# ADJOINT OPERATORS ON A HILBERT SPACE

BY

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## DEDICATION

This work is dedicated to my parents:

MR JULIUS S. SHILGBA

MRS JUSTINA N. SHILGBA

for their encouragement and efforts at making me what I presently am within their God-given limits.

The ultimate dedication is to the Almighty God, for opportunities, life and enablements He has providentially bestowed on my person.

## DECLARATION

It is hereby declared that the whole content of the work now submitted, for the M.Sc Mathematics of Ahmadu Bello University, Zaria is the result of the author's research except where references are made to published literature, which are duly acknowledged. This work has not been part of any presentation for any other qualification.

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CERTIFICATION

This thesis entitled "Adjoint Operators on a Hilbert Space" by Leonard Karshima Shilgba meets the regulations governing the award of the Master of Science (M.Sc) of Ahmadu Bello University and is approved for its contribution to scientific knowledge and literary presentation.

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L. K. S.

## ABSTRACT

Hilbert spaces are some of the ABSTRACT SPACES of interest in the abstract branch of mathematics called FUNCTIONAL ANALYSIS,

The aim of the present work is to isolate linear operators on these abstract spaces (HILBERT SPACES) with the singular property of SELF-ADJ0INTNE3S for investigation. Algebraic and analytic properties of such operators come under survey. The SPECTRAL THEORY vis-a-vis these operators is explored. Furthermore, the approximate point spectrum ( $o_a\ (T)$ ) of a self

adjoint operator T on a non-trivial separable Hilbert space examined with some proposition relating it to the POINT (EXACT POINT) spectrum ( $o_a(T)$ ) of T

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# CHAPTER ONE

## 1.0 INTRODUCTION

In the study of linear operators on a Hilbert space, (complete inner product space - complete with respect to the induced norm) it is possible and fascinating to isolate a class of operators for study and wider investigation. In this work, in particular, operators with the singular property of SYMMETRY or SELF-ADJOINTNESS will be isolated and investigated in greater depth than arbitrary operators in BANACH SPACES (which Hilbert spaces are, but much more). These operators (with symmetry property) play a vital role in analysis and theoretical physics.

Here in this write-up, we shall examine the concept ADJOINT OPERATORS both as it relates to BOUNDED and UNBOUNDED operators, but in both cases, LINEAR. Furthermore, we shall concentrate our study on both finite dimensional and infinite dimensional Hilbert spaces.

In this work, effort will be made to investigate both ALGEBRAIC and ANALYTIC properties of operators under survey and study. This style gains attention from the unavoidable fact that, in FUNCTIONAL ANALYSIS (from which this study originates), best results are when we marry the ALGEBRAIC and ANALYTIC structures together. Needless, to say that every normed space - by definition - has a built-in algebraic structure.

Mention should be made from the on set that of all ADJOINT

OPERATORS that will be under focus in this intellectual effort, only one or two will attract elaborate attention. Interesting theorems will be stated, some proved while others left out. Those left without proof will only be relied upon as supporting poles to build up relevant results. Though this effort lays no claim to originality or seminality of thought, however, care in simplifying layout of proofs and explanation of important pertinent concepts, without loss of originality will be taken. Moreover, attempts at fusion of inter-related ideas as gathered from available books and papers will not be ignored.

Besides, we shall explore SPECTRAL THEORY of ADJOINT OPERATORS in HILBERT SPACES, but only in so far as it relates to SELF-ADJOINT operators. Generally speaking, this theory is concerned with certain inverse operators, viz:

(i) their general peoperties,

(ii) their relations to the original operators.

Such inverse operators arise quite naturally in connection with the problem of solving equations (systems of linear algebraic equations, differential equations, integral equations) just like the mathematical quantities - $i, \pi, e$ , appear quite naturally and sometimes unexpectedly in analytical problem-solving. Infact, the quantities - $i, \pi, e$ can be called the TRINITY OF MATHEMATICS, and without them it would be extremely difficult to tackle some problems, not only in mathematics, but

in other branches of physical science. This conveys the importance, and may we say, the indispensability of certain inverse operators in analysis.

As we shall see, the spectra theory of operators is very important for an understanding of the operators themselves.

Let us now settle down to the pot of porridge before us.

## 1.1 CONCEPT OF ADJOINT OPERATORS

As earlier stated, we shall examine ADJOINT OPERATORS both bounded and unbounded.

**CASE A:**

**ADJOINT OF BOUNDED LINEAR OPERATORS IN A HILBERT SPACE:**

This operator was suggested by problems in matrices and linear differential and integral equations and also helps to define three important classes of operators (called SELF-ADJOINT, UNITARY and NORMAL operators) which have been extensively studied because of the key role they play in various applications.

**DEFINITION: (HILBERT-ADJOINT OPERATORS)**

Let H be a Hilbert space and T a bounded linear operator on H, with range in the same space [we can have two different spaces $H_1$, $H_2$ say, such that domain of T and its range can be in one or the other].

Now, consider a linear functional $f_y(x) = \langle Tx,y \rangle$ for all x (in H) and a fixed y (in H).

As a linear functional in the Hilbert space H, defined above,

Hence, there exists a unique $y^*$ (in H) such that

$$f_y(x) = \langle Tx, y\rangle = \langle x, y^*\rangle \text{ for all } x \text{ (in H)}$$

It is obvious that $f_y$ varies with y, and consequently, $y^*$ too; and then we get the operator $T^*$ defined by

$$T^*y = y^*$$

defined on H with range in the same space. The operator $T^*$ is associated with T by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \qquad .....(1.1)$$

and is called the ADJOINT OPERATOR of T in H

**UNIQUENESS** $T^*$:$T^*$ is <u>uniquely</u> defined by (1.1)

Suppose for all x, y (in H)

$\langle Tx, y\rangle = \langle x, T_1y\rangle = \langle x, T^*y\rangle$, then

$$\langle x, (T_1 - T^*)y\rangle = 0 \text{ for all } x, y \text{ (in H)}$$

or $(T_1 - T^*)y = 0$ for all y (in H)

or $T_1 - T^* = 0$

or $T_1 = T^*$

Hence, $T^*$ is uniquely defined.

Thus, we have the theorem:

**THEOREM 1.0:**

The Hilbert operator $T^*$ of T given by (1.1) above exists uniquely. Moreover,

$$\|T\| = \|T^*\|$$

**PROOF:** The first part has been proved above. We shall only show that $\|T\| = \|T^*\|$ .

Define.

$h(y,x) = \langle y, Tx \rangle$ $x,y$ (in H), which is a sesquilinear form on H x H because the inner product is sesquilinear and T is linear. Infact, conjugate linearity of the form is seen from

$$h(y, \alpha_1 x_1 + \alpha_2 x_2) = \langle y, T(\alpha_1 x_1 + \alpha_2 x_2) \rangle$$

$$= \langle y, \alpha_1 T x_1 + \alpha_2 T x_2 \rangle$$

$$= \overline{\alpha_1} \langle y, T x_1 \rangle + \overline{\alpha_2} \langle y, T x_2 \rangle$$

$$= \overline{\alpha_1}\, h(y, x_1) + \overline{\alpha_2}\, h(y, x_2)$$

h is bounded, for by Schwartz inequality,

$$|h(y,x)| = |\langle y, Tx \rangle|$$

$$\leq \|y\| \, \|Tx\|$$

$$\leq \|y\| \|x\| \|T\|$$

This implies that $\|h\| \leq \|T\|$ . Furthermore,

$$\|h\| = \sup_{\substack{X \neq 0 \\ Y \neq 0}} \frac{|h(y,x)|}{\|x\| \, \|y\|} = \sup_{\substack{X \neq 0 \\ Y \neq 0}} \frac{|\langle y, Tx \rangle|}{\|x\| \, \|y\|}$$

$$\geq \sup_{\substack{x \neq 0 \\ Tx \neq 0}} \frac{|\langle Tx, Tx \rangle|}{\|x\| \, \|Tx\|}$$

$$= \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

$$= \|T\|$$

Therefore $\|h\| \geq \|T\|$

Together, $\|h\| = \|T\|$

From Riesz representation theorem for bounded sesquilinear form h, writing $T^*$ for a bounded linear operator on H, we have

$$h(y,x) = \langle T^*y, x \rangle = \langle y, Tx \rangle$$

That is, $\|h\| = \|T^*\|$ [We have already the benefit of the Riesz representation theorem for a bounded sesquilinear form h]

Also, $\|h\| = \|T\|$

Hence, $\|T\| = \|T^*\|$

CASE B:

HILBERT - **ADJOINT OF UNBOUNDED OPERATORS ON A HILBERT SPACE:**

Now, let us consider defining the concept ADJOINT OPERATOR in the case whereby we have an arbitrary linear operator T (nothing is said about its boundlessness) from $H_1$ into $H_2$ being HILBERT SPACES. That is

$$T;\ H_1 \to H_2$$

It may be that for some y (in $H_2$), we have the representation

$\langle Tx, y\rangle = \langle x,z\rangle$, for some fixed z (in $H_1$), and for all x (in $D_T \subset H_1$), where DT stands for the domain of T, and is taken to be dense in $H_1$

We denote by $H^*$ the set of all such y in $H_2$.

That is, $H^* \subseteq H_2$

Now, define the operator $T^*$ from $H_2$ into H,

$$T^*: H_2 \to H_1$$

with domain of definition $D_{T^*} = H^* \subseteq H_2$ , by the formula

$T^*y = z(\text{in } H_1)$ .....(1.2)

That is, for each y (in $H^*$) there exists z(in $H_1$) such that (vv) holds.

This operator $T^*$ is called the ADJOINT (HILBERT ADJOINT) OPERATOR of T.

We must say that the vector z (in $H_1$) is UNIQUELY defined by the vector y (in $H^*$). For suppose we also had

$(Tx,y) = \langle x,z\rangle = \langle x,z'\rangle$ z' (in $H_1$). Then,

$\langle x, z - z'\rangle = 0$ for all x (in $D_T \subseteq H_1$ )

This implies the vector $z\,z'$ (in $H_1$) is orthogonal to the domain of T (DT). But since DT is dense in $H_1$, then there exists a sequence $(x_n)$, $x_n$ (in DT) such that

$$z - z' = \lim x_n$$

That is, $\langle x, z - z'\rangle = \langle x, \lim x_n\rangle$

$= \lim \langle x, x_n\rangle$ [continuity of inner product]

$= 0$ for all x(in DT)

Therefore, $\langle x,x_n\rangle = 0$ for all $x_n$

Then, $x_n = 0$ for all n and thus, $z - z' = 0$

If and only if $Z = Z'$

REMARK: Here, we see that the initial assumption on the density of $D_T$ in $H_1$ is very important; for in the contrariwise, it would be impossible to define the ADJOINT OPERATOR uniquely. Hence, we have the statement: The adjoint of an operator on a Hilbert space exists if and only if it is either bounded or its domain is dense.

In short, we can write:

$\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all x (in $DT \sqsubseteq H_1$),

y (in $DT^* = H^* \sqsubseteq H_2$)

$\langle Tx,y\rangle$ is an inner product in $H_2$, while $\langle x, T^*y\rangle$ is an inner product in $H_1$.

Thus, we have considered the definition of HILBERT-ADJOINT operator of an operator on a Hilbert space - bounded or unbounded. In the unbounded case, the density of the domain of definition is essential.

## 1.2 SOME EXAMPLES OF ADJOINT OPERATORS ON SOME HILBERT SPACES

(I) Let H be an arbitrary Hilbert space and I: H → H the identity operator.

Now define T: H → H by

$T = 2iI$

then $T^* = -2iI$

This is clear, since

$$\begin{aligned}\langle Tx,y\rangle &= \langle 2ix,y\rangle \\ &= \langle x, -2iy\rangle \\ &= \langle x, T^*y\rangle\end{aligned}$$

(II) Let H be the unitary space $\mathbb{C}^n$ (of finite dimension n).

If $T: \mathbb{C}^n \to \mathbb{C}^n$ is a linear operator represented by the

matrix $T = [T_{ij}]$, then $T^*$ is associated with the complex conjugate of the transpose of T, such that $T^* = [\overline{T}_{ji}]$

The above examples are for finite dimensional cases, particularly; now, let us consider an example on infinite-dimensional Hilbert space.

(III) Here, we shall consider an example of a Hilbert-Adjoint operator of a matrix operator on an infinite-dimensional Hilbert space H.

Let us consider the real Hilbert sequence space $l^2$ which is infinite dimensional.

For x, y (in $l^2$)

$$\mathbf{x} = (x_1, x_2, x_3, \ldots)$$

$$\mathbf{y} = (y_1, y_2, y_3, \ldots)$$

$x_i, y_i$ (in $\mathbf{R}$ )

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

$(e_n)$ are sequences all of whose terms are zero except the nth term, which is 1.

Then, the set $\{e_1, e_2, \ldots\}$ forms a basis for $l^2$.

Now, define the operator $T: l^2 \to l^2$ by $Te_n = e_{n+1}$ and

$T^*: l^2 \to l^2$ by $\qquad n = 1, 2, \ldots$

$T^{\dagger}\ em = e_{m-1}$

$m = 2, 3, \ldots$

$$\langle Te_n, e_m \rangle = \langle e_{n+1}, e_m \rangle = \delta m, n+1 = \begin{matrix} 0 \text{ if } m \neq n+1 \\ 1 \text{ if } m = n+1 \end{matrix}$$

$$\langle e_n, T^{\dagger} e_m \rangle = \langle en, e_{m-1} \rangle = \delta n, m-1 = \begin{matrix} 0 \text{ if } n \neq m-1 \\ 1 \text{ if } n = m-1 \end{matrix}$$

But m = n+1 if and only if $n = m - 1$ . Therefore

$\delta m, n+1 = \delta n, m-1$ .

Thus, $T^{\dagger}$ is the Hilbert-Adjoint of T.

$$T \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ \cdot \\ e_n \\ \cdot \\ \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} e_2 \\ e_3 \\ \cdot \\ \cdot \\ \cdot \\ e_{n+1} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

This implies that

$$T = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & . & . & . & . & . \\ 0 & 0 & 1 & . & . & 0 & . & . & . & . & . \\ 0 & 0 & 0 & 1 & 0 & . & . & . & 0 & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \end{bmatrix}$$

$$T^* = \begin{bmatrix} 0 & 0 & . & . & 0 & . & . & . & . & . \\ 1 & 0 & 0 & . & . & 0 & . & . & . & . \\ 0 & 1 & 0 & . & . & . & . & . & 0 & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \end{bmatrix}$$

(IV) We shall now look at an example of Hilbert Adjoint operator for an integral operator.

Let us consider in $L^2$ (0,1) the Fredholm operator

$$y = Kx(t) = \int_0^1 K(t,s)\,x(s)\,ds \qquad \ldots(1.3)$$

with continuous Kernel K(s,t) on the square [0,1] x [0,1]. An arbitrary linear functional f(y) on $L^2$ (0,1) has the form

$$f(y) = \int_0^1 f(s)\; y(s)\,ds \quad f \text{ (in } L^2 \text{ (0,1))}$$

Therefore,

$$f(Kx) = \int_0^1 f(s)\,\left(\int_0^1 K(t,s)\,x(t)\,dt\right) ds$$

$$= \int_0^1 x(t)\,\left(\int_0^1 K(s,t)\,f(s)\,ds\right) dt$$

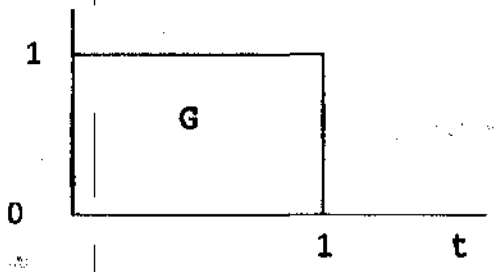
$$= \int_0^1 x(s)\left(\int_0^1 K(t,s)\,f(t)\,dt\right) ds$$

$$= \int_0^1 x(s)\,g(s)\,ds$$

with $\quad g(s) = \int_0^1 k(t,s)\,f(t)\,dt \quad (g = k^*f)$

In the above case, therefore, there corresponds an interchange of the variables in the kernel to arrive at the adjoint operator. The corresponding adjoint integral equation is the equation with the adjoint operator.

NOTE: The integral equation (1.3) above, is called a FREDHOLM EQUATION of the FIRST KIND. The kernel K of the equation is a given function on the square G = [0,1] x [0,1]




Domain of K(s,t) $(0 \leq s, t \leq 1)$

(V) Let $\tau$ be a differential operator of order n:

$$\tau = a_n(t)\left(\frac{d}{dt}\right)^n + a_{n-1}(t)\left(\frac{d}{dt}\right)^{n-1} + \ldots + a_0(t)$$

on I = [a,b] and f, g (in $L^2(I)$) such that f, g (in $C^0(I)$), $a_k$ (t) (in C (I)) k = 0,1, ...,n.

Consider the integral

$$\int_a^b (\tau f)(t)\ \overline{g(t)}\,dt$$

$$= \sum_{k=0}^{n} \int_a^b a_k(t)\left[\left(\frac{d}{dt}\right)^k f(t)\right] \overline{g(t)}\,dt \qquad \ldots(1.4)$$

We are to find $\tau^*$ such that

$$\langle \tau f, g\rangle = \langle f, \tau^* g\rangle$$

Integrating the Kth term on the right of (1.4) by parts K times, we see that

$$\int_a^b \left[\left(\frac{d}{dt}\right)^k f(t)\right] a_k(t)\ \overline{g(t)}\,dt$$

$$= -\int_a^b \left[\left(\frac{d}{dt}\right)^{k-1} f(t)\right]\left[\left(\frac{d}{dt}\right)\{a_k(t)\ \overline{g(t)}\}\right]dt$$

$$+ a_k(t)\overline{g(t)}\left(\frac{d}{dt}\right)^{k-1} f(t)\,\Big|_a^b = \cdots$$

$$= (-1)^k \int_a^b f(t)\left(\frac{d}{dt}\right)^k \{a_k(t)\ \overline{g(t)}\}\,dt$$

$$+ \sum_{i=1}^{k} (-1)^{i-1}\left[\left(\frac{d}{dt}\right)^{i-1}\{a_k(t)\ \overline{g(t)}\}\right]\left(\frac{d}{dt}\right)^{k-i} f(t)\,\Big|_a^b$$

Thus,

$$\int_a^b (\tau f)(t)\,\overline{g(t)}\,dt = \int_a^b f(t)\ \sum_{k=0}^{n} (-1)^k \left(\frac{d}{dt}\right)^k \{\,a_k(t)\ \overline{g(t)}\,\}dt$$

$$+ F_b(f,g) - F_a(f,g) \qquad \ldots(1.5)$$

where,

$$F_t(f,g) = \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{i-1} \left[ \left(\frac{d}{dt}\right)^{i-1} (a_k(t)\ \overline{g(t)}) \right] \left(\frac{d}{dt}\right)^{k-i} f(t)$$

However, by Leibniz's rule,

$$\left(\frac{d}{dt}\right)^{j} (a_k(t)\overline{g(t)} = \sum_{l=0}^{j} \binom{j}{l} \left[ \left(\frac{d}{dt}\right)^{j-l} a_k(t) \right] \left[ \left(\frac{d}{dt}\right)^{l} \overline{g(t)} \right]$$

It is hence, seen that the integral on the right in (2) may be written in the form:

$\int_a^b f(t)\overline{\tau^* g(t)}\,dt$ where $\tau^*$ is the operator

$$\tau^* = \sum_{j=0}^{n} b_j(t) \left(\frac{d}{dt}\right)^{j}$$

and

$$b_j(t) = \sum_{k=j}^{n} (-1)^k \binom{k}{j} \left(\frac{d}{dt}\right)^{k-j} \overline{a_k(t)}$$

because $\overline{\overline{a_k(t)}} = a_k(t)$

**ELEMENTARY PROPERTIES OF FORMAL DIFFERENTIAL OPERATORS**

From the point of view of applications, the most important single class of operators are the differential operators. The study of these operators is complicated by the fact that they are necessarily unbounded.

