ABSTRACT EVOLUTION EQUATIONS AND THE CAUCHY AND MIXED PROBLEMS FOR SYMMETRIC HYPERBOLIC SYSTEMS

by

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DEDICATED TO

my father
Elijah Olagunju

and my mother
Deborah Olagunju

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CHAPTER 1

INTRODUCTION

In this thesis we shall consider systems of equations of the form

$$L[u] = \sum_{i=0}^{n} A^j(x)\frac{\partial u}{\partial x_j} + B(x)u = f(x) \qquad (1.1.1)$$

where u takes values in some Banach space X and the A^j and B are linear operators on X; the A^j being symmetric and A^o positive definite for all x. Such systems are called symmetric hyperbolic. They can be transformed linearly into

$$L[u] = \bar{L}[v] = \dot{v}_t + \sum_{j=1}^{n} \bar{A}^j \frac{\partial v}{\partial x_j} + \bar{B}v = \bar{f}$$

where the operators $\bar{A}^j$ are still symmetric.

Many of the equations of mathematical physics derived from variational principles are usually symmetric hyperbolic. Also, certain second order partial differential equations can be reduced to first order equations which are symmetric hyperbolic. Generally, given a second order equation of the form

$$U_{x_o x_o} = \sum_{i,k=1}^{n} a_{ik}(x)\, U_{x_i x_k} + \sum_{i=o}^{n} b_i(x)\, U_{x_i} + w(x)$$

in Eucklidean (n+1) - space R^{n+1} with $a_{ik} = a_{ki}$, if we introduce the vector with components $U^i = U_{x_i}$ for $i=o,\ldots\ldots, n+1$, we have

$$U^o_{x_o} - \sum_{i,k=1}^{n} a_{ik}(x)\, U^i_{x_k} - \sum_{i=o}^{n} b_i(x)\, U^i = w(x)$$

$$\sum_{k=1}^{n} a_{jk}\, U^k_{x_o} = \sum_{k=1}^{n} a_{jk} U^o_{x_k} \quad \text{for } j=1,\ldots\ldots,n.$$

This is again of the form (1.1.1).

Symmetric hyperbolic systems have been treated by Friedrichs, Lax and Phillips, Kato, Fischer and Marsden, Massey III, Cordes and Moyer to mention just a few.

The usual approach to solving this problem is to derive a - priori estimates using energy inequalities for the system. This immediately yields a uniqueness theorem for a solution u. The a-priori estimates can also be made the basis of an existence proof. One can either attempt to represent a solution as a limit of solutions of problems which can be solved (e.g. by finite -difference approximations or by the theorem of Cauchy - Kowalewski for analytic data and coefficients) and establish convergence with the help of the a-priori estimates, or use a more abstract Hilbert space approach to conclude existence from uniqueness.

In recent years, this problem has been solved from the point of view of abstract evolution equations. This thesis is a description of some of the workthat has been done in this direction. It is based mainly on some of the works of Kato. Others who have studied this problem include Massey III [1] and Fischer and Marsden [1] . The last mentioned have applied this theory to give new proofs of the existence and uniqueness theorems of the Eistein equations in General Relativity due to Choquet - Bruhat and Lichnerowicz. These new proofs using first order techniques are considerably simpler than the classicial proofs based on second-order techniques. The existence results obtained are as sharp as had been previously known and their uniqueness results improve by one degree of differentiability those previously existing in the literature.

CHAPTER II

ABSTRACT EVOLUTION EQUATIONS

I. Preliminaries.

Definition 2.1.1

Let X be a Banach space and T_t, $t \geq o$ a one parameter family of bounded linear operators $\in B(X)$, the space of all bounded linear operators on X. If $\{T_t : t \geq o\}$ satisfy the conditions

$$T_t \ T_s = T_{t+s} \quad \text{for } t,s \geq o$$

$$T_o = I$$

$$\underset{t \to t_o}{\text{S-lim}} \ T_t x = T_{t_o} x \quad \text{for each } t_o \geq o \text{ and each } x \varepsilon X$$

then $\{T_t\}$ is called a semigroup of class Co. A semigroup $\{T_t\}$ of class C_o satisfies the condition $||T_t|| \leq M e^{\beta t}$ for $o \leq t < \infty$ with constants $M > 0$ and $\beta < \infty$. If $M = 1$ and $\beta = 0$, $\{T_t\}$ is called a contraction semigroup of class C_o.

Definition 2.1.2

Let $\{T_t : t \geq o\}$ be a semigroup of class C_o. The infinetesimal generator A of T_t is defined by

$$Ax = \lim_{h \to o} h^{-1}(T_h - I) x$$

ie. A is a linear operator whose domain is the set

$$D(A) = \left\{ x \varepsilon X : \lim_{h \to o} h^{-1} (T_h - I) x \text{ exists in } X \right\}$$

and for $x \varepsilon D(A)$

$$A x = \lim_{h \to o} h^{-1}(T_h - I) x$$

A is densely defined, closed and has resolvent

$$(\lambda I - A)^{-1} x = \int_0^\infty e^{-\lambda t} \ T_t \ xdt \qquad \text{for} \quad \lambda > o$$

Moreover

$$T_t\, x = \lim_{n\to\infty} \exp\ (tA\,(I - n^{-1}A)^{-1})x,\ x \in X$$

and the convergence is uniform in t for every compact interval of t. This is the representation theorem for semi-groups.

Let X be a real or complex Banach space. A linear operator $-A$ in X generates a semigroup $\{T_t\} = \{e^{-tA}\}$, $0 \leq t < \infty$ of class C_o on X if and only if A is densely defined, closed, and has resolvent satisfying

$$||(A+\lambda)^{-n}|| \leq M(\lambda-\beta)^{-n},\ \lambda>\beta \quad n = 1,2,\ldots\ldots\ldots\ldots \tag{2.1.1}$$

In this case we have

$$||e^{-tA}|| \leq Me^{\beta t}, \qquad t \geq o \tag{2.1.2}$$

$$e^{-tA} = S\text{-}\lim_{n\to\infty} (I + n^{-1}t\,A)^{-n} \qquad t \geq o \tag{2.1.3}$$

$$(A+\lambda)^{-1} = \int_o^\infty e^{-\lambda t}\, e^{-tA}\, dt \qquad \lambda>\beta \tag{2.1.4}$$

For $-A$ to generate a C_o-semigroup it suffices that (2.1.1) holds for sufficiently large λ and n. We shall denote by $G(X,M,\beta)$ the set of all A satisfying the condition (2.1.1) and write

$$G(X,M) = \bigcup_{-\infty<\beta<\infty} G(X,M,\beta) \quad \text{and}$$

$$G(X) = \bigcup_{M>0} G\ (X,M)$$

A linear operator A on $D(A) \subset X$ into X where X is a Hilbert space is called accretive (the terminology is due to K-Friedrichs and T.Kato) if $\mathrm{Re}(Au,u) \geq o$ for all $u \in D(A)$. Here $(\ ,\ )$ denotes the scalar products in X.

A is called dissipative (the terminology is due to R.S. Phillips) if $-A$ is accretive. (see Yosida [1]).

A is called m-accretive if $A+\lambda$ has range X for all λ with $\lambda>o$. This implies that $-A$ generates a C_o-semigroup of contraction in X (see Massey III [1]).

2. Integration of Equations of Evolution in Banach spaces.

By equations of evolution we mean equations in which a distinguished variable t plays the role of time propagation. This includes the differential equations of mathematical physics that describe time dependent processes like the diffusion equations, wave equations and Schrödinger equations.

It is known that the initial value problem

$$\frac{dy}{dt} = \alpha y , \quad y(o) = 1 \tag{2.2.1}$$

is satisfied by the ordinary exponentral function.
Consider the diffusion equation

$\frac{\partial u}{\partial t} = \Delta u$ where $\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian in R^m and the initial condition

$u(x,o) = f(x)$, where $f(x) = f(x_1,\ldots\ldots,x_m)$ is a given function of x. We seek a solution $u = u(x,t)$, $t \geq o$ of the Cauchy problem.
The Cauchy problem for the wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = \Delta u , \qquad -\infty<t<\infty$$

with initial data

$$u(x,o) = f(x)$$

$$\frac{\partial u}{\partial t} = g(x) \qquad \text{for } t = o$$

This may be written in vector form as follows

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} o & I \\ \Delta & o \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \quad v = \frac{\partial u}{\partial t}$$

with initial condition

$$\begin{pmatrix} u(x,o) \\ v(x,o) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}$$

These two examples show that in a suitable function space the wave equation and the diffusion equation are similar to (2.2.1). Since the solution of (2.2.1) is the exponential function, it suggests that the heat equation and the wave equation may be solved by properly defining the exponential functions for the operators Δ and $\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ in suitable function spaces. This gives a motivation for the application of semi-group theory to differential equations. In what follows we consider the integration of the equation of evolution

$$\frac{du(t)}{dt} = A(t)u(t) + f(t) \qquad a \leq t \leq b \tag{2.2.2}$$

and the associated homogeneous equation

$$\frac{d}{dt}u(t) = A(t)u(t) \tag{2.2.3}$$

Here the unknown $u(t)$ is an element in a complex Banach space X depending on a real variable t, where $f(t)$ is a given element of X and $A(t)$ is a given, in general unbounded, linear operator (the non-linear ease will be mentioned briefly later) in X, both depending on t.

The solution of (2.2.2) is formally given by

$$u(t) = U(t,a)\phi + \int_a^t U(t,s)f(s)ds, \quad u(a) = \phi \tag{2.2.4}$$

where $U(t,s)$ is a linear operator in X depending on s, t with $s \leq t$. The main task in solving equations of the form (2.2.3) therefore is proving the existence of the evolution operator $U(t,s)$.

The first fundamental contributions to non-stationary abstract evolution equations using Semigroup theory were made by Kato [1], [7]. Kato [1] showed that if $A(t) = A$ is independent of t, $U(t,s)$ is given formally by $U(t,s) = \exp[(t-s)A]$.
The rigorous definition of the exponential function has been given by Hille [1] and Yosida [1] in connection with the analytic theory of semigroups. By taking their assumptions on the infinitesimal generator A for $-A(t)$ for each t, Kato generalized some of the results to the case in which $A(t)$ actually depends on t. He then presented sufficient

conditions for the existence of $U(t,s)$ and studied its properties. Before discussing some of the results of that work we introduce the following definition -

Definition 2.2.1

A linear operator A in X is said to have property S if the following conditions are satisfied:

(1) A is a closed linear operator with domain dense in X.

