APPLICATION OF PORTFOLIO OPTIMIZATION: A STATISTICAL APPROACH

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COMPARATIVE STUDY OF THE MARKOWITZ MEAN-VARIANCE AND GINI-MEAN DIFFERENCE APPROACHES TO PORTFOLIO OPTIMIZATION

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A DISSERTATION SUBMITTED TO THE POSTGRADUATE SCHOOL, AHMADU BELLO UNIVERSITY, ZARIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE (M.Sc.) IN STATISTICS

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DECLARATION

I declare that the work in this dissertation entitled "**COMPARATIVE STUDY OF THE MARKOWITZ MEAN-VARIANCE AND MEAN-GINI DIFFERENCE APPROACHES TO PORTFOLIO OPTIMIZATION**" has been performed by me in the Department of Mathematics under the supervision of Dr H. G. Dikko and Dr. Ibrahim Abdullahi.

The information derived from literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any University or Institution.

Hamza Maigari Ahmadu	____________________	____________________
Name of Student	Signature	Date

CERTIFICATION

The dissertation entitled (**COMPARATIVE STUDY OF THE MARKOWITZ MEAN-VARIANCE AND GINI-MEAN DIFFERENCE APPROACHES TO PORTFOLIO OPTIMIZATION**) by MAIGARI Ahmadu Hamza (MSC/SCI/41811/2012-2013) meets the regulations governing the award of the degree of Master of Science of the Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

Dr. H. G. Dikko Chairman, Supervisory Committee	____________ Signature	______________ Date
Dr. I. Abdullahi Member, Supervisory Committee	___________ Signature	______________ Date
Dr. H. G. Dikko Head of Department	___________ Signature	______________ Date
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Prof. Kabiru Bala Dean, School of Postgraduate Studies	___________ Signature	______________ Date

DEDICATION

This dissertation is dedicated to my late brother Mal. Murtala Maigari Ahmadu, former HOD Mathematics/Statistics/Computer Science department, Taraba State University Jalingo, May jannatul firdaus be your abode.

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ABSTRACT

The main objective of this thesis is to look at how the Markowitz Mean-Variance assets selection model performs with distribution free model, Gini-Mean Difference model and highlight statistical approach to portfolio optimization in terms of risk reduction; interrelationships of diversified assets, assets rebalancing and stochastic dominance etc. In the study, the Mean-Variance model tends to slightly outperform the Gini-Mean Difference model in return/risk characteristics. More sophisticated investors can use the Gini-Mean Difference model if they posses the skills as it involves more complex computations in the procedure, and further captures the data than Mean-Variance approach since its distribution free model.

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CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

Nigeria stock market is a place where investors buy shares of assets or securities mostly for them to earn reward (returns) at a given level of 'tolerable' risk. The word 'portfolio' is used to mean a mix or combinatory pool of assets, securities or investment of financial or physical nature, in which an investor holds to meet his/her set objectives. On the other hand, portfolio selection is choosing the most suitable combination of assets by risk averse or rational investor to maximize return while at the same time minimizes risk.

Due to abrupt unpleasant surprises of global financial crisis, and inferior market downturn experienced by financial markets in different times, investors are becoming highly concerned about the risk of their investments coupled seeking ways to achieve more attractive risk/return characteristics and better capital protection in difficult environments. The main plan of portfolio management is to form diverse securities in a portfolio that meets the needs of investors and thereafter manage the portfolio in other to obtain required goals (Vigdis *et al*., 2011). Many players in the financial market have been frequently looking for strategic ways of investing which is capable of meeting this aforesaid protest of the market.

Modern Portfolio Theory (MPT) pioneered by Markowitz (1952, 1970, 1987) seminal work specifically solves the tradeoff between risk and return using formed curve in graph called efficient frontier. This frontier solves the problem by considering the risk, return of invested assets and the correlation that exist between the asset return. The curve identifies those that are at maximum return for a given level of risk, or at their minimum risk for a given level of return.

Markowitz's work to a great extent changes the behavior of investors and financial managers by providing more insight on the issue of assets selection. Nowadays, players in the market are embracing the framework viewed as the most standard model for today's investment management even with the model's unrealistic questionable assumptions.

Markowitz (1991), as well as Elton and Gruber (1997) talk more on the main issue an investor faces when investing, of them is how to distribute resources among different assets alternatives. Almost all investors are being posed with this same problem, with the added sufferings and complications needed to clearly include the properties of the liabilities in the analysis (Bodie *et al.* 2004). The problems are different in terms of structure, but that can be categorized to the portfolio theory.

MPT can be confidently termed as the opposite to traditional asset picking brought into light by economist persevering to look and understand the dynamics of the market in it totality, rather than basing attention at things that make an investment opportunity unique. The problem of spreading one's fund into different asset alternatives is one of the forefront basic concerns of financial theory (Cohen and Natoli 2003). Risk as well as asset allocation are important parts of the MPT, with investment being explained statistically with respect to their long-term return rate and their short-term expected volatility.

Risk (volatility) is the likelihood average bad years of an investment. The objective and aim of an investor is to find acceptable risk tolerance of his investment, while at the same time identify portfolio with maximum expected return with that level of risk (Elton and Gruber 1997).

As pointed out by Elton and Gruber (1997), market players or investors intend to form 'perfect investment' and attribute it to have high return with no risk coming with it. This type of investment in reality is almost unachievable. With no surprise, these investors spend much of their time and energy coming up with methods and theories that come close to the ‘‘perfect investment’’. Among all the theories and methods none is with high popularity and power of the MPT. That been said, it would be highly important that market players and professionals get themselves familiar with how to make use of that theory to develop portfolio which best suit client wishes and risk tolerance. It would also be highly beneficial if financial managers understand the forces that drive risk and return of portfolio and know how they can be amended to tune with clients aspirations for maximum benefit.

Kristein *et al*. (2006) stated that modern portfolio theory holds that spreading funds among different assets which is termed as diversification increases return at given level of risk or at minimum level of risk. The theory uses the volatility of returns implied by market price fluctuations as the composite of risks. It clearly solved the dilemma on how risk-averse financial investors can form master assets (portfolio) in such a way that optimize market risk for a given expected returns, with emphasis that volatility or risk is an inherent aspect of higher reward (return).

Diversification is one of the strategies financial investors use when constructing low risky portfolio (Bodie *et al*. 2004). It is a wisely offensive style to the market swings or a defensive technique to investment risk. As the popular adage “don’t put all your eggs into one basket” is quite simple to say but more hard to practically perform in real life. Diversification is highly regarded these days due to the heat coming with global financial crisis and as proof of this; Harry Markowitz was awarded the famous Nobel Prize in Economics in recognition of his outstanding research (Markowitz, 1991).

1.2 Statement of the Problem

During global financial crisis at different point in time, investors face the risk of loosing their capital not to talk of the excess return they garners. Due to this reoccurring problem at different point in time, investors are always in search of ways to secure their capital or at least the ways their investments would come with tolerable risk. The question that arises is how do these investors spread their/diversify their wealth in presence of multiple investments to choose from that have attributes of their expectations? And when they have successfully achieved it, how do they maintain it so that the market does not take them by surprise? We will compare two approaches, Markowitz mean-variance and Gini-Mean approaches to portfolio optimization, using variance and gini index as the measures of risk to see the superiority of one over the other, and thereafter derive some useful statistics risk-averse investors would be interested to know.

1.3 Purpose of the Study

The set target of this study is to examine whether there is any possible improvement when applying MPT (Markowitz mean-variance) investing strategy than using the naïve index investing. On doing that, we would be exploring the statistical analysis of portfolio diversification, correlation among market assets and Gini-Mean Difference statistic as a substitute for variance (risk) statistic in Markowitz mean-variance model.

1.4 Significance of the study

The significance of the study is its highly advocacy of the MPT as a way of yielding good investment returns at an acceptable risk as against the traditional (naïve) index investing strategy that high return stocks always come with high risk.

1.5 Aim and Objectives of the Study

The aim of this research is to compare the two portfolio optimization approaches, Markowits Mean-variance and Gini-Mean difference models while the objectives include:

i. Use of Variance and Gini index as two risk measures in the objective functions of the two approaches.
ii. Assets selection rebalancing in the optimization engine using five different sectors of the Nigeria Stock Market.

1.6 Limitation of the Study

As a result of major concern of resources constraints, all the parameters cannot be taken into consideration when examining the assets portfolio performance. The main point of view is solely performance and power results using the Nigerian Capital Market in asset selection. Also, other constraints such as tax-efficiency, positive and negative leverage and are overlooked in the study

As mentioned above, no amount of money going to be invested rather the real performance is to be examined from selected assets market index as its going to help the investor in achieving a nicer and more all-encompassing outcome of the findings. For us to be able to get more true observational result in the study, the benchmark index is narrowed down. The index is the general market performance index for the thirty most capitalized stocks in the Nigerian Stock Exchange; this index mimics the market movement of the stocks in total. The difficulty comes from the fact that as the change in market condition happens, the risk and expected return of the various stocks change; and as a result of this constantly shifting market conditions, we only used short period of market history that encompass good and bad market cycles to look at the effectiveness of the approaches.

CHAPTER TWO

LITERATURE REVIEW

There is ever growing interest in portfolio theory today as a result of the huge blow investors are encountering due to time to time global financial protest leading to investors losing their resources. Markowitz (1952) has received and still receiving applause for his outstanding pioneering work in the field later popularized by other people such as Sharpe (1995), D'Ambrosio (1976) and Chandra and Shadel (2007).

Numerous authors are of the view that there are two stages involved in portfolio selection model. Markowitz (1952) mentioned that the first begins from examination, going forward he ended the principle with the view about the possible or probable performance of available securities. While in the second stage he began the principle with highlighting the important views on the possible assets performance with portfolio selection coming at the end of the stage. The main area of Harry Markowitz study focused on the second stage of portfolio selection. In the area, he came up with good model that works on the system of expected rate of return and risk of portfolio.

Markowitz *et al,* (1979) in their work on estimating utility by mean return and variance of return function of 145 mutual funds, they found out that dictating portfolios using mean-variance rule was almost equal to the order gotten when using expected utility. The mean-variance formula gives a good approximation to expected utility functions using both monthly and semi-annual return data as stated by Pulley (1981) in his study. The study also pointed out that investors can calmly depend on mean-variance optimization with attitude towards local changes in portfolio value reflected by the local relative risk-aversion.

Ulucan (2007) when checking the optimal holding period (investment horizon) for the classical mean-variance model, utilized the historical transaction record of Istanbul Stock Exchange, ISE-

100 index stocks data for empirical analysis. In the study, he found out the portfolio returns with changing holding period has a convex structure with an optimal holding period.

On the other hand, when checking the optimal portfolio allocation with the absence or without including treasury securities, Bomfim (2001) crosschecked the limit/extent on decisions of investors' portfolio allocation are likely to be affected by the retirement of all federal government debt. He found out that if the existing projections of future budget surpluses come into being, putting resources in treasury securities; could at last be something of the past. Therefore, in such circumstances, those that are to be less affected by the removal of treasuries from the investable portfolio are highly conservative and aggressive investors.

