SCALE PARAMETER ESTIMATION OF THE LOG-LOGISTIC DISTRIBUTION UNDER THE ASSUMPTION OF SOME SELECTED INFORMATIVE PRIORS

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January, 2017

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B.Sc. (A.B.U. 2011)

P13SCMT8049

A DISSERTATION SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES,

AHMADU BELLO UNIVERSITY, ZARIA

NIGERIA

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF THE DEGREE OF MASTER OF SCIENCE (M.Sc.) IN STATISTICS

January, 2017.

DECLARATION

I declare that the work in this dissertation entitled "**SCALE PARAMETER ESTIMATION OF THE LOG-LOGISTIC DISTRIBUTION UNDER THE ASSUMPTION OF SOME SELECTED INFORMATIVE PRIORS**" has been carried out by me in the Department of Mathematics under the supervision of Dr. A. Yahaya and Prof. B.M. Yakasai.

The information derived from literature has been duly acknowledged in the text and a list of reference provided. No part of this thesis was previously presented for another degree or diploma at any University or Institution.

Mustapha Muhammad DEWU	_____	_____
Name of Student	Signature	Date

CERTIFICATION

The dissertation entitled "**SCALE PARAMETER ESTIMATION OF THE LOG-LOGISTIC DISTRIBUTION UNDER THE ASSUMPTION OF SOME SELECTED INFORMATIVE PRIORS**" by Mustapha Muhammad DEWU (P13SCMT8049) meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

Dr. A. Yahaya **Chairman Supervisory committee**	**Signature**	**Date**
Dr. S.I.S. Doguwa **Member Supervisory Committee**	**Signature**	**Date**
Dr. H. G. Dikko **Head of Department**	**Signature**	**Date**

DEDICATION

This dissertation is dedicated to Allah(SWT) for his infinite blessings and mercy throughout this study.

ACKNOWLEDGMENT

My profound gratitude goes to Allah(SWT). I also acknowledge the work of previous authors and papers cited. My sincere appreciation goes to my supervisors Dr. A. Yahaya for his relentless effort to make this work a success, his suggestions and criticisms have really brought out the best in me. I also wish to acknowledge Dr. S.I.S. Doguwa for his efforts.

I want to use this medium to appreciate my lecturers especially Dr. A. Yahaya, Dr. H. G. Dikko Mal. Aliyu Yakubu, Mal. Tasi'u, Mal. Ibrahim and Mr. C.N. Nnamani for their guidance, constructive criticisms and direction all through this work. My special thanks goes to my parents and to my entire family. My sincere appreciation also goes to my beloved wife Khadijah. Also, my friends; Ibrahim A. Isa, Abdul Rasheed A. Hassan, Aliyu(GeoAli), Bello Adamu, Michael O. Akinwande, kaileNasir, Umar(teacher), Reuben David, Aminu Mukhtar and many more too numerous to mention. Finally, to the entire PG (2013-2014) students of Statistics Department, I appreciate you all.

ABSTRACT

Log-logistic distribution is a continuous probability distribution that is used in many fields such as hydrology, economics and much more, it has two parameters(scale and shape parameters), in 2015, Nasir *et. al.* estimated the scale parameter of the distribution using two non-informative priors wherein they concluded that the use of jeffrey's prior produced better estimate of the scale parameter and on the other hand Precautionary loss function performed better than squared error loss function but their work is limited to the use of non-informative priors only thereby excluding the performance of informative priors when estimating this parameter. Therefore, using the Bayesian estimation technique we estimated the scale parameter of log-logistic distribution using some selected informative priors. Posterior distributions were derived using Gamma, Chi-square, Maxwell and Rayleigh priors under squared error loss function (SELF) and precautionary loss function (PLF). Fairly extensive Monte Carlo simulations were carried out to obtain the Bayes estimates and their corresponding posterior risks after which a performance comparison was carried out on the different priors and loss functions. It was observed that the estimates obtained under the assumption of informative priors outperforms those obtained using non-informative priors and the PLF yields less posterior risk than the SELF. Based on the obtained results, this study recommends the use of Rayleigh prior under PLF for the estimation of the scale parameter of log-logistic distribution.

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Abbreviations

MOM……...Method of moments

MLE……Maximum likelihood estimator

SELF……...Squared error loss function

PLF……. Precautionary loss function

MCMC……. Markov Chain Monte-Carlo

MSEs……Mean square errors

RMSEs……. RootMean square errors

QQLF…… Quasi-quadratic loss function

SLELF…… Squared log error loss function

QLF……. Quadratic loss function

CHAPTER ONE: INTRODUCTION

1.1 Background of the Study

Log-Logistic Distribution also referred to as the Fisk distribution in Economics is a continuous probability distribution for a non-negative random variable with two parameters (scale and shape parameters). The Log-logistic distribution is a real life distribution used in many fields of human endeavors comprising Hydrology, Economics, Survival Analyses and many more. Its features makes it an alternative to the log-normal distribution (Akhter and Khan, 2014). It is a distribution that was derived from the logistic distribution originally developed by Verhulst (1838) to model population growth (Aryal, 2013). Practically, if T is a random variable having a log-logistic distribution, then $Y = \log T$ is a Random variable that follows a logistic distribution (Santana *et al.*, 2012).

In Bayesian inference, the Bayes' theorem is the focal point for all derivations and analyses. This theorem is named after Rev. Thomas Bayes, an 18^{th} century mathematician who derived it and Bayes' findings were published in 1763 after his death (Brethorst, 1990).

Bayesian parameter estimation is a technique that provides an avenue whereby we can incorporate our prior knowledge/belief about a parameter into an already existing information about that parameter which is in form of a likelihood function. In this regards, our initial belief/knowledge about the parameter is represented by the prior distribution while the likelihood function defines the information contained in the data. Thus, by merging the prior distribution and likelihood function we can obtain our posterior distribution. The parameter is seen as a random variable when considering the Bayesian approach whereas the classical/frequentist approach (method of moments (MOM), maximum likelihood estimation (MLE), etc.) views the

parameter as a fixed but unknown variable, the parameter is random in the sense that we can assign to it a probability at our discretion based on our belief about the parameter.

The Bayesian approach comes in handy for implementing scientific method, wherein a prior probability distribution expresses our initial beliefs/knowledge about the parameter of interest, a likelihood function defines a relevant sample data of the parameter and the posterior probability distribution defines our updated belief about such population parameter based on the collected data and prior knowledge of that parameter. All inferences about the parameter are made by computing appropriate summaries of the posterior probability distribution (Ramachandran, 2009).

In practice one of the three under-listed situation is possible:

i. The posterior distribution derived appears to be one of the standard distributions we are familiar with, hence working with the posterior is quite simple, this often happens when the prior and likelihood form a conjugate family. That is, when the posterior is in the same family of distributions as the prior.
ii. The posterior is not recognized, but we are still able to calculate an exact expression for it. In this case, calculations of the posterior mean or credible interval and other quantities can either be done directly or through simulation on a computer.
iii. The posterior density is not recognizable and cannot even be written in a closed form whereby we are unable to calculate the normalizing constant of the derived distribution.

Because of formidable theoretical and computational challenges, the Bayesian approach has found relatively limited use but thanks to recent advances in Bayesian analyses combined with the growing power of computers that simplifies the job thereby making Bayesian methods

increasingly popular and practical. Currently, the most widely used simulation methods are Markov Chain Monte Carlo (MCMC) and a package in R called R2WinBUGS.

Simplicity in answering scientific questions, flexibility in making inference on a wide range of models, incorporation of prior information/knowledge are among other practical reasons that drive the recent popularity of Bayesian analyses(Ramachandran, 2009).

1.2 Statement of the Problem

From the Bayesian paradigm, the priors formally express degrees of personal(subjective) belief and one of the main criticisms of Bayesian inference is aimed at the subjectiveness of prior choices. Thus, a formal procedure for specifying prior may be divided into finding priors called non-informative, reflecting a desire to remove subjectiveness and to obtain informative priors which are compatible with partial prior knowledge (Figueiredo, 2004). In earlier studies including Feroze *et al*. (2012), Dey (2012) and Nasir *et al*. (2015) different distributions were studied using the Bayesian approach in which the parameters of those distributions were estimated using both informative and non-informative priors, after which an assessment of the Bayes estimates derived using the two classes of priors were carried out and suggestions/recommendations were given as to which combination of loss function and prior is preferred based on the results obtained.

However, Nasir *et al.* (2015) made use of only the non-informative priors to study the scale parameter of log-logistic distribution; but the question of what will be the performance of the estimators of the scale parameter using informative priors remains unanswered.

1.3 Aim and Objectives of the Study

The aim of the research is to estimate the scale parameter of log-logistic distribution using some selected informative priors comprising Chi-square, Maxwell, Rayleigh and Gamma distributions. The above stated aim will be achieved through the following objectives especially by:

i. Deriving the posterior distributions of scale parameter of the log-logistic distribution under the assumption of chi-square, Maxwell, Rayleigh and Gamma priors.
ii. Obtaining the Bayes estimators and Bayes posterior risks from the resultant posterior distributions obtained in (i) under the SELF and PLF.
iii. Conducting a Simulation study to assess and compare the performances of the Bayes estimates obtained.

1.4 Motivation of the study

The work of Feroze and Aslam (2012) used both informative and non-informative priors to obtain results that suggested the use of entropy loss function under informative priors for the estimation of the scale parameter of error function distribution. Nasir *et al.* (2015) made use of only the non-informative priors for the estimation of the scale parameter of log-logistic distribution. In view of the foregoing, we developed an interest in assessing the performance of scale parameter estimates for the log-logistic distribution under some informative priors (Gamma, Maxwell, Rayleigh and chi-square).

1.5 limitation of the study

One of the major controversy in Bayesian analyses is the choice of the prior distribution. In theory, this reflects prior beliefs about the parameter of interest and selecting meaningful prior distributions (and loss functions) is often difficult. Accordingly, a limitation of this study is the subjectiveness in our choice of priors coupled with the fact that we cannot exhaust the existing

informative priors thereby making the conclusion of these study limited to only the priors used. However, future works can be undertaken to investigate the sensitivity of the Bayes estimates to other informative priors.

1.6 Significance of the study

As is apparent in many researches, the use of informative priors in the estimation of parameters of several distributions provides a comprehensive assessment of such parameters, Nasir *et al.* (2015) studied the scale parameter of the log-logistic distribution using only the non-informative priors (Uniform and Jeffrey's priors). Hence this study extended such work by studying the scale parameter of the log-logistic distribution using informative priors which further broadened the knowledge of how the selected informative priors used in this research influenced the results of the Bayes estimate and posterior risk of the scale parameter of log-logistic distribution. Furthermore, the results obtained from this work will assist in choosing the combination of prior and loss function having a minimum risk and hence producing the best result/estimate.

1.7 Definition of Terms

Scale parameter: Defines how spread out the data of the distribution is, a larger scale value stretches the distribution while a smaller value shrinks the data distribution.

Prior distribution: Prior distribution is a key part of the Bayesian inference that represents the information about an uncertain parameter which is combined with the probability distribution of the new data to yield the posterior distribution. In practice one can check the dependence on prior distributions by comparing posterior inferences under different reasonable choices of prior distributions (Gelman, 2002).

Informative priors: An informative prior is that prior which is not dominated by the likelihood of the distribution and therefore has an impact on the posterior distribution. It is based on the information gathered from previous study, past experience and/or the experimenter's own opinion or belief.

Non-informative priors: A prior is said to be non-informative if it is flat relative to the likelihood function. Thus, a prior is non-informative if it has minimal or no impact on the posterior distribution. Typical examples of non-informative priors include the Uniform and Jeffrey's priors.

Posterior distribution: The posterior distribution summarizes the current state of knowledge about all the uncertain quantities in a Bayesian analyses. Analytically, the posterior density is the product of the prior density and the likelihood function of the distribution.

Loss and Risk functions: The loss function simply measures how far an estimated value of a parameter is from its true value; whereas the risk function is the average measure of loss and it is calculated by taking the expected value of the loss function. In this study we intend to make use of the SELF which is classified as a symmetric loss function and the PLF which is classified as an asymmetric loss function.

Markov Chain Monte-Carlo: It is a numerical simulation method that samples a given distribution function by building a Markov chain that has the desired distribution as its equilibrium distribution.

Hyper-parameters: These are simply parameters of the adjoining prior distributions that are found in the posterior distributions.

CHAPTER TWO: LITERATURE REVIEW

According to Ramachandran and Tsokos (2009), Bayesian procedures are becoming increasingly popular in building statistical models for real-world problems. In recent years, the Bayesian statistical methods have been used in the scientific fields. Bayesian inference is a method of analyses that combines information collected from the experiment with the knowledge already existing prior to performing such experiment. Bayesian statistics provides a natural method for updating uncertainty in the light of evidence, this approach seeks to optimally merge information from a prior information and likelihood functions of the distribution. Basically, the prior distribution represents our initial belief, whereas the information in the data is expressed by the likelihood function. Therefore, combining prior distribution and likelihood function, we can obtain the posterior distribution which expresses our revised uncertainty in light of the data. The Bayesian approach provides a convenient model for implementing the scientific method, the appropriate summaries of the posterior probability distribution are calculated to make inferences about the parameter of interest. The exact solution of the mathematical expression is often very challenging, but in recent times computers have been employed to carry out such mathematics using some complicated simulation techniques such as the MCMC method.

