BAYESIAN ESTIMATIION OF SHAPE PARAMETER OF GENERALIZED INVERSE EXPONENTIAL DISTRIBUTION UNDER THE NON-INFORMATIVE PRIORS

BY

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Declaration

I declare that the work in this dissertation entitled "Bayesian estimation of the shape parameter of Generalized Inverse Exponential Distribution under the non-informative priors" has been performed by me in the Department of Statistics, Faculty of Physical Science, Ahmadu Bello University, Zaria under the supervision of Dr. S. I. S. Doguwa and Dr. H. G. Dikko. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other Institution.

YUSUF RASHIDAT ADARA ______________________ ______________________

Signature Date

Certification

This dissertation entitled "Bayesian estimation of the shape parameter of Generalized Inverse Exponential Distribution under the non-informative prior" by YUSUF RASHIDAT ADARA meets the regulation governing the award of the degree of Master of Science in Statistics of the Ahmadu Bello University, and is approved for its contributions to knowledge and literary presentation.

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Dedication

This research work is dedicated to Almighty Allah who has seen me through the programme successfully.

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Abbreviations

Abbreviation	**Translation**
BPR	Bayes Posterior Risk
CDF	Cumulative Distribution Function
ELF	Entropy Loss Function
GELF	General Entropy Loss Function
GIED	Generalized Inverse Exponential Distribution
LINEX	Linear Exponential
MLEs	Maximum Likelihood Estimate(s)
MSE	Mean Square Error
Pdf	probability density function
PLF	Precautionary Loss Function
PLFU	Precautionary Loss Function under the Uniform prior
PLFJ	Precautionary Loss Function under the Jeffrey's prior
QLF	Quadratic Loss Function
SELF	Squared Error Loss Function
SELFU	Squared Error Loss Function under the Uniform prior
SELFJ	Squared Error Loss Function under the Jeffrey's prior
SLLF	Squared Logarithmic Loss Function
WBLF	Weighted Balance Loss Function
WLF	Weighted Balance Loss function

Symbols

Symbols	**Full meaning**
_	Minus
+	Plus
/	divide
>	Greater than
Π	Product
ln	Natural Logarithm
Σ	Summation
~	Distributed
∫	Integral
α	Scale parameter
β	Shape parameter

Abstract

In this research, the shape parameter of the Generalized Inverse Exponential Distribution (GIED) was estimated using maximum likelihood and Bayesian estimation techniques. The Bayes estimates were obtained under the squared error loss function and precautionary loss function under the assumption of two non-informative priors. An extensive Monte Carlo simulation study was carried out to compare the performances of the Bayes estimates with that of the maximum likelihood estimates at different sample sizes. It was found out that the maximum likelihood have the same estimate with the Jeffrey's prior using the squared error loss function, and also performed better than the Bayes estimates under the Jeffrey's prior using the precautionary loss function and uniform prior using both loss function but performed lesser than the Extended Jeffrey's prior under both loss functions. The Extended Jeffrey's prior was observed to have estimated the shape parameter of the GIED better when compared with the maximum likelihood estimator and other Bayes estimate at all sample sizes using their mean squared error. Also the squared error loss function under the Extended Jeffrey's prior has the best estimate when compared with other Bayes estimates using their posterior risk. Hence the Bayes estimate under the Extended Jeffrey's using the squared error loss function has the best estimator for estimating the shape parameter of the GIED.

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

In the past, many generalized univariate continuous distribution have been proposed. The generalization of these distributions is important in order to make its shape more flexible to capture the diversity present in the observed dataset. One of such generalizations is the Generalized Inverse exponential distribution (GIED) proposed by Abouammoh and Alshangiti (2009), in which the shape parameter was added to make the distribution more flexible. As a result, this parameter has to be estimated using the appropriate estimation technique. One of such techniques is the Bayesian method of estimation which combines the prior knowledge with new observations to come up with updated information.

Researchers have estimated the parameter of different distributions using the Bayesian technique because of its advantage over other methods of estimation. Some of this research includes the work of Farhad *et al.,* (2013) which studied the scale parameter of inverse weibull distribution. Also, Dey (2015) studied the inverted exponential distribution using this technique.

Although the GIED has been studied using this technique under the assumption of the informative prior, but there are situations where we do not have information about the prior as such there will be need to study it under the non-informative prior. It is in the light of this that, this research intends to study the estimation of the shape parameter of the GIED under the non-informative priors using two loss functions with the assumption that the scale parameter is known.

1.2 Generalized Inverse Exponential Distribution

One of the simplest and most widely discussed distributions that is used for life testing is the one parameter exponential distribution. The distribution plays a vital role in the development of theories. One of the limitations of this distribution is that its applicability is restricted to a constant hazard rate. This is because there is hardly any system that has time independent hazard rate. As a result, a number of generalizations of the exponential distribution have been proposed in earlier literatures, for example the gamma distribution which is sum of independent exponential variates.

One of the extension of the exponential distribution is the inverted exponential distribution proposed by Killer and Kamath (1982) which possess the inverted bathtub hazard rate and has cumulative distribution function (CDF) expressed as

$$F(x,\alpha) = e^{-(\frac{\alpha}{x})} \tag{1.1}$$

and probability density function (pdf) as

$$f(x,\alpha) = \frac{\alpha}{x^2} \mathrm{e}^{-(\frac{\alpha}{x})}; x > 0, \alpha > 0 \tag{1.2}$$

The generalized inverse exponential distribution has cumulative distribution function express as;

$$F(x,\alpha,\beta) = 1 - \left(1 - e^{-\frac{\alpha}{x}}\right)^{\beta} \tag{1.3}$$

and probability density function (pdf),

$$f(x,\alpha,\beta) = \frac{\alpha\beta}{x^2} e^{-\frac{\alpha}{x}} \left(1 - e^{-\frac{\alpha}{x}}\right)^{\beta-1}; x > 0, \alpha > 0, \beta > 0 \tag{1.4}$$

where α is the scale parameter.

β is the shape parameter.

1.3 Statement of Problem

Bayesian estimation involves choosing the appropriate prior for the parameters. Although there is no way one can say one prior is better than the other. It all depends on the prior chosen. If there is no information about the parameter of interest then it is more preferable to estimate the parameter using non-informative prior. Otherwise the informative prior will be better. As a result there is need for us to find the appropriate prior for estimating the shape parameter of the Generalized Inverse Exponential Distribution when there is little or no information about the prior.

1.4 Aim and Objectives

The aim of this research is to estimate the shape parameter of Generalized Inverse Exponential Distribution (GIED) using Bayesian approach.

The aim is to be achieved through the following specific objectives

1. obtain the posterior distribution under the uniform, Jeffrey and Extended Jeffrey's priors;
2. determine the Bayes estimator and Bayes posterior risk of the shape parameter using the afore mentioned priors under square error loss function (SELF) and precautionary loss function (PLF);
3. conduct simulation study in order to find the most appropriate combination of the loss functions and prior for the estimation of the shape parameter of the posterior distribution.

4. determine which prior estimate the shape parameter of the generalized inverse exponential distribution with a minimum risk.
5. compare the maximum likelihood estimate and the Bayes estimate.

1.5 Significance and Justification of the study

The research will be of much importance in Statistics, since statistical decision theory deals with situation in which decision have to be made with some level of uncertainty. The Bayesian approach offers a method of formalizing a prior belief and combining them with available observation with the aim of allowing a rational deviation of optimal decision criteria. Important reason behind the choice of prior belief is that inferential problem can be naturally viewed as a special case of decision problem. As a result all the conceptual tools of Bayesian decision theories are incorporated into inference criteria.

This study will help to determine which prior estimate the shape parameter of the generalized inverse exponential distribution with a minimum risk.

1.6 Definition of Terms

1.4.1 Likelihood function

The likelihood of a parameter θ given x is define as the joint probability density function assumed for the observed outcomes given the parameter value.

1.4.2 Prior distribution

The prior distribution is the assumed distribution of the parameter before any data is observed. There are three different types of prior:

The informative is used when there is previous knowledge about the parameter to be estimated. This distribution favors certain value of the parameter.

The non- informative prior is used when there is a general lack of knowledge about the parameter to be estimated. This prior does not favor any value of the parameter.

The conjugate prior has the same functional form with the posterior distribution. The rationale behind the use of this prior is that of easing computational difficulties and also can have a close form expression for the distribution.

For this research the non- informative prior will be used in order to allow the data speak for itself and also to have the prior distribution contribute minimally, since it was stated by Arnold and Press (1983) that there is no way one can say one prior is better than the other. It all depends on the prior chosen.

1.4.3 Posterior distribution

Under the Bayesian approach, prior beliefs about parameter of interest are combine with the sample information to give an updated information about the parameter.

The posterior distribution is define as the distribution of the parameter after taking into consideration the prior and the observed data. It summarizes available probabilistic information on the parameter in the form of prior distribution and sample information which are contained in the likelihood function.

CHAPTER TWO

LITERATURE REVIEW

2.1 Baseline Distribution

A one-parameter Inverse Exponential Distribution introduced by Keller and Kamath (1982) has an inverted bathtub failure rate and it can compare competitively with exponential distribution. It is one of the distributions that is used in modelling lifetime data. Recently, several generalization of inverse-exponential distribution were obtained. One of which is the Generalized Inverse Exponential distribution (GIED) introduced by Abouammoh and Alshangiti (2009). They have investigated its statistical properties and its reliability functions. This distribution can be used to represent different shapes of failure rates and hence different shapes of aging criteria.

2.2 Bayesian Concept

The Bayesian concept was introduced by Reverend Thomas Bayes in the 1970s. In this concept, we combine any new information that is available with the prior information we have, to form the basis for the statistical procedure. The Bayesian approach seeks to optimally merge information from two sources namely the; the knowledge that is known from theory or opinion formed at the beginning of the research obtained in the form of prior and information contained in the data in the form of likelihood function. This two combine together can be used to obtain the posterior distribution. The main difference between the Bayesian approach and the classical approach is that in Bayesian approach, the parameters are viewed as random variable, whereas the classical concept consider the parameters to be fixed but unknown.

Different researcher have used the Bayesian approach to estimate the shape and (or) scale parameter of different distributions and compare with the classical approach. Some of this research includes;

Feroze (2012) discussed the Bayesian analysis of the scale parameter of inverse Gaussian distribution using different priors and loss function. He used both informative (exponential, gamma and chi-square) and non-informative (uniform and Jeffreys) priors with eight loss functions namely; squared error loss function (SELF), quadratic loss function (QLF), entropy loss function (ELF), weighted loss function (WLF), squared logarithmic loss function (SLLF), linear exponential (LINEX) loss function, precautionary loss function (PLF) and weighted balanced loss function (WBLF). It was deduced from the study that, the performance of the estimates under uniform prior is better than those under Jeffreys prior for most of the cases. While in case of informative priors used, the performance of estimates using exponential prior is the best in terms of Bayes risks. Similarly, in comparison of informative and non informative priors, the informative priors give better results. Although, the estimates under informative priors converge to the estimates under non-informative priors as the values of hyper-parameters approach zero. It is also indicated that the estimates under ELF are associated with the minimum risks using each prior.

Farhad *et al.,* (2013) studied the classical and Bayesian approach of estimating the scale parameter of Inverse Weibull distribution when the shape parameter was known under the assumption of quasi, gamma and uniform priors using square error loss function, entropy loss function and precautionary loss function. It was observed that Bayes method of estimation for gamma prior is superior to the Maximum Likelihood Estimates (MLEs) method. Also in the case of the gamma prior, the Bayes estimator relative to the precautionary loss function have the smallest mean square error when compared with the Bayes estimator relative to the square error loss function or the Bayes estimator under entropy loss function or the MLEs.

Yahgmaei *et al.*, (2013) proposed classical and Bayesian approaches for estimating the scale parameter in the inverse Weibull distribution when shape parameter is known. The Bayes estimators for the scale parameter of the Inverse Weibull distribution were derived, by considering Quasi, Gamma and uniform priors under squared error, entropy and precautionary loss function.The results shows that the Bayes method of estimation for gamma prior is superior to the MLE method. Also, in the case of gamma prior, the Bayes estimators related to precautionary loss function have the smallest MSE as compared with the Bayes estimators related to square error loss function or the Bayes estimators under entropy loss function or the MLEs. Furthermore, in the case of uniform prior, the Bayes estimators under square error loss function are doing better than the Bayes estimators under precautionary loss function.

Azam and Ahmed (2014) estimated the scale parameter of Nakagami distribution using Bayesian approach. The study revealed that the scale parameter was estimated under three prior distributions, namely; Uniform, Inverse Exponential and Levy priors and three loss functions namely; Squared Error Loss Function, Quadratic Loss Function and Precautionary Loss Function. The result of their analysis showed that the Precautionary Loss Function produces the least posterior risk when uniform prior is used while Squared Error Loss Function is the best when inverse exponential and Levy Priors are used.