We are given-to begin with - a "formal differential operator" That is, an expression of the form:

$$\tau = a_n(t)\left(\frac{d}{dt}\right)^n + a_{n-1}(t)\left(\frac{d}{dt}\right)^{n-1} + \ldots + a_0(t)$$

There is an obvious sense in which such an expression can be "applied" to a function f. if say, f (in $C^n$). Thus, we can define an operator whose domain is $C^n$ (but whose range is not in $C^\infty$, only in C). We might however, have "applied" the formal differential operator to a larger or smaller class of functions to begin with, and defined operators with larger or smaller domains.

First, in order to be able to apply the abstract Hilbert - space theory, it is necesary to relate all operators to a

Hilbert space of square - integrable functions. Once this is done, the question of ADJOINTS can be expected to assume paramount importance. That is, having settled on any preliminary domain of functions to which the formal differential operator T can be applied, we must attempt to discover the adjoint of the resulting unbounded operator in Hilbert space. All these problems permit satisfactory solutions. The above example (V) gives us the adjoint operator required.

# CHAPTER TWO

## 2.1 SELF-ADJOINT OPERATORS:

As stated in the projected scope of this work, the class of SELF-ADJOINT operators will be of much interest.

A bounded linear operator T: H → H on a Hilbert space H is said to be SELF-ADJOINT or HERMITIAN if it coincides with its adjoint.

That is, $T = T^*$ or

$\langle Tx,y\rangle = \langle x, Ty\rangle$ for all x,y(in H).

We have other classes of linear operators of great practical importance that can be defined by the use of the Hilbert-adjoint operator:

Already, we have defined th concept SELF-ADJOINTNESS of a linear operator T on the Hilbert space H.

T: H → H,where H is a Hilbert space has the following important characterizations:

(1) $T = T^*$ means T is SELF-ADJOINT

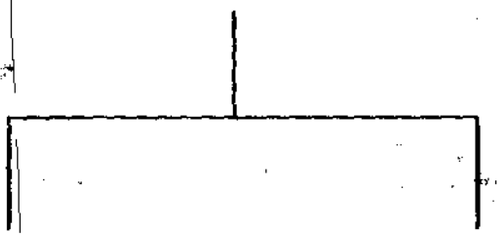
(2) T is said to be UNITARY if T is bijective and $T^* = T^{-1}$

That is $T^*T = TT^* = I$

(3) T is NORMAL if $T^*T = TT^*$

If T is self-adjoint or unitary, then it is trivially Normal.

Consider the line - diagram below:




**Line Diagram - Fig 2**

Of course, a normal operator need not be self-adjoint or unitary. Let's consider this example:

If I: H → H is the identity operator on the Hilbert space H, then

$T = 2iI$ is normal since $T^* = 2iI$, so that

$TT^* = T^*T = 4I$ and neither is $T^* = T$ nor

$T^* = T^{-1}$ $(= -\frac{1}{2}iI)$

Let us consider the following theorem:

**THEOREM 2.1:**

If T and A are self-adjoint operators on a Hilbert space H, then

i) $A + T$ and (ii) $\alpha T$ (*or* $\alpha A$) where $\alpha$ (in $\mathbb{R}$) are also self-adjoint operators.

**PROOF:** By definition, $\langle AX, y\rangle = \langle X, Ay\rangle$ and

$\langle TX, y\rangle = \langle X, TY\rangle$ for all X, y (in H).

But $(A + T)^* = A^* + T^* = A+T$

and $(\alpha T)^* = \overline{\alpha} T^* = \alpha T$ $(\overline{\alpha} = \alpha\ (in\ \mathbf{R}),\ T = T^*)$ Hence, the proof.

From the above theorem, we have the following generated result:

The set of all self-adjoint operators on a Hilbert space H forms a REAL NORMED LINEAR SPACE with the operator norm Topology.

If A and T are self-adjoint, then the composition AT, in general, is not self-adjoint, except when A and T commute. That is where AT=TA.

Thus, we can conclude that the set of commuting self-adjoint operators forms an ALGEBRA over the real field $\mathbf{R}$ .

## 2.2 CHARACTERISTIC OF SELF-ADJOINT OPEATORS:

**DEFINITION:** A self adjoint operator T is said to be POSITIVE or strictly POSITIVE in a Hilbert space H if $\langle X, TX\rangle \geq 0$ and $\langle X, TX\rangle > 0$ for all X(in H), respectively.

It should be noted that, from the definition of self-adjoint operators, it follows that if T is self-adjoint, then $\langle TX, X\rangle = \langle X, TX\rangle = \overline{\langle TX, X\rangle}$ for all X in the Hilbert space H. This implies that $\langle X, TX\rangle$is real-valued ... which is an important characteristic OF SELF -ADJOINT operators.

**THEOREM 2.2 (SEQUENCES OF SELF-ADJOINT OPERATORS)**

Let (Tn) be a sequence of bounded self-adjoint operators

(linear) Tn: H → H on a Hilbert space H. Suppose that (Tn) converges, say, Tn → T.

In other words, $\|Tn - T\| \to 0$ , where $\|\cdot\|$ is the norm on B(H,H) = space of all bounded linear operators on the Hilbert space H. Then the limit operator T is a bounded self-adjoint linear operator on H.

PROOF: We must show that $T^* = T$. This will follow from

$$\|T - T^*\| = 0$$

using the statement $\|Tn^* - T^*\| = \|(Tn - T)^*\|$

$$= \|Tn - T\|$$

and by triangular inequality

$$\|T - T^*\| = \|T - Tn + Tn - Tn^* + Tn^* - T^*\|$$

$$\leq \|T - Tn\| + \|Tn - Tn^*\| + \|Tn^* - T^*\|$$

$$= \|T - Tn\| + 0 + \|Tn - T\| = 2\|Tn - T\| \to 0 \quad (n \to \infty)$$

Hence, $\|T - T^*\| = 0$ and as such $T^* = T$.

The boundedness of T is clear from the boundedness of the sequence elements - since $\|T\| = \|\lim Tn\| = \lim \|Tn\|$

We have the following result from the above:

The real-normed linear space of self-adjoint operators on a Hilbert space H is closed in B(H,H).

It may be of interest to state that products (composites) of self-adjoint operators appear quite often in applications. Consequently, the following theorem will be useful.

**THEOREM 2.3: (SELF-ADJOINTNESS OF PRODUCT)**

The product of two bounded self-adjoint linear operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute.

**PROOF:**

We shall assume the use of an unproved but easy-to-be proved statment:

$(ST)^* = T^*S^* = TS$

Hence,

$ST = (ST)^*$ if and only if $ST = TS$

This is the needed proof.

**LEMMA 2.4:**

If T: H → H is a self-adjoint operator on Hilbert space H, then so also is $T^2, T^3, T^4 \ldots T^n, \ldots$

**PROOF:** Let X,Y be in H; and it is known that $T^* = T$ Then,

$$\begin{aligned} \langle T^2x, y\rangle &= \langle T(Tx), y\rangle \\ &= \langle Tx, T^*y\rangle \\ &= \langle x, T^{*2}y\rangle \\ &= \langle x, (T^*)^2y\rangle \\ &= \langle x, T^2y\rangle \end{aligned}$$

Hence, $(T^2)^* = (T^*)^2 = T^2$ and $T^2$ is also self-adjoint. We show by induction that $T^3$, $T^4$ ...$T^n$ ... are also self - adjoint. Suppose that $T^{n-1}$ is self-adjoint. Then,

$$(T^{n-1})^* = (T^*)^{n-1} = T^{n-1}$$

Hence,

$$\langle T^n x, y\rangle = \langle T^{n-1}(Tx), y\rangle$$

$$= \langle Tx, T^{n-1}y\rangle$$

$$= \langle x, T^*(T^{n-1}y)\rangle \quad (T = T^*)$$

$$= \langle x, T^n y\rangle$$

Therefore, $(T^n)^* = T^n = (T^*)^n$

It should be noted that if T: H → H is a bounded self-adjoint operator (linear) and $T \neq 0$, then $T^n \neq 0$ for every n (in **N**)

**NOTE:**

We can prove Theorem 2.2 for a sequence (Tn) of self-adjoint linear operators such that Tn → T where nothing is said about boundedness.

We prove even the stronger assertion:

If operators Tn are self adjoint and the sequence (Tn) is point wise convergent to the operator T, then T is also a self - adjoint operator. (usually, the pointwise convergence of linear operators in the Hilbert space is called STRONG CONVERGENCE)

Infact, because of the continuity of the inner product (scalar product), we have for any x,y (in H).

$\langle Tx,y\rangle = \langle \lim TnX,y\rangle$

$= \lim \langle Tnx,y\rangle = \lim \langle x, Tny\rangle$

$= \langle X, \lim Tny\rangle$

$= \langle X, Ty\rangle$

## 2.3 EXAMPLES OF SELF-ADJOINT OPERATORS

**EXAMPLES (I)** Let $H = \mathbb{C}^n$ (The n- dimensional space of complex numbers) and T be a linear operator on $\mathbb{C}^n$ represented by $T= [T_{ij}]$. Then, $T^*$ is associated with the complex conjugate of the transpose of T:

$T^* = [\overline{T_{ji}}]$ . T is self adjoint (Hermitian) if and only if

$\overline{T}ji = Tij$ .

**EXAMPLE (II):** The operator T: H → H, H being any Hilbert space, defined by

$Tx = \alpha x$ for all x (in H) and a fixed scalar $\alpha$ ,is Hermitian or self-adjoint if and only if $\alpha$ is real, since

$(\alpha T)^* = \overline{\alpha} T^*$ and

$T^t x = \overline{\alpha} x = \alpha x$

**EXAMPLE (III)**

The Fredholm operator

$$Au = \int_0^1 K(x,\epsilon)\, u(\epsilon)\, d\epsilon \ ,$$

A: $C^0(0,1) \to C^0(0,1)$ is self-adjoint if and only if the continuous Kernel $K(x, \epsilon) = K(\epsilon, x)$.

(More on this operator later)

**EXAMPLE (IV)**

It is a well-known result, essentially due to Weyl in his paper [27] that if the functions p and q are defined on the real axis $\mathbf{R}$, P>0 and q is bounded below, then the differential operator:

$$LU = -(PU')' + qU \quad \ldots(2.1) \text{ restricted to } C_0^\infty(\mathbf{R})$$

is essentially SELF-ADJOINT in the Hilbert space $L^2(\mathbf{R})$. This result is usually stated for the situation where P and q are

defined on $[0,\infty)$ , in which event, L will be in the limit point case. However, the two cases are equivalent via the Kodaira index relations.

A more detailed appraisal of this operator shall engage the appropriate attention of this work.

**EXAMPLE (V)**

We shall now consider the essential self-adjointness of shrodinger - type operators:

We shall consider the essential self-adjointness of the strongly elliptical oprators of the form

$$LU = -\sum_{j,k=1}^{n} (\partial j - ib_j)\, a_{jk}(\partial k - ib_k)\, u + qu \quad \ldots (2.2)$$ in

$\mathbb{R}^n$, $n \geq 1$ acting on $C_0^\infty(\mathbb{R}^n)$

where the matrix $(a_{jk})$ is real and symmetric, $b_j$ and q are real (q is a real potential), $a_{jk}$ and $b_j$ are sufficiently smooth and q (in $L^2_{10c}$)

Ural' ceva in his paper [24] has shown that weyl's result (stated in example IV) does not persist for higher dimensions. For $n \geq 3$ she has given examples with $b_k = 0$ and $q = 1$, which show that if the coefficents of the principal part rise too

rapidly, then the restriction of L to $C_0^\infty(\mathbb{R})$ may not necessarily be self-adjoint.

For n = 3, another example has been given by Laptev [19] for the case where $b_k = 0$, $q = 1$ and $a_{jk}(x) = a(x)\delta_{jk}$. This is indeed surprising since when $a_{jk}(x) = a(|x|)\delta_{jk}$, it can be rather easily shown that Weyl's theorem remains true.

Nevertheless, Weyl's conclusion for operators of the form (2.2) is valid for a large class of coefficients. For example, it is now a classical result that if the coefficients $a_{jk}$, $b_j$ and q are sufficiently smooth, upon setting

$$\lambda(r) = \sup\left\{\sum_{jk} a_{jk}(x)\xi_j\xi_k;\ |\xi| = 1,\ |x| = r\right\}$$

$$\text{If } \int_0^\infty \frac{dr}{\lambda(r)^{\frac{1}{2}}} = \infty \qquad \ldots\ldots(1)$$

and if q is bounded below, then the restriction of L to $C_0^\infty(\mathbb{R}^n)$ is essentially self-adjoint.

Considerably more proof is given in [12] and [23]

It should be noted that condition (2.3) above is a

condition on the growth of the principal coefficients at infinity; and self-adjointness of L is a result of this condition rather thanthe fact that the operator is being cosidered in $L^2(\mathbb{R}^n)$ . A similar condition on the growth of the principal coefficients at the boundary of a domain in $\mathbb{R}^n$ which may be bounded will also yield self-adjointness - see the following literature [1], [23], [26].

This is in contrast to Weyl's theorem where it is essential that the underlying domain has infinite Lebesgue measure.

Kato [16] has considered operators of the form (2.2) with

$$aj_k = \delta j_k, \quad b_k \text{ (in } C^1(\mathbb{R}^n)\text{ ) and } q \text{ (in } L^2_{loc}).$$

He has shown - among other things - that if $q(x) \geq -K|x|^2$ , then the minimal operator associated with L is self - adjoint. His principal new tool is a differential distributional inequality which allows him to work with a minimal smoothness condition on the potential, namely that, it is in $L^2_{loc}$.

It should be noted that A. DEVINATZ [9] showed that Weyl's theorem for the operator (2.2) is 'close' to being true in the following sense:

If the principal coefficients are" well-behaved" within a sequence of closed shells which go to infinity, and if q is bounded below in $L^2_{loc}$, then the minimal operator is self-adjoint. These conditions - generally speaking -do not impose any global growth restrictions, as,for instance, in (2.3),but rather impose only local growth restrictions within each of the afore mentioned shells. This result is almost an immediate consequence of kato's in equality and a generalised maximum principle for solutions of elliptic inequalities due to Littman [20].

Kato's inequality when taken in conjunction with the results developed by Littman appears to be the appropriate tool for handling the problems considered by A. DEVINATZ under a minimal smoothness assumption on the potential.

## 2.4. SPECTRAL ANALYSIS OF LINEAR SELF-ADJOINT OPERATORS.

Spectral analysis of linear operators is basically a study of the behaviour of linear operators with special regard to the existence of certain inverses. In particular, if T (in L (H,H)) [set of linear operators on Hilbert space H], and if $\alpha$ is a scalar, we are concerned with the existence of the inverse of the operator $(\alpha I - T)$ , where I is the identity operator.

In finite- dimensional spaces, the situation is clear - either $(\alpha I - T)^{-1}$

exists or it does not. If for a given scalar $\alpha$ , it does not exist, then $\alpha$ is called an EIGEN VALUE of T; and if dim (H) = n, $(n < \infty)$ , there are at most n (distinct) eigen values.

However, if H is infinite - dimensional, there may be infinitely many and indeed a continuum of scalars $\alpha$ such that $(\alpha I - T)^{-1}$ does not exist.

If $(\alpha I - T)^{-1}$ exists, the question arises:

Is it bounded or not? Moreover,whether its Domain (of $(\alpha I - T)^{-1}$ ) is dense in H. Neither of the questions arises in a finite - dimensional case.

These questions above considered are in the 'province' of the so-called SPECTRAL THEORY of linear operators.

The problem of finding scalars $\alpha$ (in $\mathbb{C}$ ) such athat there exists x in domain of T ( $x \neq 0$ ), satisfying the equation $(\alpha I - T)x = 0$ is called an EIGEN-VALUE PROBLEM associated with

the linear operator.

Our emphasis here - to an elaborate extent - is to consider the SPECRAL THEORY with respect to SELF-ADJOINT OPERATORS on a Hilbert space H.

**CASE 1: BOUNDED SELF-ADJOINT LINEAR OPERATORS. (ON FINITE DIMENSIONAL SPACES)**

**THEOREM 2.5 (EIGEN VALUES, EIGEN VECTORS)**

Let $T: H \to H$ be a bounded self-adjoint linear operator on a complex Hilbert space H Then,

(i) all the eigen values of T (if they exist) are real

(ii) eigen vectors corresponding to (numerically) different eigen values of T are orthogonal.

PROOF: (i) Let $\alpha$ be an eigenvalue of T and X eigen vector ( $x \neq 0$ ) Then

$Tx = \alpha x$ . Using the self-adjointness of T, we obtain:

$$\alpha\langle x, x\rangle = \langle \alpha x, x\rangle$$

$$= \langle Tx, x\rangle$$

$$= \langle x, Tx\rangle$$

$$= \overline{\alpha}\ \langle x, x\rangle$$

Now, $\langle x,x\rangle = \|x\|^2 \neq 0$ since $x \neq 0$ ; and division by $\langle x,x\rangle$

gives $\alpha = \bar{\alpha}$ . Hence is real.

This completes the proof of (i)

(ii) Let $\upsilon$ and $\lambda$ be eigen values of T and X, y be corresponding eigen vectors, respectively;

$\upsilon \neq \lambda$

Then, $Tx = \upsilon x$ and $Ty = \lambda y$ . since T is self-adjoint $\upsilon$ and $\lambda$ are real.

$$\upsilon\langle x,y\rangle = \langle \upsilon x,y\rangle$$

$$= \langle Tx,y\rangle$$

$$= \langle x, Ty\rangle$$

$$= \langle x,\lambda y\rangle$$

$$= \lambda\langle x,y\rangle$$

Since $\upsilon \neq \lambda$ , we must have $\langle x,y\rangle = 0$,which implies orthogonality of x and y.

All eigen vectors corresponding to an eigen value $\lambda$ ;

that is $x \neq 0$ such that

$Tx = \lambda x$ x (in H) and T; H → H [H is a Hilbert space] and the zero vector form a vector sub space of H which is called the EIGEN SPACE of T corresponding to $\lambda$ . The set $\sigma(T)$ of all eigen values of T is called the SPECTRUM of T. Its complement $\rho(T) = \phi - \sigma(T)$ in the complex place is called the RESOLVENT SET OF T.

It should be noted that a linear operator on a finite dimensional complex normed space $x \neq (0)$ has at least one eigen value.

**CASE 11: INFINITE DIMENSIONAL SPACES**

Here, we consider Hilbert spaces of any dimension, and as we shall see, in infinite dimensional spaces, spcectral theory becomes more complicated.

**Now,**

T: D(T) → H is a linear operator on the Hilbert space defined on the domain D(T) C H. We associate with the operator T the operator $T\lambda = T - \lambda I$

where $\lambda$ is a complex number and I is the identity operator on D(T). If $T_\lambda$ has an inverse, we denote it by $R_\lambda(T)$, that is,

$R\lambda(T) = T_\lambda^{-1} = (T - \lambda I)^{-1}$ called the RESOLVENT OPERATOR of T or, simply the RESOLVENT of T.

The investigation of $R_\lambda(T)$ - its properties - will basically lead to an understanding of T itself. And it is natural that many properties of $T_\lambda$ and $R_\lambda$ depend on $\lambda$, and, as earlier mentioned, SPECTRA THEORY is concerned with those properties:

- For instance, we shall be interested in the set of all $\lambda$ in the complex plane such that $R_\lambda$ exists - EXISTENCE of $R_\lambda$.

