EXAMINATION OF THE LAGRANGIAN EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL ELLIPTIC RESTRICTED THREE-BODY PROBLEM WITH ZONAL HARMONICS OF THE SECONDARY

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NIGERIA

SEPTEMBER 2016

DECLARATION

I hereby declare that the work in this dissertation entitled "Examination of the Lagrangian Equilibrium Points in the Photogravitational Elliptic Restricted Three-Body Problem with Zonal Harmonics of the Secondary" has been performed by me in the Department of Mathematics under the supervision of Prof. J. Singh and Prof. D. Singh, the information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any other institution.

Rukkayat Suleiman

Name of student	Signature	Date

CERTIFICATION

This dissertation entitled "EXAMINATION OF THE LAGRANGIAN EQUILIBRIUM POINTS IN THE PHOTOGRAVITATIONAL ELLIPTIC RESTRICTED THREE-BODY PROBLEM WITH ZONAL HARMONICS OF THE SECONDARY" by Suleiman, Rukkayat (M. Sc./Sci./1933/2011-2012) meets the regulations governing the award of Master of Science of Ahmadu Bello University, Zaria, and is approved for its contributions to knowledge and literary presentation.

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DEDICATION

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ABSTRACT

This dissertation examines the Lagrangian equilibrium points in the photogravitational elliptic restricted three-body problem with Zonal Harmonics of the secondary under the assumption that the primary is an intense emitter of radiation and the secondary has the shape of an oblate spheroid up to J_4 (octupolar mass moment), with the primaries moving in elliptic orbits of small eccentricity e. The positions and stability of collinear equilibrium points areobtained both analytically and numerically, the binary systems (Zera Cygni, Procyon A/B and 54 Piscium) are used to analyzed the result obtained numerically using the software package mathematica 5.0, the combined effect of oblateness (J_2 and J_4), (e) and (q_1)also shows that the size of the region of stability has a general reduction.Despite the introductions of perturbations the points were also found to remain unstable in the Lyapunov sense and still remain on the line joining the primaries. Apart fromthis points, the positions and linear stability of the triangular points are obtained analytically under the influence of the perturbing agents. The results obtained are then analyzed numerically with the same binary systems, the points are found to exists and formsscalene triangles with the line joing the primaries unlike the classical case which forms equilateral triangles. Thus the points are conditionally stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq$ ½. It was seen that the size of the region of stability decreases with increase in eccentricity (e=0.3, 0.25 and 0.2), While in the absence of radiation pressure the position and linear stability of the triangular points are obtained using other binaries (NLTT 11748, LP400-22 and J1257+5428), it was observed that eccentricity does not affect the $\xi_{coordinate}$ and ceases to exist in the quasi-parabolic case, but in the $\eta_{coordinate}$ there is a shift away from the primaries but reverse is the case for both coordinates with the presence of semi-major (a). The mass ratio of the binaries used are outside the range $0 < \mu < \mu_c$, thus making the triangular points unstable and they also cease to exist, as semi- major axis approaches unity. The Quadruple and octupolar mass moment causes a shift towards the origin and away from the line joining the primaries respectively on both coordinates.

Table of Contents

Title Page i

Fly Leaf ii

Cover Page ii

Declaration iii

Certification iv

Acknowledgement v

Dedication vi

Table of Contents viii

List of Figure xi

List of Notations xiv

CHAPTER ONE 1

GENERAL INTRODUCTION 1

1.0 Introduction 1

1.1Statement of the Problem 4

1.2Aim and Objectives of Study 5

1.3 Significant / Justification of the Study 5

1.4 Preliminary Ideas 6

1.4.1 Circular Restricted Three-Body Problem 6

1.4.2 Elliptic Restricted Three Body Problem 7

1.4.3 Equilibrium/Lagrangian/Stationary/Libration points. 7

1.5Theoretical Framework 8

1.5.1 Stability 8

1.5.2 Oblateness 10

1.5.3 Zonal Harmonics 11

1.5.4 Ellipsoid 12

1.5.5 Radiation Pressure 12

1.6Definition of Terms 13

1.6.1 Solar Mass ($\boldsymbol{M}\odot$). 13
1.6.2 Solar Radius ($\boldsymbol{R}\odot$). 13
1.6.3 Jupiter Mass ($\boldsymbol{Mj}$) 13
1.6.4 Solar Luminosity($\boldsymbol{L}\odot$) 13
1.6.5 Astronomical Unit ($\boldsymbol{AU}$). 14
1.6.6 Parsec ($\boldsymbol{pc}$). 14
1.6.7 Extrasolar Objects. 14
1.6.8 Interferometry. 14
1.6.9 Eccentricity (e) 14
1.6.10 Centimetre–Gram–Second System of Units (CGS or cgs). 14
CHAPTER TWO 15
2.0 LITERATURE REVIEW 15
2.1 Binary Systems 26
2.2 Binary Classifications 27
2.2.1 Zeta Cygni (ζ Cyg) 27
2.2.2 Procyon 28
2.2.3 54 Piscium 30
3.1 INTRODUCTION 33
3.2 Equation of Motion 33
3.3 Positions of Equilibrium Points 34
3.3.1 Position of Collinear Libration Points 35
3.3.2 Positions of Triangular Points 52
CHAPTER FOUR 64
STABILITY OF LIBRATION POINTS 64
4.1 Introduction 64
4.2 Variational and Characteristic Equations 64
4.3.1 Stability of Collinear Points 67

4.3.2 Stability of Triangular Points 74
CHAPTER FIVE 160
RESULTS AND DISCUSSION 160
5.1 INTRODUCTION 160
5.2 Numerical Results of Collinear Points 161
5.3 Numerical Results of Triangular Points 167
CHAPTER SIX 178
SUMMARY, CONCLUSION AND RECONMMENRATIONS 178
6.1 Summary 178
6.2 Conclusion 179
6.3Recommendations/Future Direction 180
6.4 Contribution to Knowledge 181
Refrences 182

LIST OF FIGURE

Figure 1. 1: Configuration of the R3BP 6
Figure 1. 2: The five Lagrangian points in CR3BP (Sun-Earth-Moon system) 8
Figure 1. 3: Diagram showing stable and unstable position of a simple pendulum 9
Figure 1. 4: An Ellipsoid 12

Figure 2. 1: Image showing Transition curve of Archid - I 25
Figure 2. 2: Image showing Transition curve of Luyten - II 26
Figure 2. 3: Image showing location of Zeta Cygni between NGC 6992 and NGC 6995 28
Figure 2. 4: Image showing positon of Procyon from the winter triangle 30
Figure 2. 5: Image showing 54 Piscium A and the brown dwarf 54 Piscium B (circled). 32
Figure 5. 1: Effect of J_2 on Zeta Cygni 164
Figure 5. 2: Effect of J_4 on 54 Piscium 165
Figure 5. 3: Effect of Semi-Major Axis on Procyon A/B 165
Figure 5. 4: Effect of Eccentricity on Procyon A/B 166
Figure 5. 5: Effect of Semi-Major Axis on 54 Piscium 166
Figure 5. 6: Effect of Eccentricity on 54 Piscium 167
Figure 5. 7: Effect of A_2 and B_2 on $L_{4,5}$ of Zeta Cygni 170
Figure 5. 8: Effect of A_2 and B_2 on $L_{4,5}$ Procyon A/B 171
Figure 5. 9: Effect of A_2 and B_2 on L4,5 for 54 Piscium 171
Figure 5. 10: Effect of eccentricity on $\boldsymbol{\xi}$- co-ordinates of Zeta Cygni on L_4 for A_2= 0.1, B_2 = -0.00001 172
Figure 5. 11: Effect eccentricity on $\boldsymbol{\eta}$ – co-ordinates of Zeta-Cygni on L_4 for A_2= 0.1, B_2 = -0.00001 172
Figure 5. 12: Effect of Semi-major axis (a) on $L_{4,5}$ of Zeta Cygni for A_2= 0.1, B_2 = -0.00001 and e=0.2 173
Figure 5. 13: Surface representation of the effect of eccentricity on L_4 of Zeta Cygni 173
Figure 5. 14: Effects of eccentricity and Semi-major axis on the triangular points of NLTT11748 174
Figure 5. 15: Effect of B_2 on $L_{4,5}$ of J1257+1548 for A_2=0.1, e=0.8 and a=0.7. 174
Figure 5. 16: Surface representation of the effect of the quadruple mass moment for e=0.6, a=0.8 and B_2=-0.00001 of LP400-22 174
Figure 5. 17: Effect of Radiation Pressure Force (q) on Critical Mass (μ_c) varying eccentricity 175