As the name implies, the log-logistic distribution is a derivative of the logistic distribution that was originally developed by Verhulst (1838) to model population growth, since the inception of the logistic growth curve there have been several contributions suggesting alternative functional forms for growth whilst retaining the sigmoid and asymptotic property of the Verhulst logistic curve. The use of the logistic distribution for economic and demographic purposes was very popular in the nineteenth century and it is also known by names such as growth function, autocatalytic curve and so on depending on its applications. The importance of the logistic

distribution is evident as it applies to many areas of human endeavor including biology, epidemiology, psychology, technology, energy among others (Aryal, 2013). In practice, if T has a log-logistic distribution, then $Y = \log T$ has a logistic distribution (Santana *et al.*, 2012). The log-logistic distribution is widely used in practice and it is an alternative to the log-normal distribution since it presents a failure rate function that increases, reaches a peak after some finite period and then declines gradually, its properties makes it an alternative to the log-normal and Weibull distributions in the analyses of survival data (Collet, 2003). Recent study by Dey *et al.* (2010) helps to point out the differences between the log-normal and log-logistic distributions, the log-logistic distribution has also been used to model stream flow and precipitation (Shoukri*et al.*, 1988) and (Ashkar and Mahdi, 2006). In addition,Gupta *et al.*(1999) made a study of log-logistic model in survival analyses and collet (2003) suggested the log-logistic distribution for modelling the time following a heart transplantation. Aryal(2013) proposed and studied a generalization of the log-logistic distribution which he called the transmuted log-logistic distribution. The log-logistic distribution also referred to as the fisk distribution in economics is a continuous probability distribution for a non-negative random variable as stated in Akhter and Khan (2014), in addition the log-logistic distribution has two parameters namely: scale and shape parameter.

Gupta and Kundu (2000) obtained the Bayes estimators of the unknown parameters of generalized exponential distribution under the assumptions of gamma prior on both the scale and shape parameters. Approximate Byes estimators were computed using lindley's idea because the estimators could not be obtained in explicit form. They also proposed the Gibb's sampling procedure to generate samples from the posterior distributions and in turn computed the Bayes estimators. A comparison was made between the approximate Bayes estimators obtained with

that of the Maximum likelihood estimators(MLE) using Monte-Carlo simulation method. They also analyzed one real data set to demonstrate how the proposed methods can be used in practice. They compared the MLEs with the approximate Bayes estimates in terms of biases and root mean squared errors(RMSE) for different sample sizes and for different parameter values. As expected, it was observed that the performances of all the estimators became better when the sample sizes increases, it was also observed that both in terms of biases and RMSE that for large sample sizes, the Bayes estimates and the MLEs become closer. The Bayes estimates of the scale parameter(λ) performs better than the MLEs, in general the Bayes estimates of the shape parameter(α) also perform better than the MLEs for $\alpha \leq 1$ and vice-versa for $\alpha > 1$. While the performance of Bayes estimates under the assumption of non-informative (Uniform and Jeffrey's) priors and that of the MLEs are quite similar.

Pandey and Rao (2009) made use of the Quasi, inverted gamma and uniform prior distributions to obtain the Bayes estimators of the shape parameter of generalized Pareto distribution under the LINEX, precautionary and entropy loss functions. These results were compared with the corresponding Bayes estimators under the squared error loss functions. The Bayes estimators are seen to depend upon the parameters of prior distribution and from the results obtained, it was clear that no one of the estimators uniformly dominates any other, therefore they recommended that the estimators be chosen according to the value of *d* when Quasi-density is used as the prior distribution and this choice in turn depends on the situation at hand.

Pandey *et al.* (2011) considered the estimation of the scale parameter of two parameter Weibull distribution with known shape parameter using both maximum likelihood and Bayes estimation methods. The Bayes estimates were obtained using Jeffrey's prior under LINEX loss function, relative efficiency of the estimators were also calculated in small and large samples for over-

estimation and under-estimation using simulated data sets, at the end they concluded that the Bayes estimator performs better than the maximum likelihood estimator when sample size is small.

Pradhan and Kundu (2011) considered the Bayes estimates of the two-parameter gamma distribution and in their study it was assumed that the scale parameter has a gamma prior while the shape parameter has a log-concave prior and they are independently distributed, under the mentioned priors they adopted the Gibb's sampling technique to generate samples from the posterior density function. Also they computed the approximated Bayes estimates using lindley's approximation under the assumption of gamma priors for the shape parameter, furthermore they made use of the Monte-Carlo simulation technique to compare the performances of the Bayes estimators with that of the classical estimators. The results of this study showed that when informative priors are used, the proposed Bayes estimators behave better than the classical maximum likelihood estimators but with the use of non-informative priors their behaviors and performances are almost the same, and again it was observed that the proposed Gibbs sampling procedure can be used quite effectively for the posterior predictive density of a future observation based on the current sample and also for constructing the associated predictive interval.

Ramin*et al.* (2011) focused on the study of Bayesian and non-Bayesian(classical) estimators for the shape parameter, reliability and failure rate functions of the Kumaraswamy distribution in the cases of progressively type II censored samples. They obtained the maximum likelihood estimators and Bayes estimators of the shape parameter and also obtained the reliability and failure rate functions using gamma distribution as the prior. Bayes estimators and maximum likelihood estimators(MLEs) under general progressively type II censoring and progressively

type II right censoring using square error, LINEX and precautionary loss functions were obtained. They conducted a simulation study wherein the mean squared errors using generated random samples of different sizes were computed for each estimator in order to assess the statistical performances of these estimators, the simulation study was executed using the MATLAB software. From their study Bayes approach under squared error loss function for the shape parameter estimation of Kumaraswamy distribution was suggested.

Saemeh and Hassan (2011) obtained Bayesian estimators of the shape parameter of the minimax distribution using grouped and ungrouped data. In the Bayesian estimation under the assumption of Gamma prior, they considered two types of loss functions; the squared error and precautionary loss functions, they considered both interval and point estimation. A simulation study was conducted with the sole aim of comparing these estimators in terms of biases and mean square errors (MSEs) which were calculated based on Monte-Carlo simulation procedures. From the results obtained, Bayes estimates under precautionary loss function have the smallest estimated MSE as compared with the Bayes estimates under squared error loss function, thus they suggested the use of Bayes approach under precautionary loss function for estimating the shape parameter of the distribution for both grouped and ungrouped data.

Dey (2012) took into account the Jeffrey's prior to obtain the Bayes estimators of the inverse Rayleigh distribution under the SELF which is symmetric in nature and the LINEX loss function which is asymmetric, the reliability function of the distribution was also discussed. From the simulation study, it was observed that Bayes estimators under the LINEX loss function exhibit better performance than the squared error loss function and in the case of reliability function, it is worth mentioning that estimation under the LINEX loss function is superior to the SELF with respect to the measure of RMSE.

Feroze and Aslam (2012a) estimated the scale parameter of log-gamma distribution under Bayesian and maximum likelihood framework. They assumed the uniform and Jeffrey's priors for posterior analyses and the Bayes estimators and associated risks were derived under squared error, precautionary, entropy, weighted and weighted balanced loss functions. The credible intervals and highest posterior density intervals were also constructed under each prior. In this study, they affirmed that the performance of the Bayes estimation is better than maximum likelihood estimation. In comparison of priors, the performance of the Jeffrey's prior is found to be better. Similarly, the estimates under entropy loss function are associated with the minimum risks and the magnitude of these risks is independent of parametric values choice. The Bayesian interval estimation further strengthened the findings of the point estimation. So, in order to estimate the scale parameter of the log-gamma distribution, the Bayesian estimation can be preferred over maximum likelihood estimation. While under the Bayesian estimation, the entropy loss function can effectively be employed.

Feroze and Aslam (2012b) considered the Bayesian analyses of the scale parameter of error function distribution. A class of informative (Maxwell, Rayleigh, Chi-square and Normal) and non-informative (Uniform and Jeffrey's) priors were assumed to derive the corresponding posterior distributions. The Bayes estimators and associated risks were also calculated under different loss functions and they also constructed the Bayesian credible intervals under each prior. They undertook a comprehensive simulation study to evaluate and compare the performance of the Bayes estimators. The purpose is to find out the combination of a loss function and a prior having the minimum Bayes risk and hence producing the best results. The study depicts that in order to estimate the said parameter using the Bayesian point estimation

technique, the use of entropy loss function under Maxwell prior can be preferred. While for interval estimation, the chi-square prior can affectively be employed.

Feroze (2012) carried out a Bayesian analyses of the scale parameter of inverse Gaussian distribution wherein he derived the Bayes estimators along with their corresponding posterior risks under the assumption of Uniform, Jeffrey's, exponential, gamma and chi-square priors using squared error, quadratic, entropy, weighted, squared logarithmic, LINEX, precautionary and weighted balance loss functions. He conducted a simulation study to evaluate the behavior and performance of the derived estimators by making a comparison in terms of the magnitude of their posterior risks under the various loss functions and priors. From the study it was obvious that Bayes estimates of the scale parameter converges to the true value of the parameter by increasing the sample size. Using LINEX loss function, the estimated value of the parameter appeared to be always less than the estimates obtained under other loss functions and the Bayes estimates obtained under entropy and weighted loss functions are same for each prior. Again, it was observed that the risk associated with estimates under entropy loss function is the minimum for each estimate, while the maximum risk is associated with estimates under squared error loss function for each prior while the risks for estimates under quadratic, square log and weighted balanced loss functions are similar for each prior especially for large samples. Finally, he concluded that the entropy loss function is preferred for obtaining the Bayes estimates of the scale parameter of inverse Gaussian distribution for each of the priors used, as the risks associated with these estimates are always the minimum whereas, the performance of informative priors for each of the adopted loss function was seen to be better. Hence, entropy loss function under informative priors provides the most efficient point estimator for the scale parameter of inverse Gaussian distribution.

Guure*et al.* (2012) obtained the Bayes and maximum likelihood estimators for the two-parameter Weibull failure time distribution. The Bayes estimator was obtained using the lindley's approximation technique using an extension of Jeffrey's prior under linear exponential (LINEX) loss, generalized entropy loss and squared error loss functions. They carried out a simulation study to compare the performances of the different estimators obtained and it was observed from the study that the unknown parameters, the reliability and hazard functions are best estimated using Bayes estimation technique under LINEX loss function with the proposed prior followed by general entropy loss function.

Mahmood and Aijaz (2012) employed the MCMC method to estimate the parameters of generalized exponential distribution. The MCMC method had been shown to be easy in computational implementation, the estimates always exist and are statistically consistent and their probability intervals are also convenient to construct. R functions were developed to study the statistical properties of the distribution and the output analyses of MCMC samples were generated from open BUGS.

Sindhu *et al.* (2013) derived posterior distributions under informative (exponential, gamma and inverse levy) and non-informative(uniform and Jeffrey's) priors in the presence of left censoring, they also provided the Bayes estimators and corresponding posterior risks under SELF, weighted squared error loss function(WSELF), Quasi-Quadratic loss function(QQLF), squared log error loss function(SLELF) and entropy loss function(ELF), credible intervals for type II left censored data under the different priors were also presented. Elicitation of the hyper-parameters method via prior predictive approach was also discussed and in addition they derived the posterior predictive distribution as well as posterior predictive intervals.

They undertook a simulation study which displayed some interesting properties of the Bayes estimates. The risks of the estimates seem to be larger for the larger values of the parameter and vice versa, except under quasi-quadratic loss function. However, the risks under the said loss functions reduce as the sample size increases. Another interesting remark concerning the risks of the estimates is that increasing (decreasing) the value of the parameter reduces (increases) the risks of the estimates under quasi-quadratic loss function. The performance of squared-log error loss function and entropy loss function is independent of choice of parametric value. The increased values of the parameter result in higher levels of overestimation/underestimation of the parameters. In comparison of non-informative priors, the Jeffrey's prior provides the better estimates as the corresponding risks are smaller under WSELF, PLF and SELF. While the uniform prior turns out to perform better under QQLF, SLELF and ELF. On the other hand, in comparison of informative and non-informative priors, gamma prior turns out to perform better under the said loss functions except QQLF and SELF, therefore it produces more efficient estimates as compared to other informative or non-informative priors. In addition, estimates under quasi-quadratic loss function give the minimum risks among all loss functions for each prior. It was also observed that the performance of estimates under informative priors is better than those under non-informative priors. The credible intervals are in accordance with the point estimates, that is, the width of credible interval is inversely proportional to sample size while, it is directly proportional to the parametric value. The results also revealed that the credible intervals assuming inverse Levy prior are much narrower than the credible intervals assuming the rest of informative and non-informative priors. The Bayesian predictive intervals of the future observation assuming informative and non-informative priors were also constructed. The Inverse Levy prior produced more precise predictive intervals than (its competitors) other

informative and non-informative priors. The results they obtained showed that the gamma prior has the least posterior risk as compared to other priors under majority of the loss functions except QQLF and SELF. The real life example depicts the similar behavior of the estimates of parameter under study, which they have observed under simulation study. So result of simulation study and real data set are similar in all aspects.

Yahgmaei*et al.* (2013) obtained the maximum likelihood and Bayes estimators of the scale parameter of inverse Weibull distribution, they considered quasi, gamma and uniform priors under the square error, entropy and precautionary loss functions. The MCMC simulation method was used to compare the estimators obtained and from the results of the study it was clear that the Bayes method of estimation gives the best estimate for the scale parameter of the inverse Weibull distribution. Also, in the case of gamma prior, the Bayes estimators related to precautionary loss function have the smallest mean squared error as compared with the estimators under entropy and squared error loss function while in the case of uniform prior, the Bayes estimators under square error loss function are doing better than those under precautionary loss function.