- Suppose $R_\lambda$ exists, for complex numbers $\lambda$, then, we further would like to know the complex numbers for

which $\mathbf{R}_\lambda$ is bounded - That is, BOUNDEDNESS of $\mathbf{R}_\lambda$ is another property that is essential. We have now narrowed the size of the set of complex numbers $\lambda$ for which $\mathbf{R}_\lambda$ exists.

- Furthermore, we shall also ask: for what $\lambda$'s is the domain $D(\mathbf{R}_\lambda)$ of $\mathbf{R}_\lambda$ DENSE in H?

Now, the RESOLVENT SET

$= \{\lambda \in \mathbb{C} : \mathbf{R}\lambda$ exists, is bounded and $D(\mathbf{R}\lambda)$ is dense in H $\}$. Its complement $\sigma(T) = \mathbb{C} - \rho(T)$ in the complext plane $\mathbb{C}$ is called the SPECTRUM of T, and a $\lambda$ (in $\sigma(T)$ ) is called a SPECTRAL VALUE of T. Furthermore, the spectrum $\sigma(T)$ is partitioned into three disjoint sets as follows:

(i) POINT SPECTRUM or <u>discrete spectrum</u> $\sigma_p(T)$ is the

set such that $R_\lambda(T)$ does not exist. A $\lambda$ (*in* $\sigma_p(T)$ is called an EIGEN VALUE of T.

(ii) CONTINUOUS SPECTRUM $\sigma_c(T)$ is the set such that

- $R_\lambda(T)$ EXISTS.
- $D(R_\lambda(T))$ is DENSE in H but $R_\lambda(T)$ is NOT BOUNDED.

(iii) RESIDUAL SPECTRUM $\sigma_r(T)$ is the set such that

- $R_\lambda(T)$ EXISTS (bounded or not)
- $D(R_\lambda(T))$ is NOT DENSE in H.

We have an EXISTENCE problem here. For instance, in the finite dimensional case $\sigma_c(T) = \sigma_r(T) = \emptyset$ . That is, in finite dimensional case, SPECTRUM of the operator T is the set of all EIGEN VALUES - PURE POINT SPECTRUM.

NOTE:

$$\mathbb{C} = \rho(T) \cup \sigma_r(T)$$

$$\rho(T) \cup \delta_p(T) \cup \delta_c(T) \cup \delta_r(T)$$

and the definition of eigen value is still the same in the finite dimensional case as in the infinite dimensional case. That is,

$\lambda$ is an <u>eigen value</u> of T if there exists a vector $x \neq 0$ such that

$(T - \lambda I)x = 0$ , and x is called an EIGEN-VECTOR (or EIGEN FUNCTION of T if H is a function space)

If H is infinite dimensional, then T can have spectral values which are not eigen values.

Let us consider an example of an operator with a spectral value which is not an eigen value:

On the Hilbert sequence space $H = l^2$, we define the operator (linear)

$T: l^2 \to l^2$ by

$$(\alpha_1, \alpha_2 \ldots) \to (0, \alpha_1, \alpha_2, \ldots) \quad \ldots(2.4)$$

where $x = (\alpha_i)$ (in $l^2$). The operator T is called the RIGHT-SHIFT OPERATOR. T is bounded (and $\|T\| = 1$ ) because

$$\|Tx\|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2 = \|x\|^2$$

The operator $R_0(T) = (T - 0I)^{-1} = T^{-1}: T(H) \to H$ exists; and is the LEFT-SHIFT OPERATOR given by

$$(\alpha_1, \alpha_2 \ldots) \to (\alpha_2, \alpha_3, \ldots)$$

But $R_0(T)$ does not have its domain T(H) as a dense set in H; indeed, T(H) is the sub-space L consisting of all $y = (\beta_1)$ with $\beta_1 = 0$. Hence, by definition, $\lambda = 0$ is a spectral value of T.

Moreover, $\lambda = 0$ is NOT an eigen value since Tx = 0 implies x = (0, 0, 0 ...0) from definition of T in (2.4) above; and the vector x = 0 (zero vector) is NOT an eigen vector.

**THEOREM 2.6 (RESOLVENT SET)**

Let T: H → H be a bounded self-adjoint linear operator on a complex Hilbert space H. Then a number $\lambda$ belongs to the resolvent set $\rho(T)$ if and only if there exists a c > 0 such that for every x (in D(T) CH)

$$\|T\lambda x\| = \geq C\|x\|$$

where $T\lambda = T - \lambda I$

PROOF: If $\lambda$ (in $\rho(T)$ ), then

$\mathbf{R}\lambda = T\lambda^{-1}: H \to H$ exists and is bounded, say, $\|\mathbf{R}\lambda\| \leq k$

where k > 0, since $\mathbf{R}\lambda \neq 0$ . Now, $I = \mathbf{R}\lambda T\lambda$ , so that for every

x (in D(T) $\subseteq$ H), we have

$$\|x\| = \|\mathbf{R}_\lambda T_\lambda x\| \leq \|R_\lambda\| \|T_\lambda x\| \leq K\|T_\lambda x\|$$

This gives $\|T\lambda x\| \geq C\|x\|$ , where $C = \frac{1}{k}$

Conversely, suppose $\|T\lambda x\| \geq C\|x\|$ for all x (in D(T) CH)

and some C > 0.

Let us prove that:

(i) $T\lambda: D(T) \to T\lambda(H)$ is bijective.

(ii) $T\lambda(H)$ is dense in H

(iii) $T\lambda(H)$ is closed in H

So that $T\lambda(H) = H$ and $R\lambda = T\lambda^{-1}$ is bounded, from the fact that: Putting $T\lambda x = y$ , $\|x\| = \|R\lambda y\| \leq C^{-1}\|y\|$ since $\|T\lambda x\| \geq C\|x\|$ for some C > 0 and all x (in D(T).

Also →[BOUNDED INVERSE THEOREM – That if $T\lambda : D(T) \rightarrow T\lambda(H)$ is bijective, $T\lambda^{-1}$ is continuous and thus bounded.]

(i) We must show that $T_\lambda x_1 = T_\lambda x_2$ implies $x_1 = x_2$, $x_1$, $x_2$ (in D(T)). Since $T\lambda$ is linear and from $\|T\lambda\| \geq C\|x\|$ ,

$$0 = \|T\lambda x_1 - T\lambda x_2\| = \|T\lambda(x_1 - x_2)\| \geq C\|x_1 - x_2\|$$

Since C > 0, this implies $\|x_1 - x_2\| = 0$ . Hence, $x_1 = x_2$.

$x_1$, $x_2$ are arbitrary, so that the operator $T\lambda$: D(T) → $T\lambda(H)$ is bijective.

(ii) We show that $\overline{T\lambda(H)} = H$ by showing that $x_0 \perp \overline{T\lambda(H)}$

implies $x_0 = 0$.

Let $x_0 \perp \overline{T\lambda(H)}$ . Then $x_0 \perp T\lambda(H)$ . Hence, for every

$y = T\lambda x (in\ T\lambda(H))$ .

$$0 = \langle T\lambda x,\ x_0\rangle = \langle Tx, x_0\rangle - \lambda\langle x, x_0\rangle$$

Thus, for all x (in D(T))

$$\langle Tx,\ x_0\rangle = \langle x, \bar{\lambda} x_0\rangle$$

By definition of the Hilbert-adjoint operation this shows that

$x_0$ (in $D(T^*)$) and

$$T^* x_0 = \bar{\lambda} x_0$$

But since T is self-adjoint, $D(T^*) = D(T)$ and $T^* = T$; thus

$$Tx_0 = \bar{\lambda} x_0$$

$x_0 \neq 0$ implies that $\bar{\lambda}$ is an eigen value of T, and then

$\bar{\lambda} = \lambda$ must be real [from Teorem 2.5(i)]. Hence, $Tx_0 = \lambda x_0$ ,

that is, $T\lambda x_0 = 0$ . But now, we have that

$\|T\lambda x\| \geq C\|x\|$ for c > 0 and all x (in D(T)). Consequently $T\lambda x_0 = 0$ yields a contradiction.

$0 = \|T\lambda x_0\| \geq C\|x_0\|$ . This, therefore, to avoid the contradiction, implies that $\|x_0\| = 0$

It follows that $\overline{T\lambda(H)}^{\perp} = \{0\}$ , so that $\overline{T\lambda(H)} = H$ ... That is $T\lambda(H)$ is dense in H.

(iii) Finally, we'll prove that y (in $\overline{T\lambda(H)}$ ) implies y (in $T\lambda(H)$ ), so that, $T\lambda(H)$ is closed, and $T\lambda(H) = H$ by (ii).

Let $y_0$ (in $\overline{T\lambda(H)}$ ). Then there is a sequence $(y_n)$ in $T\lambda(H)$ which converges to $y_0$. Since $y_n$ (in $T\lambda(H)$ ). We have $y_n = T\lambda x_n$ for $x_n$ in $D(T_\lambda) = D(T)$ . Then,

$$\|x_n - x_m\| \leq \frac{1}{C}\|T\lambda(x_n - x_m)\| = \frac{1}{C}\|y_n - y_m\|$$

Since $(y_n)$ converges, this shows that $(x_n)$ in cauchy. Since H is complete $(x_n)$ converges, say, $x_n \to x_0$. T is continuous, and so also is $T\lambda$, and $y_n = T\lambda x_n \to T\lambda x_0$. By definition $t_\lambda x_0$ $(in\ T\lambda(H))$.

The limit being unique, implies that, $T\lambda x_0 = y_0$. So that $y_0$ (in $T\lambda(H)$). Hence $T\lambda(H)$ is closed because $y_0(in\ \overline{T\lambda(H)})$ is arbitraty. We thus have $T\lambda(H) = H$, from (ii). This means that $R\lambda = T\lambda^{-1}$ is defined on all of H, and is bounded [from BOUNDED INVERSE THEOREM] or directly from the fact that: $C\|x\| \leq \|T\lambda x\|$ $C > 0$ for all x (in H). Hence, $\lambda$ (in $\rho(T)$).

The above theorem [THEOREM 2.6] beautifully characterizes the RESOLVENT SET $\rho(T)$ of the bounded linear self-adjoint operator T: H → H.

We thus have the following corollary:

COROLLARY 2.7:

A complex number $\lambda$ belongs to the RESOLVENT set $\rho(T)$ of the bounded linear self-adjoint operator T: H → H if and only if there exists a constant c > 0 such that

$$-\|T\| + C \leq |\lambda| \ .$$

PROOF: From [THEOREM 2.6]. $\lambda$ (in $\rho(T)$ ) if and only if there exists some c > 0 such that $\|T\lambda x\| \geq C\|x\|$ for all x (in D(T)).

or if and only if

$$\|Tx - \lambda x\| \geq C\|x\|$$

NOTE: $\|Tx - \lambda x\| = \|Tx + (-\lambda)x\|$

$$\leq \|Tx\| + \|(-\lambda)x\|$$

$$\leq \|T\|\|x\| + \lambda\|x\|$$

or $\|T\|\|x\| + |\lambda|\|x\| \geq C\|x\|$

or if and only if

$$\|T\| + |\lambda| \geq C \quad \text{for} \quad \|x\| \neq 0$$

if and only if

$$-\|T\| + C \leq |\lambda|$$

**REMARK:** Considering the whole set $\rho(T)$, we can say the following:

- We can find the greatest $C > 0$ such that $-\|T\| + C \leq |\lambda|$ for all $\lambda$ (in $\rho(T)$)

- $-\|T\|$ is NOT the greatest lower bound of $\rho(T)$. Infact, $|\lambda| > \|T\|$ (or $|\lambda| > -\|T\|$ for all $\lambda$ (in $\rho(T)$)

**THEOREM 2.8:** SPECTRUM $\sigma(t)$ of a bounded self-adjoint linear operator T: H → H on a complex Hilbert space H is real.

**PROOF:**

Already, in the finite dimensional case, we have seen that the spectrum (coinciding with POINT SPECTRUM) is real. That is, the eigen values are real.

We only need to show that a $\lambda = \alpha + i\beta$ ($\alpha, \beta$ real) with $\beta \neq 0$ must belong to $\rho(T)$, so that $\sigma(T) \subset \mathbb{R}$.

For every $x \neq 0$ in H we have

$$\langle T\lambda x, x\rangle = \langle Tx, x\rangle - \lambda\langle x, x\rangle$$

and, since $\langle x, x\rangle$ and $\langle Tx, x\rangle$ are real

$$\overline{\langle T\lambda x, x\rangle} = \langle Tx, x\rangle - \bar{\lambda}\langle x, x\rangle$$

Here, $\bar{\lambda} = \alpha - i\beta$ . By subtraction

$$\overline{\langle T\lambda x, x\rangle} - \langle T_\lambda x, x\rangle = (\lambda - \bar{\lambda})\langle x, x\rangle = 2i\beta\|x\|^2$$

The left side is $-2i\ Im\ \langle T\lambda x, x\rangle$ , where Im denotes the imaginary part.

$$-2i\ Im\ \langle T\lambda x, x\rangle = 2i\beta\|x\|^2$$

That is,

$$-i\ Im\ \langle T\lambda x, x\rangle = i\beta\|x\|^2$$

$$|\beta|\|x\|^2 = |Im\ \langle T\lambda x, x\rangle| \leq |\langle T\lambda x, x\rangle| \leq \|T\lambda x\|\|x\|$$

Division by $\|x\| \neq 0$ gives $|\beta|\|x\| \leq \|T\lambda x\|$ .

If $\beta \neq 0$ , then $\lambda$ (in $\rho(T)$ ) by THEOREM 2.6. Hence for $\lambda \in \delta(T)$ we must have $\beta = 0$ , that is, $\sigma(T)$ is REAL.

We have shown above that the specrum $\sigma(T)$ of a bounded self-adjoint linear operator T is real. It is interesting, moreover, to give a more detailed characterization of the spectrum $\sigma(T)$ . Without proof, we wish to state that the spectrum $\sigma(T)$ of a bounded linear operator T: X → X on any complex Banach space X is compact and lies in the disk given by $|\lambda| \leq \|T\|$ and the resolvent set $\rho(T)$ and spectrum $\sigma(T)$ are not empty.

Furthermore,

THEOREM 2.9: The spectrum $\sigma(T)$ of a bounded self-adjoint linear operator T: H → H on a complex Hilbert space H lies in the closed interval [m, M] on the real axis, where

$$m = \inf_{\|x\| = 1} \langle Tx, x \rangle \quad \text{and} \quad M = \sup_{\|x\| = 1} \langle Tx, x \rangle$$

**PROOF:** We have already shown from previous work that $\sigma(T)$ lies on the real axis. Now, we shall show that any real $\lambda = M + C \;\; C > 0$ belongs to the resolvent set $\rho(T)$ .

For every $x \neq 0$ and $V = \|x\|^{-1}x$ , we have $x = \|x\|V$ and

$$\langle Tx, x\rangle = \|x\|^2 \langle Tv, v\rangle \leq \|x\|^2 \sup_{\|\overline{V}\| = 1} \langle T\overline{V}, \overline{V}\rangle$$

$$= \langle x, x\rangle M$$

That is, $\langle Tx, x\rangle \leq \langle x, x\rangle M$

Hence, $-\langle Tx, x\rangle \geq -\langle x, x\rangle M$ , and by Schwartz in equality, we obtain

$$\|T\lambda x\|\|x\| \geq -\langle T\lambda x, x\rangle = -\langle Tx, x\rangle + \lambda\langle x, x\rangle$$

$$\geq (-M + \lambda)\langle x, x\rangle$$

$$= C\|x\|^2$$

where $C = \lambda - M > 0$ by assumption. Division by $\|x\|$ gives the inequality

$\|T\lambda x\| \geq C\|x\|$ . Hence $\lambda$ (in $\rho(T)$ ) by THEOREM 2.6

Similarly, we shall show that for any real $\lambda = m - C$ , C > 0 belongs to the resolvent set $\rho(T)$ .

Using the same $V = \|x\|^{-1}x$, $x = V\|x\|$ as above.

$$\langle Tx, x\rangle = \|x\|^2 \langle Tv, v\rangle \geq \|x\|^2 m = \langle x, x\rangle m$$

That is $\langle Tx, x\rangle \geq \langle x, x\rangle m$

Now, $\|T\lambda x\|\|x\| \geq \langle T\lambda x, x\rangle$

$$\langle Tx, x\rangle - \lambda\langle x, x\rangle$$

$$\geq (m - \lambda)\langle x, x\rangle$$

$$= C\|x\|^2$$

By division by $\|x\| \neq 0$ ,

$$\|T\lambda x\| \geq C\|x\| - C > 0$$

Thus, $\lambda$ (in $\rho(T)$). We are done.

An interesting question arises: How do we relate m and M to the norm of T; H → H, H being a complex Hilbert space?

**STATEMENT ON NORM:** For any bounded self-adjoint linear operator T on a complex Hilbert space H, we have

$$\|T\| = max(|m|, |M|) = \sup_{\|x\| = 1} |\langle Tx, x\rangle|$$

**PROOF:** By the Schwartz inequality,

$$\sup_{\|x\| = 1} |\langle Tx, x\rangle| \leq \sup_{\|x\| = 1} \|Tx\|\|x\| = \|T\|$$

That is, $K \leq \|T\|$, where K denotes the expression on the left.

We'll show that $\|T\| \leq K$, then we shall have been done.

If TZ = 0 for all Z of norm 1, then $T \equiv 0$ and we'll be done with the proof [for $\|T\| = 0$]. Otherwise for any Z of norm 1 such that $TZ \neq 0$, let

$$V = \sqrt{|TZ|}Z \quad \text{and} \quad w = \frac{TZ}{\sqrt{|TZ|}}$$

Thus,

$$\|V\|^2 = \|W\|^2 = |TZ|$$

Now, set $y_1 = v + w$ and $y_2 = v - w$. Then,

$$\begin{aligned}
\langle Ty_1, y_1\rangle - \langle Ty_2, y_2\rangle = \quad & \langle Tv,v\rangle + \langle Tv,w\rangle + \langle Tw,v\rangle + \langle Tw,w\rangle - \\
& \langle Tv,v\rangle + \langle Tv,w\rangle + \langle Tw,y\rangle - \langle Tw,w\rangle \\
= \ & 2(\langle Tv,w\rangle + \langle Tw,v\rangle) \\
= \ & 2(\langle Tz,Tz\rangle + \langle T^2z,z\rangle) \\
= \ & 2(\langle Tz,Tz\rangle + \langle Tz,Tz\rangle) \\
& \text{[self-adjointness of T]} \\
= \ & 4\ \|Tz\|^2 \quad \ldots(2.5)
\end{aligned}$$

Now, for every $y \neq 0$ and $x = \|y\|^{-1}y$, we have $y = \|y\|x$ and

$$|\langle Ty,y\rangle| = \|y\|^2|\langle Tx,x\rangle| \leq \|y\|^2 \sup_{\|\bar{x}\| = 1} |\langle T\bar{x},\bar{x}\rangle|$$

$$= k\,\|y\|^2$$

So, by triangular inequality and further calculation

$$|\langle Ty_1,y_1\rangle - \langle Ty_2,y_2\rangle| \leq |\langle Ty_1,y_1\rangle| + |\langle Ty_2,y_2\rangle|$$

$$\leq k\,(\|y_1\|^2 + \|y_2\|^2)$$

$$= k\,(\|v+w\|^2 + \|v-w\|^2)$$

$$= 2k\,(\|v\|^2 + \|w\|^2)$$

$$= 4k\,(\|Tz\|) \quad \ldots(2.6)$$

Comparing (2.5) with (2.6) above, we have $= 4\|Tz\|^2 \leq 4k\|Tz\|$. Hence $\|Tz\| \leq K$. But taking the supremum over all z of norm 1, we obtain $\|T\| \leq K$. Together with $K \leq \|T\|$, we have $\|T\| = K$. That is,

$$\|T\| = \max(|m|, |M|) = \sup_{\|x\| = 1} |\langle tx, x\rangle|$$

so that $\lambda$ cannot be an eigen value of T.

