NONCOMMUTATIVE ERGODIC THEOREMS ON VON NEUMANN ALGEBRAS.

BY

MADUGU ABDU (B.Sc.)

JUNE, 2008

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DEDICATION

This thesis is dedicated to my late father Alhaji Madugu, my mother Mallama Aminat and my beloved children (Yahaya, Mariyam, Fatima, Abdullahi and Khadizat).

Madugu Abdu

Signature Date

DECLARATION

I hereby declare that this thesis has been carried out by me and that to the best of my knowledge, it has never been submitted to Ahmadu Bello University or any other institution of higher learning for the award of any degree. All resources used outside the primary sources have been duly acknowledged in the form of reference

CERTIFICATION

This thesis titled Non-Commutative ERGODIC THEOREMS ON VON NEUMANN ALGEBRA by MADUGU Abdu meets the regulation governing the award of the degree of master of science (M.Sc) of Ahmadu Bello University, Zaria and is approved for its contribution to scientific knowledge and literary presentation.

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ABSTRACT

This work studies Ergodic Theorems on Non commutative L_p spaces over a semifinite von Neumann Algebras. Using the non commutative analogue of Marcinkiewicz interpolation theorem, we prove Stein's maximal ergodic inequality for symmetric contractions and using this result we obtain the corresponding individual (Point wise) ergodic theorem.

CHAPTER ONE

INTRODUCTION

1.1 The *-operation on a Linear Space of Operators

Let H be a (Complex) Hilbert space with an inner product <•, •> and let B(H) be the Banach space of all bounded linear operators on H with the operator norm $\|\bullet\|$ given by $\|a\| = \sup\{\|ax\| : a \in H, \|x\| = 1, a \in B(H)\}$. For any $a \in B(H)$, the relation <ax, y> = < x, a*y >, x, y $\in$H defines an operator a* ϵB(H), called the adjoint of a. The mapping B(H) $\ni$ a $\rightarrow$a * $\in$ B(H) is called the canonical involution on B(H). Thus B(H) becomes an involutive algebra or a * algebra in a canonical way[1].

1.2 The Strong Operator Topology on B (H)

It is a locally convex topology induced by the family of seminorms {P_x: xϵH}, defined by B(H) $\ni a \rightarrow || ax ||$, $x \in H$, $p_x(a) = \|ax\|$

The strong operator topology on B (H) has a base of neighbourhoods of an operator a_o consisting of sets of the form.

{aϵB(H): $|| (a - a_0) x_i || < \varepsilon$, i = 1, 2....., n} where $x_1, x_2, ..., x_n$ are in H and ε is positive. Thus a net $(a_i)_{i \in I}$ of operators in B(H) is strong operator convergent to a if and only if $\|(a_i - a)x\|$ converges to zero, for each x in H,

as i↑, i.e if and only if the net ($a_i x$) of vectors in H converges to ax, for each x in H, in the norm topology on H.

1.3 The Weak Operator Topology on B(H)

It is a locally convex topology induced by the family of seminorms $\{P_{x,,y}: x,y \epsilon H\}$defined by

B (H)э $a \rightarrow |< ax, y >|, x, y \in H$, $p_{x,y}(a) = |< ax, y >|$

In this topology the family of sets of the form

$$\{a \in B(H) : |< (a - a_0)x_i, y_i >| < \varepsilon, i = 1,2,3.....,n\},$$

where $\varepsilon>0$ and x_1, x_2,..., x_n, y_1, y_2,..., y_n are in H, forms a base of neighbourhoods of a_0 thus a net (a_i) of operators in B(H) is weak operator convergent to a iff $|$ < (a_i – a) x, y > $|$ converges to zero for all x y, ϵ H as i ↑ [1].

1.4 The utraweak (or σ – weak) operator topology on B (H)

It is a locally convex topology induced by the family of seminorms $\{P_{xn, yn}: x_n, y_n \epsilon H\}$, with ($x_n$), ($y_n$) ⊂ H, $\sum || x_n ||^2 < \infty$, $\sum || y_n ||^2 < \infty$, defined by B (H) э a → $\sum |< a\, x_n, y_n >|$. Here a net $(a_i)_{i\epsilon I}$ of operators in B(H) is σ – weak operator convergent to a iff $\sum |$ < (a_i-a) x_n, y_n, > $|$ converges to zero as i ↑, for all (x_n), (y_n)⊂H with $\sum ||y_n||^2 < \infty$, $\sum || x_n ||^2 < \infty$ [1].

1.5 The Ultrastrong (or σ-strong) Operator Topology on B(H).

It is locally convex topology induced by the family of seminorms, $\{P_{Xn}: x_n \in H\}$ with $(x_n) \subset H$, $\sum ||x_n||^2 < \infty$ defined by

$B(H) \ni a \to (\sum || ax_n ||^2)^{½}$. In this topology a net $(a_\propto)_{\propto \in I}$ of operators in B(H) is σ strongly operator convergent to a if and only if whenever $(x_i) \subset H$ with $\sum ||x_i||^2 < \infty$ implies $\sum || (a_\propto - a) x_i||^2$ converges to zero as $\propto \uparrow$[1].

1.6 Remark

Since the seminorms defining the σ strong topology are uniform limits of strongly continuous seminorms then the σ-strong operator topology is finer (stronger) than the strong operator topology. σ – strong operator topology is also stronger than σ – weak operator topology. Similarly, the σ weak operator topology is finer than the weak operator topology.

The topology given by the norm $||\bullet||$ is called the uniform (or norm) operator topology. The relation between these various topologies is given as

uniform < σ-strong < σ-weak

^ ^

Strong < weak

Where < means “finer than”.

All the topologies defined above have some interesting features with respect to algebraic operations.

1.7 Theorem

A linear functional ϕ on B(H) is weak operator continuous iff it is strong operator continuous.[1]

1.8 Definition

The space of all σ- weakly continuous linear functionals on B(H) is called the predual of B (H) and is denoted by $(B(H))^*$

1.9 Definition

By von Neumann algebra, we mean a sub-algebra of B(H) which is self adjoint, closed in the weak operator topology and contains identity I.

1.10 Definitions

An operator p in B(H) is said to be a projection if $P^* = P$ and $P^2 = P$. Note that if P is a projection then its orthogonal complement $(1-P) = P^{\perp}$ is also a projection. If P is in B(H) is a projection then pH is a closed

subspace of H called the projection subspace of P. Conversely to any closed substance K⊂H their corresponds a unique projection P in B(H) such that pH = K. For a in B(H) we denote by

n(a) - the projection onto the kernel of a; {x∈H: ax=0),

ℓ(a) - the projection onto the closure of the range of a : a$\overline{H}$ i.e l(a) is the smallest projection p with pa = a.

r(a) = 1-n(a). i.e r(a) is the smallest projection q with aq = a, it is projection onto the closure of range of a*. We call ℓ(a) the left support of a and r(a) the right support of a, if a is self-adjoint then we write S(a) = ℓ(a) = r(a) and call S(a) the support of a. An operator u in B(H) is said to be a partial isometry if there exists a closed subspace K⊂H such that

$\|ux\| = \|x\|, \forall\, x \in k, ux = 0, \forall\, x \in k^{\perp}$.

The closed subspace k is called the initial subspace of u while the subspace uk is called the final subspace of u, note r(u) = u*u = ℓ(u) = uu*.

The projection u*u is called the initial projection of u while the projection uu* is called the final projection of u.

