho_{32}}$

$$\rho_{23}^2 = (\xi+\upsilon-1)^2 + \eta^2 + \zeta^2$$

The parameters remained as previously defined in problem **3.6**.When $\psi = \varphi = 1$, the small perturbations in the Coriolis and centrifugal forces are ineffective and both equations of motion (3.244) and (3.245) coincide fully with those in problem **3.6.**

## 3.7.2 Locations of equilibrium points

The positions of the EPs are found by solving the equations $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$. That is

$$\psi+\kappa-1\ \xi - \frac{\kappa\upsilon\ (\xi+\upsilon-1)}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}} = 0$$

$$\left[\psi+\kappa-1\ - \frac{\kappa\upsilon}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}}\right]\eta = 0 \tag{3.246}$$

$$\left[\kappa-1\ - \frac{\kappa\upsilon}{\left[(\xi+\upsilon-1)^2+\eta^2+\zeta^2\right]^{\frac{3}{2}}}\right]\zeta = 0$$

From the last equation in (3.246), if $\kappa = 1$ and $\zeta \neq 0$, we'll have

$$\frac{\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}+\zeta^{2}\ \right]^{\frac{3}{2}}}=0,$$

which is not possible; so we must have $\zeta=0$ when $\kappa=1$.

Now, if in equations (3.246), we suppose that $\zeta=0$, we have

$$\psi+\kappa-1\ \xi-\frac{\kappa\upsilon\ \ \xi+\upsilon-1}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\ \right]^{\frac{3}{2}}}=0$$

$$\psi+\kappa-1\ -\frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\ \right]^{\frac{3}{2}}}=0,\quad \eta\neq 0 \tag{3.247}$$

From the first equation of (3.247), we get

$$\left[\psi+\kappa-1-\frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\ \right]^{\frac{3}{2}}}\right]\xi+\frac{\kappa\upsilon\ \ 1-\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\ \right]^{\frac{3}{2}}}=0 \tag{3.248}$$

Using the second equation of (3.247) in (3.248), yields

$$\frac{\kappa\upsilon\ \ 1-\upsilon}{\left[\ \xi+\upsilon-1\ ^{2}+\eta^{2}\ \right]^{\frac{3}{2}}}=0 \tag{3.249}$$

But $\kappa\upsilon\ \ 1-\upsilon\ \neq 0$, since $0<\kappa<\infty$ and $0<\upsilon<1$. Therefore, equation (3.249) is a contradiction and so we must have $\eta=0$. Hence, we search for the solutions of equations (3.246) on the $\xi\zeta-$ plane.

**Equilibrium point near the center of the shell**

The EP located near the center of the first primary which is a spherical shell is found by solving the first equation of (3.246) with $\eta = \zeta = 0$ to get

$$\psi + \kappa - 1\ \xi - \frac{\kappa\upsilon\ \ \xi + \upsilon - 1}{\left[\ \ \xi + \upsilon - 1\ ^2\right]^{\frac{3}{2}}} = 0$$

$$\Rightarrow\ \psi + \kappa - 1\ \xi + \frac{\kappa\upsilon}{1 - \upsilon - \xi\ ^2} = 0 \tag{3.250}$$

Now, when there is no deflection due to the centrifugal force, the only solution of (3.250) is $\xi = -\upsilon$ as observed from problem **3.6**. Therefore, when a small perturbation is effective, we can assume the solution of (3.250), to be

$$\xi_p = -\upsilon + \chi\ ,\ \chi << 1 \tag{3.251}$$

Substituting $\xi_p = \xi$ in (3.250), we get

$$\psi + \kappa - 1\ \ -\upsilon + \chi\ + \frac{\kappa\upsilon}{\left[1 - \upsilon - \ -\upsilon + \chi\ \right]^2} = 0$$

$$\Rightarrow\ -\upsilon\ \psi + \kappa - 1\ + \chi\ \psi + \kappa - 1\ + \frac{\kappa\upsilon}{1 - \chi\ ^2} = 0$$

$$\Rightarrow\ -\upsilon\ \psi + \kappa - 1\ + \kappa\chi + \chi\ \psi - 1\ + \frac{\kappa\upsilon}{1 - \chi\ ^2} = 0$$

We ignore the product $\chi\ \psi - 1$ , since they are very small, that is $\chi << 1$ and $\psi - 1\ << 1$; to get

$$-\upsilon\ \psi + \kappa - 1\ + \kappa\chi + \frac{\kappa\upsilon}{1 - \chi\ ^2} = 0$$

$$\Rightarrow\ -\upsilon\ 1 - \chi\ ^2\ \psi + \kappa - 1\ + \kappa\ 1 - \chi\ ^2\ \chi + \kappa\upsilon = 0$$

$$\Rightarrow \quad -\upsilon\ (1-2\chi+\chi^2\ \ \psi+\kappa-1\ )+\kappa\ (1-2\chi+\chi^2\ \ \chi+\kappa\upsilon=0$$

Ignoring higher powers in $\chi$, we get

$$-\upsilon\ (1-2\chi\ \ \psi+\kappa-1\ )+\kappa\chi+\kappa\upsilon=0$$

$$\Rightarrow \quad -\kappa\upsilon\ (1-2\chi\ )-\upsilon\ (1-2\chi\ \ \psi-1\ )+\kappa\chi+\kappa\upsilon=0$$

Ignoring the product $\chi\ \psi-1$ gives

$$-\upsilon\ (\psi-1\ )-\kappa\upsilon\ (1-2\chi\ )+\kappa\chi+\kappa\upsilon=0$$

$$\Rightarrow \quad -\upsilon\ (\psi-1\ )-\kappa\upsilon+2\kappa\upsilon\chi+\kappa\chi+\kappa\upsilon=0$$

$$\Rightarrow \quad -\upsilon\ (\psi-1\ )+\kappa\chi\ (2\upsilon+1\ )=0$$

$$\Rightarrow \quad \kappa\chi\ (2\upsilon+1\ )=\upsilon\ (\psi-1)$$

Therefore

$$\Rightarrow \quad \chi=\frac{\upsilon\ (\psi-1}{\kappa\ (1+2\upsilon)} \tag{3.252}$$

Hence, substituting equation (3.252) in (3.251), yields

$$\xi_p=-\upsilon+\bar{q}\ (\psi-1 \tag{3.253}$$

where $\bar{q}=\dfrac{\upsilon}{\kappa\ (1+2\upsilon)}$

These points lie to the right or left of the center of the shell depending on whether the perturbation is positive or negative. Further, due to the centrifugal force and the arbitrary constant $\kappa\ \ 0<\kappa<\infty$ , every point on the line joining the centers of the primaries is an EP, provided these points lie within the shell. When $\kappa=1$, the coordinate is same as that found by Hallan and Rana (2001a).

**Non-collinear equilibrium points**

The non-collinear EPs of the autonomized systems are the solutions of (3.246), when $\eta = 0, \zeta \neq 0$ and $\kappa > 1$. That is, we solve the equations

$$\psi + \kappa - 1\ \xi - \frac{\kappa\upsilon\ \ \xi + \upsilon - 1}{\left[\ \xi + \upsilon - 1\ ^{2} + \zeta^{2}\right]^{\frac{3}{2}}} = 0$$

$$\kappa - 1 - \frac{\kappa\upsilon}{\left[\ \xi + \upsilon - 1\ ^{2} + \zeta^{2}\right]^{\frac{3}{2}}} = 0 \tag{3.254}$$

$$\rho_{23}^{2} = \ \xi + \upsilon - 1\ ^{2} + \zeta^{2} \tag{3.255}$$

From the second equation of (3.254), we solve to get

$$\rho_{23} = \left(\frac{\kappa\upsilon}{\kappa - 1}\right)^{1/3} \tag{3.256}$$

Substituting equation (3.256) in the first equation of (3.254), we have

$$\psi + \kappa - 1\ \xi - \frac{\kappa\upsilon\ \ \xi + \upsilon - 1\ \ \kappa - 1}{\kappa\upsilon} = 0$$

$$\Rightarrow \quad \psi + \kappa - 1\ \xi - \ \xi + \upsilon - 1\ \ \kappa - 1\ = 0$$

$$\Rightarrow \quad \psi + \kappa - 1\ \xi - \ \kappa - 1\ \xi + \ 1 - \upsilon\ \ \kappa - 1\ = 0$$

$$\Rightarrow \quad \psi + \kappa - 1 - \kappa + 1\ \xi + \ 1 - \upsilon\ \ \kappa - 1\ = 0$$

$$\Rightarrow \quad \psi\xi + \ 1 - \upsilon\ \ \kappa - 1\ = 0$$

$$\Rightarrow \quad \xi = -\frac{1 - \upsilon\ \ \kappa - 1}{\psi} \tag{3.257}$$

This gives the abscissa of the non-collinear equilibrium points.

Next, we use equations (3.257) and (3.256) in (3.254), to get

$$\left[\frac{\kappa\upsilon}{\kappa-1}\right]^{\frac{2}{3}} = \xi+\upsilon-1\ ^2+\zeta^2$$

Using equation (3.257), yields

$$\Rightarrow \quad \left[\frac{\kappa\upsilon}{\kappa-1}\right]^{\frac{2}{3}} = \left[-\frac{1-\upsilon\quad\kappa-1}{\psi}+\upsilon-1\right]^2+\zeta^2$$

$$\Rightarrow \quad \zeta^2 = \left[\frac{\kappa\upsilon}{\kappa-1}\right]^{\frac{2}{3}} - \left[\frac{-\ 1-\upsilon\quad\kappa-1}{\psi}+\upsilon-1\right]^2$$

$$\Rightarrow \quad \zeta^2 = \left[\frac{\kappa\upsilon}{\kappa-1}\right]^{\frac{2}{3}} - \left[\frac{-\ 1-\upsilon\quad\kappa-1\ +\ \upsilon-1\ \psi}{\psi}\right]^2$$

$$\Rightarrow \quad \zeta^2 = \left[\frac{\kappa\upsilon}{\kappa-1}\right]^{\frac{2}{3}} - \frac{1}{\psi^2}\left[-\ 1-\upsilon\quad\kappa-1\ +\psi\ \ \upsilon-1\ \right]^2$$

$$\Rightarrow \quad \zeta^2 = \left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \frac{1}{\psi^2}\left[-\ 1-\upsilon\quad\psi+\kappa-1\ \right]^2$$

$$\Rightarrow \quad \zeta^2 = \left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \frac{1}{\psi^2}\ 1-\upsilon\ ^2\ \ \psi+\kappa-1\ ^2$$

Simplifying and ignoring higher order terms in $\psi-1$, we get

$$\zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \frac{\kappa\ 1-\upsilon\ ^2\left[2\ \psi-1\ +\kappa\right]}{\psi^2}}$$

Hence, the coordinates of the non-collinear EPs are

$$\xi = -\frac{1-\upsilon\quad\kappa-1}{\psi}, \quad \zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{\frac{2}{3}} - \frac{\kappa\ 1-\upsilon\ ^2\left[2\ \psi-1\ +\kappa\right]}{\psi^2}} \tag{3.258}$$

This gives the position of a pair of EPs which lie in the $\xi\zeta$ – plane, and depend on the mass ratio, small perturbation in the centrifugal force and kappa. Numerically, it is observed that these equilibrium points exist in the Earth-Moon system only when $1.0036 \le \kappa \le 1.01$ for any value of the perturbation in the centrifugal force chosen very small. We note that these points do not exist in any study of the Robe (1977) problem when the density parameter is zero.

The EP near the center of the shell and the non-collinear points for the non-autonomous system are sought using the MT and differ from those of the autonomized system with constant coefficient only by the function $R\ t$ . We express them as

$$x^{1}\ t = -\xi_p R\ t\ , x^{2}\ t = \xi^{2} R\ t\ ,\ z^{2,3}\ t = \zeta^{2,3} R\ t$$

where $\xi_p$, $\xi^{2}$ and $\zeta^{2,3}$ are the solutions found for the autonomized system.

### 3.7.3. Stability of equilibrium points

The variational equations of the autonomized system under effects of small perturbations in the Coriolis and centrifugal forces when motion takes place in the $\xi\zeta$ – plane are:

$$\begin{aligned} u'' - 2\phi v' &= \Omega^0_{\xi\xi}\ u + \Omega^0_{\xi\eta}\ v + \Omega^0_{\xi\zeta}\ w, \\ v'' + 2\phi u' &= \Omega^0_{\xi\eta}\ u + \Omega^0_{\eta\eta}\ v + \Omega^0_{\eta\zeta}\ w, \\ w'' &= \Omega^0_{\xi\zeta}\ u + \Omega^0_{\eta\zeta}\ v + \Omega^0_{\zeta\zeta}\ w. \end{aligned} \tag{3.259}$$

where $u, v, w$ are small displacements and the derivatives are evaluated at the EPs.

Now from the force function in equations (3.245), the second order derivatives in this case are

$$\Omega_{\xi\xi} = \psi + \kappa - 1\ - \frac{\kappa\upsilon}{\left[\ \xi + \upsilon - 1\ ^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} + \frac{3\kappa\upsilon\ \ \xi + \upsilon - 1\ ^2}{\left[\ \xi + \upsilon - 1\ ^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}}$$

$$\Omega_{\eta\eta} = \psi + \kappa - 1 - \frac{\kappa\upsilon}{\left[ \xi + \upsilon - 1 \right]^2 + \eta^2 + \zeta^2 \Big]^{\frac{3}{2}}} + \frac{3\kappa\upsilon\eta^2}{\left[ \xi + \upsilon - 1 \right]^2 + \eta^2 + \zeta^2 \Big]^{\frac{5}{2}}}$$

$$\Omega_{\zeta\zeta} = \kappa - 1 - \frac{\kappa\upsilon}{\left[ \xi + \upsilon - 1 \right]^2 + \eta^2 + \zeta^2 \Big]^{\frac{3}{2}}} + \frac{3\kappa\upsilon\zeta^2}{\left[ \xi + \upsilon - 1 \right]^2 + \eta^2 + \zeta^2 \Big]^{\frac{5}{2}}} \tag{3.260}$$

$$\Omega_{\xi\eta} = \frac{3\kappa\upsilon \ \xi + \upsilon - 1 \ \eta}{\left[ \xi + \upsilon - 1 \right]^2 + \eta^2 + \zeta^2 \Big]^{5/2}} \quad , \ \Omega_{\eta\zeta} = \frac{3\kappa\upsilon\eta\zeta}{\left[ \xi + \upsilon - 1 \right]^2 + \eta^2 + \zeta^2 \Big]^{5/2}}$$

$$\Omega_{\xi\zeta} = \frac{3\kappa\upsilon \ \xi + \upsilon - 1 \ \zeta}{\left[ \xi + \upsilon - 1 \right]^2 + \eta^2 + \zeta^2 \Big]^{5/2}}$$

**Equilibrium points near the center of the shell**

Following problem **3.6**, the obtained characteristic equation in this case is

$$\lambda^4 - \left( \Omega^0_{\xi\xi\xi} + \Omega^0_{\eta\eta} - 4\varphi^2 \right) \lambda^2 + \Omega^0_{\xi\xi\xi}\Omega^0_{\eta\eta} - \left( \Omega^0_{\xi\eta} \right)^2 = 0 \tag{3.261}$$

The superscript $0$ indicates that the second order partial derivatives will be computed at the EP $\xi_p, 0, 0$ near the center of the shell.

Now, we substitute $\eta = \zeta = 0$ in the equations (3.260), to get

$$\Omega^0_{\xi\xi\xi} = \psi + \kappa - 1 - \frac{\kappa\upsilon}{\left[ \xi + \upsilon - 1 \right]^2 \Big]^{\frac{3}{2}}} + \frac{3\kappa\upsilon \left( \xi + \upsilon - 1 \right)^2}{\left[ \xi + \upsilon - 1 \right]^2 \Big]^{\frac{5}{2}}}$$

$$\Rightarrow \quad \Omega^0_{\xi\xi\xi} = \psi + \kappa - 1 - \frac{\kappa\upsilon}{\left| \xi + \upsilon - 1 \right|^3} + \frac{3\kappa\upsilon}{\left| \xi + \upsilon - 1 \right|^3}$$

$$\Rightarrow \quad \Omega^0_{\xi\xi\xi} = \psi + \kappa - 1 + \frac{2\kappa\upsilon}{\left| \xi + \upsilon - 1 \right|^3}$$

Similarly, we have

$$\Omega^0_{\eta\eta} = \psi + \kappa - 1 - \frac{\kappa\upsilon}{\left|\xi + \upsilon - 1\right|^3}$$

$$\Omega^0_{\zeta\zeta} = \kappa - 1 - \frac{\kappa\upsilon}{\left|\xi + \upsilon - 1\right|^3}$$

$$\Omega^0_{\xi\zeta} = 0,\ \Omega^0_{\eta\zeta} = 0,\ \Omega^0_{\xi\eta} = 0$$

Substituting the EP in equation (3.253) in the above, gives

$$\Omega^0_{\xi\xi} = \psi + \kappa - 1 + \frac{2\kappa\upsilon}{\left[\bar{q}\ \psi - 1\ -1\right]^3}\ ,\quad \Omega^0_{\eta\eta} = \psi + \kappa - 1 - \frac{\kappa\upsilon}{\left[\bar{q}\ \psi - 1\ -1\right]^3} \tag{3.262}$$

$$\Omega^0_{\zeta\zeta} = \kappa - 1 - \frac{\kappa\upsilon}{\left[\bar{q}\ \psi - 1\ -1\right]^3}\ ,\quad \Omega^0_{\xi\zeta} = 0,\ \Omega^0_{\eta\zeta} = 0,\ \Omega^0_{\xi\eta} = 0$$

These are the partial derivatives computed at the point (3.253), when $\psi = 1$, these partial derivatives (3.262) reduce fully to those calculated at the EP located at the center of the shell in problem **3.6**. Further, when $\kappa = 1$, in the presence of effective perturbation in the centrifugal force, the partial derivatives fully coincide with those of Hallan and Rana (2001a) and reduce to that in Robe (1977) when $\psi = 1 = \kappa$.

Now, substituting the required equations of (3.262) in the first equation of the variational equations of motion (3.259), we get

$$u'' - 2\varphi v' = \left[\frac{\psi + \kappa - 1\ \left[-\bar{q}\ \psi - 1\ +1\right]^3 + 2\kappa\upsilon}{\left[-\bar{q}\ \psi - 1\ +1\right]^3}\right] u$$

For simplicity, we let $\psi - 1 = \in'$ so that

$$u'' - 2\varphi v' = \left[\frac{\kappa + \in'\quad 1 - \bar{q}\ \in'^{\,3} + 2\kappa\upsilon}{1 - \bar{q}\ \in'^{\,3}}\right] u$$

Using Binomial expansions and ignoring second and higher terms in $\in'$, we at once have

$$u'' - 2\varphi v' = \left[\ \kappa + \in' \ \ 1 - 3\bar{q} \in' \ + 2\kappa\upsilon\right] 1 - \bar{q} \in'^{-3} u$$

$$\Rightarrow \quad u'' - 2\varphi v' = \left[\ \kappa + \in' \ \ 1 - 3\bar{q} \in' \ + 2\kappa\upsilon\right] 1 + 3\bar{q} \in' \ u$$

$$\Rightarrow \quad u'' - 2\varphi v' = \ \kappa - 3\kappa\bar{q} \in' + \in' + 2\kappa\upsilon \ \ 1 + 3\bar{q} \in' \ u$$

$$\Rightarrow \quad u'' - 2\varphi v' = \ \kappa + 3\kappa\bar{q} \in' - 3\kappa\bar{q} \in' + \in' + 2\kappa\upsilon + 6\kappa\upsilon\bar{q} \in' \ u$$

$$\Rightarrow \quad u'' - 2\varphi v' = \ \kappa + 2\kappa\upsilon + \in' + 6\kappa\upsilon\bar{q} \in' \ u$$

Hence,

$$u'' - 2\varphi v' = \left[\kappa \ 1 + 2\upsilon \ + \ 1 + 6\kappa\upsilon\bar{q} \ \in'\right] u$$

Similarly, substituting (3.262) in the second and third equations of (3.259); using Binomial expansions and ignoring second and higher terms in $\in'$, we get

$$v'' + 2\varphi u' = \left[\kappa \ 1 - \upsilon \ + \ 1 - 3\kappa\upsilon\bar{q} \ \in'\right] v$$

$$w'' = \left[\kappa - 1 - \kappa\upsilon \ 1 + 3\bar{q} \in' \ \right] w$$

And so the variational equations of motion for the points near the center of the shell are

$$u'' - 2\varphi v' = \left[\kappa \ 1 + 2\upsilon \ + \ 1 + 6\kappa\upsilon\bar{q} \ \in'\right] u$$

$$v'' + 2\varphi u' = \left[\kappa \ 1 - \upsilon \ + \ 1 - 3\kappa\upsilon\bar{q} \ \in'\right] v \qquad (3.263)$$

$$w'' = \left[\kappa - 1 - \kappa\upsilon \ 1 + 3\bar{q} \in' \ \right] w$$

Now, the characteristic equation corresponding to the first two equations of (3.263) is

$$\lambda^4 + P\lambda^2 + Q = 0 \qquad (3.264)$$

where

$$P = 4 + 8\epsilon - \left(2 + \frac{3\upsilon^2}{1+2\upsilon}\right)\epsilon' - \kappa\ \ 2+\upsilon\ ,\ Q = \kappa^2\ \ 1-\upsilon\ \ \ 1+2\upsilon\ \ + \kappa\epsilon'\left[\left(2+\upsilon+\frac{3\upsilon^2}{1+2\upsilon}\ \ 1-4\upsilon\ \right)\right].$$

Clearly, $Q > 0$ while $P <=> 0$ when $\kappa <=> \dfrac{4\ \ 1+2\upsilon\ \ +8\epsilon\ \ 1+2\upsilon\ \ -\ \ 3\upsilon^2+4\upsilon+2\ \ \epsilon'}{1+2\upsilon\ \ \ 2+\upsilon}$,

respectively.

The roots are

$$\lambda_{1,2}^2 = \frac{-\left[4+8\epsilon-\kappa\ \ 2+\upsilon\ \ -\ \ 2+3\kappa\upsilon\bar{q}\ \ \epsilon'\right] \pm \sqrt{D}}{2}$$

where

$$D = -16\ \ \kappa-1\ \ +\kappa\upsilon\ \ 9\kappa\upsilon-8\ \ +16\ \ 4-2\kappa-\kappa\upsilon\ \ \epsilon + \frac{2\epsilon'}{1+2\upsilon}\ \ 27\kappa\upsilon^3 - 12\kappa\upsilon^2 - 16\upsilon - 8 \quad (3.265)$$

We solve equation (3.265) for $\upsilon$ when the discriminant vanishes to get the critical mass parameter:

$$\upsilon_{C\kappa p} = \frac{4}{9\kappa}\left[1+\sqrt{9\kappa-8} + \frac{9\kappa-16+2\sqrt{9\kappa-8}\ \ \epsilon}{\sqrt{9\kappa-8}} + \frac{27\kappa^2-120\kappa+128-48\kappa\sqrt{9\kappa-8}+56\sqrt{9\kappa-8}\ \ \epsilon'}{2\kappa\sqrt{9\kappa-8}\ \ 9\kappa+8\sqrt{9\kappa-8}}\right]$$

(3.266)

When $\kappa = 1$, $\epsilon' = \epsilon = 0$; the critical mass is the same as that obtained by Robe (1977).

Now, the nature of the roots of the characteristic equation depends on the discriminant, small perturbations in the Coriolis and centrifugal forces, mass ratio $\upsilon$, and the constant $\kappa$ of a particular integral of the GMP. We observe that when $\upsilon_{c\kappa p} < \upsilon < 1$, $D > 0$ and $P > 0$, the roots of

the characteristic equation (3.264) are all distinct pure imaginary. In this case, the EP is stable; otherwise it is unstable.

**Non-collinear equilibrium points**

We follow the same procedure used in problem **3.6**. The second order partial derivatives computed at the perturbed non-collinear EPs (3.258), have forms:

$$\Omega^0_{\xi\xi\xi} = 1+\epsilon' + \frac{3\kappa\ (1-\upsilon)^2\ (\kappa-1)^{5/3}\ (\kappa+2\epsilon')}{\kappa\upsilon^{2/3}},\quad \Omega^0_{\eta\eta\eta} = 1+\epsilon',\quad \Omega^0_{\eta\zeta\zeta} = \Omega^0_{\xi\eta\zeta} = 0$$

$$\Omega^0_{\zeta\zeta\zeta} = 3\ (\kappa-1)\left[1-\frac{3\kappa\ (1-\upsilon)^2\ (\kappa-1)^{2/3}\ (\kappa-2\kappa\epsilon'+2\epsilon')}{\kappa\upsilon^{2/3}}\right] \tag{3.267}$$

$$\Omega^0_{\xi\xi\zeta} = -\frac{3\zeta\ (\kappa-\kappa\epsilon'+\epsilon')\ (1-\upsilon)\ (\kappa-1)^{5/3}}{\kappa\upsilon^{2/3}}$$

When the there are no perturbations, all these derivatives are same as those found in problem **3.6.** Now, substituting equations (3.267) in the variational equations (3.259), we obtain the characteristic equation as

$$\lambda^6 - p_1\lambda^4 + p_2\lambda^2 - p_3 = 0 \tag{3.268}$$

where

$$p_1 = 3\kappa - 5 - 8\epsilon + 2\epsilon'\left[1+\frac{3\kappa^2\ (1-\upsilon)^2\ (\kappa-1)^{\frac{5}{3}}}{\kappa\upsilon^{\frac{2}{3}}}\right]$$

$$p_2 = 1+2\epsilon' + \frac{3\kappa\ (1-\upsilon)\ (\kappa-1)^{\frac{5}{3}}\ (\kappa-1-\upsilon)\ \epsilon'}{\kappa\upsilon^{\frac{2}{3}}} + \frac{18\kappa\ (1-\upsilon)^2\ (\kappa-1)^{\frac{1}{3}}\epsilon'}{\kappa\upsilon^{\frac{2}{3}}} + \frac{36\kappa^4\ (1-\upsilon)^2\ (\kappa-1)^{\frac{13}{3}}\epsilon'}{\kappa\upsilon^{\frac{4}{3}}}$$

$$-\frac{18\kappa^3\ (1-\upsilon)^4\ (\kappa-1)^{\frac{13}{3}}}{\kappa\upsilon^{\frac{4}{3}}} + \frac{3\kappa^2\ (1-\upsilon)^2\ (\kappa-1)^{\frac{5}{3}}}{\kappa\upsilon^{\frac{2}{3}}} - \frac{18\kappa^2\ (1-\upsilon)^2\ (\kappa-1)^{\frac{8}{3}}}{\kappa\upsilon^{\frac{2}{3}}} + \frac{18\kappa^4\ (1-\upsilon)^4\ (\kappa-1)^{\frac{10}{3}}}{\kappa\upsilon^{\frac{4}{3}}}$$

$$p_3 = 7 - 6\kappa - \frac{3\kappa^2\ (\kappa-1)^{\frac{5}{3}}}{(\kappa\upsilon)^{\frac{2}{3}}}\left[\frac{3\zeta^2\ (\kappa-1)^{\frac{5}{3}}}{(\kappa\upsilon)^{\frac{2}{3}}} - 2\ (1-\upsilon)^2 - 3\ (1-\upsilon)^2\ (\kappa-1) + \frac{3\kappa^2\ (1-\upsilon)^4\ (\kappa-1)^{\frac{5}{3}}}{(\kappa\upsilon)^{\frac{2}{3}}}\right]$$

$$-24\in\left[\kappa-1+\frac{\kappa^2\ (1-\upsilon)^2\ (\kappa-1)^{\frac{5}{3}}}{(\kappa\upsilon)^{\frac{2}{3}}}\right]+\in'\left[\frac{3\kappa^2\ (1-\upsilon)^{\frac{5}{3}}}{(\kappa\upsilon)^{\frac{4}{3}}}+3\kappa-4-\frac{3\ (5-6\kappa\ (1-\upsilon)^2\ (\kappa-1)^{\frac{5}{3}}}{(\kappa\upsilon)^{\frac{2}{3}}}\right.$$

$$+\frac{18\kappa\ (1-\upsilon)^2\ (\kappa-1)^{\frac{8}{3}}}{(\kappa\upsilon)^{\frac{2}{3}}}+\frac{18\kappa^4\ (1-\upsilon)^4\ (\kappa-1)^{\frac{10}{3}}}{(\kappa\upsilon)^{\frac{4}{3}}}-\frac{36\kappa^3\ (1-\upsilon)^4\ (\kappa-1)^{\frac{10}{3}}}{(\kappa\upsilon)^{\frac{4}{3}}}-\frac{18\kappa\zeta^2\ (\kappa-1)^{\frac{10}{3}}}{(\kappa\upsilon)^{\frac{4}{3}}}$$

$$+\frac{18\kappa^2\zeta^2\ (\kappa-1)^{\frac{10}{3}}}{(\kappa\upsilon)^{\frac{4}{3}}}$$

The roots are

$$\lambda_{1,2} \to \pm\left[\frac{p_1}{3}+\frac{p_1^2-p_2}{3}\left(\frac{2}{N}\right)^{1/3}+\frac{N^{1/3}}{32^{1/3}}\right]^{1/2}$$

$$\lambda_{3,4} \to \pm\left[\frac{p_1}{3}-\frac{p_1^2}{32^{2/3}N^{1/3}}+\frac{ip_1^2}{2^{2/3}\sqrt{3}N^{1/3}}+\frac{p_2}{2^{2/3}N^{1/3}}-\frac{ip_2\sqrt{3}}{2^{2/3}N^{1/3}}+\frac{N^{1/3}}{22^{1/3}\sqrt{3}}+\frac{N^{1/3}}{62^{1/3}}\right]^{1/2}$$

$$\lambda_{5,6} \to \pm\left[\frac{p_1}{3}-\frac{p_1^2}{32^{2/3}N^{1/3}}-\frac{ip_1^2}{2^{2/3}\sqrt{3}N^{1/3}}+\frac{p_2}{2^{2/3}N^{1/3}}+\frac{i\sqrt{3}p_2}{2^{2/3}N^{1/3}}+\frac{N^{1/3}}{22^{1/3}\sqrt{3}}-\frac{N^{1/3}}{62^{1/3}}\right]^{1/2}$$

where

$$N = 2p_1^3 - 9p_1p_2 + 27p_3 + \sqrt{4\ (3p_2 - p_1^2)^3 + (2p_1^3 - 9p_1p_2 + 27p_3)^2}$$

These roots depend on the values of the coefficients. In line with problem **3.6**, we found that the roots contain at least a positive root and a positive real part which consequently makes the non-collinear EPs unstable.

## 3.8 Robe's Restricted Problem with Non-Isotropic Mass Variation of the Bodies

In this section, we study the motion of the third body in the frame of the Robe (1977) formulations when the masses of the three bodies vary with time at the same rate in a non-isotropic manner (arbitrary mass variations). The motivation for this problem arose because of the challenge we encountered in the case of the isotropic mass flow where the non-autonomous systems do not fully transform to the autonomized systems in the presence of fluid. However, in this problem, the first primary is assumed to be a fluid which forms a medium for the third body while the second primary is located away from the first. We also assumed that the densities of the fluid and the third body vary at the same rate as the masses while the volume is kept constant. We derive the equations of motion of the system and also study the existence and stability of all the equilibrium points. We use this study to discuss the dynamic problem in the Submarine-Earth-Moon system.

### 3.8.1 Equations of motion of the third body

Let $m_1$ be the mass of the first primary which is a fluid in the shape of a sphere of radius $\Re$ with center at $M_1$ having density $\delta_1$ and volume $V_1$. Also, let $m_2$ be the mass of the second primary with center at $M_2$ which describes a circular orbit around the first one. Both masses are assumed to vary with time as they travel in a static medium which acts as a sink or source of mass. Now, let $m_3$ be the mass of the third body whose mass is very small compared with the masses of the primaries, with center at $M_3$, having density $\delta_1$ and volume $V_3$. We suppose that its mass varies with time, also as it moves about in the fluid, it gain or loss mass to the medium.

Let the positions vector between the center of the fluid and the centers of the second primary and the third body be $\vec{r}_{12}$ and $\vec{r}_{13}$, respectively, and let that between the third body and the second primary be $\vec{r}_{23}$. Following Robe (1977) and knowing that the masses, distances, and densities vary with time, the forces acting on the third body are

i. The force of attraction of $m_2$

$$\overrightarrow{F_{m_2}} = -\frac{Gm_2m_3\overrightarrow{M_2M_3}}{r_{23}^3};$$

ii. The gravitational force exerted by the fluid

$$\overrightarrow{F_A} = -\frac{4\pi\Re^3 G}{3}\frac{m_3\delta_1\overrightarrow{M_1M_3}}{r_{13}^3}$$

iii. The buoyancy force exerted by the fluid

$$\overrightarrow{F_B} = \frac{4\pi\Re^3 G}{3}\frac{m_3\delta_1^2\overrightarrow{M_1M_3}}{r_{13}}.$$

where $G$ is the gravitational constant.

Following the derivations of the Robe (1977) problem and those of the R3BP with variable masses (both are given in Sect**. 1.8.3** and Sect. **1.8.7**, respectively in the introduction); the equations of motion of the third body, taking into account the force of attraction of $m_2$; the gravitational and buoyancy forces exerted by the fluid; in a rotating coordinate system $Oxyz$ with origin at the center of mass, $O$, of the primaries, $Ox$ pointing towards the second primary, and $oxy$ being the orbital plane of $m_2$, have the form:

$$\ddot{x}-2\omega\dot{y}=\omega^2 x+\dot{\omega}y-\frac{4\pi G\Re^3\delta_1}{3r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right)\left(x-x_1\right)-\frac{\mu_2\left(x-x_2\right)}{r_{23}^3}-\frac{\dot{m}_3}{m_3}\left(x-\omega y\right)$$

$$\ddot{y}+2\omega\dot{x}=\omega^2 y-\dot{\omega}x-\frac{4\pi G\Re^3 y\delta_1}{3r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right)-\frac{\mu_2 y}{r_{23}^3}-\frac{\dot{m}_3}{m_3}\left(y-\omega x\right) \tag{3.269}$$

$$\ddot{z}=-\frac{4\pi G\Re^3 z\delta_1}{3r_{13}^3}\left(1-\frac{\delta_1}{\delta_3}\right)-\frac{\mu_2 z}{r_{23}^3}-\frac{\dot{m}_3\dot{z}}{m_3}$$

where $r_{13}^2=\left(x-x_1\right)^2+y^2+z^2$, $r_{23}^2=\left(x-x_2\right)^2+y^2+z^2$

The barycentric coordinates $x_1$ and $x_2$ are connected with the distance between the primaries by the following equations:

$$\mu_1\ t\ =\mu\frac{x_2}{r},\ \mu_2\ t\ =-\mu\frac{x_1}{r} \tag{3.270}$$

where $\mu_1\ t\ =Gm_1\ t\ ,\ \ \mu_2\ t\ =Gm_2\ t\ ;m_1\ t\ =\delta_1 V_1;\ \ V_1=\frac{4\pi\Re^3}{3}$ and the over dot denotes differentiation with respect to time $t$.

Now, in order to obtain useful dynamical predictions, we transform $x,y,z,t$ to the autonomous form $\xi,\eta,\zeta,\tau$ . Following the autonomization procedure done in Section **1.8.8** in the case when all the three bodies vary with time, in addition, we assume that the radius $\Re$ of the fluid varies at the same rate as the distances while the densities of the fluid and the third body vary at the same rate as their masses, respectively, since the volume is constant. That is, we take

$$\Re=\ell\left(\frac{\mu_0}{\mu_*}\right)^3,\ \delta_1\ t\ =\delta_{01}\frac{\mu_*}{\mu_0},\quad \delta_3\ t\ =\delta_{03}\frac{\mu_*}{\mu_0} \tag{3.271}$$

where $\ell$, $\delta_{01}$ and $\delta_{03}$ are the radius of the fluid, the density of the fluid and the density of the third body, respectively, at initial time $t_0$; $\mu_*$ and $\mu_0$ are defined in equations (1.95).

Therefore, the system of equations (3.269) is cast to the autonomized forms:

$$\xi'' - 2\omega_0\eta' = \frac{\partial\Omega}{\partial\xi}, \qquad \eta'' + 2\omega_0\xi' = \frac{\partial\Omega}{\partial\eta}, \quad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{3.272}$$

where $\Omega\ (\xi,\eta,\zeta) = \dfrac{\kappa\omega_0^2\ \xi^2+\eta^2+\zeta^2}{2} + \dfrac{\omega_0^2\zeta^2}{2} + \dfrac{\mu_{02}}{\rho_{23}} - G\dfrac{D_p}{2}\left[\ \xi-\xi_1\ ^2+\eta^2+\zeta^2\right]$

$$\rho_{13}^2 = \ \xi-\xi_1\ ^2+\eta^2+\zeta^2, \quad \rho_{23}^2 = \ \xi-\xi_2\ ^2+\eta^2+\zeta^2, \tag{3.273}$$

$$\xi_1 = -\frac{\mu_{02}}{\mu_0}\rho_{12},\ \xi_2 = \frac{\mu_{01}}{\mu_0}\rho_{12}, \quad D_p = \frac{4\pi\ell^3\delta_{01}}{3\rho_{13}^3}\left(1-\frac{\delta_{01}}{\delta_{03}}\right) \tag{3.274}$$

The parameters $\rho_{12}$, $\kappa$, $\mu_0$, $\mu_{01}$ $\mu_{02}$ are constants and the dashes signify differentiation with respect to the new time $\tau$.

Now, we choose units for the distance and time, such that at initial time, $\rho_{12}=1,\ \omega_0=C=1$, respectively, while for the unit of sum of the masses, we get $\mu_0 = G = \kappa$.

Next, without loss of generality, we introduce the mass parameter defined as $\upsilon = \dfrac{\mu_{20}}{\mu_0}$ $0<\upsilon<1$ and also assume that the pressure field of the fluid of density $\delta_{01}$ maintains a spherical symmetry around the center of the fluid such that $\rho_{13}=\ell$. Also, we have $\xi_1 = -\upsilon$ and $\xi_2 = 1-\upsilon$.With the help of these units, the system of (3.272) takes the following form:

$$\xi'' - 2\eta' = \frac{\partial\Omega}{\partial\xi}, \ \ \eta'' + 2\xi' = \frac{\partial\Omega}{\partial\eta}, \qquad \zeta'' = \frac{\partial\Omega}{\partial\zeta} \tag{3.275}$$

where $$\Omega = \frac{\kappa\ (\xi^2+\eta^2}{2} + \frac{\kappa-1\ \zeta^2}{2} + \frac{\kappa\upsilon}{\rho_{23}} - \frac{\kappa D_p}{2}\rho_{13}^2 \tag{3.276}$$

$$\rho_{13}^2 = \ \xi+\upsilon\ ^2+\eta^2+\zeta^2, \quad \rho_{23}^2 = \xi+\upsilon-1^2+\eta^2+\zeta^2, \tag{3.277}$$

$$D_p = \frac{4\pi\delta_{01}}{3}\left(1-\frac{\delta_{01}}{\delta_{03}}\right) \tag{3.278}$$

Equations (3.275 to 3.278) give the autonomized equations of motion of the third body of our `problem. These equations are different from those in Bekov (1988) and analogous to the equations in Hallan and Rana (2001b) only differing due to the second term that appears in the force function and the parameter $\kappa$ .

## 3.8.2. Locations of the equilibrium points

The EPs represent stationary solutions of the R3BP. These solutions are the singularities of the manifold of the components of the velocity and the coordinates and are found by setting $\xi' = \eta' = \zeta' = \xi'' = \eta'' = \zeta'' = 0$ in the equations of motion (3.275). That is, they are the solutions of the equations $\Omega_\xi = \Omega_\eta = \Omega_\zeta = 0$, which are

$$\xi - \frac{\upsilon\ \xi+\upsilon-1}{\rho_{23}^3} - D_p\ \ \xi+\upsilon\ = 0$$

$$\left(1-\frac{\upsilon}{\rho_{23}^3}-D_p\right)\eta = 0, \tag{3.279}$$

$$\left(\frac{\kappa-1}{\kappa}-\frac{\upsilon}{\rho_{23}^3}-D_p\right)\zeta = 0,$$

The solutions are categorized as follows.

1. The solutions of first equation of (3.279) with $\eta = \zeta = 0$ yield the collinear EPs which lie on the line joining the center of the first and second primary.

2. When the first and the second equation of (3.279) are solved with $\zeta = 0$, we get the circular EPs. These points lie in the spherical fluid and form circles.

3. The solutions of first and the third equations of (3.279) with $\eta = 0$ results in the non-collinear EPs. These solutions are valid provided they lie inside the fluid. We shall consider them in the following subsections below.

**Collinear equilibrium points.**

These points are found by solving system (3.279), with $\eta = \zeta = 0$. That is

$$\xi - \frac{\upsilon\ \xi+\upsilon-1}{\rho_{23}^3} - D_p\ \ \xi+\upsilon\ = 0\,. \Rightarrow \qquad \xi - \frac{\upsilon\ \xi+\upsilon-1}{\xi+\upsilon-1^{\ 3}} - D_p\ \ \xi+\upsilon\ = 0$$

$$\Rightarrow \qquad \xi + \frac{\upsilon}{1-\upsilon-\xi^{\ 2}} - D_p\ \ \xi+\upsilon\ = 0$$

Multiplying this equation throughout by $1-\upsilon-\xi^{\ 2}$, we have

$$1-\upsilon-\xi^{\ 2}\ \xi+\upsilon - D_p\ \ \xi+\upsilon\ \ 1-\upsilon-\xi^{\ 2} = 0$$

Expanding, we get

$$1-\upsilon-\xi-\upsilon+\upsilon^2+\upsilon\xi-\xi+\upsilon\xi+\xi^2\ \ \xi+\upsilon - D_p\ \ \xi+\upsilon\ \ 1-\upsilon-\xi-\upsilon+\upsilon^2+\upsilon\xi-\xi+\upsilon\xi+\xi^2\ = 0$$

$$\Rightarrow\ 1-2\upsilon-2\xi+\upsilon^2+2\upsilon\xi+\xi^2\ \ \xi+\upsilon - D_p\ \ \xi+\upsilon\ \ 1-2\upsilon-2\xi+\upsilon^2+2\upsilon\xi+\xi^2\ = 0$$

$$\Rightarrow \xi-2\upsilon\xi-2\xi^2+2\upsilon\xi^2+\upsilon^2\xi+\xi^3+\upsilon - D_p\ \ \xi+\upsilon\ \ 1-2\upsilon-2\xi+\upsilon^2+2\upsilon\xi+\xi^2\ = 0$$

$$\Rightarrow \xi+\upsilon+\xi^3+\upsilon\xi^2-2\xi^2-2\upsilon\xi+\upsilon\xi^2+\upsilon^2\xi - D_p\ \ \xi+\upsilon\ \ 1-2\upsilon-2\xi+\upsilon^2+2\upsilon\xi+\xi^2\ = 0$$

$$\Rightarrow \xi+\upsilon+\xi^2\ \ \xi+\upsilon\ -\ 2\xi-2\upsilon+\upsilon\xi+\upsilon^2\ \ \xi - D_p\ \ \xi+\upsilon\ \ 1-2\upsilon-2\xi+\upsilon^2+2\upsilon\xi+\xi^2\ = 0$$

$$\Rightarrow \ \xi+\upsilon\ +\xi^2\ \ \xi+\upsilon\ -\ 2-\upsilon\ \ \xi+\upsilon\ \xi-D_p\ \ \xi+\upsilon\ \ 1-2\upsilon-2\xi+\upsilon^2+2\upsilon\xi+\xi^2\ \ =0$$

Simplifying, we get

$$\xi+\upsilon\ \left[\ 1-D_p\ \ \xi^2+\ \upsilon-2+2D_p-2D_p\upsilon\ \ \xi+1+2D_p\upsilon-D_p\upsilon^2-D_p\ \right]=0$$

Therefore, $\xi+\upsilon=0$, which implies that

$$\xi=-\upsilon \tag{3.280}$$

Hence, equation (3.280) is always a solution whether $D_p=0$ or is not, for $\kappa>0$.

Now, when $1-D_p\ \ \xi^2+\ \ -2+\upsilon+2D_p-2D_p\upsilon\ \ \xi+1-D_p+2D_p\upsilon-D_p\upsilon^2=0$, we multiply this equation throughout by negative to get

$$D_p-1\ \ \xi^2-\ \ -2+\upsilon+2D_p-2D_p\upsilon\ \ \xi+D_p-1-2D_p\upsilon+D_p\upsilon^2=0 \tag{3.281}$$

Equation (3.281) is a quadratic equation in $\xi$ and its roots are

$$\xi_{2,3}=\frac{-2+\upsilon+2D_p\ \ 1-\upsilon\ \ \pm\sqrt{D}}{2\ \ D_p-1} \tag{3.282}$$

where

$$D=\upsilon\ \ 4D_p+\upsilon-4\ \ .$$

Now, the solutions of (3.282) exist only for $D_p>0$, however, the second solution

$$\xi_2=\frac{-2+\upsilon+2D_p\ \ 1-\upsilon\ \ +\sqrt{\upsilon\ \ 4D_p+\upsilon-4}}{2\ \ D_p-1}$$

is greater than $1-\upsilon$ and consequently will lie outside the shell, so we ignore it. The third solution is

$$\xi_3=\frac{-2+\upsilon+2D_p\ \ 1-\upsilon\ \ -\sqrt{\upsilon\ \ 4D_p+\upsilon-4}}{2\ \ D_p-1} \tag{3.283}$$

and is less than $1-\upsilon$, the $\xi$-coordinate of the second primary.

Hence, there are two collinear equilibrium points inside the first primary and they lie on the line joining the centers of the primaries. Therefore, the coordinate $-\upsilon,0,0$ is always an EP. For $D_p>1$, there exist an equilibrium point $\xi_3,0,0$ which lies to the left or right of the first primary depending upon whether $D_p<1+2\upsilon$ or $D_p=1+2\upsilon$.

When $D_p=1+2\upsilon$, equation (3.283) becomes

$$\xi_3=\frac{-2+\upsilon+2\ 1+2\upsilon\ \ 1-\upsilon\ -\sqrt{\upsilon\left[4\ 1+2\upsilon\ +\upsilon-4\right]}}{2\ 1+2\upsilon-1}$$

$$\Rightarrow\qquad \xi_3=\frac{-2+\upsilon+2\ 1-\upsilon+2\upsilon-2\upsilon^2\ -\sqrt{\upsilon\ 4+8\upsilon+\upsilon-4}}{4\upsilon}$$

$$\Rightarrow\qquad \xi_3=\frac{-2+\upsilon+2-2\upsilon+4\upsilon-4\upsilon^2-\sqrt{9\upsilon^2}}{4\upsilon}$$

$$\Rightarrow\qquad \xi_3=\frac{3\upsilon-4\upsilon^2-3\upsilon}{4\upsilon}$$

Hence, we have

$$\xi_3=-\upsilon$$

Therefore, when $D_p=1+2\upsilon$, the only equilibrium point is the center of the shell.

**Circular equilibrium points**

These solutions are found by solving the first and second equations of (3.279) with $\zeta=0$. That is we solve the equations

$$\xi-\frac{\upsilon\ \ \xi+\upsilon-1}{\rho_{23}^3}-D_p\ \ \xi+\upsilon\ =0$$

$$1-\frac{\upsilon}{\rho_{23}^3}-D_p=0,\quad \eta\neq 0 \tag{3.284}$$

From the first equation of (3.283), we get

$$\xi\left(1-\frac{\upsilon}{\rho_{23}^3}-D_p\right)-\frac{\upsilon\ \upsilon-1}{\rho_{23}^3}-D_p\upsilon=0$$

Using the second equation of (3.284), we at once have

$$\frac{\upsilon\ \upsilon-1}{\rho_{23}^3}=D_p\upsilon$$

So that

$$D_p=1-\frac{\upsilon}{\rho_{23}^3} \tag{3.285}$$

Substituting equation (3.285) in the first equation of (3.284), we have

$$\xi+\ D_p-1\ \ \xi+\upsilon-1\ -D_p\ \ \xi+\upsilon\ =0$$

$$\Rightarrow\quad \xi+D_p\xi+D_p\upsilon-D_p-\xi-\upsilon+1-D_p\xi-D_p\upsilon=0$$

$$\Rightarrow\quad -\upsilon+1-D_p=0$$

$$\Rightarrow\quad D_p=1-\upsilon \tag{3.286}$$

On using (3.286) in the second equation of (3.284), we have

$$1-\frac{\upsilon}{\rho_{23}^3}-\ 1-\upsilon\ =0$$

$$\Rightarrow\quad 1-\frac{\upsilon}{\rho_{23}^3}-1+\upsilon=0$$

$$\Rightarrow\quad \frac{\upsilon}{\rho_{23}^3}=-\upsilon$$

$$\Rightarrow\quad \rho_{23}^3=1$$

Hence, we have the solution

$$\xi+\upsilon-1\ ^{2}+\eta^{2}=1,\quad D_{p}=1-\upsilon \tag{3.287}$$

which gives the coordinate of any point on the circle (3.287) with center $1-\upsilon,0,0$ which is the center of the second primary and radius one which is the distance between the centers of the fluid and the second primary. Thus, the solution gives us an infinite number of EPs, provided they lie inside the fluid.

**Non-collinear points**

The non-collinear EPs are found by solving first and third equations of (3.279) with $\eta=0$. That is the equations

$$\xi-\frac{\upsilon\ \ \xi+\upsilon-1}{\rho_{23}^{3}}-D_{p}\ \ \xi+\upsilon\ =0$$

$$\left(\frac{\kappa-1}{\kappa}-\frac{\upsilon}{\rho_{23}^{3}}-D_{p}\right)\zeta=0, \tag{3.288}$$

$$\rho_{23}^{2}=\ \xi+\upsilon-1\ ^{2}+\zeta^{2}$$

From the second equation of (3.288), since $\zeta\neq 0$, we have

$$\frac{\kappa-1}{\kappa}-D_{p}=\frac{\upsilon}{\rho_{23}^{3}}$$

$$\Rightarrow\quad \frac{\kappa-1-\kappa D_{p}}{\kappa}=\frac{\upsilon}{\rho_{23}^{3}}\ )$$

$$\Rightarrow\quad \rho_{23}^{3}=\frac{\kappa\upsilon}{\kappa-1-\kappa D_{p}} \tag{3.289}$$

When $D_{p}=0$, we get $\rho_{23}^{3}=\frac{\kappa\upsilon}{\kappa-1}$

Now substituting equation (3.289) in the first equation of system (3.288), results in

$$\xi - \frac{\upsilon\ \xi+\upsilon-1\ \ \kappa-1-\kappa D_p}{\kappa\upsilon} - D_p\ \ \xi+\upsilon\ = 0$$

$$\Rightarrow\quad \kappa\xi - \ \xi+\upsilon-1\ \ \kappa-1-\kappa D_p\ -\kappa D_p\ \ \xi+\upsilon\ = 0$$

$$\Rightarrow\quad \kappa\xi - \xi\ \ \kappa-1-\kappa D_p\ +\ 1-\upsilon\ \ \ \kappa-1-\kappa D_p\ -\kappa D_p\xi - \kappa D_p\upsilon = 0$$

$$\Rightarrow\quad \xi\ \ \kappa-\kappa+1+\kappa D_p - \kappa D_p\ +\ 1-\upsilon\ \ \ \kappa-1-\kappa D_p\ -\kappa D_p\upsilon = 0$$

$$\Rightarrow\quad \xi+\ 1-\upsilon\ \ \ \kappa-1-\kappa D_p\ -\kappa D_p\upsilon = 0$$

$$\Rightarrow\quad \xi+\kappa-1-\kappa D_p - \kappa\upsilon+\upsilon+\kappa D_p\upsilon - \kappa D_p\upsilon = 0$$

$$\Rightarrow\quad \xi+\kappa-1-\kappa D_p - \kappa\upsilon+\upsilon = 0$$

$$\Rightarrow\quad \xi+\ \kappa-1\ -\upsilon\ \ \kappa-1\ -\kappa D_p = 0$$

$$\Rightarrow\quad \xi = \ \kappa-1\ \ 1-\upsilon\ +\kappa D_p \qquad (3.290)$$

This case coincides fully with the result of Hallan and Rana (2001b) and different from the case which results to $\xi = 0$. Now, substituting equations (3.289) and (3.290) in the third equation of system (3.288), we have

$$\left(\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right)^{2/3} = \left[-\ \kappa-1\ \ 1-\upsilon\ +\kappa D_p+\upsilon-1\right]^2 + \zeta^2$$

$$\Rightarrow\quad \zeta^2 = \left(\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right)^{2/3} - \left[-\ \kappa-1\ \ 1-\upsilon\ +\kappa D_p+\upsilon-1\right]^2$$

So that $\zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right)^{2/3} - \left[-\ \kappa-1\ \ 1-\upsilon\ +\kappa D_p+\upsilon-1\right]^2}$

$$\Rightarrow \qquad \zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right)^{2/3} - \left[-\ \kappa-1-\kappa\upsilon+\upsilon\ +\kappa D_p+\upsilon-1\right]^2}$$

$$\Rightarrow \qquad \zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right)^{2/3} - \left[-\kappa+1+\kappa\upsilon-\upsilon+\kappa D_p+\upsilon-1\right]^2}$$

Simplifying, we get

$$\zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right)^{2/3} - \kappa^2\ \ 1-\upsilon-D_p\ \ ^2}$$

Hence, the coordinates of the non-collinear EPs are

$$\xi = \ \kappa-1\ \ 1-\upsilon\ +\kappa D_p, \quad \zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right)^{2/3} - \kappa^2\ \ 1-\upsilon-D_p\ \ ^2} \tag{3.291}$$

Equations (3.291) give the position $\xi, 0, \zeta$ of real non-collinear EPs provided $\kappa > \frac{1}{1-D_p}$.

## 3.8.3 Stability of equilibrium points

**Collinear equilibrium points**

In this study, we found two EPs collinear with the centers of the fluid and the second primary. Robe (1977) discussed the stability of the EP at the center of the shell, and we have discussed it also in problem **3.**, hence, we shall discuss here the stability of the second EP $\xi_3, 0, 0$ near the center of the fluid represented in equation (3.283).

The variational and characteristic equations are same with those given in equation (3.229) and (3.231), respectively.

Now from the potential of the dynamical system (3.276), we get the following second derivatives:

$$\Omega_{\xi\xi} = \kappa - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{3/2}} + \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ ^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}} - \kappa D_p$$

$$\Omega_{\eta\eta} = \kappa - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{3/2}} + \frac{3\kappa\upsilon\eta^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}} - \kappa D_p$$

$$\Omega_{\zeta\zeta} = \ \kappa-1\ - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{3/2}} + \frac{3\kappa\upsilon\zeta^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}} - \kappa D_p \qquad (3.292)$$

$$\Omega_{\xi\eta} = \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ \ \eta}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}},\ \Omega_{\eta\zeta} = \frac{3\kappa\upsilon\eta\zeta}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}}$$

$$\Omega_{\xi\zeta} = \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ \ \zeta}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}}$$

The values of the second order partial derivatives computed at the EP $\xi_3, 0, 0$ , is found by substituting equation (3.283) and $\eta = \zeta = 0$ in equations (3.292) to get

$$\Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon}{\xi_3+\upsilon-1\ ^3} + \frac{3\kappa\upsilon}{\xi_3+\upsilon-1\ ^3} - \kappa D_p$$

$$\Rightarrow \qquad \Omega^0_{\xi\xi\xi} = \kappa\left[1 - \frac{\upsilon}{\xi_3+\upsilon-1\ ^3} + \frac{3\upsilon}{\xi_3+\upsilon-1\ ^3} - D_p\right]$$

$$\Rightarrow \qquad \Omega^0_{\xi\xi\xi} = \kappa\left[1 + \frac{2\upsilon}{\xi_3+\upsilon-1\ ^3} - D_p\right] \qquad (3.293)$$

Substituting equation (2.283) in equation (3.293), we have

$$\Omega^0_{\xi\xi} = \kappa\left[1+\frac{2\upsilon}{\left\{\frac{-2+\upsilon+2D_p\ \ 1-\upsilon\ -\sqrt{\upsilon\ \ 4D_p+\upsilon-4}}{2\ \ D_p-1}+\upsilon-1\right\}^3}-D_p\right]$$

$$\Rightarrow \quad \Omega^0_{\xi\xi} = \kappa\left[1-D_p+\frac{2\upsilon}{\left\{\frac{-2+\upsilon+2D_p\ \ 1-\upsilon\ -\sqrt{\upsilon\ \ 4D_p+\upsilon-4}\ +2\ \ D_p-1\ \ \ \upsilon-1}{2\ \ D_p-1}\right\}^3}\right]$$

$$\Rightarrow \quad \Omega^0_{\xi\xi} = \kappa\left[1-D_p+\frac{16\upsilon\ \ D_p-1\ ^3}{-2+\upsilon+2D_p-2D_p\upsilon-\sqrt{\upsilon\ \ 4D_p+\upsilon-4}\ +2D_p\upsilon-2D_p-2\upsilon+2\ ^3}\right]$$

Simplifying results in

$$\Omega^0_{\xi\xi} = \kappa\left[1-D_p+\frac{16\upsilon\ \ D_p-1\ ^3}{-\upsilon-\sqrt{\upsilon\ \ 4D_p+\upsilon-4}\ ^3}\right]$$

or $$\Omega^0_{\xi\xi} = \kappa\left[1-D_p+\frac{16\upsilon\ \ D_p-1\ ^3}{\left\{\left[-\upsilon-\sqrt{\upsilon\ \ 4D_p+\upsilon-4}\ \right]^2\right\}^{3/2}}\right]$$

$$\Rightarrow \quad \Omega^0_{\xi\xi} = \kappa\left[1-D_p+\frac{16\upsilon\ \ D_p-1\ ^3}{\upsilon+\sqrt{\upsilon\ \ 4D_p+\upsilon-4}\ ^3}\right]$$

Hence,

$$\Omega^0_{\xi\xi} = \kappa\ 1 - D_p + 4\bar{A} \tag{3.294}$$

Similarly

$$\Omega^0_{\eta\eta} = \kappa\left[1 - D_p - \frac{\upsilon}{\xi + \upsilon - 1^{\ 2}}\right]$$

$$\Omega^0_{\eta\eta} = \kappa\left[1 - D_p - \frac{\upsilon}{\left\{\frac{-2 + \upsilon + 2D_p\ \ 1 - \upsilon\ - \sqrt{\upsilon\ \ 4D_p + \upsilon - 4}}{2\ \ D_p - 1} + \upsilon - 1\right\}^3}\right]$$

$$\Omega^0_{\eta\eta} = \kappa\left[1 - D_p - \frac{8\upsilon\ \ D_p - 1^{\ 3}}{-2 + \upsilon + 2D_p - 2D_p\upsilon - \sqrt{\upsilon\ \ 4D_p + \upsilon - 4\ } + 2D_p\upsilon - 2D_p - 2\upsilon + 2^{\ 3}}\right]$$

Simplifying, gives

$$\Omega^0_{\eta\eta} = \kappa\left[1 - D_p - \frac{8\upsilon\ \ D_p - 1^{\ 3}}{\upsilon + \sqrt{\upsilon\ \ 4D_p + \upsilon - 4}^{\ 3}}\right]$$

Hence, $\Omega^0_{\eta\eta} = \kappa\ 1 - D_p - 2\bar{A}$

Doing similar calculations, we get the second order partial derivatives in this case:

$$\Omega^0_{\xi\xi} = \kappa\ 1 - D_p + 4\bar{A}\ ,\ \Omega^0_{\eta\eta} = \kappa\ 1 - D_p - 2\bar{A}\ \ ,\ \Omega^0_{\zeta\zeta} = \kappa\left(\frac{\kappa - 1}{\kappa} - 2\bar{A} - D_p\right)$$

$$\Omega^0_{\xi\zeta} = \Omega^0_{\eta\zeta} = \Omega^0_{\xi\eta} = 0 \tag{3.295}$$

where

$$\bar{A} = \frac{4\upsilon\ \ D_p - 1^{\ 3}}{\left[\upsilon + \sqrt{\upsilon\ \ 4D_p + \upsilon - 4}\ \right]^3}$$

When $\kappa = 1$, the equations in system (3.295) fully coincide with those of Hallan and Rana (2001b).

Substituting (3.295) in the variational equations (3.229), at once gives

$$\begin{aligned} u'' - 2v' &= \kappa\ \ 1 - D_p + 4\bar{A}\ \ u, \\ v'' + 2u' &= \kappa\ \ 1 - D_p - 2\bar{A}\ \ v, \end{aligned} \tag{3.296}$$

$$w'' = \kappa\left(\frac{\kappa - 1}{\kappa} - D_p - 2\bar{A}\right)w. \tag{3.297}$$

Now, (3.297) is independent of (3.296). If $\kappa = 1$, the motion parallel to the $\zeta$-axis is stable. If $\kappa < 1$, the same analysis follows. However, when $\kappa > 1$, the stability of the motion parallel to the $\zeta$-axis depends on whether $\frac{1-\kappa}{\kappa} + 2\bar{A}_1 + D_p > 0$

$$\Rightarrow \quad 1 - \kappa + 2\kappa\bar{A}_1 + \kappa D_p > 0$$

$$\Rightarrow \quad \kappa - 1 - 2\kappa\bar{A}_1 - \kappa D_p < 0$$

$$\Rightarrow \quad \kappa - 2\kappa\bar{A}_1 - \kappa D_p < 1$$

$$\Rightarrow \quad \kappa\ \ 1 - 2\bar{A}_1 - D_p\ \ < 1$$

$$\Rightarrow \quad \kappa < \frac{1}{1 - 2\bar{A}_1 - D_p}$$

Hence, the motion is stable along the $\zeta$-axis provided $1 \le \kappa < \frac{1}{1 - 2\bar{A} - D_p}$.

Now, substituting equations (3.295) in the characteristic equation (3.231) and solving gives

$$\lambda^4 + \lambda^2 \left\{ 2\kappa \left( \frac{2}{\kappa} - 1 + D_p - \bar{A} \right) \right\} + \kappa^2 \ D_p - 1 - 4\bar{A} \ D_p - 1 + 2\bar{A} \ = 0 \tag{3.298}$$

where $D_p > 1$, $\bar{A} > 0$

The roots of (3.298) are

$$\Lambda_n = \frac{-P \pm \sqrt{D}}{2}, \quad n = 1, 2 \tag{3.299}$$

where

$$P = 2\kappa \left( \frac{2}{\kappa} - 1 + D_p - \bar{A} \right) \tag{3.300}$$

$D$ is the discriminant and given by the equation

$$D = \left[ 2\kappa \left( D_p - 1 - \bar{A}_1 + \frac{2}{\kappa} \right) \right]^2 - 4\kappa^2 \ 1 - D_p + 4\bar{A}_1 \ 1 - D_p + 2\bar{A}_1$$

Simplifying, we get

$$D = 4 \ 4\kappa \ D_p - \bar{A} \ - 4 \ \kappa - 1 \ + 9\kappa^2 \bar{A}^2 \tag{3.301}$$

Equation (3.301) gives the value of the discriminant of (3.298) and is always positive for any $\kappa$ .When $\kappa = 1$, we have

$$D = 4 \left[ 4 \ D_p - \bar{A}_1 \ + 9\bar{A}_1^2 \right] \tag{3.302}$$

which is same as that obtained by Hallan and Rana (2001b).

Now, since the discriminant is positive, if $P > 0$, which occurs when

$$0 < \kappa < \frac{2}{1 - D_p + \bar{A}} \tag{3.303}$$

We observe that

$$P > \sqrt{D} \tag{3.304}$$

Hence, as a consequence, we have from equation (3.299) that

$$-P \pm \sqrt{D} < 0 \tag{3.305}$$

This implies that both values of $\Lambda_n$ given by equation (3.299) are negative and consequently, the roots are distinct pure imaginary. Therefore, the collinear EP $\xi_3, 0, 0$ is stable provided equation (3.303) holds; otherwise, it is unstable.

**Circular equilibrium points**

These equilibrium points exist only for $D_p = 1-\upsilon$. The coordinates of any point on the circle $\xi+\upsilon-1\ ^2+\eta^2=1,\ \zeta=0$ have the form $1-\upsilon-\cos\theta,\ \sin\theta,\ 0$ .

The partial derivatives at these points are obtained from equations (3.294), with $\zeta = 0$, to get

$$\Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2\ \right]^{3/2}} + \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ ^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2\ \right]^{5/2}} - \kappa D_p\ ,\ \Omega^0_{\eta\zeta\zeta}=0\ ,\ \Omega^0_{\xi\zeta\zeta}=0$$

$$\Omega^0_{\eta\eta\eta} = \kappa - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2\ \right]^{3/2}} + \frac{3\kappa\upsilon\eta^2}{\left[\ \xi+\upsilon-1\ ^2+\eta^2\ \right]^{5/2}} - \kappa D_p\ ,\ \Omega^0_{\xi\eta\eta} = \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ \ \eta}{\left[\ \xi+\upsilon-1\ ^2+\eta^2+\zeta^2\ \right]^{5/2}}$$

$$\Omega^0_{\zeta\zeta\zeta} = \ \kappa-1\ - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\eta^2\ \right]^{3/2}} - \kappa D_p \tag{3.306}$$

Substituting $\xi+\upsilon-1\ ^2+\eta^2=1, \xi=1-\upsilon-\cos\theta$ and $\eta=\sin\theta$, in equations of (3.306), we get

$$\Omega^0_{\xi\xi\xi} = \kappa - \kappa\upsilon + 3\kappa\upsilon\left[1-\upsilon-\ \sqrt{1-\upsilon-\cos\theta}\ \right]^2 - \kappa D_p$$

$$\Omega^0_{\xi\xi\xi} = \kappa - \kappa\upsilon + 3\kappa\upsilon\ 1-\upsilon-1+\upsilon+\cos\theta\ ^2 - \kappa D_p$$

$$\Omega^0_{\xi\xi\xi} = \kappa - \kappa\upsilon + 3\kappa\upsilon\cos^2\theta - \kappa D_p$$

But $D_p = 1-\upsilon$,

$$\Rightarrow \Omega^0_{\xi\xi} = \kappa - \kappa\upsilon + 3\kappa\upsilon\cos^2\theta - \kappa\ 1-\upsilon$$

$$\Rightarrow \Omega^0_{\xi\xi} = \kappa - \kappa\upsilon + 3\kappa\upsilon\cos^2\theta - \kappa + \kappa\upsilon$$

$$\Rightarrow \Omega^0_{\xi\xi} = 3\kappa\upsilon\cos^2\theta$$

Next,

$$\Omega^0_{\eta\eta} = \kappa - \kappa\upsilon + 3\kappa\upsilon\sin^2\theta - \kappa D_p$$

$$\Rightarrow \Omega^0_{\eta\eta} = \kappa - \kappa\upsilon + 3\kappa\upsilon\sin^2\theta - \kappa\ 1-\upsilon$$

$$\Rightarrow \Omega^0_{\eta\eta} = 3\kappa\upsilon\sin^2\theta$$

Similarly,

$$\Omega^0_{\zeta\zeta} = \ \kappa - 1\ - \kappa\upsilon - \kappa D_p$$

$$\Rightarrow \Omega^0_{\zeta\zeta} = \ \kappa - 1\ - \kappa\upsilon - \kappa\ 1-\upsilon$$

$$\Rightarrow \Omega^0_{\zeta\zeta} = \kappa - 1 - \kappa\upsilon - \kappa + \kappa\upsilon$$

$$\Rightarrow \Omega^0_{\zeta\zeta} = -1$$

Lastly,

$$\Omega^0_{\xi\eta} = \frac{3\kappa\upsilon\ \ \xi + \upsilon - 1\ \ \eta}{\left[\ \ \xi + \upsilon - 1\ ^2 + \eta^2\ \right]^{5/2}}$$

Put $\xi + \upsilon - 1\ ^2 + \eta^2 = 1$ and $\eta = \sin\theta$, we have

$$\Rightarrow \Omega^0_{\xi\eta} = 3\kappa\upsilon\ \ \xi + \upsilon - 1\ \sin\theta$$

With $\xi = 1 - \upsilon - \cos\theta$, we get

$$\Omega^0_{\xi\eta} = -3\kappa\upsilon\left[1 - \upsilon - \ 1 - \upsilon - \cos\theta\ \right]\sin\theta$$

$$\Rightarrow \quad \Omega^0_{\xi\eta} = -3\kappa\upsilon\ 1-\upsilon-1+\upsilon+\cos\theta\ \sin\theta$$

$$\Rightarrow \quad \Omega^0_{\xi\eta} = -3\kappa\upsilon\cos\theta\sin\theta$$

Hence, the partial derivatives computed at the circular points are

$$\Omega^0_{\xi\xi} = 3\kappa\upsilon\cos^2\theta,\ \Omega^0_{\eta\eta} = 3\kappa\upsilon\sin^2\theta,\ \ \Omega^0_{\zeta\zeta} = -1 \quad , \quad \Omega^0_{\xi\eta} = -3\kappa\upsilon\cos\theta\sin\theta \tag{3.307}$$

$$\Omega^0_{\eta\zeta} = \Omega^0_{\xi\zeta} = 0$$

When equations (3.307) are substituted in the variational equations (3.229) with $\zeta = 0$, we have

$$\begin{aligned} u'' - 2v' &= 3\kappa\upsilon\cos\theta\ \ u\cos\theta - v\sin\theta\ , \\ v'' + 2u' &= -3\kappa\upsilon\sin\theta\ \ u\cos\theta - v\sin\theta\ , \\ w'' &= -w. \end{aligned} \tag{3.308}$$

The last equation of (3.308) shows that motion is stable along the $\zeta$ -axis.