At a glance:

**SELF-ADJOINTNESS:**

$$\langle tX, Y\rangle = \int_0^1 tx(t)\ \overline{y(t)}\,dt$$

$$= \int_0^1 x(t)\ \overline{ty(t)}\,dt = \langle x, Ty\rangle$$

$$T = T^*$$

**BOUNDEDNESS:** Integral is definite as $\|Tx\|^2 = \int_0^1 t^2|x(t)|^2dt$

$$\leq \|x\|^2$$

We shall consider some properties of two unbounded linear operators, namely, the operator of multiplication by independent variable and a differentiation operator:

(1) Consider the operator:

$$T: D(T) \to L^2(-\infty, \infty)$$

$$x \to tx \qquad \ldots(3.1)$$

where $D(T)\ C\ L^2(-\infty, +\infty)$

The domain D(T) consists of all x (in $L^2(-\infty,+\infty)$ ), such that we have Tx (in $L^2(-\infty,+\infty)$ , that is

$$\int_{-\infty}^{+\infty} t^2|x(t)|^2dt < \infty \quad ...(3.2)$$

This implies that $D(T) \neq L^2(-\infty,+\infty)$ . For instance, an x(in $L^2(-\infty,+\infty)$ ) not satisfying (3.2) is given by

$$x(t) = \begin{cases} \frac{1}{t} & \text{if } t \geq 1 \\ 0 & \text{if } t < 1 \end{cases}$$

Hence $x \notin D(T)$ .

The multiplication operator defined by (i) is NOT BOUNDED:

This can be seen if we pick a particular case as a counter example. We take

$$x_n(t) = \begin{cases} 1 & \text{if } n \leq t < n+1 \\ 0 & \text{elsewhere} \end{cases}$$

Clearly, $\|x_n\| = 1$ since $\int_n^{n+1} dt = 1$ , and

$$\|Tx_n\|^2 = \int_n^{n+1} t^2 dt$$

$$= \frac{1}{3}(3n^2 + 3n + 1)$$

$$> n^2$$

This implies that $\frac{\|Tx_n\|}{\|x_n\|} > n$, where we choose $n \in \mathbf{N}$ as large as we please. It should be noted that the unboundedness results from the fact that we are dealing with functions on an infinite interval.

SELF-ADJOINTNESS: T is symmetric because using $t = \bar{t}$, we have

$$\langle Tx, y\rangle = \int_{-\infty}^{+\infty} tx\,\overline{y(t)}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\,\overline{ty(t)}\,dt$$

$$= \langle x, Ty\rangle$$

Therefore, since the set $CL^2$ of continuous square-integrable functions is everywhere dense in the space $L^2$, T is

self-adjoint.

Important spectral properties of the operator T are as follows:

1.0 T has no eigen values

1.1 $\sigma(T)$ is all of $\mathbb{R}$ .

PROOF: (1.0)

For any $\lambda$ , let x (in D(T)) be such that $Tx = \lambda x$ . Then

$(T - \lambda I)x = 0$ . Hence, by definition of T,

$$0 = \|(T - \lambda I)x\|^2 = \int_{-\infty}^{\infty} |t - \lambda|^2 |x(t)|^2 dt$$

Since $|t - \lambda| > 0$ for all $t \neq \lambda$ , we have x(t) = 0 for almost all t (in $\mathbb{R}$ ), that is x = 0. This shows that x is not an eigen vector, and $\lambda$ is not an eigen value of T. Since $\lambda$ was chosen arbitrarily, T has no eigen values.

(1.1) We have spectrum $\sigma(T)$ of T in $\mathbb{R}$ [from THEOREM 2.8]

Let $\lambda$ $(in\ \mathbf{R})$ . We define

$$w_n(t) = \begin{cases} 1 & if\ \lambda - \frac{1}{n} \leq t \leq \lambda + \frac{1}{n} \\ 0 & elsewhere \end{cases}$$

and consider $x_n = \|w_n\|^{-1} w_n$ . Then $\|x_n\| = 1$ . Writing

$T\lambda = T - \lambda I$ , as usual, we have from definition of T,

$$\|T\lambda x_n\|^2 = \int_{-\infty}^{+\infty} (t - \lambda)^2 |x_n(t)|^2 dt$$

$$\leq \frac{1}{n^2} \int_{-\infty}^{+\infty} |x_n(t)|^2 dt = \frac{1}{n^2}$$

We thus have

$$\|T\lambda x_n\| \leq \frac{1}{n} \qquad \ldots(3.3)$$

Since T has no eigen values, the resolvent $R\lambda = T\lambda^{-1}$ exists, and $T\lambda x_n \neq 0$ since $x_n \neq 0$ . The vector

$$y_n = \frac{1}{\|T\lambda x_n\|} T\lambda x_n$$

are in the range of $T\lambda$ - which is the domain of $R\lambda$ , and have norm I. Applying $R\lambda$ and using (3.3) above, we set:

$\|R\lambda y_n\| = \frac{1}{\|T\lambda x_n\|} \|x_n\| \geq n$ ... which shows that $R\lambda$ is unbounded; hence $\lambda(in\ \sigma(T))$ . Since $\lambda(in\ \mathbf{R})$ was arbitrary, $\sigma(T) = \mathbf{R}$ .

The other operator we shall consider in this section is the DIFFERENTIAL OPERATOR

$$D:\ D(D) \to L^2(-\infty,\ +\infty)$$

$$x \to ix'$$

where $x' = \frac{dx}{dt}$ and i helps to make D self-adjoint

To show that D is self-adjoint, we shall use the tools from the theory of Lebesgue integration and can be found, e.g in G.

Helmberg (1969) **INTRODUCTION TO SPECTRAL THEORY IN HILBERT SPACE**; New York: American Elsevier. Mention will finally be made that D is unbounded; has no eigen values; and the spectrum $\sigma(D)$ is all of $\mathbb{R}$ .

## 3.2 INTEGRAL HERMITIAN (SELF-ADJOINT) OPERATORS AND THEIR SPECTRUM

Refer to chapter one, section 1.2, Where we looked at some examples of ADJOINT OPERATORS on some Hilbert spaces. We gave an example of an integral operator - The FREDHOLM operator.

Let us turn again to the integral operator of the form:

$$y(s) = \int_a^b k(s,t)x(t)\,dt \quad \ldots(3.4)$$

arising from the fredholm equation (mentioned earlier). we denote this operator by the letter K and then, the relation (3.4) has the new form:

$$y = Kx \quad \ldots(3.5)$$

As formerly, we assume that the Kernel K(s,t) is a continuous function defined in the square G = [a,b] x [a,b]. That is $a \le s,\ t \le b$ .

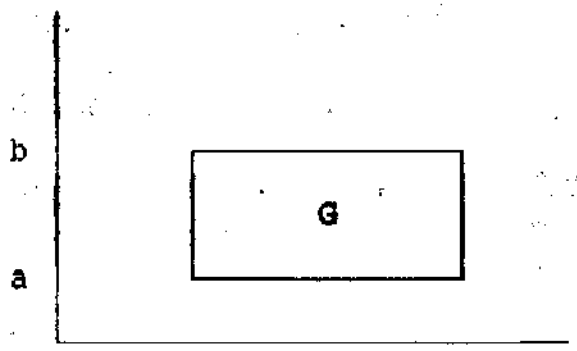



Fig. 3

As earlier done, the operator K is given over the function space $L^2[a,b]$. Moreover, the values kx are continuous functions which can be considered as the elements of the space $L^2$. That is, kx belongs to $CL^2$ [continuous and in $L^2$ [a,b]].

The operator K is additive (That is, $K(x_1 + x_2) = kx_1 + kx_2$). We shall show that K is bounded in $L^2$ [a,b].

Let $M = \max |k(s,t)|$ ... This exists

$$a \le s,\ t \le b$$

because K(s,t) is continuous on the bounded square G. Then, according to the inequality - Cauchy - Bunia Kovskii inequality - for Scalar product in $L^2$,

$$|y(s)| = \left| \int_a^b k(s,t)\, x(t)\, dt \right|$$

$$\leq M\sqrt{b-a}\,\|x\| \quad \ldots(3.6)$$

and

$$\|y\|^2 = \int_a^b y^2(s)\,ds \leq M^2(b-a)^2\|x\|^2$$

Consequently, the operator k is bounded and

$$\|k\| \leq M(b-a)$$

**REMARK:** The Fredholm operator k is bounded because of the FINITE interval [a,b], otherwise, it will be unbounded.

**NOTE:** The Cauchy - Bunia Kovskii inequality for the scalar product in $L^2$ takes the form of the usual integral Buniakovskii inequality:

$$|\langle x,y\rangle| = \left|\int_a^b x(t)\,y(t)\,dt\right|$$

$$\leq \sqrt{\int_a^b x^2(t)}\,\sqrt{\int_a^b y^2(t)\,dt} = \|x\|\|y\|$$

But if y (in $CL^2$) and $|y(t)| \leq M$, then it follows from the above Bunia Kovskii inequality that

$$\left|\int_a^b x(t)\,y(t)\,dt\right| \le M\sqrt{b-a}\,\|x\|$$

where x, y (in $L^2$).

The Scalar product in $L^2$ [a,b] can be written in the form:

$\langle x,y\rangle = \int_a^b x(t)\,y(t)\,dt$ . Hence,

$$\|x\|^2 = \langle x,x\rangle = \int_a^b x^2(t)\,dt$$

The kernel is symmetric if k(t,s) = k(s,t). We shall thus, show that Fredholm operator K is self-adjoint if the kernel K(s,t) is symmetric.

For any two continuous functions x(t) and z(t),

$$\langle kx,z\rangle = \int_a^b \left(\int_a^b k(s,t)\,x(t)\,dt\right) z(s)\,ds$$

$$= \int_a^b \int_a^b k(s,t)\,x(t)\,z(s)\,ds\,dt$$

where we have the double integral on G, on the right hand side. On the other hand,

$$\langle x,kz\rangle = \int_a^b \int_a^b k(s,t)\,z(t)\,x(s)\,ds\,dt$$

Interchanging, in the last integral, the notations of the variable s and t and assuming that k(s,t) = k(t,s), we find that

$$\langle x, kz \rangle = \int_a^b \int_a^b k(s,t)\, z(s)\, x(t)\, ds\, dt$$

That is,

$$\langle kx, z \rangle = \langle x, kz \rangle$$

Now, let x and z be arbitrary elements in $L^2[a,b]$. Since the set $C_{L^2}$ [set of continuous functions in $L^2$] is everywhere dense in the space $L^2$, we represent

$$x = \lim x_n$$

$$z = \lim z_n$$

where, $x_n$, $z_n$ (in $C_{L^2}$ )

Then, as a consequence of continuity of the scalar product and continuity of the operator k,

$$\langle kx, z \rangle = \lim \langle kx_n, z_n \rangle$$

$$= \lim \langle x_n, kz_n \rangle$$

$$= \langle x, kz \rangle$$

Hence, k is Hermitian (self-adjoint).

**SPECTRUM OF THE FREDHOLM OPERATOR K**

The Fredholm operator k on $L^2[a,b]$ with the symmetrical

theory of integral equations. In this theory, the homogenous Fredholm equation is usually written in the form;

$$x(s) = u \int_a^b k(s,t)x(t)\,dt$$

where u is a parameter.

In our operator notations, this equation has the form:

$$x = u\,kx \qquad \ldots.(3.8)$$

For u = 0 the equation (3.8), obviously, has only zero solution x = 0. These values - certainly different and distinct from zero, (and they exist) - for which the equation (3.8) has solution $x \neq 0$ are called CHARACTERISTIC NUMBERS of the equation (3.8).

For $u \neq 0$, the equation (3.8) may be written in the equivalent form:

$$kx = \frac{1}{u}x \qquad \ldots.(3.9)$$

Comparing equation (3.7) and (3.9) we have the consequence: To every characteristic number $u \neq 0$ of the equation (3.8) there corresponds the eigen value $\lambda \neq 0$ of the

operator k and conversely, to every eigen value $\lambda \neq 0$ of the operator k, there corresponds the characteristic number $\frac{1}{\lambda}$ of equation (3.8).

Moreover, the eigen values of the operator k coincide with the non-zero solution of equation (3.8), which in the theory of integral equations are called the EIGEN FUNCTIONS.

We have, therefore, conclusively seen that the point spectrum $\sigma_p(K)$ of the Fredholm operator k is non-empty. Since the Fredholm operator k with symmetric kernel k(s,t) on the square $G = [a,b] \times [a,b] = a \leq s, t \leq b$ has been proved to be bounded and self-adjoint then, its spectrum $\sigma(K)$ lies in the set

$[ - M(b-a), M (b-a)]$, where

$\|k\| = M(b - a)$ and

$$M = \max_{a \leq s,t \leq b} |k(s,t)|$$

## 3.2.1 STURM-LIOUVILLE OPERATOR

In the solution of many problems of mathematical physics, an important role is played by the differential equations of the form:

$$-\frac{d}{dt}\left(p(t)\frac{dx}{dt}\right) - q(t)x = \lambda x(t) \qquad \ldots.(3.10)$$

called STURM-LIOVILLE EQUATIONS.

We assume here that the functions p(t) and q(t) are continous on the segment $[a,b] = a \le t \le b$, and that p(t) has a continous derivative of the first order; $\lambda$ is a real parameter.

Suppose that the functions we are looking for, satisfy some boundary conditions

$$x(a) = x(b) = 0 \qquad \ldots.(3.11)$$

The further derivatives are valid also for some other boundary conditions. In that way, we shall seek the solution of the equation (3.10) among functions continuous on the segment [a,b], with their derivations of the first and second order satisfying the conditions (3.11). We denote the set of such functions by the letter M and consider it as a subset of the space $L^2$ constructed on the segment [a,b]. It is clear that the subset M is linear.

The left hand side of the equation (3.10) is the result of the application of a certain differential operator to the

function x(t). We denote this operator by the letter U, and we shall take it to be given on the set M. The values of the operator, Ux are continuous functions; but we shall treat them as elements of the space $L^2$. We can re-write the equation (3.10) in the form:

$$Ux = \lambda x$$

It is clear that the operator U is additive and homogeneous.

We shall show that the operator U is symmetric; that is $\langle Ux,y\rangle = \langle x,Uy\rangle$ for arbitrary x,y (in M).

$$\langle Ux, y\rangle = -\int_a^b [(px')' + qx]\, y\, dt$$

$$-\int_a^b (px')'y\, dt - \int_a^b qxy\, dt \quad .....(3.12)$$

We transform the first integral twice applying integration by parts. Taking into account the boundary conditions (3.11) which the functions x(t) and y(t) satisfy, we obtain;

$$\int_a^b (px')'y\, dt$$

$$= [(px')y]_a^b - \int_a^b (px')y'\, dt$$

$$= -\int_a^b x'(py') \, dt$$

$$= [-x(py')]_{t=a}^{t=b} + \int_a^b x(py')' \, dt$$

Hence,

$$\langle ux, y \rangle = -\int_a^b x(py')' dt - \int_a^b qxy \, dt$$

$$= -\int_a^b x[(py')' dt + qy] \, dt = \langle x, Uy \rangle$$

$$\therefore \langle Ux, y \rangle = \langle x, Uy \rangle \quad \ldots\ldots(3.13)$$

Now, we shall assume that $p(t) \geq 0$ and $q(t) \leq 0$ .

We calculate <Ux,x> for arbitrary x (in M), making use of the expression (3.12) and integrating by parts the first integral. We get:

$$\langle Ux, x \rangle = -\int_a^b (px')' x dt - \int_a^b qx^2 \, dt$$

$$= \int_a^b px^2 dt - \int_a^b qx^2 \, dt \quad \ldots\ldots(3.14)$$

Hence, it follows that

$$\langle Ux, x\rangle \geq 0 \quad \ldots.(3.15)$$

That is, U is positive.

It should be stated here that if x is an eigen element of a symmetric operator A corresponding to an eigen value $\lambda$ and y is an eigen element corresponding to an eigen value $\mu$ and $\lambda \neq \mu$, then $x \perp y$. Therefore, from the above established properties of the operator U, eigen elements of the operator U corresponding to the distinct eigen values are mutually orthogonal. Moreover, if A is a symmetric operator and $\langle Ax, x\rangle \geq 0$ for all $x(InD)$ (domain of A), then every eigen value of the operator A is non-negative. Infact, if $\lambda$ is an eigen value of A and x is a corresponding eigen element, then

$$\lambda\|x\|^2 = \lambda\langle x, x\rangle$$

$$= \langle \lambda x, x\rangle$$

$$= \langle Ax, x\rangle \geq 0$$

Hence, it follows that $\lambda \geq 0$ because $\|x\|^2 \neq 0$ .

Thus, taking into account the assumptions concerning the signs of the functions p(t), q(t) and by (3.15) we can assert that all eigen values of the operator U are non-negative.

The eigen values of the operator U are those values of the parameter $\lambda$ for which the equation (3.10) has solutions different from zero and satisfying the conditions (3.11); and the eigen elements are solutions. In that way we have obtained some properties of the solutions of the Sturm-Liouville equation, important for mathematical physics.

We consider on the segment $0 \leq t \leq \pi$ the equation

$$-x'' = \lambda x \quad \ldots.(3.16)$$

with the boundary conditions

$$x(0) = x(\pi) = 0$$

This equation can be obtained from (3.10) if we put $p(t) \equiv 1$ , $q(t) \equiv 0$ . Since all the eigen values $\lambda \geq 0$ then assuming $\lambda = \mu^2$ where $\mu \geq 0$ , we reduce the equation (3.16) to the

form

$$x'' + \mu^2 x = 0 \ .$$

For $\mu > 0$ the general solution of this equation has the form:

$$x = C_1 \cos \mu t + C_2 \sin \mu t$$

Since $x(0) = C_1$, then according to the first boundary condition $C_1 = 0$. Consequently, $x(t) = C_2 \sin \mu t$ and then $x(\pi) = C_2 \sin \mu\pi$ ; according to the second boundary condition, $x(\pi) = C_2 \sin \mu\pi = 0$ . But if we assume $C_2 = 0$, then we shall obtain $x(t) \equiv 0$ . In order to obtain a non-zero solution, we have $C_2$ arbitrary and require that $\sin \mu\pi = 0$ . This is possible provided $\mu$ is equal to any integer n. In this way we see that the eigen values are $(n > 0)$ $\lambda_n = n^2$ and we can assign to each eigen value an eigen function, that is, the solution $x_n(t) = \sin nt$ (We assume that $C_2$ is equal to 1).

These eigen functions form the well known system of sines, orthogonal on the segment $[0,\pi]$ .

It is interesting to verify that in the case we have considered, $\lambda = 0$ is not an eigen value. That is, for $\lambda = 0$ , among the solutions of the equation (3.16) only the zero element satisfies the given boundary conditions.

For $\lambda = 0$ , equation (3.16) becomes

$$-x'' = 0$$

Solving the above, we have

$x(t) = C$, where C is a constant.

Since $x(0) = x(\pi) = 0$, $C = 0$ .

Hence, only the zero element satisfies the given boundary conditions.

