ALGEBRAIC STUDY OF RHOTRIX SEMIGROUP

By

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M.Sc/SCIE/5406/2010-2011

DEPARTMENT OF MATHEMATICS,

AHMADU BELLO UNIVERSITY,

ZARIA, NIGERIA

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A THESIS SUBMITTED TO THE POSTGRADUATE SCHOOL, AHMADU BELLO UNIVERSITY, ZARIA, NIGERIA IN PARTIAL FULFILMENT FOR THE AWARD OF MASTER OF SCIENCE DEGREE IN MATHEMATICS

DEPARTMENT OF MATHEMATICS, AHMADU BELLO UNIVERSITY, ZARIA NIGERIA

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DECLARATION

I declare that the work in this thesis entitled "ALGEBRAIC STUDY OF RHOTRIX SEMIGROUP" has been performed by me in the Department of Mathematics under the supervision of Dr. A. Mohammed and Dr. B. Sani. The information derived from literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any University or Institution.

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Name of student	Signature	Date

CERTIFICATION

This thesis entitled "ALGEBRAIC STUDY OF RHOTRIX SEMIGROUP" by BALARABE, Musa meets the regulations governing the award of the degree of Masters of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This Thesis is dedicated to The Almighty Allah (SWT) and our Beloved Prophet Muhammad (SAW). Also to my lovely parents (Alh. Balarabe and Malama Maimuna), my lovely wife and children (Farida, Abulkhair and Khadijah) and my brothers and sisters.

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ABSTRACT

In this thesis, we present an algebraic study of '*rhotrix semigroup*'. We identify the properties of this semigroup and introduce some new concepts such as rhotrix rank and rhotrix linear transformation in order to characterize its Green's relations. Furthermore, as comparable to regular semigroup of square matrices, we show that the rhotrix semigroup is also a regular semigroup.

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NOMENCLATURE

$R_n(F)$	- A set of all $n \times n$ rhotrices over a field F.
$h(R)$	- Heart of rhotrix R
$\langle x^{ni} \rangle$	- A rhotrix row vector of size n
$\langle x^{nj} \rangle$	- A rhotrix column vector of size n
$M_n(F)$	- A set of all $n \times n$ matrices over a field F

CHAPTER ONE

GENERAL INTRODUCTION

1.1 INTRODUCTION

The theory of Rhotrix is a relatively new area of mathematical discipline dealing with algebra and analysis of array of numbers in mathematical rhomboid form. The theory began from the work of (Ajibade, 2003), when he initiated the concept, algebra and analysis of rhotrices as an extension of ideas on matrix-tersions and matrix-noitrets proposed by (Atanassov and Shannon, 1998). Ajibade gave the initial definition of rhotrix of size 3 as a mathematical array that is in some way, between two-dimensional vectors and 2×2 dimensional matrices. Since the introduction of the theory in 2003, many authors have shown interest in the usage of rhotrix set, as an underlying set, for construction of algebraic structures.

Following Ajibade's work, (Sani, 2004) proposed an alternative method for multiplication of rhotrices of size three, based on their rows and columns, as comparable to matrix multiplication, which was considered to be an attempt to answer the question of 'whether a transformation can be made to convert a matrix into a rhotrix and vice versa' posed in the concluding section of the initial article on rhotrix. This method of multiplication is now referred to as 'row-column based method for rhotrix multiplication'. Unlike Ajibade's method of multiplication that is both commutative and associative, Sani's method of rhotrix multiplication is non-commutative but associative.

It was shown in (Sani, 2004) that there exists an isomorphic relationship between the group of all invertible rhotrices of size n and the group of all invertible $w \times w$ dimensional matrices, where $w = \frac{n+1}{2}$ and $n \in 2Z^{+} + 1$. The row-column method for multiplication of base rhotrices was later generalized to include rhotrices of size of n by (Sani, 2007).

Thus, two methods for multiplication of rhotrices are presently available in the literature of rhotrix theory. From now on, we shall refer to the method for multiplication of rhotrices defined by Ajibade as "commutative method for rhotrix multiplication" and the row-column method for multiplication of rhotrices defined by Sani as "non-commutative method for rhotrix multiplication".

Mohammed (2007a) adopted the commutative method for rhotrix multiplication to propose classification of rhotrices and their expression as algebraic structures of groups, semigroups, monoids, rings and Boolean algebras.

Based on non-commutative method for rhotrix multiplication, the aim of transforming rhotrix to a matrix and vice-versa was completely achieved in (Sani, 2008), where he proposed a method of converting rhotrix to a special form of matrix called '*coupled matrix'*. This coupled matrix was used to solve two different systems of linear equations simultaneously, where one is an $n \times n$ system while the other one is an $(n-1)\times(n-1)$.

Following this idea, (Sani, 2009) presented the solution of two coupled matrices by extending the idea of a coupled matrix presented in his earlier work to a general case involving $m \times n$ and $(m-1)\times(n-1)$ matrices.

It is noteworthy to mention that any research work by interested author(s) in the literature of rhotrix theory is based on either commutative method or non-commutative method for rhotrix multiplication. So in the presentation of our algebraic study of rhotrix semigroup, we shall adopt the non-commutative technique for multiplication of rhotrices having the same size. The reason behind our choice is that an algebraically non-commutative semigroup offers an exciting platform for carrying out mathematical research in semigroup theory.

One of the well known areas of Mathematics is semigroup theory. It deals with the study of algebra of a set that is closed under an associative binary operation. Semigroup theory has been well developed by researchers, since before the twentieth century. Many concepts in semigroup theory were analogous to group theory, but the concept of Green's relations and many others are developed independently. This makes semigroup theory a well deserved area of research.

The concept of Green's relations was first initiated by Green in 1951. These are five equivalence relations defined on a semigroup and they have played a vital role in the development of semigroup theory. Since the introduction of these equivalence relations, they became standard tools for investigating the structure of any given semigroup. In fact, these relations are so important that, on encountering a new class of semigroups, almost the first question one asks is what are the Green's relations like? In certain classes of semigroups, these five equivalence relations turn out to be equal. For instance, in a commutative semigroup, the five relations reduce to one.

This research work is dealing with the algebraic study of rhotrix semigroup. A rhotrix set, $R_n(F)$ of size n over a field F was considered, together with the binary operation of non-commutative method for rhotrix multiplication, in order to construct a certain algebraic system termed as '*Rhotrix Semigroup*'. Properties of this semigroup were identified and characterize its Green's relations. Furthermore, as comparable to regular semigroup of square matrices, we show that the rhotrix semigroup is also a regular semigroup. Toward achieving the characterization of Green's relations in the rhotrix semigroup, it was found necessary to introduce two concepts; rank of a rhotrix and rhotrix linear transformation.

1.2 RESEARCH AIM AND OBJECTIVES

The aim of this research is to initiate the concept of rhotrix semigroup. The following objectives were set:

a) To develop the basic fundamental algebra necessary for studying the concept of 'rhotrix semigroup' as new paradigm of science.

b) To identify and study the properties of rhotrix semigroup as analogous to other types of semigroups in the literature.

c) To characterize Green's relations in the rhotrix semigroup.

d) To investigate the existence of any isomorphic relationship between certain rhotrix semigroup and certain matrix semigroup.

1.3 RESEARCH METHODOLOGY

The method adopt in this thesis is to consult all necessary and relevant papers in the literature on fundamentals of Rhotrix theory, Matrix theory and Semigroup theory in order to obtain background information for developing the theory of rhotrix semigroup. These papers are thoroughly reviewed to cover major works done on rhotrix. In the thesis also, the non-commutative method for rhotrix multiplication was adopted.

In the first stage of the work, review of development made on rhotrix theory was documented. This will serve as reference for further research works.

Next, focuses on the algebraic study of rhotrix semigroup, in which we construct and show that the set of all rhotrices of size n, together with the non-commutative rhotrix multiplication operation forms a semigroup. The properties of this rhotrix semigroup were identified and characterized its Green's relations. Towards achieving that, the concept of rhotrix rank and rhotrix linear transformation was introduced and presented at the final stage of the work.

1.4 DEFINITION OF TERMS

The following definitions are useful in the subsequent chapters:

Definition 1 (Matrix Set)

A matrix set of all $M_{n\times m}(C)$ is a collection of rectangular arrays, called $n\times m$ dimensional matrices with entries from a set of all complex numbers. Thus,

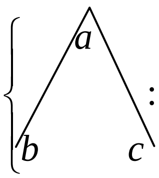
$$A_{n\times m}(C)=\left\{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & \cdots & \cdots & a_{1n} & \vdots \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & \cdots & \cdots & a_{2n} & \vdots \\ \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{mn} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & \vdots & \vdots \end{bmatrix} : a_{11}, a_{12}, \ldots, a_{mn}, \ldots \in C\right\} \qquad (1.1)$$

Definition 2 (Matrix-tertion and Matrix-noitret)

Matrix-tertion and Matrix-noitret can be defined as mathematical arrays that are in some way between 2-dimensional vectors and 2×2-dimensional matrices introduced by (Atanassov and Shannon, 1998). Matrix-tertion and Matrix-noitret are denoted by T and N and respectively defined as

$$T=\left\{\begin{array}{ccc} a & & b \\ & c & \end{array} : a, b, c \in C\right\}$$

and

$N = \{$ 

$: a, b, c \in C \}$

Definition 3 (Semigroup)

A nonempty set S together with a binary operation $\circ$ defined on S is called a semigroup if $(S, \circ)$ satisfied the following properties:

S1: Closure property, that is, for all $x, y \in S$, $x \circ y \in S$.

S2: Associative property, that is, for all $x, y, z \in S$, $(x \circ y) \circ z = x \circ (y \circ z)$.

Definition 4 (Monoid and Zero elements)

The semigroup $(S, \circ)$ is called a *Monoid* if it has an identity element. That is, if there exists $e \in S$ such that $x \circ e = e \circ x = x$ for all $x \in S$. An element $0 \in S$ is called *zero element* of S if $x \circ 0 = 0 \circ x = 0$ for all $x \in S$ and $(S, \circ)$ is called a semigroup with zero. If $(S, \circ)$ has no identity or zero element, then it is easy to adjoin an extra identity or zero to S, in order to form a monoid or semigroup with zero respectively. We write S^1 and S^0 to respectively denote the semigroup with identity or zero adjoined if necessary. We defined $1 \circ s = s \circ 1 = s$, $1 \circ 1 = 1$ and $0 \circ s = s \circ 0 = 0 \circ 0 = 0$ for all $s \in S$. Thus,

$$S^1 = \begin{cases} S & \text{if } S \text{ has identity element,} \\ S \cup \{1\} & \text{otherwise} \end{cases} \tag{1.2}$$

and

$$S^0 = \begin{cases} S & \text{if } S \text{ has zero element,} \\ S \cup \{0\} & \text{otherwise} \end{cases} \tag{1.3}$$

Definition 5 (ideal)

A nonempty subset I of a semigroup S is called a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$, and a (two-sided) *ideal* if it is both a left and a right ideal.

Alternatively, a nonempty subset $I \subseteq S$ is

i. a left ideal of S, if for all $a \in I$ and $s \in S$, $sa \in I$;

ii. a right ideal of S, if for all $a \in I$ and $s \in S$, $sa \in I$;

iii. an ideal of S, if for all $a \in I$ and $s \in S$, $sa, as \in I$.

Definition 6 (Subsemigroups)

A subset H of a semigroup $(S, \circ)$ is called a *subsemigroup* of $(S, \circ)$, if $(H, \circ)$ is also a semigroup under the same binary operation.

Definition 7 (Idempotent)

An element *a* of a semigroup S is called an idempotent element if $a^2 = a$.