LIST OF TABLES

Table 1. 1: Physical Characteristics and Orbital Properties of Planets 11
Table 2. 1: The 54 Piscium Planetray System 32
Table 5. 1: Numerical Data 160
Table 5. 2: Numerical data 160
Table 5. 3: Effect of J_2 on the roots of the collinear points of Zeta Cygni, Procyon A/B and 54 Piscium 161
Table 5. 4: Effect of J_4 on the roots of the collinear points of Zeta Cygni, Procyon A/B and 54 Piscium 162
Table 5. 5: Effect of J_2 on the locations of the collinear points of Zeta Cygni, Procyon A/B and 54 Piscium 162
Table 5. 6: Zeta Cygni showing effect of Semi-Major axis A_2 =0.001 and B_2=-0.000001 162
Table 5. 7: Zeta Cygni showing effect of Eccentricity A_2=0.01 and B_2=-0.0000001 .. 163
Table 5. 8: Procyon A/B showing effect of Semi-Major axis A_2=0.001 and B_2=-0.000001 163
Table 5. 9: Procyon A/B showing effect of Eccentricity A_2=0.01 and B_2=-0.0000001 163
Table 5. 10: 54 Piscium showing effect of Semi-Major axis B_1=0.001 and B_2=-0.000001 163
Table 5. 11: 54 Piscium showing effect of Eccentricity A_2=0.01 and B_2=-0.0000001 164
Table 5. 12: Locations of Triangular Points of Zeta Cygni, Procyon A/B and 54 Piscium 167
. Table 5. 13: Effect of eccentricity on $L_{4,5}$ of Zeta Cygni for A_2= 0.1, B_2 = -0.00001 168
Table 5. 14: Effect of semi-major axis on $L_{4,5}$ of Zeta Cygni for A_2= 0.1, B_2 = -0.00001 and e=0.2 168
Table 5. 15: Effects of A_2 and B_2 for e=0.6, a=0.8; B_2=-0.00001 and B_1=0.1on LP400-22. 169
Table 5. 16: Effects of eccentricity and semi-major axis for e=0.6; a=0.8; A_2=0.01 and B_2=-0.00001on LP400-22. 169
Table 5. 17: Effects of A_2 and B_2 for e=0.8, a=0.7; B_1=0.1 and B_2=-0.00001 on J1257+5428 169
Table 5. 18: Effects of eccentricity and semi-major axis for e=0.8; a=0.7; A_2=0.01 and B_2=-0.00001 on J1257+5428 169

Table 5. 19: Effects of B_1 and B_2 for e=0.3, a=0.7; A_2=0.1 and B_2=-0.00001 on NLTT 11748. 169
Table 5. 20: Effects of eccentricity and semi-major axis for e=0.3; a=0.8; A_2=0.01 and B_2=-0.00001 on NLTT11748. 170
Table 5. 21: Effect of radiation (q) on □***c*** varying eccentricity 170

LIST OF NOTATIONS

m_1	Mass of the first primary
m_2	Mass of the second primary
m_3	Mass of the infinitesimal body
J_2	Expression for first zonal harmonic
J_4	Expression for second zonal harmonic
q_1	Radiation of the first primary
n	Mean motion
e	Eccentricity
A_2&B_2	Oblateness coefficient of the second primary
$M_\odot$	Solar Mass
M_j	Jupiter Mass
$R_\odot$	Solar Radius
$L_\odot$	Solar Luminosity
AU	Astronomical Unit
cgs	centimeter-gram-second system of unit
α, β and γ	Small displacement
λ	Constant
pc	parsec
K	Kelvin

CHAPTER ONE

GENERAL INTRODUCTION

1.0 Introduction

The most important technological achievement of mankind in the field of Space Dynamics is attaining an advancement in space study. It has made possible modern communication, exploration of earth resources, weather forecast, accurate positioning, as well as several other tasks which are part of our lives. It also play an important role in development and enable us to plan how to launch and control a space vehicle in order to obtain the results needed (Antonio *et al* 2009). Celestial mechanics is the study of the motions of astronomical bodies as they move under the influence of their mutual gravitational attraction. The main problem in the field of celestial mechanics is the three-body problem, which has several relevance in different scientific areas such as astrophysics and astrodynamics.Whittaker 1937 described the general three-body problem, as the most challenging, celebrated problems and important aspect of space dynamics, which deals with three particles moving in space under their mutual gravitational attraction; given their initial conditions, and determine their subsequent motion (Grutzelius 2004).They are dynamical systems that are rich in mathematical intricacy, practical applicability, and all its masses are non-zero and their initial positions and velocities are not arranged in any particular way and have been the focus of many investigations and remains unsuccessful despite several years of exploring. The study of particular or even approximate solutions of the three-body problem appears to be useful in mission design, although it is non-integrable (Farquhar *et al*., 1970; Lo *et al*., 1998), thus a special form of the general three body problem called the restricted three-body problem is then introduced, based on the assumption that the mass of one of the bodies is negligible as well as the interaction forces between the masses of the other

two bodies (Musielak and Quarles, 2015). The larger bodies are referred to as the primaries and the third is an infinitesimal mass. Typical examples include the sun-planet-planet, sun-planet-moon or sun-planet asteroid systems. The R3BP, is described based on the shape of the orbit of the primaries been focused, when two bodies with dominant masses moving around their common center of mass along either circular/elliptic orbits and parabolic/hyperbolic orbits. The most extensively studied R3BP is the CR3BP.

The circular restricted three-body problem (***CR3BP***) is the special case of the R3BP in which the primaries are confined to a circular path about their common center of mass or barycenter, such that the infinitesimal mass does not influence the motion of the primaries (approximated example the sun-earth-moon system). While the Elliptic restricted three-body problem (***ER3BP***) is a natural generalization of the (***CR3BP***) described based on three-dimensional motion of a small particle called the third body (*infinitesimal mass*) under the gravitational attraction of the primaries that moves in elliptic paths relative to each other.

The main difference between the CR3BP and ER3BP is that, in the latter the primaries are constrained to move on elliptic paths relative to each other and the orbits of the primaries share a common value of eccentricity (e), however the primaries revolve along circular paths about their center of mass or barycenter, the motion of the third body is subject of the CR3BP (Lisa 1992 and Amadeu *et al*. 2015).

The photogravitational restricted three-body problem is special case of restricted three-body problem where at least one or both of the primaries are intense emitter of radiation; introduced by Radzievskii (1950) for specific three-body problem of Sun-Particle and dust particle on the assumption that particle moves under the influence of

central forces only: gravitation and radiation pressure. The effect of radiation pressure has been the subject of many studies.

The orbits of most planets and stars are elliptic rather than circular; as a result, therefore making the study of the elliptic restricted three-body problem (ER3BP) significant. When the primaries' orbit is elliptic, a non-uniformly rotating-pulsating coordinate system is commonly used. These new coordinates have the felicitous property that, the positions of the primaries are fixed; however, the Hamiltonian is explicitly time-dependent (Szebehely 1967; Narayan and Shrivastana 2014). Such an oscillating coordinate system has been introduced by using the variable distance between the primaries as a unit of length of the system by which distances are divided.