Azam and Akhter (2014) obtained Bayesian estimates of the scale parameter of Nakagami Distribution. For the posterior distribution of the scale parameter, they considered Uniform, inverse Exponential and Levy priors under SELF, QLF and precautionary PLF. The performance of the estimators was assessed on the basis of their relative posterior risks using Monte-Carlo simulation techniques. Based on the results obtained, it was discovered that the precautionary loss function(PLF) produces the least posterior risk when uniform prior is used while squared error loss function (SELF) performed better when Inverse exponential and Levy priors are used.

Nasir *et al.* (2015) used the Bayes approach to study the scale parameter of the log-logistic distribution. They derived the posterior distribution using uniform and Jeffrey's prior which are all non-informative priors. The Bayes estimators and their corresponding risks were also derived using squared error, precautionary, simple precautionary and weighted loss functions. After all, a simulation study was carried out to check the performances of the priors and loss functions that were used in the study and from the results, it was concluded that Bayes estimators approach their true value and their posterior risks decreases with increasing sample size, again it was noticed that Jeffrey's prior performed better than the uniform prior as it's posterior risks is minimum, in the same vein precautionary loss function exhibited a minimal posterior risk compared to other loss functions used. Therefore, they suggested the use of Jeffrey's prior under precautionary loss function for the estimation of the scale parameter of log-logistic distribution.

Nasir and Aslam (2015) studied the shape parameter of the Frechet distribution using the Bayesian paradigm wherein they derived the posterior distribution using Gumbel type-II and levy priors, square error, simple precautionary, weighted balanced and minimum expected loss functions were used to obtain the Bayes estimators and their corresponding posterior risks. In order to compare the performances of the estimators, a simulation study was undertaken and from their result, Gumbel type-II performs better than the levy prior while weighted balanced loss function(WBLF) performs better when compared to the other three loss functions. Hence, they recommended the use of Gumbel type-II as a prior distribution under weighted balanced loss function(WBLF) when estimating the shape parameter of the Frechet distribution.

Li (2015) studied the Bayes estimation of the unknown shape parameter of Minimax distribution. The prior distribution used here is the non-informative quasi-prior of the parameter. Bayes estimators were derived under squared error loss function and three asymmetric loss functions,

which are the LINEX loss, precaution loss and entropy loss functions. To compare the Bayes estimators obtained in this study, MCMC simulations were used to generate random samples of different sizes where the MSEs were used to compare the performance of different Bayes estimates. The Simulation results show that the Bayes estimates under the squared error loss function have the largest MSEs as compared with estimates under other loss functions and it is also obvious that MSEs decrease as sample size increases. When the sample size n is large, such as $n > 50$, the MSEs are almost the same.

Sultan and Ahmad (2015) studied the estimation of shape parameter of Kumaraswamy Distribution using various Bayesian approximation techniques like normal approximation, Lindley's Approximation, Tierney and Kadane (T-K) Approximation. Different informative (Jeffrey's and modified Jeffrey's) and non-informative priors (Gamma and inverse Levy) were also used to obtain the Bayes' estimate of the parameter under different approximation techniques. To examine the performance of Bayesian estimates for shape parameter of kumaraswamy distribution under different approximation techniques, a simulation study was carried out Using R-software and estimates were presented along with posterior standard deviation and MSE in case of Lindley's approximation. From the findings of their study it is observed that in all cases normal approximation, Lindley's approximation, T-K approximation, Bayesian estimates under informative priors are better than those under non-informative priors especially the Inverse levy distribution which proves to be efficient with minimum posterior standard deviation and mean square error in case of Lindley's approximation. In case of non-informative priors modified Jeffrey's prior proves to be more efficient. It is also observed that under informative as well as non- informative priors, the normal approximation behaves well than T-K approximation, although the posterior variances in case of T-K approximation are very

close to that of normal approximation. Further we conclude that the posterior standard deviation based on different priors tends to decrease with the increase in sample size. It implies that the estimators obtained are consistent. It can also be observed that the performance of Bayes estimates under informative priors (inverse levy) is better than non-informative prior.

Yaseen and Feroze (2015) considered inverse gamma prior and uniform prior for the analyses of the scale parameter of Nakagami distribution under WLF, WBLF and PLF. They conducted a simulation study using the MCMC simulation method to evaluate the behavior and performance of the estimators obtained, their comparisons were based on the magnitude of the posterior risks, on the basis of the analyses it was concluded that in order to estimate the scale parameter of Nakagami distribution under a Bayesian framework, the use of inverse gamma prior under WBLF can be preferred.

Aliyu and Abubakar (2015) estimated the shape parameter of the generalized Rayleigh distribution using the Bayesian method of estimation and the method of maximum likelihood estimation (MLE). In the Bayesian approach, they made use of the extended Jeffrey's prior under squared error, entropy as well as precautionary loss function to obtain the posterior distributions, Bayes estimators and their corresponding risks. Extensive Monte-Carlo simulations were employed to compare the performances of the estimators with respect to biases and MSEs using different sample sizes. From their results, the MSEs of the Bayesian estimates are lower than that of the MLEs. Hence, the Bayesian estimators are more stable than the MLEs, it was further observed that the Bayesian estimators under entropy loss function perform better than the estimates under the precautionary and squared error loss functions as well as the maximum likelihood estimates.

CHAPTER THREE: METHODOLOGY

3.1 Bayesian setup

In the Bayesian parametric approach, several quantities are modelled. Firstly, the behavior of the data given the parameter α is modelled, this is the same as the likelihood in frequentist/classical models and it is written as $p(x|\alpha)$, x here denotes the vector of the observed data $x=\{x_1,x_2,...,x_n\}$ in this case it is assumed that our data is a random sample (iid) and hence $p(x|\alpha)=\prod_{i=1}^{n} p(x_i|\alpha)$. Thus, all inferences on α are given by the posterior distribution:

$$p(\alpha|x)=\frac{p(x|\alpha)\pi(\alpha)}{\int_{-\infty}^{\infty} p(x|\alpha)\pi(\alpha)d\alpha} \tag{3.1}$$

and

$$p(\alpha|x)\propto p(x|\alpha)\pi(\alpha)$$

The posterior distribution is a way in which we combine the data observed with our previous/prior knowledge and/or beliefs about the parameter of interest.

Given a data $x=\{x_1,x_2,...,x_n\}$, Bayesian inference is carried out in the following manner;

i. A prior distribution $\pi(\alpha)$ for the parameter α is chosen which describes our belief about the parameter α before the data is examined.

ii. From the observed data x, a model density $p(x|\alpha)$ is selected in form of a likelihood function to describe the distribution of x given the parameter α.

iii. We combine the prior distributions and the observed data through the calculation of the posterior distribution $\pi(\alpha \mid x)$ given in equation (1).

iv. Bayes estimates and their corresponding risks under Squared error loss function(SELF) and precautionary loss function(PLF) will be calculated.

v. Finally, a simulation study is carried out to provide estimates of the parameter using the different prior distributions under SELF and PLF.

3.2 Model and Likelihood function.

The probability density function(PDF) of log-logistic distribution is given by:

$$f(x;\alpha,\beta) = \frac{\alpha x^{\alpha-1}}{\beta^{\alpha}\left[1+\left(\frac{x}{\beta}\right)^{\alpha}\right]^2} \quad where\, x,\alpha,\beta > 0 \tag{3.2}$$

The cumulative distribution function(CDF) is given by:

$$F(x;\alpha,\beta) = \frac{1}{1+\left(\frac{x}{\beta}\right)^{-\alpha}} \quad where\, x,\alpha,\beta > 0 \tag{3.3}$$

Here α is the scale parameter while β is the shape parameter.

The likelihood function of the distribution is derived as follows;

$$L(x;\alpha,\beta) = \frac{\left(\alpha x_1^{\alpha-1}\right)\times\left(\alpha x_2^{\alpha-1}\right)\times\ldots\times\left(\alpha x_n^{\alpha-1}\right)}{\left(\beta^{\alpha}\left[1+\left(\frac{x_1}{\beta}\right)^{\alpha}\right]^2\right)\times\left(\beta^{\alpha}\left[1+\left(\frac{x_2}{\beta}\right)^{\alpha}\right]^2\right)\times\ldots\left(\beta^{\alpha}\left[1+\left(\frac{x_n}{\beta}\right)^{\alpha}\right]^2\right)}$$

$$L(x;\alpha,\beta)=\frac{\left(\alpha\prod_{i=1}^{n}x_i^{\alpha-1}\right)}{\left(\beta^{n\alpha}\prod_{i=1}^{n}\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]^2\right)}$$

Transforming the likelihood function into exponential form

By letting:

$$X=\prod_{i=1}^{n}x_i^{\alpha-1}$$

$$Y=\beta^{n\alpha}$$

$$Z=\prod_{i=1}^{n}\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]^2$$

And taking the log of $X, Y \text{ and } Z$ each, we get;

$$X=\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}}$$

$$Z=\ell^{2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}$$

$$Y=\ell^{n\alpha\ln\beta}$$

Hence:

$$L(x;\alpha,\beta)=\frac{\alpha^n\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}}}{\ell^{n\alpha\ln\beta}\ell^{2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}$$

$$L(x;\alpha,\beta)=\alpha^n\times\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}}\times\ell^{-n\alpha\ln\beta}\times\ell^{-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}$$

$$L(x;\alpha,\beta)=\alpha^n\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \quad (3.4)$$

3.3 Prior distribution

An appropriate choice of prior(s) is indispensable for Bayesian analyst. However, very often, researchers select prior(s) based on their subjective knowledge and beliefs. Nevertheless, if we have enough information about the parameter(s), we should use informative prior(s), otherwise it is better to consider non-informative or vague prior(s), see Dey(2012).

In this work we consider the following informative priors for the estimation of the scale parameter of log-logistic distribution:

i. Gamma prior is;

$$\pi(\alpha) \propto \alpha^{a-1}\ell^{-b\alpha} \quad where\ a,b,\alpha > 0 \tag{3.5}$$

ii. Chi-square prior is;

$$\pi(\alpha) \propto \alpha^{k-1}\ell^{\frac{\alpha^2}{2}} \quad where\ k,\alpha > 0 \tag{3.6}$$

iii. Maxwell prior is;

$$\pi(\alpha) \propto \alpha^{2}\ell^{-\frac{\alpha^2}{a}} \quad where\ a,\alpha > 0 \tag{3.7}$$

iv. Rayleigh prior is;

$$\pi(\alpha) \propto \alpha\ell^{\frac{\alpha^2}{b^2}} \quad where\ b,\alpha > 0 \tag{3.8}$$

3.4 Loss and Risk function

Taking into account the stated priors, we consider two different loss functions for the model, the SELF and PLF. From a Bayesian perspective, choice of loss functions is an essential part in estimation and prediction problems. Since there is no specific analytical procedure to identify the appropriate choice of loss function, most often for convenience, researchers consider the

underlying loss function to be the squared error which is symmetric in nature. However, indiscriminate use of SELF is not appropriate especially in cases where the losses are not symmetric. Thus in order to make statistical inferences more practical and applicable, we often incorporate an asymmetric loss function. In this study we consider both symmetric(SELF) and asymmetric(PLF) loss functions for better comprehension of our Bayesian analysis.

Below are the stated loss functions and their corresponding risks:

i. The squared error loss function is given by;

$$L(\hat{\alpha},\alpha)=(\hat{\alpha}_{sl}-\alpha)^2 \tag{3.9}$$

Where $\hat{\alpha}_{sl}$ denotes the Bayes estimator of α relative to the SELF and it's given by:

$$\hat{\alpha}_{sl}=E(\alpha\mid x) \tag{3.10}$$

The risk function of the SELF is derived as follows:

By taking the expectation of the SELF

$$E\left[(\hat{\alpha}_{sl}-\alpha)^2\right]=E(\hat{\alpha}^2-2\hat{\alpha}\alpha+\alpha^2)$$
$$E\left[(\hat{\alpha}_{sl}-\alpha)^2\right]=\hat{\alpha}^2-2\hat{\alpha}E(\alpha)+E(\alpha^2)$$
substituting $\hat{\alpha}=E(\alpha)$
$$E\left[(\hat{\alpha}_{sl}-\alpha)^2\right]=\left[E(\alpha)\right]^2-2\left[E(\alpha)\right]^2+E(\alpha^2)$$
$$E\left[(\hat{\alpha}_{sl}-\alpha)^2\right]=E(\alpha^2)-\left[E(\alpha)\right]^2$$

Hence the risk associated with SELF is given by:

$$R\hat{\alpha}_{sl}=E(\alpha^2)-\left[E(\alpha)\right]^2 \tag{3.11}$$

The squared error loss function is classified as a symmetric loss function which assigns equal importance to the losses incurred as a result of overestimation and/or underestimation of a parameter. However, this is impractical in real life situations since overestimation of a parameter can lead to a more severe or less severe consequence than under estimation or vice versa as in the estimation of reliability or failure rate function. Hence, the loss function which associate greater importance to overestimation or underestimation and serves as an alternative to symmetric loss function is the asymmetric loss function. (Aliyu and Abubakar, 2015).

ii. Precautionary loss function introduced byNordstrom (1996) is an asymmetric loss function which is given as:

$$L\left(\hat{\alpha},\alpha\right)=\frac{\left(\hat{\alpha}_{pl}-\alpha\right)^{2}}{\hat{\alpha}_{pl}} \tag{3.12}$$

Where $\hat{\alpha}_{pl}$ denotes the Bayes estimator of α relative to the PLF and it is given by:

$$\hat{\alpha}_{pl}=\sqrt{E\left(\alpha^{2}\mid x\right)} \tag{3.13}$$

The risk function of the PLF is derived as follows:

By taking the expectation of the PLF

$$E\left[\frac{\left(\hat{\alpha}-\alpha\right)^{2}}{\hat{\alpha}}\right]=\left[\frac{\hat{\alpha}^{2}-2\hat{\alpha}\alpha+\alpha^{2}}{\hat{\alpha}}\right]$$

$$\textit{substituting } \hat{\alpha}_{pl}=\sqrt{E\left(\alpha^{2}\mid x\right)}$$

$$E\left[\frac{\left(\hat{\alpha}-\alpha\right)^{2}}{\hat{\alpha}}\right]=\frac{\left[\sqrt{E\left(\alpha^{2}\right)}\right]^{2}-E\left(\alpha^{2}\right)}{\sqrt{E\left(\alpha^{2}\right)}}-2E(\alpha)$$

$$E\left[\frac{(\hat{\alpha}-\alpha)^2}{\hat{\alpha}}\right]=2\left[\sqrt{E(\alpha^2)}\right]-2E(\alpha)$$

Hence the risk associated with the PLF is given by:

$$R\hat{\alpha}_{pl}=2\left[\sqrt{E(\alpha^2)}-E(\alpha)\right] \tag{3.14}$$

Note:

$$E(\alpha)=\int_0^\infty \alpha\, p(\alpha\mid x)\,d\alpha \tag{3.15}$$

$$E(\alpha^2)=\int_0^\infty \alpha^2\, p(\alpha\mid x)\,d\alpha \tag{3.16}$$

3.5 Bayesian Analysis under the assumption of Gamma prior

Let the Gamma prior to be;

$$\pi(\alpha)\propto \alpha^{a-1}\ell^{-b\alpha} \ \ where\ a,b,\alpha>0$$

Adopting equation (1), The posterior distribution of the scale parameter for a given data $\left(X=\{x_1,x_2,...,x_n\}\right)$ is derived as;

$$p(\alpha\mid x)=\frac{\alpha^n \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right]}\times\alpha^{a-1}\ell^{-b\alpha}}{\int_0^\infty \alpha^n \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right]}\times\alpha^{a-1}\ell^{-b\alpha}d\alpha}$$

$$p(\alpha\mid x)=\frac{\alpha^{n+a-1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right]-b\alpha}}{\int_0^\infty \alpha^{n+a-1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right]-b\alpha}d\alpha} \tag{3.17}$$

Bayes estimate of the scale parameter using Gamma prior under SELF;

Where

$$\hat{\alpha}_{sl} = E(\alpha \mid x) = \int_0^\infty \alpha\, p(\alpha \mid x)\, d\alpha$$

$$E(\alpha \mid x) = \int_0^\infty \frac{\alpha^{n+a-1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha} d\alpha} \times \alpha\, d\alpha$$

$$E(\alpha \mid x) = \int_0^\infty \frac{\alpha^{n+a}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha} d\alpha} d\alpha \qquad (3.18)$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{\alpha}_{sl} = E(\alpha^2) - \left[E(\alpha)\right]^2$$

$$R\hat{\alpha}_{sl} = \int_0^\infty \frac{\alpha^{n+a}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha} d\alpha} \times \alpha\, d\alpha - \left[\int_0^\infty \frac{\alpha^{n+a}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha} d\alpha} d\alpha\right]^2$$

$$R\hat{\alpha}_{sl} = \int_0^\infty \frac{\alpha^{n+a+1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha} d\alpha} d\alpha - \left[\int_0^\infty \frac{\alpha^{n+a}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1}\, \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - b\alpha} d\alpha} d\alpha\right]^2 \qquad (3.19)$$

Bayes estimate under PLF;

Where

$$\hat{\alpha}_{pl} = \sqrt{E\left(\alpha^2 \mid x\right)} = \sqrt{\int_0^\infty \alpha^2 p\left(\alpha \mid x\right) d\alpha}$$

$$\sqrt{E\left(\alpha^2 \mid x\right)} = \sqrt{\int_0^\infty \frac{\alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha} d\alpha} \times \alpha^2 d\alpha}$$

$$\sqrt{E\left(\alpha^2 \mid x\right)} = \sqrt{\int_0^\infty \frac{\alpha^{n+a+1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha} d\alpha} d\alpha} \quad (3.20)$$

Posterior risk of the estimate under PLF;

$$R\hat{\alpha}_{pl} = 2\left[\sqrt{E\left(\alpha^2\right)} - E(\alpha)\right]$$

$$R\hat{\alpha}_{pl} = 2\left[\sqrt{\int_0^\infty \frac{\alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha} d\alpha} \times \alpha^2 d\alpha} - \int_0^\infty \frac{\alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha} d\alpha} \times \alpha d\alpha\right]$$

$$R\hat{\alpha}_{pl} = 2\left[\sqrt{\int_0^\infty \frac{\alpha^{n+a+1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha} d\alpha} d\alpha} - \int_0^\infty \frac{\alpha^{n+a} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha}}{\int_0^\infty \alpha^{n+a-1} \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - b\alpha} d\alpha} d\alpha\right] \quad (3.21)$$

3.6 Bayesian analysis under the assumption of Maxwell prior

Let the Maxwell prior be;

$$\pi(\alpha) \propto \alpha^2 \ell^{-\frac{\alpha^2}{a}} \text{ where } a, \alpha > 0$$

The posterior distribution is derived as follows;

$$p(\alpha \mid x) = \frac{\alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \times \alpha^2 \ell^{-\frac{\alpha^2}{a}}}{\int_0^{\infty} \alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \times \alpha^2 \ell^{-\frac{\alpha^2}{a}} d\alpha}$$

$$p(\alpha \mid x) = \frac{\alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}}}{\int_0^{\infty} \alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} d\alpha} \tag{3.22}$$

Bayes estimate of the scale parameter using Maxwell prior under SELF;

$$E(\alpha \mid x) = \int_0^{\infty} \frac{\alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} \times \alpha}{\int_0^{\infty} \alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} d\alpha} d\alpha$$

$$E(\alpha \mid x) = \int_0^{\infty} \frac{\alpha^{n+3} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} \times \alpha}{\int_0^{\infty} \alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} d\alpha} d\alpha \tag{3.23}$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{\alpha}_{sl} = \int_0^{\infty} \frac{\alpha^{n+3} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} \times \alpha}{\int_0^{\infty} \alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} d\alpha} d\alpha - \left[\int_0^{\infty} \frac{\alpha^{n+3} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}}}{\int_0^{\infty} \alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] - \frac{\alpha^2}{a}} d\alpha} d\alpha \right]^2$$

$$R\hat{\alpha}_{sl}=\int_0^{\infty}\frac{\alpha^{n+4}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}\times\alpha\,d\alpha-\left[\int_0^{\infty}\frac{\alpha^{n+3}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha\right]^2 \quad (3.24)$$

Bayes estimate under PLF;

$$\sqrt{E\left(\alpha^2\mid x\right)}=\sqrt{\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}\times\alpha^2\,d\alpha}$$

$$\sqrt{E\left(\alpha^2\mid x\right)}=\sqrt{\int_0^{\infty}\frac{\alpha^{n+4}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}\times\alpha^2\,d\alpha} \quad (3.25)$$

Posterior risk of the estimate under PLF;

$$R\hat{\alpha}_{pl}=2\left[\sqrt{\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}\times\alpha^2\,d\alpha}-\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}\times\alpha\,d\alpha\right]$$

$$R\hat{\alpha}_{pl}=2\left[\sqrt{\int_0^{\infty}\frac{\alpha^{n+4}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}\,d\alpha}-\int_0^{\infty}\frac{\alpha^{n+3}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha\right] \quad (3.26)$$

3.7 Bayesian analysis under the assumption of Rayleigh prior

Let the Rayleigh prior be;

$$\pi(\alpha) \propto \alpha \ell^{\frac{\alpha^2}{b^2}} \quad where\ b, \alpha > 0$$

The posterior distribution is derived as follows;

$$p(\alpha \mid x) = \frac{\alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \times \alpha \ell^{\frac{\alpha^2}{b^2}}}{\int_0^{\infty} \alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \times \alpha \ell^{\frac{\alpha^2}{b^2}} d\alpha}$$

$$p(\alpha \mid x) = \frac{\alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] + \frac{\alpha^2}{b^2}}}{\int_0^{\infty} \alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] + \frac{\alpha^2}{b^2}} d\alpha} \tag{3.27}$$

Bayes estimate of the scale parameter using Rayleigh prior under SELF;

$$E(\alpha \mid x) = \int_0^{\infty} \frac{\alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] + \frac{\alpha^2}{b^2}} \times \alpha}{\int_0^{\infty} \alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] + \frac{\alpha^2}{b^2}} d\alpha} d\alpha$$

$$E(\alpha \mid x) = \int_0^{\infty} \frac{\alpha^{n+2} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] + \frac{\alpha^2}{b^2}}}{\int_0^{\infty} \alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right] + \frac{\alpha^2}{b^2}} d\alpha} d\alpha \tag{3.28}$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{\alpha}_{sl}=\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}\times\alpha}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}d\alpha}d\alpha-\left[\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}d\alpha}d\alpha\right]^2$$

$$R\hat{\alpha}_{sl}=\int_0^{\infty}\frac{\alpha^{n+3}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha-\left[\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha\right]^2 \quad (3.29)$$

Bayes estimate under PLF;

$$\sqrt{E\left(\alpha^2\mid x\right)}=\sqrt{\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}\times\alpha^2}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}d\alpha}d\alpha}$$

$$\sqrt{E\left(\alpha^2\mid x\right)}=\sqrt{\int_0^{\infty}\frac{\alpha^{n+4}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{b^2}}d\alpha}d\alpha} \quad (3.30)$$

Posterior risk under PLF;

$$R\hat{\alpha}_{pl}=2\left[\sqrt{\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}\times\alpha^2}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha}-\int_0^{\infty}\frac{\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}\times\alpha}{\int_0^{\infty}\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha\right]$$

$$R\hat{\alpha}_{pl} = 2\left[\sqrt{\int_0^\infty \frac{\alpha^{n+4}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - \frac{\alpha^2}{a}}}{\int_0^\infty \alpha^{n+1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - \frac{\alpha^2}{a}} d\alpha} d\alpha} - \int_0^\infty \frac{\alpha^{n+2}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - \frac{\alpha^2}{a}}}{\int_0^\infty \alpha^{n+1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] - \frac{\alpha^2}{a}} d\alpha} d\alpha\right] \quad (3.31)$$

3.8 Bayesian analysis under the assumption of chi-square prior

Let chi-square prior be;

$$\pi(\alpha) \propto \alpha^{k-1}\ell^{\frac{\alpha^2}{2}} \text{ where } k, \alpha > 0$$

The posterior distribution is derived as follows;

$$p(\alpha \mid x) = \frac{\alpha^n \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right]} \times \alpha^{k-1}\ell^{\frac{\alpha^2}{2}}}{\int_0^\infty \alpha^n \ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right]} \times \alpha^{k-1}\ell^{\frac{\alpha^2}{2}} d\alpha}$$

$$p(\alpha \mid x) = \frac{\alpha^{n+k-1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] + \frac{\alpha^2}{2}}}{\int_0^\infty \alpha^{n+k-1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] + \frac{\alpha^2}{2}} d\alpha} \quad (3.32)$$

Bayes estimate of the scale parameter using chi-square prior under SELF;

$$E(\alpha \mid x) = \int_0^\infty \frac{\alpha^{n+k-1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] + \frac{\alpha^2}{2}}}{\int_0^\infty \alpha^{n+k-1}\ell^{\sum_{i=1}^n \ln x_i^{\alpha-1} - n\alpha\ln\beta - 2\sum_{i=1}^n \ln\left[1+\left(\frac{x_i}{\beta}\right)^\alpha\right] + \frac{\alpha^2}{2}} d\alpha} \times \alpha d\alpha$$

$$E(\alpha \mid x)=\int_0^{\infty}\frac{\alpha^{n+k}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha \tag{3.33}$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{\alpha}_{sl}=\int_0^{\infty}\frac{\alpha^{n+k}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}\times\alpha}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha-\left[\int_0^{\infty}\frac{\alpha^{n+k}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha\right]^2$$

$$R\hat{\alpha}_{sl}=\int_0^{\infty}\frac{\alpha^{n+k+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha-\left[\int_0^{\infty}\frac{\alpha^{n+k}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]-\frac{\alpha^2}{a}}d\alpha}d\alpha\right]^2 \tag{3.34}$$

Bayes estimate under PLF;

$$\sqrt{E\left(\alpha^2 \mid x\right)}=\sqrt{\int_0^{\infty}\frac{\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}\times\alpha^2}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha}$$

$$\sqrt{E\left(\alpha^2 \mid x\right)}=\sqrt{\int_0^{\infty}\frac{\alpha^{n+k+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha} \tag{3.35}$$

Posterior risk under PLF;

$$R\hat{\alpha}_{pl}=2\left[\sqrt{\int_0^{\infty}\frac{\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}\times\alpha^2 d\alpha}-\int_0^{\infty}\frac{\alpha^{n+k}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha\right]$$

$$R\hat{\alpha}_{pl}=2\left[\sqrt{\int_0^{\infty}\frac{\alpha^{n+k+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha}-\int_0^{\infty}\frac{\alpha^{n+k}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}}{\int_0^{\infty}\alpha^{n+k-1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]+\frac{\alpha^2}{2}}d\alpha}d\alpha\right] \quad (3.36)$$

3.9 Bayesian analysis under the assumption of Uniform prior

The uniform prior as a non-informative prior is assumed to be:

$$\pi(\alpha)\propto 1 \; where \;\; 0<\alpha<\infty \quad (3.37)$$

The posterior of the scale parameter is:

$$p(\alpha\mid x)=\frac{\alpha^n\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty}\alpha^n\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha} \quad (3.38)$$