1.11 Definition

Two projection p and q in a von Neumann algebra M are said to be equivalent if there exists a partial isometry u∈M such that u*u = p and uu*

= q. We denote this relation by p ~ q. If $p \in \wp_m$ is equivalent to a projection $e \in \wp_m$ with $e \leq q \in \wp_m$ we write $p \leq q$ or $q \geq p$.

If $p \leq q$ and $p \neq q$ then we write $p < q$. The relation ~ is an equivalence relation on $\wp_m$.

1.12 Definition

A projection $e \in M$ is called central projection if it belongs to the center Z of M. The set of all central projection $p \in Z \ni p\,a = a$, $a \in M$, is called the central support of a and is denoted by c(a). The central support of a projection p in M is the smallest central projection in M majorizing p and is denoted by c(p).

1.13 Definition

A projection p in a von Neumann algebra M is said to be abelian if the reduced algebra pMp is commutative. A projection $p \in M$ is said to be finite if there exists an equivalent sub projection q of p such that p = q (i.e if $\exists\ q \in \wp_m \ni q \leq p$, $p \sim q \Rightarrow p = q$). Otherwise p is said to be infinite. A projection $p \in M$ is said to be purely infinite if there is no non zero finite sub-projection of p in M. A projection p in M is said to be properly infinite if e is infinite for every central projection e in M with $ep \neq 0$, equivalently, p is

properly infinite if ep finite implies ep = 0 for every central projection e in M.

1.14 Definition:

Let $M \subset B(H)$ be a von Neumann algebra. Then:

i. M is said to be finite, properly infinite or purely infinite if the identity projection I is finite, properly finite or purely infinite.

ii. M is said to be semi-finite if any non zero central projection contains a non zero finite projection.

iii. M is said to be of type I if any non zero central projection contains a non zero abelian projection.

iv. M is said to be of type II if it is semi-infinite and it does not contain any non zero abelian projection.

v. M is said to be type III if it does not contain any non zero finite projection

vi. A finite type II von Neumann algebra is said to be type II_1.

vii. A properly infinite type II von Neumann algebra is said to be of type $II\infty$.

1.15 Definition

M_+ is the positive cone with the values in the extended positive reals $[0, \infty]$ on a semifinite von algebra M

1.16 Definition

A trace on a von Neumann algebra M is a function τ on the positive cone M_+ with values in the extended positive reals $[0, +\infty]$ satisfying the following conditions.

i. $\tau(x+y) = \tau(x) + \tau(y)$, $x, y \in M_+$;

ii. $\tau(\lambda x) = \lambda\tau(x)$, $x \in M_+$, $\lambda \geq 0$;

iii. $\tau(x^*x) = \tau(xx^*)$, $x \in M_+$

with usual convention $0 (+\infty) = 0$.

Semi- infinite if every nonzero $x \in M_+$ majorises some non zero $y \geq 0$ with $\tau(y) < +\infty$, finite if $\tau(I) < +\infty$, normal if $\tau(\sup x_i) = \sup\tau(x_i)$ for every bounded increasing net $\{x_i\}$ in M_+.

With this property τ is called a faithful normal semi-finite trace on M_+

1.17 Definition

A weight on a von Neumann algebra M is a function $\varphi: M_+ \to [0,\infty]$ such that the following conditions hold

i. $\phi(x+y) = \phi(x) + \phi(y)$, $x, y \in M_+$

ii. $\phi(\lambda x) = \lambda\phi(x)$ for $x \in M_+, \lambda \geq 0$.

Von Neumann algebra is called σ - finite if all collection of mutually orthogonal projections in M has at most a countable cardinality.

1.18 Definition:

Let M and N be von Neumann algebras. A mapping π: $M \to N$ is called a * homomorphism if it satisfies the following conditions.

R_1: $\pi(a+b) = \pi(a) + \pi(b)$

R_2 : $\pi(\lambda a) = \lambda\pi(a)$

R_3 : $\pi(ab) = \pi(a)\ \pi(b)$

R_4: $\pi(a^*) = \pi(a)^*$, for all $a, b \in M$, $\lambda \geq 0$. A representation of M is a * homomorphism π of M into B(H). We shall denote this representation by (H, π). The space H is called the representation space. The operators $\pi(a)$ are called the representatives M. The representation (H, π) is said to be non degenerate if for any non zero $x \in H$ there exists an element $a \in M$ such that $\pi(a)x \neq 0$. It is said to be faithful iff Ker $\pi = \{a \in M: \pi(a) = 0\} = \{0\}$. A representation (H, π) of M is said to be cyclic if there exists an element $x \in H$ such that $[\pi(M)x] = H$, where $[\pi(a)x]$ is the closed subspace of H generated by $\{\pi(a)x : a \in M\}$. Such an element $x \in H$ is called a cyclic a vector

of (H,π). We denote this type of a representation by a GNS triple (H,π,Ω). A triple (H,π,Ω) is called a cyclic representation if (H,π) is a representation of M and $\Omega \in H$ is cyclic to vector of (H, π).

CHAPTER TWO

ON NON COMMUTATIVE L_p SPACES

There are various types of non commutative L_p spaces on von Neumann algebra but in this chapter we will discuss the Segal's L_p spaces.

2.1 Definition

A gauge space Γ is a system (H, *M*, τ) composed of a Hilbert space H, a semi-finite von Neumann algebra *M* and a linear functional $\tau : M_+ \to [0,\infty]$ that satisfies these conditions

i. $\tau(x+y) = \tau(x) + \tau(y), x, y \in M_+$

ii. $\tau(\lambda x) = \lambda\tau(x), x \in M_+, \lambda \geq 0$

iii. $\tau(x^*x) = \tau(xx^*),\ x \in M_+$

iv. It is faithful if $\tau(x) = 0 \Rightarrow x = 0$

v. Normal if $\tau(x) = \sup\tau(x_i)$, for every bounded increasing net $\{x_i\}$ in M_+ such that x_n increases to x.

vi. Semi finite if every nonzero $x \in M_+$ majorises some non zero $y \geq 0$ with $\tau(y) < \infty$.

2.2 Definition .

An operator T on H is called measurable with respect to M if

i. T η *M* (T is affiliated to M)

ii. T has a strongly dense domain

iii. T is closed.

2.3 Remark

If S and T are operators on a Hilbert space, then S + T is defined as the operators with domain D = D (S) ∩ D(T) that is given by the equation (S + T) x = Sx + Tx; ST is defined as the operator with domain { x ∈ D(T) : Tx ∈ D(S) and (ST)x = S(Tx)}.

2.4 Definition

If S and T are measurable operators with the respect to von Neumann algebra on a Hilbert space H, the closures of S + T and ST are called strong Sum and strong product of S and T respectively and are denoted by S + T and S•T.

2.5 Remark

Note that the collection of all operators that are measurable with respect to a particular operator on a Hilbert space is an algebra with involution, relative to the strong sum and Product, [12].

2.6 Definition

A sequence $\{T_n\}$ of operators on a Hilbert space H each of which is measurable with respect to a von Neumann algebra M on H is said to converge nearly everywhere (n.e.) (relative to M) to a measurable operator T, if for every positive ε there exists a sequence $\{P_n\}$ of projections in M such that $P_n \uparrow I$ as $n \uparrow \infty$ and

$\|(T_n - T).P_n\| < \varepsilon$ $(n=1,2,...)$.

We have the following theorem from [12].

2.7 Theorem

Let $\{S_n\}$ and $\{T_n\}$ be sequences of measurable operators converging nearly everywhere to the measurable operators S and T respectively. Then the sequence $\{S_n + T_n\}$ converges nearly everywhere to S + T.