(2) The resolvent set of A includes all positive reals and $||(I - \lambda A)^{-1}|| \leq 1$ for $\lambda > 0$.

The following are the main assumptions on $A(t)$. (A_1) $A(t)$ is defined for $a \leq t \leq b$ and has property S for each t.

(A_2) (1) The domain D of $A(t)$ is independent of t [which implies that $B(t,s) = [I-A(t)][I-A(s)]^{-1}$ is a bounded linear operator for each s,t]

(2) $B(t,s)$ is uniformly bounded, that is, there is a $M>0$ such that $||B(t,s)|| \leq M$ for every s,t

(3) $B(t,s)$ is of bounded variation in t in the sense that there is a $N \geq 0$ such that

$$\sum_{j=1}^{n} || B(t_j,s) - B(t_{j-1},s)|| \leq N \text{ for every partition}$$

$a = t_o < t_1 < \ldots\ldots < t_n = b$ of the interval (a,b) at least for some s

(A_3) $B(t,s)$ is weakly continuous in t at least for some s

(A_4) $B(t,s)$ is weakly differentiable in t and $\frac{\partial}{\partial t} B(t,s)$ is strongly continuous in t, at least for some s.

If conditions A_1 and A_2 are satisfied Kato proved that the evolution operator $U(t,s)$ exists. If in particular X is reflexive then $U(t.s)\, D \subset D$ and in this case $U(t,s)$ is uniquely determined.

If X is an arbitrary Banach space and if conditions A_1, A_2, A_3, A_4 are satisfied then $U(t,s)$ satisfies $U(t,s)D \subset D$,

$u(t)$ is strongly differentiable, (here $u(t) = U(t,o)\phi$),

$\frac{d}{dt} u(t) = A(t)\, u(t)$ and $A(t)u(t)$ is strongly continuous.

Furthermore if $\phi \in D$ and if $t \to f(t)$ is a function on (a,b) in D such that $[A(r) - I]\ f(t)$ is strongly continuous in t for some fixed r, the expression (2.2.4) is well defined, belongs to D for each t, is strongly differentiable in t and satisfies the differential equation (2.2.2) with initial condition $u(a) = \phi$.

Elliot [1] has studied the same problem and proved the existence and uniqueness of the evolution operator using an entirely different method from Kato's. Her condition on $A(t)$ is Kato's first condition, a ssumption S. She relaxes Kato's condition that the domain of $A(t)$ is independent of t. Her results, however, are too restrictive for applications.

Employing another different method. Goldstein [1] has also proved the existence theorem for $U(t,s)$. Like Elliot, he allows the domain of $A(t)$ to vary with t. His method has the advantage that his smoothness condition on $A(t)$ is easy to verify in practice. However his commutativity hypothesis restricts the applicability of his results to concrete problems.

In recent years Kato has given new sufficient conditions for the existence of the evolution operator that are useful for applications to symmetric hyperbolic equations. We shall give a detailed discussion of some of his results in the next section.

We conclude this section by mentioning briefly some of the work that has been done on non-linear evolution equations.

There has appeared in recent years many works on non-linear evolution equations in Hilbert or Banach spaces. See, for example, Browder [1], Komura [1] Kato [5], [6] , Yosida [1], and Segal [1].

Some of these, like Browder's, consider the semi-linear case

$$\frac{d}{dt} u(t) = A(t)u(t) + f(t,u) \qquad (2.2.5)$$

where $A(t)$ is a linear unbounded operator and $f(t,.)$ a non-linear perturbation.

Komura [1], studied the Cauchy problem for equations of the form

$$\frac{d}{dt} u(t) \in A\, u(t) \quad \text{for a.e. } t \in [0,1] \qquad (2.2.6)$$

where $A:H \to H$ is a multivalued non-linear mapping and H a Hilbert space. His consideration of multi-valued operators in a Hilbert space as the infinitesimal generators of semigroups of non-linear contraction operators, however, is rather general and complicates the theory. (See Komura [1] for a definition of non-linear semigroups.).

By restricting the theory to single valued operators in a Banach space X, Kato [5] constructed the semigroups generated by them in a more elementary fashion. He then considers time-dependent non-linear equations of the form

$$\frac{du}{dt} + A(t)\,u = o \qquad o \leq t \leq T \qquad (2.2.7)$$

where the unknown $u(t)$ is an X-valued function and where $\{A(t)\}$ is a family of non-linear operators with domains and ranges in X. He then proves the existence and uniqueness of the solution of (2.2.7) for a given initial data. His basic assumptions are that the adjoint space X^* is uniformly convex and that $A(t)$ are m-monotonic operators, together with some smoothness condition for $A(t)$ as a function of t. (see Kato [5] for a definition of m-monotonicity). No explicit assumptions are made on the continuity of the operators $A(t)$.

A formulation of the underlying equation as $\frac{du}{dt} = A\,u + K(u)$ was studied by Segal [1]. Here A is a skew-adjoint operator in a Hilbert space and K a given non-linear operator. He developed a simple and general theory of the global existence of solutions of hyperbolic equations that is relevant to relativistic physics. He then studied the Cauchy and scattering problems for such equations.

3. Linear evolution equations of "hyperbolic" type.

In this section we list some of the results which pertain to the integration of equations of evolution in Banach spaces that will be needed in the sequel. We shall consider the Cauchy problems for Linear evolution equations

$$\frac{du}{dt} + A(t)u = f(t) \quad , o \leq t \leq T \ , \quad u(o) = \phi \qquad (2.3.1)$$

where $-A(t)$ are infinitesimal generators of C_o semigroups. This is the so called "hyperbolic" type. This problem has been extensively investigated by Kato [2] and [3]. He weakens the assumption in Kato [1] by only requring that the domain $D(A(t))$ of $A(t)$ satisfy $D(A(t)) \subset Y \not\subset X$ for some Banach space Y and for all t.

Definition 2.3.1.

Let Y be a Banach space densely and continuously embedded in X, and $A \in G(X,M,\beta)$. Y is said to be admissible with respect to A (or simply A-admissible) if $\{e^{-tA}\}$ leaves Y invariant and forms a semigroup of class C_o on Y.

Definition 2.3.2.

Let X be a Banach space and $A(t) \in G(X)$. $\{A(t)\}$ is said to be stable if there are constants M,β (called constants of stability) such that

$$\left\| \prod_{j=1}^{k} (A(t_j) + \lambda)^{-1} \right\| \leqslant M(\lambda-\beta)^{-k} \quad \lambda>\beta \tag{2.3.2}$$

For any finite family $\{t_j\}$ with $o \leqslant t_1 \leqslant \ldots\ldots \leqslant t_k \leqslant T$, $k = 1,2,\ldots\ldots$ The product Π is time-ordered in the sense that a factor with larger t_j stands to the left of ones with smaller t_j.

Proposition 2.3.3.

For each t let $\|\ \ \|_t$ be a new norm on X equivalent to the original one, depending on t smoothly in the sense that

x≠0 $\|x\|_t / \|x\|_s \leqslant e^{c|t-s|}$, $x \in X$, $s,t \in [o,T]$. For each t let $A(t) \in \cancel{G(X}_t$ $G(X_t,1,\beta)$ where X_t stands for the space X with norm $\|\ \ \|_t$. Then $\{A(t)\}$ is stable with $M = e^{2cT}$ with respect to $\|\ \ \|_t$ for any $t \in [o,T]$.

Proof

Let $\|\ \ \|_t$ also denote the associated operator norm, so that

$$\|(A(t_j) + \lambda)^{-1}\|_t \leqslant (\lambda-\beta)^{-1}, \quad \lambda>\beta \quad \text{by hypothesis.}$$

Now

$$\left\| \prod_{j=1}^{k} (A(t_j) + \lambda)^{-1} x \right\|_T \leqslant e^{c(T-t_k)} \left\| \prod_{j=1}^{k} (A(t_j)+\lambda)^{-1} x \right\|_{t_k}$$

$$\leqslant e^{c(T-t_k)} (\lambda-\beta)^{-1} \left\| \prod_{j=1}^{k-1} (A(t_j)+\lambda)^{-1} x \right\|_{t_k}$$

. . . .

$\leq (\lambda-\beta)^{-k} e^{c(T-t_k)} e^{c(t_k-t_{k-1})} \ldots\ldots e^{ct_1} \|x\|_0$

$\leq (\lambda-\beta)^{-k} e^{2cT} \|x\|_T$

So (2.3.1) is satisfied for $\| \ \|_T$ with $M = e^{2cT}$ and β. The same is true for $\| \ \|_T$ replaced by $\| \ \|_t$.

Proposition 2.3.4.

Let $\{A(t)\}$ be stable with constants M, β and let $B(t) \in B(X)$ be uniformly bounded ($\|B(t)\| \leq K$). Then $A(t) + B(t) \in G(X)$ and forms a stable family with constants M and $\beta + MK$.
(For the proof see Hille - Phillips [1], Theorem 13.2.1)

We now state without proof sufficient conditions for the existence of the evolution operator in the following

Theorem 2.3.5

Let X and Y be Banach spaces such that Y is densely and continuously embedded in X. Let $A(t) \in G(X)$, $0 \leq t \leq T$, and assume that

(i) $A(t)$ is stable with constants M, β

(ii) Y is $A(t)$-admissible for each t. If $\tilde{A}(t) \in G(Y)$ is the restriction of $A(t)$ to Y, $\{\tilde{A}(t)\}$ is stable, say with constants $\tilde{M}$, $\tilde{\beta}$.

(iii) $Y \subset D(A(t))$ so that $A(t) \in B(Y,X)$ the space of bounded linear maps on Y to X, for each t, and the map $t \to A(t)$ is norm continuous.

Under these conditions, there exists a unique evolution operator $U(t,s) \in B(X)$ defined for $0 \leq s \leq t \leq T$, with the following properties

(a) $U(t,s)$ is strongly continuous (X) in s,t with $U(s,s) = I$ and $\|U(t,s)\| \leq M e^{\beta(t-s)}$

(b) $U(t,r) = U(t,s)U(s,r) \qquad r \leq s \leq t$

(c) $(D_t^+ U(t,s)y)_{t=s} = -A(s)y, \quad y \in Y \qquad 0 \leq s \leq t \leq T.$

(d) $\frac{d}{ds} U(t,s)y = U(t,s)A(s)y, \qquad y \in Y, \quad 0 \leq s \leq t \leq T.$

Here D_t^+ denotes right derivative, and $\frac{d}{ds}$ derivative in the strong sense of X.