It is easily verified that the operator

$Ux = -x''$ corresponding to the equation (3.16) is not BOUNDED. Infact, for the eigen element $x_n$ we have

$Ux_n = n^2x_n$. Consequently,

$\|Ux_n\| = n^2\|x_n\|$ . Therefore, a constant C for which the

inequality $\|Ux\| \leq C\|x\|$ would be satisfied for all x (in D - domain), does not exist. Thus, generalising, Sturn Liouville operator is NOT BOUNDED.

The SPECTRUM, therefore, of the operator $Ux = -x''$ coincides with the point spectrum; and it is the set

$\{1, 4, 9, \ldots\} = \{n^2: \quad n \in IN\}$

## SPECTRUM OF STURM-LIOUVILLE OPERATOR U

The domain M of the operator U is the set of continuous functions in $L^2$ [a,b], which is everywhere dense in $L^2$[a,b]. Hence, the symmetric operator U is Hermitian (or SELF-ADJOINT).

The SPECTRUM of U is the POINT SPECTRUM $\sigma_p(U)$ which consists of non-negative real numbers. The eigen values are determined by the functions p and q.

## 3.2.2 NUMBER AND DISTRIBUTION OF THE EIGEN VALUES OF THE STURM-LIOVILLE BOUNDARY VALUE PROBLEM

For the following investigation we assume the equation (3.10) to have the general form:

$$-\frac{d}{dt}\left[p(t)\frac{dx}{dt}\right] - [q(t) + \lambda r(t)]x(t) = 0 \quad \ldots(3.17)$$

since this still reduces to (3.10) if r(t) = 1. And in

general, we can write linear homogeneous boundary conditions in the form

$$\begin{bmatrix} k_1x(a) + k_2x'(a) = 0 \ldots (i) \\ l_1x(b) + l_2x'(b) = 0 \ldots (ii) \end{bmatrix} \quad \ldots (3.18)$$

As in equation (1), $\lambda$ is a parameter, $a \le t \le b$ is the interval over which the functions p(t), q(t), r(t) are continuous. $k_1$ and $k_2$ are given real constants not BOTH zero, and so are $l_1$ and $l_2$.

We see directly from (3.17) and (3.18) that fof every number $\lambda$, the problem has the trivial solution $x \equiv 0$, that is, x(t) = 0 for all $t \in [a,b]$. A number $\lambda$ for which the problem has a solution $x \neq 0$ [if such a number exists] is called an EIGEN VALUE of the problem, and this solution x is called an EIGEN FUNCTION of the problem CORRESPONDING TO THIS EIGEN VALUE $\lambda$.

In the example (3.16) we considered earlier, we have seen that there are inifinitely many eigen values in the case where $\lambda = \mu^2 > 0$. On the other hand, we do not convey the

impression that there always exist many eigen values. If we consider the problem

$$y'' - u^2 y = 0$$

$y(0) = 0$ , $l_1 y(t_2) + l_2 y'(t_2) = 0$ we see immediately that there are difficulties in solving this problem. The differential equation has the general solution.

$$y = c_1 e^{ut} + c_2 e^{-ut}$$

and it follows from the first boundary condition that $c_1 = -c_2$. This gives for the second boundary condition after division by $c_1$

$$l_1(e^{ut_2} - e^{-ut_2}) + ul_2(e^{ut_2} + e^{-ut_2}) = 0$$

or

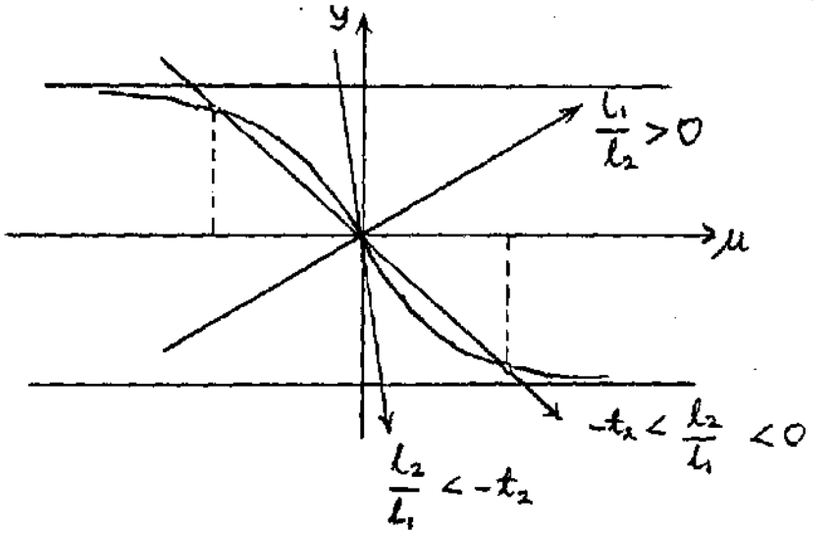
$$\frac{l_2}{l_1} u = -\tanh ut_2$$

In order to solve this equation geometrically, we have to intersect the line $y = \left(\frac{l_2}{l_1}\right)u$ with the curve $y = -\tanh ut_2$ in a

u,y- co-ordinate system. The slope of $y = \left(\frac{l_2}{l_1}\right)u$ is $\frac{l_2}{l_1}$ and

the slope of - tanhut$_1$ at u = 0 is $-t_2$. As is apparent, u = 0 is a solution under all circumstances but the corresponding solution is the trivial solution y = 0. The only possibility of obtaining a non-trivial solution is for $-t_2 < \frac{l_2}{l_1} < 0$. In

this case the two values of u differ only in sign; ie, we obtain only one real eigen value $u^2$, see figure below:




Solution diagram on $\mu y$ -axis

Fig 4

Fig. 4

In the case just considered i.e $\lambda = -u^2 < 0$ only one or no eigen value at all exists.

The situation which prevails in general and which is quite clearly reflected in the above example is described in the following two theorems [*THEOREM 3.1 and THEOREM 3.3*]

**THEOREM 3.1** Let p(t), q(t) and r(t) be continuous, p(t) differentiable and p(t) > 0, r(t) > 0 in $a \leq t \leq b$. Then there exists a value $\lambda 0$ such that for all $\lambda < \lambda_0$, the Sturm-Liouville Problem (3.17), (3.18) has no solution, with $t_1$, $t_2$ $\epsilon$ [a,b].

We also have an important Lemma which will turn out to be essential for our investigation:

Let us consider the differential equation

$$\frac{d}{dt}[p(t)x'] + q(t)x = 0 \quad \ldots(3.19), \text{ where}$$

p(t) and q(t) are assumed to be continuous and that p(t) is differentiable in a certain closed interval $a \leq t \leq b$. We will furthermore, assume that

$p(t) > 0$ in $a \leq t \leq b$ . [In case $p(t) < 0$ in $a \leq t \leq b$ , we multiply the equation (3.19) by $-1$. The case where $p(t) = 0$ at certain points of the interval has to be excluded].

We can write (3.19) as a system of two differential equations of the first order by introducing

$y(t) = p(t)x'(t)$ ...(3.20). Then, we have

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{1}{p(t)}y \\ \frac{dy}{dt} &= -q(t)x \end{aligned}\right\} \quad ....(3.21)$$

And we can apply to this system the standard existence and uniqueness theorems for ordinary differential equations. Under not too severe conditions concerning $p(t)$ and $q(t)$ we can establish the existence and uniqueness of the solutions of (3.21).

It is of considerable help in investigating the nature of the solutions of (3.19) and especially their zeros to introduce polar co-ordinates [This method was introduced by H. Pruefer, MATHEMATISCHE ANNALEN 95 (1926)].

Let

$$\left.\begin{aligned} x &= r(t)\cos\phi \\ y &= r(t)\sin\phi \end{aligned}\right\} \quad ...(3.22)$$

NOTE: This r(t) is different from the one in (3.17).

Thus, we obtain for (3.21)

$$r'\cos\theta - r\sin\phi.\phi' = \frac{1}{p}r\sin\phi$$

$$r'\sin\phi - r\cos\phi.\phi' = qr\cos\phi$$

Solving for $r'$ and $\phi'$ yields

$$\frac{d\phi}{dt} = -\left(\frac{1}{p(t)}\sin^2\phi + q(t)\cos^2\phi\right) \quad ...(3.23)$$

$$\frac{dr}{dt} = \left(\frac{1}{p(t)} - q(t)\right)r\sin\phi\cos\phi \quad ...(3.24)$$

Since (3.23) does not contain r(t), we can solve for ø independent of (3.24), substitute this ø into (3.24), separate the variables, and solve for r(t).

$$r(t) = \int_{\phi}^{\frac{1}{2}\int(\frac{1}{p} - q)\sin 2\theta dt} \quad .....(3.25)$$

r(t) = e

It is sufficient for our purposes to concentrate our investigatiions on the solution ø of (3.23). We can, first of all, see that if ø(t) is a solution of (3.23), then ø(t) + kπ

will also be a solution since, according to (3.22)

NOTE: $k \in Z$

$$r \cos(\phi + k\pi) = (-1)^k r \cos\phi = (-1)^k x$$

$$r \sin(\phi + k\pi) = (-1)^k r \sin\phi = (-1)^k y = (-1)^k px';$$

That is, if k is odd $\phi + k\pi$ will give the solution $-x(t)$ instead of $x(t)$ and if k is even, we obtain $x(t)$ itself. Since $-x$ is, in the case of a homogeneous differential equation, a solution if and only if x is a solution, we can consider

$\phi(t) + k\pi$ for all $k = 0, \pm1, \pm2, \ldots$as one and the same solution. Hence, we can always assume that at a given point $t_0$ $\phi(t)$ satisfies the inequality

$$\frac{\pi}{2} + (k-1)\pi < \phi(t\phi) \leq \frac{\pi}{2} + k\pi$$

for any integer k.

We shall next prove the following lemma, which we said earlier will turn out to be essential for subsequent investigations;

**LEMMA** 3.2 The solution $x(t)$ of (3.19) has a zero at $t = t_1$ if and only if $\phi(t_1) = \left(\frac{\pi}{2}\right) + k\pi$ . (We call such points $t_1$ the $\frac{\pi}{2}$ -intercepts of $\phi(t)$.) **PROOF:** If $\theta(t_1) = \left(\frac{\pi}{2}\right) + k\pi$ , it

follows from (3.22) that $x(t_1) = 0$. Let us now assume that $x(t_1) = 0$ to prove the necessity of the condition.

It follows then from (3.22) that

$$r(t_1) \cos \phi (t_1) = 0$$

since $r(t_1) \neq 0$ for $t_1$ in $a \leq t \leq b$ in view of (3.25) above,

it follows that $\cos \phi(t_1) = 0$ and from this $\phi(t) = \frac{\pi}{2} + k\pi$ .

We should have in mind what we are interested in - the investigation of the number and distribution of the eigen values of the Sturm-Liouville boundary value problem. All we have done so far is to lay a foundation for our work.

We shall not prove THEOREM 3.1 but rather use it. However, it suffices to say that if we let $M > 0$ to be any number, specifically chosen and choose a $\lambda o$ (which depends on M) such that

$$q(t) + \lambda o\, r(t) < -M \text{ in } a \leq t \leq b ,$$

and since $p(t)$ is continuous in $a \leq t \leq b$ , there exists an $m$ such that

$p(t) \geq m$ in $a \leq t \leq b$ , Also, $\max_{[a,b]} |q(t)|, \min_{[a,b]} r(t)$ exist, and

$$-M - \max_{[a,b]} |q(t)| \qquad \text{[They are continuous in [a,b,]]}$$

$$\lambda_0 < \frac{-M - \max_{[a,b]} |q(t)|}{\min_{[a,b]} r(t)}$$

We shall show that the spectrum of Sturm-Liouville operator is <u>discrete</u>. To do that, we shall prove that the eigen values of the Sturm-Liouville boundary value problem (3.17), (3.18) given earlier can be ordered. Let us consider the theorem below:

**THEOREM** 3.3 Let p(t), q(t), r(t) be continuous, p(t) differentiable, and p(t) > 0, r(t) > 0 in $a \le t \le b$. Then, there are infinitely many values $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ (eigen values) for which the Sturm-Liouville Problem (3.17), (3.18) has a non-trivial solution. If $x(t,\lambda_k)$ is the solution corresponding to the $k^{th}$ eigen value $\lambda_k$, then $x(t,\lambda_k)$ has one more zero in $t_1 \le t \le t_2$ than $x(t,\lambda_{k-1})$, which is the solution corresponding to the (k-1)th eigen value $\lambda_{k-1}$ and $\lim_{k \to \infty} \lambda_k = \infty$

To prove this theorem, we need the following two lemmas:

**LEMMA 3.4** Let $\phi(t,\lambda)$ be the ø-curve corresponding to the solution $x(t,\lambda)$ (3.17) which satisfies the boundary condition (3.18) (i) at $t_1$ ie $k_1x(t_1) + k_2x'(t_1) = 0$ for any $\lambda$ . Then, under the conditions stated in *THEOREM 3.3*, $\phi(t_2,\lambda)$ is strictly monotonically decreasing in $\lambda$ , ie

$$\phi(t_2,\lambda_1) > \phi(t_2,\lambda_2) \quad \text{if} \quad \lambda_1 < \lambda_2. \quad (t_1 < t_2)$$

**PROOF:** Let $x(t,\lambda)$ be a solution of (3.17) for any $\lambda$ which satisfies the boundary condition (3.18) (i) at $t_1$. Then, the corresponding ø-curve $\phi(t, \lambda)$ satisfies the boundary condition ( 3.26(i) ) at $t_1$ given below:

$$\left.\begin{aligned} \phi(a) &= \alpha + v\pi \ldots\ldots (i) \\ \phi(b) &= \beta + u\pi \ldots\ldots (ii) \end{aligned}\right\} \ldots 3.26u, v = 0,1,2,\ldots$$

NOTE: Since the t, ø-diagram proved to be of considerable help in our investigations of the problem (1) earlier, if we employ it by introducing the function y(t) according to (3.20) the boundary conditions (3.18) will be transformed to the form:

$$k_1 x(a) + \frac{k_2}{p(a)} y(a) = 0$$

$$l_1 x(b) + \frac{l_2}{p(b)} y(b) = 0$$

Let us now introduce four constants ó, ß, A, B according to

$$k_1 = A \sin ó, \quad l_1 = B \sin ß$$

$$\frac{k_2}{p(a)} = -A \cos \alpha, \quad \frac{l_2}{p(b)} = -B \cos \beta$$

ie, $$A = \sqrt{k_1^2 + \frac{k_2^2}{p^2(a)}}, \quad B = \sqrt{l_1^2 + \frac{l_2^2}{p^2(b)}}$$

$$\alpha = -\tan^{-1}\left\{\frac{k_1 p(a)}{k_2}\right\}, \quad \beta = -\tan^{-1}\left\{\frac{l_1 p(b)}{l_2}\right\}$$

[The sign of A and B depend on whether we use the principal values α and ß or α + π and ß + π]. In terms of these quantities, the boundary conditions become:

$$\sin \alpha \cos \phi(a) - \cos \alpha \sin \phi(a) = 0$$

$$\sin ß \cos \phi(b) - \cos ß \sin \phi(b) = 0$$

or $\sin \phi(a) \cos \alpha - \cos \phi(a) \sin \alpha = 0$

$\sin \phi(b) \cos \beta - \cos \phi(b) \sin \beta = 0$

That is,

$$\sin [\phi(a) - \alpha] = 0$$

$$\sin [\phi(b) - \beta] = 0$$

Thus we have:

$$\phi(a) = \alpha + v\pi \ldots\ldots (i)$$
$$\phi(b) = \beta + u\pi \ldots\ldots (ii) \qquad u, v = 0, 1, 2, 3, \ldots$$

Now, let

$\lambda_1 < \lambda_2$ . Then

$q(t) + \lambda_1 r(t) < q(t) + \lambda_2 r(t)$ *in* $a \le t \le b$ and in view

of the following comparison lemma:

If $x_1(t)$ is a solution of

$$\frac{d}{dt}[p_1(t)x'] + q_1(t)x = 0$$

and $x_2(t)$ is a solution of

$$\frac{d}{dt}[p_2(t)x'] + q_2(t)x = 0$$

where $p_1(t) \ge p_2(t)$, $q_1(t) \le q_2(t)$, [p(t) and q(t) may have different or equal signs] and if $\phi_1(t)$ is the $\phi$-curve of $x_1(t)$

and $\phi_2(t)$ the $\phi$-curve of $x_2(t)$ and for a certain $t_0$ in $a \le t \le b$.

$\phi_1(t_0) = \phi_2(t_0)$, then for all $t > t_0$

$\phi_2(t) \le \phi_1(t)$

(In case $p_1 \ge p_2$, $q_1 < q_2$ hold, $\phi_2 < \phi_1$) - The modification of this comparison lemma is:

$\phi_2(t) < \phi_1(t)$ for all $t > t_0$ if $p_1(t) \ge p_2(t)$ and $q_1(t) < q_2(t)$, and since

$$\phi(t_1, \lambda_1) = \phi(t_1, \lambda_2) = \alpha + \nu\pi, \quad \nu = 0,1,2,\ldots$$

and $q_1(t) = [q(t) + \lambda_1 r(t)] < q_2(t) = [q(t) = \lambda_2 r(t)]$

$\phi(t, \lambda_1) > \phi(t, \lambda_2)$ for all $t > t_1$.

Since $t_2 > t_1$,

$\phi(t_2, \lambda_1) > \phi(t_2, \lambda_2)$

Having considered one of the lemmas [*LEMMA 3.4*] that will help us prove (*THEOREM 3.3*), let us consider the second one.

**LEMMA 3.5:**

Let $\phi(t, \lambda)$ be the $\phi$-curve corresponding to the solution $x(t,\lambda)$ of (3.17) which satisfies the boundary condition (3.18) (i) at $t_1$ for any $\lambda$.

Then,

$$\lim_{\lambda \to \infty} \phi(t_2, \lambda) = -\infty \qquad t_2 > t_1$$

We shall not prove this lemma but proceed to prove *THEOREM 3.3.*

First we note, that $x(t,\lambda)$ which is a solution of (3.17) and satisfies the boundary condition (3.18) (i) at $t_1$, is continuous in $\lambda$ See [22] pp. 364 and consequently, its $\phi$-curve $\phi(t, \lambda)$ with $\phi(t_1,\lambda) = \alpha$ is continuous in $\lambda$ . Now we choose a $\lambda_o$ such that

$q(t) + \lambda_o r(t) > 0$ . The curve $\phi(t, \lambda_o)$ of the solution $x(t,\lambda_0)$ which satisfies (3.18) (i) will intersect the line t = $t_2$ at some point $\phi_0$. If $\phi_0 = \beta$, then $\phi(t_2, \lambda_o) = \beta$; ie $x(t,\lambda_0)$ satisfies also (3.18) (ii); hence $\lambda = \lambda_o$ is the first eigen value.