In his study, Kisaka *et al*. (2015) aimed at finding out the relationship that exist between portfolio risk and size, the size of the optimal portfolio of the Nairobi Stock Exchange and the limit/level of risk reduction that is achieved through diversification. Findings in the study showed that diversification led to risk reduction benefits. Also, the risk of the portfolio reduced significantly as the number of assets in the pool increased. Further findings in the research reveal that at the beginning, risk reduction drastically go down by 40% with eight assets in the portfolio. But as eight more assets are incorporated into the pool, a further 5% reduction in the risk of the portfolio is achieved while subsequent 14 assets added yield only in 2% of risk reduced.

In another study, carried out in Ghana insurance company, Boah *et al*. (2015) applied a programming model that is linear to find optimal portfolio for the Ghana multi-grow insurance company which received GH200,000 but were finding it hard to find how much to put in each of the 5 investable areas so as to maximize return. Due to the approach that is used, optimal portfolio mix was achieved for the insurance company.

Bekkers *et al*. (2009) tried to find out which asset classes add value to a traditional portfolio of stocks, bonds and cash by using 10 different investment categories simultaneously in a mean-variance analysis as well as a market portfolio approach. The study further sought to determine the optimal weights of all asset classes in the optimal. In the study, the mean-variance analysis suggested that real estate, commodities and high yield add most value to the traditional asset mix of stocks, bonds and cash. The authors went further to opine that adding these three asset classes come close to an all asset portfolio. The portfolio with all assets showed a diversification benefit along the efficient frontier that varies between 0.40% and 0.93% in the volatility range of 7% and 20%. Based on their analysis, they concluded that the proportion of non-traditional asset classes appearing in the market portfolio is relatively small, and thus, investors must determine their own individual constraints, while the market portfolio and the portfolio optimized by mean-variance are considered as the boundaries for the asset classes.

Bhuyan *et al*. (2014) in their study, examined and develop a sensitivity analysis for differential risk premium in REIT stocks and the effects in determining an optimal portfolio mix by applying mean-variance analysis in the U.S financial markets. The study used the mean-variance approach to maximize the utility of the optimal portfolio with varying degrees of risk aversion. Two different risk premiums between stocks and bonds, such as 0.006% and 0.012% were applied to examine the portfolio choices. Their result showed that when the risk premium of REITs and stocks was 1.5%, investors with risk aversion equal to 1 to 6 were better off investing almost all capital in REITs.

Among the suggestions in the study included that the group of investors can short sell their bonds and put a very small weight in stocks. Furthermore, investors can also derive the same benefit even when the risk premium of REITs is 2.0% with a risk aversion of 1 to 9. The study went further to say that in spite of this, when the risk premium of REITs and stock is 2.5%, the investor's risk aversion factor does not matter, and it suggests that investors can short bonds and invest in REITs having a larger weight in the portfolio.

Yahaya (2012) in his study, developed a procedure for obtaining optimal solution to Markowitz mean-variance portfolio selection problem based on already developed model called black model. He finds out that there exist efficient tool within the reach of Markowitz investor to make an intelligent decision of allocating investment funds to the assets that make up the portfolio. It is also learnt that the portfolio on the efficient frontier are also non-denominated in the sense that, any portfolio in the efficient frontier offers a better return than any other (off-efficient frontier) portfolio having the same degree of portfolio.

In another study, Yahaya *et al*. (2011) looked at the effects and advantages of constructing a reasonably diversified portfolio from a pool of assets while giving emphasis on the interrelationship existing among portfolio constituent assets. The study found out that it is possible to diversify away a portion of total portfolio risk; the other part of risk which cannot be removed by diversification is as a result of market-wide sources. Also, the study observed that, the degree of relationship (correlation) existing among various securities in the market plays a vital role in portfolio's risk reduction; stating that if assets are correlated, the diversifiable risk cannot be reduced lower than the average covariance, while if portfolio's constituent assets are uncorrelated, the diversifiable risk can be completely eliminated, and the only risk an investor has to contend with is the non-diversifiable (market risk).

Furthermore, Yahaya (2015) presented a short tale statistical procedure for allocating investment capital to constituent assets in a two-asset portfolio mix, with the aim of achieving minimum risk (variance) while taking into account the type and degree of linear relationship (correlation) existing between the assets returns. The study showed how to intelligently allocate investment fund between two stocks that make up a portfolio with the primary aim of obtaining the minimum risk portfolio, which is always the main goal of a strictly risk-averse investor.

Bodie *et al.* (2004) in his findings states that when systematic risk of portfolio is been controlled by manipulating the average beta of the component securities, the number of securities is of no consequences. But in the case of non-systematic risk, the number of securities involved is more important than the firm-specific variance of the securities. Sufficient diversification can virtually eliminate firm-specific risk. They went further to state that understanding this distinction is essential to understanding the role of diversification in portfolio construction.

Ross *et al.* (2008) are of the view that the process of spreading an investment across assets (and there by forming portfolio) is called diversification. They stated that the principle of diversification tells us spreading an investment across many assets will eliminate some of the risk.

Yamazaki and Konno (1991) used mean absolute deviation risk statistic to eliminate the difficulty attached to Markowitz model and at the same time maintain it merit over equilibrium models. On doing that, he showed that portfolio optimization of large stocks can be solved in real time basis. They validate their findings by using the data of 225 securities to show that the model formed portfolio similar to that of Markowitz model in a lesser time period needed to solve the later.

Offiong *et al.* (2016) aimed at developing optimal portfolio of three assets in Nigeria stock exchange. In the study, they used utility function test coupled with two methodological approaches − matrix algebra and lagrangian methods in other to obtain the best that will give the highest expected return at a minimum risk. Among the three selected assets: First Bank of Nigeria plc, Guinness Nig. Plc and Cadbury Nig. Plc, Guinness recorded the best value of 0.031 with least risk of 4.268. The lagrangian method of optimization was found to be highly effective in getting global minimum variance as well as the efficient frontier of the portfolio. And matrix algebra was helpful in calculating the portfolio weights.

Jamaan *et al*. (2011a) in their own study of comparative analysis implored Markowitz Mean-variance and Gini-mean models in comparing performance composition of the formed portfolio, based on the two models, by utilizing Malaysian capital market data. The result obtained indicated mean-gini model portfolio outperformed Markowitz mean-variance portfolio. From the result obtained, they are suggesting mean-gini model as the better candidate for risk-averse investors.

Ogryzak and Ruszezynski(1998) in a different study, use two risk measures: mean- gini absolute difference or for simplicity mean-gini and mean absolute deviation statistics in other to reexamine their consistency with second degree stochastic dominance–an ordinality test of assets return. The author obtained that both models are to some extent consistent with second degree stochastic dominance rules with the condition that the coefficient is bounded by certain constant. He also obtain that the consistency of gini-mean is much more stronger than that of mean-absolute deviation.

Agouram and Ghizlane (2015), used the capital market turbulent period of 2011-2013 to also compare mean-variance and mean-gini as in the study by Jamaan *et al*. (2011b). The difference is that in Agouram and Ghizlane (2015) compared the two models with value-at-risk (Var) and conditional value-at-risk (CVar) models in the said period of financial protest.

Nwakana *et al*, (2014) in their study to investigate the performance of two diversification strategies: Talmud and Markowitz diversification strategies, draw data from Nigeria Capital market consisting of quarterly data of 17 companies for seven years, utilizing t-test in testing in testing the difference s between independent sample means after formulating three different hypothesis. They stated that this means diversification can reduce investment risk. As they pointed out, they did this in other to check which of the strategies provide better performance in terms maximizing returns and minimizing risk. In the findings, they recommended Talmud over Markowitz diversification strategy as there is significant uplift in output in the strategy.

Yitzhaki (2003) states that among the popular statistics used in variability measure, the variance is the most pronounced and popular. In his research, he argues the superiority of gini-mean difference (GMD) over variance in that the GMD shares the same properties with the variance and possesses more advantageous properties than the variance like: ordinality, exchangeability in marginal distributions whenever index number is severe (i.e when it comes to selection of the base to use for marginal distribution comparison, when the base is likely to determine the direction of the result). He also affirmed that GMD is advantageous in the aspect of stratification: GMD is sensitive to stratification among sub-groups of the population when the whole distribution is composed of sup-populations.

In other to study the statistics of sharp ratio−a measure of performance of different investing strategies in portfolio optimization, (Andrew, 2002) using standard asymptotic theory on several sets of assumption like; stationarity returns and time aggregation derive explicitly, derive explicitly the expression of the statistical distribution of the sharp ratio. He fin d out that because of serial correlation presence, the sharp ratio for hedge fund can be overstated by as much as

65%. He went further to affirm that this serial correlation is handled; rankings of hedge funds based on sharp ratios can change dramatically.

The outstanding advantage of the mean-variance portfolio optimization strategy is its absolute simplicity of deriving the portfolio frontier, while its disadvantage lies in its distribution restriction which is the normality assumption. Shalit and Yitzhaki (2005) presents an alternative measure of variability: mean-gini and mean-extended gini to obtain the frontier of the two measures using optimization algorithms. They compare the frontiers of both measures with the classical mean-variance efficient frontier using Ibbotson (2000) monthly returns. They observed the two measures to be more diversified than the mean-variance portfolio efficient frontier. When there is short selling restriction constrain, i.e. when the constraint is included in the objective function of the optimization model, investor risk aversion tends to have different patterns portfolio diversification; a result less obvious when short selling is allowed. In the findings, mean-gini frontier derivation is identical to that of mean-variance model, and the consequences paid for digging mean-gini efficient frontier is the information loss of the assets distribution.

Merton (1972) discussed in a lengthy write up attributes of mean-variance portfolio frontier, derived explicitly efficient frontier and justifies truth of its characteristics. He also states and gives the proof important derivation of the characteristics, which is the separation technique in the settings of mutual fund theorem and also, show under certain requirement or requisite that the classical graphical method for obtaining the efficient portfolio frontier is not correct.

Yitzhaki *et al.* (1984) present a paper on th approach of mean-gini model comparing uncertain and risky prospects. In the proposed method, they also affirmed the simplicity of of the model as equal with that of Markowitz mean-variance model (i.e. in derivation of population characteristics) with the addition of stochastic dominance; a rule for assets ordering. In the study, they apply mean-gini model to capital market data and also derived security valuation theorem as an accepted relationship between return and risk of assets. They further continued to involve a degree of risk aversion that can be estimated from capital market data.

That been said, we will compare mean-gini and Markowitz mean-variance model on different sectors of the Nigeria capital market and see the improved result output if at any in terms of risk minimization, return maximization, and portfolio diversification. And also derive explicitly and more comprehending steps of some good choices a risk-averse investor would be interested to know. Based on the different literature reviewed, we would try to see how we can further minimixe risk using assets selected from different market sectors of the Nigeria stock exchange.

CHAPTER THREE

THEORETICAL FRAMEWORK AND METHODOLOGY

3.1 Expected Return

Sharpe (2000) states that a portfolio expected return is the weighted average of the expected return of the individual assets. Depending on the weight of an individual asset this asset will have a larger or smaller impact on the return of the portfolio. Alternative asset differ in their terms of expected return, but the expected return is only a part of the assets' future performance. What may influence the expected return is how volatile the asset is (Gibson, 2000).

Suppose we have portfolio of two assets A and B made up by resources or value put in each asset.

