TRANSMUTATION OF WEIBULL PARETO DISTRIBUTION

BY

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DECLARATION

I declare that the work in this dissertation entitled **"Transmutation of Weibull-Pareto Distribution"** has been performed by me in the Department of Statistics, Ahmadu Bello University, Zaria under the supervision of Prof. A. Isah and Dr A. Yahaya.

The information derived from literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any University or Institution.

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Name of Student	Signature	Date

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Last but not the least, I will like to thank my mother and brother for their prayers and support, my wife for her patience and never-ending support and MrOladeleKehinde for his support and constant interest in whatever I do.

DEDICATION

In memory of my late father

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ABSTRACT

A generalization of the New Weibull-Pareto so called Transmuted Weibull Pareto Distribution is proposed and studied using the quadratic rank transmutation map (QRTM). The structural properties which include explicit expression for the quantiles, moment and order statistics of the proposed distribution were derived. Its parameters were estimated using the method of Maximum likelihood and least square estimation. The Transmuted Weibull Pareto Distribution was applied to real life data set to examine its flexibility over the New Weibull Pareto distribution. The result obtained shows that the Transmuted Weibull Pareto Distribution outperforms the New-Weibull Pareto distribution. The Performance criteria used are the AIC and BIC values

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CHAPTER ONE

INTRODUCTION

1.1 Background to the study

Inthe field of Statistics and reliability engineering, the quality of the procedures used in statistical analysis depends heavily on the assumed probability distribution. Due to this fact, significant efforts have been made by many researchers in the development of standard probability distributions which are different from the known classical probability distribution. The standard probability distributions are obtained by generalizing the classical probability distribution such as exponential, Weibull, Pareto and Beta distributions. Application of probability distributions in engineering, medicine, finance, ICT among others, have further shown that many data sets do not follow the existing classical distributions. As a result of this, there is the need for development of standard probability distributions by generalization of some well-known classical distributions.

These distributions are derived by adding one or more parameters to the baseline model of continuous distributions. These families provide more flexibility in modelling and in analyzing real life data in many applied areas. For instance, the generalized transmuted-G family proposed by Nofal*et al*(2015), Transmuted Weibull Distribution: A Generalization of the Weibull Probability Distribution proposed by Aryal and Tsokos (2011), the transmuted geometric-G family introduced by Afify*et al*(2016), the transmuted exponentiated generalized-G class of distributions defined by Yousof*et al*(2015), transmuted exponential distribution proposed by Enahoro*et al*(2015)and the Kumaraswamy transmuted-G family introduced by Afify*et al* (2016).

Over the years, several attempts have been made to generalize the Weibull distribution by adding new parameters into the distribution which has led to the development ofnew distributions. For instance theExponentiated Weibull distribution (Pal *et al* 2003), Transmuted Weibull distribution (Gokarnal and Chris, 2011), Lomax-Weibul distribution (Almheidat et al 2015) ,Beta Weibull Distribution (Cordeiro et al 2012), New Weibull-Pareto distribution (Suleiman and Albert 2015).These distributions have been found to be more flexible than the Weibull distribution when applied to real life data sets.

1.2 Weibull Distribution

. The Weibull distribution is a well-known distribution named after WaloddiWeibull, He developed the distribution in 1939 and applied it to analyze the breaking strength of materials.

A random Variable X is said to have a Weibull distribution with pdf

$$f(x) = \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)\alpha}$$

where$x > 0, \alpha > 0,\ \lambda > 0$

The distribution is mostly used in reliability engineering for studying the fatigue and endurance life in devices and materials. There is the fact that the Weibull distribution cannot capture the behavior of life time data sets that exhibit bathtub or upside down bathtub failure rate that are usually encountered in reliability engineering. This also led to several generalization of Weibull distribution which have been proposed and studied to address this limitation of Weibull.

1.3 Pareto Distribution

The Pareto distribution was developed in the 19th century by Italian economist Vilfredo Pareto to model the distribution of income over a population. Let x be a random variable from a Pareto distribution with its probability density function given by

$$f(x;\theta,k)=\frac{k\theta^k}{x^{k+1}}$$

where $\theta > 0$ is a scale parameter and $k > 0$ is the shape parameter

`Pareto distribution is often used to model life time of a manufactures item.

1.4New Weibull Pareto Distribution

According to Suleiman and Albert (2015), a life random variable X is said to have a New Weibull Pareto distribution denoted by NWPD if its distribution function has the form:

$$G(x)=1-e^{-\delta(\frac{x}{\theta})^{\beta}} \quad (1)$$

And its probability density function is

$$g(x)=\frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1}e^{-\delta(\frac{x}{\theta})^{\beta}} \quad (2)$$

where$0 < x < \infty, \beta > 0, \theta > 0, \delta > 0$

$\theta$is the scale parameter, β and δ are shape parameters.

The New Weibull Pareto distribution (NWPD) is suitable for modelling components that wears faster with time or component that wears slower with time. Another usefulness of NWPD is that it can be used in the characterization of the survival time of a given system because of its analytical structure (Suleiman and Albert 2015).

1.5Statement of the Problem

Many real life data sets are heavily skewed and cannot be fitted by theexisting New Weibull Pareto Distribution. As a result of this, there is the need to generalize the New Weibull

Pareto distribution by introducing a new parameter that will increase the flexibility of the existing New Weibull-Pareto distribution that will provide a better fit than the New Weibull-Pareto Distribution.

1.6 Significance of the Study

The introduction of a transmuted parameter to the New Weibull Pareto Distribution will provide more flexibility in modeling and analyzing real life datasets in many applied areas and greatly improve the sensitivity and efficiency of the statistical tests associated with the distribution. Some real life data sets do not follow the existing New Weibull-Pareto distribution, as a result of this there is the need to generalize the New Weibull-Pareto distribution by adding a new parameter to it so as to increase its flexibility.

1.7Aim and Objectives of the Research

The aim of this research is to propose a new distribution called Transmuted Weibull Pareto Distribution using the Quadratic Rank Transmutation Map and to derive its properties.

The objectives are to:

ii. Derive the Transmuted Weibull Pareto Distribution using Quadratic Rank Transmutation Map(QRTM)

ii. Determine the reliability behavior (survival and hazard function) and obtain various structural properties.

iii. Estimate the parameters of the transmuted distribution through the Maximum Likelihood Estimation and Least Square Estimation.

iv. Compare the performance of the Transmuted Weibull Pareto Distribution to New Weibull Pareto Distribution (Suleiman et al 2015),Weibull-Pareto Distribution (Tahir *et al* 2015) and Pareto Distribution.

1.8 Definition of Terms and Relevant Abbreviations

AIC:Akaike Information Criterion

BIC: Bayesian Information Criterion

CDF: Cumulative Distribution Function

Classical Probability Distribution: These are the common probability distributions such as Exponential, Weibull, Pareto andp Beta distributions.

Common events: These are events that occur with a high frequency

PDF: Probability Density Function

Real life Data: This is a data set obtained through an experiment. It is a non-simulated data set

Rare events: These are events that occur with a low frequency

Standard Probability Distribution: These are probability distributions that are obtained by adding one or more parameters into the common probability distributions.