Bhupendra and Reetu (2015) studied the maximum and Bayesian estimation of the inverse exponential distribution based on type-II censored sample under the assumption of conjugate prior using square, Generalize Entropy Loss Function (GELF) and LINEX loss functions. The maximum likelihood was compared with Bayesian estimates using Monte Carlo simulations. The result showed that the Bayesian estimation based on Squared Error Loss Function (SELF), LINEX and GELF was more precise than that of the maximum likelihood estimation.

Dey (2015) studied the inverted exponential distribution as a life distribution from a Bayesian viewpoint. The Bayes estimators for the parameter of the distribution base on the SELF and LINEX loss function were derived. It observed that the LINEX was more appropriate than the SELF.

Nasir *et al.,* (2015) studied Bayesian estimation of the scale parameter of log logistic distribution using square error loss function, precautionary loss function, simple precautionary loss function and weighted loss function with two non-informative priors(uniform and Jeffery). The study shows that Bayes estimators approach to their true value and posterior risks decreases as sample size increases. From the study, Jeffery prior performed better than the uniform prior and also precautionary loss function perform better than the other loss functions. Therefore, Jeffery with precautionary loss function provides minimum posterior risks as compared to other loss functions and priors.

Wasif and Navid (2015) studied the posterior analysis of Nakagami distribution under the assumptions of uniform and inverse gamma prior using Weighted, Weighted Balance and precautionary loss functions. The Bayesian estimator of the scale parameter of Nakagami distribution was obtained. The result of the analysis showed that the performance of inverse gamma prior is better than the uniform prior and that weighted balance loss function are associated with least amount of posterior risk under each prior.

Yahia *et al* (2015) developed the maximum likelihood and Bayesian estimation based on pooled sample of two independent type-II censored samples from the inverse exponential distribution.

The conjugate gamma prior was used with square error, LINEX and general entropy loss functions. A simulation study was carried out using Monte Carlo simulation to evaluate the performance of the maximum likelihood estimate and all the Bayesian estimates. It was observed that the Bayesian estimates returned smaller values of loss functions than those of maximum likelihood estimates.

Aliyu and Abubakar (2016) studied the Bayesian estimation of the shape parameter of generalized Raleigh distribution under the assumption of extended Jeffrey's prior with three loss functions; SELF, ELF and PLF, compared the performance of the MLEs and the Bayes estimator and found out that Bayes estimator under the entropy loss function is better than the Bayes estimators under the square error and precautionary loss functions and that of MLEs.

Kaisar *et al* (2016) studied the classical and Bayesian approach of scale parameter of Nakagami distribution under the assumption of Jeffrey, Extended Jeffrey and Quasi priors using quadratic, Al-Bayyati and entropy loss functions. The estimate of the scale parameter using simulated data set was obtained.

Sanjay *et al.,* (2013) studied the maximum likelihood estimates (MLEs) of the parameters of generalized inverted exponential distribution in case of type-II censoring scheme with binomial removal under the assumption of gamma prior and two loss functions. The result of the analysis showed that the estimated risk of estimator decreases as effective sample size increases and Bayes estimates have the smallest estimated risk when compared with their corresponding maximum likelihood estimates.

Therefore, with the available knowledge and insight of what others have done, this research intends to estimate the shape parameter of a generalized inverted exponential distribution using

Bayesian viewpoint under three non-informative priors and two loss functions and compare with the maximum likelihood estimate.

CHAPTER THREE

RESEARCH METHODOLOGY

Bayesian estimation procedure will be used to estimate the shape parameter of GIED assuming the non-informative priors under the squared error loss function which is classified as symmetric loss function and the precautionary loss function which is classified as asymmetric loss function.

3.1 Likelihood Function

Let $(x_1, x_2, ..., x_n)$ be random sample of size n, then the likelihood function is defined as;

$$L(x_i / \beta) = \prod_{i=1}^{n} f(x_i / \beta) \quad (3.1)$$

The likelihood plays an important role in the estimation of parameter from a set of statistics.

Let $(x_1, x_2, ..., x_n)$ be random sample of size n drawn from GIED having pdf given as;

$$f(x;\alpha,\beta) = \frac{\alpha\beta}{x^2} e^{-\frac{\alpha}{x}} (1 - e^{-\frac{\alpha}{x}})^{\beta-1}; x > 0, \alpha > 0, \beta > 0 \quad (3.2)$$

then, taking equation (3.2) into (3.1), the likelihood of the GIED is given by:

$$L(x,\alpha,\beta) = \prod_{i=1}^{n} \frac{\alpha\beta}{x_i^2} e^{-\frac{\alpha}{x_i}} (1 - e^{-(\frac{\alpha}{x_i})})^{\beta-1} \quad (3.3)$$

which can also be expressed as;

$$L(x;\alpha,\beta) = \alpha^n \beta^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i}^{n} \frac{1}{x_i^2} (1 - e^{-(\frac{\alpha}{x_i})})^{\beta-1} \quad (3.4)$$

3.2 Maximum Likelihood Estimation

The value of the statistic which maximizes the likelihood function is called the maximum likelihood estimate and it is obtained as follows; taking the natural logarithm of equation (3.3), we have

$$\ln L\left(x_i;\alpha,\beta\right)=n\ln(\alpha)+n\ln(\beta)-\sum_{i=1}^{n}\ln\left(x_i^2\right)-\alpha\sum_{i=1}^{n}\frac{1}{x_i}+(\beta-1)\sum_{i=1}^{n}\ln(1-e^{-\frac{\alpha}{x_i}}) \quad (3.5)$$

differentiating equation (3.5) w.r.t β and setting it to zero

$$\frac{d\ln L\left(x;\alpha,\beta\right)}{d\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln(1-e^{-\frac{\alpha}{x_i}})=0 \quad (3.6)$$

solving for β in equation (3.6), gives

$$\hat{\beta}_{MLE}=-\frac{n}{\sum_{i=1}^{n}\ln(1-e^{-\frac{\alpha}{x_i}})}$$

Which can also be expressed as;

$$\hat{\beta}_{MLE}=\frac{n}{\sum_{i=1}^{n}\ln(1-e^{-\frac{\alpha}{x_i}})^{-1}} \quad (3.7)$$

3.3 Prior Distribution

The three non-informative priors used in this research are the uniform, Jeffrey's and extended Jeffrey's priors.

3.3.1 Uniform prior

One of the most famous non-informative priors is a uniform prior, it can be expressed as:

$$\pi(\beta) = k \text{ , where } k \text{ is a constant} \tag{3.8}$$

3.3.2 Jeffrey's prior

Another approach use to elicit a non-informative prior is the Jeffrey's prior whose principle leads to specifying that a prior should be proportional to the $\left[I\left(\beta\right)\right]^{\frac{1}{2}}$ .i.e.

$$\pi(\beta) \propto \left[I\left(\beta\right)\right]^{\frac{1}{2}} \tag{3.9}$$

where

$$I\left(\beta\right) = -nE\left[\frac{\partial^2 \log f\left(x;\alpha,\beta\right)}{\partial\beta^2}\right] \tag{3.10}$$

and $f\left(x;\alpha,\beta\right)$ denotes the conditional pdf for x given the parameter β. Now the $I(\beta)$ for the Generalized Inverse Exponential Distribution is obtain as follows:

Taking the natural logarithm of equation (3.2) we have

$$\ln f(x;\alpha,\beta) = \ln\alpha + \ln\beta - \log x^2 - \frac{\alpha}{x} + (\beta-1)\ln(1-e^{-\frac{\alpha}{x}}) \tag{3.11}$$

Taking the first and second derivatives w.r.t β we have

$$\frac{\partial\beta \ln f(x;\alpha,\beta)}{\partial\beta} = \frac{1}{\beta} + \ln(1-e^{-\frac{\alpha}{x}})$$

$$I(\beta) = \frac{\partial^2\beta \ln f(x;\alpha,\beta)}{\partial\beta^2} = -\frac{1}{\beta^2} \tag{3.12}$$

taking the expectation of equation (3.12) we have;

$$I(\beta) = -nE\left[\frac{\partial^2 \log f(x;\alpha,\beta)}{\partial\beta^2} / \beta\right] = -nE\left[-\frac{1}{\beta^2}\right]$$

$$I(\beta) = \frac{n}{\beta^2} \tag{3.13}$$

If the constant of proportionality is assumed to be one, then the Jeffrey's prior defined in (3.9) is given as

$$\pi(\beta) = \frac{\sqrt{n}}{\beta} \tag{3.14}$$

3.3.3 Extended Jeffrey's prior.

The extended Jeffrey's prior proposed by Al-Kutubi (2005) is defined by

$$\pi(\beta) \propto (I(\beta))^r \tag{3.15}$$

Where $(I(\beta))$ is as defined in equation (3.10).

Substituting (3.13) into (3.15) we obtained the extended Jeffrey's prior as

$$\pi(\beta) = \frac{n^r}{\beta^{2r}} \tag{3.16}$$

3.4 Posterior Distribution

The posterior distribution is given as;

$$P(\beta / x) = \frac{L(x_1, x_2, \ldots x_n, \alpha, \beta)\pi(\beta)}{\int_0^\infty L(x_1, x_2, \ldots x_n, \alpha, \beta)\pi(\beta)d\beta} \tag{3.17}$$

where $L(x_i, x_2, ..., x_n / \beta) = \prod_{i=1}^{n} f(x_i, \beta)$

and $\pi(\beta)$ is the prior distribution.

However, we shall use the GIED as sampling distribution with the non-informative priors to derive the posterior distribution.

3.4.1 Posterior distribution under the uniform prior

The posterior distribution under the uniform prior can be obtained by substituting 3.4 and 3.8 into 3.17 and it is given as;

$$p(\beta / x) = \frac{\alpha^n \beta^n e^{-\alpha \sum_{i=1}^{n} \frac{1}{x_i}} \prod_{i}^{n} \frac{1}{x_i^2} [1 - e^{-\frac{\alpha}{x_i}}]^{\beta - 1} .k}{\int_0^{\infty} \alpha^n \beta^n e^{-\alpha \sum_{i=1}^{n} \frac{1}{x_i}} \prod_{i}^{n} \frac{1}{x_i^2} [1 - e^{-\frac{\alpha}{x_i}}]^{\beta - 1} .k d\beta} \tag{3.18}$$

integrating and re-arranging the denominator of equation (3.18) gives

$$k \alpha^n e^{-\alpha \sum_{i=1}^{n} \frac{1}{x_i}} \int_0^{\infty} \beta^n \prod_{i}^{n} \frac{1}{x_i^2} [1 - e^{-\frac{\alpha}{x_i}}]^{\beta} [1 - e^{-\frac{\alpha}{x_i}}]^{-1} .d\beta \tag{3.19}$$

$$\text{Let } M = \prod_{i}^{n} \left[1 - e^{-\frac{\alpha}{x_i}} \right]^{\beta} \tag{3.20}$$

taking the natural logarithm of equation (3.20) and multiplying both sides by minus one (-1)

$$\Rightarrow -\ln(M) = \beta \sum_{i=1}^{n} \ln(1 - e^{-\frac{\alpha}{x_i}})^{-1} \tag{3.21}$$

making β the subject, we have

$$\beta = \frac{-\ln(M)}{\sum_{i}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}} \tag{3.22}$$

$$-\frac{1}{M}dM = \sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1} d\beta \tag{3.23}$$

$$d\beta = \frac{-dM}{M\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}} \tag{3.24}$$

substituting equations (3.22) and (3.24) into (3.19) gives,

$$\int_0^\infty L(\alpha,\beta;x)\pi(\beta)d\beta = k\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \int_1^0 \frac{\left(-\ln M\right)^n}{\left[\sum_{i}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right]^n} \prod_{i}^{n} \frac{1}{x_i^2} M[1-e^{-\frac{\alpha}{x_i}}]^{-1} \frac{-dM}{M\sum_{i}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}}$$

$$= -k\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \int_1^0 \frac{\left(-\ln M\right)^n}{\left[\sum_{i}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right]^{n+1}} \prod_{i=1}^{n} \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{-1} dM$$

$$= \frac{-k\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i}^{n} \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{-1}}{\left[\sum_{i}^{n} \ln[1-e^{-\frac{\alpha}{x_1}}]\right]^{n+1}} \int_1^0 \left(-\ln M\right)^n dM \tag{3.25}$$

let $y = -\ln M \Rightarrow -y = \ln M$ (3.26)

then

$$e^{-y} = M \tag{3.27}$$

differentiating equation (3.27) yields

$$-e^{-y}dy = dM \tag{3.28}$$

substituting equations (3.26) and (3.28) into equation (3.25)

$$\frac{k\alpha^{n} e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{-1}}{\left(\sum_{i}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right)^{n+1}}\int_{0}^{\infty} y^{n}e^{-y}dy$$

therefore the denominator of equation (3.18) becomes,

$$\int_0^\infty L(\alpha,\beta,x)\pi(\beta)d\beta = \frac{k\alpha^{n} e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{-1}}{\left(\sum_{i}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right)^{n+1}}\Gamma(n+1) \tag{3.29}$$

substituting equation (3.29) into (3.18) and re-arranging we have

$$p(\beta / x) = \frac{\alpha^{n}\beta^{n} e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta-1}.k.\left(\sum_{i}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right)^{n+1}}{k\alpha^{n} e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{-1}\Gamma(n+1)} \tag{3.30}$$

$$\frac{\beta^{n}\prod_{i}^{n}[1-e^{-\frac{\alpha}{x_i}}]^{\beta}\left(\sum_{i}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right)^{n+1}}{\Gamma(n+1)} \tag{3.31}$$

but from equation (3.21)

recall that $M=\prod_{i=1}^{n}(1-e^{-\frac{\alpha}{x_i}})^{\beta} \Rightarrow -\ln M=\beta\sum_{i}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}$