Let U_1 and U_2 are amount invested in asset A and B respectively. Then the total value which made up the portfolio is given by :

$$U_T = U_1 + U_2 \tag{1}$$

Let us assume that we have portfolios with positive value $(U > 0)$. If $U = 0$ then the portfolio is called arbitrage portfolio.

Now if we consider a weight (w_1) of a portfolio given by $w_1 = \frac{U_1}{U_T}$ which is a weight on asset A and $w_2 = \frac{U_2}{U_T}$ being a weight of asset B then

$$w_1 + w_2 = \frac{U_1 + U_2}{U_T} = \frac{U_T}{U_T} = 1. \tag{2}$$

Also, let us consider a portfolio with w_i as the proportion or fraction being put in asset i, then we have:

$$w_1 = \frac{U_1}{U_T}, w_2 = \frac{U_2}{U_T}, \ldots, w_n = \frac{U_n}{U_T} \quad (3)$$

$$w_1 + w_2 + \ldots + w_n = \frac{U_1 + U_2 + \ldots + U_n}{U_T} = \frac{U_T}{U_T} = 1 \quad (4)$$

therefore, for portfolio consisting of *n* assets, $w_i (i = 1, 2, \ldots, n)$,

$$w_1 + w_2 + \ldots + w_n = 1 \quad (5)$$

$$\Rightarrow \quad \sum_{i=1}^{n} w_i = 1, \quad (6)$$

$$r_p = w_1 r_1 + w_2 r_2 + \ldots + w_n r_n = \sum_{i=1}^{n} w_i r_i \quad (7)$$

$$\Rightarrow \quad E\left(r_p\right) = E\left(w_1 r_1 + w_2 r_2 + \ldots w_n r_n\right) = E\left(\sum_{i=1}^{n} w_i r_i\right) \quad (8)$$

$$\Rightarrow \overline{r_p} = w_1 \overline{r_1} + w_2 \overline{r_2} + \ldots + w_n \overline{r_n} = \sum_{i=i}^{n} w_i \overline{r_i} \quad (9)$$

This is the expected return of the portfolio, while r_i is the return of asset i.

3.2 Standard Deviation

This is the statistic that measures the volatility/risk of the portfolio; it measures how uncertain the return of the portfolio is (Sharpe, 2000). The higher the statistic the riskier the portfolio returns.

Yahaya *et al.* (2011) clearly showed the steps of getting the risk/standard deviation of the portfolio return; we let variance of the i^{th} asset be σ_i^2, covariance of i^{th} and j^{th} assets be σ_{ij} and the portfolio's risk be σ_p^2. Doing that we get:

$$\sigma_p^2 = E\left(r_p - \overline{r_p}\right)^2 \tag{10}$$

$$= E\left[\sum_{i=1}^{n} w_i r_i - \sum_{i=1}^{n} w_i \overline{r_i}\right]^2 \tag{11}$$

$$= E\left[\sum_{i=1}^{n} w_i (r_i - \overline{r_i}) \sum_{j=1}^{n} w_j (r_j - \overline{r_j})\right] \tag{12}$$

$$= E\left[\sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j (r_i - \overline{r_i})(r_j - \overline{r_j})\right] \tag{13}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j E\left[(r_i - \overline{r_i})(r_j - \overline{r_j})\right] \tag{14}$$

remember that $\sigma_{ij} = E\left[(r_i - \overline{r_i})(r_j - \overline{r_j})\right]$ (15)

$$\Rightarrow \sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \tag{16}$$

This is the relationship between portfolio variance and covariances of asset return pairs in a portfolio.

In terms of correlation that exist between assets in a pool/portfolio. The variance (risk) can be represented as;

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \sigma_i \sigma_j \tag{17}$$

since $\rho_{ij} = \dfrac{\sigma_{ij}}{\sigma_i \sigma_j}$, (18) where ρ_{ij} is the correlation coefficient between asset i and j , σ_i and σ_j are the standard deviation of asset i and j respectively.

3.3 Diversification

There a lot of studies being carried out proving the effectiveness of diversification; spreading wealth across several available assets in a portfolio.

Using the simplest case of two assets portfolio, we show the effectiveness of diversification in terms of the three scenarios of correlation coefficient.

It follows from Yahaya *et al.* (2011), if p is a portfolio consisting of two assets, having weight 1 in 1st asset and $1-w$ in asset 2 with portfolio variance σ_p^2 and correlation coefficient between asset 1 and 2 ρ_{12} ;

$$\Rightarrow \sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12} \qquad (19)$$

from equation (18)

$$\Rightarrow \rho_{12}\sigma_1\sigma_2 = \sigma_{12} \qquad (20)$$

therefore

$$\sigma_p^2 = w^2\sigma_1^2 + \sigma_2^2(1-w)^2 + 2w(1-w)\rho_{12}\sigma_1\sigma_2 \qquad (21)$$

when $\rho_{12} = 0$

$$\Rightarrow \sigma_p^2 = w^2\sigma_1^2 + \sigma_2^2(1-w)^2 \qquad (22)$$

$$\Rightarrow \sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 \qquad (23)$$

when $\rho_{12} = 1$

$$\Rightarrow \sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2 \quad (24)$$

$$= (w\sigma_1 + (1-w)\sigma_2)^2 \quad (25)$$

$$\Rightarrow \sigma_p = (w\sigma_1 + (1-w)\sigma_2) \quad (26)$$

This gives the weighted average of asset 1 and 2 respectively.

when $\rho_{12} = -1$, then

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 - 2w(1-w)\sigma_1\sigma_2 \quad (27)$$

$$= [w\sigma_1 - (1-w)\sigma_2]^2 \quad (28)$$

$$\sigma_p = w\sigma_1 - (1-w)\sigma_2 \quad (29)$$

when $\rho_{12} < 1$, it implies that

$$\sigma_p^2 < w\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_1\sigma_2 \quad (30)$$

$$< w\sigma_1 + (1-w)\sigma_2 \quad (31)$$

Clearly showing the reduction of portfolio risk significantly even when there is positive coefficient of correlation, proving that covariance or correlation between assets have greater impact when it comes to risk reduction of portfolio than the reduction due to variance of portfolio that contributes very little.

And lastly

when $\rho_{12} < -1$, the risk as seen below drastically reduces to:

$$\sigma_p^2 < w^2\sigma_1^2 + (1-w)\sigma_2^2 - 2w(1-w)\sigma_1\sigma_2 \quad (32)$$

$$< [w\sigma_1 - (1-w)\sigma_2]^2 \quad (33)$$

$$\sigma_p = w\sigma_1 - (1-w)\sigma_2 \quad (34)$$

The last equation emphasizes the importance of one to invest in assets having negative correlation between them.

3.4. Correlated Assets Portfolio

For clarity, let us look at the characteristics of portfolio of $n_i(i = 1, 2, ..., n)$ assets and for simplicicity when we have two correlated assets i and j , $\rho_{ij} \neq 0$ (i.e. $\sigma_{ij} \neq 0$) , assuming the n assets in the portfolio are equally-weighted i.e. $w_i = 1/n \; (i = 1, 2, ..., n)$. It can be seen that typical variance terms is $(1/n)^2\sigma_{ii}$ (having n total number of variance terms), with covariance terms $(1/n)^2\sigma_{ii}(i \neq j)$ (having $n^2 - n$ total number of covariance terms in the portfolio). Then the risk of the portfolio from equation (16) is:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} \quad (35)$$

$$= \sum_{i=1}^{n}(1/n)^2\sigma_{ii} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}(1/n)^2\sigma_{ij}$$

(36)

$$=\left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right)+\left(\frac{n^2-n}{n^2}\right)\left(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}\right) \quad (37)$$

$$\sigma_p^2=\left(\frac{1}{n}\right)\text{(average variance) + }\left(\frac{n^2-n}{n^2}\right)\text{(average covariance)} \quad (38)$$

when n is increased to a very large value $(n\to\infty)$

$$\sigma_p^2=\left(\frac{1}{\infty}\right)\text{+(average variance) + }\left(1-\frac{1}{\infty}\right)\text{(average covariance)} \quad (40)$$

$$= (0)\,\text{(average variance) + }\ (1-0)\,\text{(average covariance)} \quad (41)$$

$$\sigma_p^2 = 0\ \text{ + average covariance} \quad (42)$$

This is implying that contribution of variance terms tends insignificantly to zero while that of the covariance tends to average covariance when supposing equally-weighted portfolio (investment shared in the same proportion among the assets in the portfolio, and asset i and j to be correlated i.e $\rho_i \neq 0$). (Yahaya *et al*. 2011).

3.5. Uncorrelated Assets Portfolio

On the other hand, when looking at uncorrelated assets in a portfolio i.e when assuming that the i^{th} and j^{th} assets in a portfolio of $n_i(i=1,2,...,n)$ assets are completely uncorrelated, i.e, $\rho_{ij}=0$; with the same assumption of equally-weighted portfolio assets in the portfolio, the variance/risk will be:

$$\sigma_p^2=\sum_{i=1}^{n}\sum_{j=1}^{n}w_iw_j\sigma_{ij} \quad (43)$$

$$= \sum_{i=1}^{n} (1/n)^2 \sigma_{ii} + \sum_{i=1}^{n} \sum_{j \neq i}^{n} (1/n)^2 \sigma_{ij} \quad (44)$$

$$= \sum_{i=1}^{n} \frac{\sigma_i^2}{n^2} + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \frac{\sigma_{ij}}{n^2} \quad (45)$$

$$= \sum_{i=1}^{n} \left(\frac{1}{n} \right)^2 \sigma_i^2 + 0 \quad \text{(since } \sigma_{ij} = \rho_{ij} \sigma_i \sigma_j \text{)} \quad (46)$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_i^2}{n^2} = \frac{1}{n} \text{(average variance)} \quad (47)$$

as n tends to be very large i.e $n \to \infty$

$$\lim_{n \to \infty} \sigma_p^2 = \lim_{n \to \infty} \frac{1}{n} \text{(average variance)} \quad (48)$$

$$= 0 \quad \text{(Yahaya } et\ al. \text{ 2011)} \quad (49)$$

implying that as we have portfolio constituted of uncorrelated assets, the assets portfolio risk tends to zero (0).

3.6 Efficient Frontier

The efficient frontier answers the questions of how to determine the level of diversification. The idea of an efficient frontier can be applied in a number of ways. In essence, an efficient frontier is a curve on a graph representing the relationship between risk and return for a set of portfolios. For a portfolio to be on the efficient frontier, the portfolio most maximizes return for a given level of risk (Manganelli, 2002).

In the idea, it means that risk and return have good interrelationship between them, and therefore it would be necessary to find a way of obtaining the degree of risk needed for various levels of assets return. Hagstrom (2001) states that it is very difficult to get high returns without endangering yourself to some sort of risk. Markowitz (1959) created what he termed as the efficient frontier, a trade-off graph fixing return on one axis and risk on the other. It is a graphical curve representing all portfolios that are able to maximize expected return for given risk level. The efficient frontier is a line that is drawn from the bottom left to the top right with each point on the line representing an intersection between reward and corresponding risk level and efficient portfolio gives the highest return for a given portfolio risk level.