Some significant studies related to the effect of oblateness up to the quadruple (J_2) and Octupolar (J_4) mass moments in either one or both primaries of the restricted three-body problem have been carried out recentlyby some scientists; Abouelmagd (2012) considered the effect of oblateness (J_2 and J_4) of the more massive primary on the locations and linear stability of triangular points in the planar circular restricted three-body problem, Singh and Taura (2013) examined the effects of oblateness of both primaries up to Zonal Harmonics, together with gravitational potential from circular cluster of material points on the existence and linear stability of triangular equilibrium points, also in (2014a) they examined the effect of radiation and oblateness up to $J_4$of the primaries and potential created by circular cluster on the linear stability of libration locations on the infinitesimal mass, in addition Singh and Taura (2014b) they examined the effects of oblateness up to $J_4$of the less massive primary and gravitational potential from the circum-binary belt on the linear stability of the triangular points in the CR3BP, Many researchers have given great consideration to the behavior of equilibrium points

in ER3BP. Some of these works are by Danby (1964), Bennett (1965), Szebebely (1967), Markeev (1978), Selaru and Cucu-Dumitrescu (1995), and Hallan and Rana (2001). The influence of the eccentricity of the orbits of the primaries with or without radiation pressure on the existence and stability of the equilibrium points was studied by Gyorgyey (1985), Grebenikov (1964), Kumar and Choudhry (1987,1990), Markellos *et al.,* (1992) and Markellos(1996), Sahoo and Ishwar (2000), Roberts (2002), Zimovshchikov and Tkhai (2004), Ammar (2008), E´rdi *et al*. (2009), Kumar and Ishwar (2011), Singh and Umar (2012a,b, 2013a,b, 2014), Usha *et al*. (2014), and Narayan and Singh (2014a, 2014b, 2014c, 2015).

1.1 Statement of the Problem

Various studies on R3BP have been carried out by different researchers based on the shapes of their orbit. Singh and Umar (2012b) carried out a study on the photogravitational elliptical restricted three-body problem with oblateness of both primaries and Singh and Taura (2013) study was on the photogravitational restricted three-body problem with oblateness up to zonal harmonics (J_4) with the circular cluster material in the circular case, however the combination of the two modification is examined in different perspective.

Thus, our study is on photogravitational restricted three-body problem with zonal harmonics of the secondary in the elliptical case, when the primary is an intense emitter of radiation and the secondary is an oblate spheroid up to zonal harmonics (J_4) and the third body is assumed to be a test particle having infinitesimal mass and influenced by the gravitational attraction of the primaries, i.e: the radiation pressure of the primary, quadruple and octupolar mass moment of the secondary.

1.2 Aim and Objectives of Study

The aim of this study is to investigate the motion of an infinitesimal mass in the framework of elliptic restricted three-body problem, when the primary is an intense emitter of radiation and the secondary has the shape of an oblate spheroid up to J_4(octupolar mass moment).

The objectives of the study are to determine and study the positions and stability of: -

i) the triangular equilibrium point

ii) collinear equilibrium point

in the ER3BP under the consideration of the luminosity of the primary and the quadruple and octupolar mass moment of the secondary with applications on the binary systems Zeta Cygni, Procyon A/B, and 54 Piscium.

1.3 Significant / Justification of the Study

The restricted three-body problem is the most celebrated problem of Celestial Mechanics [Szebehely (1967), Marsden and Ross (2005)] due to its interdisciplinary applications. It has also attracted the interest of many researchers from different fields of science [e.g., Prosmiti *et al*. (1996), Kalantonis *et al*. (2001), Sano (2007)]. Various modifications of the classical problem have been recommended over the years in order to make it relevant and applicable to certain systems of Dynamical Astronomy.

The sun is a star located outside the planetary system, a source of radiation which constitute a mixture of electromagnetic waves ranging from infrared (IR) to ultraviolet rays,includes visible light, which is between IR and UV in the electromagnetic spectrum. When electromagnetic radiation from the sun strikes the Earth's atmosphere, some of it is absorbed while the rest proceed to the earth's surface.

Thus, the combination of the sun together with its planetary system, including the Earth, is known as the solar system, as such any combination of the sun-planet–satellite system will constitute a good model for this study, and in the stellar systems we have the low and high masses binary systems like Zeta Cygni, Procyon A/B and 54 Piscium which are used in this work, also constitute excellent model for this problem, as a result this work will contribute a lot in the study of solar and stellar systems.

1.4 Preliminary Ideas

1.4.1 Circular Restricted Three-Body Problem

If three bodies with masses (***m₁, m₂*** & ***m₃***), assumed to be spherically symmetric with gravitational fields such that each body is modeled as a point mass, the motion of two (***m₁*** &***m₂***) of the three bodies on circular orbits restricts its effects on the third one (infinitesimal), then the ***restricted three-body problem*** (***R3BP***) is said to exist.

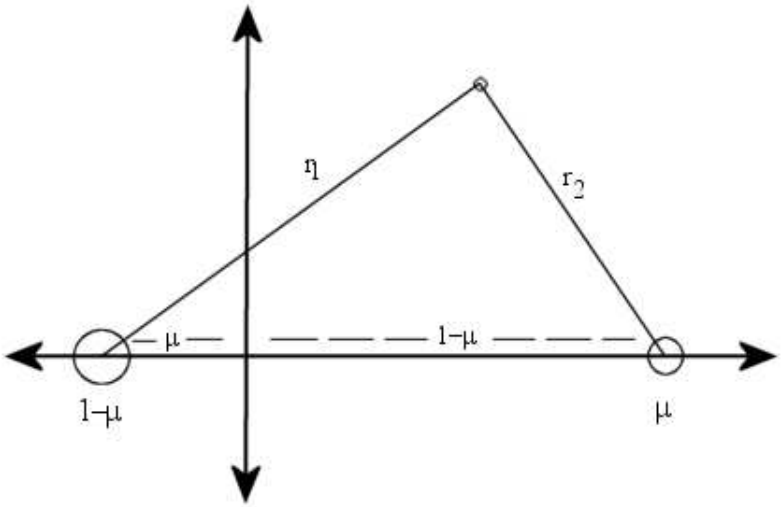


Figure 1. 1: Configuration of the R3BP

However, when two bodies of finite mass called the primaries, move on a path determined by their mutual gravitational attraction but unaffected by the third body, free

to travel in all three spatial dimensions, if these primaries are further confined to circular paths about their center of mass or barycenter, this simplified problem of the motion of the infinitesimal mass is generally known as the ***circular restricted three-body problem*** *(CR3BP).*

1.4.2 **Elliptic Restricted Three Body Problem**

The elliptic restricted three-body problem similar to the ***CR3BP***, focusses on the motion of bodies of finite mass called primaries assumed to be significantly larger than the third one ($m_1 >> m_3$ and $m_2 >> m_3$), and that the third mass has no gravitational influence on the primaries and the primaries are further constrained to move in elliptic paths relative to each other such that the orbit of the primaries will be an ellipse of eccentricity e.

1.4.3 **Equilibrium/Lagrangian/Stationary/Libration points.**

The Lagrangian points also called the *L-points or Libration points or Equilibrium points*in celestial mechanics, named after an Italian-french mathematicianJoseph – Louis Lagrange are positions of orbital configuration of two massive bodies, where a small body is affected only by the gravity, and maintains stable positions relative to the two bodies, they mark the positions where combined gravitational pull of the two massive body provides centripetal force required to orbit with them. They are solutions of the equations of motion of this system. These points are five, labeled L_1 *to* L_5, the first three L_1 L_2 L_3 are on a straight line the $(x-axis)$, joining the two massive bodies called the "collinear points" which are unstable. They were discovered by Leonard Euler in the eighteenth century; while the last two equilibrium points are L_4& L_5, are stable such that objects can orbit around them in a rotating system tied to the massive bodies, forming an equilateral triangle with the two massive bodies and are called Lagrangian points. These points were discovered by Lagrange and all the five

equilibrium points are named after him. The diagram illustrates the five equilibrium points in the Sun-Earth-Moon system.