2.8 Theorem

Let $\{T_{(i)}\}$ be a sequence of measurable operators converging nearly everywhere to the measurable operator T with respect to a von-Neumann algebra with a countable decomposable centre. Then there exists a subsequence $\{n_i\}$ of the set of integers such that $\{T_{(ni)}^*\}$ converges n.e. to T^* .

2.9a Definition

A sequence $\{T_n\}$ of measurable operators converges metrically nearly everywhere (m.n.e.) to a measurable operator T, with respect to a gauge space (H, M, τ), if it converges n.e. to T on the carrier of τ.

2.9b. Definition

A measurable operator T on the gauge space is called integrable if it is the limit m.n.e. of a sequence $\{T_n\}$ of elementary operator that is Cauchy in L_1. The integral or trace of T, denoted $\tau(T)$ is defined as $\lim_n \tau(T)$. The collection of all integrable operators is denoted L_1 (H, M, τ) or simply L_1 when the identity of the gauge space is clear from the context.

2.10 Definition

The L_1 norm, designated $||T||_{1.}$ of an elop T, with respect to the gauge space (H, *M*, τ) is defined as l.u.b. $|\tau (ST)|$ where

$\{S \text{ in } M, ||S|| \leq 1\}$. The bound of an operator $S \in M$ will be denoted by $||S||_{\infty,}$ in situation where $||S||$ might be interpreted as the L_1 norm.

2.11 Corollary

The L_1 norm on the elementary operator on a gauge space has the following properties

a. $||T||_{1,} \geq 0$ and $||T||_1 = 0$ if and only if T is τ - equivalent to 0;

b. $||S + T||_1 \leq ||S||_1 + ||T||_1$

c. $||\alpha T||_1 = |\alpha| \, || \, T \, ||_1$, for any complex α

d. $||T||_1 = ||T^*||_1 = \tau(|T|)$, where $T = U \, |T|$ is the canonical polar decomposition of T

e. If V is bounded measurable then

$||VT||_1 \leq ||V||_\infty \, ||T||_{1,}$ and $||TV||_1 = ||V||_\infty \, ||T||_1$.

It is clear from the definition of $||T||_1$ that $V \in M$ where (H, *M*, τ) is the gauge space in question, then $||VT||_1 \leq ||V||_\infty \, ||T||_{1.}$ On the other hand, $||TV||_1 = \text{l.u.b} \, |\tau(STV)| \, \{S \in M, ||S||_\infty \leq 1\} \leq ||V||_\infty \, ||T||_1$.

2.12 Remark

It should be noted that the limit just indicated necessarily exists, for if S is any elop,

$|\tau(S)| \leq ||S||$, so $|\tau(T_n) - \tau(T_m) \leq ||T_n - T_m|| \rightarrow 0$ as $m, n \rightarrow \infty$.

Note that the integral of an integrable operator is unique and the integral is a linear functional on the linear set of all integrable operators.

2.13 Corollary

If S and T are bounded measurable and integrable operators respectively, then S. T and TS are integrable and $\tau(S.T) = \tau(TS)$. Moreover, T* and |T| are integrable, and $\tau(T^*)=\tau(T)$.

2.14 Definition

An operator T is said to be a square integrable with respect to a gauge space Γ if it is measurable and if $T = R + iS$ with R and S symmetric and R^2 and S^2 integrable. The collection of all square integrable operators is designated $L_2(\Gamma)$, or simply L_2 when the identity of Γ is clear.

2.15 Corollary

For any square integrable operators X and Y, X. Y is integrable

$|\tau(X^*.Y)| \leq \tau\,(X^*Y)\ \tau\,(Y^*Y)$.

2.16 Corollary

A measurable operator T is square integrable if and only if T*T is integrable

2.17 Definition

The L_2- norm of a measurable operator T is defined as $(\tau\,(T^*T))^{1/2}$, and designated $||T||_2$.

2.18 Corollary

The L_2 – norm on the operator square integrable over a gauge space has actually the following properties.

a. $||T||_2 \geq 0$ and $||T||_2 = 0$ if and only if T is τ equivalent to 0.

b. $||S + T||_2 \leq ||S||_2 + ||T||_2$

c. $||\alpha T||_2 = |\alpha|\ ||T||_2$

d. $||T||_2 = ||T^*||_2$

e. If S is bounded measurable then

$||S.T||_2 \leq ||S||_\infty ||T||_2$ and $||TS|| \leq ||S||_\infty ||T||_2$.

Now if T = R + iS, $T^*T = R^2 + S^2 + i(R.S - S.R)$ while

$TT^* = R^2 + S^2 - i(R.S - S.R)$, so that by the corollary, $\tau(T^*T) = \tau(R^2 + S^2) = \tau(TT^*)$ and in particular $||T||_2 = ||T^*||_2$. Now if TV is square integrable as $||TV||_2^2 = \tau(V^*.\ T^*TV)$ which is finite as T*T is integrable and corollary 2.13 shows that

$\tau(V^*.T^*TV) \leq ||T^*T||_2 ||V||_\infty ||T||_2$.

Now V.T = (T*V*)*, and V* and T* are respectively in L_∞ and L_2, so $V.T \in L_2$ and $||V.T|| = ||(T^*V^*)^*||_2 = ||T^*V^*||_2 \leq ||V^*||_\infty ||T^*||_2 = ||V||_\infty ||T||_2$.

2.19 Corollary

For any hermitian measurable operator T with spectral resolution

$T = \int \lambda dE$, (where the integral is from +0 to $+\infty$).

If $|S|^p, |T|^p \in L_1$, then $|ST|^p \in L_p$

For any $1 \leq p \leq \infty, 1 \leq q \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$,

The norm of any measurable operator is given by

$||T||_p = (\tau|T|^p)^{1/p}$, $L_p(M) = \{x \in M: (\tau|T|^p) < \infty$

now for any two measurable operators S and T we have the-Minkowski inequality [11].

$||S + T||_p \leq ||S||_p + ||T||_p$

Let

$\tau = ||S + T||^p_{Lp} = ||(S + T)(S+T)^{p-1}||_p = ||S(S + T)^{p-1}||_p + ||T(S + T)^{p-1}||_p$

$\leq ||S\,(S + T)^{p-1}\,|| + ||T||_p\,||(S + T)^{p-1}||_p$

$\leq (\tau|S|^p)^{1/p}\,(\tau\,|S + T|^{(p-1)q}) + (\tau\,|T|^p)^{1/p}\,(\tau|\,S + T|^{(p-1)q})^{1/q}$

$\leq (\tau|S|^p)^{1/p}\,(\tau\,|S + T|^{(p-1)q}) + (\tau\,|T|^p)^{1/p}\,(\tau|\,S + T|^{(p-1)q})^{1/q}$

$\leq [\tau\,|S|^p)^{1/p} + \tau(|T|^p)^{1/p}]\,(|S + T|)^p)^{1/q}$

$= (\tau\,|\,S + T|^p)^{1-1/q} \leq (\tau\,|S|^p)^{1/p} + (\tau|T|^p)^{1/p}$

$= (\tau\,|S + T|^p)^{1/q} \leq (\tau\,|S|^p)^{1/p} + (\tau\,|T|^p)^{1/p}$

$||S||_p = (\tau\,|S|^p)^{1/p}$ or generally

$(\tau\,|S + T|^p)^{1/p} = ||\,S + T||_p$

2.20 Remark

i. We let $L_\infty(M, \tau) \cong M$ equipped with operator norm.

ii. We $L_0(M) = L_0(M, \tau)$ denote the space of all closed densely defined operator on H with respect to (M,τ). $L_0(M) = \{x \in M : \tau|x| < \infty\}$. For $0 < p < \infty$, we have $L_o(M) = \left\{x \in L_0(M) : \tau|x|^p < \infty\right\}$

CHAPTER THREE

MAXIMAL ERGODIC INEQUALITIES.