Substituting the partial derivatives (3.307) in the characteristic equation (3.231), we get

$$\lambda^4 - \lambda^2\left[3\kappa\upsilon\cos^2\theta + 3\kappa\upsilon\sin^2\theta - 4\right] + 9\kappa^2\upsilon^2\cos^2\theta\sin^2\theta - \ 9\kappa^2\upsilon^2\cos^2\theta\sin^2\theta\ = 0$$

(4.120)(3.309)

Simplifying, we get

$$\lambda^2\ \ \lambda^2 - \ 3\kappa\upsilon - 4\ \ = 0 \tag{3.310}$$

Therefore, $\lambda^2 = 0$ and so $\lambda_{1,2} = 0$ twice or $\lambda^2 - 3\kappa\upsilon - 4 = 0$ which gives $\lambda_{3,4} = \pm\sqrt{3\kappa\upsilon - 4}$

Hence, two of the roots of the characteristic equation (3.310) are multiple zero roots and this induces instability at the circular equilibrium points. Hence, the circular equilibrium points $\xi = 1 - \upsilon - \cos\theta$ and $\eta = \sin\theta$ are unstable.

**Non-collinear equilibrium points**

The partial derivatives computed at the non-collinear EPs are obtained by substituting equations (3.291) and $\eta = 0$ in the derivatives of equations (3.292). For the derivative $\Omega^0_{\xi\xi\xi}$, we have

$$\Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon}{\left[\ \xi+\upsilon-1\ ^2+\zeta^2\ \right]^{3/2}} + \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ ^2}{\left[\ \xi+\upsilon-1\ ^2+\zeta^2\ \right]^{5/2}} - \kappa D_p$$

But $\rho_{23}^3 = \left[\ \xi+\upsilon-1\ ^2+\zeta^2\ \right]^{3/2}$

$$\Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon}{\rho_{23}^3} + \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ ^2}{\rho_{23}^5} - \kappa D_p$$

The use of equation (3.289), reduces the above equation to

$$\Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon}{\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}} + \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ ^2}{\left[\frac{\kappa\upsilon}{\kappa-1-\kappa D_p}\right]^{5/3}} - \kappa D_p$$

$$\Rightarrow\ \Omega^0_{\xi\xi\xi} = \kappa - \frac{\kappa\upsilon\ \ \kappa-1-\kappa D_p}{\kappa\upsilon} + \frac{3\kappa\upsilon\ \ \xi+\upsilon-1\ ^2\ \ \kappa-1-\kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{5/3}} - \kappa D_p$$

$$\Rightarrow\ \Omega^0_{\xi\xi\xi} = \kappa - \kappa + 1 + \kappa D_p + \frac{3\ \ \xi+\upsilon-1\ ^2\ \ \kappa-1-\kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} - \kappa D_p$$

$$\Rightarrow\ \Omega^0_{\xi\xi\xi} = 1 + \frac{3\ \ \xi+\upsilon-1\ ^2\ \ \kappa-1-\kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}$$

$$\Rightarrow\ \Omega^0_{\xi\xi\xi} = 1 + \frac{3\ \ \kappa-1-\kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}\left[-\ 1-\upsilon\ \ \ \kappa-1\ +\kappa D_p+\upsilon-1\right]^2$$

$$\Rightarrow \Omega_{\xi\xi}^{0} = 1 + \frac{3\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} \left[\ 1 - \upsilon \quad \kappa - 1\ - \kappa D_p + 1 - \upsilon\right]^2$$

$$\Rightarrow \Omega_{\xi\xi}^{0} = 1 + \frac{3\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} \left[\ 1 - \upsilon \quad \kappa - 1 + 1\ - \kappa D_p\right]^2$$

$$\Rightarrow \Omega_{\xi\xi}^{0} = 1 + \frac{3\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} \left[\kappa\ \ 1 - \upsilon\ - \kappa D_p\right]^2$$

$$\Rightarrow \Omega_{\xi\xi}^{0} = 1 + \frac{3\kappa^2\ \ 1 - \upsilon - D_p\ ^2\ \ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}$$

For the derivative $\Omega_{\eta\eta}^{0}$, we have

$$\Omega_{\eta\eta}^{0} = \kappa - \frac{\kappa\upsilon}{\left[\ \xi + \upsilon - 1\ ^2 + \zeta^2\right]^{3/2}} - \kappa D_p$$

$$\Rightarrow \quad \Omega_{\eta\eta}^{0} = \kappa - \frac{\kappa\upsilon}{\rho_{23}^{3}} - \kappa D_p$$

Using equation (3.289) at once yields

$$\Omega_{\eta\eta}^{0} = \kappa - \frac{\kappa\upsilon}{\dfrac{\kappa\upsilon}{\kappa - 1 - \kappa D_p}} - \kappa D_p$$

$$\Rightarrow \quad \Omega_{\eta\eta}^{0} = \kappa - \frac{\kappa\upsilon\ \ \kappa - 1 - \kappa D_p}{\kappa\upsilon} - \kappa D_p$$

$$\Rightarrow \quad \Omega_{\eta\eta}^{0} = \kappa - \ \ \kappa - 1 - \kappa D_p\ \ - \kappa D_p$$

$$\Rightarrow \quad \Omega_{\eta\eta}^{0} = \kappa - \kappa + 1 + \kappa D_p - \kappa D_p$$

$$\Rightarrow \quad \Omega_{\eta\eta}^{0} = 1$$

Similarly, we get $\Omega^0_{\xi\eta} = 0, \Omega^0_{\eta\zeta} = 0$

Next for $\Omega^0_{\zeta\zeta}$, we have

$$\Omega^0_{\zeta\zeta} = \kappa - 1 - \left(\kappa - 1 - \kappa D_p\right) + \frac{3\zeta^2}{(\kappa\upsilon)^{2/3}} \left(\kappa - 1 - \kappa D_p\right)^{5/3} - \kappa D_p$$

$$\Omega^0_{\zeta\zeta} = \left(\kappa - 1\right) - \frac{\kappa\upsilon}{\left[\left(\xi + \upsilon - 1\right)^2 + \zeta^2\right]^{3/2}} + \frac{3\kappa\upsilon\zeta^2}{\left[\left(\xi + \upsilon - 1\right)^2 + \zeta^2\right]^{5/2}} - \kappa D_p$$

Using equation (3.289), we have

$$\Omega^0_{\zeta\zeta} = \left(\kappa - 1\right) - \frac{\kappa\upsilon\left(\kappa - 1 - \kappa D_p\right)}{\kappa\upsilon} + \frac{3\kappa\upsilon\zeta^2}{(\kappa\upsilon)^{5/3}} \left(\kappa - 1 - \kappa D_p\right)^{5/3} - \kappa D_p$$

$$\Rightarrow \quad \Omega^0_{\zeta\zeta} = \kappa - 1 - \left(\kappa - 1 - \kappa D_p\right) + \frac{3\zeta^2}{(\kappa\upsilon)^{2/3}} \left(\kappa - 1 - \kappa D_p\right)^{5/3} - \kappa D_p$$

$$\Rightarrow \quad \Omega^0_{\zeta\zeta} = \kappa - 1 - \kappa + 1 + \kappa D_p + \frac{3\zeta^2}{(\kappa\upsilon)^{2/3}} \left(\kappa - 1 - \kappa D_p\right)^{5/3} - \kappa D_p$$

Simplifying gives

$$\Omega^0_{\zeta\zeta} = \frac{3\zeta^2}{(\kappa\upsilon)^{2/3}} \left(\kappa - 1 - \kappa D_p\right)^{5/3}$$

Lastly, for the derivative $\Omega^0_{\xi\zeta}$, we get

$$\Omega^0_{\xi\zeta} = -\frac{3\kappa\zeta\left(1 - \upsilon - D_p\right)\left(\kappa - 1 - \kappa D_p\right)^{5/3}}{(\kappa\upsilon)^{2/3}}$$

Hence, the computed partial derivatives at the non-collinear equilibrium points are

$$\Omega^0_{\xi\xi} = 1 + \frac{3\kappa^2\left(1 - \upsilon - D_p\right)^2\left(\kappa - 1 - \kappa D_p\right)^{5/3}}{(\kappa\upsilon)^{2/3}}, \qquad \Omega^0_{\eta\eta} = 1, \Omega^0_{\eta\zeta} = \Omega^0_{\xi\eta} = 0$$

$$\Omega^0_{\zeta\zeta} = \frac{3\zeta^2\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}, \Omega^0_{\xi\xi\zeta} = -\frac{3\zeta\kappa\ 1 - \upsilon - D_p\ \ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} \tag{3.311}$$

Following the procedure used in problems **3.6** and **3.7**, we obtain the characteristic equation obtained from the variational equations and then substitute the partial derivatives (3.311) into it to get

$$\lambda^6 + p_1\lambda^4 + p_2\lambda^2 + p_3 = 0 \tag{3.312}$$

where

$$p_1 = \Omega^0_{\xi\xi\xi} + \Omega^0_{\zeta\zeta\zeta} - 3$$

$$p_2 = -3\Omega^0_{\zeta\zeta\zeta} + \Omega^0_{\xi\xi\xi}\ 1 + \Omega^0_{\zeta\zeta\zeta}\ -\ \Omega^0_{\xi\xi\zeta}\ ^2$$

$$p_3 = \Omega^0_{\xi\xi\xi}\Omega^0_{\zeta\zeta\zeta} -\ \Omega^0_{\xi\xi\zeta}\ ^2$$

Substituting the derivatives, we have

$$p_1 = 1 + \frac{3\kappa^2\ 1 - \upsilon - D_p\ ^2\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} + \frac{3\zeta^2}{\kappa\upsilon\ ^{2/3}}\ \kappa - 1 - \kappa D_p\ ^{5/3} - 3$$

$$p_1 = -2 + \frac{3\kappa^2\ 1 - \upsilon - D_p\ ^2\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} + \frac{3\zeta^2}{\kappa\upsilon\ ^{2/3}}\ \kappa - 1 - \kappa D_p\ ^{5/3}$$

$$\Rightarrow \qquad p_1 = -2 + \frac{3\kappa^2\ 1 - \upsilon - D_p\ ^2\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} + \frac{3\zeta^2}{\kappa\upsilon\ ^{2/3}}\ \kappa - 1 - \kappa D_p\ ^{5/3}$$

$$\Rightarrow \qquad p_1 = -2 + \frac{3\kappa^2\ 1 - \upsilon - D_p\ ^2\ \kappa - 1 - \kappa D_p\ ^{5/3} + 3\zeta^2\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}$$

$$\Rightarrow \qquad p_1 = -2 + \frac{3\ \kappa - 1 - \kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}\left[\kappa^2\ 1 - \upsilon - D_p\ ^2 + \zeta^2\right]$$

But $\zeta^2 = \left( \frac{\kappa\upsilon}{\kappa - 1 - \kappa D_p} \right)^{2/3} - \kappa^2 \ (1 - \upsilon - D_p)^2$

Therefore,

$$p_1 = -2 + \frac{3\ (\kappa - 1 - \kappa D_p)^{5/3}}{(\kappa\upsilon)^{2/3}} \left[ \kappa^2 \ (1 - \upsilon - D_p)^2 + \left( \frac{\kappa\upsilon}{\kappa - 1 - \kappa D_p} \right)^{2/3} - \kappa^2 \ (1 - \upsilon - D_p)^2 \right]$$

$$\Rightarrow \ p_1 = -2 + \frac{3\ (\kappa - 1 - \kappa D_p)^{5/3}}{(\kappa\upsilon)^{2/3}} \left( \frac{\kappa\upsilon}{\kappa - 1 - \kappa D_p} \right)^{2/3}$$

$$\Rightarrow \ p_1 = -2 + \frac{3\ (\kappa\upsilon)^{2/3} \ (\kappa - 1 - \kappa D_p)^{5/3}}{(\kappa\upsilon)^{2/3} \ (\kappa - 1 - \kappa D_p)^{2/3}}$$

$$\Rightarrow \ p_1 = -2 + 3\ (\kappa - 1 - \kappa D_p)$$

$$\Rightarrow \ p_1 = -5 + 3\kappa \ (1 - D_p) \qquad (3.313)$$

Similarly,

$$p_2 = -3 \left[ \frac{3\zeta^2 \ (\kappa - 1 - \kappa D_p)^{5/3}}{(\kappa\upsilon)^{2/3}} \right] + \left[ 1 + \frac{3\kappa^2 \ (1 - \upsilon - D_p)^2 \ (\kappa - 1 - \kappa D_p)^{\frac{5}{3}}}{(\kappa\upsilon)^{2/3}} \right] \left[ 1 + \frac{3\zeta^2 \ (\kappa - 1 - \kappa D_p)^{\frac{5}{3}}}{(\kappa\upsilon)^{2/3}} \right]$$

$$- \left[ - \frac{3\kappa\zeta \ (1 - \upsilon - D_p) \ (\kappa - 1 - \kappa D_p)^{5/3}}{(\kappa\upsilon)^{2/3}} \right]^2$$

$$p_2 = - \frac{9\zeta^2 \ (\kappa - 1 - \kappa D_p)^{5/3}}{(\kappa\upsilon)^{2/3}} + \left[ \frac{(\kappa\upsilon)^{2/3} + 3\kappa^2 \ (1 - \upsilon - D_p)^2 \ (\kappa - 1 - \kappa D_p)^{\frac{5}{3}}}{(\kappa\upsilon)^{2/3}} \right] \left[ \frac{(\kappa\upsilon)^{2/3} + 3\zeta^2 \ (\kappa - 1 - \kappa D_p)^{\frac{5}{3}}}{(\kappa\upsilon)^{2/3}} \right]$$

$$- \frac{3\kappa^2\zeta^2 \ (1 - \upsilon - D_p)^2 \ (\kappa - 1 - \kappa D_p)^{10/3}}{(\kappa\upsilon)^{4/3}}$$

$$p_2 = -\frac{9\zeta^2\ \kappa-1-\kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}} - \frac{9\kappa^2\zeta^2\ 1-\upsilon-D_p\ ^2\ \kappa-1-\kappa D_p\ ^{10/3}}{\kappa\upsilon\ ^{4/3}} + \frac{\left[\ \kappa\upsilon\ ^{2/3}+3\zeta^2\ \kappa-1-\kappa D_p\ ^{5/3}\right]}{\kappa\upsilon\ ^{4/3}}$$

$$\times\left[\ \kappa\upsilon\ ^{2/3}+3\kappa^2\ 1-\upsilon-D_p\ ^2\ \kappa-1-\kappa D_p\ ^{5/3}\right]$$

$$p_2 = -\frac{9\zeta^2\ \kappa-1-\kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{2/3}}\left[1+\frac{\kappa^2\ 1-\upsilon-D_p\ ^2\ \kappa-1-\kappa D_p\ ^{\frac{5}{3}}}{\kappa\upsilon\ ^{2/3}}\right]$$

$$+\ \kappa\upsilon\ ^{\frac{2}{3}}\frac{\left[\ \kappa\upsilon\ ^{\frac{2}{3}}+3\kappa^2\ 1-\upsilon-D_p\quad\kappa-1-\kappa D_p\ ^{\frac{5}{3}}\right]}{\kappa\upsilon\ ^{4/3}}$$

$$+\frac{3\zeta^2\ \kappa-1-\kappa D_p\ ^{5/3}}{\kappa\upsilon\ ^{4/3}}\left[\ \kappa\upsilon\ ^{2/3}+3\kappa^2\ 1-\upsilon-D_p\ ^2\ \kappa-1-\kappa D_p\ ^{5/3}\right]$$

Substituting for $\zeta^2$ and simplifying, we get

$$p_2 = 1+\frac{3\ \kappa-1-\kappa D_p}{\upsilon^{2/3}}\ 2\upsilon^{2/3}-3\kappa^{4/3}\ 1-\upsilon-D_p\ ^2\ \kappa-1-\kappa D_p\ ^{2/3} \tag{3.314}$$

A similar calculation gives

$$p_3 = 3\ 1+\kappa D_p-\kappa\left[1-\frac{\kappa^{4/3}\ \kappa-1-\kappa D_p\quad 1-\upsilon-D_p\ ^2}{\upsilon^{2/3}}\right] \tag{3.315}$$

The roots of equation (3.312) are

$$\lambda_{1,2}\rightarrow\pm\left[\frac{p_1}{3}+\frac{p_1^2-p_2}{3}\left(\frac{2}{N}\right)^{1/3}+\frac{N^{1/3}}{32^{1/3}}\right]^{1/2}$$

$$\lambda_{3,4}\rightarrow\pm\left[\frac{p_1}{3}-\frac{p_1^2}{32^{2/3}N^{1/3}}+\frac{ip_1^2}{2^{2/3}\sqrt{3}N^{1/3}}+\frac{p_2}{2^{2/3}N^{1/3}}-\frac{ip_2\sqrt{3}}{2^{2/3}N^{1/3}}+\frac{N^{1/3}}{22^{1/3}\sqrt{3}}+\frac{N^{1/3}}{62^{1/3}}\right]^{1/2} \tag{3.316}$$

$$\lambda_{5,6} \to \pm\left[\frac{p_1}{3} - \frac{p_1^2}{32^{2/3}N^{1/3}} - \frac{ip_1^2}{2^{2/3}\sqrt{3}N^{1/3}} + \frac{p_2}{2^{2/3}N^{1/3}} + \frac{i\sqrt{3}p_2}{2^{2/3}N^{1/3}} + \frac{N^{1/3}}{22^{1/3}\sqrt{3}} - \frac{N^{1/3}}{62^{1/3}}\right]^{1/2}$$

where $N = 2p_1^3 - 9p_1p_2 + 27p_3 + \sqrt{4\ 3p_2 - p_1^2\ ^3 + \ 2p_1^3 - 9p_1p_2 + 27p_3\ ^2}$

A positive root or a cmplex root with positive real part will bring about instability around the non-collinear EPs.

## 3.9 Conclusion

This chapter of the thesis studied the R3BP and the Robe's R3BP witn variable masses under different characterizations. In the case of the R3BP with variable masses, we assumed that mass variations of the primaries occur according to the UML and their motion dictated by the GMP under radiation pressure forces, oblateness, perturbations in the Coriolis and centrifugal forces and the presence of a circumbinary disc. In the case when the first primary is an oblate spheroid and the second one a radiation emitter under effects of small perturbations in the Coriolis and centrifugal force; it is seen that there exist three collinear, two triangular and four out-of-plane EPs. When both primaries are radiation source and oblate spheroids coupled with small perturbations in the Coriolis and centrifugal forces, there exist five collinear, two triangular and four out-of-plane EPs. In the presence of disc, we also established four out-of-plane EPs using series expansion about the disc mass. The collinear and triangular EPs of the autonomized systems are stable while the out-of-plane EPs points are unstable. However, the EP of the non-autonomous equations are unstable. The small change in the Coriolis and centrifugal forces and oblateness of the primaries has both stabilizing and destabilizing tendencies while the radiation pressure forces are always destabilizing.

In the study of the periodic orbits, it is seen that the orbits can be ellipses, parabolas or hyperbolas. The frequencies, orientation, eccentricities and lengths of the semi axes of the short and long period orbits are different from those of the study conducted by Szebehely (1967a), Meyer and Schmidt (1971), Sharma (1976), Elipe and Lara (1997), Perdios (2003), Perdios and Kalantonis (2006), AbdulRaheem and Singh (2008), Singh and Begha (2011b) and, Singh and Haruna (2014b).

In the case of the Robe's R3BP with variable masses we considered the cases when both the primaries vary isotropically in accordance with the UML and their equation of motion of the is described by the GMP (when rigid shell is empty: problem **3.6** amd **3.7**) and when the three-bodies vary their masses at the same rate in a non-isotropic manner (problem **3.8**) in which case the first primary is a fluid . There is an EP at the center and a pair of non-collinear EP when the shell is empty. When small perturbations are involved, every point (inside the shell) on the line joining the centers of the primaries and a pair of non-collinear points, are the EPsof the system. In the case when the three masses vary with time with the first primary being a fliud, an equilibrium point at the center, another near the center, circular points and pairs of non-collinear EPs exist. The points on the line inside the shell are stable while the circular points and the non-collinear EPs of the autonomized systems are unstable. The EPs of the non-autonomous system are unstable according to the Lyapunov's theorem.

# CHAPTER FOUR

## 4.0 RESTRICTED THREE-BODY PROBLEM WITH CONSTANT MASSES AND DISC UNDER DIFFERENT CHARATERIZATIONS

### 4.1. Introduction

Circumbinary disc are commonly observed around binary systems and play important role in their evolution. Trilling *et al*. (2007) detected debris discs in many main-sequence stellar binary systems using the *Spitzer Space Telescope*. The importance of the problem in astronomy has been addressed by Jiang and Yeh (2004, 2006) where it was shown that the influence from the disc can change the locations of the EPs and new EPs can exist under some conditions.

In this chapter, we study the R3BP in the presence of a circumbinary disc under various characterizations of the bodies. We also examine the periodic orbits in the range of linearly stable triangular EPs for the problems. This chapter contains five sections. Section **4.1** gives the introduction, Sect. **4.2** deals with the problem when the first primary is a radiation source while the second primary and the third body are modeled as oblate spheroids. Section **4.3** investigates the motion around the triangular EPs when the primary star is a triaxial star and the second one is a radiating-oblate star coupled with small perturbations in the Coriolis and centrifugal forces. In particular, we apply the study to the case when the first primary is a depleted pulsating post-AGB star, IRAS 11472-0800 while its companion is a white dwarf star, G29-38. Section **4.4** generalize studies of the R3BP with constant masses. In this section, the three bodies are assumed to have the shape of an oblate spheroid while both primaries are radiation emitters under small perturbations in the Coriolis and centrifugal forces. Section **4.5** is the last section and concludes the chapter.

## 4.2 Motion in a Modified Restricted Three-Body Problem with Oblateness and Disc

In this section, the restricted problem of three bodies is modified to include a case when the passively gravitating third body is an oblate spheroid while the first primary is a source of radiation and the second one is an oblate spheroid, under the condition that both primaries are enclosed by a disc of dust. The EPs are found and the linear stability of the points is explored. Illustrative numerical exploration is rendered in the Sun–Jupiter and Sun–Earth systems where we show that in reality, for some values of the system parameters, the additional EPs do not in general exist even when there is a disc to interact with.

### 4.2.1 Equations of motion

Let $m_1$ and $m_2$ be the masses of the first and second primary, respectively, and let $m_3$ be the mass of the third body having infinitesimal. The gravitational potential energy acting on the third body when the both primaries have the shape of an oblate spheroid is given (Singh and Ishwar 1999) by

$$V = -Gm_3\left[m_1q_1\left(\frac{1}{r_1}+\frac{\alpha_1}{2r_1^3}\right)+m_2q_2\left(\frac{1}{r_2}+\frac{\alpha_2}{2r_2^3}\right)\right] \tag{4.1}$$

where $r_1^2 = (x-x_1)^2 + y^2 + z^2$, $r_2^2 = (x-x_2)^2 + y^2 + z^2$

Here, $r_1$ and $r_2$ are the distances of the third body from the first and second primary, respectively, $G$ is the gravitational constant; $q_1$ and $q_2$ are radiation factors of the first and second primary, respectively, while $\alpha_1$ and $\alpha_2$ are the parameters representing the oblateness of the first primary and second primary, respectively, and are such that (McCuskey 1963, Singh and Ishwar 1999)

$$\alpha_i = \frac{AE_i^2 - AP_i^2}{5R^2}, \alpha_i << 1: \ i = 1,2$$

Now, we suppose that the third body is a satellite whose shape is built in the form of an oblate spheroid. Then, the potential (4.1) takes the form (Abouelmagd and EL-Shaboury, 2012)

$$V = -Gm_3\left[m_1\left(\frac{q_1}{r_1} + \frac{\alpha_1 q_1}{2r_1^3} + \frac{\alpha_3}{2r_1^3}\right) + m_2\left(\frac{q_2}{r_2} + \frac{\alpha_2 q_2}{2r_2^3} + \frac{\alpha_3}{2r_2^3}\right)\right] \tag{4.2}$$

where $\alpha_3$ is the oblateness of the third body and defined as $\alpha_3 = \frac{AE_3^2 - AP_3^2}{5R^2}, \alpha_3 << 1$

Now, in this configuration, we assume that the primaries are enclosed by a disc of dust and so we use a model that best explains a flattened potential given by Miyamoto and Nagai (1975); which describes the gravitational potential within a stellar system (or subsystem) and written as

$$\bar{V}\ (r,z\ ) = -\frac{GM_d}{\sqrt{r^2 + \ a + \sqrt{z^2 + b^2}\ ^2}} \tag{4.3}$$

where $M_d$ is the mass of the disc; $a$ and $b$ are parameters which determine the density profile of the disc. $a$ is the flatness parameter while $b$ is the core parameter. This flattened system reduces to the Plummer model if $a = 0$ and the Kuzmin disc if $b = 0$. When $a = b = 0$, the potential equals to the one by a point mass. The behavior of the two parameters having dimension of distance determines the two limiting cases. If $a = 0$ the axial symmetry is reduced to its special case (the spherical one) while for $b = 0$ the system has a collapse. In our problem, we consider motion in the $xy-$plane and so $z = 0$. Therefore, equation (4.3) reduces to

$$\bar{V}\ r,z\ = -\frac{GM_d}{\sqrt{r^2+\ a+b\ ^2}} \tag{4.4}$$

Now, we intend to study the motion of the oblate satellite in the Sun-Earth and Sun-Jupiter model of the primaries, in which the first primary (Sun) is only a radiating source and the second primary (Earth/Jupiter) is an oblate spheroid. Then, the potential acting on the third body under the combined actions of oblateness, radiation and disc mass have the forms

$$U=-Gm_3\left[m_1\left(\frac{q_1}{r_1}+\frac{\alpha_3}{2r_1^3}\right)+m_2\left(\frac{1}{r_2}+\frac{\alpha_2 q_2}{2r_2^3}+\frac{\alpha_3}{2r_2^3}\right)+\frac{M_d}{r^2+\bar{T}^2\ ^{1/2}}\right] \tag{4.5}$$

where $r_1^2=\ x-x_1\ ^2+y^2, r_2^2=\ x-x_2\ ^2+y^2$ and $\alpha_i=\frac{AE_i^2-AP_i^2}{5R^2}, \alpha_i<<1:\ i=2,3$ .

The last term in the potential $U$, is the gravitational potential due to the mass $M_d$ of the circumbinary disc (Kushvah 2008a). $r=\sqrt{x^2+y^2}$ is the radial distance of the third body and $\bar{T}=a+b$ is the parameter which defines the density profile of the belt.

Now, we choose unit of mass such that, the sum of the masses of the primaries is taken as unity, thus we take $m_1=1-\mu$ and $m_2=\mu:\ 0<\mu=\frac{m_2}{m_1+m_2}<1$. For the unit of length, we take the distance between the primaries as unity, that is $R=1$ and the unit of time is also selected such that $G=1$.

Now, the equations of motion of a passively gravitating third body in a barycentric rotating coordinate system, in the gravitational field of the primaries, based on the potential (4.5), have the form:

$$\ddot{x}-2n\dot{y}=n^2x-\frac{q_1\ 1-\mu\quad x-\mu}{r_1^3}-\frac{\mu\ \ x-\mu+1}{r_2^3}-\frac{3\alpha_2\mu\ \ x-\mu+1}{2r_2^5}-\frac{3\alpha_3\ \ 1-\mu\quad x-\mu}{2r_1^5}$$
$$-\frac{3\alpha_3\mu\ \ x-\mu+1}{2r_2^5}-\frac{M_d x}{r^2+\bar{T}^2\ \ ^{3/2}} \tag{4.6}$$

$$\ddot{y}+2n\dot{x}=\left\{n^2-\frac{q_1\ 1-\mu}{r_1^3}-\frac{\mu}{r_2^3}-\frac{3\alpha_2\mu}{2r_2^5}-\frac{3\alpha_3\ \ 1-\mu}{2r_1^5}-\frac{3\alpha_3\mu}{2r_2^5}-\frac{M_d}{r^2+\bar{T}^2\ \ ^{3/2}}\right\}y$$

$$\text{where } n^2=1+\frac{3}{2}\alpha_2+\frac{2M_d r_c}{r_c^2+\bar{T}^2\ \ ^{\frac{3}{2}}} \text{ (Kushvah, 2008a)} \tag{4.7}$$

$$r_1^2=(x-\mu)^2+y^2,\quad r_2^2=(x-\mu+1)^2+y^2 \tag{4.8}$$

Here, $\mu$ is the mass ratio; $n$ is the mean motion of the primaries and is defined by the oblateness of the first primary, the mass of the disc and its components which represents the density profile of the disc and the radial distance $r_c$ of the third body.

Equations (4.6) differ from those of Kushvah (2008a) only due to the oblateness of the third body and this system of equations admits the Jacobi integral

$$J+\ \dot{x}^2+\dot{y}^2\ =2U \tag{4.9}$$

where

$$U=\frac{n^2\ \ x^2+y^2}{2}+\frac{q_1\ 1-\mu}{r_1}+\frac{\mu}{r_2}+\frac{\alpha_2\mu}{2r_2^3}+\frac{\alpha_3\ \ 1-\mu}{2r_1^3}+\frac{\alpha_3\mu}{2r_2^3}+\frac{M_d}{r^2+\bar{T}^2\ \ ^{1/2}} \tag{4.10}$$

and $J$ is the Jacobi constant.

## 4.2.2 Locations of equilibrium points

The equilibrium points represent stationary solutions of the R3BP. These points are the singularities of the manifold of the components of the velocity and the coordinates. Therefore, they can be found by equating the R.H.S of equations (4.6) to zero. That is we have to solve the following equations

$$n^2 x - \frac{q_1\ 1-\mu\quad x-\mu}{r_1^3} - \frac{\mu\ \ x-\mu+1}{r_2^3} - \frac{3\alpha_2\mu\ \ x-\mu+1}{2r_2^5} - \frac{3\alpha_3\ \ 1-\mu\quad x-\mu}{2r_1^5} - \frac{3\alpha_3\mu\ \ x-\mu+1}{2r_2^5}$$
$$- \frac{M_d x}{r^2+\bar{T}^2\ ^{3/2}} = 0 \qquad (4.11)$$

$$\left\{ n^2 y - \frac{q_1\ 1-\mu}{r_1^3} - \frac{\mu}{r_2^3} - \frac{3\alpha_2\mu}{2r_2^5} - \frac{3\alpha_3\ \ 1-\mu}{2r_1^5} - \frac{3\alpha_3\mu}{2r_2^5} - \frac{M_d}{r^2+\bar{T}^2\ ^{3/2}} \right\} = 0$$

When equation (4.11) is solved for $y = 0$, then we only need to find $x$. If however, $y \neq 0$, then we have to solve both equations of (4.11) for $x$ and $y$.

**Collinear equilibrium points**

The collinear equilibrium points are found by solving equation (4.11) for $x$ when $y = 0$, so that we have

$$\left[ n^2 - \frac{M_d}{x^2+\bar{T}^2\ ^{3/2}} \right] x - \frac{q_1\ 1-\mu\quad x-\mu}{r_1^3} - \frac{\mu\ \ x-\mu+1}{r_2^3} - \frac{3\alpha_2\mu\ \ x-\mu+1}{2r_2^5} - \frac{3\alpha_3\ \ 1-\mu\quad x-\mu}{2r_1^5}$$
$$- \frac{3\alpha_3\mu\ \ x-\mu+1}{2r_2^5} = 0$$

We denote this equation by $f\ \ x\ $, so that

$$f(x) = \left[ n^2 - \frac{M_d}{(x^2 + \bar{T}^2)^{3/2}} \right] x - \frac{q_1 (1-\mu)(x-\mu)}{r_1^3} - \frac{\mu (x-\mu+1)}{r_2^3} - \frac{3\alpha_3 (1-\mu)(x-\mu)}{2r_1^5} - \frac{3\mu (x-\mu+1)(\alpha_2+\alpha_3)}{2r_2^5} = 0 \quad (4.12)$$

where $r_1 = |x-\mu|$ and $r_2 = |x-\mu+1|$

The abscissas of the collinear points are the roots of the equation $f(x) = 0$.

Observe that

$$\frac{df(x)}{dx} = n^2 - \frac{M_d x}{(x^2 + \bar{T}^2)^{3/2}} + \frac{3M_d x^2}{(x^2 + \bar{T}^2)^{5/2}} + \frac{2q_1 (1-\mu)}{|x-\mu|^3} + \frac{2\mu}{|x-\mu+1|^3} + \frac{6\alpha_3 (1-\mu)}{|x-\mu|^5} + \frac{6\mu (\alpha_2+\alpha_3)}{|x-\mu+1|^5} \quad (4.13)$$

And so $\frac{df(x)}{dx} > 0$ if

$$n^2 + \frac{3M_d x^2}{(x^2 + \bar{T}^2)^{5/2}} + \frac{2q_1 (1-\mu)}{|x-\mu|^3} + \frac{2\mu}{|x-\mu+1|^3} + \frac{6\alpha_3 (1-\mu)}{|x-\mu|^5} + \frac{6\mu (\alpha_2+\alpha_3)}{|x-\mu+1|^5} > \frac{M_d x}{(x^2 + \bar{T}^2)^{3/2}}$$

Now, when $x = \pm\infty$, we have $\frac{df(x)}{dx} = n^2$. When $x = \mu - 1$ and $x = \mu$, equation (4.13) in both cases give $\frac{df(x)}{dx} = \infty$.

Now, from equation (4.12), if $x = \mu - 2$, we observed that $f(\mu-2) < 0$, for, $M_d << 1$, $\alpha_{2,3} << 1$ and $q_1 \approx 1$. When $x = 0$, we see found that $f(0) > 0$, while for $x = \mu + 1$, $f(\mu+1) > 0$

Therefore, the real roots of the equation $f\ x\ =0$ will lie in the opens interval $\mu-2,\mu-1$ , $\mu-1,0$ and $\mu,\mu+1$ . These roots correspond to the collinear points and we shall denote them appropriately when they are found.

Now, we express equation (4.12) as

$$f\ x\ =P\ x\ +Q\ x \tag{4.14}$$

where

$$P\ x\ =n^2x-\frac{1-\mu\quad x-\mu}{\left|x-\mu\right|^3}\left\{q_1+\frac{3}{2}\frac{\alpha_3}{\left|x-\mu\right|^2}\right\}-\frac{\mu\ \ x-\mu+1}{\left|x-\mu+1\right|^3}\left\{1+\frac{3}{2}\frac{\alpha_2+\alpha_3}{\left|x-\mu+1\right|^2}\right\} \tag{4.15}$$

$$Q\ x\ =-\frac{M_d x}{x^2+\bar{T}^2\ ^{3/2}} \tag{4.16}$$

Now, observe that equation (4.15) can have the following forms depending on positions of the primaries. When $x<\mu-1,\ \mu-1<x<\mu$ and $\mu<x$, then equation (4.15), give respectively

$$P\ x\ =n^2x+\frac{1-\mu}{x-\mu\ ^2}\left\{q_1+\frac{3}{2}\frac{\alpha_3}{x-\mu\ ^2}\right\}+\frac{\mu}{x-\mu+1\ ^2}\left\{1+\frac{3}{2}\frac{\alpha_2+\alpha_3}{x-\mu+1\ ^2}\right\} \tag{4.17}$$

$$P\ x\ =n^2x+\frac{1-\mu}{x-\mu\ ^2}\left\{q_1+\frac{3}{2}\frac{\alpha_3}{x-\mu\ ^2}\right\}-\frac{\mu}{x-\mu+1\ ^2}\left\{1+\frac{3}{2}\frac{\alpha_2+\alpha_3}{x-\mu+1\ ^2}\right\} \tag{4.18}$$

$$P\ x\ =n^2x-\frac{1-\mu}{x-\mu\ ^2}\left\{q_1+\frac{3}{2}\frac{\alpha_3}{x-\mu\ ^2}\right\}-\frac{\mu}{x-\mu+1\ ^2}\left\{1+\frac{3}{2}\frac{\alpha_2+\alpha_3}{x-\mu+1\ ^2}\right\} \tag{4.19}$$

Hence, in order to investigate the position of collinear equilibrium points, we divide the orbital plane *Oxy* into three parts with respect to the primaries: $x<\mu-1,\ \mu-1<x<\mu$ and $\mu<x$.

The first case $x < \mu - 1$, implies that $x - \mu + 1 < 0$ and if this happens, then $x - \mu < 0$ follows. In the second case, when $\mu - 1 < x < \mu$, we have $x - \mu < 0$ and $x - \mu + 1 > 0$. The third case when $\mu < x$, we have $x - \mu > 0$ and so $x - \mu + 1 > 0$ also follows. We shall consider these one after the other.

**Case I**

In the first region when $x < \mu - 1$, this means that the collinear point will lie in the interval $x \in -\infty, \mu - 1$ ; in this case, if we take the first derivative of equation (4.17) with respect to $x$, we get,

$$P'(x) = n^2 + \frac{2(1-\mu)}{|x-\mu|^3}\left\{q_1 + \frac{3}{2}\frac{\alpha_3}{|x-\mu|^2}\right\} + \frac{2\mu}{|x-\mu+1|^3}\left\{1 + \frac{3}{2}\frac{\alpha_2 + \alpha_3}{|x-\mu+1|^2}\right\} \tag{4.20}$$

Evidently, for $0 < \mu \leq \frac{1}{2}$, $\alpha_{2,3} << 1$ and $q_1 \approx 1$ we see that $P'(x) > 0$

Knowing that equation (4.20) is positive and since $\lim_{x \to -\infty} P(x) = -\infty$ and $\lim_{x \to \mu - 1} P(x) = \infty$; then, $P(x)$ is strictly increasing in the interval. Similarly, from equation of (4.16), we see that $\lim_{x \to -\infty} Q(x) = 0$ , $\lim_{x \to \mu - 1} Q(x) > 0$ and $Q'(x) < 0$; it follows that $Q(x)$ is strictly decreasing function in the interval $-\infty, \mu - 1$ . Hence, $\lim_{x \to -\infty} f(x) < 0$ and $\lim_{x \to \mu - 1} f(x) > 0$. This shows that there exists a point say $x_1 \in -\infty, \mu - 1$ at which $f(x_1) = 0$. We denote this equilibrium point as $L_1$.

**Case II**

In this case, the collinear point lies in the open interval $\mu-1,\ \mu$ . Here, two cases arise: the first one is when $x\in\ \mu-1,\ 0$ and secondly, when it lies in the interval $0,\ \mu$ . We discuss them separately.

For the case when $x\in\ \mu-1,\ 0$ , we have $f\ \ 0\ =\frac{1-\mu}{\mu^2}\left\{q_1+\frac{3}{2}\frac{\alpha_3}{\mu^2}\right\}-\frac{\mu}{1-\mu\ ^2}\left\{1+\frac{3}{2}\frac{\alpha_2+\alpha_3}{1-\mu\ ^2}\right\}$

But $0<\mu\leq\frac{1}{2},\psi\approx1,\ \alpha_{2,3}<<1$ and $q_1\approx1$, therefore $f\ \ 0\ <0$. Since $f\ \ 0\ <0$ and

$\lim_{x\to\ \mu-1}\ f\ \ x\ >0$, then, there exist a point say $x_2\in\ \mu-1,\ 0$ at which $f\ \ x_2\ =0$. We denote this point by $L_2$.

Next, for the case when $x\in\ 0,\ \mu$ ; if we suppose that $\bar{T}<\sqrt{2}\mu$, then $\frac{\bar{T}}{\sqrt{2}}<\mu$ and so, we consider two cases when $x\in\left(0,\ \frac{\bar{T}}{\sqrt{2}}\right)$ and when $x\in\left(\frac{\bar{T}}{\sqrt{2}},\ \mu\right)$. Now, in the first case, if $P\left(\frac{\bar{T}\sqrt{2}}{2}\right)+Q\left(\frac{\bar{T}\sqrt{2}}{2}\right)>0$ , then $f\left(\frac{\bar{T}\sqrt{2}}{2}\right)>0$ and since $f\ \ 0\ <0$, then, there exist a point say $x_{21}\in\left(0,\ \frac{\bar{T}}{\sqrt{2}}\right)$ at which $f\ \ x_{21}\ =0$. We denote this point by $L_{21}$.

In the second case when $x\in\left(\frac{\bar{T}}{\sqrt{2}},\ \mu\right)$; if $P\left(\frac{\bar{T}\sqrt{2}}{2}\right)+Q\left(\frac{\bar{T}\sqrt{2}}{2}\right)>0$, then $f\left(\frac{\bar{T}\sqrt{2}}{2}\right)>0$ and because $\lim_{x\to\mu}P\ \ x\ =-\infty$ and $\lim_{x\to\mu}Q\ \ x\ >0$, we have $\lim_{x\to\mu}f\ \ x\ <0$. Then, we must have a unique

point, say $x_{22} \in \left( \frac{\bar{T}}{\sqrt{2}}, \mu \right)$ in this region, where $f\ x_{22} = 0$. We label it as $L_{22}$. We note that, if $P\left( \frac{\bar{T}\sqrt{2}}{2} \right) + Q\left( \frac{\bar{T}\sqrt{2}}{2} \right) < 0$, no EP exists in the interval $0, \mu$ because $f\left( \frac{\bar{T}\sqrt{2}}{2} \right) \neq 0$.

The points $L_{21}$ and $L_{22}$ depend only on the mass parameter, the mass of the disc and the parameters which determine the density profile of the disc.

**Case III**

In this region, the point lies in the open interval $\mu, \infty$. We observe from equation of (4.19) that $P'\ x > 0$ and since $\lim_{x \to \mu} P\ x = -\infty$ and $\lim_{x \to \infty} P\ x = \infty$, this means $P\ x$ is a strictly increasing function in this interval. Also, because $Q'\ x > 0$, $\mu Q\ x < 0$ and $\lim_{x \to \infty} Q\ x = 0$, $Q\ x$ is also a strictly increasing function for $x \in \mu, \infty$. Hence, $\lim_{x \to \mu} f\ x < 0$ and $\lim_{x \to \infty} f\ x > 0$; so there exist a unique point say $x_3$ in this open interval at which $f\ x_3 = 0$. We call this point $L_3$.

Thus, in addition to the usual three collinear EPs of the classical R3BP, due to the presence of the potential from the disc and the mass parameter, there exist two additional collinear EPs $L_{21}$ and $L_{22}$. Hence, we have the points $L_1$ and $L_2$ located to the left of the center of mass while $L_{21}$, $L_{22}$ and $L_3$ are positioned at the right.

In order to make a comprehensive numerical analysis, we represent the collinear points respectively as:

$$x_1 = \mu - 1 - \xi_1, \quad x_2 = \mu - 1 + \xi_2, \quad x_3 = \mu + \xi_3 \tag{4.21}$$

where $\xi_i = \ i = 1, 2, 3$ are the roots of equation (4.12).

Substituting equation (4.21) in (4.12) and simplifying, we get a polynomial for the point $L_1$:

$$\xi_1^9 + a_{11}\xi_1^8 + a_{12}\xi_1^7 + a_{13}\xi_1^6 + a_{14}\xi_1^5 - a_{15}\xi_1^4 - a_{16}\xi_1^3 - a_{17}\xi_1^2 - a_{18}\xi_1 - a_{19} = 0$$

where $a_{11} = \frac{5+4\upsilon}{1+\upsilon}$, $a_{12} = \frac{10+6\upsilon}{1+\upsilon}$, $a_{13} = \frac{2\ \ 2\upsilon+5}{1+\upsilon} - \frac{\upsilon+q_1}{n^2-A\ \ 1+\upsilon}$,

$$a_{14} = \frac{n^2-A\ \ 5+\upsilon\ \ -2\ \ \upsilon+q_1}{n^2-A\ \ 1+\upsilon}, a_{15} = \frac{3\ \ 4+\alpha_2+\alpha_3\ \ \upsilon+3\alpha_3+2q_1-2\ \ n^2-A}{2\ \ n^2-A\ \ 1+\upsilon}$$

$$a_{16} = \frac{2\upsilon\left[2+3\ \ \alpha_2+\alpha_3\ \right]}{n^2-A\ \ 1+\upsilon}, a_{17} = \frac{\upsilon\left[1+9\ \ \alpha_2+\alpha_3\ \right]}{n^2-A\ \ 1+\upsilon}, a_{18} = \frac{6\upsilon\ \ \alpha_2+\alpha_3}{n^2-A\ \ 1+\upsilon}$$

$$a_{19} = \frac{3\upsilon\ \ \alpha_2+\alpha_3}{2\ \ n^2-A\ \ 1+\upsilon}$$

For the point $L_2$, we have

$$\xi_2^9 - a_{11}\xi_2^8 + a_{12}\xi_2^7 - a_{13}\xi_2^6 + a_{14}\xi_2^5 - a_{15}\xi_2^4 + a_{16}\xi_2^3 - a_{17}\xi_2^2 + a_{18}\xi_2 - a_{19} = 0$$

where $a_{11} = \frac{5+4\upsilon}{1+\upsilon}$, $a_{12} = \frac{10+6\upsilon}{1+\upsilon}$, $a_{13} = \frac{n^2-A\ \ 4\upsilon+10\ \ +q_1+\upsilon}{n^2-A\ \ 1+\upsilon}$

$$a_{14} = \frac{n^2-A\ \ 5+\upsilon\ \ +2q_1+4\upsilon}{n^2-A\ \ 1+\upsilon}, \quad a_{15} = \frac{2\ \ n^2-A+q_1+6\upsilon\ \ +3\alpha_2\upsilon+3\alpha_3\ \ 1+\upsilon}{2\ \ n^2-A\ \ 1+\upsilon}$$

$$a_{16} = \frac{2\upsilon\left[2+3\ \ \alpha_2+\alpha_3\ \right]}{n^2-A\ \ 1+\upsilon}, \quad a_{17} = \frac{\upsilon\left[1-9\ \ \alpha_2+\alpha_3\ \right]}{n^2-A\ \ 1+\upsilon}, \quad a_{18} = \frac{6\upsilon\ \ \alpha_2+\alpha_3}{n^2-A\ \ 1+\upsilon}$$

$$a_{19} = \frac{3\ \ \alpha_2+\alpha_3}{2\ \ n^2-A\ \ 1+\upsilon}, A = \frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}, r_c^2 = 1-\mu+\mu^2$$

The polynomials for the collinear EP $L_3$ can be obtained analogously as above from the third equation of (4.21). Hence, the numbers and values of roots is influenced by the oblateness

parameters, small perturbation in the centrifugal force, the radiation pressure of the first primary and the effect occurring due to the gravitational potential from the disc.

**Triangular equilibrium points**

The triangular points lie on the $xy-$plane, and denoted by $L_4$ and $L_5$. We get the coordinates by first solving system (4.11) for $r_1$ and $r_2$, to get

$$r_1 = 1-\frac{1}{2}\alpha_2 - \frac{1}{3}\left(1-q_1\right) + \frac{1}{2}\alpha_3 - \frac{M_d}{3\left(r_c^2+\bar{T}^2\right)^{3/2}}\left(2r_c-1\right)$$

$$r_2 = 1-\frac{1}{2}\alpha_2 + \frac{1}{2}\left(\alpha_2+\alpha_3\right) - \frac{M_d}{3\left(r_c^2+\bar{T}^2\right)^{3/2}}\left(2r_c-1\right) \tag{4.22}$$

Now, the exact coordinates of the triangular point is found by solving equations (4.8) simultaneously for $x$ and $y$, to get

$$x = \mu - \frac{1}{2} + \frac{r_2^2-r_1^2}{2}, \qquad y = \pm\sqrt{\frac{r_1^2+r_2^2}{2} - \frac{1}{4} - \frac{\left(r_2^2-r_1^2\right)^2}{2}} \tag{4.23}$$

Now, from equations (4.22), finding $r_2^2-r_1^2$ and $r_2^2+r_1^2$; and substituting in equations (4.23), we get

$$x = \mu - \frac{1}{2} + \left\{1-\frac{M_d\left(2r_c-1\right)}{3\left(r_c^2+\bar{T}^2\right)^{3/2}}\right\}\left[\frac{1}{3}\left(1-q_1\right) - \frac{\alpha_2}{2}\right] \tag{4.24}$$

$$y = \pm\frac{\sqrt{3}}{2}\left\{1-\frac{2}{9}\left(1-q_1\right) + \frac{2}{3}\alpha_2 - \frac{1}{3}\left(\alpha_2+2\alpha_3\right) + \frac{M_d\left(2r_c-1\right)}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\right\}$$

These are the coordinates of the triangular equilibrium points of the restricted problem of three oblate bodies under radiation pressure forces of the primaries, small change in the Coriolis and centrifugal forces and the force due to the potential from the disc.

### 4.2.3 Stability of equilibrium points

In the R3BP, for all values of the free parameters, the triangular equilibrium points $L_{4,5}$ may be stable. Looking at small deviations from these points, we investigate their stability. To do this, we displace the dust particle is a little from the equilibrium points, by applying a small velocity. If the particle rapidly departs from the vicinity of the triangular equilibrium point, then the equilibrium points is an unstable one. If however, it oscillates about this point, we can then say the point is stable. Because the mass distribution of the general body is now taken into account, the stability of the equilibrium points, where are expected to deviate from the R3BP. We introduce $x = x_0 + \xi$ and $y = y_0 + \eta$, where $(\xi, \eta)$ is a small displacement, and substitute it in (4.6), to get the variational equations:

$$\ddot{\xi} - 2n\dot{\eta} = U_{xx}^0 \xi + U_{xy}^0 \eta$$

$$\ddot{\eta} + 2n\dot{\xi} = U_{xy}^0 \xi + U_{yy}^0 \eta$$

The partial derivatives $U_{xx}^0, U_{yy}^0$ and $U_{xy}^0$ are to be computed at the equilibrium points under study.

Now the characteristic equation is given by

$$\lambda^4 - \left(U_{xx}^0 + U_{yy}^0 - 4n^2\right)\lambda^2 + U_{xx}^0 U_{yy}^0 - \left(U_{xy}^0\right)^2 = 0 \qquad (4.25)$$

**Collinear equilibrium points**

And the partial derivatives at the collinear point $x,0$ are

$$U_{xx}^0 = n^2 + \frac{2\ 1-\mu\ q_1}{\left|x_0-\mu\right|^3} + \frac{2\mu}{\left|x_0+1-\mu\right|^3} + \frac{6\ 1-\mu\ \alpha_3}{\left|x_0-\mu\right|^5} + \frac{6\mu\ \alpha_2+\alpha_3}{\left|x_0-\mu+1\right|^5} + \left[\frac{3x_0^2}{x_0^2+\bar{T}^2\ ^{5/2}} - \frac{1}{x_0^2+\bar{T}^2\ ^{3/2}}\right]M_d$$

$$U_{xy}^0 = 0 \tag{4.26}$$

$$U_{yy}^0 = n^2 - \frac{2\ 1-\mu}{\left|x_0-\mu\right|^3}\left[q_1 + \frac{3\alpha_3}{\left|x_0-\mu\right|^3}\right] - \frac{\mu}{\left|x_0+1-\mu\right|^3}\left[1+\frac{3\ \alpha_2+\alpha_3}{\left|x_0+1-\mu\right|^2}\right] - \frac{M_d}{x_0^2+\bar{T}^2\ ^{3/2}}$$

Now, an inspection of the first partial derivative in equations (4.26) revealed it is positive while the derivative is negative, for $0<\mu\le\frac{1}{2}, M_d << 1,\ \alpha_{2,3} << 1$ and $q_1 \approx 1$.

Now substituting equations of (4.26) in the characteristic equation (4.25), we get

$$\lambda^4 + P_1\lambda^2 + Q_1 = 0 \tag{4.27}$$

where

$$P_1 = n^2 - U_{xx}^0 - U_{yy}^0,\ Q_1 = U_{xx}^0 U_{yy}^0 < 0$$

The four roots of equation (4.27) are

$$\lambda_{1,2,3,4} = \pm\sqrt{\frac{P_1 \pm \sqrt{D}}{2}} \tag{4.28}$$

where $D$ is the discriminant of equation (4.27) and is expressed such that $D = P_1^2 - Q_1$

Now, the kinds of roots given by equation (4.28) will depend on the nature of the discriminant coupled with the changes in $P_1$ since $Q_1$ is always negative. Therefore, there is at least one

positive root of (4.27) which induces instability at the collinear EPs. Hence, the collinear points are unstable in this case.

**Triangular equilibrium points**

Now, from the potential given by equation (4.10), the second order partial derivatives are:

$$U_{xx} = n^2 - \frac{q_1\ 1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3q_1\ 1-\mu\quad x-\mu\ ^2}{r_1^5} + \frac{3\mu\ \ x-\mu+1\ ^2}{r_2^5} - \frac{3}{2}\frac{1-\mu\ \ \alpha_3}{r_1^5} - \frac{3}{2}\frac{\mu\ \ \alpha_2+\alpha_3}{r_2^5}$$
$$+\frac{15}{2}\frac{1-\mu\quad x-\mu\ ^2\alpha_3}{r_1^7} + \frac{15}{2}\frac{\mu\ \ x-\mu+1\ ^2\ \ \alpha_2+\alpha_3}{r_1^7} - \frac{M_d}{r^2+\bar{T}^2\ ^{3/2}} + \frac{3M_d x^2}{r^2+\bar{T}^2\ ^{5/2}}$$

$$U_{yy} = n^2 - \frac{q_1\ 1-\mu}{r_1^3} - \frac{\mu}{r_2^3} + \frac{3q_1\ 1-\mu\ \ y^2}{r_1^5} + \frac{3\mu y^2}{r_2^5} - \frac{3}{2}\frac{1-\mu\ \ \alpha_3}{r_1^5} - \frac{3}{2}\frac{\mu\ \ \alpha_2+\alpha_3}{r_2^5} + \frac{15}{2}\frac{1-\mu\ \ y^2\alpha_3}{r_1^7}$$
$$+\frac{15}{2}\frac{\mu y^2\ \ \alpha_2+\alpha_3}{r_1^7} - \frac{M_d}{r^2+\bar{T}^2\ ^{3/2}} + \frac{3M_d y^2}{r^2+\bar{T}^2\ ^{5/2}} \qquad 4.29$$

$$U_{xy} = \frac{3q_1\ 1-\mu\quad x-\mu\ \ y}{r_1^5} + \frac{3\mu\ \ x-\mu+1\ \ y}{r_2^5} + \frac{15}{2}\frac{1-\mu\quad x-\mu\ \ y\alpha_3}{r_1^7} + \frac{15}{2}\frac{\mu\ \ x-\mu+1\ \ y\ \ \alpha_2+\alpha_3}{r_1^7}$$
$$+\frac{3M_d xy}{r^2+\bar{T}^2\ ^{5/2}}$$

From the equations of (4.22), we find that

$$r_1^{-3} = 1+\frac{3}{2}\alpha_2 + \frac{3}{2}\ 1-q_1\ \ -\frac{3}{2}\alpha_3 + \frac{M_d\ \ 2r_c-1}{r_c^2+\bar{T}^2\ ^{3/2}},$$

$$r_2^{-3} = 1+\frac{3}{2}\alpha_2 - \frac{3}{2}\ \ \alpha_2+\alpha_3\ \ + \frac{M_d\ \ 2r_c-1}{r_c^2+\bar{T}^2\ ^{3/2}}$$

$$r_2^{-5} = 1+\frac{5}{2}\alpha_2 - \frac{5}{2}\ \ \alpha_2+\alpha_3\ \ + \frac{5M_d\ \ 2r_c-1}{3\ \ r_c^2+\bar{T}^2\ ^{3/2}},$$

$$r_1^{-7} = 1 + \frac{7}{2}\alpha_2 + \frac{7}{3}\left(1 - q_1\right) - \frac{7}{2}\alpha_3 + \frac{7M_d}{3}\frac{\left(2r_c - 1\right)}{\left(r_c^2 + \bar{T}^2\right)^{3/2}}$$

$$r_2^{-7} = 1 + \frac{7}{2}\alpha_2 - \frac{7}{2}\left(\alpha_2 + \alpha_3\right) + \frac{7M_d}{3}\frac{\left(2r_c - 1\right)}{\left(r_c^2 + \bar{T}^2\right)^{3/2}}$$

Now substituting these equations and the triangular points $L_4$ (4.24) in the partial derivatives (4.29), and solve correctly, we get

$$U_{xx}^0 = \frac{3}{4} + \frac{3}{8}\alpha_2 - 3\alpha_2\mu + \frac{5}{4}\frac{M_d}{\left(r_c^2 + \bar{T}^2\right)^{3/2}}\left(2r_c - 1\right) - \frac{1}{2}\left(1 - q_1\right) + \frac{3\mu}{2}\left(1 - q_1\right) - \frac{3M_d}{\left(r_c^2 + \bar{T}^2\right)^{3/2}}\left(\mu - 1/2\right)^2$$
$$\times\left[1 + \frac{1}{\mu - 1/2}\left\{\frac{1}{3}\left(1 - q_1\right) + \frac{1}{2}\alpha_2\right\}\right]$$

$$U_{yy}^0 = \frac{9}{4} + \frac{33}{8}\alpha_2 + 3\alpha_3 + \frac{1 - q_1}{2} - \frac{3}{2}\mu\left(1 - q_1\right) + \frac{7}{4}\frac{M_d}{\left(r_c^2 + \bar{T}^2\right)^{3/2}}\left(2r_c - 1\right) + \frac{9M_d}{4\left(r_c^2 + \bar{T}^2\right)^{5/2}} \tag{4.30}$$

$$U_{xy}^0 = \sqrt{3}\left[-\frac{3}{4} + \frac{1}{6}\left(1 - q_1\right) - \frac{7}{8}\alpha_2 - \frac{1}{2}\alpha_3 - \frac{11}{12}\frac{M_d}{\left(r_c^2 + \bar{T}^2\right)^{3/2}}\left(2r_c - 1\right) + \mu\left\{\frac{3}{2} + \frac{1}{6}\left(1 - q_1\right) + \frac{13}{4}\alpha_2 + \alpha_3\right.\right.$$
$$\left.\left. + \frac{11}{6}\frac{M_d}{\left(r_c^2 + \bar{T}^2\right)^{3/2}}\left(2r_c - 1\right)\right\} + \frac{3}{4}\frac{M_d}{\left(r_c^2 + \bar{T}^2\right)^{5/2}}\left(2\mu - 1\right)\right]$$

Now, substituting the equations of (4.30) in the characteristic equation (4.25), we simplify and ignoring product and higher order terms in $\alpha_{2,3}$, $1 - q_1$ and $M_d$; to get

$$\lambda^4-\left[-4-6\alpha_2+3\left\{1+\frac{3\alpha_2}{2}+\alpha_3+\alpha_2\mu+\frac{2r_c-1\ M_d}{r_c^2+\bar{T}^2\ ^{3/2}}+\frac{M_d\mu^2}{r_c^2+\bar{T}^2\ ^{3/2}}-\frac{M_d\mu}{r_c^2+\bar{T}^2\ ^{3/2}}+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\right\}\right]\lambda^2$$

$$+\frac{3}{4}\mu\ 1-\mu\ \left[9+39\alpha_2+12\alpha_3\ \ +2\ 1-q_1\ +\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\frac{9}{r_c^2+\bar{T}^2}+22\ \ 2r_c-1\right\}\right]=0 \qquad 4.31$$

The roots are

$$\lambda_{1,2}=\pm s_1 i\ ,\ \lambda_{3,4}=\pm s_2 i \qquad (4.32)$$

where

$$s_{1,2}=\frac{1}{2}\ -b\pm\sqrt{D}$$

$$b=1+\frac{3}{2}\ \alpha_2+2\alpha_3\ -3\mu\alpha_2+\frac{3M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\frac{8r_c}{3}-2r_c-1-\frac{\mu^2-\mu+1}{r_c^2+\bar{T}^2}\right\}$$

$$D=3\mu^2\left[9+39\alpha_2+12\alpha_3+2\ 1-q_1\ -\frac{22\ 1-2r_c\ M_d}{r_c^2+\bar{T}^2\ ^{\frac{3}{2}}}+\frac{7M_d}{r_c^2+\bar{T}^2\ ^{\frac{5}{2}}}\right]+\frac{2M_d}{r_c^2+\bar{T}^2\ ^{\frac{3}{2}}}\left\{2r_c+3-\frac{3}{r_c^2+\bar{T}^2}\right\}$$

$$-3\mu\left[9+41\alpha_2+12\alpha_3+2\ 1-q_1\ -M_d\left\{\frac{22\ 1-2r_c}{r_c^2+\bar{T}^2\ ^{3/2}}-\frac{7}{r_c^2+\bar{T}^2\ ^{5/2}}\right\}\right]+1+3\alpha_2-6\alpha_3$$

$D$ is the discriminant of equation (4.31) and its four roots (4.32) play a crucial role in determining the form of orbits in the basin of equilibrium points. Evidently, values of $D$ when $\mu=0$ and $\mu=\frac{1}{2}$ have opposite signs. Hence, there is a value of $\mu$ in the interval $0<\mu\le\frac{1}{2}$ at which $D=0$. Hence, we solve $D=0$ for $\mu$, and simplify to get

$$\mu_C = \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)-\frac{1}{9}\left(\frac{13}{\sqrt{69}}-1\right)\alpha_2-\frac{22\alpha_3}{9\sqrt{69}}-\frac{2\ 1-q_1}{27\sqrt{69}}+\frac{2M_d}{27\sqrt{69}\ \ r_c^2+\bar{T}^2\ ^{\frac{3}{2}}}$$
$$\times\left[38-4r_c-\frac{61}{2\ r_c^2+\bar{T}^2}\right] \tag{4.33}$$

We call the expression in equation (4.33) the critical mass function. The first term represents Routh's critical mass ratio; the second, third and fourth expressions are the effects arising from oblateness of the second primary, oblateness of the third body and radiation of the first primary, respectively, while the last expression indicates the force due to gravitational potential from the circumbinary disc.

i. When $0<\mu<\mu_C$, $D>0$; and the roots (4.32) are distinct pure imaginary and motion is bounded, and composed of two oscillatory solutions given as (Szebehely 1967a):

$$\begin{aligned}\xi &= C_1\,\mathrm{coss}_1 t + S_1\sin s_1 t + C_2\,\mathrm{coss}_2 t + S_2\sin s_2 t\\ \eta &= \bar{C}_1\,\mathrm{coss}_1 t + \bar{S}_1\sin s_1 t + \bar{C}_2\,\mathrm{coss}_2 t + \bar{S}_2\sin s_2 t\end{aligned} \tag{4.34}$$

where $s_i$ ◖$=1,2$ ‾ are the frequencies for the long periodic and short periodic orbit respectively, while the coefficients $C_1, S_1, \bar{C}_1$ , $\bar{S}_1$ and $C_2, S_2, \bar{C}_2$ , $\bar{S}_2$ are the long and short periodic terms respectively. Hence, the triangular point is stable in this case.

ii. When $\mu_C<\mu\le\frac{1}{2}$, $D<0$; the real parts of two of the roots are positive and the triangular points are unstable.

iii. When $\mu=\mu_C$, $D=0$; the roots are double and this leads to instability of the triangular point.

Therefore, we conclude that the triangular EP is stable for $0<\mu<\mu_C$ other it is unstable.

## 4.2.4 Periodic orbits around triangular equilibrium points

**Frequency and Orientation of the Orbits**

To study the periodic orbits, we know that in the stable region $0 < \mu < \mu_C$, where $\mu_C$ is the critical mass parameter which depends on the radiation pressure of the first primary, the gravitational potential from the disc, oblateness of the third body and that of the second primary; the characteristic equation (4.31) has pure imaginary roots. Solving equation (4.32), we get

$$s_1 = \left[ \frac{2A}{r_c^2 + \bar{T}^2} + \mu\ 1-\mu\ \left\{ \frac{27}{4} + \frac{117}{4}\alpha_2 + 9\alpha_3 + \frac{3}{2}\ 1-q_1\ - \left( \frac{33}{2} - 33r_c - \frac{19}{4\ r_c^2 + \bar{T}^2} \right) A \right\} \right]^{1/2} \quad (4.35)$$

$$s_2 = \left[ 1 - \frac{27}{4}\mu\ 1-\mu\ + \left( \frac{3}{2} - \frac{129}{4}\mu + \frac{117}{4}\mu^2 \right)\alpha_2 - \ 3 - 9\mu + 9\mu^2\ \ \alpha_3 + \frac{3}{2}\ 1-q_1\ \ \mu\ 1-\mu \right.$$
$$\left. + \left\{ 3 + 2r_c - \frac{2}{r_c^2 + \bar{T}^2} + \left( \frac{33}{2} - 33r_c - \frac{69}{12\ r_c^2 + \bar{T}^2} \right)\mu + \left( -\frac{33}{2} + 33r_c + \frac{69}{12\ r_c^2 + \bar{T}^2} \right)\mu^2 \right\} A \right]^{1/2} \quad 4.36$$

The frequency $s_1$ is for the long periodic orbit while $s_2$ is for the short periodic orbit.