Definition 8 (Regular semigroup)

An element a of a semigroup S is called *regular* if there exists $x \in S$ such that $axa = a$. The semigroup S is called *regular semigroup* if all its elements are regular, that is

$$\forall a \in S, \exists x \in S \ni axa = a. \tag{1.4}$$

The semigroup $M(n,F)$ of all $n \times n$ matrices over a field F with respect to matrix multiplication is an example of regular semigroup, that is for every $A \in M(n,F)$ there exists $B \in M(n,F)$ such that $ABA = A$.

A regular semigroup must contain idempotent elements. It follows from (1.4) that both ax and xa are idempotents.

Definition 9 (Inverse semigroup)

An element b of the semigroup S is an *inverse* of $a \in S$ if $aba = a$ and $bab = b$. A semigroup is called an *inverse semigroup* if every element of the semigroup has a unique inverse.

Notice that, an element with an inverse is necessarily regular. Less obviously, every regular element has an inverse for if there exists x such that $axa = a$, then define $b = xax$ and observe that

$$aba = a(xax)a = (axa)xa = axa = a$$

and

$$bab = (xax)a(xax) = x(axa)(xax) = xaxax = x(axa)x = xax = b$$

An element a may well have more than one inverse. So, the idea of inverse under discussion here is substantially more general than a group inverse.

Definition: 10 (Green's relations)

Let S be an arbitrary semigroup. The equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{H}$ and $\mathcal{D}$ are defined on S as follows:

$a \,\mathcal{L}\, b$ *if* and only if $(\exists x, y \in S^1) a = xb$ and $b = ya$;

$a \,\mathcal{R}\, b$ if and only if $(\exists u, v \in S^1) a = bu$ and $b = av$;

$a \,\mathcal{J}\, b$ if and only if $(\exists x, y, u, v \in S^1) a = xby$ and $b = uav$;

$a \,\mathcal{D}\, b$ if and only if $(\exists c \in S)$ aLc and $c\Re b$;

and

$$\mathcal{H} = \mathcal{L} \cap \mathcal{R}$$

For all $a, b \in S$.

These five relations on S are called Green's relations. See (Howie, 1995).

1.5 OUTLINE OF THE THESIS

The outline of the thesis is as follows:

Chapter one presents the general introduction of the thesis, the aim and objectives of the study, the methodology for carrying out the research, definition of terms and then finally, the outline for the thesis.

Chapter two focuses on a review of developments in the literature of rhotrix theory, starting from the year 2003, when the concept of rhotrix was introduced up to the end of 2013. Furthermore, a review of certain concepts in semigroup and Green's relations were discussed.

Chapter three considers the rhotrix set $R_n(F)$ of size n over a field F and together with Sani's row-column based method for rhotrix multiplication, in order to initiate the concept of *non-commutative rhotrix semigroup*. We identify the properties of this semigroup and characterize its Green's relations. Furthermore, as comparable to regular semigroup of square matrices, we showed that the rhotrix semigroup is also a *regular semigroup.*

Chapter four introduces two concepts; rank of a rhotrix and rhotrix linear transformation, as two necessary tools required for achieving our aim of characterization of Green's relations in rhotrix semigroup. Furthermore, some properties of this rank and a necessary and sufficient condition under which a linear transformation can be represented by a rhotrix were also presented in the chapter.

Chapter five gives the summary for the whole thesis, its conclusion and recommendations for future research direction.

CHAPTER TWO

LITERATURE REVIEW

2.1 RHOTRIX THEORY

Rhotrix theory was initiated by (Ajibade, 2003) and a rhotrix was defined as a rhomboidal form of representing array of numbers. The concept is an extension of ideas of Matrix-tersions and Matrix-noitrets proposed by (Atanassov and Shannon, 1998). (Ajibade, 2003) presented the initial concept, analysis and algebra on rhotrices, where he defined an operation of multiplication of rhotrices of size three. This operation of multiplication is known as *heart-oriented multiplication* and is commutative. (Sani, 2004) proposed an alternative method for multiplication of rhotrices of size three and later generalized the idea to rhotrices of size *n*. This alternative operation of multiplication is known as *row-column based method for rhotrix multiplication* and is non-commutative but associative.

Therefore, in the literature of rhotrix theory, two methods for multiplication of rhotrices having the same size are currently available and each method provides enabling environment to explore the usefulness of rhotrices as tools for carrying out mathematical research.

Based on this, we shall have our review of developments of rhotrix theory in systemic form, starting with the review of commutative rhotrix theory followed by the review of non-commutative rhotrix theory.

2.2 COMMUTATIVE RHOTRIX THEORY

A set of rhotrices of size three was defined by (Ajibade, 2003) as:

$$R_3(\Re) = \left\{ R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \in \Re \right\} \qquad (2.1)$$

Where $h(R) = c$ is called the heart of R. Extension of size of R from *3* to *n* was also considered possible. Thus, by a rhotrices of size *n* denoted by $R(n)$ or R_n, we mean a rhomboidal array having $\frac{1}{2}(n^2 + 1)$ entries and of size $n \in 2Z^+ + 1$.

The operations of addition ($+$), scalar multiplication (m) and multiplications ($\circ$) were also defined in (Ajibade, 2003) recorded as below

Let $R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$ and $Q = \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & i \\ & j & \end{array} \right\rangle$ be any two rhotrices of size three and *m* a scalar, then

$$R+Q=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & i \\ & j & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+i \\ & e+j & \end{array} \right\rangle,$$

$$mR=m\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & ma & \\ mb & mh(R) & md \\ & me & \end{array} \right\rangle$$

and

$$R\circ Q=\left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle \circ \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & i \\ & j & \end{array} \right\rangle$$

$$=\left\langle \begin{array}{ccc} & ah(Q)+fh(R) & \\ bh(Q)+gh(R) & h(R)h(Q) & dh(Q)+ih(R) \\ & eh(Q)+jh(R) & \end{array} \right\rangle.$$

Remark 1

The above operation of multiplication is commutative and $(R_3(\Re),+)$ is an Abelian group i.e a commutative group with identity

$$0=\left\langle \begin{array}{ccc} & 0 & \\ 0 & 0 & 0 \\ & 0 & \end{array} \right\rangle$$

Ajibade (2003) also, determined the identity and inverse of the rhotrix $R\in R_3(\Re)$ as

$$I=\left\langle \begin{array}{ccc} & 0 & \\ 0 & 1 & 0 \\ & 0 & \end{array} \right\rangle$$

and

$$R^{-1} = \frac{-1}{(h(R))^2}\left\langle \begin{array}{ccc} & a & \\ b & -h(R) & d \\ & e & \end{array} \right\rangle,$$

where $h(R) \neq 0$, respectively.

Ajibade (2003) also established certain relationships between a rhotrix and its heart recorded as follows:

Theorem 2

A rhotrix *R* is invertible if and only if $h(R) \neq 0$

Theorem 3

For any rhotrix $R \neq 0$, $R^2 = 0$ if and only if $h(R) = 0$

The set of rhotrices of size *3* and the operation of rhotrix multiplication defined in (Ajibade, 2003) were later extended to rhotrices of size *n* by (Mohammed, 2011) recorded as follows:

$$\hat{R}(n) = \left\{ \left\langle \begin{array}{ccccccccc} & & & & r_1 & & & & \\ & & & r_2 & r_3 & r_4 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ r_{\left\{\frac{\frac{1}{2}(n^2+1)+1}{2}\right\}-n/2} & \cdots & \cdots & \cdots & r_{\left\{\frac{\frac{1}{2}(n^2+1)+1}{2}\right\}} & \cdots & \cdots & \cdots & r_{\left\{\frac{\frac{1}{2}(n^2+1)+1}{2}\right\}+n/2} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & r_{\frac{1}{2}(n^2-5)} & r_{\frac{1}{2}(n^2-3)} & r_{\frac{1}{2}(n^2-1)} & & & \\ & & & & r_{\frac{1}{2}(n^2+1)} & & & & \end{array} \right\rangle : r_1, \ldots, r_{\frac{1}{2}(n^2+1)} \in \Re \right\} \qquad (2.2)$$

Where $h(R) = r_{\frac{w+1}{2}}$ is the heart of the rhotrix, $n\backslash 2$ is the integer value obtained on division of n by 2 and $w = \frac{1}{2}(n^2+1)$.

Thus for $n = 3$, we get $R(3) = \left\langle \begin{array}{ccc} & r_1 & \\ r_2 & h(R) & r_4 \\ & r_5 & \end{array} \right\rangle$ and so on.

Furthermore, if A and B are any two rhotrices of the same size n, then $A \circ B$ is the resultant rhotrix C defined as:

$$C(n) = A(n) \circ B(n) = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_2 & a_3 & a_4 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{\left\{\frac{w+1}{2}\right\}-n/2} & \cdots & \cdots & \cdots & a_{\left\{\frac{w+1}{2}\right\}} & \cdots & \cdots & \cdots & a_{\left\{\frac{w+1}{2}\right\}+n/2} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_{w-3} & a_{w-2} & a_{w-1} & & & \\ & & & & a_w & & & & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_2 & b_3 & b_4 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{\left\{\frac{w+1}{2}\right\}-n/2} & \cdots & \cdots & \cdots & b_{\left\{\frac{w+1}{2}\right\}} & \cdots & \cdots & \cdots & b_{\left\{\frac{w+1}{2}\right\}+n/2} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & b_{w-3} & b_{w-2} & b_{w-1} & & & \\ & & & & b_w & & & & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccccccc} & & & & a_1 h(B) + b_1 h(A) & & & & \\ & & & a_2 h(B) + b_2 h(A) & a_3 h(B) + b_3 h(A) & a_4 h(B) + b_4 h(A) & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{\left\{\frac{w+1}{2}\right\}-n/2} h(B) + b_{\left\{\frac{w+1}{2}\right\}-n/2} h(A) & \cdots & \cdots & \cdots & h(A)\ h(B) & \cdots & \cdots & \cdots & a_{\left\{\frac{w+1}{2}\right\}+n/2} h(B) + b_{\left\{\frac{w+1}{2}\right\}+n/2} h(A) \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_{w-3} h(B) + b_{w-3} h(A) & a_{w-2} h(B) + b_{w-2} h(A) & a_{w-1} h(B) + b_{w-1} h(A) & & & \\ & & & & a_w h(B) + b_w h(A) & & & & \end{array} \right\rangle,$$

Where $n \in 2Z^+ + 1$, $w = \frac{1}{2}(n^2 + 1)$, $h(A) = a_{\left\{\frac{w+1}{2}\right\}}$, $h(B) = b_{\left\{\frac{w+1}{2}\right\}}$ and $n\backslash 2$ is the integer value obtained on division of *n* by 2.

Mohammed (2007a) adopted the rhotrix operations defined in (Ajibade, 2003) for presentation of various imaginations of rhomboidal arrays forming algebraic structures such groups, semigroups, monoids and rings using rhotrix set as an underlying set. Also, in his work, various special types of rhotrices such as symmetric rhotrix, diagonal rhotrix, lower and upper triangular rhotrix, zero heart rhotrix, unit heart rhotrix, odd and even heart rhotrix, nonzero heart rhotrix, odd and even rhotrix and hearty rhotrix were presented.

The theorem on rhotrix exponent rule was first given without proof in (Mohammed, 2007a), thereafter, (Mohammed, 2007b) established and characterized the theorem on rhotrix exponent rule and extended the result to special series and polynomial equations over rhotrices.

We record the following theorem from (Mohammed, 2007b)

Theorem 4

Let $R = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle$ be any rhotrix of size *3*, then for any integer value *m*,

$R^m = (h(R))^{m-1} \left\langle \begin{matrix} & ma & \\ mb & h(R) & md \\ & me & \end{matrix} \right\rangle$. In particular, R^0 and R^{-1} are the identity and inverse of R respectively, provided $h(R)$ is non-zero.