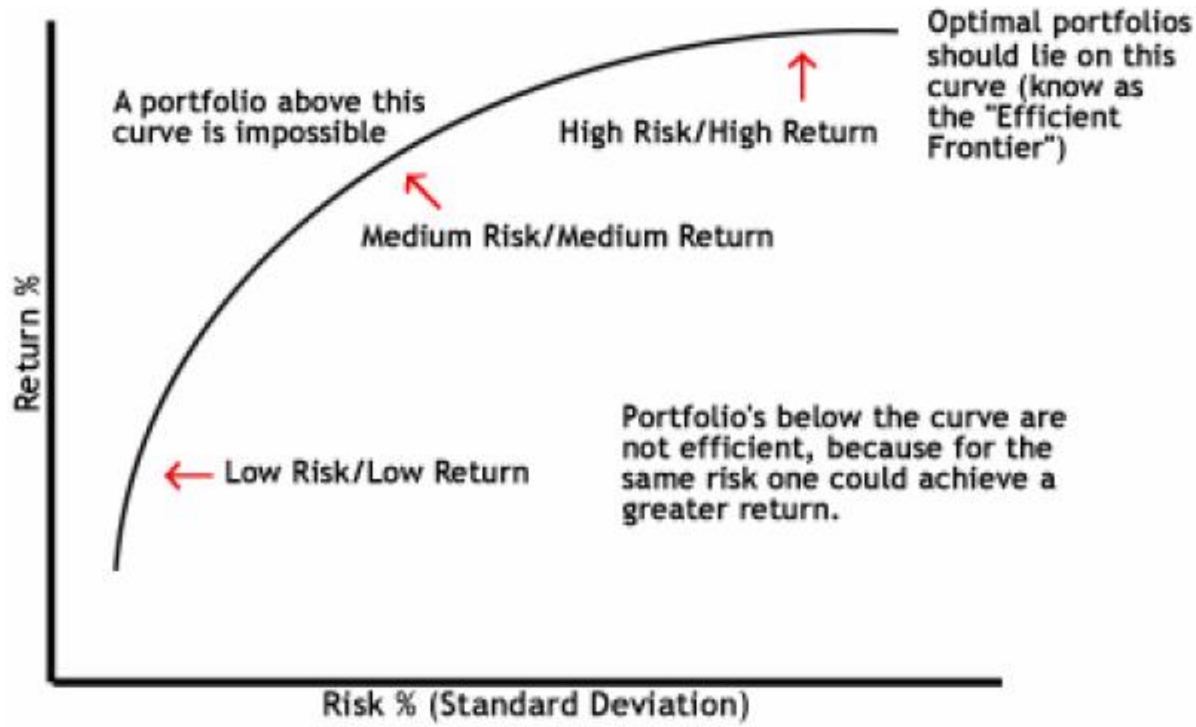

Figure 3: Efficient Frontier

3.7. Methodology

Acting on the orientation, objective and the literature review of this study, we collect data of daily closing prices of five (5) assets from different market sectors (to see effect of diversification across the different market segments) of the Nigeria Stock Exchange and compare it with the benchmark (GMI). The five assets (companies: 7up bottling company, Ashaka cement plc, Julius Berger Nigeria plc and Flourmill Nig. plc) are among the thirty most capitalized stocks in the market; the data covers the period (2007-2009) that included the turbulent period of the global financial crisis of 2008 to see the effectiveness of Markowitz strategy in minimizing risks and maximizing returns over the benchmark.

As a result of time and resources constraints to take every parameter into consideration when evaluating the performance of the portfolio, the perspective is the power and performance outcome exclusively in the Nigeria capital market.

There is no actual amount of money will be invested in the portfolio, instead the performance of the portfolio will be displayed in percentage, to be able to give the investor a clearer and more comprehensive conclusion of the findings.

The market index that is used as the benchmark is the general market performance index of the Nigeria capital market. This index mimics the general market movement in total. The challenge arises from the fact that as market condition slight, the risk and expected return of the various securities change. Due to the constantly shifting market conditions, only a relatively short period of market history which comprises good and bad market conditions can be used. Also, the portfolio is rebalanced every four months to see the updated performance of the strategy (Kristian *et al*, 2006).

The basic sources of data used are secondary data. The data analytical tools employed in this study are the mean, variance, standard deviation, coefficient of variation, optimization engine, covariance of return.

3.7.1 Mean-Variance Portfolio Selection without Short Sales

Optimization of portfolio has posed to be a big challenge in quantitative risk management, going back from the Markowitz work as the pioneer in the field. The main assumption being that assets return is a random variable of which expected mean and variance are to be estimated using their trading day's historical data. Risk and reward are correspondingly. Risk and reward are variance and return respectively. The problem of optimizing portfolio can then be constructed as: when we have set of financial assets with expected mean, variance and covariances, we find the optimal weights of the asset set that will result in smallest risk of the portfolio at a given level of the portfolio return. On setting the optimization model, we explicitly evaluate the optimal weights of the portfolio. We further extend to the analytical derivation of some useful measures an investor would be interested to know, these include: the efficient frontier, its risk and return values, tangency portfolio/maximum sharp ratio, its corresponding risk and return and its asset weight or asset allocation of the tangency portfolio.

3.7.2 Asset and Portfolio

A portfolio is defined as a pool of N assets A_n, with corresponding returns R_n, $n = 1, 2, 3, ..., N$, and available invested wealth W. We will denote the amount that is invested in the n^{th} asset by W_n. Having W as the total invested wealth, then:

$$\sum_{n=1}^{N} W_n = W \tag{50}$$

$$w_n = \frac{W_n}{W} \Rightarrow \sum_{n=1}^{N} w_n = 1 \tag{51}$$

and actual portfolio return is $\sum_{n=1}^{N} w_n R_n = R$ (52)

Therefore, expected return of the portfolio is given as

$$\phi = E[R] = E\left[\sum_{n=1}^{N} w_n R_n\right] = \sum_{n=1}^{N} w_n r_n \tag{53}$$

$r_n = E\left[R_n\right]$ is the expected return of the n^{th} asset, $n = 1, 2, ..., N$. We use portfolio covariance matrix **S**:

$$\mathbf{S} = \begin{pmatrix} s_{11} & s_{12} \cdots & s_{13} \\ s_{21} & s_{22} \cdots & s_{1N} \\ \vdots & \vdots & \vdots \\ s_{N1} & s_{N2} \cdots & s_{NN} \\ \end{pmatrix} \tag{54}$$

$$s_{ij} = s_{ji} = E\left[(R_i - r_i)(R_j - r_j)\right] \tag{55}$$

to be able to evaluate the dispersion from the expected return and also finding the risk of the portfolio. The portfolio variance in matrix is formulated as:

$$s^2 = E\left[\left(R - \phi\right)^2\right] = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j s_{ij} = \mathbf{w}^T \mathbf{S} \mathbf{w} \tag{56}$$

$\mathbf{w} = \left[w_1, w_2, ..., w_N\right]^T$ is the weights vector.

3.7.3 Optimization of N Risky Assets

We would now explicitly construct the portfolio optimization procedure for N risky assets and further analytically derive some of the useful statistics a risk-averse investor would be interested to know.

We say a portfolio is optimal when for a given target expected return, the portfolio has the smallest variance/risk, s^2. To achieve such a portfolio, we need to solve the following constrained mathematical quadratic optimization model (Dorfman 1979).

$$\mathbf{w} = \arg\min_{\mathbf{w}}\left[\mathbf{w}^T \mathbf{S} \mathbf{w}\right] \tag{57}$$

subject to $$\mathbf{w}^T \mathbf{u} = \sum_{n=1}^{N} w_n = 1 \tag{58}$$

and $$\mathbf{w}^T\mathbf{r} = \sum_{n=1}^{N} w_n r_n = \phi \qquad (59)$$

$\mathbf{w}^T\mathbf{r} = \sum_{n=1}^{N} w_n = 1$ means the total available wealth must be invested all.

where $\mathbf{u} = [1,1,...,1]^T$ and $\mathbf{r} = [r_1, r_2, ..., r_N]^T$

Utilizing lagrangian multiplier method,

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \mathbf{w}^T\mathbf{S}\mathbf{w} -_1 \lambda_1(\mathbf{w}^T\mathbf{u} - 1) - \lambda_2(\mathbf{w}^T\mathbf{r} - \phi) \qquad (60)$$

λ_1 and λ_2 are the lagrangians which we need to find, therefore

$$\nabla_{\mathbf{w}} L(\mathbf{w}, \lambda_1\lambda_2) = 2\mathbf{S}\mathbf{w} - \lambda_1\mathbf{u} - \lambda_2\mathbf{r} = 0 \qquad (61)$$

$$\frac{\partial L}{\partial \lambda_1} = \mathbf{w}^T\mathbf{r} - \phi = 0$$

(62)

$$\mathbf{w} = \frac{1}{2}\mathbf{S}^{-1}(\lambda_1\mathbf{u} + \lambda_2\mathbf{r}) = 2 \qquad (63)$$

$$\mathbf{u}^T\mathbf{S}^{-1}\mathbf{u}\lambda_1 + \mathbf{r}^T\mathbf{S}^{-1}\mathbf{u}\lambda_2 = 2 \qquad (64)$$

$$\mathbf{u}^T\mathbf{S}^{-1}\mathbf{r}\lambda_1 + \mathbf{r}^T\mathbf{S}^{-1}\mathbf{r}\lambda_2 = 2\phi \qquad (65)$$

putting in mind that

$$\mathbf{u}^T\mathbf{S}^{-1}\mathbf{r} = \mathbf{r}\mathbf{S}^{-1}\mathbf{u} \qquad (66)$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 2\begin{pmatrix} 1 \\ \phi \end{pmatrix} \qquad (67)$$

where

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} \mathbf{u}^T\mathbf{S}^{-1}\mathbf{u} & \mathbf{r}^T\mathbf{S}^{-1}\mathbf{u} \\ \mathbf{r}^T\mathbf{S}^{-1}\mathbf{u} & \mathbf{r}^T\mathbf{S}^{-1}\mathbf{r} \end{pmatrix} \qquad (68)$$

Obviously the system (64) and (65) will have a define solution iff :

$$d = a_{11}a_{22} - a_{21}^2 \neq 0 \quad (69)$$

To show that $d \neq 0$, we know that $\mathbf{S}$ is a positive definite matrix, $\Rightarrow \mathbf{S}^{-1}$ is also positive definite. Meaning $\mathbf{y}\mathbf{S}^{-1}\mathbf{y} > 0, \forall \mathbf{y} \neq 0$. Clearly $a_{11} > 0$ and $a_{22} > 0$:

$$\left(a_{12}\mathbf{u} - a_{12}\mathbf{r}\right)^T \mathbf{S}^{-1}\left(a_{12}\mathbf{u} - a_{11}\mathbf{r}\right) = a_{12}a_{12}a_{11} - a_{11}a_{12}a_{12} + a_{11}a_{11}a_{22} \quad (70)$$

$$= a_{11}\left(a_{11}a_{22} - a_{12}^2\right) \quad (71)$$

$$= a_{11}d > 0 \quad (72)$$

and because $a_{11} > 0$ we know that $d > 0$

$$\Rightarrow d \neq 0$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{2}{d}\begin{pmatrix} a_{22} - a_{12}\phi \\ -a_{12} + a_{11}\phi \end{pmatrix} \quad (73)$$

$$\Rightarrow \lambda_1 = \frac{2}{d}\left(a_{22} - a_{12}\phi\right) \text{ and } \lambda_2 = \frac{2}{d}\left(-a_{12} + a_{11}\phi\right) \quad (74)$$

substituting the values of λ_1 and λ_2 in $\mathbf{w}$ we have the optimal portfolio weight as:

$$\mathbf{w}(\phi) = \mathbf{f} + \phi\mathbf{g} \quad (75)$$

$$\mathbf{f} = \frac{1}{d}\mathbf{S}^{-1}\left(a_{22}\mathbf{u} - a_{12}\mathbf{r}\right) \text{ and } \mathbf{g} = \frac{1}{d}\mathbf{S}^{-1}\left(-a_{12}\mathbf{u} - a_{11}\mathbf{r}\right) \quad (76)$$