Transmuted Probability Distribution: These are probability distributions obtained using the Quadratic Rank Transmutation Map

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

Many distributions have been proposed by adding one or more parameters to the baseline distribution with relevant statistical methodologies. As a result of this, statistical literature contains a good number of transmuted probability distributions.

2.2 Transmuted Probability Distributions

Shaw and Buckley (2007) introduced a technique for introducing skewness parameter into a symmetric or other distribution through “transmutation" map, which is the functional composition of the cumulative distribution function of one distribution with the inverse cumulative distribution (quantile) function of another.

Aryaland Tsokos (2011) introduced a new generalization of the Weibull distribution called the transmuted Weibull distribution. The Weibull distribution is a well-known probability distribution and widely used in reliability engineering for analyzing lifetime data. The Transmuted Weibull distribution was developed using the quadratic rank transmutation map and taking the 2-parameter Weibull distribution as the base distribution. Various mathematical properties of the Transmuted Weibull distribution were obtained; the Transmuted Weibull distribution was applied on two data sets to examine its flexibility over Weibull distribution. It was observed that the Transmuted Weibull Distribution performed better than the Weibull distribution.

Ashour and Eltehiwy(2013) studied a new generalization of exponentiatedTransmuted Weibull distribution called the transmuted exponentiatedTransmuted Weibull distribution. The subject distribution was derived by using the quadratic rank transmutation map and taking the exponentiatedTransmuted Weibull distribution as the baseline distribution. Various structural properties such as the moments, quantile, and moment generating function were also obtained. The least square method of parameter estimation was used for the parameter estimation.

Faton*et al*(2014) proposed a new distribution called transmuted generalized inverse Weibull distribution which extends the generalized inverse Weibull distribution. Various statistical properties were obtained and the parameters of the Transmuted generalized inverse Weibull distribution were estimated through Maximum likelihood estimation. An application of the TGIW distribution to real data showed that the TGIW distribution performed better than the generalized inverse Weibull distribution.

In addition, Fatonp*et al* (2014) also proposed a new model called the transmuted Pareto distribution which extends the Pareto distribution in the analysis of data with real support. Various properties such as expectation, variance, moments and moment generating function were derived. The model parameters of the transmuted Pareto were estimated using Maximum Likelihood Estimation and the likelihood ratio statistic was used to compare their model with the baseline model.

Similarly, Ahmed *et al*(2016) proposed a new four parameter distribution called Transmuted Weibull-Pareto distribution by extending the Weibull-Pareto distribution introduced by Tahir *et al* (2015). Some mathematical properties of the distribution were provided and the Maximum Likelihood Estimation was adopted for the estimation of model parameters.

Furthermore, Enahoro*et al*(2015) studied the performance of the Transmuted Exponential distribution with respect to Beta Exponential, Generalized Exponential and Exponentiated Exponential. It was observed that the shape of the Transmuted Exponential distribution could be decreasing or unimodal depending on the value of the parameters. The Transmuted Exponential distribution provides a better fit than the Beta Exponential, Generalized Exponential and Exponentiated Exponential distribution in terms of flexibility when applied to two real life data sets.

Oguntunde*et al* (2015) defined a two parameter transmuted Inverse exponential distribution as a generalization of the one parameter inverse Exponential distribution. Explicit expressions were provided for the rth moment and the moment generating function. It was observed that the moment of the transmuted Inverse exponential distribution only exist when r<1.

Yassmen(2016) proposed a new model called the transmuted weighted exponential distribution which extends the weighted exponential distribution. The new model provides larger flexibility in modeling reallife data. Various properties were obtained and the parameters of the transmuted weighted exponential distribution wereestimated through the method of maximum likelihood.

Ehsan *et al* (2016) transmuted a two parameters Rayleigh distribution by using the quadratic rank transmutation map. The transmuted Rayleigh distribution was found to be more flexible than the two parameters Rayleigh distribution. Some properties of the transmuted Rayleigh distribution were obtained such as the moments, moment generating function, mean, variance, median, quantile function and reliability function. Themethod of least square estimation was adopted for the parameter estimation of transmuted Rayleigh distribution

Abdul-Moniem and Seham(2015) introduced a new distribution called the transmuted Gompertz distribution (TGD). Various properties of the transmuted Gompertz distribution such as the rth moment, TL-moments, L-moments are derived. The method of maximum likelihood estimation was adopted for the parameter estimation. The Transmuted Gompertz distribution was applied on a real life data to examine its flexibility over the Gompert distribution.

Many authors estimated the parameters of transmuted distributions using the method of maximum likelihood estimation. In this study, estimation of parameters of the Transmuted Weibull Pareto distribution will be carried out using both the method of maximum likelihood and least square estimation.

The objective of this dissertation is to use the Quadratic Rank Transmutation Map (QRTM) introduced by Shaw and Buckley (2007) to provide a further modification of the Weibull-Pareto distribution, another version of which was also introduced by Suleiman and Albert (2015).

CHAPTER THREE

METHODOLOGY

3.1 Introduction

In this chapter, the Quadratic Rank Transmutation Map obtained by Shaw and Buckley(2007) will be applied to derive the probability density function (pdf) and the cumulative density function (cdf) of the proposed distribution. According to Shawand Buckley (2007), a random variable X is said to have a transmuted distribution function if its pdf and cdf are respectively given by;

$$f(x) = g(x)[1+\lambda-2\lambda G(x)] \quad (3.1)$$

$$F(x) = (1+\lambda)G(x)-\lambda[G(x)]^2 \quad (3.2)$$

where; $x>0$ and $|\lambda|\leq 1$ is the transmuted parameter, $G(x)$ is the cdf of any given baseline distribution and $g(x)$ is the pdf of the given baseline distribution.

Observe that if $\lambda = 0$; Equation (3.1) and (3.2) reduces to the pdf and cdf of any baseline distribution.

3.2 The Transmuted Weibull Pareto Distribution

The Transmuted Weibull Pareto Distribution is introduced by generalizing the New Weibull-Pareto distribution proposed by Suleiman and Albert (2015) using the Quadratic Transmutation Map introduced byShaw and Buckley (2007).

3.2.1 Probability Density Function of Transmuted Weibull Pareto Distribution

The Probability density function and cumulative density function of Transmuted Weibull Pareto Distribution will be derived by using the Quadratic Transmutation Map introduced by Shaw and Buckley (2007).

According to Suleiman and Albert (2015), a life random variable X is said to have a New Weibull Pareto distribution denoted by NWPD if its distribution function has the form:

$$G(x) = 1 - e^{-\delta(\frac{x}{\theta})^{\beta}} \qquad (3.3)$$

and its probability density function as

$$g(x) = \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^{\beta}} \qquad (3.4)$$

Theprobability density function of the Transmuted Weibull Pareto Distribution is derived by substituting equation (3.3) and (3.4) in equation (3.1)

$$f(x) = \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^{\beta}} [1 + \lambda - 2\lambda\{1 - e^{-\delta(\frac{x}{\theta})^{\beta}}\}]$$

$$f(x) = \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^{\beta}} [1 + \lambda - 2\lambda + 2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]$$

$$f(x) = \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^{\beta}} [1 - \lambda + 2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}] \qquad (3.5)$$

for$0 < x < \infty, \beta > 0, \theta > 0, \ \delta > 0, |\lambda| \leq 1$

where θ is the scale parameter, β , δ and λ are shape parameters.