Remark 5

For the general case, $R(n)$, where *n* is odd,

$$R^{m}(n)=(h(R))^{m-1}\left\langle \begin{array}{c} mr_1 \\ mr_2 \quad mr_3 \quad mr_4 \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ mr_{\left\{\frac{w+1}{2}\right\}-n/2} \quad \cdots \quad \cdots \quad \cdots \quad h(R) \quad \cdots \quad \cdots \quad \cdots \quad mr_{\left\{\frac{w+1}{2}\right\}+n/2} \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ mr_{r_{w-3}} \quad mr_{r_{w-2}} \quad mr_{r_{w-1}} \\ mr_{r_w} \end{array} \right\rangle \qquad (2.3)$$

Where $n \in 2Z^{+}+1$, $w=\frac{1}{2}\left(n^{2}+1\right)$, $h(R)=r_{\left\{\frac{w+1}{2}\right\}}$.

Mohammed (2009) presented a remark on classification of rhotrices as abstract structures. In his work, rhotrix ring, rhotrix integral domain and rhotrix field were constructed with certain conditions.

Aminu (2009), initiated the study of linear systems over rhotrices. In his work, Aminu investigated the system of linear equations arising form the rhotrix equation $A \circ B = C$, where *A, B and C* are rhotrices of the same size.

We record the following proposition establishing a necessary and sufficient condition for the solvability of the system of linear equations arising form the rhotrix equation $A \circ B = C.$

Proposition 6

Let A, B and C be three rhotrices of the same size with entries in $\Re$. Then, the system of linear equations resulting from $A \circ B = C$ has

a) a unique solution if and only if $h(A) \neq 0$ and $h(C) \neq 0$

b) an infinite solution if and only if $h(A) = h(C) = 0$

c) no solution if and only if $h(A) = 0$ and $h(C) \neq 0$

In line with the work of (Aminu, 2009) on linear system of equations arising from rhotrix equation $A \circ X = C,$ where one of the equations was treated and a number of solvability conditions were suggested, (Aminu, 2012a) extended the problem to the case when all the systems were considered to be solved simultaneously.

Usaini and Tudunkaya (2011a) extended the work by (Mohammed, 2009) to construct certain field of fractions over rhotrices. The construction was done step by step, where at each step a particular algebraic property was shown. But it was later discovered by (Usaini and Tudunkaya, 2012) that the field presented by (Mohammed, 2009) can only be possible if the underlying rhotrix set is a set of all hearty rhotrices of the same size defined in (Mohammed, 2007a).

As an extension to the work of (Mohammed, 2011), where the generalization of heart based method for multiplication of rhotrices of size n was presented, (Mohammed *et al.*, 2011) presented an algorithmic implementation for the generalized heart based method for multiplication of rhotrices of size n. Thereafter, (Absalom *et al.*, 2011a) proposed a simplified version of the rhotrix expression generalization proposed by (Mohammed, 2011) for heart based rhotrices of size *n*.

Mohammed and Sani (2011) extended the concept of graph theory to rhotrix theory through their introduction of rhomtrees of order $m = \frac{1}{2}(n^2 + 1)$ as a graphical representation of rhotrices of size *n*, where $n \in 2Z^+ + 1$. The rhomtrees were shown in their work to have relationship with certain real world problems such as topology of computing network, methane compound and certain product of sets.

Mohammed and Tijjani (2011) defined metric or distance function from a rhotrix set to the set of all real numbers. They extended their work to the construction of metric topological spaces over rhotrices.

Tudunkaya and Makanjuola (2010) presented a method of constructing finite fields over rhotrices. The cardinality of these finite fields was also given through concrete examples.

Tudunkaya and Makanjuola (2012a) proposed certain quadratic extension, as an extension to the work of (Mohammed, 2007b), where a note on rhotrix exponent rule and its applications to special series and polynomial equations defined over rhotrices was presented. Thereafter, rhotrix polynomial and polynomial rhotrices were proposed by (Tudunkaya, 2013) as another further extension to the work of Mohammed given in 2007b.

As an extension to their work in 2010, (Tudunkaya and Makanjuola, 2012b), further presented properties of certain finite fields constructed over rhotrices. (Usaini and Tudunkaya, 2011b) presented a note on rhotrices and construction of finite fields as an extension to the work of (Tudunkaya and Makanjuola, 2010).

Mohammed and Tella (2012) presented rhotrix sets and rhotrix spaces categorized over real and complex fields.

The following section present the review of developments of rhotrix theory based on "*non-commutative method for rhotrix multiplication*".

2.3 NON-COMMUTATIVE RHOTRIX THEORY

Sani (2004) extended the concept of row-column multiplication of two dimensional matrices to propose an alternative method for multiplication of rhotrices of size three. This is in an attempt to answer the question of 'whether a transformation can be made to convert a matrix into a rhotrix and vice versa' posed by Ajibade in the concluding remark of his work.

Sani (2004) defined multiplication of two rhotrices as follows:

$$R \circ Q = \left\langle \begin{matrix} & \mathrm{a} & \\ \mathrm{b} & \mathrm{h(R)} & d \\ & e & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & \mathrm{f} & \\ \mathrm{g} & \mathrm{h(Q)} & i \\ & j & \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} & af + dg & \\ bf + eg & h(R)h(Q) & ai + dj \\ & bi + ej & \end{matrix} \right\rangle$$

This multiplication is non-commutative, but it is associative. The method of rhotrix multiplication was used to establish some relationship between rhotrices of size *3* and matrices of dimension *2*.

Sani (2004) further determined the identity, inverse, determinant and transpose of the rhotrix *R(3)* respectively as follows:

$$I = \left\langle \begin{matrix} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{matrix} \right\rangle,$$

$$R^{-1} = \left\langle \begin{matrix} & \frac{e}{ae - bd} & \\ \frac{-b}{ae - bd} & \frac{1}{h(R)} & \frac{-d}{ae - bd} \\ & \frac{a}{ae - bd} & \end{matrix} \right\rangle,$$

$$\det(R) = h(R)(ae - bd),$$

and

$$R^{T} = \left\langle \begin{matrix} & a & \\ d & h(R) & b \\ & e & \end{matrix} \right\rangle$$

In 2007, Sani extended the non-commutative rhotrix multiplication method for base rhotrices Sani in 2004 to rhotrices having size n as follows:

$$\mathrm{R}(n) \circ \mathrm{S}(n) = \left\langle a_{i_1 j_1}, c_{l_1 k_1} \right\rangle \circ \left\langle b_{i_2 j_2}, d_{l_2 k_2} \right\rangle$$

$$= \left\langle \sum_{i_2 j_1 = 1}^{w} (a_{i_1 j_1}, b_{i_1 j_1}), \sum_{l_2 k_1 = 1}^{w-1} (c_{l_2 k_2}, d_{l_2 k_2}) \right\rangle, \qquad (2.4)$$

Where $w = \frac{1}{2}(n+1)$,

$$\mathrm{R}(n) = \left\langle a_{ij}, c_{lk} \right\rangle = \left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{21} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{w1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1w} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{ww-2} & c_{w-1w-2} & a_{w-1w-1} & c_{w-2w-1} & a_{w-2w} & & \\ & & & a_{ww-1} & c_{w-1w-1} & a_{w-2w} & & & \\ & & & & a_{ww} & & & & \end{array} \right\rangle$$

and

$$\mathrm{S}(n) = \left\langle a_{ij}, c_{lk} \right\rangle = \left\langle \begin{array}{ccccccccc} & & & & b_{11} & & & & \\ & & & b_{21} & d_{11} & b_{12} & & & \\ & & b_{31} & d_{21} & b_{21} & d_{12} & b_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ b_{w1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1w} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & b_{ww-2} & d_{w-1w-2} & b_{w-1w-1} & d_{w-2w-1} & b_{w-2w} & & \\ & & & b_{ww-1} & d_{w-1w-1} & b_{w-2w} & & & \\ & & & & b_{ww} & & & & \end{array} \right\rangle$$

The elements $a_{ij}\,(i,\, j = 1, 2, ..., w)$ and $c_{kl}\,(k, l = 1, 2, ..., w-1)$ are called the major and minor entries of $R(n)$ respectively. Similarly, the elements $b_{ij}\,(i,\, j = 1, 2, ..., w)$ and $d_{kl}\,(k, l = 1, 2, ..., w-1)$ are the major and minor entries of $S(n)$ respectively.

Also Sani (2007), generalised the definition of the transpose, determinant, identity and inverse of rhotrix $R(n)$ of size *n*, (provided $R(n) \neq 0$). (Sani, 2007) further established some interesting relationships between invertible *n*-size rhotrices and invertible $w \times w$ dimensional matrices, where $w = \frac{1}{2}(n+1)$, $n \in 2Z^{+} + 1$.

Kaurangini and Sani (2007) presented the concept of Hilbert matrix and its relationship with a special rhotrix, where they constructed a special form of rhotrix, termed as *'Hilbert rhotrix'* of size *5* coupling two Hilbert matrices of dimensions 3×3 and 2×2.

The question of transforming rhotrix to matrix and vice-versa posed by (Ajibade, 2003) was completely resolved by (Sani, 2008), when he proposed a method of converting rhotrix to a special form of matrix called '*coupled matrix'.* This is done by rotating the rhotrix R of size $n \in 2Z^{+} + 1$ through 45^{0} in anti-clockwise direction. This is a special form of matrix with missing values. For example, the coupled matrix of rhotrix $R(5)$ is as follows:

$$R_5^{T/2} = \left\langle \begin{matrix} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{matrix} \right\rangle^{T/2} = \begin{bmatrix} a_{11} & & a_{12} & & a_{13} \\ & c_{11} & & c_{12} & \\ a_{21} & & a_{22} & & a_{23} \\ & c_{21} & & c_{22} & \\ a_{31} & & a_{32} & & a_{33} \end{bmatrix} \quad (2.5)$$

Where $T/2$ indicates a rotation through 45^0 in anti-clockwise direction. The special matrix in (2.5) is a coupling of 3×3 matrix with a 2×2 matrix, hence, the name 'a coupled matrix'. Therefore, in general we have:

$$R_n^{T/2} = \langle a_{ij}, c_{kl} \rangle^{T/2} = [a_{ij}, c_{kl}] = [Ac](n). \quad (2.6)$$

That rotation results into a coupled matrix, consisting of two matrices of dimensions $w\times w$ and $(w-1)\times(w-1)$.

Two coupled matrices $[Ac]_n$ and $[Bd]_n$ can be multiplied together by simply filling the missing spaces with zeros, after the multiplication, we removed the zeros in order to have the result in filled coupled matrix form.

We record the following result from (Sani, 2008):

Theorem 7

If a coupled matrix $[Ac]_n$ is completed with zeros, then its determinant is the product of the determinants of the matrices $[A]_{w\times w}$ and $[c]_{(w-1)\times(w-1)}$, where $w = \frac{1}{2}(n+1)$.

Remark 8

The determinant of a coupled matrix $[Ac]_n$ can be obtained in the same way as the determinant of a rhotrix of size *n*, i.e. $\det[Ac]_n = \det[A]_{w\times w} \det[c]_{(w-1)\times(w-1)}$.

The idea of a coupled matrix can be used to solve problems involving two different systems simultaneously, where one is a $w \times w$ system, $\mathbf{A}X = b$ while, the other is a $(w-1)\times(w-1)$ system, $cY = d$. The two system $\mathbf{A}X = b$ and $cY = d$ can be coupled together as

$$\begin{bmatrix} a_{11} & & a_{12} & & \cdots & \cdots & \cdots & & a_{1w} \\ & c_{11} & & c_{12} & \cdots & \cdots & \cdots & c_{1w-1} & \\ a_{21} & & a_{22} & & \cdots & \cdots & \cdots & & a_{2w} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{w-11} & & a_{w-12} & & \cdots & \cdots & \cdots & & a_{w-1w} \\ & c_{w-11} & & c_{w-12} & \cdots & \cdots & \cdots & c_{w-1w-1} & \\ a_{w1} & & a_{w2} & & \cdots & \cdots & \cdots & & a_{ww} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ \cdots \\ \cdots \\ \cdots \\ x_{w-1} \\ y_{w-1} \\ x_w \end{bmatrix} = \begin{bmatrix} b_1 \\ d_1 \\ y_2 \\ \cdots \\ \cdots \\ \cdots \\ b_{w-1} \\ d_{w-1} \\ b_w \end{bmatrix} \qquad (2.7)$$

If we filled the missing spaces of equation 2.7 with zeros, we get $n \times n$ matrix which can then be solved to get solution of the two systems simultaneously.