Portfolio that minimizes variance for given expected return is " frontier portfolio". All portfolio $\mathbf{w}(\phi)$ can expressed as linear combination of these two efficient portfolios $\mathbf{f}$ and $\mathbf{g}$. Therefore the frontier portfolio becomes:

$$s^2(\phi) = \mathbf{w}^T(\phi)\mathbf{S}\mathbf{w}(\phi) \quad (77)$$

$$= \phi^2\mathbf{g}^T\mathbf{S}\mathbf{g} + \phi(\mathbf{g}^T\mathbf{S}\mathbf{f} + \mathbf{f}^T\mathbf{S}\mathbf{g}) + \mathbf{f}^T\mathbf{S}\mathbf{f} \quad (78)$$

$$= \frac{a_{11}}{d}\left(\phi - \frac{a_{12}}{a_{11}}\right)^2 + \frac{1}{a_{11}} \tag{79}$$

$$s^2(\phi) = \frac{1}{d}(a_{11}\phi^2 - 2a_{12}\phi + a_{22}) \tag{80}$$

This equation stands for the "efficient frontier" $-$a curve or hyperbola in $(s,\phi)-$plane that gives the minimum risk for a given level of expected return.

To get the weights of the minimum variance portfolio, we set the first derivative of the efficient frontier, $s^2(\phi)$, to zero and solve for ϕ and then substitute the expression in the "efficient frontier" :

$$\frac{\partial s^2(\phi)}{\partial \phi} = 0 \tag{81}$$

$$\Rightarrow \frac{\partial s^2(\partial)}{\partial \phi} = \frac{1}{d}(2a_{11}\phi_{mvp} - 2a_{12}) = 0 \tag{82}$$

$$\frac{1}{d}(2a_{11}\phi_{mvp} - 2a_{12}) = 0 \tag{83}$$

dividing through by $\frac{2}{d}$:

$$\Rightarrow a_{11}\phi_{mvp} - a_{12} = 0 \tag{84}$$

$$\Rightarrow \phi_{mvp} = \frac{a_{12}}{a_{11}} \tag{85}$$

weights vector of the minimum variance portfolio is therefore:

$$\mathbf{w}_{mvp}(\phi) = \mathbf{f} + \phi_{mvp}\mathbf{g} \tag{86}$$

$$\Rightarrow \mathbf{w}_{mvp}(\phi) = \mathbf{f} + \frac{a_{12}}{a_{11}}\mathbf{g} \tag{87}$$

On substituting the minimum variance return ϕ_{mvp} in $s^2(\phi)$, we get the corresponding risk of the efficient frontier:

$$s^2_{mvp}(\phi) = \frac{1}{\sqrt{a_{11}}} \tag{88}$$

One of the important choices an investor would be interested in is the portfolio that gives the maximum return among the efficient portfolios. This portfolio is known as "Tangency Portfolio" and it is the portfolio with the maximum sharp ratio. The sharp ratio is the statistic that measures the performance of a portfolio; it is the expected return per unit of risk (Dorfman, 1979).

Dorfman (1979) states in a geometric perspective, the point on the efficient frontier that moves down through the origin in $(s, p)-$plane is the "tangent portfolio". To get the tangent point (s_{TGP}, ϕ_{TGP}), the slope of the tangent line:

$$\frac{s_{TGP} - 0}{\phi_{TGP} - 0} = \frac{\sqrt{\frac{1}{d}\left(a_{11}\phi^2_{TGP} - 2a_{12}\phi_{TGP} + a_{22}\right)}}{\phi_{TGP}} \tag{89}$$

should be equal with derivative of the efficient frontier at that point.

$$\Rightarrow \frac{\partial s(\phi)}{\partial \phi} = \frac{a_{11}\phi_{TGP} - a_{12}}{d\sqrt{\frac{1}{d}\left(a_{11}\phi^2_{TGP} - 2a_{12}\phi_{TGP} + a_{22}\right)}} \tag{90}$$

$$1 = \frac{\sqrt{\frac{1}{d}\left(a_{11}\phi^2_{TGP} - 2a_{12}\phi_{TGP} + a_{22}\right)}}{\phi_{TGP}} \times \frac{d\sqrt{\frac{1}{d}\left(a_{11}\phi^2_{TGP} - 2a_{12}\phi_{TGP} + a_{22}\right)}}{a_{11}\phi_{TGP} - a_{12}} \tag{91}$$

$$1 = \frac{d\left(\sqrt{\frac{1}{d}\left(a_{11}\phi^2_{TGP} - 2a_{12}\phi_{TGP} + a_{22}\right)}\right)^2}{\phi_{TGP}\left(a_{11}\phi_{TGP} - a_{12}\right)} \tag{92}$$

$$\Rightarrow \phi_{TGP}\left(a_{11}\phi_{TGP} - a_{12}\right) = \frac{d}{d}\left(a_{11}\phi_{TGP}^2 - 2a_{12}\phi_{TGP} + a_{22}\right) \tag{93}$$

$$\Rightarrow a_{11}\phi_{TGP}^2 - a_{12}\phi_{TGP} = a_{11}\phi_{TGP}^2 - 2a_{12}\phi_{TGP} + a_{22} \tag{94}$$

$$\Rightarrow 2a_{12}\phi_{TGP} - a_{12}\phi_{TGP} = a_{22} \tag{95}$$

$$\Rightarrow a_{12}\phi_{TGP} = a_{22} \tag{96}$$

$$\Rightarrow \phi_{TGP} = \frac{a_{22}}{a_{12}} = \text{tangency portfolio return} \tag{97}$$

In other to get the its corresponding risk, we substitute ϕ_{TGP} in the efficient frontier $s^2(\phi)$

$$\therefore \qquad s_{TGP}^2(\phi) = \frac{1}{d}\left(a_{11}\left(\frac{a_{22}}{a_{12}}\right)^2 - 2a_{12}\left(\frac{a_{22}}{a_{12}}\right) + a_{22}\right) \tag{98}$$

$$= \frac{1}{d}\left(\frac{a_{11}a_{22}^2}{a_{12}^2} - 2a_{22} + a_{22}\right) \tag{99}$$

$$= \frac{1}{d}\left(\frac{a_{11}a_{22}^2 - a_{22}a_{12}^2}{a_{12}^2}\right) \tag{100}$$

remember that $d = a_{11}a_{22} - a_{12}^2$

$$\Rightarrow s_{TGP}^2(\phi) = \frac{1}{a_{12}} \tag{101}$$

and

$$s_{TGP} = \frac{1}{\sqrt{a_{12}}} \text{ is tangency portfolio risk} \tag{102}$$

While to get the corresponding assets allocation of the tangency portfolio, we substitute ϕ_{TGP} (tangency portfolio return) in $\mathbf{w}(\phi)$ (efficient frontier optimum weights),

$$\Rightarrow \mathbf{w}_{TGP} = \mathbf{f} + \phi_{TGP}\mathbf{g} \tag{103}$$

$$\Rightarrow \mathbf{w}_{TGP} = \mathbf{f} + \frac{a_{22}}{a_{12}}\mathbf{g}\text{, is the tangency portfolio weights} \tag{104}$$

3.8. Mean-Gini Portfolio Selection without Short Sales

The Gini's mean difference (GMD or MG) measure is a statistic broadly used in calculating income inequality. Yitzhaki (1982) come up with the MG model by employing Gini's mean difference as an alternative risk measure in capital market portfolio optimization. It is mathematically define as:

$$\tau = \frac{1}{2}\int_a^b\int_a^b |X - Y| \, f(X) f(Y) dXdY \tag{105}$$

or using mathematical expectation we have

$$\tau = \frac{1}{2} E\left(|X - Y|\right), \tag{106}$$

where $f(X)$and $f(Y)$are probability density functions of X and Y respectively

The gauges for judgement for MG efficiency are distribution F dominates distribution G whenever $\mu_Y \geq \mu_Z$ and $\mu_Y - \tau_Y \geq \mu_Z - \tau_Z$ where μ being the mean and τ stands for the half Gini mean difference. These two benchmarks are essential orders for first and second degree stochastic dominance (SSD) and the model has upper merits over mean-variance model. In a study carried out by Yitzhaki (1982), he obtained that the mean-gini efficient set is consistent with the SSD rule. The model efficient set belongs to the set of SSD efficient set. The second degree stochastic dominance efficient set is very perfect for risk-averse financial investors due to the reason that there are no limits on the distribution of the assets return and financial investors demonstrate risk-averse utility function (Hadar *et al*. 1969). The approach of the mean-gini is equally clear and simple to that of the mean-variance as it also uses two statistics measures: mean and Gini's mean difference. More so, the model procedure can also be utilized in analytical derivation of the capital asset pricing models (Shalit and Yitzhaki 1984).

We define $R_1, R_2, \dots R_n$ to be the random returns of n assets. The random return of a portfolio p, being a linear combination of the individual random returns, is:

$$R_p = \sum_{i=1}^{n} w_i R_i \tag{107}$$

$w_1, w_2, \dots, w_n$ stand for the portfolio weights (i.e., the proportions of investments funds as allocated to the individual assets) such that:

$$\sum_{i=1}^{n} w_i = 1 \tag{108}$$

and

$$w_i \geq 0, \text{ for } i = 1,2, \dots, n. \tag{109}$$

The above constraints mean that the available investment fund are completely distributed among the selected assets that are being looked at and also assets short selling is disallowed. Furthermore, we define $\mu_i = E(R_i)$ to be expected return of asset i,($i = 1,2, \dots, n$). The expected return of the portfolio is:

$$\mu_p = E\left(R_p\right) = \sum_{i=1}^{n} w_i \mu_i \ , \tag{110}$$

meaning portfolio expected return is a linear combination of individual expected returns of the assets.

The portfolio's Gini coefficient as proved in the appendix can be stated as

$$\tau_p = 2\operatorname{cov}\left[R_p, F\left(R_p\right)\right], \tag{111}$$

$F(R_p)$ stands for the portfolio cumulative returns. The concept means that, when given any set of portfolio weights, portfolio random return is a linear combination of the portfolio individual assets returns with given probability distributions, the Gini coefficient of the portfolio can be formed utilizing the constructive probability distribution of returns of the portfolio.

A similar form of equation (111), stated by Shalit and Yitzhaki (1984), is

$$\tau_p = 2\operatorname{cov}\left[\left(\sum_{i=1}^{n} w_i R_i\right), F\left(R_p\right)\right] = 2\sum_{i=1}^{n} w_i \operatorname{cov}\left[R_i, F\left(R_p\right)\right] \qquad (112)$$

This equivalent form indicates that the portfolio's Gini coefficient is twice the weighted average of the covariances between the individual asset returns and the portfolio's cumulative distribution.