<u>Validity of the Probability Density Function</u>

$$\int_0^{\infty} f(x)\partial x = \int_0^{\infty} \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^{\beta}} [1 - \lambda + 2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]\partial x$$

Expanding the equation, we have

$$\int_0^{\infty} f(x)\partial x = \int_0^{\infty} \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^{\beta}} \partial x - \lambda\int_0^{\infty} \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^{\beta}} \partial x + 2\lambda\int_0^{\infty} \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-2\delta(\frac{x}{\theta})^{\beta}} \partial x \quad (3.6)$$

Let $y=\delta(\frac{x}{\theta})^{\beta}$ and $x=\theta(\frac{y}{\delta})^{\frac{1}{\beta}}$, $\frac{dy}{dx}=\frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1}, dx=\frac{dy\theta}{\beta\delta(\frac{x}{\theta})^{\beta-1}}$

Substituting for x and y in equation (3.6)

$$\int_0^\infty e^{-y}\partial y-\lambda\int_0^\infty e^{-y}\partial y+2\lambda\int_0^\infty e^{-2y}\partial y$$

Integrating with respect to y, we have

$$-e^{-y}+\lambda e^{-y}-\lambda e^{-y}$$

Replacing y in terms of x and substituting the range of x

$$1-\lambda+\lambda=1$$

3.2.2 The Cumulative Density Function

By substituting the cdf of New Weibull Pareto Distribution in equation (3.3) into equation (3.2), the cdf of theTransmuted Weibull Pareto Distribution is derived as shown below:

$$F(x)=(1+\lambda)1-e^{-\delta(\frac{x}{\theta})^{\beta}}-\lambda[1-e^{-\delta(\frac{x}{\theta})^{\beta}}]^2 \quad (3.7)$$

$$F(x)=(1+\lambda)1-e^{-\delta(\frac{x}{\theta})^{\beta}}-\lambda[1-2e^{-\delta(\frac{x}{\theta})^{\beta}}+e^{-2\delta(\frac{x}{\theta})^{\beta}}]$$

$$F(x)=[1-e^{-\delta(\frac{x}{\theta})^{\beta}}+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}-\lambda e^{-2\delta(\frac{x}{\theta})^{\beta}}]$$

$$F(x)=[1-e^{-\delta(\frac{x}{\theta})^{\beta}}][1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}] \quad (3.8)$$

for$0<x<\infty, \beta>0, \theta>0,\ \delta>0, |\lambda|\leq 1$

where θ is the scale parameter, β , δ and λ are shape parameters.

Figure 1and 2 illustrates some of the possible shapes of the pdf and cdf of theTransmuted Weibull Pareto Distribution for selected values of parameters $\beta, \delta, \theta, \lambda$

Note: $\beta = a, \delta = b, \theta = c \; and \; \lambda = d$

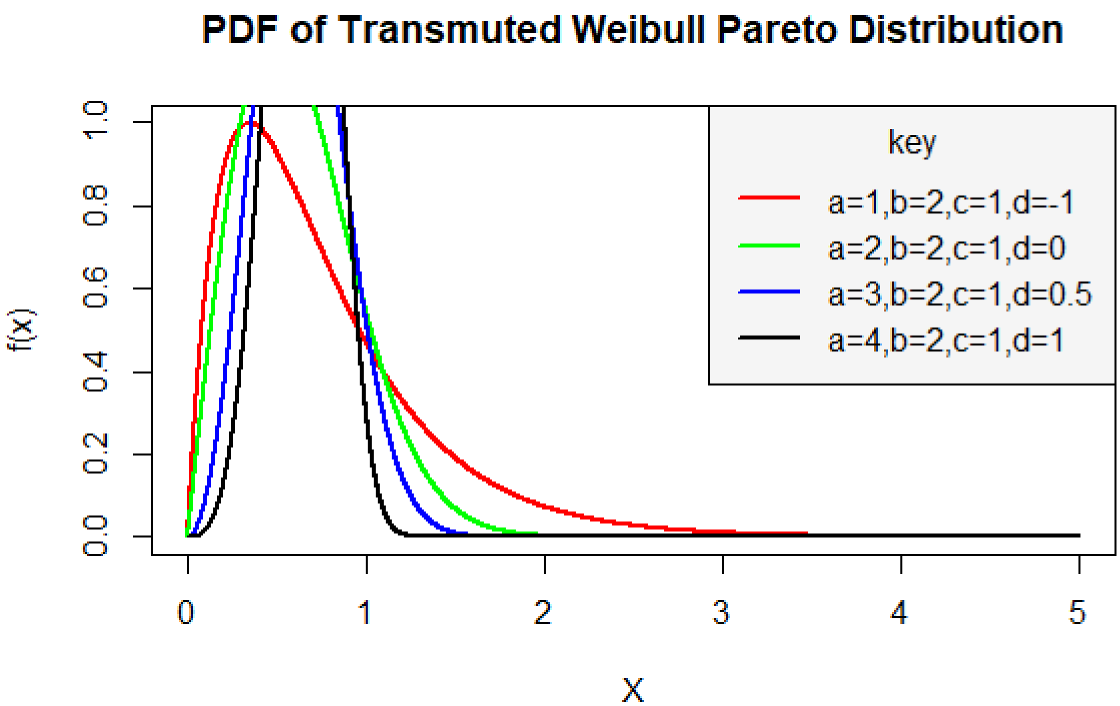

Figure-1: The behavior of PDF of Transmuted Weibull Pareto Distribution for some selected parameter values