If we assume further that

(iv) Y is reflexive

(v) The family $\{\tilde{A}(t)\}$ satisfies the condition of proposition (2.3.3) in which all the norms $\| \ \|_t$ are uniformly convex

and if we replace (ii) by

(ii′) there is a family $\{S(t)\}$ of isomorphisms of Y onto X such that S(t) is strongly continuously differentiable (Y,X) and that $S(t)A(t)S(t)^{-1} = A(t) + B(t)$, $B(t) \in B(X)$ where B(t) is strongly continuous (X); then the family U(t,s) satisfies the following additional properties.

(e) $U(t,s)Y \subset Y$, $\| U(t,s)\|_Y \leq \tilde{M} e^{\beta(t-s)}$ and U(t,s) is weakly continuous (Y) in s,t.

(f) U(t,s) is strongly continuous (Y) for fixed t

(g) For each fixed $y \in Y$ and s, U(t,s) is strongly right-continuous (Y) in t for all $t \geq s$, and strongly continuous (Y) in t

(h) For each fixed $y \in Y$ and s, $\frac{d}{dt} U(t,s)y$ exists for all $t \geq s$, equals $-A(t)U(t,s)y$, and is strongly continuous (X) in t.

(f') U(t,s) is strongly continuous jointly in s,t.

If the assumptions of the above theorem are satisfied and $B(t) \in B(X)$ $o \leq t \leq T$ is strongly continuous (X) in t, then there exists a unique evolution operator $\{V(t,s)\}$ for the family $\{A(t)+B(t)\}$ with properties corresponding to (a) to (d).

If the evolution operator U(t,s) exists for $\{A(t)\}$ and satisfy conditions (a) to (h) and (f') of theorem 2.3.5 then the solution of (2.3.1) is formally given by

$$u(t) = U(t,o)\phi + \int_0^t U(t,s)f(s)ds \qquad (2.3.3)$$

A solution u must satisfy (2.3.3) if it is continuous (X) differentiable (X) ,$u(t) \in D(A(t))$ and satisfy (2.3.1) This is obvious from the following heuristic argument. From (d) of theorem 2.3.5 we have

$$\frac{d}{ds}[U(t,s)u(s)] = \frac{d}{ds}U(t,s)u(s) + U(t,s)\frac{d}{ds}u(s)$$
$$= U(t,s)A(s)u(s) + U(t,s)[f(s) - A(s)u(s)]$$
$$= U(t,s)f(s)$$

Integrating on $[o,t]$, we have

$U(t,t)u(t) - U(t,o)u(o) = \int_0^t U(t,s)f(s)ds.$

which gives

$u(t) = U(t,o)\phi + \int_0^t U(t,s)f(s)ds.$

Although (2.3.3) makes sense for any $\phi \in X$ and any continuous f, the resulting u may not be differentiable. For u to be a genuine solution of (2.3.1) certain conditions on f must be satisfied.

Theorem 2.3.6.

Suppose the evolution operator $\{U(t,s)\}$ for $\{A(t)\}$ exists and satisfies conditions (a) to (h) and (f') of theorem 2.3.5. Let $\phi \in Y$ and let $f(t)$ be continuous (Y) in t. If U is given by (2.3.3) then $u(t) \in Y$, $u(t)$ is strongly continuously differentiable (X) for all $t \in [o,T]$ and it satisfies (2.3.1).

Proof

By (h) of theorem 2.3.5

$$\frac{d(U(t,o)\phi)}{dt} = \frac{d}{dt}U(t,o)\phi \qquad t \geq o$$
$$= -A(t)U(t,o)\phi \qquad t \geq o$$

Giving

$$\frac{d}{dt}(U(t,o)\ ) + A(t)U(t,o)\phi = 0$$

Now, $U(t,o)\phi = \phi$ for $t = 0$, so the first term on the right hand side of (2.3.3) satisfies the homogeneous equation and the initial ccondition.

We now show that the second term on the right hand side of (2.3.3) satisfies the inhomogeneous equation with vanishing initial data.

By condition (f') of theorem 2.3.5

$U(t,s)f(s)$ is continuous (Y) jointly in s,t;

$A(t)U(t,s)f(s)$ is continuous (X) in s,t and

$$\frac{d}{dt}\int_0^t U(t,s)f(s)ds = \int_0^t \frac{d}{dt} U(t,s)f(s)ds + U(t,t)f(t)$$

$$= \int_0^t - A(t)U(t,s)f(s)ds + f(t)$$

for $t > o$ by (h) of theorem 2.3.5. Thus we have

$$\frac{d}{dt}\int_0^t U(t,s)f(s)ds + A(t)\int_0^t U(t,s)f(s)ds = f(s)$$

This shows that the second term on the right hand side of (2.2.3) has the desired property. By superposition we see that u as given in (2.3.3) satisfies (2.3.1)

Uniqueness follows from the uniqueness of the evolution operator.

In order to be able to apply this theory to non-linear problems we need some approximation theorems for equation (2.3.1). This is done by generalizing the notion of stability given in above and then proving a new existence theorem for the evolution operator.

Definition 2.3.7

Let X be a Banach space. A family $A = \{A(t)\}$ of elements of $G(X)$ is said to be quasi-stable if

$$\left\|\prod_{j=1}^{k}(A(t_j) + \lambda_j)^{-1}\right\| < M\prod_{j=1}^{k}(\lambda_j - \beta(t_j))^{-1} \qquad (2.3.4)$$

for every finite family of real numbers $\{t_j, \lambda_j\}$ such that $o \leq t_1 \ldots \leq t_k \leq T$, $\lambda_1 > \beta(t_1)$, $\lambda_2 > \beta(t_2), \ldots\ldots\ldots, \lambda_k > \beta(t_k)$, where M is a constant and β is a real valued upper - integrable function (in the Lebesgue sense) defined a.e on I. $\{M,\beta\}$ is called a stability index for A and the product on the left of (2.3.4) is time-ordered in increasing time. Here $I = [o,T]$.

Theorem 2.3.8 (See Kato [3])

Let a family $A = \{A(t)\}$ satisfy the following

(i) $A: I \to G(x)$ is quasi-stable with index $\{M,\beta\}$

(ii) There is a Banach space Y, continuously and densely embedded in X and a family $S = \{S(t)\}$ of isomorphisms of Y onto X, such that $S(t)A(t)S(t)^{-1} = A(t) + B(t)$, $B(t) \in B(X)$ for a.e $t \in I$, where $B:I \to B(X)$ a.e is strongly measurable with $||B(t)||_X$ upper integrable. Furthermore there is a strongly measurable function $\mathring{S}:I \to B(Y,X)$ a.e with $||\mathring{S}(.)||_{Y,X}$ upper integrable.on I, such that S is equal to an indefinite strong integral of $\mathring{S}$.

(iii) $Y \subset D(A(t))$, $t\in I$ and $A:I \to B(Y,X)$ is norm continuous.

Then there exists a unique evolution operator $U = \{U(t,s)\}$ defined on the triangle $\Delta: T \geq t \geq s \geq o$, with the following properties

(a) U is strongly continuous on Δ to $B(X)$ with $U(s,s) = I$

(b) $U(t,s)\, U(s,r) = U(t,r)$

(c) $U(t,s)\, Y \subset Y$, and U is strongly continuous on Δ to $B(Y,X)$

(d) $\frac{d}{dt} U(t,s) = - A(t)\, U(t,s)$,

$\frac{d}{ds} U(t,s) = U(t,s)A(s)$, which exist in the strong sense in $B(Y,X)$ and are strongly continuous on Δ to $B(Y,X)$.

If the assumptions of theorem 2.3.8 are satisfied, then the mild solution of (2.3.1) is again given by (2.3.3). For simplicity we abbreviate (2.3.3) to

$$u = U(t,o)\phi + \int_0^t U(t,s)f(s)ds \equiv U(\delta\otimes\phi) + U f$$

$$\equiv U(\delta\otimes\phi \oplus f)$$

As was mentioned above u is not a solution in the strict sense. However, we have the following results regarding the map U.

Let $L(X) = L^1(I,X)$, $C(X) = C(I,X)$ etc.

Theorem 2.3.9.

U maps (a) $\delta\otimes X \oplus L(X)$ into $C(X)$

(b) $\delta\otimes Y \oplus L(Y)$ into $C(Y)$

(c) $\delta\otimes Y \oplus L(Y)\, C(X)$ into $C(Y)\cap C^1(X)$

The map U is continuous in each case and we have for (2.3.3)

$$|| u ||_{\infty,X} \leqslant || U ||_{\infty,X} (|| \phi ||_X + ||f||_{1,X}) \text{ in case (a)} \qquad (2.3.5)$$

$$|| u ||_{\infty,Y} \leqslant || U ||_{\infty,Y} (|| \phi ||_Y + || f ||_{1,Y}) \text{ in cases (b) and (c)} \quad (2.3.6)$$

$$|| \frac{du}{dt} ||_{\infty,X} \leqslant || f ||_{\infty,X} + || A ||_{\infty\, Y,X} || U ||_{\infty,Y} (|| \phi ||_Y + || f ||_{1,Y}) \qquad (2.3.7)$$

in case (c) u is a solution of (2.3.1) in the strict sense.
(Here we have $|| f ||_{\infty,X} = \int_I^* ||f(t)||_X dt$ etc. and $f:T \to X$ a.e.
For an operator valued function $F:I \to B(Y,X)$ a.e we define $|| F ||_{\infty,Y,X}$ and $|| F||_{I,Y,X}$ similarly with $|| F ||_{\infty,X} = || F ||_{\infty,X,X}$ and so on and $\int^*$ denotes upper integral).

Proof of theorem 2.3.9

If $\phi \in X$ and $f \in C(X)$ then $u \in C(X)$ and (2.3.5) holds from (2.3.3). Similarly we prove (b) and (2.3.6) To prove (c) we use the result of theorem 2.3.6 namely, if $\phi \in Y$ and $f \in C(Y)$, their $u \in C(Y) \cap C^1(X)$ and solves the initial value problem. (2.3.7) then follows from (2.3.1) and (2.3.6). The required result follows since $C(Y)$ is dense in $L(Y) \cap C(X)$.

