INVESTIGATION OF THE STABILITY OF EQUILIBRIUM POINTS IN THE RELATIVISTIC RESTRICTED THREE-BODY PROBLEM WITH PERTURBATIONS

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DECLARATION

I declare that the work in this thesis titled **INVESTIGATION OF THE STABILITY OF EQUILIBRIUM POINTS IN THE RELATIVISTIC RESTRICTED THREE-BODY PROBLEM WITH PERTURBATIONS** has been performed by me under the supervision of Prof. J. Singh, Prof. B. K. Jha, and Dr. A. O. Ajibade in the Department of Mathematics, Ahmadu Bello University, Zaria. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any other institution.

Bello Nakone	___________________	____________________
Name of student	**Signature**	**Date**

CERTIFICATION

This Thesis titled INVESTIGATION OF THE STABILITY OF EQUILIBRIUM POINTS IN THE RELATIVISTIC RESTRICTED THREE-BODY PROBLEM WITH PERTURBATIONS by Bello NAKONE meets the regulation governing the award of the degree of Doctor of Philosophy in Mathematics of Ahmadu Bello University, and is approved for its' contribution to knowledge and literary presentation.

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To my late father Nakone Mayaki and my late grand mother Tacorgo Koussougou.

ABSTRACT

In the work by Bhatnagar and Hallan (1998), linear stability of the relativistic triangular L_4 *and* L_5 points was studied and it was shown that these points were unstable for the whole range $0 \leq \mu \leq \frac{1}{2}$ despite the well-known fact that the non-relativistic L_4 and L_5 are stable for $\mu < \mu_0$, where μ_0 is the Routh critical mass ratio. The same problem was later investigated by Douskos and Perdios (2002) and Ahmed *et al*. (2006) and they obtained two different ranges of mass ratios in which the relativistic triangular points are linearly stable in contradiction with the result of Bhatnagar and Hallan (1998). In this thesis we reconsider and generalize the problem investigated by these authors in that perturbations in the Coriolis and centrifugal forces, radiation pressure, oblateness and triaxiality factors of the primaries have been considered in our investigation. The locations of equilibrium points are obtained and their stability are analyzed by using variational method and Lyapunov's criteria. The triangular points of the relativistic three-body problem (R3BP) are studied from various aspects of perturbations such as oblateness, triaxiality and radiation pressure of the primaries as well as the small perturbations in the centrifugal and Coriolis forces. It is found that the locations of the triangular points are affected by the asphericity of the primaries, the relativistic terms and a small change in the centrifugal force. It is also found that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c < \mu \leq \frac{1}{2}$, where μ_c is the critical mass parameter depending on the perturbation parameters and relativistic terms. It is further found that the Coriolis force has stabilizing tendency, while the centrifugal force, radiation pressure forces, oblateness, triaxiality of the primaries and relativistic terms have destabilizing effects. The motion of an infinitesimal mass near the collinear equilibrium points when the smaller primary is a triaxial body is also studied. It is observed that the positions of the collinear points are affected by the relativistic and triaxiality factors. The collinear points are found to remain unstable. Numerical studies in this connection with the Sun-Earth, Sun-Pluto and Earth-Moon systems have been carried out to show the relativistic and triaxiality effects. The motion of an infinitesimal mass near the collinear equilibrium points when the smaller primary is oblate is also investigated. The collinear points are found to be unstable. A numerical exploration in this connection, with some members of our solar system reveals that the locations of the

collinear points L_1, L_2 are affected prominently by the relativistic factor in the absence of oblateness and they are also affected significantly by the oblateness factor in the absence of relativistic terms. It is also found that in most of the cases, the position of L_3 is negligibly affected by the relativistic and oblateness factors. More specifically, all parameters involved have no effect on the position of L_3 of the Sun-Mars system. The results show that the oblateness and relativistic factors have the same but separate effect on the position of L_1 of the Sun-Uranus system and have also the same effect on the position of L_1 of the Sun-Neptune system. It is also found that in the presence of relativistic terms, the effect of oblateness on the Sun-Planet pairs does not show physically. Also, the frequencies of the long and short orbit of the periodic motion, eccentricities, axes and the orientation of the orbits around the stable triangular points when the bigger primary is triaxial are determined and found to be affected by the triaxiality and relativistic effects. The results of this study generalise the classical relativistic restricted three-body problem (R3BP) and the results of Douskos and Perdios (2002) can be deduced from this study while the present results differ with the results of Bhatnagar and Hallan (1998) and differ also with the results of Ahmed *et al*. (2006).

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μ	Mass ratio
μ_c	Critical value of the mass parameter
W	Potential force function
m_1	Mass of the bigger primary
m_2	Mass of the smaller primary
A_1	Oblateness of the bigger primary
A_2	Oblateness of the smaller primary
ρ_1	Distance from the third body to the bigger primary
ρ_2	Distance from the third body to the smaller primary
$q_1 = 1 - \delta_1$	Radiation coefficient of the bigger primary where δ_1 is the ratio of the radiation pressure force to the gravitational force of the bigger primary
$q_2 = 1 - \delta_2$	Radiation coefficient of the smaller primary where δ_2 is the ratio of the radiation pressure force to the gravitational force of the smaller primary
n	Mean motion
n_d	Dimensionless mean motion
c	Speed of light
c_d	Dimensionless speed of light
$\sigma_i \left(i = 1, 2 \right)$	Triaxiality coefficients of the bigger primary
$\sigma_i' \left(i = 1, 2 \right)$	Triaxiality coefficients of the smaller primary
$\psi = 1 + \varepsilon_1$	Centrifugal force where ε_1 is the perturbation in the centrifugal force.
$\varphi = 1 + \varepsilon_2$	Coriolis force where ε_2 is the perturbation in the Coriolis force

CHAPTER ONE

INTRODUCTION

1.1 Background of the Study

The approximately circular motion of the planets around the sun and the small masses of the asteroids and satellites compared to planetary masses suggested formulation of the circular restricted three-body problem (CR3BP).

The restricted three-body problem (R3BP) represents a very wealthy treasure of dynamical system, since the discovery of its non-integrability due to the pioneer Poincare (1892-1899). This problem concerns with the motion of a test particle moving under the gravitational effect of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the test mass does not influence the motion of the primaries. Although R3BP is insoluble, or non-integrable, some families of particular solutions exist. Euler (1765) found a collinear solution for the restricted three-body problem that assumes one of the three bodies is a test mass. Soon after his solution was extended for a general three-body by Lagrange, who also found an equilateral triangular solution in 1772. Now the solutions for the restricted three-body problem are called Lagrange points. These solutions are corresponding to the five singular points of Jacobian function. These families of solution can be found by setting the derivatives of the potential to zero.

These five points are classified as three equilibrium collinear points $L_i\left(i=1,2,3\right)$ (straight solution) or Eulerian points due to Euler and the other two equilibrium $L_i\left(i=4,5\right)$ represent equilateral triangle with the primaries.

On the line joining the primaries, the equivalent potential at the Eulerian points is a maximum, and therefore these equilibrium points are unstable, the points are actually saddle points.

The theory of the general relativity is currently the most successful gravitational theory describing the nature of space and time, and well confirmed by observations. In application to astrophysics, the general relativity enables one to analyse phenomena not compatible within the framework of Newtonian concepts. General relativity made it possible to calculate the binary pulsar motion (as a problem of relativistic celestial). Binary pulsars observations confirmed the conclusion of general relativity about the energy loss due to gravitational radiation. Krefetz (1976) computed the post-Netonian deviations of the triangular Lagrange points from their classical positions in a fixed frame of reference for the first time, but without explicitly stating the equations of motion. After a decade, Brumberg (1972, 1991) studied the problem in more details and collected most of the important results on relativistic celestial mechanics. The author did not only obtain the equations of motion for general problem of three bodies but also deduced the equations of motion for the restricted problem of three bodies. Contopolous (1976) treated the relativistic R3BP in rotating coordinates. The author derived Lagrangian of the system and the deviations of the triangular points as well. Maindl and Dvorak (1994) derived the equation of motion for the relativistic R3BP using post-Newtonian approximation of relativity. Abd EL-Salam and Abd EL-Bar (2011) derived the equations of motion of the relativistic restricted three-body problem in the post-Newtonian formalism.

1.2 Theoretical Framework

The fundamental laws of mechanics, as given by Newton in his Principia in 1687, may be stated as:

First law: If there are no external forces, an object will maintain its state of motion, that is it will stay at rest or continue rectilinear motion at constant velocity.

Second law: The rate of change of the momentum of an object is proportional to the applied force F.

Third law: If a body A exerts a force F, on a body B, the body B will exert a force F, on body A.

1.2.1 The Newtonian law of gravitation

Every two particles in the universe attract each other with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them that is if the bodies A and B have masses M_A and M_B, respectively, and if their mutual distance is r, A will act on B with a force that is directed towards A and has a magnitude $G\frac{M_A M_B}{r^2}$, where G is a constant of gravity, the value of which depends on the units chosen.

1.2.2 Kepler's laws of planetary motion

i. The planets move in ellipses with the Sun at one focus

ii. A radius vector from the Sun to a planet sweeps out equal area in equal times.

iii. The square of the orbital period of a planet is proportional to the cube of its semi-major axis.

1.2.3 Three-body problem

The three-body problem refers to three bodies of arbitrary masses which move under their mutual gravitational attraction.

1.2.4 The circular restricted three-tody problem

If two of the three bodies move in circular and coplanar orbits around their common barycenter, and additionally, the third body's mass is small compared to the other two masses so that the third does not affect the movement of the other bodies, one speaks of the circular restricted three-body problem. The two finite, massive bodies usually denoted by m_1 and m_2, are called primaries and the third body (infinitesimal mass) is called a test particle. The equations of motion of the classical restricted three-body problem in the (x, y, z) coordinate system with m_1 and m_2, and ρ_1 and ρ_2, as the masses and the distance to the bigger and smaller primaries respectively are:

$\ddot{x} - 2\dot{y} = W_x$, $\ddot{y} + 2\dot{x} = W_y$, $\ddot{z} = W_z$, where the force function is

$$W = \frac{x^2 + y^2}{2} + \frac{1-\mu}{\rho_1} + + \frac{\mu}{\rho_2}$$

$$\rho_i^2 = (x - x_i)^2 + y^2 + z^2, (i = 1,2),$$

$$x_1 = -\mu,\ \ x_2 = 1-\mu,\ 0 < \mu \le \frac{1}{2},\ \mu = \frac{m_2}{m_1 + m_2}.$$

1.2.5 Equilibrium points

Although the circular restricted three-body problem is not integrable, we can find a number of special solutions. This can be achieved by searching for points where the particle has zero velocity and zero acceleration in the rotating frame. Such points are called equilibrium points of the system.

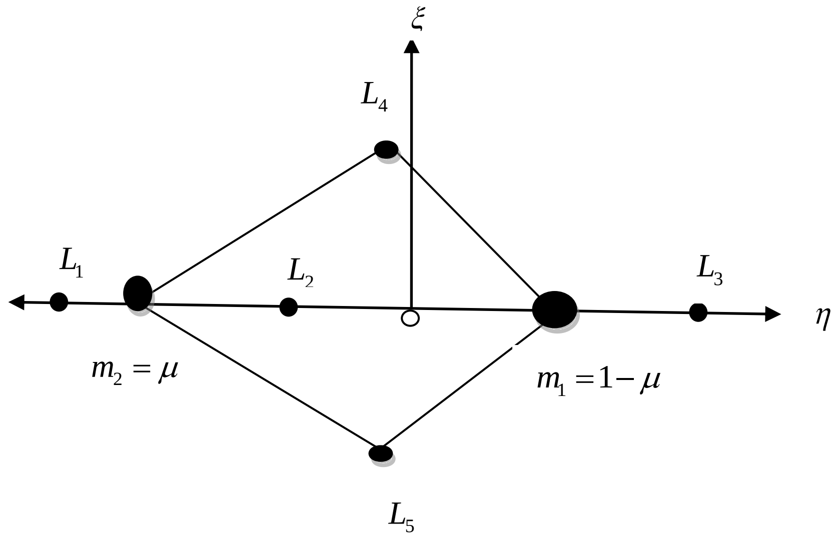


Figure 1.1: The location of the equilibrium points $L_i \left(i = 1, 2, ..., 5\right)$

1.2.6 Stability of equilibrium points

Five particular solutions of the motion of the infinitesimal body have been found. If the infinitesimal body is displaced a very little from the exact points of the solutions and given a small velocity it will either oscillate around these respective points at least for a considerable time, or it will rapidly depart from them. In the first case the particular solution from which the displacement is made is said to be stable; in the second case, it is said to be unstable.

Let us have a dynamical system with s degree of freedom which is described by a set of a differential equation as:

$\dot{\bar{x}} = \bar{g}\left(\bar{x}\right)$ where $\bar{x} \in R^s$ and $\bar{g} = \left(g_1, g_2, ...g_s\right)$ is a vector function from R^s into itself. Now, let J be the Jacobian Matrix defined as:

$$J(\bar{x})D\bar{g}(\bar{x})=\begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \bullet & \bullet & \bullet & \frac{\partial g_1}{\partial x_s} \\ \bullet & \bullet & & & \bullet \\ \bullet & & \bullet & & \bullet \\ \bullet & & & \bullet & \bullet \\ \frac{\partial g_s}{\partial x_1} & \bullet & \bullet & \bullet & \frac{\partial g_s}{\partial x_s} \end{pmatrix}$$

In the neighborhood an equilibrium location $\bar{x}=\bar{x}_0$, we refer to $\dot{\bar{\chi}}=J(\bar{x}_0)\bar{\chi}, \bar{\chi}\in R^s$ as the linearized system. According to the stability in Lyapunov sense solution $\bar{x}=\bar{x}(t)$ is stable if for any $\varepsilon>0$ there exists $\delta=\delta(\varepsilon)>0$ such that if any solution $\bar{y}=\bar{y}(t)$ satisfy $|\bar{x}(t_0)-\bar{y}(t_0)|<\delta$ at the initial time $t=t_0$ then $|\bar{x}(t)-\bar{y}(t)|<\varepsilon$ for $t\geq t_0$, the solution is said to be asymptotically stable if for $|\bar{x}(t_0)-\bar{y}(t_0)|<\delta$ one has $\lim|\bar{x}(t)-\bar{y}(t)|=0.$ as $t\longrightarrow\infty$

When the solution coincides with an equilibrium point, its linear stability can be inferred from the analysis of the eigenvalues of the Jacobian Matrix. Let $\bar{\upsilon}_i$ is the eigenvector associated to the eigenvalue $\lambda_i\,(i=1,2,...s)$ where, $\lambda_i^{'}s$ are the roots of the characteristic equation $\det(J-\lambda I)=0$ such that I is identity matrix of order $s\times s$ and $\upsilon_i\in R^s$ then, the solution of the linearized system can written in the form:

$\bar{\chi}(t)=\sum_{i=1}^{s}k_i\bar{\upsilon}_i e^{\lambda_i t}$ for some real or complex coefficient k_i which are determined by the initial conditions.

The stability properties for $t\geq t_0$, of the linearized system may be stated simply:

(A) For complex roots of characteristic equation we have the following properties.

a. When the characteristic roots all have negative real parts, the equilibrium point is asymptotically stable. This is true also when some of the roots are multiple.
b. When some or all the characteristic roots have positive real parts, the equilibrium point is unstable. This is true also when some of the roots are multiple.

(B) For pure imaginary roots the motion is oscillatory and the solution is stable though it is not asymptotically stable. If there are multiple roots the solution contains mixed (periodic and secular) terms and the equilibrium point is unstable.

(C) If the roots are real and all negative, the solution is stable, if any of the roots is positive the point is unstable. These statements are also true for multiple roots.

Assuming for simplicity that $s = 2,$ the equilibrium points are characterized according to the nature of the eigenvalues λ_1 and λ_2 as follows:

1. If λ_1 and λ_2 are real negative (positive) numbers, the equilibrium point is a stable (unstable) node.
2. If λ_1 and λ_2 are real numbers with opposite signs, then the equilibrium point is a sadde.
3. If λ_1 and λ_2 are complex numbers with negative (positive) real parts, then the equilibrium is a stable (unstable) focus or spiral
4. If λ_1 and λ_2 are purely imaginary, then the equilibrium is a center or vortex.

1.2.7 Periodic orbits

A dynamical system in which the same configuration is repeated at regular interval of time is said to exhibit a periodic motion. Same can be said of motions which are repeated in a relative sense.

The periodic orbits have great significance generally in celestial mechanics and especially in space dynamics.

For any particular solution of the restricted three-body problem, one can always find a periodic solution. We can also get periodic orbits through linearized solution for the motion of infinitesimal body around equilibrium point. Periodic orbits play a vital role in separating the various classes of orbits and reduce the dimensionality of the problem in phase space. In addition they can be used as reference orbits.

1.2.8 Ellipse

An ellipse is the locus of a point that moves such that the sum of its distances from two fixed points called the foci is a constant.

1.2.9 Eccentricity

The orbital eccentricity of an astronomical body denoted by e, is the amount by which its orbit deviates from a perfect circle. In other words the ratio of the distance between the foci to length of the string is called the eccentricity e of the ellipse.

In other words the ratio of the distance between the foci to length of the string is called the eccentricity e of the ellipse. From analytic geometry we know that when:

$e < 1$, the orbit is an ellipse

$e = 1$, the orbit is a parabola

$e > 1$, the orbit is a hyperbola

e=0 , the orbit is a circle

1.2.10 Special relativity

In 1905, Albert Einstein introduced the special theory of relativity in his paper 'On the Electrodynamics of Moving Bodies'. Special relativity, as it is usually called, postulates two things: First, any physical law which is valid in one reference frame is also valid for any frame moving uniformly relative to the first. A frame for which this holds is referred to as an inertial frame. Second, the speed of light in vacuum is the same in all inertial reference frames, regardless of how the light source may be moving.

The first postulate implies there is no preferred set space and time coordinates. For instance, suppose you are sitting at rest in a car moving at constant speed. While looking straight out a side window, everything appears to be moving so quickly!

Trees, building, and even people are flashing by faster than you can focus on them.

However, an observer outside of your vehicle would say that you are the one who appears to be moving. In this case, how should we define the coordinates of you in your car and the observer outside of your car? We could say that the outside observer was simply mistaken, and that you were definitely not moving. Thus, his spatial coordinates were changing while you remained stationary. However, the observer could adamantly argue that you definitely were moving, and so it is your spatial coordinates that are changing. Hence, there is no absolute coordinate system that could describe every event in the universe for which all observers would agree and we see that each observer has his own way to measure distances relative to the frame of reference he is in.

It is important to note that special relativity only holds for frames of reference moving uniformly relative to the other, that is, constant velocities and no acceleration. We can illustrate this with a simple example. Imagine a glass of water sitting on a table.

According to special relativity, there is no difference in that glass sitting on a table in your kitchen and any other frame with uniform velocity, such as a car traveling at constant speed. The glass of water in the car, assuming a smooth, straight ride with no shaking, turning or bumps, will follow the same laws of physics as it does in your kitchen. In this case, the water in each glass is undisturbed within the glass as time goes on. However, if either reference frame underwent acceleration, special relativity would no longer hold. For instance, if in your car, you were to suddenly stop, then the water in your glass would likely spill out and you would be forced forward against your seat belt.

Special relativity was not extended to include acceleration until Einstein published 'The Foundation of the General Theory of Relativity' in 1916. In special relativity, observers in different inertial frames cannot agree on distances, and they certainly cannot agree on forces depending on the distance between two objects. Such is the case with Newtonian gravitation, as it describes gravity as an instantaneous force between two particles dependent on their distance from one another. With this in mind, Einstein desired to formulate gravity so that observers in any frame would agree on the definition, regardless of how they were moving in relation to each other. Einstein accomplished this by defining gravity as a curvature of spacetime rather than a force.

In physics, special relativity is the generally accepted and experimentally well confirmed physical theory regarding the relationship between space and time. In Einstein's original pedagogical treatment, it is based on two postulates:

1. That the laws of physics are invariant (i.e. identical) in all inertial systems (non accelerating frames of reference).
2. That the speed of light in a vacuum is the same for all observers regardless of the motion of the light source.

The theory is "special" in that it only applies in the special case where the curvature of space-time due to gravity is negligible.

1.2.11 General relativity

There are three essential ideas underlying general relativity. The first is that space- time may be described as a curved, four- dimensional mathematical structure called a pseudo-Riemannian manifold. In brief, time and space together comprise a curved four-dimensional non-Euclidean geometry. Consequently, the practitioner of general relativity must be familiar with the fundamental geometrical properties of curved space-time. In particular, the laws of physics must be expressed in a form that is valid independently of any coordinate system used to label points in space-time.

The second essential idea underlying general relativity is that at every space-time point there exist locally inertial reference frames, corresponding to locally flat coordinates carried by freely falling observers, in which the physics of general relativity is locally indistinguishable from that of special relativity. This is Einstein's famous strong equivalence principle and it makes general relativity an extension of special relativity to a curved space-time. The third key idea is that mass curves space-time.

These three ideas have been demonstrated by contrasting general relativity with Newtonian gravity. In the Newtonian view, gravity is a force accelerating particles through Euclidean space, while time is absolute. From the viewpoint of general relativity as a theory of curved space-time, there is no gravitational force. Rather, in the absence of electromagnetic and other forces, particles follow the straightest possible paths (geodesics) through a space-time curved by mass. Freely falling particles define locally inertial reference frames. Time and space are not absolute but are combined into the four- dimensional manifold called space-time.

In special relativity there exist global inertial frames. This is no longer true in the presence of gravity. However, there are local inertial frames in general relativity, such that within a suitably small space-time volume around an event, one may choose coordinates corresponding to a nearly-flat space-time. Thus, the local properties of special relativity carry over to general relativity.

General relativity generalizes special relativity and Newton's law of Universal gravitation, providing a unified description of gravity as a geometric property of space and time, or space time. General relativity predicts that the path of light is bent in the gravitational field; light passing a massive body is deflected towards that body. This effect has been confirmed by observing the light of stars or distant quasars being deflected as it passes the Sun.

1.2.12 Radiation pressure forces

Cosmic radiation is the term that is used to describe the radiation that reaches us from stars. While solar radiation is the term used to describe the radiation from our own Sun. On the other hand the radiation pressure is the pressure exerted upon any surface exposed to electromagnetic radiation.

So the dimensions of the actual body are very important when considering radiation pressure coefficient. A large body can have, a large surface area, but if its diameter or atomic diameters are not close to the incident radiation wavelength it will have a lower radiation pressure coefficient, and thus a lower momentum will be imparted to it. Generally if a beam of electromagnetic waves strikes a surface, waves will be reflected, some will be absorbed and some keep going straight through, depending on the target's properties and frequency. The radiation pressure coefficient is equal to the ratio of the momentum acquired by the target to the electromagnetic momentum of the waves before impact. For example when a beam of electromagnetic radiation is incident upon a surface and complete transmission happened, the radiation pressure coefficient equal zero. But its value equals one for total absorption and two when all incident waves are reflected. Radiation pressure is dissipative force, the consequences of this force are loss of energy and angular momentum of the satellite. A dissipative force leads to reduction in the semi-major axis of the satellite.

As the solar radiation pressure force F_p changes with the distance by the same law as gravitational attraction force F_g and acts opposite to it, it is possible to consider that the result of the action of this force will lead to reducing the effective mass of the massive particle. Since the effect of reducing the mass of a particle depends upon the properties, it is acceptable to speak about a reduced mass of the particle.

Thus the resultant force on the particle is

$$F = F_g - F_p = F_g\left(1 - \frac{F_p}{F_g}\right) = qF_g$$

Where $q = 1 - \frac{F_p}{F_g}$, is the mass reduction factor constant for a particle which depends on the size and shape of the third body.

Chernikov (1970) has given the expression $q = 1 - \frac{5.6\times 10^{-5}}{\rho\, d}\kappa$ where ρ is the particle radius and d its density, while κ is a radiation-pressure efficiency factor (in the C.G.S. system).

If the solar radiation flood fluctuations and a shadow effect of the planet are neglected, then q is assumed to be constant. Depending upon the value of q, the reduced particle mass is positive, negative or zero. In this thesis we consider the case when the gravitation prevails, i.e. $q > 0$. We denote the radiation factors as $q_i\,(i = 1,2)$ for the bigger and smaller primaries respectively, and they are given by

$q_i = 1 - \left(\frac{F_{pi}}{F_{g_i}}\right) = 1 - \delta_i\left(i = 1,2\right)$ where δ_i is the ratio of radiation pressure force to the gravitational force of the primary such that $0 < 1 - q_i << 1\ \left(i = 1, 2\right)$

1.2.13 Ellipsoid

An ellipsoid is a closed type of quadratic surface that is a higher dimensional analogue of an ellipse. The equation of a standard axis-aligned ellipsoid body in an xyz Cartesian coordinate system is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{d^2} = 1$, where a and b are the equatorial radii (along the x and y axes) and d is the polar radius (along the $z - axis$), all of which are fixed positive real numbers determining the shape of the ellipsoid. If all three radii are equal, the solid body is a sphere; if two radii are equal, the ellipsoid is a spheroid that is

$a = b = d$:Sphere

$a = b > d$:Oblate spheroid (disk-shaped)

$a = b < d$:Prolate spheroid (like a rugby ball)

$a > b > d$:Scalene ellipsoid (three unequal sides)

The points $(a,0,0)$, $(0,b,0)$ and $(0,0,d)$ lie on the surface and the line segments from the origin to these points are called the semi-principal axes. These correspond to the semi-major axis and semi-minor axis of the appropriate ellipses. Scalene ellipsoid are frequently called "triaxial ellipsoid", the implication being that all three axes need to be specified to define the shape.

Any plane's cross section passing through the center of ellipsoid forms an ellipse on its surface with the possible special case of a circle as the equator of revolving ellipsoid.

1.2.14 Oblateness

From a physical point of view, it is unreasonable to consider all objects as being points masses with no physical dimensions. This is in conflict with the real cases for celestial bodies. It is known that the effect of rotation causes deformation in the shape of the objects at the equator as might be expected. In short the oblateness is the departure of planets and celestial objects from spherical form because of the centrifugal force of the rotation. For this reason most objects may be treated to a good approximation as oblate spheroids (triaxial ellipsoid with two equal long axes and one short axis).

We denote $A_i \ (i=1,2)$ for the oblateness coefficients of the bigger and smaller primaries respectively, such that $0 < A_i << 1$ (McCuskey, 1963) and $A_i = \frac{AE_i^2 - AP_i^2}{5R^2} (i=1,2)$, where AE_1 and AE_2 are the equatorial radii, AP_1 and AP_2, the polar radii of the bigger and smaller primaries respectively and R the distance between the primaries.

1.2.15 Inertial and synodic coordinate systems

An inertial frame is a coordinate frame in which Newton's laws hold true. If we have an absolute frame of the whole universe, at least all the frames moving at constant velocity with respect to this absolute frame would be inertial. The concept of such an absolute frame has, however, turned out to be rather problematic. Yet we can define that a frame is inertial if experiments show that Newton's laws are valid in that frame.

The system of coordinates (ξ, η) such that the plane rotates in the positive direction with angular velocity equal to that of the common velocity of one primary with respect to the other keeping the origin fixed, then that coordinate system is known as synodic system.

The primaries appear at rest in the synodic or rotaing system (ξ, η) having the origin at the centre of mass rotating along them and are placed on the $\xi -$axis. The plane $\xi - \eta$ is the plane of motion of the primaries. The coordinates are sometimes called synodical. In this system (ξ, η), the primaries m_1 and m_2 are located at $(-\mu, 0), (1-\mu, 0)$ respectively and have zero velocity. The advantage of this system is that m_1 and m_2 have fixed positions, so that the equations of motion are time-independent and therefore it is the easiest way to obtain the stationary solutions.

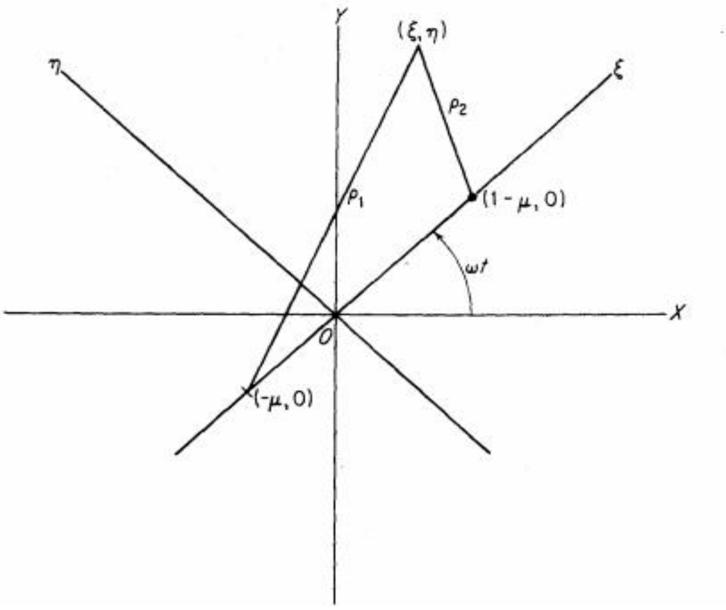


Figure 1.2: The rotating (synodic) coordinate system $(0\,\xi\,\eta)$ with angular velocity $\omega = n$ relative to the inertial (fixed) frame $(0\,X\,Y)$

1.2.16 Coriolis and centrifugal forces

In physics the Coriolis effect is a deflection of moving objects when they are viewed in a rotating reference frame. In a reference frame with clockwise rotation the deflection is to the left of the motion of the object, and for counter-clockwise rotation the deflection is to the right. The Newton's laws of motion govern the motion of an object in a non-accelerating inertial frame of reference. When Newton's laws are transformed to a uniformly rotating frame of reference, the Coriolis and centrifugal forces appear. Both the Coriolis and Centrifugal forces are proportional to the mass of the object. The Coriolis force is proportional to the rotation rate and the centrifugal force is proportional to its square. The Coriolis force acts in the direction perpendicular to the rotation axis. The centrifugal force acts outwards in the radial direction and is proportional to the distance of the body from the axis of the rotation frame. It is important to note that these forces do not arise from any physical agency. They arise solely as a result of rotation of the coordinate system. Thus, if the angular velocity is reduced to zero, both the Coriolis and centrifugal forces are referred to as “fictitious or inertial” forces.

1.3 Statement of the Problem

The classical restricted three-body problem concerns the study of the motion of a particle of infinitesimal mass (third body) in the gravitational field of two other massive bodies, say $m_1, m_2 (m_1 \geq m_2)$ (conventionally called primaries). The classical restricted three-body problem possesses five libration points (equilibrium points). The three libration points L_1, L_2, L_3 are collinear with the two primaries. The collinear points are not stable in general for any value of the mass parameter $\mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2}$. The two libration points L_4, L_5 define an equilateral triangle with primaries. They are called triangular points. The triangular points are stable for the mass ratio $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass parameter. This critical mass value is used to study and analyze the size of the stability region. There exist also stable periodic orbits of long and short periods at each of equilibrium point.

In order that the restricted three-body problem be realistic in real situations, the classical problem was generalized by considering various aspects of perturbing forces such as the shape of the bodies, radiation, Coriolis and centrifugal forces, and so on.

As in classical R3BP, in the relativistic restricted three-body problem also, the two primaries have dominant masses and move around their center of mass; however the third body is very small and its gravitational influence on the primaries is negligible. The equations describing motion of the third body in the relativistic R3BP were originally obtained by Brumberg (1972), who used a synodic (rotating) frame of reference with the origin at the centre of mass and the primaries fixed.

The equilibrium points are the relativistic counterparts of the collinear $L_1, L_2, \textit{and } L_3$, and triangular $L_4 \textit{ and } L_5$ points.

Linear stability of the relativistic collinear points was investigated by Ragos et al. (2001) and Douskos and Perdios (2002), who showed that all these points were unstable, which is consistent with the results obtained for the non-relativistic collinear points.

In the work by Bhatnagar and Hallan (1998), linear stability of the relativistic triangular $L_4 \textit{ and } L_5$ points was studied and it was demonstrated that these points were unstable for the whole range $0 \le \mu \le \frac{1}{2}$, despite the well-known fact that the non-relativistic $L_4 \textit{ and } L_5$ are stable for $\mu < \mu_0$, where $\mu_0 = 0.038521...$ is the Routh critical mass ratio. The problem was later revisited by Douskos and Perdios (2002) and Ahmed *et al*.

(2006) who found that the relativistic triangular points are linearly stable in the range of mass ratios $0 \le \mu < \mu_r$ where $\mu_r = \mu_0 - \frac{17\sqrt{69}}{486c_d^2}$ (Douskos and Perdios, 2002), and $\mu_r = 0.03840$ (Ahmed *et al.*, 2006).

Hence due to the differences between the previous authors' results and also from our knowledge no work has been done on the stability of equilibrium points in the relativistic R3BP with perturbations, it raised a curiosity in our minds to generalize the same model problem considering the problem "Investigation of stability of equilibrium points in the relativistic relativistic restricted three-body problem with perturbations" to comfirm or refute the previous authors' results. The problem is perturbed and generalized in the sense that we considered:

1. Triaxiality of the bigger primary
2. Triaxiality of the smaller primary
3. Radiation pressures and triaxiality of the primaries
4. Triaxiality of the bigger primary and oblateness of the smaller primary
5. Small perturbations in the Coriolis and centrifugal forces with oblateness of the bigger primary
6. Small perturbations in the Coriolis and centrifugal forces with triaxial bigger primary
7. Oblateness of the smaller primary

1.4 Justification/Significance

The general theory of relativity was developed by Einstein a century ago. Since then, it has become the standard theory of gravity, especially important to the field of fundamental astrometry, astrophysics, cosmology and experimental gravitational physics. Today, the application of general relativity is also essential for many practical purposes involving astrometry, navigation, geodesy and synchronization.

The problem of computing relativistic effect in solar system is of increasing importance for study of low perihelia and low semi major axis populations. The theory of general relativity is currently the most successful gravitational theory of describing the nature of space and time, and well confirmed by observations. Especially, it has passed "classical tests", such as the deflection of light, the perihelion shift of Mercury and the Shapiro time delay, and also a systematic test using the remarkable binary pulsar "PSR" 1913+16 (Will, 1993). It is worthwhile to examine the three-body (or more generally, N-body) problem in general relativity, N-body dynamics in the general relativity gravity plays important roles in astrophysics. For instance, the formation of massive black holes in star clusters is tackled mostly by Newton N-body simulations (Zwart *et al*., 2004). In addition, future space astrometric missions such as Space Interferometry Mission (SIM) and Galactic Astometric Instrument for Astrophysics (GAIA) require a general relativistic modeling of the solar system within the accuracy of a microarc-second (Klioner, 2003).

The Equations of motion of the relativistic R3BP by including various perturbing parameters have been generalized.

The locations and stability of equilibrium points in the relativistic R3BP under the following characterizations have been also:

- When the primaries are triaxial and sources of radiation. It was found that the relativistic terms, triaxiality and radiation factors affect the locations and reduce the size of stability region.

- When the bigger primary is oblate with small perturbations in the Coriolis and centrifugal forces. It was found that the lacations are affected by the parameters involved. It was also found that the relativistic terms, radiation factors and centrifugal force all reduce the size of stability region and all have destabilizing tendencies, while the stability behavior of the Coriolis force remains unchanged.

- When the bigger primary is triaxial with small perturbations in the Coriolis and centrifugal forces. It was found that the locations and stability region are all affected by the parameters involved. All the parameters involved are found to have destabilizing tendencies except the Coriolis force.

The collinear points L_1 and L_2 are the most interesting for space mission design due to their fast instability. This instability has already been used in consuming small expenditure of fuel presumably needed to keep a spacecraft there. Lagrange's triangular points L_4 and L_5 for the Sun-Jupiter system are stable and indeed the Trojan asteroids are located there.

1.5 Aim and Objectives of the Study

The aim of this research is to investigate the existence and linear stability of the equilibrium points, together with periodic orbits around stable triangular points in the relativistic R3BP with perturbations.

The aim has been achieved through the following objectives:

i. Generalization of the model of the relativistic restricted three-body problem.
ii. Determination of the locations of the equilibrium points
iii. Examination of the stability of equilibrium points
iv. Determination of long and short periodic orbits, their eccentricities and axes around the triangular points.

1.6 Methodology

The method that is used to study the stability of equilibrium points is stated in the following sequence:

i. The subject of equilibrium points is approached by considering the problem of finding location of the points where a particle (infinitesimal mass) could be placed, with the appropriate velocity in the inertial frame, where it remains stationary in the rotating frame.
ii. In order to investigate the stability of equilibrium points we obtain the variational equations of motion using the classical approach.
iii. Then the characteristic equation and its roots are obtained. Using Lyapunov's criteria, the nature of stability of the point under consideration is determined.
iv. Subsequently, the value of the critical mass parameter is computed.
v. Finally, the region of stability is established.

1.7 Outline of the Thesis

This thesis is divided into six chapters. The first and second are the introductory chapter and the literature review respectively. The main body of the work is divided into two chapters (chapter three and four): Oblateness, radiation and triaxiality of the primaries are studied in chapter three while small perturbations in the Coriolis and centrifugal forces together with oblateness and triaxiality are studied in chapter four. Finally, the discussions of the results are given in chapter five and summary, conclusion, recommendations are given in chapter six.

CHAPTER TWO

LITERATURE REVIEW

The general three-body problem is the problem of motion of three celestial bodies under their mutual gravitational attraction. The restricted three-body is a simplified form of the general three-body problem, in which one of the bodies is of infinitesimal mass, and therefore does not influence the motion of the remaining two massive bodies called the primaries (Bruno, 1994;Valtonen & Karttunen, 2006).