Following this idea, (Sani, 2009) presented the solution of two coupled matrices by extending the idea of a coupled matrix presented in his earlier work to a generalized case involving $m \times n$ and $(m-1)\times(n-1)$ matrices as follows:

Suppose $A=\lfloor a_{ij}\rfloor$ is $m\times n$ matrix and $b=[b_{lk}]$ is $(m-1)\times(n-1)$ matrix then these two matrices can be coupled together to form the matrix:

$$\begin{bmatrix} a_{11} & & a_{12} & & \cdots & \cdots & \cdots & & a_{1n} \\ & b_{11} & & b_{12} & \cdots & \cdots & \cdots & b_{1n-1} & \\ a_{21} & & a_{22} & & \cdots & \cdots & \cdots & & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m-11} & & a_{m-12} & & \cdots & \cdots & \cdots & & a_{m-1n} \\ & b_{m-11} & & b_{m-12} & \cdots & \cdots & \cdots & b_{m-1n-1} & \\ a_{m1} & & a_{m2} & & \cdots & \cdots & \cdots & & a_{mn} \end{bmatrix} = [a_{ij}\ , b_{kl}] = [Ab]_{(2m-1)\times(2n-1)}$$

Completing the above equation with zeros, we obtain a $(2m-1)\times(2n-1)$ dimensional matrix whose properties could be deduced from those of the two separate matrices. This concept is used to solve problems involving $m\times n$ and $(m-1)\times(n-1)$ matrices simultaneously.

The next theorem recorded from (Sani, 2009) shows that, the two systems above could be coupled together which will lead to a method of solving two systems simultaneously.

Theorem 9

If $\mathbf{A}X=c$ is an $s\times w$ system of linear equations, whose solution is the vector X and $bY=d$ is an $(s-1)\times(w-1)$ system whose solution is the vector Y, then the following coupled system gives the solution of the two systems

$$\begin{bmatrix} a_{11} & 0 & a_{12} & 0 & \ldots & \ldots & \ldots & 0 & a_{1w} \\ 0 & b_{11} & 0 & b_{12} & \ldots & \ldots & \ldots & b_{1w-1} & 0 \\ a_{21} & 0 & a_{22} & 0 & \ldots & \ldots & \ldots & 0 & a_{2w} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{s-11} & 0 & a_{s-12} & 0 & \ldots & \ldots & \ldots & 0 & a_{s-1w} \\ 0 & b_{s-11} & 0 & b_{s-12} & \ldots & \ldots & \ldots & b_{s-1w-1} & 0 \\ a_{s1} & 0 & a_{s2} & 0 & \ldots & \ldots & \ldots & 0 & a_{sw} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ \ldots \\ \ldots \\ \ldots \\ x_{w-1} \\ y_{w-1} \\ x_w \end{bmatrix} = \begin{bmatrix} c_1 \\ d_1 \\ c_2 \\ \ldots \\ \ldots \\ \ldots \\ c_{w-1} \\ d_{s-1} \\ c_s \end{bmatrix}$$

Aminu (2010a) adopted the concept of rows and columns of rhotrix suggested by (Sani, 2007) to define the concept of rhotrix row and column vectors of size n as

$$\left\langle \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & 0 & 0 & & & \\ & & 0 & 0 & 0 & 0 & 0 & & \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ a_{w1} & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 0 \\ & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \\ & & a_{ww-2} & 0 & 0 & 0 & 0 & & \\ & & & a_{ww-1} & 0 & 0 & & & \\ & & & & a_{ww} & & & & \end{array} \right\rangle$$

and

$$
\left\langle
\begin{array}{ccccccccc}
 & & & & a_{11} & & & & \\
 & & & a_{21} & 0 & 0 & & & \\
 & & a_{31} & 0 & 0 & 0 & 0 & & \\
 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
a_{w1} & \dots & \dots & \dots & \dots & \dots & \dots & \dots & 0 \\
 & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
 & & 0 & 0 & 0 & 0 & 0 & & \\
 & & & 0 & 0 & 0 & & & \\
 & & & & 0 & & & &
\end{array}
\right\rangle
$$

respectively.

The difference between $w \times w$ dimensional matrix vectors and n-size rhotrix vectors is that, there is a unique representation of any $w \times w$ dimensional matrix vector, while any n-size rhotrix vector can be represented in w different ways, where $w = \frac{1}{2}(n+1)$, $n \in 2Z^{+} + 1$.

The representations of the main rhotrix column and row vectors are respectively

$$\left\langle x^{nj} \right\rangle \text{ and } \left\langle x^{in} \right\rangle. \tag{2.8}$$

Similarly, other columns and rows which are minor as

$$\left\langle x^{n-1k} \right\rangle \text{ and } \left\langle x^{\mathrm{ln}-1} \right\rangle \tag{2.9}$$

Where $i, j = 1, 2, \ldots, w$, $k, l = 1, 2, \ldots, w-1$ and $w = \frac{1}{2}(n+1)$.

The following theorem from (Aminu, 2010a) presents properties of rhotrix vector space.

Theorem 10

Let $A(n)$, $B(n)$ and $C(n)$ be n-dimensional rhotrix vectors with the same representation. If α and β are scalars, then:

a) $A(n) + O = A(n)$

b) $0\, A(n) = 0$

c) $A(n) + B(n) = B(n) + A(n)$

d) $(A(n) + B(n)) + C(n) = A(n) + (B(n)) + C(n))$

e) $\alpha\,(A(n) + B(n)) = \alpha\, A(n) + \alpha\, B(n)$

f) $(\alpha + \beta)\, A(n) = \alpha\, A(n) + \beta\, A(n)$

g) $(\alpha\,\beta)\, A(n) = \alpha\,(\beta\, A(n))$

Note that *0* and ***O*** denotes the usual zero and zero rhotrix respectively. *O* is the neutral element under addition, and for convenience, we use *O* to denote any rhotrix vector having every component as 0.

Aminu (2010a) further constructed and presented one-sided system of the form

$$R_n(X) = b, \tag{2.10}$$

where R_n is an *n*-dimensional rhotrix, X the unknown *n*-dimensional rhotrix vector and b the right hand side rhotrix vector. The necessary and sufficient condition for the solvability of the system of rhotrix equation $R_n\langle x^{ni}\rangle = \langle b^{nj}\rangle$ was indicated.

Any system of the form in equation (2.10) is called a system of *n* rhotrix equations. The following theorem was recorded in (Aminu, 2010a).

Theorem 11

Let $R_n = \langle a_{ij}, c_{kl}\rangle$ be an *n*-dimensional rhotrix. A necessary and sufficient condition for the solvability of the system $R_n\langle x^{nj}\rangle = \langle b^{nj}\rangle$ is that, the corresponding system of equations, $Ax^{wj} = b^{wj}$, is solvable, where $A = (a_{ij}) \in \Re^{w \times w}$, $x^{wj}, b^{wj} \in \Re^{w \times w}$ and $w = \frac{1}{2}(n+1)$.

Aminu (2010a) further presented the concepts of rhotrix eigenvector and eigenvalue problems. The rhotrix eigenvalue problem (REP) is the following. Given $R_n = \langle a_{ij}, c_{kl}\rangle$, find all $\lambda \in \Re$ (eigenvalues) and an n-dimensional rhotrix column vector $\langle x^{nj}\rangle$, $\langle x^{nj}\rangle \neq 0$ (eigenvectors) such that

$$R_n\langle x^{nj}\rangle = \langle b^{nj}\rangle. \qquad (2.11)$$

In order to solve REP, we have the following result from Aminu (2010a)

Theorem 12

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an *n*-dimensional rhotrix. Then, $\lambda \in \Re$, is a rhotrix eigenvalue of R_n if and only if $\det(A - \lambda I) = 0$, where $A = (a_{ij}) \in \Re^{w \times w}$ and $w = \frac{1}{2}(n+1)$.

Corollary 13

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an *n*-dimensional rhotrix and $A = (a_{ij}) \in \Re^{w \times w}$ be matrix generated from $R(n)$ $w = \frac{1}{2}(n+1)$. Then $\langle v^{nj} \rangle$ is the rhotrix eigenvector corresponding to the eigenvalue λ if the system $(A - \lambda I) = 0$.

Aminu (2010b) adopted the alternative method for multiplication of rhotrices by (Sani, 2007) to introduce the concept of rhotrix vector spaces. This is given as follows:

'A rhotrix vector space $\langle v \rangle$ with entries from the set of all real numbers is a non-empty set of rhotrix vectors with two operations, addition and scalar multiplication which satisfy the axioms of a vector space'.

The properties of these vector spaces were discussed. Furthermore, it was shown that the set of all rhotrices of size-*n* forms a vector space.

Following this, (Aminu, 2010c) extended the concept of linear mapping to rhotrices, where he considered the linear mapping $T: U \to V$, such that U and V are rhotrix vector spaces and present its properties. It was also shown in his work that the proposed method of converting a rhotrix to a special matrix called ‘coupled matrix’ suggested by (Sani, 2009) is a linear mapping.

A note on relationship between invertible rhotrices and associated invertible matrices was presented by (Sharma and Kanwar, 2011).

Absalom *et al.* (2011) presented an algorithm design for the implementation of row-column based (also called non-commutative) method for multiplication of rhotrices of size *n* proposed by (Sani, 2007).

Usaini (2012a) presented elementary row operations on rhotrix, due to the vital roles they played in matrix theory. These operations can be used to determine rhotrix inverses and solve a system of *n* rhotrix equations.

Aminu (2012a) extended the concept of determinant method (one of the well-known methods that is formulated and proved in linear algebra on matrices), to the concept of rhotrix. Here, a rhotrix system of linear equations was solved using determinant method.

The concept of involutory matrix was also extended to rhotrices. That is, a matrix which is its own inverse. Such matrices are of great importance in matrix theory and algebraic cryptography. (Usaini, 2012a) extended this concept of involution to rhotrices and present their properties. A method of constructing involutory rhotrices was also indicated in his work. Thereafter, extension on the concept of involutory rhotrix was considered by (Sharma and Kanwar, 2013) to give certain theorems of involution in the context of rhotrices. Also, the concept of Pascal rhotrix and its related properties were presented.

Aminu (2012b), extended the concept of Cayley-Hamilton, one of the well-known theorems that is formulated and proved in linear algebra on matrices to the concept of rhotrix and also present some properties that are attached to it. This extension was also considered by (Sharma and Kanwar, 2012a).

Following the row-column based method for multiplication of rhotrices defined by (Sani, 2007), (Chinedu, 2012) identified some various methods of representing an arbitrary rhotrix. One of the methods - the row-wise method - has been chosen as it is observed to be flexible in analyzing rhotrices for mathematical enrichment. A relationship between the location of the heart of a rhotrix and the dimension of the rhotrix and also a relationship between the location of the heart of a rhotrix and the order of the principal matrix of the rhotrix have been determined. The flexibility of the representation has paved way for two formulae, one for row-column multiplication of arbitrary rhotrices and the other for heart-based multiplication of arbitrary rhotrices. (Chinedu, 2012) further give some examples as a way of demonstrating the application of the proposed formulae.

Sharma and Kanwar (2012b) used the concept of adjoint of matrix and present the concept of adjoint of rhotrix and its basic properties. They described adjoint of a rhotrix, and also proved some related analogous results of matrices in the context of rhotrices.

Sharma and Kanwar (2012c) introduced the concept of inner product and bilinear forms over real rhotrices.