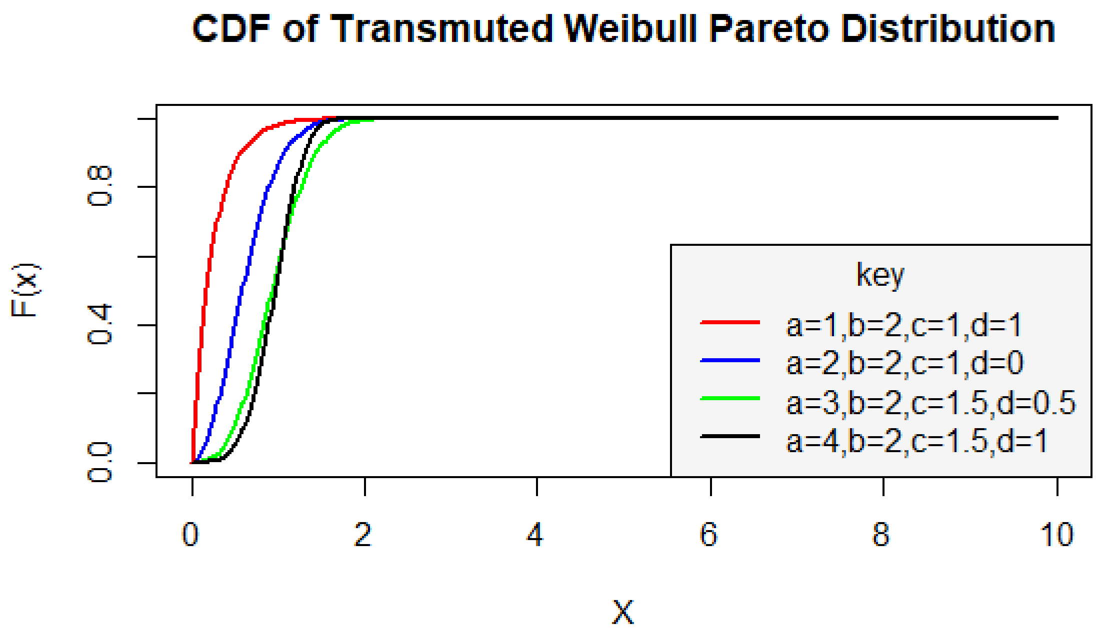

Figure-2: The behavior of CDF of Transmuted Weibull Pareto Distribution for some selected parameter values

3.3 RELIABILITY ANALYSIS

3.3.1Survival Function of Transmuted Weibull Pareto Distribution

The Survival function of any distribution which is probability of survival beyond time *x*is defined by $S(x)=1-F(x)$.The Survival function of the Transmuted Weibull Pareto distribution is given as; $S(x)=1-F(x)$,where F(x) is the CDF of the Transmuted Weibull Pareto distribution

$$S(x)=1-[1-e^{-\delta(\frac{x}{\theta})^{\beta}}][1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]$$

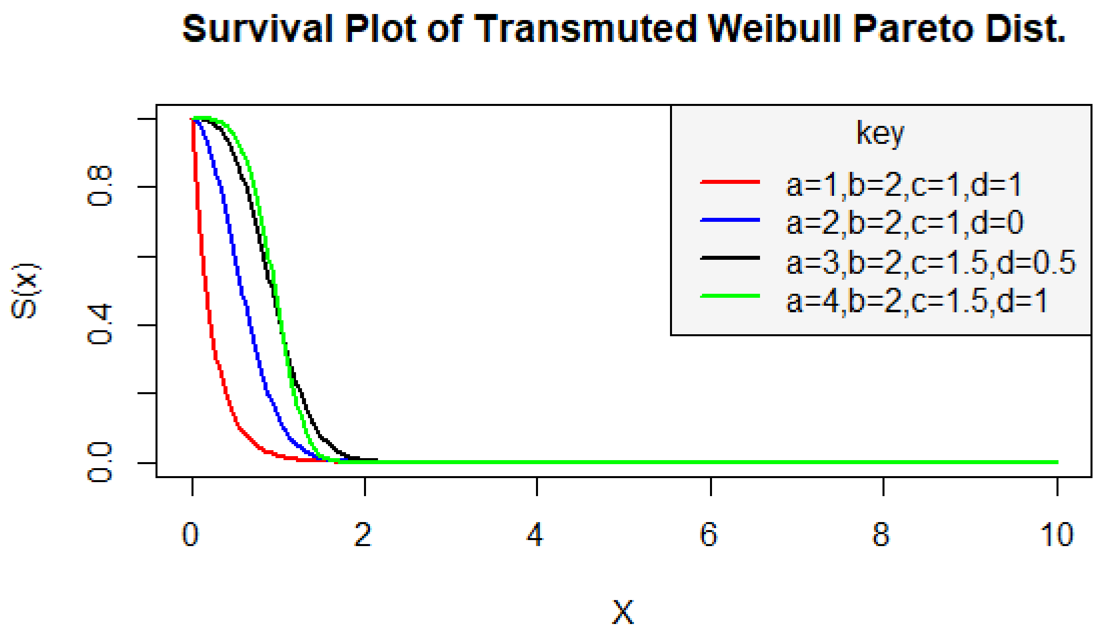

Figure-3: The behavior of Survival function of Transmuted Weibull Pareto Distribution for some selected parameter values

3.3.2Hazard Function of Transmuted Weibull Pareto Distribution

The hazard rate can be interpreted as the conditional probability of failure of an item given that the item has survived to the time (x).The hazard rate function of a random variable x with the pdf $f(x)$ and survival function $s(x)$ is given by $H(x)=\frac{f(x)}{S(x)}$, using this relation, the Hazard rate function of the Transmuted Weibull Pareto distribution is given as;

$H(x)=\frac{f(x)}{S(x)}$, where $f(x)$ is the pdf of Transmuted Weibull Pareto Distribution and $S(x)$ is the cdf of the Transmuted Weibull Pareto Distribution

$$H(x) = \frac{\frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1}[1-\lambda+2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]}{[\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}+1-\lambda]}$$

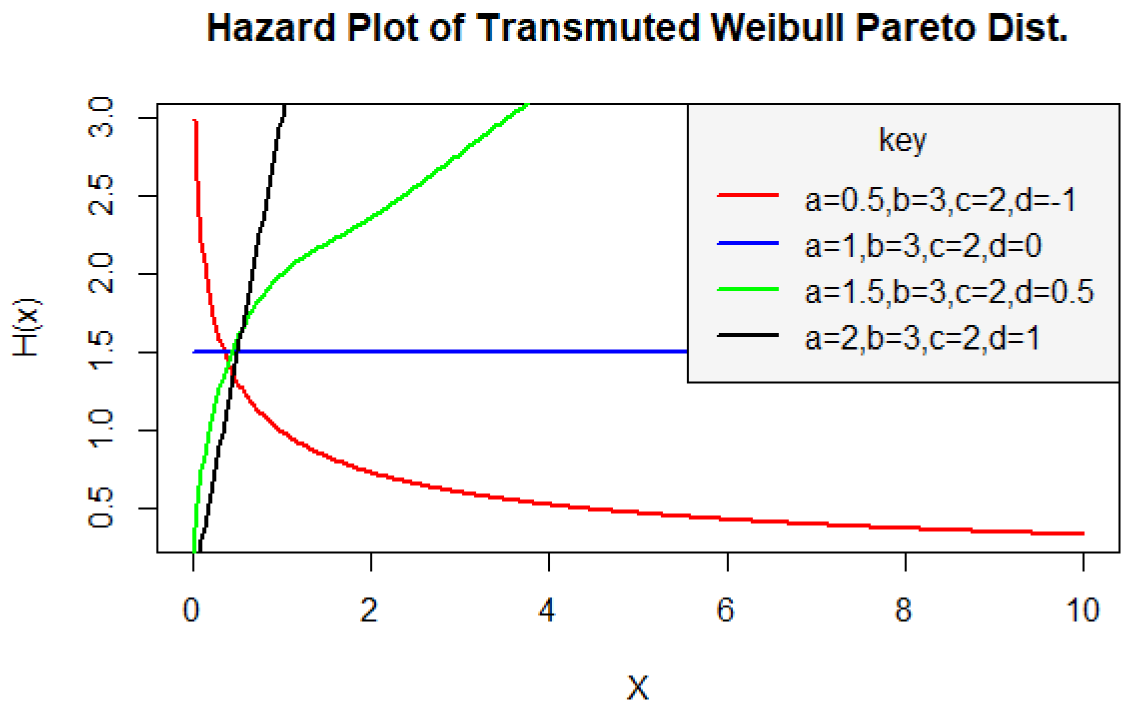

Figure-4: The behavior of Hazard function of Transmuted Weibull Pareto Distribution for some selected parameter values