If $\phi_0 \neq \beta$ , we can assume without loss of generality that $\beta - \pi \leq \phi_0 < \beta$. Now let $\lambda$ increase and consequently, $\phi(t_1, \lambda)$ will decrease monotonically in view of *LEMMA 3.4* and continually until for a certain $\lambda = \lambda_1$ , $\phi(t_1, \lambda_1) = \beta - \pi$. Then, $\lambda_1$ is the first eigen value. We let $\lambda$ continue to increase until for some $\lambda = \lambda_2$ , $\phi(t_1, \lambda_1) = \beta - 2\pi$; ie $\lambda_2$ is the second eigen value, and so on. Clearly, there is no end to this process because

$$\lim_{\lambda \to \infty} \phi(t_1, \lambda) = -\infty \quad t_2 > t_1 \text{ in view of } \textit{LEMMA 3.5}$$

Thus,we obtain successive values $\lambda_k$ such that

$$\phi(t_1, \lambda_k) = \beta - k\pi \text{ [See Fig. 4 below]}$$

Furthermore, since

$$\phi(t_1, \lambda_k) = \beta - k\pi \text{ while}$$

$$\phi(t_2, \lambda_{k-1}) = \beta - (k-1)\pi,$$

it follows that $\phi(t, \lambda_k)$ has to have one more $\frac{\pi}{2}$ - intercept than $\phi(t, \lambda_{k-1})$ in $t_1 \leq t \leq t_2$. Hence, $x(t,\lambda_k)$ has to have one more zero in $t_1 \leq t \leq t_2$ than $x(t,\lambda_{k-1})$.

Thus, we can order the eigen values:

$\lambda_1 < \lambda_2 < \ldots$ which implies that the Spectrum of Sturm-Liouville operator is <u>discrete</u> and there exists $\lambda_o$ such that $\lambda_o < \lambda_1 < \lambda_2 < \ldots\ldots$ and (3.17) (3.18) has no solution for all $\lambda < \lambda_o$ [FROM THEOREM 3.1]

We still have to show that

$\lim_{k \to \infty} \lambda_k = \infty$ : suppose that $\lim_{k \to \infty} \lambda_k = M < \infty$

Then, there exists for any arbitrary small $\epsilon > 0$ an $N\epsilon$

such that

$$|\lambda_n - \lambda_m| < \epsilon \quad \text{for all} \quad m, n > N\epsilon$$

Since $\phi(t_2, \lambda)$ is continuous in $\lambda$, we have for a $\epsilon_1 > 0$ a $\delta_{\epsilon_1} > 0$ such that

$$|\phi(t_2, \lambda_n) - \phi(t_2, \lambda_m)| < \epsilon_1, \text{for } |\lambda_n - \lambda_m| < \delta_{\epsilon_1}$$

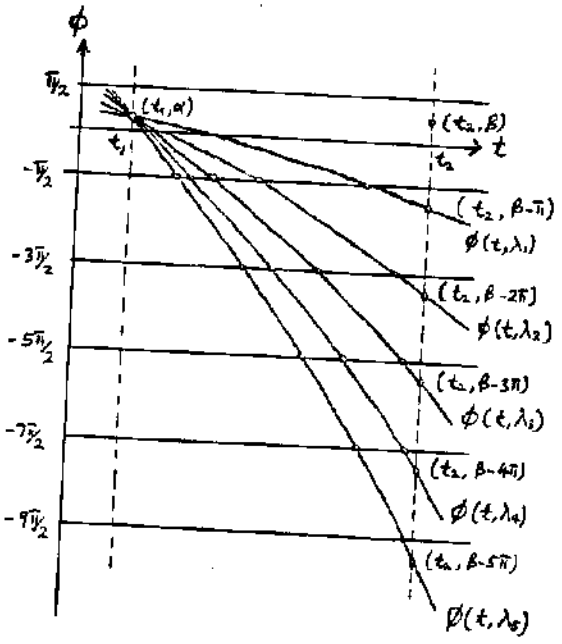
Let $\epsilon_1 < \pi$ and choose $\epsilon = \delta_{\epsilon_1}$. Then

$$|\phi(t_2, \lambda_n) - \phi(t_2, \lambda_m)| < \pi$$

However,

$$|\phi(t_2, \lambda_n) - \phi(t_2, \lambda_m)| \geq \pi$$

Since $\phi(t_2, \lambda_n) = \beta - n\pi$ and $\phi(t_2, \lambda_m) = \beta - m\pi$




Eigen value distribution diagram
Fig 5

JOSEPH LIOUVILLE (1809-1882) who investigated this equation together with boundary conditions with great success. This equation includes many differential equations of mathematical physics as special cases. We shall consider some of these special cases:

**(I) THE LEGENDRE EQUATION**

Legendre's differential equation, so called in honour of the French mathematician ADRIEN MARIE LEGENDRE (1752-1833) is of the form

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \ldots S_1$$

and arises in numerous physical problems, particularly in boundary value problems for spheres. The parameter n is a given real number. Any solution of $S_1$ is called a LEGENDRE FUNCTION.

We can re-write $S_1$ as:

$$[(1-x^2)y']' + \lambda y = 0, \quad \lambda = n(n+1) .$$

Hence, it is a Sturm-Liouville equation with

$p = 1-x^2$, $q = 0$, $r = 1$. Since $p = 0$ when $x = \pm 1$, no boundary conditions are needed to form a Sturm-Liouville problem on the interval $-1 \leq x \leq 1$].

**(II) BESSEL'S EQUATION**

One of the most important differential equations in applied mathematics is BESSEL'S DIFFERENTIAL EQUATION in respect of the German astronomer and mathematician FRIEDRICH WILHEIM BESSEL (1784-1846). This equation is given by

$x^2y'' + xy' + (\lambda x^2 - v^2)y = 0$ .....$S_2$

and $\lambda$ is a parameter (a given number), $v$ is a fixed value.

We can re-write $S_2$ in STURM-LIOUVILLE form thus:

$$[xy']' + \frac{(\lambda x^2 - v^2)}{x}y = 0$$

$$p = x; \quad q = \frac{-v^2}{x} \; ; \quad r = x$$

(III) **EULER-CAUCHY EQUATION**

This equation is given in the form

$x^2y'' + 2xy' + \lambda y = 0$ .....$S_3$ ( $\lambda$ is a constant).

This can be expressed in STURM-LIOUVILLE form as:

$$[x^2y']'' + [(x^2y')' + \lambda y = 0]$$

$p = x^2$; $r = 1$; $q = 0$

The above are just some of the differential equations of second order expressible as STURM-LIOUVILLE equations.

# CHAPTER FOUR

## 4.1 POSITIVE OPERATORS

In chapter two section two, we mentioned the condition for positivity of a self-adjoint operator T, say, on a Hilbert space H; that is, if $\langle Tx, x\rangle \geq 0$ for all x (in H). Again, we said that $\langle Tx, x\rangle$ is real. As a consequence, we may introduce a PARTIAL ORDERING $\leq$ on the set of all BOUNDED SELF-ADJOINT LINEAR operators on the complex Hilbert space H, and denote theset by $B_s(H)$. We define this partial ordering $\leq$ as follows on $B_s(H)$.

$T_1 \leq T_2$ if and only if $\langle T_1x, x\rangle \leq \langle T_2x, x\rangle$ for all x (in H).

We can simply write $T \geq 0$ instead of $\langle Tx, x\rangle \geq 0$ to mean that T is POSITIVE, for all x (in H).

The set $B_s(H)$, therefore, is a POSET with the PARTIAL ORDERING $\leq$ ; This POSET can be canonically represented by

$(B_s(H), \leq)$ .

On $(B_s(H), \leq)$, let us define two binary operations $\wedge$ (meet) and $\vee$ (join) - according to the concept in SEMI GROUP studies - Such that:

$$|T_1 \wedge T_2| = Min\,(|T_1|, |T_2|)$$

$$|T_1 \vee T_2| = Max\,(|T_1|, |T_2|)$$

It is clear that these binary operations - $\wedge$ and $\vee$ satisfy the following:

(1) $T_1 \wedge T_2 = T_2 \wedge T_1$; $T_1 \vee T_2 = T_2 \vee T_1$ (commutativity)

(2) $T_1 \wedge (T_2 \wedge T_3) = (T_1 \wedge T_2) \wedge T_3$; $T_1 \vee (T_2 \vee T_3) = (T_1 \vee T_2) \vee T_3$

(Associativity)

(3) $T_1 \wedge (T_1 \vee T_2) = T_1 = T_1 \vee (T_1 \wedge T_2)$

(4) $T_1 \wedge T_1 = T_1 \vee T_1$

$T_1$, $T_2$, $T_3$ are in $B_s(H)$.

Then, $(B_s(H) \leq, \wedge, \vee)$ is called a LATTICE.

Furthermore,

$(B_s(H) \leq, \wedge,)$ is called a LOWER SEMI LATTICE

and

$(B_s(H) \leq, \vee,)$ is called an UPPER SEMI LATTICE

There exists a simple relationship between the partial ordering of two members, say, $T_1$, $T_2$ of $B_s(H)$ and the definition that:

$T \geq 0$ if and only if $\langle Tx, x\rangle \geq 0$ for all x (in H) and T (in Bs (H)). The relationship is that:

$T_1 \leq T_2$ if and only if $0 \leq T_2 - T_1$ .

That is, $T_1$ and $T_2$ can be partially ordered in the sense $T_1 \leq T_2$ if and only if $\langle T_1x, x\rangle \leq \langle T_2x, x\rangle$ for all x (in H) if and only if $T_2 - T_1$ is POSITIVE.

DEFINITION: (IDEMPOTENT OPERATOR)

Let T be a bounded linear operator on a Hilbert space H. T is said to be IDEMPOTENT if $T^2 = T$

EXAMPLE

Let Y be a proper closed subspace of a Hilbert space H. Then, given x(in H), we can have the expression.

$x = y + y^{\perp}$ , where y (in Y) and $y^{\perp}$ (in $Y^{\perp}$ ) ( $Y^{\perp}$ is the ORTHOGONAL COMPLEMENT of Y; that is, $H = Y \oplus Y^{\perp}$ and $\langle y, y^{\perp} \rangle = 0$ for all y (in Y) and $y^{\perp}$ (in $Y^{\perp}$ )

Now, define

P: H → Y where

px = y,

We shall show that:

(i) P is linear

(ii) P is bounded (and $\|P\| = 1$ )

(iii) P is IDEMPOTENT.

(i) **LINEARITY:** Let $x_1 = y_1 + y_1^{\perp}$

$$x_2 = y_2 + y_2^{\perp}$$

Then,

$$p(\alpha_1 x_1 + \alpha_2 x_2)$$

$$= p(\alpha_1 y_1 + \alpha_2 y_2 + \alpha_2 y_2^{\perp} + \alpha_1 y_1^{\perp})$$

$= \alpha_1 y_1 + \alpha_2 y_2$

$= \alpha_1 p(x_1) + \alpha_2 (p(x_2)$

(ii) **BOUNDEDNESS:** Given $x = y + y^{\perp}$ , and by PYTHAGOREAN theorem,

$$\|x\|^2 = \|y\|^2 + \|y^{\perp}\|^2$$

Therefore,

$$\|px\|^2 = \|y\|^2 \leq \|x\|^2$$

Hence,

$$\|p\| \leq 1$$

Infact, $\|p\| = 1$ (P acting on x (in Y)

(iii) **IDEMPOTENT:**

For $x = y + y^{\perp}$

$p^2 x = p(px)$

$= py$

$= y$

$= px$

Thus, $p^2 = p$

P is called PROJECTION (or PROJECTION OPERATOR).

Later, we shall show that for non-trivial idempotents, that is, $T \neq 0$, $T \neq I$ (identity operator) the spectrum

$\sigma(T) = \{0,1\}$ .

In what, we choose to call, FUNCTIONAL SEMI GROUP, there exists the following result:

Every lower semi lattice is a commutative BAND and vice versa.

**DEFINITION:** A BAND is a semi group of idempotents. Thus a commutative BAND is semi group of commuting idempotents.

Let us, now, prove the stated result.

**PROOF:** From the definition of a semi lattice, (given in this work), every semi lattice is a commutative band.

Conversely, let S be a commutative band.

First, we show that S is a poset.

Define partial ordering $\leq_p$ in S by $x \leq_p y$ if and only if

$xy = x$

Then $\leq_p$ is a partial ordering, since

(i) $x \leq_p x$ ($x \cdot x = x$ as x is idempotent, for all x (in S) )

(ii) $x \leq_p y$ and $y \leq_p x$ implies that

$xy = x$ and $yx = y$

That is, $x = xy$

$= yx$ (commutativity property in S)

$= y$ x,y (in S)

iii) If $x \leq_p y$ and $y \leq_p z$

then, $xy = x$ and $yz = y$, so that

$x = x(yz) = (xy)z$

$= xz$

That is, $x \leq_p z$

Thus, $\leq_p$ is a partial ordering

Then,

(i) $x \wedge x = xx = x$

(ii) $x \wedge (y \wedge z) = x \,.\, (y \,.\, z)$

$= (x \,.\, y)z$

$= (x \wedge y) \wedge z$

iii) $x \wedge y = xy = yx = y \wedge x$

We are done!

How do we apply the above result to our pertinent study on the set of bounded self- adjoint linear operators $B_s(H)$ on the complex Hilbert space?

We can say then, that a class of commuting idempotents in $B_s(H)$ can be partially ordered in the sense that, for $T_1$, $T_2$ in $B_s(H)$, $T_1^2 = T_1$ and $T_2^2 = T_2$, $T_1T_2 = T_2T_1$ ,

$$T_1 \leq T_2 \quad \text{if and only if } \langle T_1x,x\rangle = \langle T_2x,x\rangle.$$

And the binary operation $\wedge$ can be imposed on the class such that

$$\|T_1 \wedge T_2\| = \min \{\|T_1\|, \|T_2\|\}$$

Such is a very interesting piece of fusion between algebra and analysis which holds some attraction in FUNCTIONAL ANALYSIS.

Futhermore, if the COMPOSITION operation on two members $T_1$, $T_2$ of $B_s(H)$ satisfies the properties of the binary operation $\wedge$ (meet), then, we can conclusively say that the operators on which such properties are fulfilled are IDEMPOTENTS.

THE PROJECTION OPERATOR P defined earlier, is SELF-ADJOINT and POSITIVE.

SELF-ADJOINTNESS: Let $x_1 = y_1 + y_1^{\perp}$ and $x_2 = y_2 + y_2^{\perp}$ where

$y_1$, $y_2$ (in Y) and $y_1^{\perp}, y_2^{\perp}$ (*in* $Y^{\perp}$), $H = Y \oplus y^{\perp}$

Then,

$$\langle px_1, x_2 \rangle = \langle y_1, y_2 + y_2^{\perp} \rangle$$

$$= \langle y_1, y_2 \rangle + \langle y_1 y_2^{\perp} \rangle$$

$$= \langle y_1, y_2 \rangle \quad (\langle y_1 y_2^{\perp} \rangle = 0)$$

$$\langle x_1, px_2 \rangle = \langle y_1 + y_1^{\perp}, y_2 \rangle$$

$$= \langle y_1, y_2 \rangle \quad (\langle y_1 y_2^{\perp} \rangle = 0)$$

Therefore,

$\langle px,y \rangle = \langle x,py \rangle$ for all x,y (in H) and hence

P is SELF-ADJOINT

POSITIVENESS: Let x be in H. Then

$\langle px,x \rangle = \langle p^2x,x \rangle$

$= \langle px, px \rangle$

$\|px\|^2 \geq 0$

Therefore, P is positive.

Infact, generally speaking, if an idempotent operator T on a Hilbert space H has an adjoint operator $T^*$, then $T^*$ is necessarily idempotent.

**PROOF:** Suppose T has the adjoint $T^*$, then by definition,

$\langle Tx,y\rangle = \langle x,T^*y\rangle$ for all x,y (in H).

Now,

$\langle x,T^*y\rangle = \langle Tx,y\rangle = \langle T^2x,y\rangle = \langle x,T^{*2}y\rangle$ for all x,y (in H)

Therefore,

$T^* = T^{*2}$ which implies that $T^*$ is idempotent.

Let us look at other examples of idempotent operators on a Hilbert space which are self-adjoint:

(1) Let $H = \mathbb{R}^2$ and $T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

It is clear that $\langle Tx,y\rangle = \langle x,Ty\rangle$ and $T^2 = T$.

(2) Given $T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$ on $\mathbb{R}^2$

$T = T^2$ and $\langle Tx,y\rangle = \langle x,Ty\rangle$

Now, the question is this: can we now generalize - since

all the idempotents we have looked at are self-adjoint - that every idempotent is self-adjoint in any Hilbert space? This question, we suppose, will be of tremendous interest to students of functional analysis. Like we have stated earlier, the spectrum of any idempotent operator $T \neq 0$ or I consists of the reals 0 and 1. That is for T: $T^2 = T$ and $T \neq 0$, $T \neq I$, $\sigma(T) = \{0,1\}$. We already have proved that the spectrum $\sigma(T)$ of a self-adjoint operator is real. Can we then say that the converse holds? If the spectrum $\sigma(T)$ of an operator T in the Hilbert space H is real, does it necessarily follow that: $\langle Tx,y\rangle = \langle x,Ty\rangle$ for all x,y (in H)?

To answer the question, we would like to say that if we can find an idempotent operator T on a Hilbert space H that is NOT self-adjoint then we will have succeeded in proving by just one COUNTER EXAMPLE that, the generalisation that every idemponent operator is self-adjoint does not hold. Since it is easier and possible to disprove the generalisation above stated by a counter example, let us follow this logical line of argument.

Consider, therefore, the following counter example:

For any real number 'a' let $T = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$

$a \neq 0$

and $H = \mathbf{R}^2$ . Clearly, $T^2 = T$ since

$$\begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$$

For any x,y in $\mathbf{R}^2$ , where

$x = (x_1, x_2)$ and $y = (y_1, y_2)$. We can express x and y in column vector form; that is, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x_i, y_i (in\ \mathbf{R})$

$i = 1,2.$

$$\langle x, y \rangle = \sum_{i=1}^{2} x_i y_i$$

Now,

$$\langle Tx, y\rangle = \left\langle x_2 \begin{pmatrix} a \\ 1 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle$$

$$= x_2[ay_1 + y_2]$$

$$\langle x, Ty\rangle = \left\langle y_2 \begin{pmatrix} a \\ 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle$$

$$= y_2[ax_1 + x_2]$$

Thus,

$\langle Tx, y\rangle \neq \langle x, Ty\rangle$ for all x,y (in $\mathbf{R}^2$ ).

However, for x = y

$\langle Tx, x\rangle = \langle x, Tx\rangle$ which implies that though T is not self-adjoint $\langle Tx, x\rangle$ is real for all x (in $\mathbf{R}^2$ ) ...(4.1)

The question, therefore, arises from the result (4.1) in the space $\mathbf{R}^2$ and $T = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$ : can we say that for every idempotent $a \neq 0$ operator T in a Hilbert space H, that $\langle Tx, x\rangle = \langle x, Tx\rangle$ for all x (in H)? The answer this time is: YES for all self-adjoint idempotents; And: "Yes" for some idempotents

which are NOT self-adjoint, like the operator $T = \begin{pmatrix} 0 & a \\ 0 & 1 \end{pmatrix}$ a (in

$\mathbb{R}^2$ ) given above and

$$a \neq 0$$

the matrix operator given below:

Let us consider another matrix operator on $\mathbb{R}^2$ which is idempotent but not self-adjoint.

Let $T = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$ for any real 'b' $(b \neq 0)$

Then,

$$\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$$

For any x,y (in $\mathbb{R}^2$ ) where $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

i.e $-\lambda(1 - \lambda) = 0$

or $\lambda(\lambda - 1) = 0$

Therefore, either $\lambda = 0$ or $\lambda = 1$

Hence, the spectrum $\sigma(T)$ of T is {0,1}. We shall prove now that, for all bounded idempotent operators T, the spectrum $\sigma(T)$ = {0,1}. Before that, let this statement be noted:

The spectrum of an operator T in a Hilbert space H may be real but it does not imply that T is self-adjoint. We have already seen that from the foregoing.

THEOREM: If T is a bounded linear operator on the complex Hilbert space H, and T is an idempotent, that is, $T^2 = T$, then its spectrum $\sigma(T) = \{0,1\}$.