In this chapter we intend to study the Maximal ergodic inequalities in non commutative case. The spaces required for this research work is more general than that of Segal in chapter two. Here we have to deal with sequences, as such we need to use the non commutative spaces $L_p(M;\ell_\infty)$. This space was introduced by Pisier [9], for von Neumann algebras . Additionally, we also have to use or study Marcinkiewicz type interpolation theorem for $L_p(M;\ell_\infty)$, because it is the key to the proof of non commutative ergodic inequalities.

3.1 Definition

Given $1 \leq p \leq \infty$, $L_p(M;\ell_\infty)$ is defined as the space of all sequences $(x_n)_{n\geq 0}$ in $L_p(M)$ which admit a factorization of the form:

a, b $\in L_{2p}(M) and (y_n) \subset L_\infty(M)$ such that

$(x_n) = ay_nb$, for all $n \geq 0$.

We then define

$\|x\|_{L_p(M;\ell_\infty)} = \inf\left\{\|a\|_{2p}\sup_{n\geq 0}\|y_n\|_\infty\|b\|_{2p}\right\}$, where the infimum runs over all factorization as above. It is known that such $L_p(M;\ell_\infty)$ equipped with the norm is a Banach space.

3.2 Remark

ℓ_p = spaces of all sequences (x_n) s. t $\sum_{n=1}^{\infty}|x_n|^p < \infty$

ℓ_∞ = space of all bounded sequences with sup norm

c = space of all convergent sequences

c_o = space of all sequences that converge to 0.

To gain a very first understanding on $L_p(M;\ell_\infty)$,

3.3 Proposition [6]

Let us consider a positive sequence $x=(x_n)$. Then one can show that

$x \in L_p(M;\ell_\infty)$ iff there are $a\in L_p^+(M)$ and $y_n \in L_\infty^+(M)$ such that

$x_n = a^{\frac{1}{2}}y_n a^{\frac{1}{2}}$ for all $n\geq 0$.

Proof

We can clearly assume that y_n are positive contractions. Thus if x has a factorization as above, then $x_n \leq a$ for all n. Conversely, if $x_n \leq a$ for some $a \in L_p^+(M)$, then $x_n^{\frac{1}{2}} = u_n a^{\frac{1}{2}}$ *for* a contraction $u_n \in M$, and so $x_n = a^{\frac{1}{2}} u_n^* u_n a^{\frac{1}{2}}$. Thus $x \in L_p(M; \ell_\infty)$.

In summary, a positive sequence x belongs to $L_p(M; \ell_\infty)$ if and only if there is a $a \in L_p^+(M)$ such that $x_n \leq a$ for all n, and moreover,

$$\|x\|_{L_p(M;\ell_\infty)} = \inf \left\{ \|a\|_p : a \in L_p^+(M) s.t. x_n \leq a, \forall n \geq 0 \right\}$$

It is seen that $L_p(M; \ell_\infty)$ is a dual space for every $p > 1$. Its predual is $L_{p'}(M; \ell_1)$ (p' being the index conjugate to p) [7].

3.4 Definition

Given $1 \leq p \leq \infty$, a sequence (x_n) belongs to $L_p(M; \ell_1)$ if there are u_{kn}, $v_{kn} \in L_{2p}(M)$ such that

$x_n = \sum_{k \geq 0} u_{kn}^* v_{kn}$ for all n and

$$\sum_{k \geq 0} u_{kn}^* v_{kn} \in L_p(M), \qquad \sum_{k \geq 0} u_{kn}^* v_{kn} \in L_p(M).$$

Here all series are required to be convergent in $L_p(M)$ (relative to the w* topology in the case of $p = \infty$). $L_{p'}(M; \ell_1)$ is also a Banach space when equipped with norm.

$$\|x\|_{Lp(M:\ell\infty)} = \inf\left\{\left\|\sum_{k,n\geq 0} u_{kn}^* u_{kn}\right\|_p^{\frac{1}{2}} \left\|\sum_{k,n\geq 0} v_{kn}^* v_{kn}\right\|_p^{\frac{1}{2}}\right\},$$

where the infimum is taken over all (u_{kn}) and (v_{kn}) as above. It is clear that finite sequences are dense in $L_p(M;\ell_1)$ if $p < \infty$. The duality between $L_p(M;\ell_{\infty 1})$ and $L_{p'}(M;\ell_1)$ is given by $\langle x, y\rangle = \sum_{n\geq 0} \tau(x_n y_n)$. In fact, a positive sequence (x_n) belongs to $L_p(M;\ell_1)$ if and only if

$\sum x_n \in L_p(M)$. If this is the case,

$$\|x\|_{2p(M;\ell 1)} = \left\|\sum x_n\right\|_p$$

We now consider some preliminary properties of these spaces described above in the following proposition. We denote the subspace of $L_p(M;\ell_\infty)$ by $L_p(M;\ell_\infty^{n+1})$ which consists of all finite sequences (x_0, x_1, ... $x_{n,,}$ 0, …). Similarly the subspace $L_p(M;\ell_1^{n+1})$ for $L_p(M;\ell_1)$

3.5 Proposition [6]

i. Each element in the unit ball of $L_p(M;\ell_\infty)$ (resp. $L_p(M;\ell_1)$ is a sum of sixteen (resp. eight) positive elements in the same ball.

ii. A sequence (x_n) in $L_p(M)$ belongs to $L_p(M;\ell_\infty)$ if and only if $\sup_{n\geq 0}\left\|\sup_{0\leq k\leq n}{}^{+} x_k\right\|_p < \infty$. If this is the cases, then $\left\|\sup_n{}^{+} x_n\right\|_p = \sup_{n\geq 0}\left\|\sup_{0\leq k\leq n}{}^{+} x_k\right\|_p$

iii. Let (x_n) be a positive sequence in $L_p(M;\ell_\infty)$. Then

$$\left\|\sup_n{}^{+} x_n\right\|_p = \sup\left\{\sum_n \tau(x_n y_n) : y_n \in L^+_{p'}(M)\ \&\ \left\|\sum y_n\right\|_{p'} \le 1\right\}.$$

iv. We have the following Cauchy-Schwarz type inequality: For any sequences (x_n) and (y_n) in $L_{2p}(M)$

$$\left\|\sup_n{}^{+} x_n^* y_n\right\|_p \le \left\|\sup_n{}^{+} x_n^* x_n\right\|_p^{\frac{1}{2}} \left\|\sup_n{}^{+} y_n^* y_n\right\|_p^{\frac{1}{2}}.$$

Proof. (i)

Note that both $L_p(M;\ell_\infty)$ and $L_p(M;\ \ell_\infty)$ are closed with respect to involution. Thus we need only to consider self adjoint elements. Let x be a self adjoint elements (i.e $x_n = x_n^*$ for all n) in the unit ball of $L_p(M;\ \ell_\infty)$. Write a factorization of x:

$x_n = a^* y_n b$ with $\|a\|_{2p} \le 1, \|b\|_{2p} \le 1$ and $\sup_n \|y_n\|_\infty \le 1.$

Then by standard polarization argument,

$$x_n = \frac{1}{4}\sum_{k=0}^{3} i^{-k}\left(a + i^k b\right)^* y_n \left(a + i^k b\right) = \frac{1}{4}\sum_{k=0}^{3}\left(a + i^k b\right)^* 2n\left(a + i^k b\right)$$

$$= \sum_{k=0}^{3}\left(\frac{a + i^k b}{2}\right)^* z_n^+\left(\frac{a + i^k b}{2}\right) - \sum_{k=0}^{3}\left(\frac{a + i^k b}{2}\right) z_n^-\left(\frac{a + i^k b}{2}\right),$$

where

$$z_n = \frac{i^{-k} y_n + \left(i^{-k} y_n\right)^*}{2}$$

Hence, the assertation concerning $L_p(M; \ell_\infty)$ follows.

ii. It is trivial that

$\sup_{n\geq 0} \left\| \sup_{0\leq k\leq n}{}^{+} x_k \right\|_p \leq \left\| \sup_n{}^{+} x_n \right\|_p$. For the converse, we introduce the subspace $L_p(M; \ell_\infty^0)$ of $L_p(M; \ell_1)$, which consists of all finite sequences x admitting a factorization as in the definition above of $L_p(M; \ell_1)$, but with finite families u_{kn} and v_{kn} only. Note that for p finite, $L_p(M; \ell^0{}_1)$ is dense in $L_p(M; \ell_1)$. Now let x be a sequence such that $\sup_{n\geq 0} \left\| \sup_{0\leq k\leq n}{}^{+} x_k \right\|_p = 1.$ (i.e. $p' < \infty$), by the duality l can be identified with an element of $L_p(M; \ell_\infty)$. This element must be x, and so we are done in this case.

iii. For p > 1. This is already proved [6]

We use duality. Let (u_{kn}) and (v_{kn}) be two finite families in $L_{2p}(M)$. Then by the Cauchy – Schwarz inequality.

$$\left| \sum_{k,n} \tau\left(x_n^* y_n u_{kn}^* v_{kn}\right) \right| \leq \left(\tau \sum x_n u_{kn}^* u_{kn} x_n^*\right)^{\frac{1}{2}} \left(\tau \sum_{k,n} y_n u_{kn}^* u_{kn}{}^* \right)^{\frac{1}{2}}$$

$$\left(\tau \sum_{k,n} y_n u_{kn}^* u_{kn} y_n^* \right)^{\frac{1}{2}} \leq \left\| \sup_n{}^{+} x_n^* x_n \right\|_p^{\frac{1}{2}} \left\| \sup_n{}^{+} y_n^* y_n \right\|_p^{\frac{1}{2}} \left\| \sum_{k,n} v_{kn}^* v_{kn} \right\|_{p;}^{\frac{1}{2}} \left\| \sum_{k,n} u_{kn}^* u_{kn} \right\|_{p'}^{\frac{1}{2}}.$$

Whence the desired inequality.

3.6 Remark

i. From the proof of part (ii) above one sees that the infimum defining the norm $\left\|\sup_n x_n\right\|_p$ is attained for any $x \in L_p(M;\ell_\infty)(1 \le p \le \infty)$. The same proof shows that L_1 $(M:\ell_\infty)$ is identified as an isometric subspace of the dual of L_∞ $(M:\ell_1^\infty)$.

ii. We have a statement similar to proposition 3.5 (ii), for L_p $(M:\ell_1)$. On the other hand, let $L_p(M:c_0)$ be the closure of finite sequences in L_p $(M:\ell_\infty)$ for $1 \le p < \infty$. Then one can see that the dual space of $L_p(M:c_0)$ is equal to $L_{p'}(M:\ell_1)$ isometrically.

iii. Neither L_p $(M:\ell_\infty)$ nor L_p $(M:\ell_1)$ is stable under the operation $(x_n)_n \to (|x_n|)_n$. Thus.

$\left\|\sup^+ x_n\right\|_p \neq \left\|\sup^+ |x_n|\right\|_p$ in general.

3.7 Remark

If $N \subset M$ is a von Neumann sub algebra such that the trace τ restricted to N is semi finite on N, then we have a natural isometric inclusion $L_p(N) \subset L_p(M)$. This extends to isometric inclusion.

$L_p(N:\ell_\infty) \subset L_p(M:\ell_\infty)$ and $L_p(N:\ell_1) \subset L_p(M:\ell_1)$.

Indeed, by the definition of $L_p(M:\ell_\infty)$ and $L_p(M:\ell_1)$, the inclusion above are contractive. On the other hand, they are both isometric.

3.8 Proposition [6]

Let $1 \le p_0 < p \le \infty$ and $0 < \theta < 1$. Then we have isometrically.

$$L_p(M;\ell_1) = \left(L_{p_0}(M:\ell_1), L_{p_1}(M:\ell_1)\right)\theta, \left(L_p(M;\ell_\infty)\right) = \left(L_{p_0}(M:\ell_\infty), L_{p_1}(M;\ell_\infty)\right)\theta$$

Where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Before the interpolation theorem, we first introduce some definitions.

3.9 Definition.

For every integer $n \ge 0$, assume given a map:

$S_n : L_p^{+}(M) \to L_0^{+}(M)$. Set $S = (S_n)_{n\ge 0}$. Thus S is a map which sends positive operators to sequences of positive operators. We say that S is of weak type

(P,P) (P < ∞) if there is a positive constant C such that for any $\left(x \in L_p^+(M)\right)$ and any λ > 0 there is a projection e∈M such that $\tau\left(e^{\perp}\right) \leq \left[C\frac{\|x\|_p}{\lambda}\right]^p$ and e (S_n(x)e ≤ λ.

Similarly, we say that S is of type (p, p) (this time p may be equal to ∞) if there is a positive constant C such that any $x \in L_p^+(M)$ there is a $\in L_p^+(M)$satisfying .

$$\|a\| \leq C\|x\|_p \, and (S_n(x)) \leq a, \forall n \geq 0.$$

$$\|S(x)\|_{Lp(M;\ell\infty)} \leq \|x\|_p \, \forall x \in L_p^+(M).$$

3.10 Theorem

Let $1 \leq P_0 < P \leq \infty$.

Let S = $(S_n)_{n\geq 0}$ be a sequence of maps from $L_{p_0}^+(M) + L_{p_1}^+(M)$ into $L_0^+(M)$. Assume that S is sub additive in the sense that S_n (x + y) ≤ S_n (x) + S_n (y) for all n ∈ N. If S is of weak type (P_o, P_o) with constant C_o and of type (P_1 P_1) will constant C_1, then for any $P_0 < P < P_1$, S is of type (P, P) with constant C_p satisfying.

$C_p \le CC_0^{1-\theta} C_1^{\theta} \left(\frac{1}{p_0} - \frac{1}{p} \right)^{-2}$ where θ is determined by $\frac{(1-\theta)}{p_0} + \frac{\theta}{p_1}$ and C is a universal constant. This is the non commutative analogue of the classical Marcinkiewicz interpolation theorem.

We now start the main central theme of this chapter i.e. Maximal ergodic inequalities. Before then, we may wish to add additional notation as follows.