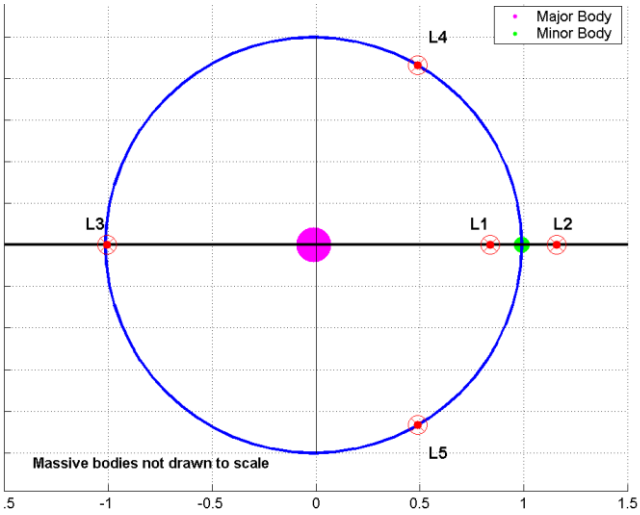


Figure 1. 2: The five Lagrangian points in CR3BP (Sun-Earth-Moon system)

The L_1 lies on a straight line the$(x-axis)$, defined by two bodies (*major and minor bodies*)and between them, where the gravitational attraction of the (*minor body*)partially cancels the gravitational attraction of (*major body*), the L_2lies on the same axis as L_1, but located after the minor body, where the gravitational forces of the two large masses balance the centrifugal effect on the body at L_2. Point L_3lies on the straight line defined for the two bodies but before the major body; while L_4&L_5lies at the third corners of the equilateral triangles in the plane of orbit whose common base is the line between the centers of the two masses, such that the point lies behind L_5 or ahead of L_4 of the smaller mass with regard to the orbits around the larger mass.

1.5 Theoretical Framework

1.5.1 Stability

An equilibrium point of a dynamic system is a solution that does not change with time. In any dynamic system, the equilibrium points are those points which satisfies $\dot{x} = 0$.

Equilibria are sometimes called fixed points or steady states; equilibrium points are in two forms, the stable and unstable equilibrium

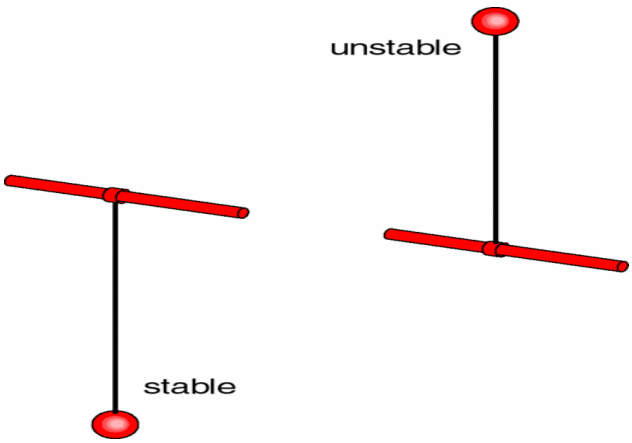


Figure 1. 3: Diagram showing stable and unstable position of a simple pendulum

A body is in stable equilibrium if after a slight displacement it returns to its original position, while a body is said to be unstable if after slight displacement causes the body to move to a new position as time elapses and does not return to its state, a mechanical example is a simple pendulum, if the pendulum is at the downward position, the system remains there forever, thus the downward position is said to be in a stable state of an equilibrium point. On the contrary, the upward position of the system is in an unstable position. In dynamical systems, stability of trajectories does not change under small perturbations.

Due to perturbations, the position of the infinitesimal body would be displaced a little from the libration point. If the resultant motion of the infinitesimal body is a rapid departure from the vicinity of the point, we call such a position of libration point an "unstable one" if however, the body merely oscillates about the libration point, it is said to be a "stable position" (in the sense of Lyapunov).

1.5.2 Oblateness

Oblateness is the degree to which the plane rotation causes its equator to bulge and poles to flatten. Numerically, it is the ratio of the difference between the equatorial and polar radii to the equatorial radius. Any celestial body that rotates tends to bulge at its equator by an amount that depends both on its speed of rotation and on whether it is solid or fluid in composition. It is the approximated shape of many planets and celestial bodies, including Saturn, Jupiter and to a lesser extent the Earth. Oblateness is usually an indication of how fast the body is rotating. Oblateness is therefore a good indicator of the rotational speed of a star or planet.

Table 1. 1: Physical Characteristics and Orbital Properties of Planets

Planet	*Oblateness* $[= De - Dp/De]$	*Eccentricity* e	*Zonal Harmonics* *J Values*
Mercury	0.000	0.2056	0.00006
Venus	0.000	0.0068	0.000027
Earth	0.00335	0.0167	[0.0010826269-0.000025323-0.000016204]
Mars	0.00648	0.0934	[0.001964-0.000036]
Jupiter	0.06487	0.0484	[0.01475 0-0.00058]
Saturn	0.09796	0.0542	[0.01645 0-0.001]
Uranus	0.02293	0.0472	0.013
Neptune	0.01708	0.0086	0.004
Pluto	0.000	0.2488	

Where D_e= Equatorial Diameter and D_p=Polar Diameter

The R3BP assumes that the masses concerned are spherically symmetrical in homogenous layers, but it is found that celestial bodies, such as Saturn and Jupiter, are sufficiently oblate (Beatty *et al*. 1999). The minor planets (e.g., Ceres) and meteoroids have irregular shapes (Millis *et al*. 1978; Norton & Chitwood 2008). Several researchers SubbaRao & Sharma (1975), Elipe & Ferrer (1985), Singh & Ishwar (1999), Sharma et al. (2001), AbdulRaheem & Singh (2006), Vishnu et al. (2008), Mital *et al*. (2009), Singh and Begha (2011); Singh & Umar (2012a, b), Abdoulmegd (2012); Singh & Leke (2013a, b), Singh & Umar (2013); have included oblateness of one or both primaries in their studies.

1.5.3 Zonal Harmonics

In mathematical study, they are special spherical harmonic polynomials that assume constant values on the axis of symmetry, which are characterized by single-valued harmonic functions connected through domains of rotational symmetry. In physical situations, a harmonic functions is a function 'f'that satisfies the Laplace's equation

$\nabla^2 f = 0$ and can be represented as polynomials with an infinite number of terms, which describes equilibrium conditions such as the temperature or electrical charge distribution over a region in which the value at each point remains constant. They are sometimes referred to as solid zonal harmonics and as surface zonal harmonics. Researchers like Singh and Taura (2013); Singh and Omale (2014); Abouelmagd (2012) and Abouelmagd *et al*. (2015) have included the effect of zonal harmonics in their studies.

1.5.4 Ellipsoid

An ellipsoid is a closed geometric three-dimensional surface characterized by lengths a, b, c corresponding to semi-major axis and semi-minor axis of appropriate ellipses, whose plane sections are ellipses or circles. When two of the axes of an ellipsoid appears to be the same and one of the axes rotates an ellipse it said to be an ellipsoid of revolution or spheroid (i.e. a = b>c oblate spheroid & a=b<c prolate spheroid) but when the lengths of all the three axes are the same, it's a sphere (a=b=c sphere).

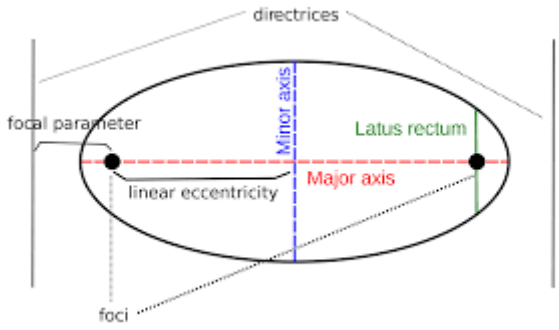


Figure 1. 4: An Ellipsoid

1.5.5 Radiation Pressure

Radiation pressure is said to occur when electromagnetic radiations or sound waves exerts minute pressure on everything it encounters, it's the transfer of momentum from

photons as they strike the surface of the object. In everyday situation this pressure is negligible except in the environs of stars due to the large quantities of photons emitted, however, they do play a crucial role in some settings, such as astronomy and astrodynamics. It is exerted by solar radiation on objects within the solar system.