The objective function of MG model is to minimize the Gini's mean difference subject to similar constraints of the mean-variance objective function described earlier. The MG mathematical model is presented as:

Minimize $\quad 2\sum_{i=1}^{n} w_i \operatorname{cov}\left[R_i, F\left(R_p\right)\right]$

subject to

$$\sum_{i=1}^{n} w_i \mu_i = \mu_{required} \ , \ \sum_{i=1}^{n} w_i = 1$$

$\mu_{required}$ is the given target expected return by the investor

3.9. Research Approach

One of the ill sides of optimization is that in many cases, it results in a very narrow portfolio that leads to some concentration of risks and to over exposure to risks which the model will have difficulty in estimating, these include fraud, disaster and lots more. To prevent the portfolio against such risks, we use the aforementioned constraints and covariance matrix as inputs for the optimization procedure with excel solver (one of leading optimization software) in order to force some level of diversification (Gao, 2004).

The main objective of the study is use of variance and Gini index as two risk measures in the objective functions of the two approaches, and also, asset selection rebalancing in the optimization engine using five different sectors of the Nigeria Stock Market. We use quantitative method because measurements and interpretations of numerical data are employed to form index prices in the sense that historical data are used to forecast future outcome. The data collected from Nigeria financial market are used to illustrate best explanation of the study objectives.

We will form portfolio consisting of five indices combined in a particular manner to the theory and historical data from five indices (7Up, Ashaka cement, Julius Berger Nig. Plc, flourmill plc and Mobil Nig. plc) of same period 2007-2009 provided by www.cashcraft.com.

The benchmark we compared the portfolio with is the general market index (GMI) of Nigeria stock exchange. The index reflects the period (2007-2009) status and meaningful distortion in the market and consists of thirty most capitalized companies.

The estimation period for each quarter is held constant of the three years of return and in every quarter, new estimates of the individual assets are calculated and the optimal risky portfolio is formed in every quarter during the whole period of the study.

CHAPTER FOUR

RESULT ANALYSIS AND DISCUSSION

4.1 Introduction

As shown in the literature review, investors have different reasons and motives when it comes to investing, but many are risk averse. For that, I illustrate the following tables and graphs to give an insight of the various computations done in order to give a succinct evaluation of the strategies.

The chapter is composed of tabular and graphical illustrations of said objectives of the study. The first table is the estimated variance-covariance matrix used in computing mean-variance optimal portfolio and efficient frontier. In the next tables are the optimal weights of the two models given different level of expected return (port return) employing excel solver (one of leading optimization software).

The unrebalanced mean-gini assets table helps us in looking at the merits of portfolio rebalancing of weights – running the optimization at a frequent periods to avoid assets concentration due to investment risk. After carrying out all necessary computations, the efficient frontier is plotted graphically to have a clear understanding of explained in the methodology.

Lastly, the return and risk tables for both strategies are plotted for the whole period under study to gauge the performance of the portfolio compared to general market index (GMI)

4.2 Portfolio Composition

An investor can reduce risk of his investment by spreading his fund over a number of different available securities. The question arises then that if investor is able to form a diversified portfolio and not constrained to a single security, how then will he allocate his fund among the various alternatives. In this study mean-variance (MV) model is employed to construct optimal portfolios from 5 selected firms subject to the said constraints in the methodology. Table 4.2 and 4.3 show the optimal portfolio compositions of MV and MG model after estimating means, variances and covariance matrices in excel solver discussed in the methodology.

Table 4. 1: 1st quarter 2007 estimated covariance matrix

	7UP	ASHAKA	JBERGER	FLUORML	MOBIL
7UP	9.84E-05	9.85E-05	0.000312	5.77E-05	0.000292
ASHAKA	9.85E-05	0.000155	0.000171	6.29E-05	0.000342
JBERGER	0.000312	0.000171	0.00219	-0.00027	0.00079
FLUORML	5.77E-05	6.29E-05	-0.00027	0.000262	0.000181
MOBIL	0.000292	0.000342	0.00079	0.000181	0.00091

Above is the output estimated covariance matrix of the first 4 months data, a step for computing weights at different expected returns given by the investor. We use it to illustrate the idea of efficient frontier utilizing the first 4 months data (Jan-Apr 2007).

Table 4. 2: 1st quarter 2007 Summary Statistics of MV Optimal Portfolio (%)

Mean Return	0.06	0.1	0.5	0.96	1.42	1.88	2.43
Risk	0.98	0.98	0.96	0.98	1.07	1.17	1.29
Sharpe Ratio	6.12	10.2	52.08	97.96	132.71	160.68	188.37

Table 4. 3: 1st quarter 2007 Summary Statistics of MG Optimal Portfolio (%)

Mean Return	0.06	0.1	0.5	0.96	1.42	1.88	2.43
Risk	1.47	1.47	1.48	1.58	1.71	1.84	2
Sharpe Ratio	4.08	6.8	33.78	60.76	83.04	102.17	121.5

Table 4.2 and 4.3 are also obtained using the first 4 months data required in computing efficient frontier. The mean returns in the tables are the given expected mean constraints stated in the objective functions of the models; they are fixed values used in calculating the minimum risk and

weights of the portfolio. The performance (sharp ratio) is the ratio of the mean return to risk of the portfolio as mention in the methodology.

Table 4. 4: 1st quarter 2007 Assets Allocation of MV Portfolio (%)

port ret.	port std	7UP	ASHAKA	JBERGER	FLUORML	MOBIL
0.06	0.98	62.15	24.56	0	13.29	0
0.1	0.98	63.37	23.15	0	13.47	0
0.5	0.96	75.63	9.08	0	15.29	0
0.96	0.98	70.51	0	1.17	28.33	0
1.42	1.07	49.3	0	7.16	43.54	0
1.88	1.17	28.09	0	13.15	58.76	0
2.43	1.29	2.74	0	20.31	76.95	0

Table 4. 5: 1st quarter 2007 Assets Allocation of MG Portfolio (%)

port ret.	port std	7UP	ASHAKA	JBERGER	FLUORML	MOBIL
0.06	1.47	75.15	20.4	0	4.44	0
0.1	1.47	76.35	19.01	0	4.64	0
0.5	1.48	88.29	5.04	0	6.67	0
0.96	1.58	78.34	0	4.23	17.42	0
1.42	1.71	57.57	0	10.39	32.03	0
1.88	1.84	36.8	0	16.55	46.64	0
2.43	2	11.98	0	23.92	64.1	0

Table 4. 6: 2007-2009 Unrebalanced Assets Allocation of MG Portfolio (%)

port ret.	port std	7UP	ASHAKA	JBERGER	FLUORML	MOBIL
0.06	6.02	30.42	0	26.7	13.1	29.77
0.1	6.17	28.11	0	27.56	14.02	30.31
0.5	7.7	4.61	0	36.13	22.67	36.6
0.96	10.76	0	0	77.07	20.65	2.29
1.42	11.89	0	0	1	0	0
1.88	11.89	0	0	1	0	0
2.43	11.89	0	0	1	0	0

The weightings are solved by testing every possible combination of assets resulting in the given return (port ret.). The weights generating the lowest standard deviation for the given return is considered an efficient portfolio. Every efficient portfolio is then presented as a data point resulting in an efficient frontier. An interesting observation in Table 4.4 and 4.5 worth talking is even at different level of expected return, no fraction of investment recommended for Mobil oil Nigeria plc, this prompts an investor to know that something is wrong in that sector or company, one of advantages of rebalancing and active over passive management style of investment.

In Table 4-6, we refuse to rebalance (solving again the objective functions using the given constraints) the portfolio using every four months data in the period and leave it till after the whole three (3) years, to affirm the already established fact that one of the reasons of rebalancing is market conditions shift and that leads to asset concentration, making the investments highly volatile by concentrating wealth in a particular sector or some assets. Looking at the table, the resources at last concentrated towards Julius Berger Nigeria Plc and this implies that the essence of diversification is not achieved.

4.3 The Efficient Frontier

The efficient frontier is formed by holding a constant return while minimizing the standard deviation. In Table 4-1 each return and its corresponding standard deviation is listed. The weight for achieving maximum sharp ratio is given for each scenario.

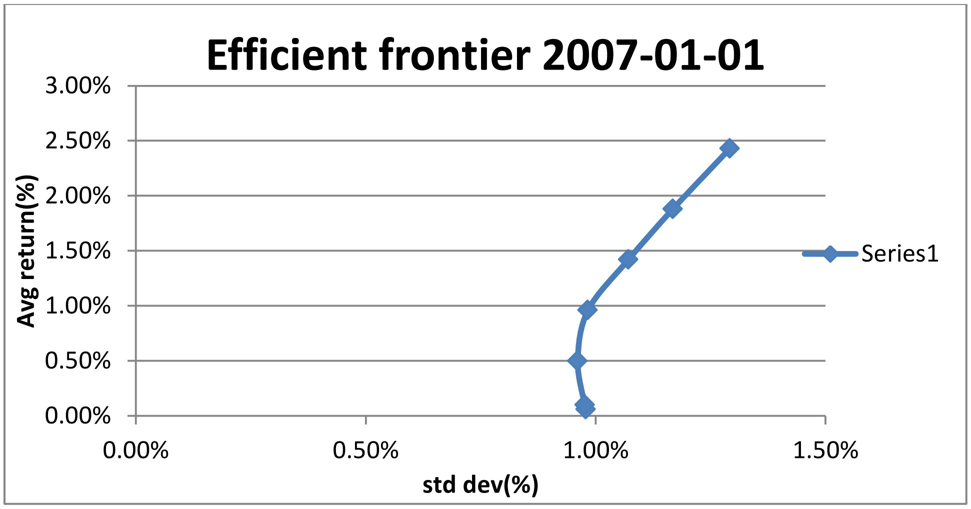

Figure 4. 1: Efficient Frontier 2007-01-01

Using the data in Table 4.3, the efficient frontier is plotted in the above Figure 4.1 to illustrate the idea. By combining the assets through different weightings of the stocks in the optimization engine (excel solver), a portfolio (series) for every point on the efficient frontier can be comprised. As illustrated in the figure above, there are no portfolio situated above the efficient frontier and all those that lie below are inferior to those situated on the efficient frontier. Each point on the efficient frontier represents a different but efficient portfolio. By investing in an efficient portfolio, the investor achieves the highest possible return for the given risk. Choosing which one to invest in, among the efficient portfolios in the efficient frontier depends on the level of risk tolerance of an investor.

Table 4. 7: Portfolio and GMI returns (%)

Dates	Portfolio	GMI
30-04-07	0.92	6.92
31-08-07	-1.43	1.44
31-12-07	4.03	3.96
30-04-08	3.68	0.86
31-08-08	-1.09	-5.19
31-12-08	-7.46	-9.64
30-04-09	-12.86	-7.5
31-08-09	6.61	3.44
31-12-09	5.18	-2.98

Table 4. 8: Portfolio and GMI risks (%)

Dates	portfolio	GMI
30-04-07	2.72	5.66
31-08-07	2.31	4.17
31-12-07	4.43	4.17
30-04-08	6.01	7.39
31-08-08	8.72	5.51
31-12-08	5.76	8.22
30-04-09	9.43	12.24
31-08-09	12.23	23.23
31-12-09	7.62	1.57

We obtained the portfolio and GMI returns in Table 4.7 by partitioning the whole period data (2007-2009) into nine parts each containing four months data, starting from January 2007 to December 2009. Since the portfolio contain five assets, the portfolio return of every four months is computed using the combined return data of the five assets in the portfolio, as shown in the analytical portfolio return calculation in the methodology. The same method is applied in obtaining the GMI return for the whole period under study, by using the general market performance index that mimics the market movement as explained earlier in the methodology. Also, same approach is used in obtaining portfolio and GMI risks in Table 4.8, all utilizing mean and standard deviation equations.