Let M be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace τ. Let $L_p(M)$ be the associated non commutative L_p – space. Let T: M → M be a linear map which satisfy some of the properties below.

a. T is a contraction on M: $||Tx||_\infty \le ||x||_\infty \ \forall x \in M$

b. T is positive : T x ≥ 0 if x ≥ 0.

c. $\tau 0 T \le \tau : \tau(T(x)) \le \tau(x)$ for all $x \in L_1(M) \cap M_+$

d. T is symmetric relative to τ: τ (T(y)*x) = τ(y* T(x)) for all x, y $\in L_2(M) \cap M$

if T satisfies properties (a) – (c), then T naturally extends to a contraction on L_p (M) for all $1 \le p < \infty$.

The extension is still denoted by T. If T additionally has property (d) then its extension is self adjoint on $L_2(M)$. Let us consider the ergodic averages of T.

$$M_n(T) = \frac{1}{n+1}\sum_{k=0}^{n} T^k \; x$$

3.11 Theorem [6]

Let $1 < p < \infty$ and T be a linear map satisfying properties (a) – (c) above. Then,

i. for any $x \in L_p(M)$ such that

$\forall n \in N, M_n(T)(x) \le a$ and $\|a\|_p \le C_p \|x\|_p$

Where C_p is a positive constant depending only on p. Moreover, $C_p \le C_{p^2}(p-1)^{-2}$ and $(p-1)^{-2}$ is the optimal order of C_p as $p \to 1$.

ii. If additionally, T satisfies (d) above, then for any $x \in L_p(M)$ with $x \ge 0$ there is a $\in L_p, (M)$ such that $T^n(x) \le a$ and $\|a\|_p \le C'_p \|x\|_p \; \forall n \in N.$, [13].

The following theorem is the main maximal ergodic inequality in non commutative L_p spaces restricted to the positive cone $L_p^+(M)$.

3.12 Theorem [12]

Let T be a linear map with properties (a) – (c) above. Let

$$M_n \equiv M_n(T) = \frac{1}{n+1}\sum_{k=0}^{n} T^k x$$

Then for every $1 < p \leq \infty$

We have.

$$\left\| \sup_n{}^+ M_n(x) \right\|_p \leq C_p \|x\|_p \; \forall x \in L_p(M) \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots * $$

Moreover, $C_p \leq C_{p2}\ (p - 1)^{-2}$ and this is the optimal order of C_p as $P \rightarrow 1$.

Proof

By decomposing an operator into a linear combination of four positive cones, we can assume $x \in L_p^+(M)$. Now consider each M_n as a map defined on $L_1^+(M) + L_\infty^+(M)$. Then M_n is positive and additive (and so additive too). Let $(M_n)_{n\geq 0}$. M is of weak type (1,1). On the other hand, M is trivially of (∞, ∞) with constant 1.

Therefore, by theorem 3.10, M is of type (p, p) for every $1 < p < \infty$ with constant C_p verifying

$C_p \le C\left(1-\frac{1}{p}\right)^{-2}$. Thus it proved equation (*). The optimality of this estimate follows from the optimal order of the best constant in the non commutative Doob maximal inequality [6].

3.13 Definition

Given $2 \le p \le \infty$ and I be an index set, we defined the space $L_p\left(M;\ell_\infty^c(I)\right)$ as the family of all $(x_i)_{\in I} \subset L_p(M)$ for which there are $a \in L_p(M)$ and $(y_i)_{i\in I} \subset L_\infty(M)$ such that

$$x_i = y_i a \text{ and } \sup_{i\in I}\|y_i\|_\infty < \infty$$

$\|x\|_{Lp(M:\ell_\infty^c(I)}$ is then defined to be the infimum $\left\{\sup_{i\in I}\|y\|_\infty \|a\|_p\right\}$ over all factorizations of (x_i) as above. It is easy to check that $\|x\|_{Lp(M:\ell_\infty^c(I)}$ is a norm, which makes $L_p\left(M;\ell_\infty^c(I)\right)$a Banach space, note that $(x_i) \in L_p\left(M;\ell_\infty^c(I)\right)$ if and only if $\left(x_i^* x_i\right) \in L_{p/2}(M;\ell_\infty(I))$. If

I = N, $L_p\left(M;\ell_\infty^c(I)\right)$ is simply denoted by $L_p\left(M;\ell_\infty^c\right)$.

The improvement mentioned above will be a consequence of the following of theorem 3.12.

3.14 Corollary

Let T be as in theorem 3.12 above and $2 \le p \le \infty$. Then

$$\left\|\left(M_n(x)_{n\ge 0}\right)\right\|_{Lp\left(m;\ell_\infty^c\right)} \le \sqrt{C_{p/2}}\|x\|_p, \forall x \in L_p(M)$$

Proof.

Let $x \in L_p(M)$. By decomposing x into its real and imaginary parts, we can assume x self adjoint. Since T is positive, so is M_n for every n. Thus by the Kadison inequality [6], we have

$$(M_n(x))^2 \le M_n\left(x^2\right).$$

Thus applying theorem 3.12 to $x^2 \in L_{p/2}(M)$ we get $b \in L^+_{p/2}(M)$ such that $\|b\|_{p/2} \le C_{p/2}\left\|x^2\right\|_{p/2}$ and $M_n\left(x^2\right) \le b, \forall n \ge 0.$ Hence $(M_n(x)^2) \le b$. It then follows that for each n there is a contraction $y \in M$ such that $M_n(x) = y_n b^{\frac{1}{2}}$. This gives the desired factorization of $(M_n(x))_{n\ge 0}$ as an element in $L_p\left(M;\ell_\infty^c\right)$

CHAPTER FOUR

Individual (Point Wise) Ergodic Theorems

Introduction

In this chapter, we apply the maximal inequalities proved in chapter three to study the point wise ergodic theorem. And to do this we need some notation and also need an appropriate analogue for the non commutative setting of the usual almost every convergence. This is the almost uniform convergence introduced by Lance [8]

4.1 Theorem

Let $1 < p < \infty$ and T satisfy properties (a) – (c) of chapter three above. Let F be the projection on to the fixed point subspace of T considered as a map on $L_p(M)$. Then $(M_n(x) - F(x))_n \in L_p(M; c_o)$ for each $x \in L_p(M)$. If additionally T has properly (d) also in chapter three then $(T^n(x) - F(x))_n \in L_p(M; c_o)$.

The almost uniform convergence which was introduced by Lance as mentioned above is as follow:

4.2 Definition

Let M be a von Neumann algebra equipped with a semi finite normal faithful trace τ. Let $x_n, x \in L_0(M)$.

i. x_n is said to converge bilaterally almost uniformly (b.a.u) to x if for every $\varepsilon > 0$ there is a projection $e \in M$ such that $\tau(e^{\perp}) < \varepsilon$ and $\lim_{n\to\infty} \|e(x_n - x)e\| = 0$, [2].

ii. (x_n) is said to converge almost uniformly (a.u) to x if for every $\varepsilon > 0$ there is a projection $e \in M$ such that $\tau(e^{\perp}) < \varepsilon$ and $\lim_{n\to\infty} \|(x_n - x)e\| = 0$, [7]

In the commutative case, both convergences in the definition above are equivalent to the usual almost everywhere convergence by virtue of Egorov's theorem. However, they are different in the non commutative setting.