Now, the expansion of the force function $U$ around the triangular point $L_4$ is expressed as

$$U = U^0 + U_{xx}^0\left( \frac{\xi^2}{2} \right) + U_{xy}^0 \xi\eta + U_{yy}^0\left( \frac{\eta^2}{2} \right) + o\ \ \xi^3, \eta^3 \quad (4.37)$$

Clearly, since equation (4.37) is in quadratic form, this indicates that the periodic orbits around $L_4$ are elliptical. Equation (4.37) can be expressed in the form

$$U = K\xi^2 + L\xi\eta + M\eta^2 + N \quad (4.38)$$

where

$$K=\frac{3}{8}+\left(\frac{3}{16}+\frac{3}{2}\mu\right)\alpha_2+\left(-\frac{1}{4}+\frac{3}{4}\mu\right)\ 1-q_1\ +\left\{\frac{3}{8\ r_c^2+\bar{T}^2}+\frac{5}{4}r_c-\frac{5}{8}-\frac{3\mu\ 1-\mu}{2\ r_c^2+\bar{T}^2}\right\}A$$

$$L=\sqrt{3}\left[-\frac{3}{4}+\frac{3}{2}\mu-\left(\frac{7}{8}-\frac{13}{4}\mu\right)\alpha_2-\left(\frac{1}{2}+\mu\right)\alpha_3+\left(\frac{1}{6}+\frac{1}{6}\mu\right)\ 1-q_1\ +\left\{\left(\frac{11}{12}-\frac{11}{6}\mu\right)-\left(\frac{11}{6}-\frac{11}{3}\mu\right)r_c\right.\right.$$
$$\left.\left.-\frac{3}{r_c^2+\bar{T}^2}+\frac{3\mu}{2\ r_c^2+\bar{T}^2}\right\}A\right]$$

$$M=\frac{9}{8}+\frac{33}{16}\alpha_2+\frac{3}{2}\alpha_3+\left(\frac{1}{4}-\frac{3}{4}\mu\right)\ 1-q_1\ +\left\{\frac{9}{8\ r_c^2+\bar{T}^2}+\frac{7}{4}r_c-\frac{7}{8}\right\}A \tag{4.39}$$

$$N=\frac{3}{2}+\left(\frac{3}{4}+\frac{1}{2}\mu\right)\alpha_2+\frac{1}{2}\alpha_3-\ 1-\mu\ \ 1-q_1\ +M_d\Delta\,,\Delta=\frac{1}{r_c^2+\bar{T}^{2\ 1/2}}\left\{1+\frac{r_c}{r_c^2+\bar{T}^2}\right\}$$

We introduce the variables $\bar{\xi}$ and $\bar{\eta}$ by the transformation

$$\begin{aligned}\xi&=\bar{\xi}\cos\alpha-\bar{\eta}\sin\\ \eta&=\bar{\xi}\sin\alpha+\bar{\eta}\cos\alpha\end{aligned} \tag{4.40}$$

This is equivalent to the rotation of the coordinate system $\xi$ and $\eta$ through angle $\alpha$ . Therefore, we select $\alpha$ such that the terms containing $\bar{\xi}\bar{\eta}$ in the force function $U$ vanishes. The new force function in its quadratic has the form:

$$U=\bar{K}\bar{\xi}^2+\bar{L}\bar{\eta}^2+\bar{M} \tag{4.41}$$

where

$$\bar{K} = \frac{3}{8} + \frac{3}{4}\sin^2\alpha - \frac{3\sqrt{3}}{8}\ (1-2\mu)\ \sin 2\alpha + \left[\left(-\frac{1}{4} + \frac{3}{4}\mu\right)\cos 2\alpha + \frac{\sqrt{3}}{2}\left(\frac{1}{6} + \frac{1}{6}\mu\right)\sin 2\alpha\right]\ (1-q_1)$$
$$+\left[\frac{3}{16} + \frac{3}{2}\mu + \left(\frac{15}{8} - \frac{3}{2}\mu\right)\sin^2\alpha - \frac{\sqrt{3}}{2}\left(\frac{7}{8} - \frac{13}{4}\mu\right)\sin 2\alpha\right]\alpha_2 + \left[\frac{3}{2}\sin^2\alpha - \frac{\sqrt{3}}{2}\left(\frac{1}{2} + \mu\right)\sin 2\alpha\right]\alpha_3$$
$$+\left[\left\{\frac{5}{4}r_c - \frac{5}{8} + \frac{3}{8\ r_c^2 + \bar{T}^2} - \frac{3\mu\ (1-\mu)}{2\ r_c^2 + \bar{T}^2}\right\} + \left\{\frac{1}{2}r_c - \frac{1}{4} + \frac{3}{4\left(r_c^2 + \bar{T}^2\right)} + \frac{3\mu\ (1-\mu)}{2\left(r_c^2 + \bar{T}^2\right)}\right\}\sin^2\alpha\right.$$
$$\left.+\frac{\sqrt{3}}{2}\left\{\left(\frac{11}{12} - \frac{11}{6}\mu\right) - \left(\frac{11}{6} - \frac{11}{3}\mu\right)r_c - \frac{3}{\left(r_c^2 + \bar{T}^2\right)} + \frac{3\mu}{2\left(r_c^2 + \bar{T}^2\right)}\right\}\sin 2\alpha\right]A$$

$$\bar{L} = \frac{3}{8} + \frac{3}{4}\cos^2\alpha + \frac{3\sqrt{3}}{8}\ (1-2\mu)\ \sin 2\alpha + \left[\left(\frac{1}{4} - \frac{3}{4}\mu\right)\cos 2\alpha - \frac{\sqrt{3}}{2}\left(\frac{1}{6} + \frac{1}{6}\mu\right)\sin 2\alpha\right]\ (1-q_1)$$
$$+\left[\frac{3}{16} + \frac{3}{2}\mu + \left(\frac{15}{8} - \frac{3}{2}\mu\right)\cos^2\alpha + \frac{\sqrt{3}}{2}\left(\frac{7}{8} - \frac{13}{4}\mu\right)\sin 2\alpha\right]\alpha_2 + \left[\frac{3}{2}\cos^2\alpha + \frac{\sqrt{3}}{2}\left(\frac{1}{2} + \mu\right)\sin 2\alpha\right]\alpha_3$$
$$+\left[\left\{\frac{5}{4}r_c - \frac{5}{8} + \frac{3}{8\ r_c^2 + \bar{T}^2} - \frac{3\mu\ (1-\mu)}{2\ r_c^2 + \bar{T}^2}\right\} + \left\{\frac{1}{2}r_c - \frac{1}{4} + \frac{3}{4\left(r_c^2 + T^2\right)} + \frac{3\mu\ (1-\mu)}{2\left(r_c^2 + \bar{T}^2\right)}\right\}\cos^2\alpha\right.$$
$$\left.-\frac{\sqrt{3}}{2}\left\{\left(\frac{11}{12} - \frac{11}{6}\mu\right) - \left(\frac{11}{6} - \frac{11}{3}\mu\right)r_c - \frac{3}{\left(r_c^2 + \bar{T}^2\right)} + \frac{3\mu}{2\left(r_c^2 + \bar{T}^2\right)}\right\}\sin 2\alpha\right]A$$

$$\bar{M} = \frac{3}{2} + \left(\frac{3}{4} + \frac{1}{2}\mu\right)\alpha_2 + \frac{1}{2}\alpha_3 - \ (1-\mu)\ \ (1-q_1)\ + M_d\Delta$$

The principal axes of the curve of zero velocity are oriented according to

$$\tan 2\alpha = \sqrt{3}\left[1 - 2\mu + \left(\frac{11}{3} - \frac{7}{3}\mu - 4\mu^2\right)\alpha_2 - \left(\frac{4}{3} - \frac{16}{3}\mu\right)\alpha_3 + \left(\frac{8}{9} - \frac{28}{9}\mu + 4\mu^2\right)\ (1-q_1)\right.$$
$$\left.-\left\{1 - \frac{16}{9}r_c - \frac{2}{r_c^2 + T^2} - \frac{4\mu\ (1-2r_c)}{9\ r_c^2 + T^2} - \frac{4\mu^2}{r_c^2 + T^2}\left(\frac{3}{2} - \mu\right)\right\}A\right] \qquad (4.42)$$

**Eccentricities of the orbits**

Now, the corresponding characteristics equation of the Jacobi integral $J = 2U$ is

$$\lambda^2 - \left[3 + \left(\frac{9}{2} + 3\mu\right)\alpha_2 + 3\alpha_3 + \left\{-3 + \frac{3}{r_c^2 + \bar{T}^2} + \frac{3\mu\ (1-\mu)}{4\ r_c^2 + \bar{T}^2} + 6r_c\right\} A\right]\lambda$$
$$+\mu\ (1-\mu)\left[\frac{27}{4} + \frac{117}{4}\alpha_2 + 9\alpha_3 + \frac{3}{2}\ (1-q_1)\ - \left\{\frac{33}{2} - \frac{27}{4\ r_c^2 + \bar{T}^2} - 33r_c\right\} A\right] = 0 \qquad (4.43)$$

Equation (4.43) is quadratic in $\lambda$ and has the roots:

$$\lambda_1 = 3 - \frac{9}{4}\mu\ (1-\mu)\ + \left(\frac{9}{2} - \frac{27}{4}\mu + \frac{39}{2}\mu^2\right)\alpha_2 + \ (3 - 3\mu + 3\mu^2\ \ \alpha_3 - \frac{1}{2}\mu\ (1-\mu)\ \ (1-q_1$$
$$+A\mu\ (1-\mu)\ \left\{\frac{11}{2} - \frac{9}{4\ r_c^2 + \bar{T}^2} - 33r_c\right\} \qquad (4.44)$$

$$\lambda_2 = \mu\ (1-\mu)\left[\frac{9}{4} + \frac{39}{4}\alpha_2 + 3\alpha_3 + \frac{1}{2}\ (1-q_1\ - \left\{3 - 6r_c - \frac{3}{r_c^2 + \bar{T}^2} + \frac{11}{2}\mu\ (1-\mu)\ - \frac{6\mu\ (1-\mu)}{r_c^2 + \bar{T}^2}\right\} A\right]$$

The eccentricities are given (Szebehely 1967a) by the equations

$$e_i = \left(-\chi_i^2\right)^{1/2} \qquad (4.45)$$

where

$$\chi_i = \frac{2s_i}{s_i^2 + \lambda_1},\ \left(=1,2\right) \qquad (4.46)$$

Substituting equations (4.35) and (4.36) and first equation of (4.44) in (4.46), ignoring products and higher powers of very small quantities, we get the respective equations

$$\chi_1^2 = 3\mu - 12\mu^2 + 4\ \mu - 22\mu^2\ \ \alpha_2 - 2\ \ \mu + 11\mu^2\ \ \alpha_3 + \frac{2}{3}\mu\ 1-\mu\ \ 1-q_1$$
$$+\left[\frac{8}{9\ r_c^2+\bar{T}^2} - \left\{\frac{22}{3} - \frac{11}{3}r_c + \frac{41}{9\ r_c^2+\bar{T}^2}\right\}\mu + \left\{\frac{154}{2} - \frac{143}{3}r_c - \frac{61}{9\ r_c^2+\bar{T}^2}\right\}\mu^2\right]A \quad (4.47)$$

and

$$\chi_2^2 = \frac{1}{4} - \frac{9}{16}\mu\ 1-\mu\ \ - \frac{243}{32}\mu^2 - \left(\frac{3}{8} - \frac{57}{16}\mu + \frac{294}{4}\mu^2\right)\alpha_2 - \left(\frac{3}{4} + \frac{15}{8}\mu - \frac{273}{16}\mu^2\right)\alpha_3$$
$$+\frac{1}{4}\mu\ 1-\mu\ \ 1-q_1\ + \frac{127}{32}\mu^2\ 1-q_1\ + \left[\frac{3}{8} + \frac{1}{4}r_c - \frac{1}{4\ r_c^2+\bar{T}^2} + \left\{\frac{233}{32} + \frac{63}{16}r_c - \frac{35}{8\ r_c^2+\bar{T}^2}\right\}\mu\right.$$
$$\left.+\left\{\frac{437}{16} - \frac{1251}{16}r_c - \frac{283}{32\ r_c^2+\bar{T}^2}\right\}\mu^2\right]A \quad (4.48)$$

Hence, the eccentricity of the long and short period orbit is found respectively by substituting equations (4.47) and (4.48) in (4.45), to get

$$e_1 = 1 - \frac{3}{2}\mu + \mu^2 - 2\ \mu - 22\mu^2\ \ \alpha_2 + \ \mu + 11\mu^2\ \ \alpha_3 - \frac{1}{3}\mu\ 1-\mu\ \ 1-q_1$$
$$+\left[\frac{4}{9\ r_c^2+\bar{T}^2} - \left\{\frac{11}{3} - \frac{11}{6}r_c + \frac{41}{18\ r_c^2+\bar{T}^2}\right\}\mu + \left\{\frac{77}{3} - \frac{143}{6}r_c - \frac{61}{18\ r_c^2+\bar{T}^2}\right\}\mu^2\right]A \quad (4.49)$$

and

$$e_2 = \frac{\sqrt{3}}{2}\left[1 + \frac{3}{8}\mu\ 1-\mu\ + \frac{81}{16}\mu^2 + \left(\frac{1}{4} - \frac{19}{8}\mu + 49\mu^2\right)\alpha_2 + \left(\frac{1}{2} + \frac{5}{4}\mu - \frac{91}{8}\mu^2\right)\alpha_3\right.$$
$$-\frac{1}{6}\ 1-\mu\ \ 1-q_1\ - \frac{127}{48}\mu^2\ 1-q_1\ + \left[\frac{1}{6\ r_c^2+T^2} - \frac{1}{6}r_c - \frac{1}{4} - \left\{\frac{233}{48} + \frac{21}{8}r_c - \frac{35}{18\ r_c^2+\bar{T}^2}\right\}\mu\right. \quad (4.50)$$
$$\left.\left.-\left\{\frac{437}{24} - \frac{417}{8}r_c - \frac{283}{48\ r_c^2+\bar{T}^2}\right\}\mu^2\right]A\right]$$

**The Semi-axes of the orbits**

The lengths of the semi-major axis, $a_i$ and the semi-minor axis, $b_i$ of the long and short period are obtained from the respective relations below

$$a_i = \frac{1}{\chi_i^2}\left(\xi_0^2\chi_i^2 + \eta_0^2\right)^{1/2},\ b_i = \left(\xi_0^2\chi_i^2 + \eta_0^2\right)^{1/2} \quad (i=1,2) \tag{4.51}$$

where $\xi_0, \eta_0$ are given in (4.24) at initial conditions.

From equation (4.24), we get

$$\xi_0^2 = \frac{1}{4}\left[1-4\mu+4\mu^2 + \left(-2+4\mu\right)\alpha_2 + \left(-\frac{4}{3}+\frac{8}{3}\mu\right)\left(1-q_1\right)\right] \tag{4.52}$$

$$\eta_0^2 = \frac{3}{4}\left[1-\frac{2}{3}\alpha_2+\frac{4}{3}\alpha_3-\frac{4}{9}\left(1-q_1\right)-\frac{8}{9}\left(2r_c-1\right)A\right]$$

Substituting $\chi_i^2 \ (i=1,2)$ and equations (4.52) in equations (4.51), we get the following equations respectively

$$a_1 = \frac{\sqrt{5}}{2}\left[1+\frac{1}{10\mu}-\frac{2}{5}\mu\left(1-\mu\right)+\left(\frac{37}{15}-\frac{1}{5\mu}+\frac{2}{5}\mu\right)\alpha_2+\left(\frac{19}{15}+\frac{1}{5\mu}\right)\alpha_3-\left(\frac{13}{45}+\frac{1}{5\mu}-\frac{4}{15}\mu\right)\left(1-q_1\right)\right.$$
$$\left.+\left\{\left(-\frac{61}{45}+\frac{1}{3\mu}\right)+\left(\frac{79}{90}-\frac{3}{10\mu}\right)r_c+\left(\frac{61}{135}+\frac{41}{270\mu}-\frac{4}{135\mu^2}\right)\left(r_c^2+\bar{T}^{2\ -1}\right)\right\}A\right]$$

$$a_2 = \frac{\sqrt{13}}{2}\left[1+\frac{23}{26}\mu\ 1-\mu\ +\left(\frac{4}{13}-\frac{185}{26}\mu-\frac{243}{26}\mu^2\right)\alpha_2+\left(2+\frac{63}{13}\mu-\frac{162}{13}\mu^2\right)\alpha_3\right.$$
$$-\left(\frac{10}{39}+\frac{32}{39}\mu-\frac{81}{13}\mu^2\right)\ 1-q_1\ +\left\{-\frac{19}{117}-\frac{32}{117}r_c+\frac{2}{13\ r_c^2+\bar{T}^2}-\left(\frac{249}{52}+\frac{47}{26}r_c-\frac{35}{13\ r_c^2+\bar{T}^2}\right)\mu\right.$$
$$\left.\left.-\left(\frac{237}{52}-\frac{1451}{26}r_c-\frac{283}{52\ r_c^2+\bar{T}^2}\right)\mu^2\right\}A\right] \qquad (4.53)$$

$$b_1 = \frac{\sqrt{3}}{2}\left[1+\frac{1}{2}\mu+\left(-\frac{1}{3}-\frac{1}{3}\mu-\frac{343}{3}\mu^2\right)\alpha_2+\left(\frac{2}{3}-\frac{1}{3}\mu-\frac{7}{3}\mu^2\right)\alpha_3-\left(\frac{2}{9}+\frac{5}{9}\mu-\frac{31}{9}\mu^2\right)\ 1-q_1\right.$$
$$\left.+\left\{\frac{4}{9}-\frac{8}{9}r_c+\frac{4}{27\ r_c^2+\bar{T}^2}-\left(\frac{11}{19}-\frac{11}{18}r_c+\frac{1}{6\ r_c^2+\bar{T}^2}\right)\mu+\left(\frac{121}{9}-\frac{181}{18}r_c+\frac{45}{18\ r_c^2+\bar{T}^2}\right)\mu^2\right\}A\right]$$

$$b_2 = \frac{\sqrt{13}}{4}\left[1-\frac{25}{26}\mu\ 1-\mu\ +\frac{18}{13}\mu^2+\left(-\frac{23}{13}+\frac{179}{13}\mu-\frac{2223}{26}\mu^2\right)\alpha_2+\left(2+\frac{19}{13}\mu+\frac{345}{13}\mu^2\right)\alpha_3\right.$$
$$+\left(-\frac{40}{39}+\frac{40}{39}\mu+\frac{23}{52}\mu^2\right)\ 1-q_1\ +\left\{\frac{78}{39}-\frac{241}{78}r_c-\frac{2}{13\ r_c^2+\bar{T}^2}+\left(\frac{185}{52}+\frac{47}{26}r_c-\frac{3}{52\ r_c^2+\bar{T}^2}\right)\mu\right.$$
$$\left.\left.+\left(\frac{5}{26}-\frac{1487}{26}r_c-\frac{245}{52\ r_c^2+\bar{T}^2}\right)\mu^2\right\}A\right]$$

Equations (4.53) gives the lengths of the semi-major and semi-minor axes of the long and short period orbits. They are defined by the potential from the disc, radiation force, the mass ratio, oblateness of the second primary and the third body.

## 4.3. Analytic Treatment of Motion of Dust Grain Particle around Triangular Equilibrium Points with Post-AGB Binary Star and Disc

This section investigates the motion around the triangular equilibrium points, of a passively gravitating dust particle in the gravitational field of a low-mass post-AGB binary system, surrounded by circumbinary disc. The two bodies of the binary are modeled as a triaxial star and radiating-oblate star. Due to small deviation of disc stars on circular orbits, we have assumed that the Coriolis and centrifugal forces of the stars are slightly perturbed.

### 4.3.1 Equations of motion

Let $m_1$ and $m_2$ be the masses of the first primary and its companion (second primary), respectively. Then, the gravitational potential energy acting on the third body which is a dust grain particle, when the first primary is a triaxial star and the second one, is a radiating oblate star, have the form (Sharma *et al.* 2001; Singh and Begha 2011a):

$$V = -Gm_3\left[ m_1\left( \frac{1}{r_1} + \frac{1}{2m_1 r_1^3}\left( I_1 + I_2 + I_3 - 3I \right) \right) + m_2 q_2 \left( \frac{1}{r_2} + \frac{1}{2m_2 r_2^3}\left( I_1' + I_3' - 3I' \right) \right) \right] \qquad (4.54)$$

$I_i \,(i = 1,2,3)$ are the principal moments of inertia of the triaxial star and its center of mass, with $a, b, c$ as lengths of its semi-axes, while $I$ is the moment of inertia about the line joining the center of $m_1$ and $m_3$ . $I_i' \,(i = 1,3)$ are the principal moments of inertia of the radiating oblate star at its centre, with $a_{21}, a_{22}$ being its equatorial and polar radii, respectively. $I'$ is the moment of inertia about the line joining the center of $m_2$ and $m_3$ while $R$ and $G$ are the dimensional distance between the primaries and the gravitational constant, respectively.

Next, we consider the influence of the gravitational potential from the disc. Following the lead of Sharma *et al* (2001) and Kushvah (2008a), the potential (4.54), is cast into the form:

$$U = -Gm_3\left[ m_1\left( \frac{1}{r_1} + \frac{1}{2r_1^3}\ 2\sigma_1 - \sigma_2\ \right) + m_2 q_2\left( \frac{1}{r_2} + \frac{\alpha_2}{2m_2 r_2^3} \right) + \frac{M_d}{r^2 + \bar{T}^2\ ^{1/2}} \right] \tag{4.55}$$

$$\sigma_1 = \frac{a^2 - b^2}{5R^2}, \sigma_2 = \frac{b^2 - c^2}{5R^2}, \alpha_2 = \frac{a_{21}^2 - a_{22}^2}{5R^2}, \sigma_{1,2} << 1,\ \alpha_2 << 1$$

The last term in the potential $U$, is the gravitational potential from the disc of mass $M_d$ disc (Kushvah 2008a). $r = \sqrt{x^2 + y^2}$ is the radial distance of the particle; $\bar{T}$ and $r_c$ are the parameters which defines the density profile of the disc and the radial distance of the particle in the classical R3BP, respectively. The parameters $\sigma_1$ and $\sigma_2$ represent the triaxiality of the primary star while $\alpha_2$ is the oblateness parameter of the secondary star.

Choosing units, we select the sum of the masses of the stars to be unity, thus we take $m_2 = \mu$, $0 < \mu = \frac{m_2}{m_1 + m_2} < 1$. For the unit of length, we take the distance between the stars $R = 1$ and the unit of time is also selected such that $G = 1$. Therefore, the equations of motion of the particle in a barycentric rotating coordinate system, based on the potential (4.55), under effects of small change in Coriolis and centrifugal forces, have the form:

$$\ddot{x} - 2\varphi n\dot{y} = U_x\ ,\ \ddot{y} + 2\varphi n\dot{x} = U_y \tag{4.56}$$

where

$$U = \frac{n^2\psi\ \ x^2 + y^2}{2} + \frac{1-\mu}{r_1} + \frac{q_2\mu}{r_2} + \frac{1-\mu\ \ 2\sigma_1 - \sigma_2}{2r_1^3} + \frac{\alpha_2 q_2 \mu}{2r_2^3} - \frac{3\ 1-\mu\ \ \sigma_1 - \sigma_2\ \ y^2}{2r_1^5} + \frac{M_d}{r^2 + \bar{T}^2\ ^{1/2}}$$

$$n^2 = 1 + \frac{3}{2}\ 2\sigma_1 - \sigma_2\ + \frac{3}{2}\alpha_2 + \frac{2M_d r_c}{r_c^2 + \bar{T}^2\ ^{\frac{3}{2}}} \tag{4.57}$$

$$r_1^2 = \left(x - \mu\right) + y^2, r_2^2 = \left(x - \mu + 1\right) + y^2$$

Here, $\mu$ is the mass ratio; $n$ is the mean motion of the stars and is defined by the triaxiality of the primary star, oblateness of the secondary star, the mass of the circumbinary disc, the parameter which represents the density profile of the disc, and the radial distance. $r_1$ and $r_2$ are distances of the particle from the stars. The parameters $\varphi$ and $\psi$ represent the small change in Coriolis and the centrifugal forces, respectively, and are such that $\varphi - 1 << 1$ and $\psi - 1 << 1$. Equations of motion (4.56), admits the Jacobi integral

$$J + \dot{x}^2 + \dot{y}^2 = 2U \tag{4.58}$$

where $C$ is the Jacobi constant.

## 4.3.2 Location of triangular equilibrium points

The equilibrium points represent stationary solutions of the R3BP. These points are the singularities of the manifold of the components of the velocity and the coordinates. Therefore, they can be found by equating the R.H.S of equations (4.56) to zero. Here, our interest is in the triangular equilibrium solutions. To get the solutions, we must solve the following equations

$$n^2\psi x - \frac{1-\mu\ \ x-\mu}{r_1^3} - \frac{q_2\mu\ \ x-\mu+1}{r_2^3} - \frac{3}{2}\frac{2\sigma_1-\sigma_2\ \ 1-\mu\ \ x-\mu}{r_1^5} - \frac{3}{2}\frac{\alpha_2 q_2\mu\ \ x-\mu+1}{r_2^5}$$
$$+ \frac{15}{2}\frac{\sigma_1-\sigma_2\ \ 1-\mu\ \ x-\mu\ \ y^2}{r_1^7} - \frac{M_d x}{r^2+\bar{T}^2\ ^{3/2}} = 0 \tag{4.59}$$

$$\left[ n^2\psi - \frac{1-\mu}{r_1^3} - \frac{q_2\mu}{r_2^3} - \frac{3}{2}\frac{4\sigma_1-3\sigma_2\ \ 1-\mu}{r_1^5} - \frac{3}{2}\frac{\alpha_2 q_2\mu}{r_2^5} + \frac{15}{2}\frac{\sigma_1-\sigma_2\ \ 1-\mu\ \ y^2}{r_1^7} - \frac{M_d}{r^2+\bar{T}^2\ ^{3/2}} \right] y = 0$$

when $y \neq 0$

To do this analytically, we use perturbation method to first obtain the solutions of (4.59), when the parameters representing, the radiation factor, oblateness, triaxiality and the potential from the

disc are null, and then treat the case when they are present, as a perturbation. In other words, if the stars have the shape of a sphere, the secondary star being a non luminous one and there is no circumbinary disc. Then $\alpha_2 = \sigma_1 = \sigma_2 = M_d = 0$ and $q_2 = 1$. Evidently, the solutions of equations (4.59) in this case are $r_1 = r_2 = \psi^{-1/3}$ which consequently result to the solutions:

$$x = \mu - \frac{1}{2} \text{ and } y = \pm\sqrt{\psi^{-2/3} - \frac{1}{4}}$$

Clearly, the triangular solutions of the classical case of Szebehely (1967a) can be recovered from the above equations when $r_1 = r_2 = \psi^{-1/3} = 1$.

Now, when these parameters are valid, that is, if the primary star is triaxial and the secondary one, is oblate and radiating, with disc mass, whose gravitational potential affect the dynamic system; then $0 < \alpha_2 << 1, 0 < 1 - q_2 << 1, 0 < \sigma_{1,2} << 1$ and $0 < M_d << 1$. In this case, the values of $r_i$ $(= 1,2)$ will change by very small quantities say, $\varepsilon_i$ $(= 1,2)$, so that now

$$r_1 = \psi^{-1/3} + \varepsilon_1, \quad r_2 = \psi^{-1/3} + \varepsilon_2 : |\varepsilon_i| << 1 \tag{4.60}$$

and the coordinates will be

$$x = \mu - \frac{1}{2} - \psi^{-1/3} \quad \varepsilon_1 - \varepsilon_2 \quad , \quad y = \pm\frac{\sqrt{4 - \psi^{2/3}}}{2\psi^{1/3}}\left[1 + \frac{2\psi^{1/3}}{4 - \psi^{2/3}} \quad \varepsilon_1 + \varepsilon_2 \right] \tag{4.61}$$

where

$$\varepsilon_1 = \frac{1}{3\psi^{1/3}}\left[\frac{3}{2}\alpha_2 - \left(\frac{15}{8}\psi^{4/3} - 3\psi^{2/3} - 3\right)\sigma_1 + \left(\frac{15}{8}\psi^{4/3} + \frac{9}{2}\psi^{2/3} - \frac{3}{2}\right)\sigma_2 + \frac{2\psi r_c - 1 \;\; M_d}{\psi \;\; r_c^2 + \bar{T}^2 \;\; ^{3/2}}\right]$$

$$\varepsilon_2 = \frac{1}{3\psi^{1/3}}\left[ 1-q_2 \ +\frac{3}{2}\alpha_2 \ 1-\psi^{2/3} \ -\left(\frac{3\psi^{2/3}}{2\mu}-\frac{3\psi^{2/3}}{2}-3\right)\sigma_1+\left(\frac{3\psi^{2/3}}{2\mu}-\frac{3\psi^{2/3}}{2}-\frac{3}{2}\right)\sigma_2 \right.$$
$$\left. +\frac{2\psi r_c-1 \ M_d}{\psi \ r_c^2+\bar{T}^2 \ ^{3/2}}\right]$$
$$r_c = \sqrt{1-\mu+\mu^2}$$

The two points denoted by $L_4\left(x, y\right)$ and $L_5\left(x,-y\right)$ given by (4.61), form triangles with the stars and are called triangular equilibrium points. The positions depend on, a small change in the centrifugal force, triaxiality of the post-AGB primary star, oblateness and radiation of the post-AGB secondary star; and the gravitational potential from the disc mass $M_d$ . However, these points are not influenced by a small change in the Coriolis force

### 4.3.3. Stability of triangular equilibrium points

We follow same methodology and compute the partial derivatives to get

$$U_{xx}^0 = \frac{3}{4}\psi^{5/3}\left[1+\left\{\frac{5 \ 2r_c\psi-1}{3\psi \ r_c^2+\bar{T}^2 \ ^{3/2}}+\frac{4\psi^{2/3}\mu^2-4\psi^{2/3}\mu+1}{\psi^{5/3} \ r_c^2+\bar{T}^2 \ ^{5/2}}\right\}M_d+\left(11-\frac{15}{2}\psi^{2/3}+\frac{5}{4}\psi^{4/3}\right.\right.$$
$$\left.-\frac{2}{\mu}+5\mu\psi^{2/3}-\frac{5}{4}\mu\psi^{4/3}\right)\sigma_1-\left(\frac{13}{2}-\frac{15}{2}\psi^{2/3}+\frac{5}{4}\psi^{4/3}-\frac{2}{\mu}+4\mu+5\mu\psi^{2/3}-\frac{5}{4}\mu\psi^{4/3}\right)\sigma_2 \quad (4.62)$$
$$\left.+\frac{1}{2} \ 1+8\mu \ \alpha_2+\frac{2}{3} \ 2\psi^{-2/3}+\mu \ 1-4\psi^{-2/3} \quad 1-q_2\right]$$

$$U_{yy}^{0}=\frac{3}{4}\psi^{5/3}\Bigg[\;4\psi^{-2/3}-1\;+\left\{\frac{4\psi^{-2/3}-1}{\psi^{5/3}\;\left(r_c^2+\bar{T}^2\right)^{5/2}}+\frac{2\psi r_c-1\;\;20\psi^{-2/3}-12}{3\psi\;\left(r_c^2+\bar{T}^2\right)^{3/2}}\right\}M_d$$

$$-\left(11-12\psi^{-2/3}-\frac{15}{2}\psi^{2/3}+\frac{5}{4}\psi^{4/3}-\frac{2}{\mu}+5\mu\psi^{2/3}-\frac{5}{4}\mu\psi^{4/3}\right)\sigma_1+\left(\frac{21}{2}-6\psi^{-2/3}-\frac{15}{2}\psi^{2/3}\right.$$

$$\left.+\frac{5}{4}\psi^{4/3}-\frac{2}{\mu}+5\mu\psi^{2/3}-\frac{5}{4}\mu\psi^{4/3}\right)\sigma_2+\frac{1}{2}\;12\psi^{-2/3}-1\;\alpha_2-\frac{2}{3}\;2-\mu\;4\psi^{-2/3}-1\;\;(1-q_2)\Bigg]$$

$$U_{xy}^{0}=\frac{3\psi^{4/3}\sqrt{4-\psi^{2/3}}}{8}\Bigg[2-4\mu+\frac{8-\psi^{2/3}+6\psi^{2/3}\mu-32\mu\;\;\alpha_2}{4-\psi^{2/3}}+\frac{4\;\;4-2\psi^{2/3}-\mu\psi^{4/3}\;\;1-q_2}{3\psi^{2/3}\;\;4-\psi^{2/3}}$$

$$+\frac{2M_d}{\psi\;\;r_c^2+\bar{T}^2\;^{3/2}}\left\{\frac{\psi^{-2/3}}{r_c^2+\bar{T}^2}+\frac{2\psi r_c-1\;\;16-5\psi^{2/3}}{3\;\;4-\psi^{2/3}}+\mu\left(\frac{2\psi^{-2/3}}{r_c^2+\bar{T}^2}+\frac{2\;\;2\psi r_c-1\;\;16-5\psi^{2/3}}{3\;\;4-\psi^{2/3}}\right)\right\}$$

$$+\frac{4\sigma_1}{4-\psi^{2/3}}\left\{2\psi^{2/3}-6-\frac{5}{2}\psi^{4/3}+\frac{5}{8}\psi^2+\frac{1}{\mu}\left(2-\psi^{2/3}\right)\mu\left(16-8\psi^{2/3}+\frac{15}{4}\psi^{4/3}-\frac{5}{8}\psi^2\right)\right\}$$

$$+\frac{4\sigma_2}{\left(4-\psi^{2/3}\right)}\left\{\frac{3}{4}\psi^{2/3}+\frac{5}{2}\psi^{4/3}-\frac{5}{8}\psi^2-\frac{1}{\mu}\left(2-\psi^{2/3}\right)\mu\left(8-\frac{13}{2}\psi^{2/3}+\frac{15}{4}\psi^{4/3}-\frac{5}{8}\psi^2\right)\right\}\Bigg]$$

The superscript 0, means that the partial derivatives have been calculated using the triangular EPs given in equations (4.61).

Now, substituting equation (4.57) in (4.25) and expressing $\varphi=1+\;\varphi-1$ and $\psi=1+\;\psi-1$ , where $\varphi-1\;<<1$ and $\;\psi-1\;<<1$. We simplify, ignoring product and higher order terms in $\alpha_2$, $(1-q_2)$, $\sigma_i\;(i=1,2)$, $\;\varphi-1$ , $\;\psi-1\;$ and $M_d$. Hence, the characteristic equation (4.25) becomes

$$\lambda^4+\left[1+3\sigma_1-\left(\frac{9}{2}-3\mu\right)\sigma_2+\left(\frac{3}{2}-3\mu\right)\alpha_2-3\ \psi-1\ +8\ \varphi-1\ +\frac{2\psi r_c-1\ M_d}{\psi\ \left(r_c^2+\bar{T}^2\right)^{3/2}}\right.$$

$$\left.+\frac{3\ \mu^2-\mu+1\ M_d}{\left(r_c^2+\bar{T}^2\right)^{5/2}}\right]\lambda^2+\frac{3}{4}\mu\ 1-\mu\ \left[9-\frac{3}{4}\ 10-99\mu+89\mu^2\ \sigma_1+\frac{3}{4}\ 10-47\mu+37\mu^2\ \sigma_2\right.$$

$$\left.+39\alpha_2+\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{-\frac{22}{\psi}+44r_c+\frac{9}{r_c^2+\bar{T}^2}\right\}+22\ \psi-1\ +2\ 1-q_2\ \right]=0 \qquad (4.63)$$

Its roots are given by equation (4.32) but here, we have

$$b=1+3\sigma_1-\left(\frac{9}{2}-3\mu\right)\sigma_2+\left(\frac{3}{2}-3\mu\right)\alpha_2-\frac{3M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{\frac{2\psi r_c-1}{\psi}-\frac{8}{3}r_c+\frac{\mu^2-\mu+1}{r_c^2+\bar{T}^2}\right\}-3\ \psi-1$$

$$+8\ \varphi-1$$

$$D=1+\frac{57}{2}\sigma_1-\frac{63}{2}\sigma_2+3\alpha_2+16\ \alpha-1\ -6\ \psi-1\ +\frac{2M_d}{\left(r_c^2+\bar{T}^2\right)^{5/2}}\left\{2r_c+\frac{3}{\psi}-\frac{3}{r_c^2+\bar{T}^2}\right\}$$

$$+3\mu^2\left[9+\frac{267}{4}\sigma_1-\frac{111}{4}\sigma_2+39\alpha_2+22\ \psi-1\ +2\ 1-q_2\ -\frac{22\ 1-2\psi r_c\ M_d}{\psi\ \left(r_c^2+\bar{T}^2\right)^{3/2}}+\frac{7M_d}{\left(r_c^2+\bar{T}^2\right)^{5/2}}\right]$$

$$-3\mu\left[9+\frac{297}{4}\sigma_1-\frac{149}{4}\sigma_2+41\alpha_2+22\ \psi-1\ +2\ 1-q_2\ -M_d\left\{\frac{22\ 1-2\psi r_c}{\psi\ \left(r_c^2+\bar{T}^2\right)^{3/2}}-\frac{7}{\left(r_c^2+\bar{T}^2\right)^{5/2}}\right\}\right]$$

Solving $D=0$ for $\mu$, and putting $\varphi-1=\in$, $\psi-1=\in'$, we get the critical mass

$$\mu_C=\frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)+\left(\frac{5}{6}+\frac{59}{9\sqrt{69}}\right)\sigma_1-\left(\frac{19}{18}+\frac{85}{9\sqrt{69}}\right)\sigma_2+\frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)\alpha_2+\frac{4\ 36\in-19\in'}{27\sqrt{69}}$$

$$-\frac{2\ 1-q_2}{27\sqrt{69}}+\frac{2M_d}{27\sqrt{69}\ \left(r_c^2+\bar{T}^2\right)^{3/2}}\left[38-4r_c-\frac{61}{2\ r_c^2+\bar{T}^2}\right] \qquad (4.64)$$

Therefore, when $0 < \mu < \mu_C$, $D > 0$; the roots of equation (4.62) are distinct pure imaginary numbers and the triangular points are stable, otherwise it is unstable.

## 4. 3. 4. Periodic orbits around triangular equilibrium points

**Frequency and orientation of the Orbits**

To study the periodic orbits, we know that motion is stable in the region $0 < \mu < \mu_C$, we estimate the frequency of the long and short orbits, to be

$$s_1 = \left[ \mu \left\{ \frac{27}{4} + \frac{891\sigma_1}{16} - \frac{423\sigma_2}{16} + \frac{117\alpha_2}{4} + \frac{3}{2}\ 1-q_2\ + \frac{33\in'}{2} - \frac{M_d}{r_c^2 + T^{2\ 3/2}} \left( \frac{33}{2} - 33r_c - \frac{19}{4\ r_c^2 + T^2} \right) \right\} \right.$$

$$-\mu^2 \left\{ \frac{27}{4} + \frac{801}{16}\sigma_1 - \frac{333}{16}\sigma_2 + \frac{117}{4}\alpha_2 + \frac{3}{2}\ 1-q_2\ + \frac{33}{2}\in' - \frac{M_d}{r_c^2 + T^{2\ 3/2}} \left( \frac{33}{2} - 33r_c - \frac{19}{4\ r_c^2 + T^2} \right) \right\}$$

$$\left. + \frac{2M_d}{r_c^2 + T^{2\ 5/2}} - \frac{45}{8}(\sigma_1 - \sigma_2) \right]^{1/2} \qquad (4.65)$$

$$s_2 = \left[ 1 + \frac{69}{8}\sigma_1 - \frac{81}{8}\sigma_2 + \frac{3}{2}\alpha_2 + \frac{M_d}{r_c^2 + \bar{T}^{2\ 3/2}} \left\{ 3 + 2r_c - \frac{2}{r_c^2 + \bar{T}^2} \right\} + 8\in -3\in' - \mu \left\{ \frac{27}{4} + \frac{891\sigma_1}{16} - \frac{471\sigma_2}{16} \right.\right.$$

$$+ \frac{129}{4}\alpha_2 - \frac{M_d}{r_c^2 + \bar{T}^{2\ 3/2}} \left( \frac{33}{2} - 33r_c - \frac{69}{12\ r_c^2 + \bar{T}^2} \right) + \frac{3}{2}\ 1-q_2\ + \frac{33}{2}\in' \left. \right\} + \mu^2 \left\{ \frac{27}{4} + \frac{801}{16}\sigma_1 - \frac{333}{16}\sigma_2 \right.$$

$$\left.\left. + \frac{117}{4}\alpha_2 + \frac{3}{2}\ 1-q_2\ + \frac{33}{2}\in' - \frac{M_d}{(r_c^2 + \bar{T}^{2\ 3/2})} \left( \frac{33}{2} - 33r_c - \frac{69}{12(r_c^2 + \bar{T}^2)} \right) \right\} \right]^{1/2} \qquad (4.66)$$

Equation (4.37) in this case is expressed as

$$U = \bar{J} + K\xi^2 - L\xi\eta + M\eta^2 \qquad (4.67)$$

where

$$J=\frac{3}{2}+\left(\frac{3}{4}+\frac{1}{2}\mu\right)\alpha_2+\left(\frac{11}{8}+\frac{\mu}{8}\right)\sigma_1-\left(\frac{1}{8}+\frac{5\mu}{8}\right)\sigma_2+\frac{M_d}{r_c^2+\bar{T}^2\ ^{1/2}}\left\{1+\frac{r_c}{r_c^2+\bar{T}^2}\right\}+\frac{1}{2}\in'$$
$$-\mu\ 1-q_2$$

$$K=\frac{3}{8}+\left(\frac{57}{32}+\frac{45\mu}{32}-\frac{3}{4\mu}\right)\sigma_1-\left(\frac{3}{32}+\frac{93\mu}{32}-\frac{3}{4\mu}\right)\sigma_2+\left(\frac{3}{16}+\frac{3\mu}{2}\right)\alpha_2+\frac{5}{8}\in'-\left(\frac{1}{2}-\frac{3\mu}{4}\right)1-q_2$$
$$+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\frac{3}{8\ r_c^2+\bar{T}^2}+\frac{5r_c}{4}-\frac{5}{8}-\frac{3\mu\ 1-\mu}{2\ r_c^2+\bar{T}^2}\right\}$$

$$L=\sqrt{3}\left[\frac{3}{4}-\frac{3\mu}{2}+\left(\frac{47}{16}-\frac{89\mu}{16}-\frac{1}{2\mu}\right)\sigma_1-\left(\frac{9}{16}-\frac{37\mu}{16}-\frac{1}{4\mu}\right)\sigma_2+\left(\frac{7}{8}-\frac{13\mu}{4}\right)\alpha_2+\left(\frac{11}{12}+\frac{11\mu}{6}\right)\in'\right.$$
$$\left.+\left(\frac{1}{3}-\frac{\mu}{6}\right)1-q_2\ -\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\left(\frac{11}{12}-\frac{11\mu}{6}\right)-\left(\frac{11}{6}-\frac{11\mu}{3}\right)r_c-\frac{3}{r_c^2+\bar{T}^2}+\frac{3\mu}{2\ r_c^2+\bar{T}^2}\right\}\right]$$

$$M=\frac{9}{8}+\left(\frac{87}{32}-\frac{45\mu}{32}+\frac{3}{4\mu}\right)\sigma_1-\left(\frac{21}{32}-\frac{45\mu}{16}+\frac{3}{4\mu}\right)\sigma_2+\frac{33}{16}\alpha_2+\frac{7}{8}\in'-\left(\frac{1}{2}-\frac{3}{4}\mu\right)1-q_2$$
$$-\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\frac{7}{8}-\frac{7}{4}r_c-\frac{9}{8\ r_c^2+\bar{T}^2}\right\}$$

We introduce the variables $\bar{\xi}$ and $\bar{\eta}$ by the transformation (4.40), the new force function analogous to equation (4.41) has the form:

$$\bar{U}=\bar{J}+\bar{K}\bar{\xi}^2+\bar{L}\bar{\eta}^2 \qquad (4.68)$$

where

$$\bar{K}=\frac{3}{8}+\frac{3}{4}\sin^2\alpha-\frac{3\sqrt{3}}{8}\ 1-2\mu\ \sin 2\alpha+\left[\left(\frac{1}{4}-\frac{3\mu}{4}\right)\cos 2\alpha-\frac{\sqrt{3}}{6}\left(1-\frac{\mu}{2}\right)\sin 2\alpha\right]\ 1-q_2$$

$$+\left[\frac{57}{32}+\frac{15}{16}\sin^2\alpha-\left(\frac{3}{4\mu}-\frac{45}{32}\mu\right)\cos 2\alpha-\frac{\sqrt{3}}{4}\left(\frac{47}{8}-\frac{89\mu}{8}-\frac{1}{\mu}\right)\sin 2\alpha\right]\sigma_1$$

$$+\left[-\frac{3}{32}-\frac{9}{16}\sin^2\alpha+\left(\frac{3}{4\mu}-\frac{45}{32}\mu\right)\cos 2\alpha-\frac{3\mu}{2}\cos^2\alpha+\frac{\sqrt{3}}{4}\left(\frac{9}{8}-\frac{37\mu}{8}-\frac{1}{\mu}\right)\sin 2\alpha\right]\sigma_2$$

$$+\left[\frac{3}{16}+\frac{3\mu}{2}+\left(\frac{15}{8}-\frac{3\mu}{2}\right)\sin^2\alpha-\frac{\sqrt{3}(7-26\mu)}{16}\sin 2\alpha\right]\alpha_2+\left[\frac{5}{8}+\frac{1}{4}\sin^2\alpha-\frac{11\sqrt{3}(1-2\mu)}{24}\sin 2\alpha\right]\in'$$

$$+\left[\left\{\frac{5}{4}r_c-\frac{5}{8}+\frac{3}{8(r_c^2+\bar{T}^2)}-\frac{3\mu(1-\mu)}{2(r_c^2+\bar{T}^2)}\right\}+\left\{\frac{1}{2}r_c-\frac{1}{4}+\frac{3}{4(r_c^2+\bar{T}^2)}+\frac{3\mu(1-\mu)}{2(r_c^2+\bar{T}^2)}\right\}\sin^2\alpha\right.$$

$$\left.+\frac{\sqrt{3}}{2(r_c^2+\bar{T}^2)^{3/2}}\left\{\left(\frac{11}{12}-\frac{11}{6}\mu\right)-\left(\frac{11}{6}-\frac{11}{3}\mu\right)r_c-\frac{3}{(r_c^2+\bar{T}^2)}+\frac{3\mu}{2(r_c^2+\bar{T}^2)}\right\}\sin 2\alpha\right]M_d$$

$$\bar{L}=\frac{3}{8}+\frac{3}{4}\cos^2\alpha+\frac{3\sqrt{3}}{8}\ 1-2\mu\ \sin 2\alpha+\left[\left(-\frac{1}{2}+\frac{3\mu}{4}\right)\cos 2\alpha+\frac{\sqrt{3}}{2}\left(\frac{1}{3}-\frac{\mu}{6}\right)\sin 2\alpha\right]\ 1-q_2$$

$$+\left[\frac{57}{32}+\frac{15}{16}\cos^2\alpha+\left(\frac{3}{4\mu}-\frac{45}{32}\mu\right)\cos 2\alpha+\frac{\sqrt{3}}{4}\left(\frac{47}{8}-\frac{89\mu}{8}-\frac{1}{\mu}\right)\sin 2\alpha\right]\sigma_1$$

$$+\left[-\frac{3}{32}-\frac{9}{16}\cos^2\alpha-\left(\frac{3}{4\mu}-\frac{45}{32}\mu\right)\cos 2\alpha-\frac{3\mu}{2}\sin^2\alpha-\frac{\sqrt{3}}{4}\left(\frac{9}{8}-\frac{37\mu}{8}-\frac{1}{\mu}\right)\sin 2\alpha\right]\sigma_2$$

$$+\left[\frac{3}{16}+\frac{3\mu}{2}+\left(\frac{15}{8}-\frac{3}{2}\mu\right)\cos^2\alpha+\frac{\sqrt{3}(7-26\mu)\sin 2\alpha}{16}\right]\alpha_2+\left[\frac{5}{8}+\frac{1}{4}\cos^2\alpha+\frac{11(1-2\mu)\sqrt{3}}{24}\sin 2\alpha\right]\in'$$

$$+\frac{M_d}{(r_c^2+\bar{T}^2)^{3/2}}\left[\left\{\frac{5}{4}r_c-\frac{5}{8}+\frac{3}{8(r_c^2+\bar{T}^2)}-\frac{3\mu(1-\mu)}{2(r_c^2+\bar{T}^2)}\right\}+\left\{\frac{1}{2}r_c-\frac{1}{4}+\frac{3}{4(r_c^2+\bar{T}^2)}+\frac{3\mu(1-\mu)}{2(r_c^2+\bar{T}^2)}\right\}\cos^2\alpha\right.$$

$$\left.+\frac{\sqrt{3}}{2}\left\{\left(\frac{11}{12}-\frac{11}{6}\mu\right)-\left(\frac{11}{6}-\frac{11}{3}\mu\right)r_c-\frac{3}{(r_c^2+\bar{T}^2)}+\frac{3\mu}{2(r_c^2+\bar{T}^2)}\right\}\sin 2\alpha\right]$$

The principal axes of the curve of zero velocity are oriented according to

$$\tan 2\alpha = \sqrt{3}\left[1+\left(\frac{20}{3}-\frac{8}{3\mu}-\frac{7\mu}{6}-\frac{45\mu^2}{6}\right)\sigma_1-\left(4-\frac{8}{3\mu}+\frac{25\mu}{6}-\frac{23\mu^2}{6}\right)\sigma_2-\left(\frac{4}{3}-\frac{8\mu}{3}+4\mu^2\right)\alpha_2\right.$$
$$-\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{1-\frac{16}{9}r_c+\frac{4\mu\ \ 2r_c-1\ \ -18-18\mu^2\ \ 3-2\mu}{9\ \ r_c^2+\bar{T}^2}\right\}+\left(\frac{8}{9}-\frac{16}{9}\mu\right)\in'-2\mu$$
$$\left.+\left(\frac{16}{9}-\frac{44}{9}\mu+4\mu^2\right)\ 1-q_2\ \right] \qquad (4.69)$$

**Eccentricities of the orbits**

Now, the corresponding characteristics equation of the Jacobi integral is

$$\lambda^2-\left[3\ \ 1+\in'\ \ +9\sigma_1-\left(\frac{3}{2}+3\mu\right)\sigma_2+\left(\frac{9}{2}+3\mu\right)\alpha_2+\frac{3M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{2r_c-1+\frac{4+\mu-\mu^2}{4\ \ r_c^2+\bar{T}^2}\right\}\right]\lambda \qquad (4.70)$$
$$+\mu\ \ 1-\mu\ \left[\frac{27}{4}+\frac{117\alpha_2}{4}-\frac{3}{2\ \ r_c^2+\bar{T}^2\ \ ^{3/2}}\left\{11-2r_c-\frac{9}{2\ \ r_c^2+\bar{T}^2}\right\}M_d+\frac{3\ \ 1-q_2}{2}+\frac{33\in'}{4}\right]$$
$$-\left(\frac{45}{8}-\frac{891\mu}{16}+\frac{891\mu^2}{16}\right)\sigma_1+\left(\frac{45}{8}-\frac{423\mu}{16}+\frac{333\mu^2}{16}\right)\sigma_2=0$$

The roots of equation (5.70) are

$$\lambda_1=3-\frac{9}{4}\mu\ \ 1-\mu\ \ +\left(\frac{87}{8}-\frac{297\mu}{16}+\frac{267\mu^2}{16}\right)\sigma_1-\left(\frac{27}{8}-\frac{93\mu}{16}+\frac{111\mu^2}{16}\right)\sigma_2+\frac{3}{2}\left(1-\frac{9\mu}{2}+13\mu^2\right)\alpha_2$$
$$+\frac{3\mu\ \ 1-\mu}{2\ \ r_c^2+\bar{T}^2\ \ ^{3/2}}\left\{\frac{11}{3}-22r_c-\frac{3}{2\ \ r_c^2+\bar{T}^2}\right\}M_d+\left(3-\frac{11\mu}{2}+\frac{11}{2}\mu^2\right)\in'-\frac{1}{2}\mu\ \ 1-\mu\ \ \ 1-q_2 \qquad (4.71)$$

$$\lambda_2=\mu\ \ 1-\mu\ \left[\frac{9}{4}+\frac{39}{4}\alpha_2-\frac{3M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{\frac{11\mu\ \ 1-\mu}{6}+1-2r_c-\frac{4+7\mu-7\mu^2}{4\ \ r_c^2+\bar{T}^2}\right\}+\frac{11}{2}\in'+\frac{1}{2}\ \ 1-q_2\ \right]$$
$$-\left(\frac{15}{8}-\frac{297\mu}{16}+\frac{267\mu^2}{16}\right)\sigma_1+\left(\frac{15}{8}-\frac{141\mu}{16}+\frac{111\mu^2}{16}\right)\sigma_2$$

With the help of the first root above, the eccentricities of the ellipse can be obtained by first calculating equation (4.46), to get

$$\chi_1^2 = 3\mu\ 1-4\mu\ +\left(\frac{5}{2}-\frac{63}{4}\mu-\frac{521}{4}\mu^2\right)\sigma_1-\left(\frac{5}{2}-\frac{55}{4}\mu-\frac{241}{4}\mu^2\right)\sigma_2+4\ \mu-22\mu^2\ \alpha_2$$
$$+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left[\frac{8}{9\ r_c^2+\bar{T}^2}-\left\{\frac{22}{3}-\frac{11}{3}r_c+\frac{41}{9\ r_c^2+\bar{T}^2}\right\}\mu+\left\{\frac{154}{2}-\frac{143}{3}r_c-\frac{61}{9\ r_c^2+\bar{T}^2}\right\}\mu^2\right] \quad (4.72)$$
$$+\frac{2}{3}\mu\ 1-7\mu\ \ 1-q_2\ +\frac{4}{3}\mu\ 1-34\mu\ \in'$$

$$\chi_2^2=\frac{1}{4}-\frac{9}{16}\mu\left(1-\frac{25}{2}\mu\right)-\left(\frac{9}{32}-\frac{1377}{64}\mu+\frac{4713}{32}\mu^2\right)\sigma_1-\left(\frac{27}{32}+\frac{1269}{64}\mu-\frac{2685}{32}\mu^2\right)\sigma_2$$
$$-\left(\frac{3}{8}-\frac{57\mu}{16}+\frac{294\mu^2}{4}\right)\alpha_2-\frac{1}{8}\mu\left(1+\frac{25\mu}{2}\right)\ 1-q_2\ +\left(1+\frac{63\mu}{4}-\frac{63\mu^2}{4}\right)\in-\left(\frac{3}{4}+\frac{19\mu}{8}+\frac{259\mu^2}{8}\right)\in'$$
$$+\left[\frac{3}{8}+\frac{r_c}{4}-\frac{1}{4\ r_c^2+\bar{T}^2}+\left\{\frac{233}{32}+\frac{63r_c}{16}-\frac{35}{8\left(r_c^2+\bar{T}^2\right)}\right\}\mu+\left\{\frac{437}{16}-\frac{1251r_c}{16}-\frac{283}{32\left(r_c^2+\bar{T}^2\right)}\right\}\mu^2\right] \quad (4.73)$$
$$\times\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}$$

Now, substituting for $\chi_i^2$ in equation (4.45), we get the eccentricity of the long and short period orbits respectively:

$$e_1=1-\frac{3\mu}{2}\ 1-4\mu\ -\left(\frac{5}{4}-\frac{63\mu}{8}-\frac{521\mu^2}{8}\right)\sigma_1+\left(\frac{5}{4}-\frac{55\mu}{8}-\frac{241\mu^2}{8}\right)\sigma_2-2\mu\ 1-22\mu\ \alpha_2$$
$$+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left[\frac{4}{9\ r_c^2+\bar{T}^2}-\left\{\frac{11}{3}-\frac{11r_c}{6}+\frac{41}{18\ r_c^2+\bar{T}^2}\right\}\mu+\left\{\frac{77}{3}-\frac{143r_c}{6}-\frac{61}{18\ r_c^2+\bar{T}^2}\right\}\mu^2\right]$$
$$-\frac{2}{3}\mu\ 1-34\mu\ \in'-\frac{1}{3}\mu\ 1-7\mu\ \ 1-q_2 \quad (4.74)$$

$$e_2 = \frac{\sqrt{3}}{2}\left[1+\frac{3\mu}{8}+\frac{75\mu^2}{16}+\left(\frac{3}{16}-\frac{459\mu}{32}+\frac{1571\mu^2}{16}\right)\sigma_1+\left(\frac{9}{16}+\frac{423\mu}{32}-\frac{895\mu^2}{16}\right)\sigma_2+\left(49\mu^2+\frac{1}{4}\right.\right.$$
$$\left.-\frac{19\mu}{8}\right)\alpha_2+\left(\frac{1}{12}\mu+\frac{25\mu^2}{24}\right)\ 1-q_2\ -\left(\frac{2}{3}+\frac{21\mu}{2}-\frac{21\mu^2}{2}\right)\in+\left(\frac{1}{2}+\frac{19\mu}{6}+\frac{259\mu^2}{12}\right)\in'+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}$$
$$\left.\times\left\{\frac{1}{6\ r_c^2+\bar{T}^2}-\frac{1}{6}r_c-\frac{1}{4}-\left(\frac{233}{48}+\frac{21}{8}r_c-\frac{35}{18\left(r_c^2+\bar{T}^2\right)}\right)\mu-\left(\frac{437}{24}-\frac{417}{8}r_c-\frac{283}{48\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\right\}\right] \quad (4.75)$$

**The semi-axes of the orbits**

The lengths of the semi-major axis, $a_i$ and the semi-minor axis, $b_i$ of the long and short period are sought from the equations given in (4.51), where

$$\xi_0^2 = \frac{1}{4}\left[1-4\mu+4\mu^2+\left(\frac{9}{2}-\mu-\frac{2}{\mu}\right)\sigma_1-\left(\frac{5}{2}+3\mu-\frac{2}{\mu}\right)\sigma_2-2\left(-2\mu\,\tilde{\alpha}_2+\left(\frac{4}{3}-\frac{8}{3}\mu\right)\right)-q_2\right]$$

$$\eta_0^2 = \frac{3}{4}\left[1-\left(\frac{23}{6}-\frac{2}{3\mu}\right)\sigma_1+\left(\frac{19}{6}-\frac{2}{3\mu}\right)\sigma_2-\frac{2}{3}\alpha_2-\frac{4}{9}\ 1-q_2\ -\frac{8}{9}\in'-\frac{8\ 2r_c-1\ M_d}{9\ r_c^2+\bar{T}^2\ ^{3/2}}\right]$$

Now, substituting equations (4.72) and (4.73) and the above equations in (4.51), we get the following results respectively:

$$a_1 = \frac{\sqrt{5}}{2}\left[1-\frac{2}{5}\mu\ 1-\mu\ +\frac{1}{10\mu}+\left(\frac{391}{120}-\frac{1}{10}\mu+\frac{5}{24\mu}-\frac{1}{5\mu^2}\right)\sigma_1-\left(\frac{119}{120}+\frac{3}{10}\mu+\frac{5}{24\mu}-\frac{1}{5\mu^2}\right)\sigma_2\right.$$
$$+\left(\frac{37}{15}+\frac{2}{5}\mu-\frac{1}{5\mu}\right)\alpha_2+\left(\frac{1}{9}-\frac{4}{15}\mu-\frac{1}{15\mu}\right)\ 1-q_2\ -\left(\frac{2}{9}+\frac{2}{15\mu}\right)\in'-\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}$$
$$\left.\times\left\{\left(\frac{61}{45}-\frac{1}{3\mu}\right)-\left(\frac{79}{90}-\frac{3}{10\mu}\right)r_c-\left(\frac{61}{135}+\frac{41}{270\mu}-\frac{4}{135\mu^2}\right)\left(r_c^2+\bar{T}^2\right)^{-1}\right\}\right] \quad (4.76)$$

$$a_2=\frac{\sqrt{13}}{2}\left[1+\frac{23\mu}{26}+\frac{683\mu^2}{52}-\left(\frac{2}{13}+\frac{\mu}{3}-\frac{3\mu^2}{13}\right)1-q_2\ -\left(\frac{5}{13}-\frac{3}{13\mu}+\frac{3649\mu}{104}-\frac{23103\mu^2}{104}\right)\sigma_1\right.$$
$$+\left(\frac{29}{13}-\frac{3}{13\mu}+\frac{3237\mu}{104}-\frac{11835\mu^2}{104}\right)\sigma_2+\left(\frac{4}{13}-\frac{185}{26}\mu+\frac{3303}{26}\mu^2\right)\alpha_2-\left(\frac{24}{13}+\frac{378\mu}{13}-\frac{378\mu^2}{13}\right)\in$$
$$+\left(\frac{38}{39}+\frac{102\mu}{13}+\frac{627\mu^2}{13}\right)\in'-\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{\frac{19}{117}+\frac{32r_c}{117}-\frac{2}{13\left(r_c^2+\bar{T}^2\right)}+\left(\frac{249}{52}+\frac{47r_c}{26}-\frac{35}{13\left(r_c^2+\bar{T}^2\right)}\right)\mu\right.$$
$$\left.\left.+\left(\frac{237}{52}-\frac{1451r_c}{26}-\frac{283}{52\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\right\}\right] \qquad (4.77)$$

$$b_1=\frac{\sqrt{3}}{2}\left[1+\frac{1}{2}\mu-4\mu^2-\left(\frac{5}{2}-\frac{1}{3\mu}-\frac{47\mu}{24}+\frac{457\mu^2}{24}\right)\sigma_1+\left(\frac{13}{6}-\frac{1}{3\mu}-\frac{31\mu}{24}+\frac{65\mu^2}{24}\right)\sigma_2\right.$$
$$-\left(\frac{1}{3}+\frac{1}{3}\mu+\frac{34}{3}\mu^2\right)\alpha_2-\left(\frac{4}{9}-\frac{2}{9}\mu+\frac{76}{9}\mu^2\right)\in'-\left(\frac{2}{9}-\frac{7}{9}\mu+\frac{47}{9}\mu^2\right)1-q_2\ +\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}$$
$$\left.\times\left\{\frac{4}{9}\frac{1-2r_c}{}+\frac{4}{27\ r_c^2+\bar{T}^2}-\left(\frac{11}{19}-\frac{11r_c}{18}+\frac{1}{6\ r_c^2+\bar{T}^2}\right)\mu+\left(\frac{121}{9}-\frac{181r_c}{18}+\frac{45}{18\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\right\}\right] \qquad (4.78)$$

$$b_2=\frac{\sqrt{13}}{4}\left[1-\frac{25}{104}\mu+\frac{121}{208}\mu^2-\left(\frac{305}{208}-\frac{3}{13\mu}-\frac{2171\mu}{416}+\frac{16959\mu^2}{416}\right)\sigma_1+\left(\frac{221}{208}-\frac{3}{13\mu}-\frac{1911\mu}{416}\right.\right.$$
$$\left.+\frac{11463\mu^2}{416}\right)\sigma_2-\left(\frac{23}{52}-\frac{115}{104}\mu+\frac{1239}{104}\mu^2\right)\alpha_2+\left(\frac{2}{13}+\frac{47}{26}\mu-\frac{299}{26}\mu^2\right)\in-\left(\frac{41}{78}+\frac{7}{26}\mu+\frac{131}{52}\mu^2\right)\in'$$
$$-\left(\frac{2}{13}+\frac{37}{156}\mu+\frac{11}{8}\mu^2\right)\left(1-q_2\right)\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{\frac{73}{156}-\frac{61}{78}r_c-\frac{1}{26\left(r_c^2+\bar{T}^2\right)}\right.$$
$$\left.\left.+\left(\frac{185}{208}+\frac{47}{104}r_c-\frac{3}{208\left(r_c^2+\bar{T}^2\right)}\right)\mu-\left(\frac{5}{104}+\frac{1487}{104}r_c+\frac{245}{208\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\right\}\right] \qquad (4.79)$$

From the above equations (4.76 to 4.79), the lengths of the semi-major axis $a_1$ and minor axis, $b_1$ of the long orbits are affected by triaxiality, radiation, oblateness, change in the centrifugal force and the potential from the disc, but are not affected by a small change in the Coriolis force, while those for the short period orbits, $a_2$ and $b_2$ are affected by all the system parameters.

## 4.4 A Generalized Restricted Problem of Three-Oblate Bodies with Radiation Pressure and Disc

This problem generalized most studies of the existence of equilibrium points, the linear stability and investigations of the periodic orbits around stable triangular equilibrium points of the R3BP. In this problem, we considered the three bodies in the configuration as having the shape of an oblate spheroid with further assumptions that both primaries are radiating and enclosed by a disc of dust coupled with effects of small perturbations in the Coriolis and centrifugal forces.

Recently, Singh and Haruna (2014a, 2014b) extended the work of AbdulRaheem and Singh (2006, 2008) to include the oblateness of the third body. Here, we extend the model by considering the effects of disc mass in the configuration. This study can equally be assessed as an extension of the restricted problem studied in section **4.2** of this chapter.

### 4.4.1Equations of motion

Let $m_1$ and $m_2$ be the masses of the first and second primary, respectively, and let $m_3$ be the mass of the third body having infinitesimal mass compared to the masses of the primaries. We assume that the three bodies are in the shape of an oblate spheroid with both primaries radiating and surrounded by a disc. The modified potential in this case is

$$U=-Gm_3\left[m_1\left(\frac{q_1}{r_1}+\frac{\alpha_1 q_1}{2r_1^3}+\frac{\alpha_3}{2r_1^3}\right)+m_2\left(\frac{q_2}{r_2}+\frac{\alpha_2 q_2}{2r_2^3}+\frac{\alpha_3}{2r_2^3}\right)+\frac{M_d}{r^2+\bar{T}^2\ ^{1/2}}\right] \tag{4.80}$$

where $r_1^2=\ x-x_1\ ^2+y^2, r_2^2=\ x-x_2\ ^2+y^2$ and $\alpha_i=\frac{AE_i^2-AP_i^2}{5R^2}, \alpha_i<<1:\ i=1,2,3$ .

$G$ is the gravitational constant; $q_1$, $q_2$ are radiation factors of the first and second primary, respectively, while $\alpha_1, \alpha_2$ and $\alpha_3$ are the parameters representing the oblateness of the first primary and second primary and the third body, respectively. $r_1$ and $r_2$ are the distances of the body from the first and second primary, respectively, and last term in the potential, is the gravitational potential due to the mass $M_d$ of the disc; $r = \sqrt{x^2 + y^2}$ is the radial distance of the third body and $\bar{T} = a + b$ is the parameter which defines the density profile of the belt.