Taking exponential of both sides we have

$$M=e^{-\beta\left[\sum_{i}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right]}$$

therefore the posterior distribution is given as

$$p(\beta / x)=\frac{\beta^{n}N^{n+1}e^{-N\beta}}{\Gamma(n+1)} \qquad (3.32)$$

$$\text{where } N=\left(\sum_{i}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}\right) \qquad (3.33)$$

Therefore the posterior distribution under the uniform prior is given as equation (3.32), which is gamma distribution with parameters $(n+1)$ and N

3.4.2 Posterior distribution under the Jeffrey's prior

The posterior distribution under the Jeffrey's prior can be obtain by substituting equations (3.4) and (3.14) into (3.17) and it is given as;

$$p(\beta / x)=\frac{\alpha^{n}\beta^{n}e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}}\prod\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta-1}.\frac{\sqrt{n}}{\beta}}{\int_{0}^{\infty}\alpha^{n}\beta^{n}e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}}\prod\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta-1}.\frac{\sqrt{n}}{\beta}d\beta} \qquad (3.34)$$

re-arranging the denominator of equation (3.34) gives

$$\int_0^{\infty} L(x;\alpha,\beta)\pi(\beta)d\beta = \sqrt{n}\alpha^n e^{-\frac{\alpha}{\sum_{i=1}^{n} x_i}} \int_0^{\infty} \beta^{n-1} \prod_i^n \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta}[1-e^{-\frac{\alpha}{x_i}}]^{-1} d\beta \qquad (3.35)$$

Let $M = \prod_i^n [1-e^{-\frac{\alpha}{x}}]^{\beta}$

$$-\ln M = \beta \sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}$$

making β subject of formula we have

$$\beta = \frac{-\ln M}{\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}} \qquad (3.36)$$

$$d\beta = \frac{-dM}{M\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}}$$

therefore equation (3.35) becomes

$$\int_0^{\infty} L(x,\alpha,\beta)\pi(\beta)d\beta$$

$$= \sqrt{n}\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \int_1^0 \frac{\left(-\ln M\right)^{n-1}}{\left[\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^{n-1}} \prod_{i=1}^{n} \frac{1}{x_i^2} M[1-e^{-\frac{\alpha}{x}}]^{-1} \frac{-dM}{M\left[\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]}$$

$$= \frac{-\sqrt{n}\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i=1}^{n} \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x}}]^{-1}}{\left[\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^{n}} \int_1^0 (-\ln M)^{n-1} dM \qquad (3.37)$$

Let $y = -\ln M \Rightarrow -y = \ln M$ (3.38) taking the exponential of equation (3.38)

$$e^{-y} = M \tag{3.39}$$

differentiating equation (3.39) we have,

$$-e^{-y}dy = dM \tag{3.40}$$

substituting into the integral part of equation (3.37) we have

$$\int_0^\infty (-\ln M)^{n-1} dM = -\int_0^\infty y^{n-1} e^{-y} dy = -\Gamma(n)$$

The denominator of equation (3.34) becomes

$$\int_0^\infty L(x,\alpha,\beta)\pi(\beta)d\beta = \frac{\sqrt{n}\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i=1}^{n} \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x}}]^{-1}}{\left[\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^n} \Gamma(n) \tag{3.41}$$

Substituting equation (3.41) into (3.34) we derive the posterior using the Jeffrey's prior as

$$p(\beta / x) = \frac{\alpha^n \beta^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i=1}^{n} \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta-1} . \frac{\sqrt{n}}{\beta} . \left[\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^n}{\sqrt{n}\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i=1}^{n} \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x}}]^{-1}\Gamma(n)}$$

$$= \frac{\beta^{n-1} \prod_{i=1}^{n} (1-e^{-\frac{\alpha}{x_i}})^{\beta} \left[\sum_{i=1}^{n} \ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^n}{\Gamma(n)} \tag{3.42}$$

$$= \frac{\beta^{n-1} N^n e^{-N\beta}}{\Gamma(n)} \tag{3.43}$$

where N is as define in equation (3.33).

Therefore the posterior distribution is given as equation (3.43), which is a gamma distribution with parameters n and N.

3.4.3 Posterior distribution under the extended Jeffrey's prior.

The posterior distribution under the Jeffrey's prior can be obtain by substituting equations (3.4) and (3.16) into (3.17) and it is given as

$$p(\beta / x) = \frac{\alpha^n \beta^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta-1}.\frac{n^r}{\beta^{2r}}}{\int_0^\infty \alpha^n \beta^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta-1}.\frac{n^r}{\beta^{2r}}d\beta} \tag{3.44}$$

Integrating the denominator of equation (3.44)

$$\int_0^\infty L(x;\alpha,\beta)\pi(\beta)d\beta = n^r\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}}\int_0^\infty \beta^{n-2r}\prod_i^n \frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta}[1-e^{-\frac{\alpha}{x_i}}]^{-1}d\beta \tag{3.45}$$

Let $M = \prod_i^n [1-e^{-\frac{\alpha}{x}}]^{\beta}$

$$-\ln M = \beta\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}$$

Making β subject of formula we have

$$\beta = \frac{-\ln M}{\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}} \tag{3.46}$$

$$d\beta = \frac{-dM}{M\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x_i}}]^{-1}}$$

Therefore equation (3.45) becomes

$$= n^r \alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \int_1^0 \frac{\left(-\ln M\right)^{n-2r}}{\left[\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^{n-2r}} \prod_{i=1}^{n}\frac{1}{x_i^2} M[1-e^{-\frac{\alpha}{x}}]^{-1} \frac{-dM}{M\left[\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]}$$

$$= \frac{-n^r \alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i=1}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x}}]^{-1}}{\left[\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^{n-2r+1}} \int_1^0 (-\ln M)^{n-2r} dM \tag{3.47}$$

Let $y = -\ln M \Rightarrow -y = \ln M$ (3.48)

$$e^{-y} = M \tag{3.49}$$

Substituting into the integral part of equation (3.47) we have,

$$\int_0^\infty (-\ln M)^{n-2r} dM = -\int_0^\infty y^{n-2r} e^{-y} dy = -\Gamma(n-2r+1)$$

Therefore, the denominator of equation (3.47) becomes

$$= \frac{n^r \alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}} \prod_{i=1}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x}}]^{-1}}{\left[\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^{n-2r+1}} \Gamma(n-2r+1) \tag{3.50}$$

Substituting equation (3.50) into (3.44) we derive the posterior under the extended Jeffrey's prior as

$$p(\beta/x)=\frac{\alpha^n\beta^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}}\prod_{i=1}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x_i}}]^{\beta-1}\cdot\frac{n^r}{\beta^{2r}}\cdot\left[\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^{n-2r+1}}{n^r\alpha^n e^{-\alpha\sum_{i=1}^{n}\frac{1}{x_i}}\prod_{i=1}^{n}\frac{1}{x_i^2}[1-e^{-\frac{\alpha}{x}}]^{-1}\Gamma(n-2r+1)}$$

$$=\frac{\beta^{n-2r}\prod_{i=1}^{n}(1-e^{-\frac{\alpha}{x_i}})^{\beta}\left[\sum_{i=1}^{n}\ln[1-e^{-\frac{\alpha}{x}}]^{-1}\right]^{n-2r+1}}{\Gamma(n-2r+1)}$$

$$=\frac{\beta^{n-2r}N^{n-2r+1}e^{-N\beta}}{\Gamma(n-2r+1)} \tag{3.51}$$

Therefore the posterior distribution is given as equation (3.51), which is a gamma distribution with parameters (n-2r+1) and N.

3.5 Loss Function

From a decision theoretic view point, to select the best estimator, a loss function must be specified, which is used to present the penalty associated with each of the possible Bayes estimates. Two loss functions namely squared error loss function and precautionary loss function shall be used to estimate the penalty associated with each of the possible Bayes estimates of the GIED parameter.

3.5.1 Squared error loss function

The squared error loss function (SELF), classified under the symmetric loss function associates' greater importance to both over and under estimation and it is defined as:

$$L(\beta, \hat{\beta}) = (\beta - \hat{\beta})^2 \quad (3.52)$$

The Bayes estimator of β relative to the SELF is denoted by

$$\hat{\beta} = E(\beta / x) \quad (3.53)$$

where $$E(\beta / x) = \int_0^\infty \beta p(\beta / x) d\beta \quad (3.54)$$

Equation (3.54) can be obtain by minimizing the expected loss ($E[L(\beta, \hat{\beta})]$) over β with respect to the posterior distribution ($p(\beta / x)$) i.e.

$$E[L(\beta, \hat{\beta})] = \int_0^\infty L(\beta, \hat{\beta}) P(\beta / x) d\beta$$

$$= \int_0^\infty (\hat{\beta} - \beta)^2 P(\beta / x) d\beta$$

Differentiating with respect to $\hat{\beta}$ and equating to zero, we obtain

$$2\int_0^\infty (\hat{\beta} - \beta) P(\beta / x) d\beta = 0$$

$$\int_0^\infty \hat{\beta} P(\beta / x) d\beta - \int_0^\infty \beta P(\beta / x) d\beta = 0$$

$$\int_0^\infty \hat{\beta} P(\beta / x) d\beta = \int_0^\infty \beta P(\beta / x) d\beta$$

but $\int_0^\infty P(\beta / x) d\beta = 1$

This implies that

$$\widehat{\beta} = \int_0^\infty \beta P(\beta / x) d\beta$$

and the Bayes posterior risk is given as:

$$V(\beta / x) = E(\beta^2 / x) - \left[E(\beta / x)\right]^2 \tag{3.55}$$

3.5.1.1 Bayes estimate and posterior risk relative to squared error loss function under uniform prior.

Let denote the Bayes estimator of β by $\widehat{\beta}_{selfu}$ relative to the squared error loss function under the uniform prior.