Bayes estimate of the scale parameter using Uniform prior under SELF;

$$E(\alpha\mid x)=\int_0^{\infty}\frac{\alpha^n\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}\times\alpha}{\int_0^{\infty}\alpha^n\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha}d\alpha$$

$$E\left(\alpha \mid x\right)=\int_0^{\infty}\frac{\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty}\alpha^{n}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha}d\alpha \tag{3.39}$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{\alpha}_{sl}=\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty}\alpha^{n}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha}d\alpha-\left[\int_0^{\infty}\frac{\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty}\alpha^{n}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha}d\alpha\right]^2 \tag{3.40}$$

Bayes estimate under PLF;

$$\sqrt{E\left(\alpha^2 \mid x\right)}=\sqrt{\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty}\alpha^{n}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha}d\alpha} \tag{3.41}$$

Posterior risk under PLF;

$$R\hat{\alpha}_{pl}=2\left[\sqrt{\int_0^{\infty}\frac{\alpha^{n+2}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty}\alpha^{n}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha}d\alpha}-\int_0^{\infty}\frac{\alpha^{n+1}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty}\alpha^{n}\ell^{\sum_{i=1}^{n}\ln x_i^{\alpha-1}-n\alpha\ln\beta-2\sum_{i=1}^{n}\ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}d\alpha}d\alpha\right] \tag{3.42}$$

3.10 Bayesian analysis under the assumption of Jeffrey's prior

Jeffrey's prior as another non-informative prior is assumed to be:

$$\pi(\alpha) \propto \frac{1}{\alpha} \ \textit{where} \ \ 0 < \alpha < \infty \tag{3.43}$$

The posterior of the scale parameter is:

$$p(\alpha \mid x) = \frac{\alpha^{n-1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty} \alpha^{n} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} \tag{3.44}$$

Bayes estimate of the scale parameter using Jeffrey's prior under SELF;

$$E(\alpha \mid x) = \int_0^{\infty} \frac{\alpha^{n-1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \times \alpha}{\int_0^{\infty} \alpha^{n} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha$$

$$E(\alpha \mid x) = \int_0^{\infty} \frac{\alpha^{n} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty} \alpha^{n} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha \tag{3.45}$$

Corresponding posterior risk of the estimate under SELF;

$$R\hat{\alpha}_{sl} = \int_0^{\infty} \frac{\alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty} \alpha^{n} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha - \left[\int_0^{\infty} \frac{\alpha^{n} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty} \alpha^{n} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha\right]^2 \tag{3.46}$$

Bayes estimate under PLF;

$$\sqrt{E\left(\alpha^2 \mid x\right)} = \sqrt{\int_0^{\infty} \frac{\alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \times \alpha}{\int_0^{\infty} \alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha}$$

$$\sqrt{E\left(\alpha^2 \mid x\right)} = \sqrt{\int_0^{\infty} \frac{\alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} \times \alpha}{\int_0^{\infty} \alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha} \tag{3.47}$$

Posterior risk under PLF;

$$R\hat{\alpha}_{pl} = 2\left[\sqrt{\int_0^{\infty} \frac{\alpha^{n+1} \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty} \alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha} - \int_0^{\infty} \frac{\alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]}}{\int_0^{\infty} \alpha^n \ell^{\sum_{i=1}^{n} \ln x_i^{\alpha-1} - n\alpha \ln \beta - 2\sum_{i=1}^{n} \ln\left[1+\left(\frac{x_i}{\beta}\right)^{\alpha}\right]} d\alpha} d\alpha\right] \tag{3.48}$$

3.11 Simulation study

In this research, we used the wolfram mathematica 10.3 software to generate random numbers from the log-logistic distribution with different values of shape (β) and scale (α) parameters to investigate the performance of priors and loss functions. The Bayesian estimates and posterior risks were obtained under a reasonably high number of replications, different sample sizes were used to investigate the effect of small medium and large sample size on the estimators.

CHAPTER FOUR: ANALYSIS AND DISCUSSION

In previous studies such as;(Feroze and Aslam, 2012 and Nasir *et. al.*, 2015) simulation studies were performed to obtain numerical estimates of parameters which enables researchers to measure the loss attached to the value of the estimated parameter against the true value of that parameter. This chapter entails the results obtained from the simulation analysis we undertook. Extensive Monte-Carlo simulation was carried out to compare the performance of the different priors and loss functions discussed in this work with respect to their corresponding posterior risks for different sample sizes $(n=10,30,50,70)$ and different values of shape $(1,2,3,5\&7)$ and scale $(0.5,1.5,2\&4)$ parameters of the log-logistic distribution. In this study we used the obtained posterior risks to assess the performance of each prior under the two loss functions (SELF and PLF). Below are the tables presenting the results where tables (4.1-4.7) shows results obtained under squared error loss function(SELF) while tables (4.8-4.14) depicts results obtained under precautionary loss function(PLF).

Abbreviations used in the tables are as follows;

N → sample size

UP → Uniform prior

JP → Jeffrey's prior

GP → Gamma prior

RP → Rayleigh prior

MP → Maxwell prior

CP → Chi-square prior

Table 4.1:*Bayes estimates and Risks(in parenthesis) of the scale parameter of log-logistic distribution using different loss functions under SELF where* $\alpha = 2, \beta = 3$.

Prior		$\alpha = 2\ \beta = 3$				
N		10	30	50	70	100
UP		2.3144 (0.3823)	2.0891 (0.1017)	2.0572 (0.0590)	2.03613 (0.0414)	2.02347 (0.0286)
JP		2.3084 (0.3764)	2.0803 (0.1006)	2.02211 (0.0581)	2.03505 (0.0413)	2.03019 (0.0288)
GP	$a = 1.5, b = 1$	2.2945 (0.3751)	2.0835 (0.1010)	2.0129 (0.0564)	2.0241 (0.0409)	2.0294 (0.0287)
	$a = 1, b = 1.5$	2.2707 (0.3652)	2.0995 (0.1030)	1.97117 (0.0546)	2.04251 (0.0416)	2.03028 (0.0289)
	$a = 2, b = 2.5$	2.2821 (0.3656)	2.0998 (0.1027)	1.9438 (0.0524)	2.0397 (0.0416)	2.0341 (0.0289)
RP	$b = 1$	2.3046 (0.3735)	2.1052 (0.1036)	1.8857 (0.0449)	2.0430 (0.0417)	2.0305 (0.0288)
	$b = 0.9$	2.3274 (0.3876)	2.1050 (0.1032)	1.8282 (0.0416)	2.0407 (0.0416)	2.0254 (0.0287)
	$b = 0.5$	2.3191 (0.3771)	2.0945 (0.1022)	1.5269 (0.0263)	2.0471 (0.0418)	2.0284 (0.0288)
MP	$a = 1$	2.3030 (0.3748)	2.1001 (0.1028)	1.8981 (0.0448)	2.0392 (0.0415)	2.0271 (0.0287)
	$a = 0.9$	2.2858 (0.3737)	2.0990 (0.1026)	1.8801 (0.0436)	2.0400 (0.0416)	2.0307 (0.0288)
	$a = 0.5$	2.3042 (0.3708)	2.1047 (0.1031)	1.7600 (0.0365)	2.0457 (0.0418)	2.0216 (0.0286)
CP	$k = 1$	2.2981 (0.3695)	2.1017 (0.1028)	1.9468 (0.0504)	2.0524 (0.0420)	2.0273 (0.0287)
	$k = 0.9$	2.3062 (0.3745)	2.0889 (0.1014)	1.9294 (0.0496)	2.0415 (0.0416)	2.0336 (0.0289)
	$k = 0.5$	2.2819 (0.3652)	2.0969 (0.1022)	1.9254 (0.0498)	2.0359 (0.0414)	2.0280 (0.0287)

From this table, we can see that Rayleigh prior outperforms the other prior distributions used for estimation of the scale parameter. It is also obvious that using the hyper-parameter ($b = 0.9$) gives better estimates than the other assumed hyper-parameter values.

Table 4.1:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under SELF where* $\alpha = 0.5, \beta = 1$.

Prior		$\alpha = 0.5\ \beta = 1$				
N		10	30	50	70	100
UP		0.5827 (0.0241)	0.519337 (0.0063)	0.5190 (0.0038)	0.5096 (0.0026)	0.5077 (0.0018)
JP		0.5770 (0.0234)	0.527253 (0.0065)	0.5143 (0.0037)	0.5138 (0.0026)	0.5051 (0.0018)
GP	$a = 1.5, b = 1$	0.5661 (0.0226)	0.5218 (0.0064)	0.51089 (0.0036)	0.5107 (0.0026)	0.5071 (0.0018)
	$a = 1, b = 1.5$	0.5715 (0.0230)	0.5246 (0.0064)	0.5117 (0.0037)	0.5095 (0.0026)	0.5086 (0.0018)
	$a = 2, b = 2.5$	0.5814 (0.0237)	0.5243 (0.0064)	0.513349 (0.0037)	0.5094 (0.0026)	0.5088 (0.0018)
RP	$b = 1$	0.5686 (0.0228)	0.5218 (0.0064)	0.5169 (0.0037)	0.5113 (0.0026)	0.5067 (0.0018)
	$b = 0.9$	0.5734 (0.0233)	0.5205 (0.0063)	0.5116 (0.0037)	0.5128 (0.0026)	0.5059 (0.0018)
	$b = 0.5$	0.5690 (0.0228)	0.5225 (0.0064)	0.5120 (0.0037)	0.5103 (0.0026)	0.5084 (0.0018)
MP	$a = 1$	0.5727 (0.0229)	0.5278 (0.5278)	0.5149 (0.0037)	0.5089 (0.0026)	0.5093 (0.0018)
	$a = 0.9$	0.5816 (0.0239)	0.5263 (0.0065)	0.5127 (0.0037)	0.5121 (0.0026)	0.5088 (0.0018)
	$a = 0.5$	0.5805 (0.0238)	0.5252 (0.0064)	0.5516 (0.0037)	0.5075 (0.0026)	0.5069 (0.0018)
CP	$k = 1$	0.5655 (0.0224)	0.5203 (0.0063)	0.5154 (0.0037)	0.5123 (0.0026)	0.5068 (0.0018)
	$k = 0.9$	0.5803 (0.0237)	0.5240 (0.0064)	0.5112 (0.0037)	0.5114 (0.0026)	0.5087 (0.0018)
	$k = 0.5$	0.5762 (0.0236)	0.5235 (0.0064)	0.5108 (0.0036)	0.5088 (0.0026)	0.5081 (0.0018)

Using $\alpha = 0.5, \beta = 1$,as seen from the posterior risk values of the different estimates, no significant difference is recorded between the different prior distributions used. It is also obvious that increase in sample sizes results in better estimate of the scale parameter.

Table 4.2:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under SELF where* $\alpha = 1.5, \beta = 2$.

Prior		$\alpha = 1.5, \beta = 2$				
N		10	30	50	70	100
UP		1.5711 (0.2089)	1.5711 (0.0574)	1.5432 (0.0333)	1.5268 (0.0232)	1.5236 (0.0162)
JP		1.7026 (0.2047)	1.5773 (0.0580)	1.5294 (0.0328)	1.5273 (0.0233)	1.5238 (0.0162)
GP	$a = 1.5, b = 1$	1.7031 (0.2030)	1.5871 (0.0588)	1.5425 (0.0333)	1.5312 (0.0234)	1.5248 (0.0162)
	$a = 1, b = 1.5$	1.7329 (0.2109)	1.5754 (0.05769)	1.5453 (0.0334)	1.5245 (0.0232)	1.5190 (0.0161)
	$a = 2, b = 2.5$	1.7253 (0.2098)	1.5755 (0.0578)	1.5425 (0.0333)	1.5347 (0.0236)	1.5249 (0.0162)
RP	$b = 1$	1.7325 (0.2109)	1.5123 (0.0568)	1.5450 (0.0334)	1.5254 (0.0232)	1.5189 (0.0161)
	$b = 0.9$	1.7323 (0.2113)	1.5549 (0.0564)	1.5423 (0.0332)	1.5340 (0.0235)	1.5253 (0.0163)
	$b = 0.5$	1.7003 (0.2025)	1.5776 (0.0580)	1.5483 (0.0335)	1.5226 (0.0231)	1.5242 (0.0163)
MP	$a = 1$	1.7159 (0.2087)	1.5705 (0.0574)	1.5413 (0.0332)	1.5323 (0.0234)	1.5193 (0.0161)
	$a = 0.9$	1.7475 (0.2166)	1.5790 (0.0581)	1.5410 (0.0332)	1.5266 (0.0233)	1.5223 (0.0162)
	$a = 0.5$	1.7219 (0.2092)	1.55696 (0.0564)	1.5358 (0.0329)	1.5232 (0.0232)	1.5208 (0.0161)
CP	$k = 1$	1.7172 (0.2082)	1.57763 (0.5810)	1.5386 (0.0331)	1.5322 (0.0234)	1.5191 (0.0162)
	$k = 0.9$	1.7208 (0.2085)	1.5646 (0.0570)	1.5450 (0.0334)	1.5376 (0.0236)	1.5122 (0.0160)
	$k = 0.5$	1.7158 (0.2067)	1.5665 (0.0573)	1.5471 (0.0334)	1.5287 (0.0234)	1.5142 (0.0160)

The above table as with all the other tables shows that Rayleigh prior yielded estimates with the minimum risks thereby appearing to be the best prior for the estimation against other prior distributions used in this studies.