In order to deduce the individual ergodic theorems from the corresponding maximal ergodic inequalities it is convenient to use the subspace $L_p(M;c_0)$ of $L_p(M; \ell_\infty)$, $L_p(M; c_0)$ is defined as the space of all sequences $(x_n) \subset L_p(M)$ such that there are $a, b \in L_2(M)$ and $(y_n) \subset M$ verifying $x_n = ay_nb$ and $\lim_{x\to\infty} \|y_n\|_\infty = 0$. It is easy to check that $L_p(M; c_o)$ is a closed subspace of $L_p(M; \ell_\infty)$ and where the infimum runs

over all factorization of (x_n) as above we define similarly the subspace $L_p(M; c_0^c)$ of $L_p(M; \ell_\infty^c)$. Note that $L_p(M; c_0)$ (resp $L_p(M; c_0^c)$) is just the closure in $L_p(M: \ell_\infty)$ (resp $L_p(M; \ell_\infty^c)$) of finite sequences in $L_p(M)$ for $1 \leq p < \infty$.

4.3 Lemma

i. If $(x_n) \in L_p(M; c_o)$ with $1 \leq p < \infty$, then x_n converges bilaterally almost uniform to 0.

ii. If $2 \leq p < \infty$ and $(x_n) \in L_p(M; c_o^c)$, then x_n converges a.u. to 0.

Proof

i. Let $(x_n) \in L_p(M; c_0)$. Then there are $a, b \in L_{2p}(M)$ and $y_n \in M$ such that $x_n = ay_nb$ and $\|a\|_{2p} < 1$. $\|b\|_{2p} < 1, \ldots \lim_{x\to\infty} \|y_n\|_\infty = 0$

We can clearly assume $a.b \geq 0$. Let e be a spectral projection of a such that $\tau(e^{\perp}) \leq \tau(e'^{\perp}) < \frac{\varepsilon}{2}$ and $\|e'a\| \infty \leq \left(\frac{2}{\varepsilon}\right)^{\frac{1}{2p}}$

Similarly, we find a spectral projection e' of b. set $e = e' \wedge e$. Then

$\tau(e^{\perp}) \leq \tau(e'^{\perp}) + \tau(e'^{\perp}) < \varepsilon$

And $\|ex_ne\|_\infty \leq \|ea\|_\infty \|y_n\|_\infty \|be\|_\infty \leq \|y_n\|_\infty \|e'a\|_\infty \|be'\|_\infty \leq \left(\frac{2}{\varepsilon}\right)^{\frac{1}{p}} \|y_n\|\infty$

Thus $\lim_n \|ex_n\| \infty = 0$, and so $x_n \rightarrow 0$ b.a.u.

ii. The proof of this part is similar.

Now let T be a linear map satisfying the conditions (a) – (c). Let $(M_n)_n$ denote the ergodic averages of T. Recall that F_p denotes the fixed point subspace of T in L_p (M) and F the projection from L_p (M) onto F_p.

4.4 Theorem [6]

Let T be a map on M satisfying (a) – (c) in the chapter three, let $1 < p < \infty$ and $x \in L_p$ (M). Then $(M_n(x) - F(x))_n \in L_p(M; c_o)$. Moreover, if p>2, $(M_n (x) - F(x))_n \in L_p(M; c_0^c)$.

Proof

Let $x \in L_p$ (M). Since (I – T)($L_1(M) \cap L_\infty(M)$) is dense in $F_p^{\perp}$ there are $x_k \in (I - T) (L_1 (M) \cap L_\infty (M))$ such that

$$\lim_{k\to\infty} \left\| x - F(x) - x_k \right\|_p = 0$$

$$\left\| M_n(x) - F(x) - M_n(x_k)_n \right\|_{Lp(M;\ell\infty)} \leq C_p \left\| x - F(x) - x_k \right\|_p$$

Thus

$\lim_{k\to\infty}\left(M_n(x_k)\right)_n = \left(M_n(x) - F(x)\right)_n \; in L_p(M;\ell_\infty).$

Since L_p (M; c_0) is closed in L_p (M; ℓ_∞) it is suffices to show

$(M_n(x_k))_n \in L_p(M;c_o)$ for every k. To this end consider an arbitrary

$z \in (I - T)(L_1(M) \cap L_\infty(M))$. Let $y \in L_1(M) \cap L_\infty(M)$ such that

$z = y - T(y)$.

Then

$$M_n(z) = \frac{1}{n+1}\left[y - T^{n+1}(y)\right]$$

Since $z \in L_p(M)$ for any $1 < q < \infty$ we deduce from corllary 3.14 that $(M_n(z))_n$ belongs to L_q (M; ℓ_∞). Choose $aq \in (1,p)$. Then by proposition 3.8 for any $m < n$.

$$\left\|\sup_{m\le j\le n}{}^{+} M_j(z)\right\|_p \le \sup\left\|M_j(z)\right\|_\infty^{1-\frac{q}{p}} \left\|\sup_{m\le j\le n}{}^{+} M_j(z)\right\|_p^{q/p}$$

$$\le \left[\frac{2\|y\|_\infty}{m+1}\right]^{1-\frac{q}{p}} \left\|\sup_{m\le j\le n}{}^{+} M_j(z)\right\|_p^{q/p}$$

Let $z^{(k)}$ denote the finite sequence ($M_o(z)$ … $M_k(z)$, 0, …). The inequality above shows that the sequence $(z^{(k)})_{z\ge 0}$ converges to $(M_n(z))_n$ in L_q (M; ℓ_∞) as $k \to \infty$. Thus $(M_n(z))_n \in L_q$ (M; c_0), as required.

The second part can be similarly proved. Now we use corollary 3.14 and the analogue for the spaces $L_p(M;c_0^c)$ of Proposition 3.8

The following is an extension of Yeadon's non commutative individual ergodic theorem [14] to all $L_p(M)$ with $1 < p < \infty$.

4.5 Corollary:

Let T be a map satisfying the conditions (a) – (c) of chapter three. Let $1 < p < \infty$ and $z \in L_p(M)$. Then $(M_n(z))_n$ converges to F(x) b.a.u for $1 < p \leq 2$ and a.u. for $2 < p < \infty$

Proof

This is an immediate consequence of Lemma 4.3 and Theorem 4.4

4.6 Remark

I. Corollary 4.5 can also be proved by using Yeadon's theorem [14]. This is however not the case for the multiple individual ergodic theorem.

II. Again using Yeadon's theorem [14], one can prove that the convergence in theorem 4.4 is a.u. for p = 2.

4.7 Theorem [6]

Let $T_1, \ldots T_d$ be maps satisfying (a) – (c) as usual.

Let $1 < p < \infty$ and $x \in L_p(M;c_0(N^d))$.

Then

$$M_{n_1,\ldots,n_d}(x) - F_d \ldots F_1(x))_{n_1,\ldots n_d \geq 1} \in L_p(M;c_0(N^d)).$$

And if $p > 2$,

$$M_{n_1,\ldots,n_d}(x) - F_d \ldots F_1(x))_{n_1,\ldots n_d \geq 1} \in L_p(M;c_0(N^d)).$$

Consequently,

$M_{n_1,\ldots,n_d}(x)$ converges b.a.u. to $F_d \ldots F_1(x)$ as $n_1, \ldots, n_d$ tend to ∞.

Moreover, the convergene is a.u. in the case of $p > 2$.