$\widehat{\beta}_{selfu}$ can be obtained by substituting equation (3.32) into (3.54)

$$\widehat{\beta}_{selfu} = \int_0^\infty \frac{\beta\beta^n N^{n+1} e^{-N\beta}}{\Gamma(n+1)} d\beta \tag{3.56}$$

$$= \frac{N^{n+1}}{\Gamma(n+1)} \int_0^\infty \beta^{n+1} e^{-N\beta} d\beta \tag{3.57}$$

let $h = N\beta$

then $dh = Nd\beta$

and $\beta = \dfrac{h}{N}$

therefore,

$$\beta_{selfu} = \frac{N^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \frac{h^{n+1}}{N^{n+2}} e^{-h} dh \tag{3.58}$$

$$= \frac{1}{\overline{|(n+1)}.N} \int_0^{\infty} h^{n+1} e^{-h} dh \tag{3.59}$$

$$= \frac{\Gamma(n+2)}{\Gamma(n+1).N} \tag{3.60}$$

$$\widehat{\beta}_{selfu} = \frac{(n+1)}{N} \tag{3.61}$$

also $E(\beta^2 / x) = \int_0^{\infty} \beta^2 P(\beta / x) d\beta$

$$\int_0^{\infty} \frac{\beta^2 \beta^n N^{n+1} e^{-N\beta}}{\Gamma(n+1)} d\beta \tag{3.62}$$

$$= \int_0^{\infty} \frac{\beta^{n+2} N^{n+1} e^{-N\beta}}{\Gamma(n+1)} d\beta$$

$$= \frac{N^{n+1}}{\Gamma(n+1)} \int_0^{\infty} \frac{h^{n+2}}{N^{n+3}} e^{-h} dh \tag{3.63}$$

$$= \frac{1}{\Gamma(n+1) N^2} \int_0^{\infty} h^{n+2} e^{-h} dh$$

$$= \frac{\Gamma(n+3)}{\Gamma(n+1) N^2}$$

therefore, $E\left(\beta^2 / x\right) = \frac{(n+2)!}{n! N^2}$

$$= \frac{(n+2)(n+1)}{N^2} \tag{3.64}$$

Therefore the Bayes estimate of the parameter β under uniform prior is given as (3.61) while the Bayes posterior risk corresponding to the estimate is obtained as;

$$\beta PR_{selfu} = \frac{(n+1)(n+2)}{N^2} - \left[\frac{(n+1)}{N}\right]^2$$

$$\beta PR_{selfu} = \frac{(n+1)}{N^2} \tag{3.65}$$

3.5.1.2 Bayes estimate and posterior risk relative to squared error loss function under Jeffrey's prior.

The estimate can be obtain by substituting equation (3.43) into (3.54)

$$\hat{\beta}_{selfj} = \int_0^\infty \frac{\beta\beta^{n-1}N^n e^{-N\beta}}{\Gamma(n)} d\beta$$

$$\frac{N^n}{\Gamma(n)} \int_0^\infty \beta^n e^{-N\beta} d\beta$$

$$= \frac{N^n}{\Gamma(n)} \int_0^\infty \frac{h^n}{N^{n+1}} e^{-h} dh \tag{3.66}$$

where $h = N\beta$

$$= \frac{1}{\Gamma(n)N} \int_0^\infty h^n e^{-h} dh \tag{3.67}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n)N}$$

$$= \frac{n!}{(n-1)!N}$$

therefore,

$$\hat{\beta}_{selfj} = \frac{n}{N} \tag{3.68}$$

also $E\left(\beta^2 / x\right) = \int_0^\infty \frac{\beta^2 \beta^{n-1} N^n e^{-N\beta}}{\Gamma(n)} d\beta$

$$= \frac{N^n}{\Gamma(n)} \int_0^\infty \frac{h^{n+1}}{N^{n+2}} e^{-h} dh$$

where $h = N\beta$

$$= \frac{1}{\Gamma(n) N^2} \int_0^\infty h^{n+1} e^{-h} dh$$

$$= \frac{\Gamma(n+2)}{\Gamma(n) N^2}$$

$$= \frac{(n+1)n(n-1)!}{N^2 (n-1)!}$$

$$= \frac{n(n+1)}{N^2} \tag{3.69}$$

$$BPR_{selfj} = E\left(\beta^2 / x\right) - \left[E\left(\beta / x\right)\right]^2$$

$$= \frac{n\left(n+1\right) - \left(n\right)^2}{N^2}$$

$$BPR_{selfj} = \frac{n}{N^2} \tag{3.70}$$

3.5.1.2 Bayes estimate and posterior risk relative to squared error loss function under extended Jeffrey's prior.

The estimate can be obtain by substituting equation (3.51) into (3.54)

$$\hat{\beta}_{selfex} = \int_0^\infty \frac{\beta\beta^{n-2r} N^{n-2r+1} e^{-N\beta}}{\Gamma(n-2r+1)} d\beta$$

$$\frac{N^{n-2r+1}}{\Gamma(n-2r+1)} \int_0^\infty \beta^{n-2r+1} e^{-N\beta} d\beta$$

$$= \frac{N^{n-2r+1}}{\Gamma(n-2r+1)} \int_0^\infty \frac{h^{n-2r+1}}{N^{n-2r+2}} e^{-h} dh \tag{3.71}$$

where $h = N\beta$

$$= \frac{1}{\Gamma(n-2r+1)N} \int_0^\infty h^{n-2r+1} e^{-h} dh \tag{3.72}$$

$$= \frac{\Gamma(n-2r+2)}{\Gamma(n-2r+1)N}$$

$$= \frac{(n-2r+1)(n-2r)!}{(n-2r)!N}$$

therefore,

$$\hat{\beta}_{selfex} = \frac{n-2r+1}{N} \tag{3.73}$$

also $E\left(\beta^2/x\right)=\int_0^\infty \frac{\beta^2\beta^{n-2r}N^{n-2r+1}e^{-N\beta}}{\Gamma(n-2r+1)}d\beta$

$$=\frac{N^{n-2r+1}}{\Gamma(n-2r+1)}\int_0^\infty \frac{h^{n-2r+2}}{N^{n-2r+3}}e^{-h}dh$$

where $h=N\beta$

$$=\frac{1}{\Gamma(n-2r+1)N^2}\int_0^\infty h^{n-2r+2}e^{-h}dh$$

$$=\frac{\Gamma(n-2r+3)}{\Gamma(n-2r+1)N^2}$$

$$=\frac{(n-2r+2)(n-2r+1)(n-2r)!}{N^2(n-2r)!}$$

$$=\frac{(n-2r+1)(n-2r+2)}{N^2} \tag{3.74}$$

$$BPR_{selfj}=E\left(\beta^2/x\right)-\left[E\left(\beta/x\right)\right]^2$$

$$=\frac{(n-2r+1)(n-2r+2)-\left(n-2r+1\right)^2}{N^2}$$

$$BPR_{selfex}=\frac{n-2r+1}{N^2} \tag{3.75}$$

Therefore the Bayes estimate is given as equation (3.73) and its posterior risk is given as (3.75)

3.5.2 Precautionary loss function

Precautionary loss function is defied by Norstrom (1996) as;

$$L(\hat{\beta},\beta)=\frac{(\beta-\hat{\beta})^2}{\hat{\beta}} \tag{3.76}$$

And the Bayes estimator of β denoted by $\hat{\beta}_p$ relative to PLF is given as;

$$\hat{\beta}_p=\left[E(\beta^2/x)\right]^{\frac{1}{2}} \tag{3.77}$$

$$\text{where } E(\beta^2/x)=\int_0^\infty \beta^2 p(\beta/x)d\beta \tag{3.78}$$

Equation (3.77) can be obtain by minimizing the expected loss ($E[L(\beta,\hat{\beta})]$) over β with respect to the posterior distribution ($p(\beta/x)$) i.e.

$$R(\beta,\hat{\beta})=\int_0^\infty \frac{(\hat{\beta}-\beta)^2}{\hat{\beta}}P(\beta/x)d\beta$$

$$=\int_0^\infty (\hat{\beta}-\beta)^2\hat{\beta}^{-1}P(\beta/x)d\beta$$

Differentiating w.r.t $\hat{\beta}$ and equating to zero

$$\int_0^\infty \left[2(\hat{\beta}-\beta)\hat{\beta}^{-1}-\hat{\beta}^{-2}(\hat{\beta}-\beta)^2\right]P(\beta/x)d\beta=0$$

$$\int_0^\infty \left[2\left(\frac{(\hat{\beta}-\beta)}{\hat{\beta}}\right)-\left(\frac{(\hat{\beta}-\beta)}{\hat{\beta}}\right)^2\right]P(\beta/x)d\beta=0$$

$$\int_0^\infty \left[\left(\frac{(\hat{\beta}-\beta)}{\hat{\beta}}\right)\left(2-\frac{(\hat{\beta}-\beta)}{\hat{\beta}}\right)\right]P(\beta/x)d\beta=0$$

$$\int_0^\infty \left[\left(\frac{(\hat{\beta}-\beta)}{\hat{\beta}}\right)\left(2-1+\frac{\beta}{\hat{\beta}}\right)\right]P(\beta/x)d\beta=0$$

$$\int_0^\infty \left[\left(\frac{(\hat{\beta}-\beta)}{\hat{\beta}}\right)\left(1+\frac{\beta}{\hat{\beta}}\right)\right]P(\beta/x)d\beta=0$$

$$\int_0^\infty \left[\left(1-\frac{\beta}{\hat{\beta}}\right)\left(1+\frac{\beta}{\hat{\beta}}\right)\right]P(\beta/x)d\beta=0$$

$$\int_0^\infty \left[\left(1-\frac{\beta^2}{\hat{\beta}^2}\right)\right]P(\beta/x)d\beta=0$$

$$\int_0^\infty P(\beta/x)d\beta-\int_0^\infty \frac{\beta^2}{\hat{\beta}^2}P(\beta/x)d\beta=0$$

$$1-\frac{1}{\hat{\beta}^2}\int_0^\infty \beta^2 P(\beta/x)d\beta=0$$

$$1=\frac{1}{\hat{\beta}^2}\int_0^\infty \beta^2 P(\beta/x)d\beta$$

$$1=\frac{1}{\hat{\beta}^2}E\left(\beta^2/x\right)$$

$$\hat{\beta}^2=E\left(\beta^2/x\right)$$

$$\hat{\beta}=\left[E\left(\beta^2/x\right)\right]^{\frac{1}{2}}$$

and the Bayes posterior risk is given as

$$V(\beta / x) = 2\left[\sqrt{E\left(\beta^2 / x\right)} - E\left(\beta / x\right)\right] \tag{3.79}$$

3.5.2.1 Bayes estimate relative to the precautionary loss function under the uniform prior

Let denote the Bayes estimator of β by $\widehat{\beta}_{plfu}$ relative to the precautionary loss function under the uniform prior.

$\widehat{\beta}_{plfu}$ can be obtained by substituting equation (3.32) into (3.78)

$$= \int_0^\infty \frac{\beta^2 \beta^n N^{n+1} e^{-N\beta}}{\Gamma(n+1)} d\beta \tag{3.80}$$

$$= \frac{N^{n+1}}{\Gamma(n+1)} \int_0^\infty \beta^{n+2} e^{-N\beta} d\beta \tag{3.81}$$

let $h = N\beta$

$$\widehat{\beta}^2_{plfu} = \frac{N^{n+1}}{\Gamma(n+1) N^{n+3}} \int_0^\infty h^{n+2} e^{-h} dh \tag{3.82}$$

$$= \frac{\Gamma(n+3)}{\Gamma(n+1) N^2} \tag{3.83}$$

$$= \frac{(n+1)(n+2)}{N^2}$$

$$\widehat{\beta}_{plfu} = \frac{\sqrt{(n+2)(n+1)}}{N} \tag{3.84}$$

Therefore the Bayes estimator of $\widehat{\beta}_{plfu}$ is given as equation (3.84)

The posterior risk is obtained as

$$BPR_{plfu} = \frac{2\left[\sqrt{(n+2)(n+1)} - (n+1)\right]}{N} \tag{3.85}$$

3.5.2.2 Bayes estimate relative to precautionary loss function under the Jeffrey's prior

Let denote the Bayes estimator of β by $\hat{\beta}_{Plfj}$ relative to the precautionary loss function under the Jeffrey's prior.

$\hat{\beta}_{Plfj}$ can be obtained by substituting equation (3.43) into (3.78)

$$\hat{\beta}_{plfj} = \left[\int_0^\infty \frac{\beta^2 \beta^{n-1} N^n e^{-N\beta}}{\Gamma(n)} d\beta\right]^{\frac{1}{2}} \tag{3.86}$$

where N is as defined earlier

$$\hat{\beta}^2_{plfj} = \frac{N^n}{\Gamma(n)} \int_0^\infty \beta^{n+1} e^{-N\beta} d\beta \tag{3.87}$$

let $h = N\beta$

then $$\hat{\beta}^2_{plfj} = \frac{1}{\Gamma(n) N^2} \int_0^\infty h^{n+1} e^{-h} dh \tag{3.88}$$

$$= \frac{\Gamma(n+2)}{\Gamma(n) N^2}$$

$$= \frac{n(n+1)}{N^2}$$

$$\text{and } \hat{\beta}_{plfj} = \frac{\sqrt{n(n+1)}}{N} \tag{3.89}$$

Therefore the Bayes estimate relative to the precautionary loss function under the Jeffrey's prior is given as equation (3.89)

The posterior risk is obtained as

$$BPR_{plfj} = \frac{2\left[\sqrt{n(n+1)} - (n)\right]}{N} \tag{3.90}$$

3.5.2.3 Bayes estimate relative to precautionary loss function under the Extended Jeffrey's prior

Let denote the Bayes estimator of β by $\widehat{\beta}_{Plfex}$ relative to the precautionary loss function under the extended Jeffrey's prior.

$\widehat{\beta}_{Plfex}$ can be obtained by taking the square root of equation (74)

$$\widehat{\beta}_{plfex} = \left[\int_0^\infty \frac{\beta^2 \beta^{n-2r} N^{n-2r+1} e^{-N\beta}}{\Gamma(n-2r+1)} d\beta\right]^{\frac{1}{2}} \tag{3.91}$$

$$= \left[\frac{(n-2r+1)(n-2r+2)}{N^2}\right]^{\frac{1}{2}}$$

$$= \frac{\sqrt{(n-2r+1)(n-2r+2)}}{N}$$

$$\hat{\beta}_{plfex} = \frac{\sqrt{(n-2r+1)(n-2r+2)}}{N} \tag{3.92}$$

Therefore the Bayes estimate relative to the precautionary loss function under the Extended Jeffrey's prior is given as equation (3.92)

The posterior risk is obtained as

$$\beta PR_{plfex} = 2\left[\frac{\sqrt{(n-2r+1)(n-2r+2)}-(n-2r+1)}{N}\right] \tag{3.93}$$

3.6 Transformation of the random variable N and its distribution.

Recall that the random variable N in equation (33) was define as;

$$N = \sum_{i=1}^{n} \ln(1-e^{-\frac{\alpha}{x_i}})^{-1}$$

where x_i's are random sample of size n drawn from Generalized Inverse Exponential Distribution (GIED) with probability density function given in equation (3.2).