The circular restricted three-body problem (CR3BP) possesses five stationary solutions called Lagrangian points. Three are collinear with the primaries and the other two are in equilateral triangular configuration with the primaries. The three collinear points $L_{1,2,3}$ are unstable, while the triangular points $L_{4,5}$ are stable for the mass ratio $\mu = \frac{m_2}{m_1 + m_2} < 0.03852..., m_1 \geq m_2$ being the masses of the primaries (Szebehely, 1967a). (Wintner,1941;Contopoulos, 2002) have shown that the stability of the triangular equilibrium points is due to the existence of the Coriolis terms in the equations of motion when these equations are written in rotating coordinate system. In the classical problem, the effects of the perturbations have been ignored. Perturbations can well arise from the causes such as from the lack of sphericity, or the triaxiality, oblateness,and radiation forces of the bodies, variation of masses, the atmospheric drag, the solar wind, Poynting Robertson effect and the action of the other bodies. The most striking example are perturbations due to oblateness in the solar system is the orbit of the fifth satellite of Jupiter, Amalthea. This planet is very oblate and the satellite's orbit is very small that its line of apsides advances about 900^0 in one year (Moulton, 1914). Such oblateness-driven effects are competing disturbing effects for qualitatively similar general relativistic effects (Iorio, 2009; Iorio *et al*., 2013;Iorio, 2006;Renzetti, 2012b).The Kirkwood gaps in the ring of the asteroid's orbits lying between the orbits of the Mars and Jupiter are examples of the perturbations produced by Jupiter on an asteroid. This enables many researchers to study the restricted problem by taking into account the effects of small perturbations in the Coriolis and the centrifugal forces ,radiation, oblateness and triaxiality of the bodies (Szebehely, 1967a; Schuerman, 1972; SubbRao and Sharma ,1975; Bhatnagar and Hallan ,1978; Bhatnagar and Hallan, 1979; Schuerman, 1980; AbdulRaheem and Singh ,2006; Oberti and Vienne, 2003; AbdulRaheem and Singh, 2008, Singh, 2011a,b; Singh and Begha, 2011;Singh, 2013; Abouelmagd, 2013). Szebehely (1967b) investigated the stability of triangular points by keeping the centrifugal force constant and found that the Coriolis force is a stabilizing force.

The bodies in the R3BP are strictly spherical in shape, but in nature, celestial bodies are not perfect spheres. They are either oblate or triaxial. The lack of sphericity, triaxiality or oblateness of the celestial bodies causes large perturbation in a two body orbit. It must be noted here that the asphericity of the primary has a number of applications in several fields of astronomical sciences (e.g testing alternative gravities) and also in fundamental physics (Renzetti, 2012a; Iorio, 2009). The asphericity issue inspired several researchers (Subbarao and Sharma, 1975; Elipe and Ferrer, 1985; El-Shaboury and El-Tantawy, 1993) to include non-spherity of the bodies in their studies of the R3BP. When one or both primaries are triaxial bodies this problem was discussed by El-Shaboury *et al*. (1991), Khanna and Bhatnagar (1999), Hallan *et al*. (2001), Sharma et al. (2001a, 2001b), Singh (2013). Khanna and Bhatnagar (1999) have discussed the stationary solutions of the planar restricted three-body problem when the smaller primary is a triaxial rigid body with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. The bigger primary is taken as an oblate spheroid and its equatorial plane coinciding with the plane of motion. They have shown that there exist five libration points, two triangular and three collinear. The collinear points are unstable, while the triangular points are stable for the mass parameter $0 \leq \mu < \mu_{crit}$ (the critical mass parameter) and the triangular points have long or short periodic elliptical orbits in the same range of μ. Sharma et al. (2001b) have discussed the existence of the libration points in the restricted three-body problem when both the primaries are triaxial rigid bodies and shown that there exist five libration points, two triangular and three collinear. Sharma and Rao (1975) investigated numerically the collinear libration points, by taking the oblateness of the primaries in consideration for 19 systems. They found that in some of the systems the shifts are significant. These equilibria are shown to be unstable in general, though the existence of conditional, infinitesimal (linearized) periodic orbits around them can be established, in the usual way. They also showed that the eccentricity and synodic period of these orbits are functions of oblateness. Numerical study, in this connection, with the above systems, revealed that the orbits around the libration point which is farthest from the primary whose oblateness effect is included, exhibit a different trend from those around the other two points. Singh and Umar (2014) investigated the motion of a test particle in the vicinity of a binary made of a triaxial primary and spherical companion moving along elliptic orbits about their common barycenter in the neighborhood of the collinear libration points. Their positions and stability are found to be affected by triaxiality of the bigger primary and by the semi-major axis and the eccentricity of the binary's orbits as well. They obtained the analytical results and applied it to binary neutron stars consisting of a bigger triaxial primary and a spherical companion.

Abouelmagd and El-Shaboury (2012) studied the existence of libration points and their linear stability when three participating bodies are axisymmetric and the primaries are radiating, they found that the collinear points remain unstable, and the triangular points are stable for region $0 \le \mu < \mu_c$; the range of stability for these points decreases. They studied periodic orbits around the triangular points and found that these orbits are elliptical; the frequencies of long and short orbits of the periodic motion are affected by terms which involve the parameters that characterize the oblateness and radiation repulsive forces; they deduced that the period of long periodic orbits adjusts with the change in its frequency while the period of short periodic orbits will decrease.

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time, and well confirmed by observations (Will,2014) . Regarding the three–body relativistic effects we may also cite: The geodesic precession of the orbit of two-body system which is about a third mass in general relativity (Renzetti, 2012a) and for the post-Newtonian tidal effects (Iorio, 2014).

In 1967, Krefetz computed the post-Newtonain deviations of the triangular Lagrangian points from their classical positions in a fixed frame of reference for the first time, but without explicitly stating the equations of motion. After a decade, Contopoulos (1976) treated the relativistic R3BP in rotating coordinates. He derived the Lagrangian of the system and the deviations of the triangular points as well.

Then, Brumberg (1972, 1991) studied the relativistic n-body problem of three bodies in more detail and collected most of the important results on relativistic celestial mechanics. He did not only obtain the equations of motion for the general problem of three bodies but also deduced the equations of motion for the restricted problem of three bodies. Maindl and Dvorak (1994) derived the equation of motion for the relativistic R3BP using the post-Newtonian approximation of relativity. They applied this model to the computation of the advance of Mercury's perihelion in the solar system and found that they are compatible with published data.

Bhatnagar and Hallan (1998) studied the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP, and found that $L_{4,5}$ are always unstable in the whole range $0 \le \mu \le \frac{1}{2}$ in contrast to the classical R3BP where they are stable for $\mu < \mu_0$, where μ is the mass ratio and $\mu_0 = 0.03852...$ is the Routh's value.

In the beginning of the 21st century, Ragos *et al*. (2001) investigated numerically the linear stability of the collinear libration points $L_{1,2,3}$ in the relativistic R3BP for several solar system cases, and found that the points $L_{1,2,3}$ are unstable. Douskos and Perdios (2002) examined the stability of the triangular points in the relativistic R3BP and contrary to the result of Bhatnagar and Hallan (1998), they obtained a region of linear stability in the parameter space as $0 \le \mu < \mu_0 - \frac{17(69)^{\frac{1}{2}}}{486 c_d{}^2}$, where $\mu_0 = 0.03852...$. is Routh's value. They also determined the positions of the collinear points and showed that they are always unstable.

Later, Wanex (2003) studied the chaotic amplification in the relativistic R3BP and noticed that the difference between Newtonian and post-Newtonian trajectories for the restricted three-body problem is greater for chaotic trajectories than it is for trajectories that are not chaotic. He also discussed the possibility of using this chaotic amplification effect as a novel test of general relativity.

Ahmed et al. (2006) investigated also the stability of the triangular points in the relativistic R3BP. In contrast to the previous result of Bhatnagar and Hallan (1998), they obtained a region of linear stability as $0 < \mu < 0.03840$, where μ_0 is the Routh's value.

There after, Abd El-Salam and Abd El-Bar(2011) derived the equations of motion of the relativistic three-body in the post-Newtonian formalism.

Yamada and Asada (2012) discussed the post-Newtonian effects on Lagrange's equilateral triangular solution for the three-body problem .For three finite masses,it is found that a triangular configuration satisfies the post-Newtonian equation of motion in general relativity ,if and only if it has the relativistic corrections to each side length .This post-Newtonian configuration for three finite masses is not always equilateral and it recovers previous results for the restricted-problem when one mass goes to zero. They also found that for the same masses and angular velocity,the post-Newtonian triangular is always smaller than the Newtonian one. Yamada and Asada (2011) investigated collinear solutions to general relativistic three-body problem. They showed that the equation determining the ratio among the three masses, which has been obtained as a seventh-order polynomial has at most three positive roots, which apparently provide three cases of the distance ratio. It is found however, that even for such cases, there exists one physically reasonable root and only one, because the remaining two positive roots do not satisfy the slow motion assumption in the post-Newtonian approximation are discarded. This means that especially for the restricted three-body problem, exactly three positions of a third body are true even at the post-Newtonian order. Abd El-Bar and Abd El-Salam (2012) investigated the relativistic effects on the equilibrium point of the relativistic R3BP. They obtained approximate locations of collinear and triangular points. Abd El-Bar and Abd El-Salam (2013) computed the locations of collinear points in the photogravitational relativistic R3BP. Series forms of these locations are obtained as new results. Lastly, Abd El-Salam and Abd El-Bar (2014) studied the photogravitational Restricted Three-Body Problem within the framework of the post-Newtonian approximation. The mass of the primaries are assumed to change under the effect of continuous radiation process. The locations of the triangular points are computed. Series forms of the locations are obtained as new analytic results.

From literature above and also to our present knowledge, no work has been done on the stability of equilibrium points on the relativistic R3BP with the perturbations, hence, it raised a curiosity in our minds to study the effects of various perturbations on the above mentioned problem .

CHAPTER THREE

OBLATENESS, RADIATION AND TRIAXIALITY OF THE PRIMARIES

3.1 Introduction

In this chapter, the equations of motion taking into consideration the oblateness, triaxiality and radiation of the primaries are presented. The locations are obtained and the study of the stability of the equilibrium points is carried out. The periodic orbits around the stable triangular points when the bigger primary is triaxial are also examined.

3.2 Triangular Points with Radiation and Triaxial Primaries

In this section, the locations and stability of triangular points when the primaries are triaxial and radiating sources are studied.

3.2.1 Equations of motion

The equations of motion of the infinitesimal mass in the relativistic restricted three-body problem in a barycentric synodic coordinate system (ξ, η) with origin at the centre of mass of the primaries are given by Brumberg (1972) and Bhatnagar and Hallan (1998) and Ragos *et al*. (2001):

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$

$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \tag{3.1}$$

where,

$$W=\frac{1}{2}n^2\left(\xi^2+\eta^2\right)+\gamma\left(\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}\right)+\frac{1}{c^2}\left[\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2n(\xi\dot{\eta}-\eta\dot{\xi})+n^2\left(\xi^2+\eta^2\right)\right\}^2+\frac{3\gamma}{2}\left(\frac{m_1}{\rho_1}+\frac{m_2}{\rho_2}\right)\right.$$

$$\left\{\dot{\xi}^2+\dot{\eta}^2+2n(\xi\dot{\eta}-\eta\dot{\xi})+n^2\left(\xi^2+\eta^2\right)\right\}-\frac{\gamma^2}{2}\left(\frac{m_1^2}{\rho_1^2}+\frac{m_2^2}{\rho_2^2}\right)+\frac{\gamma\, m_1m_2}{M}\left\{na\left(4\dot{\eta}+\frac{7}{2}n\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right.$$

$$\left.\left.-\frac{n^2a^2\eta^2}{2M}\left(\frac{m_2}{\rho_1^3}+\frac{m_1}{\rho_2^3}\right)+n^2a^2\left(-\frac{a}{\rho_1\rho_2}+\frac{m_2-2m_1}{2M\rho_1}+\frac{m_1-2m_2}{2M\rho_2}\right)\right\}\right] \tag{3.2}$$

$\gamma =$ constant of gravitation

$$n=\text{mean motion}=\left\{1-\frac{3\gamma M}{2c^2a}\left(1-\frac{m_1m_2}{3M^2}\right)\right\}\frac{\sqrt{\gamma M}}{a^{\frac{3}{2}}} \tag{3.3}$$

$m_1, m_2 =$ masses of the primaries

$\rho_1, \rho_2 =$ distances of the infinitesimal mass from the primaries

$c =$ velocity of light

$M = m_1 + m_2$

$a =$ distance between the primaries.

The above mentioned dimensional quantities are transformed to dimensionless ones when the units of mass, length and time are chosen such that $M=1$, $a=1$ and $\gamma=1$ respectively.

Then, if we denote by $\mu, 0<\mu\le\frac{1}{2}$, the mass of the less massive primary, the mass of the other primary is equal to $1-\mu$ and their coordinates are $(1-\mu,0)$ and $(-\mu,0)$ respectively.

In dimensionless system, the equations of motion given by (3.1) remain the same, but the values of W and n given in equation (3.2) and equation (3.3) become respectively

$$W=\frac{1}{2}\left(\xi^2+\eta^2\right)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}+\frac{1}{c_d^2}\left[-\frac{3}{2}\left(1-\frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2)+\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})\right.\right.$$

$$\left.+\xi^2+\eta^2\right\}^2+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\left(\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+\xi^2+\eta^2\right)-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right)$$

$$\left.+\mu(1-\mu)\left\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)-\frac{\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\left(-\frac{1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\right\}\right] \tag{3.4}$$

and

$$n_d=1-\frac{3}{2{c_d}^2}\left(1-\frac{(1-\mu)\mu}{3}\right),\quad c_d=\frac{c}{\sqrt{\frac{\gamma(m_1+m_2)}{a}}}, \qquad \text{(Ragos } et\ al\text{, 2001)} \tag{3.5}$$

where,

$$\rho_1^2=(\xi+\mu)^2+\eta^2$$

$$\rho_2^2=(\xi+\mu-1)^2+\eta^2 \tag{3.6}$$

The triaxiality and radiation factors of the bigger and smaller primaries are included with the help of the parameters $\sigma_i,\ \sigma_i'\ ,\delta_i\ (i=1,2)$, respectively.

The pertinent equations of motion of an infinitesimal mass in the relativistic R3BP in a barycentric synodic coordinate system (ξ,η) and dimensionless variables can be written as:

$$\ddot{\xi}-2n_d\dot{\eta}=\frac{\partial W}{\partial\xi}-\frac{d}{dt}\left(\frac{\partial W}{\partial\dot{\xi}}\right)$$

$$\ddot{\eta}+2n_d\dot{\xi}=\frac{\partial W}{\partial\eta}-\frac{d}{dt}\left(\frac{\partial W}{\partial\dot{\eta}}\right) \tag{3.7}$$

with

$$W=\frac{1}{2}\left\{1+\frac{3}{2}\left(2\sigma_1-\sigma_2\right)+\frac{3}{2}\left(2\sigma_1'-\sigma_2'\right)\right\}(\xi^2+\eta^2)+\frac{q_1(1-\mu)}{\rho_1}+\frac{q_2\mu}{\rho_2}+\frac{q_1(1-\mu)\left(2\sigma_1-\sigma_2\right)}{2\rho_1^3}$$
$$+\frac{3q_1(1-\mu)\left(2\sigma_1-\sigma_2\right)\eta^2}{2\rho_1^5}+\frac{q_2\mu\left(2\sigma_1'-\sigma_2'\right)}{2\rho_2^3}+\frac{3q_2\mu\left(2\sigma_1'-\sigma_2'\right)\eta^2}{2\rho_2^5}+\frac{1}{{c_d}^2}\left[-\frac{3}{2}\left(1-\frac{1}{3}\mu(1-\mu)\right)(\xi^2+\eta^2)\right.$$
$$+\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\right\}^2+\frac{3}{2}\left(\frac{q_1(1-\mu)}{\rho_1}+\frac{q_2\mu}{\rho_2}\right)\left(\dot{\xi}^2+\dot{\eta}^2+2(\xi\dot{\eta}-\eta\dot{\xi})+(\xi^2+\eta^2)\right)$$
$$-\frac{1}{2}\left(\frac{q_1^2(1-\mu)^2}{\rho_1^2}+\frac{q_2^2\mu^2}{\rho_2^2}\right)+q_1q_2(1-\mu)\mu\left\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)-\frac{\eta^2}{2}\left(\frac{q_2\mu}{\rho_1^3}+\frac{q_1(1-\mu)}{\rho_2^3}\right)+\left(\frac{-1}{\rho_1\rho_2}\right.\right.$$
$$\left.\left.\left.+\frac{q_2\mu-2q_1(1-\mu)}{2\rho_1}+\frac{q_1(1-\mu)-2q_2\mu}{2\rho_2}\right)\right\}\right]$$

(3.8)

$$n_d=1+\frac{3}{4}\left(2\sigma_1-\sigma_2\right)+\frac{3}{4}\left(2\sigma_1'-\sigma_2'\right)-\frac{3}{2{c_d}^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \quad (3.9)$$

$$\rho_1^2=(\xi+\mu)^2+\eta^2$$
$$\rho_2^2=(\xi+\mu-1)^2+\eta^2 \quad (3.10)$$

where $0<\mu\leq\frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; n_d is the mean motion of the primaries; c_d is the dimensionless speed of light.

$\sigma_1=\frac{h^2-f^2}{5R^2}$, $\sigma_2=\frac{b^2-f^2}{5R^2}$, $\sigma_1'=\frac{h'^2-f'^2}{5R^2}$, $\sigma_2'=\frac{b'^2-f'^2}{5R^2}$, (McCuskey, 1963) and $\sigma_i<<1,\ \sigma_i'<<1\ (i=1,2)$ characterize the triaxiality of the bigger and smaller primary with h,b,f as lengths of the semi-axes of the bigger primary and h',b',f' as those of the smaller primary. The radiation factor $q_i(i=1,2)$ is given by $F_{pi}=F_{gi}(1-q_i)$ such that $0\leq(1-q_i)<<1$ (Radzievskii, 1950), where F_{gi} and F_{pi} are respectively the gravitational and radiation pressure.

Here as Katour *et al*. (2014), the parameters $\sigma_i,\ \sigma_1'\ \ (i=1,2)$ are not included in the relativistic part of W since the magnitude of these terms is very small due to ${c_d}^{-2}$.

3.2.2 Locations of the triangular points

The libration points are obtained from equation (3.7) after putting $\dot{\xi} = \dot{\eta} = \ddot{\xi} = \ddot{\eta} = 0.$

These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta} \text{ with } \dot{\xi} = \dot{\eta} = 0.$$

This is equivalent to.

$$\xi - \frac{q_1(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{q_2\mu(\xi-1+\mu)}{\rho_2^3} + \left\{3\left(\sigma_1+\sigma_1'\right) - \frac{3}{2}\left(\sigma_2+\sigma_2'\right)\right\}\xi - \frac{3q_1(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2\rho_1^5}$$
$$-\frac{15q_1(1-\mu)(\xi+\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7} - \frac{3q_2\mu(\xi-1+\mu)(2\sigma_1'-\sigma_2')}{2\rho_2^5} - \frac{15q_2\mu(\xi-1+\mu)(\sigma_2'-\sigma_1')\eta^2}{2\rho_2^7}$$
$$+\frac{1}{c_d^{\,2}}\left[-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\} + \frac{1}{2}\xi(\xi^2+\eta^2) - \frac{3}{2}(\xi^2+\eta^2)\left\{\frac{q_1(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{q_2\mu(\xi-1+\mu)}{\rho_2^3}\right\}\right.$$
$$+3\left(\frac{q_1(1-\mu)}{\rho_1} + \frac{q_2\mu}{\rho_2}\right)\xi + \frac{q_1^2(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{q_2^2\mu^2(\xi-1+\mu)}{\rho_2^4} + q_1q_2\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)\right.$$
$$+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) + \frac{3}{2}\eta^2\left(\frac{q_2\mu(\xi+\mu)}{\rho_1^5} + \frac{q_1(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) + \frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3}$$
$$\left.\left.-\frac{\left(q_2\mu-2q_1(1-\mu)\right)(\xi+\mu)}{2\rho_1^3} - \frac{\left(q_1(1-\mu)-2q_2\mu\right)(\xi-1+\mu)}{2\rho_2^3}\right\}\right] = 0$$

and (3.11)

$$\eta F = 0,$$

with

$$F = \left(1 - \frac{q_1(1-\mu)}{\rho_1^3} - \frac{q_2\mu}{\rho_2^3}\right) + 3\left(\sigma_1+\sigma_1'\right) - \frac{3}{2}\left(\sigma_2+\sigma_2'\right) + \frac{3q_1(1-\mu)}{\rho_1^5}\left(\frac{3}{2}\sigma_2 - 2\sigma_1\right) - \frac{15q_1(1-\mu)\left(\sigma_2-\sigma_1\right)\eta^2}{2\rho_1^7}$$
$$+\frac{3q_2\mu}{\rho_2^5}\left(\frac{3}{2}\sigma_2' - 2\sigma_1'\right) - \frac{15q_2\mu\left(\sigma_2'-\sigma_1'\right)\eta^2}{2\rho_2^7} + \frac{1}{c_d^{\,2}}\left[-3\left(1-\frac{\mu(1-\mu)}{3}\right) + \frac{1}{2}(\xi^2+\eta^2) + 3\left(\frac{q_1(1-\mu)}{\rho_1} + \frac{q_2\mu}{\rho_2}\right)\right.$$
$$-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{q_1(1-\mu)}{\rho_1^3} + \frac{q_2\mu}{\rho_2^3}\right) + \frac{q_1^2(1-\mu)^2}{\rho_1^4} + \frac{q_2^2\mu^2}{\rho_2^4} + q_1q_2\mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \left(\frac{q_2\mu}{\rho_1^3} + \frac{q_1(1-\mu)}{\rho_2^3}\right)\right.$$
$$\left.\left.+\frac{3}{2}\eta^2\left(\frac{q_2\mu}{\rho_1^5} + \frac{q_1(1-\mu)}{\rho_2^5}\right) + \left(\frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{\left(q_2\mu-2q_1(1-\mu)\right)}{2\rho_1^3} - \frac{\left(q_1(1-\mu)-2q_2\mu\right)}{2\rho_2^3}\right)\right\}\right]$$

The triangular points are the solutions of the system (3.11) with $\eta \neq 0$. Since $\frac{1}{{c_d}^2} << 1$ and in the case $\frac{1}{{c_d}^2} \to 0$ and in the absence of triaxiality and radiation (i.e. $\sigma_i = \sigma_i' = \delta_i = 0, i = 1, 2$), one can obtain $\rho_1 = \rho_2 = 1$, and we assume in the relativistic R3BP with radiation pressure and triaxiality that $\rho_1 = 1 + x$ and $\rho_2 = 1 + y$ where $x, y << 1$ may be depending upon the radiation, triaxiality, and relativistic terms. Substituting these values in equation (3.10), solving them for ξ, η, and ignoring terms of second and higher powers of x and y, the following solutions are obtained as

$$\xi = x - y + \frac{1-2\mu}{2},$$

$$\eta = \pm\left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}}\right). \tag{3.12}$$

Now substituting the values $\rho_1, \rho_2, \xi, \eta$ in equations (3.11) with $\eta \neq 0$ and for simplicity, putting $q_i = 1 - (1 - q_i) = 1 - \delta_i$ and neglecting second and higher terms in $x, y, \frac{1}{{c_d}^2}, \sigma_i, \sigma_i', \delta_i$ (i=1,2) and their products, the following system is obtained as

$$\left\{\frac{3}{2}(1-\mu) + \frac{\delta_1}{2}(\mu-1) + \delta_2\mu\right\}x + \left\{\frac{3\mu}{2} + \delta_1(1-\mu) - \frac{\delta_2\mu}{2}\right\}y + \frac{\delta_1(1-\mu)}{2} - \frac{\delta_2\mu}{2} + \frac{(57\sigma_2 - 69\sigma_1)}{16}\mu$$
$$+\frac{45}{16}(\sigma_1 - \sigma_2) + \frac{(57\sigma_2' - 69\sigma_1')}{16}\mu + \frac{3}{4}(2\sigma_1' - \sigma_2') + \frac{1}{{c_d}^2}\left(-\frac{9\mu}{16} + \frac{27\mu^2}{16} - \frac{9\mu^3}{8}\right) = 0$$

(3.13)

$$\{3(1-\mu) + 3(\mu-1)\delta_1\}x + \{3\mu - 3\delta_2\mu\}y + (1-\mu)\delta_1 + \delta_2\mu + \frac{3}{8}(\sigma_1 + 3\sigma_2)\mu + \frac{21}{8}(\sigma_1 + \sigma_2)$$
$$-\frac{3}{2}(\sigma_1' + 3\sigma_2')\mu + \frac{3}{2}(2\sigma_1' - \sigma_2') + \frac{1}{{c_d}^2}\left\{\frac{21}{8}\mu(1-\mu)\right\} = 0$$

Solving these equations for x and y, the following solutions are obtained as

$$x = -\frac{\mu(2+3\mu)}{8{c_d}^2} - \frac{\delta_1}{3} + \frac{11}{8}(\sigma_2 - \sigma_1) - \left(1 - \frac{\mu}{2(1-\mu)}\sigma_1'\right) + \frac{1}{2}\left(1 - \frac{\mu}{(1-\mu)}\right)\sigma_2'$$

$$y = -\frac{(1-\mu)(5-3\mu)}{8{c_d}^2} - \frac{\delta_2}{3} + \left(\frac{1}{2\mu} - \frac{3}{2}\right)\sigma_1 + \left(-\frac{1}{2\mu} + 1\right)\sigma_2 - \frac{11}{8}\sigma_1' + \frac{11}{8}\sigma_2' \qquad (3.14)$$

Thus , the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 ,respectively,are

$$\xi = \frac{1-2\mu}{2}\left(1 + \frac{5}{4{c_d}^2}\right) + \left(\frac{1}{8} - \frac{1}{2\mu}\right)\sigma_1 - \left(\frac{1}{2\mu} + \frac{3}{8}\right)\sigma_2 + \left(\frac{3}{8} - \frac{\mu}{2(1-\mu)}\right)\sigma_1'$$
$$-\left(\frac{\mu}{2(1-\mu)} + \frac{7}{8}\right)\sigma_2' - \frac{1}{3}(\delta_2 - \delta_1)$$

$$\eta = \pm\frac{\sqrt{3}}{2}\left[1 + \frac{1}{12{c_d}^2}\left(-5 + 6\mu - 6\mu^2\right) + \frac{2}{3}\left\{\left(-\frac{23}{8} + \frac{1}{2\mu}\right)\sigma_1 + \left(\frac{19}{8} - \frac{1}{2\mu}\right)\sigma_2\right\}\right.$$
$$\left. + \frac{2}{3}\left\{\left(-\frac{19}{8} + \frac{\mu}{2(1-\mu)}\right)\sigma_1' + \left(\frac{15}{8} - \frac{\mu}{2(1-\mu)}\right)\sigma_2'\right\} - \frac{2}{9}(\delta_2 + \delta_1)\right]$$

(3.15)

3.2.3 Stability of $L_{4,5}$

3.2.3.1 The Variational equations

In order to study the motion near any of the equilibrium points $L(\xi_0, \eta_0, \dot{\xi}_0, \dot{\eta}_0) = L(a, b, 0, 0)$, we may write $\xi = a + \alpha, \eta = b + \beta, (\alpha, \beta << 1)$.

The function W may be expanded in a Taylor series around L giving

$$W = W^0 + W_\xi^0\alpha + W_\eta^0\beta + W_{\dot{\xi}}^0\dot{\alpha} + W_{\dot{\eta}}^0\dot{\beta} + \frac{1}{2!}\left\{W_{\xi\xi}^0\alpha^2 + W_{\eta\eta}^0\beta^2 + W_{\dot{\xi}\dot{\xi}}^0\dot{\alpha}^2 + W_{\dot{\eta}\dot{\eta}}^0\dot{\beta}^2 + 2W_{\xi\eta}^0\alpha\beta + W_{\xi\dot{\xi}}^0\alpha\dot{\alpha}\right.$$
$$\left. + W_{\xi\dot{\eta}}^0\alpha\dot{\beta} + W_{\eta\dot{\eta}}^0\beta\dot{\beta} + W_{\dot{\xi}\dot{\eta}}^0\dot{\alpha}\dot{\beta} + W_{\eta\dot{\xi}}^0\beta\dot{\alpha}\right\} + \ldots$$

(3.16)

The superscript 0 means that these derivatives are calculated at the equilibrium point $(\xi_0, \eta_0, 0, 0)$.

Then, the following equations are obtained

$$W_{\xi} = W^0_{\xi\xi}\alpha + W^0_{\xi\eta}\beta + W^0_{\xi\dot{\xi}}\dot{\alpha} + W^0_{\xi\dot{\eta}}\dot{\beta} \qquad = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta} \tag{3.17}$$

where, $A = W^0_{\xi\xi}, B = W^0_{\xi\eta}, C = W^0_{\xi\dot{\xi}}, D = W^0_{\xi\dot{\eta}}$

$$W_{\dot{\xi}} = W^0_{\dot{\xi}} + W^0_{\dot{\xi}\dot{\xi}}\dot{\alpha} + W^0_{\xi\dot{\xi}}\alpha + W^0_{\eta\dot{\xi}}\beta + W^0_{\dot{\eta}\dot{\xi}}\dot{\beta} \tag{3.18}$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right) = W^0_{\dot{\xi}\dot{\xi}}\ddot{\alpha} + W^0_{\xi\dot{\xi}}\dot{\alpha} + W^0_{\eta\dot{\xi}}\dot{\beta} + W^0_{\dot{\xi}\dot{\eta}}\ddot{\beta} = F\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta} \tag{3.19}$$

where,

$F = W^0_{\xi\dot{\xi}}, B_2 = W^0_{\eta\dot{\xi}}, C_2 = W^0_{\dot{\xi}\dot{\xi}}, D_2 = W^0_{\dot{\xi}\dot{\eta}}$

$$W_{\eta} = W^0_{\eta\eta}\beta + W^0_{\xi\eta}\alpha + W^0_{\eta\dot{\eta}}\dot{\beta} + W^0_{\eta\dot{\xi}}\dot{\alpha} = E\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta} \tag{3.20}$$

where,

$E = W^0_{\xi\eta}, B_1 = W^0_{\eta\eta}, C_1 = W^0_{\eta\dot{\xi}}, D_1 = W^0_{\eta\dot{\eta}}$

$$W_{\dot{\eta}} = W^0_{\dot{\eta}} + W^0_{\dot{\eta}\dot{\eta}}\dot{\beta} + W^0_{\xi\dot{\eta}}\alpha + W^0_{\eta\dot{\eta}}\beta + W^0_{\dot{\xi}\dot{\eta}}\dot{\alpha} \tag{3.21}$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) = W^0_{\dot{\eta}\dot{\eta}}\ddot{\beta} + W^0_{\xi\dot{\eta}}\dot{\alpha} + W^0_{\eta\dot{\eta}}\dot{\beta} + W^0_{\dot{\xi}\dot{\eta}}\ddot{\alpha} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta} \tag{3.22}$$

where,

$A_3 = W^0_{\xi\dot{\eta}}, B_3 = W^0_{\eta\dot{\eta}}, C_3 = W^0_{\dot{\xi}\dot{\eta}}, D_3 = W^0_{\dot{\eta}\dot{\eta}}$

Since the nature of the linear stability about the point L_5 will be similar to that about L_4, it will be sufficient to consider here the stability only near L_4.

Let (a,b) be the coordinates of the triangular point L_4

In equations (3.7), we set $\xi = a + \alpha, \eta = b + \beta, (\alpha, \beta << 1)$.

First, the terms of the R.H.S of the above equations are computed, neglecting second and higher order terms, using equation (3.17), the following expression is obtained as

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta} \tag{3.23}$$

where,

$$A = \frac{3}{4}\left\{1 + \frac{1}{2c_d^{\,2}}\left(2 - 19\mu + 19\mu^2\right)\right\} + \frac{3\left(15\mu^2 + 19\mu - 8\right)}{16\mu}\sigma_1 - \frac{3\left(31\mu^2 + \mu - 8\right)}{16\mu}\sigma_2$$
$$+ \frac{3\left(15\mu^2 - 49\mu + 26\right)}{16(1-\mu)}\sigma_1' - \frac{3\left(31\mu^2 - 63\mu + 24\right)}{16(1-\mu)}\sigma_2' + \frac{1}{2}(3\mu - 1)\delta_1 - \left(\frac{3\mu}{2} - 1\right)\delta_2,$$

$$B = \frac{3\sqrt{3}}{4}(1-2\mu)\left(1 - \frac{2}{3c_d^{\,2}}\right) - \frac{\sqrt{3}\left(89\mu^2 - 47\mu + 8\right)}{16\mu}\sigma_1 + \frac{\sqrt{3}\left(37\mu^2 - 9\mu + 8\right)}{16\mu}\sigma_2$$
$$+ \frac{\sqrt{3}\left(89\mu^2 - 131\mu + 50\right)}{16(1-\mu)}\sigma_1' - \frac{\sqrt{3}\left(37\mu^2 - 65\mu + 36\right)}{16(1-\mu)}\sigma_2' - \frac{\sqrt{3}}{6}(1+\mu)\delta_1 + \frac{\sqrt{3}}{6}(2-\mu)\delta_2,$$

$$C = \frac{\sqrt{3}}{2c_d^{\,2}}(1-2\mu),$$

$$D = \frac{6 - 5\mu + 5\mu^2}{2c_d^{\,2}}.$$

Similarly, using equation (3.20), the following equation is written as

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = E\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta} \tag{3.24}$$

where,

$$E = \frac{3\sqrt{3}}{4}(1-2\mu)\left(1 - \frac{2}{3c_d^{\,2}}\right) - \frac{\sqrt{3}\left(89\mu^2 - 47\mu + 8\right)}{16\mu}\sigma_1 + \frac{\sqrt{3}\left(37\mu^2 - 9\mu + 8\right)}{16\mu}\sigma_2$$
$$+ \frac{\sqrt{3}\left(89\mu^2 - 131\mu + 50\right)}{16(1-\mu)}\sigma_1' - \frac{\sqrt{3}\left(37\mu^2 - 65\mu + 36\right)}{16(1-\mu)}\sigma_2' - \frac{\sqrt{3}}{6}(1+\mu)\delta_1 + \frac{\sqrt{3}}{6}(2-\mu)\delta_2,$$

$$B_1 = \frac{9}{4}\left\{1 + \frac{7}{6{c_d}^2}\left(-2 + 3\mu - 3\mu^2\right)\right\} - \frac{3\left(15\mu^2 - 29\mu - 8\right)}{16\mu}\sigma_1 + \frac{3\left(15\mu^2 - 7\mu - 8\right)}{16\mu}\sigma_2$$
$$- \frac{3\left(15\mu^2 - \mu - 22\right)}{16(1-\mu)}\sigma_1' + \frac{3\mu\left(15\mu - 23\right)}{16\mu}\sigma_2' + \frac{1}{2}(1-3\mu)\delta_1 + \left(\frac{3\mu}{2} - 1\right)\delta_2,$$

$$C_1 = \frac{1}{2{c_d}^2}\left(-4 + \mu - \mu^2\right),$$

$$D_1 = -\frac{\sqrt{3}\left(1-2\mu\right)}{2{c_d}^2}.$$

Using equation (3.19), the following equation is written as

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha,\ \eta=b+\beta} = F\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta} \tag{3.25}$$

where,

$$F = \frac{\sqrt{3}}{2{c_d}^2}\left(1-2\mu\right),$$

$$B_2 = -\frac{1}{2{c_d}^2}\left(4 - \mu + \mu^2\right),$$

$$C_2 = \frac{1}{4{c_d}^2}\left(17 - 2\mu + 2\mu^2\right),$$

$$D_2 = -\frac{\sqrt{3}}{4{c_d}^2}\left(1-2\mu\right).$$

Similarly , using equation (3.22) the following equation is written as

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha,\eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta} \tag{3.26}$$

where,

$$A_3 = \frac{1}{2c_d{}^2}\left(6 - 5\mu + 5\mu^2\right),$$

$$B_3 = -\frac{\sqrt{3}}{2c_d{}^2}\left(1 - 2\mu\right),$$

$$C_3 = -\frac{\sqrt{3}}{4c_d{}^2}\left(1 - 2\mu\right),$$

$$D_3 = \frac{3\left(5 - 2\mu + 2\mu^2\right)}{4c_d{}^2}.$$

Thus the variational equations of motion corresponding to equations (3.7), on utilizing equation (3.9), can be written as:

$$p_1\ddot{\alpha} + p_2\ddot{\beta} + p_3\dot{\alpha} + p_4\dot{\beta} + p_5\alpha + p_6\beta = 0, \tag{3.27}$$

$$q_1\ddot{\alpha} + q_2\ddot{\beta} + q_3\dot{\alpha} + q_4\dot{\beta} + q_5\alpha + q_6\beta = 0.$$

where,

$$p_1 = 1 + C_2,\, p_2 = D_2,\, p_3 = F - C$$
$$p_4 = \left\{B_2 - 2\left(1 + \frac{3}{4}\left(2\sigma_1 - \sigma_2\right) + \frac{3}{4}\left(2\sigma_1{}' - \sigma_2{}'\right) - \frac{3}{2c_d{}^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\right) - D\right\},$$
$$p_5 = -A, P_6 = -B$$

$$q_1 = C_3, q_2 = 1 + D_3,\; q_3 = 2\left(1 + \frac{3}{4}\left(2\sigma_1 - \sigma_2\right) + \frac{3}{4}\left(2\sigma_1{}' - \sigma_2'\right) - \frac{3}{2c_d{}^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\right)$$
$$-C_1 + A_3,\, q_4 = B_3 - D_1,$$
$$q_5 = -E, q_6 = -B_1$$

3.2.3.2 Characteristic equation

The system (3.27) can be written in matrix form as follows:

Let

$$\begin{aligned} \alpha_1 &= \alpha \\ \alpha_2 &= \dot{\alpha} \\ \beta_1 &= \beta \\ \beta_2 &= \dot{\beta} \end{aligned} \tag{3.28}$$

Substitute equation (3.28) in equation (3.27), to obtain

$$p_1\dot{\alpha}_2 + p_2\dot{\beta}_2 + p_3\alpha_2 + p_4\beta_2 + p_5\alpha_1 + p_6\beta_1 = 0, \tag{3.29}$$

$$q_1\dot{\alpha}_2 + q_2\dot{\beta}_2 + q_3\alpha_2 + q_4\beta_2 + q_5\alpha_1 + q_6\beta_1 = 0.$$