In order to develop the approximation theory that we need we introduce some primitive constants that describe the behaviour of the family $\{A(t)\}$ and then estimate $||U||_{\infty,X}, ||U||_{\infty,Y}$ in terms of these constants. We shall call primitive constants for $\{A(t)\}$ the following system of numbers (or any system majorizing them.).

$M, ||\beta||_1, ||S||_{\infty,Y,X}, ||S^{-1}||_{\infty,X,Y} ||B-C||_{1,X}$ where $\{M,\beta\}$ is the stability index for A. $C(t) = \dot{S}(t)S(t)^{-1} \in B(X)$ for a.e $t \in I$. From the construction of U we have

$$||U||_{\infty,X} \leqslant M \exp||\beta||_1 \qquad (2.3.8)$$

$$||U||_{\infty,Y} \leqslant ||S||_{\infty,Y,X} ||S^{-1}||_{\infty,X,Y} ||W||_{\infty,X} \qquad (2.3.9)$$

where $||W||_{\infty,X} \leqslant ||U||_{\infty,X} \exp(||B-C||_{1,X} ||U||_{\infty,X})$. (See Kato [3]).

Suppose we have another equation of the same type

$$\frac{du'}{dt} + A'(t)u' = f'(t),\ o \leq t \leq T,\ u'(o) = \phi' \qquad (2.3.10)$$

in the same Banach space X. For $A'(t)$ we make the same basic assumptions (i),(ii),(iii) of theorem 2.3.8 and we use the obvious notations such as $S'(t), B'(t)$ etc. The space $Y \subset X$ is assumed to be common to the two systems. Then there exists the evolution operator U' for the new system and the mild solution of (2.3.10) is given by

$$u' = U'(\delta \otimes \phi' \oplus f')$$

We now obtain an estimate for $u'-u$ in the X-norm, where $u = U(\delta \otimes \phi \oplus f)$ is a solution of (2.3.1).

Theorem 2.3.10

Let $\phi \in Y$, $f \in L(Y)$, $\phi' \in X$ and $f' \in L(X)$.

Then
$$||u'-u||_{\infty,X} \leq K'[||\phi' - \phi||_X + ||f' - f||_{1,X} + ||(A'-A)u||_{1,X}]$$

where K' is a constant depending only on the primitive constants for A' and $A'-A$ is regarded as a multiplication operator on $C(Y)$ to $L(X)$. (By theorem 2.3.9(b) $u \in C(Y)$).

Proof

From (d) of theorem 2.3.8 we have

$$\frac{d}{ds} U'(t,s)U(s,r)y = \left(\frac{d}{ds} U'(t,s)\right)U(s,r)y + U'(t,s)\frac{d}{ds} U(s,r)y$$

$$= U'(t,s)A(s)U(s,r)y - U'(t,s)A(s)U(s,r)y$$

Integrating we have

$$[U'(t,s)U(s,r)y]_r^t = \int_r^t U'(t,s)\,[A'(s) - A(s)]\,U(s,r)y\,ds.$$

or

$$U(t,r)y - U'(t,r)y = \int_r^t U'(t,s)[A'(s) - A(s)]U(s,r)y\,ds.$$

So, we have the following identity

$$U'(t,r)y - U(t,r)y = -\int_r^t U'(t,s)[A'(s)-A(s)]U(s,r)y\,ds$$

Setting $r=o$ and $y=\phi$ we have

$$(U'-U)(\delta \otimes \phi) = -U'(A'-A)U(\delta \otimes \phi)$$

Setting $y = f(r)$ and integrating in r gives

$(U'-U)\ f = -\ U'(A'-A)Uf$. Hence

$$(U'-U)(\delta \otimes \phi \oplus f) = -\ U'(A'-A)u \qquad (2.3.11)$$

On the other hand we have

$$u'-u = U'[\delta \otimes \phi'-\phi)\oplus(f'-f)] + (U'-U)(\delta \otimes \phi \oplus f).$$
$$= U'[\delta \otimes (\phi'-\phi)\oplus(f'-f)] - U'(A'-A)u$$

using (2.3.11). So we, have

$$||u'-u||_{\infty,X} \leq ||U'||_{\infty,X}\,[||\phi'-\phi||_X + ||f'-f||_{1,X} + ||(A'-A)u||_{1,X}],$$

which gives the desired result with $K' = ||U'||_{\infty,X}$

4. Function spaces.

In this section we give a brief discussion of various function spaces that will be required in the sequel. The proofs that will be given later lean heavily on some aspects of the theory of Sobole spaces. Hence, we present without proofs, a number of technical lemmas on embeddings of Sobolev spaces. Proofs and more extensive discussions can be found in Hormander [1], Lion -Magenes [1], Kato [4] and Yosida [1].

In what follows P denotes a fixed Hilbert space, with the inner product (,) and norm $|\ \ |$. $B(P)$ denotes the Banach algebra of all bounded linear operators on P to P, with the operator norm also denoted by $|\ \ |$. Unless otherwise stated the topology of $B(P)$ will be the one induced by the norm.

Definition 2.4.1

For integers $k \geq o$ $C_b^k(R^m,P)$ denotes the space of all P-valued functions on R^m that have continuous and bounded derivatives of order $\leq k$, with the usual sup norm.

Definition 2.4.2

For $p \geq 1$, $L^p(R^m,P)$ denotes the space of all (equivalence classes of) p-valued measurable functions u on R^m such that

$||u||_{L^p} = (\int |u(x)|^p\, dx)^{\frac{1}{p}} < \infty$ (where integration is over R^m) with the usual convention when $p=\infty$.

We define in a similar way $C_b^k(R^m,B(P))$ and $L^p(R^m,B(P))$

For the space L^2 we shall write $\| \ \|$ for $\| \ \|_{L^2}$.

Definition 2.4.3

Let $f(x)$ be a P-valued function defined on R^m. By the support of f we mean the smallest closed set of R^m outside which f vanishes identically.

Definition 2.4.4

By $C_o^k(R^m,P)$, $o \leq k \leq \infty$, k an integer, we mean the set of functions $\varepsilon\ C^k(R^m,P)$ whose supports are compact subsets of R^m.

The Sobolev Space H^s.

For each integer $s \geq o$, $H^s(R^m,P)$ denotes the space of all $u \in L^2(R^m,P)$ such that all distribution derivatives $D^\alpha u$ with $|\alpha| \leq s$ belong to $L^2(R^m,P)$.

$H^s(R^m,P)$ is a Hilbert space under the norm

$$\|u\|_s^2 = \|(I-\Delta)^{s/2} u\|^2 = \sum_{|\alpha| \leq s} C_\alpha \| D^\alpha u \|^2 \qquad (2.4.1)$$

Where C_α are certain positive constants. We denote the associated inner product by $((\ ,\ ))_s$.

$H^s(R^m,B(P))$ is a Banach space with the norm $\|u\|_s = \Sigma\ C_\alpha \|D^\alpha u\|^2$

with C_α as above. (2.4.2)

In the following embedding theorems the inclusion symbol $\subset$ implies the continuity of the embedding. We shall simply write H^s for $H^s(R^m,P)$ or $H^s(R^m,B(P))$.

Lemma 2.4.5

(a) $D^\alpha H^s \subset H^{s-|\alpha|}$ $\qquad |\alpha| \leq s$

(b) If $s < m/2$, then $H^s \subset L^p$ for any P with $\frac{1}{2} - s/m < 1/p < \frac{1}{2}$

If $s = m/2$, then $H^s \subset C_b^k$ for $k = s - m/2 - 1$

If $s > m/2$, then $H^s \subset C_b^k$ for $k = s - m/2 - 1$

(c) If $r = \min\{s,t,s+t - m/2 - 1\} > 0$, then $H^s H^t \subset H^r$. (2.4.3)

The left member of (2.4.3) is the set of all functions ab(pointwise product) where $a \in H^s$ and $b \in H^t$, a and b may both be B(P)-valued then ab is too. Or a may be B(P)-valued and b P-valued then ab is P-valued. These remarks will not be repeated below.

Corollary 2.4.6

$H^s(R^m, B(P))$ is a Banach algebra if $s > m/2$

Definition 2.4.7

For $1 \leq p < \infty$ we denote by $\tilde{L}^p(R^m, P)$ the set of all (equivalence classes of) P-valued measurable functions u on R^m such that

$$||u||_{\tilde{L}^p} \equiv \sup_{x \in R^m} \left(\int_{|y-x|<1} |u(y)|^p dy \right)^{1/p} < \infty \qquad (2.4.4)$$

We define similarly $\tilde{L}^p(R^m, B(P))$. $\tilde{L}^p$ is a Banach space with norm (2.4.4) and $L^p \subset \tilde{L}^p$.

Definition 2.4.8 - For each integer $s \geq 0$, we denote by $\tilde{H}^s(R^m, P)$ The set of all $u \in \tilde{L}^2(R^m, P)$ such that all distribution derivatives $D^\alpha u$ of order $|\alpha| \leq s$ are in $\tilde{L}^2$. Similarly we define $\tilde{H}^s(R^m, B(P))$.

$\tilde{H}^s$ is a Banach space by the norm $|||u|||_s = \sup_{|\alpha| \leq s} ||D^\alpha u||_{\tilde{L}^2}$ and $H^s \subset \tilde{H}^s$.

Lemma 2.4.9.

Lemma 2.4.5 is true when H^s and L^p are replaced by $\tilde{H}^s$ and $\tilde{L}^p$, respectively.

Corollary 2.4.10

Let $s > m/2$. Then the elements of $\tilde{H}^s(R^m, B(P))$ are multipliers for $\tilde{H}^t(R^m, P)$ and $\tilde{H}^t(R^m, B(P))$ for any $t \leq s$. In particular $\tilde{H}^s(R, B(P))$ is a Banach algebra with unit $e : e(x) = I$ (the identity operator on P)

Lemma 2.4.11

Let s,t,r be as in lemma 2.4.5 (c), then

$$\tilde{H}^s H^t \subset H^r, \quad H^t \tilde{H}^s \subset H^r \qquad (2.4.5)$$

Corollary 2.4.12

Let $s > m/2$. Then the elements of $\tilde{H}^s(R^m, B(P))$ are multipliers for $H^t(R^m, P)$ and $H^t(R^m, B(P))$ for any $t \leq s$. In particular $H^s(R^m, B(P))$ is a two-sided algebraic ideal of $\tilde{H}^s(R^m, B(P))$.