The need to know the probability distribution of a random variable $N = \phi(x)$ where ϕ is some known function when the probability distribution of the random variable X is known arise in many statistical applications. Various methods for finding the distribution of a transformed random variable have been developed. These include distribution method, Transformation method, Convolution method, etc.

Transformation method and convolution methods will be used in this work to find the distribution of N.

Theorem 3.1: let X be a continuous random variable with probability density function f(x). let $n = T(x)$ be an increasing (or decreasing) function. Then, the density function of the random variable $n = T(X)$ is given by

$$g(n) = \left|\frac{dx}{dn}\right| f\left(w(n)\right) \tag{3.94}$$

where $x = w(n)$ is the inverse of $T(x)$

Proof

Suppose $N = T(x)$ is an increasing function. The distribution function $G(n)$ of N is given by

$$G(n) = P(N \leq n)$$

$$P(T(x) \leq n)$$

$$P\left(X \leq w(n)\right)$$

$$G(n) = \int_{-\infty}^{w(n)} f(x)dx$$

On differentiating, the density function of N is obtained as

$$g(n) = \frac{dG(n)}{dn} = \frac{d}{dn}\left(\int_{-\infty}^{w(n)} f(x)dx\right)$$

$$= f\left(w(n)\right)\frac{dw(n)}{dn}$$

$$= f\left(w(n)\right)\frac{dx}{dn} \qquad \text{since } x = w(n) \tag{3.95}$$

If on the other hand, $n = T(x)$ is a decreasing function, then the distribution of N is given by

$$G(n) = P(Y \leq n)$$

$$P(T(x) \le n)$$

$$P\left(X \ge w(n)\right)$$

$$G(n) = 1 - \int_{-\infty}^{w(n)} f(x)dx$$

On differentiating, the density function of N is given by

$$G(y) = -f\left(w(y)\right)\frac{dx}{dy} \tag{3.96}$$

Combining (3.88) and (3.89) the distribution of Y is given by

$$G(n) = \left|\frac{dx}{dn}\right| f\left(w(n)\right) \tag{3.97}$$

Theorem 3.2: let the joint density function of the random variable X and N be $f(x,n)$. The probability density of $X+N$ is given by

$$h_{x+n}(V) = \int_{-\infty}^{\infty} f(u, v-u)du \tag{3.98}$$

Proof

Let $U = X$ and $V = X + N$, so that $X = R(U,V) = U$ and $N = S(U,V) = V - U$. Hence, the jacobian is given by

$$j = \frac{\partial x}{\partial u}\frac{\partial n}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial n}{\partial u} = 1$$

The joint density function of U and V is given by

$$g(u,v)=\left|j\right| f\left(R(u,v),S(u,v)\right)$$

$$f\left(R(u,v),S(u,v)\right)$$

$$f(u,v-u)$$

Hence, the marginal density of $V = X + N$ is given by

$$h_{x+n}(V)=\int_{-\infty}^{\infty} f(u,v-u)du$$

If X and N are independent and have pdf $f(x)$ and $g(n)$ respectively, then

$$h_{x+n}(V)=\int_{-\infty}^{\infty} g(n)f(z-n)dn$$

3.6.1 Convolution

Let f and g be two real valued functions, the convolution of f and g is defined as

$$(f*g)(z)=\int_{-\infty}^{\infty} f(z-n)g(n)dn=\int_{-\infty}^{\infty} f(z-x)g(x)dx \tag{3.99}$$

Hence, the convolution of f and g is equal to the convolution of g and f (i.e. $f*g=g*f$).

Now, if $N=\sum_{i=1}^{n}\ln(1-e^{-\frac{\alpha}{x_i}})^{-1}$ and $X \sim GIED(\alpha,\beta)$ then, the distribution of N can be obtained as follows:

Let $i=1$, then $N_1=\ln(1-e^{-\frac{\alpha}{x_i}})^{-1}$

then, $x_1 = \frac{\alpha}{\ln(1-e^{-N_1})^{-1}}$

$$\frac{dx_1}{dN_1} = \frac{\alpha\left[\frac{e^{-N_1}}{(1-e^{-N_1})}\right]}{\left[\ln(1-e^{-N_1})^{-1}\right]^2}$$

$$= \frac{\alpha e^{-N_1}}{(1-e^{-N_1})\left[\ln(1-e^{-N_1})^{-1}\right]^2}$$

the jacobian $\left|\frac{dx_1}{dN_1}\right| = \frac{\alpha e^{-N_1}}{(1-e^{-N_1})\left[\ln(1-e^{-N_1})^{-1}\right]^2}$

Therefore, the pdf of N_1 if X_1 has the pdf (1.4) is given by

$$f(N_1) = f\left(w(x_1)\right)\left|\frac{dx_1}{dN_1}\right| \tag{3.100}$$

Where $f\left(w(x_1)\right) = \frac{\beta}{\alpha}\left[\ln(1-e^{-N_i})^{-1}\right]^2 e^{\ln(1-e^{-N_i})}\left[1-e^{\ln(1-e^{-N_i})}\right]^{\beta-1}$

$$= \frac{\beta}{\alpha}\left[\ln(1-e^{-N_1})^{-1}\right]^2 e^{\ln(1-e^{-N_1})}\left[1-e^{\ln(1-e^{-N_1})}\right]^{\beta-1}$$

$$= \frac{\beta}{\alpha}\left[\ln(1-e^{-N_i})^{-1}\right]^2 (1-e^{-N_i})\left[1-(1-e^{-N_i})\right]^{\beta-1}$$

$$= \frac{\beta}{\alpha}\left[\ln(1-e^{-N_i})^{-1}\right]^2 (1-e^{-N_i})e^{-N_i(\beta-1)}$$

then $f(N_1) = \frac{\beta}{\alpha}\left[\ln(1-e^{-N_i})^{-1}\right]^2 (1-e^{-N_i})e^{-N_i(\beta-1)} \times \frac{\alpha e^{-N_1}}{(1-e^{-N_1})\left[\ln(1-e^{-N_1})^{-1}\right]^2}$

$$\left[\ln A^{-1}\right]^2 = \left[-\ln A\right]^2 = \left[\ln A\right]^2$$

therefore

$$f(N_1) = \beta e^{-N_1}.e^{-N_1(\beta-1)}$$

$$= \beta e^{-N_1 - N_1(\beta-1)}$$

$$= \beta e^{-\beta N_1} \quad (3.101)$$

Following the same technique, for *i = 2, 3, ..., n* the distribution of $x_2, x_3, ..., x_n$ are respectively given by $f(N_2) = \beta e^{-\beta N_2}$, $f(N_3) = \beta e^{-\beta N_3}$, … , $f(N_n) = \beta e^{-\beta N_n}$. However, our interest is to find the distribution of $N_1 + N_2 + ... + N_n$ and since $N_1, N_2, ..., N_n$ are independent, the convolution method will be used to find the distribution of $N_1 + N_2 + ... + N_n$. First the density of the random variable $z = N_1 + N_2$ is the convolution of N_1 with N_2 that is

$$h(z) = (f * g)(z) = \int_{-\infty}^{\infty} f(z - N_1) g(N_1) dN_1 \quad (3.102)$$

Note that the $z = N_1 + N_2$ is between 0 and ∞ and $0 < N_1 < z$, hence (3.102) becomes

$$h(z) = (f * g)(z) = \int_0^z \beta e^{-\beta(z-N_1)} \beta e^{-\beta N_1} dN_1 \quad (3.103)$$

$$= \int_0^z \beta^2 e^{-\beta z + \beta N_1 - \beta N_1)} dN_1$$

$$= \beta^2 e^{-\beta z} \int_0^z dN_1 = \beta^2 e^{-\beta z} \left[N_1\right]_0^z$$

$$= \beta^2 z e^{-\beta z} \tag{3.104}$$

Following the same technique the distribution of $z = N_3 + N_4$ is

$$h(N_3 + N_4) = (f * g)(z) = \int_0^z \beta e^{-\beta(z-N_3)} \beta e^{-\beta N_3} dN_3$$

$$= \beta^2 z e^{-\beta z} \tag{3.105}$$

Following the same procedure, the distribution of $z = N_1 + N_2 + N_3 + N_4$ is obtain by

$$h(z) = (f * g)(z) = \int_0^z f(z - z_1) g(z_1) dz_1 \tag{3.106}$$

$$= \int_0^z \beta^2 (z - z_1) e^{-\beta(z-z_1)} \beta^2 z_1 e^{-\beta z_1} dz_1$$

$$= \int_0^z \beta^4 z_1 (z - z_1) e^{-\beta z} dz_1$$

$$= \beta^4 e^{-\beta z} \int_0^z z_1 z - z_1{}^2) dz_1$$

$$= \beta^4 e^{-\beta z} \left[\frac{z_1{}^2 z}{2} - \frac{z_1{}^3}{3} \right]_0^z$$

$$\beta^4 e^{-\beta z} \left[\frac{z^3}{2} - \frac{z^3}{3} \right]$$

$$= \frac{\beta^4 z^3 e^{-\beta z}}{6} \tag{3.107}$$

Careful examination of (3.104), (3.105) and (3.107) indicates that

$$f(N_1) = \beta e^{-\beta N_1} \sim Gamma(1, \beta)$$

$$f(z) = \beta^2 z e^{-\beta z} \sim Gamma(2, \beta) \text{ with } z = N_1 + N_2 \text{ and}$$

$$f(z) = \frac{\beta^4 z^3 e^{-\beta z}}{6} \sim Gamma(4, \beta) \text{ with } z = N_1 + N_2 + N_3 + N_4$$

Hence by induction $f(N_1 + N_2 + ... + N_n)$ is given by

$$f(N_1 + N_2 + ... + N_n) = f(N) = \frac{\beta^n N^{n-1} e^{-\beta N}}{\Gamma(n)} \sim Gamma(n, \beta) \tag{3.108}$$

therefore if $N = \sum_{i=1}^{n} \ln(1 - e^{-\frac{\alpha}{x_i}})^{-1}$ and $X \sim GIED(\alpha, \beta)$ then $z = \sum_{i=1}^{n} N_i \sim Gamma(n, \beta)$.

3.6.2 Variance and mean square error of estimates under the uniform prior and Jeffrey's prior using the various loss functions

From equation (3.108), it can be shown that

$$E(N^r) = \int_0^\infty N^r f(N) dN$$

$$= \int_0^\infty N^r \frac{\beta^n N^{n-1} e^{-\beta N}}{\Gamma(n)} dN$$

$$= \frac{\beta^n}{\Gamma(n)} \int_0^\infty N^{n+r-1} e^{-N\beta} dN$$

let $h = N\beta \Rightarrow N = \frac{h}{\beta}$ and $dN = \frac{1}{\beta} dh$

then,

$$E\left[N^r\right] = \frac{\beta^{-r}}{\Gamma(n)} \int_0^\infty \frac{h}{\beta^{n+r}}^{n+r-1} e^{-h} dh$$

$$= \frac{\beta^{-r}}{\Gamma(n)} \Gamma(r+n)$$

$$E(N^r) = \frac{\Gamma(r+n)}{\beta^r \Gamma(n)} \qquad (3.109)$$

when r=-1 and r=-2 in equation (91), we have

$$E(N^{-1}) = E\left(\frac{1}{N}\right) = \frac{\beta\Gamma(n-1)}{\Gamma(n)}$$

$$E(N^{-1}) = \frac{\beta}{(n-1)} \qquad (3.110)$$

$$E(N^{-2}) = E\left(\frac{1}{N^2}\right) = \frac{\beta^2\Gamma(n-2)}{\Gamma(n)}$$

$$= \frac{\beta^2}{(n-1)(n-2)} \qquad (3.111)$$

and variance of $\frac{1}{N}$ is obtained as follows:

$$\text{var}\left(\frac{1}{N}\right) = E\left(\frac{1}{N}\right)^2 - \left(E\left(\frac{1}{N}\right)\right)^2$$

$$= \frac{\beta^2}{(n-1)(n-2)} - \frac{\beta^2}{(n-1)^2}$$

$$= \frac{\beta^2}{(n-1)^2(n-2)} \tag{3.112}$$

while the MSE is given as

$$MSE\left(\widehat{\beta}\right) = \text{var}\left(\beta\right) + Bias^2 \tag{3.113}$$