Solving for derivatives for $\alpha_1, \alpha_2, \beta_1, \beta_2$ and noting that $p_3 = q_4 = 0$, the following system is obtained as

$$\dot{\alpha}_2 = -\frac{p_4q_2}{p_1q_2 - q_1p_2}\beta_2 + \frac{p_2q_3}{p_1q_2 - q_1p_2}\alpha_2 + \frac{p_2q_5 - p_5q_2}{p_1q_2 - q_1p_2}\alpha_1 + \frac{p_2q_6 - p_6q_2}{p_1q_2 - q_1p_2}\beta_1$$

$$\begin{aligned} \dot{\beta}_2 &= \frac{q_1p_4}{p_1q_2 - q_1p_2}\beta_2 - \frac{p_1q_3}{p_1q_2 - q_1p_2}\alpha_2 + \frac{q_1p_5 - p_1q_5}{p_1q_2 - q_1p_2}\alpha_1 + \frac{q_1p_6 - p_1q_6}{p_1q_2 - q_1p_2}\beta_1 \\ \dot{\alpha}_1 &= \alpha_2 \\ \dot{\beta}_1 &= \beta_2 \end{aligned} \tag{3.30}$$

The system in equation (3.30) corresponds to matrix form:

$$\begin{bmatrix} \dot{\alpha}_2 \\ \dot{\beta}_2 \\ \dot{\alpha}_1 \\ \dot{\beta}_1 \end{bmatrix} = \begin{bmatrix} \frac{p_2q_3}{p_1q_2 - q_1p_2} & -\frac{p_4q_2}{p_1q_2 - q_1p_2} & \frac{p_2q_5 - p_5q_2}{p_1q_2 - q_1p_2} & \frac{p_2q_6 - p_6q_2}{p_1q_2 - q_1p_2} \\ -\frac{p_1q_3}{p_1q_2 - q_1p_2} & \frac{q_1p_4}{p_1q_2 - q_1p_2} & \frac{q_1p_5 - p_1q_5}{p_1q_2 - q_1p_2} & \frac{q_1p_6 - p_1q_6}{p_1q_2 - q_1p_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \alpha_1 \\ \beta_1 \end{bmatrix} \tag{3.31}$$

The characteristic equation of the system is given as:

$$\begin{vmatrix} \frac{p_2q_3}{p_1q_2-q_1p_2}-\lambda & -\frac{p_4q_2}{p_1q_2-q_1p_2} & \frac{p_2q_5-p_5q_2}{p_1q_2-q_1p_2} & \frac{p_2q_6-p_6q_2}{p_1q_2-q_1p_2} \\ -\frac{p_1q_3}{p_1q_2-q_1p_2} & \frac{q_1p_4}{p_1q_2-q_1p_2}-\lambda & \frac{q_1p_5-p_1q_5}{p_1q_2-q_1p_2} & \frac{q_1p_6-p_1q_6}{p_1q_2-q_1p_2} \\ 1 & 0 & 0-\lambda & 0 \\ 0 & 1 & 0 & 0-\lambda \end{vmatrix}=0 \tag{3.32}$$

It is important to note that

$$q_1 = p_2, q_3 = -p_4, q_5 = p_6$$

Hence the characteristic equation can be written in the form

$$(p_1q_2 - p_2q_1)\lambda^4 + (p_1q_6 + p_5q_2 - p_6q_1 - p_2q_5 - p_4q_3)\lambda^2 + p_5q_6 - p_6q_5 = 0 \tag{3.33}$$

Substituting the values of $p_i, q_i, i = 1, 2..., 6$ and neglecting second and higher powers of small quantities, the characteristic equation of the variational equations of motion corresponding to equation (3.7) can be expressed as:

$$\lambda^4 + b\lambda^2 + d = 0 \tag{3.34}$$

where,

$$b = \left(1 - \frac{9}{{c_d}^2}\right) + 3\sigma_1 + \frac{3}{2}(2\mu - 3)\sigma_2 + 3\sigma_1' + \left(3\mu + \frac{3}{2}\right)\sigma_2'$$

$$\begin{aligned} d = {} & \frac{27\mu(1-\mu)}{4} + \frac{9\mu\left(-65 + 77\mu - 24\mu^2 + 12\mu^3\right)}{8{c_d}^2} + \frac{9\left(-89\mu^2 + 99\mu - 10\right)}{16}\sigma_1 \\ & + \frac{9\left(37\mu^2 - 47\mu + 10\right)}{16}\sigma_2 - \frac{9\mu\left(89\mu - 79\right)}{16}\sigma_1' + \frac{9\mu\left(37\mu - 27\right)}{16}\sigma_2' \\ & + \frac{3}{2}\mu(\delta_1 + \delta_2) - \frac{3}{2}\mu^2(\delta_1 + \delta_2) \end{aligned}$$

For $\frac{1}{c_d{}^2} \to 0$ and when the primaries are non-luminous and spherical ($i.e.$ $\delta_1 = \delta_2 = \sigma_1 = \sigma_2 = \sigma_1' = \sigma_2' = 0$), equation (3.34) reduces to its well-known classical restricted problem form (see e.g. Szebehely, 1967a):

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0.$$

The discriminant of equation (3.34) is

$$\Delta = \frac{-54}{c_d{}^2}\mu^4 + \frac{108}{c_d{}^2}\mu^3 + \left(27 + 6\delta_1 + 6\delta_2 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 + \frac{801}{4}\sigma_1' - \frac{333}{4}\sigma_2' - \frac{693}{2c_d{}^2}\right)\mu^2$$
$$+\left(-27 - 6\delta_1 - 6\delta_2 - \frac{891}{4}\sigma_1 + \frac{447}{4}\sigma_2 - \frac{771}{4}\sigma_1' + \frac{219}{4}\sigma_2' + \frac{585}{2c_d^2}\right)\mu + 1 + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2$$
$$+ 6\sigma_1' - 3\sigma_2' - \frac{18}{c_d^2}$$

(3.35)

The roots of equation (3.34) are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \tag{3.36}$$

where,

$$b = \left(1 - \frac{9}{c_d{}^2}\right) + 3\sigma_1 + \frac{3}{2}(2\mu - 3)\sigma_2 + 3\sigma_1' + \left(3\mu + \frac{3}{2}\right)\sigma_2'$$

From (3.35), the followings are obtained

$$\frac{d\Delta}{d\mu} = \frac{-216}{c_d{}^2}\mu^3 + \frac{324}{c_d{}^2}\mu^2 + 2\left(27 + 6\delta_1 + 6\delta_2 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 + \frac{801}{4}\sigma_1' - \frac{333}{4}\sigma_2' - \frac{693}{2c_d{}^2}\right)\mu$$
$$+\left(-27 - 6\delta_1 - 6\delta_2 - \frac{891}{4}\sigma_1 + \frac{447}{4}\sigma_2 - \frac{771}{4}\sigma_1' + \frac{219}{4}\sigma_2' + \frac{585}{2c_d{}^2}\right) < 0 \;\; \forall \;\; \mu \in \left(0, \frac{1}{2}\right].$$

(3.37)

$$\frac{d^2\Delta}{d\mu^2} = -\frac{648}{c_d^{\ 2}}\mu^2 + \frac{648}{c_d^{\ 2}}\mu + 2\left(27 + 6\delta_1 + 6\delta_2 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 + \frac{801}{4}\sigma_1' - \frac{333}{4}\sigma_2' - \frac{693}{3c_d^{\ 2}}\right) > 0$$

$$\forall \ \mu \in \left(0, \frac{1}{2}\right] \tag{3.38}$$

This implies that $\frac{d\Delta}{d\mu}$ is monotonic increasing in $\left(0, \frac{1}{2}\right]$

But

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=0} = -27 - 6\delta_1 - 6\delta_2 - \frac{891}{4}\sigma_1 + \frac{447}{4}\sigma_2 - \frac{771}{4}\sigma_1' + \frac{219}{4}\sigma_2' + \frac{585}{2c_d^{\ 2}} > 0 \ \ \forall \ \ \mu \in \left(0, \frac{1}{2}\right] \tag{3.39}$$

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} = -\frac{45}{2}\sigma_1 + \frac{57}{2}\sigma_2 + \frac{15}{2}\sigma_1' - \frac{57}{2}\sigma_2' \tag{3.40}$$

and

$$(\Delta)_{\mu=0} = 1 + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2 + 6\sigma_1' \ - 3\sigma_2' \ - \frac{18}{c_d^{\ 2}} > 0$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4} - \frac{525}{16}\sigma_1 + \frac{57}{16}\sigma_2 - \frac{525}{16}\sigma_1' \ + \frac{57}{16}\sigma_2' \ - \frac{3}{2}(\delta_1 + \delta_2) + \frac{207}{4c_d^{\ 2}} < 0 \tag{3.41}$$

In order to study the monotonicity of Δ, two cases are considered:

Case 1: $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} \le 0$

Fot this case, the table of variation of Δ is given in table below

Table 3.1: Variation of Δ

μ	0	1/2
$\frac{d^2\Delta}{d\mu^2}$	+	
$\frac{d\Delta}{d\mu}$	−	
Δ	$(\Delta)_{\mu=0}$ ↘	$(\Delta)_{\mu=1/2}$

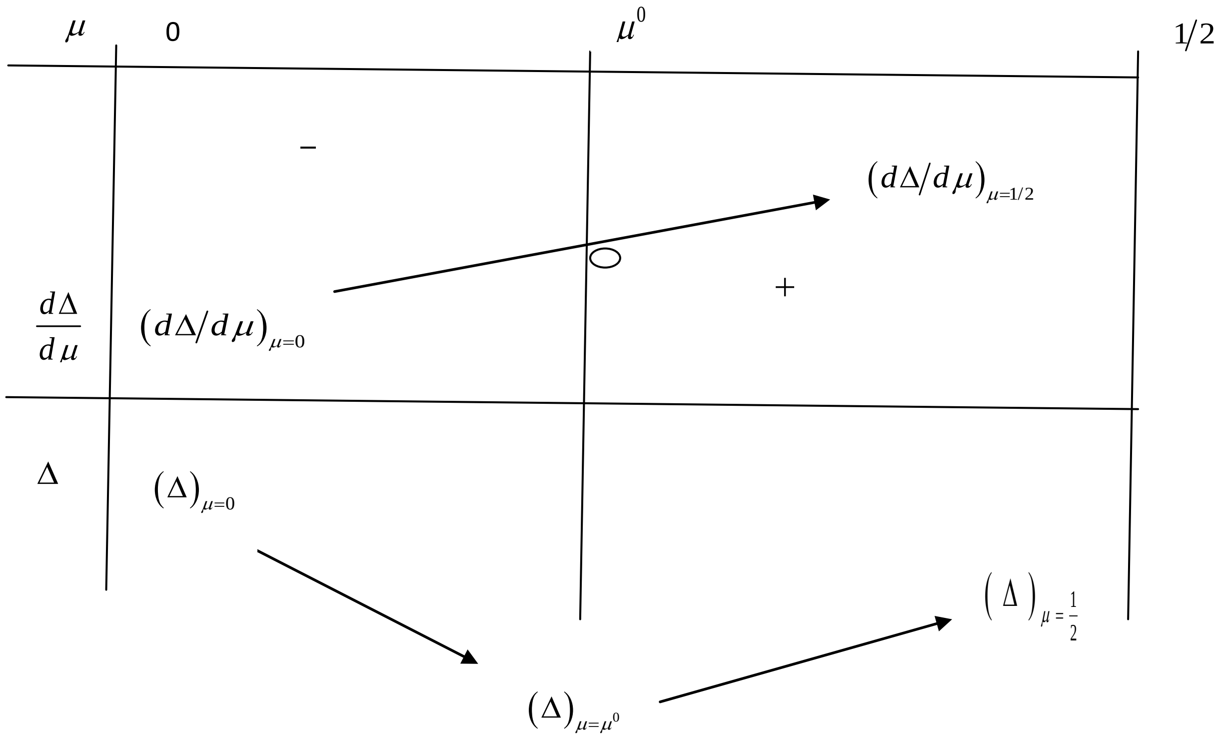
From the above table it can be seen that Δ is monotonic decreasing in $\left(0,\frac{1}{2}\right]$. Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone and continuous and by intermediate value property there is one value of μ say. μ_c' in $\left(0,\frac{1}{2}\right]$ for which $\Delta = 0$.

Case 2: $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$

Since from equation (3.38), $\left(\frac{d\Delta}{d\mu}\right)$ is monotonic increasing in $\left(0,\frac{1}{2}\right]$ and $\left(\frac{d\Delta}{d\mu}\right)_{\mu=0} < 0$ and $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$, this implies that there exists $\mu^0 \in \left(0,\frac{1}{2}\right]$ such that $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\mu^0} = 0$, hence $\frac{d\Delta}{d\mu} \le 0 \;\forall\; \mu \in \left(0,\, \mu^0\right]$ and $\frac{d\Delta}{d\mu} \ge 0 \;\forall\; \mu \in \left[\mu^0, \frac{1}{2}\right]$

Hence the following table of variation of Δ is shown below

Table 3.2: Variation of Δ



Since $(\Delta)_{\mu=0} > 0$ and $(\Delta)_{\mu=\frac{1}{2}} < 0$, it can be concluded that $(\Delta)_{\mu=\mu^0} < 0$. Since Δ is continuous and monotone in $\left(0, \mu^0\right]$ and by the intermediate value property there is one value of μ say. μ_c'' in $\left(0, \mu^0\right]$ for which $\Delta = 0$. Hence $\mu_c = \mu_c' = \mu_c''$. Solving the equation $\Delta = 0$ using equation (3.35), the critical value of the mass parameter is obtained as

$$\mu_c = \frac{1}{2} - \frac{1}{18}\sqrt{69} - \frac{17\sqrt{69}}{486{c_d}^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 - \frac{1}{2}\left(\frac{5}{6} - \frac{59}{9\sqrt{69}}\right)\sigma_1' + \frac{1}{2}\left(\frac{19}{18} - \frac{85}{9\sqrt{69}}\right)\sigma_2' - \frac{2}{27\sqrt{69}}(\delta_1 + \delta_2) \tag{3.42}$$

$$\mu_c = \mu_0 - \frac{17\sqrt{69}}{486{c_d}^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 - \frac{1}{2}\left(\frac{5}{6} - \frac{59}{9\sqrt{69}}\right)\sigma_1' + \frac{1}{2}\left(\frac{19}{18} - \frac{85}{9\sqrt{69}}\right)\sigma_2' - \frac{2}{27\sqrt{69}}(\delta_1 + \delta_2) \tag{3.43}$$

where $\mu_0 = 0.03852...$ is the Routh's value.

The following three regions of the values of μ are considered separately.

i. When $0 < \mu < \mu_c$, $\Delta > 0$, the values of λ^2 given by equation (3.36) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

ii. When $\mu_c < \mu \le \frac{1}{2}, \Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

iii. When $\mu = \mu_c, \Delta = 0$, the values of λ^2 given by equation (3.36) are the same. Therefore the solutions contain secular terms.This induces instability of the triangular points.

Hence, the stability region is

$$0 < \mu < \mu_0 + p \tag{3.44}$$

with

$$p = -\frac{17\sqrt{69}}{486c^2} - \frac{2}{27\sqrt{69}}(\delta_1 + \delta_2) + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 - \frac{1}{2}\left(\frac{5}{6} - \frac{59}{9\sqrt{69}}\right)\sigma_1' + \frac{1}{2}\left(\frac{19}{18} - \frac{85}{9\sqrt{69}}\right)\sigma_2'$$

3.3 Triangular Points with Bigger Triaxial Primary and Smaller Oblate Primary

In this section the same methodology as in section 3.2 is followed.

The triaxiality of the bigger primary and oblateness of the smaller primary are introduced by means of parameters $\sigma_i \ (i = 1,2)$ and A_2 respectively with $\sigma_i << 1 (i = 1,2)$, $A_2 << 1$, where $\sigma_1 = \frac{h^2 - f^2}{5R^2}$, $\sigma_2 = \frac{b^2 - f^2}{5R^2}$. $A_2 = \frac{(AE^2 - AP^2)}{5R^2}$, (McCuskey,1963). Here σ_1, σ_2 characterize the triaxiality of the bigger primary with h, b, f as lengths of its semi-axes, AE and AP are the equatorial and polar radii of the smaller primary, and R is the distance between the primaries.

3.3.1 Equations of motion

Neglecting second and higher power of $\sigma_i\,(i=1{,}2)$, A_2 and also their products, the equations of motion can be written as:

$$\ddot{\xi}-2n_d\dot{\eta}=\frac{\partial W}{\partial \xi}-\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$

$$\ddot{\eta}+2n_d\dot{\xi}=\frac{\partial W}{\partial \eta}-\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \tag{3.45}$$

where W is the potential-like function of the relativistic R3BP. As Katour *et al*. (2014), we do not include the parameters $\sigma_i\,(i=1{,}2)$, A_2 in the relativistic part of W since the magnitude of these terms is so small due to c_d^{-2}.

Hence,

$$\begin{aligned}W=&\frac{1}{2}\left(1+\frac{3}{2}A_2+\frac{3}{2}\left(2\sigma_1-\sigma_2\right)\right)(\xi^2+\eta^2)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\left(1+\frac{A_2}{2\rho_2^2}\right)+\frac{1-\mu}{2\rho_1^3}\left(2\sigma_1-\sigma_2\right)\\&+\frac{3(1-\mu)\eta^2}{2\rho_1^5}\left(\sigma_2-\sigma_1\right)+\frac{1}{c_d^{\,2}}\left[-\frac{3}{2}\left\{1-\frac{1}{3}\mu(1-\mu)\right\}(\xi^2+\eta^2)+\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2\left(\xi\dot{\eta}-\eta\dot{\xi}\right)\right.\right.\\&\left.+(\xi^2+\eta^2)\right\}^2+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\left\{\dot{\xi}^2+\dot{\eta}^2++2\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+(\xi^2+\eta^2)\right\}-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right)\\&\left.+\mu(1-\mu)\left\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)-\frac{\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\left(-\frac{1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\right\}\right],\end{aligned} \tag{3.46}$$

and n_d the perturbed mean motion of the primaries is given by

$$n_d=1+\frac{3}{4}\left(2\sigma_1-\sigma_2\right)+\frac{3}{4}A_2-\frac{3}{2c_d^{\,2}}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{3.47}$$

3.3.2 Locations of the triangular points

The libration points are obtained from equation (3.45) after putting $\dot{\xi}=\dot{\eta}=\ddot{\xi}=\ddot{\eta}=0$.

These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi}=0=\frac{\partial W}{\partial \eta} \text{ with } \dot{\xi}=\dot{\eta}=0.$$

That is

$$\begin{aligned}
&\xi-\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{\mu(\xi-1+\mu)}{\rho_2^3}+\frac{3}{2}A_2\left\{\xi-\frac{\mu(\xi-1+\mu)}{\rho_2^5}\right\}+\left(3\sigma_1-\frac{3}{2}\sigma_2\right)\xi-\frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2\rho_1^5}\\
&-\frac{15(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)\eta^2}{2\rho_1^7}+\frac{1}{{c_d}^2}\left[-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\}+\frac{1}{2}\xi(\xi^2+\eta^2)-\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}\right.\right.\\
&\left.+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\}+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\xi+\frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4}+\frac{\mu^2(\xi-1+\mu)}{\rho_2^4}+\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}\right.\right.\\
&\left.+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\frac{(\xi+\mu)}{\rho_1^3\rho_2}+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}\\
&\left.\left.-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right\}\right]=0
\end{aligned}$$

and

$$\eta F=0, \tag{3.48}$$

with

$$\begin{aligned}
F=&\left(1-\frac{1-\mu}{\rho_1^3}-\frac{\mu}{\rho_2^3}\right)+\frac{3}{2}A_2\left(1-\frac{\mu}{\rho_2^5}\right)+\left(3\sigma_1-\frac{3}{2}\sigma_2\right)+\frac{3(1-\mu)}{\rho_1^5}\left(\frac{3}{2}\sigma_2-2\sigma_1\right)-\frac{15(1-\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7}\\
&+\frac{1}{{c_d}^2}\left[-3\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{1}{2}(\xi^2+\eta^2)+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)+\left(\frac{(1-\mu)^2}{\rho_1^4}+\frac{\mu^2}{\rho_2^4}\right)\right.\\
&\left.+\mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{1-\mu}{\rho_2^5}\right)+\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right\}\right]
\end{aligned}$$

The triangular points are the solutions of equations (3.48) with $\eta\neq 0.$ Since $\frac{1}{{c_d}^2}<<1$ and in the case $\frac{1}{{c_d}^2}\to 0$ and in the absence of triaxiality and oblateness factors $(i.e.\ \sigma_1=\sigma_2=A_2=0)$, one can obtain $\rho_1=\rho_2=1$; we assume in the relativistic R3BP that $\rho_1=1+x$ and $\rho_2=1+y$ where, $x, y<<1.$ may be depending upon the relativistic and triaxiality factors. Substituting these values in the equations (3.6), solving them for ξ, η and ignoring terms of second and higher powers of x and y, the solutions are obtained as

$$\xi = x - y + \frac{1-2\mu}{2},$$

$$\eta = \pm\left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}}\right). \tag{3.49}$$

Substituting the values of $\rho_1, \rho_2, \xi, \eta$ from the above equations (3.48) with $\eta \neq 0$, and neglecting the terms of second and higher powers in $x, y, \frac{x}{c_d^{\ 2}}, \frac{y}{c_d^{\ 2}}, \sigma_1, \sigma_2, A_2$ and their products, the following system is obtained as

$$\frac{3}{2}(1-\mu)x - \frac{3\mu}{2}y + \frac{3}{4}A_2(1-\mu) + \frac{(57\sigma_2 - 69\sigma_1)}{16}\mu + \frac{45}{16}(\sigma_1 - \sigma_2) + \frac{1}{c_d^{\ 2}}\left(-\frac{9\mu}{16} + \frac{27\mu^2}{16} - \frac{9\mu^3}{8}\right) = 0$$

$$3(1-\mu)x + 3\mu y + \frac{3}{2}A_2(1-\mu) + \frac{3}{8}(\sigma_1 + 3\sigma_2)\mu + \frac{21}{8}(\sigma_1 - \sigma_2) + \frac{21(\mu - \mu^2)}{8c_d^{\ 2}} = 0 \tag{3.50}$$

Solving these equations for x and y, the solutions are

$$x = -\frac{\mu(2+3\mu)}{8c_d^{\ 2}} - \frac{1}{2}A_2 + \frac{11}{8}(\sigma_2 - \sigma_1)$$

$$y = -\frac{(1-\mu)(5-3\mu)}{8c_d^{\ 2}} + \left(\frac{1}{2\mu} - \frac{3}{2}\right)\sigma_1 + \left(-\frac{1}{2\mu} + 1\right)\sigma_2 \tag{3.51}$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 respectively are,

$$\xi = \frac{1-2\mu}{2}\left(1+\frac{5}{4{c_d}^2}\right)-\frac{1}{2}A_2+\left(\frac{1}{8}-\frac{1}{2\mu}\right)\sigma_1+\left(\frac{1}{2\mu}+\frac{3}{8}\right)\sigma_2$$

$$\eta = \pm\left\{\frac{\sqrt{3}}{2}\left[1+\frac{1}{12{c_d}^2}\left(-5+6\mu-6\mu^2\right)+\frac{2}{3}\left\{\left(-\frac{23}{8}+\frac{1}{2\mu}\right)\sigma_1+\left(\frac{19}{8}-\frac{1}{2\mu}\right)\sigma_2\right\}\right]-\frac{\sqrt{3}}{6}A_2\right\}. \tag{3.52}$$

3.3.3 Stability of L_4

In this section the same methodology as in section (3.2.3) is followed.

Let (a,b) be the coordinates of the triangular point L_4

Setting $\xi = a+\alpha, \eta = b+\beta, (\alpha, \beta << 1)$, in the equations (3.45)

The terms of their R.H.S. are computed, neglecting second and higher order terms, the following equations are obtained

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta} \tag{3.53}$$

where,

$$A = \frac{3}{4}\left\{1+\frac{1}{2{c_d}^2}\left(2-19\mu+19\mu^2\right)\right\}+\left(\frac{3+24\mu}{8}\right)A_2+\frac{3\left(15\mu^2+19\mu-8\right)}{16\mu}\sigma_1-\frac{3\left(31\mu^2+\mu-8\right)}{16\mu}\sigma_2,$$

$$B = \frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3{c_d}^2}\right)+\left\{-\frac{\sqrt{3}(-7+26\mu)}{8}\right\}A_2-\frac{\sqrt{3}\left(89\mu^2-47\mu+8\right)}{16\mu}\sigma_1+\frac{\sqrt{3}\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2,$$

$$C = \frac{\sqrt{3}}{2{c_d}^2}(1-2\mu),$$

$$D = \frac{6-5\mu+5\mu^2}{2{c_d}^2}.$$

Similarly,

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha,\eta=b+\beta} = E\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta} \tag{3.54}$$

where,

$$E = \frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3{c_d}^2}\right) - \left\{\frac{\sqrt{3}(-7+26\mu)}{8}\right\}A_2 - \frac{\sqrt{3}\left(89\mu^2-47\mu+8\right)}{16\mu}\sigma_1$$
$$+\frac{\sqrt{3}\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2,$$

$$B_1 = \frac{9}{4}\left\{1+\frac{7}{6{c_d}^2}\left(-2+3\mu-3\mu^2\right)\right\} + \frac{33}{8}A_2 - \frac{3\left(15\mu^2-29\mu-8\right)}{16\mu}\sigma_1 + \frac{3\left(15\mu^2-7\mu-8\right)}{16\mu}\sigma_2,$$

$$C_1 = \frac{1}{2{c_d}^2}\left(-4+\mu-\mu^2\right),$$

$$D_1 = -\frac{\sqrt{3}(1-2\mu)}{2{c_d}^2}.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha,\eta=b+\beta} = F\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta} \tag{3.55}$$

where,

$$F = \frac{\sqrt{3}}{2{c_d}^2}(1-2\mu),$$

$$B_2 = \frac{1}{2{c_d}^2}\left(-4+\mu-\mu^2\right),$$

$$C_2 = \frac{1}{4{c_d}^2}\left(17-2\mu+2\mu^2\right),$$

$$D_2 = -\frac{\sqrt{3}}{4{c_d}^2}(1-2\mu).$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta} \tag{3.56}$$

where,

$$A_3 = \frac{1}{2{c_d}^2}\left(6-5\mu+5\mu^2\right),$$

$$B_3 = -\frac{\sqrt{3}}{2{c_d}^2}\left(1-2\mu\right),$$

$$C_3 = -\frac{\sqrt{3}}{4{c_d}^2}\left(1-2\mu\right),$$

$$D_3 = \frac{3\left(5-2\mu+2\mu^2\right)}{4{c_d}^2}.$$

Following as in section (3.2), the variational equations of motion corresponding to equations (3.45), on making use of equation (3.47), can be obtained as

$$\begin{aligned} &p_1\ddot{\alpha} + p_2\ddot{\beta} + p_3\dot{\alpha} + p_4\dot{\beta} + p_5\alpha + p_6\beta = 0, \\ &q_1\ddot{\alpha} + q_2\ddot{\beta} + q_3\dot{\alpha} + q_4\dot{\beta} + q_5\alpha + q_6\beta = 0. \end{aligned} \tag{3.57}$$

where

$$p_1 = 1 + C_2,\, p_2 = D_2,\, p_3 = F - C,\; p_4 = \left\{B_2 - 2\left(1+\frac{3}{4}\left(2\sigma_1-\sigma_2\right)+\frac{3}{4}A_2-\frac{3}{2{c_d}^2}\left(1-\frac{1}{3}\mu(1-\mu)\right)\right)-D\right\},$$
$$p_5 = -A,\, p_6 = -B$$

$$q_1 = C_3, q_2 = 1 + D_3, q_3 = 2\left(1+\frac{3}{4}\left(2\sigma_1-\sigma_2\right)+\frac{3}{4}A_2-\frac{3}{2{c_d}^2}\left(1-\frac{1}{3}\mu(1-\mu)\right)\right)+A_3-C_1, q_4 = B_3 - D_1,$$
$$q_5 = -E, q_6 = -B_1$$

Then, the corresponding characteristic equation is

$$(p_1q_2 - p_2q_1)\lambda^4 + (p_1q_6 + p_5q_2 + p_3q_4 - p_6q_1 - p_2q_5 - p_4q_3)\lambda^2 + p_5q_6 - p_6q_5 = 0 \qquad (3.58)$$

Substituting the values of $p_i, q_i, i = 1, 2, ..., 6$ in equation (3.58), the characteristic equation (3.58) after normalizing becomes

$$\lambda^4 + b\lambda^2 + d = 0 \qquad (3.59)$$

where,

$$b = \left(1 - \frac{9}{c_d^{\ 2}}\right) + \left(-3\mu + \frac{3}{2}\right)A_2 + 3\sigma_1 + \frac{3}{2}(2\mu - 3)\sigma_2,$$

$$d = \frac{27\mu(1-\mu)}{4} + \frac{9\mu\left(-65 + 77\mu - 24\mu^2 + 12\mu^3\right)}{8c_d^{\ 2}} + \frac{117}{4}\mu(1-\mu)A_2 + \frac{9\left(-89\mu^2 + 99\mu - 10\right)}{16}\sigma_1$$
$$+ \frac{9\left(37\mu^2 - 47\mu + 10\right)}{16}\sigma_2.$$

When $\frac{1}{c_d^{\ 2}} \to 0$ and in the absence of the triaxiality and oblateness $(i.e.\ \sigma_1 = \sigma_2 = A_2 = 0)$, equation (3.59) reduces to its well-known classical restricted problem form (see e.g. Szebehely, 1967a):

$$\lambda^4 + \lambda^2 + \frac{27\mu(1-\mu)}{4} = 0$$

The discriminant of equation (3.59) is

$$\Delta = \frac{-54}{c_d^{\ 2}}\mu^4 + \frac{108}{c_d^{\ 2}}\mu^3 + \left(27 + 117A_2 + \frac{801}{4}\sigma_1 - \frac{333}{4}\sigma_2 - \frac{693}{2c_d^{\ 2}}\right)\mu^2$$
$$+\left(-27 - 123A_2 - \frac{891}{4}\sigma_1 + \frac{447}{4}\sigma_2 + \frac{585}{2c_d^{\ 2}}\right)\mu + 1 + 3A_2 - \frac{18}{c_d^{\ 2}} + \frac{57}{2}\sigma_1 - \frac{63}{2}\sigma_2 \qquad (3.60)$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \qquad (3.61)$$

where,

$$b=\left(1-\frac{9}{{c_d}^2}\right)+\left(-3\mu+\frac{3}{2}\right)A_2+3\sigma_1+\frac{3}{2}(2\mu-3)\sigma_2,$$

From (3.60), we have

$$\frac{d\Delta}{d\mu}=\frac{-216}{{c_d}^2}\mu^3+\frac{324}{{c_d}^2}\mu^2+2\left(27+117A_2+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2-\frac{693}{2{c_d}^2}\right)\mu$$
$$\left(-27-123A_2-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2+\frac{585}{2{c_d}^2}\right)<0\ \forall\ \mu\in\left(0,\frac{1}{2}\right]. \quad (3.62)$$

$$\frac{d^2\Delta}{d\mu^2}=-\frac{648}{{c_d}^2}\mu^2+\frac{648}{{c_d}^2}\mu+2\left(27+117A_2+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2-\frac{693}{2{c_d}^2}\right)>0\ \forall\ \mu\in\left(0,\frac{1}{2}\right]$$

(3.63)

This implies that $\frac{d\Delta}{d\mu}$ is monotonic increasing in $\left(0,\frac{1}{2}\right]$

But

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=0}=-27-123A_2-\frac{711}{4}\sigma_1+\frac{219}{4}\sigma_2+\frac{585}{2{c_d}^2}<0\ \forall\ \mu\in\left(0,\frac{1}{2}\right] \quad (3.64)$$

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}}=-6A_2-\frac{45}{2}\sigma_1+\frac{57}{2}\sigma_2-\frac{162}{{c_d}^2} \quad (3.65)$$

$$(\Delta)_{\mu=0}=1+3A_2+\frac{57}{2}\sigma_1-\frac{63}{2}\sigma_2-\frac{18}{{c_d}^2}>0$$
$$(\Delta)_{\mu=\frac{1}{2}}=-\frac{23}{4}+\frac{207}{4{c_d}^2}-\frac{117}{4}A_2-\frac{525}{16}\sigma_1+\frac{57}{16}\sigma_2<0 \quad (3.66)$$

In order to study the monotonicity of Δ , two cases are considered:

Case 1: $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}}\leq 0$

For this case ,the table of variation is given below:

Table 3.3: Variation of Δ

μ	0		$1/2$
$\dfrac{d^2\Delta}{d\mu^2}$		$+$	
$\dfrac{d\Delta}{d\mu}$		$-$	
Δ	$(\Delta)_{\mu=0}$	$\searrow$	$(\Delta)_{\mu=1/2}$

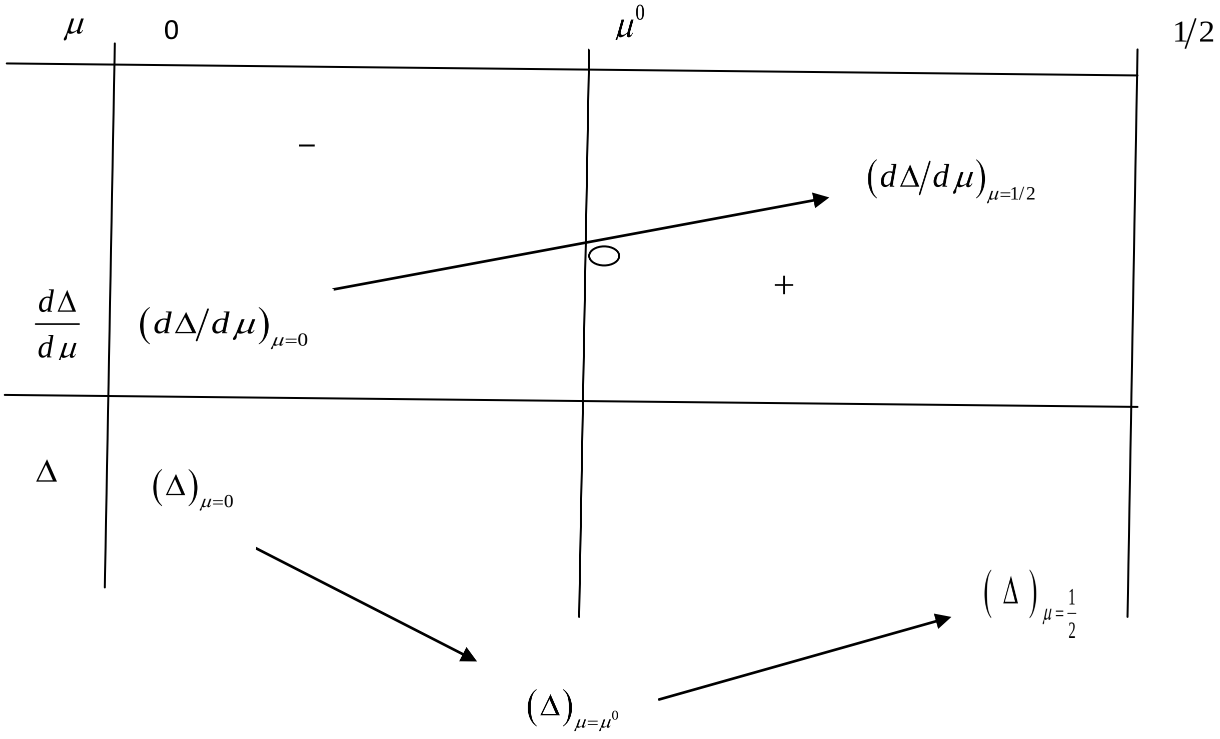
From the above table it can be seen that Δ is monotonic decreasing in $\left(0,\dfrac{1}{2}\right]$.

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone and continuous and by the intermediate value property, there is one value of μ say. μ_c' in the interval $\left(0,\dfrac{1}{2}\right]$ for which $\Delta = 0$.

Case 2: $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$

Since from equation (3.63), $\left(\dfrac{d\Delta}{d\mu}\right)$ is monotonic increasing in $\left(0,\dfrac{1}{2}\right]$ and $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=0} < 0$ and $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$, this implies that there exists $\mu^0 \in \left(0,\dfrac{1}{2}\right]$ such that $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=\mu^0} = 0$, hence $\dfrac{d\Delta}{d\mu} \le 0 \ \forall \ \mu \in \left(0,\mu^0\right]$ and $\left(\dfrac{d\Delta}{d\mu}\right) \ge 0 \ \forall \ \mu \in \left[\mu^0,\dfrac{1}{2}\right]$, hence the following table of variation is shown below.

Table 3.4: Variation of Δ



Since $(\Delta)_{\mu=0} > 0$ and $(\Delta)_{\mu=\frac{1}{2}} < 0$, it can be concluded from the above table that $(\Delta)_{\mu=\mu^0} < 0$, hence since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\mu^0}$ are of opposite signs, and Δ is monotone decreasing and continuous in $\left(0, \mu^0\right]$ and by the intermediate value property, there is one value of μ say. μ_c'' in $\left(0, \mu^0\right]$ for which $\Delta = 0$. Hence $\mu_c = \mu_c' = \mu_c''$.

Solving the equation $\Delta = 0$, using equation (3.60), the critical value of the mass parameter is obtained as

$$\mu_c = \frac{1}{2} - \frac{1}{18}\sqrt{69} - \frac{17\sqrt{69}}{486 c_d^{\,2}} + \frac{1}{9}\left(1 - \frac{13}{\sqrt{69}}\right)A_2 + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right) - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 \tag{3.67}$$

$$\mu_c = \mu_0 - \frac{17\sqrt{69}}{486 c_d^{\,2}} + \frac{1}{9}\left(1 - \frac{13}{\sqrt{69}}\right)A_2 + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2$$

where $\mu_0 = 0.03852...$ is the Routh's value.