3.4 STATISTICAL PROPERTIES

3.4.1 Quantile Function and Median

The quantile function of theTransmuted Weibull Pareto Distributionis solution of the following equation x_q

$$x_q = \frac{\theta}{\delta}\left\{-\ln\left[1-\left(\frac{1+\lambda+\sqrt{\lambda^2+1+2\lambda-4\lambda q}}{2\lambda}\right)\right]\right\}^{\frac{1}{\beta}} \text{, where } 0<q<1 \tag{3.9}$$

for $0 < x < \infty, \beta > 0, \theta > 0,\ \delta > 0, |\lambda| \leq 1$

where θ is the scale parameter, β , δ and λ are shape parameters.

Proof

The q_{th} quantile x_q of Transmuted Weibull Pareto Distributionis defined as

$$\text{q}th = F(x_q)$$

From equation (3.7),

$F(x) = (1+\lambda)\{1-e^{-\delta(\frac{x}{\theta})^\beta}\} - \lambda\{1-e^{-\delta(\frac{x}{\theta})^\beta}\}^2$, therefore we have the q_{th} quantile x_q of Transmuted Weibull Pareto Distribution defined as

$q_{th}= F(x_q) = (1+\lambda)\{1-e^{-\delta(\frac{x_q}{\theta})^\beta}\} - \lambda\{1-e^{-\delta(\frac{x_q}{\theta})^\beta}\}^2$ which can be written as

$$\lambda\{1-e^{-\delta(\frac{x_q}{\theta})^\beta}\}^2 - (1+\lambda)\{1-e^{-\delta(\frac{x_q}{\theta})^\beta}\} + q = 0$$

Considering this as a quadratic in$1-e^{-\delta(\frac{x_q}{\theta})^\beta}$, we have

$$1-e^{-\delta(\frac{x_q}{\theta})^\beta} = \frac{1+\lambda \pm \sqrt{\lambda^2+1+2\lambda-4\lambda q}}{2\lambda}$$

Solving for x_q, we have

$$x_q = \frac{\theta}{\delta}\left\{-\ln\left[1-\left(\frac{1+\lambda+\sqrt{\lambda^2+1+2\lambda-4\lambda q}}{2\lambda}\right)\right]\right\}^{\frac{1}{\beta}}, \text{ where } 0<q<1 \quad (3.10)$$

3.4.2 The Median of Transmuted Weibull Pareto Distribution

If $q = 0.5$ in Equation (3.9) we can get the median of Transmuted Weibull Pareto Distribution as

$$x_{0.5} = \frac{\theta}{\delta}\left\{-\ln\left[1-\left(\frac{1+\lambda-\sqrt{\lambda^2+1}}{2\lambda}\right)\right]\right\}^{\frac{1}{\beta}} \quad (3.11)$$

for$0 < x < \infty, \beta > 0, \theta > 0,\ \delta > 0, |\lambda| \leq 1$, where θ is the scale parameter, β, δ and λ are shape parameters.**3.4.3 Random Number Generation**

The random number generation of the Transmuted Weibull Pareto Distributionis defined by the relation below if we substitute u rather than q, where u follows standard uniform variate

$$x_q = \frac{\theta}{\delta}\left\{-\ln\left[1-\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2+4\lambda u}}{2\lambda}\right)\right]\right\}^{\frac{1}{\beta}} \quad (3.12)$$

where u is a uniform distribution with interval (0,1).

for $0 < x < \infty, \beta > 0, \theta > 0,\ \delta > 0, |\lambda| \leq 1$, where θ is the scale parameter, β and δ are shape parameters, λ is the transmuted parameter.

3.4.4rth Non Central Moments

The*rth* non central moment of the Transmuted Weibull Pareto Distribution(u_r) is given below

$$u_r = \theta^r (\frac{1}{\delta})^{\frac{r}{\beta}} \Gamma(\frac{r}{\beta}+1)[1-\lambda+\lambda 2^{-\frac{r}{\beta}}]$$

Proof:

$$u_r = \int_0^\infty x^r f(x)dx$$, where $f(x)$ is the p.d.f of the Transmuted Weibull Pareto Distribution

$$= \int_0^\infty x^r \frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1} e^{-\delta(\frac{x}{\theta})^\beta}[1-\lambda+2\lambda e^{-\delta(\frac{x}{\theta})^\beta}]dx \qquad (3.13)$$

Let $y = \delta(\frac{x}{\theta})^\beta$ and $x = \theta(\frac{y}{\delta})^{\frac{1}{\beta}}$, $\frac{dy}{dx} = \beta\delta(\frac{x}{\theta})^{\beta-1}$, $dx = \frac{\theta dy}{\beta\delta(\frac{x}{\theta})^{\beta-1}}$

Substituting for x and y in equation (3.13)

$$u_r = \int_0^\infty (\theta(\frac{y}{\delta})^{\frac{1}{\beta}})^r e^{-y}[1-\lambda+2\lambda e^{-y}]dy$$

$$u_r = \theta^r (\frac{1}{\delta})^{\frac{r}{\beta}} \int_0^\infty y^{\frac{r}{\beta}} e^{-y}[1-\lambda+2\lambda e^{-y}]dy$$

$$u_r = \theta^r (\frac{1}{\delta})^{\frac{r}{\beta}} [1-\lambda \int_0^\infty y^{\frac{r}{\beta}} e^{-y} dy + 2\lambda \int_0^\infty y^{\frac{r}{\beta}} e^{-2y} dy] \qquad (3.14)$$

where $\int_0^\infty y^{\frac{r}{\beta}} e^{-y} dy = \Gamma\frac{r}{\beta}+1$ and $\int_0^\infty y^{\frac{r}{\beta}} e^{-2y} dy = 2^{-(\frac{r}{\beta}+1)} \Gamma\frac{r}{\beta}+1$

Equation (3.14) becomes;

$$u_r = \theta^r (\frac{1}{\delta})^{\frac{r}{\beta}} \Gamma(\frac{r}{\beta}+1)[1-\lambda+\lambda 2^{-\frac{r}{\beta}}] \tag{3.15}$$

From equation (3.12), when $r=1$, the mean of Transmuted Weibull Pareto Distribution becomes

$$E(x) = \theta\delta^{-\frac{1}{\beta}} \Gamma(\frac{1}{\beta}+1)[1-\lambda+\lambda 2^{-\frac{1}{\beta}}]$$