Now, the equations of motion of a passively gravitating third body in a barycentric rotating coordinate system, in the gravitational field of the primaries, based on the potential (5.80), have the form:

$$\ddot{x} - 2n\dot{y} = n^2 x - \frac{q_1\ 1-\mu\ \ x-\mu}{r_1^3} - \frac{q_2\mu\ \ x-\mu+1}{r_2^3} - \frac{3\alpha_1 q_1\ 1-\mu\ \ x-\mu}{2r_1^5} - \frac{3\alpha_2 q_2\mu\ \ x-\mu+1}{2r_2^5}$$
$$- \frac{3\alpha_3\ 1-\mu\ \ x-\mu}{2r_1^5} - \frac{3\alpha_3\mu\ \ x-\mu+1}{2r_2^5} - \frac{M_d x}{r^2 + \bar{T}^2\ ^{3/2}} \qquad 4.81$$

$$\ddot{y} + 2n\dot{x} = n^2 y - \frac{q_1\ 1-\mu\ \ y}{r_1^3} - \frac{q_2\mu y}{r_2^3} - \frac{3\alpha_1 q_1\ 1-\mu\ \ y}{2r_1^5} - \frac{3\alpha_2 q_2\mu y}{2r_2^5} - \frac{3\alpha_3\ 1-\mu\ \ y}{2r_1^5} - \frac{3\alpha_3\mu y}{2r_2^5}$$
$$- \frac{M_d y}{r^2 + \bar{T}^2\ ^{3/2}}$$

where $n^2 = 1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 + \frac{2M_d r_c}{r_c^2 + \bar{T}^2\ ^{\frac{3}{2}}}$ (4.82)

Here, $\mu$ is the mass ratio; $n$ is the mean motion of the stars and is defined by the oblateness of the primaries, the mass of the circumbinary disc, the parameter which represents the density profile of the disc, and the radial distance $r_c$ of the third body in the classical R3BP.

Now, due to small deviation of disc primaries on circular orbits, we have assumed that the Coriolis and centrifugal forces of the stars are slightly perturbed and so the system of equations of motion (4.81) becomes

$$\ddot{x} - 2\varphi n\dot{y} = U_x \quad , \quad \ddot{y} + 2\varphi n\dot{x} = U_y \tag{4.83}$$

where

$$U = \frac{n^2\psi\ \ x^2 + y^2}{2} + \frac{q_1\ \ 1-\mu}{r_1} + \frac{q_2\mu}{r_2} + \frac{\alpha_1 q_1\ \ 1-\mu}{2r_1^3} + \frac{\alpha_2 q_2\mu}{2r_2^3} + \frac{\alpha_3\ \ 1-\mu}{2r_1^3} + \frac{\alpha_3\mu}{2r_2^3} + \frac{M_d}{r^2 + \bar{T}^2\ \ ^{1/2}}$$

The parameters $\varphi$ and $\psi$ represent the small change in Coriolis and the centrifugal forces, respectively, and are such that $\varphi - 1 << 1$ and $\psi - 1 << 1$. Equations of motion (4.83), equally admits the Jacobi integral

$$J + \ \dot{x}^2 + \dot{y}^2\ = 2U \tag{4.84}$$

where $J$ is the Jacobi constant and differs from that in (4.9) because of the difference in the potential here and that in equation (4.10).

## 4.4.2 Locations of equilibrium points

The EPs $x, y$ are found by setting the velocity and acceleration in equation (5.83), to zero to get

$$n^2\psi x - \frac{q_1\ \ 1-\mu\ \ x-\mu}{r_1^3} - \frac{q_2\mu\ \ x-\mu+1}{r_2^3} - \frac{3\alpha_1 q_1\ \ 1-\mu\ \ x-\mu}{2r_1^5} - \frac{3\alpha_2 q_2\mu\ \ x-\mu+1}{2r_2^5}$$
$$- \frac{3\alpha_3\ \ 1-\mu\ \ x-\mu}{2r_1^5} - \frac{3\alpha_3\mu\ \ x-\mu+1}{2r_2^5} - \frac{M_d x}{r^2 + \bar{T}^2\ \ ^{3/2}} = 0 \tag{4.85}$$

$$n^2\psi y - \frac{q_1\ \ 1-\mu\ \ y}{r_1^3} - \frac{q_2\mu y}{r_2^3} - \frac{3\alpha_1 q_1\ \ 1-\mu\ \ y}{2r_1^5} - \frac{3\alpha_2 q_2\mu y}{2r_2^5} - \frac{3\alpha_3\ \ 1-\mu\ \ y}{2r_1^5} - \frac{3\alpha_3\mu y}{2r_2^5} - \frac{M_d y}{r^2 + \bar{T}^2\ \ ^{3/2}} = 0$$

Now, we have to solve this pair of equations for $x$ and $y$.

**Collinear equilibrium points**

The collinear equilibrium points are found by solving for $x$ when $y = 0$ in equation (4.85) so that we have

$$\left[ n^2\psi - \frac{M_d}{\left(x^2 + \bar{T}^2\right)^{3/2}} \right] x - \frac{q_1 \ 1-\mu \quad x-\mu}{r_1^3} - \frac{q_2\mu \ \ x-\mu+1}{r_2^3} - \frac{3\alpha_1 q_1 \ 1-\mu \quad x-\mu}{2r_1^5} - \frac{3\alpha_2 q_2 \mu \ \ x-\mu+1}{2r_2^5}$$
$$- \frac{3\alpha_3 \ 1-\mu \quad x-\mu}{2r_1^5} - \frac{3\alpha_3 \mu \ \ x-\mu+1}{2r_2^5} = 0$$

We denote this equation by $f\ \ x\ $, so that

$$f\ \ x\ = \left[ n^2\psi - \frac{M_d x}{\left(x^2 + \bar{T}^2\right)^{3/2}} \right] x - \frac{q_1 \ 1-\mu \quad x-\mu}{r_1^3} - \frac{q_2\mu \ \ x-\mu+1}{r_2^3} - \frac{3 \ 1-\mu \quad x-\mu \quad \alpha_1 q_1 + \alpha_3}{2r_1^5}$$
$$- \frac{3\mu \ \ x-\mu+1 \quad \alpha_2 q_2 + \alpha_3}{2r_2^5} = 0 \qquad 4.86$$

where $r_1 = |x-\mu|$ and $r_2 = |x-\mu+1|$

The abscissas of the collinear points are the roots of (4.86).

Now, observe that

$$\frac{df\ \ x}{dx} = n^2\psi - \frac{M_d x}{\left(x^2 + \bar{T}^2\right)^{3/2}} + \frac{3M_d x^2}{\left(x^2 + \bar{T}^2\right)^{5/2}} + \frac{2q_1 \ 1-\mu}{|x-\mu|^3} + \frac{2q_2\mu}{|x-\mu+1|^3} + \frac{6 \ 1-\mu \quad \alpha_1 q_1 + \alpha_3}{|x-\mu|^5}$$
$$+ \frac{6\mu \ \ \alpha_2 q_2 + \alpha_3}{|x-\mu+1|^5} \qquad 4.87$$

And so $\frac{df\ \ x}{dx} > 0$ if

$$n^2\psi + \frac{3M_d x^2}{\left(x^2 + \bar{T}^2\right)^{5/2}} + \frac{2q_1 \ 1-\mu}{|x-\mu|^3} + \frac{2q_2\mu}{|x-\mu+1|^3} + \frac{6 \ 1-\mu \quad \alpha_1 q_1 + \alpha_3}{|x-\mu|^5} + \frac{6\mu \ \ \alpha_2 q_2 + \alpha_3}{|x-\mu+1|^5} > \frac{M_d x}{\left(x^2 + \bar{T}^2\right)^{3/2}}$$

Observe that, when $x = \pm\infty$, we have $\frac{df\ x}{dx} = n^2\psi$ .

When $x = \mu - 1$ and $x = \mu$, equation (4.87) in both cases give $\frac{df\ x}{dx} = \infty$.

Now, from equation (4.86), if $x = \mu - 2$, we observed that $f\ \mu - 2\ < 0$, for $\psi \approx 1, M_d << 1$, $\alpha_i << 1$ and $q_i \approx 1$. When $x = 0$, we see found that $f\ 0\ > 0$, while for $x = \mu + 1$, $f\ \mu + 1\ > 0$

Therefore, the real roots of the equation (4.86) will lie in the opens interval $\mu - 2, \mu - 1$ , $\mu - 1, 0$ and $\mu, \mu + 1$ . These roots correspond to the collinear points and we shall denote them appropriately when they are found.

Let us express equation (4.86) as

$$f\ x\ = P\ x\ + Q\ x \tag{4.88}$$

where

$$P\ x\ = n^2\psi x - \frac{1-\mu\quad x-\mu}{\left|x-\mu\right|^3}\left\{q_1 + \frac{3}{2}\frac{\alpha_1+\alpha_3}{\left|x-\mu\right|^2}\right\} - \frac{\mu\ \ x-\mu+1}{\left|x-\mu+1\right|^3}\left\{q_2 + \frac{3}{2}\frac{\alpha_2+\alpha_3}{\left|x-\mu+1\right|^2}\right\} \tag{4.89}$$

$$Q\ x\ = -\frac{M_d x}{x^2 + \overline{T}^{2\ 3/2}} \tag{4.90}$$

Now, observe that equation (4.89) can have the following forms depending on positions of the primaries. When $x < \mu - 1, \mu - 1 < x < \mu$ and $\mu < x$, then equation (4.89), give respectively

$$P\ x\ = n^2\psi x + \frac{1-\mu}{x-\mu\ ^2}\left\{q_1 + \frac{3}{2}\frac{\alpha_1+\alpha_3}{x-\mu\ ^2}\right\} + \frac{\mu}{x-\mu+1\ ^2}\left\{q_2 + \frac{3}{2}\frac{\alpha_2+\alpha_3}{x-\mu+1\ ^2}\right\} \tag{4.91}$$

$$P\ x\ = n^2\psi x + \frac{1-\mu}{x-\mu\ ^2}\left\{q_1 + \frac{3}{2}\frac{\alpha_1+\alpha_3}{x-\mu\ ^2}\right\} - \frac{\mu}{x-\mu+1\ ^2}\left\{q_2 + \frac{3}{2}\frac{\alpha_2+\alpha_3}{x-\mu+1\ ^2}\right\} \tag{4.92}$$

$$P\ x\ = n^2\psi x - \frac{1-\mu}{x-\mu\ ^2}\left\{q_1 + \frac{3}{2}\frac{\alpha_1+\alpha_3}{x-\mu\ ^2}\right\} - \frac{\mu}{x-\mu+1\ ^2}\left\{q_2 + \frac{3}{2}\frac{\alpha_2+\alpha_3}{x-\mu+1\ ^2}\right\} \tag{4.93}$$

Hence, in order to investigate the position of collinear equilibrium points, we divide the orbital plane *Oxy* into three parts with respect to the primaries: $x < \mu - 1, \mu - 1 < x < \mu$ and $\mu < x$.

The first case $x < \mu - 1$, implies that $x - \mu + 1 < 0$ and if this happens, then $x - \mu < 0$ follows. In the second case, when $\mu - 1 < x < \mu$, we have $x - \mu < 0$ and $x - \mu + 1 > 0$. The third case when $\mu < x$, we have $x - \mu > 0$ and so $x - \mu + 1 > 0$ also follows. We shall consider these one after the other.

**Case I**

In the first region, when $x < \mu - 1$ this means that the collinear point will lie in the interval $x \in -\infty, \mu - 1$. In this case, if we take the derivative of equation (4.91) with respect to $x$, to get,

$$P'\ x\ = n^2\psi + \frac{2\ 1-\mu}{\left|x-\mu\right|^3}\left\{q_1 + \frac{3}{2}\frac{\alpha_1+\alpha_3}{\left|x-\mu\right|^2}\right\} + \frac{2\mu}{\left|x-\mu+1\right|^3}\left\{q_2 + \frac{3}{2}\frac{\alpha_2+\alpha_3}{\left|x-\mu+1\right|^2}\right\} \tag{4.94}$$

Evidently, for $0 < \mu \le \frac{1}{2}$, $\psi \approx 1, \alpha_i << 1$ and $q_i \approx 1$ we see that $P'\ x\ > 0$

Knowing that equation (4.94) is positive and since $\lim_{x\to-\infty} P\ x\ = -\infty$ and $\lim_{x\to\ \mu-1} P\ x\ = \infty$; then, $P\ x$ is strictly increasing in the interval. Similarly, from equation of (4.90), we see that $\lim_{x\to-\infty} Q\ x\ = 0$ , $\lim_{x\to\ \mu-1} Q\ x\ > 0$ and $Q'\ x\ < 0$; it follows that $Q\ x$ is strictly decreasing function in the interval $-\infty, \mu - 1$. Hence, $\lim_{x\to-\infty} f\ x\ < 0$ and $\lim_{x\to\ \mu-1} f\ x\ > 0$. This shows that

there exists a point say $x_1 \in -\infty, \mu-1$ at which $f\ x_1 = 0$. We denote this equilibrium point as $L_1$.

**Case II**

In this case, the collinear point lies in the interval $\mu-1, \mu$ . Here, two cases arise: the first one is when $x \in \mu-1, 0$ and when it lies in the interval $0, \mu$ . We discuss them separately.

When $x \in \mu-1, 0$ , we have

$$f\ 0 = \frac{1-\mu}{\mu^2}\left\{q_1 + \frac{3}{2}\frac{\alpha_1+\alpha_3}{\mu^2}\right\} - \frac{\mu}{1-\mu^{\ 2}}\left\{q_2 + \frac{3}{2}\frac{\alpha_2+\alpha_3}{1-\mu^{\ 2}}\right\}$$

Now, and knowing that $0 < \mu \le \frac{1}{2}, \alpha_i << 1$ and $q_i \approx 1$, we see that $f\ 0 < 0$. Therefore, since $f\ 0 < 0$ and $\lim_{x \to \mu-1} f\ x > 0$, then there exist a point say $x_2 \in \mu-1, 0$ at which $f\ x_2 = 0$. We denote this point by $L_2$.

Next, when $x \in 0, \mu$ ; if we suppose that $\bar{T} < \sqrt{2}\mu$, then $\frac{\bar{T}}{\sqrt{2}} < \mu$ and so, we consider the two cases when $x \in \left(0, \frac{\bar{T}}{\sqrt{2}}\right)$ and when $x \in \left(\frac{\bar{T}}{\sqrt{2}}, \mu\right)$. Now, in the first case, if $P\left(\frac{\bar{T}\sqrt{2}}{2}\right) + Q\left(\frac{\bar{T}\sqrt{2}}{2}\right) > 0$ , then $f\left(\frac{\bar{T}\sqrt{2}}{2}\right) > 0$ and since $f\ 0 < 0$, then, there exist a point say $x_{21} \in \left(0, \frac{\bar{T}}{\sqrt{2}}\right)$ at which $f\ x_{21} = 0$. We denote this point by $L_{21}$.

In the second case when $x \in \left( \frac{\bar{T}}{\sqrt{2}}, \mu \right)$; if $P\left( \frac{\bar{T}\sqrt{2}}{2} \right) + Q\left( \frac{\bar{T}\sqrt{2}}{2} \right) > 0$, then $f\left( \frac{\bar{T}\sqrt{2}}{2} \right) > 0$ and because $\lim_{x \to \mu} P\ x = -\infty$ and $\lim_{x \to \mu} Q\ x > 0$, we have $\lim_{x \to \mu} f\ x < 0$. Then, we must have a unique point, say $x_{22} \in \left( \frac{\bar{T}}{\sqrt{2}}, \mu \right)$ in this region, where $f\ x_{22} = 0$. We label it as $L_{22}$. We note that, if $P\left( \frac{\bar{T}\sqrt{2}}{2} \right) + Q\left( \frac{\bar{T}\sqrt{2}}{2} \right) < 0$, no EP exists in the interval $0, \mu$ because $f\left( \frac{\bar{T}\sqrt{2}}{2} \right) \neq 0$.

The points $L_{21}$ and $L_{22}$ depend only on the mass parameter, the mass of the disc and the parameters which determine the density profile of the disc.

**Case III**

In this region, the point lies in the open interval $\mu, \infty$. We observe from equation of (4.93) that $P'\ x > 0$ and since $\lim_{x \to \mu} P\ x = -\infty$ and $\lim_{x \to \infty} P\ x = \infty$, this means $P\ x$ is a strictly increasing function in this interval. Also, because $Q'\ x > 0, \mu Q\ x < 0$ and $\lim_{x \to \infty} Q\ x = 0, Q\ x$ is also a strictly increasing function for $x \in \mu, \infty$. Hence, $\lim_{x \to \mu} f\ x < 0$ and $\lim_{x \to \infty} f\ x > 0$; so there is exist a unique point say $x_3$ in this open interval at which $f\ x_3 = 0$. We call this point $L_3$.

Thus, in addition to the usual three collinear equilibrium points of the classical R3BP, due to the presence of the potential from the disc and the mass parameter, there exist two additional collinear EPs $L_{21}$ and $L_{22}$. Hence, we have the points $L_1$ and $L_2$ located to the left of the center of mass while $L_{21}$, $L_{22}$ and $L_3$ are positioned at the right.

In order to make a comprehensive numerical analysis, we represent the collinear points respectively as:

$$x_1 = \mu - 1 - \xi_1 , x_2 = \mu - 1 + \xi_2 , x_3 = \mu + \xi_3 \tag{4.95}$$

where $\xi_i = \ i = 1,2,3$ are the roots of equation (4.86).

Substituting equations of (4.95) in (4.86), simplifying, we get for the point $L_1$:

$$\xi_1^9 + a_{11}\xi_1^8 + a_{12}\xi_1^7 + a_{13}\xi_1^6 + a_{14}\xi_1^5 - a_{15}\xi_1^4 - a_{16}\xi_1^3 - a_{17}\xi_1^2 - a_{18}\xi_1 - a_{19} = 0$$

where $a_{11} = \dfrac{5+4\upsilon}{1+\upsilon}$, $a_{12} = \dfrac{10+6\upsilon}{1+\upsilon}$, $a_{13} = \dfrac{2\ \ 2\upsilon+5}{1+\upsilon} - \dfrac{q_2\upsilon + q_1}{n^2\psi - A\ \ 1+\upsilon}$,

$$a_{14} = \frac{5+\upsilon}{1+\upsilon} - \frac{4q_2\upsilon + 2q_1}{n^2\psi - A\ \ 1+\upsilon}, \quad a_{15} = \frac{3\ \ 4q_2 + \alpha_2 q_2 + \alpha_3\ \ \upsilon + 2q_1 + 3\ \ \alpha_1 q_1 + \alpha_3}{2\ \ n^2\psi - A\ \ 1+\upsilon} - \frac{1}{1+\upsilon}$$

$$a_{16} = \frac{2\upsilon\left[\, 2q_2 + 3\ \ \alpha_2 q_2 + \alpha_3\ \right]}{n^2\psi - A\ \ 1+\upsilon}, a_{17} = \frac{\upsilon\left[\, q_2 + 9\ \ \alpha_2 q_2 + \alpha_3\ \right]}{n^2\psi - A\ \ 1+\upsilon}, a_{18} = \frac{6\upsilon\ \ \alpha_2 q_2 + \alpha_3}{n^2\psi - A\ \ 1+\upsilon}$$

$$a_{19} = \frac{3}{2}\frac{\upsilon\ \ \alpha_2 q_2 + \alpha_3}{n^2\psi - A\ \ 1+\upsilon}$$

For the point $L_2$, we have

$$\xi_2^9 - a_{11}\xi_2^8 + a_{12}\xi_2^7 - a_{13}\xi_2^6 + a_{14}\xi_2^5 - a_{15}\xi_2^4 + a_{16}\xi_2^3 - a_{17}\xi_2^2 + a_{18}\xi_2 - a_{19} = 0$$

where $a_{11} = \dfrac{5+4\upsilon}{1+\upsilon}$, $a_{12} = \dfrac{10+6\upsilon}{1+\upsilon}$, $a_{13} = \dfrac{n^2\psi - A\ \ 4\upsilon+10\ \ + q_1 + q_2\upsilon}{n^2\psi - A\ \ 1+\upsilon}$,

$$a_{14} = \frac{n^2\psi - A\ \ 5+\upsilon\ \ + 2q_1 + 4q_2\upsilon}{n^2\psi - A\ \ 1+\upsilon}, a_{15} = \frac{2\ \ n^2\psi - A + q_1 + 6q_2\upsilon\ \ + 3\ \ \alpha_1 q_1 + \alpha_2 q_2\upsilon\ \ + 3\alpha_3\ \ 1+\upsilon}{2\ \ n^2\psi - A\ \ 1+\upsilon}$$

$$a_{16} = \frac{2\upsilon\left[\, 2q_2 + 3\ \ \alpha_2 q_2 + \alpha_3\ \right]}{n^2\psi - A\ \ 1+\upsilon}, a_{17} = \frac{\upsilon\left[\, q_2 - 9\ \ \alpha_2 q_2 + \alpha_3\ \right]}{n^2\psi - A\ \ 1+\upsilon}, a_{18} = \frac{6\upsilon\ \ \alpha_2 q_2 + \alpha_3}{n^2\psi - A\ \ 1+\upsilon}$$

$$a_{19} = \frac{3}{2}\frac{\alpha_2 q_2 + \alpha_3}{n^2\psi - A\ \ 1+\upsilon}$$

The polynomials for the collinear equilibrium point $L_3$ can be obtained analogously as above from the third equation of (4.95). Hence, the numbers and values of roots is influenced by the oblateness parameters, small perturbation in the centrifugal force, the radiation pressure of the first primary and the effect occurring due to the gravitational potential from the disc.

**Triangular equilibrium points**

The triangular EPs are the solutions of system (4.85) with $y \neq 0$. From the first equation of (4.85), we get

$$x-\mu\left[n^2\psi-\frac{q_1\ 1-\mu\ x}{r_1^3}-\frac{q_2\mu}{r_2^3}-\frac{3\alpha_1 q_1\ 1-\mu}{2r_1^5}-\frac{3\alpha_2 q_2\mu}{2r_2^5}-\frac{3\alpha_3\ 1-\mu}{2r_1^5}-\frac{3\alpha_3\mu}{2r_2^5}\right]-\frac{M_d x}{r^2+\bar{T}^2\ ^{3/2}}$$
$$+\mu\left[n^2\psi-\frac{q_2}{r_2^3}-\frac{3\alpha_2 q_2}{2r_2^5}-\frac{3\alpha_3}{2r_2^5}\right]=0$$

Now, using the second of (4.85) in the above equation (since $y \neq 0$), we get

$$\frac{M_d}{r^2+\bar{T}^2\ ^{3/2}}\ x-\mu\ =-\mu\left[n^2\psi-\frac{q_2}{r_2^3}-\frac{3\alpha_2 q_2}{2r_2^5}-\frac{3\alpha_3}{2r_2^5}\right]+\frac{M_d}{r^2+\bar{T}^2\ ^{3/2}}$$

Simplifying, this equation yields

$$n^2\psi-\frac{q_2}{r_2^3}-\frac{3\alpha_2 q_2}{2r_2^5}-\frac{3\alpha_3}{2r_2^5}-\frac{M_d}{r^2+\bar{T}^2\ ^{3/2}}=0 \qquad (4.96)$$

Similarly, $n^2\psi-\frac{q_1}{r_1^3}-\frac{3\alpha_1 q_1}{2r_1^5}-\frac{3\alpha_3}{2r_1^5}-\frac{M_d}{r^2+\bar{T}^2\ ^{3/2}}=0$

When the primaries are not radiating and are spherical, equations (4.96) reduce to

$n^2\psi - \frac{1}{r_2^3} - \frac{3\alpha_3}{2r_2^5} - \frac{M_d}{\left(r^2+\bar{T}^2\right)^{3/2}} = 0$ and $n^2\psi - \frac{1}{r_1^3} - \frac{3\alpha_3}{2r_1^5} - \frac{M_d}{\left(r^2+\bar{T}^2\right)^{3/2}} = 0$, respectively.

Further, when the force from the disc is absence, these equations reduce to

$\psi - \frac{1}{r_1^3} = 0$ and $\psi - \frac{1}{r_2^3} = 0$ .

Therefore, the solution when these parameters are null but a slight deflection on circular orbit remains, the solution is

$$r_1 = r_2 = \frac{1}{\psi^{1/3}} \tag{4.97}$$

Hence, we use perturbation method to solve the system of equations (4.96), when all the parameters are present. To achieve this, we assume that the solutions are

$$r_1 = \frac{1}{\psi^{1/3}} + \varepsilon_1 \text{ and } r_2 = \frac{1}{\psi^{1/3}} + \varepsilon_2 , \tag{4.98}$$

where $\varepsilon_i \quad i = 1,2$ are very small quantities.

Now, from equations (4.98), we get

$$r_1^{-3} = \psi\ \left(1 - 3\psi^{1/3}\varepsilon_1\right) \ , \ r_2^{-3} = \psi\ \left(1 - 3\psi^{1/3}\varepsilon_2\right) \quad , \quad r_1^{-5} = \psi^{5/3}\ \left(1 - 5\psi^{1/3}\varepsilon_1\right) \tag{4.99}$$

$$r_2^{-5} = \psi^{5/3}\ \left(1 - 5\psi^{1/3}\varepsilon_2\right) \ ,$$

Substituting (4.82), (4.99) in the equations of system (4.96), and solving, we get

$$\varepsilon_1 = \frac{1}{3\psi^{1/3}}\left[-\frac{3}{2}\ \alpha_1+\alpha_2\ -\ 1-q_1\ +\frac{3}{2}\ \alpha_1+\alpha_3\ \ \psi^{2/3}-\frac{M_d}{r^2+\bar{T}^2\ ^{3/2}}\left(2r_c-\frac{1}{\psi}\right)\right]$$

$$\varepsilon_2 = \frac{1}{3\psi^{1/3}}\left[-\frac{3}{2}\ \alpha_1+\alpha_2\ -\ 1-q_2\ +\frac{3}{2}\ \alpha_1+\alpha_3\ \ \psi^{2/3}-\frac{M_d}{r^2+\bar{T}^2\ ^{3/2}}\left(2r_c-\frac{1}{\psi}\right)\right] \qquad (4.100)$$

Substitute equations (4.100) in equations (4.98) and simplifying, we obtain the solutions of equations (4.96):

$$r_1 = \frac{1}{\psi^{1/3}}\left[1-\frac{1}{2}\ \alpha_1+\alpha_2\ -\frac{1}{3}\ 1-q_1\ +\frac{1}{2}\ \alpha_1+\alpha_3\ \ \psi^{2/3}-\frac{M_d}{3\ \ r^2+\bar{T}^2\ ^{3/2}}\left(2r_c-\frac{1}{\psi}\right)\right]$$

$$r_2 = \frac{1}{\psi^{1/3}}\left[1-\frac{1}{2}\ \alpha_1+\alpha_2\ -\frac{1}{3}\ 1-q_2\ +\frac{1}{2}\ \alpha_2+\alpha_3\ \ \psi^{2/3}-\frac{M_d}{3\ \ r^2+\bar{T}^2\ ^{3/2}}\left(2r_c-\frac{1}{\psi}\right)\right] \qquad (4.101)$$

Now, the exact coordinate of the triangular points are

$$x=\mu-\frac{1}{2}+\frac{r_2^2-r_1^2}{2},\ y=\pm\sqrt{\frac{r_1^2+r_2^2}{2}-\frac{1}{4}-\frac{r_2^2-r_1^2\ ^2}{2}} \qquad (4.102)$$

Now, from equations (4.101), finding $r_2^2-r_1^2$ and $r_2^2+r_1^2$; and substituting in equations (4.102), we get

$$x=\mu-\frac{1}{2}+\left\{\frac{1}{\psi^{2/3}}-\frac{M_d\ \ 2\psi r_c-1}{3\psi^{5/3}\ \ r^2+\bar{T}^2\ ^{3/2}}\right\}\left[\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ -\frac{\alpha_1-\alpha_2\ \ \psi^{2/3}}{2}\right] \qquad (4.103)$$

$$y=\pm\frac{\sqrt{4-\psi^{2/3}}}{2\psi^{1/3}}\left\{1-\frac{2}{4-\psi^{2/3}}\left[\frac{1}{3}\ 1-q_1\ +\frac{1}{3}\ 1-q_2\ +\ \alpha_1+\alpha_2\ -\frac{1}{2}\psi^{2/3}\ \ \alpha_1+\alpha_2+2\alpha_3\right.\right.$$

$$\left.\left.+\frac{2M_d\ \ 2\psi r_c-1}{3\psi\ \ r^2+\bar{T}^2\ ^{3/2}}\right]\right\}$$

These are the coordinates of the triangular equilibrium points of the restricted problem of three oblate bodies under radiation pressure forces of the primaries, small change in the Coriolis and centrifugal forces and the force due to the potential from the disc.

## 4.4.3 Stability of equilibrium points

**Collinear equilibrium points**

Following the methodology used in subsection **4.2.3**, the partial derivatives at the collinear point $x,0$ are

$$U_{xx}^0=n^2\psi+\frac{2\ 1-\mu\ \ q_1}{\left|x_0-\mu\right|^3}+\frac{2q_2\mu}{\left|x_0+1-\mu\right|^3}+\frac{6\ 1-\mu\ \ \alpha_1+\alpha_3}{\left|x_0-\mu\right|^5}+\frac{6\mu\ \ \alpha_2+\alpha_3}{\left|x_0-\mu+1\right|^5}+\left[\frac{3x_0^2}{x_0^2+\bar{T}^2\ ^{5/2}}\right.$$

$$\left.-\frac{1}{x_0^2+\bar{T}^2\ ^{3/2}}\right]M_d$$

$$U_{xy}^0=0 \qquad (4.104)$$

$$U_{yy}^0=n^2\psi-\frac{2\ 1-\mu}{\left|x_0-\mu\right|^3}\left[q_1+\frac{3\ \ \alpha_1+\alpha_3}{\left|x_0-\mu\right|^3}\right]-\frac{\mu}{\left|x_0+1-\mu\right|^3}\left[q_2+\frac{3\ \ \alpha_2+\alpha_3}{\left|x_0+1-\mu\right|^2}\right]-\frac{M_d}{x_0^2+\bar{T}^2\ ^{3/2}}$$

Now, $U_{xx}^0>0$ and $U_{yy}^0<0$ for $0<\mu\le\frac{1}{2}, M_d<<1, \alpha_i<<1\ \ i=1,2,3\ \ , \psi\approx 1$ and $q_i\approx 1\ \ i=1,2$ .

The characteristic equation in this case is

$$\lambda^4-\ U_{xx}^0+U_{yy}^0-4\varphi^2n^2\ \ \lambda^2+U_{xx}^0U_{yy}^0-\ U_{xy}^0\ ^2=0 \qquad (4.105)$$

Since, $U_{xy}^0 = 0 \text{ and } U_{xx}^0 U_{yy}^0 < 0$, the roots of (4.105) contains at least a positive root and so the collinear equilibrium points are unstable according to Lyapunov's theorem (Krasnov *et al*. 1983).

**Triangular equilibrium points**

To from the potential in equation (4.83), we take partial derivative up to second order to get

$$U_{xx} = n^2\psi - \frac{q_1\ 1-\mu}{r_1^3} - \frac{q_2\mu}{r_2^3} + \frac{3q_1\ 1-\mu\ \ x-\mu\ ^2}{r_1^5} + \frac{3q_2\mu\ \ x-\mu+1\ ^2}{r_2^5} - \frac{3}{2}\frac{1-\mu\ \ \alpha_1 q_1+\alpha_3}{r_1^5}$$
$$-\frac{3}{2}\frac{\mu\ \ \alpha_2 q_2+\alpha_3}{r_2^5} + \frac{15}{2}\frac{1-\mu\ \ x-\mu\ ^2\ \ \alpha_1 q_1+\alpha_3}{r_1^7} + \frac{15}{2}\frac{\mu\ \ x-\mu+1\ ^2\ \ \alpha_2 q_2+\alpha_3}{r_1^7} - \frac{M_d}{r^2+\bar{T}^2\ ^{3/2}}$$
$$+\frac{3M_d x^2}{r^2+\bar{T}^2\ ^{5/2}}$$

$$U_{yy} = n^2\psi - \frac{q_1\ 1-\mu}{r_1^3} - \frac{q_2\mu}{r_2^3} + \frac{3q_1\ 1-\mu\ \ y^2}{r_1^5} + \frac{3q_2\mu y^2}{r_2^5} - \frac{3}{2}\frac{1-\mu\ \ \alpha_1 q_1+\alpha_3}{r_1^5} - \frac{3}{2}\frac{\mu\ \ \alpha_2 q_2+\alpha_3}{r_2^5}$$
$$+\frac{15}{2}\frac{1-\mu\ \ y^2\ \ \alpha_1 q_1+\alpha_3}{r_1^7} + \frac{15}{2}\frac{\mu y^2\ \ \alpha_2 q_2+\alpha_3}{r_1^7} - \frac{M_d}{r^2+\bar{T}^2\ ^{3/2}} + \frac{3M_d y^2}{r^2+\bar{T}^2\ ^{5/2}}$$

$$U_{xy} = \frac{3q_1\ 1-\mu\ \ x-\mu\ \ y}{r_1^5} + \frac{3q_2\mu\ \ x-\mu+1\ \ y}{r_2^5} + \frac{15}{2}\frac{1-\mu\ \ x-\mu\ \ y\ \ \alpha_1 q_1+\alpha_3}{r_1^7} + \frac{3M_d xy}{r^2+\bar{T}^2\ ^{5/2}}$$
$$+\frac{15}{2}\frac{\mu\ \ x-\mu+1\ \ y\ \ \alpha_2 q_2+\alpha_3}{r_1^7} \qquad 4.106$$

From the equations of (5.101), we find that

$$r_i^{-3} = \psi\left[1+\frac{3}{2}\ \alpha_1+\alpha_2\ +\frac{3}{3}\ 1-q_i\ -\frac{3}{2}\ \alpha_i+\alpha_3\ \ \psi^{2/3} + \frac{M_d\ \ 2\psi r_c - 1}{\psi\ \ r^2+\bar{T}^2\ ^{3/2}}\right]$$

$$r_i^{-5} = \psi^{5/3}\left[1+\frac{5}{2}\ \alpha_1+\alpha_2\ +\frac{5}{3}\ 1-q_i\ -\frac{5}{2}\ \alpha_i+\alpha_3\ \ \psi^{2/3} + \frac{5M_d\ \ 2\psi r_c - 1}{3\psi\ \ r^2+\bar{T}^2\ ^{3/2}}\right]$$

$$r_i^{-7} = \psi^{5/3}\left[1+\frac{7}{2}\ \alpha_1+\alpha_2\ +\frac{7}{3}\ 1-q_i\ -\frac{7}{2}\ \alpha_i+\alpha_3\ \ \psi^{2/3}+\frac{7M_d\ \ 2\psi r_c-1}{3\psi\ \ r^2+\bar{T}^2\ ^{3/2}}\right] \tag{4.107}$$

With the help of equations (4.82), (4.103) and (4.107), the partial derivatives (4.106) are

$$U_{xx}^0 = \psi^{5/3}\left[\frac{3}{4}+\frac{27}{8}\alpha_1+\frac{3}{8}\alpha_2-3\mu\ \ \alpha_1-\alpha_2\ \ +\frac{5}{4\psi}\frac{M_d\ \ 2\psi r_c-1}{r^2+\bar{T}^2\ ^{3/2}}\right]+\psi\left[\frac{\psi^{2/3}-2}{2}\ \ 1-q_1\ \ +\ 1-q_2\right.$$

$$\left.-\frac{\mu}{2}\ \ \psi^{2/3}-4\ \ \ 1-q_1\ \ +\frac{\mu}{2}\ \ \psi^{2/3}-4\ \ \ 1-q_2\ \ \right]-\frac{3M_d}{r^2+\bar{T}^2\ ^{3/2}}\ \ \mu-1/2\ ^2\left[1+\frac{1}{\psi^{2/3}\ \ \mu-1/2}\left\{\frac{1}{3}\ 1-q_1\right.\right.$$

$$\left.\left.+\frac{1}{3}\ 1-q_2\ \ -\frac{1}{2}\psi^{2/3}\ \ \alpha_1-\alpha_2\ \right\}\ \right] \tag{4.108}$$

$$U_{yy}^0 = \frac{3M_d\ \ 4-\psi^{2/3}}{4\psi^{\frac{2}{3}}\ \ r^2+\bar{T}^2\ ^{\frac{5}{2}}}+\psi\left[\frac{3\ \ 4-\psi^{2/3}}{4}+\frac{3\ \ 12-\psi^{2/3}\ \ \ \alpha_1+\alpha_2}{8}+3\alpha_3\psi^{\frac{2}{3}}+\frac{1-q_1\ \ \ 2-\psi^{2/3}}{2}\right.$$

$$\left.-\ 1-q_2\ +\frac{20-13\psi^{2/3}}{4\psi}\frac{M_d\ \ 2\psi r_c-1}{r^2+\bar{T}^2\ ^{3/2}}+\mu\left\{\frac{4-\psi^{2/3}}{2}\ \ 1-q_2\ \ -\frac{4-\psi^{2/3}}{2}\ \ 1-q_1\ \right\}\right]$$

$$U_{xy}^0 = \psi^{4/3}\sqrt{4-\psi^{2/3}}\left[-\frac{3}{4}+\frac{4-4\psi^{2/3}+\psi^{4/3}\ \ \ 1-q_1}{2\psi^{2/3}\ \ 4-\psi^{2/3}}+\frac{\psi^{2/3}-2\ \ 1-q_2}{\psi^{2/3}\ \ 4-\psi^{2/3}}+\frac{3}{8}\frac{5\psi^{2/3}-24}{4-\psi^{2/3}}\alpha_1\right.$$

$$+\frac{3}{8}\frac{\psi^{2/3}-8}{4-\psi^{2/3}}\alpha_2-\frac{3}{2}\frac{\psi^{2/3}}{4-\psi^{2/3}}\alpha_3-\frac{16-5\psi^{2/3}}{4\psi\ \ 4-\psi^{2/3}}\frac{2\psi r_c-1\ \ M_d}{r^2+\bar{T}^2\ ^{3/2}}+\mu\left\{\frac{3}{2}-\frac{\psi^{2/3}-2}{2\ \ 4-\psi^{2/3}}\ \ 1-q_1\right.$$

$$\left.-\frac{\psi^{2/3}-2}{2\ \ 4-\psi^{2/3}}\ \ 1-q_2\ \ -\frac{3\left(3\psi^{2/3}-16\right)}{4\left(4-\psi^{2/3}\right)}\left(\alpha_1+\alpha_2\right)\frac{3\psi^{2/3}}{\left(4-\psi^{2/3}\right)}\alpha_3+\frac{\left(16-5\psi^{2/3}\right)\left(2\psi r_c-1\right)M_d}{2\psi\left(4-\psi^{2/3}\right)\left(r^2+\bar{T}^2\right)^{3/2}}\right\}$$

$$\left.+\frac{3}{4\psi^{5/3}}\frac{\left(2\mu-1\right)M_d}{\left(r^2+\bar{T}^2\right)^{5/2}}\right]$$

Now, substituting these derivatives in the characteristic equation (4.105), expressing $\varphi = 1 + \ \varphi - 1$ and $\psi = 1 + \ \psi - 1$ , where $\varphi - 1 \ll 1$ and $\psi - 1 \ll 1$ and simplifying while ignoring product and higher order terms in $\alpha_i \ \ i = 1,2,3$ , $1 - q_1$ , $1 - q_2$ , $\varphi - 1$ , $\psi - 1$ and $M_d$ ; we get

$$\lambda^4 - \left[ 3\ \psi - 1\ - 4 - 6\ \alpha_1 + \alpha_2\ - 8\ \varphi - 1\ 1 + 3\left\{ 1 + \frac{5}{2}\alpha_1 + \frac{3}{2}\alpha_2 + \alpha_3 - \mu\ \alpha_1 - \alpha_2\ + \frac{2\psi r_c - 1\ M_d}{\psi\ \left( r_c^2 + \bar{T}^2 \right)^{3/2}} \right.\right.$$

$$\left.\left. + \frac{M_d \mu^2}{\psi\ \left( r_c^2 + \bar{T}^2 \right)^{3/2}} - \frac{M_d \mu}{\left( r_c^2 + \bar{T}^2 \right)^{3/2}} + \frac{M_d}{\left( r_c^2 + \bar{T}^2 \right)^{3/2}} \right\} \right] \lambda^2 + \frac{3}{4}\mu\ 1 - \mu \left[ 9 + 39\ \alpha_1 + \alpha_2\ + 12\alpha_3 + 22\ \psi - 1 \right.$$

$$\left. + 2\ 1 - q_1\ + 2\ 1 - q_2\ + \frac{M_d}{\left( r_c^2 + \bar{T}^2 \right)^{3/2}} \left\{ \frac{9}{r_c^2 + \bar{T}^2} + \frac{22}{\psi}\ 2\psi r_c - 1 \right\} \right] = 0$$

where

$$\lambda_{1,2} = \pm s_1 i\ ,\ \lambda_{3,4} = \pm s_2 i \qquad (4.109)$$

$$s_{1,2} = \frac{1}{2}\ -b \pm \sqrt{D}$$

$$b = 1 - 3\ \psi - 1\ + 8\ \alpha - 1\ - \frac{3}{2}\ \alpha_1 - \alpha_2 - 2\alpha_3\ + 3\mu\ \alpha_1 - \alpha_2\ + \frac{3M_d}{\left( r_c^2 + \bar{T}^2 \right)^{3/2}} \times$$

$$\left\{ \frac{8r_c}{3} - \frac{2\psi r_c - 1}{\psi} - \frac{\mu^2 - \mu + 1}{r_c^2 + \bar{T}^2} \right\}$$

$$D = 3\mu^2\left[9+39\ \alpha_1+\alpha_2\ +12\alpha_3+22\ \psi-1\ +2\ 1-q_1\ +2\ 1-q_2\ -\frac{22\ 1-2\psi r_c\ M_d}{\psi\ r_c^2+\bar{T}^2\ ^{3/2}}+\frac{7M_d}{r_c^2+\bar{T}^2\ ^{5/2}}\right]$$

$$-3\mu\left[9+37\alpha_1+41\alpha_2+12\alpha_3+22\ \psi-1\ +2\ 1-q_1\ +2\ 1-q_2\ -M_d\left\{\frac{22\ 1-2\psi r_c}{\psi\ r_c^2+\bar{T}^2\ ^{3/2}}-\frac{7}{r_c^2+\bar{T}^2\ ^{5/2}}\right\}\right]$$

$$+1+16\ \alpha-1\ -6\ \psi-1\ -3\ \alpha_1-\alpha_2\ -6\alpha_3+\frac{2M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{2r_c+\frac{3}{\psi}-\frac{3}{\left(r_c^2+\bar{T}^2\right)}\right\}$$

$D$ is the discriminant and the four roots in equation (4.109) determine whether the equilibrium point is stable. Now, since the value of $D$ when $\mu = 0$ and $\mu = \frac{1}{2}$ have opposite signs. Then, we must have a critical value of $\mu$ in the interval $0 < \mu \le \frac{1}{2}$ at which $D = 0$.

Solving $D = 0$ for $\mu$, we get

$$\mu_C = \frac{1}{2}-\frac{1}{2}\sqrt{\frac{23}{27}}-\frac{1}{9}\left(\frac{13}{\sqrt{69}}+1\right)\alpha_1-\frac{1}{9}\left(\frac{13}{\sqrt{69}}-1\right)\alpha_2+\frac{4\ 36\in-19\in'}{27\sqrt{69}}-\frac{2\ 1-q_1}{27\sqrt{69}}-\frac{2\ 1-q_2}{27\sqrt{69}}$$

$$-\frac{22}{9\sqrt{69}}\alpha_3+\frac{2M_d}{27\sqrt{69}\ r_c^2+\bar{T}^2\ ^{3/2}}\left[38-4r_c-\frac{61}{2\ r_c^2+\bar{T}^2}\right] \qquad 4.110$$

$\mu_C$ is the critical mass parameter and the first term represents Routh's critical mass ratio (Szebehely 1967a); the second and third expressions are the effects arising from oblateness of the primaries, while fourth, fifth, sixth, represent the effects arising from small change in Coriolis and centrifugal forces, radiation pressure force of the first and second primary, respectively. The second to the last and the last expression stand for the effect of oblateness of third body and the gravitational potential from the circumbinary disc on the critical mass, respectively.

## 4.4.4 Periodic orbits around the triangular equilibrium points

### Frequency and orientation of the orbits

The frequencies of the long and short period orbits $s_i \;\; i=1,2$ are found from equation (4.109).

Now, we have

$$s_1=\left[\frac{1}{2}\left\{1-3\epsilon'+8\epsilon-\frac{3}{2}\;\alpha_1-\alpha_2+2\alpha_3\;+\frac{8M_d}{r_c^2+\bar{T}^2\;^{3/2}}-\frac{3M_d\;\;2r_c-1}{r_c^2+\bar{T}^2\;^{3/2}}+3\mu\;\;\alpha_1-\alpha_2\;+\frac{M_d\;\;\mu^2-\mu+1}{r_c^2+\bar{T}^2\;^{5/2}}\right.\right.$$

$$-\left\{3\left[9+39\;\;\alpha_1+\alpha_2\;\;+12\alpha_3+2\;\;1-q_1\;\;+2\;\;1-q_2\;\;+22\epsilon'+\frac{M_d}{r_c^2+\bar{T}^2\;^{3/2}}\left\{\frac{7}{r_c^2+\bar{T}^2}+44r_c-22\right\}\right]\mu^2\right.$$

$$-3\mu\left[9+39\;\;\alpha_1+\alpha_2\;\;-2\;\;\alpha_1-\alpha_2\;\;+12\alpha_3+2\left(1-q_1\right)2\left(1-q_2\right)22\epsilon'+\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{\frac{7}{\left(r_c^2+\bar{T}^2\right)}\right.\right.$$

$$\left.\left.\left.+44r_c-22\right\}\right]+1-6\epsilon'+16\epsilon-3\left(\alpha_1-\alpha_2+2\alpha_3\right)\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}+\left\{\frac{6}{\left(r_c^2+\bar{T}^2\right)}-4r_c-6\right\}\right\}^{\frac{1}{2}}\left.\right]\Bigg\}^{\frac{1}{2}}$$

Simplifying, using Binomial expansion and retaining only linear terms of the very small quantities, we get

$$s_1=\left\{\mu\;\;1-\mu\;\left[\frac{27}{4}+\frac{33\epsilon'}{2}+\frac{3\;\;1-q_1}{2}+\frac{3\;\;1-q_2}{2}+9\alpha_3+\frac{117}{4}\;\;\alpha_1+\alpha_2\;\;-\frac{M_d}{r_c^2+\bar{T}^2\;^{\frac{3}{2}}}\left\{\frac{33}{2}-33r_c\right.\right.\right.$$

$$\left.\left.\left.-\frac{19}{4\;\;r_c^2+\bar{T}^2}\right\}\;\;\right]+\frac{2M_d}{r_c^2+\bar{T}^2\;^{5/2}}\Bigg\}^{1/2} \qquad (4.111)$$

Similarly, for the frequency of the short period orbits, we get

$$s_2 = \left[ 1-3\in'+8\in -\frac{3}{2}\ 1-2\mu \quad \alpha_1+\alpha_2 \ -3\alpha_3 + \frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}} \left\{ 3+2r_c - \frac{2}{r_c^2+\bar{T}^2} \right\} \right.$$

$$+\mu\ 1-\mu\ \left\{ \frac{27}{4}+\frac{117}{4}\ \alpha_1+\alpha_2\ +9\alpha_3 + \frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}} \left( \frac{33}{2} - 33r_c - \frac{69}{12\ r_c^2+\bar{T}^2} \right) + \frac{3}{2}\ 1-q_1 \right.$$

$$\left. \left. +\frac{3}{2}\ 1-q_2\ +\frac{33}{2}\in' \right\} \right]^{1/2} \qquad (4.112)$$

$s_1$ is the frequency of the long periodic orbits while $s_2$ is for the short periodic orbits.

By eliminating the long or short period orbit, we have the elliptic orbit. Expanding the potential $U$ around the triangular points, we have

$$U = U^0 + \frac{1}{2}U_{xx}^0\xi^2 + U_{xy}^0\xi\eta + \frac{1}{2}U_{yy}^0\eta^2 + o\ \ \xi^3,\eta^3 \qquad (4.113)$$

Equation (4.113) is in quadratic form, and shows that the orbits around the triangular points are ellipses.

A more symmetric form of the potential in equation (4.83) is

$$U^0 = \frac{n^2\psi}{2}\left[\ 1-\mu\ \ r_1^2+\mu r_2^2\ \right] + \frac{q_1\ 1-\mu}{r_1} + \frac{q_2\mu}{r_2} + \frac{\alpha_1q_1+\alpha_3\ \ 1-\mu}{2r_1^3} + \frac{\alpha_2q_2+\alpha_3\ \ \mu}{2r_2^3} + \frac{M_d}{r_c^2+\bar{T}^2\ ^{1/2}}$$

Using equations (4.82), (4.101) and (4.107) in the above and reducing to its simplest form, we get

$$U^o = \frac{3}{2} + \left(\frac{5}{4} - \frac{1}{2}\mu\right)\alpha_1 + \left(\frac{3}{4} + \frac{1}{2}\mu\right)\alpha_2 + \frac{1}{2}\alpha_3 - \ 1-\mu \ \ 1-q_1 \ - \mu \ 1-q_2 \ + \frac{1}{2}\in' + \frac{M_d}{r_c^2 + \bar{T}^2 \ ^{1/2}}\left\{1 + \frac{r_c}{r_c^2 + \bar{T}^2}\right\} \tag{4.114}$$

Putting equation (4.114) in (4.113) gives

$$U = K\xi^2 - M\xi\eta + N\eta^2 + L_* \tag{4.115}$$

where $L_*$ is the expression given in equation (4.114) and

$$K = \frac{3}{8} + \left(\frac{27}{16} - \frac{3}{2}\mu\right)\alpha_1 + \left(\frac{3}{16} + \frac{3}{2}\mu\right)\alpha_2 - \left(\frac{1}{4} - \frac{3}{4}\mu\right)\ 1-q_1 \ + \left(\frac{1}{2} - \frac{3}{4}\mu\right)\ 1-q_2 \ + \frac{5}{8}\in' + \frac{M_d}{r_c^2 + \bar{T}^2 \ ^{3/2}}\left\{\frac{3}{8\ r_c^2 + \bar{T}^2} + \frac{5r_c}{4} - \frac{5}{8} - \frac{3\mu\ 1-\mu}{2\ r_c^2 + \bar{T}^2}\right\}$$

$$M = \sqrt{3}\left[\frac{3}{4} - \frac{3\mu}{2} + \left(\frac{19}{8} - \frac{13\mu}{4}\right)\alpha_1 + \left(\frac{7}{8} - \frac{13\mu}{4}\right)\alpha_2 + \left(\frac{1}{2} + \mu\right)\alpha_3 - \left(\frac{1}{6} + \frac{\mu}{6}\right)\ 1-q_1 \ + \left(\frac{1}{3} + \frac{\mu}{6}\right)\ 1-q_2 \right.$$
$$\left. + \left(\frac{11}{12} - \frac{11}{6}\mu\right)\in' - \frac{M_d}{r_c^2 + \bar{T}^2 \ ^{3/2}}\left\{\left(\frac{11}{12} - \frac{11\mu}{6}\right) - \left(\frac{11}{6} - \frac{11\mu}{3}\right)r_c - \frac{3}{r_c^2 + \bar{T}^2} + \frac{3\mu}{2\ r_c^2 + \bar{T}^2}\right\}\right]$$

$$N = \frac{33}{16}\ \alpha_1 + \alpha_2 \ + \frac{3}{2}\alpha_3 + \left(\frac{1}{4} - \frac{3}{4}\mu\right)\ 1-q_1 \ - \left(\frac{1}{2} - \frac{3}{4}\mu\right)\ 1-q_2 \ - \frac{M_d}{r_c^2 + \bar{T}^2 \ ^{3/2}}\left\{\frac{7}{8} - \frac{7r_c}{4} - \frac{9}{8\ r_c^2 + \bar{T}^2}\right\}$$

$$+ \frac{1}{8}\ 9 + 7 \in'$$

Next, when we introduce the variables $\bar{\xi}$ and $\bar{\eta}$ by the transformation given in equation (4.40) and we get the new force function in its simplified quadratic form, such that

$$\bar{U} = \bar{B}_1\bar{\xi}^2 + \bar{B}_2\bar{\eta}^2 + L_* \tag{4.116}$$

where

$$\bar{B}_1 = \frac{3}{8}+\frac{3}{4}\sin^2\alpha-\frac{3\sqrt{3}}{8}\ (1-2\mu)\ \sin 2\alpha+\left[\frac{27}{16}-\frac{3}{2}\mu+\left(\frac{3}{8}+\frac{3}{2}\mu\right)\sin^2\alpha-\frac{\sqrt{3}}{2}\left(\frac{19}{8}-\frac{13}{4}\mu\right)\sin 2\alpha\right]\alpha_1$$
$$+\left[\frac{3}{16}+\frac{3\mu}{2}+\left(\frac{15}{8}-\frac{3\mu}{2}\right)\sin^2\alpha-\frac{\sqrt{3}}{2}\left(\frac{7}{8}-\frac{13\mu}{4}\right)\sin 2\alpha\right]\alpha_2+\left[\frac{3}{2}\sin^2\alpha-\frac{\sqrt{3}}{2}\left(\frac{1}{2}+\mu\right)\sin 2\alpha\right]\alpha_3$$
$$+\left[\left(-\frac{1}{4}+\frac{3\mu}{4}\right)\cos 2\alpha+\frac{\sqrt{3}}{2}\left(\frac{1}{6}+\frac{\mu}{6}\right)\sin 2\alpha\right](-q_1)+\left[\left(\frac{1}{2}-\frac{3\mu}{4}\right)\cos 2\alpha+\frac{\sqrt{3}}{2}\left(-\frac{1}{3}+\frac{\mu}{6}\right)\sin 2\alpha\right](-q_2)$$
$$+\left[\frac{5}{8}+\frac{1}{4}\sin^2\alpha-\frac{11\sqrt{3}(-2\mu)}{24}\sin 2\alpha\right]\in'+\left[\left\{\frac{5}{4}r_c-\frac{5}{8}+\frac{3}{8(r_c^2+\bar{T}^2)}-\frac{3\mu(-\mu)}{2(r_c^2+\bar{T}^2)}\right\}+\left\{\frac{r_c}{2}-\frac{1}{4}+\frac{3}{4(r_c^2+\bar{T}^2)}\right.\right.$$
$$\left.\left.+\frac{3\mu(-\mu)}{2(r_c^2+\bar{T}^2)}\right\}\sin^2\alpha+\frac{\sqrt{3}}{2(r_c^2+\bar{T}^2)^{3/2}}\left\{\left(\frac{11}{12}-\frac{11\mu}{6}\right)-\left(\frac{11}{6}-\frac{11\mu}{3}\right)r_c-\frac{3}{(r_c^2+\bar{T}^2)}+\frac{3\mu}{2(r_c^2+\bar{T}^2)}\right\}\sin 2\alpha\right]M_d$$

$$\bar{B}_2 = \frac{3}{8}+\frac{3}{4}\cos^2\alpha+\frac{3\sqrt{3}}{8}\ (1-2\mu)\ \sin 2\alpha+\left[\frac{27}{16}-\frac{3}{2}\mu+\left(\frac{3}{8}+\frac{3}{4}\mu\right)\cos^2\alpha+\frac{\sqrt{3}}{2}\left(\frac{19}{8}-\frac{13}{4}\mu\right)\sin 2\alpha\right]\alpha_1$$
$$+\left[\frac{3}{16}+\frac{3}{2}\mu+\left(\frac{15}{8}-\frac{3}{2}\mu\right)\cos^2\alpha+\frac{\sqrt{3}}{2}\left(\frac{7}{8}-\frac{13}{4}\mu\right)\sin 2\alpha\right]\alpha_2+\left[\frac{3}{2}\cos^2\alpha+\frac{\sqrt{3}}{2}\left(\frac{1}{2}+\mu\right)\sin 2\alpha\right]\alpha_3$$
$$+\left[\left(\frac{1}{4}-\frac{3\mu}{4}\right)\cos 2\alpha-\frac{\sqrt{3}}{2}\left(\frac{1}{6}+\frac{\mu}{6}\right)\sin 2\alpha\right](-q_1)+\left[\left(\frac{1}{2}-\frac{3\mu}{4}\right)\cos 2\alpha-\frac{\sqrt{3}}{2}\left(\frac{\mu}{6}-\frac{1}{3}\right)\sin 2\alpha\right](-q_2)$$
$$+\left[\frac{5}{8}+\frac{1}{4}\cos^2\alpha+\frac{11(-2\mu)\sqrt{3}}{24}\sin 2\alpha\right]\in'+\frac{M_d}{(r_c^2+\bar{T}^2)^{3/2}}\left[\left\{\frac{5}{4}r_c-\frac{5}{8}+\frac{3}{8(r_c^2+\bar{T}^2)}-\frac{3\mu(-\mu)}{2(r_c^2+\bar{T}^2)}\right\}\right.$$
$$+\left\{\frac{1}{2}r_c-\frac{1}{4}+\frac{3}{4(r_c^2+\bar{T}^2)}+\frac{3\mu(-\mu)}{2(r_c^2+\bar{T}^2)}\right\}\cos^2\alpha+\frac{\sqrt{3}}{2}\left\{\left(\frac{11}{12}-\frac{11\mu}{6}\right)-\left(\frac{11}{6}-\frac{11\mu}{3}\right)r_c-\frac{3}{(r_c^2+\bar{T}^2)}\right.$$
$$\left.\left.+\frac{3\mu}{2(r_c^2+\bar{T}^2)}\right\}\sin 2\alpha\right]$$

Now, to get the orientation of the orbits, we set the term containing $\xi\eta$ to zero, in equation (4.115). We do this by substituting terms in $\xi^2$ for those in $\eta^2$; and the principal axes of the curve of zero velocity have the orientation given by

$$\tan 2\alpha = \frac{4\sqrt{3}}{3}\left[\frac{3}{4}-\frac{3}{2}\mu+\left(\frac{2}{3}-\frac{4\mu}{3}\right)\in' + \left(2-4\mu+3\mu^2\right)\alpha_1 + \left(-1+2\mu-3\mu^2\right)\alpha_2 - \left(1-4\mu\right)\alpha_3\right.$$

$$-\left(\frac{2}{3}-\frac{7\mu}{3}+3\mu^2\right)\left(1-q_1\right)+\left(\frac{4}{3}-\frac{11\mu}{3}+3\mu^2\right)\left(1-q_2\right)-\left\{1-\frac{16r_c}{9}+\frac{4\mu\left(2r_c-1\right)\left(-18-18\mu^2\right)\left(3-2\mu\right)}{9\left(r_c^2+\bar{T}^2\right)}\right\}$$

$$\left.\times\frac{3M_d}{4\left(r_c^2+\bar{T}^2\right)^{3/2}}\right] \qquad (4.117)$$

Equation (4.117) is the orientation of the elliptic orbits in terms of the parameters, representing the radiation pressure forces of the primaries, oblateness of three bodies, components of the disc and the small change in the centrifugal force.

**Eccentricities of the orbits**

The equation given by (4.115) admits the Jacobi constant $J = 2U$. We get the associated characteristic equation such that

$$\lambda^2-\left[3+3\in'+\left(\frac{15}{2}-3\mu\right)\alpha_1+\left(\frac{9}{2}+3\mu\right)\alpha_2+3\alpha_3+\frac{3M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{2r_c-1+\frac{4+\mu-\mu^2}{4\left(r_c^2+\bar{T}^2\right)}\right\}\right]\lambda+\frac{27}{4}\mu\left(1-\mu\right)$$

$$+\mu\left(1-\mu\right)\left[\frac{117\left(\alpha_1+\alpha_2\right)}{4}+9\alpha_3+\frac{3\left(1-q_1\right)}{2}+\frac{3\left(1-q_2\right)}{2}+\frac{33\in'}{4}-\frac{3}{2\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{11-2r_c-\frac{9}{2\left(r_c^2+\bar{T}^2\right)}\right\}M_d\right]=0$$

The two roots are given as

$$\lambda_1 = 3-\frac{9}{4}\mu\left(1-\mu\right)+\left(\frac{15}{2}-\frac{51\mu}{4}+\frac{39\mu^2}{4}\right)\alpha_1+\left(\frac{9}{2}-\frac{27\mu}{2}+\frac{39}{4}\mu^2\right)\alpha_2+\left(3-3\mu+3\mu^2\right)\alpha_3$$

$$+\frac{3\mu\left(1-\mu\right)}{2\left(r_c^2+\bar{T}^2\right)^{3/2}}\left\{\frac{11}{3}-22r_c-\frac{3}{2\left(r_c^2+\bar{T}^2\right)}\right\}M_d+\left(3-\frac{11\mu}{2}+\frac{11}{2}\mu^2\right)\in'-\frac{1}{2}\mu\left(1-\mu\right)\left(1-q_1\right)$$

$$-\frac{1}{2}\mu\left(1-\mu\right)\left(1-q_2\right)$$

$$\lambda_2 = \mu\ 1-\mu\ \left[\frac{9}{4}+\frac{39}{4}\ \alpha_1+\alpha_2\ +3\alpha_3+\frac{11}{2}\in'+\frac{1}{2}\ 1-q_1\ +\frac{1}{2}\ 1-q_2\ -\frac{11M_d}{2\ r_c^2+\bar{T}^2\ ^{3/2}}-\frac{21M_d}{4\ r_c^2+\bar{T}^2\ ^{5/2}}\right]$$
$$-\frac{3M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{1-2r_c-\frac{1}{r_c^2+\bar{T}^2}\right\}$$

In order to get the eccentricities of the ellipses, we use the relations given by Szebehely (1967a) and given in equation (4.45). Then,

$$s_i^2+\lambda_1 = 3\left[1+\frac{3}{2}\mu\ 1-\mu\ +\left(1+\frac{11\mu}{3}+\frac{11}{3}\mu^2\right)\in'+\left(\frac{5}{2}+\frac{11}{2}\mu-\frac{13}{2}\mu^2\right)\alpha_1+\left(\frac{3}{2}+\frac{15}{2}\mu-\frac{13}{2}\mu^2\right)\alpha_2\right.$$
$$+\ 1+2\mu-2\mu^2\ \alpha_3+\frac{1}{3}\mu\ 1-\mu\ \ 1-q_1\ +\frac{1}{3}\mu\ 1-\mu\ \ 1-q_2\ +\frac{\mu\ 1-\mu}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\frac{5}{6\ r_c^2+\bar{T}^2}-\frac{11}{3}\right\}M_d$$
$$\left.+\frac{2M_d}{3\ r_c^2+\bar{T}^2\ ^{5/2}}\right]$$

Therefore,

$$s_i^2+\lambda_1\ ^{-2} = \frac{1}{9}\left[1-3\mu\ 1-\mu\ +\left(-2-\frac{22\mu}{3}+\frac{22\mu^2}{3}\right)\in'+\ -5-11\mu+13\mu^2\ \alpha_1+\ -3-15\mu+13\mu^2\ \alpha_2\right.$$
$$+\ -2-4\mu+4\mu^2\ \alpha_3-\frac{2}{3}\mu\ 1-\mu\ \ 1-q_1\ -\frac{2}{3}\mu\ 1-\mu\ \ 1-q_2\ +\frac{\mu\ 1-\mu\ M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\frac{22}{3}-\frac{5}{3\ r_c^2+\bar{T}^2}\right\}$$
$$\left.-\frac{4M_d}{3\ r_c^2+\bar{T}^2\ ^{5/2}}\right]$$

Now substituting in equation (4.46), and simplifying gives

$$\chi_1^2 = 3\mu - 12\mu^2 - 2\ \mu + 35\mu^2\ \alpha_1 + 4\ \mu - 22\mu^2\ \alpha_2 - 2\ \mu + 11\mu^2\ \alpha_3 + \frac{2}{3}\mu\ 1-7\mu\ \ 1-q_1$$

$$+\frac{2}{3}\mu\ 1-7\mu\ \ 1-q_2\ + \frac{4}{3}\mu\ 1-34\mu\ \in' + \frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left[\frac{8}{9\ r_c^2+\bar{T}^2} - \left\{\frac{22}{3} - \frac{11r_c}{3} + \frac{41}{9\ r_c^2+\bar{T}^2}\right\}\mu\right.$$

$$\left. + \left\{\frac{154}{3} - \frac{143r_c}{3} - \frac{61\mu\ 1-\mu}{9\ r_c^2+\bar{T}^2}\right\}\mu^2\right] \qquad (4.118)$$

Similarly,

$$\chi_2^2 = \frac{1}{4} - \frac{9}{16}\mu\ 1-\mu\ - \frac{243}{32}\mu^2 + \left(1 + \frac{63\mu}{4} - \frac{63\mu^2}{4}\right)\in + \left(-\frac{3}{4} - \frac{19\mu}{4} - \frac{259\mu^2}{8}\right)\in' \qquad 4.119$$

$$-\left(\frac{9}{8} - \frac{27\mu}{16} + \frac{507}{8}\mu^2\right)\alpha_1 - \left(\frac{3}{8} - \frac{57\mu}{16} + \frac{294\mu^2}{4}\right)\alpha_2 - \left(\frac{3}{4} + \frac{15\mu}{8} - \frac{273\mu^2}{16}\right)\alpha_3 + -\frac{27}{16}\mu^2\ 1-q_1$$

$$-\frac{1}{8}\mu\ 1-\mu\ \ 1-q_1\ -\frac{27}{16}\mu^2\ 1-q_2\ -\frac{1}{8}\mu\ 1-\mu\ (-q_2)\frac{M_d}{(r_c^2+\bar{T}^2)^{3/2}}\left[\frac{3}{8} + \frac{1}{4}r_c - \frac{1}{4(r_c^2+\bar{T}^2)}\right.$$

$$\left. + \left\{\frac{233}{32} + \frac{63r_c}{16} - \frac{35}{8(r_c^2+\bar{T}^2)}\right\}\mu + \left\{\frac{233}{32} + \frac{63r_c}{16} - \frac{35}{8(r_c^2+\bar{T}^2)}\right\}\mu + \left\{\frac{437}{16} - \frac{1251r_c}{16} - \frac{283}{32(r_c^2+\bar{T}^2)}\right\}\mu^2\right]$$

We observe that the values given by equation (4.118) are affected by the mass ratio, small change in the centrifugal force, oblateness parameters of the three bodies and radiation pressure forces of both primaries, while equation (4.119) is also affected by all these parameters and the small change in the Coriolis force.