Table 4.3:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under SELF where* $\alpha = 2, \beta = 5$.

Prior		$\alpha = 2, \beta = 5$				
N		10	30	50	70	100
UP		2.2984 (0.3744)	2.0975 (0.1023)	2.0576 (0.0591)	2.0320 (0.0412)	2.0265 (0.0288)
JP		2.3370 (0.3879)	2.0910 (0.1018)	2.0565 (0.0591)	2.03903 (0.0416)	2.0314 (0.0288)
GP	$a = 1.5, b = 1$	2.2724 (0.3610)	2.08745 (0.10164)	2.0421 (0.0582)	2.04535 (0.0418)	2.01975 (0.0285)
	$a = 1, b = 1.5$	2.2826 (0.3654)	2.1034 (0.1029)	2.0655 (0.0597)	2.0281 (0.0411)	2.0299 (0.0288)
	$a = 2, b = 2.5$	2.2801 (0.3662)	2.0987 (0.1025)	2.0566 (0.0591)	2.0361 (0.0414)	2.0270 (0.0287)
RP	$b = 1$	2.2917 (0.3714)	2.0927 (0.1019)	2.0657 (0.0597)	2.0376 (0.0414)	2.0431 (0.0291)
	$b = 0.9$	2.2613 (0.3606)	2.0802 (0.1008)	2.0423 (0.0583)	2.0412 (0.0416)	2.0234 (0.0286)
	$b = 0.5$	2.3281 (0.3817)	2.1036 (0.1033)	2.0579 (0.0592)	2.0429 (0.0416)	2.0249 (0.0286)
MP	$a = 1$	2.2718 (0.3646)	2.1155 (0.1045)	2.0529 (0.0589)	2.0578 (0.0423)	2.0288 (0.0288)
	$a = 0.9$	2.3023 (0.3720)	2.0954 (0.1021)	2.0572 (0.0590)	2.0398 (0.0415)	2.0244 (0.0286)
	$a = 0.5$	2.2992 (0.3706)	2.0758 (0.1001)	2.0747 (0.0602)	2.0539 (0.0421)	2.0286 (0.0288)
CP	$k = 1$	2.3047 (0.3738)	2.0966 (0.1021)	2.0519 (0.0588)	2.0466 (0.0418)	2.03073 (0.0289)
	$k = 0.9$	2.2978 (0.3704)	2.1040 (0.1031)	2.0554 (0.0591)	2.0462 (0.0418)	2.0288 (0.0288)
	$k = 0.5$	2.2982 (0.3697)	2.0947 (0.1022)	2.0566 (0.0591)	2.0408 (0.0416)	2.0176 (0.0284)

Again from this table it is obvious that Rayleigh prior with the hyper-parameter value ($b = 0.9$) yielded the best estimate for the scale parameter. Among the non-informative priors Jeffrey's prior yielded better estimates for the parameter of interest.

Table 4.5:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under SELF where* $\alpha = 2, \beta = 7$.

Prior		$\alpha = 2, \beta = 7$				
N		10	30	50	70	100
UP		2.2910 (0.3685)	2.1035 (0.1030)	2.0498 (0.0587)	2.0475 (0.0419)	2.0231 (0.0286)
JP		2.2385 (0.1495)	2.0650 (0.0473)	2.0538 (0.0285)	2.0248 (0.0201)	2.0120 (0.0140)
GP	$a = 1.5, b = 1$	1.8076 (0.1094)	1.9258 (0.0427)	1.9579 (0.0266)	1.9750 (0.0193)	1.9803 (0.0136)
	$a = 1, b = 1.5$	1.7953 (0.1086)	1.9314 (0.0428)	1.9554 (0.0265)	1.9738 (0.0193)	1.9820 (0.0136)
	$a = 2, b = 2.5$	1.8124 (0.1096)	1.9361 (0.0430)	1.9514 (0.0265)	1.9721 (0.0193)	1.9830 (0.0137)
RP	$b = 1$	1.0349 (0.0443)	1.3793 (0.0239)	1.5328 (0.0171)	1.6220 (0.0135)	1.7121 (0.0104)
	$b = 0.9$	1.0364 (0.0443)	1.3829 (0.0240)	1.5341 (0.0172)	1.6268 (0.0135)	1.7135 (0.0104)
	$b = 0.5$	1.0334 (0.0442)	1.3844 (0.0240)	1.5391 (0.0172)	1.6247 (0.0172)	1.7098 (0.0104)
MP	$a = 1$	1.7969 (0.0957)	1.9262 (0.0402)	1.9506 (0.0255)	1.96427 (0.0187)	1.9762 (0.0133)
	$a = 0.9$	1.7825 (0.0952)	1.9260 (0.0401)	1.9559 (0.0255)	1.9645 (0.0187)	1.9744 (0.0133)
	$a = 0.5$	1.8010 (0.0959)	1.9323 (0.0404)	1.9499 (0.0255)	1.9681 (0.0187)	1.9827 (0.0134)
CP	$k = 1$	1.7979 (0.0959)	1.9180 (0.0401)	1.9442 (0.0254)	1.9740 (0.0188)	1.9792 (0.0134)
	$k = 0.9$	1.7995 (0.0958)	1.9273 (0.0403)	1.9502 (0.0255)	1.9611 (0.0187)	1.9820 (0.0134)
	$k = 0.5$	1.7834 (0.0952)	1.9374 (0.0405)	1.9603 (0.0256)	1.9664 (0.0187)	1.9807 (0.0013)

Also, this table shows a steady improvement of the parameter estimates with increase in sample size, among the informative priors Rayleigh prior resulted the best estimates for the scale parameter while Jeffrey's prior yielded better estimates against uniform prior in the case of non-informative priors.

Table 4.4:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under SELF where* $\alpha = 4, \beta = 5$.

Prior		$\alpha = 4, \beta = 5$				
N		10	30	50	70	100
UP		4.6568 (1.5244)	4.1735 (0.4056)	4.1352 (0.2389)	4.0758 (0.1656)	4.0421 (0.1142)
JP		4.6125 (1.4916)	4.1512 (0.4005)	4.1344 (0.2395)	4.0678 (0.1651)	4.0645 (0.1154)
GP	$a = 1.5, b = 1$	4.5993 (1.4890)	4.1881 (0.4076)	4.1367 (0.2393)	4.0823 (0.1665)	4.0657 (0.1156)
	$a = 1, b = 1.5$	4.6679 (1.5316)	4.1519 (0.4014)	4.1279 (0.2386)	4.0655 (0.1648)	4.0438 (0.1142)
	$a = 2, b = 2.5$	4.6367 (1.5107)	4.1906 (0.4095)	4.0954 (0.2347)	4.0816 (0.1664)	4.0435 (0.1144)
RP	$b = 1$	4.6786 (1.5536)	4.1905 (0.4096)	4.1163 (0.2370)	4.0507 (0.1636)	4.0678 (0.1155)
	$b = 0.9$	4.5911 (1.4866)	4.1590 (0.4031)	4.1079 (0.2356)	4.0728 (0.1657)	4.0603 (0.1153)
	$b = 0.5$	4.6198 (1.5063)	4.1847 (0.4075)	4.1372 (0.2388)	4.0678 (0.1650)	4.0575 (0.1151)
MP	$a = 1$	4.5470 (1.4509)	4.2290 (0.4164)	4.0984 (0.2344)	4.0893 (0.1668)	4.0557 (0.1148)
	$a = 0.9$	4.6316 (1.5182)	4.1926 (0.4085)	4.1214 (0.2369)	4.0768 (0.1660)	4.0699 (0.1159)
	$a = 0.5$	4.5795 (1.4879)	4.2003 (0.4117)	4.1221 (0.2373)	4.0828 (0.1667)	4.0421 (0.1142)
CP	$k = 1$	4.6336 (1.5295)	4.1823 (0.4069)	4.1166 (0.2370)	4.0916 (0.1671)	4.0690 (0.1158)
	$k = 0.9$	4.6180 (1.5039)	4.209 (0.4142)	4.1478 (0.2406)	4.0847 (0.1663)	4.0615 (0.1153)
	$k = 0.5$	4.6502 (1.5284)	4.1897 (0.4083)	4.1167 (0.2364)	4.0793 (0.1662)	4.0751 (0.1161)

This table also shows that Rayleigh prior performs better than the other prior distributions used for estimation of the scale parameter. It is also obvious that using the hyper-parameter $(b = 0.9)$ gives better estimates than the other assumed hyper-parameter values $(b = 0.5 \, and \, b = 1)$.

Table 4.5:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under SELF where* $\alpha = 4, \beta = 7$.

Prior		$\alpha = 4, \beta = 7$				
N		10	30	50	70	100
UP		4.7126 (1.5660)	4.1310 (0.3979)	4.1479 (0.2407)	4.0560 (0.1642)	4.0591 (0.1152)
JP		4.6098 (1.4966)	4.1827 (0.4080)	4.1229 (0.2380)	4.1025 (0.1679)	4.0358 (0.1140)
GP	$a = 1.5, b = 1$	4.5838 (1.4721)	1.1475 (0.4010)	4.0977 (0.2345)	4.0778 (0.1661)	4.0573 (0.1151)
	$a = 1, b = 1.5$	4.6112 (1.5034)	4.1803 (0.4072)	4.1123 (0.2363)	4.0785 (0.1661)	4.0715 (0.1158)
	$a = 2, b = 2.5$	4.6544 (1.5369)	4.1930 (0.4092)	4.1163 (0.2367)	4.0926 (0.1673)	4.0693 (0.1157)
RP	$b = 1$	4.5529 (1.4463)	4.1767 (0.4070)	4.1072 (0.2358)	4.1003 (0.1680)	4.0468 (0.1146)
	$b = 0.9$	4.6406 (1.5166)	4.1906 (0.4086)	4.0782 (0.2324)	4.0905 (0.1671)	4.0487 (0.1146)
	$b = 0.5$	4.6635 (1.5463)	4.1954 (0.4115)	4.1113 (0.2363)	4.0617 (0.1647)	4.0311 (0.1136)
MP	$a = 1$	4.6380 (1.5180)	4.1826 (0.4067)	4.1153 (0.2367)	4.0781 (0.1661)	4.0384 (0.1139)
	$a = 0.9$	4.5792 (1.4822)	4.1966 (0.4098)	4.1109 (0.2361)	4.0721 (0.1653)	4.0505 (0.1146)
	$a = 0.5$	4.6203 (1.4998)	4.1973 (0.4110)	4.1037 (0.2350)	4.0603 (0.1647)	4.0523 (0.1149)
CP	$k = 1$	4.6453 (1.5150)	4.2004 (0.4112)	4.1243 (0.2381)	4.0733 (0.1656)	4.0585 (0.1152)
	$k = 0.9$	4.5562 (1.4559)	4.1965 (0.4097)	4.1135 (0.2365)	4.0900 (0.1670)	4.0584 (0.1150)
	$k = 0.5$	4.6481 (1.5204)	4.1732 (0.4058)	4.1180 (0.2368)	4.1023 (0.1680)	4.0411 (0.1142)

Rayleigh and Gamma priors here resulted better estimates than all the other priors used in this study, it is also evident that an increase in sample size results in better estimate of the parameter, again it can be seen that all the informative priors here outperforms the non-informative priors used.

Table 4.6:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under PLF where* $\alpha = 2, \beta = 3$.

Prior		$\alpha = 2, \beta = 3$				
N		10	30	50	70	100
UP		2.4190 (0.1498)	2.1298 (0.0476)	2.0749 (0.0283)	2.0527 (0.0201)	2.02748 (0.0140)
JP		2.2092 (0.1477)	2.0719 (0.0475)	2.0450 (0.0283)	2.0283 (0.0201)	2.0189 (0.0140)
GP	$a = 1.5, b = 1$	2.2977 (0.3723)	2.1010 (0.1026)	2.02719 (0.0275)	2.0397 (0.0416)	2.0314 (0.0288)
	$a = 1, b = 1.5$	1.9208 (0.1227)	1.9719 (0.0445)	1.9663 (0.0266)	1.9736 (0.0193)	1.9778 (0.0136)
	$a = 2, b = 2.5$	1.8138 (0.1098)	1.9346 (0.0429)	1.9668 (0.0267)	1.9678 (0.0192)	1.9899 (0.0137)
RP	$b = 1$	1.0401 (0.0444)	1.3846 (0.0240)	1.8819 (0.0234)	1.6302 (0.0136)	1.7126 (0.0104)
	$b = 0.9$	1.5266 (0.0697)	1.3799 (0.0239)	1.5342 (0.0171)	1.6259 (0.0135)	1.7098 (0.0103)
	$b = 0.5$	1.0343 (0.0443)	1.3890 (0.0240)	1.5327 (0.0171)	1.6241 (0.0135)	1.7047 (0.0103)
MP	$a = 1$	1.7663 (0.0945)	1.9171 (0.0401)	1.9588 (0.0244)	1.9714 (0.0187)	1.9797 (0.0134)
	$a = 0.9$	1.6423 (0.0702)	1.9194 (0.0401)	1.9588 (0.0255)	1.9744 (0.0188)	1.9823 (0.0134)
	$a = 0.5$	1.3745 (0.0565)	1.9269 (0.0402)	1.9652 (0.0256)	1.9656 (0.0187)	1.9729 (0.0133)
CP	$k = 1$	1.7848 (0.0953)	1.9158 (0.0401)	1.9546 (0.0255)	1.9722 (0.0188)	1.9819 (0.0134)
	$k = 0.9$	1.8014 (0.0959)	1.9152 (0.4006)	1.9592 (0.0259)	1.9608 (0.0187)	1.9732 (0.0133)
	$k = 0.5$	1.7414 (0.0960)	1.9218 (0.0402)	1.9589 (0.0256)	1.9670 (0.0187)	1.9748 (0.0133)

This table also reveals the realization of better estimates for the scale parameter with increasing sample sizes, again the use of Rayleigh prior for the estimation resulted in better parameter estimates.