Hence, solar radiation pressure is exerted by solar radiation on objects within the solar system, while it acts on all bodies within the system, affecting the smaller bodies. Solar radiation pressure influences trajectory of spacecraft with large area to mass ratio as a result of emission of electromagnetic radiation from the sun. Solar radiation pressure affects bodies throughout the solar system. Thus, small bodies are affected more than large ones because of their lower mass and inertial properties; Spacecraft are affected along natural bodies (comet, asteroids, dust grain, gas molecules). It is used in the design of solar sail

Thus, when classical restricted three-body problem has radiation parameter in either one or both of the primaries, another problem called photogravitational problem arises discovered by Radzievskii (1950, 1953).

1.6 Definition of Terms

1.6.1 Solar Mass ($M_{\odot}$):- is a standard unit of mass used in astronomy , it is used to indicate the masses of other stars, as well as clusters, nebulae and galaxies.

1.6.2 Solar Radius ($R_{\odot}$):- is a unit of distance to express the size of stars in astronomy, defined as the radius to the layer in the sun's photosphere.

1.6.3 Jupiter Mass (M_j):- is the unit of mass equal to the total mass of the planet Jupiter. Jupiter mass is used to describe masses of the gas giants, such as outer planet and extrasolar planets, also used to descrine brown dwarfs.

1.6.4 Solar Luminosity($L_{\odot}$) :- is a unit of radiative (power emitted in the form of photons) conventionally used by astronomers to measure the luminosity of stars.

1.6.5 Astronomical Unit (AU) :- is a *unit* of length, roughly the distance from Earth to the Sun. However, that distance varies as Earth orbits the Sun, from a maximum (aphelion) to a minimum (perihelion) and back again once a year.

1.6.6 Parsec (pc) :- a unit of length used to measure large distances to objects outside our solar systems.

1.6.7 Extrasolar Objects: - is an astronomy objects that exists outside the solar system.

1.6.8 Interferometry: - a kind of techniques in which waves, usually electromagnetic, are superimposed in order to exact information about the waves.

1.6.9Eccentricity (e): -shows how much a conic section (a circle, ellipse, parabola or hyperbola) varies from being circular, thus it's the deviation from a circular path or orbit

1.6.10Centimetre–Gram–Second System of Units (CGS or cgs): - The cgs replaced the MKS system and SI units, in mechanics, the CGS and SI systems of units are built in an identical way. The only difference is in the scale of two out of the three base units (centimetre versus metre and gram versus kilogram, respectively), while the third unit (second as the unit of time) is the same in both systems.

CHAPTER TWO

2.0 LITERATURE REVIEW

Despite many years of exploration to solve the three-body-problem, in which different techniques are applied but complete solution to it (**3BP**) has remain a formidable challenge till date (Julia 1991); unlike the two body problem which has a known analytic solution and **R3BP** which has an equilibrium solution. The **R3BP** is the simplest form of general **3BP**, where one of the masses of the bodies ("third body") is negligible with respect to the mass of the two bodies (the "primaries"). There are several literatures devoted to works that have been done in the field of the restricted three-body problem. This include both analytic and numerical developments. A review of some published works as related to the study is given in this chapter.

The restricted three-body problem (R3BP) is of considerable importance today and constitutes a major source of interesting theories in lunar and planetary sciences with its application to dynamical astronomy, celestial and space mechanics (Topputo *et al*. 2005; Belbruno *et al*. 2008; Romagnoli and Circi 2009; Bazso 2012), though it is unsolvable and non-integrable, the use of synodic reference system makes the primaries to be fixed with respect to uniformly rotating axes, thereby the Hamiltonian does not depend on time explicitly (Narayan and Singh 2014). Some particular solutions in relation to equilibrium came up in the R3BP, these solutions correspond to five singular points or Jacobian function, which are classified into three collinear points L_1, L_2, L_3 (straight line solutions) of Eulerian points determined by Euler (1767) and the other two points denoted by L_4, L_5 representing equilateral triangle with the primaries usually called the lagrangian or libration points named after it discoverer Lagrange (1772) (Katour *et al.* 2014).

The bodies of the CR3BP are assumed to be spherically symmetrical in homogeneous layers, some celestial bodies tends to act like CR3BP, such as Saturn and Jupiter, are sufficiently oblate, while the minor planets like Ceres and Meteoroids have irregular shapes etc., hence making it different from the CR3BP. Recently, various perturbing factors such as oblateness, triaxiality and radiation forces of the primaries, Coriolis and centrifugal forces, variation of the masses of the primaries and the infinitesimal mass, the atmospheric drag, the solar wind, Poynting-Robertson effect and the action of other bodies etc. have been included in various research of R3BP. Some of which are reviewed here:

Vidyakin (1974), established the location of five (5) equilibrium points and their stability when both primaries are oblate spheroids whose equatorial plane coincides with the plane of motion.SubbaRao and Sharma (1975) studied the same problem as Szebehely (1967), under the consideration that one or both primaries are oblate with their equatorial planes coinciding with the plane of motion. They discovered that oblateness of the primary resulted in increase in both the Coriolis and centrifugal force respectively, also found that the range of stability of triangular points decreases¸ hence concluding that the Coriolis force is not always a stabilizing force. Sharma (1987) investigated the linear stability of triangular points in the R3BP when the more massive primary is a source of radiation and the smaller primary an oblate spheroid. He considered the positive values of the radiation pressure only and discovered that the collinear points remain unstable and the triangular points linearly stable and also discovered that the critical mass value decreases with increase in oblateness and radiation pressure factor.

Radiation pressure factor plays an important role in the generalized 3BP, and its importance in celestial bodies has been recognized by many scientists in the restricted three-body problem when either one or both primaries are luminous¸ some of these works are: Kozai (1962), McCraken and Alexander (1968), Simmons *et al*. (1985), Vkrouhlixky (1993), Murray C.D. (1994), Ragos and Zafiropoulos F.A. (1995).

The photogravitational restricted three-body problem was developed by Radzievskii (1950) as special form of the classical problem when one of the interacting bodies is an intense emitter of radiation, he models it for three specific bodies sun, planet and dust particle. In 1950, he considered the planar case, where he studied libration points of the photogravitational problem, while in 1953, the non-planar case was considered, where he established two additional libration points called 'out of plane' libration points, denoted by $L_6$and L_7.The gravitational and light repulsive force is generally characterized by the mass reduction factor *q*.

Ishwar and Singh (1999) in their study discovered that the triangular points are stable for $0 \leq \square \leq \square_c$ and unstable for $\square_c < \square \leq ½$ when they considered both primaries to be sources of radiation as well as oblate spheroids. They discovered that L_4 is stable for all mass ratios in the range of linear stability except for those mass ratios depending upon oblateness co-efficient and mass reduction factors. The same problem under the influence of small perturbations in the Coriolis and centrifugal forces was studied by AbdulRaheem and Singh (2008). Also, Singh (2005) examined the stability of the collinear points under the effects of both oblateness and radiation of both primaries. Sahoo & Ishwar (2000) also studied the stability of collinear libration points in the generalized photogravitational elliptic restricted three-body problem when the smaller primary is an oblate spheroid and the bigger primary an intense emitter of radiation, it was discovered that these points are unstable as in the classical case. Sharma *et al*.

(2001) discovered that the collinear libration points are unstable, while the triangular points remain stable under the condition $0 < \mu \leq \mu_c$, when the bigger primary is radiating and also a rigid triaxial body with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion. The critical mass value μ_c is obtained as

$$\mu_c = 0.0385208965 \ldots - 0.089174706P + 0.81126474\sigma_1 - 1.09626653\sigma_2$$

where

$$P = \frac{Radiation\ pressure\ due\ to\ bigger\ primary}{Gravitational\ force\ due\ to\ bigger\ primary} \ll 1.$$

Kushvah and Ishwar (2006), examined the triangular equilibrium points in their study of generalized photogravitational restricted three body problem with Poynting-Robertson drag, taking the primary as a source of radiation and the secondary assumed to be an oblate spheroid. It was discovered that the equations of motions are affected by the parameter and thus concluded that the position of triangular equilibrium points are affected by mass reduction factors, P-R drag and oblateness coefficient.