3.4.5 Order Statistics

In statistics, often times sample values such as the smallest, largest or middle observation from a random sample provide important information. Let X_1 denote the smallest of { $X_{1,} X_{2,} X_{3,...,} X_n$ }, X_2 denote the second smallest of { $X_{1,} X_{2,} X_{3,...,} X_n$ } and similarly X_j denote the j^{th} smallest of { $X_{1,} X_{2,} X_{3,...,} X_n$ }. Then the random variables $X_{1,} X_{2,} X_{3,...,} X_n$ called the order statistics of the sample $X_{1,} X_{2,} X_{3,...,} X_n$ has probability density function of the j^{th} order statistics as:

$$f_{X(j)}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_x(x)]^{j-1}[1-F_X(x)]^{n-j} \tag{3.16}$$

for $j=1,2,3,...n$

Substituting the pdf and cdf of the Transmuted Weibull Pareto Distributionin equation 3.5 into equation 3.16, we have the pdf of the j^{th} order statistics for the Transmuted Weibull Pareto Distributiongiven as:

$$f_{X(j)}(x)=\frac{n!}{(j-1)!(n-j)!}\frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1}e^{-\delta(\frac{x}{\theta})^{\beta}}\left[1-\lambda+2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}\right]\times\left[\left\{1-e^{-\delta(\frac{x}{\theta})^{\beta}}\}\{1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}\right\}\right]^{j-1}\times$$

$$\left[1-\left\{(1-e^{-\delta(\frac{x}{\theta})^{\beta}}\}\{1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}})\right\}\right]^{n-j}$$

3.5Parameter Estimation

For the Transmuted Weibull Pareto Distribution, the two known method of estimation namely Least Square Method and the Maximum Likelihood estimation method will be used for the parameter estimation

3.5.1 Least Square Estimation

Let $X_1 < X_2 < ... < X_n$ be n ordered random sample of any distribution with CDF $F(x)$, we have

$$E(F(x))=\frac{i}{n+1} \quad (3.17)$$

The Least Square estimates of any distribution are obtained by minimizing the expression in equation (3.17)

$$G(\alpha,\lambda)=\sum_{i=1}^{n}\left(F(x_i)-\frac{i}{n+1}\right)^2 \quad (3.18)$$

Substituting the CDF of Transmuted Weibull Pareto Distribution in equation (3.18), we have

$$G(\delta,\beta,\theta,\lambda)=\sum_{i=1}^{n}\left(\left\{[1-e^{-\delta(\frac{x}{\theta})^{\beta}}][1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]\right\}-\frac{i}{n+1}\right)^2 \quad (3.19)$$

To minimize equation (3.19), we need to differentiate it with respect to $\delta, \beta, \theta, \lambda$ which gives the following equation

$$\frac{dG}{d\delta}=\sum_{i=1}^{n}\left(\left\{[1-e^{-\delta(\frac{x}{\theta})^{\beta}}][1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]\right\}\right)\times\left(\frac{x}{\theta}\right)^{\beta}e^{-\delta(\frac{x}{\theta})^{\beta}}\left[1-\lambda+2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}\right]$$

$$\frac{dG}{d\beta}=\sum_{i=1}^{n}\left(\left\{[1-e^{-\delta(\frac{x}{\theta})^{\beta}}][1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]\right\}\right)\times\ln\left(\frac{x}{\theta}\right)\delta(\frac{x}{\theta})^{\beta}e^{-\delta(\frac{x}{\theta})^{\beta}}\left[1-\lambda+2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}\right]$$

$$\frac{dG}{d\theta}=\sum_{i=1}^{n}\left(\left\{[1-e^{-\delta(\frac{x}{\theta})^{\beta}}][1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]\right\}\right)\times\frac{\beta\delta}{\theta^{\beta+1}}x^{\beta}e^{-\delta(\frac{x}{\theta})^{\beta}}\left[1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}-\lambda\right]+\left[1-e^{-\delta(\frac{x}{\theta})^{\beta}}\right]$$

$$\frac{dG}{d\lambda}=\sum_{i=1}^{n}\left(\left\{[1-e^{-\delta(\frac{x}{\theta})^{\beta}}][1+\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]\right\}\right)\times\left[1-e^{-\delta(\frac{x}{\theta})^{\beta}}\right]e^{-\delta(\frac{x}{\theta})^{\beta}}$$

The above likelihood equation cannot be solved analytically,therefore we canresort to Newton Raphson Algorithm to get the solution

3.5.2 Newton Raphson Algorithm

The Newton Raphson Algorithm is an iterative procedure that is widely used in solving non-linear partial differential equations. It starts by constructing a quadratic approximation to the likelihood function of interest around some initial parameter values, then adjusts the parameter values to that which maximizes the likelihood function. This procedure is iterated until the parameter values stabilizes(" The Newton Raphson Algorithm for Function Optimization,"n.d.).

3.5.3 Maximum Likelihood Estimation

Let $x_1, x_2, ..., x_n$ be a random sample from Transmuted Weibull Pareto Distribution. The Pdf of Transmuted Weibull Pareto Distributionis given as

$$f(x)=\frac{\beta\delta}{\theta}(\frac{x}{\theta})^{\beta-1}e^{-\delta(\frac{x}{\theta})^{\beta}}[1-\lambda+2\lambda e^{-\delta(\frac{x}{\theta})^{\beta}}]\ (3.20)$$

for $0 < x < \infty, \beta > 0,\ \theta > 0,\ \delta > 0, |\lambda| \leq 1$

where θ is the scale parameter, β , δ and λ are shape parameters.

The Log-likelihood function of probability density function is given by;

$$L = n\log\beta + n\log\delta - n\log\theta - \delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} + (\beta-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right) + \sum_{i=1}^{n}\log\left[1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\right] \qquad (3.21)$$

Hence, the parameters are obtained by differentiating equation (3.21) with respect to the parameters. The maximum likelihood differential equations are;

$$\frac{dL}{d\delta} = \frac{n}{\delta} - \sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta} - 2\lambda\sum_{i=1}^{n}\frac{e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\left(\frac{x_i}{\theta}\right)^{\beta}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)\beta}} = 0$$

$$\frac{dL}{d\beta} = \frac{n}{\beta} - \delta\sum_{i=1}^{n}\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right) + \sum_{i=1}^{n}\ln\left(\frac{x_i}{\theta}\right) + 2\lambda\sum_{i=1}^{n}\frac{e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}\delta\left(\frac{x_i}{\theta}\right)^{\beta}\ln\left(\frac{x_i}{\theta}\right)}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)\beta}} = 0$$

$$\frac{dL}{d\lambda} = \sum_{i=1}^{n}\frac{2e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}} - 1}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)\beta}} = 0$$

$$\frac{dL}{d\theta} = -\frac{n}{\theta} + \frac{\beta\delta}{\theta^{(\beta+1)}}\sum_{i=1}^{n}x_i^{\beta} + \frac{n(1-\beta)}{\theta} - \frac{2\lambda\beta\delta}{\theta^{(\beta+1)}}\sum_{i=1}^{n}\frac{e^{-\delta\left(\frac{x_i}{\theta}\right)^{\beta}}x_i^{\beta}}{1-\lambda+2\lambda e^{-\delta\left(\frac{x_i}{\theta}\right)\beta}} = 0$$

The above equations are not in explicit form, the estimated values can be obtained numerically using iteratively based procedures such as the Newton-Raphson Algorithm.