The following three regions of the values of μ are considered separately.

i. When $0<\mu<\mu_c, \quad \Delta>0,$ the values of λ^2 given by equation (3.61) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

ii. When $\mu_c<\mu\le\frac{1}{2}, \Delta<0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

iii. When $\mu=\mu_c, \Delta=0$, the values of λ^2 given by equation (3.61) are the same. Therefore the solutions contain the secular terms.This induces instability of the triangular points.

Hence, the stability region is

$$0<\mu<\mu_0-\frac{17\sqrt{69}}{486c_d^{\,2}}+\frac{1}{9}\left(1-\frac{13}{\sqrt{69}}\right)A_2+\frac{1}{2}\left(\frac{5}{6}+\frac{59}{9\sqrt{69}}\right)\sigma_1-\frac{1}{2}\left(\frac{19}{18}+\frac{85}{9\sqrt{69}}\right)\sigma_2 \qquad (3.68)$$

3.4 Periodic Orbits Around Stable Triangular Points when the Bigger Primary is Triaxial

The study of periodic orbits is considered as one of the most important aspect in the restricted three-body problem; and is a starting point for attacking the problem of classifying solutions. Furthermore, if one has a particular solution for the restricted problem, then he can always get a periodic solution. The importance of studying these orbits lies in their significant appearing in nature, they provide us exciting informations about orbital resonances and spin orbits and they can be used as reference orbits to approximate quasi-periodic trajectories.

In this section the results of the problem (3.3) when smaller primary is spherical (i.e. $A_2=0$) are used

Hence the triangular Lagrangian points $L_{4,5}\,(\xi_0,\pm\eta_0)$ are given from equation (3.52) as:

$$\begin{aligned}\xi_0&=\frac{1-2\mu}{2}\left(1+\frac{5}{4c_d^{\,2}}\right)+\left(\frac{1}{8}-\frac{1}{2\mu}\right)\sigma_1+\left(\frac{1}{2\mu}+\frac{3}{8}\right)\sigma_2\\ \eta_0&=\pm\frac{\sqrt{3}}{2}\left[1+\frac{1}{12c_d^{\,2}}\left(-5+6\mu-6\mu^2\right)+\frac{2}{3}\left\{\left(-\frac{23}{8}+\frac{1}{2\mu}\right)\sigma_1+\left(\frac{19}{8}-\frac{1}{2\mu}\right)\sigma_2\right\}\right]\end{aligned} \qquad (3.69)$$

In section 3.2, it is stated that the triangular point are linearly stable in the range $0 < \mu < \mu_c$ and the characteristic equation has four pure imaginary roots. Thus, the motion in this region is bounded and made up of two harmonic motion with frequencies s_1 and s_2 given by:

$$\begin{aligned} \alpha &= C_1 \cos s_1 t + D_1 \sin s_1 t + C_2 \cos s_2 t + D_2 \sin s_2 t \\ \beta &= \bar{C}_1 \cos s_1 t + \bar{D}_1 \sin s_1 t + \bar{C}_2 \cos s_2 t + \bar{D}_2 \sin s_2 t \end{aligned} \tag{3.70}$$

with

$$\begin{aligned} \bar{C}_i &= \Gamma_i \left(2ns_i D_i - W^0_{\xi\eta} C_i\right) \qquad (i = 1, 2) \\ \bar{D}_i &= -\Gamma_i \left(2ns_i C_i - W^0_{\xi\eta} D_i\right) \qquad (i = 1, 2) \end{aligned} \tag{3.71}$$

and $\Gamma_i = \dfrac{s_i^2 + W^0_{\xi\eta}}{4n^2 s_i^2 + \left(W^0_{\xi\eta}\right)^2} = \dfrac{1}{s_i^2 + W^0_{\eta\eta}} \qquad (i = 1,2)$ (Szebehely,1967a)

where s_1 and s_2 are the frequencies with respect to long and short periodic orbits respectively, the terms with coefficients $C_1, D_1, \bar{C}_1, \bar{D}_1$ are the long periodic terms while the coefficients $C_2, D_2, \bar{C}_2, \bar{D}_2$ are the short periodic terms.

The four roots of the characteristic equation (3.59) when $A_2 = 0$ are written as:

$$\begin{aligned} &\lambda_{1,2} = \pm i\ s_1 \\ &and \\ &\lambda_{3,4} = \pm i\ s_2 \end{aligned} \tag{3.72}$$

From equation (3.61) when $A_2 = 0$ we get

$$s_1^2 = \frac{27\mu(1-\mu)}{4} + \left(\frac{-45}{8} + \frac{891}{16}\mu - \frac{801}{16}\mu^2\right)\sigma_1 + \left(\frac{45}{8} - \frac{423}{16}\mu + \frac{333}{16}\mu^2\right)\sigma_2$$
$$+\frac{1}{8c_d^{\ 2}}\left(108\mu^4 - 216\mu^3 + 693\mu^2 - 585\mu\right)$$
$$s_2^2 = 1 - \frac{27\mu(1-\mu)}{4} + \left(\frac{69}{8} - \frac{891}{16}\mu + \frac{801}{16}\mu^2\right)\sigma_1 + \left(-\frac{81}{8} + \frac{471}{16}\mu - \frac{333}{16}\mu^2\right)\sigma_2 \quad (3.73)$$
$$-\frac{1}{8c_d^{\ 2}}\left(108\mu^4 - 216\mu^3 + 693\mu^2 - 585\mu + 72\right)$$

3.4.1 Elliptic orbits

The Taylor expansion of the function of the potential W around the triangular points $L_{4,5}$ can be written in the following form:

$$W = W^0 + \frac{1}{2}W^0_{\xi\xi}\alpha^2 + W^0_{\xi\eta}\alpha\beta + \frac{1}{2}W^0_{\eta\eta}\beta^2 + 0\left(\alpha^3, \beta^3\right) \quad (3.74)$$

where the terms containing the third and higher powers of α and β are neglected and

$$W^0 = \frac{1}{2}\left(\mu^2 - \mu + 3\right) + \left(\frac{11}{8} - \frac{11}{8}\mu + \frac{3}{2}\mu^2\right)\sigma_1 - \frac{1}{8}\left(6\mu^2 - \mu + 1\right)\sigma_2$$
$$-\frac{1}{8c_d^{\ 2}}\left(3\mu^4 - 6\mu^3 - 2\mu^2 + 5\mu + 3\right), \quad (3.75)$$

$$W^0_{\xi\xi} = \frac{3}{4}\left\{1 + \frac{1}{2c_d^{\ 2}}\left(2 - 19\mu + 19\mu^2\right)\right\} + \frac{3\left(15\mu^2 + 19\mu - 8\right)}{16\mu}\sigma_1 - \frac{3\left(31\mu^2 + \mu - 8\right)}{16\mu}\sigma_2, \quad (3.76)$$

$$W^0_{\eta\eta} = \frac{9}{4}\left\{1 + \frac{7}{6c_d^{\ 2}}\left(-2 + 3\mu - 3\mu^2\right)\right\} - \frac{3\left(15\mu^2 - 29\mu - 8\right)}{16\mu}\sigma_1 + \frac{3\left(15\mu^2 - 7\mu - 8\right)}{16\mu}\sigma_2, \quad (3.77)$$

$$W^0_{\xi\eta} = \frac{3\sqrt{3}}{4}\left(1 - 2\mu\right)\left(1 - \frac{2}{3c_d^{\ 2}}\right) - \frac{\sqrt{3}\left(89\mu^2 - 47\mu + 8\right)}{16\mu}\sigma_1 + \frac{\sqrt{3}\left(37\mu^2 - 9\mu + 8\right)}{16\mu}\sigma_2, \quad (3.78)$$

Equation (3.74) represents a quadratic form and indicates that the periodic orbits around the triangular points are ellipse.

3.4.2 Orientation of the principal axes of the ellipse

With the transformation

$\begin{pmatrix}\bar{\alpha}\\ \bar{\beta}\end{pmatrix}=\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, the directions of the principal axes are given by:

$$\tan\ 2\theta=\frac{2W_{\xi\eta}^{0}}{W_{\xi\xi}^{0}-W_{\eta\eta}^{0}} \quad \text{[Abouelmagd and El-Shaboury (2012)]} \tag{3.79}$$

Substituting (3.76), (3.77) and 3.78) in (3.79), we obtain

$$\tan 2\theta=\pm\sqrt{3}\left\{(1-2\mu)-\frac{10}{3c_d{}^2}(1-2\mu)\left(1-3\mu+3\mu^2\right)+\left(\frac{45\mu^3+7\mu^2-40\mu+16}{6\mu}\right)\sigma_1\right.$$
$$\left.-\left(\frac{69\mu^3-25\mu^2-24\mu+16}{6\mu}\right)\sigma_2\right\}, \tag{3.80}$$

3.4.3 Eccentricities of the ellipse

The Taylor expansion of the function of potential W around the triangular points $L_{4,5}$ is given by equation (3.74), but the Jacobian constant $C=2W$ implies that

$$C=\left(\mu^2-\mu+3\right)+\left(\frac{11}{4}-\frac{11}{4}\mu+3\mu^2\right)\sigma_1-\frac{1}{4}\left(1-\mu+6\mu^2\right)\sigma_2-\frac{1}{4c_d{}^2}\left(3\mu^4-6\mu^3-2\mu^2+5\mu+3\right)$$
$$+\left\{\frac{3}{4}\left(1+\frac{1}{2c_d{}^2}\right)\left(2-19\mu+19\mu^2\right)+3\left(\frac{15\mu^2+19\mu-8}{16\mu}\right)\sigma_1-3\left(\frac{31\mu^2+\mu-8}{16\mu}\right)\sigma_2\right\}\alpha^2$$
$$+\left\{\frac{3\sqrt{3}}{2}(1-2\mu)\left(1-\frac{2}{3c_d{}^2}\right)-\sqrt{3}\left(\frac{89\mu^2-47\mu+8}{8\mu}\right)\sigma_1+\sqrt{3}\left(\frac{37\mu^2-9\mu+8}{8\mu}\right)\sigma_2\right\}\alpha\beta$$
$$+\left\{\frac{9}{4}\left(1+\frac{7}{6c_d{}^2}\left(-2+3\mu-3\mu^2\right)\right)-3\left(\frac{15\mu^2-29\mu-8}{16\mu}\right)\sigma_1+3\left(\frac{15\mu^2-9\mu+8}{8\mu}\right)\sigma_2\right\}\beta^2. \tag{3.81}$$

The characteristic equation of the associated matrix of equation (3.81) is

$$\lambda^2-\left\{3+9\sigma_1-\left(3\mu+\frac{3}{2}\right)\sigma_2+\frac{3}{4c_d{}^2}\left(-6+\mu-\mu^2\right)\right\}\lambda+\frac{27}{4}\mu(1-\mu)+\left(-\frac{45}{8}+\frac{891}{16}\mu-\frac{801}{16}\mu^2\right)\sigma_1$$
$$+\left(\frac{45}{8}-\frac{423}{16}\mu+\frac{333}{16}\mu^2\right)\sigma_2-\frac{153}{8c_d{}^2}\mu(1-\mu)=0 \tag{3.82}$$

Its roots are

$$\lambda_1 = 3 - \frac{9\mu(1-\mu)}{4} + \left(\frac{87}{8} - \frac{297}{16}\mu + \frac{267}{16}\mu^2\right)\sigma_1 + \left(-\frac{27}{8} + \frac{93}{16}\mu - \frac{111}{16}\mu^2\right)\sigma_2 + \frac{3}{8{c_d}^2}\left(-12 + 19\mu - 19\mu^2\right)$$

$$\lambda_2 = \frac{9\mu(1-\mu)}{4} + \left(-\frac{15}{8} + \frac{297}{16}\mu - \frac{267}{16}\mu^2\right)\sigma_1 + \left(\frac{15}{8} - \frac{141}{16}\mu + \frac{111}{16}\mu^2\right)\sigma_2 + \frac{51}{8{c_d}^2}\left(\mu^2 - \mu\right)$$

(3.83)

The eccentricities of the ellipse are given by (Szebehely, 1967a)

$$e_i = \left(1 - \alpha_i^2\right)^{\frac{1}{2}}, \quad \alpha_i = \frac{2s_i}{s_i^2 + \bar{\lambda}} \quad (i = 1,2) \tag{3.84}$$

where $\bar{\lambda}$ is one of the roots of equation (3.82). For $i = 1,2$ and making use of equation (3.73) and $\bar{\lambda} = \lambda_1$, we obtain

$$\alpha_1^2 = 3\mu(1-\mu) - 9\mu^2(1-\mu)^2 + \left(-\frac{5}{2} + \frac{87}{4}\mu - \frac{671}{4}\mu^2 + 282\mu^3 - \frac{267}{2}\mu^4\right)\sigma_1 + \left(\frac{5}{2} - \frac{95}{4}\mu + \frac{391}{4}\mu^2 - 132\mu^3 + \frac{111}{2}\mu^4\right)\sigma_2 + \frac{1}{2{c_d}^2}\left(-47\mu + 518\mu^2 - 1032\mu^3 + 741\mu^4 - 270\mu^5 + 90\mu^6\right)$$

$$\alpha_2^2 = \frac{1}{4} - \frac{9\mu(1-\mu)}{8} - \frac{243\mu^2(1-\mu)^2}{32} + \left(-\frac{9}{32} + \frac{1377}{64}\mu - \frac{4713}{32}\mu^2 + \frac{3807}{16}\mu^3 - \frac{7209}{64}\mu^4\right)\sigma_1 + \left(-\frac{27}{32} - \frac{1269}{64}\mu + \frac{2685}{32}\mu^2 - \frac{1755}{16}\mu^3 + \frac{2997}{64}\mu^4\right)\sigma_2 - \frac{1}{32{c_d}^2}\left(72 + 414\mu - 20679\mu^2 + 41364\mu^3 - 29187\mu^4 + 10206\mu^5 - 3402\mu^6\right).$$

(3.85)

Hence,

$$e_1 = 1 - \frac{3}{2}\mu(1-\mu) + \frac{9}{2}\mu^2(1-\mu) + \left(\frac{5}{4} - \frac{87}{8}\mu + \frac{671}{8}\mu^2 - 141\mu^3 + \frac{267}{4}\mu^4\right)\sigma_1 + \left(-\frac{5}{4} + \frac{95}{8}\mu - \frac{391}{8}\mu^2 + 66\mu^3 - \frac{111}{4}\mu^4\right)\sigma_2 + \frac{1}{4{c_d}^2}\left(47\mu - 518\mu^2 + 1032\mu^3 - 741\mu^4 + 270\mu^5 - 90\mu^6\right)$$

(3.86)

$$e_2=\frac{\sqrt{3}}{2}\left\{1+\frac{3\mu(1-\mu)}{4}+\frac{81\mu^2(1-\mu)^2}{16}+\left(\frac{3}{16}-\frac{459}{32}\mu+\frac{1571}{16}\mu^2-\frac{1269}{8}\mu^3+\frac{2403}{32}\mu^4\right)\sigma_1\right.$$
$$+\left(\frac{9}{16}+\frac{423}{32}\mu-\frac{895}{16}\mu^2+\frac{585}{8}\mu^3-\frac{999}{32}\mu^4\right)\sigma_2+\frac{1}{64{c_d}^2}\left(24+138\mu-6893\mu^2\right.$$
$$\left.\left.+13788\mu^3+9729\mu^4+3402\mu^5-1134\mu^6\right).\right\}$$

3.4.4 Semi-major and semi-minor axes

The lengths of the semi-major axis, a_i and the semi-minor axis, b_i of the long and short period are obtained from

$$a_i=\left(\xi_0^2+\frac{\eta_0^2}{\alpha_i^2}\right)^{\frac{1}{2}}$$
$$and \qquad (3.87)$$
$$b_i=\left(\alpha_i^2\xi_0^2+\eta_0^2\right)^{\frac{1}{2}}$$

respectively, with ξ_0 and η_0 as initial conditions (Szebehely, 1967a).

To obtain the semi-major axes of the orbit of the long period and that of the short period, the equations (3.69) and (3.85) are used in equation (3.87), when $i=1$, respectively. Thus, the followings are obtained

$$a_1=\frac{\sqrt{5}}{2}\left\{\left(1-\frac{1}{10\mu}+\frac{7\mu^2}{10}\right)+\left(\frac{493}{120}-7\mu+\frac{33}{10}\mu^2-\frac{25}{24\mu}+\frac{3}{20\mu^2}\right)\sigma_1+\left(-\frac{221}{120}+2\mu-\frac{9\mu^2}{10}+\frac{25}{24\mu}-\frac{3}{20\mu^2}\right)\sigma_2\right.$$
$$\left.+\frac{1}{60{c_d}^2}\left(-517+1020\mu+\frac{42}{\mu}-156\mu^2+324\mu^3-108\mu^4\right)\right\}.$$

$$a_2=\frac{\sqrt{13}}{2}\left\{\left(1+\frac{25}{13}\mu+\frac{629}{52}\mu^2-\frac{729}{26}\mu^3+\frac{729}{52}\mu^4\right)+\left(\frac{4}{13}-\frac{4135}{104}\mu+\frac{3}{13\mu}+\frac{21573}{104}\mu^2-\frac{16767}{52}\mu^3\right.\right.$$
$$\left.+\frac{8019}{52}\mu^4\right)\sigma_1+\left(\frac{323}{52}-\frac{1884}{13}\mu-\frac{3}{13\mu}-\frac{58563}{104}\mu^2+\frac{37017}{52}\mu^3-\frac{31347}{104}\mu^4\right)\sigma_2$$
$$\left.+\frac{1}{104{c_d}^2}\left(402+2312\mu-124901\mu^2+251640\mu^3-180495\mu^4+65610\mu^5-21870\mu^6\right)\right\}$$
$$(3.88)$$

Similarly, the semi-minor axes of the orbits of the long and short periods are obtained as

$$b_1 = \frac{\sqrt{3}}{2}\left\{\left(1+\frac{\mu}{2}-4\mu^2+\frac{39}{3}\mu^3-\frac{43}{2}\mu^4+18\mu^5-6\mu^6\right)+\left(-\frac{10}{3}+\frac{277}{24}\mu+\frac{1}{3\mu}-\frac{477}{8}\mu^2+\frac{1151}{6}\mu^3\right.\right.$$
$$\left.-\frac{1991}{6}\mu^4+\frac{557}{2}\mu^5-89\mu^6\right)\sigma_1+\left(3-\frac{261}{24}\mu-\frac{1}{3\mu}+\frac{1039}{24}\mu^2-\frac{215}{2}\mu^3+\frac{943}{6}\mu^4-\frac{241}{2}\mu^5\right.$$
$$\left.+\frac{111}{3}\mu^6\right)\sigma_2+\frac{1}{12c_d^2}\left(-52+209\mu+204\mu^2-2714\mu^3+6296\mu^4-6822\mu^5+3954\mu^6\right.$$
$$\left.\left.-1440\mu^7+360\mu^8\right)\right\}$$

and

$$b_2 = \frac{\sqrt{13}}{4}\left\{\left(1-\frac{17}{52}\mu-\frac{31}{208}\mu^2+\frac{585}{104}\mu^3-\frac{1015}{208}\mu^4+\frac{729}{52}\mu^5-\frac{243}{52}\mu^6\right)+\left(-\frac{161}{208}+\frac{1889}{416}\mu-\frac{3507}{104}\mu^2\right.\right.$$
$$\left.+\frac{11337}{104}\mu^3-\frac{22533}{104}\mu^4+\frac{45117}{208}\mu^5-\frac{7209}{104}\mu^6\right)\sigma_1+\left(\frac{205}{208}-\frac{1701}{416}\mu-\frac{4}{13\mu}+\frac{8055}{416}\mu^2-\frac{4443}{208}\mu^3\right.$$
$$\left.+\frac{987}{52}\mu^4-\frac{8565}{208}\mu^5+\frac{2997}{104}\mu^6\right)\sigma_2+\frac{1}{416{c_d}^2}\left(-312-422\mu+21353\mu^2-114036\mu^3+247209\mu^4\right.$$
$$\left.\left.-263250\mu^5+151254\mu^6-54432\mu^7+13608\mu^8\right)\right\}$$

(3.89)

3.5 Collinear Equilibrium Points in the Relativistic Restricted Three-Body Problem with a Smaller Triaxial Primary.

In this section, the analytical and numerical locations of the collinear points are obtained and also the study of their stability when the smaller primary is triaxial is carried out.

3.5.1 Equations of motion

The triaxiality factors of the smaller primary are introduced with the help of the parameters $\sigma_i << 1,\ i=1,2$ where $\sigma_1 = \frac{h^2-f^2}{5R^2}$, $\sigma_2 = \frac{b^2-f^2}{5R^2}$ (McCuskey, 1963) with h, b, f as lengths of its semi-axes and R is the dimensional distance between the primaries.

Ignoring second and higher powers of σ_i and neglecting also their product, the equations of motion are taken as:

$$\ddot{\xi} - 2n\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$

(3.90)

$$\ddot{\eta} + 2n\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)$$

where W is the potential like function of the relativistic R3BP. As Katour *et al*. (2014), the parameters $\sigma_i\,(i=1,2)$ are not included in the relativistic part of W since the magnitude of these terms is so small due to c_d^{-2} where c_d is the dimensionless speed of light.

Hence,

$$\begin{aligned} W &= \frac{1}{2}\left(1+\frac{3}{2}\left(2\sigma_1-\sigma_2\right)\right)(\xi^2+\eta^2)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}+\frac{\mu}{2\rho_2^3}\left(2\sigma_1-\sigma_2\right)+\frac{3\mu\eta^2}{2\rho_2^5}\left(\sigma_2-\sigma_1\right) \\ &\quad +\frac{1}{c_d^{\,2}}\left[-\frac{3}{2}\left\{1-\frac{1}{3}\mu(1-\mu)\right\}(\xi^2+\eta^2)+\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+(\xi^2+\eta^2)\right\}^2\right. \\ &\quad +\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\left\{\dot{\xi}^2+\dot{\eta}^2+2\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+(\xi^2+\eta^2)\right\}-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right) \\ &\quad \left.+\mu(1-\mu)\left\{\left(4\dot{\eta}+\frac{7}{2}\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)-\frac{\eta^2}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\left(\frac{-1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\right\}\right], \end{aligned} \tag{3.91}$$

and n_d is the perturbed mean motion of the primaries and is given by

$$n_d = 1+\frac{3}{4}\left(2\sigma_1-\sigma_2\right)-\frac{3}{2c_d^{\,2}}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{3.92}$$

3.5.2 Locations of collinear points

Equilibrium points are those points at which no resultant force acts on the third infinitesimal body. Therefore, if it is placed at any of these points with zero velocity, it will stay there. In fact all derivatives of the coordinates with respect to the time are zero at these points. Therefore, the equilibrium points are solutions of equations

$$W_\xi = 0 \text{ and } W_\eta = 0 \tag{3.93}$$

W_ξ and W_η may be written as

$$W_\xi = \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \left(3\sigma_1 - \frac{3}{2}\sigma_2\right)\xi - \frac{3\mu(\xi-1+\mu)(2\sigma_1-\sigma_2)}{2\rho_2^5} - \frac{15\mu(\xi-1+\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_2^7}$$
$$+\frac{1}{c_d{}^2}\left[-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\} + \frac{1}{2}\xi(\xi^2+\eta^2) - \frac{3}{2}(\xi^2+\eta^2)\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\} + 3\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)\xi\right.$$
$$+\frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4} + \mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right)\right.$$
$$\left.\left.+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) + \frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3} - \frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right\}\right]$$

and

$$W_\eta = \eta F\ ,$$

with

$$F = \left(1 - \frac{1-\mu}{\rho_1^3} - \frac{\mu}{\rho_2^3}\right) + \left(3\sigma_1 - \frac{3}{2}\sigma_2\right) + \frac{3\mu}{\rho_2^5}\left(\frac{3}{2}\sigma_2 - 2\sigma_1\right) - \frac{15\mu(\sigma_2-\sigma_1)\eta^2}{2\rho_2^7}$$
$$\frac{1}{c_d{}^2}\left[-3\left(1-\frac{\mu(1-\mu)}{3}\right) + \frac{1}{2}(\xi^2+\eta^2) + 3\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) - \frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right) + \left(\frac{(1-\mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4}\right)\right.$$
$$\left.+\mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3}\right) + \frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5} + \frac{1-\mu}{\rho_2^5}\right) + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{(3\mu-2)}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3}\right\}\right].$$

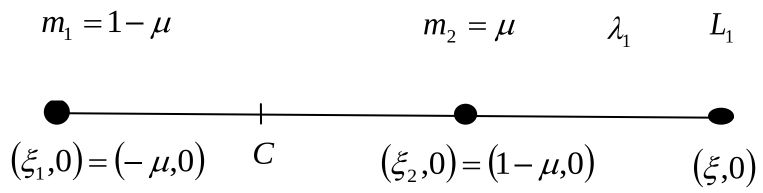
In order to find the collinear points, we put $\eta = 0$ in equation (3.93). Their abscissae are the roots of the equation

$$
\begin{aligned}
g(\xi) = {} & \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \left(3\sigma_1 - \frac{3}{2}\sigma_2\right)\xi - \frac{3\mu(\xi-1+\mu)(2\sigma_1-\sigma_2)}{2\rho_2^5} \\
& + \frac{1}{{c_d}^2}\Bigg[-3\xi\left(1-\frac{\mu(1-\mu)}{3}\right) + \frac{1}{2}\xi^3 - \frac{3}{2}\xi^2\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\} + 3\left(\frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}\right)\xi \\
& + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4} + \mu(1-\mu)\Bigg\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) \\
& + \frac{\xi-1+\mu}{\rho_1\rho_2^3} + \frac{\xi+\mu}{\rho_1^3\rho_2} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3} - \frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\Bigg\}\Bigg] = 0
\end{aligned}
\tag{3.94}
$$

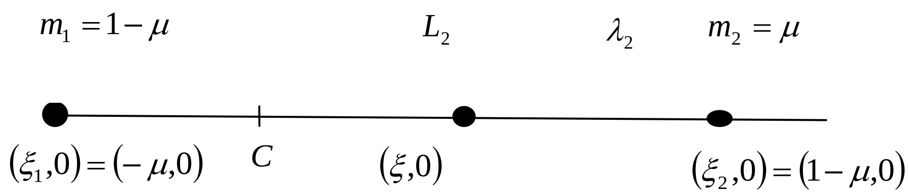
with $\rho_1 = |\xi + \mu|,\ \rho_2 = |\xi - 1 + \mu|$

To locate the collinear points on the $\xi-$axis, the orbital plane is divided into three parts: $\xi < \xi_1,\ \xi_1 < \xi < \xi_2$ and $\xi_2 < \xi$ with respect to the primaries where $\xi_1 = -\mu$ and $\xi_2 = 1-\mu$

(a)



(b)



(c)

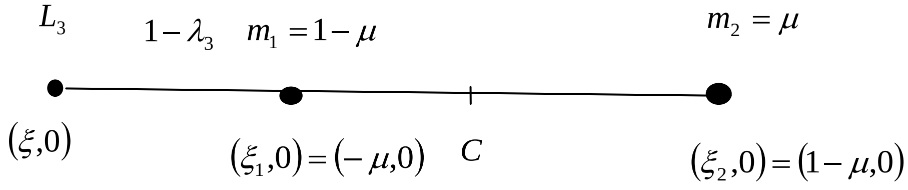


Figure 3.1: Reference parameter for collinear Lagrangian points

Case1. Position of $L_1(\xi > \xi_2)$ (see Fig. 3.1 (a))

Let $\xi - \xi_2 = \lambda_1;\ \xi - \xi_1 = 1 + \lambda_1 \Rightarrow \xi = 1 + \lambda_1 + \xi_1;$ since the distance between the primaries is unity, i.e. $\xi_2 - \xi_1 = 1 \Rightarrow \xi_1 = -\mu$ and $\xi_2 = 1 - \mu$ then

$$\xi = 1 + \lambda_1 - \mu; \rho_1 = 1 + \lambda_1;\ \rho_2 = \lambda_1 \text{ with } \rho_i > 0\ (i = 1,2) \tag{3.95}$$

Now substituting equation (3.95) in equation (3.94), the following equation is obtained

$$\begin{aligned}
&\lambda_1^{10} + (6 - 3\mu)\lambda_1^9 + \left(9 + 2c_d^2 - 13\mu - 3c_d^2\sigma_2 + \mu^2 + 6c_d^2\sigma_1\right)\lambda_1^8 + \left(24c_d^2\sigma_1 + 2\mu^2 - 12c_d^2\sigma_2\right.\\
&\left.+3\mu c_d^2\sigma_2 - 16\mu + 8c_d^2 - 1 + \mu^3 - 6\mu c_d^2\sigma_1 - 2\mu c_d^2\right)\lambda_1^7 + \left(-18c_d^2\sigma_2 + 9\mu c_d^2\sigma_2 + 3\mu^3 - 18\mu c_d^2\sigma_1\right.\\
&\left.-6\mu c_d^2 + 12c_d^2 - 12 + 36c_d^2\sigma_1\right)\lambda_1^6 + \left(-7\mu^2 + 24c_d^2\sigma_1 - 18\mu c_d^2\sigma_1 + 6c_d^2 + 16\mu - 6\mu c_d^2 - 9 + 3\mu^3\right.\\
&\left.-12c_d^2\sigma_2 + 9\mu c_d^2\sigma_2\right)\lambda_1^5 + \left(16\mu - 6\mu c_d^2\sigma_1 - 14\mu^2 - 3c_d^2\sigma_2 + 6c_d^2\sigma_1 + 3\mu c_d^2\sigma_2 - 6\mu c_d^2 + 3\mu^3\right)\lambda_1^4\\
&+\left(3\mu c_d^2\sigma_2 - 12\mu^2 + 15\mu - 6\mu c_d^2 + 3\mu^3 - 6\mu c_d^2\sigma_1\right)\lambda_1^3 + \left(5\mu + 9\mu c_d^2\sigma_2 - 2\mu c_d^2\right.\\
&\left.-18\mu c_d^2\sigma_1 + \mu^3\right)\lambda_1^2 + \left(-18\mu c_d^2\sigma_1 + 9\mu c_d^2\sigma_2 + 2\mu^2\right)\lambda_1 + \left(3\mu c_d^2\sigma_2 - 6\mu c_d^2\sigma_1\right) = 0
\end{aligned} \tag{3.96}$$

In the presence of triaxiality effect only, the corresponding equation is obtained as

$$\begin{aligned}
&(6\sigma_1 - 3\sigma_2 + 2)\lambda_1^7 + (-6\sigma_1\mu + 18\sigma_1 - 9\sigma_2 + 3\sigma_2\mu - 2\mu + 6)\lambda_1^6 + (-12\sigma_1\mu + 18\sigma_1\\
&-4\mu - 9\sigma_2 + 6 + 6\sigma_2\mu)\lambda_1^5 + (-2\mu + 6\sigma_1 - 6\sigma_1\mu - 3\sigma_2 + 3\sigma_2\mu)\lambda_1^4 - 4\mu\lambda_1^3\\
&+(-2\mu + 3\sigma_2\mu - 6\sigma_1\mu)\lambda_1^2 + (-12\sigma_1\mu + 6\sigma_2\mu)\lambda_1 + 3\sigma_2\mu - 6\sigma_1\mu = 0
\end{aligned} \tag{3.96a}$$

Case 2. Position of $L_2(\xi_1 < \xi < \xi_2)$ (see Fig. 3.1 (b))

Let $\xi_2 - \xi = \lambda_2;\ \xi - \xi_1 = 1 - \lambda_2 \Rightarrow \xi = 1 - \lambda_2 - \mu;\ \ \rho_1 = 1 - \lambda_2;\qquad \rho_2 = \lambda_2$ with

$$\rho_i > 0\ (i = 1,2) \tag{3.97}$$

Substituting equation (3.97) in equation (3.94), yields

$$\lambda_2^{10}+\left(-6+3\mu\right)\lambda_2^{9}+\left(\mu^2-13\mu+2c_d^{\,2}+9+6c_d^{\,2}\sigma_1-3c_d^{\,2}\sigma_2\right)\lambda_2^{8}+\left(-2\mu^2-24c_d^{\,2}\sigma_1+1+6\mu c_d^{\,2}\sigma_1+2\mu c_d^{\,2}+22\mu-8c_d^{\,2}\right.$$
$$\left.-3\mu c_d^{\,2}\sigma_2+12c_d^{\,2}\sigma_2-\mu^3\right)\lambda_2^{7}+\left(3\mu^3-18\mu c_d^{\,2}\sigma_1+36c_d^{\,2}\sigma_1+9\mu c_d^{\,2}\sigma_2-6\mu c_d^{\,2}-18c_d^{\,2}\sigma_2+12c_d^{\,2}-18\mu-12\right)\lambda_2^{6}$$
$$+\left(9+12c_d^{\,2}\sigma_2+2\mu c_d^{\,2}+18\mu c_d^{\,2}\sigma_1+8\mu-\mu^2-\mu^3-9\mu c_d^{\,2}\sigma_2-24c_d^{\,2}\sigma_1-6c_d^{\,2}\right)\lambda_2^{5}+\left(6\mu c_d^{\,2}-3\mu^3-16\mu-6\mu c_d^{\,2}\sigma_1\right.$$
$$\left.+18\mu^2-3c_d^{\,2}\sigma_2+6c_d^{\,2}\sigma_1+3\mu c_d^{\,2}\sigma_2\right)\lambda_2^{4}+\left(-6\mu c_d^{\,2}\sigma_1-24\mu^2+3\mu c_d^{\,2}\sigma_2+15\mu-6\mu c_d^{\,2}+3\mu^3\right)\lambda_2^{3}+\left(-5\mu+2\mu c_d^{\,2}\right.$$
$$\left.+18\mu c_d^{\,2}\sigma_1-9\mu c_d^{\,2}\sigma_2-\mu^3+12\mu^2\right)\lambda_2^{2}+\left(-18\mu c_d^{\,2}\sigma_1+9\mu c_d^{\,2}\sigma_2-2\mu^2\right)\lambda_2+6\mu c_d^{\,2}\sigma_1-3\mu c_d^{\,2}\sigma_2=0 \quad (3.98)$$

In the presence of triaxiality effect only, the corresponding equation is obtained as

$$\left(3\sigma_2-6\sigma_1-2\right)\lambda_2^{7}+\left(18\sigma_1-2\mu-9\sigma_2+3\sigma_2\mu-6\sigma_1\mu+6\right)\lambda_2^{6}+\left(-18\sigma_1+4\mu+12\sigma_1\mu+9\sigma_2\right.$$
$$\left.-6\sigma_2\mu-6\right)\lambda_2^{5}+\left(2\mu+6\sigma_1-6\sigma_1\mu-3\sigma_2+3\sigma_2\mu\right)\lambda_2^{4}-4\mu\lambda_2^{3}+\left(6\sigma_1\mu-3\sigma_2\mu+2\mu\right)\lambda_2^{2}$$
$$+\left(6\sigma_2\mu-12\sigma_1\mu\right)\lambda_2+6\sigma_1\mu-3\sigma_2\mu=0 \quad (3.98a)$$

Case 3. Position of $L_3\left(\xi<\xi_1\right)$ (see Fig. 3.1 (c))

Let the distance of the point L_3 from the bigger primary be $1-\lambda_3$.

Since $\xi_2-\xi_1=1\Rightarrow\xi_1-\xi=1-\lambda_3;\xi_2-\xi=2-\lambda_3$ and

$$\xi=\lambda_3-\mu-1;\ \rho_1=1-\lambda_3;\ \rho_2=2-\lambda_3 \text{ with } \rho_i>0\ \left(i=1,2\right) \quad (3.99)$$

Substituting equation (3.99) in equation (3.94), we obtain

$$-\lambda_3^{10}+\left(3\mu+14\right)\lambda_3^{9}+\left(-\mu^2-6c_d^{\,2}\sigma_1-41\mu+3c_d^{\,2}\sigma_2-81-2c_d^{\,2}\right)\lambda_3^{8}+\left(2\mu c_d^{\,2}+72c_d^{\,2}\sigma_1+240\mu\right.$$
$$\left.+6\mu c_d^{\,2}\sigma_1+247+14\mu^2+24c_d^{\,2}-3\mu c_d^{\,2}\sigma_2-36c_d^{\,2}\sigma_2-\mu^3\right)\lambda_3^{7}+\left(11\mu^3-404-22\mu c_d^{\,2}-372c_d^{\,2}\sigma_1\right.$$
$$\left.-66\mu c_d^{\,2}\sigma_1+186c_d^{\,2}\sigma_2-84\mu^2-124c_d^{\,2}+33\mu c_d^{\,2}\sigma_2-784\mu\right)\lambda_3^{6}+\left(255+306\mu c_d^{\,2}\sigma_1+277\mu^2-51\mu^3\right.$$
$$\left.+1080c_d^{\,2}\sigma_1-153\mu c_d^{\,2}\sigma_2-540c_d^{\,2}\sigma_2+102\mu c_d^{\,2}+358c_d^{\,2}+1562\mu\right)\lambda_3^{5}+\left(387\mu c_d^{\,2}\sigma_2+131\mu^3\right.$$
$$\left.-1926c_d^{\,2}\sigma_1-1950\mu-262\mu c_d^{\,2}+963c_d^{\,2}\sigma_2+248-624c_d^{\,2}-542\mu^2-774\mu c_d^{\,2}\sigma_1\right)\lambda_3^{4}$$
$$+\left(636\mu^2-1080c_d^{\,2}\sigma_2+2160c_d^{\,2}\sigma_1-616-205\mu^3+1507\mu+410\mu c_d^{\,2}+1146\mu c^2\sigma_1+656c_d^{\,2}-573\mu c_d^{\,2}\sigma_2\right)\lambda_3^{3}$$
$$+\left(480-673\mu-398\mu c_d^{\,2}-990\mu c_d^{\,2}\sigma_1-432\mu^2-384c_d^{\,2}+744c_d^{\,2}\sigma_2-1488c_d^{\,2}\sigma_1+199\mu^3+495\mu c_d^{\,2}\sigma_2\right)\lambda_3^{2}$$
$$+\left(158\mu^2-231\mu c_d^{\,2}\sigma_2+462\mu c_d^{\,2}\sigma_1-288c_d^{\,2}\sigma_2-144+224\mu c_d^{\,2}+128\mu-112\mu^3+96c_d^{\,2}+576c_d^{\,2}\sigma_1\right)\lambda_3$$
$$+\left(-56\mu c_d^{\,2}-96c_d^{\,2}\sigma_1+12\mu-28\mu^2+28\mu^3-90\mu c_d^{\,2}\sigma_1+48c_d^{\,2}\sigma_2+45\mu c_d^{\,2}\sigma_2\right)=0 \quad (3.100)$$

In the presence of triaxiality only, the corresponding equation is obtained as

$$\left(6\sigma_1-3\sigma_2+2\right)\lambda_3^{\,7}+\left(-66\sigma_1-2\mu+33\sigma_2-22-6\sigma_1\mu+3\sigma_2\mu\right)\lambda_3^{\,6}+\left(306\sigma_1+20\mu+60\sigma_1\mu\right.$$
$$\left.-153\sigma_2+102-30\sigma_2\mu\right)\lambda_3^{\,5}+\left(-82\mu-256-246\sigma_1\mu+387\sigma_2-774\sigma_1+123\sigma_2\mu\right)\lambda_3^{\,4}+$$
$$\left(1152\sigma_1+528\sigma_1\mu+180\mu-264\sigma_2\mu-576\sigma_2+368\right)\lambda_3^{\,3}+\left(309\sigma_2\,\mu-288-618\sigma_1\mu-1008\sigma_1+\right.$$
$$\left.504\sigma_2-230\mu\right)\lambda_3^{\,2}+\left(372\sigma_1\mu+96-186\sigma_2\mu-240\sigma_2+168\mu+480\sigma_1\right)\lambda_3^{\,3}+\left(45\sigma_2\,\mu-56\mu+\right.$$
$$\left.48\sigma_2-96\sigma_1-90\sigma_1\mu\right)=0$$

(3.100a)

It should be noted that some of the above equations have more than one positive root but in each case there is only one physically acceptable root.This is confirmed by Yamada and Asada (2011) in the absence of triaxiality. It is also here pointed out that we have not considered the higher order relativistic corrections because $\xi_i\,(i=1,\,2),\ \frac{1}{c_d^{\,2}}<<1$.