PROOF:

From spectral mapping theorem for polynomials $p(T) = T^2 - T = 0$; as such,

$\lambda^2 - \lambda = 0$ which holds only for $\lambda = 0,1$ .

Hence, for $\lambda \in \sigma(T)$, $\sigma(T) = \{0,1\}$ .

We can prove the above in yet another way:

The resolvent $R\lambda(T)$ , where T is bounded and linear operator on the Hilbert space H, $\lambda \neq 0$ can be represented as:

$$R_\lambda = (T - \lambda I)^{-1}$$

$$= -\frac{1}{\lambda}\left(I - \frac{T}{\lambda}\right)^{-1}$$

$$= -\frac{1}{\lambda}\sum_{i=0}^{\infty}\left(\frac{1}{\lambda}T\right)^i$$

where the series on the right hand side converges for all $\lambda$ such that

$$\left\|\frac{1}{\lambda}T\right\| = \frac{\|T\|}{|\lambda|} < 1 \quad \text{that is} \quad |\lambda| > \|T\|$$

Now, in our case T is idempotent, that is, $T^2 = T$.

For $|\lambda| > 1$ , we obtain

$$R\lambda(T) = -\lambda^{-1}[I + \lambda^{-1}T + \lambda^{-2}T + \lambda^{-3}T + \ldots] \quad (T^2 = T)$$

and hence $T^n = T$ for any n (in $\mathbf{N}$).

That is,

$$R\lambda(T) = -\lambda^{-1}[I - T + T + \lambda^{-1}T + \lambda^{-2}T + \ldots]$$

$$= -\lambda^{-1}[(I - T) + T(1 + \lambda^{-1} + \lambda^{-2} + \ldots)]$$

$$= -\lambda^{-1}\left[I - T + T\left(1 - \frac{1}{\lambda}\right)^{-1}\right]$$

$$= -\frac{1}{\lambda}(I - T) - (\lambda - 1)^{-1}T \ ,$$

which holds for any $\lambda \neq 0, 1$ .

Hence, the spectrum $\sigma(T)$ of T is the set $\{0,1\}$.

COROLLARY: IDEMPOTENT OPERATORS T ARE NON-INVERTIBLE, that is $T^{-1}$ is not existent.

This is clear since $0 \in \sigma(T)$, $\sigma(T) = \{0,1\}$

**IDEMPOTENT MATRICES ON FINITE-DIMENSIONAL SPACES:**

Since linear operators on finite dimensional vector spaces can be represented in terms of matrices, matrices become the most important tools for studying linear operators in finite

dimensional case. Thus, it will be interesting to obtain some classes of idempotent matrices on some finite dimensional Hilbert spaces.

But first, let us show that the matrix

$$T_B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

represents an idempotent operator $T: \mathbb{R}^3 \to \mathbb{R}^3$ with respect to an orthonormal basis B. We shall also determine the spectrum - which of course, is {0,1}. We shall also find eigen vectors and eigen spaces corresponding to the eigen values in the spectrum $\sigma(T_B)$ of $T_B$ .

Clearly,

$$T_B^2 = T_B$$

Let $\lambda = \sigma(T_B)$ Then,

$$|T_B - \lambda I| = \begin{bmatrix} \frac{1}{2} - \lambda & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} - \lambda & 0 \\ 0 & 0 & 1 - \lambda \end{bmatrix}$$

That is, $\lambda(\lambda - 1)^2 = 0$

Therefore,

$\sigma(T_B) = \{0, 1\}$, $\lambda = 1$ has multiplicity 2

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen element corresponding to $\lambda = 0$

and $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$ the eigen element corresponding to $\lambda = 1$

For $\lambda = 0$ ,

$$T_B x = 0$$

i.e $\frac{1}{2}x_1 + \frac{1}{2}x_2 = 0 \Rightarrow x_1 = -x_2$

and

$$x_3 = 0$$

Therefore,

$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigen vector of $T_B$

corresponding to $\lambda = 0$ .

For $\lambda = 1$

$$T_B y = y$$

$$\frac{1}{2}y_1 + \frac{1}{2}y_2 = y_1$$

$$\frac{1}{2}y_1 + \frac{1}{2}y_2 = y_2$$

$$y_3 = y_3$$

which implies $y_1 = y_2$

From the above, the third entry in y assumes any value in $\mathbb{R}$ , while the first and second must attain the same value.

That is, the eigen vectors corresponding to $\lambda = 1$ are $\begin{pmatrix} a \\ a \\ b \end{pmatrix}$

where a, b (in $\mathbf{R}$ ).

Conclusively,

The eigen space $E_0$ corresponding to the eigen value

$\lambda = 0$ is: $\left\{ \alpha \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} : \alpha \in \mathbf{R} \right\}$ generated by the eigen vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

While, the eigen space $E_1$ corresponding to the eigen value

$\lambda = 1$ is:

$\left\{ \begin{pmatrix} a \\ a \\ b \end{pmatrix} : a, b \in \mathbf{R} \right\}$ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is one eigen vector in this space.

Let $M_3$ represent the class of idempotent matrices on $\mathbf{R}^3$ .

Then, we can obtain the following non-trivial idempotents:

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Infact for every every $n \in \mathbf{N}$ we have

$$\begin{pmatrix} \frac{1}{n} & \frac{1}{n} & \dots \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots \frac{1}{n} \\ \frac{1}{n} & \frac{1}{n} & \dots \frac{1}{n} \end{pmatrix}$$ as idempotent operator in $\mathbf{R}^n$ .

We can, therefore, state that in every finite dimensional Hilbert space H, there is an operator T: H → H such that

$$T^2 = T$$

## 4.2. SQUARE ROOTS OF A POSITIVE OPERATOR

If T is self-adjoint, then $T^2$ is positive since

$\langle T^2x, x\rangle = \langle Tx, Tx\rangle$

$= \|Tx\|^2 (\geq 0)$ for all x in domain D(T) of T.

Conversely, given a positive operator T, the problem will be to find a self-adjoint A such that $A^2 = T$.

**DEFINITION (POSITIVE SQUARE ROOT)**

Let T: H → H be a positive bounded self-adjoint linear operator. A is called a SQUARE ROOT OF T if

$$A^2 = T$$

If, in addition, $A \geq 0$, then A is called a POSITIVE ROOT of T and is denoted by

$$A = T^{\frac{1}{2}}$$

$T^{\frac{1}{2}}$ exists and is unique.

**THEOREM:**

Every positive bounded self-adjoint linear operator T: H → H on a complex Hilbert space H has a positive square root A which is unique. This operator A commutes with every bounded linear operator on H which commutes with T.

**PROOF:**

• 1 We shall, first of all, show that if the theorem holds under the additional assumption $T \leq I$, it also holds without that assumption.

• 2 We obtain the EXISTENCE of the operator $A = T^{\frac{1}{2}}$ from

$A_n x \to Ax$ where $A_0 = 0$ and

$$A_{n+1} = A_n + \frac{1}{2}(T - A_n^2) \quad n = 0,1,2,\ldots \quad (4.3)$$

and we also prove the commutativity stated in the theorem.

• 3 Finally, we shall prove the uniqueness of the square root.

• 1 If T = 0, we can take $A = T^{\frac{1}{2}} = 0$ . Let $T \neq 0$ . By the Schwartz inequality,

$$|\langle Tx, x\rangle| \leq \|T\|\|x\|^2$$

But since $\langle Tx, x\rangle$ is real,

$$\langle Tx, x\rangle \leq \|T\|\|x\|^2$$

Dividing through by $\|T\| \neq 0$ , get,

$$\langle Qx, x\rangle \leq \langle Ix, x\rangle \quad \text{where,} \quad Q = \frac{1}{\|T\|}T ,$$

Thus, $Q \leq I$

Assuming that Q has a unique positive square root $B = Q^{\frac{1}{2}}$, we have $B^2 = Q$ and we see that a square root of $T = |T|Q$ is $|T|^{\frac{1}{2}}Q^{\frac{1}{2}} = |T|^{\frac{1}{2}}B$ since

$$(|T|^{\frac{1}{2}}B)^2 = |T|B^2 = |T|Q = T$$

clearly, the uniqueness of $Q^{\frac{1}{2}}$ implies uniqueness of the POSITIVE square root of T.

Hence, if we can prove the theorem under the additional assumption that $T \leq I$, it would suffice.

• 2 EXISTENCE: From (4.3) above, since $A_0 = 0$, we have:

$A_1 = \frac{1}{2}T$, $A_2 = T - \frac{1}{8}T^2$, $A_3 = \frac{3}{2}T - \frac{5}{8}T^2 + \frac{1}{8}T^3 - \frac{1}{32}T$ etc. That is, each $A_n$ is a polynomial P(T) in T and is self-adjoint. All the $A_n$'s commute and with every operator commuting with T.

* $A_n \le I \quad n = 0,1,\ldots.$:

$A_0 \le I$ . Let n > 0

Since $I - A_n - 1$ is self-adjoint [being polynomial in T] then,

$(I - A_{n-1})^2 \ge 0$ . Again, $T \le I$ implies $I - T \ge 0$ .

Hence,

$$0 \le \frac{1}{2}(I - A_{n-1})^2 + \frac{1}{2}(I - T)$$

$$= I - A_{n-1} - \frac{1}{2}(T - A_{n-1}^2)$$

$$= I - A_n$$

Therefore, $A_n \le I \quad n = 0,1,\ldots.$ (2)

* $A_n \le A_{n+1} \quad n = 0,1,\ldots.$

By induction, and using (1),

$$0 = A_0 \le A_1 = \frac{1}{2}T$$

Suppose $A_{n-1} \le A_n$ for any fixed n.

Then,

$$A_{n+1} - A_n = A_n + \frac{1}{2}(T - A_n^2) - A_{n-1} - \frac{1}{2}(T - A_{n-1}^2)$$

$$= A_n - A_{n-1} - \frac{1}{2}(A_n^2 - A_{n-1}^2)$$

$$= A_n - A_{n-1} - \frac{1}{2}[A_n - A_{n-1}][A_n + A_{n-1}]$$

$$= (A_n - A_{n-1})[I - \frac{1}{2}(A_n + A_{n-1})]$$

By hypothesis, $A_n - A_{n-1} \geq 0$ and since

$0 \leq I - A_n$ $n = 0,1....$ and

$$A_n = \frac{1}{2}A_n + \frac{1}{2}A_n \geq \frac{1}{2}A_n + \frac{1}{2}A_{n-1} = \frac{1}{2}(A_n + A_{n-1})$$

Hence $0 \leq I - A_n \leq I - \frac{1}{2}(A_n + A_{n-1})$

Consequently,

$$A_{n+1} - A_n \geq 0$$

That is,

$$A_n \leq A_{n+1} \quad n = 0,1,\ldots \quad (4.5)$$

* $A_n x \to A_x, \quad A = T^{\frac{1}{2}}$

From (4.5) $(A_n)$ is monotone increasing and $A_n \leq I$ (From (4.4) above). Hence, there exists a bounded self-adjoint linear operator A such that $A_n x \to Ax$ for all x (in H). Since $(A_n x)$ converges, from (4.3),

$$A_{n+1}x - A_n x = \frac{1}{2}(Tx - A_n^2 x) \to 0 \text{ as } n \to \infty .$$

Thus,

$Tx - A_n^2 x = 0$ for all x, that is, $T = A^2$. Also, $A \geq 0$ because $0 = A_0 \leq A_n$ by (4.5), that is $\langle A_n x, x\rangle \geq 0$ for x (in H) which implies $\langle Ax, x\rangle \geq 0$ for every x (in H) by the

continuity of the inner product.

* $ST = TS \rightarrow AS = SA$ ....(4.6)

where S is a bounded linear operator on H.

$ST = TS$ *implies* $AS = SA_n$ because, $A_n$'s are polynomials in T.

Now, we have

$A_nSx = SA_nx$ for all x (in H). As $n \rightarrow \infty$

$ASx = SAx$ for all x (in H)

Thus, AS = SA

The astericked sub proofs prove the existence and commutativity of A with all bounded linear operators S which commute with T.

• 3 **UNIQUENESS:**

Suppose A and B are both positive square roots of T. Then,

$A^2 = B^2 = T$. Also,

$BT = BB^2 = B^2B = TB$ [AT = TA], so that AB = BA ... from (4.6).

Let x (in H) be arbitrary and y = (A-B)x.

Then,

$\langle Ay, y\rangle \geq 0$ and $\langle By, y\rangle \geq 0$ since $A \geq 0$, $B \geq 0$ , using AB

= BA and $A^2 = B^2$ we get

$$\langle Ay, y\rangle + \langle By, y\rangle = \langle (A + B)y, y\rangle = \langle (A^2 - B^2)x, y\rangle = 0$$

Hence,

$\langle Ay, y\rangle = \langle By, y\rangle = 0$ . Since $A \geq 0$ and A is self-adjoint, it has itself a positive square root C, that is, $C^2 = A$ and C is self-adjoint. We thus obtain

$$0 = \langle Ay, y\rangle = \langle C^2y, y\rangle = \langle Cy, Cy\rangle = \|Cy\|^2 \quad \text{and } Cy = 0.$$

Also, $Ay = C^2y = C(Cy) = 0$, similarly $By = 0$.

Hence,

$(A - B)y = 0$. Using $y = (A - B)x$, we thus have for all x (in H).

$$\|Ax - Bx\|^2 = \langle (A - B)^2x, x\rangle = \langle (A - B)y, x\rangle = 0$$

This shows that $Ax - Bx = 0$ for all x (in H) and proves that $A = B$.

- SPECTRUM $\sigma(T^{\frac{1}{2}})$ of the square root $A = T^{\frac{1}{2}}$ of a bounded self-adjoint linear operator $T: H \to H$ on complex Hilbert space H lies in the finite interval.

$$[-\|T\|^{\frac{1}{2}}, \|T\|^{\frac{1}{2}}]$$

# CHAPTER 5

## 5.1 FURTHER EXAMINATION OF SPECTRA OF SELF ADJOINT OPERATORS

So far, we have examined the concept of ADJOINT OPERATOR within/on a Hilbert space with bias towards SELF-ADJOINT operators and their spectral theory. We shall now explore another dimension in this pertinent study.

Let H be a separable complex Hilbert space, B(H) the set of all (bounded) operators on H and T an element in B(H). A complex number $\lambda$ is called an APPROXIMATE EIGEN VALUE OF T if for each $\epsilon > 0$ there exists an <u>$\epsilon$-eigen vector</u> of T corresponding to $\lambda$ -that is, a unit vector X such that

$$\|Tx - \lambda x\| < \epsilon$$

The set of all approximate eigen values of T is called the APPROXIMATE POINT SPECTRUM of T, and is denoted by $\sigma_a(T)$, and is a closed subset of the complex space $\mathbb{C}$ [That is, $\sigma_a(T) \neq \Phi$ ]; if $b > 0$ is a bound of T, then $|\lambda| \leq b$, for each $\lambda$

(in $\sigma_a(T)$ ). If T is a self-adjoint operator, then $\sigma_a(T) \subset \mathbb{R}$ (if T is a self-adjoint operator, we have proved that its spectrum $\sigma(T)$ is real). If T is POSITIVE (in the sense, $\langle Tx, x \rangle \geq 0$ for each x in H), then each element of $\sigma_a(T)$ is non-negative. We have proved related cases ealier but with respect to POINT SPECTRUM $\sigma_p(T)$ of T and Spectrum $\sigma(T)$ of the self-adjoint T- in specific terms - for the Sturm - Liouville operator, which is a self adjoint operator (though not bounded).

The Resolvent set of an element T of B(H) is

$R(T) = \{\lambda \in \mathbb{C}: T - \lambda I$ has an inverse in B(H)$\}$

R(T) is an open subset of $\mathbb{C}$ .

The SPECTRUM of T is

$\sigma(T) = \sim R(T) = \{\lambda \in \mathbb{C}: \lambda \neq \xi$ for each $\xi \in R(T)\}$

$\sigma(T)$ is a closed subset of $\mathbb{C}$ and (as we shall show later) includes $\sigma_a(T)$ .

It is relatively easy to prove classically that $\sigma(T)$ is compact; also that, if T is self-adjoint, then either $\|T\|$ or $-\|T\|$ belongs to $\sigma(T)$, where $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$ and if T is normal then $\sigma_a(T)$ and $\sigma(T)$ coincide [7, § 31, 32, 34].

From the foregoing, it becomes clear that the statements vis-a-vis EXACT POINT SPECTRUM of $T \in B(H)$ still hold in APPROXIMATE POINT SPECTRUM.

The fact, therefore, generates an imprtant proposition. Before stating it,may we consider the concept of SCHAUDER BASIS.

**DEFINITION (SCHAUDER BASIS)**

Let us use the concept of convergence of series to define a BASIS as follows:

If a normed space x contains a sequence $(e_n)$ with the property that for every x(in X) there exists a unique sequence of scalars $(\alpha_n)$ such that

$$\|x - (\alpha_1 e_1 + \ldots\ldots + \alpha_n e_n)\| \to 0 \quad (as\ n \to \infty)$$

then $(e_n)$ is called a SCHAUDER BASIS for X. The series

$$\sum_{k=1}^{\infty} \alpha_k e_k$$

which has athe sum x is then called the expansion of x with respect to $(e_n)$, and we write

$$x = \sum_{k=1}^{\infty} \alpha_k e_k$$ That is, the sequence $(x_n)$ $\left( x_n = \sum_{k=1}^{n} \alpha_k e_k \right)$

converges to x.

**PROPOSITION:** Given that H is a separable complex Hilbert space and T (in B(H)). Then, for any eigen value $\lambda$ of T corresponding to an eigen vector x, there exists a sequence $(\lambda_n)$ of approximate eigen-values of T corresponding to $\epsilon_n$ - eigen vectors $(x_n)$ So that $(X_n) \to X$; $\lambda_n \to \lambda$ and

$$\|Tx_n - \lambda_n x_n\| \to 0 \quad (as\ n \to \infty)$$

Also $\epsilon_n \to 0$

**PROOF:** Given any eigen value $\lambda$ of T coresponding to eigen vector x, we can have the sequence $(\lambda, \lambda, \ldots\ldots\lambda, \ldots\ldots) \to \lambda$ and $(x, x, \ldots, x, \ldots) \to x$ such that $(\epsilon_n) = \left(\frac{1}{n}\right)$. So, $\epsilon_n \to 0$ and

$$|Tx - \lambda x| < \epsilon_n \quad \text{for each n.}$$

and $|Tx - \lambda x| \to 0$

So then, we have proved the existence or possibility of such sequences.

Conversely,if there exists a sequence $(\lambda_n)$ of approximate eigen values of T corresponding to the $\epsilon_n$ - eigen vectors $(x_n)$ (ie $\epsilon_n > 0$ ) and $(\epsilon_n) \to 0$ , then there exists an eigen value $\lambda$ of T corresponding to an eigenvector X such that $\lambda = lim\lambda_n$ and $x = \lim x_n$.

**PROOF:**

By definition,for each n = 1,2...

$\|Tx_n - \lambda_n x_n\| < \epsilon_n$ .

Taking the limit on both sides.

$\lim \|Tx_n - \lambda_n x_n\| < \lim \epsilon_n$

From the continuity of norm and $\lim \epsilon_n = 0$

$\|\lim Tx_n - \lim \lambda_n x_n\| < 0$ which implies that

$\|\lim Tx_n - \lim \lambda_n x_n\| \to 0$

Hence, we have

$\lim Tx_n = Tx$ , x (in H) and $\lim \lambda_n x_n = \lambda x$

$x = \lim x_n$ $\lambda = \lim \lambda_n$ .

and $\|Tx - \lambda x\| = 0$ [Tx, $\lambda$ exist finitely!]