where the Bias is given as

$$Bias = E(\widehat{\beta}) - \beta \tag{3.114}$$

the variance of the estimates under the uniform prior for the various loss functions are obtained as

$$\text{var}(\beta_{selfu}) = \text{var}\left(\frac{(n+1)}{N}\right) = (n+1)^2 \,\text{var}\left(\frac{1}{N}\right) \tag{3.115}$$

substituting equation (3.112), the variance of β_{selfu} is obtained as

$$\text{var}(\beta_{selfu}) = \frac{(n+1)^2}{(n-1)^2(n-2)}\beta^2 \tag{3.116}$$

and the Bias is obtain as follows

$$Bias = E\left(\widehat{\beta}_{selfu}\right) - \beta$$

$$= E\left[\frac{(n+1)}{N}\right] - \beta$$

$$=(n+1)E\left[\frac{1}{N}\right]-\beta$$

$$=n+1\left[\frac{\beta}{(n-1)}\right]-\beta$$

$$=\left[\frac{(n+1)-(n-1)}{(n-1)}\right]\beta$$

$$=\frac{2}{(n-1)}\beta \qquad (3.117)$$

therefore

$$MSE_{selfu}=\frac{(n+1)^2}{(n-1)^2(n-2)}\beta^2+\left[\frac{2}{(n-1)}\beta\right]^2$$

$$=\frac{(n+1)^2}{(n-1)^2(n-2)}\beta^2+\frac{4}{(n-1)^2}\beta^2$$

$$=\frac{\beta^2}{(n-1)^2}\left[\frac{(n+1)^2}{(n-2)}+4\right]$$

$$=\frac{\beta^2}{(n-1)^2}\left[\frac{(n+1)^2+4(n-2)}{(n-2)}\right]$$

$$=\frac{\beta^2}{(n-1)^2}\left[\frac{(n+7)(n-1)}{(n-2)}\right]$$

$$MSE_{selfu}=\frac{(n+7)}{(n-1)(n-2)}\beta^2 \qquad (3.118)$$

The $\operatorname{var}(\beta_{plfu})$ is obtained as

$$\operatorname{var}(\beta_{plfu}) = Var\left(\frac{\sqrt{(n+1)(n+2)}}{N}\right)$$

$$= (n+1)(n+2)Var(\frac{1}{N})$$

$$\operatorname{var}(\beta_{plfu}) = \frac{(n+2)(n+1)}{(n-1)^2(n-2)}\beta^2 \tag{3.119}$$

The bias is obtained as

$$Bias = E\left(\hat{\beta}_{plfu}\right) - \beta$$

$$= E\left[\frac{\sqrt{(n+1)(n+2)}}{N}\right] - \beta$$

$$= \sqrt{(n+1)(n+2)}E\left[\frac{1}{N}\right] - \beta$$

$$= \sqrt{(n+1)(n+2)}\left[\frac{\beta}{(n-1)}\right] - \beta$$

$$= \left[\frac{\sqrt{(n+1)(n+2)} - (n-1)}{(n-1)}\right]\beta \tag{3.120}$$

therefore

$$MSE_{plfu} = \frac{(n+1)(n+2)}{(n-1)^2(n-2)}\beta^2 + \left[\frac{\sqrt{(n+1)(n+2)}-(n-1)}{(n-1)}\beta\right]^2$$

$$= \frac{\beta^2}{(n-1)^2}\left[\frac{(n+1)(n+2)}{(n-2)} + \left[\sqrt{(n+1)(n+2)}-(n-1)\right]^2\right] \tag{3.121}$$

the variance of the estimates under the Jeffrey's prior for the various loss functions are obtained as

$$\text{var}(\beta_{selfj}) = \text{var}\left(\frac{n}{N}\right) = n^2 \text{ var}\left(\frac{1}{N}\right)$$

$$= \frac{n^2}{(n-1)^2(n-2)}\beta^2 \tag{3.122}$$

and the bias is obtained as follows

$$Bias = E\left(\widehat{\beta}_{selfj}\right) - \beta$$

$$= E\left[\frac{n}{N}\right] - \beta$$

$$= nE\left[\frac{1}{N}\right] - \beta$$

$$= n\left[\frac{\beta}{(n-1)}\right] - \beta$$

$$= \left[\frac{n-(n-1)}{(n-1)}\right]\beta$$

$$= \frac{\beta}{(n-1)} \quad (3.123)$$

therefore

$$MSE_{selfj} = \frac{n^2}{(n-1)^2(n-2)}\beta^2 + \left[\frac{\beta}{(n-1)}\right]^2$$

$$= \frac{\beta^2}{(n-1)^2}\left[\frac{n^2}{(n-2)} + 1\right]$$

$$= \frac{\beta^2}{(n-1)^2}\left[\frac{n^2 + (n-2)}{(n-2)}\right]$$

$$= \frac{\beta^2}{(n-1)^2}\left[\frac{(n+2)(n-1)}{(n-2)}\right]$$

$$= \frac{(n+2)}{(n-1)(n-2)}\beta^2 \quad (3.124)$$

The variance of β_{plfj} is obtained as

$$\text{var}(\beta_{plfj}) = Var\left(\frac{\sqrt{n(n+1)}}{N}\right)$$

$$= n(n+1)Var\left(\frac{1}{N}\right)$$

$$\text{var}(\beta_{plfj}) = \frac{n(n+1)}{(n-1)^2(n-2)}\beta^2 \quad (3.125)$$

and the bias is obtain as follows

$$Bias = E\left(\hat{\beta}_{plfj}\right) - \beta$$

$$= E\left[\frac{\sqrt{n(n+1)}}{N}\right] - \beta$$

$$= \sqrt{n(n+1)}E\left[\frac{1}{N}\right] - \beta$$

$$= \sqrt{n(n+1)}\left[\frac{\beta}{(n-1)}\right] - \beta$$

$$= \left[\frac{\sqrt{n(n+1)} - (n-1)}{(n-1)}\right]\beta \tag{3.126}$$

therefore,

$$MSE_{plfj} = \frac{\beta^2}{(n-1)^2}\left[\frac{n(n+1)}{(n-2)} + \left[\sqrt{n(n+1)} - (n-1)\right]^2\right] \tag{3.127}$$

the variance of the estimates under the Extended Jeffrey's prior for the various loss functions are obtained as

$$\text{var}(\beta_{selfex}) = \text{var}\left(\frac{n-2r+1}{N}\right) = (n-2r+1)^2 \text{ var}\left(\frac{1}{N}\right)$$

$$= \frac{(n-2r+1)^2}{(n-1)^2(n-2)}\beta^2 \tag{3.128}$$

and the bias is obtained as follows

$$Bias = E\left(\widehat{\beta}_{selfex}\right) - \beta$$

$$= E\left[\frac{n-2r+1}{N}\right] - \beta$$

$$= (n-2r+1)E\left[\frac{1}{N}\right] - \beta$$

$$= (n-2r+1)\left[\frac{\beta}{(n-1)}\right] - \beta$$

$$= \left[\frac{(n-2r+1)-(n-1)}{(n-1)}\right]\beta$$

$$= \frac{\beta(2-2r)}{(n-1)} \qquad (3.129)$$

therefore

$$MSE_{selfex} = \frac{(n-2r+1)^2}{(n-1)^2(n-2)}\beta^2 + \left[\frac{\beta(2-2r)}{(n-1)}\right]^2$$

$$= \frac{\beta^2}{(n-1)^2}\left[\frac{(n-2r+1)^2}{(n-2)} + (2-2r)^2\right]$$

$$= \frac{\beta^2}{(n-1)^2}\left[\frac{(n-2r+1)^2 + (n-2)(2-2r)^2}{(n-2)}\right]$$

The variance of β_{plfex} is obtained as

$$\operatorname{var}(\beta_{plfex}) = Var\left(\frac{\sqrt{(n-2r+1)(n-2r+2)}}{N} \right)$$

$$= (n-2r+1)(n-2r+2)Var\left(\frac{1}{N} \right)$$

$$\operatorname{var}(\beta_{plfex}) = \frac{(n-2r+1)(n-2r+2)}{(n-1)^2(n-2)}\beta^2 \qquad (3.130)$$

and the bias is obtain as follows

$$Bias = E\left(\hat{\beta}_{plfex}\right) - \beta$$

$$= E\left[\frac{\sqrt{(n-2r+1)(n-2r+2)}}{N} \right] - \beta$$

$$= \sqrt{(n-2r+1)(n-2r+2)}E\left[\frac{1}{N} \right] - \beta$$

$$= \sqrt{(n-2r+1)(n-2r+2)}\left[\frac{\beta}{(n-1)} \right] - \beta$$

$$= \left[\frac{\sqrt{(n-2r+1)(n-2r+2)} - (n-1)}{(n-1)} \right]\beta \qquad (3.131)$$

therefore,

$$MSE_{plfex} = \frac{\beta^2}{(n-1)^2}\left[\frac{(n-2r+1)(n-2r+2)}{(n-2)} + \left[\sqrt{(n-2r+1)(n-2r+2)} - (n-1) \right]^2 \right]$$

also the variance of $\hat{\beta}_{MLE}$ of equation (3.7) is obtained as

$$Var\left(\hat{\beta}_{MLE}\right)=Var\left(\frac{n}{N}\right)=n^2Var\left(\frac{1}{N}\right)$$

$$=\frac{n^2.\beta^2}{(n-1)^2(n-2)} \tag{3.132}$$

and the mean squared error as

$$MSE_{MLE}=\frac{(n+2)}{(n-1)(n-2)}\beta^2 \tag{3.133}$$

3.7 Simulation Study

There are different methods of simulating data under the Monte Carlo's method some of which are inverse-transformed method, alais method etc. (Reuven and Dirk 2007) For this work, the inverse-transform method will be use to generate our random numbers.

3.7.1 Inverse-transformation

Let X be a random variable with cumulative distribution function (cdf) F. where F is a decreasing function, the inverse function F^{-1} may be define as

$$F^{-1}(y)=\inf\left\{x:F(x)\geq y\right\};0\leq y\leq 1 \tag{3.134}$$

It is easy to show that if $U\sim U(0,1)$ then,

$$X=F^{-1}(U)$$

has cdf F since F is invertible and $P(U\leq u)=u$ we have

$$P(X \le x) = P(F^{-1}(U) \le x) = P(U \le F(x)) = F(x)$$

Thus, to generate a random variable X with cdf F, draw $U \sim U(0,1)$ and set $X = F^{-1}(U)$

For the distribution used in this work,

$$U = 1-(1-e^{-\frac{\alpha}{x_i}})^{\beta} \quad (3.135)$$

Thus our $X = F^{-1}(U)$ is obtain by making X subject of formula from equation

Hence, $$X = \frac{\alpha}{\ln(1-(1-u)^{\frac{1}{\beta}})^{-1}} \quad (3.136)$$

3.8 Monte-Carlo test.

The Monte Carlo test introduced by Barnard (1963) has attracted attractable attention recently. In order to test models against data we have to make use of Monte Carlo test.

The method is straight forward. Quite generally, let b_1 be the observed value of a statistic B and let b_j , j=2, …,s, be the corresponding values generated by independent random sampling from the distribution of B under the simple hypothesis H_o. Let $b_{(k)}$ be the k^{th} order statistic (denote the k^{th} largest, among $b_{(k)}$), k=1,2,…s. then under H_o

$P(b_1 = b_{(j)}) = 1/s$, j=1,2,…,s

And rejection of H_o on the basis that b_1 ranks r largest or highest (or lower or smallest) given an exact one side test of size $\alpha = \frac{r}{s}$. The test is exact in the sense that the type I error is precisely α

For a two sided Monte Carlos test, r is chosen such that $\alpha = \frac{2r}{s}$. It is expected that the value of the fixed parameter b_1 should fall within the s=99 simulation bands. This is done to enable us have 99% confidence bands that the fixed parameter fall between the minimum and maximum values of the parameters obtained. i.e. $\hat{\beta}_{\min} < \beta_1 < \hat{\beta}_{\max}$

CHAPTER FOUR

ANALYSIS AND DISCUSSION OF RESULT

In this section, an extensive Monte Carlo simulation was carried out to obtain and compare the performance of the different estimators for different sample sizes (n=15, 35, 75 and 100) against different shape parameter (β) values of 0.5, 1.0, 1.5 and 2.0 with the assumption that the scale parameter is known. The Monte Carlo simulation were replicated 10,000 times and averaged over.