Furthermore, AbdulRaheem and Singh (2006) study was carried out under the influence of small perturbations in the Coriolis and Centrifugal Forces, under the consideration that both primaries are oblate spheroid as well as source of radiation, it was found that the collinear points are unstable and triangular points are stable which depends on the parameters above. It was further observed that the Coriolis force has stabilizing tendency while the others (centrifugal force, radiation and oblateness) of the primaries have destabilizing effects, hence the overall effects of the range of stability of the triangular points decreases.The triangular points were discovered to be unstable with the presence of Poynting-Robertson drag effects in the study carried out by Singh and

Balogun (2014), where the primary is luminous and the secondary is a triaxial rigid body with one of the axes as the axis of symmetry and its equatorial plane coinciding with the plane of motion, under Poynting-Robertson (P-R) drag effect as a result of the radiating primary and conditionally stable in the absence of (P-R) drag effects. Similarly, in the study of Singh and Amuda (2014) under the consideration that the primary is an oblate's spheroid and secondary is a source of radiation with the presence of Poynting- Robertson (P-R) drag effects, it was noticed that the triangular points remain unstable due to the positive roots obtained in the Lyapunov sense when the Poynting-Robertson effects is considered against conditional stability that appears in the absence of the P-R drag effects. Recently; Singh and Taura studies carried outin the circular restricted three-body problem under the effect of oblateness up to zonal harmonics of (either primary) are as follows: in (2013a) a reformed type of the clasical R3BP was studied where both primaries are oblate spheroid and are enclosed by a homogenous circular planar cluster of material points centre at mass center of he system, thus th effect on the number and linear stability if the equilibrium points of the infiniesmial body was examined. It was discovere tht aside from the usual five equilibrium points two more collinear points (*Ln1*, *Ln2*) were discovered due to the potential from the circular cluster, the collinear points still remain unsaable while mumerical result showed that *Ln1* moves away from the more massive primary and *Ln2* moves towards it as the mass of the circular cluster material points increases, for oblateness of the body reverse is the case, the triangular points was still stable in range of interval $0 < \mu < \mu_c$ and $\mu_c \leq \mu \leq$ ½. The oblateness and Circular Cluster of material points have destabilizing effects been influenced by the critical mass ratio μ_c. Whilein 2013b the effects of oblateness of both primaries up to zonal harmonics J_4 of the secondary together with gravitational potential from the circular cluster of material

points on the existence and linear stability of the triangular equilibrium points was examined, it was found that the triangular points are stable in the range $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu < ½$, the effect of stability was as result of the co-efficient J_4, while J_2 and the potential from the circular cluster material points caused destabilizing effects.Singh and Taura 2014a examined the effects of radiation and oblateness up to J_4 of the primaries and the potential created by the circular cluster on the linear stability of the liberations loations of the test particle, it was found that the libration points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu < ½$, the combined effect of the perturbations on the stability of the triangular libration points is that it has stabilizing propensity. Furthermore, the influence of oblateness up to J_4 of the secondary as well as gravitational potential from the circum-binary belt and the primary as a source of radiation influencing other bodies of the system on the linear stability of triangular points in the CR3BP was investigated by Singh and Taura 2014b. It was observed that the triangular points are stable in the range of interval $0 < \mu < \mu_c$ and $\mu_c \leq \mu \leq ½$, so also it moves towards the line joining the primaries in the presence of any of the perturbations while in the presence of oblateness up to J_4 the points move away from the line joining the primaries. The critical mass ratio are affected by oblateness up to J_4 of the secondary, electromagnetic radiation of the primary, potential form the and all have destabilizing tendencies except with the co-efficient of J_4 and potential from the belt. In addition, Singh and Taura 2015 investigatd a refined type of classical restricted three body problem was examined where the primary is radiating and the secondary is an oblate spheroid up to zonal harmonics, thus the effect of oblateness of the secondary and radiation of the primary with gravitational potential generated from the belt, has been proved numerically in addition to the three collinear points $L_i(i = 1,2,3)$ in the classical restricted three-body problem, four more collinear points $L_{ni}(i = 1,2,3,4)$

came up due to the potential from the belt we have L_{n1} and L_{n2} while L_{n3} and L_{n4} occurred due to oblateness up to zonal harmonics J_4 of the smaller primary. Due to the perturbations forces L_1 and L_3 come nearer to the primaries while L_{n3} move away from the primaries and $L_2$and L_{n1}tend towards the smaller primary, whereas L_{n2} and L_{n4} draw closer to the bigger primary. Hence the collinear libration points L_i $(i = 1,2,3)$ and L_{n2} are linearly unstable whereas the L_{n1}, L_{n3} and L_{n4} are linearly stable.

Similarly, Abouelmagd (2012) determined the influence of oblateness up to $J_4$of the primary on the locations and linear stability of triangular points, it was discovered that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq ½$, also the numerical values obtained for the positions of the triangular using some planets systems showed that the range of stability decreases, but in some instance the range is not affected by the existence of J_4.In addition Abouelmagd (2013) examined the existence of triangular points and linear stability of a generalized restricted three-body problem under the consideration that the effect of oblateness is up to 10^{-6}. The locations of the triangular points were seen to be affected by the perturbing forces on the primaries, also the points are stable for $0 \leq \mu \leq \mu_c$ and unstable for $\mu_c \leq \mu \leq ½$. The critical mass (μ_c) value depends on terms which involve parameters that characterize the oblateness and radiation repulsive forces.

The primaries in elliptic restricted three-body problem moves in elliptical Keplarian orbits as such making the location of the primaries to be fixed because of the introduction of non-uniformly rotating and pulsating co-ordinate systems. The elliptic restricted three-body problem is a generalization of the CR3BP despite the orbits of the primaries being elliptical, some effective and major properties of the circular model exists or can be adapted. Some problem of photogravitational elliptic restricted three

body problem in which either or both primaries emit light energy, have been considered in various aspects of ER3BP, some of these works have been introduced by Kumar and Choudry (1987, 1990), Markellos *et al*., (1992), Sahoo and Ishwar (2000), Kunitsyn (2001)and Floria (2004), Zimovshchikov and Tkhai (2004), Ammar (2008), Kumar and Ishwar (2009), Singh and Umar (2012a,b, 2013a,b, 2014a,b,c), Narayan and Ramesh (2011a), Narayan and Shrivastava (2014), Usha *et al*., (2014), Narayan *et al*., (2015), have also devoted their efforts exclusively in these aspects of R3BP. Singh and Taura (2014 and 2015) and Abouelmagd *et al*. (2015) applied the effect of zonal harmonics in their study.

The investigation of stability of collinear points of an infinitesimal mass in the photogravitational elliptic restricted three-body problem under the influence of radiation pressures of the primaries, together with the gravitational attraction force was carried out by Singh and Umar (2010), and discovered that collinear points are unstable while the inner (L_2) collinear points appears to be stable due to the interval in the mass reduction factor of the bigger primary under certain condition conditions depending on the mass ratio, eccentricity and semi-major axis of the orbit. Singh and Umar (2012a) carried out study on the motion of an infinitesimal mass around the seven equilibrium points within the framework of ER3BP under the assumption that the primary is non-luminous, oblate and the secondary is luminous. The triangular points appears to be conditional stable for $0 < \square \leq \square_c$, under the combined influence of radiation effects, oblateness, eccentricity and semi-major axis of the elliptic orbits. Due to increase of these parameters, it resulted to a destabilizing effects on the orbits of the test particles. The overall effect shows that the size of the region of stability decreases when the value of the parameters increases. In addition, Singh and Umar (2013a) examined and located the of 'out of plane' equilibrium points,($L_{6,7}$) of an infinitesimal body in the field of

stellar-oblate binary systems moving in elliptic orbits around their common center of mass. Oblateness and radiation co-efficient of the primaries and eccentricity of the orbits was used to determine the positions and stability. Furthermore, the study on the positions and stability of the collinear equilibrium points, $L_{1,2,3}$ of an infinitesimal body in the elliptic restricted three-body problem (ER3BP) was carried out by Singh and Umar (2013) when both primaries of the system are luminous and oblate spheroids moving in elliptic orbits around their common centre of mass. It was observed the radiation pressure forces, oblateness and the eccentricity of the orbits affects the positions, but the stability character remains unchanged and are unstable.