Lemma 2.4.13

Let $s,t > m/2$. Then

$\tilde{H}^s H^t \subset L^2$, $H^t \tilde{H}^s \subset L^2$.

Lemma 2.4.14.

Let $s > m/2$ and $a \in \tilde{H}^s(R, B(P))$. In order that a be invertible within the algebra $\tilde{H}^s$, it is necessary and sufficient that $a(x)^{-1} \in B(P)$ exists for all $x \in R^m$ and $|a(x)^{-1}| < \delta^{-1} < \infty$; for some $\delta > 0$

In this case we have

$$|||a^{-1}|||_s \leq c\,\delta^{-1}(1 + \delta^{-s}|||a|||_s^s) \tag{2.4.6}$$

where c depends only on m and s.

For $t \leq s > m/2$ we may estimate a^{-1} as a multiplier for H^t or $\tilde{H}^t$ by

$$|||a^{-1}b||| \leq c\delta^{-1}(1 + \delta^{-t}\,|||a|||_s^t)\;|||b|||_t \tag{2.4.7}$$

Lemma 2.4.15.

Each closed ball in $H^s(R^m,P)$ or $H^s(R^m,B(P))$ is a complete metric space under the metric induced by the H^o-norm. Similarly for $\tilde{H}^s$ with the $\tilde{H}^o$ - norm.

Lemma 2.4.16

Let $s > m/2$. Then the map $a,b \to ab$ is continuous in the $\tilde{H}^o$-topologies provided a and b stay in bounded subsets of $\tilde{H}^s$.

The map $a,u \to au$ is continuous from the $\tilde{H}^o \times H^o$ - topology to the H^o - topology provided a and b stay in bounded subsets of $\tilde{H}^s$ and H^s, respectively.

Lemma 2.4.17

Let $s > m/2$. The map $a \to a^{-1}$ is continuous in the $\tilde{H}^o$-topologies provided a^{-1} stays in bounded subsets of $\tilde{H}^s(R^m,B(P))$.

Lemma 2.4.18.

If $||a_n - a||_o \to o$ with $||a_n||_s$ bounded, then $||a_n - a||_t \to o$ for $t<s$.

If $|||a_n - a|||_o \to o$ with $|||a_n|||_s$ bounded, then $|||a_n - a|||_t \to o$ for $t<s$

CHAPTER III

SYMMETRIC HYPERBOLIC SYSTEMS

We consider the equation

$$a_o(t,x,u)\frac{\partial u}{\partial t} + \sum_{j=1}^{m} a_j(t,x,u)\frac{\partial u}{\partial x_j} = f(t,x,u)$$

$o\leq t\leq T \quad x\varepsilon R^m, \; u(o,x) = u_o(x).$ (*)

where the unknown u, the initial value u_o, and the right member f take values in a real or complex Hilbert space P, while the coefficients a_j take values in B(P). We assume that the values of the a_j are symmetric operators and in particular that those of a_o are uniformly positive definite.

The proof is based on a standard method for handling quasi-linear equations. (see Courant-Hilbert [1]). Firstly we linearize the non-linear problem and solve it. We then define a non-linear mapping and look for its fixed point. This fixed point if it exists gives the solution of the non-linear problem. The solution is propagated only for a small time interval $[o,T']\subset[o,T]$ depending on u_o, f and the a_j but u will exist on the whole of R^m.

We seek a solution $u(t,.)\varepsilon\, C[o,T; H^s(R^m,P)]$, where s is an integer $> m/2 + 1$. We assume that

$u_o\varepsilon\; H^s$. We will show that the solution $u(t,.)\;\varepsilon\; H^s$ of the linear problem exists if $a_j(t,.)\;\varepsilon\;\tilde{H}^s\;(R^m,B(P))$ and $f(t)\;\varepsilon\; H^s(R^m,P)$ with certain mild assumptions in their t dependence. We then assume that in (*) the a_j have the property that whenever $v(t,.)\;\varepsilon\; H^s(R^m,P)$, $a_j(t,.,v(t))$ [resp. $f(t,.,v(t))$] are in $\tilde{H}^s(R^m,B(P))$ [resp. $H^s(R^m,P)$] with some additional assumptions on their continuity. Under these conditions we can prove the existence, uniqueness, and continuous dependence in the H^s-norm on u_o, of the solution u of (*) on a short time interval.

1. The Cauchy problem for Linear Symmetric Hyperbolic Systems.

We consider the linear problem

$$a_o(t,x)\frac{\partial u}{\partial t} + \sum_{j=1}^{m} a_j(t,x)\frac{\partial u}{\partial x_j} + a_{m+1}(t,x)\, u = f(t,x)$$

$$o\leqslant t\leqslant T,\ x\varepsilon R^m,\quad u(o,x) = u_o(x) \qquad (3.1.1)$$

We shall regard $u = u(t,x)$ as a function of t with values in a function space over the space variable $x\varepsilon R^m$. We will write $u(t)$ for $u(t,.)$ and adopt the same convention for the a_j and f. (For the uniformly local Sobolev space $\tilde{H}^s$ and the associated norm $|||\ \ |||_s$ see the preceding section.)

Theorem 3.1.1

Let s be an integer such that $s>m/2 + 1$, and let $1\leqslant s'\leqslant s$. Assume that

$$a_j \ \varepsilon\ C[o,T;\ \tilde{H}^o(R^m,B(P))] \ ,\quad o\leqslant j\leqslant m+1 \qquad (3.1.2)$$

$$|||a_o(t)|||_s \leqslant K_o,\quad |||a_j(t)|||_s \leqslant K \quad o\leqslant t\leqslant T,\ 1\leqslant j\leqslant m+1 \qquad (3.1.3)$$

$$|||a_o(t') - a_o(t)|||_{s-1} \leqslant L|t' - t|,\quad o\leqslant t,\ t'\leqslant T \qquad (3.1.4)$$

$$a_j(t,x) \text{ is symmetric for } o\leqslant t\leqslant T,\ x\varepsilon R^m,\ o\leqslant j\leqslant m \qquad (3.1.5)$$

$$a_o(t,x) \geqslant 1 \text{ for all } t,\ x \text{ (uniformly positive definite)} \qquad (3.1.6)$$

$$f\ \varepsilon\ L^1[o,T;\ H^{s'}(R^m,P)]\cap C[o,T;\ H^{s'-1}(R^m,P)] \qquad (3.1.7)$$

$$u_o\ \varepsilon\ H^{s'}(R^m,P) \qquad (3.1.8)$$

Then (3.1.1) has a solution u such that

$$u\ \varepsilon\ C[o,T;\ H^{s'}(R^m,P)]\cap C^1[o,T;\ H^{s'-1}(R^m,P)] \qquad (3.1.9)$$

The solution u is unique in the larger class.

$C[o,T;H^1(R^m,P)]\cap C^1[o,T;H^o(R^m,P)]$.

In addition we have the estimates

$$||u(t) - u_{\infty}||_r \leqslant Ce^{c(L+K+K k_o^r)T}[||u_o-u_\infty||_r + c(1+k_o^r)(F_r^1+KT||u_\infty||_{r+1})] \ . \qquad (3.1.10)$$

$$||\frac{du}{dt}||_{r-1} \leqslant c(1+K_o^{r-1})(||f(t)||_{r-1}+K||u(t)||_r), \quad o\leqslant t\leqslant T \qquad (3.1.11)$$

Here $o\leqslant r\leqslant s'$ in (3.1.10) and $1\leqslant r\leqslant s'$ in (3.1.11); (

u_{oo} is an arbitrary element in $H^{r+1}(P^m,P)$.

$$C = ||a_o(o)||^{\frac{1}{2}}_{L^\infty} \leqslant \cdot\ |||a_o(o)\ |||^{\frac{1}{2}}_{s-1} \qquad (3.1.12)$$

$$F_r^1 = \int_o^T ||f(t)||_r dt \qquad (3.1.13)$$

and c denotes various contants depending on m and s.

Lemma 3.1.2.

Assumptions (3.1.2) and (3.1.3) imply that

$$a_j \varepsilon\ C[o,T;\tilde{H}^{s-1}(R^m,B(P))] \subset C\ [o,T;\ C_b\ (R^m,B(P))]$$

Proof

From assumptions (3.1.2) and (3.1.3)
$|||a_j(t+h) - a_j(t)|||_o \to o$ as $h\to o$ and $|||a_j(t)|||_s$
is bounded. Therefore from lemma 2.4.18 we have
$|||a_j(t+h) - a_j(t)|||_t \to o$ as $h\to o$ for $t<s$. In particular
$|||a_j(t+h) - 1_j(t)|||_{s-1} \to o$ as $h\to o$. So
$a_j \in C[o,T;\ \tilde{H}^{s-1}(R^m,B(P))]$ The desired result follows from lemma 2.4.9.

The same is true for a_o^{-1}. This follows from lemma 2.4.14 and the fact that in a Banach algebra, the map $a\to a^{-1}$ is continuous.

We do not assume that the $a_j(t)$ are continuous in t in the $\tilde{H}^s$-norm nor that $f(t)$ is continuous in the $H^{s'}$ norm. Thus the a_j, for example, need not be continuous on $[o,T]$ to $C^1(R^m,B(P))$. This weakening is necessary for application to non-linear problems.