Table 4.7:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under PLF where* $\alpha = 0.5, \beta = 1$.

Prior		$\alpha = 0.5, \beta = 1$				
N		10	30	50	70	100
UP		0.5754 (0.0235)	0.5227 (0.0064)	0.5147 (0.0037)	0.5128 (0.0026)	0.5067 (0.0018)
JP		0.5615 (0.0375)	0.5161 (0.0118)	0.5120 (0.0071)	0.5032 (0.0050)	0.5088 (0.0035)
GP	$a = 1.5, b = 1$	0.5792 (0.0338)	0.5284 (0.0116)	0.5166 (0.0070)	0.5111 (0.0050)	0.5066 (0.0035)
	$a = 1, b = 1.5$	0.5790 (0.0339)	0.5199 (0.0114)	0.5177 (0.0070)	0.5143 (0.0050)	0.5071 (0.0035)
	$a = 2, b = 2.5$	0.5802 (0.0340)	0.5216 (0.0114)	0.6165 (0.0069)	0.514638 (0.0050)	0.5059 (0.0035)
RP	$b = 1$	0.5298 (0.0274)	0.5126 (0.0107)	0.5109 (0.0067)	0.5069 (0.0048)	0.5028 (0.0034)
	$b = 0.9$	0.5336 (0.0275)	0.5162 (0.0108)	0.5107 (0.0067)	0.5083 (0.0048)	0.5043 (0.0034)
	$b = 0.5$	0.5242 (0.0271)	0.5165 (0.0108)	0.5108 (0.0067)	0.5062 (0.0048)	0.5055 (0.0034)
MP	$a = 1$	0.5774 (0.0351)	0.5229 (0.0116)	0.5149 (0.0070)	0.5127 (0.0050)	0.5089 (0.0035)
	$a = 0.9$	0.5800 (0.0354)	0.5260 (0.0117)	0.5141 (0.0070)	0.5124 (0.0050)	0.5076 (0.0035)
	$a = 0.5$	0.5766 (0.0353)	0.5209 (0.0116)	0.5153 (0.0070)	0.5102 (0.0050)	0.5063 (0.0035)
CP	$k = 1$	0.5802 (0.0353)	0.5272 (0.0118)	0.5160 (0.0070)	0.5106 (0.0050)	0.5093 (0.0035)
	$k = 0.9$	0.5771 (0.0351)	0.5258 (0.0117)	0.5136 (0.0070)	0.5113 (0.0050)	0.5077 (0.0035)
	$k = 0.5$	0.5716 (0.0349)	0.5241 (0.0117)	0.5152 (0.0070)	0.5105 (0.0050)	0.5099 (0.0035)

Again on the table above we can see that the use of Rayleigh prior for the estimation of the parameter resulted in better estimates against the other estimates used, and in addition when we compare the results obtained using the precautionary loss function against those obtained using the squared error loss function, we can see an improvement of the estimates under the use of precautionary loss function.

Table 4.8:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under PLF where* $\alpha = 1.5, \beta = 2$.

Prior		$\alpha = 1.5, \beta = 2$				
N		10	30	50	70	100
UP		1.7585 (0.2202)	1.5707 (0.0574)	1.5375 (0.0331)	1.5226 (0.0231)	1.5181 (0.0161)
JP		1.6845 (0.1123)	1.5505 (0.0356)	1.5428 (0.0214)	1.5183 (0.0151)	1.5152 (0.0105)
GP	$a = 1.5, b = 1$	1.4387 (0.0860)	1.4841 (0.0328)	1.4925 (0.0202)	1.4954 (0.0146)	1.4926 (0.0103)
	$a = 1, b = 1.5$	1.4796 (0.0886)	1.5010 (0.0331)	1.4942 (0.0202)	1.4946 (0.0146)	1.4984 (0.0103)
	$a = 2, b = 2.5$	1.4537 (0.0872)	1.4858 (0.0328)	1.5000 (0.0203)	1.4986 (0.0146)	1.5037 (0.0103)
RP	$b = 1$	0.9525 (0.0419)	1.1944 (0.0217)	1.2796 (0.0150)	1.3306 (0.0116)	1.3730 (0.0087)
	$b = 0.9$	0.9557 (0.0420)	1.1956 (0.0217)	1.2820 (0.0150)	1.3361 (0.0116)	1.3741 (0.0087)
	$b = 0.5$	0.9573 (0.0420)	1.1871 (0.0216)	1.2756 (0.0149)	1.3283 (0.0116)	1.3747 (0.0087)
MP	$a = 1$	1.4648 (0.0815)	1.4999 (0.0322)	1.5065 (0.0200)	1.4935 (0.0144)	1.5013 (0.0102)
	$a = 0.9$	1.4843 (0.0824)	1.4987 (0.0321)	1.5003 (0.0200)	1.5004 (0.0145)	1.5012 (0.0102)
	$a = 0.5$	1.4702 (0.0818)	1.4987 (0.0321)	1.4978 (0.0199)	1.5008 (0.0145)	1.5026 (0.0102)
CP	$k = 1$	1.4853 (0.0825)	1.4958 (0.0321)	1.4998 (0.0199)	1.5027 (0.0145)	1.5011 (0.0102)
	$k = 0.9$	1.4682 (0.0817)	1.4928 (0.0320)	1.4959 (0.0199)	1.5046 (0.0145)	1.5015 (0.0102)
	$k = 0.5$	1.4578 (0.0811)	1.4839 (0.0318)	1.5012 (0.0200)	1.4896 (0.0144)	1.5058 (0.0103)

From this table we can deduce that the use of informative priors resulted in better estimates against the use of non-informative priors and also this table shows that Rayleigh prior produced the best estimates as regards the estimation of the scale parameter.

Table 4.9:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under PLF where* $\alpha = 2, \beta = 5$.

Prior		$\alpha = 2, \beta = 5$				
N		10	30	50	70	100
UP		2.3342 (0.3876)	2.1106 (0.1035)	2.0594 (0.0593)	2.0427 (0.0416)	2.0282 (0.0288)
JP		2.2239 (0.1482)	2.0808 (0.0477)	2.0374 (0.0282)	2.0257 (0.0201)	2.0241 (0.0141)
GP	$a = 1.5, b = 1$	1.8029 (0.1091)	1.9228 (0.0426)	1.9485 (0.0264)	1.9615 (0.0192)	1.9846 (0.0137)
	$a = 1, b = 1.5$	1.8089 (0.1094)	1.9487 (0.0432)	1.9517 (0.0265)	1.9700 (0.0193)	1.9751 (0.0136)
	$a = 2, b = 2.5$	1.8107 (0.1095)	1.9490 (0.0432)	1.9577 (0.0266)	1.9715 (0.0193)	1.9853 (0.0137)
RP	$b = 1$	1.0385 (0.0444)	1.3844 (0.0240)	1.5367 (0.0172)	1.6222 (0.0135)	1.7092 (0.0103)
	$b = 0.9$	1.0367 (0.0443)	1.3855 (0.0240)	1.5358 (0.0172)	1.6247 (0.0135)	1.7153 (0.0104)
	$b = 0.5$	1.0371 (0.0443)	1.3832 (0.0239)	1.5353 (0.0172)	1.6215 (0.0135)	1.7095 (0.0103)
MP	$a = 1$	1.7842 (0.0952)	1.9273 (0.0403)	1.9597 (0.0256)	1.9682 (0.0187)	1.9830 (0.0134)
	$a = 0.9$	1.8157 (0.0965)	1.9268 (0.0402)	1.9526 (0.0255)	1.9550 (0.0186)	1.9851 (0.0134)
	$a = 0.5$	1.7897 (0.0955)	1.9136 (0.0401)	1.9549 (0.0255)	1.9621 (0.0187)	1.9688 (0.0133)
CP	$k = 1$	1.7803 (0.0950)	1.9191 (0.0401)	1.9572 (0.0256)	1.9683 (0.0187)	1.9774 (0.0134)
	$k = 0.9$	1.7825 (0.0953)	1.9202 (0.0401)	1.9449 (0.0254)	1.9744 (0.0188)	1.9736 (0.0133)
	$k = 0.5$	1.7685 (0.0947)	1.9309 (0.0403)	1.9561 (0.0256)	1.9612 (0.0186)	1.9877 (0.0134)

The results on the table above also reveals that Rayleigh prior performed best when compared to the other prior distributions used for the estimation of the scale parameter. Again it can also be seen that increase in sample size results in better estimates.

Table 4.12:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under PLF where* $\alpha = 2, \beta = 7$.

Prior		$\alpha = 2, \beta = 7$				
N		10	30	50	70	100
UP		2.2944 (0.3711)	2.1038 (0.1031)	2.0621 (0.0593)	2.0396 (0.0416)	2.0312 (0.0288)
JP		2.3351 (0.3866)	2.0814 (0.1010)	2.0388 (0.0581)	2.0386 (0.0415)	2.0286 (0.0287)
GP	$a = 1.5, b = 1$	2.2933 (0.3693)	2.0952 (0.1020)	2.0517 (0.0589)	2.0399 (0.0416)	2.0288 (0.0288)
	$a = 1, b = 1.5$	2.3023 (0.3771)	2.1260 (0.1051)	2.0550 (0.0549)	2.0332 (0.0412)	2.0235 (0.0286)
	$a = 2, b = 2.5$	2.2883 (0.3741)	2.0914 (0.1018)	2.0647 (0.0595)	2.0301 (0.0411)	2.0208 (0.0285)
RP	$b = 1$	2.3226 (0.3813)	2.0883 (0.1018)	2.0714 (0.0600)	2.0396 (0.0415)	2.0316 (0.0288)
	$b = 0.9$	2.3313 (0.3837)	2.0863 (0.1013)	2.0682 (0.0598)	2.0419 (0.0417)	2.0320 (0.0289)
	$b = 0.5$	2.2823 (0.3637)	2.1058 (0.1037)	2.0382 (0.0580)	2.0446 (0.0417)	2.0310 (0.0288)
MP	$a = 1$	2.3036 (0.3730)	2.1136 (0.1040)	2.0624 (0.0594)	2.0255 (0.0410)	2.0367 (0.0290)
	$a = 0.9$	2.2674 (0.3625)	2.0997 (0.1028)	2.0561 (0.0591)	2.0373 (0.0414)	2.0420 (0.0291)
	$a = 0.5$	2.3089 (0.3750)	2.0969 (0.1023)	2.0560 (0.0591)	2.0294 (0.0411)	2.0306 (0.0288)
CP	$k = 1$	2.3168 (0.3825)	2.0892 (0.1017)	2.0526 (0.0588)	2.0361 (0.0414)	2.0330 (0.0289)
	$k = 0.9$	2.2921 (0.3691)	2.0973 (0.1025)	2.0486 (0.0587)	2.0498 (0.0419)	2.0240 (0.0287)
	$k = 0.5$	2.3374 (0.3877)	2.1029 (0.1032)	2.0624 (0.0595)	2.0505 (0.0420)	2.0285 (0.0287)

As with all the other tables, the hyper-parameter value $(b = 0.9)$ of the Rayleigh distribution produced better estimates than other assumed hyper-parameter values $(b = 1)\ and\ (b = 0.5)$. As is apparent the use of informative priors resulted in better estimates than when non-informative priors are used.

Table 4.10:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under PLF where* $\alpha = 4, \beta = 5$.

Prior		$\alpha = 4, \beta = 5$				
N		10	30	50	70	100
UP		4.6690 (1.5366)	4.1907 (0.4096)	4.1114 (0.2366)	4.0893 (0.1670)	4.0586 (0.1151)
JP		4.4178 (0.2946)	4.1011 (0.0941)	4.0640 (0.0563)	4.0704 (0.0403)	4.0370 (0.0281)
GP	$a = 1.5, b = 1$	2.8003 (0.1776)	3.4742 (0.0783)	3.6908 (0.0506)	3.7572 (0.0370)	3.8233 (0.0265)
	$a = 1, b = 1.5$	2.8207 (0.1788)	3.4556 (0.0779)	3.6530 (0.0501)	3.7599 (0.0370)	3.8290 (0.0265)
	$a = 2, b = 2.5$	2.7976 (0.1772)	3.4661 (0.0781)	3.6660 (0.0502)	3.7706 (0.0371)	3.8266 (0.0265)
RP	$b = 1$	1.1592 (0.0481)	1.7375 (0.0278)	2.0728 (0.0209)	2.3009 (0.0171)	2.5494 (0.0138)
	$b = 0.9$	1.1597 (0.0481)	1.7384 (0.0279)	2.0720 (0.0209)	2.2997 (0.0171)	2.5496 (0.0138)
	$b = 0.5$	1.1600 (0.0481)	1.7429 (0.0279)	2.0731 (0.0209)	2.3006 (0.0171)	2.5442 (0.0137)
MP	$a = 1$	2.5035 (0.1215)	3.1785 (0.0603)	3.4273 (0.0413)	3.5650 (0.0317)	3.6701 (0.0235)
	$a = 0.9$	2.5049 (0.1215)	3.1931 (0.0604)	3.4339 (0.0414)	3.5551 (0.0316)	3.6717 (0.0235)
	$a = 0.5$	2.5099 (0.1216)	3.1715 (0.6006)	3.4428 (0.0414)	3.5802 (0.0318)	3.6805 (0.0236)
CP	$k = 1$	2.5179 (0.1218)	2.2058 (0.0606)	3.4517 (0.0416)	3.5600 (0.0317)	3.6832 (0.0236)
	$k = 0.9$	2.5166 (0.1218)	3.1870 (0.0603)	3.4314 (0.0414)	3.5730 (0.0317)	3.6803 (0.0236)
	$k = 0.5$	2.5029 (0.1215)	3.1900 (0.0604)	3.4478 (0.0415)	3.5718 (0.0317)	3.6831 (0.0236)

The above table also shows that Rayleigh prior resulted in better estimates for the scale parameter, and looking at all the results presented in the tables, we can conclude that Bayes estimates under precautionary loss function are associated with minimum posterior risks when compared to those obtained under squared error loss function with same parametric values.