Narayan and Ramesh (2011a) in their study of effects of photogravitational and oblateness on the triangular Lagrangian points in the elliptical restricted three-body problem using the method of averaging by Grebenikov to analysis stability of equilibrium points, shows that the range of stability decreases as the oblateness and radiation pressure parameter increases, when the primary and secondary are an oblate spheroid, as well as the primary been a source of radiation. A similar research conducted by Narayan and Ramesh (2011b) indicated that the characteristics exponent is determined based on the Floquet's theory of the variational equations of motion as such the stability of the libration points was obtained by using simulation technique to draw transition curves, the system appears to be unstable whereas the region outside the curves are stable. Usha *et al*. (2014) also investigated the effects of radiation and triaxiality of primaries on the triangular libration points in elliptic restricted three-body problem, where the primary is radiating and the secondary a rigid triaxial body, the method of averaging developed by Grebnikov (Nauka, Moscow revised 1986) was used, it was discovered that the critical mass ratio depends on the radiation pressure, triaxiality, eccentricity, semi-major axis of the elliptic orbits and the range of stability

decreases as the radiation parameter increases. Abd El-Salam (2015) examined the elliptic restricted three-body problem with oblate and triaxial primaries using pulsating coordinates, it was found that the stability regions depend on the eccentricity of the orbits, oblateness coefficient and triaxial of the primaries. The value of e=0.25 showed that the stability is destroyed completely for the Earth-Moon system.

The location of triangular equilibrium points as well as collinear points in the generalized photogravitational elliptic restricted three-body problem was examined by Prakash and Ramshankar (2013), the primary is taken as an oblate spheroid and the secondary as a source of radiation. The result obtained for the triangular ad collinear libration points are different from those of the classical case but when $\rho = A_1 = B_2 = 0$ and $A = 1$ the same result of the classical case is obtained as in Mc Cuskey. Also, Narayan and Shrivastana (2014), analyzed the existence of the triangular equilibrium points of the planar elliptical restricted three-body problem when both primaries are oblate spheroid as well as source of radiation under the assumption that eccentricity is zero, the effect of oblateness and radiation on the primaries affects the location and resonance stability of the elliptical R3BP when $e = 0$ and near the resonance frequency $\omega_2 = ½$, they concluded that there is shift on the locations and resonance stability of the triangular points, due to changes in oblateness and radiation parameters around some well-known binary systems.

Singh and Umar (2014) investigated the positions and stability of collinear points when the primary is a triaxial rigid body and the secondary is an oblate spheroid, it was observed that the positions and stability are affected by the triaxiality and oblateness as well as the semi-major-axis and eccentricity, the numerical analysis shows that the positions of the collinear points on some binary system (PSR J1518+4904, PSR

B1534+12, PSR B1914+16, and PSR B2127+11c) are affected by the triaxiality of the bigger primary and the eccentricity of the orbits, however the stability behavior remains unchanged and are unstable in the Lyapunov sense due to the presence of real roots, despite the introduction of triaxiality and oblateness parameters. Narayan and Ramesh (2014) examined the motion and stability of the Elliptical Restricted Three Body Problem under the consideration that the primary is an oblate spheroid as well as radiating and the secondary also an oblate spheroid under the assumption that the eccentricity of the orbit of the gravitating bodies is small. The radiation parameters of the primary affect the motion of the infinitesimal body but not the motion of the secondary due to its large mass.

Furthermore, Narayan et al., (2015) studied the effects of oblateness and radiation of both primaries on the stability of the infinitesimal mass within the framework of ER3BP using some binary system, the region stability changed with variations in eccentricity, oblateness and radiation pressures. It was also observed that the equilibrium points remain stable in the shaded region of the transition curves and unstable in the region outside the shade portion, for the binary system (Archid I and Luyten II).

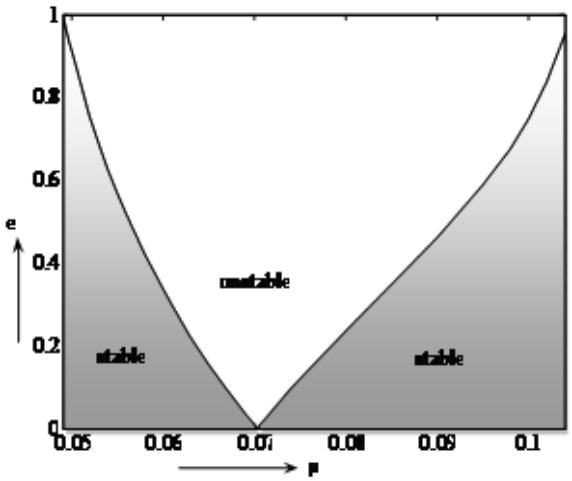


Figure 2. 1: Image showing Transition curve of Archid - I

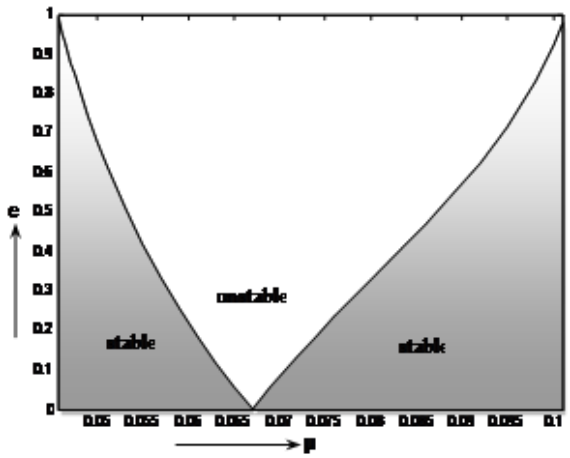


Figure 2. 2: Image showing Transition curve of Luyten - II

Singh and Omale (2014) considered the effects of zonal harmonics on the out of plane equilibrium points of Robe's CR3BP when the hydrostatic equilibrium shape of the primary is an oblate spheroid, the shape of the secondary is an oblate spheroid up to the second zonal harmonics and full buoyancy of the fluid is considered. The size of oblateness and zonal harmonics affect the positions of the out -of –plane equilibrium points L_6 and L_7 and they appear to be unstable within the possible region of motion.

2.1 Binary Systems

Most single points of lights that we see visibly at night in the sky are combination of two or more stars orbiting together, the most common of them are binary stars, i.e. systems of only two stars together. The term binary was first used by Sir William Herschel (1802), where he listed about 700 pairs of stars with reference to double stars. He said that two stars should be placed close to each other and at the same time separated such that they will not be affected by the attractions of the neighboring stars, but compose separate system, and remain united by the bond of their own gravitation towards each other, called a real double star.Thus, binary systems are systems of two objects in space (usually stars, also brown dwarfs, planets, neutron stars, black holes,

galaxies or asteroids) which are close enough that their gravitational movement causes them to orbit around a shared mass.

2.2 Binary Classifications

Binary star is a star system made up two stars that orbit around where mass is most concentrated. There are several subcategories of binary stars, based on their visual properties. Visually viewed by observation; spectroscopically, by periodic changes in spectral lines; photometrically, by changes in brightness caused by an eclipse; or astrometrically, by measuring deviation on stars' position caused by an unseen companion. They are denoted by suffixes *A* and *B* attached to the system's keyword, *A* denotes the primary while *B* denotes the secondary, but suffix *AB* are used for system (e.g. the binary star α Centauri AB consists of the stars α Centauri A and α Centauri B) and letters *C*, D, etc. are used for systems with more than two stars.