Now, with the help of these equations, we can get the eccentricities of the ellipses. Now, substituting (4.118) in (4.45), the eccentricity of the long and short period orbits are found to be

$$e_1 = 1-\frac{3\mu}{2}\ 1-4\mu\ +\ \mu+35\mu^2\ \alpha_1-2\ \mu-22\mu^2\ \alpha_2+\ \mu+11\mu^2\ \alpha_3-\frac{2}{3}\mu\ 1-34\mu\ \in'$$
$$+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left[\frac{4}{9\ r_c^2+\bar{T}^2}-\left\{\frac{11}{3}-\frac{11r_c}{6}+\frac{41}{18\ r_c^2+\bar{T}^2}\right\}\mu+\left\{\frac{77}{3}-\frac{143r_c}{6}-\frac{61}{18\ r_c^2+\bar{T}^2}\right\}\mu^2\right] \quad (4.120)$$
$$-\frac{1}{3}\mu\ 1-7\mu\ \ 1-q_1\ -\frac{1}{3}\mu\ 1-7\mu\ \ 1-q_2$$

Equation (4.120) is the eccentricity of the long period orbits of the ellipses. We can also find that of the short period orbit in the same way by substituting equation (4.119) in (4.45). Hence, the eccentricity of the short period orbits

$$e_2 = \frac{\sqrt{3}}{2}\left[1+\frac{3\mu}{8}+\frac{75\mu^2}{16}+\left(\frac{3}{4}-\frac{9}{8}\mu+\frac{169}{4}\mu^2\right)\alpha_1+\left(\frac{1}{4}-\frac{19\mu}{8}+49\mu^2\right)\alpha_2+\left(\frac{1}{2}+\frac{5}{4}\mu-\frac{91}{8}\mu^2\right)\alpha_3\right. \quad 4.121$$
$$+\left(\frac{1}{12}\mu+\frac{25\mu^2}{24}\right)1-q_1\ +\left(\frac{1}{12}\mu+\frac{25\mu^2}{24}\right)1-q_2\ -\left(\frac{2}{3}+\frac{21\mu}{2}-\frac{21\mu^2}{2}\right)\in+\left(\frac{1}{2}+\frac{19\mu}{6}+\frac{259\mu^2}{12}\right)\in'$$
$$\left.+\frac{M_d}{r_c^2+\bar{T}^2\ ^{\frac{3}{2}}}\left\{\frac{1}{6\ r_c^2+\bar{T}^2}-\frac{r_c}{6}-\frac{1}{4}-\left(\frac{233}{48}+\frac{21r_c}{8}-\frac{35}{18\left(r_c^2+\bar{T}^2\right)}\right)\mu-\left(\frac{437}{24}-\frac{417r_c}{8}-\frac{283}{48\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\right\}\right]$$

These two respective equations are the values of the eccentricity of the long and short period orbits, respectively. Clearly, eccentricity of the long periodic orbits depends on the mass ratio, small change in the centrifugal force, oblateness parameters of the three bodies and radiation pressure forces of both primaries, while that for the short period orbit is affected by all the system parameters.

**Semi axes of the orbits**

Following same methodology in the previous problem in section **4.2** and 4**.3**, the lengths of the semi axes are

$$a_1 = \frac{\sqrt{5}}{2}\left[1-\frac{2}{5}\mu\ 1-\mu\ +\frac{1}{10\mu}+\left(\frac{34}{15}-\frac{2}{5}\mu\right)\alpha_1+\left(\frac{37}{15}+\frac{2}{5}\mu-\frac{1}{5\mu}\right)\alpha_2+\left(\frac{19}{15}+\frac{1}{5\mu}\right)\alpha_3\right.$$
$$+\left(\frac{7}{45}-\frac{4}{15}\mu+\frac{1}{5\mu}\right)\ 1-q_2\ +\left(\frac{1}{9}-\frac{4}{15}\mu-\frac{1}{15\mu}\right)\ 1-q_2\ -\left(\frac{2}{9}+\frac{2}{15\mu}\right)\in' \qquad (4.122)$$
$$\left.+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\left(-\frac{61}{45}+\frac{1}{3\mu}\right)+\left(\frac{79}{90}-\frac{3}{10\mu}\right)r_c+\left(\frac{61}{135}+\frac{41}{270\mu}-\frac{4}{135\mu^2}\right)\left(r_c^2+\bar{T}^2\right)^{-1}\right\}\right]$$

$$a_2 = \frac{\sqrt{13}}{2}\left[1+\frac{23\mu}{26}+\frac{683\mu^2}{52}+\left(\frac{24}{13}-\frac{103}{26}\mu+\frac{2817}{26}\mu^2\right)\alpha_1+\left(\frac{4}{13}-\frac{185}{26}\mu+\frac{3303}{26}\mu^2\right)\alpha_2\right.$$
$$+\left(2+\frac{63}{13}\mu-\frac{162}{13}\mu^2\right)\alpha_3-\left(\frac{24}{13}+\frac{378\mu}{13}-\frac{378\mu^2}{13}\right)\in+\left(\frac{38}{39}+\frac{102\mu}{13}+\frac{627\mu^2}{13}\right)\in'$$
$$-\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}\left\{\frac{19}{117}+\frac{32r_c}{117}-\frac{2}{13\ r_c^2+\bar{T}^2}+\left(\frac{249}{52}+\frac{47r_c}{26}-\frac{35}{13\left(r_c^2+\bar{T}^2\right)}\right)\mu\right. \qquad (4.123)$$
$$\left.\left.+\left(\frac{237}{52}-\frac{1451r_c}{26}-\frac{283}{52\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\right\}+\left(-\frac{10}{39}-\frac{5\mu}{39}+\frac{3\mu^2}{13}\right)\left(1-q_1\right)\left(-\frac{2}{13}-\frac{\mu}{3}+\frac{3\mu^2}{13}\right)\left(1-q_2\right)\right]$$

$$b_1 = \frac{\sqrt{3}}{2}\left[1+\frac{1}{2}\mu-4\mu^2\left(-\frac{1}{3}+\frac{2}{3}\mu-\frac{49}{3}\mu^2\right)\alpha_1-\left(\frac{1}{3}+\frac{1}{3}\mu+\frac{34}{3}\mu^2\right)\alpha_2+\left(\frac{2}{3}-\frac{1}{3}\mu-\frac{7}{3}\mu^2\right)\alpha_3\right.$$
$$+\left(-\frac{2}{9}-\frac{5}{9}\mu+\frac{25}{9}\mu^2\right)\ 1-q_1\ +\left(-\frac{2}{9}+\frac{7}{9}\mu-\frac{47}{9}\mu^2\right)\ 1-q_2\ +\left(-\frac{4}{9}+\frac{2}{9}\mu-\frac{76}{9}\mu^2\right)\in'+\frac{M_d}{r_c^2+\bar{T}^2\ ^{3/2}}$$
$$\left.\times\left\{\frac{4\ 1-2r_c}{9}+\frac{4}{27\ r_c^2+\bar{T}^2}-\left(\frac{11}{19}-\frac{11r_c}{18}+\frac{1}{6\left(r_c^2+\bar{T}^2\right)}\right)\mu+\left(\frac{121}{9}-\frac{181r_c}{18}+\frac{45}{18\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\right\}\right] \qquad (4.124)$$

$$b_2=\frac{\sqrt{13}}{4}\Bigg[1-\frac{25}{104}\mu+\frac{121}{208}\mu^2+\left(-\frac{21}{52}+\frac{65}{104}\mu-\frac{1374}{104}\mu^2\right)\alpha_1+\left(-\frac{23}{52}+\frac{115}{104}\mu-\frac{1239}{64}\mu^2\right)\alpha_2$$
$$+\left(\frac{1}{2}+\frac{9}{52}\mu+\frac{345}{52}\mu^2\right)\alpha_3+\left(-\frac{10}{39}+\frac{31}{156}\mu+\frac{109}{104}\mu^2\right)\,1-q_1\,+\left(-\frac{2}{13}-\frac{37}{156}\mu-\frac{11}{8}\mu^2\right)\,1-q_2$$
$$+\left(\frac{2}{13}+\frac{47}{26}\mu-\frac{299}{26}\mu^2\right)\in-\left(\frac{41}{78}+\frac{7}{26}\mu+\frac{131}{52}\mu^2\right)\in'+\frac{M_d}{\left(r_c^2+\bar{T}^2\right)^{3/2}}\Bigg\{\frac{73}{156}-\frac{61}{78}r_c-\frac{1}{26\left(r_c^2+\bar{T}^2\right)}$$
$$+\left(\frac{185}{208}+\frac{47}{104}r_c-\frac{3}{208\left(r_c^2+\bar{T}^2\right)}\right)\mu-\left(\frac{5}{104}+\frac{1487}{104}r_c+\frac{245}{208\left(r_c^2+\bar{T}^2\right)}\right)\mu^2\Bigg\}\Bigg] \qquad (4.125)$$

Equations (4.122), (4.123), (4.124) and (4.125) represent the lengths of the, semi-major axis of the long period orbits, semi-major axes of the short period term, semi-minor axes of the long period orbit and semi-minor axes of the short orbits. From these equations, we observe that, the lengths of the semi-major axis $a_1$ and minor axis, $b_1$ of the long period orbits are affected by the oblateness of the bodies, radiation pressure forces of the primaries, small change in the centrifugal force and the components of the disc, but are not affected by a small change in the Coriolis force, while those for the short period orbits, $a_2$ and $b_2$ are affected by all the system parameters.

## 4.5. Conclusion

In this chapter, we investigated the R3BP when the masses of the involved bodies are kept constant with the view that the motion of the third body occurs in the presence of a circumbinary disc. We considered various characterizations of the bodies ranging from, slight perturbations in the Coriolis and centrifugal forces, to when one or all the bodies have the shape of an oblate spheroid or triaxial. It is seen that, two additional collinear EPs exist and depend on the mass parameter of the system and the mass of the disc. In the case of the triangular points, we used

perturbation method to find these points for each of the problems studied. Their locations are strongly affected by the involved system parameters of each problem.

The triangular EPs are conditionally stable while the collinear EPs are unstable. In the range of the linearly stable triangular EPs, we examined the periodic orbits around these points and found that the orbits are ellipses having long and short period orbits. The orientation, eccentricities, and the semi axes of the orbits are found. These orbits around the triangular Eps are in general stable using the PSS and are important for scientific investigations. Since, a number of space missions have been completed and some are being proposed by various space agencies.

# CHAPTER FIVE

## 5.0 RESULTS AND DISCUSSION

In this chapter, we give the results and discussion of the findings of the thesis accordingly.

## 5.1 Numerical Treatments of the Restricted Problem of Three-bodies with Variable Masses

The thesis studied the motion of a third body in the gravitational fields of two primaries using the restricted three-body problem. We classified the study of this work into two main categories; when the masses vary with time and when they are constant. In the case when the masses vary with time, we use the models of the restricted problem as described by Szebehely (1967a) and the Robe's (1977) formulation.

### 5.1.1 Restricted three-body with isotropic mass variations

The R3BP with variable masses, when the masses of the primaries vary isotropically in accordance with the unified Mestschersky (1952) law and their motion governed by the Gylden-Mestschersky problem (Gylden 1884; Mestschersky 1902) under various characterizations, have been studied. We derived the equations of motion of the restricted problems, located and studied the linear stability of the EPs. The periodic orbits around linearly stable triangular EPs have also been studied. The system (3.14) of equations of motion is unlike those obtained by Bekov (1988), Luk'yanov (1989, 1990) and Singh and Leke (2010) due to the presence of radiation of the second primary, oblate shape of the first primary and small perturbations in the Coriolis and centrifugal forces.

Because the system of equations (3.14) is a non-integrable differential equations with variable coefficients, we recast it to the autonomized system (3.17), using the Mestschersky (1902)

transformation (1.46); the UML (1.47); the particular solutions of the GMP and in view of the variable oblateness of the first primary and further introduced a transformation (3.16) that took care of the variable oblateness of the first primary. We apply the problem to the binary Cen X-4, which is a low mass X-ray binary consisting of a neutron star and a white dwarf secondary star.

Numerically, the binary Cen X-4, which is a low mass x-ray binary consisting of a neutron star with a mass of $1.4M_{\Theta}$, an equatorial radius of $R_{eq} = 10km$ with a dwarf secondary of mass $0.2M_{\Theta}$ with parametric values such that $\upsilon = 0.125, \alpha_1 = 0.02, q_2 = 0.9985$ and $\in' = 0.002$ for $0 < \kappa < \infty$; is taken as our model. Using the software package *Mathematica*, we compute the positions of the collinear EP $L_{2\kappa}$ for different $\kappa$ in the table **5.1** below.

**Table 5.1:** Numerical computations of the collinear point $L_{2\kappa}$

| $\kappa$ | $L_{2_\kappa}$ | $L_{2_\kappa}$ $(\alpha_1 = 0)$ | $L_{2_\kappa}$ $(\alpha_1 = 0,\ q_2 = 1)$ |
|---|---|---|---|
| $\kappa = 0.001$ | -0.76272 | -0.75367 | -0.75367 |
| $\kappa = 0.01$ | -0.99687 | -0.99442 | -0.99443 |
| $\kappa = 0.1$ | -1.04684 | -1.04553 | -1.04554 |
| $\kappa = 0.5$ | -1.05185 | -1.05065 | -1.05067 |
| $\kappa = 0.9$ | -1.05242 | -1.05123 | -1.05125 |
| $\kappa = 1$ | -1.05249 | -1.05130 | -1.05132 |
| $\kappa = 2$ | -1.05281 | -1.05163 | -1.05164 |
| $\kappa = 10$ | -1.05230 | -1.05189 | -1.0519 |
| $\kappa = 20$ | -1.05230 | -1.05192 | -1.05194 |
| $33 \le \kappa \le 65$ | -1.05231 | -1.05193 to -1.05194 | -1.05195 to -1.05196 |
| $66 \le \kappa < \infty$ | -1.05231 | -1.05194 to -1.05195 | -1.05196 to -1.05197 |

It is seen that there exist only three collinear EPs in the presence or absence of the oblateness of the first primary, radiation of the second one and small perturbation in the centrifugal forces. The collinear EPs $L_{1\kappa}$ and $L_{3\kappa}$ are also of seventh degree and can be determined analogously as the case above.

The coordinates of the triangular EPs are given by equation (3.42). These points are determined by small perturbation in the centrifugal force, radiation pressure of the second primary, oblateness of the first primary and the constant of a particular integral of the GMP. The positions of the triangular EPs exist only when $\psi - 1 < \kappa < \infty$, When there is no perturbation in the centrifugal force i.e., $\psi = 1$, these points will exist for $0 < \kappa < \infty$ and are affected only by the radiation pressure, and oblateness coefficient . The small perturbation given in the centrifugal force is very important, as this allows the existence of the parameter kappa; so there can be different solutions for different $\kappa$. When $\alpha_1 = 0$, equations (3.42) are different from the ones obtained by Singh and Leke (2010)

Also, four out-of-plane EPs given in equations (3.62) and (3.63) exist only for $\kappa > 1$ provided $\psi\xi < \upsilon$ and $\psi\xi + \kappa - 1 \quad 1 - \upsilon > 0$, respectively and these are denoted by $L_{6,7}$, and $L_{8,9}$, respectively. Results of Bekov (1988, 2005), Luk'yanov (1989, 1990) can be deduced from our results when $\alpha_1 = 0,\ q_2 = 1,$ and $\psi = 1$; while the results here are similar to those in Singh and Leke (2010). The EPs $L_i \quad i = 1, 2...9$ of the system of equations of the non-autonomous system are given in equation (3.64) using the Mestschersky's transformation (1.46).

The characteristic equation (3.68) of the collinear EPs is different from those of Singh and Leke (2010) due to the oblateness of the first primary. It is seen that the coefficients can take either

positive or negative values because of the presence of the arbitrary constant which takes values between zero and infinity, and the other parameter involved. Consequently, a stable collinear point is possible under some conditions.

In the case when both primaries are radiating and oblate spheroid as well under effects of small perturbations in the Coriolis and centrifugal forces, the polynomial (3.95) is of degree nine because we have assumed that both primaries are oblate spheroids. Notice that the polynomial has degree seven when one of the primaries is an oblate spheroid (Equation 3.21). We compute the roots of the polynomials corresponding to each collinear EP numerically (Tables **5.2, 5.3** and **5.4**), using the software package *Mathematica* (Wolfram 2003). To do this, we consider a yellow supergiant eclipsing binary system which consists of a pair of recently found stars orbiting each other so closely and having the appearance of a giant peanut in space. Both stars are very bright, yellow oblate stars and are mostly same in size. Therefore, we choose $\upsilon = 0.48785$, for range of $\kappa$ in the interval $0 < \kappa < \infty$. Our numerical exploration reveals that aside the three collinear points $L_{1,2,3}$ two additional EPs $L_{21}$ and $L_{22}$ exist around the point $L_2$ due to the small perturbation in the centrifugal force and $\kappa$ provided $\psi + \kappa < 1$. That is, these EPs exist when simultaneously $\kappa + \in' < 0$ and both primaries are oblate. In the case when the first condition holds but one of the primaries is not an oblate spheroid, no additional EPs exists. As an example, when $\kappa = 0.001$ and $\in' = -0.002$ and choosing values for the oblateness with or without radiation. In this case, the polynomial (3.95) has three distinct real roots viz. $-3.15339, -1.29282$ and $1.06768$; each of which corresponds to a collinear EPs; the first and second lying to the right of the second primary, while the third lies to the left of the center of mass (Table **5.4**). Hence, we have **five** collinear equilibrium points different from those in problem **3.2** and other studies of the R3BP, where there are only **three**.

**Table 5.2:** Numerical computations of the collinear points for $\upsilon = 0.48785$, $q_1 = 0.9988$ $q_2 = 0.9855, \in' = 0.002$, $\alpha_1 = 0.024$ and $\alpha_2 = 0.02$

| $\kappa$ | $L_{1_\kappa}$ | $L_{2_\kappa}$ | $L_{3_\kappa}$ |
|---|---|---|---|
| 0.001 | −1.0378 | −1.9621 | 1.20344 |
| 0.01 | −1.0356 | −1.9775 | 1.21827 |
| 0.1 | −1.09476 | −1.98566 | 1.24195 |
| 0.5 | −1.10071 | −1.98652 | 1.24446 |
| 0.9 | −1.10127 | −1.98661 | 1.24469 |
| 1 | −1.10146 | −1.98663 | 1.24477 |
| 1.065 | −1.10151 | −1.98664 | 1.24479 |
| 10 | −1.10214 | −1.98673 | 1.24506 |
| $11 \le \kappa \le 411$ | −1.10215 to −1.10221 | −1.98674 | 1.24506 to 1.24509 |
| $412 \le \kappa < \infty$ | −1.10221 to −1.10222 | −1.98675 | 1.24509 to 1.24509 |

**Table 5.3:** Numerical computations of the collinear points for $\upsilon = 0.48785$, $q_1 = q_2 = 1$, $\in' = 0.002$, $\alpha_1 = 0.024$ and $\alpha_2 = 0.02$

| $\kappa$ | $L_{1_\kappa}$ | $L_{2_\kappa}$ | $L_{3_\kappa}$ |
|---|---|---|---|
| 0.001 | −1.03868 | −1.96210 | 1.20344 |
| 0.01 | −1.03605 | −1.97760 | 1.21846 |
| 0.1 | −1.09523 | −1.98610 | 1.24217 |
| 0.5 | −1.10119 | −1.98659 | 1.24468 |
| 0.9 | −1.10185 | −1.98669 | 1.24496 |
| 1 | −1.10194 | −1.98670 | 1.24499 |
| 1.065 | −1.10198 | −1.98671 | 1.24501 |
| 10 | −1.10262 | −1.98680 | 1.245282 |
| $11 \le \kappa < \infty$ | −1.10263 to −1.10269 | −1.9868 to −1.98681 | 1.24528 to 1.24531 |

**Table 5.4:** Numerical computations of the collinear points when $\in' = 0$ and $\kappa + \in' < 0$

| When $\in' = 0$, $\kappa$ vanishes | $L_{1_\kappa}$ | $L_{2_\kappa}$ | $L_{3_\kappa}$ |
|---|---|---|---|
| $\alpha_1 = \alpha_2 = 0$, $q_1 = 0.9988$, $q_2 = 0.9985$, $\in' = 0$ | −1.01534 | 0.74687 | 1.15338 |
| $\alpha_1 = \alpha_2 = 0$, $q_1 = q_2 = 1$, $\in' = 0$ | −1.01575 | 0.74692 | 1.15355 |
| $\kappa + \in' < 0$, $\in' = -0.002$, $\kappa = 0.001$ | −1.78292 | −3.15339, −1.29282, 1.06768 | −1.02728 |

The characteristic equation of the collinear points is given in equation (3.132) and the nature of the roots of this equation depends on the discriminant coupled with the changes in $P_1$ and $Q_1$ which depends solely on $\kappa$, while the values are affected by all the involved parameters. The only possibility of a stable collinear point, is when simultaneously $D_1 > 0$, $P_1 > 0$ and $Q_1 > 0$. In this case, all $\lambda_i$ will be distinct pure imaginary, but this occurs numerically only when $\kappa = 0.0001$ and $0.25 < \delta_1$, otherwise, $P_1$ is always negative while $Q_1$ can be positive or negative depending solely on $\kappa$. To verify our claims, we perform a numerical example taking $\upsilon = 0.48785$, $q_1 = 0.9988$, $q_2 = 0.9985$, $\alpha_1 = 0.024$, $\alpha_2 = 0.02$, $\phi = 1.001$, $\psi = 1.002$, $0 < \kappa < \infty 0$ and $0 < \delta_1 < 0.25$. It is seen that $P_1$ is always negative while $Q_1$ changes sign. In this case, there exist at least a positive root and the collinear point is unstable. However, when simultaneously $0.25 < \delta_1$ and $\kappa = 0.0001$ with the same chosen values of the involved parameters, we simultaneously have $P_1 > 0$ and $Q_1 > 0$; and this results in four distinct pure imaginary roots. Consequently, the motion is bounded and the collinear point is stable. The same procedure shows that all the five collinear points are conditionally stable with respect to the involved parameters.

Now, the triangular EPs are stable when $0<\upsilon<\upsilon_{C_\kappa}$ and if the discriminant of the governing characteristic equation is positive. In this case, the roots of the characteristic equation (3.136) are distinct pure imaginary and the motion is bounded. The critical mass values under combined effects of radiation, perturbation, oblateness and mass variation have been obtained in equation (3.139). When $\kappa=1,2,3$, the critical masses are respectively

$$\upsilon_{c_1}=\frac{1}{2}\left(1-\frac{\sqrt{23}}{3\sqrt{3}}\right)-\frac{2\ 1-q_1}{27\sqrt{69}}-\frac{2\ 1-q_2}{27\sqrt{69}}-\frac{76\in'}{27\sqrt{69}}+\frac{16\in}{3\sqrt{69}}-\frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)\alpha_1+\frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)\alpha_2$$

$$\upsilon_{c_2}=\frac{1}{2}\left(1-\frac{\sqrt{23}}{3\sqrt{3}}\right)-\frac{2\ 1-q_1}{27\sqrt{69}}-\frac{2\ 1-q_2}{27\sqrt{69}}+\frac{28\in'}{27\sqrt{69}}-\frac{8\in}{3\sqrt{69}}+\frac{1}{9}\left(1+\frac{5}{\sqrt{69}}\right)\alpha_1-\frac{1}{9}\left(1-\frac{5}{\sqrt{69}}\right)\alpha_2$$

$$\upsilon_{c_3}=\frac{1}{2}\left(1-\frac{\sqrt{143}}{9\sqrt{3}}\right)-\frac{50\ 1-q_1}{81\sqrt{429}}-\frac{50\ 1-q_2}{81\sqrt{69}}+\frac{260\in'}{9\sqrt{429}}-\frac{80\in}{9\sqrt{429}}+\frac{5}{27}\left(1+\frac{7}{\sqrt{429}}\right)\alpha_1$$
$$-\frac{5}{27}\left(1-\frac{7}{\sqrt{429}}\right)\alpha_2$$

The first equation fully coincides with those of AbdulRaheem and Singh (2006). Results of Singh and Ishwar (1999), Sharma (1982) Bhatnagar and Chawla (1979), Bhatnagar and Hallan (1978), SubbaRao and Sharma (1975), and Szebehely (1967a, 1967b) can be deduced from it. However, when $\kappa\neq1$ for instance, when $\kappa=2,3$ the values obtained are different from those presented in other works. When the first primary is a black body and the second one is a sphere, the critical masses (3.139) reduce to those in equations (3.72).

The table **5.5** below gives the numerical computations of $\upsilon_{C_\kappa}$ and $\upsilon_{0_\kappa}$ in equations (3.72) for $\kappa$ in the interval $0.7145311,\ 9.952136$ and $\delta_2=0.0015$, $\alpha_1=0.02$, $\in=0.001, \in'=0.002$

**Table 5.5:** Numerical computations of the critical mass parameter $\upsilon_{C_\kappa}$

| $\kappa$ | $\upsilon_0$ | $\delta_2 = 0.0015$ | $\in = 0.001$ | $\in' = 0.002$ | $\alpha_1 = 0.02$ | $\upsilon_{C_\kappa}$ |
|---|---|---|---|---|---|---|
| 0.7145311 | Complex | Imaginary | Imaginary | Imaginary | Complex | Complex |
| 0.7145312 | 0.49982 | -0.23884 | 7.41089 | -6.19740 | -15.9952 | -14.5207 |
| 0.7145319 | 0.49895 | -0.03998 | 1.24065 | -1.03750 | -2.68068 | -2.01856 |
| 0.71459 | 0.49058 | -0.00442 | 0.13725 | -0.11477 | -0.29977 | 0.20887 |
| 0.71460 | 0.48981 | -0.00408 | 0.12688 | -0.10610 | -0.27740 | 0.22911 |
| 0.72 | 0.40991 | -0.00044 | 0.01400 | -0.01161 | -0.03427 | 0.37758 |
| 0.75 | 0.28010 | -0.00015 | 0.00503 | -0.00398 | -0.01558 | 0.26541 |
| 0.99999 | 0.03852 | -0.00001 | 0.00077 | -0.00044 | -0.00619 | 0.03264 |
| 1.0000 | 0.03852 | -0.00001 | 0.00077 | -0.00044 | -0.00619 | 0.03263 |
| 4/3 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0.03852 | -0.00001 | -0.00038 | 0.00009 | 0.019655 | 0.05787 |
| 3 | 0.11643 | -0.00004 | -0.00051 | 0.00006 | 0.072830 | 0.18877 |
| 4 | 0.18085 | -0.00007 | -0.00055 | 0.00004 | 0.159606 | 0.33987 |
| 5 | 0.23402 | -0.00011 | -0.00058 | 0.00002 | 0.287590 | 0.52094 |
| 6 | 0.28010 | -0.00015 | -0.00628 | 0.000016 | 0.469235 | 0.74291 |
| 7 | 0.32235 | -0.00020 | -0.00069 | 0.000002 | 0.728478 | 1.04994 |
| 8 | 0.36391 | -0.00028 | -0.00081 | -0.00001 | 1.123410 | 1.48622 |
| 9 | 0.40991 | -0.00044 | -0.00112 | -0.00001 | 1.875780 | 2.28410 |
| 9.5 | 0.43943 | -0.00067 | -0.00159 | -0.00003 | 2.808190 | 3.24532 |
| 9.94 | 0.49028 | -0.00428 | -0.00956 | 0.00022 | 15.57210 | 16.0581 |
| 9.95 | 0.49592 | -0.01022 | -0.02278 | -0.00054 | 36.26200 | 36.7244 |
| 9.9521 | 0.49947 | -0.07933 | -0.17674 | -0.00419 | 276.8030 | 277.042 |
| 9.952135 | 0.49993 | -0.67735 | -1.50894 | -0.03584 | 2358.090 | 2356.37 |
| 9.952136 | Imaginary | Imaginary | Imaginary | Imaginary | Complex | Complex |

We observe that for $0.7145311 \le \kappa < 0.71459$ and $\kappa > 9.952135$ with the allowance for oblateness of the first primary, $\upsilon_{C_\kappa}$ is either complex or negative; therefore the physically possible range of $\kappa$ is $0.71459 \le \kappa \le 9.952135$. However, in the absence of oblateness,

i.e., $\alpha_1 = 0$, the range becomes $0.7145311 < \kappa \leq 9.9521$. For example when $\kappa = 0.7145312$, we get $\upsilon_{C_\kappa}$ = -14.5207 which is unrealistic; when the first primary is spherical, the critical mass becomes $\upsilon_{C_\kappa} = 1.28339$. Below in Fig **5.1** is a graphical representation of $\upsilon_{C_\kappa}$ as a function of $\kappa$ in the interval $0.71459 \leq \kappa < 9.952136$ for the chosen parametric values.

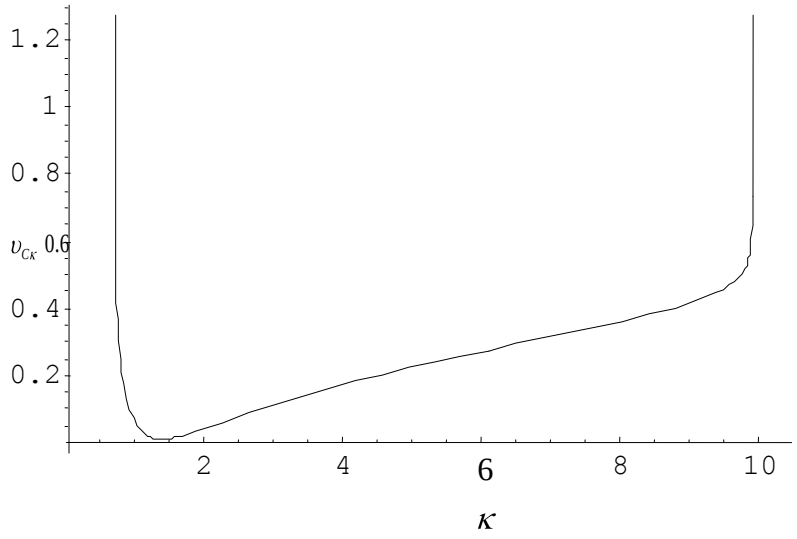



**Fig. 5.1:** Graph of $\kappa$ and the critical mass parameters $\upsilon_{C_\kappa}$

If in (3.72), we annul the effect of the small perturbations and radiation pressure and oblateness of the first primary (i.e. $\in = \in' = 0$, $q_2 = 1, \alpha_1 = 0$) and $\kappa = 1, 2$; then $\upsilon_{c_{1,2}} = \upsilon_{0_{1,2}} = 0.038520$, this fully coincide with the Routhian value (Szebehely1967a, 1967b). We observe from system (3.73) that, for $0.7145312 \leq \kappa < \frac{4}{3}$ the Coriolis force has a stabilizing tendency while the centrifugal force, radiations pressure and oblateness of the first primary have destabilizing behaviors in this range of $\kappa$. When $\kappa = \frac{4}{3}$, all these parameters are annulled and have no effect (eqn. (3.74)). For $\frac{4}{3} < \kappa \leq 9.952135$ the Coriolis force and the radiation pressure of the second primary remains all

through a destabilizing parameter, while here, the oblateness of the first primary assumes the role of a stabilizing parameter. The small perturbation in the centrifugal force plays a stabilizing role only when $\frac{4}{3} < \kappa \leq 7$ and is destabilizing outside this range of kappa; while for $\frac{4}{3} < \kappa \leq 5$, oblateness of the second primary has a destabilizing tendencies which afterwards becomes stabilizing for $6 \leq \kappa \leq 9.952135$. Hence the stabilizing or destabilizing behaviors of the Coriolis and centrifugal forces and oblateness is controlled by kappa. It is seen that (except for the case when $\kappa = \frac{4}{3}$), the radiation pressure always has a destabilizing tendency. This behavior gets weaker with increase in kappa in the interval $0.7145312 \leq \kappa < \frac{4}{3}$; and has no effect for $\kappa = \frac{4}{3}$ and interestingly grows stronger for $\frac{4}{3} < \kappa \leq 9.952135$. Our numerical computation reveals that for any $|\in| << 1$, $|\in'| << 1$, $1 - q_2 << 1$ and $0.71460 \leq \kappa < \frac{4}{3}$, every $\upsilon_{c_\kappa} < \upsilon_{0_\kappa}$; consequently the region of stability is decreasing. This is so because of the inclusion of oblateness of the first primary. If this is ignored i.e., the first primary is spherical, we'll have $\upsilon_{c_\kappa} > \upsilon_{0_\kappa}$ which implies that the region of stability is increasing. When $\kappa = \frac{4}{3}$, it coincides and equal zero; while for $\frac{4}{3} < \kappa \leq 9.952135$, due to oblateness every $\upsilon_{c_\kappa} > \upsilon_{0_\kappa}$ and this implies that the region of stability of the triangular EP is increasing. If $\alpha_1 = 0$, reverse is the case as observed in the work of Singh *et al.* (2010). The results of Szebehely (1967b), SubbaRao and Sharma (1975), Bhatnagar and Hallan (1978), Bekov (1988) and Singh *et al.* (2010) can be confirmed here from our results.

Our results show that a condition for stable triangular EPs is when the mass ratio is in the region $0 < \upsilon < \upsilon_{c_\kappa}$. Numerically, we must have $0.002 < \kappa < 1.31169451$. If $8 \in -3 \in' = 0$, $\alpha_1 = 0$ then, $0 < \kappa < \frac{4}{3}$ and agrees with Singh and Leke (2010). The decrease or increase in the region of stability of the triangular EPs depends on the arbitrary constant $\kappa$ and oblateness $\alpha_1$ of the first primary and the signs of the small perturbations. In the case of the out-of-plane EPs, using the software package *Mathematica* (Wolfram 2003), we compute the coefficients $d_1$, $d_2$ and $d_3$ of the characteristic equation (3.144) as shown in Table **5.6** below; for $\kappa > 1$, $q_1 = 0.9988$, $q_2 = 0.9855$, $\in' = 0.002$, $\in = 0.001$, $\alpha_1 = 0.024$, $\alpha_2 = 0.02$ and $\upsilon = 0.48785$.

**Table 5.6:** Numerical approximations of out-of-plane points

| $\kappa$ | $\xi$ | $\pm\zeta$ | $d_1$ | $d_2$ | $d_3$ | Nature of Roots |
|---|---|---|---|---|---|---|
| 1.01 | 0.00002 | 4.65558 | −2.16995 | 0.99956 | −0.03429 | ±0.17903, ±0.8509*i*, ±1.2157*i* |
| 1.1 | 0.00102 | 2.27414 | −1.86411 | 0.326961 | −0.36140 | Same nature as above |
| 1.5 | 0.01187 | 1.76928 | 0.26311 | −4.33501 | −2.64084 | Same nature as above |
| 2 | 0. 02999 | 1.92301 | 5.47421 | −15.7303 | −8.24540 | Same nature as above |
| 3 | 0.06680 | 2.17926 | 21.7148 | −51.6019 | −25.2522 | Same nature as above |
| 10 | 0.22366 | 1.28777 | 43.304 | −159.301 | 39.7766 | ±6.83353, ±0.484552, ±1.90471*i* |
| 12 | 0.24839 | 1.20259 | 46.8809 | −205.131 | 98.1067 | Same nature as above |
| 15 | 0.27726 | 1.1228 | 52.9158 | −294.477 | 226.456 | Same nature as above |
| 20 | 0.31138 | 1.04938 | 63.9677 | −499.145 | 573.993 | Same nature as above |
| 30 | 0.35226 | 0.983404 | 87.7016 | −1122.55 | 1929.88 | Same nature as above |
| 100 | 0.42520 | 0.905583 | 263.755 | −ve | +ve | Same nature as above |
| 200 | 0.44396 | 0.891344 | 518.453 | −ve | +ve | Same nature as above |
| 1000 | 0.45995 | 0.880602 | 2560.24 | −ve | +ve | Same nature as above |
| 10000 | 0.46368 | 0.878265 | 25536.5 | −ve | +ve | Same nature as above |
| 10^6 | 0.46409 | 0.87801 | 2562980 | −1.46222 ×1012 | 5.55437 ×10^16 | Same nature as above |

Our numerical exploration shows that there exist at least one positive root and a complex root with positive real part, which induces instability in the out-of-plane EPs.

In section **3.4**, we studied the motion of the third body around out-of plane EPs, when there is a disc mass and the masses vary with time with respect to the UML. The governing equations of motion are given by the equations (3.145) and (3.147). The coordinates of the out-of plane EPs have been presented in equations (3.160) and (3.161). The first pair (equation 3.160) is denoted by $L_{6,7}$ and exist for $\kappa > 1$ while the second pair is denoted by $L_{8,9}$ and exist simultaneously, when $\kappa > 1$ and $\xi < \upsilon\ \kappa - 1$ . When $M_d = 0$, the coordinates are fully analogous with those obtained by Singh and Leke (2010).

Using the *Mathematica* software program, with some chosen parametric values for the mass of the disc and the parameter kappa, we compute numerically in Table **5.7**, the EPs of an extrasolar planet PSR B1620-26b, whose mass is too small compared to masses of the host stars. The binary system PSR B1620-26 in the constellation of Scorpius is surrounded by a disc and, composed of a pulsar PSR B1620-26 A with mass $1.35 M_\Theta$ and a white dwarf star WD B1620-26 having mass $0.34 M_\Theta$. We also compute in same Table **5.7**, the out-of-plane EPs of an extrasolar planet Kepler-16b in the circumbinary orbit, hosted by a binary star system Kepler-16, in the constellation Cygnus. The system consists of a primary Kepler-16A and a companion Kepler-16B. The primary has a mass of $0.6897 \pm 0.0035\ M_\Theta$ and the secondary's mass is $0.20255 \pm 0.00065\ M_\Theta$. Kepler-16A is a K-type main sequence star, while B is M-type red dwarf. The mass of Kepler-16b is also too small compared to the masses of the host stars. The graphs are shown in Figures **5.2**, **5.3**, **5.4** and **5.5**, for different parametric values.

**Table 5.7:** Out-of-plane equilibrium points for PSR B1620-26b and Kepler-16b in the binary systems PSR B1620-26 $\mu = 0.201183$ and Kepler-16 $\mu = 0.22701$, respectively.

| $\kappa$ | $M_d$ | $\xi$ (PSR B1620-26b) | $\pm\zeta$ | $\xi$ (Kepler-16b) | $\pm\zeta$ |
|---|---|---|---|---|---|
| 1.000001 | 0 | 0.00001 | 1.44228 | $1.4371\times10^{-43}$ | 1.45629 |
| 1.1 | | 0.00001 | 1.44228 | $1.18768\times10^{-8}$ | 1.45629 |
| 1.5 | | 0.00049 | 1.44256 | 0.00049 | 1.45655 |
| 2 | | 0.03214 | 1.45962 | 0.03174 | 1.47270 |
| 3.2 | | 1.02418 | 1.63325 | 0.95922 | 1.63817 |
| 4 | | 2.36620 | 0.51145 | 2.18377 | 0.85321 |
| 4.1 | | 2.55987 | Imaginary | 2.36019 | 0.44619 |
| 4.2 | | 2.75904 | Imaginary | 2.5416 | Imaginary |
| 1.000001 | 0.02 | $-4.17933\times10^{-48}$ | $1.05566\times10^{15}$ | $-3.52701\times10^{-48}$ | $9.60804\times10^{14}$ |
| 1.1 | | $1.85997\times10^{-8}$ | 318.050 | $2.06612\times10^{-8}$ | 289.731 |
| 1.5 | | 0.00054 | 3.97906 | 0.00055 | 3.71496 |
| 2 | | 0.03394 | 1.12254 | 0.03383 | 1.22139 |
| 3.2 | | 1.04047 | 1.61377 | 0.72738 | 0.82565 |
| 4 | | 2.38286 | 0.45645 | 1.39506 | 1.52009 |
| 4.1 | | 2.57634 | Imaginary | 2.3765 | 0.38172 |
| 4.2 | | 2.77531 | Imaginary | 2.55769 | Imaginary |
| 1.000001 | 0.04 | $-2.08964\times10^{-48}$ | $1.49293\times10^{15}$ | $-1.76348\times10^{-48}$ | $1.35878\times10^{15}$ |
| 1.1 | | $4.24811\times10^{-8}$ | 449.789 | $7.93509\times10^{-8}$ | 409.742 |
| 1.5 | | 0.00060 | 5.44996 | 0.00062 | 5.05851 |
| 2 | | 0.03595 | 0.64111 | 0.03621 | 0.90959 |
| 3.2 | | 1.05729 | 1.59763 | 0.99325 | 1.52690 |
| 4 | | 2.39975 | 0.39224 | 2.21709 | 0.79320 |
| 4.1 | | 2.59303 | Imaginary | 2.39304 | 0.30195 |
| 4.2 | | 2.79177 | Imaginary | 2.57397 | Imaginary |
| 1.000001 | 0.06 | $-1.39308\times10^{-48}$ | $1.82845\times10^{15}$ | $-1.17565\times10^{-48}$ | $1.66416\times10^{15}$ |
| 1.1 | | $1.49597\times10^{-7}$ | 550.863 | $-4.31118\times10^{-8}$ | 501.822 |
| 1.5 | | 0.00067 | 6.61298 | 0.00071 | 6.12650 |
| 2 | | 0.03822 | imaginary | 0.03894 | 0.43560 |
| 3.2 | | 1.07466 | 1.58428 | 1.01119 | 1.49170 |
| 4 | | 2.41688 | 0.31315 | 2.23414 | 0.76030 |
| 4.1 | | 2.60993 | Imaginary | 2.40981 | 0.18823 |
| 4.2 | | 2.80843 | Imaginary | 2.59047 | Imaginary |
| 1.000001 | 0.08 | $-1.04481\times10^{-48}$ | $2.11132\times10^{15}$ | $-8.81735\times10^{-49}$ | $1.92161\times10^{15}$ |
| 1.1 | | $-2.70935\times10^{-8}$ | 636.091 | $-1.69511\times10^{-8}$ | 579.455 |
| 1.5 | | 0.00076 | 7.615 | 0.00084 | 7.05148 |
| 2 | | 0.04079 | Imaginary | 0.04213 | Imaginary |
| 3.2 | | 1.09261 | 1.57323 | 1.02979 | 1.46695 |
| 4 | | 2.43425 | 0.20233 | 2.25145 | 0.72507 |
| 4.1 | | 2.62705 | Imaginary | 2.42682 | Imaginary |
| 4.2 | | 2.82529 | Imaginary | 2.60718 | Imaginary |

From Table **5.7**, real out-of-plane EPs for PSR B1620-26b, exist for $1<\kappa\leq 4$ when $0\leq M_d\leq 0.04$ , while it is not entirely so when $M_d=0.06$ and $M_d=0.08$ . There exists at least a point (when $\kappa=2$) where the points do not exist as $\zeta$ is imaginary. In the case of Kepler-16b, the same trend is noticed; however, here real out-of-plane EPs exist for $1<\kappa\leq 4.1$, when $0\leq M_d\leq 0.04$. In our numerical exercise, we observe that the out-of-plane EPs of both extra-solar planets, do not for any values of the involved parameters, reduce to a point collinear with the stars, because $\zeta$ doesn't vanish.

Fig.**5.2** and **5.3** give the graphs of out-of-plane EPs for PSR B1620-26b and Kepler-16b in the binary system PSR B1620-26 and Kepler-16; respectively, when $M_d=0$ and $1<\kappa\leq 4.2$ while Fig. **5.4** shows the variations in their locations when $M_d=0.02$. Fig. **5.5** explains the effect of the mass of the disc on the location of PSR B1620-26b .

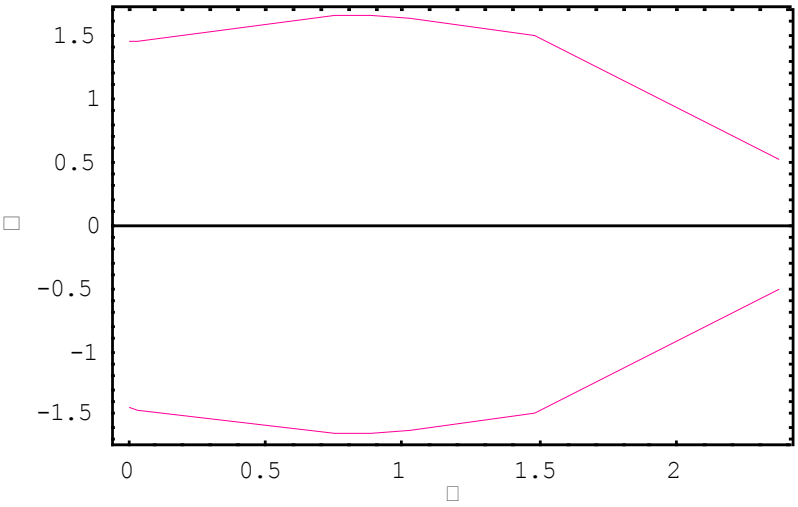



**Fig 5.2:** Out-of-plane equilibrium points for PSR B1620-26b in the binary system PSR B1620-26 when $M_d=0$ ; $1<\kappa\leq 4.2$

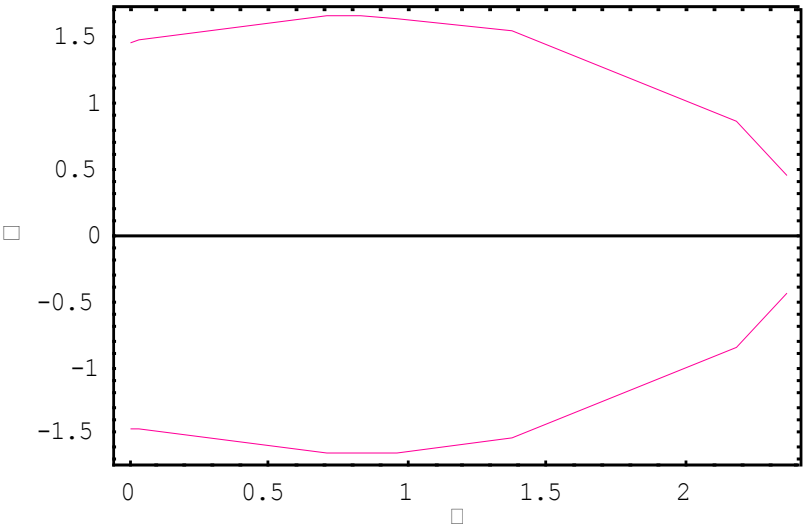



**Fig 5.3:** Out-of-plane equilibrium points for Kepler-16b in the binary system Kepler-16; when $M_d = 0$ ; $1 < \kappa \leq 4.2$

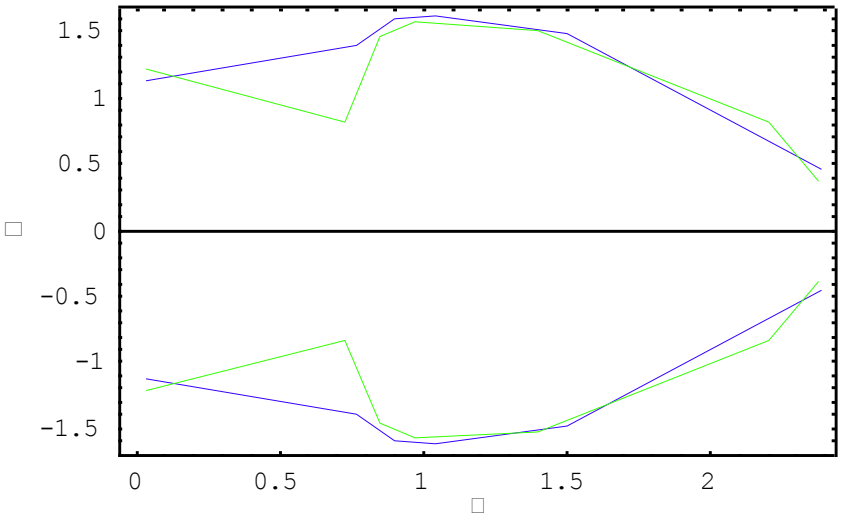



**Fig. 5.4:** Out-of-plane equilibrium points for $1 < \kappa \leq 4.2$ of PSR B1620-26b (blue) and Kepler-16b (green) in the binary systems PSR B1620-26 & Kepler-16, respectively; with $M_d = 0.02$

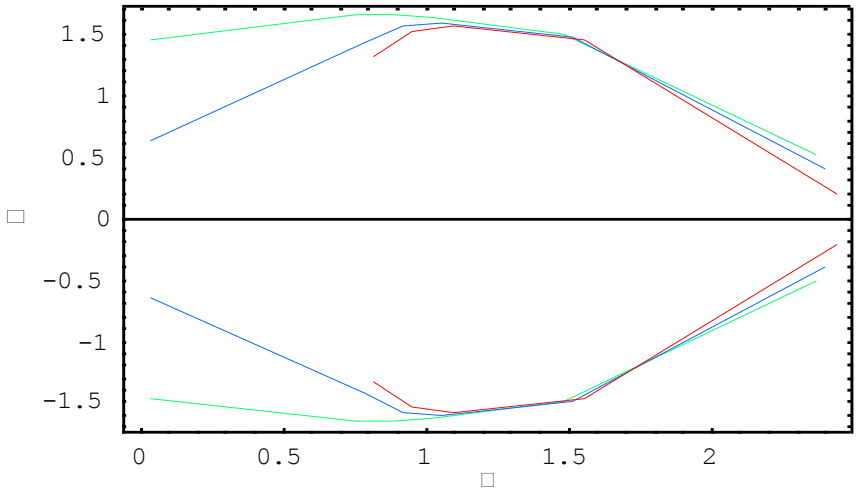



**Fig 5.5:** Out-of-plane equilibrium points for $1 < \kappa \leq 4.2$ of PSR B1620-26b in the binary system PSR B1620-26, with $M_d = 0$ (green); $M_d = 0.04$ (blue) and $M_d = 0.08$ (red).

The motion of extra-solar planets around the out-of-plane EPs, when the host stars are surrounded by disc and their masses vary isotropically in accordance with the UML has been studied. The system (3.145) of equations of motion are different from those obtained by Gelf'gat (1973), Bekov (1988), Luk'yanov (1990), and Singh and Leke (2010) due to the introduction of the mass of the circumbinary disc. Since the set of equations (3.145) are non-autonomous, we transformed them using the MT (1.46), the UML (1.47), the particular solutions and integral of the GMP. However, we also had to assume that the mass of the disc $M$ varies at the same rate as the masses of the primaries and the parameters $a$ and $b$ which determine the density profile of the disc also vary at the same rate as the distances, thus, we introduced the transformation (3.146). The autonomized systems (3.147) are different from those of Jiang and Yeh (2004, 2005, 2006), Yeh and Jiang (2006), Kushvah (2008a, 2009), Singh and Taura (2013).

The coordinates of the out-of-plane EPs are determined using series expansion about the disc mass of the system. Four out-of-plane EPs are found and the solutions are unique, even when the

initial condition is altered. For some chosen parametric values, these points do not exist and do not for any value of the parameters, reduce to points collinear with the stars. The behavior of motion of the extra-solar planets around the out-of-plane EPs is unstable due to a positive root and a complex root with positive real part, of the characteristic equation. Those for the non-autonomous are also unstable around the EPs. Results of Bekov (1988), Luk'yanov (1989, 1990), Douskos and Markellos (2006), and Singh and Leke (2010) are in agreement with our result.

**Effects of mass variations on the periodic orbits around triangular equilibrium points**

It is known that the periodic orbits play a dominant role in understanding the dynamics of a system, because they determine critically the structure of the phase space. In this way, we can detect the regions where stable points could exist. These are the regions close to the stable periodic orbits, where a real planetary system could be trapped. Many extra-solar multiplanet systems seem to move on Keplerian ellipses with periods that satisfy rational relations. Such a type of resonant motion is a consequence of the existence of periodic orbits, and, particularly, stable periodic orbits of the same resonance. Computation of periodic orbits can be used as a method to determine regular motion.

The differential equations of motion of the R3BP with variable masses are described by the equations (3.89) and (3.91), while the triangular EPs and the partial derivatives are presented in equations (3.112) and (3.134), respectively. The characteristic equation under the combined effects is given by the equation (3.165), while the frequency of the long and short period orbits have been presented in equations (3.168) and (3.169), respectively. The orientation (3.178) of the orbits depends on all the other parameter representing shape, radiation and small change in the centrifugal force, but is not affected by the small change in the Coriolis force. Also, the eccentricity, lengths of the long and short orbits are given in equations (3.188), (3.191), (3.193),

(3.194), (3.195) and (3.196), respectively. If we set $\kappa = 1$, the equations (3.165), (3.168), (3.169), (3.178), (3.188), (3.191) to equations (3.196) reduce respectively, to the equations

$$\lambda^4 + \left[ 4\varphi^2 - 3\psi - \frac{3}{2}\ 3\psi + 2\psi^{5/3}\ \ \alpha_1 - \frac{9\psi}{2}\alpha_2 + 6\varphi^2\ \ \alpha_1 + \alpha_2\ \ + 3\ \alpha_1 - \alpha_2\ \ \upsilon\psi^{5/3} \right] \lambda^2$$

$$+ \frac{3\psi^{8/3}}{4} \left[ 3\ 16 - 3\psi^{2/3}\ \ \upsilon\ 1 - \upsilon\ \ \ \alpha_1 + \alpha_2\ \ + 3\ \ 4 - \psi^{2/3}\ \ \upsilon\ 1 - \upsilon\ \ + \frac{2\upsilon\ \ 5 - 3\psi^{2/3} - \psi^{4/3}}{\psi^{2/3}}\ \ 1 - q_1 \right.$$

$$\left. + 2\upsilon\ \ 1 - q_2\ \ - 2\upsilon^2\ \ 2 - \psi^{2/3}\ \ \ 1 - q_1\ \ - 2\upsilon^2\psi^{2/3}\ \ 1 - q_2 \right] = 0 \qquad (6.1)$$

$$s_1 = \left[ \frac{\upsilon\ 1 - \upsilon}{4\varphi^2 - 3\psi} \left\{ \frac{9\ \ 4\psi^{8/3} - \psi^{10/3}}{4} + \frac{9\ \ 16\psi^{8/3} - 3\psi^{10/3}}{4}\ \ \alpha_1 + \alpha_2\ \ + \frac{3\upsilon\psi^2}{2\ \ 1 - \upsilon}\ \ 1 - q_1\ \times \right. \right.$$

$$\left. \left. \left[ 5 - 3\psi^{2/3} - \psi^{4/3} - \upsilon\ \ 2\psi^{2/3} - \psi^{4/3}\ \ \right] + \frac{3\psi^{8/3}\ \ 1 - \upsilon\psi^{2/3}}{2\ \ 1 - \upsilon}\ \ 1 - q_2\ \right\} \right]^{1/2} \qquad (5.2)$$

and

$$s_2 = \left[ 4\varphi^2 - 3\psi - \left[ \frac{9\psi + 6\psi^{5/3} - 12\varphi^2}{2} + \left\{ \frac{180\psi^{8/3} - 27\psi^{10/3} - 48\alpha^2\psi^{5/3}}{4\ \ 4\varphi^2 - 3\psi} \right\} \upsilon - \left\{ \frac{144\psi^{8/3} - 27\psi^{10/3}}{4\ \ 4\varphi^2 - 3\psi} \right\} \upsilon^2 \right] \alpha_1 \right.$$

$$- \left[ \frac{9\psi}{2} - 6\varphi^2 + \left\{ \frac{108\psi^{8/3} - 27\psi^{10/3} + 48\varphi^2\psi^{5/3}}{4\ \ 4\varphi^2 - 3\psi} \right\} \upsilon - \left\{ \frac{144\psi^{8/3} - 27\psi^{10/3}}{4\ \ 4\varphi^2 - 3\psi} \right\} \upsilon^2 \right] \alpha_2 - \frac{\upsilon\ (1 - \upsilon)}{(4\varphi^2 - 3\psi)} \times$$

$$\left. \left\{ \frac{9\ (4 - \psi^{2/3})^{8/3}}{4} + \frac{3\psi^2}{2\ (1 - \upsilon)} \left[ 5 - 3\psi^{2/3} - \psi^{4/3} - \upsilon\ (2\psi^{2/3} - \psi^{4/3}) \right] (1 - q_1)\ \frac{3\psi^{8/3}\ (1 - \upsilon\psi^{2/3})}{(1 - \upsilon)} (1 - q_2) \right\} \right]^{\frac{1}{2}} \qquad (6.3)$$

$$\tan 2\alpha = \frac{4\psi^{1/3}\sqrt{4-\psi^{2/3}}}{3\ \ 2-\psi^{2/3}}\left[\frac{3}{4}-\frac{3}{2}\upsilon+\left\{\frac{9-3\psi^{2/3}}{8-6\psi^{2/3}+\psi^{4/3}}+\frac{3\upsilon^2\psi^{2/3}}{2-\psi^{2/3}}-\frac{\upsilon\left[6+9\psi^{2/3}-3\psi^{4/3}\right]}{8-6\psi^{2/3}+\psi^{4/3}}\right\}\alpha_1\right.$$

$$+\left\{\frac{-3}{8-6\psi^{2/3}+\psi^{4/3}}-\frac{3\upsilon^2\psi^{2/3}}{2-\psi^{2/3}}+\frac{\upsilon\left[-6+15\psi^{2/3}-3\psi^{4/3}\right]}{8-6\psi^{2/3}+\psi^{4/3}}\right\}\alpha_2+\left\{\frac{-9+5\psi^{4/3}}{2\left(8-6\psi^{2/3}+\psi^{4/3}\right)\upsilon^{2/3}}\right.$$

$$\left.+\frac{\upsilon\left[28\psi^{2/3}-16\psi^{4/3}+2\psi^2\right]}{2\left(8-6\psi^{2/3}+\psi^{4/3}\right)\upsilon^{2/3}}-\frac{\upsilon^2\left[4-\psi^{2/3}\right]}{\left(2-\psi^{2/3}\right)}\right\}\left(1-q_1\right)\left\{\frac{\left[8-\psi^2-4\psi^{2/3}+5\psi^{4/3}\right]}{2\left(8-6\psi^{2/3}+\psi^{4/3}\right)\upsilon^{2/3}}-\frac{\upsilon^2\left[4-\psi^{2/3}\right]}{\left(2-\psi^{2/3}\right)}\right.$$

$$\left.\left.+\frac{\upsilon\left[24\psi^{2/3}-2\psi^2\right]}{2\left(8-6\psi^{2/3}+\psi^{4/3}\right)\upsilon^{2/3}}\right\}\left(1-q_2\right)\right] \qquad 5.4$$

$$e_1 = 1-\frac{4-\psi^{2/3}\ \ \upsilon\psi^{2/3}}{2\ \ 4\varphi^2-3\psi}+\frac{4-\psi^{2/3}\ \ \upsilon^2\psi^{2/3}}{2\ \ 4\varphi^2-3\psi}\left[1+\frac{4-\psi^{2/3}\ \ \psi^{2/3}\ \ 3\psi-2\varphi^2}{4\varphi^2-3\psi}\right]+\ A_{11}\upsilon+A_{12}\upsilon^2\ \ \alpha_1$$

$$+\ B_{11}\upsilon+B_{12}\upsilon^2\ \ \alpha_2+\ C_{11}\upsilon-C_{12}\upsilon^2\ \ 1-q_1\ -\ D_{11}\upsilon+D_{12}\upsilon^2\ \ 1-q_2 \qquad 5.5$$

where

$$A_{11}=\frac{\psi^{2/3}\ \ 4\psi^{2/3}-2-\psi^{4/3}}{4\varphi^2-3\psi},\quad B_{11}=-\frac{2\psi^{2/3}}{4\varphi^2-3\psi}$$

$$A_{12}=\frac{\psi^{2/3}}{4\varphi^2-3\psi}^{\ 2}\ 216\psi^{5/3}+9\psi^3-90\psi^{7/3}-160\varphi^2\psi^{2/3}+64\varphi^2\psi^{4/3}-6\varphi^2\psi^2+8\varphi^2-6\psi\Big]$$

$$B_{12}=\frac{\psi^{2/3}}{4\varphi^2-3\psi}^{\ 2}\ 180\psi^{5/3}-81\psi^{7/3}+9\psi^3-112\psi^{2/3}+52\varphi^2\psi^{4/3}-6\varphi^2\psi^2+8\varphi^2-6\psi\Big]$$

$$C_{11}=\frac{5-3\psi^{2/3}-\psi^{4/3}}{3\ \ 4\varphi^2-3\psi},\quad D_{11}=\frac{\psi^{2/3}}{3\ \ 4\varphi^2-3\psi}$$

$$C_{12}=\frac{1}{3\ \left(4\varphi^2-3\psi\right)^2}\Big[2\psi^{2/3}-\psi^{4/3}+60\psi-40\varphi^2+9\psi^{5/3}-6\varphi^2\psi^{2/3}-54\psi^{7/3}+36\varphi^2\psi^{4/3}-2\varphi^2\psi^{8/3}$$
$$+3\psi^{11/3}\Big]$$

$$D_{12}=\frac{\psi^{4/3}\ \left(21\psi-12\varphi^2-6\psi^{5/3}+4\varphi^2\psi^{2/3}\right)}{3\ \left(4\varphi^2-3\psi\right)^2}$$

$$e_2=\frac{\sqrt{\left(4\varphi^2\ \left(\varphi^2-1\right)+3\psi\right)}}{2\varphi^2}\Bigg[1+\frac{1}{\left[\left(4\varphi^2\ \left(\varphi^2-1\right)+3\psi\right)\right]}\Bigg\{K_1\upsilon-K_2\upsilon^2-\left(K_3+\frac{A_{21}\upsilon}{2}+\frac{A_{22}\upsilon}{2}\right)\alpha_1$$
$$-\left(K_4+\frac{B_{21}\upsilon}{2}+\frac{B_{22}\upsilon^2}{2}\right)\alpha_2-\frac{1}{2}\ \left(C_{21}\upsilon+C_{22}\upsilon^2\right)\ \left(1-q_1\right)-\frac{1}{2}\ \left(D_{21}\upsilon+D_{22}\upsilon^2\right)\ \left(1-q_2\right)\Bigg\}\Bigg] \tag{5.6}$$

where

$$K_1=\frac{3\psi^{5/3}}{8\ \left(4\varphi^2-3\psi\right)}\ \left(12\psi+192\varphi^2\psi-72\psi^2-128\varphi^4-3\psi^{5/3}+32\varphi^4\psi^{2/3}-48\varphi^2\psi^{5/3}+18\psi^{8/3}\right]$$

$$K_2=\frac{3\psi^{5/3}}{16\ \left(4\varphi^2-3\psi\right)}\ \left(24\psi-144\psi^2+384\varphi^2\psi-256\varphi^4-144\psi+72\psi^{5/3}-6\psi^{5/3}+36\psi^{8/3}\right.$$
$$\left.+64\varphi^4\psi^{2/3}-96\varphi^2\psi^{5/3}-9\psi^{7/3}\right]$$

$$A_{21}=\frac{9\psi^{5/3}}{4\ \left(4\varphi^2-3\psi\right)}\Big[64\varphi^4+48\varphi^2+6\psi^{5/3}-40\psi-8\varphi^2\psi^{2/3}-8\varphi^2\psi^{5/3}+36\psi^2+15\psi^{8/3}-96\varphi^2\psi$$
$$-16\varphi^4\psi^{2/3}+8\varphi^2\psi^{7/3}-6\psi^{10/3}\Big]$$

$$A_{22}=\frac{3\psi^{2/3}}{4\ \left(4\varphi^2-3\psi\right)}\Big[288\psi^{10/3}+120\varphi^2\psi^{5/3}-27\psi^4-192\varphi^4-693\psi^{8/3}+288\varphi^2\psi-108\psi^2-18\psi^{5/3}$$
$$+108\psi-48\varphi^2\psi^{7/3}+48\varphi^4\psi^{2/3}+24\varphi^2\psi^{2/3}-128\varphi^2\Big]$$

$$B_{21} = \frac{3\psi^{5/3}}{4\ \ 4\varphi^2 - 3\psi}\left[90\psi^{8/3} - 348\psi^2 - 72\varphi^2\psi^{5/3} - 36 + 304\varphi^2\psi + 108\psi^3 - 27\psi^{11/3}\right]$$

$$B_{22} = \frac{3\psi^{5/3}}{8\ \ 4\varphi^2 - 3\psi}\left[48\varphi^2\psi^{2/3} - 256\varphi^2 - 36\psi^{5/3} + 216\psi + 468\psi^{10/3} - 954\psi^{8/3} - 336\varphi^2\psi^{5/3} - 384\varphi^4 + 576\varphi^2\psi - 216\psi^2 - 54\psi^4 + 48\varphi^2\psi^{7/3} + 96\varphi^4\psi^{2/3}\right]$$

$$C_{21} = \frac{\psi}{2\ \ 4\varphi^2 - 3\psi}\left[27\psi^{5/3} + 9\psi^{7/3} - 45\psi + 40\varphi^2 - 24\varphi^2\psi^{2/3} - 8\varphi^2\psi^{4/3}\right]$$

$$C_{22} = \frac{1}{4\ \ 4\varphi^2 - 3\psi}^2\left[207\psi^{13/3} + 9\psi^5 - 288\psi^{11/3} + 64\varphi^4\psi^{7/3} - 128\varphi^4\psi^{5/3} + 240\varphi^2\psi^{8/3} - 120\varphi^2\psi^{10/3} - 9\psi^{17/3}\right]$$

$$D_{21} = \frac{\psi^{5/3}\ \ 8\varphi^2 - 9\psi}{2\ \ 4\varphi^2 - 3\psi}$$

$$D_{22} = \frac{\psi^{4/3}}{4\ \ 4\varphi^2 - 3\psi}^2\left[120\varphi^2\psi - 54\psi^2 - 36\psi^{5/3} - 64\varphi^4 + 9\psi^5\right]$$

$$a_1 = \frac{\sqrt{L_1}}{2\psi^{2/3}}\left[1 - \frac{2\upsilon\psi^{4/3}}{L_1} + \frac{2\upsilon^2\psi^{4/3}}{L_1} + \frac{4\varphi^2 - 3\psi}{2\upsilon L_1} + \frac{2\alpha_1 L_2\psi^{4/3}}{L_1} + \frac{2\alpha_2 L_3\psi^{4/3}}{L_1} + \frac{2L_4\psi^{4/3}\ \ 1-q_1}{L_1} + \frac{2L_5\psi^{4/3}\ \ 1-q_2}{L_1}\right]$$

where

$$L_1 = 4\varphi^2 - 3\psi + 12\psi^{5/3} - 8\varphi^2\psi^{2/3} + \psi^{4/3} - 3\psi^{7/3} + 2\varphi^2\psi^{4/3}$$

$$L_2 = \frac{1}{2\ \ 4-\psi^{2/3}\ \ \psi^{4/3}}\Big[216\psi^{5/3}+9\psi^3-90\psi^{7/3}+8\varphi^2-6\psi-3\psi^{5/3}+18\psi^{7/3}+6\psi^{7/3}-6\varphi^2\psi^2$$
$$-3\psi^3-24\psi^{5/3}+64\varphi^2\psi^{4/3}-160\varphi^2\psi^{2/3}+4\varphi^2\psi^{4/3}-8\varphi^2\psi^{4/3}-12\varphi^2\psi^{4/3}+16\varphi^2\psi^{2/3}+2\varphi^2\psi^2$$
$$+4\psi^{4/3}-\psi^2\Big]-\upsilon+\frac{4\varphi^2-3\psi\quad 5\psi^{2/3}-4-\psi^{4/3}}{2\upsilon\ \ 4-\psi^{2/3}\ \ \psi^{4/3}}$$

$$L_3 = \frac{1}{2\ \ 4-\psi^{2/3}\ \ \psi^{4/3}}\Big[159\psi^{5/3}-63\psi^{7/3}-112\psi^{2/3}+8\varphi^2\ \ +6\psi^3+40\varphi^2\psi^{4/3}-6\psi-4\varphi^2\psi^2-4\psi^{4/3}$$
$$+12\varphi^2\psi^{2/3}+\psi^2\Big]+\upsilon-\frac{4\varphi^2-3\psi}{2\upsilon\psi^{4/3}}$$

$$L_4 = \frac{1}{6\ \ 4-\psi^{2/3}\ \ \psi^{4/3}}\Big[2-\psi^{2/3}+60\psi^{1/3}-51\psi-48\varphi^2+3\psi^{5/3}-3\psi^{7/3}+66\psi-75\psi^{5/3}+3\psi^{7/3}+3\psi^3$$
$$-8\psi^{2/3}+2\psi^{4/3}+34\varphi^2-40\varphi^2\psi^{-2/3}+2\varphi^2\psi^{2/3}-2\varphi^2\psi^{4/3}-2\varphi^2\psi^{4/3}+50\varphi^2\psi^{2/3}-2\varphi^2\psi^2\Big]+\frac{2\upsilon}{3\psi^{2/3}}$$
$$-\frac{4\varphi^2-3\psi\quad 5-\psi^{2/3}-\psi^{4/3}}{6\upsilon\ \ 4-\psi^{2/3}\ \ \psi^2}$$

$$L_5 = \frac{1}{6\ \ 4-\psi^{2/3}\ \ \psi^{4/3}}\ \ 6\psi-3\psi^{5/3}+8\psi^{2/3}-2\psi^{4/3}+4\varphi^2\psi^{2/3}-8\varphi^2\Big]-\frac{2\upsilon}{3\psi^{2/3}}-\frac{4\varphi^2-3\psi}{2\upsilon\ \ 4-\psi^{2/3}\ \ \psi^{4/3}}$$