In order to prove theorem 3.1.1 we write (3.1.1) as an abstract evolution equation

$$\frac{du}{dt} + A(t)u = \hat{f}(t), \quad o\leqslant t\leqslant T, \quad u(o) = u_o \qquad (3.1.14)$$

where $A(t) = \sum_{j=1}^{m} \hat{a}_j(t,x) \frac{\partial}{\partial x_j} + \hat{a}_{m+1}(t,x)$ (3.1.15)

and $\hat{a}_j = a_o^{-1} a_j$, $\hat{f} = a_o^{-1} f$

The following results are immediate from Corollaries 2.4.10, 2.4.12 and lemmas 2.4.14 and 3.1.2.

$$|||\hat{a}_j|||_s \leqslant c|||a_o(t)^{-1}|||_s|||a_j(t)|||_s$$
$$\leqslant c(1 + K_o^s)K \text{ , } t \in [o,T] \quad j=1,\dots, m+1 \qquad (3.1.16)$$

$$\hat{a}_j \in C\,[o,T; \tilde{H}^{s-1}] \subset C[o,T; C_b] \text{ ,} j=1,\dots,m+1 \qquad (3.1.17)$$

$$\hat{f} \in L^1\,[o,T;H^{s'}] \cap C[o,T; H^{s'}] \qquad (3.1.18)$$

We solve (3.1.14) by applying the results of section 3 of chapter 1 where we take $X = H^o(R^m,P)$ and $Y = H^{s'}(R^m,P)$. X is a Hilbert space with the L^2 inner product. We have to show that for each t, $A(t)$ generates a C_o-semigroup on X and that conditions (i), (ii), and (iii) of theorem 2.3.8 are satisfied.

Proof of Theorem 3.1.1

We begin by proving the following

Lemma 3.1.3

If $A(t)$ is defined in a natural way as a closed linear operator in X, $-A(t)$ generates a C_o-semigroup. More precisely, $A(t)+cK$ is m-accretive in X_t, the space X endowed with the inner product $((f,g))_{(t)} = ((a_o(t)f,g))$ and the associated norm $||\ \ ||_{(t)}$ ($||\ \ ||_{(t)}$ is equivalent to the old norm).

Proof

From the symmetry of the $a_j(t,x)$ for $j \leqslant m$ and integration by parts we have

$$Re((A(t)g,\ g))_{(t)} = Re(([a_{m+1} - \tfrac{1}{2} \sum_{j=1}^{m} \frac{\partial a_j}{\partial x_j}]\ g,g))$$

$$\geq -{}^{*}\left|\left|\left|a_{m+1} - \tfrac{1}{2}\sum_{j=1}^{m}\frac{\partial a_j}{\partial x_j}\right|\right|\right|_{s-1}||g||^2 \geq - cK\ ||g||^2_{(t)}$$

where we have used the fact that $||g|| \leq ||g||_{(t)}$ by $a_o \geq 1$.
It follows that the closed operator $-A(t)$ is the generator of a quasi-contraction semigroup on X_t (i.e. $A(t) \in G(X_t, 1, cK)$). Hence it is the generator of a $\mathcal{C}_o$-semigroup on X.

Lemma 3.1.4

$\{A(t);\ o \leq t \leq T\}$ is a stable family with stability constants $M = ||a_o(o)||^{\frac{1}{2}}_{L^\infty} e^{cLT}$ and $\beta = cK$.

Proof

Since $A(t) \in G(X_t, 1, cK)$ by lemma 3.1.3 it suffices to show that the norm $||\ \ ||_{(t)}$ depends on t smoothly. Now

$$\begin{aligned} ||g||^2_{(t')} - ||g||^2_{(t)} &= ((a_o(t')g,g)) - ((a_o(t)g,g)) \\ &= (([a_o(t') - a_o(t)]g,g)) \\ &\leq c|||a_o(t') - a_o(t)|||_{s-1}||g||^2 \\ &\leq c\ L|t'-t|\ ||g||^2_t \end{aligned}$$

by lemma 2.4.15, (3.1.4) and $||g|| \leq ||g||_{(t)}$.
Hence $||g||_{(t')}/||g|| \leq 1 + cL|t'-t| \leq e^{cL|t'-t|}$.

It follows from proposition 2.3.3 that $\{A(t)\}$ is stable with constants $M = e^{cLT}$ and $\beta = cK$ with respect to the norm $||\ \ ||_{(o)}$. To return to the original norm $||\ \ ||$ we have to multiply M by $||a_o(o)||^{\frac{1}{2}}_{L^\infty}$. So condition (i) of theorem 2.3.5 and a fortiori (i) of theorem 2.3.8 is satisfied.

We verify condition (ii) with $S(t) = (I-\Delta)^{s'/2}$ which is an isomorphism of Y onto X. Since S is independent of t it suffices to prove the following lemma

Lemma 3.1.5

$SA(t)S^{-1} = A(t) + B(t),\ B(t) \in B(X),\ o \leq t \leq T$ (3.1.19)

with B(t) unfiformly bounded and strongly measurable in t.

Proof

B(t) is formally given by

$$B(t) = \sum_{j=1}^{m}[S,\hat{a}_j(t)]\left(\frac{\partial}{\partial x_j}\right)S^{-1} + [S,\hat{a}_{m+1}(t)]S^{-1} \quad (3.1.20)$$

where $[S,T] = ST - TS$ denotes the commutator.

Case (i) s' an even integer

In this case $S = P(D)$ with P a polynomial of degree s'. Thus

$$[S,\hat{a}_j(t)] = \sum_{0\leq|\alpha|\leq s} \frac{1}{\alpha!}(D^{\alpha}\hat{a}_j(t))\, P^{(\alpha)}(D) \quad (3.1.21)$$

by Leibniz formula. A typical contribution to B(t) from (2.1.21) is

$$(D^{\alpha}\hat{a}_j(t))\, P^{(\alpha)}(D)\left(\frac{\partial}{\partial x_j}\right) S^{-1} \quad (3.1.22)$$

Since $P^{(\alpha)}(D)$ is a differential operator of order $s'-|\alpha|$, $P^{(\alpha)}(D)(\frac{\partial}{\partial x_j})S^{-1}$ maps $X = H^{s}$ into $H^{|\alpha|-1}$ (note that $|\alpha|\geq 1$). Since $D^{\alpha}\hat{a}_j(t) \in \tilde{H}^{s-|\alpha|}$ by (3.1.16) we see that (3.1.22) maps X into itself (see Lemma 2.4.11 noting that $s > m/2 + 1$), with the operator norm not exceeding $c\,K(1 + K_o^{s'})$. A similar argument applies to the last term of (3.1.20) giving $||B(t)||_X \leq c\,K(1 + K_o^{s'})$ (3.1.23)

For later applications we note also that

$$||B(t)||_{X_t} \leq cK(1 + K_o^{s'}) \quad (3.1.24)$$

Case (ii) s' an odd integer

Write $S = \Lambda P(D)$, where $\Lambda = (1-\Delta)^{\frac{1}{2}}$ and P is a polynomial of degree s'-1. Thus

$$[S,\hat{a}_j(t)] = [\Lambda,\hat{a}_j(t)]P(D) + \Lambda[P(D), \hat{a}_j(t)] \quad (3.1.25)$$

The first term on the right contributes a bounded operator to B(t). $[\Lambda, a_o^{-1}a_j] = a_o^{-1}[\Lambda,a_j] - a_o^{-1}[\Lambda,a_o]a_o^{-1}a_j$ is a bounded operator since $[\Lambda,a_j]$ and $[\Lambda,a_o]$ are bounded with bounds $\leq cK$ and cK_o respectively. So the contribution of the first term on the right of (3.1.25) to (3.1.23) is $\leq cK(1+K_o)$. The contribution from the second term contains terms similar to (3.1.22) except for the factor Λ, which is compensated by the fact that now P is of degree $s'-1$. So we have (3.1.23) and (3.1.24).

In order to show that $B(\cdot)$ is strongly measurable we show that it is weakly continuous (which implies strong measurability) (

If $u,v \in C_0^\infty(R^m,P)$, then

$(B(t)u, v) = (A(t)S^{-1} u, Sv) - (A(t)u,v)$ by (3.1.20) $\quad$ ~~(3.1.20)~~

where we regard S as an unbounded self adjoint operator in X. Since u and $S^{-1}u$ are in Y, $(B(t)u,v)$ is continuous if we have $D(A(t)) \supset Y$ and $t \to A(t) \in B(Y,X)$ is norm continuous. These last two conditions are condition (iii) of theorem 2.3.8 and it follows from (3.1.17) and $s > m/2 + 1$, $s' \geq 1$. Having thus verified conditions (i),(ii) and (iii) of theorem 2.3.8 we invoke the results of the t heorem, namely that there exists a unique evolution operator $U = \{U(t,t')\}$ for $A = A(t)$ and the solution of (3.1.14) is given by

$$u(t) = U(t,o)\, u_o + \int_0^t U(t,t')f(t')dt' \qquad (3.1.26)$$

Since $u_o \in H^{s'} = Y$ and $f \in L^1[o,T;Y]$ it follows from theorem 2.3.9 that $u \in C[o,T;Y]$. Since this implies that $\frac{\partial u}{\partial x_j} \in C[o,T;H^{s'-1}]$ it follows from (3.1.14) (3.1.17) and (3.1.18) that $u \in C^1[o,T;H^{s'-1}]$ and this proves (3.1.9). The uniqueness follows from the existence of the evolution operator.

To derive the estimates of (3.1.10) and (3.1.11) we establish the following

Lemma 3.1.6

$$||U||_{\infty,X} = \sup_{o\leq t'\leq t} ||U(t,t')||_X \leq Ce^{c(L+K)T} \qquad (3.1.27)$$

$$||U||_{\infty,Y} \leq Ce^{c(L+K+KK_o^{s'})T}, \quad C = ||a_o(o)||_{L^\infty}^{1/2} \qquad (3.1.28)$$

Proof

$||U||_{\infty,X} \leq Me^{\beta T}$ from (2.3.5) where M,β are the stability constants for $\{A(t)\}$. But $M = C\, e^{cLT}$ and $\beta = cK$ so we have (3.1.27).