3.5.3 Stability of collinear points

The stability of an equilibrium configuration is examined, that is its ability to restrain the body motion in its vicinity. To do so we displace the infinitesimal body a little from an equilibrium point with small velocity. If its motion is rapid departure from vicinity of the point, we call such a position of equilibrium an unstable one. If the body oscillates about the point, it is said to be a stable position.

In order to study the stability of the collinear points, we consider the characteristic equation:

$$(p_1q_2-p_2q_1)\omega^4+(p_1q_6+p_5q_2+p_3q_4-p_6q_1-p_2q_5-p_4q_3)\omega^2+p_5q_6-p_6q_5=0 \qquad (3.101)$$

where,

$$p_1 = 1 + W^0_{\dot\xi\dot\xi},\, p_2 = W^0_{\dot\eta\dot\xi},\, p_3 = W^0_{\xi\dot\xi} - W^0_{\dot\xi\xi} = 0,\, p_4 = W^0_{\eta\dot\xi} - 2n - W^0_{\xi\dot\eta},$$

$$p_5 = -W^0_{\xi\xi},\; p_6 = -W^0_{\xi\eta}, q_1 = W^0_{\dot\xi\dot\eta}, q_2 = 1 + W^0_{\dot\eta\dot\eta}, q_3 = 2n + W^0_{\xi\dot\eta} - W^0_{\eta\dot\xi},$$

$$q_4 = W^0_{\eta\dot\eta} - W^0_{\eta\dot\eta} = 0,\; q_5 = -W^0_{\xi\eta},\; q_6 = -W^0_{\eta\eta}.$$

The second order partial derivative of W are denoted by subscripts. The superscript 0 indicates that the derivative is to be evaluated at the collinear equilibrium points (ξ_0, η_0) under consideration.

In order to study the stability of the collinear points, it requires to study the motion in the proximity of these points, hence in this case the second order derivatives evaluated at (ξ_0, η_0) are

$$\begin{aligned}
W^0_{\xi\xi} &= 1 + \frac{3(1-\mu)(\xi+\mu)^2}{\rho_1^5} - \frac{(1-\mu)}{\rho_1^3} + \frac{3\mu(\xi-1+\mu)^2}{\rho_2^5} - \frac{\mu}{\rho_2^3} + \left(3\sigma_1 - \frac{3}{2}\sigma_2\right) + \frac{15\mu(2\sigma_1-\sigma_2)(\xi-1+\mu)^2}{2\rho_2^7} \\
&- \frac{3\mu(2\sigma_1-\sigma_2)}{2\rho_2^5} + \frac{1}{{c_d}^2}\Bigg[\left(\mu(1-\mu)-3\right) + \frac{3}{2}\xi^2 + \left(\frac{9(1-\mu)(\xi+\mu)^2}{2\rho_1^5} - \frac{3(1-\mu)}{2\rho_1^3} - \frac{3\mu}{2\rho_2^3} + \frac{9\mu(\xi-1+\mu)^2}{2\rho_2^5}\right)\xi^2 \\
&+ 2\left(\frac{-3(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{3\mu(\xi-1+\mu)}{\rho_2^3}\right)\xi + \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2} - \frac{4(1-\mu)^2(\xi+\mu)^2}{\rho_1^6} + \frac{(1-\mu)^2}{\rho_1^4} - \frac{4\mu^2(\xi-1+\mu)^2}{\rho_2^6} \\
&+ \frac{\mu^2}{\rho_2^4} + \mu(1-\mu)\Bigg\{-\frac{7(\xi+\mu)}{\rho_1^3} + \frac{7(\xi-1+\mu)}{\rho_2^3} + \frac{7}{2}\left(\frac{3(\xi+\mu)^2}{\rho_1^5} - \frac{1}{\rho_1^3} + \frac{1}{\rho_2^3} - \frac{3(\xi-1+\mu)^2}{\rho_2^5}\right)\xi - \frac{3(\xi+\mu)^2}{\rho_1^5\rho_2} \\
&- \frac{2(\xi+\mu)(\xi-1+\mu)}{\rho_1^3\rho_2^3} + \frac{1}{\rho_1^3\rho_2} - \frac{3(\xi-1+\mu)^2}{\rho_1\rho_2^5} + \frac{1}{\rho_1\rho_2^3} + \frac{3(3\mu-2)(\xi+\mu)^2}{2\rho_1^5} - \frac{(3\mu-2)}{2\rho_1^3} + \frac{3(1-3\mu)(\xi-1+\mu)^2}{2\rho_2^5} \\
&- \frac{(1-3\mu)}{2\rho_2^3}\Bigg\}\Bigg]
\end{aligned} \tag{3.102}$$

$$\begin{aligned}
W^0_{\eta\eta} &= 1 - \frac{(1-\mu)}{\rho_1^3} - \frac{\mu}{\rho_2^3} + \left(3\sigma_1 - \frac{3}{2}\sigma_2\right) + \frac{3\mu(3\sigma_2-4\sigma_1)}{2\rho_2^5} + \frac{1}{{c_d}^2}\Bigg[\mu(1-\mu) - 3 + \frac{1}{2}\xi^2 + \left(-\frac{3(1-\mu)}{2\rho_1^3} - \frac{3\mu}{2\rho_2^3}\right)\xi^2 \\
&+ \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2} + \frac{(1-\mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4} + \mu(1-\mu)\Bigg\{\frac{7}{2}\left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right)\xi - \frac{\mu}{\rho_1^3} - \frac{(1-\mu)}{\rho_2^3} + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} \\
&- \frac{(3\mu-2)}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3}\Bigg\}\Bigg]
\end{aligned} \tag{3.103}$$

$$W^0_{\xi\eta} = 0 \tag{3.104}$$

$$W^0_{\dot{\xi}\dot{\xi}} = \frac{1}{{c_d}^2}\left[\frac{1}{2}\xi^2 + \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2}\right] \tag{3.105}$$

$$W^0_{\dot{\eta}\dot{\eta}} = \frac{1}{{c_d}^2}\left[\frac{3}{2}\xi^2 + \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2}\right] \tag{3.106}$$

$$W^0_{\dot{\eta}\dot{\xi}} = 0 \tag{3.107}$$

$$W^0_{\xi\dot{\xi}} = 0 \tag{3.108}$$

$$W^0_{\eta\dot{\eta}} = 0 \tag{3.109}$$

Now it will be shown that the discriminant Δ of equation (3.101) is positive at the collinear points $L_i\,(i = 1,2,3)$

To show Δ is positive it is noticed that

$$T = -4(p_1q_2 - p_2q_1)(p_5q_6 - p_6q_5) > 0 \tag{3.110}$$

as shown below

T can also be written as

$$T = -4W^0_{\xi\xi}W^0_{\eta\eta}\left(1 + W^0_{\dot{\xi}\dot{\xi}}\right)\left(1 + W^0_{\dot{\eta}\dot{\eta}}\right) \tag{3.111}$$

From equations (3.105) and (3.106) it clear that $\left(1 + W^0_{\dot{\xi}\dot{\xi}}\right) > 0$ and $\left(1 + W^0_{\dot{\eta}\dot{\eta}}\right) > 0$

Now we will study the signs of $W^0_{\xi\xi}$ and $W^0_{\eta\eta}$ at the collinear points $L_i\,(i = 1,2,3)$

Firstly we will do this at L_1, since the coordinate of this point is $\left(1+\lambda_1-\mu, 0\right)$, then $\rho_1 = 1+\lambda_1$ and $\rho_2 = \lambda_1$ where $0 < \lambda_1 << 1$, hence we can write $W_{\xi\xi}^0$ and $W_{\eta\eta}^0$ as a function in λ_1 say $h(\lambda_1)$ and $f(\lambda_1)$, respectively. Therefore, in this case from (3.102), $h(\lambda_1) \cong h(0^+) = +\infty$ and from (3.103), $f(\lambda_1) \cong f(0^+) = -\infty$, hence $W_{\xi\xi}^0 > 0$ and $W_{\eta\eta}^0 < 0$ in which $W_{\xi\xi}^0 W_{\eta\eta}^0 < 0$ and consequently $T > 0$. Hence the discriminant of the equation (3.101) is positive, and the characteristic roots can be written as $\omega_{1,2} = \pm\sigma$, $\omega_{3,4} = \pm i\tau$ where σ and τ are real.

Thus $\omega_{1,2}$ are real and $\omega_{3,4}$ are pure imaginary, hence the motion around the collinear point L_1 is unbounded and the solution is unstable.

Similarly, it can be shown that the points L_2, L_3 are also unstable.

CHAPTER FOUR

PERTURBATIONS IN CORIOLIS AND CENTRIFUGAL FORCES, OBLATENESS AND TRIAXIALITY OF THE PRIMARIES

4.1 Introduction

In this chapter the locations are determined and the stability of the triangular points taking into consideration the asphericity of the primaries and small perturbations in Coriolis and centrifugal forces is studied.

4.2 Triangular Points with Perturbations in Coriolis and Centrifugal Forces with the Bigger Primary as an Oblate Spheroid

In this section, the locations are obtained and stability region of the triangular points when the bigger primary is an oblate spheroids with small perturbations in the Coriolis and centrifugal forces is also examined.

4.2.1 Equations of motion

The small perturbations in the centrifugal and Coriolis forces are introduced by parameters $\psi = 1+\varepsilon_1;\ |\varepsilon_1| << 1, \varphi = 1+\varepsilon_2; |\varepsilon_2| << 1,$ and oblateness of the bigger primary by the parameter $A_1 = \dfrac{AE^2 - AP^2}{5R^2} << 1$ (McCuskey, 1963) where AE and AP are the equatorial and polar radii of the bigger primary, and R is the distance between the primaries. Neglecting second and higher powers of A_1, the equations of motion can be written as:

$$\ddot{\xi} - 2\varphi n_d \dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$

$$\ddot{\eta} + 2\varphi n_d \dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \tag{4.1}$$

$$\begin{aligned}
W = {} & \frac{1}{2}\psi\left(1+\frac{3}{2}A_1\right)(\xi^2+\eta^2) + \frac{(1-\mu)}{\rho_1}\left(1+\frac{3}{2\rho_1^2}A_1\right) + \frac{\mu}{\rho_2} + \frac{1}{{c_d}^2}\Bigg[-\frac{3}{2}\left\{1-\frac{1}{3}\mu(1-\mu)\right\}\psi\left(1+\frac{3}{4}A_1\right)(\xi^2+\eta^2) \\
& \quad + \frac{1}{8}\left\{\varphi\left(\dot{\xi}^2+\dot{\eta}^2\right) + \left(2+\frac{3}{2}A_1\right)\left(\xi\dot{\eta}-\eta\dot{\xi}\right)\varphi + \psi\left(1+\frac{3}{2}A_1\right)(\xi^2+\eta^2)\right\}^2 \\
& + \frac{3}{2}\left(\frac{1-\mu}{\rho_1}\left(1+\frac{3}{2\rho_1^2}A_1\right) + \frac{\mu}{\rho_2}\right)\left\{\varphi\left(\dot{\xi}^2+\dot{\eta}^2\right) + \left(2+\frac{3}{2}A_1\right)\varphi\left(\xi\dot{\eta}-\eta\dot{\xi}\right) + \left(1+\frac{3}{2\rho_1^2}A_1\right)\psi(\xi^2+\eta^2)\right\} \\
& - \frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}\left(1+\frac{A_1}{2\rho_1^2}\right)^2 + \frac{\mu^2}{\rho_2^2}\right) + \mu(1-\mu)\Bigg\{\left(\left(4+3A_1\right)\varphi\dot{\eta} + \left(\frac{7}{2}+\frac{21}{4}A_1\right)\psi\xi\right)\left(\frac{1}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right) - \frac{1}{\rho_2}\right) \\
& - \frac{1}{2}\left(1+\frac{3A_1}{2}\right)\psi\eta^2\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right) + \left(1+\frac{3A_1}{2}\right)\psi\left(\frac{-1}{\rho_1\rho_2} + \frac{\mu}{2\rho_1} - \frac{1-\mu}{\rho_1}\left(1+\frac{A_1}{2\rho_1^2}\right) + \frac{1-3\mu}{2\rho_2}\right)\Bigg\}\Bigg]
\end{aligned} \tag{4.2}$$

and n_d , the perturbed mean motion of the primaries is given by

$$n_d = 1 + \frac{3}{4}A_1 - \frac{3}{2{c_d}^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right) \tag{4.3}$$

4.2.2 Locations of the triangular points

The libration points are obtained from equations (4.1) after putting $\dot{\xi} = \dot{\eta} = \ddot{\xi} = \ddot{\eta} = 0.$

These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta} \text{ with } \dot{\xi} = \dot{\eta} = 0.$$

That is, substituting the values of $\psi = 1 + \varepsilon_1$ and $\varphi = 1 + \varepsilon_2$ in the above equations and neglecting second and higher orders terms of $\varepsilon_1, \varepsilon_2, A_1$ and their products, the following system is obtained as

$$
\begin{aligned}
&\xi-\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{\mu(\xi-1+\mu)}{\rho_2^3}+\frac{3}{2}A_1\left\{\xi-\frac{(1-\mu)(\xi+\mu)}{\rho_1^5}\right\}-\varepsilon_1\xi+\frac{1}{c_d{}^2}\left[-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\}+\frac{1}{2}\xi(\xi^2+\eta^2)\right.\\
&-\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\}+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\xi+\frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4}+\frac{\mu^2(\xi-1+\mu)}{\rho_2^4}\\
&+\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}\right.\\
&\left.+\frac{(\xi+\mu)}{\rho_1^3\rho_2}-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right\}+\varepsilon_1\left\{-3\xi\left(1-\frac{\mu(1-\mu)}{3}\right)+\xi(\xi^2+\eta^2)\right.\\
&\left.-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\right)+3\xi\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\right\}+\mu(1-\mu)\varepsilon_1\left\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right.\\
&+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\left(\frac{(\xi+\mu)}{\rho_1^3\rho_2}+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}\right.\\
&\left.\left.-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right)\right\}+A_1\left\{-\frac{9}{4}\left(1-\frac{\mu(1-\mu)}{3}\right)\xi+\frac{3}{2}(\xi^2+\eta^2)\xi+\left(-\frac{9(1-\mu)(\xi+\mu)}{4\rho_1^5}-\frac{9(1-\mu)(\xi+\mu)}{4\rho_1^3}\right.\right.\\
&\left.\left.-\frac{9\mu(\xi-1+\mu)}{4\rho_2^3}\right)(\xi^2+\eta^2)+\left(\frac{3(1-\mu)}{2\rho_1^3}+\frac{9(1-\mu)}{2\rho_1}+\frac{9\mu}{2\rho_2}\right)\xi+\frac{2(1-\mu)^2(\xi+\mu)}{\rho_1^6}\right\}+\mu(1-\mu)A_1\left\{\frac{7}{4\rho_1^3}+\frac{21}{4}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\right.\\
&+\frac{7}{2}\xi\left(-\frac{3(\xi+\mu)}{2\rho_1^5}\right)+\frac{21}{4}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{9}{4}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\frac{3(1-\mu)(\xi+\mu)}{2\rho_1^5}\\
&\left.\left.+\frac{3(\xi+\mu)}{2\rho_1^3\rho_2}+\frac{3(\xi-1+\mu)}{2\rho_1\rho_2^3}-\frac{3\mu(\xi+\mu)}{4\rho_1^3}+\frac{3(1-\mu)(\xi+\mu)}{2\rho_1^3}-\frac{3(1-3\mu)(\xi-1+\mu)}{4\rho_2^3}\right\}\right]=0
\end{aligned}
$$

and

$$\eta F=0, \qquad (4.4)$$

where,

$$
\begin{aligned}
F ={}& \left(1-\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}+\frac{3}{2}A_1\left(1-\frac{1-\mu}{\rho_1^5}\right)+\varepsilon_1\right)+\frac{1}{c_d{}^2}\Bigg[-3\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{1}{2}(\xi^2+\eta^2)+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\\
&-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)+\left(\frac{(1-\mu)^2}{\rho_1^4}+\frac{\mu^2}{\rho_2^4}\right)+\mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{(1-\mu)}{\rho_2^5}\right)\right.\\
&\left.+\frac{1}{\rho_1\rho_2^3}+\frac{1}{\rho_1^3\rho_2}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right\}+\varepsilon_1\left\{-3\left(1-\frac{1}{3}\mu(1-\mu)\right)+(\xi^2+\eta^2)+\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}\right.\right.\\
&\left.\left.+\frac{\mu}{\rho_2^3}\right)\right\}+\mu(1-\mu)\varepsilon_1\left\{\frac{7}{2}\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{(1-\mu)}{\rho_2^5}\right)+\left(\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{(3\mu-2)}{2\rho_1^3}\right.\right.\\
&\left.\left.-\frac{(1-3\mu)}{2\rho_2^3}\right)\right\}+A_1\left\{-\frac{9}{4}\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{3}{2}(\xi^2+\eta^2)+\left(\frac{-9(1-\mu)}{4\rho_1^5}-\frac{9(1-\mu)}{4\rho_1^3}-\frac{9\mu}{4\rho_2^3}\right)(\xi^2+\eta^2)+\frac{3(1-\mu)}{2\rho_1^3}+\frac{9(1-\mu)}{2\rho_1}\right.\\
&\left.+\frac{9\mu}{2\rho_2}+\frac{2(1-\mu)^2}{\rho_1^6}\right\}+\mu(1-\mu)A_1\left\{\frac{21}{4}\xi\left(-\frac{1}{\rho_1^5}-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\frac{3}{2}\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{9}{4}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{1-\mu}{\rho_2^5}\right)+\frac{3}{2}\left(\frac{(1-\mu)}{\rho_1^5}+\frac{1-\mu}{\rho_1^3}\right.\right.\\
&\left.\left.+\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{\mu}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right)\right\}\Bigg]
\end{aligned}
$$

The triangular points are the solutions of equation (4.4) with $\eta \neq 0.$ Since $\frac{1}{c_d{}^2} << 1$ and in the case $\frac{1}{c_d{}^2} \to 0$ and in the absence of small perturbations and oblateness $(i.e.\, \varepsilon_1 = \varepsilon_2 = A_1 = 0)$, one can obtain $\rho_1 = \rho_2 = 1;$ it is assumed in the relativistic R3BP that $\rho_1 = 1 + x$ and $\rho_2 = 1 + y$ where $x, y << 1,$ may be depending upon relativistic, perturbations and oblateness factors. Substituting these values in the equations (3.6), solving them for ξ, η and ignoring terms of second and higher powers of x and y, the solutions are

$$
\begin{aligned}
\xi &= x - y + \frac{1-2\mu}{2}\\
\eta &= \pm\left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}}\right)
\end{aligned}
\qquad (4.5)
$$

Substituting the values of $\rho_1, \rho_2, \xi, \eta$ and neglecting the second and higher orders terms in $x^2, y^2, \frac{x}{{c_d}^2}, \frac{y}{{c_d}^2}, \varepsilon_1, \varepsilon_2, A_1$ in equations (4.4) with $\eta \neq 0$, the following system is obtained as

$$\left\{\frac{3}{2}(1-\mu)+\frac{3}{2}A_1\left(\frac{5}{2}-\frac{3\mu}{2}\right)\right\}x-\left\{\frac{3\mu}{2}+\frac{3\mu}{2}A_1\right\}y-\frac{3\mu}{4}A_1+\frac{1}{{c_d}^2}\left\{\left(-\frac{9\mu}{16}+\frac{27\mu^2}{16}-\frac{9\mu^3}{8}\right)+\right.$$
$$\left.\left(\frac{11}{8}-\frac{125\mu}{32}+\frac{183\mu^2}{32}-\frac{63\mu^3}{8}\right)A_1+\left(\frac{8\mu^3-18\mu^2+17\mu-4}{16}\right)\varepsilon_1\right\}=0$$
$$\left\{3(1-\mu)+\frac{15}{2}(1-\mu)A_1\right\}x+3\mu y+\frac{3\mu A_1}{2}+\frac{1}{{c_d}^2}\left\{\frac{21(\mu-\mu^2)}{8}+\left(\frac{11}{4}+\frac{53\mu}{16}-\frac{49\mu^2}{16}-\frac{3\mu^3}{2}\right)A_1\right.$$
$$\left.+\left(\frac{-5\mu^2+5\mu+2}{2}\right)\varepsilon_1\right\}=0$$

(4.6)

Solving these equations for x and y, the solutions are

$$x=-\frac{\mu(2+3\mu)}{8{c_d}^2}-\left(\frac{-44+51\mu-22\mu^2+30\mu^3}{48(\mu-1){c_d}^2}\right)A_1+\left(\frac{-8\mu^3+8\mu^2-7\mu+8}{24(\mu-1){c_d}^2}\right)\varepsilon_1,$$
$$y=-\frac{(1-\mu)(5-3\mu)}{8{c_d}^2}-\left(\frac{1}{2}+\frac{74-86\mu+51\mu^2}{48{c_d}^2}\right)A_1+\left(\frac{-8\mu^2+28\mu-27}{24{c_d}^2}\right)\varepsilon_1 \qquad (4.7)$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 respectively are,

$$\xi=\frac{1-2\mu}{2}\left(1+\frac{5}{4{c_d}^2}\right)+\left(\frac{1}{2}-\frac{(30-109\mu+115\mu^2-21\mu^3)}{48(\mu-1){c_d}^2}\right)A_1+\left(\frac{-18\mu^2+33\mu-14}{24(\mu-1){c_d}^2}\right)\varepsilon_1,$$
$$\eta=\pm\left\{\frac{\sqrt{3}}{2}\left(1+\frac{1}{12{c_d}^2}\left(-5+6\mu-6\mu^2\right)\right)+\left(-\frac{\sqrt{3}}{6}-\frac{\sqrt{3}\left(-118+211\mu-159\mu^2+81\mu^3\right)}{144(\mu-1){c_d}^2}A_1\right)\right.$$
$$\left.+\left(\frac{-2\sqrt{3}}{9}+\frac{-6\mu^2+6\mu+5}{54{c_d}^2}+\frac{\sqrt{3}\left(-10\mu^3+32\mu^2-51\mu+30\right)}{72(\mu-1){c_d}^2}\right)\varepsilon_1\right\}$$

(4.8)

4.2.3 Stability of L_4

In this section, the same methodology as in section (3.2.3) is used.

Let (a,b) be the coordinates of the triangular points L_4

We set $\xi = a+\alpha, \eta = b+\beta, (\alpha, \beta << 1)$ in the equations (4.1).

First, the terms on their R.H.S, are computed, neglecting second and higher order terms, the followings are obtained as

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha,\eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta} \tag{4.9}$$

where,

$$A = \frac{3}{4}\left\{1+\frac{1}{2c_d^{\,2}}\left(2-19\mu+19\mu^2\right)\right\} + \left\{-\frac{3(8\mu-9)}{8} - \frac{\left(226-1036\mu+1758\mu^2-1056\mu^3+87\mu^4\right)}{32(\mu-1)c_d^{\,2}}\right\}A_1$$
$$+\left\{\frac{5}{4}+\frac{-31+168\mu-264\mu^2+126\mu^3}{8(\mu-1)c_d^{\,2}}\right\}\varepsilon_1,$$

$$B = \frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c_d^{\,2}}\right) + \left\{-\frac{\sqrt{3}\left(26\mu-19\right)}{8} - \frac{\sqrt{3}\left(-46+393\mu-599\mu^2-135\mu^3+417\mu^4\right)}{96(\mu-1)c_d^{\,2}}\right\}A_1$$
$$+\left\{\frac{11\sqrt{3}(1-2\mu)}{12} - \frac{\sqrt{3}(34-279\mu+635\mu^2-528\mu^3+132\mu^4)}{144(\mu-1)c_d^{\,2}}\right\}\varepsilon_1,$$

$$C = \frac{\sqrt{3}}{2c_d^{\,2}}(1-2\mu) + \left\{-\frac{\sqrt{3}\left(46\mu-35\right)}{24c_d^{\,2}}\right\}A_1 + \left\{\frac{5\sqrt{3}(1-2\mu)}{18c_d^{\,2}}\right\}\varepsilon_1 + \left\{\frac{\sqrt{3}(1-2\mu)}{2c_d^{\,2}}\right\}\varepsilon_2,$$

$$D = \frac{6-5\mu+5\mu^2}{2c^2} + \left\{\frac{\left(22-33\mu+45\mu^2\right)}{8c^2}\right\}A_1 + \left(\frac{4-15\mu+15\mu^2}{6c^2}\right)\varepsilon_1 + \left(\frac{6-5\mu+5\mu^2}{2c^2}\right)\varepsilon_2.$$

Similarly,

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha,\eta=b+\beta} = E\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta} \tag{4.10}$$

where,

$$E = \frac{3\sqrt{3}}{4}(1-2\mu)\left(1-\frac{2}{3c_d{}^2}\right) + \left\{-\frac{\sqrt{3}(26\mu-19)}{8} - \frac{\sqrt{3}\left(-46+393\mu-599\mu^2-135\mu^3+417\mu^4\right)}{96(\mu-1)c_d{}^2}\right\}A_1$$
$$+\left\{\frac{11\sqrt{3}(1-2\mu)}{12} - \frac{\sqrt{3}(34-279\mu+635\mu^2-528\mu^3+132\mu^4)}{144(\mu-1)c_d{}^2}\right\}\varepsilon_1,$$

$$B_1 = \frac{9}{4}\left\{1+\frac{7}{6c_d{}^2}\left(-2+3\mu-3\mu^2\right)\right\} + \left\{\frac{33}{8} + \frac{\left(290-1292\mu+2028\mu^2-1221\mu^3+111\mu^4\right)}{96(\mu-1)c_d{}^2}\right\}A_1$$
$$+\left\{\frac{7}{4} + \frac{55-168\mu+216\mu^2-102\mu^3}{8c_d{}^2}\right\}\varepsilon_1,$$

$$C_1 = \frac{1}{2c_d{}^2}\left(-4+\mu-\mu^2\right) + \left\{-\frac{\left(20-13\mu+9\mu^2\right)}{8c_d{}^2}\right\}A_1 + \left(\frac{-2+\mu-\mu^2}{2c_d{}^2}\right)\varepsilon_1 + \left(\frac{-4+\mu-\mu^2}{2c_d{}^2}\right)\varepsilon_2,$$

$$D_1 = -\frac{\sqrt{3}(1-2\mu)}{2c_d{}^2} + \left\{\frac{\sqrt{3}(46\mu-35)}{24c_d{}^2}\right\}A_1 + \left\{\frac{-5\sqrt{3}(1-2\mu)}{18c_d{}^2}\right\}\varepsilon_1 + \left\{\frac{-\sqrt{3}(1-2\mu)}{2c_d{}^2}\right\}\varepsilon_2.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha,\eta=b+\beta} = F\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta} \tag{4.11}$$

where,

$$F = \frac{\sqrt{3}}{2c_d{}^2}(1-2\mu) + \left\{-\frac{\sqrt{3}(46\mu-35)}{24c_d{}^2}\right\}A_1 + \left\{\frac{5\sqrt{3}(1-2\mu)}{18c_d{}^2}\right\}\varepsilon_1 + \left\{\frac{\sqrt{3}(1-2\mu)}{2c_d{}^2}\right\}\varepsilon_2,$$

$$B_2 = -\frac{1}{2c_d{}^2}\left(4-\mu+\mu^2\right) + \left\{-\frac{\left(20-13\mu+9\mu^2\right)}{8c_d{}^2}\right\}A_1 + \left(\frac{-2+\mu-\mu^2}{2c_d{}^2}\right)\varepsilon_1 + \left(\frac{-4+\mu+\mu^2}{2c_d{}^2}\right)\varepsilon_2,$$

$$C_2 = \frac{1}{4c_d^{\,2}}\left(17-2\mu+2\mu^2\right)+\left\{\frac{\left(23-10\mu+6\mu^2\right)}{8c_d^{\,2}}\right\}A_1+\left(\frac{1-\mu+\mu^2}{2c_d^{\,2}}\right)\varepsilon_1+\left(\frac{10-\mu+\mu^2}{2c_d^{\,2}}\right)\varepsilon_2,$$

$$D_2 = -\frac{\sqrt{3}}{4c_d^{\,2}}\left(1-2\mu\right)+\left\{\frac{\sqrt{3}\left(14\mu-13\right)}{24c_d^{\,2}}\right\}A_1+\left\{\frac{\sqrt{3}(1-2\mu)}{9c_d^{\,2}}\right\}\varepsilon_1+\left\{\frac{\sqrt{3}(1-2\mu)}{2c_d^{\,2}}\right\}\varepsilon_2.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = A_3\dot{\alpha}+B_3\dot{\beta}+C_3\ddot{\alpha}+D_3\ddot{\beta} \tag{4.12}$$

where,

$$A_3 = \frac{1}{2c_d^{\,2}}\left(6-5\mu+5\mu^2\right)+\left\{\frac{\left(22-33\mu+45\mu^2\right)}{8c_d^{\,2}}\right\}A_1+\left(\frac{4-15\mu+15\mu^2}{6c_d^{\,2}}\right)\varepsilon_1+\left(\frac{6-5\mu+5\mu^2}{2c_d^{\,2}}\right)\varepsilon_2,$$

$$B_3 = -\frac{\sqrt{3}}{2c_d^{\,2}}\left(1-2\mu\right)+\left\{\frac{\sqrt{3}\left(46\mu-35\right)}{24c_d^{\,2}}\right\}A_1+\left\{-\frac{5\sqrt{3}(1-2\mu)}{18c_d^{\,2}}\right\}\varepsilon_1-\left\{\frac{\sqrt{3}(1-2\mu)}{2c_d^{\,2}}\right\}\varepsilon_2,$$

$$C_3 = -\frac{\sqrt{3}}{4c_d^{\,2}}\left(1-2\mu\right)+\left\{\frac{\sqrt{3}\left(14\mu-13\right)}{24c_d^{\,2}}\right\}A_1+\left\{\frac{\sqrt{3}(1-2\mu)}{9c_d^{\,2}}\right\}\varepsilon_1-\left\{\frac{\sqrt{3}(1-2\mu)}{4c_d^{\,2}}\right\}\varepsilon_2,$$

$$D_3 = \frac{3\left(5-2\mu+2\mu^2\right)}{4c_d^{\,2}}+\left\{\frac{\left(25-30\mu+18\mu^2\right)}{8c_d^{\,2}}\right\}A_1+\left(\frac{7-3\mu+3\mu^2}{6c_d^{\,2}}\right)\varepsilon_1-\left\{\frac{\sqrt{3}(1-2\mu)}{4c_d^{\,2}}\right\}\varepsilon_2.$$

Thus, the variational equations of motion corresponding to equation (4.1), on making use of equation (4.3), can be expressed as

$$\begin{aligned} &p_1\ddot{\alpha}+p_2\ddot{\beta}+p\dot{\alpha}+p_4\dot{\beta}+p_5\alpha+p_6\beta=0, \\ &q_1\ddot{\alpha}+q_2\ddot{\beta}+q_3\dot{\alpha}+q_4\dot{\beta}+q_5\alpha+q_6\beta=0. \end{aligned} \tag{4.13}$$

where,

$$p_1 = 1 + C_2,\, p_2 = D_2,\, p_3 = F - C,\; p_4 = \left\{ B_2 - 2\left(1 + \frac{3}{4}A_1 + \varepsilon_2 - \frac{3}{2{c_d}^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)(1+\varepsilon_2)\right) - D \right\},$$

$$p_5 = -A\;,\, p_6 = -B,$$

$$q_1 = C_3, q_2 = 1 + D_3, q_3 = 2\left(1 + \frac{3}{4}A_1 + \varepsilon_2 - \frac{3}{2{c_d}^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)(1+\varepsilon_2)\right) - C_1 + A_3\,, q_4 = B_3 - D_1,$$

$$q_5 = -E, q_6 = -B_1$$

Then, the corresponding characteristic equation is

$$(p_1q_2 - p_2q_1)\lambda^4 + (p_1q_6 + p_5q_2 + p_3q_4 - p_6q_1 - p_2q_5 - p_4q_3)\lambda^2 + p_5q_6 - p_6q_5 = 0 \quad (4.14)$$

Substituting the values of $p_i, q_i, i = 1, 2, ..., 6$ in equation (4.14) and neglecting second and higher powers of small quantities, the characteristic equation (4.14) becomes

$$\lambda^4 + b\lambda^2 + d = 0 \qquad (4.15)$$

where,

$$b = \left(1 - \frac{9}{{c_d}^2}\right) + \left\{ -\frac{3}{2} + 3\mu + \frac{\left(-80 - 108\mu + 105\mu^2 - 18\mu^3\right)}{8{c_d}^2} \right\} A_1 + \left\{ -3 + \frac{68 - 25\mu + 25\mu^2}{4{c_d}^2} \right\} \varepsilon_1$$
$$+ \left\{ 8 + \frac{-147 + 30\mu - 30\mu^2}{2{c_d}^2} \right\} \varepsilon_2,$$

$$d = \frac{27\mu(1-\mu)}{4} + \frac{9\mu\left(-65 + 77\mu - 24\mu^2 + 12\mu^3\right)}{8{c_d}^2} + \left\{ \frac{117\mu(1-\mu)}{4} + \right.$$
$$\left. \frac{3\left(80 - 7245\mu + 9624\mu^2 - 3366\mu^3 + 846\mu^4\right)}{64{c_d}^2} \right\} A_1 +$$
$$\left\{ \frac{33\mu(1-\mu)}{2} - \frac{3\mu(1867 - 2082\mu + 540\mu^2 - 336\mu^3}{32{c_d}^2} \right\} \varepsilon_1 + \left\{ \frac{-243\mu + 324\mu^2 - 162\mu^3 + 81\mu^4}{4{c_d}^2} \right\} \varepsilon_2.$$

For $\frac{1}{c_d{}^2} \to 0$ and in the absence of small perturbations in the centrifugal and Coriolis forces and oblateness $(i.e.\ \varepsilon_1 = \varepsilon_2 = A_1 = 0)$, equation (4.15) reduces to its well-known classical restricted problem form (see e.g. Szebehely, 1967a):

$$\lambda^4 + \lambda^2 + \frac{27}{4}\mu(1-\mu) = 0.$$

The discriminant of equation (4.15) is

$$\Delta = \left(\frac{-54}{c_d{}^2} - \frac{1269}{8c_d{}^2}A_1 - \frac{126}{c_d{}^2}\varepsilon_1 - \frac{81}{c_d{}^2}\varepsilon_2 \right)\mu^4 + \left(\frac{108}{c_d{}^2} + \frac{5013}{8c_d{}^2}A_1 + \frac{405}{2c_d{}^2}\varepsilon_1 - \frac{126}{c_d{}^2}\varepsilon_2 \right)\mu^3 + \Big(27 + 117A_1 + 66\varepsilon_1$$
$$- \frac{7113}{4c_d{}^2}A_1 - \frac{3073}{4c_d{}^2}\varepsilon_1 - \frac{354}{c_d{}^2}\varepsilon_2 - \frac{693}{2c_d{}^2}\Big)\mu^2 + \left(-27 - 111A_1 - 66\varepsilon_1 + \frac{20439}{16c_d{}^2}A_1 + \frac{5501}{8c_d{}^2}\varepsilon_1 + \frac{273}{c_d{}^2}\varepsilon_2 + \frac{585}{2c_d{}^2} \right)\mu$$
$$+1 - 3A_1 - 6\varepsilon_1 + 16\varepsilon_2 - \frac{8}{c_d{}^2}A_1 + \frac{88}{c_d{}^2}\varepsilon_1 - \frac{291}{c_d{}^2}\varepsilon_2 - \frac{18}{c_d{}^2}. \tag{4.16}$$

Its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \tag{4.17}$$

where,

$$b = \left(1 - \frac{9}{c_d{}^2}\right) + \left\{ -\frac{3}{2} + 3\mu + \frac{\left(-80 - 108\mu + 105\mu^2 - 18\mu^3\right)}{8c_d{}^2} \right\} A_1 + \left\{ -3 + \frac{68 - 25\mu + 25\mu^2}{4c_d{}^2} \right\}\varepsilon_1$$
$$+ \left\{ 8 + \frac{-147 + 30\mu - 30\mu^2}{2c_d{}^2} \right\}\varepsilon_2,$$