Therefore, $\lambda$ is an eigen value of T corresponding to the eigen vector x.

Thus, we have the following statement;

**STATEMENT:** A complex number $\lambda$ is said to be an eigen

value of a bounded operator T: H → H, where H is separable complex Hilbert space, corresponding to an eigen vector x if and only if, there exists a sequence $(\lambda_n)$ of approximate $\epsilon_n$ -eigen vectors $(x_n)$ such that $\lambda_n \to \lambda$, $x_n \to x \in H$ and $\epsilon_n \to 0$

**NOTE:** The separability of the complex Hilbert space H is essential to the statement above.

**EXAMPLE:** On the space $\mathbb{R}^2$ consider the operator $T: \mathbb{R}^2 \to \mathbb{R}^2$ ,

where $T = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ . we have the sequence $\left(\frac{1}{n}\right)$ of approximate

eigen values of T corresponding to $\epsilon_n$ - eigen vectors $\begin{pmatrix} \frac{n+1}{n} \\ \frac{-1-n}{n} \end{pmatrix}$

where the sequence $\epsilon_n = \left(\frac{2(n+1)}{n^2}\right)$

Then,

$$\left\| T\begin{pmatrix} \frac{n+1}{n} \\ \frac{-1-n}{n} \end{pmatrix} - \frac{1}{n}\begin{pmatrix} \frac{n+1}{n} \\ \frac{-1-n}{n} \end{pmatrix} \right\|$$

$$= \frac{1}{n}\begin{pmatrix} \frac{n+1}{n} \\ \frac{-1+n}{n} \end{pmatrix}$$

$$= \sqrt{2}\,\frac{(n+1)}{n^2} < \frac{2(n+1)}{n^2}$$

Clearly, $\epsilon_n \to 0$

$\left(\frac{1}{n}\right) \to 0$ [eigen value of T corresponding to the

eigen- vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$(x_n) = \begin{pmatrix} \frac{n+1}{n} \\ \frac{-1-n}{n} \end{pmatrix} \to \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ one of the eigen vectors of T. We can easily find another sequence $(x'_n)$ of $\epsilon'_n$ -eigen vectors corresponding to approximate eigen values $\lambda'_n$ such that

$$\epsilon_n \to 0, \quad x_n \to \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad \lambda'_n \to 0$$

**MOVTIVATIONAL SOURCE OF EXPERIMENT:**

The definition of a SCHAUDER BASIS on a normed space x (as given in this chapter) constitutes an important tool of motivation to think, with regards to APPROXIMATE EIGEN VALUE concept, in "similar" chords of intellectual thoughts.

It is easy to prove that if a normed space X has a schauder basis, then X is separable. Conversely Banach himself, overfifty years ago, raised a famous question: Does EVERY separable Banach space have a schauder basis? He never answered this question until about thirty-five years later (1973) when P. Enflo was able to construct a separable Banach space which has no schauder basis. Thus, though surprisingly,

the answer to this famous question is "no" !

We would, therefore, be interested if any one would give an example of a separable complex Hilbert space H and an operator $T \in B(H)$ with an eigen value $\lambda$ corresponding to an eigen vector x that does not satisfy our proposition. May be an "Enflo" can do it!

## 5.2 MOTIVATIONAL SOURCE OF EXPERIMENT

We use $\rho$ to denote the metric on any metric space. Two elements x,y of a metric space X are distinct if $\rho(x,y) > 0$ , in which case we write $x \# y$ . If S is a subset of X, then its complement is

$$\sim S \equiv \{x \in X: \forall\, s \in S\ (x \# s)\}$$

and its metric complement is

$$-S \equiv \{x \in X: \exists\, r > 0\ \forall\, s \in S\ (\rho(x,s) \geq r)\}$$

If S is located (in X), in the sense that

$$\rho(x,S) \equiv inf\{\rho(x,s) : s \in S\}$$

exists (is computable) for each x in X, then

$$-S \equiv \{x \in X: \rho(x,S) \geq 0\}$$

**DEFINITION:** We say that the metric space X is COMPACT if it is totally bounded and complete: and LOCALLY COMPACT if each bounded set in X is contained in a compact set.

NOTE: Following Bishop [2] we require totally bounded sets to be non-void, so a compact or locally compact set is located).

We shall consider some important definitions and concepts which are relevant to the pertinent study at hand.

TEST FUNCTION: A TEST FUNCTION on a locally compact metric space X is a uniformly continous mapping $f:X \to \mathbf{R}$ such that f(x)=0 for all X in the metric complement of some compact set (called a COMPACT SUPPORT of f); the set of test functions on X is denoted by C(X).

The sup norm

$\|f\|_X = \|f\| = sup\{|f(x)| : x \in X\}$ of such a function exists.

**POSITIVE MEASURE:** A POSITIVE MEASURE on X is a non-zero linear mapping $\mu$ of C(X) into $\mathbf{R}$ such that $\mu(f) \geq 0$ when ever $f \geq 0$ .

As in classical analysis, a positive measure gives rise to a set $L_1(\mu)$ of integrable functions. A subset F of X is said to be Full if it is the domain of an integrable function.

**A COMPLEMENTED SET in X:**

such that K' is compact and $K^0$ is the metric complement of K'in X; we usually then identify K with K' and -K with $K^0$ . If K is a compact set, and A a complemented set such that KCA', then $A^0$ C - K and so K < A [1, ch, 6, (6,6)].

An intergrable function f gives rise to the complemented sets:

$$[f \geq t] \equiv (\{x:f(x) \geq t\},\{x:f(x) < t\})$$

$$[f > t] \equiv (\{x:f(x) > t\},\{x:f(x) \leq t\})$$

$$[f = t] \equiv (\{x:f(x) = t\},\{x:f(x) \neq t\})$$

NOTE that these are complemented sets, as all functions f, considered in intergration theory are STRONGLY EXTENSIONAL: that is, if

$$|f(x) - f(y)| > 0 \text{ , then } \rho(x,y) > 0$$

The following is a fundamental, and hard-to-prove theorem of constructive intergration theory [ 3, ch.6, (4.11)].

If f is an intergrable function, then there exists a sequence $(t_n)$ of positive numbers such that

(1) If t is a positive real number distinct from each tn - in which case, we say that t is ADMISSIBLE FOR f (relative to

u) - then the complemented sets $[f \geq t]$ and $[f > t]$ are intergrable and have the same measure; and

(2) for each admissible t and each $\epsilon > 0$ there exists $\sigma > 0$ such that , if t' is admossible and $|t - t'| < \sigma$ , then,

$$|u([f \geq t]) - u([f \geq t])| < \epsilon$$

In the case where the measure u is finite on X, then the notation of admissibility, and the related properties described in the foregoing theorem, extend in the obvious way to real, not necessarily positive number t.

**DEFINITION:**

A compact subset K of X is STRONGLY INTEGRABLE if there exists a constant C such that the following conditions hold:

For each $\epsilon > 0$ there exists $\delta > 0$ such that if $f \in C(X)$, $0 \leq f \leq 1$, $f(x) = 1$ for all $x \in K'$ and $f(x) = 0$ whenever $\rho(X,k) \geq \delta$ , then $|\int f du - C| < \epsilon$ . In that case, K is integrable and U(k) = C [ 3, ch.6 (6.2)]

A real-valued function f defined on full subset of X is ESSENTIALLY BOUNDED (relative to u) if there exists a

nonnegative number b, called a BOUND of f, such that $|f| \leq b$ on a full set. Two such functions are equal if they are equal pointwise on a full set. Taken with this notion of equality, and the usual pointwise algebraic operations, the set of all essentially bounded measurable funtions relative to $\mu$ is a Linear Space over $\mathbf{R}$, denoted by $L^\infty(u)$ ; or when there is no likelihood of confusion, by $L^\infty$ . Since the standard classical norm on $L^\infty$ is not everywhere computable, the constructive topology of $L^\infty$ is defined, not by norm, but by a family of semi norms, as follows.

Let $\mathbf{A}$ denote the set of all complemented sets that are integrable and have positive measures relative to U, where two elements of $\mathbf{A}$ are equal if their characteristics functions are equal elements of $l_1(u)$. If

$A \in \mathbf{A}$, $f \in L_\infty$ and $b \geq 0$ is a bound of f, then the semi norm

$$\|f\|_A = u(A)^{-1}\int_A |f|du$$

exists and is at most b. If $\|f\|_\infty = sup\{\|f\|_A : A \in \mathbf{A}$ exists, then f is said to be a NORMABLE ELEMENT of $L^\infty$ .

Everything discussed above - whichlooks like a digression from the topic of discourse in this work - is to give us some light to facilitate our work. Hence, the measures that will interest us are those associated with the functional calculus of a self adjoint operator on a non-trivial seperable Hilbert space H. These measures are introduced in the SPECTRAL THEOREM:

Let T be a self adjoint operator, with bound b>o on H; and let $(e_i)$ be an erthonormal basis of H. The there exists a positive measure u on [-b,b], and a bound-preserving homomorphism f → f(T) of $L_\infty(u)$ onto an algebraof commuting self adjoint operators on H, such that

(1) if $f(t) = \sum_{i=0}^{n} C_i t^i$ is a polynomial function on [-b,b] then

$$f(T) = \sum_{i=0}^{n} c_i T^i \quad \text{and}$$

$$(2)\quad u(f) = \sum_{i=0}^{\infty} 2^{-n} \langle f(T) e_n, e_n \rangle \quad \text{for each} \quad f \in L_\infty(u)$$

Moreover, if $(f_n)$ is a bounded sequence of elements of Loo (u) that converges in measure to an element f of $L_\infty(u)$ , then the sequence $(f_n(T)$ converges strongly to f(T) [3, ch.7, (8.22)].

The mapping f → f(T) in the spectral thorem is called the FUNCTIONAL CALCULUS for T, and $\mu$ is called the FUNCTIONAL CALCULUS MEASURE for T corresponding to the ORTHONORMAL basis (en). It is proved in [6] that any two functional calculus measures of a givenself adjoint operatorare equivalent (each is absolutely continous relative to the other), and that thefunctional calculus is independent of the orthonormal basis $(e_n)$ and the coresponding measure u. It follows that the property of integrability of functions, and the coresponding property of admissibility of real numbers, are independent of

the functional calculus for a given self adjoint operator. Accordingly, and by abuse of language, we say that a real number t is admissible for the self adjoint operator T if it is admissible for the identity function $x \to x$ relative to any, and therefore each functional calculas measure for T.

## 5.3 THE APPROXIMATE POINT SPECTRUM OF A SELF-ADJOINT OPERATOR.

H will denote a non-trivial separable complex Hilbert space.

In this section, we prove that the approximate point spectrum of a self adjoint operator T on H supports the functional calculus measures for T, and we give a necesary and sufficient condition for $\sigma_a(T)$ [approximate point spectrum of T] to be compact.

**LEMMA** I.0: Let T be a self adjoint operator on H with bound $b > 0$; let $f \in C[-b, b]$, and let $\lambda \in \sigma_a(T)$. Then for each $\epsilon > 0$ there exists a unit vector x such a that

$$\|f(T)x\| > |f(\lambda)| - \epsilon$$

**PROOF:** Given $\epsilon > 0$, choose a polynomial P such that $\|F - p\| < \frac{\epsilon}{3}$. Since $\lambda \in \sigma_a(T)$, there exists a unit vector

x such that $|\langle P(t)X,X\rangle - \langle P(\lambda)x,x\rangle| < \frac{\epsilon}{3}$

Then,

$$|f(\lambda)| \leq |\langle f(T)x,x\rangle| + |\langle (P(T) - f(T)x,x\rangle| + \frac{2\epsilon}{3}$$

$$\leq \|f(T)x\| + \|(P(T) - f(T)x\| + \frac{2\epsilon}{3}$$

$$\leq \|f(T)x\| + \|p - f\| + \frac{2\epsilon}{3}$$

$$\|F(t)X\| + \epsilon$$

Hence, $\|f(T)X\| > |f(\lambda)| \epsilon$

**THEOREM:**

Let T be a self adjoint operator on H with bound b > o. Then T is functionally normable if and only if $\sigma_a(T)$ is compact; in which case $\|f(T)\| = \|f\|_{\sigma_a(T)}$ for each f in C[-b,b].

For the proof of the above, we can refer to [7] pp 15.

COROLLARY of the above theorem:

If T is a functionally normable self adjoint operator on H, then for each $\epsilon > 0$ there exists $\lambda \in \sigma_a(T)$ such that

$$\|T\| - \epsilon < |\lambda| \leq \|T\|$$

If also T is positive, then $\|T\| \in \sigma_a(T)$

In proving the above corollary, the above theorem will be applied with f(t) = t, recalling that $\sigma_a(T)$ is closed and that if T is positive then $\lambda \geq 0$ for each $\lambda \in \sigma_a(T)$ .

**FURTHER PROPERTIES OF SPECTRA OF A SELF ADJOINT OPERATOR.**

What is the relationship between the spectrum and the approximate point spectrum of a self adjoint operator? We shall explore this relationship - for there exists such.

**LEMMA 1.1** **For** any $T \in B(H)$, $\sigma_a(T) \subset \sigma(T)$ and $R(T) \subset \sigma(T) \subset -\sigma_a(T)$ , where R(T) is the resolvent set of T asdefined earlier:

**PROOF:** If $\lambda \in R(T)$ , then $\lambda \notin R(T)$ and $\lambda \notin \sigma_a(T)$ . It follows from the openess of R(T) that $\lambda \in -\sigma(T) \cap -\sigma_a(T)$ .

Hence, $\sigma_a(T)\ C \sim -\sigma_a(T)C \sim R(T) = \sigma(T)$

**LEMMA 1.2:** If T is a seolf adjoint operator on H, then $R(T) = -\sigma(T) = -\sigma_a(T)$ .

**PROOF:** Let $\xi \in -\sigma_a(T)$ and compute $r > 0$ such that $|\xi - \lambda| \geq 2r$ for each $\lambda \in \sigma_a(T)$ . Let $b > 0$ be a bound for T, let $f(x) = x - \xi$ for all $[-b,b\}$, and construct $g \in C[-b,b]$ , by such that $g(x) = \frac{1}{x - \xi}$ whenever $|x - \xi| \geq r$ ,

Then, $[\xi - r, \xi + r]$ is a compact support for the function

$$h = fg - 1 \quad |f\ h(T) \neq 0$$

$[\xi - r, \xi + r] \cap \sigma_a(T)$ is inhibited [7] (P.P11 prop 2.3) by Doughlas & Hajime]

Since this contradicts our choice of r, we conclude that h (T) = 0 and therefore, f(T) g(T) = I.

Thus,

$T - \xi I$ has inverse $g(T) \in B(H)$ and therefore, $\xi \in R(T)$ , that is, $-\sigma_a(T) \subset R(T)$ ...(5.1).

Reference to LEMMAN 1.1 above completes the proof since $R(T) \subset -\sigma_a(T)$...(5.2) combining (5.1) and (5.2) we have

$$R(T) = -\sigma_a(T) = -\sigma(T)$$

**COROLLARY 1.3**: If T is a self adjoint operator on H, then $\sigma(T) \subset \mathbb{R}$ ; if also T is positive, then each element of $\sigma(T)$ is non-negative.

**PROOF**: If $\lambda \in \mathbb{C}$ has no n-zero imaginary part, then as $\sigma(T) \subset \mathbb{R}$ , $\lambda$ belongs to $-\sigma_a(T)$ and therefore, by Lemma 1.2, to R(t). And consequently, $\lambda \notin \sigma(T)$ . If also T is positive, then a similar argument shows that each negative real number belongs to R(T). The desired conclusion then follows.

**PROPOSITION 1.4**: If T is a self adjoint operator on H such that $\sigma_a(T)$ is compact, then $\sigma(T) = \sigma_a(T)$ .

**PROOF**: From Lemma 1.1, it will be enoughto prove that

$\sigma(T) \subset \sigma_a(T)$ . Now, let $\xi \in \sigma(T)$ . If $\rho(\xi, \sigma_a(T) > 0$ , then

(1250)

by Lemma 1.2, $\xi \in -\sigma_a(T) = R(T)$ , a contradiction. Hence, $I(\xi, \sigma_a(T)) = 0$ , so $\xi$ belongs to the closed set $\sigma_a(T)$ .

Thus,

$$\sigma(T) = \sigma_a(T) \ .$$

**REMARK!** We cannot replace compactness of $\sigma_a(T)$ by compactnes of $\sigma(T)$ in the Lypothesis of the above proposition. [15] [Refer Brouwerian Example 1.3]. Here,we se that if T is a normable self adjoint operator on $L^2$ with compact spectrum then

$$\sigma(T) \subset \sigma_a(T)$$

**COROLLARY 1.5:** If T is a compact self adjoint operator on H, then $\sigma(T) = \sigma_a(T)$ and $\sigma(T)$ is compact.

**PROOF:** By using polynomial approximations to continuous functions it is straight forward to prove that a compact self adjoint operatoris functionally normable, see [13,14]. The

desired conclusions now follow from Theorem:

Let T be a self adjoint operator on H with bound b > 0. Then T is functionally normable if and only if $\sigma_a(T)$ is compact; in which case $\|f(T)\| = \|f\|_{\sigma_a(T)}$ for each f in C [-b,b].

[See [15] p.p 14,15]

**DEFINITION:** A self - adjoint operator with bound b>o is FUNCTIONALLY NORMABLE if $\|f(T)\|$ exists for each f in C [-b,b].

and proposition 1.4 above.

**SPECTRA OF COMPACT SELF ADJOINT OPERATORS**

The spectrum of a compact self adjoint operator on a finite dimensional Hilbert space is the closure of finitely enumerable sets; and the spectrum of a compact self adjoint operator with infinite-dimensional range is the closureof a sequence of approximate eigen values converging to zero.

If T is a self adjoint operator on a finite-dimensional Hilbert space, then

$$\sigma_a(T) = \{\lambda : det(T - \lambda I) = 0\}$$

If N is a positive integer, and T is a self-adjoint operator on an N-dimnsional Hilbert space, then here exist real numbers $\lambda_1 \ldots\ldots\ldots \lambda_N$ such that $\sigma_a(T) = \{\lambda_1, \ldots\ldots\ldots \lambda_N\}$ .

PROOF: By the Fundamental theorem of algebra,there exist complex numbers $\lambda_1 \ldots\ldots\ldots \lambda_N$ such that

$$det(T - \lambda I) = (\lambda - \lambda_1) \ldots\ldots (\lambda - \lambda_N)$$

The desired conclusion follows immediately.

If T is a compact self adjoint operator on H with infinite - dimensional range, then for each r > o here exists $\lambda$ in $\sigma(T)$ such that $0 < |\lambda| \leq r$ . See (*) above. [see elaborate proof in [R*]].

Let us look at our main result about compact self adjoint operators

THEOREM 1.6:

Let T be a compact self adjoint operator on a Hilbert space H, such that ran(T) is infinite dimensional. The there exists an increasing sequence $(H_n)$ of finite-dimensional invariant subspacesfor T and a sequence $(\lambda_n)$ of non-zero elements of $\sigma(T)$ , such that

(1) $\overline{\bigcup_{n=1}^{\infty} H_n}$ is dense in ran (T)

(2) the restriction of T to $H_n$ is a one-to-one compact self adjoint operator, mapping $H_n$ onto $H_n$, with a compact self adjoint inverse;

(3) $\lambda_n \to 0$ *as* $n \to \infty$

(4) $\sigma(t) = \{\lambda_n : n \geq 1\}$

If T is a compact self adjoint operator on a Hilbert space with infinite dimensional range, then R(T) is dense in $\mathbf{R}$ and hence located.

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