Since condition (ii) of theorem 2.3.8 implies (ii) of theorem 3.2.5 we have $Y=H^{s'}$ is $A(t)$ - admissible, and if $\tilde{A}(t)$ is the part of $A(t)$ in Y, then $\{\tilde{A}(t)\}$ is also stable with constants $\tilde{M},\tilde{\beta}$ say. It can be shown that $M=M_1$ and $\tilde{\beta} = \beta_1$ where M_1 and β_1 are the stability constants for $\{A_1(t)\} = \{A(t)\}+ B(t)\}$ in X. Since $A(t) + cK$ is m-accretive in $X_{(t)}$ and $||B(t)||_{X_{(t)}} \leq cK(1 + K_o^{s'})$ it follows that $A(t) + B(t) + cK + cK(1 + K_o^{s'})$ is m -accretive in $X_{(t)}$. This implies that $A(t) \varepsilon G(X_{(t)}, 1, cK (1 + K_o^{s'}))$. Since $||\ ||_{(t)}$ is smooth in t we may take $M_1 = e^{cLT}$, $\beta_1 = cK(1 + K_o^{s'})$ with respect to $||\ ||_{(o)}$. To obtain the stability constants with respect to the original norm in X we multiply M by C. So since $||U||_{\infty,Y} \leq \tilde{M} e^{\tilde{\beta}T}$ we have (3.1.28)

To prove (3.1.10), let $u_{\infty} \varepsilon H^{s'+1}$ and set $u(t) = w(t) + u_{\infty}$. w satisfies the differential equation $\frac{dw}{dt} + A(t)w = \hat{f}(t) - A(t)u_{\infty}$ and the initial condition $w(o) = u_o - u_{\infty}$. By (3.1.26)

$$w(t) = U(t,o)(u_o - u_{\infty}) + \int_0^t U(t,t')(\hat{f}(t') - A(t') U_{\infty})dt' \quad (3.1.29)$$

and we have the following estimates

$$||\hat{f}(t)||_Y = ||a_o(t)^{-1}f(t)||_{s'} \leq c(1 + K_o^{s'})||f(t)||_{s'}$$

$$||A(t)u_{\infty}||_Y \leq ||a_o(t)^{-1} \ \Sigma a_j(t)\frac{\partial u}{\partial x_j}||_{s'} \leq c(1 + K_o^{s'})K||U_{\infty}||_{s'+1}$$

In view of estimate (3.1.28) we have from (3.1.29) the desired result (3.1.10) for r=s'.

For $1 \leq r \leq s'-1$, we obtain (3.1.10) by simply applying the above result to s' replaced by r. For r=o (3.1.10) follows by estimating $||w(t)||_X$ by using (3.1.27). Finally we obtain (3.1.11) from (3.1.14) in the same way. This concludes the proof of theorem 3.1.1.

2. The Cauchy problem for quasi-linear Symmetric Hyperbolic systems.

We are now prepared to solve the quasi-linear system (*) given at tthe beginning of chapter 3. As in the linear case u takes values in the Hilbert space P. The a_j (resp. f) are functions on $[o,T] \times \Omega$ to B(P) (resp P), where Ω is a subset of R^m. For convenience we write (*) as follows

$$G_o(t)[u(t)]\frac{\partial u}{\partial t} + \sum_{j=1}^{m} G_j(t)[u(t)]\frac{\partial u}{\partial x_j} = F(t)[u(t)] \qquad (3.2.1)$$

Each $G_j(t)$ (resp. F(t)) is a non-linear operator, depending on t, that sends certain P-valued functions on R^m into B(P) - valued (res. P-valued) functions on R^m. (*) is a special case of (3.2.1) in which $G_j(t)$ and F(t) act pointwise, namely

$$G_j(t)[v](x) = a_j(t,x,v(x)), \quad F(t)[v]=f(t,x,v(x)) \qquad (3.2.2)$$

Theorem 3.2.1

Let $s > m/_2 + 1$. Let D be a bounded open subset of $H^s(R^m,P)$, Let $G_j(t)$, $o \leq t \leq T$, j o,1,....,m (resp F(t), $o \leq t \leq T$) be non-linear operator on D to $\tilde{H}^s(R^m,B(P))$ (resp $H^s(R^m,P)$).

Assume that

The $G_j(t)[v]$ (resp F(t)[v]) are bounded (by a constant λ) in the $\tilde{H}^s$-norm (resp H^s-norm) for $v \in D$ uniformly in t (3.2.3)

For each j and t, the map $v \to G_j(t)[v]$ (resp $v \to F(t)[v]$) is uniformly Lipschitz - continuous (with the Lipschitz constant μ) on D from the H^o-norm to $\tilde{H}^o$-norm (resp H^o-norm), uniformly in t. (3.2.4

The map $v \to G_o(t)[v]$ is also uniformly Lipschitz- continuous on D(with Lipschitz constant μ') from the H^{s-1} norm to $\tilde{H}^{s-1}$norm, uniformly in t. (3.2.5)

The maps $t \to G_j(t)[v]$(res $t \to F(t)[v]$) are continuous in the $\tilde{H}^o$-norm (resp H^o-norm) for each $v \in D$. (3.2.6)

The map $t \to G_o(t)[v]$ is Lipschitz - continuous in [o,T] (with Lipschitz-constant ν_*) in the $\tilde{H}^{s-1}$ norm, uniformly for $v \in D$ (3.2.7)

For each $v \in D$, $G_j(t)[v](x) \in B(P)$ is symmetric for each $t,x \in [o,T] \times R^m$, $o \leq j \leq m$. (3.2.8)

$G_o(t)[v](x) \geq 1$ for each t,x for each $v \in D$ (3.2.9)

$u_o \in D$ (3.2.10)

Then there is a unique solution of (3.2.1) defined on $[o,T']$ where $o<T'\leq T$, such that

$$u\epsilon C[o,T';\ D] \cap C^1[o,T';\ H^{s-1}(R^m,P)]. \tag{3.2.11}$$

T' can be chosen common to all U_o varying in a small neighbourhood of a given point in D.

The condition given in this theorem are implicit when applied to our original equation (*). For more explicit conditions see for example Kato [4], Fischer and Marsden [1].

Condition (3.2.3) implies that $G_j(t)[v] \ \epsilon\ C_b^1(R^m,B(P)$ since $s\geq m/_2 + 1$.

Proof of Theorem 3.2.1.

Lemma 3.2.2

Given $u_o\ \epsilon\ D$, there are positive numbers R, $\rho\leq R/2$ and $u_{oo}\ \epsilon\ D\cap H^{s+1}(R^m,P)$ with the following properties

$$v\epsilon H^s \text{ with } ||v-u_{oo}||_s \leq R \text{ implies } v\epsilon D \tag{3.2.12}$$

$$||u_o - u_{oo}||_s \leq \rho \tag{3.2.13}$$

$$\rho||G_o(o)[v]||^{\frac{1}{2}}_{L^\infty} \leq R/2 \text{ for } v\epsilon D \tag{3.2.14}$$

Proof

Let $R' > 0$ be the radius of the largest open ball with centre u_o contained in D, and set $R = R'/2$.

Since $||G_o(o)[v]||_{L^\infty} \leq c||G_o(o)[v]||_{s-1} \leq c\ \lambda$ for $v\epsilon D$,

THere is $\rho>o$ satisfying 3.2.14. Since $||G_o(o)[v]||_{L^\infty}\geq 1$ by (3.2.9) this implies $\rho<R/_2$.

Since H^{s+1} is dense in H^s, there is $u_{oo}\ \epsilon\ H^{s+1}$ satisfying (3.2.13).

Now $||v-u_{oo}||_s \leq R$ implies

$||v-u_{oo}||_s \leq ||v-u_{oo}||_s + ||u_{oo}-u_o||_s \leq R + \rho \leq 3\frac{R}{2} < R'$

There (3.2.12) is satisfied.

In order to set up the required contraction map, we introduce the set $\underline{S}$ of all functions v on $[o,T']$ to $H^s(R^m,P)$ such that

$$|| v(o) - u_{\infty}||_s \leq 2\rho \qquad (3.2.15)$$
$$||v(t)-u_{\infty}||_s \leq R, \quad o\leq t\leq T' \qquad (3.2.16)$$
$$||v(t)-v(t')||_{s-1} \leq L(t, -t'), \quad o\leq t'\leq t\leq T' \qquad (3.2.17)$$

where $T'\leq T$ and L' are positive constants to be determined later.

Let $v\varepsilon \underline{S}$. Since (3.2.16) implies $v(t)\varepsilon \mathcal{D}$ by (3.2.12), we can construct

$$a_j^v(t) = G_j(t)[v(t)], \ f^v(t)=F(t)[v(t)], \ o\leq t\leq T' \qquad (3.2.18)$$

We now consider the linear differential equation'

$$a_o^v(t,x) \frac{\partial u}{\partial x_j} + \sum_{j=1}^{m} a_j^v(t,x) \frac{\partial u}{\partial x_j} = f^v(t,x)$$

$$o\leq t\leq T', \quad x\varepsilon R^m, \quad u(o,x) = u_o(x) \qquad (3.2.19)$$

In order to be able to apply the result of theorem 3.1.1 to (3.2.19) we have to verify the conditions stated in the theorem. In fact (3.2.19) satisfies these assumptions with $s'=s$ and with T replaced by T', For example

$|||a_j^v (t)|||_s \leq \lambda$ by (3.2.3) so that (3.1.3) is satisfied with $K_o = K = \lambda$. We verify (3.1.2) as follows:

$$|||a_j^v(t') - a_j^v(t)|||_r < |||G_j(t')[v(t')]-G_j(t')[v(t)]|||_r$$
$$+ |||G_j(t') - G_j(t))[v(t)]|||_r \qquad (3.2.20)$$

Setting $r=o$ and letting $t'\to t$, we see by (3.2.4) (3.2.6) and (3.2.7) that (3.2.20) tends to zero. Hence $t \to a_j^v(t)$ is continuous in the $\tilde{H}^o$-norm.

In a similar way we can verify condition (3.1.4) from (3.2.5), (3.2.7) and (3.2.17) with $L=\nu + \mu'L'$.

Also we have $||f^v(t)||_s\leq \lambda$ by (3.2.3) and $t \to f^v(t)$ is continuous in the H^o-norm. Hence it is continuous in the H^{s-1}-norm, so that (3.1.7) is satisfied with $s'=s$ and with L^1 replaced by L^∞. Other assumptions in theorem 3.1.1 follow directly from (3.2.8), (3.2.9) and (3.2.10) and we can choose $s'=s$. If we now apply theorem 3.1.1 to (3.2.19) with $s'=s$ and $T = T'$, we obtain u in the class (3.1.9).

Lemma 3.2.3.

$u \in \underline{S}$ if L' and T' are chosen appropriately.