From (4.16), we have

$$\frac{d\Delta}{d\mu} = 4\left(\frac{-54}{c_d{}^2} - \frac{1269}{8c_d{}^2}A_1 - \frac{126}{c_d{}^2}\varepsilon_1 - \frac{81}{c_d{}^2}\varepsilon_2 \right)\mu^3 + 3\left(\frac{108}{c_d{}^2} + \frac{5013}{c_d{}^2}A_1 + \frac{405}{2c_d{}^2}\varepsilon_1 + \frac{162}{c_d{}^2}\varepsilon_2 \right)\mu^2 + 2\Big(27 + 117A_1 + 66\varepsilon_1$$
$$- \frac{7113}{4c_d{}^2}A_1 - \frac{3073}{4c_d{}^2}\varepsilon_1 - \frac{354}{c_d{}^2}\varepsilon_2 - \frac{693}{2c_d{}^2}\Big)\mu + \left(-27 - 111A_1 - 66\varepsilon_1 + \frac{20439}{16c_d{}^2}A_1 + \frac{5501}{8c_d{}^2}\varepsilon_1 + \frac{273}{c_d{}^2}\varepsilon_2 + \frac{585}{2c_d{}^2} \right) < 0$$

for $\mu \in \left(0, \frac{1}{2}\right]$ (4.18)

But

$$(\Delta)_{\mu=0} = 1 - \frac{18}{c_d^{\,2}} - 3A_1 - \frac{8}{c_d^{\,2}} A_1 - 6\varepsilon_1 + \frac{88}{c_d^{\,2}}\varepsilon_1 + 16\varepsilon_2 - \frac{291}{c_d^{\,2}}\varepsilon_2 > 0$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4} + \frac{207}{4c_d^{\,2}} - \frac{117}{4}A_1 + \frac{32585}{128c_d^{\,2}}A_1 + \frac{4115}{16c_d^{\,2}}\varepsilon_1 + 16\varepsilon_2 - \frac{3645}{16c_d^{\,2}}\varepsilon_2 < 0 \qquad (4.19)$$

$$\frac{d^2\Delta}{d\mu^2} = 12\left(-\frac{54}{c_d^{\,2}} - \frac{1269}{8c_d^{\,2}}A_1 - \frac{126}{c_d^{\,2}}\varepsilon_1\right)\mu^2 + 6\left(\frac{108}{c_d^{\,2}} + \frac{5013}{c_d^{\,2}}A_1 + \frac{405}{2c_d^{\,2}}\varepsilon_1 + \frac{162}{c_d^{\,2}}\varepsilon_2\right)\mu^2 +$$
$$2\left(27 + 117A_1 + 66\varepsilon_1 - \frac{7113}{4c_d^{\,2}}A_1 - \frac{3073}{4c_d^{\,2}}\varepsilon_1 - \frac{354}{c_d^{\,2}}\varepsilon_2 - \frac{693}{2c_d^{\,2}}\right) > 0 \ \forall \ \mu \in \left(0, \frac{1}{2}\right] \qquad (4.20)$$

This implies that $\frac{d\Delta}{d\mu}$ is monotonic increasing in $\left(0, \frac{1}{2}\right]$

But

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=0} = -27 - 66\varepsilon_1 - 111A_1 + \frac{5501}{8c_d^{\,2}}\varepsilon_1 + \frac{273}{c_d^{\,2}}\varepsilon_2 + \frac{585}{2c_d^{\,2}} + \frac{20439}{16c_d^{\,2}}A_1 < 0 \ \forall \ \mu \in \left(0, \frac{1}{2}\right] \qquad (4.21)$$

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} = \frac{33}{4c_d^{\,2}}\varepsilon_1 + \frac{52143}{16c_d^{\,2}}A_1 + 6A_1 \qquad (4.22)$$

In order to study the monotonicity of Δ, two cases are considered

Case 1: $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} \leq 0$

Hence for this case, the table of variation of Δ is given below

Table 4.1 : Variation of Δ

μ	0		1/2
$\frac{d^2\Delta}{d\mu^2}$		+	
$\frac{d\Delta}{d\mu}$		−	
Δ	$(\Delta)_{\mu=0}$	↘	$(\Delta)_{\mu=1/2}$

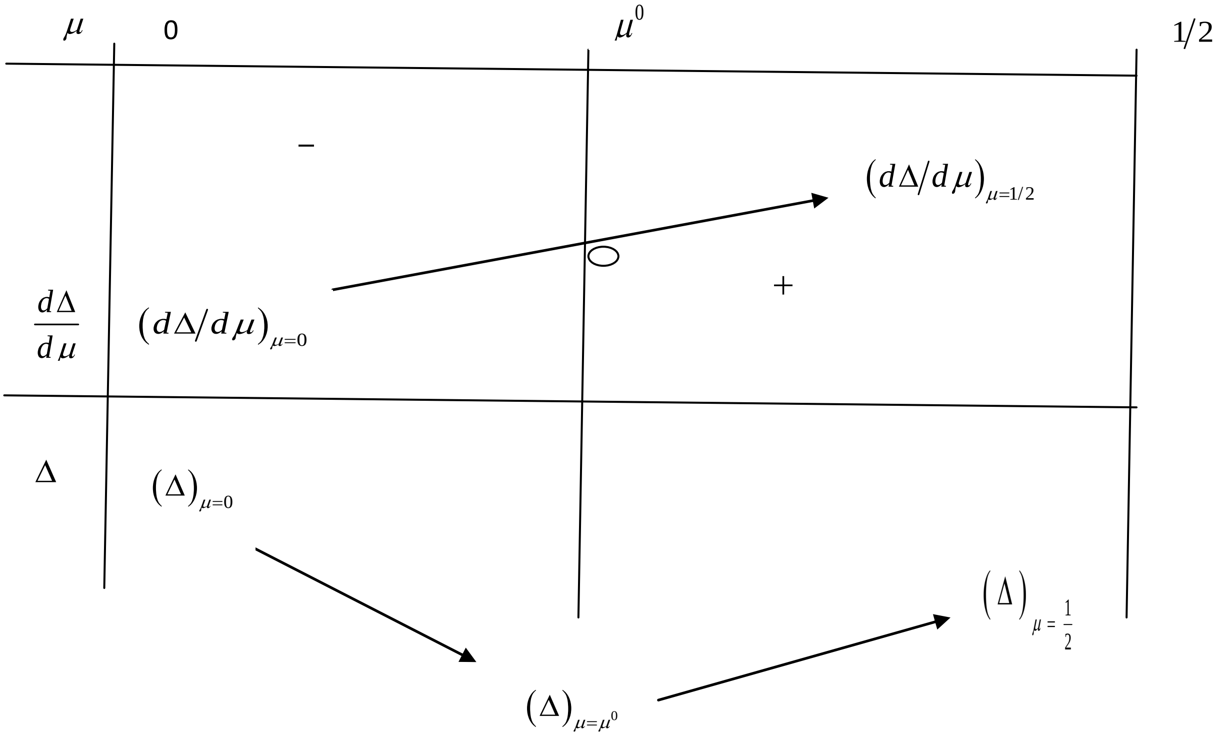
From the above table it can be seen that Δ is monotone decreasing in $\left(0,\frac{1}{2}\right]$. Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and is monotone and continuous and by the intermediate value property there is one value of μ say. μ_c' in $\left(0,\frac{1}{2}\right]$ for which $\Delta = 0$.

Case 2: $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$

Since from equation (4.20), $\frac{d\Delta}{d\mu}$ is monotone increasing in $\left(0,\frac{1}{2}\right]$ and $\left(\frac{d\Delta}{d\mu}\right)_{\mu=0} < 0$ and $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$, this implies that there xists $\mu^0 \in \left(0,\frac{1}{2}\right]$ such that $\left(\frac{d\Delta}{d\mu}\right)_{\mu=\mu^0} = 0$, hence $\frac{d\Delta}{d\mu} \le 0\ \forall\ \mu \in \left(0,\mu^0\right]$ and $\frac{d\Delta}{d\mu} \ge 0\ \forall\ \mu \in \left[\mu^0,\frac{1}{2}\right]$.

Hence, the following table of variation is given below.

Table 4.2: Variation of Δ



Since $(\Delta)_{\mu=0} > 0$ and $(\Delta)_{\mu=\frac{1}{2}} < 0$, it can be concluded from the above table that $(\Delta)_{\mu^0} < 0$, hence $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\mu^0}$ are of opposite signs and Δ is monotonic decreasing and continuous in $\left(0,\frac{1}{2}\right]$ and by the intermediate value property there is one value of μ say. μ'' in $\left(0,\mu^0\right]$ for which $\Delta = 0$. Hence $\mu_c = \mu_c' = \mu_c''$.

Solving the equation $\Delta = 0$, using equation (4.16), the critical value of the mass parameter is obtained as

$$\mu_c = \frac{1}{2} - \frac{1}{18}\sqrt{69} - \frac{17\sqrt{69}}{486{c_d}^2} - \frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1 + \left(\frac{-19733+15493\sqrt{69}}{536544{c_d}^2}\right)A_1 + \frac{4\left(36\varepsilon_2 - 19\varepsilon_1\right)}{27\sqrt{69}} + \left(\frac{34155+175301\sqrt{69}}{804816{c_d}^2}\right)\varepsilon_1 - \left(\frac{47\sqrt{69}}{81{c_d}^2}\right)\varepsilon_2 \tag{4.23}$$

There are three possible cases regarding the sign of the discriminant Δ

i. When $0<\mu<\mu_c, \Delta>0,$ the values of λ^2 given by equation (4.17) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

ii. When $\mu_c<\mu\leq\frac{1}{2}, \Delta<0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

iii. When $\mu=\mu_c, \Delta=0$, the values of λ^2 given by equation (4.17) are the same. Hence the solutions contain secular terms. This induces instability of the triangular points.

Hence, the stability region is

$$0<\mu<\mu_0-\frac{17\sqrt{69}}{486{c_d}^2}-\frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1+\left(\frac{-19733+15493\sqrt{69}}{536544{c_d}^2}\right)A_1+\frac{4\left(36\varepsilon_2-19\varepsilon_1\right)}{27\sqrt{69}}+\left(\frac{34155+175301\sqrt{69}}{804816{c_d}^2}\right)\varepsilon_1-\left(\frac{47\sqrt{69}}{81{c_d}^2}\right)\varepsilon_2, \tag{4.24}$$

where $\mu_0=0.03852...$ is Routh's value

Equation (4.24) can be written as:

$$0<\mu<\mu_0+p \tag{4.25}$$

with

$$p=-\frac{17\sqrt{69}}{486{c_d}^2}-\frac{1}{9}\left(1+\frac{13}{\sqrt{69}}\right)A_1+\left(\frac{-19733+15493\sqrt{69}}{536544{c_d}^2}\right)A_1+\frac{4\left(36\varepsilon_2-19\varepsilon_1\right)}{27\sqrt{69}}+\left(\frac{34155+175301\sqrt{69}}{804816{c_d}^2}\right)\varepsilon_1-\left(\frac{47\sqrt{69}}{81{c_d}^2}\right)\varepsilon_2.$$

4.3 Triangular Points with Perturbations in the Coriolis and Centrifugal Forces with a Triaxial Bigger Primary

In this section, the locations are obtained and stability region when the bigger primary is triaxial with small perturbations in the Coriolis and centrifugal forces is also examined.

4.3.1 Equations of motion

The small perturbations in the centrifugal and Coriolis forces and triaxiality of the bigger primary are introduced with the help of parameters $\psi = 1+\varepsilon_1; |\varepsilon_1| << 1,\ \varphi = 1+\varepsilon_2; |\varepsilon_2| << 1,\ \sigma_i\ (i=1,2)$ with $\sigma_i << 1,$ respectively, where $\sigma_1 = \dfrac{h^2 - f^2}{5R^2},\ \sigma_2 = \dfrac{b^2 - f^2}{5R^2}.$ (McCuskey, 1963) with h, b, f as lengths of its semi-axes and R is the dimensional distance between the primaries. As Katour *et al*. (2014), the triaxiality coefficients are not included in the relativistic part since the magnitude of those terms is so small due to c_d^{-2}, where c_d is the speed of light. Consequently, ignoring second and higher powers of σ_i, the equations of motion can be written as:

$$
\begin{aligned}
\ddot{\xi} - 2\varphi n_d \dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right) \\
\ddot{\eta} + 2\varphi n_d \dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)
\end{aligned}
\qquad (4.26)
$$

$$W=\frac{1}{2}\left(1+\frac{3}{2}(2\sigma_1-\sigma_2)\right)\psi(\xi^2+\eta^2)+\frac{(1-\mu)}{\rho_1}+\frac{\mu}{\rho_2}+\frac{1-\mu}{2\rho_1^3}(2\sigma_1-\sigma_2)++\frac{3(1-\mu)\eta^2}{2\rho_1^5}(\sigma_2-\sigma_1)$$
$$+\frac{1}{{c_d}^2}\left[-\frac{3}{2}\left\{1-\frac{1}{3}\mu(1-\mu)\right\}(\xi^2+\eta^2)\psi+\frac{1}{8}\left\{\varphi\left(\dot{\xi}^2+\dot{\eta}^2\right)+2\varphi\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+\psi(\xi^2+\eta^2)\right\}^2\right.$$
$$+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\left\{\varphi\left(\dot{\xi}^2+\dot{\eta}^2\right)+2\varphi\left(\xi\dot{\eta}-\eta\dot{\xi}\right)+\psi(\xi^2+\eta^2)\right\}-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\right)$$
$$\left.+\mu(1-\mu)\left\{\left(4\varphi\dot{\eta}+\frac{7}{2}\psi\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)-\frac{\eta^2}{2}\psi\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\psi\left(\frac{-1}{\rho_1\rho_2}+\frac{3\mu-2}{2\rho_1}+\frac{1-3\mu}{2\rho_2}\right)\right\}\right],$$

(4.27)

and n_d the perturbed mean motion of the primaries is given by

$$n_d=1+\frac{3}{4}(2\sigma_1-\sigma_2)-\frac{3}{2{c_d}^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{4.28}$$

4.3.2 Locations of the triangular points

The libration points are obtained from equation (4.26) after putting $\dot{\xi}=\dot{\eta}=\ddot{\xi}=\ddot{\eta}=0.$

These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi}=0=\frac{\partial W}{\partial \eta} \text{ with } \dot{\xi}=\dot{\eta}=0.$$

Substituting for simplicity, $\psi=1+\varepsilon_1$ and $\varphi=1+\varepsilon_2$ in the aforesaid equations and neglecting second and higher powers of $\varepsilon_i,\sigma_i\,(i=1,2)$, and also their products, the following system are obtained

$$\xi-\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}-\frac{\mu(\xi-1+\mu)}{\rho_2^3}+\left(3\sigma_1-\frac{3}{2}\sigma_2\right)\xi-\frac{3(1-\mu)(\xi+\mu)(2\sigma_1-\sigma_2)}{2\rho_1^5}$$

$$-\frac{15(1-\mu)(\xi+\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7}-\varepsilon_1\xi+\frac{1}{{c_d}^2}\Bigg[-3\xi\left\{1-\frac{\mu(1-\mu)}{3}\right\}+\frac{1}{2}\xi(\xi^2+\eta^2)-\frac{3}{2}(\xi^2+\eta^2)\Bigg\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}$$

$$+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\Bigg\}+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\xi+\frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4}+\frac{\mu^2(\xi-1+\mu)}{\rho_2^4}+\mu(1-\mu)\Bigg\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)$$

$$+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}+\frac{(\xi+\mu)}{\rho_1^3\rho_2}$$

$$-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\Bigg\}+\varepsilon_1\Bigg\{-3\xi\left(1-\frac{\mu(1-\mu)}{3}\right)+\xi(\xi^2+\eta^2)-\frac{3}{2}(\xi^2+\eta^2)\Bigg(\frac{(1-\mu)(\xi+\mu)}{\rho_1^3}$$

$$+\frac{\mu(\xi-1+\mu)}{\rho_2^3}\Bigg)+3\xi\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)\Bigg\}+\mu(1-\mu)\varepsilon_1\Bigg\{\frac{7}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)+\frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3}+\frac{(\xi-1+\mu)}{\rho_2^3}\right)$$

$$\frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5}+\frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right)+\left(\frac{(\xi+\mu)}{\rho_1^3\rho_2}+\frac{(\xi-1+\mu)}{\rho_1\rho_2^3}-\frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right)\Bigg\}\Bigg]=0$$

and

$$\eta F=0, \tag{4.29}$$

where,

$$F=\left(1-\frac{1-\mu}{\rho_1^3}-\frac{\mu}{\rho_2^3}\right)+\left(3\sigma_1-\frac{3}{2}\sigma_2\right)+\frac{3(1-\mu)}{\rho_1^5}\left(\frac{3}{2}\sigma_2-2\sigma_1\right)-\frac{15(1-\mu)(\sigma_2-\sigma_1)\eta^2}{2\rho_1^7}+\varepsilon_1$$

$$+\frac{1}{{c_d}^2}\Bigg[-3\left(1-\frac{\mu(1-\mu)}{3}\right)+\frac{1}{2}(\xi^2+\eta^2)+3\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)+\left(\frac{(1-\mu)^2}{\rho_1^4}+\frac{\mu^2}{\rho_2^4}\right)$$

$$+\mu(1-\mu)\Bigg\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{(1-\mu)}{\rho_2^5}\right)+\frac{1}{\rho_1\rho_2^3}+\frac{1}{\rho_1^3\rho_2}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\Bigg\}+$$

$$\varepsilon_1\Bigg\{-3\left(1-\frac{\mu(1-\mu)}{3}\right)+(\xi^2+\eta^2)+\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\right)-\frac{3}{2}(\xi^2+\eta^2)\left(\frac{1-\mu}{\rho_1^3}+\frac{\mu}{\rho_2^3}\right)\Bigg\}+\mu(1-\mu)\varepsilon_1\Bigg\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)$$

$$-\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\right)+\frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5}+\frac{(1-\mu)}{\rho_2^5}\right)+\left(\frac{1}{\rho_1\rho_2^3}-\frac{1}{\rho_1^3\rho_2}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right)\Bigg\}\Bigg]$$

The triangular points are the solutions of equations (4.29) with $\eta \neq 0$. Since $\frac{1}{c_d^{\,2}} << 1$ and in the case $\frac{1}{c_d^{\,2}} \to 0$ and in the absence of small perturbations and triaxiality $(i.e.\,\varepsilon_1 = \varepsilon_2 = \sigma_2 = \sigma_1 = 0)$, one can obtain $\rho_1 = \rho_2 = 1;$ we assume in the relativistic R3BP that $\rho_1 = 1 + x$ and $\rho_2 = 1 + y$ where $x, y << 1,$ may be depending upon relativistic, triaxiality, centrifugal and Coriolis factors. Substituting these values in the equations (3.6), solving them for ξ, η and ignoring terms of second and higher powers of x and y, and their products, the solutions are

$$\xi = x - y + \frac{1-2\mu}{2}$$
$$\eta = \pm\left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}}\right) \tag{4.30}$$

Substituting the values of $\rho_1, \rho_2, \xi, \eta$ from the above in equations (4.29) and neglecting second and higher powers of x, y, $\sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2$, the following system is written as

$$\frac{3}{2}(1-\mu)x - \frac{3\mu}{2}y + \frac{45}{16}(\sigma_1 - \sigma_2) + \frac{(57\sigma_2 - 69\sigma_1)}{16}\mu + \frac{1}{c_d^{\,2}}\left\{\left(-\frac{9\mu}{16} + \frac{27\mu^2}{16} - \frac{9\mu^3}{8}\right) - \left(\frac{8\mu^3 - 18\mu^2 + 17\mu - 4}{8}\right)\varepsilon_1\right\} = 0$$

$$3(1-\mu)x + 3\mu y + + \frac{21}{8}(\sigma_1 - \sigma_2) + \frac{3}{8}(\sigma_1 + 3\sigma_2)\mu + \frac{1}{c_d^{\,2}}\left\{\frac{21(\mu - \mu^2)}{8} + \left(\frac{-5\mu^2 + 5\mu + 2}{2}\right)\varepsilon_1\right\} = 0$$

(4.31)

Solving these equations for x and y, the solutions are

$$x = -\frac{\mu(2+3\mu)}{8c_d^{\,2}} + \frac{11}{8}(\sigma_2 - \sigma_1) + \left(\frac{-8\mu^3 + 8\mu^2 - 7\mu + 8}{24(\mu-1)c_d^{\,2}}\right)\varepsilon_1,$$
$$y = -\frac{(1-\mu)(5-3\mu)}{8c_d^{\,2}} + \left(\frac{1}{2\mu} - \frac{3}{2}\right)\sigma_1 + \left(-\frac{1}{2\mu} + 1\right)\sigma_2 + \left(\frac{-8\mu^2 + 28\mu - 27}{24c_d^{\,2}}\right)\varepsilon_1. \tag{4.32}$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 respectively are,

$$\xi = \frac{1-2\mu}{2}\left(1+\frac{5}{4c_d^{\ 2}}\right)+\left(\frac{1}{8}-\frac{1}{2\mu}\right)\sigma_1+\left(\frac{1}{2\mu}+\frac{3}{8}\right)\sigma_2+\left(\frac{-18\mu^2+33\mu-14}{24(\mu-1)c_d^{\ 2}}\right)\varepsilon_1,$$

$$\eta = \pm\left\{\frac{\sqrt{3}}{2}\left[1+\frac{1}{12c_d^{\ 2}}\left(-5+6\mu-6\mu^2\right)+\frac{2}{3}\left\{\left(-\frac{1}{2\mu}-\frac{23}{8}\right)\sigma_1+\left(\frac{19}{8}-\frac{1}{2\mu}\right)\sigma_2\right\}\right]+\left(\frac{-2\sqrt{3}}{9}\right.\right.$$
$$\left.\left.+\frac{6\mu^2+6\mu-5}{54c_d^{\ 2}}+\frac{\sqrt{3}\left(-10\mu^3+32\mu^2-51\mu+30\right)}{72(\mu-1)c_d^{\ 2}}\right)\varepsilon_1\right\}$$

(4.33)

4.3.3 Stability of L_4

Let (a,b) be the coordinates of the triangular point L_4.

Setting $\xi = a+\alpha, \eta = b+\beta, (\alpha, \beta << 1)$ in the equations (4.26).

The terms of their R.H.S. are computed, neglecting second and higher order terms, the following are obtained as

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha,\,\eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta} \qquad (4.34)$$

where,

$$A = \frac{3}{4}\left\{1+\frac{1}{2c_d^{\ 2}}\left(2-19\mu+19\mu^2\right)\right\}+\frac{3\left(15\mu^2+19\mu-8\right)}{16\mu}\sigma_1-\frac{3\left(31\mu^2+\mu-8\right)}{16\mu}\sigma_2$$
$$+\left\{\frac{5}{4}+\frac{-31+168\mu-264\mu^2+126\mu^3}{8(\mu-1)c_d^{\ 2}}\right\}\varepsilon_1,$$

$$B=\frac{3\sqrt{3}}{4}\left(1-2\mu\right)\left(1-\frac{2}{3{c_d}^2}\right)-\frac{\sqrt{3}\left(89\mu^2-47\mu-8\right)}{16\mu}\sigma_1+\frac{\sqrt{3}\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2$$
$$\left\{\frac{11\sqrt{3}(1-2\mu)}{12}-\frac{\sqrt{3}(34-279\mu+635\mu^2-528\mu^3+134\mu^4)}{144(\mu-1){c_d}^2}\right\}\varepsilon_1,$$

$$C=\frac{\sqrt{3}}{2{c_d}^2}\left(1-2\mu\right)+\left\{\frac{5\sqrt{3}(1-2\mu)}{18{c_d}^2}\right\}\varepsilon_1+\left\{\frac{\sqrt{3}(1-2\mu)}{2{c_d}^2}\right\}\varepsilon_2,$$

$$D=\frac{6-5\mu+5\mu^2}{2{c_d}^2}+\left(\frac{4-15\mu+15\mu^2}{6{c_d}^2}\right)\varepsilon_1+\left(\frac{6-5\mu+5\mu^2}{2{c_d}^2}\right)\varepsilon_2.$$

Similarly, we obtain

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha,\eta=b+\beta}=E\alpha+B_1\beta+C_1\dot{\alpha}+D_1\dot{\beta} \tag{4.35}$$

where,

$$E=\frac{3\sqrt{3}}{4}\left(1-2\mu\right)\left(1-\frac{2}{3{c_d}^2}\right)-\frac{\sqrt{3}\left(89\mu^2-47\mu+8\right)}{16\mu}\sigma_1+\frac{\sqrt{3}\left(37\mu^2-9\mu+8\right)}{16\mu}\sigma_2$$
$$+\left\{\frac{11\sqrt{3}(1-2\mu)}{12}-\frac{\sqrt{3}(34-279\mu+635\mu^2-528\mu^3+132\mu^4)}{144(\mu-1){c_d}^2}\right\}\varepsilon_1,$$

$$B_1=\frac{9}{4}\left\{1+\frac{7}{6{c_d}^2}\left(-2+3\mu-3\mu^2\right)\right\}-\frac{3\left(15\mu^2-29\mu-8\right)}{16\mu}\sigma_1+\frac{3\left(15\mu^2-7\mu-8\right)}{16\mu}\sigma_2$$
$$+\left\{\frac{7}{4}+\frac{55-168\mu+216\mu^2-102\mu^3}{8{c_d}^2}\right\}\varepsilon_1,$$

$$C_1=\frac{1}{2{c_d}^2}\left(-4+\mu-\mu^2\right)+\left(\frac{-2+\mu-\mu^2}{2{c_d}^2}\right)\varepsilon_1+\left(\frac{-4+\mu-\mu^2}{2{c_d}^2}\right)\varepsilon_2,$$

$$D_1=-\frac{\sqrt{3}\left(1-2\mu\right)}{2{c_d}^2}+\left\{\frac{-5\sqrt{3}(1-2\mu)}{18{c_d}^2}\right\}\varepsilon_1+\left\{\frac{-\sqrt{3}(1-2\mu)}{2{c_d}^2}\right\}\varepsilon_2.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)_{\xi=a+\alpha,\eta=b+\beta} = F\dot{\alpha} + B_2\dot{\beta} + C_2\ddot{\alpha} + D_2\ddot{\beta} \tag{4.36}$$

where,

$$F = \frac{\sqrt{3}}{2c_d^{\ 2}}(1-2\mu) + \left\{\frac{5\sqrt{3}(1-2\mu)}{18c_d^{\ 2}}\right\}\varepsilon_1 + \left\{\frac{\sqrt{3}(1-2\mu)}{2c_d^{\ 2}}\right\}\varepsilon_2,$$

$$B_2 = -\frac{1}{2c_d^{\ 2}}\left(4-\mu+\mu^2\right) + \left(\frac{-2+\mu-\mu^2}{2c_d^{\ 2}}\right)\varepsilon_1 + \left(\frac{-4+\mu-\mu^2}{2c_d^{\ 2}}\right)\varepsilon_2,$$

$$C_2 = \frac{1}{4c_d^{\ 2}}\left(17-2\mu+2\mu^2\right) + \left(\frac{1-\mu+\mu^2}{2c_d^{\ 2}}\right)\varepsilon_1 + \left(\frac{10-\mu+\mu^2}{2c_d^{\ 2}}\right)\varepsilon_2,$$

$$D_2 = -\frac{\sqrt{3}}{4c_d^{\ 2}}(1-2\mu) + \left\{\frac{\sqrt{3}(1-2\mu)}{9c_d^{\ 2}}\right\}\varepsilon_1 + \left\{\frac{\sqrt{3}(1-2\mu)}{2c_d^{\ 2}}\right\}\varepsilon_2.$$

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right)_{\xi=a+\alpha,\eta=b+\beta} = A_3\dot{\alpha} + B_3\dot{\beta} + C_3\ddot{\alpha} + D_3\ddot{\beta} \tag{4.37}$$

where,

$$A_3 = \frac{1}{2c_d^{\ 2}}\left(6-5\mu+5\mu^2\right) + \left(\frac{4-15\mu+15\mu^2}{6c_d^{\ 2}}\right)\varepsilon_1 + \left(\frac{6-5\mu+5\mu^2}{2c_d^{\ 2}}\right)\varepsilon_2,$$

$$B_3 = -\frac{\sqrt{3}}{2c_d^{\ 2}}(1-2\mu) + \left\{-\frac{5\sqrt{3}(1-2\mu)}{18c_d^{\ 2}}\right\}\varepsilon_1 - \left\{\frac{\sqrt{3}(1-2\mu)}{2c_d^{\ 2}}\right\}\varepsilon_2,$$

$$C_3 = -\frac{\sqrt{3}}{4c_d^{\ 2}}(1-2\mu) + \left\{\frac{\sqrt{3}(1-2\mu)}{9c_d^{\ 2}}\right\}\varepsilon_1 - \left\{\frac{\sqrt{3}(1-2\mu)}{4c_d^{\ 2}}\right\}\varepsilon_2,$$

$$D_3 = \frac{3\left(5-2\mu+2\mu^2\right)}{4c_d^{\ 2}} + \left(\frac{7-3\mu+3\mu^2}{6c_d^{\ 2}}\right)\varepsilon_1 - \left\{\frac{\sqrt{3}(1-2\mu)}{4c_d^{\ 2}}\right\}\varepsilon_2.$$

The variational equations of motion corresponding to equation (4.26), on making use of equation (4.28), can be expressed as

$$p_1\ddot{\alpha} + p_2\ddot{\beta} + p_3\dot{\alpha} + p_4\dot{\beta} + p_5\alpha + p_6\beta = 0, \tag{4.38}$$

$$q_1\ddot{\alpha} + q_2\ddot{\beta} + q_3\dot{\alpha} + q_4\dot{\beta} + q_5\alpha + q_6\beta = 0.$$

where,

$$p_1 = 1 + C_2,\ p_2 = D_2,\ p_3 = F - C,\ p_4 = \left\{B_2 - 2\left(1 + \frac{3}{4}\left(2\sigma_1 - \sigma_2\right) + \varepsilon_2 - \frac{3}{2{c_d}^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\left(1+\varepsilon_2\right)\right) - D\right\},$$

$$p_5 = -A,\ p_6 = -B,$$

$$q_1 = C_3, q_2 = 1 + D_3, q_3 = 2\left(1 + \frac{3}{4}\left(2\sigma_1 - \sigma_2\right) + \varepsilon_2 - \frac{3}{2{c_d}^2}\left(1 - \frac{1}{3}\mu(1-\mu)\right)\left(1+\varepsilon_2\right)\right) - C_1 + A_3, q_4 = B_3 - D_1,$$

$$q_5 = -E, q_6 = -B_1.$$

Then, the characteristic equation is

$$(p_1q_2 - pq_1)\lambda^4 + (p_1q_6 + p_5q_2 + p_3q_4 - p_6q_1 - p_2q_5 - p_4q_3)\lambda^2 + p_5q_6 - p_6q_5 = 0 \tag{4.39}$$

Substituting the values of $p_i, q_i, i = 1, 2, ..., 6$ in equation (4.39), the characteristic equation (4.39) after normalizing becomes

$$\lambda^4 + b\lambda^2 + d = 0 \tag{4.40}$$

where,

$$b = \left(1 - \frac{9}{{c_d}^2}\right) + 3\sigma_1 + \frac{3}{2}\left(2\mu - 3\right)\sigma_2 + \left\{-3 + \frac{68 - 25\mu + 25\mu^2}{4{c_d}^2}\right\}\varepsilon_1 + \left\{8 + \frac{-147 + 30\mu - 30\mu^2}{2{c_d}^2}\right\}\varepsilon_2,$$

$$d=\frac{27\mu(1-\mu)}{4}+\frac{9\mu\left(-65+77\mu-24\mu^2+12\mu^3\right)}{8c_d^{\ 2}}+\frac{9\left(-10+99\mu-89\mu^2\right)}{16}\sigma_1+\frac{9\left(10-47\mu+37\mu^2\right)}{16}\sigma_2$$
$$+\left\{\frac{33\mu(1-\mu)}{2}-\frac{3\mu(1867-2082\mu+540\mu^2-336\mu^3}{32c_d^{\ 2}}\right\}\varepsilon_1+\left\{\frac{-243\mu+234\mu^2-162\mu^3+81\mu^4}{4c_d^{\ 2}}\right\}\varepsilon_2.$$

For $\frac{1}{c_d^{\ 2}}\to 0$ and in the absence of small perturbations (in the centrifugal and Coriolis forces) and triaxiality ($i.e.$ $\sigma_1=\sigma_2=\varepsilon_1=\varepsilon_2=0$), equation (4.40) reduces to its well-known classical restricted problem form (see e.g. Szebehely, 1967a):

$$\lambda^4+\lambda^2+\frac{27}{4}\mu(1-\mu)=0.$$

The discriminant of equation (4.40) is

$$\Delta=\left(\frac{-54}{c_d^{\ 2}}-\frac{126}{c_d^{\ 2}}\varepsilon_1-\frac{81}{c_d^{\ 2}}\varepsilon_2\right)\mu^4+\left(\frac{108}{c_d^{\ 2}}+\frac{405}{2c_d^{\ 2}}\varepsilon_1-\frac{162}{c_d^{\ 2}}\varepsilon_2\right)\mu^3+\left(27+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2+66\varepsilon_1-\frac{3073}{4c_d^{\ 2}}\varepsilon_1\right.$$
$$\left.-\frac{354}{c_d^{\ 2}}\varepsilon_2-\frac{693}{2c_d^{\ 2}}\right)\mu^2+\left(-27-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2-66\varepsilon_1+\frac{5501}{8c_d^{\ 2}}\varepsilon_1+\frac{273}{c_d^{\ 2}}\varepsilon_2+\frac{585}{2c_d^{\ 2}}\right)\mu+1+\frac{57}{2}\sigma_1-\frac{63}{2}\sigma_2$$
$$-6\varepsilon_1+\frac{88}{c_d^{\ 2}}\varepsilon_1+16\varepsilon_2-\frac{18}{c_d^{\ 2}}-\frac{291}{c_d^{\ 2}}\varepsilon_2$$

(4.41)

Its roots are

$$\lambda^2=\frac{-b\pm\sqrt{\Delta}}{2} \qquad (4.42)$$

where,

$$b=\left(1-\frac{9}{c_d^{\ 2}}\right)+3\sigma_1+\frac{3}{2}\left(2\mu-3\right)\sigma_2+\left\{-3+\frac{68-25\mu+25\mu^2}{4c_d^{\ 2}}\right\}\varepsilon_1+\left\{8+\frac{-147+30\mu-30\mu^2}{2c_d^{\ 2}}\right\}\varepsilon_2,$$

From equation (4.41), the followings are obtained

$$\frac{d\Delta}{d\mu}=4\left(\frac{-54}{c_d{}^2}-\frac{126}{c_d{}^2}\varepsilon_1-\frac{81}{c_d{}^2}\varepsilon_2\right)\mu^3+3\left(\frac{108}{c_d{}^2}+\frac{405}{2c_d{}^2}\varepsilon_1+\frac{162}{c_d{}^2}\varepsilon_2\right)\mu^2+2\left(27+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2+66\varepsilon_1\right.$$
$$\left.-\frac{3073}{4c_d{}^2}\varepsilon_1-\frac{354}{c_d{}^2}\varepsilon_2-\frac{693}{2c_d{}^2}\right)\mu+\left(-27-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2-66\varepsilon_1+\frac{5501}{8c_d{}^2}\varepsilon_1+\frac{273}{c_d{}^2}\varepsilon_2+\frac{585}{2c_d{}^2}\right)<0 \tag{4.43}$$

for $\mu\in\left(0,\ \frac{1}{2}\right]$

But

$$(\Delta)_{\mu=0}=1+\frac{57}{2}\sigma_1-\frac{63}{2}\sigma_2-6\varepsilon_1+\frac{88}{c_d{}^2}\varepsilon_1+16\varepsilon_2-\frac{291}{c_d{}^2}\varepsilon_2-\frac{18}{c_d{}^2}>0$$
$$(\Delta)_{\mu=\frac{1}{2}}=-\frac{23}{4}+\frac{207}{4c_d{}^2}-\frac{525}{16}\sigma_1+\frac{57}{16}\sigma_2+\frac{4115}{16c_d{}^2}\varepsilon_1+16\varepsilon_2-\frac{3645}{16c_d{}^2}\varepsilon_2<0 \tag{4.44}$$

$$\frac{d^2\Delta}{d\mu^2}=12\left(-\frac{54}{c_d{}^2}-\frac{126}{c_d{}^2}\varepsilon_1-\frac{81}{c_d{}^2}\varepsilon_2\right)+6\left(\frac{108}{c_d{}^2}+\frac{405}{2c_d{}^2}\varepsilon_1+\frac{162}{c_d{}^2}\varepsilon_2\right)\mu+2\left(27+\frac{801}{4}\sigma_1-\frac{333}{4}\sigma_2\right.$$
$$\left.+66\varepsilon_1-\frac{3073}{4c_d{}^2}\varepsilon_1-\frac{354}{c_d{}^2}\varepsilon_2-\frac{693}{2c_d{}^2}\right)>0\ \forall\ \mu\in\left(0,\frac{1}{2}\right] \tag{4.45}$$

This implies that $\frac{d\Delta}{d\mu}$ is monotone increasing in $\left(0,\frac{1}{2}\right]$

But

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=0}=-27-\frac{891}{4}\sigma_1+\frac{447}{4}\sigma_2+\frac{5501}{8c_d{}^2}\varepsilon_1+\frac{273}{c_d{}^2}\varepsilon_2+\frac{585}{2c_d{}^2}-66\varepsilon_1<0\ \forall\ \mu\in\left(0,\frac{1}{2}\right] \tag{4.46}$$

$$\left(\frac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}}=\frac{33}{4c_d{}^2}\varepsilon_1-\frac{45}{2}\sigma_1+\frac{57}{2}\sigma_2 \tag{4.47}$$

In order to study the monotonicity of Δ , two cases are considered:

Case 1: $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} \leq 0$

For this case, the table of variation of Δ is given below.