Proof

This proof is similar to that of theorem 4.4. We consider only the typical case $d = 2$. Note that

$$M_{n_1,n_2} = M_{n_2}(T_2) M_{n_1}(T_1)$$

Fix $x \in L_p(M)$ and decompose x as $x = F_1(x) + y_k + u_k$ with

$$y_k \in (I - T_1)(L_1(M) \cap L_\infty(M)),\ u_k \in L_p(M)\ \|u_k\|_p \leq \frac{1}{k}.$$

Applying successively $M_{n_1,}(T_1)$ to x and $M_{n_2}(T_2)$ to $F_1(x)$, we get

$$M_{n_1,n_2}(x) = M_{n_1,n_2}(y_k) + M_{n_1,n_2}(u_k) + M_{n_2}(T_2)(z_k) + M_{n_2}(T_2)(v_k)$$

By 4.5 Corollary

$$\left\| \sup_{n_1 n_2}{}^{+} M_{n_1 n_2}(uk) \right\|_p \le C_p^2 \left\| u_k \right\|_p \le \frac{C_p^2}{k} \to 0 ask \to \infty$$

Similarly

$$\lim_{k \to \infty} \left\| Sup^{+} M_{n2}(T_2)(vk) \right\|_p = 0$$

Therefore, as in the proof of theorem 4.4 we need only to show

$$M_{n1n2}(yk)_{n1n2 \ge 1} \in L_p(M; c_0(N^2)) and (M_{n2}(zk))_{n2 \ge 1} \in L_p(M; c_0).$$

Theorem 4.5 implies the later. The former is proved by the arguments in the proof of theorem 4.4. Thus the first part of the theorem is proved.

Then applying Lemma 4.3 to multiple sequences, we deduce the announced point wise multiple ergodic convergence.

4.8 Theorem [6]

Let T be a map satisfying (a) – (d). Assume further that T is positive as an operator on L_2 (M). Let $1 < p < \infty$ and $x \in L_p$ (M). Then $(T^n(x)-F(x))_n$ belongs to L_p (M;c_0) and to L_p (M;c_0^c) if additionally

$p > 2$.

Consequently, $T^n(x)$ converges to F(x) b.a.u. for $1 < p \le 2$ and a.u. for $2 < p < \infty$.

Proof

Let us first treat the case p = 2. write the spectral decomposition of T.

$T = \int_0^1 \lambda de_\lambda$

Note that for any $x \in \overline{(I-T)L_2(M)}$..

$$\lim_{\lambda \to 1} e_\lambda(x) = x \quad in L_2(M)$$

Given $x \in L_2(M)$ choose $x_k \in (I\text{-}T)\ L_2(M)$ such that

$$\lim \left\| x - F(x) - x_k \right\|_2 = 0.$$

Then $\lim \lambda \to 1\ e\lambda\ (x_k) = x_k$. Thus replacing x_k by $e_{\lambda k}\ (x_k)$ with an appropriate $\lambda_k \in$ (a), we can assume that xk = e_λ (yk) for some yk $\in L_2(M)$.

Then

$T^n\ (x_k) = \int_0^{\lambda_k} \lambda^n de_\lambda(y_k); whence \left\| T^n(x_k) \right\|_{2k} \le \lambda_k^n \left\| y_k \right\|_2$.

It then follows that $T^n(x_k)_n \in L_2\ (M;c_0)$for every k, and so by Theorem 4.5 $(T^n(x)\text{-}F(x))_n \in L_2(M;c_0)$. To treat the general case we first claim that

$$\lim_n \left\| T^n(x) - F(x) \right\|_p = 0, \forall x \in L_2(M).$$

Indeed the preceding argument shows that this true for p = 2. Now let $2 < p < \infty$ and $x \in L_1\ (M) \cap M$ in L_p

By interpolation,

$$\left\|T^n(x)-F(x)\right\|_p \le \left\|T^n(x)-F(x)\right\|_\infty^{1-\frac{2}{p}} \left\|T^n(x)-F(x)\right\|_2^{\frac{2}{p}}$$

Where $\lim_n \left\|T^n(x)-F(x)\right\|_p = 0$. Then our claim in the case p > 2 follows from the density of L_1 (M) $\cap$M in (M). The case p < 2 is proved similarly.

Now we can easily finish the proof of the theorem. Let x $\in L_p(M)$. Fix k $\in$ *N*.

$$\lim_{k\to\infty}\left\|T^n\left(T^k(x)-F(x)\right)_n\right\|_{L_p(M;\ell_\infty)} = 0$$

Note that $T^n(T^k(x)-F(x)) = T^{n+k}(x)-F(x)$, and so the sequence $T^n(T^k(x)-F(x))_{n\ge 0}$ can be considered as the rest of $T^n(x)-F(x)_{N\ge 0}$ starting from the k- the coordinate. It follows that $(T^n(x)-F(x))_{n\ge 0} \in L_p(M;c_0)$.

4.9 Remark

If we remove the additional assumption that T is positive operator on $L_2(M)$ in Theorem, then for any $x \in L_p(M)$ the two subsequence $(T^{2n}(x))_n$ and $(T^{2n+1)}(x))_n$ still converge b.a.u; however, their limit are not equal in general.

We end this chapter with point wise ergodic theorems for semi-groups that is the application of this theorem on another fields of studies. Let $(T_1)_{t\geq 0}$ be a semi-group satisfying (a) – (d). We denote again by F the projection from L_p (M) onto the fixed point subspace of $(T_t)_{t\geq 0}$.

4.10 Theorem [6]

Let $(T_t)_{t\geq 0}$ be a semi-group with (a) – (d). Let $(M_t)_{t>0}$ denote the ergodic averages of $(T_t)_{t\geq 0}$. Let $1 < p < \infty$ and $\in L_p(M)$.

i. Then

   a. M_1 (x) converges to F(x) b.a.u. for $1 < p < 2$ and a.u. for $2 \leq p < \infty$ when $t \to \infty$.

   b. M_1 (x) converges to x b.a.u for $1 < p < 2$ and a.u. for $2 \leq p < \infty$ when $t \to 0$.

ii. Assume in addition that $(T_1)_{t\geq 0}$ satisfies (d) Then

a. $T_1(x)$ converges to $F(x)$ b.a.u. for $1 < p < 2$ and a.u for $2 \leq p < \infty$ when $t \rightarrow \infty$.

b. $T_1(x)$ converges to b.a.u. for $1 < p < 2$ and a.u. for $2 \leq p < \infty$ when $t \rightarrow 0$.

CHAPTER FIVE

SUMMARY AND CONCLUSION

5.1 Summary

In this thesis, the first chapter discusses the basic definitions and notations of von Neumann algebras. While in chapter two we constructed a non-commutative L_p space (Segal's L_p space), over a semi-finite von Neumann algebra having a faithful normal semi-finite trace. Furthermore we showed that the Minkowskin's inequality holds with respect to our defined norm. However, chapter three discussed the Maximal Ergodic inequalities through a non-commutative L_p spaces, $L_p(M;\ell_\infty)$ introduced by Pisier, [9]. Finally, with the result in chapter three and an appropriate analogue for the non commutative settings of the usual almost everywhere converges introduced by Lance [8] we studied the Point wise ergodic theorems and its various applications.

5.2 Conclusion

Today, Ergodic theory is a large and rapidly developing subject. The two major sources of ergodic theory are Mathematical Physics (especially statistical mechanics and Hamiltonian dynamics) and the theory of stationary stochastic processes, these provide a rich store of measure preserving transformation and flows. Von Neumann algebra is sufficient and rich in its content (chapter one) and similarly, as we have seen in the chapter four, some of its application on semi-groups, these properties help us to understand the manner in which a Measure Preserving Transformation (m.p.t) moves points and sets through the space on which it acts.

Hopefully, this thesis will open an avenue for future discussion and the research into this fascinating topic and understanding new relationship of ergodic theorem and von Neumann algebras.

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