2.4 SEMIGROUP AND GREEN'S RELATION

The theory of semigroup is not a new area of mathematical research. It has been well developed by researchers since before the twentieth century. Many concepts in semigroup theory were analogous to group theory, but the concept of Green's relations and many others are developed independently. This makes semigroup theory a well deserved area of research.

2.4.1 GREEN'S RELATIONS

The concept of ideals led to the study of certain equivalence relations on a semigroup known as Green's relations. These equivalences denoted by ($\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D}$ and $\mathcal{H}$) have played a fundamental role in the development of semigroup theory and they were first introduced by Green (1951). Since their introduction, they became standard tools for investigating the structure of semigroup.

An equivalence $\mathcal{L}$ on S is defined by the rule that $a \mathcal{L} b$ if and only if a and b generate the same principal left ideal, that is, if and only if $S^1a = S^1b$, for a, b elements of S. Similarly, we define the equivalence $\mathcal{R}$ by the rule that $a \mathcal{R} b$ if and only if $aS^1 = bS^1$.

The following theorem give an alternative characterisation of equivalences $\mathcal{L}$ and $\mathcal{R}$**,** recorded from Howie (1995) :

Theorem 2.4.1 [Howie (1995), Proposition 2.1.1]

Let *a, b* be elements of a semigroup S. Then $a\ \mathcal{L}\ b$ if and only if there exist x, y in S^1 such that $xa = b$ and $yb = a$. Also, $a\ \mathcal{R}\ b$ if and only if there exist u, v in S^1 such that $au = b$ and $bv = a$.

The intersection of two equivalences is again an equivalence relation and the intersection of $\mathcal{L}$ and $\mathcal{R}$ is of great importance in the development of semigroup theory, i.e, $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$**.** The join $\mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R}$ is also of great importance, and so it is denoted here as $\mathcal{D}$.

We record the following result from Howie (1995):

Theorem 2.4.2 [Howie (1995), Proposition 2.1.3**]**

The Green's relations $\mathcal{L}$ and $\mathcal{R}$ commute.

Proof

Let S be a semigroup and let *a, b* be elements of S where $(a,b) \in \mathcal{L} \circ \mathcal{R}$. Then, there exists c in S such that $a\ \mathcal{L}\ c$ and $c\ \mathcal{R}\ b$. That is, there exist x,y,u,v in S^1 such that

$$xa = c, \quad cu = b,$$

$$yc = a, \quad bv = c\ .$$

If we now write d for the element ycu of S, we see that

$$au = ycu = d, \qquad dv = ycuv = ybv = yc = a;$$

Hence, $a\,\mathcal{R}$ d. Also,

$$yb = ycu = d, \qquad xd = xycu = xau = cu = b;$$

And so $d\ \mathcal{L}\ b$, we deduce that $(a,b) \in \mathcal{R} \circ \mathcal{L}$. We have shown that $\mathcal{L} \circ \mathcal{R} \subseteq \mathcal{R} \circ \mathcal{L}$; the reverse inclusion follows in a similar way.

Our final equivalence is the two-sided analogue of $\mathcal{L}$ and $\mathcal{R}$**.** We define the equivalence $\mathcal{J}$ by the rule that $a\ \mathcal{J}\ b$ if and only if $S^1aS^1 = S^1bS^1$, that is to say, if and only if there exist $x, y, u, v \in S^1$ such that $xay = b$ and $ubv = a$.

In certain class of semigroup, these equivalences coincide as

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J} = G \times G,$$

if the semigroup turns out as a group G. In commutative semigroup, we have

$$\mathcal{H} = \mathcal{L} = \mathcal{R} = \mathcal{D} = \mathcal{J}.$$

Less trivially, we have a result from Howie (1995) which in particular, implies that $\mathcal{D} = \mathcal{J}$ in every finite semigroup:

Theorem 2.4.3 [Howie (1995), Proposition 2.1.4]:

If S is a periodic semigroup, then $\mathcal{D} = \mathcal{J}$

If U is a subsemigroup of a (not necessarily regular) semigroup S, and if $a,b \in U$, there can be ambiguity about the meaning of $a \mathcal{L} b$, since (for example) $\mathcal{L}$ may stand for the appropriate Green's equivalence either in S or in U. When confusion of this sort is likely to arise, we shall use different symbols to distinguish between the two equivalences. Thus, $(a,b) \in \mathcal{L}^{\mathrm{U}}$ or $(a,b) \in \mathcal{L}(\mathrm{U})$ means that, there exist $u,v \in U^1$ such that $ua = b$, $vb = a$, while $(a,b) \in \mathcal{L}^{\mathrm{S}}$ or $(a,b) \in \mathcal{L}(\mathrm{S})$ means that there exist $s,t \in U^1$ such that $sa = b$, $tb = a$.

We record the following useful result from Howie (1995)

Theorem 2.4.4 [Howie (1995), Proposition 2.4.2]:

If U is a regular subsemigroup of a semigroup S. Then,

(i) $\mathcal{L}(U) = \mathcal{L}(S) \cap (U \times U)$,

(ii) $\mathcal{R}(U) = \mathcal{R}(S) \cap (U \times U)$ and

(iii) $\mathcal{H}(U) = \mathcal{H}(S) \cap (U \times U)$.

Proof

Suppose that $(a,b) \in \mathcal{L}(S) \cap (U \times U)$, and let a^1 and b^1 be inverses of a and b in U respectively. Then

$$(a^1 a, a) \in \mathcal{L}(U) \subseteq \mathcal{L}(S), \qquad (b^1 b, b) \in \mathcal{L}(U) \subseteq \mathcal{L}(S)$$

And so $(a^1a, b^1b) \in \mathcal{L}(S)$. By (Howie 1996, proposition 2.3.3) each of a^1a and b^1b is a right identity for $L^S_{a^1a} = L^S_{b^1b}$; hence in particular

$$a^1ab^1b = a^1a\ , \quad b^1b\,a^1a = b^1b\ .$$

These equations involve only elements of *U*, and so may be interpreted as implying that $(a^1a, b^1b) \in \mathcal{L}(U)$. But we now have

$$a\ \mathcal{L}(U)\ a^1a, \qquad a^1a\ \mathcal{L}(U)\ b^1b, \qquad b^1b\ \mathcal{L}(U)\ b,$$

and so $a\ \mathcal{L}(U)\ b$ as required.

The proof for $\mathcal{R}$ is exactly dual, and the result for $\mathcal{H}$ is a consequence of the results for $\mathcal{L}$ and $\mathcal{R}$.

2.5 Concluding Remarks

Based on the two alternative methods for rhotrix multiplication that are available in the literature of rhotrix theory, we have studied and reviewed the developments in the theory of rhotrix for a decade, starting from the year 2003, when the concept of rhotrix was introduced up to the end of 2013. Over forty articles on rhotrix theory have been published in journals since its inception. The remarkable aspect of this literature review of these articles on rhotrix theory is that, authors following the heart based method for rhotrix multiplication enjoy the commutative property associated with the heart based method. For this reason, a number of abstract structures such as rhotrix groups, rhotrix semigroups, rhotrix rings, rhotrix Boolean algebra, rhotrix topological spaces, rhotrix metric spaces, rhotrix graphical trees called rhomtrees were developed. Furthermore, rhotrix finite fields, rhotrix exponent rule and their applications to construction of special series, polynomial equations and polynomial rings were also initiated.

The contributory authors following the non-commutative rhotrix theory focus their research predominantly on extending the properties of matrices to rhotrices. The motivation comes from the work of Sani in 2008 when he proposed the conversion of rhotrix to a special form of matrix termed as coupled matrix. The article made several authors to study properties of matrices that are analogous to rhotrices as we have seen from the non-commutative rhotrix theory.

As a step toward further in the development of rhotrix theory, we shall adopt the alternative method for multiplication of rhotrices defined by Sani (2004, 2007) which we termed as *non-commutative method for rhotrix multiplication* to initiate the algebraic study of '*rhotrix semigroup*'.

CHAPTER THREE

THE RHOTRIX SEMIGROUP

3.1 INTRODUCTION

This chapter focuses on the algebraic study of rhotrix semigroup. Using rhotrix set as an underlying set, and together with the binary operation of row-column method for rhotrix multiplication proposed by Sani (2007), we initiate certain algebraic system, which we termed as '*Rhotrix Semigroup*' and study its properties. In particular, we classify the rhotrix semigroup as a regular semigroup and characterise all its five Green's equivalence relations.

3.2 THE RHOTRIX SEMIGROUP

Let $R_n(F)$ be a set of all rhotrices of size *n* with entries from an arbitrary field F . Let the operation $\circ$ denote the rhotrix multiplication defined by Sani (2007). Then for any rhotrices $A, B, C \in R_n(F)$, we have $A \circ B \in R_n(F)$ and $(A \circ B) \circ C = A \circ (B \circ C)$. This shows that the operation $\circ$ is closed and associative. Therefore, the system $(R_n(F), \circ)$ forms a semigroup which we shall term as '*Rhotrix Semigroup*'.

It was observed that the Rhotrix Semigroup $(R_n(F), \circ)$ has the following evident properties.

i. It is non-commutative, that is, $A \circ B \neq B \circ A$, for some $A, B \in (R_n(F), \circ)$.

ii. The rhotrix semigroup $(R_n(F), \circ)$ is a monoid semigroup.

iii. It is a mortal semigroup *i.e.* a semigroup with zero.

It is worthy to note that the rhotrix semigroup $(R_n(F), \circ)$ forms an interesting algebraic structure because of its non-commutative property. For if, the operation was commutative, the concept of Green's relations will all coincide. This non-commutative property motivated us to study the rhotrix semigroup $(R_n(F), \circ)$.

Now, it is interesting for us to ask the following questions:

i. What type of semigroup is $(R_n(F), \circ)$?

ii. What are its subsemigroups?

iii. Is there any link between rhotrix semigroup $(R_n(F), \circ)$ and matrix semigroup $(M_n(F), \cdot)$?

iv. What are the Green's equivalence relations in $(R_n(F), \circ)$ like?

Throughout what follows, we shall mean $(R_n(F), \circ)$ to be a Semigroup with respect to row-column rhotrix multiplication defined by Sani (2007). We shall also sometimes write $R_n(F)$ instead of $(R_n(F), \circ)$, without any form of ambiguity.

3.3 Some Subsemigroups of $R_n(F)$

In this section, we identify certain subsemigroups of the rhotrix semigroup $(R_n(F), \circ)$ and also establish isomorphic relationship between a number of its subsemigroups. We begin with the following:

Definition 3.3.1 (Rhotrix subsemigroup)

Let $R_n(F)$ be the semigroup of rhotrices of size *n*. A subset $W_n(F)$ of $R_n(F)$ is called a subsemigroup of $R_n(F)$ if $W_n(F)$ is closed with respect to the binary operation defined on $R_n(F)$.

For instance, let

$$A(n) = \langle a_{ij}, c_{kl} \rangle = \left\langle \begin{array}{ccccccccccc} & & & & & a_{11} & & & & & \\ & & & & a_{21} & c_{11} & a_{12} & & & & \\ & & & a_{31} & c_{21} & a_{21} & c_{12} & a_{13} & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & a_{w1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1w} & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_{ww-2} & c_{w-1w-2} & a_{w-1w-1} & c_{w-2w-1} & a_{w-2w} & & & \\ & & & & a_{ww-1} & c_{w-1w-1} & a_{w-2w} & & & & \\ & & & & & a_{ww} & & & & & \end{array} \right\rangle \qquad (3.1)$$

be a rhotrix of size $n \in 2\mathrm{Z}^{+}+1$, having entries $a_{ij}\,(i, j = 1, 2, ..., w)$ and $c_{kl}\,(i, j = 1, 2, ..., w-1)$ as the major and minor entries respectively, $w = \frac{1}{2}(n+1)$.