Similarly, the length of the semi-minor axis of the long period orbit gives

$$b_1 = \frac{\sqrt{4-\psi^{2/3}}}{2\psi^{2/3}}\Bigg[1+\frac{\upsilon\psi^{4/3}}{2\ \ 4\varphi^2-3\psi}-M_1\upsilon^2+\ \ M_2\upsilon+M_3\upsilon^2+M_4\ \ \alpha_1+\ \ M_5\upsilon+M_6\upsilon^2+M_7\ \ \alpha_2$$
$$+\ \ M_8Y_6\upsilon+M_9Y_7\upsilon^2-M_{10}\quad 1-q_1\ \ +\ \ M_{11}\upsilon+M_{12}\upsilon^2-M_{13}\quad 1-q_2\ \ \Bigg] \tag{5.7}$$

where

$$M_1 = \frac{\psi^{4/3}}{2\ \left(4\varphi^2 - 3\psi\right)^2}\left[-8\varphi^2\,\psi^{2/3} + 2\varphi^2\psi^{4/3} - 3\psi^{8/3} + 12\psi^{5/3} - 15\psi + 20\varphi^2\right]$$

$$M_2 = \frac{\psi^{2/3}\ \left(\psi^{2/3} - 5\psi^{4/3} + \psi^2\right)}{4 - \psi^{2/3}\ \left(4\varphi^2 - 3\psi\right)}$$

$$M_3 = \frac{\psi^{4/3}}{2\ \left(4\varphi^2 - 3\psi\right)^2}\left[132\psi - 642\psi^{5/3} + 536\varphi^2\psi^{2/3} - 176\varphi^2 + 24\psi^{8/3} - 192\varphi^2\psi^{4/3} + 288\psi^{7/3} - 6\psi^{10/3}\right.$$
$$\left. + 16\varphi^2\psi^2 - 18\psi^3\right]$$

$$M_4 = M_7 = \frac{\psi^{2/3} - 2}{4 - \psi^{2/3}}, \quad M_5 = -\frac{\psi^{4/3}\ \left(2 - \psi^{2/3}\right)}{4 - \psi^{2/3}\ \left(4\varphi^2 - 3\psi\right)}$$

$$M_6 = \frac{\psi^{4/3}}{2\ \left(4 - \psi^{2/3}\right)\ \left[4\varphi^2 - 3\psi\right]^2}\ -12\psi - 88\varphi^2\psi^{2/3} + 8\varphi^2\psi^2 - 246\psi^{2/3} - 16\varphi^2 + 114\psi^{7/3} - 12\psi^3$$
$$\left. - 72\varphi^2\psi^{4/3} + 224\psi^{2/3}\right]$$

$$M_8 = -\frac{\psi^{4/3}\ \left(3 + \psi^{2/3} + \psi^{4/3}\right)}{3\ \left(4 - \psi^{2/3}\ \left(4\varphi^2 - 3\psi\right)\right.}$$

$$M_9 = \frac{\psi^{2/3}}{3\ \left(4 - \psi^{2/3}\ \left(4\varphi^2 - 3\psi\right)\right.^2}\left[-2\psi^{4/3} + \psi^2 + 18\psi^{5/3} - 9\psi^{7/3} + 14\varphi^2\psi^{4/3} + 9\psi^3 - 6\varphi^2\psi^2 - 2\varphi^2\psi^{8/3} + 3\psi^{11/3}\right.$$
$$\left. - 34\varphi^2\psi^2 + 40\varphi^2\psi^{4/3} + 3\psi^{11/3} - 60\psi^{7/3} + 51\psi^3 - 3\psi^{11/3} + 2\varphi^2\psi^{8/3} + 2\varphi^2\psi^{10/3} - 3\psi^{13/3} + 16\varphi^2 - 12\psi\right]$$

$$M_{10} = M_{13} = \frac{-2}{3\ \left(4 - \psi^{2/3}\right.}, \quad M_{11} = \frac{\psi^{2/3}\ \left(8 - \psi^{2/3}\right.}{3\ \left(4 - \psi^{2/3}\ \left(4\varphi^2 - 3\psi\right)\right.}$$

$$M_{12} = \frac{\psi^{2/3}}{6\ \left(4-\psi^{2/3}\right)\ \left(4\varphi^2-3\psi\right)^2}\left[54\psi^{7/3}+144\psi-204\psi^{5/3}-40\varphi^2\psi^{4/3}+144\varphi^2\psi^{2/3}-192\varphi^2\right]$$

Similarly, the length of the semi-major axes for the short period orbits are

$$a_2 = \frac{\sqrt{4\ \left(4-\psi^{2/3}\right)\ \left(\varphi^4+\psi^{2/3}\left[4\varphi^2-3\psi\right]\right)}}{2\psi^{2/3}\sqrt{4\varphi^2-3\psi}}\left[1+\frac{J_1}{2J_2}\upsilon+\frac{1}{2}J_3\upsilon^2+\frac{1}{2}\ \left(J_4+J_5\upsilon+J_6\upsilon^2\right)\ \alpha_1\right.$$
$$\left.+\frac{1}{2}\ \left(J_7+J_8\upsilon+J_9\upsilon^2\right)\ \alpha_2+\frac{1}{2}\ \left(J_{10}+J_{11}\upsilon+J_{12}\upsilon^2\right)\ \left(1-q_1\right)+\frac{1}{2}W\ \left(J_{13}+J_{14}\upsilon+J_{15}\upsilon^2\right)\ \left(1-q_2\right)\right] \quad (5.8)$$

where

$$J_1 = 144\varphi^4\psi^{8/3}-72\varphi^4\psi^{10/3}+9\varphi^4\psi^4-960\varphi^4\psi^{5/3}+768\varphi^4\psi^{7/3}-96\varphi^4\psi^3+2304\varphi^6\psi^{8/3}-54\varphi^4\psi^5$$
$$-1152\varphi^6\psi^{10/3}+144\varphi^6\psi^4-864\varphi^4\psi^{11/3}+108\psi^{11/3}+432\varphi^4\psi^{13/3}-256\varphi^6\psi^{2/3}-432\varphi^2\psi^{8/3}$$

$$J_2 = -48\varphi^4\psi^{5/3}-384\varphi^4\psi+144\varphi^2\psi^2+72\varphi^2\psi^{8/3}-27\psi^{11/3}+256\varphi^6$$

$$J_3 = \frac{-6\varphi^4\psi^{2/3}}{2\kappa^{8/3}\left[16\varphi^4-24\varphi^2-6\varphi^4\psi+9-6\varphi^4\psi^2\right]\left[16\varphi^2-3\psi^{5/3}\right]}\left[512\varphi^6-1440\varphi^4-6\varphi^4\psi+864\varphi^2\psi^2\right.$$
$$-216\psi^3+3072\varphi^8\psi-18\varphi^4\psi^{10/3}+1728\varphi^4\psi^3-27\varphi^4\psi^{17/3}+32\varphi^4\psi^5-1296\varphi^4\psi^{13/3}+144\varphi^4\psi^{8/3}$$
$$\left.-1536\varphi^8\psi^{5/3}+2304\varphi^6\psi^{8/3}+192\varphi^8\psi^{7/3}-4608\varphi^6\psi^2-288\varphi^6\psi^{10/3}+108\varphi^4\psi^2+864\varphi^4\psi^{11/3}\right]$$

$$J_4 = \frac{2}{\left[4\varphi^2-3\psi\right]\left[4N\varphi^4+\psi^{2/3}\ \left(4\varphi^2-3\psi\right)\right]}\left[4\varphi^6\psi+21\varphi^4\psi+16\varphi^4\psi^{2/3}+9\psi^{8/3}-24\varphi^2\psi^{5/3}-6\varphi^4\psi^{7/3}\right.$$
$$\left.-12\varphi^4\psi+16\varphi^4\right]$$

$$J_5 = \frac{6}{\left[4\varphi^2 - 3\psi\right]^2\left[4N\varphi^4 + \psi^{2/3}\ \ 4\varphi^2 - 3\psi\right]}\Big[32\varphi^8\psi^3 + 18\varphi^4\psi^{10/3} - 192\varphi^8\psi^{7/3} + 288\varphi^6\psi^{10/3} - 3\varphi^4\psi^4$$
$$-108\varphi^4\psi^{13/3} - 24\varphi^4\psi^{8/3} + 256\varphi^8\psi^{5/3} - 48\varphi^6\psi^4 - 384\varphi^6\psi^{8/3} + 144\varphi^4\psi^{11/3} + 18\varphi^4\psi^5 + \frac{1}{3}\ 288\varphi^4\psi$$
$$+128\varphi^6 - 216\varphi^2\psi^2 + 54\psi^3\ \ - \frac{2}{3}A_{21}\varphi^4\ \ 16\varphi^2 + 3\psi^{5/3} - 12\psi - 4\varphi^2\psi^{2/3}\ \Big\}$$

$$J_6 = \frac{3N\varphi^4\psi^{5/3}}{\kappa^{8/3}\left[4\varphi^2 - 3\psi\right]^2\left[4N\varphi^4 + \psi^{2/3}\ \ 4\varphi^2 - 3\psi\right]}\ 12\psi - 72\psi^2 - 36\psi^{8/3} + 54\psi^{10/3} - 9\psi^4 + 192\varphi^4$$
$$-6\psi^{5/3} + 64\varphi^4\psi^{2/3} - 96\varphi^2\psi^{5/3} - 288\psi^{10/3} - 120\varphi^2\psi^{5/3} + 693\psi^{8/3} - 108\psi - 288\varphi^2\psi + 108\psi^2 + 27\psi^4$$
$$+48\varphi^2\psi^{7/3} + 18\psi^{5/3} + 128\varphi^2 - 48\varphi^4\psi^{2/3} - 24\varphi^2\psi^{2/3}\Big]$$

$$J_7 = \frac{2}{\left[4\varphi^2 - 3\psi\right]\left[4N\varphi^4 + \psi^{2/3}\ \ 4\varphi^2 - 3\psi\right]}\Big[4\varphi^6\psi^{2/3} - 3\varphi^4\psi^{5/3} - 12\varphi^4\psi - 16\varphi^4\psi^{2/3} + 24\varphi^2\psi^{5/3}$$
$$+16\varphi^4 - 9\psi^{8/3}\Big]$$

$$J_8 = \frac{\psi^{5/3}}{\kappa^{4/3}\left[4\varphi^2 - 3\psi\right]^2\left[4N\varphi^4 + \psi^{2/3}\ \ 4\varphi^2 - 3\psi\right]}\Big[688\varphi^4\kappa^{4/3} + 1728\varphi^6\psi^{8/3} - 2016\varphi^4\psi^{8/3} - 1152\varphi^8\psi^{5/3}$$
$$+4032\varphi^4\psi^2 - 576\varphi^4\psi - 432\varphi^4\psi^3 + 1536\varphi^8\psi - 2304\varphi^6\psi^2 - 3648\varphi^6\psi + 1776\varphi^6\psi^{5/3} - 288\varphi^6\psi^{10/3}$$
$$+252\varphi^4\psi^{10/3} + 192\varphi^8\psi^{7/3} + 27\varphi^4\psi^{13/3} - 108\varphi^4\psi^{2/3} - 216\varphi^6\psi^{7/3} + 432\varphi^2\psi^2 - 108\psi^3\Big]$$
$$J_9 = \frac{-3N\varphi^4\psi^{5/3}}{2\left[4\varphi^2 - 3\psi\right]^2\left[4N\varphi^4 + \psi^{2/3}\ \ 4\varphi^2 - 3\psi\right]}\ 192\psi + 48\varphi^2\psi^{2/3} + 48\varphi^2\psi^{7/3} - 144\varphi^2\psi^{5/3} - 72\psi^2$$
$$+360\psi^{10/3} - 882\psi^{8/3} + 192\varphi^2\psi - 24\psi^{5/3} - 36\psi^4 - 32\varphi^4\psi^{2/3} - 256\varphi^2\Big]$$

$$J_{10} = -\frac{4\left[4\varphi^4 + 4\varphi^2 - 3\psi\right]}{3\left[4N\varphi^4 + \psi^{2/3}\ \ 4\varphi^2 - 3\psi\right]}$$

$$J_{11}=\frac{2}{3\left[4\varphi^2-3\psi\right]^2\left[4N\varphi^4+\psi^{2/3}\ \ 4\varphi^2-3\psi\right]}\Big[27\varphi^4\psi^4+408\varphi^6\psi^{5/3}+540\varphi^4\psi^2+432\varphi^2\psi^2-576\varphi^4\psi$$
$$+531\varphi^4\psi^{8/3}-9\varphi^4\psi^{10/3}+96\varphi^6\psi^{7/3}+432\varphi^4\psi^{11/3}-1152\varphi^6\psi^{8/3}+7688\varphi^8\psi^{5/3}-72\varphi^6\psi^{7/3}-24\varphi^6\psi^3$$
$$+288\varphi^6\psi^{10/3}-192\varphi^8\psi^{7/3}-108\varphi^4\psi^{13/3}-108\psi^3-224\varphi^6\Big]$$

$$J_{12}=\frac{N\varphi^4\psi^{5/3}}{\left[4\varphi^2-3\psi\right]^3\left[4N\varphi^4+\psi^{2/3}\ \ 4\varphi^2-3\psi\right]}\Big[252\kappa^2\psi^2-207\psi^{8/3}-512\varphi^6+128\varphi^4-64\varphi^4\psi^{2/3}$$
$$+120\varphi^2\psi^{5/3}+216\psi^3-288\varphi^2\psi^{8/3}+72\varphi^2\psi^{10/3}-192\varphi^2\psi+1152\varphi^4\psi-864\varphi^2\psi^2+216\psi^{11/3}-9\psi^{10/3}$$
$$-54\psi^{13/3}+9\psi^4\Big]$$

$$J_{13}=\frac{4\left[4\varphi^2\ \ 1-\varphi^2\ \ -3\psi\right]}{3\left[4N\varphi^4+\psi^{2/3}\ \ 4\varphi^2-3\psi\right]}$$

$$J_{14}=\frac{2}{3\left[4\varphi^2-3\psi\right]^2\left[4N\varphi^4+\psi^{2/3}\ \ 4\varphi^2-3\psi\right]}\Big[24\varphi^6\psi^{7/3}-96\varphi^6\psi^{5/3}+36\varphi^4\psi^{8/3}+768\varphi^8\psi^{5/3}$$
$$+288\varphi^6\psi^{10/3}-1152\varphi^6\psi^{8/3}+432\varphi^4\psi^{11/3}-9\varphi^4\psi^{10/3}-192\varphi^8\psi^{7/3}-108\varphi^4\psi^{11/3}+576\varphi^4\psi+108\psi^3$$
$$-432\varphi^2\psi^2-256\varphi^6\Big]$$

$$J_{15}=\frac{N\varphi^4\psi^{5/3}}{4\varphi^2-3\psi\ \ ^3\left[4N\varphi^4+\psi^{2/3}\ \ 4\varphi^2-3\psi\right]}\ 192\varphi^4\psi-2048\varphi^8+6144\varphi^6\psi-1152\varphi^4\psi^{8/3}$$
$$+288\varphi^4\psi^{10/3}-288\varphi^2\psi^2-6912\varphi^4\psi^2+3456\varphi^2\psi^3+108\psi^3+1728\varphi^2\psi^{11/3}-432\varphi^2\psi^{13/3}-648\psi^4$$
$$-648\psi^{14/3}+162\psi^{16/3}-120\varphi^2\psi^{5/3}+54\psi^{8/3}+64\varphi^4\psi^{2/3}+36\psi^5-9\psi^{17/3}$$

Lastly, for the length of the semi-minor axes of the short period orbits, we get

$$b_2 = \frac{\sqrt{A^*}}{4\varphi^2\psi^{1/3}}\left[1-\frac{N_1\psi^{2/3}\upsilon}{8A^*\ \ 4\varphi^2-3\psi}+\frac{N_2\psi^{2/3}\upsilon^2}{16A^*\ \ 4\varphi^2-3\psi}+\frac{\psi^{2/3}}{4A^*}\left\{\psi\ \ 4\varphi^2-6\psi^{2/3}\ \ +\frac{N_3\upsilon}{4\varphi^2-3\psi}\right.\right.$$
$$\left.+\frac{N_4\upsilon^2}{2\ \ 4\varphi^2-3\psi}\right\}\alpha_1+\frac{\psi^{2/3}}{4A^*}\left\{9\ \ 3\psi-4\varphi^2\ \ +\frac{16\varphi^4\ \ \psi^{2/3}-1}{\psi^{2/3}}+\frac{N_5\upsilon}{4\varphi^2-3\psi}+\frac{N_4\upsilon^2}{2\ \ 4\varphi^2-3\psi}\right\}\alpha_2 \qquad (5.9)$$
$$+\frac{1}{3\psi^{2/3}A^*}\ \ 2\ \ 3\psi-4\varphi^2-4\varphi^4\psi^{2/3}\Big]+\frac{N_6\upsilon}{2\left(4\varphi^2-3\psi\right)}+\frac{3\psi^{2/3}N_7\upsilon^2}{4\left(4\varphi^2-3\psi\right)}\Big\}\left(-q_1\right)$$
$$+\ \ 2\Big\{\Big[4\varphi^2-3\psi-4\varphi^4\psi^{2/3}\Big]+\frac{N_8\upsilon}{2\left(4\varphi^2-3\psi\right)}+\frac{3\psi^{2/3}N_9\upsilon^2}{4\left(4\varphi^2-3\psi\right)}\Big\}\left(-q_2\right)\Big]$$

where

$$N_1 = 36\psi^{8/3}-384\varphi^4\psi^{5/3}+576\varphi^2\psi^{8/3}+256\varphi^4-216\psi^{11/3}-9\psi^{10/3}+96\varphi^4\psi^{7/3}-144\varphi^2\psi^{10/3}$$
$$+54\psi^{13/3}-384\varphi^2\psi+144\psi^2$$

$$N_2 = 360\kappa^2\psi^{8/3}-3840\varphi^4\psi^{5/3}+5760\varphi^2\psi^{8/3}+512\varphi^4-2160\psi^{11/3}+108\psi^{13/3}+216\psi^5-90\psi^{10/3}$$
$$+960\varphi^4\psi^{7/3}-1440\varphi^2\psi^{10/3}-27\psi^{17/3}-768\varphi^2\psi+288\psi^2$$

$$N_3 = 672\varphi^4\psi^{5/3}-288\psi^{8/3}-1008\varphi^2\psi^{8/3}+64\varphi^4+378\psi^{11/3}-168\varphi^4\psi^{7/3}+36\psi^{10/3}+108\varphi^2\psi^{10/3}$$
$$+\frac{27}{2}\psi^{13/3}-96\varphi^2\psi+312\varphi^2\psi^{5/3}+36\psi-36\varphi^4\psi^{7/3}+36\varphi^2\psi^4-27\psi^5$$

$$N_4 = 936\varphi^2\psi^{7/3}-4032\varphi^2\psi^{5/3}+1728\varphi^4\psi^{5/3}-384\varphi^4-1080\psi^{10/3}+4284\psi^{8/3}+486\psi^5-2592\varphi^2\psi^{8/3}$$
$$+360\varphi^2\psi^{10/3}-342\psi^{11/3}-54\psi^{17/3}-999\psi^{13/3}+72\varphi^2\psi^4-432\varphi^4\psi^{7/3}+432\psi^{2/3}+576\varphi^2\psi-216\psi^2$$

$$N_5 = 320\varphi^4-486\psi^{8/3}-384\varphi^4\psi^{5/3}+576\varphi^2\psi^{8/3}-54\psi^{11/3}+126\psi^{10/3}+96\varphi^4\psi^{7/3}-108\varphi^2\psi^{7/3}$$
$$144\varphi^2\psi^{10/3}+\frac{27}{2}\psi^{13/3}-480\varphi^2\psi+180\psi^2-54\psi^{2/3}+456\varphi^2\psi^{5/3}$$

$$N_6 = \frac{63}{2}\psi^{10/3} - \frac{63}{2}\psi^{8/3} + 60\varphi^2\psi^{5/3} + 96\varphi^4\psi^{7/3} - 36\varphi^2\psi^{7/3} - 192\varphi^2\psi^{8/3} + 576\varphi^2\psi^{8/3} - 144\varphi^2\psi^{10/3}$$
$$-384\varphi^4\psi^{5/3} - 216\psi^{11/3} + 54\psi^{13/3} + \frac{27}{2}\psi^4 - 12\varphi^2\psi^3 + 72\psi^2 + 128\varphi^4$$

$$N_7 = \frac{1}{2\left[4\varphi^2 - 3\psi\right]}\left[207\psi^{13/3} + 9\psi^5 - 288\psi^{11/3} + 64\varphi^4\psi^{7/3} - 128\varphi^4\psi^{5/3} + 240\varphi^2\psi^{8/3} - 9\psi^{17/3}\right.$$
$$\left.-120\varphi^2\psi^{10/3}\right] - 90\psi^{8/3} + 108\psi^2 - 160\varphi^2\psi + 96\varphi^2\psi^{5/3} + 768\varphi^4\psi - 1152\varphi^2\psi^2 + 432\psi^3$$
$$-192\varphi^4\psi^{5/3} + 288\varphi^2\psi^{8/3} + 36\psi^{11/3} - 72\psi^{13/3} + 9\psi^5 - 36\psi^{10/3} + 32\varphi^2\psi^{7/3}$$

$$N_8 = 12\varphi^2\psi^{7/3} - 96\varphi^4\psi^{7/3} - 54\psi^{13/3} + 144\varphi^2\psi^{10/3} + 192\varphi^2\psi - 72\psi^2 - 36\psi^{8/3} - \frac{9}{2}\psi^{10/3}$$
$$+384\varphi^4\psi^{5/3} - 576\varphi^2\psi^{8/3} + 216\psi^{11/3} - 128\varphi^4$$

$$N_9 = \frac{\psi^{7/3}}{2\left[4\varphi^2 - 3\psi\right]}\left[120\varphi^2\psi - 54\psi^2 + 9\psi^5 - 64\varphi^4 - 36\psi^{13/3}\right] + 18\psi^{8/3} - 32\varphi^2\psi^{5/3} - 768\varphi^4\psi$$
$$\left.+192\varphi^4\psi^{5/3} + 72\psi^2 - 432\psi^3 + 1152\varphi^2\psi^2 + 72\psi^{13/3} - 288\varphi^2\psi^{8/3} - 36\psi^{11/3} - 9\psi^5\right]$$

$$A^* = 4\ \ 4 - \psi^{2/3}\ \ \varphi^4 + \psi^{2/3}\ \ 4\varphi^2 - 3\psi$$

Equations (5.1 to 5.9) represent the case when the masses of the primaries are constant. These equations should have been fully analogous to those in the study done by AbdulRaheem and Singh (2008) but they are different because we haven't ignored products of the parameters in the Coriolis and centrifugal forces with those very small parameters of oblateness and radiation of the primaries. Some previous studies of the periodic orbits such as Singh and Begha (2011b), AbdulRaheem and Singh (2008), Perdios and Kalantonis (2006), Perdios (2003), Poddar (2002 a, b), Elipe and Lara (1997), Sharma (1976), Meyer and Schmidt (1971) and Szebehely (1967a), have some similarity with ours.

Now, we treat the numerical studies of the period orbits around triangular EPs when isotropic mass variations of the primaries and the gravitational force are the governing forces. In this case, the equations (3.112), (3.165), (3.168), (3.169), (3.178), (3.188), (3.191) to equations (3.196); which represent the triangular EPs, characteristic equation, frequency of long and short orbits, orientation, eccentricity of long and short period orbits, and the lengths of the semi-axes, of the autonomized system when both primaries vary their masses, are oblate spheroid and radiating under small perturbations in the Coriolis and centrifugal forces, reduce respectively, to the equations:

$$\xi = \frac{1}{2} - \upsilon\,,\ \eta = \pm\frac{\sqrt{3}}{2} \tag{5.10}$$

$$\lambda^4 - (3\kappa - 4)\lambda^2 + \frac{27}{4}\kappa^2\upsilon(1-\upsilon) = 0 \tag{5.11}$$

$$s_1 = \frac{3\kappa}{2}\sqrt{\frac{3\upsilon\ 1-\upsilon}{3\kappa-4}}\,, \qquad s_2 = \left[3\kappa - 4 - \frac{27\kappa^2\upsilon\ 1-\upsilon}{4\ 3\kappa-4}\right]^{1/2} \tag{5.12}$$

$$\tan 2\alpha = \sqrt{3}(1-2\upsilon) \tag{5.13}$$

$$e_1 = 1 - \frac{3\upsilon}{2(4-3\kappa)} - \frac{3(1-3\kappa)\upsilon^2}{(4-3\kappa)^2} \tag{5.14}$$

$$e_2 = \frac{\sqrt{3\kappa}}{32(4-3\kappa)}\left[4 - 48\kappa - 6(8\kappa^2 - 51\kappa + 32)\upsilon + 3(63\kappa^2 - 102\kappa + 64)\upsilon^2\right] \tag{5.15}$$

$$a_1 = \frac{\sqrt{(6\kappa-1)}}{2}\left[1 - \frac{2\upsilon(1-\upsilon)}{(6\kappa-1)} + \frac{4-3\kappa}{2(6\kappa-1)\upsilon}\right] \tag{5.16}$$

$$b_1 = \frac{\sqrt{3}}{2\kappa^{1/3}}\left[1+\frac{\kappa^{2/3}\upsilon}{2(4-3\kappa)}+\frac{\kappa^{2/3}(8\kappa^{4/3}+3\kappa-14)\upsilon^2}{2(4-3\kappa)^2}\right] \tag{5.17}$$

$$a_2 = \frac{1}{2}\sqrt{\frac{16-3\kappa}{4-3\kappa}}\left[1+\frac{\kappa^{4/3}(378\kappa^3-945\kappa^2+288\kappa+256)\upsilon}{2(27\kappa^3-216\kappa^2+432\kappa-256)}\right.$$
$$\left.+\frac{(1377\kappa^3-1602\kappa^2-1440\kappa+108\kappa^{2/3}+1728\kappa^{1/3}+512)\upsilon^2}{4(16\kappa^2-27\kappa^3-432\kappa+256)}\right] \tag{5.18}$$

$$b_2 = \frac{1}{64\sqrt{16-3\kappa}\,(4-3\kappa)}\left[16\,(9\kappa^2-60\kappa+64)+2\kappa^{2/3}(162\kappa^3-603\kappa^2+672\kappa-216)\upsilon-(1863\kappa^3\right.$$
$$\left.-4878\kappa^2+3648\kappa-512)\upsilon^2\right] \tag{5.19}$$

Below we have numerically given the eccentricities and the semi-axes of the long and short period orbits using equations (5.14), (5.15) and equations (5.16 to 5.19), respectively, in Table **5.8**. From the numerical values obtained, due to the variational parameters, we observe that, the amount (eccentricity) by which the orbits of the third body around the triangular EPs deviates from a perfect circle, is greater than zero. Hence, the orbits could be elliptical $(0<e_i<1)$, parabolic $(e_i=1)$ or hyperbolic $(e_i>1)$. The case of high eccentricities occurs when small bodies orbits very large ones. Hence, the eccentricities of smaller bodies are quite high.

**Table 5.8:** Effects of $\kappa$ on the eccentricities and lengths of the semi-axes for $0 < \mu \leq 0.5$

| $\kappa$ | $\upsilon$ | $e_1$ | $e_2$ | $a_1$ | $b_1$ | $a_2$ | $b_2$ |
|---|---|---|---|---|---|---|---|
| 0.0001 | 0.0001 | 0.99996 | 0.00865 | -14144.6i | 18.658 | 1.00003 | 1.99998 |
| | 0.01 | 0.99623 | 0.00840 | -140.732i | 18.658 | 1.00009 | 2.00006 |
| | 0.1 | 0.96062 | 0.00632 | -13.3111i | 18.6583 | 1.00581 | 2.00979 |
| | 0.2 | 0.91749 | 0.00450 | -6.13942i | 18.6583 | 1.02317 | 2.03959 |
| | 0.3 | 0.87061 | 0.00320 | -3.71114i | 18.6579 | 1.05209 | 2.08938 |
| | 0.4 | 0.81999 | 0.00242 | -2.48992i | 18.6571 | 1.09259 | 2.15915 |
| | 0.5 | 0.76561 | 0.00216 | -1.76851i | 18.6561 | 1.14466 | 2.24892 |
| 0.15 | 0.0001 | 0.99995 | 0.33532 | -39690i | 1.62993 | 1.04645 | 1.97166 |
| | 0.01 | 0.99576 | 0.32672 | -396.634i | 1.63055 | 1.04602 | 1.97007 |
| | 0.1 | 0.95643 | 0.25658 | -39.0641i | 1.63397 | 1.05671 | 1.95704 |
| | 0.2 | 0.91025 | 0.19550 | -18.906 i | 1.63316 | 1.09936 | 1.94545 |
| | 0.3 | 0.86145 | 0.15219 | -2.0673i | 1.62749 | 1.17440 | 1.93691 |
| | 0.4 | 0.81003 | 0.12663 | -8.62563i | 1.61697 | 1.28183 | 1.93141 |
| | 0.5 | 0.75600 | 0.11882 | -6.5964 i | 1.60158 | 1.42166 | 1.92897 |
| 0.17 | 0.0001 | 0.99995 | 0.35698 | 87250.1 | 1.56333 | 1.05337 | 1.96785 |
| | 0.01 | 0.99569 | 0.34801 | 872.501 | 1.56398 | 1.05281 | 1.96622 |
| | 0.1 | 0.95581 | 0.27486 | 86.4500 | 1.56759 | 1.06303 | 1.95187 |
| | 0.2 | 0.90921 | 0.21125 | 42.125 | 1.56666 | 1.10669 | 1.93696 |
| | 0.3 | 0.86019 | 0.16625 | 27.0833 | 1.56053 | 1.18435 | 1.92312 |
| | 0.4 | 0.80877 | 0.13985 | 19.5125 | 1.54919 | 1.29601 | 1.91035 |
| | 0.5 | 0.75492 | 0.13204 | 15.0500 | 1.53266 | 1.44168 | 1.89867 |
| 0.5 | 0.0001 | 0.99994 | 0.61227 | 6251. | 1.09114 | 1.20410 | 1.90394 |
| | 0.01 | 0.99402 | 0.60240 | 63.4901 | 1.09244 | 1.19893 | 1.90340 |
| | 0.1 | 0.94240 | 0.52453 | 7.1600 | 1.09865 | 1.18185 | 1.88892 |
| | 0.2 | 0.88960 | 0.46310 | 3.9650 | 1.09374 | 1.22619 | 1.85253 |
| | 0.3 | 0.84160 | 0.42808 | 2.8733 | 1.07639 | 1.33718 | 1.79479 |
| | 0.4 | 0.79840 | 0.41947 | 2.3225 | 1.04661 | 1.51482 | 1.71568 |
| | 0.5 | 0.76000 | 0.43727 | 2. | 1.00439 | 1.75910 | 1.61521 |
| 0.9 | 0.0001 | 0.99988 | 0.82156 | 2192.63 | 0.89701 | 1.59920 | 1.82348 |
| | 0.01 | 0.98876 | 0.82024 | 23.3881 | 0.89998 | 1.59268 | 1.82561 |
| | 0.1 | 0.91479 | 0.83299 | 3.61371 | 0.90763 | 1.64399 | 1.83441 |
| | 0.2 | 0.88994 | 0.89946 | 2.47094 | 0.87527 | 1.93462 | 1.82188 |
| | 0.3 | 0.92544 | 1.02098 | 2.07204 | 0.79991 | 2.47117 | 1.78586 |
| | 0.4 | 1.02130 | 1.19755 | 1.86922 | 0.68153 | 3.25364 | 1.72636 |
| | 0.5 | 1.17751 | 1.42918 | 1.75292 | 0.52015 | 4.28203 | 1.64337 |
| 1 | 0.0001 | 0.99985 | 0.86605 | 1582.72 | 0.86606 | 1.80294 | 1.80280 |
| | 0.01 | 0.98560 | 0.86967 | 17.3863 | 0.87000 | 1.82109 | 1.80527 |
| | 0.1 | 0.91000 | 0.93909 | 3.10536 | 0.87468 | 2.19904 | 1.81829 |
| | 0.2 | 0.94000 | 1.09336 | 2.27052 | 0.81406 | 3.06888 | 1.81283 |
| | 0.3 | 1.09000 | 1.32881 | 1.97537 | 0.68416 | 4.41229 | 1.78639 |
| | 0.4 | 1.36000 | 1.64545 | 1.82463 | 0.48497 | 6.22928 | 1.73899 |
| | 0.5 | 1.75000 | 2.04328 | 1.73925 | 0.21650 | 8.51985 | 1.67060 |
| 1.33 | 0.0001 | 0.98589 | 1.01370 | 15.2503 | 0.79198 | 168.537 | 1.73108 |
| | 0.01 | 8.47000 | 2.56755 | 1.99668 | -1.41353 | 15507.3 | 1.55504 |
| | 0.1 | 883 | 23.3913 | 1.83336 | -262.172 | 188468. | -0.80826 |
| | 0.2 | 3559 | 60.6830 | 1.78920 | -1060.57 | 451478. | -5.04652 |
| | 0.3 | 8029 | 112.874 | 1.76020 | -2394.42 | 789045. | -10.982 |
| | 0.4 | 14293 | 179.964 | 1.74303 | -4263.71 | 1.20117*10^6 | -18.6147 |
| | 0.5 | 35689 | 1267.10 | 1.43210 | -57123.2 | 1.35781*10^8 | -29.231 |

In Fig **5.6**, **5.7** and **5.8**, we have drawn the semi-axes for values of kappa and the mass ratio.

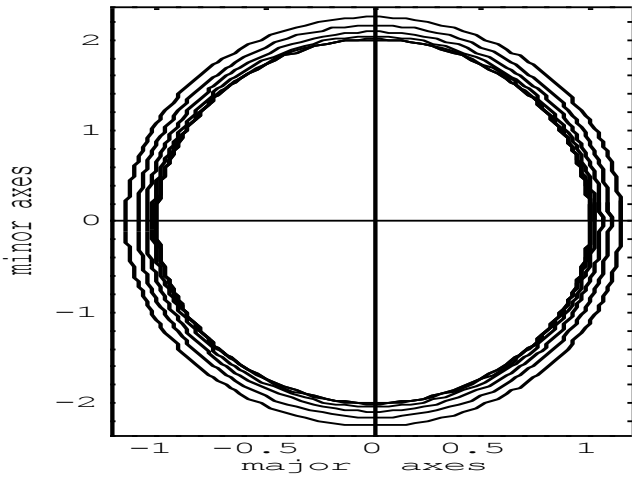



**Fig 5.6:** Lengths of the Semi axes for $\kappa = 0.0001$ and $0.0001 \leq \upsilon \leq 0.5$,

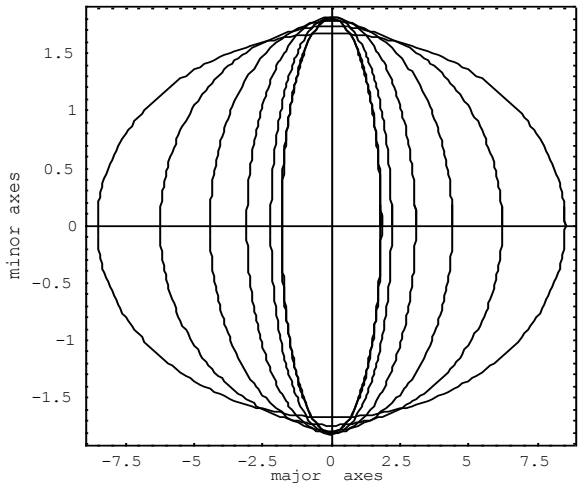



**Fig 5.7:** Lengths of the Semi axes for $\kappa = 1$ and $0.0001 \leq \upsilon \leq 0.5$

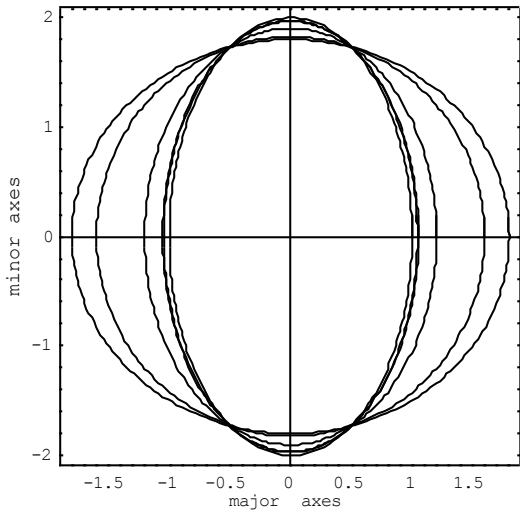
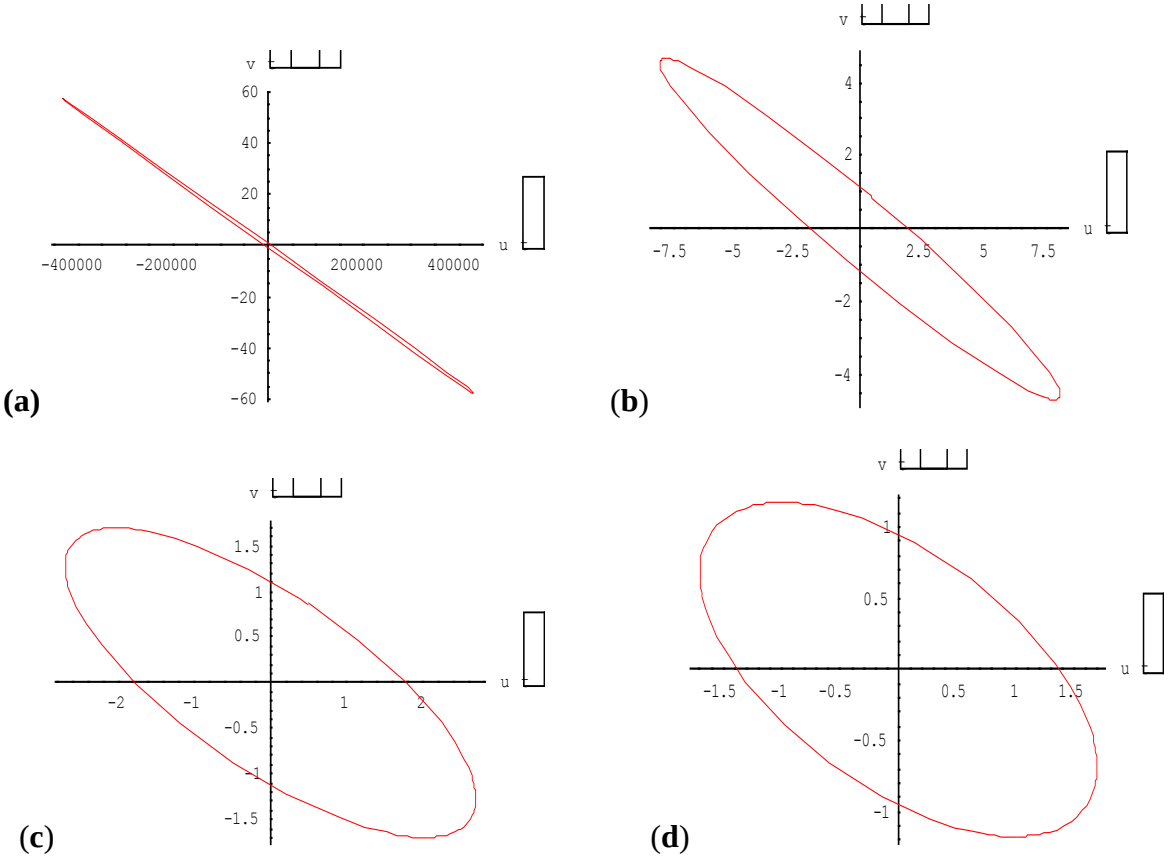
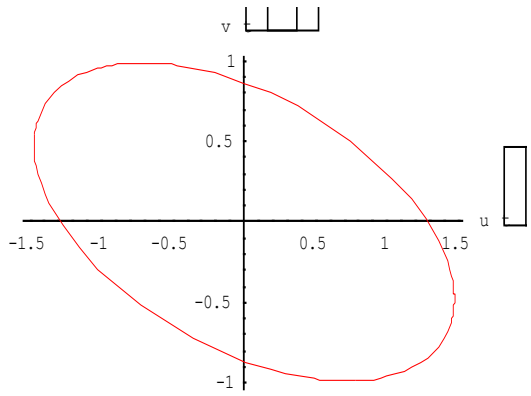


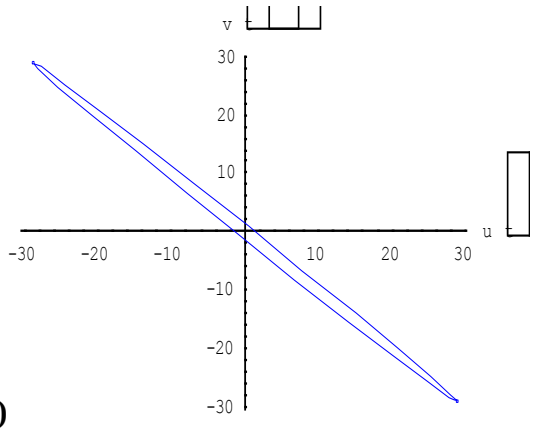

**Fig 5.8:** Lengths of the semi-axes for $\upsilon = 0.001$ and $0 < \kappa \leq 1.130.000$



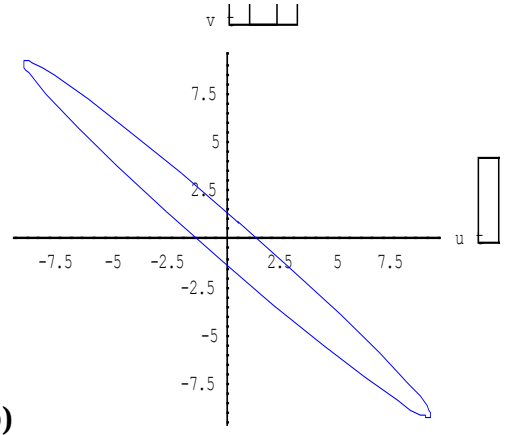



**(e)**

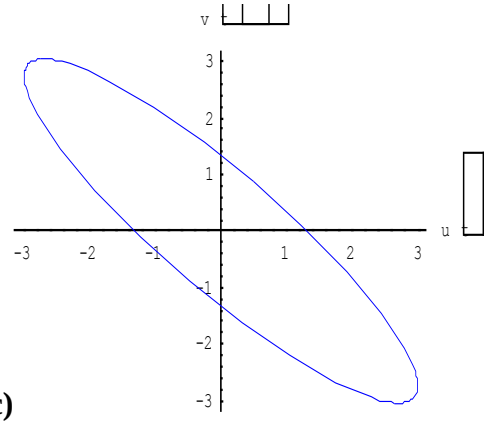
**Fig 5.9:** Periodic orbits around $L_4$ points when (a) $\kappa = 0.0001, \upsilon = 0.0001$ (b) $\kappa = 0.5, \upsilon = 0.01$ (c) $\kappa = 0.5, \upsilon = 0.1$ **(d)** $\kappa = 0.5, \upsilon = 0.3$ and **(e)** $\kappa = 0.5, \upsilon = 0.5$



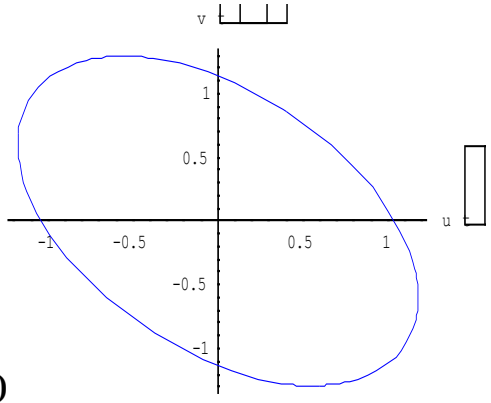

**(a)**




**(b)**




**(c)**




**(d)**

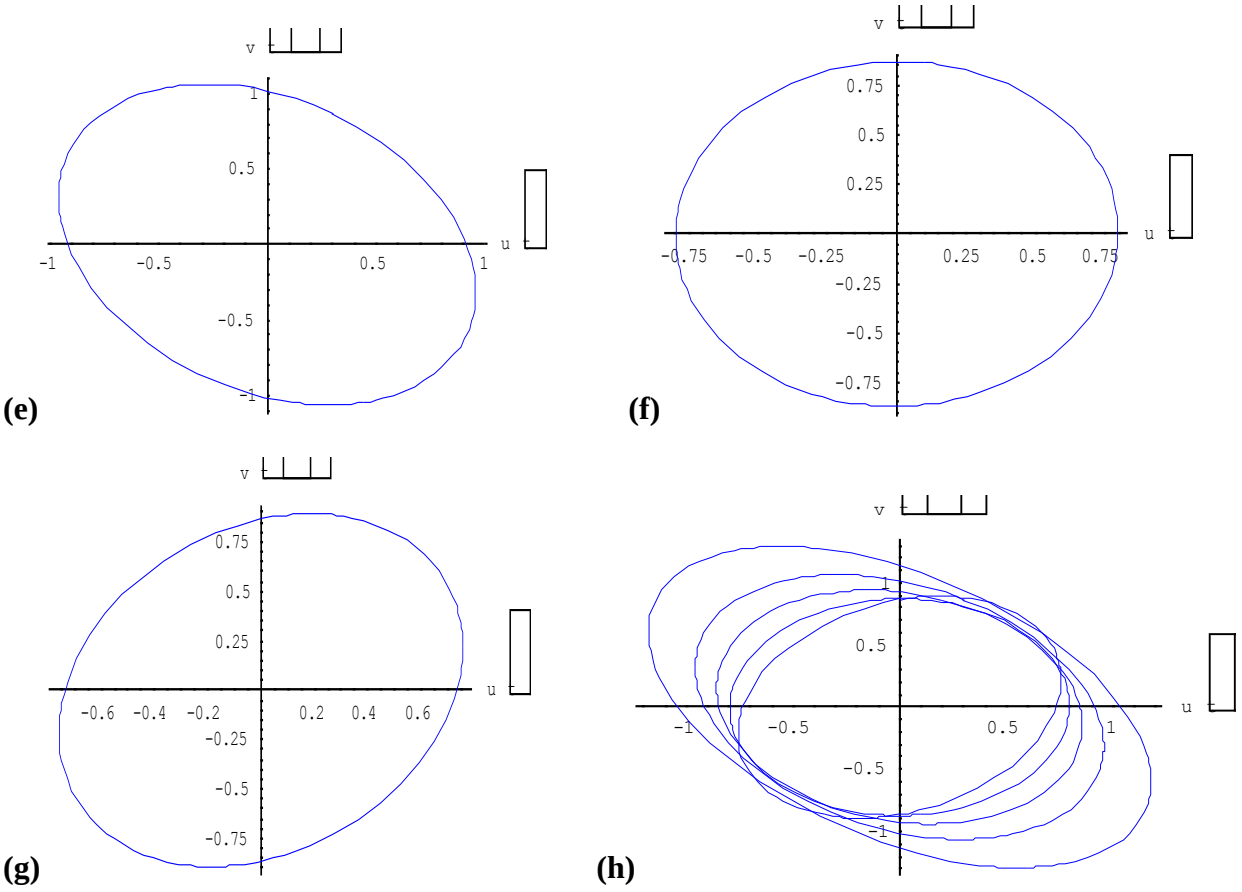



**Fig 5.10:** Effects of the mass parameter on the orbits around $L_4$ points when (**a**) $\upsilon = 0.0001$ (**b**). $\upsilon = 0.01$(**c**). $\upsilon = 0.1$, (**d**). $\upsilon = 0.2$, (**e**) $\upsilon = 0.3$, (**f**). $\upsilon = 0.4$ (**g**) $\upsilon = 0.5$(**h**) $0.0001 \leq \upsilon \leq 0.5$

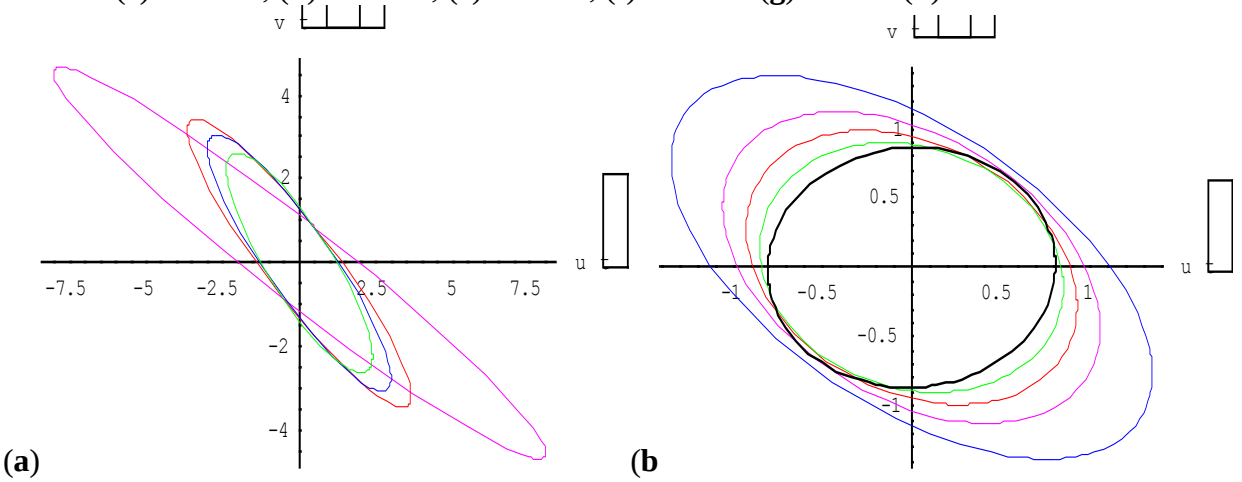



**Fig 5.11:** Orbits around triangular points $L_4$ for **(a)** $\upsilon = 0.01$ and $\kappa = 0.5$ (*purple*), $\kappa = 0.9$ (*red*), $\kappa = 1$ (*blue*) and $\kappa = 1.1$ (*green*) respectively. **(b)**. $\kappa = 0.9$ with $\upsilon = 0.1$**(***blue orbit***);** $\upsilon = 0.2$ (*purple orbit***);** $\upsilon = 0.3$ (*red orbit***);** $\upsilon = 0.4$ (*green orbit***)** and $\upsilon = 0.5$ **(***black orbit***)**

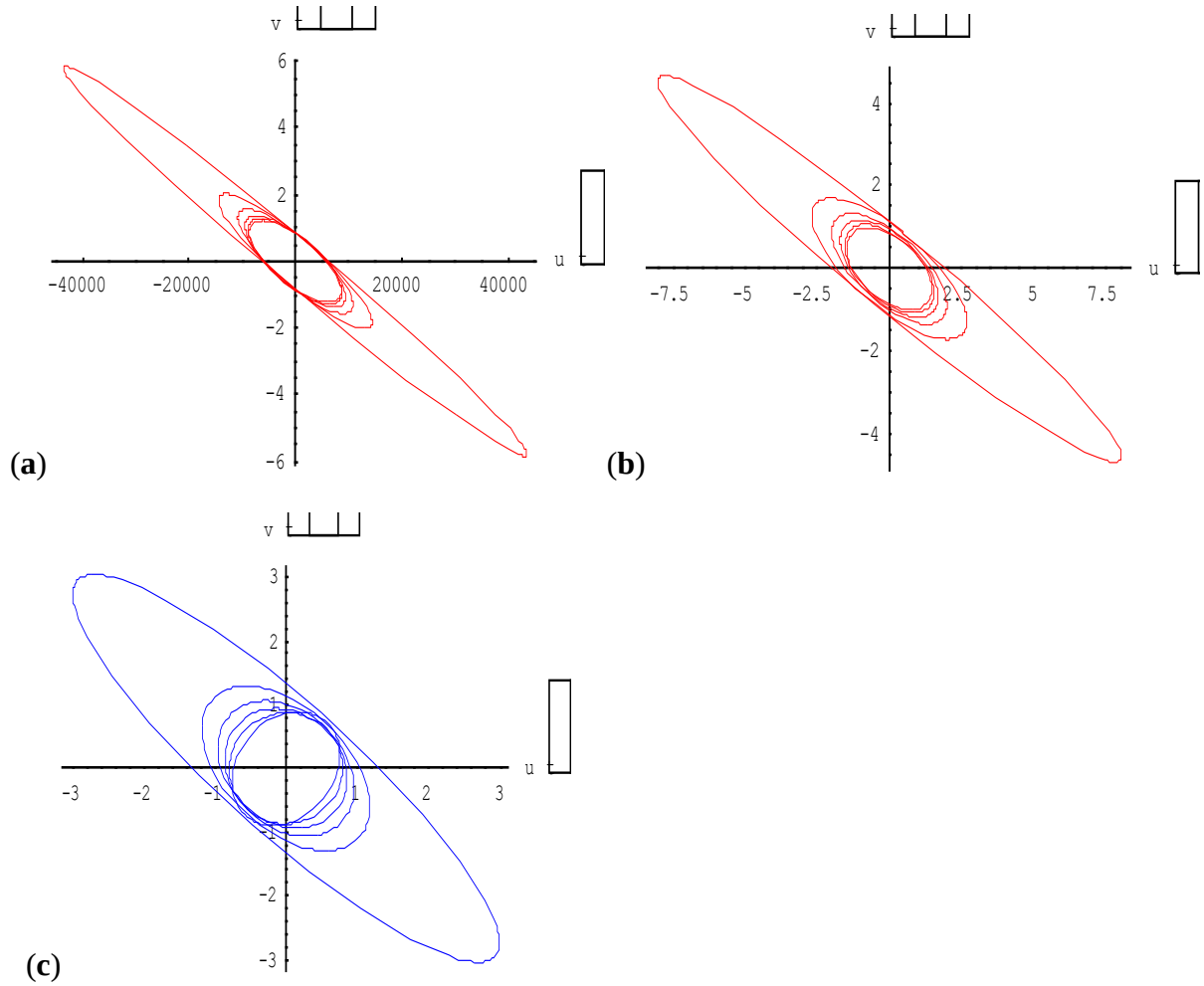
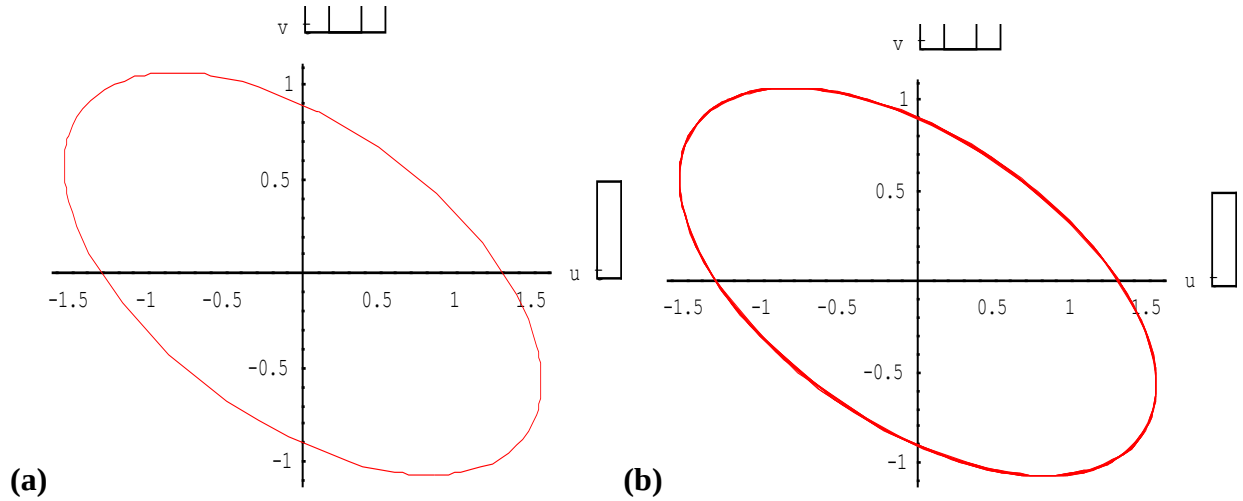
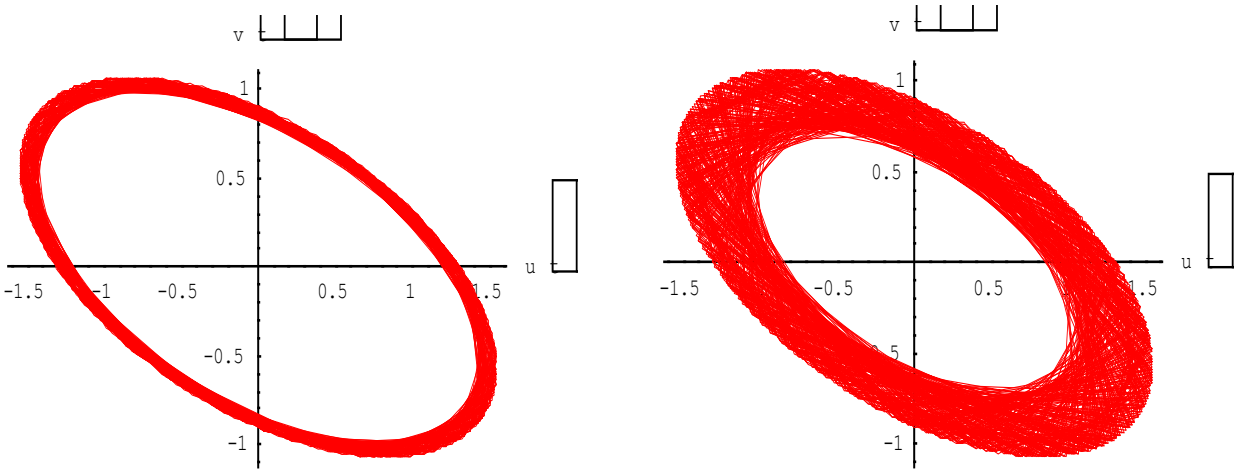



**Fig 5.12:** Orbits of the third body around triangular points $L_4$ for $0 \le \upsilon \le 0.5$ when (**a**). $\kappa = 0.0001$ (**b**). $\kappa = 0.5$ and (**c**) $\kappa = 1$, respectively.






**(c)** **(d)**

**Fig 5.13:** Orbits of the third body around triangular points $L_4$ for $\kappa = 0.5, \upsilon = 0.4$, with periods (**a**) $T^* = 15.617$ (**b**) $10T^*$ (**c**) $100T^*$ and (**d**) $200T^*$

From the above figures (Fig **5.9** to Fig **5.13**), it is seen that the triangular EP $L_4 \left(\xi = 0.5 - \upsilon, \eta = 0.86602540\right)$ is enclosed by the orbits. When the parameters $\kappa$ and $\upsilon$ are both very small, the orbit nearly completely flatten (Fig **5.9**, panel **a**) and motion around the triangular EP becomes restricted. In this premise, when $0 < \kappa < 0.17$ and $0 < \upsilon \leq \frac{1}{2}$, the length of one of the axes become imaginary (Table **5.8**). If we ignore the effect of $\kappa$ (i.e., put $\kappa = 1$, in all our results), the orbit around $L_4$ points changes and both semi-axes are not imaginary (Fig **5.10**, panel **a**.). The results in Table **5.8** are the numerical evidence of the eccentricities of long and short orbits, given by equations (3.158) and (3.159); and the lengths of the semi-axes as described by equations (3.161 to 3.164). These orbital elements are defined by the two parameters of the system. From these results, it is clear that the lengths of these axes do not exist for some values of the variational parameter $\kappa$ as they are either zero or imaginary. This obviously, does not arise when $\kappa = 1$; which corresponds to the case when the masses of the main bodies are constant.

Also, from the table, when $0<\kappa<0.9$ and $\upsilon<0.3$, the eccentricities of the orbits (both long and short) are less than one. In this particular range, the eccentricities of the long period are greater than the corresponding ones for the short period orbits. However, same trend or converse doesn't hold outside the range. When $\kappa=1$, we noticed that $e_i>e_2$ or $e_i<e_2$ and either can be less or greater than one, depending on the mass parameter. For instance, in the Table 5.6, when $\kappa=1$ and $\upsilon=0.2$; $e_1<1$ while $e_2>1$; whereas, for $\upsilon=0.3$, both eccentricity of the long and short period orbits are greater than one. Hence, the mass ratio also plays a major role alongside the parameter $\kappa$. In Fig **5.10**, we show the influence of the mass parameter on the periodic orbits around $L_4$ points. The panel (**a**) is the case when the mass parameter is very small and the periodic orbit has nearly sharp edges. The difference in their orientations, sizes, shapes and positions around the EP is described in plane (**h**). This means that, for increasing mass ratio, the orientation, size, shape and position differs.

The orbits around triangular EP $L_4$ given in Fig **5.11** show the effect of the two governing parameters of the system. In panel (**a**), we see that when $\upsilon=0.01$ and $\kappa$ varied; obviously, as $\kappa$ increases, the orbits shrink and the third body moves closer to the EP. The same behavior of the third body is observed when (panel **b**) $\kappa$ is fixed and the mass parameter is varied for $0<\upsilon\leq\frac{1}{2}$. So we can say that an increase in the parametric values produce an orbital shift which always tend towards the vicinity of the triangular points. Fig **5.13** shows the path of the third body around the triangular points when $\kappa=0.5$ and $\upsilon=0.4$; but the periods are still defined by both parameters and so, the body moves nearer to the points with increasing periods. Hence, from panel (**a**) to-panel (**d**) in Fig **5.13**, we see that the body does not depart but spirals toward the equilibrium point with increasing orbital periods.

The effects of isotropic mass variation of a binary system on the orbits of the third body around linear stable triangular EPs in the frame of the R3BP, has been studied. The motion of the main bodies is described by the GMP while their mass variation occurs in accordance with the UML. It is seen that the change in the frequencies, orbital period, eccentricities, the semi-minor and major axes, of long and short orbits, are influenced by the mass parameter $\upsilon$ of the binary and a constant $\kappa$ of a particular integral (1.66) of the GMP. Further, we observe that the triangular EPs are enclosed by the orbits, for some parametric values. In particular, for very small values of the governing parameters, the orbits almost completely collapse and motion around the EP becomes restricted. Due to the variations, the eccentricity can be less, equal to, or greater than one, and so the orbit can be elliptic, parabolic or hyperbolic types. Also, the lengths of the semi-axes do not exist for some values of the parameter $\kappa$ as they are either zero or imaginary. These results differ from the case when the main bodies have constant masses.

### 5.1.2 Robes restricted-three-body problem with variable masses

The Robe (1977) problem when the masses of the primaries vary isotropically with time at the same rate according to the UML and their motion described by the GMP has been studied. The first primary is a spherical empty shell which contains the third body. We found two kinds of EPs. The first is the one at the center of the shell, which was first found by Robe (1977). The second is a pair of non-collinear EPs, (3.228), which we have found because of inclusion of mass variations in the Robes problem. The EPs (3.228) exist for $\kappa > 1$ and lies in the $\xi\zeta-$plane (Fig.**5.14,** not drawn to scale and, Fig **5.15** and **5.16**, drawn to scale). These EPs do not have an analogy in the Robe (1977) problem when the spherical shell is empty. They are also different from those of the R3BP of Bekov (1988), Luk'yanov (1989, 1990) and, Singh and Leke (2010).