Table 4.11:*Table showing estimates of the scale parameter of log-logistic distribution using different loss functions under PLF where* $\alpha = 4, \beta = 7$.

Prior		$\alpha = 4, \beta = 7$				
N		10	30	50	70	100
UP		4.5239 (1.4320)	4.1646 (0.4041)	4.1071 (0.2360)	4.0715 (0.1655)	4.0513 (0.1146)
JP		4.4832 (0.2990)	4.1121 (0.0942)	4.0729 (0.0564)	4.0800 (0.0405)	4.0214 (0.0280)
GP	$a = 1.5, b = 1$	2.8005 (0.1772)	3.4581 (0.0780)	3.6804 (0.0504)	3.7499 (0.0369)	3.8275 (0.0265)
	$a = 1, b = 1.5$	2.7970 (0.1771)	3.5077 (0.0791)	3.6734 (0.0504)	3.7468 (0.0369)	3.8337 (0.0265)
	$a = 2, b = 2.5$	2.7966 (0.1774)	3.4593 (0.0780)	3.6701 (0.0503)	3.7602 (0.0370)	3.8302 (0.0265)
RP	$b = 1$	1.1605 (0.0481)	1.7374 (0.0278)	2.0718 (0.0209)	2.3078 (0.0172)	2.5581 (0.0138)
	$b = 0.9$	1.1603 (0.0481)	1.7369 (0.0278)	2.0734 (0.0209)	2.3022 (0.0171)	2.5527 (0.0138)
	$b = 0.5$	1.1598 (0.0481)	1.7431 (0.0279)	2.0746 (0.0209)	2.3042 (0.0172)	2.5467 (0.0138)
MP	$a = 1$	2.5130 (0.1216)	3.1780 (0.0603)	3.4341 (0.0414)	3.5495 (0.0316)	3.6656 (0.0235)
	$a = 0.9$	2.5201 (0.1221)	3.1852 (0.0603)	3.4307 (0.0414)	3.5555 (0.0317)	3.6776 (0.0237)
	$a = 0.5$	2.5067 (0.1215)	3.1847 (0.0603)	3.4443 (0.0414)	3.5707 (0.0318)	3.6851 (0.0236)
CP	$k = 1$	2.5190 (0.1219)	3.1697 (0.0601)	3.4330 (0.0414)	3.5617 (0.0317)	3.6639 (0.0235)
	$k = 0.9$	2.5055 (0.1214)	3.1854 (0.0603)	3.4353 (0.0414)	3.5634 (0.0317)	3.6734 (0.0236)
	$k = 0.5$	2.5248 (0.1222)	3.1939 (0.0604)	3.4278 (0.0413)	3.5761 (0.0318)	3.6709 (0.0235)

As expected, the results obtained from the study reveals that with increasing sample sizes, the estimated values of the scale parameter approach the true value, in addition the corresponding posterior risk associated with each estimate decreases and vice-versa. Furthermore, it is also observed that the choice of parametric values affects the estimates where increase in the parametric values results in better estimates while lower values results in poorer estimates. For the non-informative priors, Jeffrey's prior produces better estimates than the uniform prior

whereas the Rayleigh prior gives the best estimate among the informative priors. In general, it is observed that better estimates emerge under informative priors and the use of Rayleigh prior leads to better estimates under both the SELF and PLF, finally, the study further reveals that the PLF which is asymmetric yields minimum posterior risks against the SELF which is symmetric. In general, the results obtained further justifies the significance of this study in the sense that the results shows that use of informative priors indeed produced different and better results against the results obtained using non-informative priors.

CHAPTER FIVE: SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.0 Summary

As stated earlier, this study is aimed at estimating the scale parameter of the log-logistic distribution using the Bayesian method of estimation where the scale parameter is assumed to follow four informative prior distributions namely; Rayleigh, Maxwell, Gamma and chi-square prior distributions. These estimates were obtained under the squared error loss function which is symmetric in nature and precautionary loss function which is classified as asymmetric. The posterior distribution of the log-logistic distribution was derived and subsequently the posterior mean and risks were also obtained using the above mentioned priors and loss functions. Furthermore, we carried out Monte-Carlo simulation using the Mathematica software to assess the performance of the used priors and loss functions.

5.1 Conclusion

From the first chapter of this study, the main aim of this work is to estimate the scale parameter of the log-logistic distribution using informative priors whereby we were able to obtain the posterior mean and risks assuming Rayleigh, Maxwell, Gamma and Chi-square distributions as our prior distributions under squared error and precautionary loss functions. from the results we obtained, the Rayleigh prior gives the best estimate among the informative priors. In general, it is observed that better estimates emerge under informative priors and the use of Rayleigh prior leads to better estimates under both the SELF and PLF, finally, the study further reveals that the PLF which is asymmetric yields minimum posterior risks against the SELF which is symmetric.

At the end of the study, it was observed that the use of precautionary loss function(PLF) under Rayleigh prior is preferred for the Bayesian point estimation of the stated parameter, furthermore

it was also revealed that the use of informative priors yields better estimates against the use of non-informative priors.

5.2 Recommendations

From our results discussion and analysis in the preceding chapter, we offer the following recommendations when it comes to the estimation of the scale parameter of log-logistic distribution.

i. The assumption of lower values for the hyper-parameters yields better results for the estimates therefore, it poses less risk for the estimation of the parameter.
ii. When estimating the parameter in question, the assumption of Rayleigh prior under precautionary loss function is recommended.

5.4 Contribution to Knowledge

This study was able to positively contribute to the body of knowledge through the following ways;

i. This study was able to derive the posterior distributions of the scale parameter of log-logistic distribution assuming Rayleigh, Maxwell, and Gamma priors under SELF and PLF.
ii. Posterior means and risks of the said posterior distributions were also obtained in this study using Monte-Carlo simulation techniques.
iii. Based on the limitations of the assumed priors in this study, we were able to establish that scale parameter estimate of the log-logistic distribution is best obtained under the assumption of Rayleigh prior using PLF.

5.5 Areas of Further Research

Further researches in line with this study can be geared towards;

i. The elicitation of hyper-parameters, so as to obtain the best values of the hyper-parameters that can be used to obtain better estimates of the parameters.

ii. Use of other distributions as priors to obtain the posterior distributions, mean and risks of the parameter, perhaps a better prior distribution may be lying outside the assumed distributions of this study.

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Appendix

As stated in earlier, we used mathematica 10.3 software to carry out the simulation analysis of our studies and below are the codes used for this study.

For chi-square prior with hyper-parameter $(k=1)$;

fun[A_, b_, n_, sim_]:= (Postmean = RandomVariate[LogLogisticDistribution[A, b], sim];

Risk = RandomVariate[LogLogisticDistribution[A, b], sim];

For[$i = 1, i < \text{sim} + 1, i++, x =$ RandomVariate[LogLogisticDistribution[A, b], n];

$$\text{m1} = \text{NIntegrate}[(a^{n+1}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n*a\text{Log}[b] - 2*\sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - (a^2/2)])/$$

$$\text{NIntegrate}[a^{n}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n*a\text{Log}[b] - 2*\sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - (a^2/2)], \{a, 0, \text{Infinity}\}], \{a, 0, \text{Infinity}\}];$$

$$\text{m2} = \text{NIntegrate}[(a^{n+2}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n*a\text{Log}[b] - 2*\sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - (a^2/2)])/$$

$$\text{NIntegrate}[a^{n}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n*a\text{Log}[b] - 2*\sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - (a^2/2)], \{a, 0, \text{Infinity}\}], \{a, 0, \text{Infinity}\}];$$

m3 = m2 − (m1)^2;

Postmean[[i]] = m1;

Risk[[i]] = m3;

];

Return[{Mean[Postmean], Mean[Risk]}]);

For Gamma prior with hyper-parameters $(a=1.5)\,and\,(b=1)$;

fun[A_, b_, n_, sim_]:= (Postmean = RandomVariate[LogLogisticDistribution[A, b], sim];

Risk = RandomVariate[LogLogisticDistribution[A, b], sim];

For[$i = 1, i < \text{sim} + 1, i++, x =$ RandomVariate[LogLogisticDistribution[A, b], n];

$$\text{m1} = \text{NIntegrate}[(a^{n+1.5}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n*a\text{Log}[b] - 2*\sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - a])/$$

$$\text{NIntegrate}[a^{n+0.5}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n*a\text{Log}[b] - 2*\sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - a], \{a, 0, \text{Infinity}\}], \{a, 0, \text{Infinity}\}];$$

$$\text{m2} = \text{NIntegrate}[(a^{n+2.5}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n*a\text{Log}[b] - 2*\sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - a])/$$

NIntegrate[$a^{n+0.5}$Exp[$a\sum_{i=1}^{n}$ Log[$x[[i]]$] $- n * a$Log[b] $- 2 * \sum_{i=1}^{n}$ Log[$(1+(\frac{x[[i]]}{b})^{a})$] $- a$], {a, 0, Infinity}], {a, 0, Infinity}];

m3 = m2 − (m1)^2;

Postmean[[i]] = m1;

Risk[[i]] = m3;

];

Return[{Mean[Postmean], Mean[Risk]}]);

For Maxwell prior with hyper-parameter $(a=1)$;

fun[A_, b_, n_, sim_]:= (Postmean = RandomVariate[LogLogisticDistribution[A, b], sim];

Risk = RandomVariate[LogLogisticDistribution[A, b], sim];

For[$i = 1, i <$ sim $+ 1, i++, x =$ RandomVariate[LogLogisticDistribution[A, b], n];

m1 = NIntegrate[(a^{n+3}Exp[$a\sum_{i=1}^{n}$ Log[$x[[i]]$] $- n * a$Log[b] $- 2 * \sum_{i=1}^{n}$ Log[$(1+(\frac{x[[i]]}{b})^{a})$] $-(a^2/1.5)$])/

NIntegrate[a^{n+2}Exp[$a\sum_{i=1}^{n}$ Log[$x[[i]]$] $- n * a$Log[b] $- 2 * \sum_{i=1}^{n}$ Log[$(1+(\frac{x[[i]]}{b})^{a})$] $-(a^2/1.5)$], {a, 0, Infinity}], {a, 0, Infinity}];

m2 = NIntegrate[(a^{n+4}Exp[$a\sum_{i=1}^{n}$ Log[$x[[i]]$] $- n * a$Log[b] $- 2 * \sum_{i=1}^{n}$ Log[$(1+(\frac{x[[i]]}{b})^{a})$] $-(a^2/1.5)$])/

NIntegrate[a^{n+2}Exp[$a\sum_{i=1}^{n}$ Log[$x[[i]]$] $- n * a$Log[b] $- 2 * \sum_{i=1}^{n}$ Log[$(1+(\frac{x[[i]]}{b})^{a})$] $-(a^2/1.5)$], {a, 0, Infinity}], {a, 0, Infinity}];

m3 = m2 − (m1)^2;

Postmean[[i]] = m1;

Risk[[i]] = m3;

];

Return[{Mean[Postmean], Mean[Risk]}]);

For Rayleigh prior with Hyper-parameter $(b=1)$;

fun[A_, b_, n_, sim_]:= (Postmean = RandomVariate[LogLogisticDistribution[A, b], sim];

Risk = RandomVariate[LogLogisticDistribution[A, b], sim];

For[$i = 1, i <$ sim $+ 1, i++, x =$ RandomVariate[LogLogisticDistribution[A, b], n];

m1 = NIntegrate[(a^{n+2}Exp[$a\sum_{i=1}^{n}$ Log[$x[[i]]$] $- n * a$Log[b] $- 2 * \sum_{i=1}^{n}$ Log[$(1+(\frac{x[[i]]}{b})^{a})$] $- a^2$])/

NIntegrate[a^{n+1}Exp[$a\sum_{i=1}^{n}$ Log[$x[[i]]$] $- n * a$Log[b] $- 2 * \sum_{i=1}^{n}$ Log[$(1+(\frac{x[[i]]}{b})^{a})$] $- a^2$], {a, 0, Infinity}], {a, 0, Infinity}];

$$\text{m2} = \text{NIntegrate}[(a^{n+3}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n * a\text{Log}[b] - 2 * \sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - a^2])/$$

$$\text{NIntegrate}[a^{n+1}\text{Exp}[a\sum_{i=1}^{n}\text{Log}[x[[i]]] - n * a\text{Log}[b] - 2 * \sum_{i=1}^{n}\text{Log}[(1+(\frac{x[[i]]}{b})^{a})] - a^2], \{a, 0, \text{Infinity}\}], \{a, 0, \text{Infinity}\}];$$

$$\text{m3} = \text{m2} - (\text{m1})^2;$$

$$\text{Postmean}[[i]] = \text{m1};$$

$$\text{Risk}[[i]] = \text{m3};$$

];

Return[{Mean[Postmean], Mean[Risk]}];