The subsets $ZH_n(F)$ and $NZH_n(F)$, i.e the zero and non-zero heart rhotrices of size n respectively defined in equation (3.2) and (3.3) below are subsemigroups of the rhotrix semigroup $(R_n(F), \circ)$.

$$ZH_n(F) = \begin{cases} A(n) \in \mathrm{R_n}(F) : h(A) = c_{m,m} = 0, \ when \ \ n \in \{4m-1 : m \in Z^+\} \\ \\ A(n) \in \mathrm{R_n}(F) : h(A) = a_{m,m} = 0, when \ \ n \in \{4m+1 : m \in Z^+\} \end{cases} \quad (3.2)$$

and

$$NZH_n(F) = \begin{cases} A(n) \in \mathrm{R_n}(F) : h(A) = c_{m,m} \neq 0, \ when \ \ n \in \{4m-1 : m \in Z^+\} \\ \\ A(n) \in \mathrm{R_n}(F) : h(A) = a_{m,m} \neq 0, when \ \ n \in \{4m+1 : m \in Z^+\} \end{cases} \quad (3.3)$$

It is clear that if $A, B \in ZH_n(F)$, then, $A \circ B \in ZH_n(F)$, similarly if $A, B \in NZH_n(F)$, then, $A \circ B \in NZH_n(F)$.

Remark 3.3.2

The heart of the rhotrix A i.e $h(A) = a_{m,m}$ lies in the major entries of A when n, the size of the rhotrix belong to the set $\{4m + 1 : m \in Z^+\}$ and the heart of rhotrix A i.e $h(A) = c_{m,m}$ lies in the minor entries of A when n, the size of the rhotrix belong to the set $\{4m - 1 : m \in Z^+\}$ where $m = 1,2,3,\ldots$

The following theorem establishes a connection between rhotrix semigroup $(R_n(F), \circ)$ and the semigroup $(M_n(F), \cdot)$ of all $n \times n$ matrices over a field F with respect to matrix multiplication.

Theorem 3.3.3

The rhotrix semigroup $(R_n(F), \circ)$ is embedded in the matrix semigroup $(M_n(F), \cdot)$.

Proof:

Let $\langle R_n(F), \circ \rangle$ be a rhotrix semigroup with respect to non-commutative rhotrix multiplication and let $(M_n(F), \cdot)$ be a matrix semigroup with respect to usual matrix multiplication. We define a mapping $\theta : \langle R_n(F), \circ \rangle \to (M_n(F), \cdot)$ by

$$\theta\left(\left\langle \begin{array}{ccccccccc} & & & & a_{11} & & & & \\ & & & a_{21} & c_{11} & a_{12} & & & \\ & & a_{31} & c_{21} & a_{21} & c_{12} & a_{13} & & \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ a_{w1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & a_{1w} \\ & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & a_{ww-2} & c_{w-1w-2} & a_{w-1w-1} & c_{w-2w-1} & a_{w-2w} & & \\ & & & a_{ww-1} & c_{w-1w-1} & a_{w-2w} & & & \\ & & & & a_{ww} & & & & \end{array} \right\rangle\right) = \begin{pmatrix} a_{11} & 0 & a_{12} & 0 & \cdots & \cdots & \cdots & 0 & a_{1w} \\ 0 & c_{11} & 0 & c_{12} & \cdots & \cdots & \cdots & c_{1w-1} & 0 \\ a_{21} & 0 & a_{22} & 0 & \cdots & \cdots & \cdots & 0 & a_{2w} \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ a_{w-11} & 0 & a_{w-12} & 0 & \cdots & \cdots & \cdots & 0 & a_{w-1w} \\ 0 & c_{w-11} & 0 & c_{w-12} & \cdots & \cdots & \cdots & a_{w-1w-1} & 0 \\ a_{w1} & 0 & a_{w2} & 0 & \cdots & \cdots & \cdots & 0 & a_{ww} \end{pmatrix},$$

where $w = (n+1)/2$ and $n \in 2Z^+ + 1$. implying that θ mapped each rhotrix in $R_n(F)$ to its filled coupled matrix in $M_n(F)$.

Then clearly, it follows from the definition of rhotrix multiplication defined by Sani (2007) and matrix multiplication that $\theta(A \circ B) = \theta(A) \cdot \theta(B)$ for all $A, B \in (R_n(F), \circ)$ so that θ is a homomorphism.

θ is a one-to-one mapping, since no two rhotrices have the same filled coupled matrix. This completes the proof.

Remark 3.3.4

If $\theta : R_n(F) \to M_n(F)$ is the embedding in theorem 3.3.3, the image set $\theta\langle R_n(F)\rangle$ is a subsemigroup of $M_n(F)$ consisting of all filled coupled $n \times n$ matrices. Since $M_n(F)$, the semigroup of all square matrices over F is regular, then it is not difficult to see that $\theta\langle R_n(F)\rangle$ is a regular semigroup. This semigroup will be denoted by $R_n^*(F)$.

Lemma 3.3.5

Let $n = 3$ and let $F = \Re$ be the set of all real numbers. The semigroup $M_2(F)$ of all 2×2 real matrices is isomorphic to both $UH_3(\Re)$ and $ZH_3(\Re)$, the sets of all unit and zero heart rhotrices of size *3* respectively.

Proof:

Let $UH_3(\Re) = \left\{ \left\langle \begin{array}{ccc} & a & \\ b & 1 & d \\ & e & \end{array} \right\rangle : a,b,c,d,e \in \Re \right\}$ be the set of all unit heart rhotrices. We

define the mapping $\theta : M_2(\Re) \to UH_3(\Re)$ by $\theta\left(\begin{pmatrix} a & d \\ b & e \end{pmatrix}\right) = \left\langle \begin{array}{ccc} & a & \\ b & 1 & d \\ & e & \end{array} \right\rangle$

This is an isomorphism since

$$\theta\left(\begin{pmatrix} a & d \\ b & e \end{pmatrix}\begin{pmatrix} f & j \\ g & k \end{pmatrix}\right) = \theta\left(\begin{pmatrix} af+dg & aj+dk \\ bf+eg & bj+ek \end{pmatrix}\right)$$

$$= \left\langle \begin{array}{ccc} & af+dg & \\ bf+eg & 1 & aj+dk \\ & bj+ek & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a & \\ b & 1 & d \\ & e & \end{array} \right\rangle \left\langle \begin{array}{ccc} & f & \\ g & 1 & j \\ & k & \end{array} \right\rangle$$

$$= \theta\left(\begin{pmatrix} a & d \\ b & e \end{pmatrix}\right)\theta\left(\begin{pmatrix} f & j \\ g & k \end{pmatrix}\right)$$

And

$$Ker(\theta) = \left\{ I \in M_2(\Re) : \theta(I) = \theta\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \left\langle \begin{array}{ccc} & 1 & \\ 0 & 1 & 0 \\ & 1 & \end{array} \right\rangle \in R_3(\Re) \right\}.$$

Hence θ is an isomorphism.

Similarly, it is easy to verify that the mapping $\varphi : M_2(\Re) \to ZH_3(\Re)$ defined by

$$\varphi\left(\begin{pmatrix} a & d \\ b & e \end{pmatrix}\right) = \left\langle \begin{matrix} & a & \\ b & 0 & d \\ & e & \end{matrix} \right\rangle$$ is an isomorphism.

3.4. THE REGULAR SEMIGROUP

In this section, we shall establish that the rhotrix semigroup $(R_n(F), \circ)$ is regular. Recall that an element *p* of a semigroup is called regular if there exists *x* in *S* such that *pxp* = *p* and the semigroup *S* is called regular if all its elements are regular.

Theorem 3.4.1

The semigroup $(R_n(F), \circ)$ is a regular semigroup.

Proof

We are to show that for each rhotrix $A \in R_n(F)$, there exists a rhotrix $B \in R_n(F)$ such that $A \circ B \circ A = A$.

Now, let the rhotrix $A = \langle a_{ij}, c_{kl} \rangle \in R_n(F)$. Then, (a_{ij}) is an element of $M_w(F)$ and (c_{kl}) is an element of $M_{w-1}(F)$ which are respectively, the major and the minor matrices of $A \in R_n(F)$, where $w = (n+1)/2$, $n \in 2Z^+ + 1$. Since the Semigroup of all square matrices over F is regular, then, $(a_{ij}) \in M_w(F)$ and $(c_{kl}) \in M_{w-1}(F)$ are also regular.

Then by definition of regular, there exists a matrix $(b_{ij}) \in M_w(F)$ and a matrix $(d_{kl}) \in M_{w-1}(F)$ such that

$$(a_{ij})(b_{ij})(a_{ij}) = (a_{ij}) \tag{3.4}$$

and

$$(c_{kl})(d_{kl})(c_{kl}) = (c_{kl}) \tag{3.5}$$

respectively.

Now, choose a rhotrix $B = \langle b_{ij}, d_{kl} \rangle \in R_n(F)$. Then, it follows from equation (3.4) and (3.5) together with the definition of rhotrix multiplication that:

$$A \circ B \circ A = (\langle a_{ij}, c_{kl} \rangle \circ \langle b_{ij}, d_{kl} \rangle) \circ \langle a_{ij}, c_{kl} \rangle = A.$$

Hence, the semigroup $\langle R_n(F), \circ \rangle$ is a regular semigroup.

3.5 GREEN'S RELATIONS IN $(R_n(F), \circ)$

It is customary that when one encounters a new class of semigroups, almost the first question to ask, is what are the Green's relations like? Therefore, as a first step in understanding the structure of our new semigroup, $(R_n(F), \circ)$, we attempt to present in this section, a characterization of Green's relations in $(R_n(F), \circ)$. Definition of Green's relation is in definition 1.4.10.

First, we make the following observation concerning the rhotrix semigroup $(R_n(F), \circ)$ and the Green's equivalences.

Lemma 3.5.1

Let κ denote any of the five Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$, then for any

$A = \langle a_{ij}, c_{kl} \rangle$ and $B = \langle b_{ij}, d_{kl} \rangle$ in $(R_n(F), \circ)$, we have:

$$A\kappa B \text{ if and only if } (a_{ij})\kappa(b_{ij}) \text{ and } (c_{kl})\kappa(d_{kl}).$$

Proof

We prove the result for only the Green's equivalence $\mathcal{L}$ and the proofs for $\mathcal{R}, \mathcal{H}, \mathcal{D}$ and $\mathcal{J}$ follow similarly.

Let $A = \langle a_{ij}, c_{kl} \rangle$ and $B = \langle b_{ij}, d_{kl} \rangle$ in $\langle R_n(F), \circ \rangle$,

then,

$$A \,\mathcal{L}\, B \Leftrightarrow \text{there exist } X, Y \text{ such that } A = X \circ B \text{ and } B = Y \circ A$$

$$\Leftrightarrow \quad \langle a_{ij}, c_{kl} \rangle = \langle (x^1{}_{ij})(b_{ij}),\ (x^2{}_{kl})(d_{kl}) \rangle \text{ where } X = \langle x^1{}_{ij},\ x^2{}_{kl} \rangle$$

and

$$\langle b_{ij}, d_{kl} \rangle = \langle (y^1{}_{ij})(a_{ij}),\ (y^2{}_{kl})(c_{kl}) \rangle \text{ where } Y = \langle y^1{}_{ij},\ y^2{}_{kl} \rangle$$

$$\Leftrightarrow \quad a_{ij} = (x^1{}_{ij})(b_{ij}), \quad c_{kl} = (x^2{}_{kl})(d_{kl})$$

and

$$b_{ij} = (y^1{}_{ij})(a_{ij}), \quad d_{kl} = (y^2{}_{kl})(c_{kl})$$

Implying

$$a_{ij} = (x^1{}_{ij})(b_{ij}) \text{ and } b_{ij} = (y^1{}_{ij})(a_{ij}),$$

$$d_{kl} = (y^2{}_{kl})(c_{kl}) \text{ and } c_{kl} = (x^2{}_{kl})(d_{kl})$$

And by definition, we have

$$(a_{ij})\ \mathcal{L}\ (b_{ij}) \text{ and } (c_{kl})\ \mathcal{L}\ (d_{kl}) .$$

This completes the proof.