Using Table 4.7 and 4.8, we plot the chart illustrating the comparisons of both returns and risks of the portfolio and the GMI. In Figure 2, as stated in modern portfolio theory, the GMI outperformed the portfolio returns at the initial stage; while on the long run, most especially during the turbulent period of the global financial crisis of 2008-2009, the portfolio proved more effective in returns maximization. Looking at Figure 3, the GMI is clearly nowhere near the portfolio in terms of risk minimization.

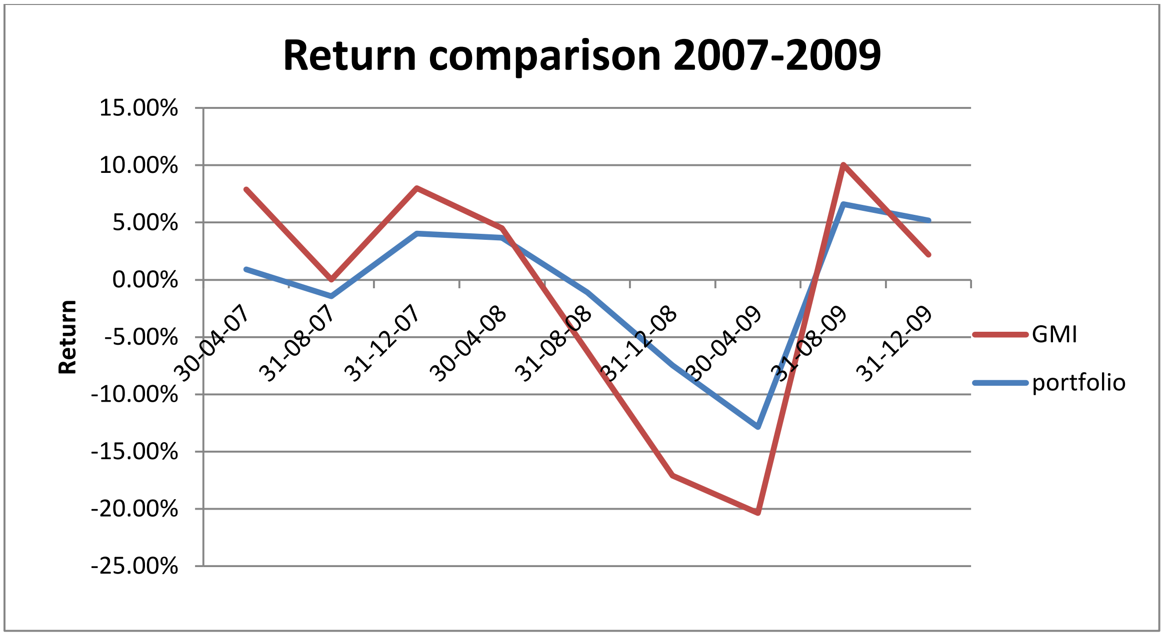

Figure 4. 2: Return Comparison

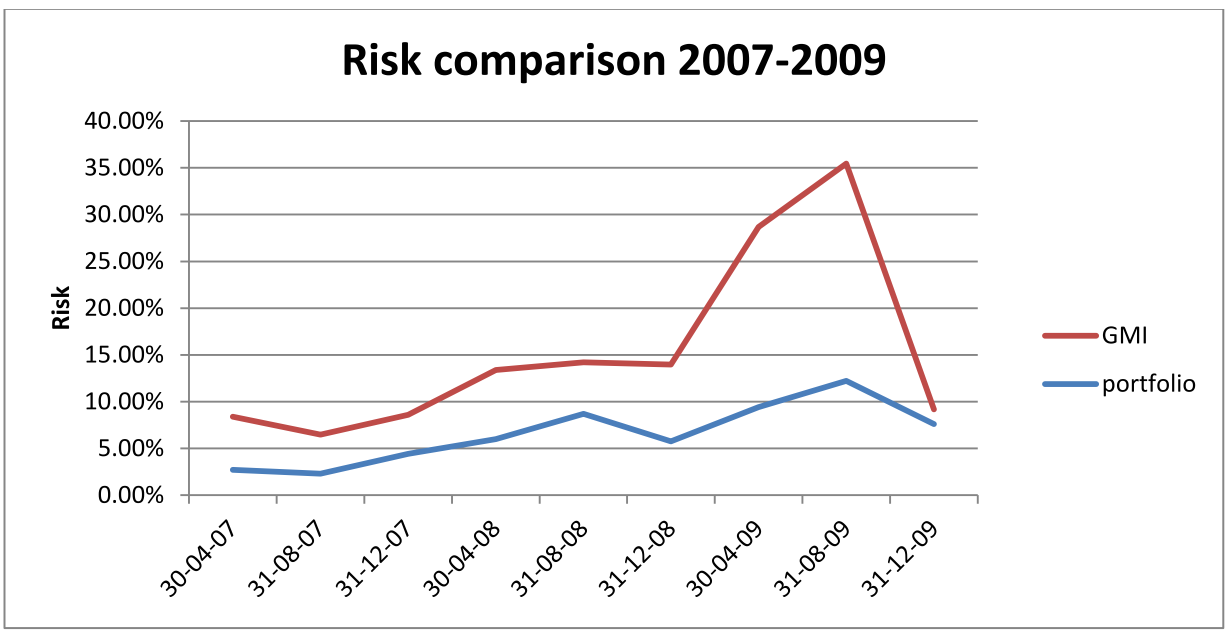

Figure 4. 3: Risk (standard deviation) Comparison

Based on the result obtained, we can confidently say that Markowitz diversification strategy provides the investor with better risk/return characteristics in the Nigeria Stock Exchange than investing in a passive index, especially in the turbulent period of the global financial crisis of 2008 seen in Table 2 and Figure 2. As expected at the initial stage, the GMI is ahead of the portfolio with attractive returns, but on the long run the portfolio outperformed the index; an indication of the power of portfolio diversification across different sectors of the Nigeria capital market.

On the other hand, as shown in Table 3 and Figure 3, the GMI is nowhere near the portfolio in terms of risk minimization during the whole period we have chosen. I chose that turbulent period of the global financial crisis as most investors are risk averse.

Conclusively, from our findings, with a careful and outstanding uncorrelated asset allocation in different sectors of the Nigeria Capital Market; low volatility investing would yield more attractive risk/return characteristics than using the naïve strategy traditional index investing.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary and Conclusion

This study has discovered that an optimal risky portfolio influenced by Modern Portfolio theory actually provides the investor with a higher return as illustrated in the efficient frontier (figure 4-1). Investing merely in a passive index cannot be considered an efficient investment alternative. However, the research questions also call for an order analysis of the portfolios as the solver can only brings out one of several optimal risky portfolios depending on the set mean return . This would greatly help in knowing those that are better.

During this study, portfolio compositions are constructed after estimating means, variances, covariances between assets coupled with Gini also as a measure of risk. This highlights the problem solved by Modern Portfolio Theory on the dilemma of how investor allocates resources among various assets alternatives.

To conclude, the portfolio approach out rightly outperformed the naïve index investing approach, evidently shown in figure 3, especially the turbulent period 2008 to early 2009. The risk in the MV framework is measured in terms of the variance of expected portfolio returns. The underlying assumption of using variances as the appropriate measure for risk is that investors weigh the probability of negative returns equally against positive returns. As argued by a number of scholars, variance is a measure that captures both the upside and downside movements of a security's returns (Fishburn, *et al,* 1977). Thus using a risk measure that strictly captures only the downside disappointment would greatly improve on this study.

5.2 Recommendation

From the finding in this study, individuals and institutional investors are recommended the followings:

- Investors should spread their investment by using the GMD or MV diversification strategy in order to minimize risk and maximize returns.
- Investors should utilize all available information about prospects before embarking on investment decisions and also employ active over passive management style.
- Investors are advised to seek the opinion of experts before committing their funds in the market.
- More ruthless investors could still adopt Markowitz and GMD strategies since they possess the skills to do so.

5.3 Areas for future research

To construct an optimal risky portfolio, that is well fully diversified the global view can be considered. It is acknowledged in research (Bodie *et al.*, 2004) that a globally diversified portfolio will provide the investor with even more valuable diversification opportunities, thereupon it would interest to investigate if a portfolio with globally diversified assets would show a different pattern of the result.

5.4 Contribution to Knowledge

The calculated diversified weights help in achieving a good results targeted at helping both risk averse individual and institutional investors willing to invest in the Nigeria Capital Market. Also, it will help researchers interested in the financial time series and risk management serving as meat and potatoes for more advanced developments. These contributions include;

- Modern Portfolio Theory also provide better risk/return characteristics in difficult market condition, evidently shown in return/risk comparison graphs.
- Helps in resource allocation in various groups of assets alternatives.
- Highlighting the importance of rebalancing, diversification and active management style.

REFERENCE

Agouram, J and Ghizlane L. (2015). A Comparative Study of Mean-Variance and Mean-Gini Portfolio Selection Using Var and CVar, *Journal of Financial Risk Management*, 4 : 72-81.

Andrew, (2002) The Statistics of Sharp Ratio. *Financial Analyst Journal*. 58 (4), 7-15.

Bhuyan, R., Kuhle, J., Ikromov, N. and Chiemeke, C. (2014) Optimal Portfolio Allocation among REITs, Stocks, and Long-Term Bonds: An Empirical Analysis of US Financial Markets. *Journal of Mathematical Finance* , 4, 104-112.

Boah, D.K., Adu, I. and Tulasi, V. (2015) Optimal Portfolio Mix for Multigrow Insurance Company in Ghana Using Linear Programming. *International Journal of Mathematics and Statistics Studies* , 3, 11-17.

Bodie, Z., Kane, A., and Marcus, A.J. (2004). *Essentials of Investments – 5th edition.* McGraw-Hill.

Bomfim, A. (2001) *Optimal Portfolio Allocation in a World without Treasury Securities*. Finance and Economics Discussion Series. https://www.federalreserve.gov/pubs/feds/2001/200111/200111pap.pdf

Chandra, S. and Shadel W.G. (2007) Crossing Disciplinary Boundaries: Applying Financial Portfolio Theory to Model the Organization of the Self-Concept. *Journal of Research Personal*, 41, 346-373. http://dx.doi.org/10.1016/j.jrp.2006.04.007

Cohen M. H and Natoli V. D. (2003). Risk and Utility in Portfolio Optimization. *Physica A: Statistical Mechanics and its Applications*. Volume 324, Issues 1-2, 81-88.

Dorfman, R., (1979) A Formula for the Gini Coefficient. *Review of Economics and Statistics*, 61:146-149.