4.1. Result 0f Analysis

Table1: average estimates, posterior risk (within parenthesis) and corresponding MSEs (green color) for n= 15

n	Method	$\beta = 0.5$	$\beta = 1.0$	$\beta = 1.5$	$\beta = 2.0$
15	MLE	0.5349772 0.02673302	1.069954 0.10693201	1.604932 0.24059733	2.139909 0.42772846
	SELFU	0.5706424 (0.02191433) 0.0393622	1.141285 (0.08765733) 0.1574489	1.711927 (0.197229) 0.3542597	2.282569 (0.3506293) 0.6297949
	SELFJ	0.5349772 (0.02054469) 0.02673302	1.069954 (0.08217874) 0.10693201	1.604932 (0.1849022) 0.24059733	2.139909 (0.328715) 0.42772846
	SELFEX	0.4279818 (0.01643575) 0.01408988	0.8559636 (0.06574299) 0.05635951	1.283945 (0.1479217) 0.12680883	1.711927 (0.262972) 0.22543800
	PLFU	0.5882047 (0.03512464) 0.04762833	1.176409 (0.07024929) 0.19051321	1.764614 (0.1053739) 0.42865497	2.352819 (0.1404986) 0.76205349
	PLFJ	0.5525221 (0.03508975) 0.03222176	1.105044 (0.07017951) 0.12888698	1.657566 (0.1052693) 0.28999571	2.210088 (0.140359) 0.51554793
	PLFEX	0.4454576 (0.03495156) 0.01445734	0.8909151 (0.06990312) 0.05782933	1.336373 (0.1048547) 0.13011607	1.78183 (0.1398062) 0.23131728

Table2: average estimates, posterior risk (within parenthesis) and corresponding MSEs for n= 35 and 75

35	MLE	0.5143571 0.008724456	1.028714 0.034897810	1.543071 0.078520071	2.057429 0.139591374
	SELFU	0.5290531 (0.00800992) 0.01047743	1.058106 (0.03203968) 0.04190972	1.587159 (0.07208928) 0.09429688	2.116212 (0.1281587) 0.16763889
	SELFJ	0.5143571 (0.00778742) 0.008724456	1.028714 (0.03114969) 0.034897810	1.543071 (0.0700868) 0.078520071	2.057429 (0.1245987) 0.139591374
	SELFEX	0.4702694 (0.007119929) 0.006701615	0.9405388 (0.02847971) 0.026806461	1.410808 (0.06407936) 0.060314521	1.881078 (0.1139189) 0.107225891
	PLFU	0.5363507 (0.01459526) 0.01159560	1.072701 (0.02919051) 0.04638238	1.609052 (0.04378577) 0.10436042	2.145403 (0.05838103) 0.18552970
	PLFJ	0.5216534 (0.01459242) 0.009515161	1.043307 (0.02918484) 0.038060660	1.56496 (0.04377726) 0.085636431	2.086613 (0.05836968) 0.152242495
	PLFEX	0.4775608 (0.01458287) 0.006759375	0.9551217 (0.02916573) 0.027037507	1.432682 (0.0437486) 0.060834343	1.910243 (0.05833147) 0.108149982

75	MLE	0.5064477 0.003655993	1.012895 0.014623960	1.519343 0.032903932	2.025791 0.058495899
	SELFU	0.5132003 (0.00351191) 0.003997911	1.026401 (0.01404767) 0.015991655	1.539601 (0.03160727) 0.035981200	2.052801 (0.0561907) 0.063966556
	SELJ	0.5064477 (0.00346570) 0.003655993	1.012895 (0.01386284) 0.014623960	1.519343 (0.03119138) 0.032903932	2.025791 (0.05545134) 0.058495899
	SELFEX	0.4861898 (0.003327081) 0.003238089	0.9723796 (0.01330832) 0.012952357	1.458569 (0.02994373) 0.029142788	1.944759 (0.05323329) 0.051809418
	PLFU	0.5165656 (0.00673056) 0.004210489	1.033131 (0.01346114) 0.016841948	1.549697 (0.0201917) 0.037894408	2.066262 (0.02692227) 0.067367793
	PLFJ	0.5098128 (0.00673027) 0.003812596	1.019626 (0.01346055) 0.015250396	1.529438 (0.02019083) 0.034313347	2.039251 (0.0269211) 0.061001525
	PLFEX	0.4895545 (0.006729351) 0.003249865	0.9791089 (0.0134587) 0.012999459	1.468663 (0.02018805) 0.029248768	1.958218 (0.0269174) 0.051997846

Table3: average estimates, posterior risk (within parenthesis) and corresponding MSEs (green color) for n= 100

n	Method	β = 0.5	β = 1.0	β = 1.5	β = 2.0
100	MLE	0.5049245 0.002680352	1.009849 0.010721407	1.514774 0.024123181	2.019698 0.042885627
	SELFU	0.5099738 (0.00260078) 0.002868258	1.019948 (0.01040314) 0.011473041	1.529921 (0.02340706) 0.025814308	2.039895 (0.04161255) 0.045892118
	SELFJ	0.5049245 (0.00257503) 0.002680352	1.009849 (0.01030014) 0.010721407	1.514774 (0.02317531) 0.024123181	2.019698 (0.04120054) 0.042885627
	SELFEX	0.4897768 0.002497783 0.002447768	0.9795536 (0.009991132) 0.009791073	1.46933 (0.02248005) 0.022029903	1.959107 (0.03996453) 0.039164286
	PLFU	0.5124922 (0.00503680) 0.002984406	1.024984 0.01007362) 0.011937613	1.537477 (0.01511043) 0.026859665	2.049969 (0.02014724) 0.047750500
	PLFJ	0.5074429 (0.00503668) 0.002766707	1.014886 (0.01007337) 0.011066834	1.522329 (0.01511006) 0.024900377	2.029772 (0.02014674) 0.044267337
	PLFEX	0.492295 (0.005036299) 0.002454301	0.9845899 (0.0100726) 0.009817201	1.476885 (0.0151089) 0.022088708	1.96918 (0.02014519) 0.039268814

4.3 Discussion of Result

As expected, it was observed that the performance of both the maximum likelihood estimates (MLEs) and the Bayes estimates become better as the sample sizes increases. Also, the MLEs and Bayes estimates becomes closer as the sample size increases.

The estimates were better at smaller value of β = 0.5 than at β = 1.0, 1.5 and 2.0 when compare in terms of their MSEs as well as in terms of the posterior risk. Hence the estimate is better at small value of β = 0.5.

The Extended Jeffrey's prior tend to perform better than the uniform and Jeffrey's priors when compared in terms of their MSEs under both loss functions used.

The uniform prior under the SELF was observed to have better estimate than the uniform prior under the PLF at all sample sizes. The Extended Jeffrey's prior under the SELF was observed to have performed better than the estimate of Extended Jeffrey's prior under the PLF. Also the Jeffrey's prior under the SELF was observed to have performed better than the estimate of Jeffrey's prior under the PLF.

But when the estimates of the Extended Jeffrey's prior under the SELF, was compare with the estimates of the Jeffrey's prior under the SELF and uniform prior under the SELF it was observed that the Extended Jeffrey's prior estimates the shape parameter with the minimum MSE and posterior risk

The MLEs performed better than the Bayes estimate under the SELFU, PLFU, and PLFJ, but performed equally with the SELF under the Jeffrey's prior. It was also observed to perform lesser than the extended Jeffrey's prior under both loss functions used.

It can also be observed that among all the Bayes estimates the SELF under the Extended Jeffrey's prior performed better than the other estimates, since SELF under the Extended Jeffrey's prior have the minimum posterior risk and mean square error.

Below is the graphical representation of the mean square error against the sample sizes at different value of β

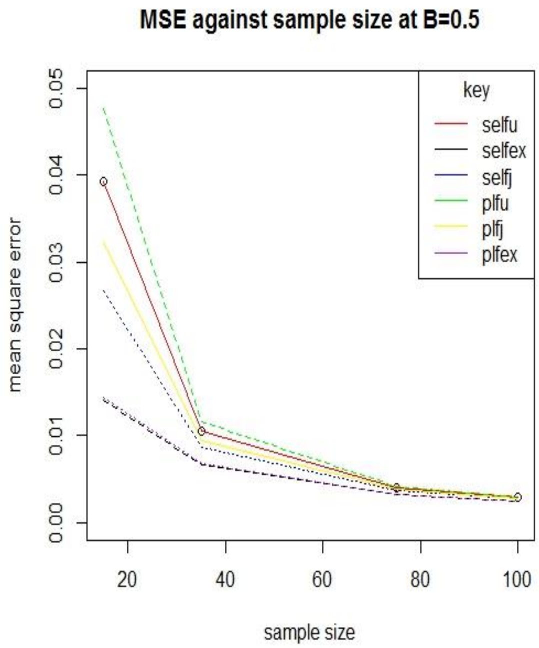


Figure 1

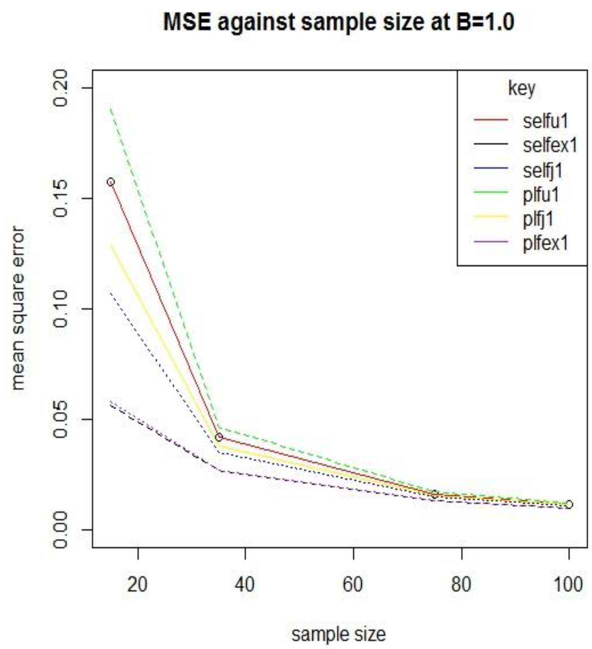


Figure 2

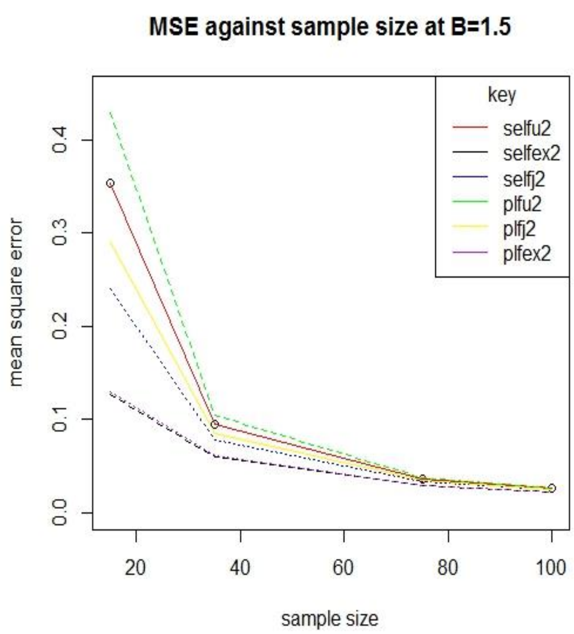


Figure 3

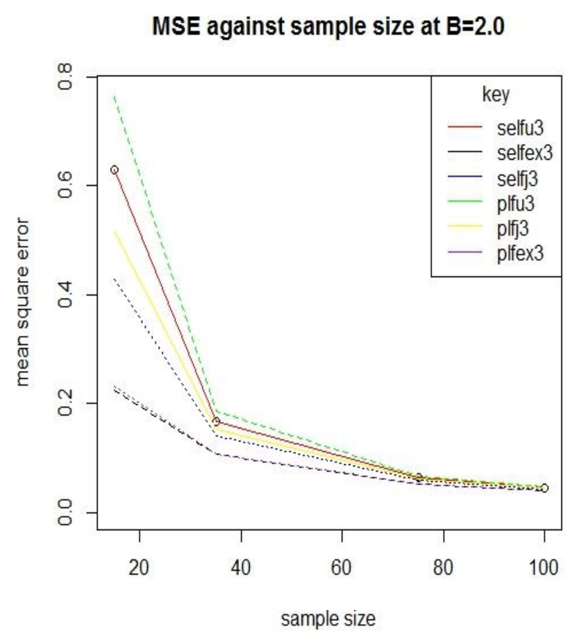
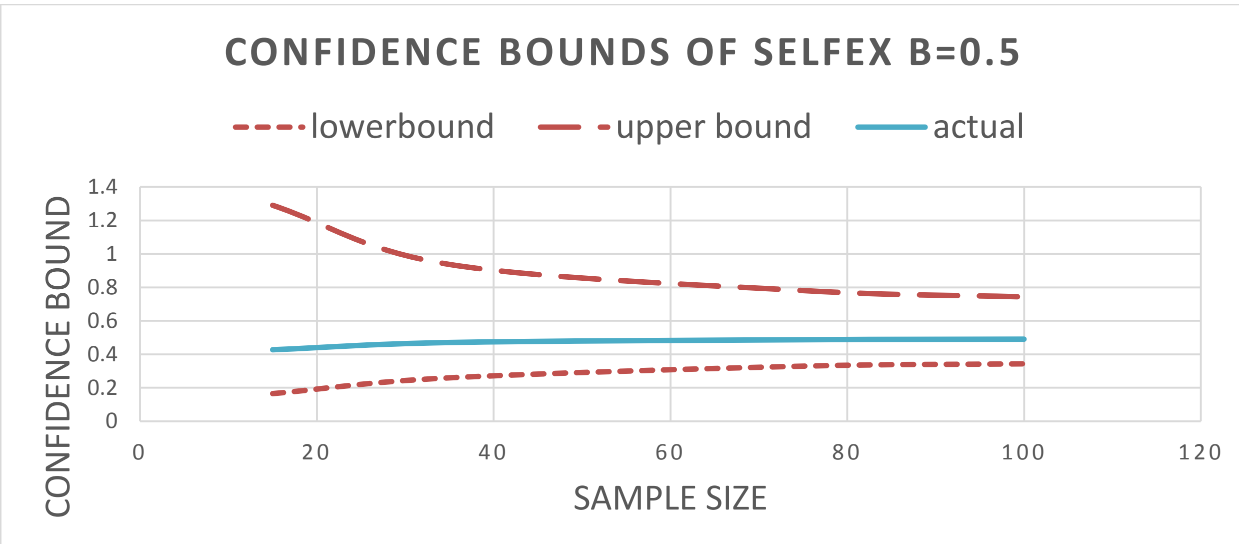