CHAPTER FOUR

APPLICATION AND DISCUSSION

4.1 Application

In this section, the flexibility of the Transmuted Weibull Pareto Distributionis illustrated on a real life data set. We shall compare the fits of the Transmuted Weibull Pareto Distributionwith New Weibull-Pareto (Suleiman and Albert 2015), Weibull-Pareto Distribution (Tahir et al 2015) and the Pareto Distribution. The density function of the New Weibull Pareto Distribution (Suleiman and Albert 2015), Weibull- Pareto Distribution (Tahir *et al* 2015) and Pareto distribution are given as follows:

The New Weibull Pareto Distribution (Suleiman and Albert 2015)

$$g(x) = \frac{\beta\delta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-\delta\left(\frac{x}{\theta}\right)^{\beta}} \quad (4.1)$$

where$0 < x < \infty, \beta > 0, \theta > 0,\ \delta > 0$

θ is the scale parameter, β and δ are shape parameters.

The Weibull-Pareto Distribution (Tahir *et al* 2015)

$$g(x) = \frac{\beta\alpha}{\theta^{\alpha}} x^{\alpha-1}\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{\beta-1} e^{-\left[\left(\frac{x}{\theta}\right)^{\alpha} - 1\right]^{\beta}}$$

where$0 < x < \infty, \beta > 0, \theta > 0,\ \alpha > 0,$

The Pareto Distribution

$$f(x) = \frac{ax_o^a}{x^{a+1}} \quad (4.2)$$

where $a, x > 0$, $x_0 = \min(x)$

a is the shape parameter.

The Transmuted Weibull Pareto Distributionwill be used to model the exceedances of flood peaks (in $m3/s$) of the Wheaton River near Carcross in Yukon Territory, Canada. The data consists of 72 exceedances for the year 1958-1984, rounded to one decimal place. Suleiman and Albert (2015) used the data to demonstrate the superiority of the New Weibull Pareto distribution over Transmuted Pareto (TP) distribution and Kumaraswamy Pareto (Kw-P) distribution.

The data set is as follows:

1.7 2.2 14.4 1.1 0.4 20.6 5.3 0.7 1.4 18.7 8.5 25.5 11.6 14.1 22.1 1.1 0.6 2.2 39.0 0.3 15.0 11.0 7.3 22.9 0.9 1.7 7.0 20.1 0.4 2.8 14.1 9.9 5.6 30.8 13.3 4.2 25.5 3.4 11.9 21.5 1.5 2.5 27.4 1.0 27.1 20.2 16.8 5.3 1.9 10.4 13.0 10.7 12.0 30.0 9.3 3.6 2.5 27.6 14.4 36.4 1.7 2.7 37.6 64.0 1.7 9.7 0.1 27.5 1.1 2.5 0.6 27.0

Table 1: Summary of data on exceedances of flood peaks (in m3/s) of the Wheaton River

Min	Q1	Q2	Q3	Mean	Skewness	Max
0.10	1.85	9.50	20.12	12.09	1.465738	64.00

In order to compare these distributions, the AIC (Akaike Information Criterion) and the BIC (Bayesian Information Criterion) will be used. These Statistics are given as:

$$AIC = -2ll + 2k$$

$$BIC = -2ll + k\ln(n)$$

where ll is the log-likelihood, K is number of parameters to be estimated and n is the number of data values.

The model with the lowest value of AIC and BIC will be regarded as the best model to fit the data.

Table 2: Performance rating of Transmuted Weibull Pareto Distribution over other models using data on exceedances of flood peaks (in m3/s) of the Wheaton River

Distribution	Estimates	-2LL	AIC	BIC
Transmuted Weibull Pareto	$\hat{\beta} = 0.1555, \hat{\delta} = 8.6231, \hat{\theta} = 0.01, \hat{\lambda} = -0.4608$	115.5854	121.5854	128.4154
New Weibull Pareto (NWPD)	$\hat{\beta} = 0.1999, \hat{\delta} = 11.7450, \hat{\theta} = 0.1$	158.3258	162.3258	166.8791
Weibull Pareto (Tahir et al 2015)	$\hat{\beta} = 4.4363,\ \hat{\delta} =\ 0.0987,\ \hat{\theta} = 0.1000$	249.3965	502.7930	507.3184
Pareto	$\hat{a}$=0.2438634, $x_0 = 0.1$	606.128	610.128	610.405

Table 2 shows the maximum likelihood estimates (MLEs) to each of the distribution fitted on the first data set (Exceedances of flood peaks (in *m*3/*s*) of the Wheaton River). The table also shows the corresponding -2LL, AIC and BIC for each model. The Log likelihood, AIC and BIC of the Transmuted Weibull Pareto Distributionappears to be the lowest which is indicative that the Transmuted Weibull Pareto Distributionperforms better than the New Weibull Pareto Distribution, Weibull Pareto (Tahir et al 2015) and the Pareto distribution on the data set. Hence, theTransmuted Weibull Pareto Distributioncan be regarded as the best model compared to other models fitted on the data set.

CHAPTER FIVE

SUMMARY AND CONCLUSION

5.1 Summary

In this research, a four parameter distribution called Transmuted Weibull Pareto Distribution is introduced which generalizes the New Weibull Pareto distribution by adding a transmuted Parameter into the New Weibull-Pareto distribution. The introduction of a transmuted parameter into the New Weibull-Pareto distribution increases its flexibility and provides a better fit than the New Weibull-Pareto distribution. Various structural properties such as explicit expressions for the reliability analysis, quantiles, rth Non central moments and order statistics of the new distribution are derived. The estimation of parameters is approached by the method of Maximum Likelihood and Least Square estimation. The usefulness of the Transmuted Weibull Pareto Distribution is examined on a real data set and the results showed that the Transmuted Weibull Pareto Distribution provides a better fit when compared with related models on the data set.

5.2 Conclusion

A probability distribution is introducedcalled the Transmuted Weibull Pareto Distributionwhich extends the New Weibull-Pareto distribution. Various structural properties such as explicit expressions for the reliability analysis, quantiles, rth Non central moments and order statistics of the new distribution are derived. The Hazard plot of the Transmuted Weibull Pareto Distributioncan be constant, increasing and decreasing, which makes the distribution suitable for modelling devices that have constant failure rate, wears out faster with time and wears out slower with time. The inherent advantages of a distribution that has an increasing, decreasing and constant hazard rate is that such distribution can be used to model an item that has an increased risk of failure over the lifetime of the item, failures that are likely to occur at early life of the item and failures that are equally likely to occur at any time in item's life. The estimation of parameters is approached by the method of maximum likelihood and least square estimation. The Transmuted Weibull Pareto Distribution fits better than the New Weibull Pareto distribution and Pareto distribution in terms of flexibility when applied on a real life data set. The criteria used for performance rating is the AIC and BIC value.

5.3 Contribution to Knowledge

This work obtained a further modification of Weibull Pareto Distribution called Transmuted Weibull Pareto Distribution which outperformed the New Weibull-Pareto distribution and the Pareto distribution using a real life data set. Various structural properties such as explicit expressions for the reliability analysis, quantiles, rth Non central moments and order statistics.