D'Ambrosio, C.A (1976) *Principles of Modern Investments*. Science Research Associates Inc., Toronto.

Elton E.J and Gruber M. J. (1997). Modern Portfolio Theory 1950 to Date. *Journal of Banking & Finance 21*, 1743-1759.

Fishburn, P.C. (1977). Mean-Risk Analysis with Risk Associated with Below-Target Returns, *The American economic Review*,67(2), 116-126.

Gao, W. (2004). *Portfolio Optimization Based on Robust Estimation Procedures*. M.S. Thesis for Professional Masters Degree in Financial Mathematics, Worcester Polytechnic Institute.

Gibson, R. (2000). *Asset Allocation : Balancing Financial Risk*. Blacklick, OH, USA: McGraw-Hill Professional Book Group.

Hadar, J., and Russell, W. R., (1969) Rules for Ordering Uncertain Prospects. *American Economic Review*, 59: 25-34.

Hagstrom, R. G. Jr. (2001). The Essential Buffett: Timeless Principles for the New Economy. John Wiley & Sons, New York.

Ibbotson, Roger G., Paul D. Kaplan. (2000). Does Asset Allocation Policy Explain 40, 90, or 100 Percent of Performance. *Financial Analysts Journal*, 56(1):26–33.

Jaaman, S.H., Lam, W.H., Isa, Z. (2011a). Higher Moment Portfolio Management with Downside risk. *American Journal of Social and Management Sciences*. 2,2: 220-224.

Jaaman, S.H., Lam, W.H., Isa, Z. (2011b). Different Downside Risk Approaches in Portfolio Optimisation. *Journal of Quality Measurement and Analysis.* 7,1: 77-84.

Kisaka, S., Mbithi. J. and Kitur, H. (2015) Determining the Optimal Portfolio Size on the Nairobi Securities Exchange. *Research Journal of Finance and Accounting*, 6, 215-229.

Kristian K., Carl L., and Jacob P. (2006). *Practical Application of Modern Portfolio Theory*. Unpublished BSc thesis. Jonkoping Internation Business School, Jonkoping University.

Manganelli, R. L. (2002). *Solving the Corporate Value Enigma : A System to Unlock Stockholder Value*. New York. AMACOM.

Markowitz H. M. (1959). *Portfolio Selection: Efficient Diversification of Investments*. John Wiley & Sons, New York.

Markowitz, H. (1987). *Mean-variance Analysis in Portfolio Choice and Capital Markets*. Basil Blackwell, New York.

Markowitz, H. (1970). *Portfolio Selection: Efficient Diversification of Investments,* Yale University Press.

Markowitz, H.M. (1952). Portfolio Selection. *Journal of Finance* 7, 77-91.

Markowitz H. M. (1991a). *Portfolio Selection - 2nd Edition*. John Wiley & Sons, New York.

Markowitz H. M. (1991b). Foundations of Portfolio Theory. *The Journal of Finance. 46*,2: 469-477.

Markowitz H. M., and Levy H. (1979) Approaximating expected utility by a function of Mean and Variance. *American Economic Review*, 69, 3.

Merton, C. R. (1972). An Analytic Derivation of the Efficient Portfolio Frontier. *Jornal of Financial and Quantitative Analysis*, 7(4), 1851-1872.

Nwakanma C. P., and Gbanador M. A. (2014) Talmud and Markowitz Diversification Strategies:" *Accounting and Finance Research*, 3(2):145-152.

Offiong, I. A., Riman, B. H., and Eyoanwan, E. E. (2016) Determining Optimal Portfolio in a Three-Asset Portfolio Mix in Nigeria. *Journal of Mathematical Finance*, 6, 524-540

Ogryczak, W. and Ruszczynski, A. (1998) *On Stochastic Dominance and Mean-Semi Deviation Models*. Research Report RRR7-98, RUTCOR, Rutgers University, Piscataway NJ.

Pulley, L.B. (1981) A General Mean-Variance Approximation to Expected Utility for Short Holding Periods. *Journal of Financial and Quantitative Analysis* , 16, 361-373. http://dx.doi.org/10.2307/2330243

Ross, S.A; Westerfield, R.W & Jordan, B.D. (2008). *Corporate Finance Fundamentals.* McGraw-Hill/Irwin. New York.

Saunders A. and Cornett M. M. (2006). *Financial Institutions Management: A risk Management Approach – 5th ed*. Irwin McGraw-Hill.

Shali, H. and Yitzhaki, S. (1984) Mean-Gini, Portfolio Theory, and the Pricing of Risky Asset. *Journal of Finance*, 39: 1449-1468.

Shalit, H. and Yitzhaki, S. (2005) The Mean-Gini Efficient Portfolio Frontier. *Journal of Financial Research* 1: 59-75.

Sharpe, W.F. (1995) *Investment*. 6th Edition, United Kingdom.

Sharpe W. F. (2000). *Portfolio Theory and Capital Markets*. McGraw-Hill, New York.

Ulucan, A. (2007) An Analysis of Mean Variance Portfolio Selection with Varying Holding Periods. *Applied Economics* , 39, 1399-1407. http://dx.doi.org/10.2307/2330243

Vigdis B., Emil B. and Zhao Z. (2011). Portfoio Optimization in a mean-semi variance framework. *Investment Management and Financial Innovations,* 8: 58-72

Yahaya, A,, Abubakar, A. H. and Garba, J.(2011) Statistical Analysis on the Advantages of Portfolio Diversification. *International Journal of Pure and Applied Sciences and Technology*, 7(2): 98-106.

Yahaya, A. (2012) On Numerical Solution for Optimal Allocation of Investment Funds in Portfolio Selection Problem. *CBN Journal of Applied Statistic*, 3,2:1-7

Yahaya, A. (2015) Statistical Analysis for Obtaining a Minimum Risk Portfolio in Two-Asset Portfolio Mix. *Nigeria Journal of Scientific Research*, 14(2), 1-15.

Yamazaki, H. and Konno, H. (1991) Mean-Absolute Deviation Portolio Optimization Model and its Application to Tokyo Stock Market. *Journal of Management Science,* 37(5) : 519-531.

Yitzhaki, S. (1982) Stochastic Dominance, Mean-Variance, Gini-Mean Difference. *American Economic Review*, 2: 178-185.

Yitzhaki, S. (2003) Gini's Mean Difference: A Superior Measure of Variability for non-normal Distributions. *International Journal of Statistics* 2: 285-316.

Yitzhaki, S.,(1984) On an extension of the Gini inequality index. *International Economic Review*, 24: 617–28.

APPENDIX

The Gini Coefficient

From the approach of Shalit and Yitzhaki (1984), we define Z_1 and Z_2 to be pair of random returns that is drawn from continuous distribution, f(z) and F(z) its density and cumulative probability density functions correspondingly and distribution extending over z=a to z=b, b>a, we get

$$\int_a^b f(z)\,dz = \int_{z=a}^{b} dF(z) = 1 \quad (113)$$

and

$$F(z) = \int_a^z f(z)\,dz \quad (114)$$

Meaning F(a)=0 and F(b)=1. If E(.) is expected value of variable (.), then Gini coefficient is given as:

$$\Gamma = \frac{1}{2} E\left(\left|Z_1 - Z_2\right|\right) \quad (115)$$

Since expected value involving absolute value difference prove to be very difficult to work with, we will follow the approach of Dorfman (1979) to state the Gini coefficient in a similar and explanatory form. An important expression that was used in the research is

$$\left|Z_1 - Z_2\right| = Z_1 + Z_2 - 2\min\left(Z_1, Z_2\right) \quad (116)$$

min(Z_1, Z_2) is the minimum between Z_1 and Z_2. Suitably, we get

$$\Gamma = \frac{1}{2}\left\{E(Z_1) + E(Z_2) - 2E\left[\min(Z_1, Z_2)\right]\right\} \tag{117}$$

where

$$E(Z_1) = E(Z_2) = \int_a^b zf(z)\,dz = \int_{z=a}^b z\,dF(z) \tag{118}$$

represents the distribution mean, μ. In other for us to be able to work with the Gini coefficient, we need to find E[min(Z_1, Z_2)] in terms of f(z) or F(z).

For this huge work we are embarking upon, we will consider z from the distribution f(z) or, similarly, F(z) being the cumulative distribution. The chance that "both Z_1 and $Z_2 > z$" and the chance that "atleast one of Z_1 and $Z_2 < z$" must be equal to 1 because the two probabilities is covering all outcomes for each of the given value of z.

$$\Pr(Z_1 \le z) = \Pr(Z_2 \le z) = F(z) \tag{119}$$

The chance that "Z_1 and Z_2 are both > z" is

$$\Pr(Z_1 > z) \times \Pr(Z_2 > z) = \left[1 - F(z)\right]^2 \tag{120}$$

Keeping in mind that "at least one of Z_1 and Z_2 is < z" is the equivalent as the chance that "minimum of Z_1 and Z_2 is < z" we get

$$\Pr\left[\min(Z_1, Z_2) \le z\right] = 1 - \Pr(Z_1 > z) \times \Pr(Z_2 > z) = 1 - \left[1 - F(z)\right]^2 \tag{121}$$

The chance Pr[min(Z_1,Z_2) ≤ z] can be looked as the value of the G(y) which is the cumulative distribution of the variable y=min(Z_1, Z_2), at y=z. the variable is extending over y=a to y=b, then

$$E\left[\min\left(Z_1, Z_2\right)\right] = \int_{y=a}^{b} y dG\left(y\right) \tag{122}$$

which is equal to

$$\int_{z=a}^{b} z dG\left(z\right) = \int_{z=a}^{b} z d\left\{1-\left[1-F\left(z\right)\right]^2\right\} = 2\int_{z=a}^{b} z\left[1-F\left(z\right)\right] dF\left(z\right) \tag{123}$$

Therefore Gini coefficient is now

$$\Gamma = \mu - 2\int_{z=a}^{b} z\left[1-F\left(z\right)\right] dF\left(z\right) = 2\int_{z=a}^{b} z\left[F\left(z\right)-\frac{1}{2}\right] dF\left(z\right) \tag{124}$$

Knowing that

$$E\left[F\left(z\right)\right] = \int_{z=a}^{b} F\left(z\right) dF\left(z\right) = \frac{1}{2}\left\{\left[F\left(b\right)\right]^2 - \left[F\left(a\right)\right]^2\right\} = \frac{1}{2} \tag{125}$$

$$\int_{z=a}^{b} \left\{F\left(z\right) - E\left[F\left(z\right)\right]\right\} dF\left(z\right) = 0 \tag{126}$$

and

$$\int_{z=a}^{b} E\left(z\right)\left\{F\left(z\right) - E\left[F\left(z\right)\right]\right\} dF\left(z\right) = E\left(z\right)\int_{z=a}^{b} \left\{F\left(z\right) - E\left[F\left(z\right)\right]\right\} dF\left(z\right) = 0 \tag{127}$$

Equivalently written as

$$\Gamma = 2\int_{z=a}^{b} \left[z - E\left(z\right)\right]\left\{F\left(z\right) - E\left[F\left(z\right)\right]\right\} dF\left(z\right) \tag{128}$$

Since the covariance of two the variables is the expected value of the multiplication of their deviations from their means, this expression is equal to

$$\Gamma = 2\operatorname{cov}\left[z, F\left(z\right)\right] \tag{129}$$