Remark 3.5.2

Lemma 3.5.1 effectively says that two rhotrices will be related under a Green's relation if and only if the corresponding filled couple matrices are related under the same Green's relation.

Now, by remarks 3.3.4 and 3.5.2, characterization of Green's relations in rhotrix semigroup $R_n(F)$ could be achieved via characterizing the relations in $R_n^*(F)$. Thus, we will concentrate on the semigroup $R_n^*(F)$ and characterize its Green's relation.

To characterize Green's relations on the semigroup $R_n^*(F)$ we take note of theorem 2.2.1.4 [Howie (1995), proposition 2.4.2] which shows that in any regular subsemigroup *U* of a semigroup *S*, the characterization of Green's relations $\mathcal{L},\ \mathcal{R},$ and $\mathcal{H}$ is the same as in *S*.

Now, since the semigroup $R_n^*(F)$ is a subsemigroup of $M_n(F)$ and $R_n^*(F)$ is also a regular semigroup then we have the following result:

Theorem 3.5.3

Let $\langle R_n(F), \circ \rangle$ be a rhotrix semigroup then:

(i) $\mathcal{L}\,(R_n^*(F)) = \mathcal{L}\,(M_n(F)) \cap R_n^*(F) \times R_n^*(F)$

(ii) $\mathcal{R}\,(R_n^*(F)) = \mathcal{R}\,(M_n(F)) \cap R_n^*(F) \times R_n^*(F)$

(iii) $\mathcal{H}\,(R_n^*(F)) = \mathcal{H}\,(M_n(F)) \cap R_n^*(F) \times R_n^*(F)$

Based on the theorem above, the characterisation of $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ relations in the semigroup $R_n^*(F)$ come in the next theorem:

Theorem 3.5.4

If $A, B \in R_n^*(F)$, then

i. $A \,\mathcal{L}\, B$ if and only if $im(A) = im(B)$

ii. $A \,\mathcal{R}\, B$ if and only if $\ker(A) = \ker(B)$

iii. $A \,\mathcal{H}\, B$ if and only if $im(A) = im(B)$ and $\ker(A) = \ker(B)$

Proof

From theorem 3.5.3, we have seen that the semigroup $R_n^*(F)$ has the same characterisation of Green's relations $\mathcal{L}$, $\mathcal{R}$, and $\mathcal{H}$ as in $M_n(F)$. But from Howie (1995), for any $A, B \in M_n(F)$,

$$A \,\mathcal{L}\, B \text{ if and only if } im(A) = im(B)$$

$$A \,\mathcal{R}\, B \text{ if and only if } \ker(A) = \ker(B).$$

Hence the proof follows.

To characterize the $\mathcal{D}$ relation in $R_n^*(F)$, we observe that $\mathcal{D}(R_n^*(F)) \subset \mathcal{D}(M_n(F))$ (properly). For example, if we consider two filled coupled matrices A and B in $R_5^*(F)$, such that

$$A=(a_{ij}, c_{kl}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$B=(b_{ij}, d_{kl}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then, $rank(A) = rank(B) = 3$. This implies $(A, B) \in \mathcal{D}(M_n(F))$.

But,

$$(a_{i,j}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ (c_{k,l}) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } (b_{i,j}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ (d_{k,l}) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So that, $rank(a_{ij}) = 2 \neq 1 = rank(b_{ij})$ and $rank(c_{kl}) = 1 \neq 2 = rank(d_{kl})$.

Therefore, by lemma 3.5.1, we can rightly say that $(A, B) \notin \mathcal{D}(R_n^*(F))$

Thus, $\mathcal{D}(R_n^*(F)) \subset \mathcal{D}(M_n(F))$ (Properly).

The next theorem Characterizes $\mathcal{D}$ and $\mathcal{J}$ Green's relation in the semigroup $R_n^*(F)$.

Theorem 3.5.5

i. If $A = (a_{ij}, c_{kl})$ and $B = (b_{ij}, d_{kl})$ belong to semigroup $R_n^*(F)$ then $A\mathcal{D}B$ if and only if $rank(a_{ij}) = rank(b_{ij}) \;\; and \;\; rank(c_{kl}) = rank(d_{kl})$.

ii. On the semigroup $R_n^*(F)$, $\mathcal{D} = \mathcal{J}$.

Proof

i. The result follows when we combine the statement of lemma 3.5.1 and the characterization of $\mathcal{D}$ Green's relation in the semigroup $M_n(F)$ from Howie (1995).

ii. Let $A = (a_{ij}, c_{kl})$ and $B = (b_{ij}, d_{kl})$ in $R_n^*(F)$ be such that $A\mathcal{D}B$. Then, by (i) above, this can occur if and only if

$$rank(a_{ij}) = rank(b_{ij}) \;\; and \;\; rank(c_{kl}) = rank(d_{kl}).$$

Since $\mathcal{J}$-relation in matrix semigroup $M_n(F)$ is characterised by the equality of rank, we have that $(a_{ij})\mathcal{J}(b_{ij})$ and $(c_{kl})\mathcal{J}(d_{kl})$ in $M_n(F)$, and by lemma 3.5.1, this is true if and only if $A\mathcal{J}B$. Hence, $\mathcal{D} = \mathcal{J}$.

Observe that, for each $A \in R_n(F)$, and it's corresponding filled couple matrix $A^* \in R_n^*(F)$, we have:

$$rank(A^*) = rank(A),\ im(A^*) = im(A) \text{ and } \ker(A^*) = \ker(A).$$

Therefore, a complete characterization of Green's relations in the rhotrix semigroup $R_n(F)$ is given in the next theorem.

Theorem 3.5.6

If $A, B \in R_n(F)$, then

i. $A\,\mathcal{L}\,B$ if and only if $im(A) = im(B)$

ii. $A\,\mathcal{R}\,B$ if and only if $\ker(A) = \ker(B)$

iii. $A\,\mathcal{H}\,B$ if and only if $im(A) = im(B)$ and $\ker(A) = \ker(B)$

iv. If $A = (a_{ij}, c_{kl})$ and $B = (b_{ij}, d_{kl})$ belong to semigroup $R_n(F)$ then $A\mathcal{D}B$ if and only if $rank(a_{ij}) = rank(b_{ij})$ *and* $rank(c_{kl}) = rank(d_{kl})$.

v. On the semigroup $R_n(F)$, $\mathcal{D} = \mathcal{J}$.

As a result of theorem 3.5.6 (*v*), we have the following observation

Remark 3.5.7

We observe that the $\mathcal{J}$ - classes in $R_n(F)$ are of the form:

$$\mathcal{J}_{r,s} = \{A = \langle a_{ij}, c_{kl} \rangle \in R_n(F) \mid rank(a_{ij}) = r, rank(c_{kl}) = s\}.$$

where $0 \le r \le w$ and $0 \le s \le w - 1$.

Thus, the rhotrix semigroup $R_n(F)$ has $\frac{(n+1)(n+3)}{4}$ $\mathcal{J}$ – classes, and the complete list of all $\mathcal{J}$ – classes is as follows:

$$\mathcal{J}_{0,0}, \mathcal{J}_{0,1}, \ldots, \mathcal{J}_{0,w-1}, \mathcal{J}_{1,0}, \ldots, \mathcal{J}_{w,0}, \mathcal{J}_{w,1}, \ldots, \mathcal{J}_{w,w-1}.$$

Where $w = \frac{n+1}{2}$.

CHAPTER FOUR

RHOTRIX LINEAR TRANSFOMATION

4.1 INTRODUCTION

In the process of achieving the main aim of the present thesis, we found it necessary to use the concept of rank of rhotrix. To the best of our knowledge, the concept of rank of rhotrix is not available in the literature of rhotrix theory. Therefore, we introduce in this chapter, the concept of rank of rhotrix and rhotrix linear transformation. Moreover, some properties of this rank and a necessary and sufficient condition under which a linear transformation can be represented by a rhotrix will be discussed in this chapter.

4.2 RANK OF RHOTRIX

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ then the entries $a_{rr}\,(1 \le r \le w)$ and $c_{ss}\,(1 \le s \le w-1)$ are the leading diagonals of the major and minor matrices of R_n respectively, where $w = \frac{n+1}{2}$ and $n \in 2Z^+ + 1$.

If all the entries to the left (right) of the leading diagonal of R_n are zeros, then, R_n is called a right (left) triangular rhotrix.

The following lemma follows trivially.

Lemma 4.2.1

A rhotrix $R_n = \langle a_{ij}, c_{kl} \rangle$ is a left (right) triangular rhotrix if and only if (a_{ij}) and (c_{kl}) are lower (upper) triangular matrices.

Proof

This follows when the rhotrix R is being rotated through 45^0 in anticlockwise direction.

In the light of this lemma, any rhotrix R_n of size *n* can be reduced to a right triangular rhotrix through reducing its major and minor matrix to echelon form using elementary row operation. Recall that, the rank of a matrix *A* denoted by *rank(A)* is the number of non – zero row(s) in its reduced row echelon form.

Definition 4.2.2

Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be a rhotrix of size *n*. Then, we define rank of the rhotrix R_n denoted by *rank* (R_n) as:

$$Rank\,(R_n) = rank(a_{ij}) + rank(c_{kl}) \qquad 4.1$$

Example 4.2.3

Let $R = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 0 & 2 & -2 & \\ 1 & -1 & 3 & 1 & 2 \\ & -2 & 1 & 1 & \\ & & 2 & & \end{array} \right\rangle$. Then, the filled coupled matrix of *A* is given by

$$m(R) = \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -2 & 0 & 2 \end{pmatrix}. \qquad 4.2$$

Now reducing 4.2 to row reduced echelon form *(rref)*, we obtain

$$rref\ (m(R)) = \begin{pmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad 4.3$$

which is a coupled matrix coupling (2×2) and (3×3) matrices,

i.e $(c_{kl}) = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ and $(a_{ij}) = \begin{pmatrix} 1 & -2 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ respectively.

Notice that, $rank(a_{ij})$ + $rank(c_{kl})$ = 2 + 2 = 4 = *rank(rref(m(R))*.

Hence, *rank(R) = 4.*

It follows from the definition of rank of rhotrix above, that many properties of rank of matrix can be extended to the rank of rhotrix. In particular, we have the following:

Theorem 4.2.4

Let $R = \langle a_{ij}, c_{kl} \rangle$, and $S = \langle b_{ij}, d_{kl} \rangle$, be any two rhotrices of size $n \in 2Z^{+} + 1$, then

i. $rank(R) \le n$;

ii. $rank(R+S) \le rank(R) + rank(S)$;

iii. $rank(R) + rank(S) - n \le rank(RS)$;

iv. $rank(RS) \leq \min\{rank(R), rank(S)\}$;

Proof

The statements *i* and *ii* follow directly from the definition. To prove the statement *iii*, we apply the corresponding inequality for matrices, that is:

$$rank(AB) \geq rank(A) + rank(B) - n$$

where A is $m \times n$ and B is $n \times p$ dimensional matrices.

Thus,

$$rank(RS) = rank[(a_{ij})(b_{ij})] + rank[(c_{kl})(d_{kl})]$$

$$\geq \left[rank(a_{ij}) + rank(b_{ij}) - \left(\frac{n+1}{2} \right) \right] + \left[rank(c_{kl}) + rank(d_{kl}) - \left(\frac{n+1}{2} \right) + 1 \right]$$

$$= rank(R) + rank(S) - n$$

For the last statement, consider

$$rank(RS) = rank[(a_{ij})(b_{ij})] + rank[(c_{kl})(d_{kl})]$$

$$\leq \min\{ rank(a_{ij}), rank(b_{ij})\} + \min\{ rank(c_{kl}), rank(d_{kl})\}$$

$$\leq \min\{ rank(a_{ij}) + rank(c_{kl}),\ rank(b_{ij}) + rank(d_{kl})\}$$

$$\leq \min\{rank(R), rank(S)\}$$

4.3 Rhotrix Linear Transformation

One of the most important concepts in linear algebra is the concept of representation of linear transformation as matrices. It turns out that certain linear transformation between odd dimensional vectors spaces can be represented by rhotrix.