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Appendix

Rcodes and Output

R-code for the PDF Plot

```
fn<-function(nwpd, a, b, c,d){(a*b/c)*((dat/c)^(a-1))*exp(-(b*(dat/c)^a))*(1-d+2*d*exp(-
(b*(dat/c)^a)))}

> dat<-seq(0,5,0.00025)

> p1<-fn(dat,a=1,b=2,c=1,d=1)

> plot(dat,p1, main="PDF of Transmuted Weibull Pareto Distribution",ylim =
c(0,1),type="n",xlab = "X",ylab = "f(x)")

> lines(dat, fn(dat,a=1,b=2,c=1,d=-1),col="red",lwd=2,lty=1)

> lines(dat, fn(dat,a=2,b=2,c=1,d=0),col="green",lwd=2,lty=1)

> lines(dat, fn(dat,a=3,b=2,c=1,d=0.5),col="blue",lwd=2,lty=1)

> lines(dat, fn(dat,a=4,b=2,c=1,d=1),col="black",lwd=2,lty=1)

> legend("topright",cex=1, title="key", c("a=1,b=2,c=1,d=-
1","a=2,b=2,c=1,d=0","a=3,b=2,c=1,d=0.5","a=4,b=2,c=1,d=1"),horiz=F,
lty=c(1,1,1,1),lwd=c(2,2,2,2), col=c("red","green","blue","black"), bg="grey96")
```

R-code for the CDF plot

```
> x<-seq(0,10,0.025)

> cd<-function(x,a,b,c,d){(1-exp(-(b*(x/c)^a)))*(1+d*(exp(-(b*(x/c)^a))))}
```

```
> c1<-cd(x,a=1,b=2,c=1,d=1)

> plot(x,c1, main="CDF of Transmuted Weibull Pareto Distribution",type="n",xlab = "X",ylab = 
"F(x)")

> lines(x, cd(x,a=1,b=2,c=1,d=1),col="red",lwd=2,lty=1)

> lines(x, cd(x,a=2,b=2,c=1,d=0),col="blue",lwd=2,lty=1)

> lines(x, cd(x,a=3,b=2,c=1.5,d=0.5),col="green",lwd=2,lty=1)

> lines(x, cd(x,a=4,b=2,c=1.5,d=1),col="black",lwd=2,lty=1)

> legend("bottomright",cex=1, title="key", 
c("a=1,b=2,c=1,d=1","a=2,b=2,c=1,d=0","a=3,b=2,c=1.5,d=0.5","a=4,b=2,c=1.5,d=1"),horiz=F, 
lty=c(1,1,1,1),lwd=c(2,2,2,2), col=c("red","blue","green","black"), bg="grey96")
```

R-code for the Survival Plot

```
> x<-seq(0,10,0.025)

> h<-function(x, a, b, c, d){(1-((1-exp(-(b*(x/c)^a)))*(1+(d*(exp(-(b*(x/c)^a)))))))}

> x1<-h(x,a=1,b=2,c=1,d=1)

> x2<-h(x,a=2,b=2,c=1,d=0)

> x3<-h(x,a=3,b=2,c=1.5,d=0.5)

> x4<-h(x,a=4,b=2,c=1.5,d=1)

> plot(x,x1, main="Survival Plot of Transmuted Weibull Pareto Dist.",ylim = 
c(0,1),type="n",xlab = "X",ylab = "S(x)")
```

```
> lines(x,x1,col="red",lwd=2,lty=1)

> lines(x,x2,col="blue",lwd=2,lty=1)

> lines(x,x3,col="black",lwd=2,lty=1)

> lines(x,x4,col="green",lwd=2,lty=1)

> legend("topright",cex=1, title="key",
c("a=1,b=2,c=1,d=1","a=2,b=2,c=1,d=0","a=3,b=2,c=1.5,d=0.5","a=4,b=2,c=1.5,d=1"),horiz=F,
lty=c(1,1,1,1),lwd=c(2,2,2,2), col=c("red","blue","black","green"), bg="grey96")
```

R-code for the Hazard Plot

```
> x<-seq(0,10,0.025)

> ha<-function(x,a,b,c,d){((a*b/c)*((x/c)^(a-1))*(1-d+2*d*exp(-(b*(x/c)^a))))/(d*exp(-
(b*(x/c)^a))+1-d)}

> x1<-ha(x,a=0.5,b=3,c=2,d=-1)

> x2<-ha(x,a=1,b=3,c=2,d=0)

> x3<-ha(x,a=1.5,b=3,c=2,d=0.5)

> x4<-ha(x,a=2,b=3,c=2,d=1)

> plot(x,x1, main="Hazard Plot of Transmuted Weibull Pareto Dist.",type="n",xlab = "X",ylab =
"H(x)")

> lines(x,x1,col="red",lwd=2,lty=1)

> lines(x,x2,col="blue",lwd=2,lty=1)
```

```
> lines(x,x3,col="green",lwd=2,lty=1)

> lines(x,x4,col="black",lwd=2,lty=1)

> legend("topright",cex=1, title="key", c("a=0.5,b=3,c=2,d=-
1","a=1,b=3,c=2,d=0","a=1.5,b=3,c=2,d=0.5","a=2,b=3,c=2,d=1"),horiz=F,
lty=c(1,1,1,1),lwd=c(2,2,2,2), col=c("red","blue","green","black"), bg="grey96"
```

R-code for the Maximum likelihood estimation

```
> x<-c(1.7,2.2,14.4,1.1,0.4,20.6, 5.3 ,0.7 ,1.4 ,18.7, 8.5, 25.5, 11.6, 14.1, 22.1, 1.1, 0.6, 2.2, 39.0,
0.3, 15.0, 11.0, 7.3, 22.9, 0.9, 1.7, 7.0, 20.1, 0.4, 2.8, 14.1, 9.9, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4,
11.9, 21.5, 1.5 ,2.5 ,27.4, 1.0, 27.1, 20.2, 16.8, 5.3 ,1.9 ,10.4, 13.0, 10.7, 12.0, 30.0, 9.3, 3.6, 2.5,
27.6, 14.4, 36.4, 1.7, 2.7, 37.6, 64.0, 1.7, 9.7, 0.1, 27.5, 1.1, 2.5, 0.6, 27.0)

> library(maxLik)

> fn<-function(a) {

+     n*log(a[1])+n*log(a[2])-n*log(a[3])-(a[2])*sum(x/a[3])^a[1]+(a[1]-
1)*sum(log(x/a[3]))+sum(log(1-a[4]+2*a[4]*exp(-(a[2]*(x/a[3])^a[1]))))

+ }

> n<-length(x)

> est<-maxLik(fn, start = c(0.155,8.6231,0.01,-0.4608))

There were 50 or more warnings (use warnings() to see the first 50)

> est
```

Maximum Likelihood estimation

Newton-Raphson maximisation, 1 iterations

Return code 3: Last step could not find a value above the current.

Boundary of parameter space?

Consider switching to a more robust optimisation method temporarily.

Log-Likelihood: -58.15352 (4 free parameter(s))

Estimate(s): 0.155 8.6231 0.01 -0.4608