Proof

$u(o) = u_o$ satisfies (3.2.15) by (3.2.13)

To verify other conditions we use (3.1.10) obtaining

$$||u(t)-u_{\infty}||_s \leq C\, e^{\alpha(L')T'}[||u_o-u_{\infty}||_s + c(1+\lambda^s)\lambda(1+||u_{\infty}||_{s+1})T'], \quad (3.2.21)$$

where $\cancel{C = ||G_o(o)||^{\frac{1}{2}}_{L^\infty}}$, $\alpha(L') = c(\nu+\mu'L' + \lambda + \lambda^{s+1})$, $C = ||G_o(o)[u_o]||^{1/2}_{L^\infty}$

using the fact that $F_s^4 \leq \lambda T'$. Also (3.1.11) gives

$$||\frac{du}{dt}||_{s-1} \leq c(1+\lambda^{s-1})\lambda(1+||u_{\infty}||_s + \beta(L',T')) \quad (3.2.22)$$

where $\beta(L',T')$ denotes the right hand side of (3.2.21).

For $u \in \underline{S}$ it is sufficient that the right members of (3.2.21) and (3.2.22) be smaller than R and L' respectively, and this can be done by a suitable choice of L' and T'.

We denote by Θ the map $v \mapsto u$. It has been shown that Θ maps $\underline{S}$ itself. In order to apply the fixed point theorem we make $\underline{S}$ into a complete metric space with the metric

$$d(v,w) = ||v-w||_{\infty,X} = \sup_{0 \leq t \leq T'} ||v(t) - w(t)||_o$$

That $\underline{S}$ is a complete metric space follows from the fact that the closed balls of $H^s(R^m,p)$ are closed subsets of H^o and lemma 2.4.15.

Lemma 3.2.4.

Θ is a contraction map of $\underline{S}$ into itself if T' is chosen sufficiently small.

Proof

We first prove that $a_j^v(t)$ and $f^v(t)$ depend on v Lipschitz-continuously. We have

$$\begin{aligned}|||a_j^v(t) - a_j^w(t)|||_o &= |||G_j(t)[v(t)] - G_j(t)[w(t)]|||_o \\ &\leq \mu||v(t) - w(t)||_o \\ &\leq \mu\, d(v,w)\end{aligned}$$

and similarly

$||f^v(t) - f^w(t)|| \leq \mu\, d(v,w)$. Hence

$|||a_o^v(t)^{-1} - a_o^w(t)^{-1}|||_o \leq |||a_o^v(t)^{-1}[a_o^w(t) - a_o^v(t)]a_o^w(t)\ \ |||_o \leq \mu\, d(v,w)$

Since $||a_o^v(t)^{-1}||_{L^\infty} \leq 1$

Setting $\hat{a}_j^v(t) = a_o^v(t)^{-1}a_j^v(t)$ and $\hat{f}^v(t) = a_o^v(t)^{-1}f^v(t)$,

we have

$$|||\hat{a}_j^v(t) - \hat{a}_j^w(t)|||_o,\ ||\hat{f}^v(t) - \hat{f}^w(t)||_o \leq (\lambda+1)\mu d(v,w) \qquad (3.2.23)$$

Let us denote the evolution equation version of (3.2.19) by (E^v) The coefficients $A^v(t)$ and the right member of (E^v) are given by (3.1.15) with the $\hat{a}_j$ and $\hat{f}$ replaced by $\hat{a}^v$ and $\hat{f}^v$, respectively. Since $s > m/_2 + 1$ and noting that $\tilde{H}^o\ H^{s-1} \subset H^o$ by (2.4.5)

$$||[A^v(t) - A^w(t)]u||_o \leq c\mu\ (\lambda+1)||u||_{s_v}\ d(v,w),\ u \in H^s = Y \qquad (3.2.24)$$

Now θv and θw are solutions of (E^v) and (E^w) respectively. It follows from theorem (2.3.10) that $d(\theta v, \theta w) = ||\theta v - \theta w||_{\infty,X}$

$$\leq ||U^w||_{\infty,X} \int_o^{T'} [||\hat{f}^v(t) - \hat{f}^w(t)||_o + ||(A^v(t) - A^w(t))\theta v(t)||_o]dt \qquad (3.2.25)$$

where $U^w = \{U^w(t,t')\}$ is the evolution operator for (E^w).

Since $||U^w||_{\infty,X}$ and $||\theta\, v(t)||_s$ are uniformly bounded by (3.1.27) and $\theta v(t) \in D$, we see from (3,2.25) (3.2.23) and (3.2.24) that θ is a contraction map of $\underline{S}$ into itself if T' is chosen sufficiently small. It follows that there is a unique fixed point $u = \theta u \varepsilon S$ of the map θ. That u is a solution of (3.2.1) satisfying (3.2.11) follows from theorem 3.1.1.

It can also be shown that the solution u depends on the initial value continuously. (see Kato [4]).

3. The mixed problem for linear symmetric hyperbolic systems.

In this section we shall consider the mixed problem for a linear symmetric hyperbolic system. F.J.Massey III[1] has shown that Kato's theory of linear evolution equations can be applied to such equations.

Consider the following system of equations

$$\frac{\partial u}{\partial t} + \sum_{j=1}^{m} a_j(t,x)\frac{\partial u}{\partial x_j} + b(t,x)\, u = f(t,x) \qquad (3.3.1)$$

$x \varepsilon \Omega$, $o \leq t \leq T$

$u(o,x) = \phi(x)$, $x\varepsilon\Omega$

$u(t,x) \; \varepsilon \; P(t,x)$, $\quad x\varepsilon\Gamma$, $o\leq t\leq T$.

The unknown $u = (u_1,\dots,u_N)$ is a real vector - valued function, the coefficients a_j and b, are real NxN matrix-valued functions and as usual the a_j are symmetric.

We assume that a_j and b are of class C^2 and C^1 on $[o,T] \times \bar{\Omega}$, respectively. Ω is a bounded open subset of R^m with boundary Γ of class C^3. The boundary matrix

$a_n(t,x) = \Sigma\, n_j(x)\, a_j(t,x)$, $x\varepsilon\Gamma$, $\quad o\leq t\leq T$,

is assumed to be non-singular, this is the regular case. Here $n=(n_1,\dots,n_m)$ is the exterior unit normal to Ω. Furthermore, we assume that the boundary space $P(t,x)$ is a linear subspace of R^N which varies in a C^3 manner with $(t,x)\varepsilon\ [o,T] \times \Gamma$, and it is maximal non-negative for each t,x in the sense that $(a_n(t,x)u,u) \geq o$, $u\ \varepsilon P(t,x)$ and $P(t,x)$ is not contained in any other subspace of R^N having this property. Here (u,v) denotes the usual inner product of $u,v \;\varepsilon\; R^N$.

In this section $X = L^2(\Omega)$, the Hilbert space of square-intergrable functions on . $H^k(\Omega)$ for $k=o, 1,2,\dots$ will denote the usual Sobolev space. if $P(x)$, $x\varepsilon\Gamma$, is a linear subspace of R^N which varies continuously with x, then $H^1_p(\Omega)$ denotes the closed subspace of $H^1(\Omega)$ consisting of those u which satisfy the boundary condition $u(x)\varepsilon p(x)$ for almost all $x\varepsilon\Gamma$.

If $u\varepsilon R^N$, then $|u|$ denotes the usual Euclidean norm of u. If a is an NxN matrix, then $||a||= \sup\{|au| : |u|= 1\}$ and a^t is the transpose of a.

Let $A_o(t)$ be the operator defined by

$A_o(t)u = \quad a_j(t,x)\frac{\partial u}{\partial x_j} + b(t,x)u,$

with domain $D(A_o(t))$, equal to $H^1_{P_t}(\Omega)$, where $P_t(x) = P(t,x)$.

Denote by $A(t)$ the closure of $A_o(t)$ regarded as an unbounded operator in X. Friendrichs , and Lax and Phillips have shown that $A(t) + \beta_t$ is m-accretive if $\beta_t = \text{Sup}\ \{|b'(t,x)|\cdot : x\varepsilon\Omega\}$ where

$b' = \frac{1}{2}(b + b^t - \Sigma\frac{\partial a_j}{\partial x_j})$. This implies that $-A(t)$ generates a C_o-semigroup in X. So, the abstract evolution version of (3.3.1) satisfies condition (i) of theorem 2.3.5.

Massey III [1], has shown that there exists an isomorphism $S(t)$ from $H^1_{P_t}(\Omega)$ onto X such that $S(t)A(t)S(t)^{-1} = A(t) + B(t)$ where $B(t) \varepsilon B(X)$ (See comments after theorem 2.3.5). The construction of $S(t)$ is however involved. Massey employed techniques involving commutator inequalities and fractional powers of operators. The existence of $S(t)$, however, means that the subspace $H^1_{P_t}(\Omega)$ is admissible with respect to $A(t)$ (see condition (ii) of theorem 2.3.5). He showed further that if $P(t,x) = P(x)$ does not vary with t, then the $S(t)$ may be chosen so that it is continuously differentiable on $[o,T]$ to $B(H^1_P(\Omega), X)$, and $B(t)$ is continuous on $[o,T]$ to $B(X)$.

So if we assume that the boundary space $P(t,x) = P(x)$ does not vary with t, then we can apply theorem 2.3.5 to the family $\{A(t)\}$,taking $H^1_P(\Omega)$ for the subspace Y. The stability condition is true with $M=1$ and $\quad = \sup_t \{\beta_t\}$ since the operators $A(t) + \beta_t$ are m-accretive.

If we now suppose that $\phi \varepsilon H^1(\Omega)$ and the map $t \to f(t,.)$ is continuous $[o,T]$ to $H^1(\Omega)$ so that $f(t,.) \varepsilon H_{P_t}{}^1(\Omega)$ for $o \leq t \leq T$. Then (3.3.1) has a unique solution $u(t,x)$ such that the map $t \to u(t,.)$ is continuously differentiable on $[o,T]$ to X and $u(t,.) \varepsilon H^1_{P_t}(\Omega)$ for $o \leq t \leq T$. If $P(t,x) = P(x)$ is independent of t this follows directly from theorem 2.3.6.

In the general case where $P(t,x)$ varies with t Massey showed that this may be reduced by an ~~orthogonal transformation~~ of the independent variable to the case $P(t,x)=P(o,x)$.

4. Conclusion.

The author is not aware that any of the following problems has been solved using Kato's theory of linear evolution equations, although it seems very likely that they might be successfully handled by extension of the arguments of Massey III.

(a) The mixed problem for linear symmetric hyperbolic systems in a general Hilbert space other than the Euclidean spaces.

(b) the mixed problem for quasi-linear Symmetric hyperbolic Systems.

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