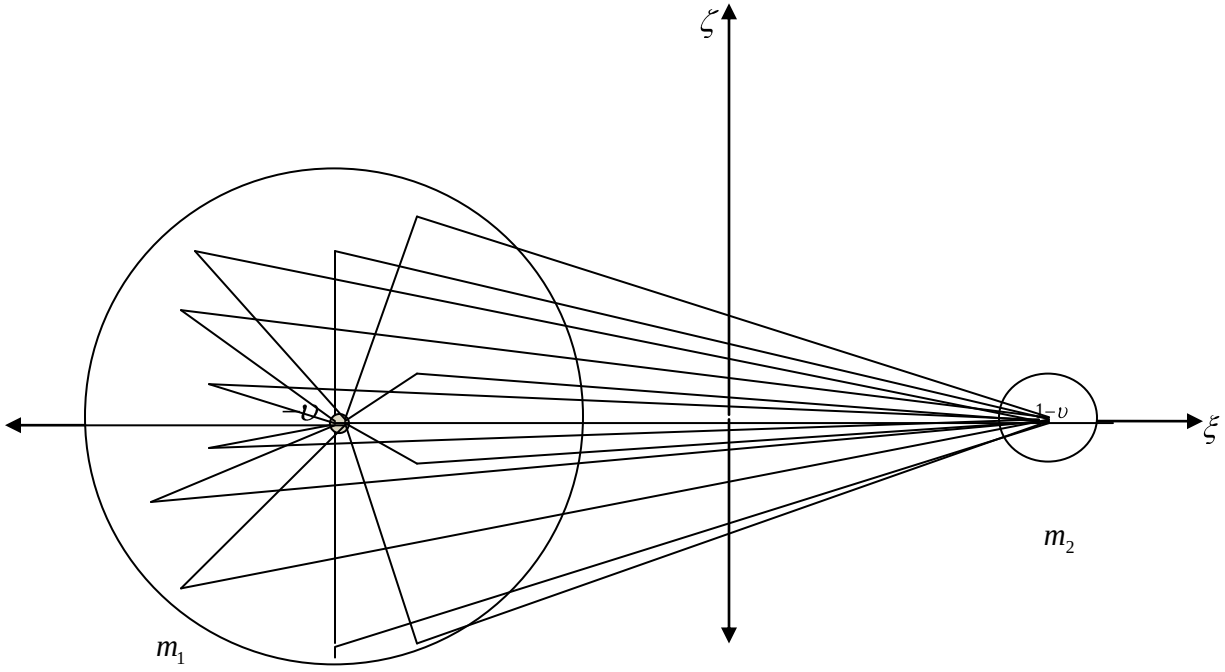



**Fig 5.14:** Non-collinear equilibrium points on the $\xi\zeta$ – plane

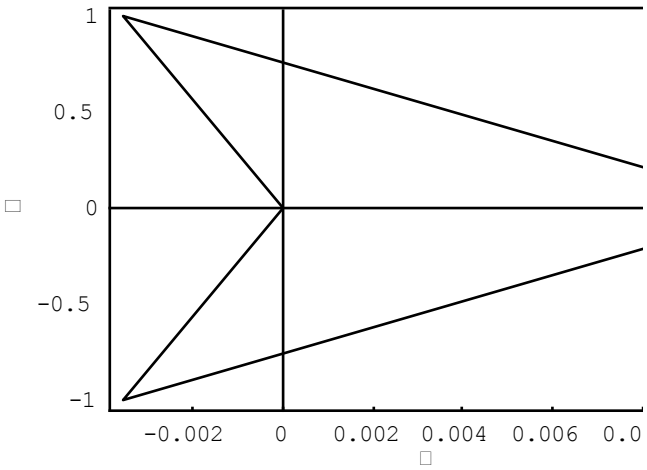



**Figure 5.15:** Non-collinear equilibrium points on the $\xi\zeta$ – plane for $\upsilon = 0.01$

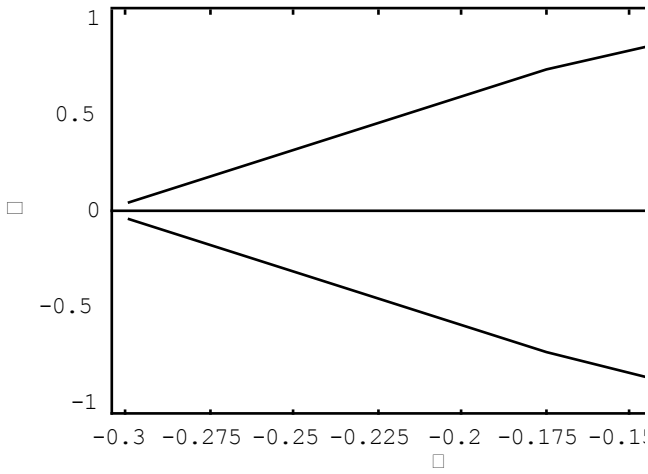



**Fig 5.16:** Non-collinear equilibrium points on the $\xi\zeta$ – plane for $\upsilon = 0.3$

We observe that the every point in the shell that is not collinear with the center of the shell is a non-collinear EPs because they form triangles with the centers of both primaries, though this depends on the parameter $\kappa$. Numerically, in the Earth–Moon system when $\upsilon = 0.01$ and $0 < \kappa < \infty$, these points exist for $1.0036 \le \kappa \le 1.01$. When $\kappa = 1.01$, we have $\xi = -0.0099$ and $\zeta = \pm 0.0827985$ which lies inside the shell. However, when $\upsilon = 0.3$, they exist for $1.115 \le \kappa \le 1.42$, (Tables **5.9** and **5.10**).

**Table 5.9:** Positions of non-collinear points and the characteristic roots $\lambda_{1,2} = \pm\omega_1$, $\lambda_{3,4} = \pm\omega_2$, $\lambda_{5,6} = \pm\omega_3$ for $\upsilon = 0.01$

| $\kappa$, | $\xi$ | $\pm\zeta$ | $p_1$ | $p_2$ | $p_3$ | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | Infinity | - | - | - | - | - | - |
| 1.001 | -0.00099 | 1.91380 | - | - | - | - | - | - |
| 1.0035 | -0.00347 | 1.01550 | - | - | - | - | - | - |
| 1.0036 | -0.00356 | 0.99680 | -1.9892 | 0.98916 | 1.11562 | 0.95342 | $1.23+0.09i$ | $-0.792-0.09i$ |
| 1.01 | -0.0099 | 0.08279 | -1.97 | 0.99959 | 1.63765 | 1.01912 | $1.24+0.08i$ | $-0.814-0.07i$ |
| $1.011 \le \kappa < \infty$ | Negative | Imaginary | - | - | - | - | - | - |

**Table 5.10**: Positions of non-collinear points and the characteristic roots $\lambda_{1,2} = \pm\omega_1, \lambda_{3,4} = \pm\omega_2$ $\lambda_{5,6} = \pm\omega_3$ for $\upsilon = 0.3$

| $\kappa$ | $\xi$ | $\pm\zeta$ | $p_1$ | $p_2$ | $p_3$ | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | Infinity | - | - | - | - | - | - |
| 1.001 | -0.0007 | 6.6598 | - | - | - | - | - | - |
| 1.16 | -0.112 | 1.00963 | - | - | - | - | - | - |
| 1.162 | -0.113 | 1.00257 | - | - | - | - | - | - |
| 1.16273 | -0.11391 | 1.00001 | - | - | - | - | - | - |
| 1.16274 | -0.11392 | 0.99997 | -1.5117 | 0.3532 | 1.26775 | 0.97960 | $1.33+0.111i$ | $-0.691-0.111i$ |
| 1.20 | -0.14 | 0.87984 | -1.4 | 0.27874 | 1.35846 | 0.99957 | $1.41+0.01i$ | $-0.66-0.09i$ |
| 1.25 | -0.175 | 0.73806 | -1.25 | 0.23421 | 1.49261 | 1.02947 | $1.53+0.07i$ | $-0.62-0.07i$ |
| 1.428 | -0.2996 | 0.03772 | -0.716 | 0.99391 | 2.26607 | 1.15512 | | |
| $1.429 \le \kappa < \infty$ | Negative | Imaginary | - | - | - | - | - | - |

When $\kappa = 1$, we have $\xi = 0$ which lies outside the shell and since $\zeta = \pm\infty$, it is seen that infinite remote EPs $L \pm \infty$ do not exist for any value of $0 < \upsilon < 1$. Evidently, from Table **5.9** above, we see that numerically, in the Earth-Moon system $\upsilon = 0.01$ and $0 < \kappa < \infty$**,** the non-collinear EPs exist for $1.0036 \le \kappa < 1.01$**,** because here $\xi < 0$ and $\zeta < 1 - \upsilon$ lies inside the shell. When $\kappa = 1.01$, we have, $\xi = -0.0099$ and $\zeta = \pm 0.0827985$ which lies inside the shell. However, for a system with mass ratio $\upsilon = 0.3$ (Table **5.10**), these points exist for values of $\kappa$ in the interval $1.16273 < \kappa \le 1.428$. When $\kappa = 1$, we have $\xi = 0$ which lies outside the shell and since $\zeta = \pm\infty$, it is seen that infinite remote EPs $L_{\pm\infty}$ do not exist for any value of the mass ratio.

The six roots of characteristic equation (3.243) and the positions of the non-collinear points (3.228) have been presented in Table **5.9** and Table **5.10** numerically for $\upsilon = 0.01$ and $\upsilon = 0.3$, respectively, with various values of the parameter $\kappa$. When $\upsilon = 0.01$ the non-collinear EPs and

the characteristic roots exist only for $1.0035 < \kappa \leq 1.01$, because here $\xi < 0$ and $\zeta < 1 - \upsilon$ lies inside the shell. Our numerical analysis reveals that when the non-collinear points exist, the coefficient $p_1$ is negative, $p_2$ is positive and $p_3$ is positive in both cases of the mass parameter. It is clear from the tables that for a specific set of values of the parameters at least one of the roots is real and positive or at one has a positive real part. We conclude that a positive root and positive real part of the complex roots induce instability at the non-collinear EPs. Hence the motion of the third body around the non-collinear EP is unstable.

In Section **3.7,** we introduced small perturbations in the Coriolis and centrifugal forces into the system of equation (3.200) and transformed it to get equations of system (3.245). These equations are different from those of Shrivastava and Garain (1991), and Hallan and Rana (2001a) due to the parameter kappa, but differs from that in equation (3.215), because of the small perturbations in the Coriolis and centrifugal forces. It is seen that due to the effects of small perturbations in the centrifugal force, the EPs (3.223) at the center of the shell drifts away from the center to the point given by equation (3.253) and Since $0 < \kappa < \infty$ and $\psi \neq 1$, every point on the line joining the centers of the primaries is an EP, provided these points lie within the shell. When $\kappa = 1$, the coordinate is same as that found by Shrivastava and Garain (1991) and, Hallan and Rana (2001a). The non-collinear EPs (3.228) and (3.258) obtained do not exist in the studies of Robe (1977), Shrivastava and Garain (1991), and Hallan and Rana (2001a). Our stability results for the EPs near the center of the shell are however similar to that worked out by Hallan and Rana (2001a). Shrivastava and Garain (1991) did not study the stability of the EPs.

The restricted problem using the model of Robe (1977) when the first primary is a fluid and the masses and densities vary with time, was examined in section **3.8**. The motivation for this problem arose because of the challenge we encountered in the case of the isotropic mass flow

where the non-autonomous systems do not fully transform to the autonomized systems in the presence of fluid. We assumed that the radius $\Re$ of the fluid varies at the same rate as the distances while the densities of the fluid and the third body vary at the same rate as their masses, respectively, since the volume is constant. Hence, we introduced the transformation:

$$\Re = \ell\left(\frac{\mu_0}{\mu_*}\right)^3,\ \delta_1\ t\ = \delta_{01}\frac{\mu_*}{\mu_0},\quad \delta_3\ t\ = \delta_{03}\frac{\mu_*}{\mu_0}$$

where $\ell$, $\delta_{01}$ and $\delta_{03}$ defined in equations (3.271) while $\mu_*$ and $\mu_0$ are defined in equations (1.95). The non-autonomous equations of motion were recast to the autonomized forms using a Gelf'gat (1973) transformation and the transformation introduced by us. In this system, an EP at the center, another near the center, circular points and pairs of non-collinear EPs, exist. The EPs established in the works of Hallan and Rana (2001a) and Robe (1977) can be validated from ours, while the EPs of the non-autonomous systems when the primaries vary differ from those of the autonomized systems only due to the function $R\ t$ but those when the three bodies vary differ by the term $\frac{\mu_0^3}{\mu_*^3}$. The linear stability outcome of the EPs of the autonomized system agrees with those of Hallan and Rana (2001b); the points on the line inside the shell are stable while the circular points and the non-collinear EPs are unstable. For the non-autonomous systems, all the EPs are unstable according to the Lyapunov's theorem. We used this problem to discuss the motion of a submarine under the gravitational attraction of the Earth and the Moon as the primaries.

The system has two collinear EPs; one at the center and the other (3.283) near the center. Circular EPs (3.287) and non-collinear EPs (3.291) also are found.

If in (3.291), we set $\kappa = 1$, we get

$$\xi = D_p \text{ and } \zeta = \pm\sqrt{\left(\frac{\upsilon}{-D_p}\right)^{2/3} - \left(1-\upsilon-D_p\right)^2}\ . \tag{5.20}$$

This fully coincides with that of Hallan and Rana (2001b).

We observe that solutions (5.20) holds provided $-\upsilon - D_p < 0 \Rightarrow D_p + \upsilon > 0$. If $D_p + \upsilon = 0$ this implies $D_p = -\upsilon$ and equation (5.20) reduces to

$\xi = -\upsilon$,

$$\zeta = \pm\sqrt{\left(\frac{\upsilon}{-(-\upsilon)}\right)^{2/3} - \left[1-\upsilon-(-\upsilon)\right]^2}$$

Simplifying, we get $\xi = -\upsilon$ and $\zeta = 0$. Hence, when $D_p = -\upsilon$ the solutions are

$$\xi = -\upsilon, \quad \zeta = 0$$

Therefore, when $D_p + \upsilon = 0$, the system has an equilibrium point $(-\upsilon, 0, 0)$ .

Next, when $-D_p + \upsilon > 0$ this implies that $D_p + \upsilon < 0$. In this case, there is no solution as

$$\left(1-\upsilon-D_p\right)^2 > 1 \text{ and } \left(\frac{\upsilon}{D_p}\right)^{2/3} < 1.$$

When observe that when $\delta_{10} = \delta_{30}$, we get $D_p = 0$, and the coordinates (3.291) of the non-collinear EPs reduce to

$$\xi = (\kappa - 1)(\upsilon - 1),$$

$$\zeta = \pm\sqrt{\left(\frac{\kappa\upsilon}{\kappa-1}\right)^{2/3} - \kappa^2(1-\upsilon)^2} \tag{5.21}$$

In this case, the EPs given by equations (5.21) is same with equation (3.228), where we claimed the existence of a rigid empty spherical shell. Hence, it is seen that the EPs of the problem when the three bodies vary their masses under the frame of the Robe (1977) formulation are fully analogous to those found by Hallan and Rana (2001b) except for the non-collinear equilibrium points which are affected by the parameter $\kappa$, the density parameter $D_p$, and the mass ratio $\upsilon$. The positions of the non-collinear points are given in Tables **5.11**, **5.12** and **5.13** and their graphical representations in Figures **5.17, 5.18**, and **5.19** in the Earth-Moon system, when the density parameter is negative, zero, and positive, respectively, while Fig **5.20** combines them.

**Table 5.11:** The non-collinear equilibrium points for $\upsilon = 0.01$ and $D_p = -0.001$

| $\kappa$ | $\xi$ | $\pm\zeta$ | Comments |
|---|---|---|---|
| 0.01 | 0.98009 | 0.02328 +0.04033i | Real non-collinear EPs do not exist |
| 0.5 | 0.4945 | Complex | ,, |
| 0.999 | Real and negative | Complex | ,, |
| 0.999001 | ,, | 215.369 | Real non-collinear EPs exist |
| 1 | -0.001 | 1.91758 | ,, |
| 1.001 | -0.011 | 1.3988 | ,, |
| 1.002 | -0.00298 | 1.12303 | ,, |
| 1.003 | -0.00397 | 0.93202 | Real non-collinear EPs exist |
| 1.004 | -0.00496 | 0.78152 | ,, |
| 1.005 | -0.00595 | 0.65245 | ,, |
| 1.006 | -0.00694 | 0.53379 | ,, |
| 1.007 | -0.00793 | 0.41635 | ,, |
| 1.008 | -0.00892 | 0.286570 | ,, |
| 1.009 | -0.00991 | 0.075576 | ,, |
| $1.009 < \kappa < \infty$ | Real and negative | Imaginary | Real non-collinear EPs exist do not exist |

**Table 5.12:** The non-collinear equilibrium points for $\upsilon = 0.01$ and $D_p = 0$

| $\kappa$ | $\xi$ | $\pm\zeta$ | Comments |
|---|---|---|---|
| 0.01 | 0.98010 | Complex | Real non-collinear EPs exist do not exist |
| 0.5 | 0.98010 | ,, | ,, |
| 0.999 | 0.00099 | ,, | ,, |
| 0.999001 | 0.00098 | ,, | ,, |
| 1 | 0 | Infinity | Infinite remote solution |
| 1.001 | -0.00099 | 1.91380 | Real non-collinear EPs exist |
| 1.002 | -0.00198 | 1.39986 | ,, |
| 1.003 | -0.00297 | 1.12416 | ,, |
| 1.004 | -0.00396 | 0.93320 | ,, |
| 1.005 | -0.00495 | 0.78277 | ,, |
| 1.006 | -0.00594 | 0.65381 | ,, |
| 1.007 | -0.00693 | 0.53532 | ,, |
| 1.008 | -0.00792 | 0.41817 | ,, |
| 1.009 | -0.00891 | 0.28904 | ,, |
| 1.0091 | -0.00900 | 0.27441 | ,, |
| 1.01 | -0.00990 | 0.08399 | ,, |
| $1.01 < \kappa < \infty$ | Real and negative | Imaginary | Real non-collinear EPs exist do not exist |

**Table 5.13**: The non-collinear equilibrium points for $\upsilon = 0.01$ and $D_p = 0.001$.

| $\kappa$ | $\xi$ | $\pm\zeta$ | Comments |
|---|---|---|---|
| 0.01 | 0.98011 | Complex | Real non-collinear EPs do not exist |
| 0.5 | 0.49550 | ,, | ,, |
| 0.999 | 0.00198 | ,, | ,, |
| 0.999001 | 0.00198 | ,, | ,, |
| 1 | 0.00100 | ,, | ,, |
| 1.001 | 0.00001 | ,, | ,, |
| 1.002 | -0.00097 | 1.92083 | Real non-collinear EPs exist |
| 1.003 | -0.00196 | 1.40162 | ,, |
| 1.004 | -0.00295 | 1.12573 | ,, |
| 1.005 | -0.00394 | 0.93471 | ,, |
| 1.006 | -0.00493 | 0.78430 | ,, |
| 1.007 | -0.00592 | 0.65542 | ,, |
| 1.008 | -0.00691 | 0.53708 | ,, |
| 1.009 | -0.00790 | 0.42023 | ,, |
| 1.0091 | -0.00799 | 0.40821 | ,, |
| 1.01 | -0.00990 | 0.29178 | ,, |
| 1.0111 | -0.00997 | 0.03934 | ,, |
| $1.0112 \le \kappa < \infty$ | Real and negative | Complex | Real non-collinear EPs do not exist |

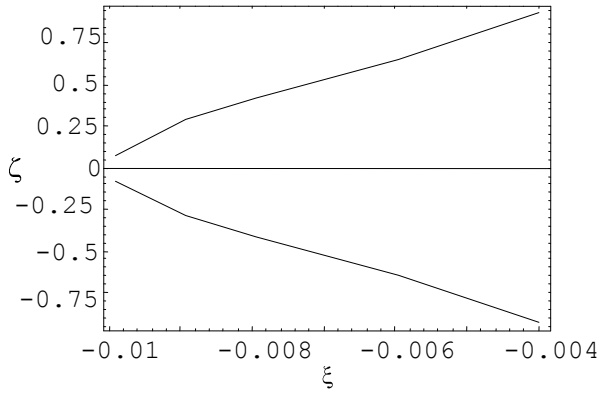



**Fig 5.17:** Non-collinear equilibrium points for, $\upsilon = 0.01$, $D_p = -0.001$.

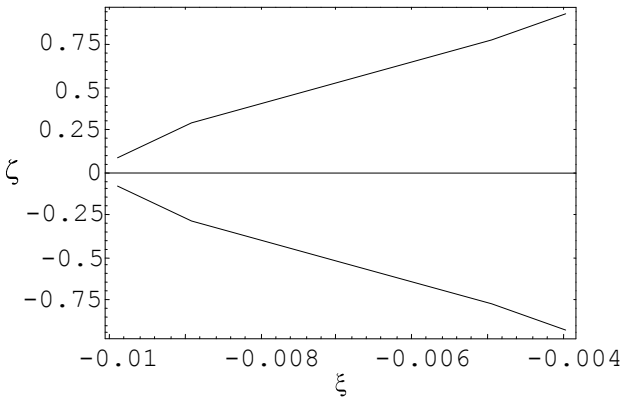



**Fig 5.18:** Non-collinear equilibrium points for, $\upsilon = 0.01,\ D_p = 0$

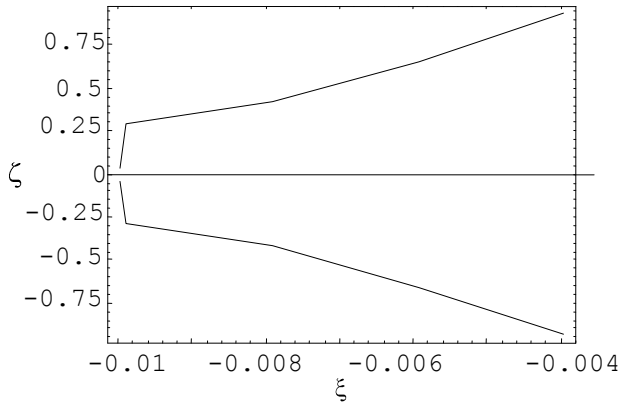



**Fig 5.19:** Non-collinear equilibrium points for $\upsilon = 0.01, D_p = 0.001$

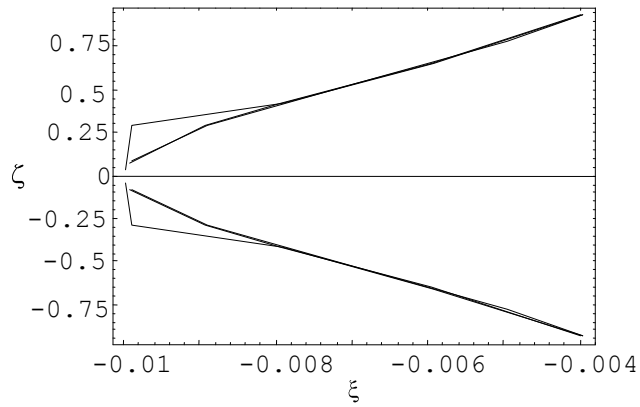



**Fig 5.20:** Non-collinear equilibrium points for, $\upsilon = 0.01,\ D_p = -0.001, D_p = 0$ and $D_p = 0.001$

In Table **5.11**, when $D_p = -0.001$ this implies that $\delta_{03} < \delta_{01}$; in this case for $0 < \kappa \leq 0.999$ and $1.009 < \kappa < \infty$, the out-of-plane EPs do not exist, but however exist in the interval $0.999001 \leq \kappa \leq 1.999$. In Table **5.12**, $\delta_{01} = \delta_{03}$, and so $D_p = 0$. In this case the non-collinear points exist only when $1.001 \leq \kappa \leq 1.01$ and do not exist in the remaining entire range of $\kappa$. Last but not the least, in Table **5.13**, we choose $D_p = 0.001$ and so $\delta_{03} > \delta_{01}$.Here, real non-collinear EPs exist, for the values of $\kappa$, in the interval $1.002 \leq \kappa \leq 1.0111$ and are nonexistent for any value of $\kappa$ outside this range. Hence, though $\kappa$ has a large range of values, however, the physically meaningful range is $\kappa \in 0.999001,\ 1.0111$ at which the non-collinear EPs exist. However, these ranges may differ for different density parameters $D_p$, which are determined by the densities $\delta_{01}$ and $\delta_{03}$ of the fluid and the third body, respectively, since $4\pi/3$ is a constant.

The roots (3.312) are computed numerically for motion of a submarine as the third body under the gravitational attraction of the Earth-Moon system when $D_p > 0$ and $D_p < 0$ and consider the values of $\kappa$ for which the out of plane points exist. Therefore, we take $\upsilon = 0.01$ and $0.999001 \leq \kappa \leq 1.0111$ with the following values in each case (Kaur and Aggarwal 2012).

When $\delta_{01} > \delta_{03}$, we have positive buoyancy force and $D_p < 0$ . In this case, we have the following:

Mass of submarine $m_3 = 1449832kg$ ,

Density of salt water $\delta_{01} = 1027kg/m^3$ ,

Density of submarine $\delta_{03} = 804.79kg/m^3$

In new units, we have

$\delta_{01} = 9317.7$ , $\delta_{03} = 7301.65$, and so $D_p = -2572.699$ .

Similarly, when $\delta_{01} < \delta_{03}$, $D_p > 0$; we have negative buoyancy force have and so we take:

Mass of submarine $m_3 = 1848632kg$,

Density of salt water $\delta_{01} = 1027kg/m^3$,

Density of submarine $\delta_{03} = 1204.39kg/m^3$

In new units, we have

$\delta_{01} = 9317.7$, $\delta_{03} = 10927.11$, and so $D_p = 1372.4$

Aside from these examples, we also consider the case when $D_p = \pm 0.001$, so that a wider generalization can be reached regarding the characteristic roots (3.312). Figure **5.21** and Figure **5.22** show the image gallery of a submarine when it has a positive buoyancy force (when it floats) and when it has a negative buoyancy force (when it submerges) while Figure **5.23** is an animation which demonstrate the mechanism of how the bouyancy force is controlled



**Fig 5.21** A floating Submarine $\delta_{01} > \delta_{03}$ (positive bouyancy) and $D_p < 0$ (**Credit**: US Navy)



**Fig 5.22** Image of a submerging submarine when $\delta_{01} < \delta_{03}$ (negative bouyancy) and $D_p > 0$

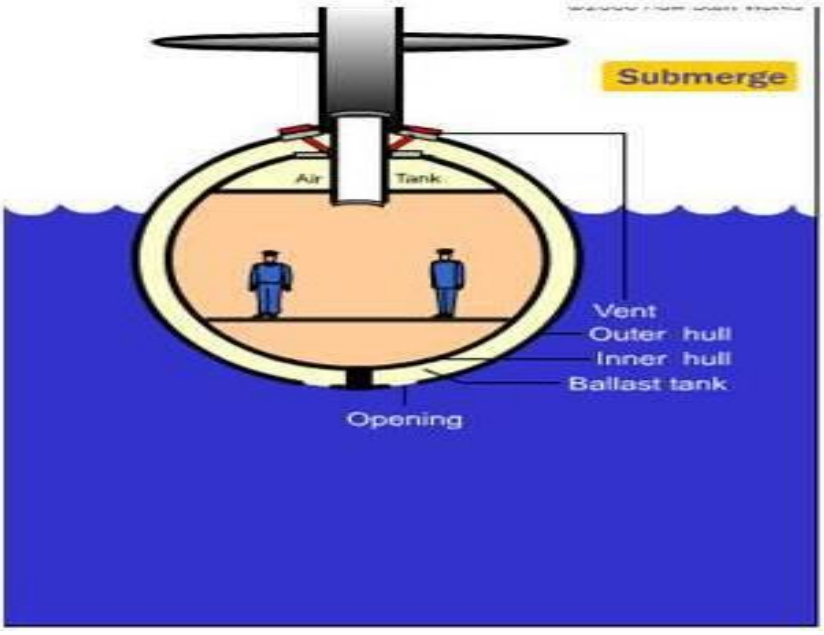



**Fig 5.23:** Mechanism of how the bouyancy force is controlled.

**Table 5.14:** The characteristic roots of the non-collinear point $\lambda_{1,2} = \pm\omega_1,\ \lambda_{3,4} = \pm\omega_2,$ $\lambda_{5,6} = \pm\omega_3$ for $\upsilon = 0.01,\ D_p = -2572.699$

| $\kappa$ | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| 0.999001 | 3526.63 +3526.09 i | 2.13681 i | 3526.63 -3526.09i |
| 1 | 3529.28 +3528.73 i | 2.13716 i | 3529.28 -3528.73i |
| 1.001 | 3531.93 +3531.38 i | 2.13752 i | 3531.93 -3531.38 i |
| 1.002 | 3534.57 +3534.03 i | 2.13787 i | 3534.57 -3534.03 i |
| 1.003 | 3537.22 +3536.67 i | 2.13823 i | 3537.22 -3536.67 i |
| 1.004 | 3539.86 +3539.32 i | 2.13859 i | 3539.86 -3539.32 i |
| 1.009 | 3553.08 +3552.53 i | 2.14036 i | 3553.08 -3552.53 i |
| 1.01 | 3555.72 +3555.17 i | 2.14071i | 3555.72 -3555.17 i |
| 1.0111 | 3558.62 +3558.08 i | 2.1411i | 3558.62 -3558.08 i |

**Table 5.15:** The characteristic roots of the non-collinear point $\lambda_{1,2} = \pm\omega_1,\ \lambda_{3,4} = \pm\omega_2, \lambda_{5,6} = \pm\omega_3$ for $\upsilon = 0.01,\ D_p = -0.001$

| $\kappa$ | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| 0.999001 | 0.00005 | $2.1781\times10^{-7} + i$ | $2.1781\times10^{-7} - i$ |
| 1 | 0.05414 | 0.02141 +1.0002i | 0.02142 -1.00021i |
| 1.001 | 0.07561 | 0.03795 +1.00065i | 0.03795 -1.00065i |
| 1.002 | 0.09134 | 0.05299 +1.00124i | 0.05299 -1.00124i |
| 1.003 | 0.10394 | 0.06709 +1.00195i | 0.06709 -1.00195i |
| 1.004 | 0.11444 | 0.08053 +1.00276i | 0.08053 -1.00276i |
| 1.009 | 0.14825 | 0.14147 +1.00796i | 0.14147 -1.00796i |
| 1.01 | 0.15247 | 0.15279 +1.00919i | 0.15279 -1.00919i |
| 1.0111 | 0.15637 | 0.16500 +1.01059i | 0.16500 -1.01059i |

**Table 5.16:** The characteristic roots of the non-collinear point $\lambda_{1,2} = \pm\omega_1,\ \lambda_{3,4} = \pm\omega_2, \lambda_{5,6} = \pm\omega_3$ for $\upsilon = 0.01,\ D_p = 0.001$

| $\kappa$ | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| 0.999001 | 0.03361 +0.9804i | 0.00021-0.07893i | 0.03229 -1.02011i |
| 1 | 0.01877 +0.98901i | 0.00004-0.05532i | 0.01836 -1.01118i |
| 1.001 | 0.00005 +0.99996i | $1.42435\times10^{-11} - 0.0020i$ | 0.00005-1.00003i |
| 1.002 | 0.05408 | 0.02136 +1.0002i | 0.02136 -1.00021i |
| 1.003 | 0.07554 | 0.03787 +1.0001i | 0.03787 -1.00065i |
| 1.004 | 0.09126 | 0.05286 +1.0012i | 0.05286 -1.00123i |
| 1.009 | 0.13755 | 0.11781 +1.00566i | 0.11781 -1.00566i |
| 1.01 | 0.14327 | 0.12961 +1.00677i | 0.12961 -1.00677i |
| 1.0111 | 0.14865 | 0.14229 +1.00805i | 0.14229 -1.00805i |

**Table 5.17:** The characteristic roots of the non-collinear point $\lambda_{1,2} = \pm\omega_1,\ \lambda_{3,4} = \pm\omega_2, \lambda_{5,6} = \pm\omega_3$ for $\upsilon = 0.01,\ D_p = 1372.4$

| $\kappa$ | $\omega_1$ | $\omega_2$ | $\omega_3$ |
|---|---|---|---|
| 0.999001 | 2426.1 +1401.08i | 1400.71 -2426.74i | 0.96216 -1.66652i |
| 1 | 2427.92 +1402.13i | 1401.76 -2428.56i | 0.96232 -1.6668i |
| 1.001 | 2429.74 +1403.18i | 1402.81 -2430.38i | 0.96248 -1.66708i |
| 1.002 | 2431.56 +1404.23i | 1403.86 -2432.20i | 0.96264 -1.66735i |
| 1.003 | 2433.38 +1405.28i | 1404.91 -2434.02i | 0.96280 -1.66763i |
| 1.004 | 2435.2 +1406.33i | 1405.96 -2435.84i | 0.96296 -1.66791i |
| 1.009 | 2444.29 +1411.58i | 1411.21 -2444.93i | 0.96376 -1.66929i |
| 1.01 | 2446.1 +1412.63i | 1412.26 -2446.74i | 0.96392 -1.66956i |
| 1.0111 | 2448.1 +1413.78i | 1413.41 -2448.74i | 0.96409 -1.66987i |

The six characteristic roots presented in Tables **5.14**, **5.15**, **5.16** and **5.17** have been computed using the software package *Mathematica* numerically for $\upsilon = 0.01$.

We sought to find the case where all the six roots are pure imaginary quantities or complex figures with negative real parts. If this happens, then the solutions will be bounded and motion

will be stable; otherwise, they will be unstable. From these tables, it is seen that that for a specific set of values of these parameters at least one of the roots among all has a positive real part or a complex root with the existence of a positive real part. Therefore, this causes the solutions to be unbounded and consequently producing unstable EPs. Hence, we conclude that the non-collinear EPs are unstable points due to a positive root and positive real part in complex roots. This agrees with the results of Douskos and Markellos (2006), Shankaran *et al* (2010) and Singh and Laraba (2012). The EPs of the non autonomous system with variable coefficients are in general unstable points according to the Lyapunov's theorem of stable solutions (Krasnov *et al* 1983).

The idea of the submarine's mass changing with time may perhaps seem impossible because it is a rigid body. However, from Archimedes' principle, a submarine floating (Fig **5.21**) or submerging (Fig **5.22**) depends on buoyancy which is controlled by the ballast tanks found between the submarine's inner and outer hulls (Fig **5.23**). Submarine resting on the water surface has positive buoyancy which means it is less dense than water. When this happens, the ballast tanks are empty. For it to submerge, vents on top of the tanks are opened and water floods in thereby making the submarine denser than the sea water. In this case, the submarine has negative buoyancy. For it to float again the water in the tanks is forced out and the submarine becomes less dense than the water and eventually floats. In this study, the buoyancy force depends on the mass of the submarine, the densities of the submarine, and the medium, as well as the volume of the submarine. And so by the simple relation, connecting mass, density, and volume, we understand that, when water is allowed into the ballast, as the mass of the submarine is changing so is the density until it becomes denser than the medium and sinks. When water is let out of the tanks, the mass and density of the submarine reduce and as a matter of fact, it will float when the

medium is denser. Neutral buoyancy is attained when the weight of the submarine equals the amount of water it displaces. The submarine neither rises nor sinks in this state. Hence, with changing mass comes a changing density while the volume remains a constant.

This study may be useful in the investigations of the dynamic problem of Earth-size planets covered completely by a water envelope (water planets), and also the study of the small oscillation of the Earth's inner core taking into account the Moon's attraction during the course of evolution. This problem is highly idealized and therefore calls for more research.

### 5.1.2 Numerical Treatments of the Restricted Problem of Three-bodies with Constant Masses

In chapter **FOUR**, we studied the R3BP in the presence of a circumbinary disc under various characterizations of the bodies and also examined the periodic orbits in the range of linearly stable triangular EPs in the cases studied in sections **4.2**, when the first primary is a radiation source while the second primary and the third body are modeled as oblate spheroids; section **4.3**, when the first primary is a triaxial star and the second one is a radiating-oblate star coupled with small perturbations in the Coriolis and centrifugal forces. In particular, we shall apply this study to the case when the first primary is a depleted pulsating post-AGB star, IRAS 11472-0800 while its companion is a white dwarf star, G29-38 and section **4.4**, when the three bodies are oblate spheroid while both primaries are radiation emitters under small change in the Coriolis and centrifugal forces.

We investigated the EPs and claimed that analytically two additional collinear EPs exist due to the gravitational potential from the disc and the mass parameter. Using the software package *Mathematica* (Wolfram 2003), the coordinates of the collinear EPs for the **Sun–Earth**, the **Sun–Jupiter** and the **Fomalhaut-Fomalhaut b** systems of primaries taking into account the free

parameters when the mass of the disc and oblateness are varied; are obtained. These are presented in Tables **5.18**, **5.19** and **5.20**, respectively for, $0 \le M_d \le 0.07$, $0 \le \alpha_2 \le 0.04$, $0 \le \alpha_3 \le 0.03$ with $\bar{T} = 0.01$ and $q_1 = 0.9985$. For these systems we have $\mu = 0.000003$, $\mu = 0.0009537$ and $\mu = 0.16883117$, respectively.

Our numerical effort reveals that when the mass parameter $\mu < 0.15463365$ for any system, there exist only three collinear EPs. When simultaneously $\mu \ge 0.15463365$ and $M_d > 0$, two additional collinear equilibrium points $L_{21}$ and $L_{22}$ exists provided $\bar{T} < \sqrt{2}\mu$. The presence of the perturbed forces due to radiation and oblateness and do not in any way result in the existence of additional collinear points but they affect the positions of these points. These additional EPs do not appear in the classical R3BP because the effects arising due to the gravitational potential from the disc is ignored.

**Table 5.18:** Numerical computations of the collinear equilibrium points for the Sun–Earth system; $\mu = 0.000003$

| $q_1$ | $\alpha_2$ | $\alpha_3$ | $M_d$ | $L_1$ | $L_2$ | $L_3$ | $L_{21}$ | $L_{22}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | −1.02167 | −0.97858 | 1.00001 | - | - |
| 0.9985 | 0 | 0 | 0 | −1.02151 | −0.97841 | 0.99951 | - | - |
| 1 | 0 | 0 | 0 | −1.02167 | −0.97858 | 1.00001 | - | - |
| | 0.02 | 0 | 0 | −1.04878 | −0.94812 | 0.99020 | - | - |
| | 0.03 | 0 | 0 | −1.05197 | −0.94312 | 0.98544 | - | - |
| | 0.04 | 0 | 0 | −1.05423 | −0.93903 | 0.98077 | - | - |
| 1 | 0 | 0.01 | 0 | −1.04558 | −0.95726 | 1.00494 | - | - |
| | | 0.02 | | −1.05280 | −0.95226 | 1.00973 | - | - |
| | | 0.03 | | −1.05793 | −0.94925 | 1.01438 | - | - |
| 0.9985 | 0.02 | 0 | 0 | −1.04781 | −0.94693 | 0.98513 | - | - |
| | 0.03 | | | −1.05104 | −0.94187 | 0.98040 | - | - |
| | 0.04 | | | −1.05334 | −0.93771 | 0.97575 | - | - |
| 0.9985 | 0.02 | 0.01 | 0 | −1.04953 | −0.94444 | 0.99014 | - | - |
| | | 0.02 | 0 | −1.05655 | −0.94042 | 0.98543 | - | - |
| | | 0.03 | 0 | −1.06105 | −0.93691 | 0.98080 | - | - |
| 0.9985 | 0.02 | 0 | 0.01 | −1.05265 | −0.94444 | 0.990143 | - | - |
| | | | 0.02 | −1.05655 | −0.94272 | 0.99500 | - | - |
| | | | 0.03 | −1.05991 | −0.94144 | 0.99972 | - | - |
| 0.9985 | 0.02 | 0.01 | 0.01 | −1.02282 | −0.97964 | 1.00333 | - | - |
| | | | 0.02 | −1.02409 | −0.98060 | 1.00663 | - | - |
| | | | 0.03 | −1.02548 | −0.98147 | 1.00991 | - | - |
| | | | 0.04 | −1.02700 | −0.98225 | 1.01317 | - | - |

**Table 5.19:** Numerical computations of the collinear equilibrium points for the Sun–Jupiter system; $\mu = 0.0009537$

| $q_1$ | $\alpha_2$ | $\alpha_3$ | $M_d$ | $L_1$ | $L_2$ | $L_3$ | $L_{21}$ | $L_{22}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | −1.06883 | −0.93237 | 1.00040 | - | - |
| 0.9985 | 0 | 0 | 0 | −1.06867 | −0.932195 | 0.99989 | - | - |
| 1 | 0 | 0 | 0 | −1.06883 | −0.93237 | 1.00040 | - | - |
| | 0.02 | 0 | 0 | −1.10427 | −0.89435 | 0.99059 | - | - |
| | 0.03 | | | −1.11073 | −0.88645 | 0.98583 | - | - |
| 1 | 0 | 0.01 | 0 | −1.09671 | −0.90799 | 1.00533 | - | - |
| | | 0.02 | | −1.10867 | −0.89896 | 1.01012 | - | - |
| | | 0.03 | | −1.11712 | −0.89318 | 1.01478 | - | - |
| 0.9985 | 0.02 | 0 | 0 | −1.10298 | −0.89332 | 0.98552 | - | - |
| | 0.03 | | | −1.10949 | −0.88543 | 0.98078 | - | - |
| | 0.04 | | | −1.11433 | −0.87912 | 0.97614 | - | - |
| 0.9985 | 0.02 | 0.01 | 0 | −1.11141 | −0.88795 | 0.99053 | - | - |
| | 0.03 | | | −1.11616 | −0.88168 | 0.98582 | - | - |
| | 0.04 | | | −1.11993 | −0.87638 | 0.98120 | - | - |
| 0.9985 | 0.02 | 0.01 | 0 | −1.11141 | −0.88795 | 0.99053 | - | - |
| | | | 0.02 | −1.1181 | −0.88404 | 0.99538 | - | - |
| | | | 0.03 | −1.12376 | −0.88099 | 1.00013 | - | - |
| 0.9985 | | 0.02 | 0.01 | −1.0699 | −0.93351 | 1.00371 | - | - |
| | | | 0.02 | −1.07101 | −0.93460 | 1.00701 | - | - |
| | | | 0.03 | −1.07215 | −0.93566 | 1.01028 | - | - |
| | | | 0.04 | −1.07331 | −0.93668 | 1.01354 | - | - |

**Table 5.20:** Numerical computations of the collinear equilibrium points for the Fomalhaut system; $\mu = 0.16883117$

| $q_1$ | $\alpha_2$ | $\alpha_3$ | $M_d$ | $L_1$ | $L_2$ | $L_3$ | $L_{21}$ | $L_{22}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | −1.27148 | −0.48832 | 1.07006 | - | - |
| 0.9985 | 0 | 0 | 0 | −1.27137 | −0.48813 | 1.06959 | - | - |
| 1 | 0 | 0 | 0 | −1.26829 | −0.48787 | 1.06559 | - | - |
| 1 | 0.02 | 0 | 0 | −1.28642 | −0.46778 | 1.06069 | - | - |
| | 0.03 | | | −1.29243 | −0.45975 | 1.05615 | - | - |
| | 0.04 | | | - 1.29776 | −0.45266 | 1.05169 | - | - |
| 1 | 0 | 0.01 | 0 | −1.28333 | −0.48168 | 1.0664 | - | - |
| | | 0.02 | | −1.29373 | −0.47654 | 1.06188 | - | - |
| | | 0.03 | | −1.30305 | −0.47240 | 1.05745 | - | - |
| 0.9985 | 0.02 | 0 | 0 | −1.28336 | −0.46723 | 1.05581 | - | - |
| | 0.03 | | | −1.28946 | −0.45922 | 1.05129 | - | - |
| | 0.04 | | | −1.29485 | −0.45215 | 1.04686 | - | - |
| 0.9985 | 0.02 | 0.01 | 0 | −1.29288 | −0.46374 | 1.06156 | - | - |
| | 0.03 | | | −1.29815 | −0.45666 | 1.05707 | - | - |
| | 0.04 | | | −1.30287 | −0.45031 | 1.05266 | - | - |
| 0.9985 | 0.02 | 0.01 | 0 | −1.29288 | −0.46374 | 1.06156 | - | - |
| | | 0.02 | | −1.30152 | −0.46088 | 1.06709 | - | - |
| | | 0.03 | | −1.30945 | −0.45847 | 1.07244 | - | - |
| | | | 0.01 | −1.27261 | −0.49103 | 1.07268 | 0.00364 | 0.01114 |
| | | | 0.02 | −1.27373 | −0.49366 | 1.0753 | 0.00152 | 0.01949 |
| | | | 0.03 | −1.27486 | −0.49624 | 1.0779 | 0.00098 | 0.02449 |
| | | | 0.04 | −1.27599 | -0.49869 | 1.0805 | 0.00073 | 0.02829 |
| 0.9985 | | | 0.01 | −1.26929 | −0.49040 | 1.06774 | 0.00363 | 0.01116 |

The coordinates of the triangular EPs of the restricted problem of oblate bodies under radiation pressure force of the first primary, under the influence of the force due to the potential from the disc, have been presented in section **4.2** and given by equation (4.24). The surface plots of the

coordinates (4.24) are presented in Fig **5.24**, **5.25**, **5.26**, **5.27**, **5.28** and Fig **5.29**. The triangular points of (2013), Kushvah (2008),, Sharma (1987), Sharma (1982), Bhatnagar and Chawla (1979), SubbaRao and Sharma (1975) and, Szebehely (1967a) can all be recovered from equation (4.24).

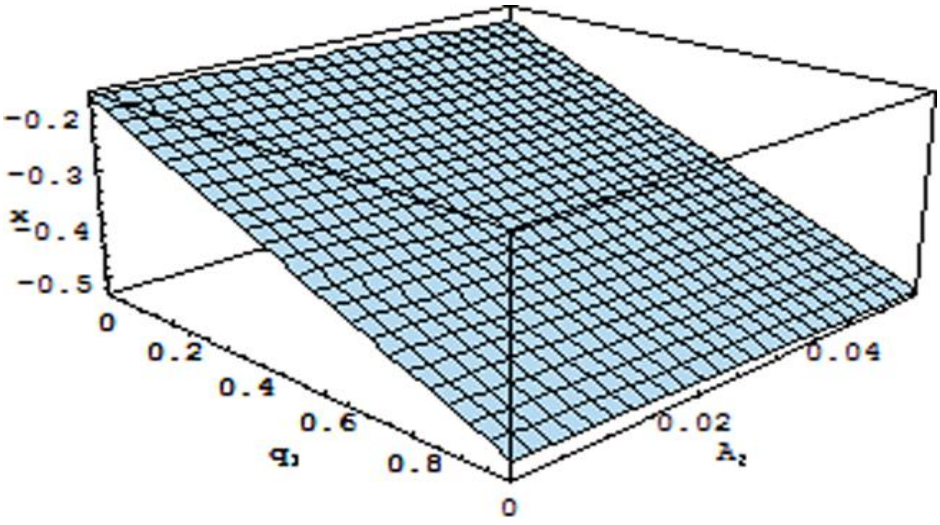



**Fig 5.24:** Surface plot of $x-$coordinate of triangular points for $\mu = 0.000003, 0 < q_1 \leq 1,$ $0 < \alpha_2 \leq 0.06$, $M_d = 0$, $r_c = 0.99997$

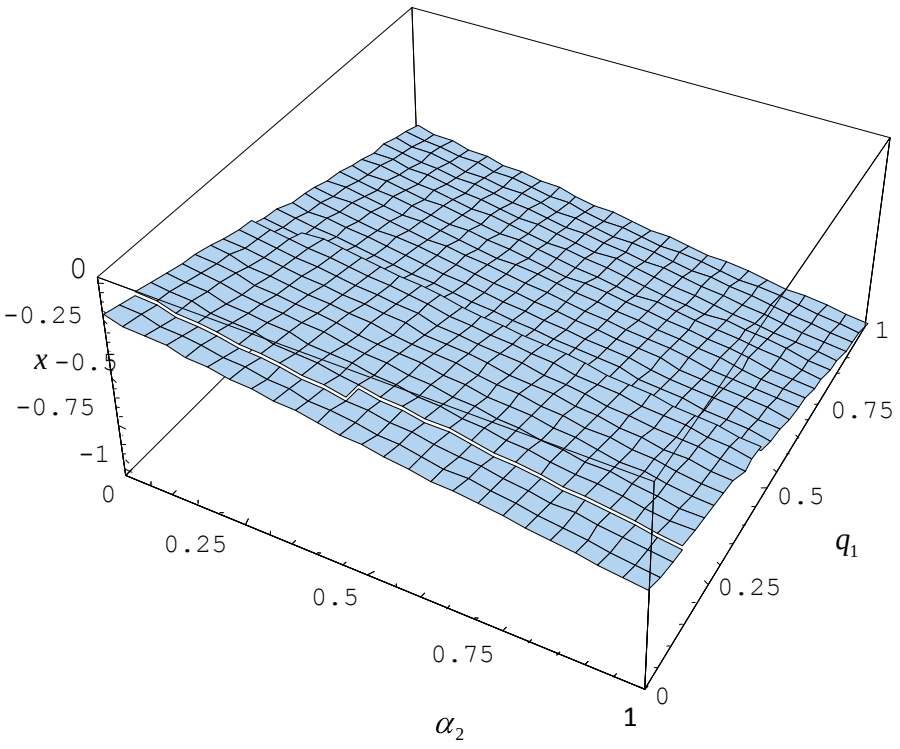



**Fig 5.25:** Surface plot of $x-$coordinate of triangular points for $\mu = 0.000003, 0 < q_1 \leq 1,$ $0 < \alpha_2 \leq 0.06$, $M_d = 0.01$, $r_c = 0.99997$

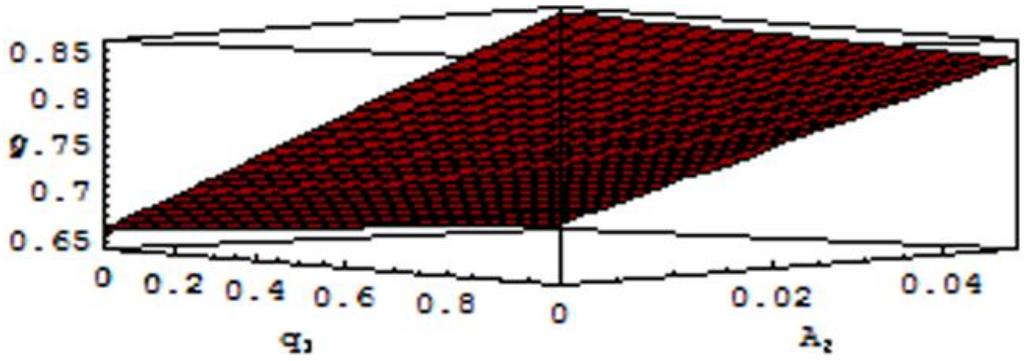



**Fig 5.26:** Surface plot of $y-$coordinate of triangular points for $\mu = 0.000003, 0 < q_1 \leq 1,$ $0 < \alpha_2 \leq 0.06$, $M_d = 0.03, r_c = 0.99997$

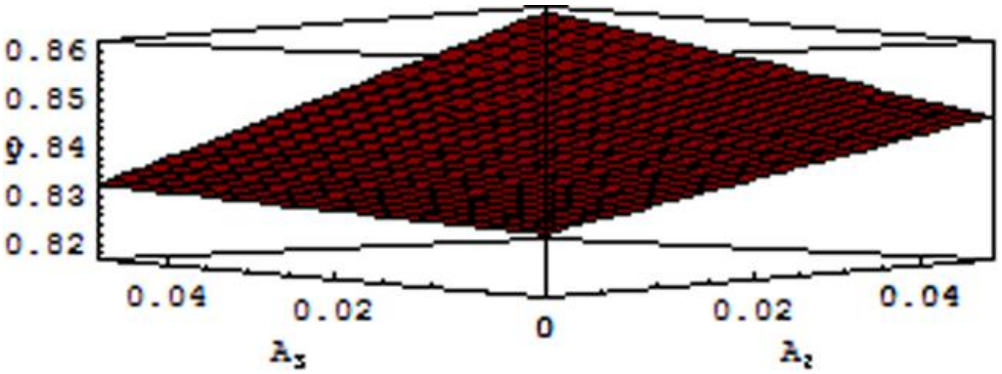



**Fig 5.27:** Surface plot of $y-$coordinate of triangular points for $\mu = 0.000003, q_1 = 0.9988$, $0 < \alpha_{2,3} \le 0.06$, $M_d = 0.01$, $r_c = 0.99997$

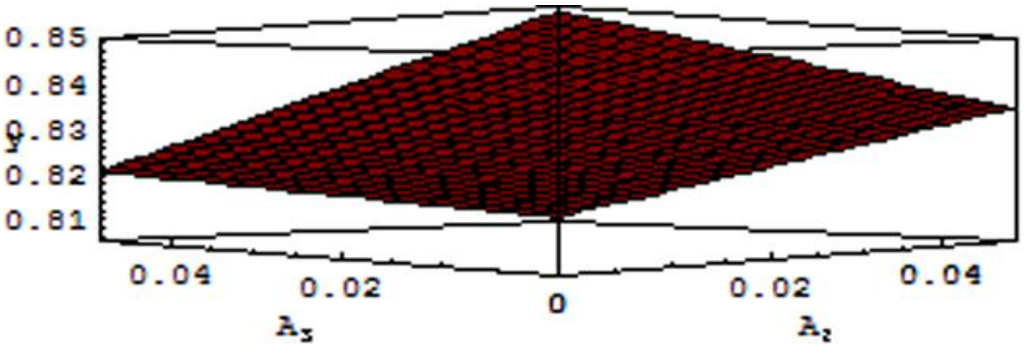



**Fig 5.28:** Surface plot of $y-$coordinate of triangular points for $\mu = 0.000003, q_1 = 0.9988$, $0 < \alpha_{2,3} \le 0.06$, $M_d = 0.03$, $r_c = 0.99997$

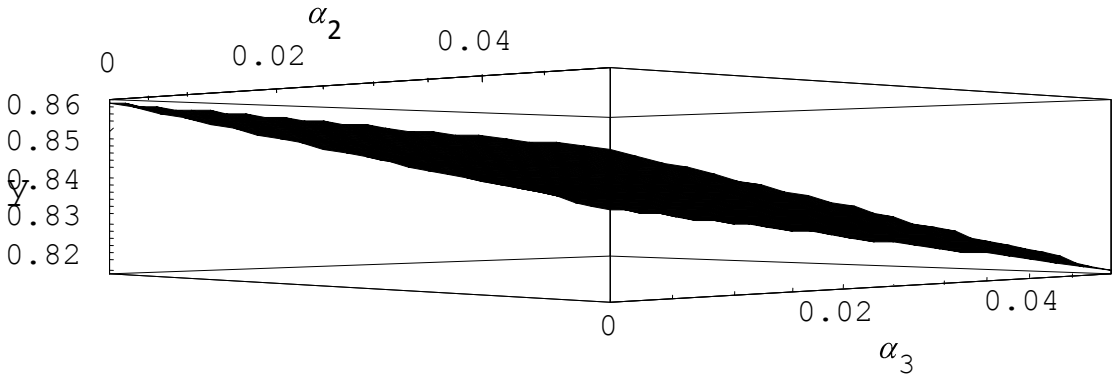



**Fig 5.29:** Surface plot of $y-$coordinate of triangular points for $\mu = 0.000003, q_1 = 0.9988$, $0 < \alpha_{2,3} \le 0.06$, $M_d = 0$, $r_c = 0.99997$

The critical mass parameter of the problem in the case of the triangular EPs is given by the equation

$$\mu_C = \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)-\frac{1}{9}\left(\frac{13}{\sqrt{69}}-1\right)\alpha_2-\frac{22\alpha_3}{9\sqrt{69}}-\frac{2\ 1-q_1}{27\sqrt{69}}+\frac{2M_d}{27\sqrt{69}\ \ r_c^2+\bar{T}^2\ ^{\frac{3}{2}}}$$

$$\times\left[38-4r_c-\frac{61}{2\ \ r_c^2+\bar{T}^2}\right]$$

The first term represents Routh's critical mass ratio; the second, third and fourth expressions are the effects arising from oblateness of the second primary, oblateness of the third body and radiation of the first primary, respectively, while the last expression indicates the force due to gravitational potential from the circumbinary disc. The coefficients of the radiation pressure of the first primary and the oblateness of the second primary also fully coincide with those in Singh and Ishwar (1999), while the coefficient of the oblateness of the body differs from that in the work of Abouelmagd and EL-Shaboury (2012) only due to the value in the numerator which is erroneous.

In the absence of oblateness of the third body (i.e., $\alpha_3 = 0$), the critical value becomes

$$\mu_C = \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)-\frac{1}{9}\left(\frac{13}{\sqrt{69}}-1\right)\alpha_2-\frac{2\ 1-q_1}{27\sqrt{69}}+\frac{2M_d}{27\sqrt{69}\ \ r_c^2+\bar{T}^2\ ^{\frac{3}{2}}}\left[38-4r_c-\frac{61}{2\ \ r_c^2+\bar{T}^2}\right]$$

and is similar to that worked by Kushvah (2008a).

Further, if the first primary is a black body and the second primary is a sphere, the critical mass will be

$$\mu_C = \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)+\frac{2M_d}{27\sqrt{69}\ \ r_c^2+\bar{T}^2\ ^{\frac{3}{2}}}\left[38-4r_c-\frac{61}{2\ \ r_c^2+\bar{T}^2}\right]$$

Evidently, from the above equation, we see that $38-4r_c-\frac{61}{2\ r_c^2+\bar{T}^2}>0$, for $r_c^2=1-\mu+\mu^2; 0<\mu\le\frac{1}{2}$ and $\bar{T}=0.01$. Hence, the stability regions expand in the sphere of influence, and hence the circumbinary disc mass is a stabilizing parameter.

Numerically, for the Sun–Earth and Sun–Jupiter systems; $\mu_{SE}=0.000003$, $\bar{T}=0.01$, , $r_c=0.99997$ and $\mu_{SJ}=0.0009537$, $\bar{T}=0.01$ and $r_c=0.999047$ respectively. Therefore, the respective critical masses are:

$$\mu_{c1}=0.03852089...-0.00089174...\ 1-q_1\ -0.0627795...\alpha_2-0.29427652...\alpha_3+0.0312085M_d$$

and

$$\mu_{c2}=0.03852089...-0.00089174...\ 1-q_1\ -0.0627795...\alpha_2-0.29427652...\alpha_3+0.0308243M_d$$

Table **5.21** below gives numerically the critical mass parameter of the Sun-Earth and Sun-Jupiter systems, as a function of the oblateness of the third body when, $\alpha_2=0.02$, $q_1=0.9988$, $M_d=0.03$, $\bar{T}=0.01$, Table **5.22** is when the critical mass is a function of the mass of the disc using same solar systems when $\alpha_2=0.02$, $\alpha_3=0.01$, $q_1=0.9988$ and $\bar{T}=0.01$

**Table 5.21:** Critical mass values $\mu_{ci}\ \ i=1,2$ for the Sun–Earth and Sun–Jupiter systems when $0 \le \alpha_3 \le 0.07$, $\alpha_2 = 0.02$, $q_1 = 0.9988$, $M_d = 0.03$, $\overline{T} = 0.01$

| $\alpha_3$ | $\mu_{c1}$ (Sun–Earth) | $\mu_{c2}$ (Sun–Jupiter) |
|---|---|---|
| 0 | 0.04114 | 0.04113 |
| 0.01 | 0.03820 | 0.03819 |
| 0.02 | 0.03526 | 0.03525 |
| 0.03 | 0.03232 | 0.03230 |
| 0.04 | 0.02937 | 0.02936 |
| 0.05 | 0.02643 | 0.02642 |
| 0.06 | 0.02349 | 0.02348 |
| 0.07 | 0.02054 | 0.02053 |

**Table 5.22:** Critical mass values for the Sun–Earth and Sun–Jupiter systems when $0 \le M_d \le 0.07$, $\alpha_2 = 0.02$, $\alpha_3 = 0.01$, $q_1 = 0.9988$, $\overline{T} = 0.01$

| $M_d$ | $\mu_{c1}$ (Sun–Earth) | $\mu_{c2}$ (Sun–Jupiter) |
|---|---|---|
| 0 | 0.03726 | 0.03726 |
| 0.01 | 0.03758 | 0.03757 |
| 0.02 | 0.03789 | 0.03788 |
| 0.03 | 0.03820 | 0.03819 |
| 0.04 | 0.03851 | 0.03850 |
| 0.05 | 0.03883 | 0.03881 |
| 0.06 | 0.03914 | 0.03911 |
| 0.07 | 0.03945 | 0.03942 |

From the tables **5.21** and **5.22**, it is seen that for both systems, $\mu_C$ is a decreasing function of the oblateness of the third body and an increasing function of the force from the disc mass $M_d$, respectively.

We know that periodic orbits exist in the stable region $0 < \mu < \mu_C$, where $\mu_C$ is the critical mass parameter. The frequency $s_1$ is for the long periodic orbits while $s_2$ is for the short periodic orbits were obtained in equations (4.35) and (4.36), respectively. Tables **5.23** and **5.24** give the numerical computations of the frequencies, the short and the long periodic orbits of the Sun-Earth and Sun-Jupiter systems respectively; for the values $0 \le M_d \le 0.07$, $\alpha_2 = 0.02$, $\alpha_3 = 0.01$, $q_1 = 0.9985$

**Table 5.23:** Computations of the periodic orbits and their terms in the Sun-Earth system

| $M_d$ | $s_1$ | $s_2$ | $S_1$ | $S_2$ | $\overline{S}_1$ | $\overline{S}_2$ | $C_1$ | $C_2$ | $\overline{C}_1$ | $\overline{C}_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2.50661 | 1.02661 | 32.5771 | 1.09136 | -24.7019 | -0.12891 | -0.23947 | -0.23947 | 0.848801 | -0.848801 |
| 0.01 | 2.50661 | 1.04112 | 22.8375 | 1.11854 | -25.2581 | -0.140158 | -0.23947 | -0.23947 | 0.85265 | -0.85265 |
| 0.02 | 2.50661 | 1.05542 | 13.0978 | 1.14573 | -25.8144 | -0.151407 | -0.23947 | -0.23947 | 0.856498 | -0.856498 |
| 0.03 | 2.50661 | 1.06954 | 3.35822 | 1.17291 | -26.3707 | -0.162655 | -0.23947 | -0.23947 | 0.860347 | -0.860347 |
| 0.04 | 2.50661 | 1.08347 | -6.38141 | 1.20009 | -26.9269 | -0.173904 | -0.23947 | -0.23947 | 0.864195 | -0.864195 |
| 0.05 | 2.50662 | 1.09722 | -16.121 | 1.22727 | -27.4832 | -0.185152 | -0.23947 | -0.23947 | 0.868044 | -0.868044 |
| 0.06 | 2.50662 | 1.1108 | -25.8607 | 1.25446 | -28.0395 | -0.1964 | -0.23947 | -0.23947 | 0.871892 | -0.871892 |
| 0.07 | 2.50662 | 1.12422 | -35.6003 | 1.28164 | -28.5957 | -0.207649 | -0.23947 | -0.23947 | 0.875741 | -0.875741 |

**Table 5.24:** Computations of the periodic orbits and their terms in the Sun-Jupiter system

| $M_d$ | $s_1$ | $s_2$ | $S_1$ | $S_2$ | $\overline{S}_1$ | $\overline{S}_2$ | $C_1$ | $C_2$ | $\overline{C}_1$ | $\overline{C}_2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 2.51032 | 1.02558 | 5.79801 | 1.09324 | -1.91967 | -0.130627 | -0.238546 | -0.238546 | 0.848801 | -0.848801 |
| 0.01 | 2.51037 | 1.04002 | 4.06241 | 1.12042 | -2.01888 | -0.141962 | -0.238546 | -0.238546 | 0.852653 | -0.852653 |
| 0.02 | 2.51043 | 1.05425 | 2.3268 | 1.1476 | -2.11809 | -0.153296 | -0.238546 | -0.238546 | 0.856505 | -0.856505 |
| 0.03 | 2.51049 | 1.0683 | 0.591197 | 1.17478 | -2.2173 | -0.16463 | -0.238546 | -0.238546 | 0.860357 | -0.860357 |
| 0.04 | 2.51055 | 1.08216 | -1.14441 | 1.20197 | -2.31651 | -0.175964 | -0.238546 | -0.238546 | 0.86421 | -0.86421 |
| 0.05 | 2.51061 | 1.09585 | -2.88001 | 1.22915 | -2.41572 | -0.187298 | -0.238546 | -0.238546 | 0.868062 | -0.868062 |
| 0.06 | 2.51067 | 1.10937 | -4.61561 | 1.25633 | -2.51493 | -0.198633 | -0.238546 | -0.238546 | 0.871914 | -0.871914 |
| 0.07 | 2.51072 | 1.12273 | -6.35122 | 1.28351 | -2.61414 | -0.209967 | -0.238546 | -0.238546 | 0.875766 | -0.875766 |

**Numerical studies of IRAS 11472-0800 Vs G29-38 system**

We devote this section to the numerical study of previously obtained analytical results of the problem studied in section **4.3**, for a particular case in which the post AGB primary star is an extremely depleted pulsating triaxial star, IRAS 11472-0800, while its stellar companion is a young weakly-radiating oblate white dwarf, G29-38. Both stars are surrounded by a circumbinary disc.

Now, for most stars in the post-AGB stage, we shall use the examples of the mass interval between 0.89 and 5.0 solar masses used in Vassiliadis and Wood (1994), for IRAS 11472-0800, so that $m_1 = 2.945 \pm 2.055\, M_\odot$, while for G29-38, we shall take $m_2 = 0.7 \pm 0.03\, M_\odot$. This helps in finding a wide range of the mass function $\mu$, which are given in table **5.8**.

**Table 5.25:** Mass functions for the binary IRAS 11472-0800 and G29-38, when $m_1 = 2.945 \pm 2.055\ M_\Theta$, $m_2 = 0.7 \pm 0.03\ M_\Theta$

| IRAS 11472-0800 $m_1 (M_\Theta)$ | $\mu (m_2 = 0.67 M_\Theta)$ | $\mu (m_2 = 0.68 M_\Theta)$ | $\mu (m_2 = 0.7 M_\Theta)$ | $\mu (m_2 = 0.73 M_\Theta)$ |
|---|---|---|---|---|
| 0.89 | 0.42948 | 0.43312 | 0.44025 | 0.45061 |
| 0.95 | 0.41358 | 0.41717 | 0.42424 | 0.43452 |
| 1 | 0.40119 | 0.40476 | 0.41176 | 0.42196 |
| 1.5 | 0.30875 | 0.31192 | 0.31818 | 0.32735 |
| 2.0 | 0.25093 | 0.25373 | 0.25925 | 0.26739 |
| 2.5 | 0.21135 | 0.21383 | 0.21875 | 0.22600 |
| 3.5 | 0.16067 | 0.16267 | 0.16666 | 0.17257 |
| 5.0 | 0.11816 | 0.11971 | 0.12280 | 0.12740 |

Now, we compute numerically in table **5.26**, the coordinates of triangular equilibrium points of a dust grain particle given by equations (4.61), for the mass function $0.118166 \le \mu \le 0.450617$ and mass of the disc, $0 \le M_d \le 0.09$ together with the chosen parametric values: $\alpha_2 = 0.02$, $q_2 = 0.9988$, $\psi = 1.002$, $\sigma_1 = 1.39 \times 10^{-4}$, $\sigma_2 = 1.32 \times 10^{-4}$ and $\bar{T} = 0.01$.

**Table 5.26:** Coordinates of triangular equilibrium points in IRAS 11472-0800 and G29-38 system, for $0.118166 \leq \mu \leq 0.450617$

| $\mu$ (IRAS 11472-0800- G29-38) | $x$ | $\pm y\ (M_d = 0)$ | $\pm y\ (M_d = 0.04)$ | $\pm y\ (M_d = 0.09)$ |
|---|---|---|---|---|
| 0.11816 | -0.37227 | 0.85923 | 0.84298 | 0.82268 |
| 0.11971 | -0.37071 | 0.85923 | 0.84297 | 0.82266 |
| 0.12280 | -0.36763 | 0.85922 | 0.84296 | 0.82262 |
| 0.12740 | -0.36303 | 0.85922 | 0.84293 | 0.82256 |
| 0.16666 | -0.32377 | 0.85922 | 0.84270 | 0.82205 |
| 0.17257 | -0.31786 | 0.85922 | 0.84267 | 0.82198 |
| 0.21135 | -0.27909 | 0.85922 | 0.84248 | 0.82155 |
| 0.25093 | -0.23951 | 0.85922 | 0.84231 | 0.82117 |
| 0.26739 | -0.22305 | 0.85922 | 0.84224 | 0.82103 |
| 0.30875 | -0.18169 | 0.85921 | 0.84210 | 0.82072 |
| 0.32735 | -0.16310 | 0.85921 | 0.84205 | 0.82060 |
| 0.40119 | -0.08925 | 0.85921 | 0.84190 | 0.82025 |
| 0.42196 | -0.06849 | 0.85921 | 0.84187 | 0.82019 |
| 0.43452 | -0.05593 | 0.85921 | 0.84186 | 0.82016 |
| 0.45061 | -0.03984 | 0.85921 | 0.84184 | 0.82013 |

From table **5.26**, the positions $L_4$ and $L_5$ of the triangular EPs when there are no circumbinary disc $(M_d = 0)$ enclosing the stars reveals in this case, that, the dust particle is located farther away from the line joining the stars. However, in the presence of disc, when $M_d$ gradually reaches up to $0.04$, the dust particle is drawn nearer to the line on which the stars lie and even goes closest, when the mass of the disc reaches close to 0.1. Hence, an increase in the mass of the circumbinary disc produces a shift in the positions of the triangular EPs and consequently makes the particle to be positioned closer to the stars. Although, this is also influenced by the

oblateness, triaxiality, small change in the centrifugal force and the radiation pressure present, in the system.