Recall that, if V and W are vector spaces of dimension m and n respectively. Then, any linear transformation T from V to W can be represented by a matrix. The matrix representation of T is called the matrix of T denoted by $m(T)$. Also, if F is a field, then any vector space V of finite dimension n over F is isomorphic to the vector space F^n. Therefore, any $n \times n$ matrix over F can be considered as a linear operator on the vector space F^n in the fixed standard basis.

Following this ideas, we study in this section, rhotrix as a linear operator on the vector space F^n . Since a rhotrix is always of odd size, it follows that in representing a linear transformation T on a vector space V by a rhotrix, the dimension of V is necessarily odd. Therefore, throughout what follows, we shall consider only odd dimensional vector spaces.

For any $n \in 2\mathrm{Z}^{+} + 1$ and F an arbitrary field. Let $w = \frac{1}{2}(n+1)$, then we define the couple (F^w, F^{w-1}) of F^w and F^{w-1} by

$$(F^w, F^{w-1}) = \{(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \beta_{w-1}, \alpha_w) \mid \alpha_i \in F \;\; and \;\; \beta_j \in F\},$$

where,

$$F^w = \{(\alpha_1, \alpha_2 ..., \alpha_w) \mid \alpha_1, ..., \alpha_w \in F\}$$

and

$$F^{w-1} = \{(\beta_1, \beta_2, ..., \beta_{w-1}) \mid \beta_1, \beta_2, ..., \beta_{w-1} \in F .$$

It is clear that (F^w, F^{w-1}) coincides with F^n. So, if $n \in 2\mathrm{Z}^+ + 1$, then any $n-$ dimensional vector V is a couple of two vector spaces V_1 and V_2 of dimensions $\frac{n+1}{2}$ and $\frac{n+1}{2} - 1$ respectively.

Less obviously, it can be seen that not every linear transformation T of F^n can be represented by a rhotrix in the standard basis. For instance, the transformation $T: F^3 \to F^3$ defined by $T(x, y, z) = (x - y,\ x + z,\ y + z)$ is a linear transformation on F^3, which cannot be represented by a rhotrix in the standard basis.

The following theorem characterises when a linear transformation T on F^n can be represented by a rhotrix.

Theorem 4.3.1

If $n \in 2\mathrm{Z}^+ + 1$ and F is a field, then a linear transformation $T: F^n \to F^n$ can be represented by a rhotrix with respect to the standard basis, if and only if T is defined as

$$T(x_1, y_1, x_2, y_2, ..., y_{w-1}, x_w) = [\alpha_1(x_1, x_2, ..., x_w), \beta_1(y_1, y_2, ..., y_{w-1}), \alpha_2(x_1, x_2, ..., x_w), \beta_2(y_1, y_2, ..., y_{w-1}), ..., \beta_{w-1}(y_1, y_2, ..., y_{w-1}), \alpha_w(x_1, x_2, ..., x_w)],$$

where, $w=\frac{n+1}{2}$, $\alpha_1,..., \alpha_w$ and $\beta_1,..., \beta_{w-1}$ are any linear transformations on F^w and F^{w-1} respectively.

Proof:

Suppose $T: F^n \to F^n$ is defined by

$$T(x_1, y_1, x_2, y_2,..., y_{w-1}, x_w) = [\alpha_1(x_1, x_2,..., x_w), \beta_1(y_1, y_2,..., y_{w-1}), \alpha_2(x_1, x_2,..., x_w), \beta_2(y_1, y_2,..., y_{w-1}), ,..., \beta_{w-1}(y_1, y_2,..., y_{w-1}), \alpha_w(x_1, x_2,..., x_w)],$$

where, $w=\frac{n+1}{2}$, $\alpha_1,..., \alpha_w$ and $\beta_1,..., \beta_{w-1}$ are any linear transformations on F^w and F^{w-1} respectively/. Consider the standard basis {(1,0,…,0), (0,1,0,…,0), …, (0,0,…,1)}. Note that, for $1 \le i \le w$ and $1 \le j \le w-1$, we have α_i, β_j as linear transformations satisfying $\alpha_i(0,...,0) = \beta_j(0,...,0) = 0$.

Thus,

$$T(1,0,...,0) = [\alpha_1(1,0,...,0), 0,..., 0, \alpha_w(1,0,...,0)]$$

$$T(1,0,...,0) = [0, \beta_1(1,0,...,0),..., \beta_{w-1}(1,0,...,0), 0]$$

… … …

… … … } (4.4)

… … …

$$T(0,...,0,1) = [0, \beta_1(0,...,0,1),..., \beta_{w-1}(0,...,0), 0]$$

$$T(0,...,0,1) = [\alpha_1(0,...,0,1), 0,..., 0, \alpha_w(0,0,...,0,1)]$$

Let

$$\alpha_{ij} = \alpha_j(0,\ldots, \underset{i^{th}-possition}{1}, \ldots, 0) \text{ for } (1 \leq i, j \leq w)$$

and

$$\beta_{kl} = \beta_l(0,\ldots, \underset{k^{th}-possition}{1}, \ldots, 0) \text{ for } (1 \leq k, l \leq w-1)$$. Then from (4.4),

we have the matrix of T is

$$\begin{pmatrix} \alpha_{11} & 0 & \alpha_{12} & \ldots & \alpha_{1w-1} & 0 & \alpha_{1w} \\ 0 & \beta_{11} & 0 & \ldots & 0 & \beta_{1w-1} & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \beta_{w-1w} & 0 & \ldots & 0 & \beta_{w-1w-1} & 0 \\ \alpha_{w1} & 0 & \alpha_{w2} & \ldots & \alpha_{ww-1} & 0 & \alpha_{ww} \end{pmatrix}. \qquad (4.5)$$

This is a filled coupled matrix from which we obtain the rhotrix representation of T as $\langle \alpha_{ij}, \beta_{kl} \rangle$.

Conversely:

Suppose $T : F^n \to F^n$ has a rhotrix representation $\langle \alpha_{ij}, \beta_{kl} \rangle$ in the standard basis. Then, the corresponding matrix representation of T is the filled coupled matrix given in (4.5) above. Thus, we obtain the system

$$T(1,0,...,0) = (\alpha_{11}, 0, \alpha_{12}, ..., \alpha_{1w-1}, 0, \alpha_{1w})$$

$$T(1,0,...,0) = (0, \beta_{1w-1}, 0, ..., \beta_{1w-1}, 0)$$

$$\left.\begin{array}{ccc} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{array}\right\} \qquad (4.6)$$

$$T(0,...,0,1) = 0, \beta_{w-1w}, 0, ..., \beta_{w-1w-1}, 0)$$

$$T(0,...,0,1) = (\alpha_{w1}, 0, \alpha_{w2}, ..., \alpha_{ww-1}, 0, \alpha_{ww})$$

From this system, it follows that for each $(x_1, y_1, x_2, y_{2,}, \ldots, y_{w-1}, x_w) \in F^n$ we have the linear transformation T defined by

$$T(x_1, y_1, x_2, y_2, ..., y_{w-1}, x_w) = [\alpha_1(x_1, x_2, ..., x_w), \beta_1(y_1, y_2, ..., y_{w-1}), \alpha_2(x_1, x_2, ..., x_w),$$
$$\beta_2(y_1, y_2, ..., y_{w-1}), ,..., \beta_{w-1}(y_1, y_2, ..., y_{w-1}), \alpha_w(x_1, x_2, ..., x_w)]\,.$$

Where, $w = \frac{n+1}{2}$, $\alpha_1, ..., \alpha_w$ and $\beta_1, ..., \beta_{w-1}$ are any linear transformations on F^w and F^{w-1} respectively, such that $\alpha_j(0, \ldots, \underbrace{1}_{i^{th}-position}, \ldots, 0) = \alpha_{ij}$ for $1 \le i \le w$ and $\beta_l(0, \ldots, \underbrace{1}_{k^{th}-position}, \ldots, 0) = \beta_{kl}$ for $1 \le j \le w-1$.

Example 4.3.2

Consider the linear transformations $T: R^3 \to R^3$ defined by

$$T(x, y, z) = (2x - z, 4y, x - 3z)$$

To find the rhotrix of T relative to the standard basis, we proceed by finding the matrix of T. Thus,

$$T(1,0,0) = (2,0,1)$$

$$T(0,1,0) = (0,4,0)$$

$$T(0,0,1) = (-1,0,-3)$$

Therefore, by definition of matrix of T with respect to the standard basis, we have

$$m(T) = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{pmatrix}$$

This is a filled coupled matrix from which we obtain the rhotrix of T on $R(3)$

$$r(T) = \left\langle \begin{matrix} & 2 & \\ -1 & 4 & 1 \\ & -3 & \end{matrix} \right\rangle .$$

Now starting with the rhotrix $r(T) = \left\langle \begin{matrix} & 2 & \\ -1 & 4 & 1 \\ & -3 & \end{matrix} \right\rangle$,

The filed coupled matrix of $r(T)$ is $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 4 & 0 \\ -1 & 0 & -3 \end{pmatrix}$

And so, defining $T : R^3 \to R^3$

$$T(1,0,0) = 2(1,0,0) + 0(1,0,0) + 1(1,0,0)$$

$$T(0,1,0) = 0(0,1,0) + 4(0,1,0) + 0(0,1,0)$$

$$T(0,0,1) = -1(0,0,1) + 0(0,0,1) - 3(0,0,1)$$

Thus, if

$$(x, y, z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

Therefore,

$$T(x, y, z) = xT(1,0,0) + yT(0,1,0) + zT(0,0,1)$$

$$= x(2,0,1) + y(0,4,0) + z(-1,0,-3)$$

$$= (2x - z, 4y, x - 3z)$$

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 SUMMARY

In this thesis, the algebraic study of rhotrix semigroup was initiated, using rhotrix set consisting of all rhotrices of size *n*, with entries from an arbitrary field (F), as the underlying set, and together with the choice of non-commutative method for rhotrix multiplication, as the binary operation. The study is termed as '*algebraic study of rhotrix semigroup'*. The study started with a complete literature survey of the developments made in the field of rhotrix theory for a decade, starting from the year 2003, when the concept of rhotrix was introduced up to the end of 2013. Next, we presented the construction of non-commutative general rhotrix semigroup and studied its properties. This was followed by investigation of its certain subgroups and characterizing its Green's relations. Parts of this are to appear in Mohammed and Balarabe (Submitted), Mohammed *et al;* (Accepted).

Towards achieving the main aim of this thesis, we have also introduced new concepts in the field of rhotrix theory, such as rhotrix rank and rhotrix linear transformation. A number of theorems developed have assisted in the quest to characterize Green's relations in the rhotrix semigroup. Part of this has appeared in Mohammed *et al;* (2012).

5.2 CONCLUSION

In conclusion, an algebraic study of rhotrix non-commutative semigroup $R_n(F)$ was initiated and presented. It was also shown that the rhotrix semigroup $R_n(F)$ is regular and embedded in the square matrix regular semigroup $M_n(F)$. As the major contribution, the study was able to characterised all the Green's relations in the rhotrix semigroup which form the basis for the development of rhotrix semigroup.

5.3 RECOMMENDATIONS

For the future research direction, it seems reasonable to consider the following topics:

1. A search for generating sets for the finite rhotrix semigroup.
2. The combinatorial aspect of rhotrix semigroup can be studied.
3. Any area of study using matrices as tool can be extended to rhotrices analogously.
4. Idempotent and the product of idempotent should be studied.

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