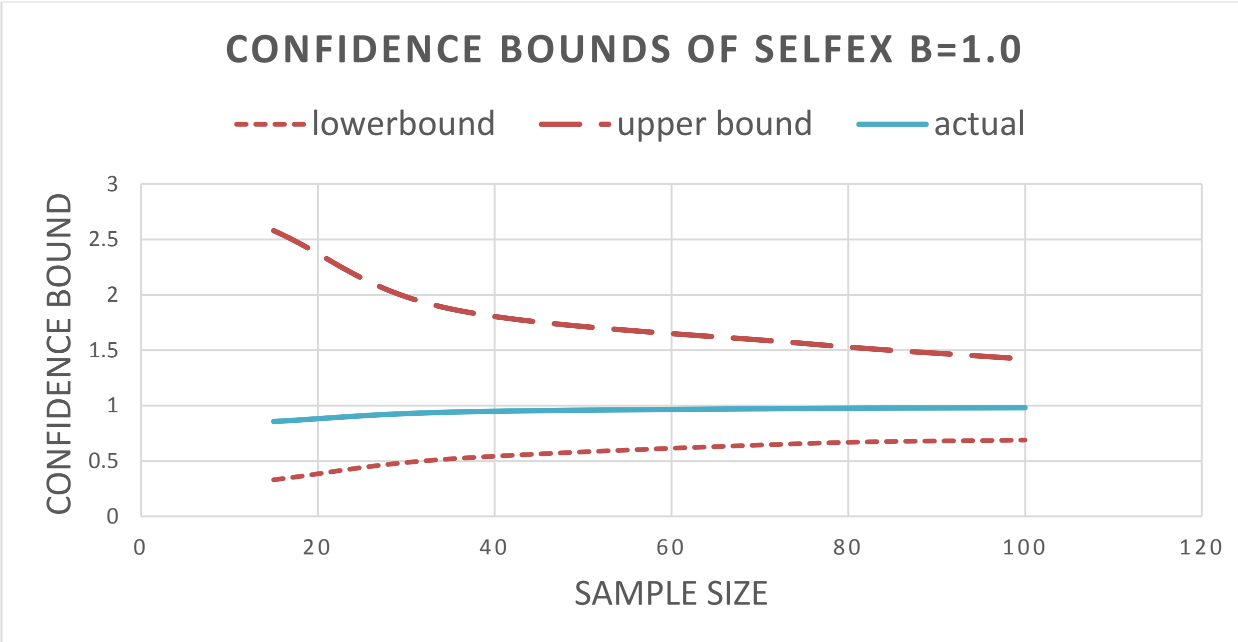
Figure 4

Graph of mean square error against sample size at different values of the shape parameter.

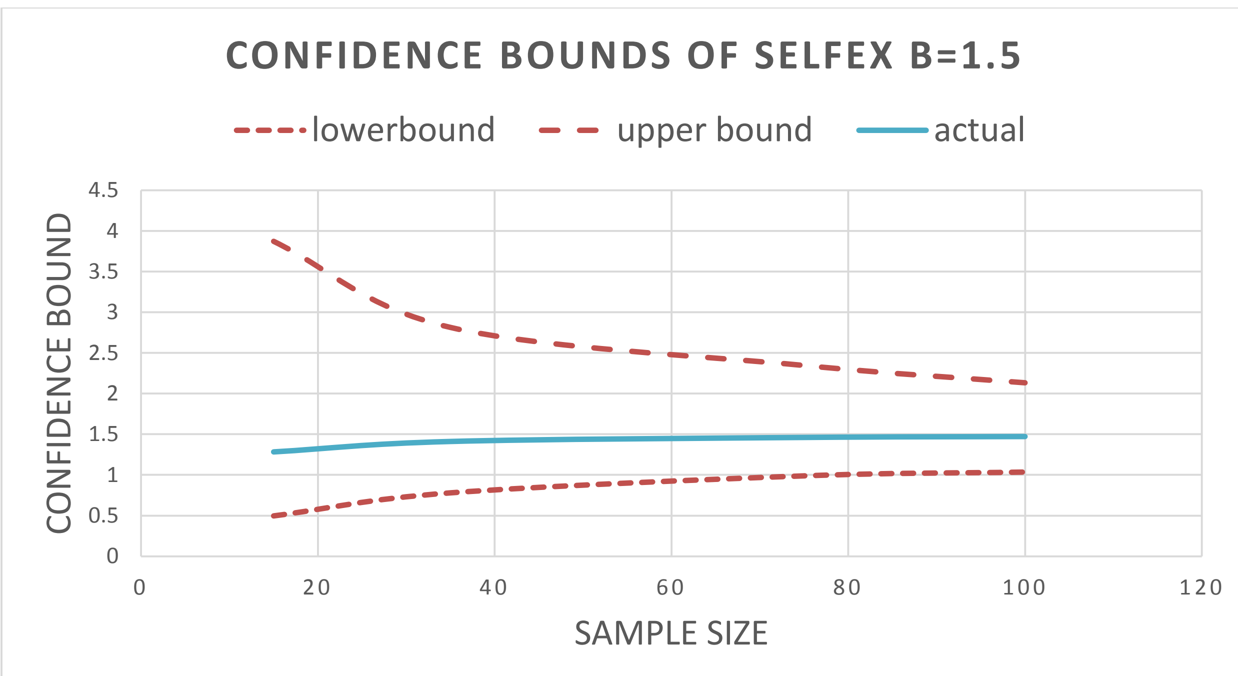
It can be seen graphically from figures 1, 2, 3 and 4 that the square error loss function under the extended Jeffrey's prior which is the black line have the best estimates for all the values of β used.

The monte carlos test show that the estimates obtain are from the Generalized inverse Exponential distribution since it falls within the 99% confidence bound and it represented graphically below;





CONFIDENCE BOUNDS OF SELFEX B=1.0
lowerbound
upper bound
actual
CONFIDENCE BOUND
3
2.5
2
1.5
1
0.5
0
0
20
40
60
80
100
120
SAMPLE SIZE



CONFIDENCE BOUNDS OF SELFEX B=1.5
lowerbound
upper bound
actual
CONFIDENCE BOUND
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
0
20
40
60
80
100
120
SAMPLE SIZE

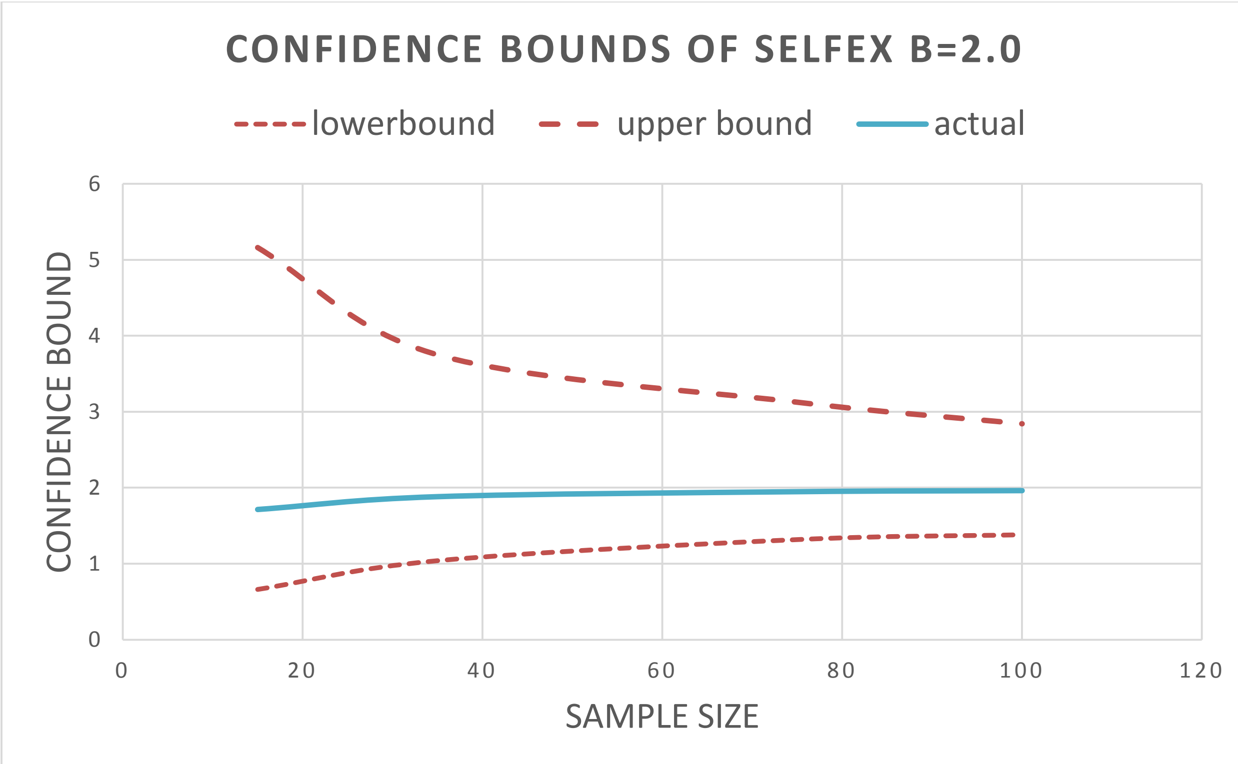


Fig 2: graph of confidence bound against sample sizes for different values of the shape parameter

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 Summary

In this research the posterior distribution of the shape parameter of the Generalized Inverse Exponential Distribution (GIED) were obtained with the assumption that the scale parameter is known. The estimates of the distribution were also obtain under three non-informative prior using the squared error loss function (SELF) and precautionary loss function (PLF) as well as that of the maximum likelihood. A Monte Carlo simulation was carried out to obtain and compare the performances of the Maximum Likelihood Estimate and the Bayes estimates using their Mean Squared Errors and also among the Bayes estimates using the posterior risk.

5.2 Conclusion

From the result of the analysis, the following conclusions were made;

The estimates become better as the sample size increases and are better at smaller value of the shape parameter (β).

The Bayes estimator under the SELF using the extended Jeffrey's prior have the best estimates when compared to the maximum likelihood and other Bayes estimators.

When all the priors were compared the extended Jeffrey's prior have better estimate than the uniform and Jeffrey's prior. Also among the Bayes estimators, the SELF under the extended Jeffrey's prior have the minimum posterior risk. Therefore the SELF under the extended Jeffrey's prior have the best estimator for estimating the shape parameter of the Generalized Inverse Exponential Distribution base on this research.

5.3 Recommendation

Based on the result obtained from this research, it is recommendation that when estimating the shape parameter of the generalized exponential distribution when the scale parameter is known, and you have little or no information about the prior distribution then, assume the Extended Jeffrey's prior using squared error loss function.

5.4 Contribution to Knowledge

i. In this research we were able to show that the Extended Jeffrey's prior is more suitable for estimating the shape parameter of the GIED than the uniform and Jeffrey's prior when we have little or no information about the prior distribution.

ii. We were also able to show that the most appropriates combination of loss function and prior for the estimation of the GIED is the Extended Jeffrey's prior using the squared error loss function when compared in terms of mean squared error and posterior risk.

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