Table 4.3. Variation of Δ

μ	0	1/2
$\dfrac{d^2\Delta}{d\mu^2}$	+	
$\dfrac{d\Delta}{d\mu}$	−	
Δ	$(\Delta)_{\mu=0}$ ↘	$(\Delta)_{\mu=1/2}$

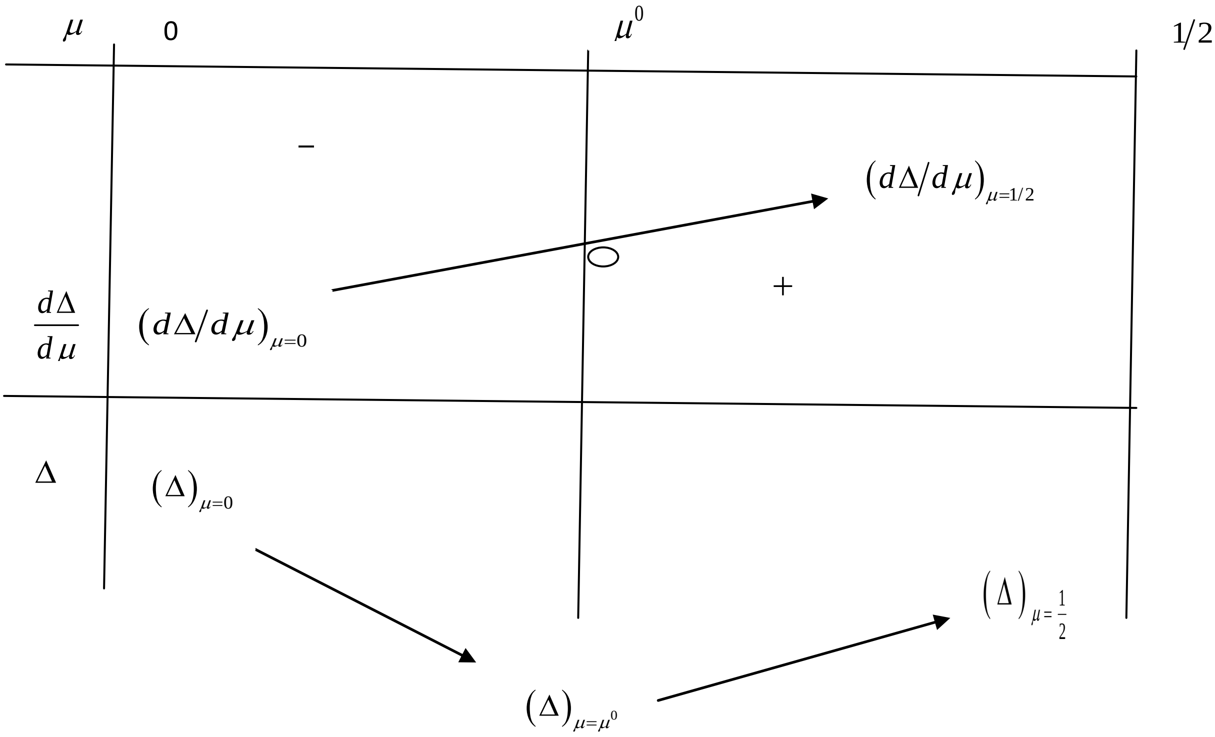
From the above table it can be seen that Δ is monotonic decreasing in $\left(0, \frac{1}{2}\right]$.

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu} = \dfrac{1}{2}$ are of opposite signs and Δ is monotone continuous and by the intermediate value poperty there is one value of μ say. μ_c' in $\left(0, \frac{1}{2}\right]$ for which $\Delta = 0$.

Case 2: $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$

Since from equation (4.45), $\dfrac{d\Delta}{d\mu}$ is monotonic increasing in $\left(0, \frac{1}{2}\right]$ and $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=0} < 0$ and $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=\frac{1}{2}} > 0$, this implies that there exists $\mu^0 \in \left(0, \frac{1}{2}\right]$ such that $\left(\dfrac{d\Delta}{d\mu}\right)_{\mu=\mu^0} = 0$, hence $\frac{d\Delta}{d\mu} \le 0 \ \forall \ \mu \in \left(0, \mu^0\right]$ and $\frac{d\Delta}{d\mu} \ge 0 \ \forall \ \mu \in \left[\mu^0, \frac{1}{2}\right]$, hence we have the following table of variation of Δ below.

Table 4.4: variation of Δ



Since $(\Delta)_{\mu=0} > 0$ and $(\Delta)_{\mu=\frac{1}{2}} < 0$, it can be concluded from the above table that $(\Delta)_{\mu=\mu^0} < 0$, hence since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\mu^0}$ are of opposite signs, and Δ is monotonic decreasing and continuous in $\left(0, \mu^0\right]$ and by the intermediate value property there is one value of μ say. μ_c'' in $\left(0, \mu^0\right]$ for which $\Delta = 0$. Hence $\mu_c = \mu_c' = \mu_c''$

Solving the equation $\Delta = 0$, using equation (4.41), the critical value of the mass parameter is obtained as

$$\mu_c = \frac{1}{2} - \frac{1}{18}\sqrt{69} - \frac{17\sqrt{69}}{486{c_d}^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 + \frac{4\left(36\varepsilon_2 - 19\varepsilon_1\right)}{27\sqrt{69}} + \left(\frac{34155 + 175301\sqrt{69}}{804816\,{c_d}^2}\right)\varepsilon_1 - \left(\frac{47\sqrt{69}}{81{c_d}^2}\right)\varepsilon_2 \tag{4.48}$$

$$\mu_c = \mu_0 - \frac{17\sqrt{69}}{486{c_d}^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 + \frac{4(36\varepsilon_2 - 19\varepsilon_1)}{27\sqrt{69}}$$
$$\left(\frac{34155+175301\sqrt{69}}{804816{c_d}^2}\right)\varepsilon_1 - \left(\frac{47\sqrt{69}}{81{c_d}^2}\right)\varepsilon_2 \quad (4.49)$$

where $\mu_0 = 0.03852....$ is the Routh's value.

The following three regions of the values of μ are consided separately.

i. When $0 < \mu < \mu_c\ \Delta > 0,$ the values of λ^2 given by equation (4.42) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.

ii. When $\mu_c < \mu \le \frac{1}{2}, \Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.

iii. When $\mu = \mu_c, \Delta = 0$, the values of λ^2 given by equation (4.42) are the same. Hence the solution contains secular terms.This induces instability of the triangular points.

Hence, the stability region is

$$0 < \mu < \mu_0 - \frac{17\sqrt{69}}{486{c_d}^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 + \frac{4(36\varepsilon_2 - 19\varepsilon_1)}{27\sqrt{69}} + \left(\frac{34155+175301\sqrt{69}}{804816{c_d}^2}\right)\varepsilon_1 - \left(\frac{47\sqrt{69}}{81{c_d}^2}\right)\varepsilon_2 \quad (4.50)$$

Equation (4.50) can be written as

$$0 < \mu < \mu_0 + p \quad (4.51)$$

with

$$p = -\frac{17\sqrt{69}}{486{c_d}^2} + \frac{1}{2}\left(\frac{5}{6} + \frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18} + \frac{85}{9\sqrt{69}}\right)\sigma_2 + \frac{4\left(36\varepsilon_2 - 19\varepsilon_1\right)}{27\sqrt{69}}$$
$$+\left(\frac{34155 + 175301\sqrt{69}}{804816{c_d}^2}\right)\varepsilon_1 - \left(\frac{47\sqrt{69}}{81{c_d}^2}\right)\varepsilon_2$$

4.4 Collinear Equilibrium Points in the Relativistic R3BP with a Smaller Oblate Primary

In this section, the analytical and numerical locations of the collinear points are obtained and also their stability when the smaller primary is oblate is examined.

4.4.1 Equations of motion

The effect of oblateness of the smaller primary is included with the help of the parameter $A_2 << 1$ and the equations of motion of an infinitesimal mass can be written as Brumberg (1972) and Bhatnagar and Hallan (1998) as:

$$\ddot{\xi} - 2n_d\dot{\eta} = \frac{\partial W}{\partial \xi} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\xi}}\right)$$
$$\ddot{\eta} + 2n_d\dot{\xi} = \frac{\partial W}{\partial \eta} - \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\eta}}\right) \qquad (4.52)$$

with

$$W=\frac{1}{2}n_d{}^2(\xi^2+\eta^2)+\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\left(1+\frac{A_2}{2\rho_2^2}\right)+\frac{1}{c_d{}^2}\left[\frac{1}{8}\left\{\dot{\xi}^2+\dot{\eta}^2+2n_d(\xi\dot{\eta}-\eta\dot{\xi})+n_d{}^2(\xi^2+\eta^2)\right\}^2\right.$$

$$+\frac{3}{2}\left(\frac{1-\mu}{\rho_1}+\frac{\mu}{\rho_2}\left(1+\frac{A_2}{2\rho_2^2}\right)\right)\left\{\dot{\xi}^2+\dot{\eta}^2+2n_d(\xi\dot{\eta}-\eta\dot{\xi})\right\}$$

$$-\frac{1}{2}\left(\frac{(1-\mu)^2}{\rho_1^2}+\frac{\mu^2}{\rho_2^2}\left(1+\frac{A_2}{2\rho_2^2}\right)^2\right)+\mu(1-\mu)\left\{n_d\left(\dot{\eta}+\frac{7}{2}n_d\xi\right)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\left(1+\frac{A_2}{2\rho_2^2}\right)\right)\right.$$

$$\left.\left.-\frac{1}{2}n_d{}^2\left(\frac{\mu}{\rho_1^3}+\frac{1-\mu}{\rho_2^3}\left(1+\frac{A_2}{2\rho_2^2}\right)\right)\eta^2+n_d{}^2\left(\frac{-1}{\rho_1\rho_2}\left(1+\frac{A_2}{2\rho_2^2}\right)+\frac{3\mu-2}{2\rho_1}+\frac{1-\mu}{2\rho_2}\left(1+\frac{A_2}{2\rho_2^2}\right)-\frac{\mu}{\rho_2}\left(1+\frac{A_2}{2\rho_2^2}\right)\right)\right\}\right],$$

(4.53)

$$n_d{}^2=1+\frac{3}{2}A_2-\frac{3}{2c_d{}^2}\left(1-\frac{1}{3}\mu(1-\mu)\right) \tag{4.54}$$

$$\rho_1^2=(\xi+\mu)^2+\eta^2$$

$$\rho_2^2=(\xi+\mu-1)^2+\eta^2$$

(4.55)

where $0<\mu\leq\frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries, ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; n_d is the perturbed mean motion of the primaries; c is the velocity of light. $A_2=\frac{(AE^2-AP^2)}{5R^2}<<1$. (McCuskey, 1963), where AE and AP are the equatorial and polar radii of the smaller primary, and R is the distance between the primaries.

It should be noted here that the second and higher powers of A_2 and $\frac{1}{c_d{}^2}$ have been ignored in writing above equations

4.4.2 Locations of collinear points

Equilibrium points are those points at which no resultant force acts on the third infinitesimal body. Therefore, if it is placed at any of these points with zero velocity, it will stay there. In fact, all derivatives of the coordinates with respect to the time are zero at these points. Therefore, the equilibrium points are solutions of equations

$$W_\xi = 0 \text{ and } W_\eta = 0 \tag{4.56}$$

where, W_ξ and W_η may be written as

$$\begin{aligned}
W_\xi &= \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \frac{3}{2}A_2\left\{\xi - \frac{\mu(\xi-1+\mu)}{\rho_2^5}\right\} + \frac{1}{c_d{}^2}\Bigg[-3\xi\left(1-\frac{1}{3}\mu(1-\mu)\right) + \frac{1}{2}\xi(\xi^2+\eta^2) \\
&-\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\} + 3\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)\xi + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4} \\
&+\mu(1-\mu)\Bigg\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) + \frac{3}{2}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) \\
&+\frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3} - \frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\Bigg\} + A_2\Bigg\{-\frac{9}{4}\left(1-\frac{\mu(1-\mu)}{3}\right)\xi + \frac{3}{2}(\xi^2+\eta^2)\xi \\
&\left(-\frac{9\mu(\xi-1+\mu)}{4\rho_2^5} - \frac{9(1-\mu)(\xi+\mu)}{4\rho_1^3} - \frac{9\mu(\xi-1+\mu)}{4\rho_2^3}\right)(\xi^2+\eta^2) + \left(\frac{3\mu}{2\rho_2^3} + \frac{9\mu}{2\rho_2} + \frac{9(1-\mu)}{2\rho_1}\right)\xi + \frac{2\mu^2(\xi-1+\mu)}{\rho_2^6}\Bigg\} \\
&+\mu(1-\mu)A_2\Bigg\{-\frac{7}{4\rho_2^3} + \frac{21}{4}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(\frac{3(\xi-1+\mu)}{2\rho_2^5}\right) + \frac{21}{4}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) \\
&+\frac{9}{4}\eta^2\left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5}\right) + \frac{3\mu(\xi-1+\mu)}{2\rho_2^5} + \frac{3(\xi+\mu)}{2\rho_1^3\rho_2} + \frac{3(\xi-1+\mu)}{2\rho_1\rho_2^3} - \frac{3(3\mu-2)(\xi+\mu)}{4\rho_1^3} \\
&-\frac{3(1-\mu)(\xi-1+\mu)}{4\rho_2^3} + \frac{3\mu(\xi-1+\mu)}{2\rho_2^3}\Bigg\}\Bigg]
\end{aligned}$$

and

$$W_\eta = \eta F$$

with

$$F = 1 - \frac{(1-\mu)}{\rho_1^3} - \frac{\mu}{\rho_2^3} + \frac{3}{2}A_2\left(1 - \frac{\mu}{\rho_2^5}\right) + \frac{1}{c_d{}^2}\left[-3\left(1 - \frac{1}{3}\mu(1-\mu)\right) + \frac{1}{2}(\xi^2+\eta^2)\right.$$
$$-\frac{3}{2}(\xi^2+\eta^2)\left\{\frac{1-\mu}{\rho_1^3} + \frac{\mu}{\rho_2^3}\right\} + 3\left(\frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2}\right) + \left(\frac{(1-\mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4}\right)$$
$$+\mu(1-\mu)\left\{\frac{7}{2}\xi\left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3}\right) - \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3}\right) + \frac{3}{2}\eta^2\left(\frac{\mu}{\rho_1^5} + \frac{1-\mu}{\rho_2^5}\right) + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{(3\mu-2)}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3}\right\}$$
$$+A_2\left\{-\frac{9}{4}\left(1 - \frac{\mu(1-\mu)}{3}\right) + \frac{3}{2}(\xi^2+\eta^2) + \left(-\frac{9(1-\mu)}{4\rho_1^3} - \frac{9\mu}{4\rho_2^3} - \frac{9\mu}{4\rho_2^5}\right)(\xi^2+\eta^2) + \frac{3\mu}{2\rho_2^3} + \frac{9(1-\mu)}{2\rho_1}\right.$$
$$\left.+\frac{9\mu}{2\rho_2} + \frac{2\mu^2}{\rho_2^6}\right\} + \mu(1-\mu)A_2\left\{\frac{21}{4}\xi\left(\frac{1}{\rho_2^5} - \frac{1}{\rho_2^3} - \frac{1}{\rho_1^3}\right) - \frac{3}{2}\left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{2\rho_2^3}\right) + \frac{9}{4}\eta^2\left(\frac{\mu}{\rho_1^5} + \frac{1-\mu}{\rho_2^5}\right)\right.$$
$$\left.\left.+\frac{3}{2}\left(\frac{\mu}{\rho_2^5} + \frac{\mu}{\rho_2^3} + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{(3\mu-2)}{2\rho_1^3} - \frac{(1-\mu)}{2\rho_2^3}\right)\right\}\right]$$

In order to find the collinear points, we put $\eta = 0$ in equation (4.56). Their abscissae are the roots of the equation

$$g(\xi) = \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \frac{3}{2}A_2\left\{\xi - \frac{\mu(\xi-1+\mu)}{\rho_2^5}\right\} + \frac{1}{c_d{}^2}\left[-3\xi\left(1 - \frac{1}{3}\mu(1-\mu)\right) + \frac{1}{2}\xi^3\right.$$
$$-\frac{3}{2}\xi^2\left\{\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3}\right\} + 3\left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\right)\xi + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4}$$
$$+\mu(1-\mu)\left\{\frac{7}{2}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) + \frac{(\xi+\mu)}{\rho_1^3\rho_2} + \frac{(\xi-1+\mu)}{\rho_1\rho_2^3} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3}\right.$$
$$\left.-\frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3}\right\} + A_2\left\{-\frac{9}{4}\left(1 - \frac{\mu(1-\mu)}{3}\right)\xi + \frac{3}{2}\xi^3 + \left(-\frac{9\mu(\xi-1+\mu)}{4\rho_2^5} - \frac{9(1-\mu)(\xi+\mu)}{4\rho_1^3} - \frac{9\mu(\xi-1+\mu)}{4\rho_2^3}\right)\xi^2\right.$$
$$\left.+\left(\frac{3\mu}{2\rho_2^3} + \frac{9\mu}{2\rho_2} + \frac{9(1-\mu)}{2\rho_1}\right)\xi + \frac{2\mu^2(\xi-1+\mu)}{\rho_2^6}\right\} + \mu(1-\mu)A_2\left\{-\frac{7}{4\rho_2^3} + \frac{21}{4}\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) + \frac{7}{2}\xi\left(\frac{3(\xi-1+\mu)}{2\rho_2^5}\right)\right.$$
$$+\frac{21}{4}\xi\left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_2^3}\right) + \frac{3\mu(\xi-1+\mu)}{2\rho_2^5} + \frac{3(\xi+\mu)}{2\rho_1^3\rho_2} + \frac{3(\xi-1+\mu)}{2\rho_1\rho_2^3} - \frac{3(3\mu-2)(\xi+\mu)}{4\rho_1^3}$$
$$\left.\left.-\frac{3(1-\mu)(\xi-1+\mu)}{4\rho_2^3} + \frac{3\mu(\xi-1+\mu)}{2\rho_2^3}\right\}\right] = 0$$

(4.57)

with $\rho_1 = |\xi+\mu|, \ \ \rho_2 = |\xi-1+\mu|.$

To locate the collinear points on the $\xi-$axis, the orbital plane is divided into three parts: $\xi < \xi_1, \xi_1 < \xi < \xi_2$ and $\xi_2 < \xi$ with respect to the primaries where $\xi_1 = -\mu$ and $\xi_2 = 1-\mu$

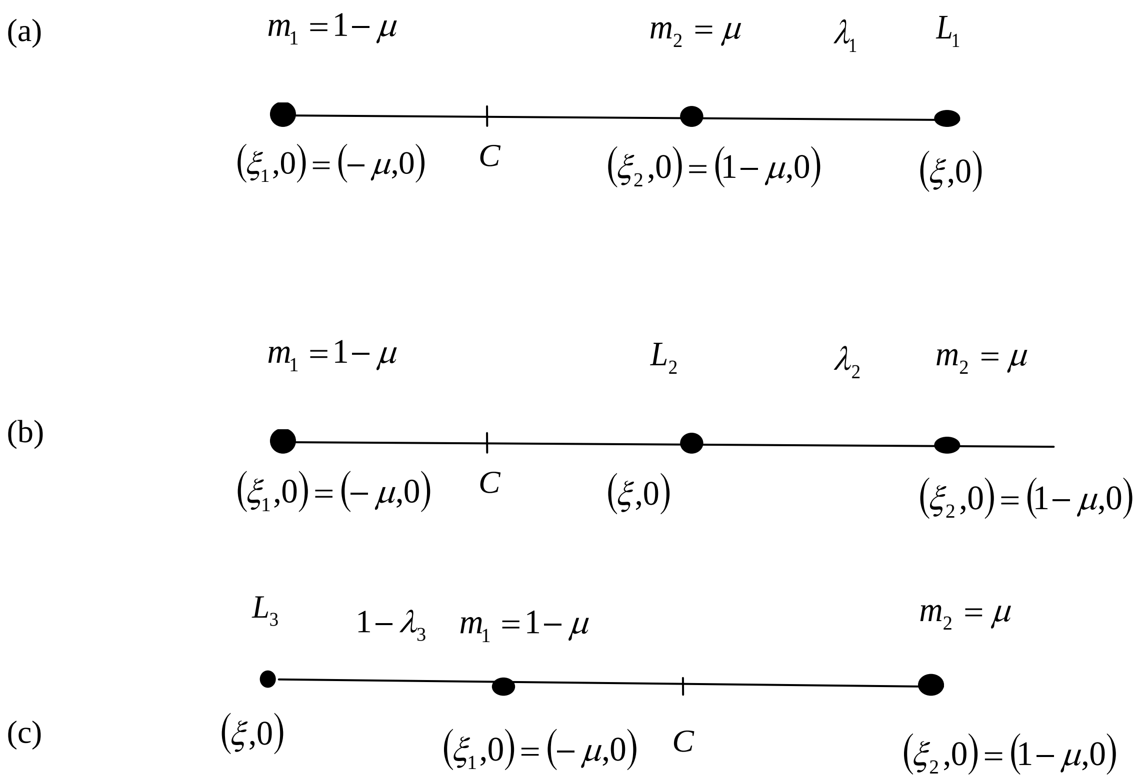


Figure 4.1: Reference parameter for collinear Lagrangian points

Case 1: Position of $L_1(\xi > \xi_2)$ (see Fig. 4.1 (a))

Let $\xi - \xi_2 = \lambda_1$; $\xi - \xi_1 = 1 + \lambda_1 \Rightarrow \xi = 1 + \lambda_1 + \xi_1$; since the distance between the primaries is unity, i.e. $\xi_2 - \xi_1 = 1 \Rightarrow \xi_1 = -\mu$ and $\xi_2 = 1-\mu$ then $\xi = 1 + \lambda_1 - \mu; \rho_1 = 1 + \lambda_1$; $\rho_2 = \lambda_1$ with $\rho_i > 0 (i = 1,2)$ (4.58)

Now substituting equation (4.58) in (4.57), we obtain

$$(6A_2+2)\lambda_1^{11}+(36A_2-6\mu-18A_2\mu+12)\lambda_1^{10}+(-26\mu+4c_d^{\,2}+6A_2c_d^{\,2}+18-87A_2\mu+81A_2+15A_2\mu^2+2\mu^2)\lambda_1^{9}$$
$$+(-6A_2c_d^{\,2}\mu-2+4\mu^2-3A_2\mu^3-32\mu+24A_2c_d^{\,2}-159A_2\mu+93A_2+16c_d^{\,2}+2\mu^3-4\mu c_d^{\,2}+57A_2\mu^2)\lambda_1^{8}$$
$$+(-12\mu c_d^{\,2}+81A_2\mu^2-24-18A_2c_d^{\,2}\mu-135A_2\mu+36A_2c_d^{\,2}+6\mu^3-9A_2\mu^3+63A_2+24c_d^{\,2})\lambda_1^{7}$$
$$+(32\mu+24A_2c_d^{\,2}-9A_2\mu^3+36A_2\mu^2-12\mu c_d^{\,2}+27A_2+12c_d^{\,2}-14\mu^2-18A_2c_d^{\,2}\mu+6\mu^3-18-39A_2\mu)\lambda_1^{6}$$
$$+(6A_2c_d^{\,2}+46A_2\mu-28\mu^2-55A_2\mu^2+6A_2+6\mu^3+3A_2\mu^3-12\mu c_d^{\,2}+32\mu-6A_2c_d^{\,2}\mu)\lambda_1^{5}$$
$$+(30\mu-6A_2c_d^{\,2}\mu+15A_2\mu^3-120A_2\mu^2+96A_2\mu+6\mu^3-12\mu c_d^{\,2}-24\mu^2)\lambda_1^{4}$$
$$+(-112A_2\mu^2-18A_2c_d^{\,2}\mu+10\mu+21A_2\mu^3-4\mu c_d^{\,2}+96A_2\mu+2\mu^3)\lambda_1^{3}$$
$$+(52A_2\mu-46A_2\mu^2+18A_2\mu^3-18A_2c_d^{\,2}\mu+4\mu^2)\lambda_1^{2}+(6A_2\mu^2+12A_2\mu-6A_2c_d^{\,2}\mu+6A_2\mu^3)\lambda_1+8A_2\mu^2=0$$

(4.59)

In the presence of oblateness effect only, the corresponding equation is

$$(3A_2+2)\lambda_1^7+(9A_2+6-3A_2\mu-2\mu)\lambda_1^6+(9A_2-6A_2\mu-4\mu+6)\lambda_1^5+(3A_2-3A_2\mu-2\mu)\lambda_1^4$$
$$-4\mu\lambda_1^3+(-3A_2\mu-2\mu)\lambda_1^2-6A_2\mu\lambda_1-3A_2\mu=0$$

(4.59a)

Case2: Position of $L_2(\xi_1<\xi<\xi_2)$ (see Fig. 4.1 (b))

Let $\xi_2-\xi=\lambda_2;\ \xi-\xi_1=1-\lambda_2\Rightarrow\xi=1-\lambda_2-\mu;\ \ \rho_1=1-\lambda_2;\rho_2=\lambda_2$ with $\rho_i>0(i=1,2)$

(4.60)

Substituting equation (4.60) in (4.57), the following equation is obtained

$$(6A_2+2)\lambda_2^{11}+(-36A_2+6\mu+18A_2\mu-12)\lambda_2^{10}+(4c_d^{\,2}-26\mu+6A_2c_d^{\,2}-87A_2\mu+18+81A_2+2\mu^2+15A_2\mu^2)\lambda_2^{9}$$
$$+(-4\mu^2+3A_2\mu^3+4\mu c_d^{\,2}-24A_2c_d^{\,2}+177A_2\mu-93A_2+44\mu+2-57A_2\mu^2-2\mu^3-16c_d^{\,2}+6A_2c_d^{\,2}\mu)\lambda_2^{8}$$
$$+(-18A_2c_d^{\,2}\mu-9A_2\mu^3-189A_2\mu+6\mu^3-36\mu-12\mu c_d^{\,2}+81A_2\mu^2+36A_2c_d^{\,2}+24c_d^{\,2}+63A_2-24)\lambda_2^{7}$$
$$+(18A_2c_d^{\,2}\mu-2\mu^2+105A_2\mu-60A_2\mu^2+18+15A_2\mu^3+4\mu c_d^{\,2}-12c_d^{\,2}-2\mu^3+16\mu-24A_2c_d^{\,2}-27A_2)\lambda_2^{6}$$
$$+(6A_2-15A_2\mu^3+36\mu^2-6A_2c_d^{\,2}\mu-6\mu^3-32\mu-64A_2\mu+12\mu c_d^{\,2}+73A_2\mu^2+6A_2c_d^{\,2})\lambda_2^{5}$$
$$+(-48\mu^2-6A_2c_d^{\,2}\mu+6\mu^3+30\mu++96A_2\mu-12\mu c_d^{\,2}-120A_2\mu^2+15A_2\mu^3)\lambda_2^{4}$$
$$+(-10\mu+128A_2\mu^2-96A_2\mu-21A_2\mu^3+18A_2c_d^{\,2}\mu-2\mu^3+24\mu^2+4\mu c_d^{\,2})\lambda_2^{3}$$
$$+(-18A_2c_d^{\,2}\mu+52A_2\mu+18A_2\mu^3-94A_2\mu^2-4\mu^2)\lambda_2^{2}+(-12A_2\mu+6A_2c_d^{\,2}\mu+42A_2\mu^2-6A_2\mu^3)\lambda_2-8A_2\mu^2=0$$

(4.61)

In the presence of oblateness effect only, the corresponding equation is

$$\left(-3A_2-2\right)\lambda_2^7+\left(9A_2+6-3A_2\mu-2\mu\right)\lambda_2^6+\left(-9A_2+6A_2\mu+4\mu-6\right)\lambda_2^5+\left(3A_2-3A_2\mu-2\mu\right)\lambda_2^4$$
$$-4\mu\lambda_2^3+\left(3A_2\mu-2\mu\right)\lambda_2^2-6A_2\mu\lambda_2+3A_2\mu=0$$

(4.61a)

Case 3: Position of $L_3\left(\xi<\xi_1\right)$ (see Fig. 4.1 (c))

Let the distance of the point L_3 from the bigger primary be $1-\lambda_3$

$\xi_2-\xi_1=1\Rightarrow\xi_1-\xi=1-\lambda_3;\xi_2-\xi=2-\lambda_3$ and $\xi=\lambda_3-\mu-1;\ \rho_1=1-\lambda_3;\ \rho_2=2-\lambda_3$ with

$\rho_i>0\left(i=1,2\right)$

(4.62)

substituting equation (4.62) in (4.57), the following equation is obtained

$$\begin{aligned}
&\left(6A_2+2\right)\lambda_3^{11}+\left(-96A_2-18A_2\mu-32-6\mu\right)\lambda_3^{10}+\left(6A_2c_d^2+681A_2+218+94\mu+273A_2\mu+15A_2\mu^2\right.\\
&\left.+4c_d^2+2\mu^2\right)\lambda_3^9+\left(-213A_2\mu^2-32\mu^2-6A_2c_d^2\mu-3A_2\mu^3-84A_2c_d^2-818-644\mu-4\mu c_d^2-2823A_2\right.\\
&\left.+-56c_d^2-1833A_2\mu+2\mu^3\right)\lambda_3^8+\left(52\mu c_d^2+2528\mu+224\mu^2+344c_d^2-26\mu^3+1796+1329A_2\mu^2+516A_2c_d^2\right.\\
&+\left(7593A_2+78A_2c_d^2\mu+7161A_2\mu+39A_2\mu^3\right)\lambda_3^7+\left(-13929A_2-438A_2c_d^2\mu-1212c_d^2-2126+146\mu^3\right.\\
&\left.-292\mu c_d^2-890\mu^2-1824A_2c_d^2-17997A_2\mu-6260\mu-4764A_2\mu^2-219A_2\mu^3\right)\lambda_3^6+\left(524-466\mu^3+17868A_2\right.\\
&\left.+1386A_2c_d^2\mu+932\mu c_d^2+687A_2\mu^3+2680c_d^2+10148\mu+2192\mu^2+10771A_2\mu^2+30374A_2\mu+4086A_2c_d^2\right)\lambda_3^5\\
&+\left(934\mu^3-10814\mu+2224-16212A_2-3808c_d^2-2694A_2c_d^2\mu-3440\mu^2-1305A_2\mu^3-34916A_2\mu-1868\mu c^2\right.\\
&\left.-15826A_2\mu^2-6012A_2c_d^2\right)\lambda_3^4+\left(3408\mu^2 10416A_2-3424+1515A_2\mu^3+14940A_2\mu^2+5808A_2c_d^2-1218\mu^3\right.\\
&\left.+3282A_2\mu c_d^2+2436\mu c_d^2+3392c_d^2+7374\mu+27212A_2\mu\right)\lambda_3^3+\left(-14060A_2\mu-4656A_2-2040c_d^2\mu-2044\mu^2\right.\\
&\left.+2208-1020A_2\mu^3-3552A_2c_d^2-1728c_d^2-8554A_2\mu^2+1020\mu^3-2442A_2c_d^2\mu-2948\mu\right)\lambda_3^2+\left(688\mu^2\right.\\
&+1014A_2c_d^2\mu-504\mu^3-576+4532A_2\mu+1248A_2c_d^2+2578A_2\mu^2+342A_2\mu^3+1344A_2+384c_d^2+488\mu\\
&\left.1008\mu c_d^2\right)\lambda_3+\left(-192A_2c_d^2-276A_2\mu^2+48\mu-112\mu^2-192A_2-36A_2\mu^3-224\mu c_d^2-180A_2\mu c_d^2\right.\\
&\left.-728A_2\mu+112\mu^3\right)=0
\end{aligned}$$

(4.63)

In the presence of oblateness effect only, the corresponding equation is

$$\left(3A_2+2\right)\lambda_3^7+\left(-33A_2-3A_2\mu-2\mu-22\right)\lambda_3^6+\left(153A_2+30A_2\mu+20\mu+102\right)\lambda_3^5+\left(-387A_2-123A_2\mu\right.$$
$$\left.-82\mu-256\right)\lambda_3^4+\left(264A_2\mu+576A_2+180\mu+368\right)\lambda_3^3+\left(-309A_2\mu-504A_2-288-230\mu\right)\lambda_3^2+\left(186A_2\mu\right.$$
$$\left.+240A_2+168\mu+96\right)\lambda_3-45A_2\mu-56\mu-48A_2=0 \tag{4.63a}$$

It is noticed that in each case there exists only one physically reasonable root.

4.4.3 Stability of collinear points

The stability of an equilibrium configuration is examined; that is, its ability to restrain the body motion in its vicinity. To do so the infinitesimal body is displaced a little from an equilibrium point with a small velocity. If its motion is rapid departure from vicinity of the point, we call such a position of equilibrium an unstable one. If the body oscillates about the point, it is said to be a stable position.

In order to study the stability of the collinear points, we consider the characteristic equation given by

$$(p_1q_2-p_2q_1)\omega^4+(p_1q_6+p_5q_2+p_3q_4-p_6q_1-p_2q_5-p_4q_3)\omega^2+p_5q_6-p_6q_5=0 \tag{4.64}$$

where,

$$p_1=1+W^0_{\dot{\xi}\dot{\xi}},\ p_2=W^0_{\dot{\eta}\dot{\xi}},\ p_3=W^0_{\xi\dot{\xi}}-W^0_{\xi\dot{\xi}}=0,\ p_4=W^0_{\eta\dot{\xi}}-2n_d-W^0_{\xi\dot{\eta}},$$

$$p_5=-W^0_{\xi\xi},\ p_6=-W^0_{\xi\eta},$$

$$q_1=W^0_{\dot{\xi}\dot{\eta}},\ q_2=1+W^0_{\dot{\eta}\dot{\eta}},\ q_3=2n_d+W^0_{\xi\dot{\eta}}-W^0_{\eta\dot{\xi}},$$

$$q_4=W^0_{\eta\dot{\eta}}-W^0_{\eta\dot{\eta}}=0,\ q_5=-W^0_{\xi\eta},\ q_6=-W^0_{\eta\eta}.$$

The second order partial derivative of W are denoted by subscripts. The superscript 0

indicates that the derivative is to be evaluated at the collinear equilibrium points (ξ_0, η_0) under consideration.