Next, haven studied the stability and calculated the expression for the critical mass value given by equation (4.64). We now compute numerically, the values of $\mu_C$ for the range of the mass ratios of the, IRAS 11472-0800 and G29-38 system, in tables **5.27** and **5.28**, for the chosen parametric values, when $M_d = 0$, $M_d = 0.02$, $M_d = 0.04$ and $M_d = 0.09$. In table **5.27**, we have taken $36 \in -19 \in' > 0$ in equation (5.64), so that $\in > 0.001056$, while the reverse is used in table **5.28**.

**Table 5.27:** Critical mass function $\mu_C$ for mass ratios in IRAS 11472-0800 and G29-38 system, when $36 \in -19 \in' > 0$.

| $\mu$ (IRAS 11472-0800- G29-38) | $\mu_C\ (M_d = 0)$ | $\mu_C\ (M_d = 0.02)$ | $\mu_C\ (M_d = 0.04)$ | $\mu_C\ (M_d = 0.09)$ |
|---|---|---|---|---|
| 0.11816 | 0.03712 | 0.03736 | 0.03761 | 0.03823 |
| 0.11971 | ---- | 0.03736 | 0.03759 | 0.03819 |
| 0.12280 | ---- | 0.03734 | 0.03756 | 0.03811 |
| 0.12740 | ---- | 0.03731 | 0.03751 | 0.03800 |
| 0.16666 | ---- | 0.03709 | 0.03706 | 0.03699 |
| 0.17257 | ---- | 0.03706 | 0.03699 | 0.03684 |
| 0.21135 | ---- | 0.03683 | 0.03653 | 0.03580 |
| 0.26739 | ---- | 0.03650 | 0.03588 | 0.03432 |
| 0.30875 | ---- | 0.03627 | 0.03542 | 0.03330 |
| 0.32735 | ---- | 0.03618 | 0.03523 | 0.03288 |
| 0.40119 | ---- | 0.03587 | 0.03463 | 0.03151 |
| 0.42196 | ---- | 0.03581 | 0.03451 | 0.03125 |
| 0.43452 | ---- | 0.03578 | 0.03445 | 0.03111 |
| 0.45061 | ---- | 0.03575 | 0.03439 | 0.03097 |

**Table 5.28:** Critical mass function $\mu_C$ for mass ratios in IRAS 11472-0800 Vs G29-38 system, when $36 \in -19 \in' < 0$.

| $\mu$ (IRAS 11472-0800- G29-38) | $\mu_C$ $(M_d = 0)$ | $\mu_C$, $(M_d = 0.02)$ | $\mu_C$ $(M_d = 0.04)$ | $\mu_C$ $(M_d = 0.09)$ |
|---|---|---|---|---|
| 0.11816 | 0.03736 | 0.03730 | 0.03755 | 0.03816 |
| 0.11971 | ---- | 0.03729 | 0.03753 | 0.03812 |
| 0.12280 | ---- | 0.03728 | 0.03750 | 0.03805 |
| 0.12740 | ---- | 0.03725 | 0.03745 | 0.03793 |
| 0.16666 | ---- | 0.03703 | 0.03700 | 0.03693 |
| 0.17257 | ---- | 0.03699 | 0.03693 | 0.03677 |
| 0.21135 | ---- | 0.03676 | 0.03647 | 0.03574 |
| 0.26739 | ---- | 0.03643 | 0.03581 | 0.03426 |
| 0.30875 | ---- | 0.03621 | 0.03536 | 0.03324 |
| 0.32735 | ---- | 0.03611 | 0.03517 | 0.03281 |
| 0.40119 | ---- | 0.03581 | 0.03456 | 0.03145 |
| 0.42196 | ---- | 0.03575 | 0.03444 | 0.03118 |
| 0.43452 | ---- | 0.03572 | 0.03438 | 0.03105 |
| 0.45061 | ---- | 0.03569 | 0.03432 | 0.03091 |

The general criterion for the triangular points to be stable is when $0 < \mu < \mu_C$. However, we see that this condition doesn't hold as we have $0 < \mu_C < \mu$ for the entire range of the critical mass functions and mass ratios. This consequently proofs that the triangular points for the IRAS 11472-0800 and G29-38 systems, are in particular, unstable.

**Numerical study of the orbits**

The equations (4.34) represent composed form of two periodic motions known as long periodic and short periodic motion with frequencies $s_1$ and $s_2$. In Figure **5.31,** we plot the graph of the frequencies (in equations 4.65 and 4.66) as a function of the mass ratios for chosen values of the other parameters. The black and red curves represent the frequencies $s_1$ and $s_2$, respectively.

Also, the numerical values and graph of the eccentricities defined by equations (4.74) and (4.75) are given in Table **5.29** and shown in Fig **5.32**, respectively. Clearly, these values increase with an increase in the mass of the disc and the mass ratios of the system. Figure **5.31** above, shows the effects of the mass distribution, oblateness, triaxiality, radiation and perturbations on the eccentricity of the long and short period orbits, respectively.

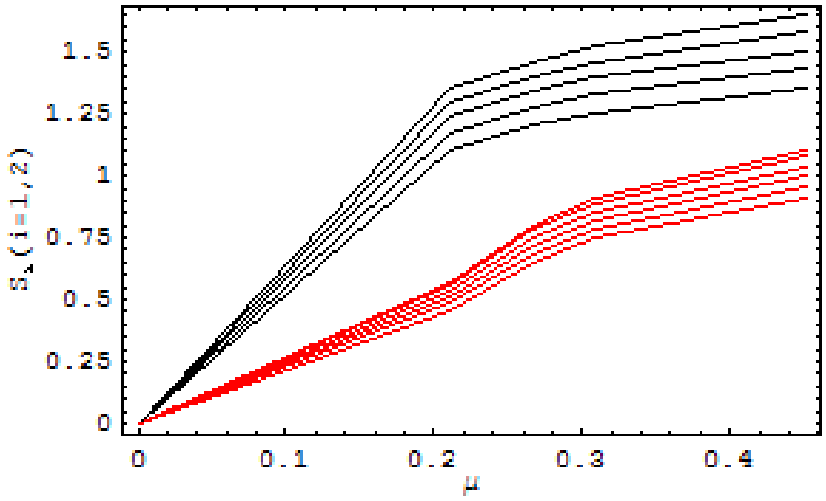



**Fig 5.30:** Variations in the frequencies $s_i$ $(=1,2)$ as a function of $\mu$

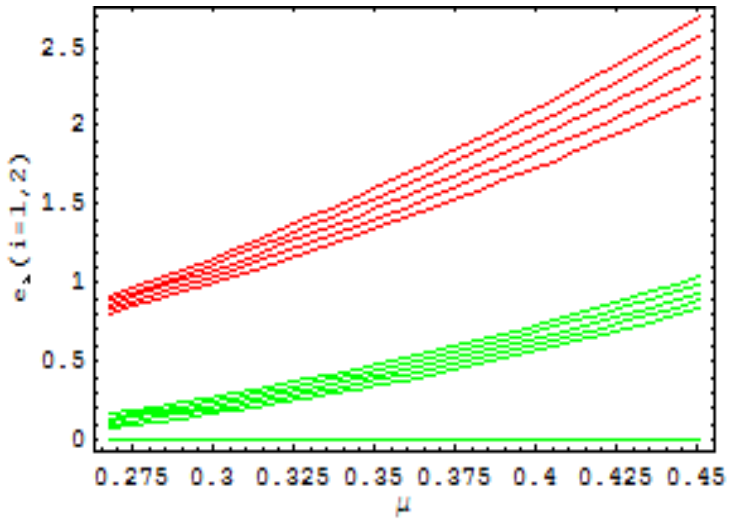



**Fig 5.31:** Variations of the eccentricities for different mass ratios and $0 \leq M_d \leq 0.09$

**Table 5.29:** Effects of the mass distributions on the eccentricities in IRAS 11472-0800 Vs G29-38 system, when $0 \le M_d \le 0.09$, $\sigma_1 = 0.000139$, $\sigma_2 = 0.000132$, $\alpha_2 = 0.02$, $q_2 = 0.9988$, $\in = 0.012, \in' = 0.002$

| $\mu$ (IRAS 11472-0800-G29-38) | $e_i\,(i=1,2)$ $(M_d = 0)$ | $e_i\,(i=1,2)$ $(M_d = 0.02)$ | $e_i\,(i=1,2)$ $(M_d = 0.04)$ | $e_i\,(i=1,2)$ $(M_d = 0.06)$ | $e_i\,(i=1,2)$ $(M_d = 0.08)$ | $e_i\,(i=1,2)$ $(M_d = 0.09)$ |
|---|---|---|---|---|---|---|
| 0.11816 | 0.04095 | 0.04194 | 0.04293 | 0.04392 | 0.04491 | 0.04541 |
| | 0.18556 | 0.17932 | 0.17307 | 0.16682 | 0.16057 | 0.15745 |
| 0.16666 | 0.05389 | 0.06047 | 0.06705 | 0.07363 | 0.08022 | 0.08351 |
| | 0.33964 | 0.33916 | 0.33869 | 0.33822 | 0.33774 | 0.33751 |
| 0.21135 | 0.18832 | 0.06409 | 0.06013 | 0.07843 | 0.08085 | 0.08707 |
| | 0.52293 | 0.53173 | 0.54054 | 0.54934 | 0.55815 | 0.56255 |
| 0.30875 | 0.30554 | 0.27866 | 0.25177 | 0.22488 | 0.19800 | 0.18455 |
| | 1.05975 | 1.10308 | 1.14640 | 1.18973 | 1.23306 | 1.25472 |
| 0.32735 | 0.38028 | 0.35045 | 0.32063 | 0.29081 | 0.26098 | 0.24607 |
| | 1.18366 | 1.23588 | 1.28809 | 1.34030 | 1.39252 | 1.41863 |
| 0.40119 | 0.74024 | 0.69863 | 0.65702 | 0.61541 | 0.57380 | 0.55300 |
| | 1.74340 | 1.83797 | 1.93253 | 2.02710 | 2.12167 | 2.16895 |
| 0.42196 | 0.85967 | 0.81479 | 0.76991 | 0.72502 | 0.68014 | 0.65770 |
| | 1.92031 | 2.02871 | 2.13711 | 2.24551 | 2.35391 | 2.40811 |
| 0.43452 | 0.93578 | 0.88894 | 0.84210 | 0.79526 | 0.74842 | 0.72500 |
| | 2.03145 | 2.14859 | 2.26574 | 2.38288 | 2.50003 | 2.55860 |
| 0.45061 | 1.03757 | 0.98826 | 0.93895 | 0.88964 | 0.84033 | 0.81567 |
| | 2.17844 | 2.30719 | 2.43593 | 2.56468 | 2.69342 | 2.75780 |

Now, equations (4.76-4.79) are lengths of the semi-major axis $a_i$ and minor axis, $b_i$ of the long period orbits. These lengths are given for chosen parametric values $\sigma_1 = 0.000139$, $\sigma_2 = 0.000132$, $\alpha_2 = 0.02$, $q_2 = 0.02$, $\in = 0.012$ and $\in' = 0.002$ in Table **5.30** and the graphical representations are given in figures **5.32**, **5.33**, **5.34**, **5.35** and **5.36**. Evidently, the lengths of the

semi-major axes increases always with increase in the mass of the disc, in the presence or absence of other parameters, while the same is not true for the semi-minor axes. This can be seen in Fig **5.34.**

**Table 5.30:** Effects of the mass of disc on the lengths of the semi-axes in IRAS 11472-0800 Vs G29-38 system, when $0.118166 \le \mu \le 0.450617, 0 \le M_d \le 0.09$, $\sigma_1 = 0.000139$, $\sigma_2 = 0.000132$, $\alpha_2 = 0.02$, $q_2 = 0.9988$, $\in = 0.012$ and $\in' = 0.002$

| $\mu$ | $a_1$ | $a_2$, | $b_1$ | $b_2$; |
|---|---|---|---|---|
| 0.11816 | 2.36561 | 1.3254 | 0.87336 | 0.85843 |
| | 2.36970 | 1.41839 | 0.86551 | 0.85517 |
| | 2.37380 | 1.60424 | 0.85766 | 0.85190 |
| | 2.37789 | 1.99009 | 0.84980 | 0.84864 |
| | 2.38199 | 1.97594 | 0.84195 | 0.84538 |
| | 2.38403 | 1.96887 | 0.83802 | 0.84375 |
| 0.21135 | 3.36284 | 1.60642 | 0.86403 | 0.78527 |
| | 3.43195 | 1.61628 | 0.84818 | 0.78729 |
| | 3.50106 | 1.62614 | 0.83233 | 0.78931 |
| | 3.57017 | 1.63600 | 0.81648 | 0.79132 |
| | 3.63928 | 1.64586 | 0.80063 | 0.79334 |
| | 3.67383 | 1.65079 | 0.79271 | 0.79435 |
| 0.26739 | 4.20113 | 1.48780 | 0.86084 | 0.71048 |
| | 4.33617 | 1.50097 | 0.83672 | 0.71785 |
| | 4.47122 | 1.51414 | 0.81261 | 0.72522 |
| | 4.60626 | 1.52732 | 0.78850 | 0.73258 |
| | 4.74131 | 1.54049 | 0.76438 | 0.73995 |
| | 4.80883 | 1.54707 | 0.75232 | 0.74363 |
| 0.30875 | 4.93465 | 1.42675 | 0.85965 | 0.64046 |
| | 5.1319 | 1.44102 | 0.82766 | 0.65292 |
| | 5.32916 | 1.45529 | 0.79567 | 0.66539 |
| | 5.52641 | 1.46956 | 0.76369 | 0.67785 |
| | 5.72367 | 1.48383 | 0.73170 | 0.69031 |
| | 5.8223 | 1.49096 | 0.71571 | 0.69654 |
| 0.40119 | 6.92705 | 1.33574 | 0.86057 | 0.43840 |
| | 7.30367 | 1.35056 | 0.80558 | 0.46561 |
| | 7.6803 | 1.36539 | 0.75059 | 0.49283 |
| | 8.05692 | 1.38021 | 0.69561 | 0.52005 |
| | 8.43355 | 1.39504 | 0.64062 | 0.54727 |
| | 8.62186 | 1.40245 | 0.61313 | 0.56087 |
| 0.45061 | 8.19215 | 1.30366 | 0.86309 | 0.30456 |
| | 8.68511 | 1.31812 | 0.79298 | 0.34122 |
| | 9.17806 | 1.33259 | 0.72287 | 0.37788 |
| | 9.67102 | 1.34706 | 0.65277 | 0.41454 |
| | 10.164 | 1.36153 | 0.58266 | 0.45121 |
| | 10.4104 | 1.36877 | 0.54760 | 0.46954 |

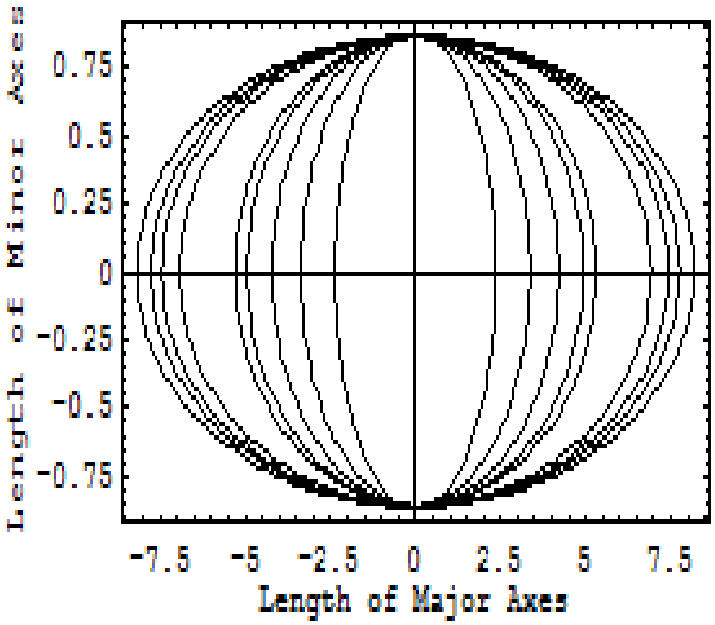



**Fig 5.32:** Effects of the parameters on the semi-axes of the orbits for (a) $\mu = 0.450617,\ M_d = 0$

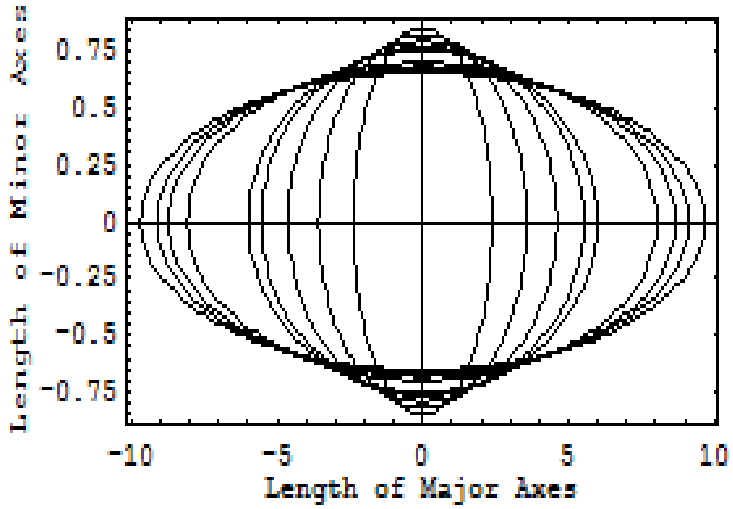



**Fig 5.33:** Effects of the parameters on the semi-axes of the orbits for $\mu = 0.450617$ and $M_d = 0.06$

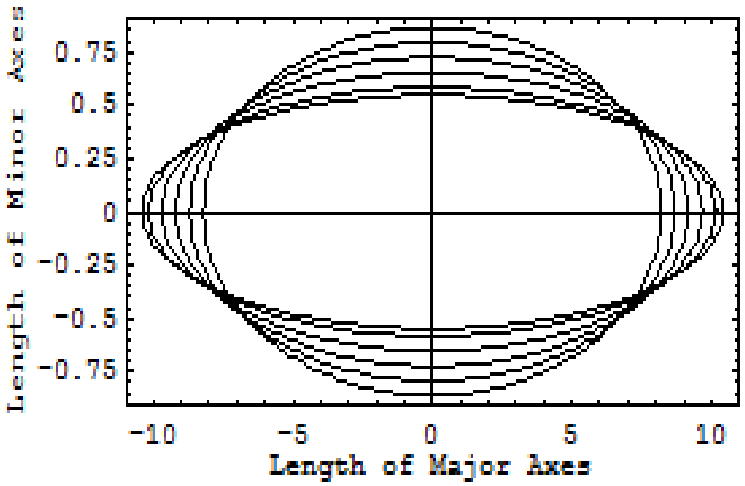



**Fig 5.34:** Effects of the parameters on the semi-axes of the orbits for $\mu = 0.450617$ and $0 \le M_d \le 0.09$

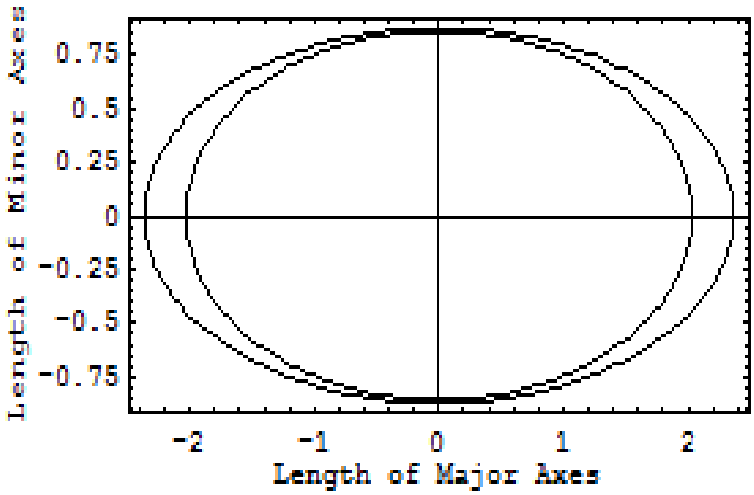



**Fig 5.35:** Lengths of the semi-axes of long and short period orbits for $\mu = 0.118166$ when $M_d = 0$.

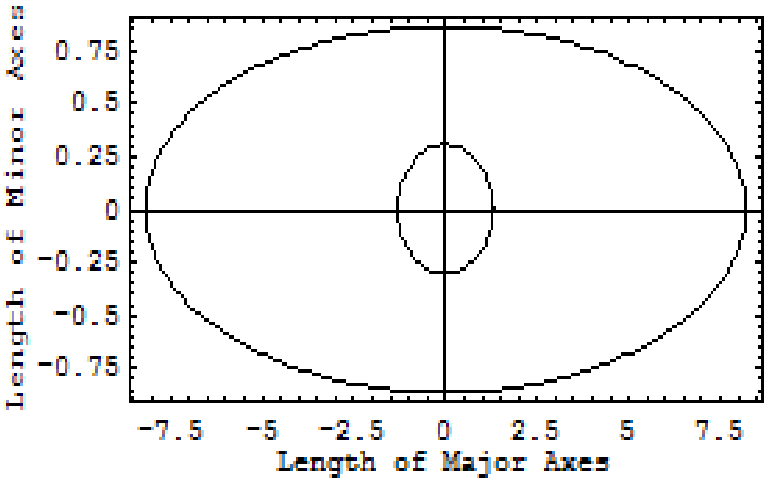
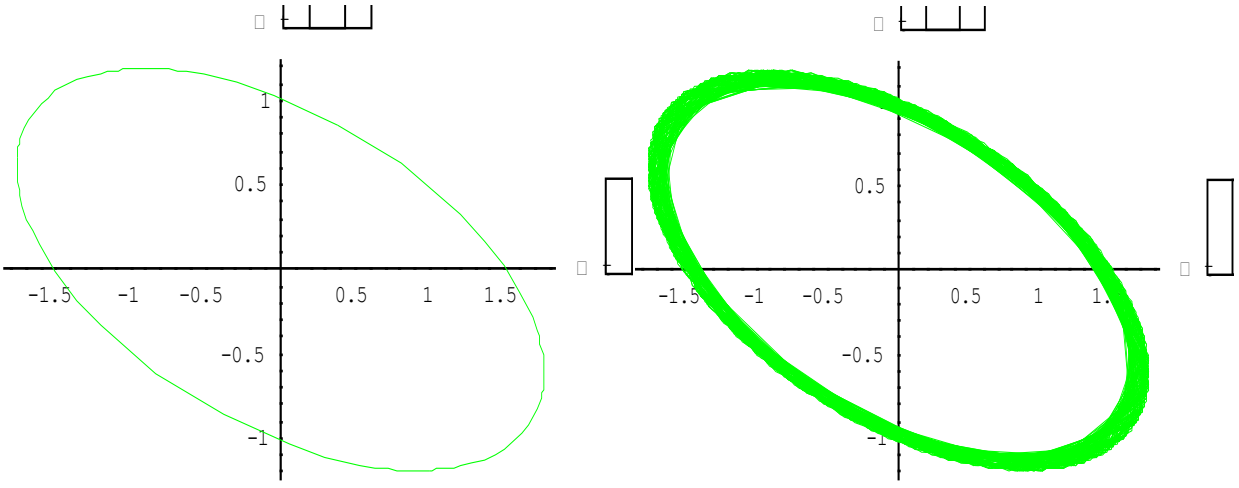



**Fig 5.36:** Lengths of the semi-axes of long and short period orbits for $\mu = 0.450617$ when $M_d = 0$.

Now, haven discussed the eccentricity and semi-axes of the orbits in the IRAS 11472-0800 and G29-38 system, we use Mathematica software (2006) to plots the orbits around the stable triangular equilibrium points with the help of equations (4.34). These orbits are shown in Fig **5.37** and Fig **5.38**. The orbits are periodic with periods $2\pi / s_1 = T_1$ and $2\pi / s_2 = T_2$; and bounded for a long period of time.

(a) (b)



(c)




**Fig 5.37:** Long periodic orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system for $\mu = 0.118166, \sigma_1 = 0.000139$ , $\sigma_2 = 0.000132$ , $\alpha_2 = 0.02$ , $q_2 = 0.9988$ , $\in = 0.012$ , $\in' = 0.002$ and $M_d = 0$ , when (a), $0 \le t \le T_1 \leftarrow 7.1688$ (b) $0 \le t \le 100T_1$ (c) $0 \le t \le 200T_1$

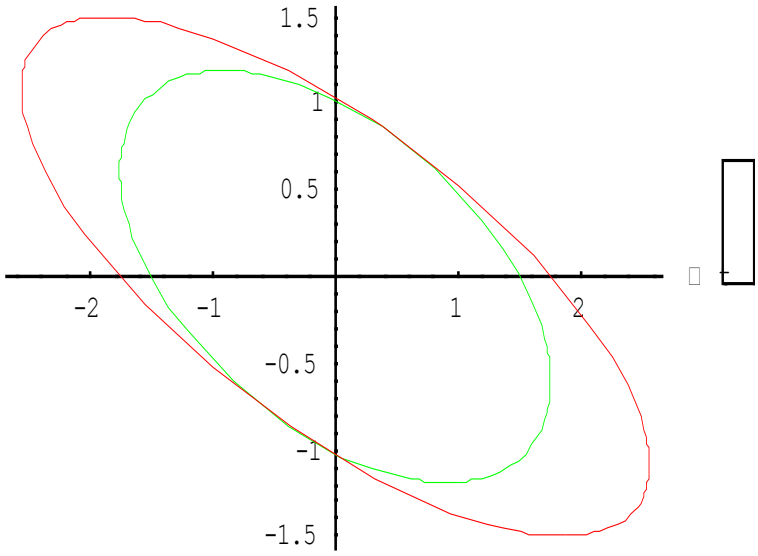



**Fig 5.38:** Short (green orbit) and long (red) periodic orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system for $0 \le t \le 12.3695$, when $\mu = 0.118166, \sigma_1 = 0.000139$ , $\sigma_2 = 0.000132$ , $\alpha_2 = 0.02$ , $q_2 = 0.9988$ , $\in = 0.012, \in' = 0.002$ and $M_d = 0$ .

Fig. **5.39** shows the deviations of the orbits for different mass ratios with our chosen parametric values. Here, the black, blue, red and green orbits are the cases when the mass ratios $\mu$ are $0.118166, 0.211356, 0.308756$ and $0.450617$, respectively. The orbits in Fig. **5.40** are plotted when $\mu = 0.450617$, for different values of the mass of disc and other parametric values. Here, the black, red, blue and green orbits are respectively the case when $M_d = 0$, $M_d = 0.04$, $M_d = 0.06$ and $M_d = 0.09$. Hence, the introduced parameters in this study, affect the periodic orbits.

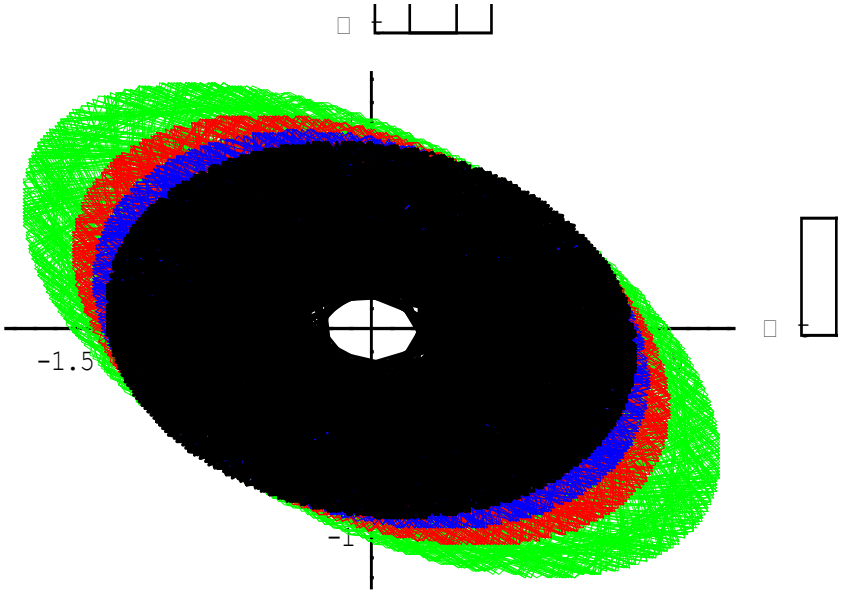



**Fig 5.39:** Variations in the orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system when $\mu = 0.118166$ (black orbit), $0.211356$ (blue orbit), $0.308756$ (red orbit), and $0.450617$ (green) for $0 \le t \le 200T^*$ $T^* = T_1 / T_2$ and the chosen parametric values.

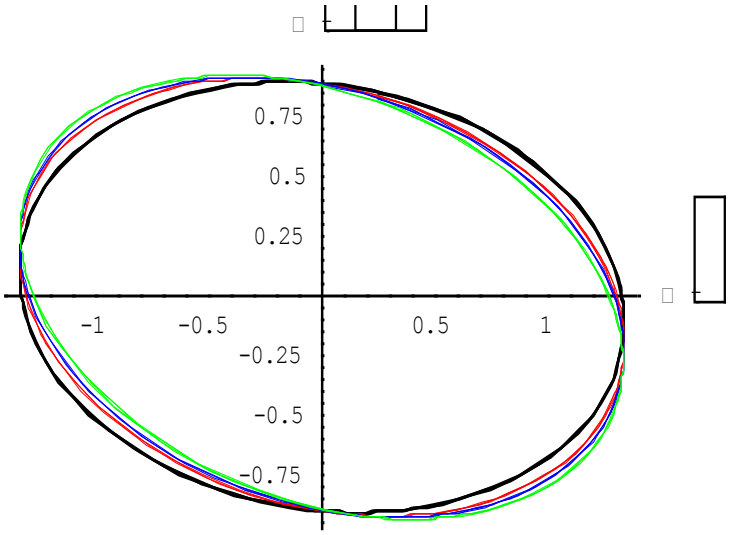



**Fig 5.40:** Variations in the orbits around $L_4$ of IRAS 11472-0800 Vs G29-38 system when $\mu = 0.450617$, $M_d = 0$ (black), $0.04$ (red), $0.06$ (blue) and $0.09$ (green); for $0 \le t \le T^*$

**Orbital stability criteria using Poincare Surfaces of Section**

Murray and Dermott (2000) have discussed a way to plot Poincaré surface of section (PSS). In order to find the orbital parameters of the third body of infinitesimal mass, we require location $(x, y)$ and the velocity coordinates $(\dot{x}, \dot{y})$ at different instant to give a 4-D dimensional phase space $(x, y, \dot{x}, \dot{y})$. With the help of the Jacobi integral (5.58), we express the variable $\dot{y}$ in terms of the other three variables and then substitute it in the equations of motion for each occurrence of $\dot{y}$, we reduce the phase space to a 3-D $(x, y, \dot{x})$ subspace of the original 4-D. Now, we follow many orbits numerically and determine the values of the remaining variables $(x, \dot{x})$ each time the orbit passes through the surface of section plane with positive $\dot{y}$. The restriction $\dot{y} > 0$ is needed so structures on the plot don't overlap. Hence, the PSS is the locus of plot of all points in the $x, \dot{x}-$plane at the location where the orbit intersect the plane. We have used the software

*Mathematica* wolfram (2006) using the method "EventLocator" and integrating orbits for very long period of time.

Figures **5.41** to **5.47** show the PSS for different chosen parametric values and the Jacobi constant $J$ (4.58). The regular islands which depicts the stability region increases or decreases depending on the Jacobi constant and the choice of the system parameters. The islands in the plane $x, \dot{x}$ correspond to the two initial conditions within the range which indicates that orbits have mean motion resonance of second order.

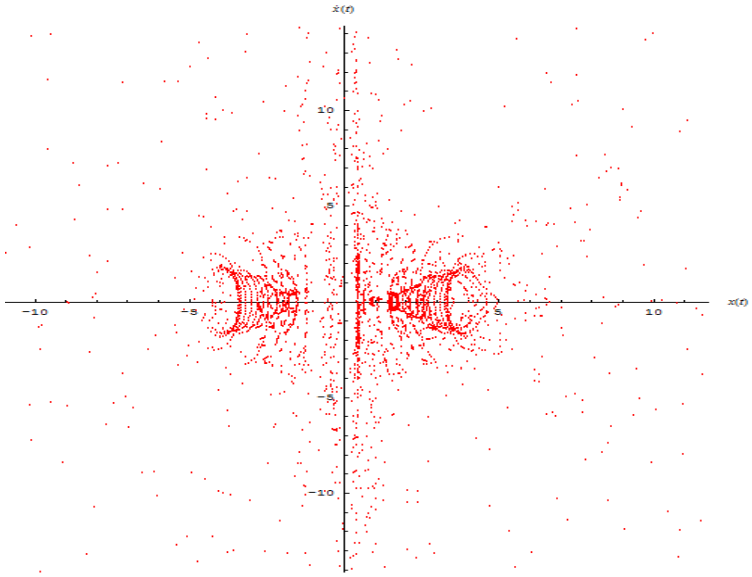



**Fig 5.41:** PSS for $\mu = 0.118166, \sigma_1 = \sigma_2 = 0, \alpha_2 = 0,\ q_2 = 1, \in = 0.012, \in' = 0.002,\ M_d = 0$ and $J = 2$

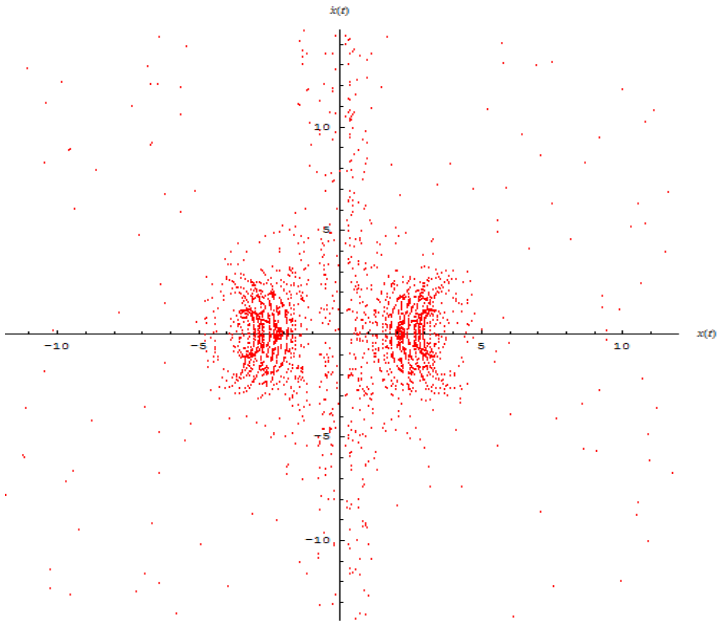



**Fig 5.42:** PSS for $\mu = 0.118166, \sigma_1 = \sigma_2 = 0$, $\alpha_2 = 0.02$, $q_2 = 1, \in = 0.012$, $\in' = 0.002, M_d = 0$ and $J = 2$

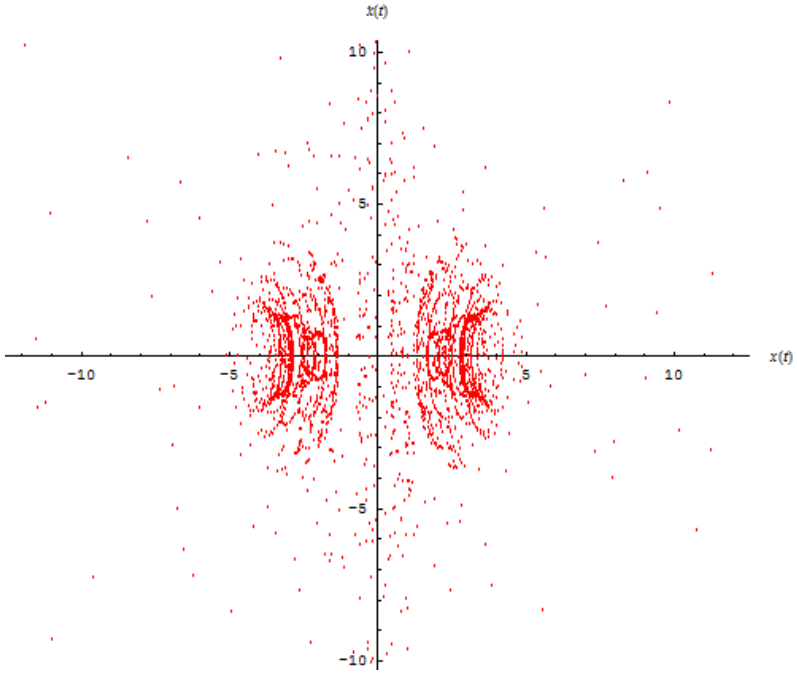



**Fig 5.43:** PSS for $\mu = 0.118166, \sigma_1 = \sigma_2 = 0, \alpha_2 = 0.02, q_2 = 0.9988, \in = 0.012$, $\in' = 0.002$ $M_d = 0, J = 2.7$

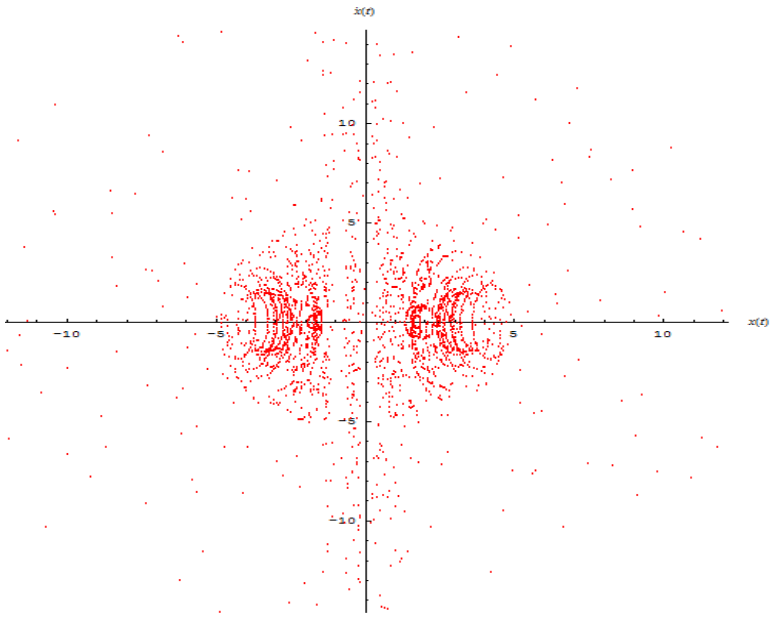



**Fig 5.44:** PSS for $\mu = 0.118166, \sigma_1 = 0.000139$ , $\sigma_2 = 0.000132$ , $\alpha_2 = 0.02$ , $q_2 = 0.9988$ , $\in = 0.012$ , $\in' = 0.002$ and $M_d = 0$ , $J = 2.7$

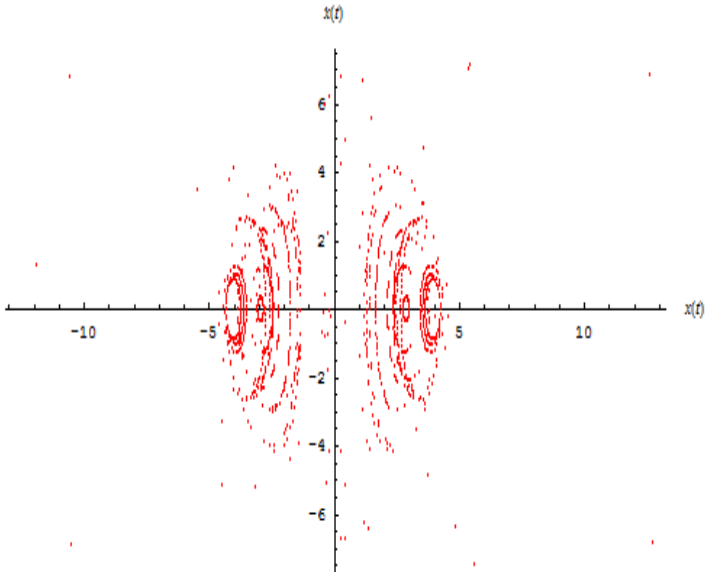



**Fig 5.45:** PSS for $\mu = 0.118166, \sigma_1 = 0.000139$ , $\sigma_2 = 0.000132$ , $\alpha_2 = 0.02$ , $q_2 = 0.9988$ , $\in = 0.012$ , $\in' = 0.002$, $M_d = 0.02$ and $J = 2.7$

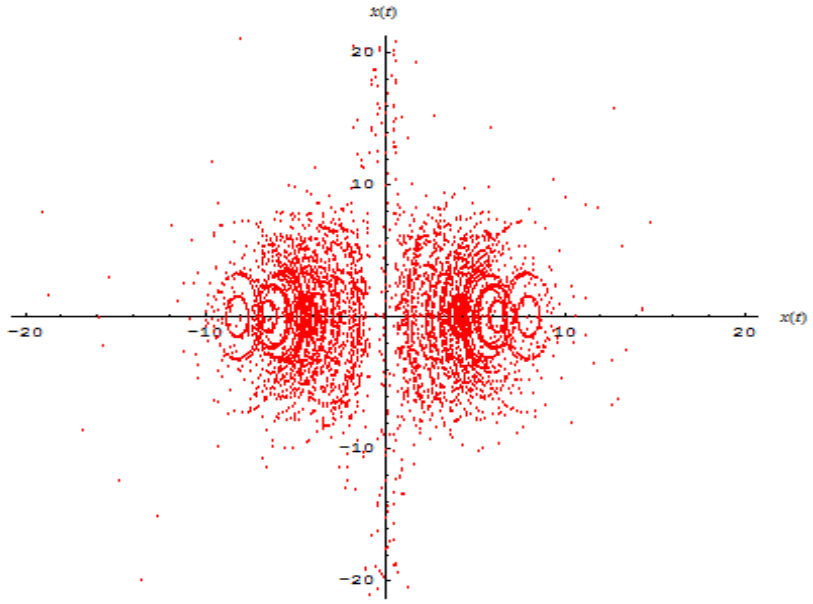



**Fig 5.46:** PSS for $\mu = 0.450617, \sigma_1 = 0.000139$ , $\sigma_2 = 0.000132$ , $\alpha_2 = 0.02$ , $q_2 = 0.9988$ , $\in = 0.012$ , $\in' = 0.002, M_d = 0.04$ and $J = 3$

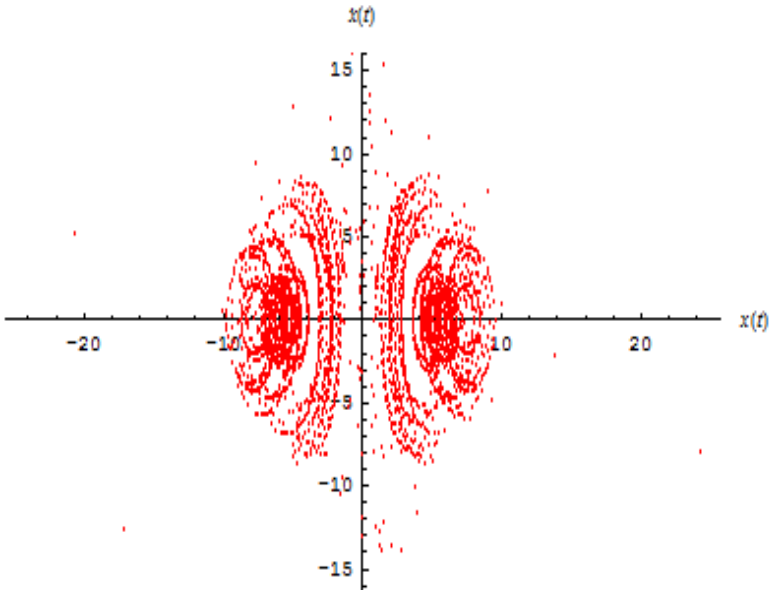



**Fig 5.47:** PSS for $\mu = 0.450617, \sigma_1 = 0.000139$ , $\sigma_2 = 0.000132$ , $\alpha_2 = 0.02$ , $q_2 = 0.9988$ , $\in = 0.012$ , $\in' = 0.002, M_d = 0.04$ and $J = 3.35$

At some points (Fig **5.43** and **5.47**), the islands begin to disappear gradually but they don't finally do as seen in the figures. Hence, the region of the orbital stability is controlled by the Jacobi constant and values of the parameters representing oblateness, radiation, triaxiality, mass of the disc and the small perturbations in the Coriolis and centrifugal forces and the orbits are stable.

In the case of the restricted problem of three oblate bodies under radiation pressure forces of the primaries, small change in the Coriolis and centrifugal forces and the force due to the mass of the disc, equations (4.103) gives the exact the coordinates of the triangular EPs. When there is no disc surrounding the oblate stars, the coordinates of triangular EPs reduces to

$$x=\mu-\frac{1}{2}+\frac{1}{\psi^{2/3}}\left[\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ -\frac{\alpha_1-\alpha_2\ \psi^{2/3}}{2}\right]$$

$$y=\pm\frac{\sqrt{4-\psi^{2/3}}}{2\psi^{1/3}}\left\{1-\frac{2}{4-\psi^{2/3}}\left[\frac{1}{3}\ 1-q_1\ +\frac{1}{3}\ 1-q_2\ +\ \alpha_1+\alpha_2\ -\frac{1}{2}\psi^{2/3}\ \alpha_1+\alpha_2+2\alpha_3\ \right]\right\}$$

and coincides with the triangular EPs worked out by Singh and Haruna (2014a). When there are no perturbations in the Coriolis and centrifugal forces, we have

$$x=\mu-\frac{1}{2}+\frac{1}{3}\ 1-q_1\ -\frac{1}{3}\ 1-q_2\ -\frac{\alpha_1-\alpha_2}{2}$$

$$y=\pm\frac{\sqrt{3}}{2}\left\{1--\frac{2}{9}\ 1-q_1\ -\frac{2}{9}\ 1-q_2\ -\frac{2}{3}\ \alpha_1+\alpha_2\ +\frac{1}{3}\ \alpha_1+\alpha_2+2\alpha_3\ \right\}$$

the points fully coincide with those of Abouelmagd and El-Shaboury (2012).

The triangular points of Singh and Taura (2013), Kushvah (2008a), AbdulRaheem and Singh (2006), Singh and Ishwar (1999), Sharma (1987), Sharma (1982), Bhatnagar and Chawla (1979), Bhatnagar and Hallan (1978), SubbaRao and Sharma (1975) and, Szebehely (1967a, 1967b) can all be recovered from equation (4.103).

Figures **5.48**, **5.49, 5.50, 5.51** and **5.52** are the 3D-surface plots of the triangular EPs (4.103).

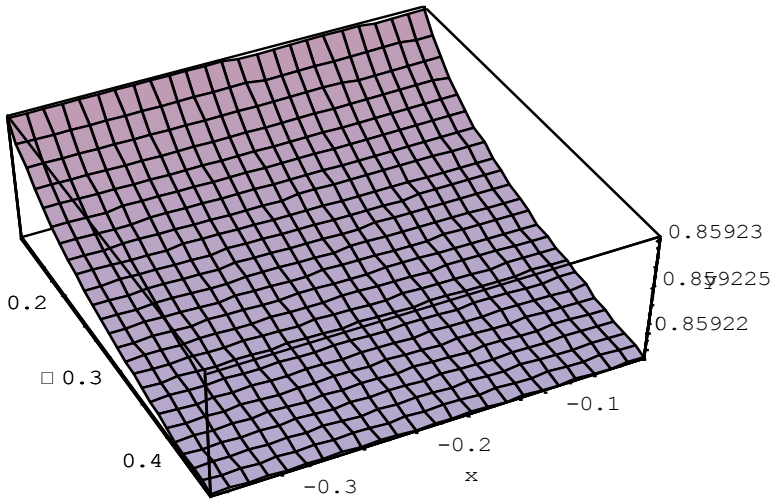



**Fig 5.48:** Surface plot of $L_4$ for $0.118166 \le \mu \le 0.450617$, $q_1 = 0.9988$, $q_2 = 0.9985$, $\psi = 1.002$, $\alpha_1 = 0.02$, $\alpha_2 = 0.018$, $\alpha_3 = 0.01$, $M_d = 0$.

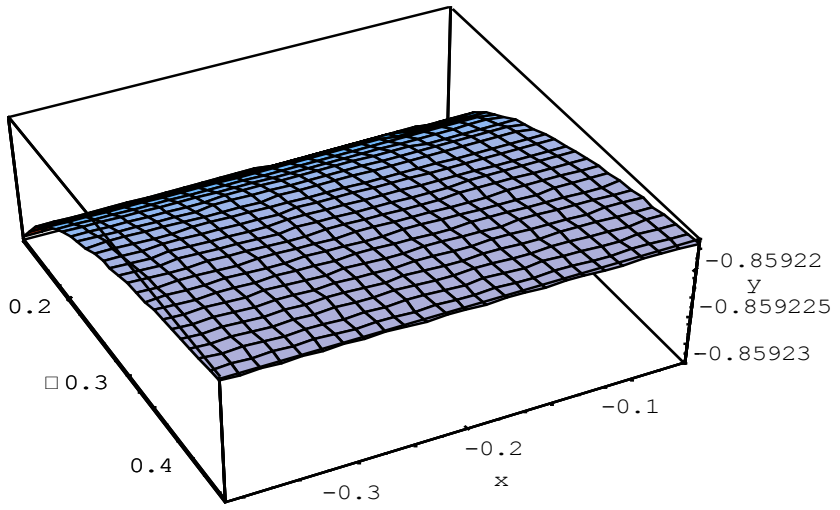



**Fig 5.49:** Surface plot of $L_5$ for $0.118166 \leq \mu \leq 0.450617, q_1 = 0.9988,\ q_2 = 0.9985,$ $\psi = 1.002, \alpha_1 = 0.02,\ \alpha_2 = 0.018,\ \alpha_3 = 0.01, M_d = 0.$

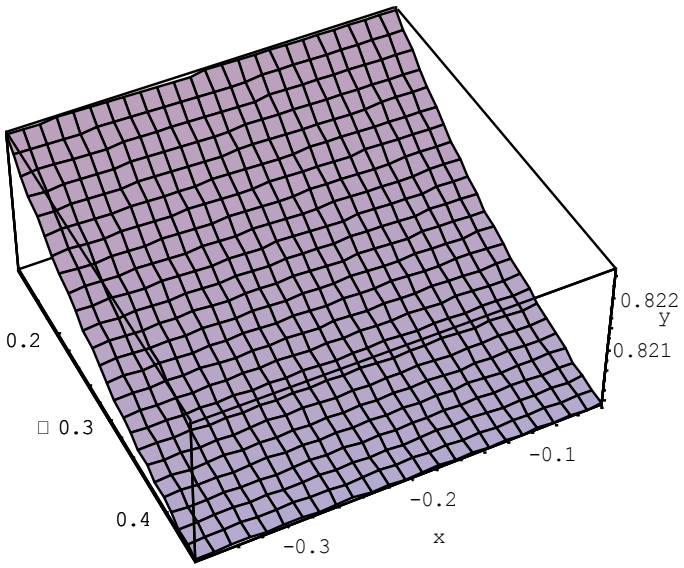



**Fig 5.50:** Surface plot of $L_4$ for $0.118166 \leq \mu \leq 0.450617, q_1 = 0.9988,\ q_2 = 0.9985,$ $\psi = 1.002\ , \alpha_1 = 0.02,\ \alpha_2 = 0.018,\ \alpha_3 = 0.01, M_d = 0.09$

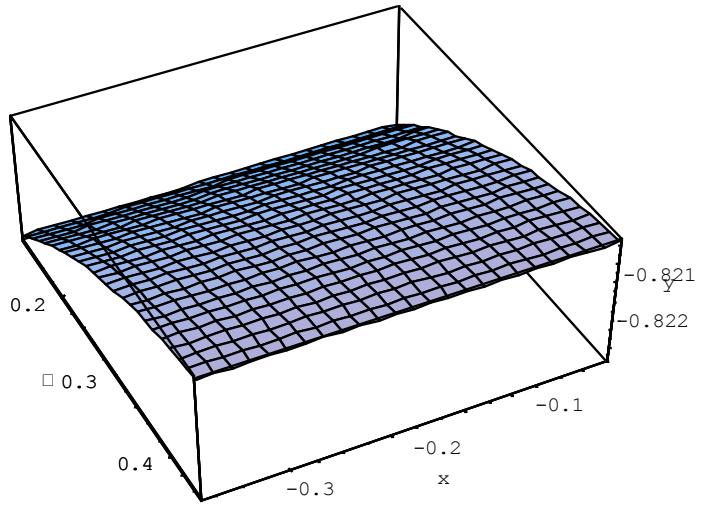



**Fig 5.51:** Surface plot of $L_5$ for $0.118166 \le \mu \le 0.450617, q_1 = 0.9988$ , $q_2 = 0.9985, \psi = 1.002$ , $\alpha_1 = 0.02$ , $\alpha_2 = 0.018$ , $\alpha_3 = 0.01, M_d = 0.09$

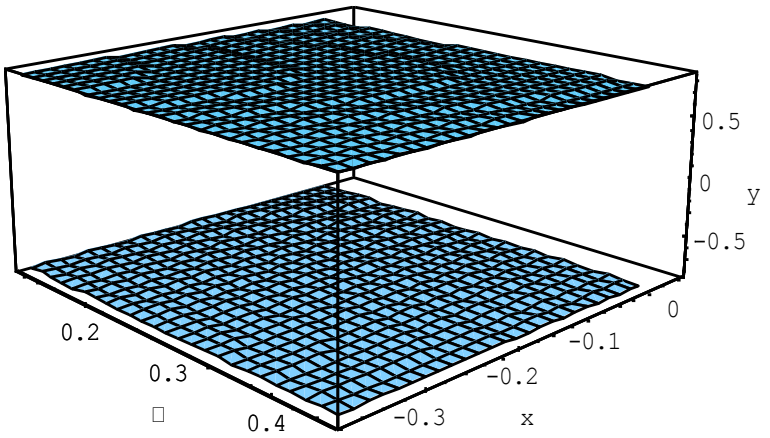



**Fig 5.52**: Surface plot of $L_{4,5}$ for $0.118166 \le \mu \le 0.450617$, $q_2 = 0.9985, \psi = 1.002, \alpha_1 = 0.02$ $\alpha_2 = 0.018$ , $M_d = 0.09$

Figures **5.48** and **5.49** show the positions $L_4$ and $L_5$ of the triangular EPs when there is no circumbinary disc $(M_d = 0)$ enclosing the stars, respectively. In this case, the dust particle is farther away from the line joining the stars. However, in the presence of disc, when $M_d$

gradually reaches up to 0,09, the dust particle is drawn nearer to the line on which the primaries lie and even goes closest, when the mass of the disc reaches close to 0.1 (Fig. **5.50** and Fig **5.51**). Hence, an increase in the mass of the circumbinary disc produces a shift in the positions of the triangular EPs and consequently makes the particle to be positioned closer to the primaries; although, this is also influenced by the oblateness and radiation pressure forces acting on the third body.

The critical mass parameter is given by the equation (4.110). The first term represents Routh's critical mass ratio (Szebehely 1967a). The second and third expressions represent the effects arising from oblateness of the primaries, respectively, while fourth, fifth, sixth and seventh term, are the effects owning to small change in Coriolis and centrifugal forces, radiation pressure force of the primaries, and oblateness of the third body, respectively. The last expression stands for the effect of gravitational potential from the cluster of material around the stars. The critical mass calculated by Singh and Haruna (2014a), Singh and Taura (2013), Abouelmagd and El-Shaboury (2012), AbdulRaheem and Singh (2006), Singh and Ishwar (1999), Sharma (1987, 1982), Bhatnagar and Chawla (1979), Bhatnagar and Hallan (1979, 1978), SubbaRao and Sharma (1975), Szebehely (1967a, 1967b), Routh (1875), and that in problem **4.2** can all be verified from equation (4.110). It is seen that the small deflection caused by the Coriolis force and the potential from the disc are both stabilizing forces while the radiation pressure of the primaries, the oblate shape of the bodies and the deflection as a result of the centrifugal force, are all destabilizing forces. The periodic orbits around stable triangular EPs in this problem is a generalized study of those performed by Singh and Haruna (2014b), Singh and Begha (2011b), AbdulRaheem and Singh (2008), Perdios and Kalantonis (2006), Perdios (2003), Poddar (2002 a, b), Elipe and Lara (1997), Sharma (1976), Meyer and Schmidt (1971) and Szebehely (1967a).

# CHAPTER SIX

## 6.0. SUMMARY, CONCLUSION AND RECOMMENDATIONS

This chapter summarizes, concludes and recommend future open problems of this thesis on the investigations of motion of a test body around equilibruim points of the circular restricted problems of three-bodies taking into account perturbing forces with variable and constant masses.

## 6.1. Summary

The problem of three bodies when the masses of the involved bodies are assumed to vary with time and when they are kept constant under various characterizations has been studied. We have carried out our study using the models of the R3BP and the Robes R3BP. The R3BP with variable masses under diverse characterizations have been studied in chapter **THREE** Investigation of the restricted problem when the masses are constant was captured in chapter **FOUR**, while the results and discussion are contained in chapter **FIVE.**

The introduction of this dissertation was given in chapter **ONE** while the review of related literature were highlighted in chapter **TWO**. Chapter **THREE** dealt with the restricted problem of three-bodies under the set up of the general R3BP and the Robe's (1977) model, when the masses are subject to variations with time. In the R3BP with variable masses, the primaries are assumed to vary isotropically in accordance with the UML and their motion determined by the GMP. Here, different characterizations of the primaries which are radiation pressure forces, oblateness, perturbations in the Coriolis and centrifugal forces and the inclusion of a circumbinary disc surrounding the primaries, were included in our study. These characterizations under the study of mass variations of the primaries gave rise to different problems which we

discussed in the sections of this chapter. In section **3.2**, we studied the effects of small perturbations in the Coriolis and centrifugal forces when the first primary is an oblate spheroid and the second one is a radiation emitter. The equations of motion were presented in subsection **3.2.1** while the search for the EPs was achieved in section **3.2.2**. In Subsections **3.2.3,** we dealt with the linear stability analysis of the EPs.

In section **3.3**, we looked at the case when both primaries are oblate spheroids and are radiating under the influence of small perturbations in the Coriolis and centrifugal forces coupled with mass variations of the primaries. In the subsections, we presented the equations of motion, locations and linear stability of the EPs and the discussion, respectively.

Section **3.4** captured investigations of the restricted problem with variable masses of the primaries when there is a variable disc present and we limited our study to the investigations of the existence and stability of the out-of-plane EPs only. The equations of motion, existence of the out-of-plane EPs, the linear stability of the out-of-plane points and the discussion, were treated. In order to give a complete treatment of our study, we examined the periodic orbits around linearly stable triangular EPs with variable masses under combined effects of radiation, perturbations and oblateness in section **3.5**.

A different description of the R3BP called the Robe's (1977) restricted model, in which the third body is located inside the first primary has had major breakthrough ever since its formulation. This problem under mass variations of the bodies is studied under various assumptions in sections **3.6**, **3.7** and **3.8**. In section **3.6,** the restricted problem when the rigid spherical shell which is the first primary is empty and the masses of the primaries vary isotropically in obedience to the UML while their motion is governed by the GMP. In section

**3.7**, we extended the problem in section **3.6** by introducing small perturbations in the Coriolis and centrifugal forces. The last problem of the chapter is dealt with in section **3.8**. This problem has the same configurations as the previous ones, but here all the masses of the participating bodies are subject to variations and the first primary is assumed to be a fluid with no rigid spherical shell. The equations of motion of the problem, the existence of the EPs, their linear stability, were given. We draw the conclusion of chapter **THREE** in section **3.9**.

In chapter **FOUR**, we investigated the R3BP when the masses of the involved bodies are kept constant with the view that the motion of the third body occurs not only in the vicinity of the primaries but also in the presence of a circumbinary disc. We considered various characterizations of the bodies, to include small perturbations in the Coriolis and centrifugal forces, oblateness and triaxiality of the bodies and radiation pressure forces of the primaries.

Section **4.2** dealt with the restricted problem when the first primary is a radiation source while the second primary and the third body are modeled as oblate spheroids. Here, the EPs, their stability and the orbits about the triangular EPs, are investigated. It is seen that when the motion of the third body takes place under the gravitational potential of the disc, two additional collinear EPs are found depending on the mass parameter of the system and the mass of the disc. Though numerical evidence showed that the existence of the two additional points depend on the system under study. They did not exist in the case of the Sun-Earth system and Sun-Jupiter system because their mass parameters are very small but existed in the case of the Fomalhaut-Fomalhaut **b** system where the mass parameter is not strictly small.

Section **4.3** studied the motion around the triangular EPs in the gravitational field of a binary system, when the primary star is a triaxial star and the second one is a radiating-oblate star

coupled with small perturbations in the Coriolis and centrifugal forces. In particular, we applied the study to the case when the first primary is a depleted pulsating post-AGB star, IRAS 11472-0800, while its companion is a white dwarf star, G29-38 . The stability of the orbits was tested using the Poincare Surfaces of Section (PSS). In section **4.4,** the generalized study of the R3BP with constant masses is solved by taking all the three bodies to have the shape of an oblate spheroid while both primaries are radiation emitters and are enclosed by a disc of dust under small perturbations in the Coriolis and centrifugal forces.

The results and discussion of the thesis have been given in Chapter **FIVE.** Here, we have discussed the outcome of our findings with those of other related works. We also gave several illustrative numerical explorations to demonstrate the beauty and significance of each of the problems we have studied. The summary, conclusion and recommendations summed up the dissertation in Chapter **SIX.**

## 6.2. Conclusion

This research studied the restricted problem of three-bodies using the models of the R3BP (Szebehely 1967a) and the Robe's (1977) R3BP under different characterizations of the primaries. In the case of the R3BP with variable masses where the masses of the primaries are assumed to vary with time isotropically in accordance with the unified Mestschersky law and their motion described by the Gylden-Mestschersky problem, we found two additional collinear EPs when both primaries are radiating and are oblate spheroids under effects of small perturbations in the Coriolis force. Two triangular and four out-of-plane EPs also exists. In the presence of disc, we also established four out-of-plane EPs which were found using series expansion about the disc mass. The collinear and triangular EPs of the autonomized systems are

stable while the out-of-plane EPs are unstable. The EPs of the non-autonomous systems differ form those of the autonomized systems by the function $R\ t$ and they are unstable. Because of the mass variations, the oblatenss, perturbations in the Coriolis and centrifugal forces of the primaries have both stabilizing and destabilizing tendencies while the radiation pressure forces are always destabilizing.

In the study of the periodic orbits, it is seen that the orbits can be elliptical, parabolic or hyperbolic in nature. The frequencies, orientation, eccentricities and lengths of the semi axes have been found and are defined by the parameters of the system.

In the case of Robe's R3BP with variable masses, when the primaries vary isotropically under the UML and their motion described by the GMP; an EP at the center of the shell and a pair of non-collinear EPs exists. Under effects of small perturbations in the Coriolis and centrifugal forces, points on the line joining the primaries are EPs, provided they are located inside the first primary. Also, a pair of non-collinear EP exists. However, in the case of the Robe's problem of three variable mass bodies where the first primary is a fluid; an EP at the center, another near the center, circular points and pairs of non-collinear EPs exist. The collinear EPs are stable while the non-collinear and circular EPs are unstable.

We observe that for the R3BP with variable masses, there are four EPs found on the $\xi\zeta-$plane while only a pair exists in the case of the Robe's R3BP with variable masses.

. In the study of R3BP with constant masses in the presence of a circumbinary disc under different characterizations, it is seen that when the motion of the third body takes place under the gravitational potential of the disc, two additional collinear EPs are found depending on the mass parameter of the system and the mass of the disc. A pair of triangular EP exists and is stable

while the collinear points are unstable. In the range of stable triangular EPs, it is seen that orbits around these points are ellipses having long and short period orbits. The orientation, eccentricities, and the semi axes of the orbits are found. The stability of the orbits is tested using the Poincare Surfaces of Section (PSS) and they are found to be stable.

Our problem can be used to study the long-term motion of planets in binary systems. The planets are taken to be test particles moving in the field of circular binary system. A simple question of celestial mechanics is, how long can the EPs keep a planet from escaping? The determination of ranges of semi-major axes may help to know when the planet is likely to escape. Further, the importance of variable mass models to global dynamics cannot be ignored. The obtained solutions may be used in different problems of stellar dynamics, and also in other astrophysical applications. For example, polar solution may be considered as a possible gravitational model of jets (astrophysical mass flow in the vicinity of young stars and galaxies) and the obtained equilibrium points may serve as indicators of the potential variations of the evolved stars.

## 6.3. Recommendations

We recommend the following problems which are still open to our knowledge.

- The restricted problem with variable masses when the primaries are charge particles may open up new research grounds and solve problems of global dynamics
- The restricted problem with quadratic and Poynting-Robertson drag when the masses are subject to variations.
- The Robe's problem with variable mass with gravitational potential from the disc
- The Robe's elliptical R3BP with variable masses.

- Robe's R3BP with variable masses when there is a porous medium and channel.

## 6.4. Contributions of The Thesis To Knowledge

1. The thesis is the first to introduce and incorporate varying oblateness of the primaries in the study of the R3BP

2. It is also the first to introduce varying mass of the circumbinary disc in the R3BP

3. We created a transformation that connects the variable and constant oblate shape of the primaries and also one which connects the disc mass.

4. This research work is the first to establish five collinear equilibrium points in the study of the R3BP with variable masses

5. The thesis also showed that oblateness and the small perturbation in the centrifugal force can have both stabilizing and destabilizing tendencies and strongly validated that the radiation pressure force is always a destabilizing force.

6. This study is the first to note that there exist a critical value of the character of mass variation which can annul the effects of the perturbing forces.

7. The thesis is the first to introduce mass variation into the Robe's R3BP

8. It is also the first to establish two non-collinear equilibrium points in the Robe's problem when there is no fluid in the shell.

9. The study is the first to discuss the effects of mass variations of the primaries on the periodic orbits of the third body of infinitesimal mass.

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