In order to study the stability of the collinear points it requires to study the motion in the proximity of these points, hence in this case, the second order derivatives evaluated at (ξ_0, η_0) are

$$
\begin{aligned}
W^0_{\xi\xi} &= 1 + \frac{3(1-\mu)(\xi+\mu)^2}{\rho_1^5} - \frac{(1-\mu)}{\rho_1^3} + \frac{3\mu(\xi-1+\mu)^2}{\rho_2^5} - \frac{\mu}{\rho_2^3} + A_2 \left\{ \frac{3}{2} + \frac{15\mu(\xi-1+\mu)^2}{2\rho_2^7} - \frac{3\mu}{2\rho_2^5} \right\} \\
&+ \frac{1}{{c_d}^2} \bigg[-3 + \mu(1-\mu) + \frac{3}{2}\xi^2 + \bigg(\frac{9(1-\mu)(\xi+\mu)^2}{2\rho_1^5} - \frac{3(1-\mu)}{2\rho_1^3} \\
&+ \frac{9\mu(\xi-1+\mu)^2}{2\rho_2^5} - \frac{3\mu}{2\rho_2^3} \bigg) \xi^2 + 2 \left(\frac{-3(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{3\mu(\xi-1+\mu)}{\rho_2^3} \right) \xi + \frac{3(1-\mu)}{\rho_1} + \frac{3\mu}{\rho_2} \\
&- \frac{4(1-\mu)^2(\xi+\mu)^2}{\rho_1^6} + \frac{(1-\mu)^2}{\rho_1^4} - \frac{4\mu^2(\xi-1+\mu)^2}{\rho_2^6} + \frac{\mu^2}{\rho_2^4} + \mu(1-\mu) \bigg\{ -\frac{7(\xi+\mu)}{\rho_1^3} + \frac{7(\xi-1+\mu)}{\rho_2^3} \\
&+ \frac{7}{2}\xi \left(\frac{3(\xi+\mu)^2}{\rho_1^5} - \frac{1}{\rho_1^3} - \frac{3(\xi-1+\mu)^2}{\rho_2^5} + \frac{1}{\rho_2^3} \right) - \frac{3(\xi+\mu)^2}{\rho_1^5\rho_2} - \frac{2(\xi+\mu)(\xi-1+\mu)}{\rho_1^3\rho_2^3} + \frac{1}{\rho_1^3\rho_2} - \frac{3(\xi-1+\mu)^2}{\rho_1\rho_2^5} \\
&+ \frac{1}{\rho_1\rho_2^3} + \frac{3(3\mu-2)(\xi+\mu)^2}{2\rho_1^5} - \frac{(3\mu-2)}{2\rho_1^3} + \frac{3(1-3\mu)(\xi-1+\mu)^2}{2\rho_2^5} - \frac{(1-3\mu)}{2\rho_2^3} + A_2 \bigg\{ \frac{3}{4}\big(\mu(1-\mu)-3\big) + \frac{9}{2}\xi^2 \\
&+ \frac{3}{4}\xi^2 \left(\frac{9(1-\mu)(\xi+\mu)^2}{\rho_1^5} - \frac{3(1-\mu)}{\rho_1^3} + \frac{9\mu(\xi-1+\mu)^2}{\rho_2^5} - \frac{3\mu}{\rho_2^3} \right) + \left(\frac{45\mu(\xi-1+\mu)^2}{4\rho_2^7} - \frac{9\mu}{4\rho_2^5} \right) \xi^2 \\
&+ 3\xi \left(-\frac{3(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{3\mu(\xi-1+\mu)}{\rho_2^3} \right) - \frac{9\mu(\xi-1+\mu)}{\rho_2^5}\xi + \left(\frac{9(1-\mu)}{2\rho_1} + \frac{9\mu}{2\rho_2} + \frac{3\mu}{2\rho_2^3} \right) - \frac{16\mu^2(\xi-1+\mu)^2}{\rho_2^8} + \frac{2\mu^2}{\rho_2^6} \bigg\} \\
&+ \mu(1-\mu) A_2 \bigg\{ \frac{21}{2} \left(\frac{\xi-1+\mu}{\rho_2^5} + \frac{\xi-1+\mu}{\rho_2^3} - \frac{\xi+\mu}{\rho_1^3} \right) + \frac{7}{2}\xi \left(\frac{-15(\xi-1+\mu)^2}{2\rho_2^7} + \frac{3}{2\rho_2^5} \right) + \frac{21}{4}\xi \bigg(\frac{3(\xi+\mu)^2}{\rho_1^5} - \frac{1}{\rho_1^3} \\
&+ \frac{1}{\rho_2^3} - \frac{3(\xi-1+\mu)^2}{\rho_2^5} \bigg) - \frac{15\mu(\xi-1+\mu)^2}{2\rho_2^7} + \frac{3\mu}{2\rho_2^5} - \frac{9(\xi+\mu)^2}{2\rho_1^5\rho_2} - \frac{3(\xi+\mu)(\xi-1+\mu)}{\rho_1^3\rho_2^3} + \frac{3}{2\rho_1^3\rho_2} + \frac{3}{2\rho_1\rho_2^3} - \frac{9(\xi-1+\mu)^2}{2\rho_1\rho_2^5} \\
&+ \frac{9(3\mu-2)(\xi+\mu)^2}{4\rho_1^5} - \frac{3(3\mu-2)}{4\rho_1^3} - \frac{3(1-3\mu)}{4\rho_2^3} + \frac{9(1-3\mu)(\xi-1+\mu)^2}{4\rho_2^5} \bigg\} \bigg]
\end{aligned}
\tag{4.65}
$$

$$W^0_{\eta\eta} = 1 - \frac{(1-\mu)}{\rho_1^3} - \frac{\mu}{\rho_2^3} + A_2\left\{\frac{3}{2} - \frac{3\mu}{2\rho_2^5}\right\}$$
$$+\frac{1}{c_d{}^2}\Bigg[-3+\mu(1-\mu)+\frac{3}{2}\xi^2+\left(\frac{-3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}\right)\xi^2+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}+\frac{(1-\mu)^2}{\rho_1^2}+\frac{3\mu}{\rho_2^4}$$
$$+\mu(1-\mu)\left\{\frac{7\xi}{2}\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\frac{\mu}{\rho_1^3}-\frac{(1-\mu)}{\rho_2^3}+\frac{1}{\rho_1^3\rho_2}+\frac{1}{\rho_1\rho_2^3}-\frac{(3\mu-2)}{2\rho_1^3}-\frac{(1-3\mu)}{2\rho_2^3}\right\}$$
$$+A_2\left\{\frac{3}{4}\big(\mu(1-\mu)-3\big)+\frac{3}{2}\xi^2+\frac{3}{2}\xi^2\left(-\frac{3(1-\mu)}{2\rho_1^3}-\frac{3\mu}{2\rho_2^3}\right)-\frac{9\mu}{4\rho_2^5}\xi^2+\frac{9(1-\mu)}{2\rho_1}+\frac{9\mu}{2\rho_2}+\frac{3\mu}{2\rho_2^3}+\frac{2\mu^2}{\rho_2^6}\right\}$$
$$+\mu(1-\mu)A_2\left\{\frac{21\xi}{4\rho_2^5}+\frac{21\xi}{4}\left(-\frac{1}{\rho_1^3}+\frac{1}{\rho_2^3}\right)-\frac{3\mu}{2\rho_1^3}-\frac{3(1-\mu)}{2\rho_2^3}+\frac{3\mu}{2\rho_2^5}+\frac{3}{2\rho_1^3\rho_2}+\frac{3}{2\rho_1\rho_2^3}-\frac{3(3\mu-2)}{4\rho_1^3}-\frac{3(1-3\mu)}{4\rho_2^3}\right\}\Bigg]$$

(4.66)

$$W^0_{\xi\eta} = 0 \tag{4.67}$$

$$W^0_{\dot\xi\dot\xi} = \frac{1}{c_d{}^2}\left[\frac{1}{2}\xi^2+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}+A_2\left(\frac{3}{4}\xi^2+\frac{3\mu}{2\rho_2^3}\right)\right] \tag{4.68}$$

$$W^0_{\dot\eta\dot\eta} = \frac{1}{c_d{}^2}\left[\frac{3}{2}\xi^2+\frac{3(1-\mu)}{\rho_1}+\frac{3\mu}{\rho_2}+A_2\left(\frac{9}{4}\xi^2+\frac{3\mu}{2\rho_2^3}\right)\right] \tag{4.69}$$

$$W^0_{\dot\eta\dot\xi} = 0 \tag{4.70}$$

$$W^0_{\xi\dot\xi} = 0 \tag{4.71}$$

$$W^0_{\eta\dot\eta} = 0 \tag{4.72}$$

It will be shown that, the discriminant Δ of (4.64) is positive at the collinear points $L_i\,(i = 1,2,3)$

To show Δ is positive it is suffices to notice that

$$M = -4(p_1q_2 - p_2q_1)(p_5q_6 - p_6q_5) > 0 \tag{4.73}$$

as shown below

M can also be written as

$$M = -4W^0_{\xi\xi}W^0_{\eta\eta}\left(1+W^0_{\dot{\xi}\dot{\xi}}\right)\left(1+W^0_{\dot{\eta}\dot{\eta}}\right) \tag{4.74}$$

From (4.68) and (4.69) it is clear that $\left(1+W^0_{\dot{\xi}\dot{\xi}}\right) > 0$ and $\left(1+W^0_{\dot{\eta}\dot{\eta}}\right) > 0$

Now, the signs of $W^0_{\xi\xi}$ and $W^0_{\eta\eta}$ are studied at the collinear points $L_i\,(i = 1,2,3)$

Firstly, we will do this at L_1, since the coordinates of this point is $(1+\lambda_1 - \mu, 0)$, then $\rho_1 = 1+\lambda_1$ and $\rho_2 = \lambda_1$ where $0 < \lambda_1 << 1$, hence we can write $W^0_{\xi\xi}$ and $W^0_{\eta\eta}$ as a function of λ_1, say $h(\lambda_1)$ and $f(\lambda_1)$, respectively. In this case from (4.65), $h(\lambda_1) \cong h(0^+) = -\infty$ and from (4.66), $f(\lambda_1) \cong f(0^+) = +\infty$, hence $W^0_{\xi\xi} < 0$ and $W^0_{\eta\eta} > 0$ contrary to the classical case where $W^0_{\xi\xi} > 0$ and $W^0_{\eta\eta} < 0$. However $W^0_{\xi\xi}W^0_{\eta\eta} < 0$ and consequently $M > 0$. Hence the discriminant of the equation (4.64) is positive, and the characteristic roots can be written as $\omega_{1,2} = \pm\sigma$, $\omega_{3,4} = \pm i\tau$ where σ and τ are real.

Thus $\omega_{1,2}$ are real and $\omega_{3,4}$ are pure imaginary, hence the motion around the collinear point L_1, is unbounded and the solution is unstable.

Similarly, it can be shown that the points L_2, L_3 are also unstable.

CHAPTER FIVE

RESULTS AND DISCUSSIONS

5.1 Introduction

In this chapter the effects of oblateness, triaxiality and radiation of the primaries as well as the effects of perturbations in the Coriolis and centrifugal forces on the locations and stability of the triangular points are discussed. The results are also validated with the classical results and other existing results from the various aspects of generalizations.

5.2 Results and Discussion of Section 3.2

The triangular libration points in the relativistic restricted three-body problem, under the assumption that the primaries are luminous and triaxial are discussed here. The positions of the triangular points in equation (3.15) are obtained. It can be seen that they are affected by the relativistic, radiation and triaxiality factors. It is important to note that these triangular libration points cease to be classical one that is they no longer form equilateral triangles with the primaries. Rather they form scalene triangles with the primaries. Equation (3.42) gives the critical value of the mass parameter μ_c of the system which depends upon relativistic factor, triaxiality parameters $\sigma_i, \sigma'_i \quad (i = 1,2)$ and radiation factors $\delta_i \quad (i = 1,2)$. In the absence of relativistic factor, the results obtained in this study are in agreement with those of Sharma *et al*. (2001b), Singh (2013) when there is no perturbations in the Coriolis and centrifugal forces $(i.e.\ \varepsilon = \varepsilon' = 0)$. When the primaries are non triaxial.

It is noticed from equation (3.44) that that radiation and triaxiality both have destabilizing effects, and therefore the size of the range of stability decreases with increase of the values of these parameters. Evidently, it can also be seen that the relativistic factor reduces the size of stability region. When the primaries are non-luminous and non-triaxial, the stability results obtained in this study are in accordance with those of Douskos and Perdios (2002) and disagree with Bhatnagar and Hallan (1998). In the absence of relativistic factor, the results obtained in this study are in agreement with those of Sharma *et al*. (2001a) and those of Singh (2013) when there is no perturbations in the Coriolis and centrifugal forces (i.e. $\varepsilon = \varepsilon' = 0$) in his study. When the primaries are oblate spheroids $(i.e.\ \sigma_1 = \sigma_2, \sigma_1' = \sigma_2')$, the results of equation (3.15) in this study differ from those of Katour *et al*. (2014) when the radiation terms are not included in the relativistic part of the potential W in their study.

By considering the primaries as triaxial rigid bodies and sources of radiation in the relativistic CR3BP, the positions of the triangular points are determined and their linear stability is studied. It is found that their positions and stability region are affected by relativistic, triaxiality and radiation factors. It is further observed that the relativistic, triaxiality and radiation factors have destabilizing tendencies resulting in a decrease in the size of the region of stability. It is noticed that the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan (1998) differ from the present study when the radiation pressure factors are absent and the primaries are spherical $(i.e.\ \delta_i = \sigma_i = \sigma_i' = 0, i = 1, 2)$; keeping in mind that the expressions denoted by A_1 and A_2 in Bhatnagar and Hallan (1998) are denoted in this study by E and F respectively. Consequently, the characteristic equation is also different. This made Bhatnagar and Hallan (1998) to infer that the triangular points are unstable, contrary to Douskos and Perdios (2002) and our results. The present results are also in disagreement with those of Katour *et al*. (2014). One major distinction is that the expression of the mean motion which they used in their study differ from our own. It seems that there is an error in their expression.

5.3 Results and Discussion of Section 3.3

The triangular equilibrium points in the relativistic R3BP are dicussed under the assumption that the bigger primary is a triaxial rigid body and the smaller one an oblate spheroid.

In analogy to corresponding problem without oblateness and triaxiality factors, the positions of analogous triangular equilibrium points in equation (3.52) are obtained. It is seen from equation (3.52) that these triangular points are affected by the relativistic, triaxiality and oblateness coefficients. It is important to note that these equilibrium points in equation (3.52) cease to be classical i.e. they no longer form equilateral triangles with the primaries because they do not meet $\rho_1 = \rho_2 = 1.$ Rather they form scalene triangles with primaries.

Equation (3.67) gives the critical value of the mass parameter μ_c of the system which depends upon relativistic, triaxiality and oblateness factors. This critical value is used to determine the size of the region of stability of the triangular points and also helps in analyzing the behavior of the parameters involved therein. It is remarkable from equation (3.68) that these parameters reduce the size of the stability.

In the absence of triaxiality and oblateness factors $(i.e.\ A_2 = \sigma_1 = \sigma_2 = 0,),$ the stability results obtained in this study are in accordance with those of Douskos and Perdios (2002) and disagree with those of Bhatnagar and Hallan (1998) and Ahmed *et al*. (2006). In the absence of relativistic terms, the present results are in accordance with of Singh and Umar (2014) when the semi-major axis is unity in the absence of eccentricity $(i.e.\ a = 1, e = 0).$

However when the bigger primary is an oblate spheroid $(i.e.\ \sigma_1 = \sigma_2 = A_1)$, there are apparent disagreements with the results of the present study with those of Katour *et al*. (2014) when the primaries are non-luminous.

By considering the bigger primary as a triaxial rigid body and the smaller one an oblate spheroid in the relativistic R3BP, the positions of the triangular points are obtained and their linear stability is studied. It is discovered that their positions and stability are affected by the relativistic, oblateness and triaxiality factors. It is also noticed that these factors reduce the size of stability region.

It is also noticed that the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan (1998) differ from the present study in the absence of oblateness and triaxiality factors $(i.e.\ A_2 = \sigma_1 = \sigma_2 = 0,)$; keeping in mind that the expressions denoted by A_1 and A_2 in Bhatnagar and Hallan (1998) are denoted in this study by E and F respectively. Consequently the expressions p_1, p_3, p_4, p_5 and the characteristic equation are also different. This led them (Bhatnagar and Hallan, 1998) to infer that the triangular points are unstable, contrary to Douskos and Perdios (2002) and the present results.

Also, the result of the present study, when the bigger primary is oblate $(i.e.\ \sigma_1 = \sigma_2 = A_1)$, differ from those of Katour *et al*. (2014) when the primaries are non-luminous. Our major distinction is that the expression of the perturbed mean motion equation (3.47) in this study differ from their own and consequently leads to different results. In addition to that, we have studied the stability of the dynamical system while they have not. It seems that there is an error in the expression of the perturbed mean motion which they have used.

5.4 Results and Discussion of Section 3.4

It can also be observed from equation (3.73) that the frequencies of the orbits of the long and short periodic motion are dependent on all the parameters involved. From equation (3.74), one can infer that the periodic orbits are elliptical. Also from equation (3.80) it can be observed that the orientation of these orbits are affected by triaxiality and relativistic factors. The eccentricities of the long and short periodic orbits are obtained in equation (3.86). It is seen that they are also affected by the triaxiality and relativistic factors.

Equations (3.88) and (3.89) provide the semi-major and semi-minor axes of the long and short periods. In the absence of relativistic and triaxiality factors i.e. when $\frac{1}{{c_d}^2} \to 0$ and $\sigma_1 = \sigma_2 = 0$, the results of the present study correspond to those of the classical case (Szebehely,1967a) and the results of equations (3.73) and (3.80) coincide with those of Abouelmagd and El- Shaboury (2012) when the oblateness and radiation factors are absent. In the absence of relativistic factor our results also validate those of Sharma *et al*. (2001b) when only the bigger primary is considered triaxial and μ tends to zero in their study.

The periodic orbit in the vicinity of the triangular points for $0 < \mu < \mu_c$ have been investigated. The frequencies, the coefficients of long and short periodic terms, the elliptic orbits, the orientation, the lengths of semi-major and semi-minor axes have also been found. They are all affected by the triaxiality and relativistic factors.

5. 5 Results and Discussion of Section 3.5

5.5.1 Numerical results

The equations (3.96), (3.98), (3.100) are used respectively to compute the positions of the collinear points L_1, L_2, L_3 in the presence of relativistic and triaxiality factors while equations (3.96a), (3.98a), (3.100a) are used to compute their positions in the presence of triaxiality factors only. For the numerical application, we use the Sun-Earth system, Earth-Moon system and Sun-Pluto system. Five different cases of different set of semi-axes in km. (h, b, f) of the smaller primary are taken. For the Sun-Earth system, we take (6400, 6400, 6400), (6400, 6390, 6380), (6400, 6380, 6360), (6400, 6370, 6340) and (6400, 6360, 6320), for the Earth –Moon system, (1738,1738,1738), (1738,1728,1718), (1738,1718,1698), (1738,1708,1678),(1738,1698,1658) and for the Sun-Pluto system, (3000,3000,3000), (3000,2990,2980), (3000,2980,2960), (3000,2970,2940), (3000,2960,2920). Some of the data has been borrowed from Ragos *et al*. (2001) and Sharma and Subba Rao (1975). In Tables 5.1, 5.2, 5.3 these positions are presented. The corresponding positions of the classical problem are also included. The corresponding positions in the presence of triaxiality effect only (second entry in table for each system) are also included.

Table 5.1: Sun-Earth system, $\mu = 0.000003003500, c_d = 10064.84, R = 149597870.61 km$

parameter	Classical	case1	case 2	case 3	case 4	case 5
σ_1		0	2.284×10^{-12}	4.561×10^{-12}	6.831×10^{-12}	9.094×10^{-12}
σ_2		0	1.141×10^{-12}	2.277×10^{-12}	3.48×10^{-12}	4.533×10^{-12}
L_1	1.0100341380 9074	1.01003413806000	1.01003413823000 1.01003413826000	1.01003413840000 1.01003413843000	1.01003413857000 1.01003413860000	1.01003413874000 1.01003413877000
L_2	0.99002657245077	0.99002657248300	0.99002657231200 0.99002657227900	0.99002657214000 0.99002657210800	0.99002657196900 0.99002657193700	0.99002657179900 0.99002657176600
L_3	-1.00000125145833	-1.00000125145831	-1.00000125145660 -1.00000125145662	-1.00000125145489 -1.00000125145491	-1.00000125145319 -1.00000125145321	-1.00000125145148 -1.00000125145151

Table 5.2: Sun-Pluto system, $\mu = 0.00000006500, c_d = 63280.18, R = 5900 \times 10^6 km$

parameter	Classical	case1	case 2	case 3	case 4	case 5
σ_1		0	$6.871588624 \times 10^{-16}$	$1.369721344 \times 10^{-15}$	$2.047687446 \times 10^{-15}$	$2.721057167 \times 10^{-15}$
σ_2		0	$3.430048837 \times 10^{-16}$	$6.825624820 \times 10^{-16}$	$1.018672795 \times 10^{-15}$	$1.351335823 \times 10^{-15}$
L_1	1.00129454074324	1.00129454074300	1.00129454074400 1.00129454074400	1.00129454074400 1.00129454074400	1.00129454074400 1.00129454074400	1.00129454074500 1.00129454074500
L_2	0.9987 0656252876	0.99870656252900	0.99870656252800 0.99870656252800	0.99870656252800 0.99870656252800	0.99870656252800 0.99870656252800	0.99870656252700 0.99870656252700
L_3	-1.0000000270833	-1.0000000270833	-1.0000000270833 -1.0000000270833	-1.0000000270833 -1.0000000270833	-1.0000000270833 -1.0000000270833	-1.0000000270833 -1.0000000270833

Table 5.3: Earth-Moon system, $\mu = 0.0121314293, c_d = 292624.8185, R = 384000km$

parameter	Classical	case1	case 2	case 3	case 4	case 5
σ_1		0	9.375000000×10^{-8}	1.864149306×10^{-7}	2.77994791×10^{-7}	3.684895833×10^{-7}
σ_2		0	4.673936632×10^{-8}	9.266493056×10^{-8}	1.37776692×10^{-7}	1.820746528×10^{-7}
L_1	1.155608442501	1.155608444240	1.155608847700 1.155608847700	1.155609249000 1.155609249000	1.155609646500 1.155609646400	1.155610040000 1.155610040000
L_2	0.837009426190	0.837009426300	0.837008972600 0.837008972800	0.837008523300 0.837008523600	0.837008078400 0.837008078900	0.837007637900 0.837007638500
L_3	-1.00505466491	-1.005054664477	-1.005054594295 -1.005054594296	-1.005054524786 -1.005054524791	-1.005054455958 -1.005054455960	-1.005054387803 -1.00505438780

5.5.2 Results and discussion

Equations (3.90)-(3.91) describe the motion of a third body under the influence of the triaxiality of the smaller primary and relativistic effect. Equations (3.96), (3.98), (3.100) give respective position of the collinear equilibrium points L_1, L_2, L_3 which are affected by the relativistic and triaxiality factors. Equations (3.96a), (3.98a), (3.100a) give their positions in the presence of triaxiality only. It can be seen in section 3.5.3 that the relativistic and triaxiality factors are unable to change the instability character of the collinear points. For the Sun-Earth system in the absence of triaxiality $(\sigma_1 = \sigma_2 = 0)$ the numerical results of the present study are in agreement with those of Ragos *et al*. (2001). This agreement can be easily seen in Table 5.1 when the notations L_1 and L_2 are interchanged.

For the Sun-Earth system it can be observed for the Sun-Earth system from Table 5.1 that L_1 moves towards the origin from the classical position due to relativistic effect only, whereas it moves from the classical position in the direction of the positive ξ-axis. The triaxiality has more shift than that of the joint effect.

L_2 moves along the positive ξ-axis from the classical position due to relativistic effect only, whereas it moves towards the origin from the classical case due to triaxiality alone or both. L_3 has a shift towards the origin due to triaxiality. It has also a shift towards the origin due to the joint effect. This shift is almost same as that of triaxiality. The relativistic shift in comparison with that of triaxiality is not remarkable.

For the Sun-Pluto system, it is observed from Table 5.2 that the point L_1 has a very small shift towards the smaller primary from the classical position due to relativistic effect. The triaxiality shifts L_1 away from the classical position. This shift is almost similar as that of joint effect. The point L_2 has a very small shift in the direction of the positive ξ-axis from the classical position due to relativistic effect alone. The triaxiality shifts L_2 towards the origin from the classical position. The similar shift is also seen due to the joint effect. The classical position of L_3 does not change due to triaxiality or relativistic or both effects.

For the Earth –Moon system, it can be seen from Table 5.3 that L_1 moves towards the origin from the classical position due to relativistic effect . The triaxiality shifts it from the classical position away from the origin. The similar shift is also observed due to the joint effect. L_2 and L_3 all move towards the origin from the classical position due to relativistic or triaxiality or both effects.

By considering the smaller primary as triaxial rigid body, the positions of collinear points are obtained and their linear stability is also examined. It is found that their positions are both affected by the relativistic terms and triaxiality parameters. This is confirmed from the tables. It is further observed that in spite of the introduction of relativistic and triaxiality coefficients, the collinear points remain unstable. Numerical investigations on this model by considering the Sun-Earth, Sun-Pluto and Earth-Moon systems have been performed to show the relativistic and triaxiality effects on collinear points

5.6 Results and Discussion of Section 4.2

Equations (4.1)-(4.3) describe the motion of a third body under the influence of oblateness of the bigger primary together with small perturbations in the Coriolis and centrifugal forces in the relativistic R3BP. Equations (4.8) give the positions of triangular equilibrium points, which are affected by the oblateness, relativistic factor and a small perturbation in the centrifugal force, but not that of Coriolis force because equation (4.8) is independent of the parameter ε_2; while equation (4.23) gives the critical value of the mass parameter μ_c of the system which depends upon small perturbations $\varepsilon_1, \varepsilon_2$ given in the centrifugal and Coriolis forces, oblateness parameter A_1 and relativistic factor.

The critical value is used to determine the size of the region of stability of the triangular points and also helps in analyzing the behavior of the parameters involved therein. Equation (4.24) describes the region of stability. It is obvious from equation (4.24) that the relativistic term, oblateness coefficient and a small perturbation $\varepsilon_1 > 0$, in the centrifugal force all shrink the stability region independently; whereas the small perturbation in the centrifugal force expands it for $\varepsilon_1 < 0$ and that of the Coriolis force expands it for $\varepsilon_2 > 0$ and shrinks it for $\varepsilon_2 < 0$. This can be explained by the presence of negative coefficients of the formers and positive coefficient of the latter.

Even on considering the coupling terms $\frac{A_1}{c_d^2}$ and $\frac{\varepsilon_i}{c_d^2}(i = 1, 2)$ which are very small quantities, from mathematical points of view it can be observed from equation (4.24) that the joint effect of relativistic and oblateness and that of relativistic and a small perturbation $\varepsilon_1 > 0$ in the centrifugal force expand the size of region of stability; whereas the joint effect of relativistic and a small perturbation ε_2 in Coriolis force reduces it for $\varepsilon_2 > 0$ and expands it for $\varepsilon_2 < 0$. Similarly, the joint effect of relativistic term and a small perturbation ε_1 in the centrifugal force reduces it for $\varepsilon_1 < 0$. This is also as a result of the positive coefficients of the coupling terms $\frac{A_1}{c_d^2}$ and $\frac{\varepsilon_1}{c_d^2}$ and negative coefficient of the coupling term $\frac{\varepsilon_2}{c_d^2}$. However, the net effect is that the size of region of stability increases or decreases or remains unchanged according as $p > 0$ or $p < 0$ or $p = 0$, respectively. In the absence of perturbations and oblateness $(\varepsilon_i = A_1 = 0, i = 1,2)$, the results of the present study are in agreement with those of Douskos and Perdios (2002) and disagree with those of Bhatnagar and Hallan (1998).

In the absence of relativistic terms and centrifugal force $\left(\frac{1}{c_d^2} \to 0, \varepsilon_1 = 0\right)$, the results coincide with those of Szebehely (1967b). In the absence of relativistic terms and oblateness $\left(\frac{1}{c_d^2} \to 0, A_1 = 0\right)$, our results are in agreement with those of Bhatnagar and Hallan (1978). In the absence of relativistic terms and perturbations $\left(\frac{1}{c_d^2} \to 0, \varepsilon_i = 0, i = 1, 2\right)$, the results of the present study coincide with those of SubbaRao and Sharma (1975). In the absence of relativistic terms, our results are in accordance with those of Abouelmagd *et al*. (2013) when the bigger primary is oblate and the mixed effect $A_1\varepsilon_i(i = 1,2)$ is ignored in their study.

By considering the bigger primary as an oblate spheroid body under the influence of small perturbations in the Coriolis and centrifugal forces in the relativistic R3BP, the positions of the triangular points are determined and their linear stability is also investigated. It is found that the effect of relativistic terms, oblateness and a small change in the centrifugal force on these positions are quite prominent. It may also be seen that relativistic terms, oblateness and a small change in the centrifugal force all reduce the size of region of stability independently, where a small perturbation in the Coriolis force expands it.

We have observed the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan (1998) differ from the present study when the oblateness and small perturbations in the Coriolis and centrifugal forces are absent $(i.e.\ \varepsilon_i = A_1 = 0, i = 1,2)$; bearing in mind that the expressions denoted by A_1 and A_2 in Bhatnagar and Hallan (1998) are denoted in this study by E and F respectively. Consequently, the expressions p_1, p_3, p_4, p_5 and the characteristic equation are also different. This led Bhatnagar and Hallan (1998) to infer that the triangular points are unstable, contrary to Douskos and Perdios (2002) and the present results.

5.7 Results and Discussion of Section 4.3

The triangular libration points in the perturbed relativistic R3BP under the assumption that the bigger primary is a triaxial rigid body are discussed. In analogy to corresponding problem without perturbations and triaxiality, the positions of analogous triangular libration points in equation (4.33) are obtained. It is important to note that these triangular libration points in equation (4.33) cease to be classical ones i.e. they no longer form equilateral triangles with the primaries as they do in the classical case. Rather they form scalene triangles with the primaries. It is seen from equation (4.33), that the positions of triangular points are affected by the relativistic effect, triaxiality and the perturbation in the centrifugal force, but not affected by that of the Coriolis force.

Equation (4.48) gives the critical value of the mass parameter μ_c of the system which depends upon triaxiality, relativistic factor and small perturbations $\varepsilon_1, \varepsilon_2$ in the centrifugal and Coriolis forces, respectively. This critical value determines the size of the region of stability of the triangular points and also helps in analyzing the behavior of the parameters involved therein. It is obvious from equation (4.50) that the relativistic and triaxiality effects reduce the size of the stability region separately where as the Coriolis effect expands it if $\varepsilon_2 > 0$ and shrinks it if $\varepsilon_2 < 0.$ Similarly the separate effect of centrifugal force expands it if $\varepsilon_1 < 0$ and shrinks it if $\varepsilon_1 > 0.$

Even on considering the coupling terms $\frac{\varepsilon_i}{{c_d}^2}\left(i=1,2\right)$ which are very small quantities, from mathematical point of view, it can be seen that from equation (4.50) that the joint effect of the relativistic term and centrifugal force that is the term containing the coupling term $\frac{\varepsilon_1}{{c_d}^2}$ expands the size of the stability region if $\varepsilon_1 > 0$ and shrinks it if $\varepsilon_1 < 0$; whereas the joint effect of the relativistic and Coriolis force i.e. the term containing the coupling term $\frac{\varepsilon_2}{c_d{}^2}$ shrinks it if $\varepsilon_2 > 0$ and expands it if $\varepsilon_2 < 0$.

From the overall analysis, it is clear that the Coriolis and centrifugal forces maintained their stabilizing and destabilizing characteristic behavior respectively.

However, it can be seen that from equation (4.51) that the net effect is that the size of the range of stability increases or decreases according as $p > 0$ or $p < 0$ where p depends upon relativistic, centrifugal and Coriolis effects.

In the absence of relativistic terms, our result coincides with those of Singh (2013) when the primaries are non-luminous and only the bigger primary is triaxial . In the absence of small perturbations and the bigger primary is oblate (i.e. $\sigma_1 = \sigma_2 = A_1$), the result of equation (4.33) are in disagreement with those of Katour *et al*. (2014) when the primaries are non-luminous and smaller one spherical.

In the absence of relativistic effect and of small perturbations $(i.e.\ \varepsilon_1 = \varepsilon_2 = 0)$, the results obtained in this study are in agreement with those of Sharma *et al*. (2001a) when the bigger primary is triaxial only; and those of Sharma *et al*. (2001b) when the primaries are non-luminous and the bigger one is triaxial only.

Under the assumption that the bigger primary is a triaxial rigid body and small perturbations $\varepsilon_1, \varepsilon_2$ are given to the centrifugal and Coriolis forces, the stability of the triangular equilibrium points in the relativistic R3BP has been examined. It is found that their positions are affected by the relativistic factor, a small change in the centrifugal force and triaxiality factors of the bigger primary. It is also observable from equation (4.48) that all the parameters involved in this study except the Coriolis force have destabilizing tendencies resulting in a decrease in the size of the region of stability.

It is also noticed that the expressions for A, D, A_2, C_2 in Bhatnagar and Hallan (1998) differ from the present unperturbed study; bearing in mind that the expressions denoted by A_1 and A_2 in Bhatnagar and Hallan (1998) are denoted in this study by E and F respectively. Consequently, the expressions p_1, p_3, p_4, p_5 and the characteristic equation are also different. This led Bhatnagar and Hallan (1998) to infer that the triangular points are unstable, contrary to Douskos and Perdios (2002) and our results.

There are striking differences between results obtained from equation (4.33) when the bigger primary is oblate $(i.e. \sigma_1 = \sigma_2 = A_1)$ and perturbations absent and those of Katour *et al*. (2014) when the primaries are non-luminous and only the bigger one is oblate. The reason is that the perturbed mean motion in equation (4.28) used in this study differs from their own. It seems that there is an error in the perturbed mean motion which they have used.

A practical application of this model could be the study of the motion of a dust grain particle near Pluto and its moon Charon.

5.8 Results and Discussion of Section 4.4

5.8.1 Numerical results

The necessary data used have been borrowed from Sharma and SubbaRao (1975) and Ragos *et al*. (2001). Some members of the solar system are used (mentioned in Table 5.4) to examine the existence and position of the collinear equilibrium points. Equations (4.59), (4.61), (4.63) and (4.59a), (4.61a), (4.63a) have been solved for the various pairs of the solar system. In Table 5.5 the positions of collinear points of the Sun-Planet pairs are presented. The corresponding positions respectively in the classical problem, classical problem with oblateness, relativistic problem and relativistic problem with oblateness as included for comparison purposes (first entry, second entry, third entry and forth entry respectively for each system).

Table 5.4: Parameters of the systems

S.No	System	c_d	μ	A_2
1	Sun-Earth	10064.84	0.000003003500	$0.0000000007\times10^{-8}$
2	Sun-Mars	12424.24	0.000000322700	$0.0000000001\times10^{-8}$
3	Sun-Jupiter	22947.35	0.000953692200	$0.0000192887\times10^{-8}$
4	Sun-Saturn	31050.90	0.000285726000	$0.0000018690\times10^{-8}$
5	Sun-Uranus	44056.13	0.000043548000	$0.0000000070\times10^{-8}$
6	Sun-Neptune	55148.85	0.000051668900	$0.0000000010\times10^{-8}$

Table 5.5: Positions of the collinear equilibrium points

S/No of the system	L_1	L_2	L_3
1	1.01003413809074	0.99002657245074	-1.00000125145833
	1.01003413809000	0.99002657245100	-1.00000125145833
	1.01003413806000	0.99002657248500	-1.00000125145831
	1.01003413806000	0.99002657248500	-1.00000125145831
2	1.00476303037278	0.99525140276082	-1.00000013445833
	1.00476303037300	0.99525140276100	-1.00000013445833
	1.00476335306300	0.99525140276100	-1.00000013445833
	1.00476335306300	0.99525140276100	-1.00000013445833
3	1.06882613997466	0.93236993769216	-1.00039737170283
	1.06882613998000	0.93236993769000	-1.00039737170270
	1.06882613992000	0.93236993764000	-1.00039737170120
	1.06882613992000	0.93236993764000	-1.00039737170120
4	1.04606932684648	0.95474919731454	-1.00011905249873
	1.04606932685000	0.95474919731000	-1.00011905249870
	1.04606932683000	0.95474921600000	-1.00011905249850
	1.04606932683000	0.95474921600000	-1.00011905249850
5	1.02454737494085	0.97576220621890	-1.00001814499999
	1.02454668140000	0.97576220622000	-1.00001814499999
	1.02454668140000	0.97576220665000	-1.00001814499998
	1.02454668140000	0.97576220665000	-1.00001814499998
6	1.02599374139930	0.97434749094956	-1.00002152870832
	1.02599374140000	0.97434749095000	-1.00002152870833
	1.02599374140000	0.97434749156000	-1.00002152870831
	1.02599374140000	0.97434749156000	-1.00002152870831

5.8.2 Results and discussion

Equations (4.52)-(4.55) describe the motion of a third body under the influence of the oblateness of the smaller primary and relativistic terms. Equations (4.59), (4.61) and (4.63) give respective positions of the collinear equilibrium points L_1, L_2, L_3 in the presence of relativistic and oblateness factors while equations (4.59a), (4.61a) and (4.63a) give their positions in the presence of oblateness factor only. It can be seen in section 4.4.3 that the relativistic and oblateness factors are unable to alter the instability behavior of the collinear points. It can be observed when comparing first and second entries of each Sun-Planet pair that the positions of L_1 and L_2 are affected by oblateness in the classical problem, while when comparing first and third entries it can be said that they are affected by the presence of the relativistic terms. It is also noticed that the oblateness effect on the position of the collinear point L_3 of the classical problem in most of the cases is negligible when comparing first and second entries except for the Sun- Jupiter system where it has a little effect, while also when comparing first and third entries it can be said that the relativistic terms have negligible effect on the position of L_3 except for the Sun- Jupiter and Sun- Saturn systems.

It is also noticed that the oblateness and relativistic factors have separately the same effect on the position of L_1 of the Sun-Uranus system and also have same separate effect on the position of L_1 of the Sun-Neptune system as shown from second and third entries of those systems.

However in all cases it is found that the third and forth entries of the relativistic problem only and relativistic problem with oblateness respectively are same up to forteen decimal places. This indicates that in the presence of relativistic terms, the effect of oblateness does not show physically on the positions of collinear points in our Solar system. It is also observed that all the parameters involved have no effect on the position of L_3 of the Sun -Mars system.

By considering the smaller primary as an oblate spheroid, the positions of the collinear points and their linear stability in the relativistic R3BP are studeid. It is found that in spite of the inclusion of relativistic and oblateness coefficients, the instability behavior of the collinear points remains unaltered. A numerical survey of some members of the Sun-Planet pairs of our solar system reveals that the positions of L_1 and L_2 are significantly affected by the oblateness in the absence of relativistic factor and by also relativistic factor in the absence of oblateness ; while the position of L_3 is negligibly affected by oblateness and relativistic factors in most of the cases and more specifically all the parameters involved have no effect on the position of L_3 of the Sun-Mars system. It is observed that the oblateness and relativistic factors have same separate effect on the position of L_1 of the Sun-Uranus system they have also same effect on the position of L_1 of the Sun-Uranus system. It is also noticed that in the presence of relativistic terms, the effect of oblateness does not show physically in our solar system as comparing third and forth entries of Table 5.5.

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Summary

In this thesis, the effects of oblateness, radiation and triaxiality of the primaries on locations and stability of the triangular points are first investigated. Firstly the case where both primaries are triaxial and sources of radiation are considered and then the case where the bigger primary is triaxial and the smaller one is oblate. The locations and stability of the collinear points when the smaller primary is triaxial are also investigated. The analytical and numerical studies in this connection with the Sun-Earth, Sun-Pluto and Earth-Moon systems have been carried out.

The frequencies, eccentricities, semi-major and semi minor axes of the periodic orbits around stable triangular points when the bigger primary is triaxial have been obtained and are found to be affected by the relativistic terms and triaxiality.

Lastly, the locations and stability of the triangular points with perturbations in Coriolis and centrifugal forces with the bigger primary as an oblate spheroid and the smaller one as spherical and then with a triaxial bigger primary and spherical smaller primary have been studied and also the locations and stability of collinear points when the smaller primary is oblate are studied.

6.2 Conclusion

The equilibrium points of the relativistic R3BP when the primaries are non-spherical as well as sources of radiation under the influence of small perturbations in the Coriolis and centrifugal forces have been examined.

It was found that, the perturbation in the centrifugal force, radiation and asphericity of the primaries and relativistic terms affect the locations of the triangular points. It was also observed that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$ where μ_c is the critical mass value which depends on the combined effects of the parameters involved. It was also found that oblateness, triaxiality, radiation pressure forces, centrifugal force and relativistic terms have destabilizing tendencies, while the stabilizing behavior of the Coriolis force remains unchanged. However, the net effect is that the range of stability region increases or decreases according as p is positive or negative, where p depends upon perturbations factors. It was also found that the stability behavior of the collinear points does not change despite the inclusion of triaxiality or oblateness. Hence, they remain unstable.

The periodic orbits around the stable triangular equilibrium points in the range $0 < \mu < \mu_c$ when the bigger primary is triaxial were studied. It was found that the long and short periods, the eccentricities, the semi-major and semi-minor axes and also their orientation were affected by the parameters involved.

The results of the present study generalize the classical and previous results obtained by previous researchers. That is, the previous results as in Douskos and Perdios (2002) can be deduced from it. There are also disagreements with some of the previous authors' results such as in Bhatnagar and Hallan (1998), Ahmed et al. (2006) and in Katour *et al.* (2014) with the present results. It seems that there are some errors therein.

6.3 Recommendations

The results of this study may be applied in space exploration programme. In this research, the stability of equilibrium points in the relativistic restricted three-body problem with constant mass and perturbations has been investigated. For further research, the same problem with variable mass can be investigated.

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