Astrophysicists find binary systems to be very important in determining the mass of the individual stars involved by using Newton's calculation for gravity and also other stellar parameters, such as radius and density, are approximated indirectly and empirical mass-luminosity (*MLR*). The relative mass of similar single stars is deduced from the data obtained from binary stars.

2.2.1 Zeta Cygni (ζ Cyg)

Zeta Cygni is a binary star system belonging to the northern constellation of Cygnus, the swan, having an overabundance of barium, as well as other heavy chemical elements in the atmosphere and sometimes called "***mild***" barium star. It hosts a high mass bodyand has an apparent visual magnitude of 3.26 and about 143 light-years (44 parsecs) away with color-index of +0.98. It has two components the primary and

secondary. The primary component, (***ζ Cyg A***) is a giant star with a spectral type of ***G8 IIIp***, with mass of 3.05$M_{\odot}$, radius of 15$R_{\odot}$, brightness of 112$L_{\odot}$, surface gravity of 2.41cgs and temperature of 4,910Kat this temperature, the star glows with yellow hue of a G-type star, while the secondary component, (***ζ Cyg B***), is a white dwarf of type ***DA4.2*** with mass 0.6$M_{\odot}$ and temperature of 12,000k, the white dwarf cannot be seen directly and estimated to have an apparent magnitude of 13.2.The pair orbit each other every 6,489 days with an eccentricity of 0.22 and semi-major axis of 0.19.

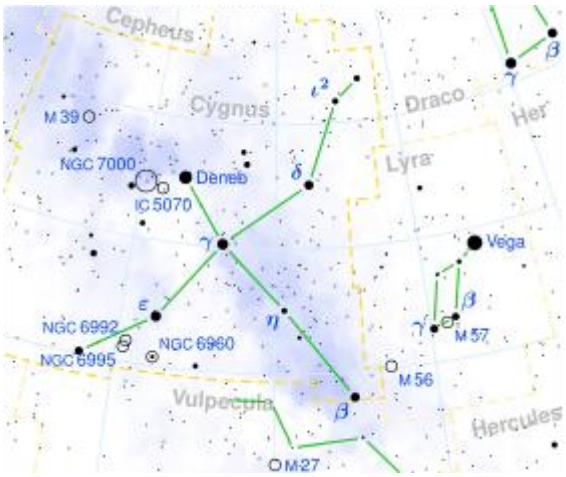


Figure 2. 3: Image showing location of Zeta Cygni between *NGC 6992* **and** *NGC 6995*

2.2.2 Procyon

Astronomers were able to discover that Procyon is not alone in space; but has small companion dubbed Procyon B as such Procyon is classified as binary star system, the most luminous star belonging to the Canis Constellation, and the eighth luminous star, having an apparent magnitude of 0.34,hosting a high mass body. In 1844 it was observed that Procyon B have great influence on the larger companion, around 1896 Procyon B was viewed to be very hard to spot because it's 15,000 times fainter than ***Procyon A***.

Thus, ***Procyon A*** is a bright component with an apparent magnitude of 0.34 consisting of white main-sequence star of spectral type F5 IV–V with mass of $1.499M_{\odot}$, radius of $2.048R_{\odot}$, luminosity of $6.93L_{\odot}$, the effective temperature of the stellar atmosphere is estimated to be 6,530K and age at 1.87Gyr, while ***Procyon B*** is a faint white dwarf companion of spectral type DQZ with mass of $0.602M_{\odot}$, radius of $0.01234R_{\odot}$, luminosity of $0.00049L_{\odot}$, the effective temperature of the stellar atmosphere is estimated to be 7,740K and age at 1.37Gyr and assumed to have carbon core. It has an orbital period of 40.82 years along an elliptical orbit with an eccentricity of 0.407 and Semi-Major axis of 4.3". The plane of their orbit is inclined at an angle of 31.1° to the line of sight with the Earth. The average separation of the two components is 15.0 AU, a little less than the distance between Uranus and the Sun, though the eccentric orbit carries them as close as 8.9 AU and as far as 21.0 AU.

The brightness of Procyon is due to its closeness to the sun and lies at a distance of 11.46 light – years (3.51 parsecs) and the nearest stellar neigbhours because which means you don't need optical aids to see it blazing in the sky. The prime period for viewing it is in late winter in the evening, culminating at midnight around January 14, forming one of the three vertices of the Winter Triangleasterism, in combination with Sirius and Betelgeuse. It has a color index of 0.42, and its hue has been described as having a faint yellow tinge to it. Its closest neighboring star is Luyten's Star, about 1.12 ly (0.34 pc) away, and the latter would appear as a visual magnitude 2.7 star in the night sky of a hypothetical planet orbiting Procyon.

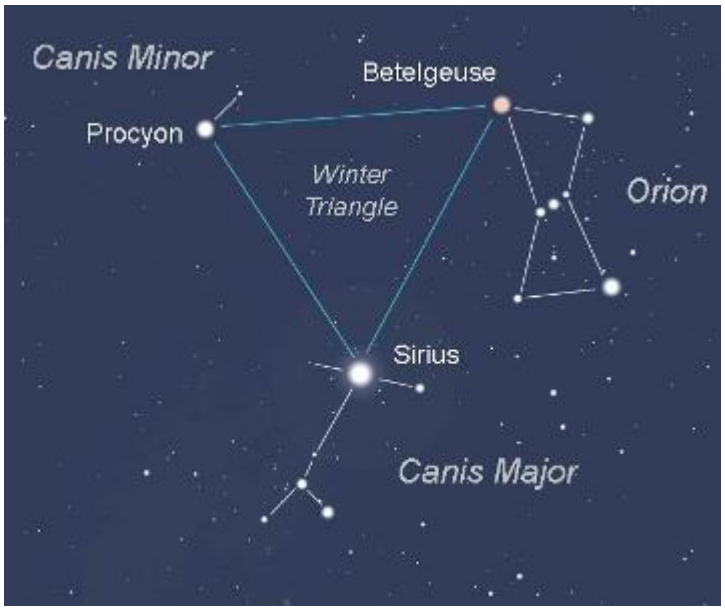


Figure 2. 4: Image showing positon of Procyon from the winter triangle

2.2.3 54 Piscium

It originated from the star catalogue of British astronomer John Flamsteed published in 1712 and listed by Greek astronomer "Ptolemy" in the 2nd century to belong to one of the largest Constellation called Pisces, which is the 14th constellation in size occupying area of 889 square degrees from the Babylonian Origin, lying between Aries Constellation to the east and Aquarius to the west, located in the first quadrant ofthe northern hemisphere (NQ1), at latitudes +90^{o} and -65^{o}.

It is the 6th magnitude star in the constellation professionally referred to as HD 3651, classified as an orange dwarf star belonging to spectral class KOV with luminosity class V indicating that it's a main sequence star generating energy at its core through thermonuclear fusion of hydrogen into helium, the value was revealed to be [Fe/H], 0.12 dex (Santos *et al*. 2004), whereas (Cenarro *et al*.2007) value published it as –0.15 dex, less massive and luminous than the sun with approximately 36 light-years away, an apparent magnitude of 5.86 which can be viewed without the use of any aid suitable for viewing. The star has a reputation for variability. It has been calculated to have

76%$M_\odot$ (0.76), it shines with 52%$L_\odot$ (0.52), the radius has been directly determined by interferometry to be 94%$R_\odot$(0.52), surface gravity of 4.61cgs, due to its brightness (it is called HR 166) giving it a magnitude of 5.87, have a fairly long rotational period of 48 days compared to Sol with greater age (Noyes et al, 1984),54 Piscium has been estimated to be 5.1 to 8.2 +4.3/-5.2 billion years old based on chromospheric activity and isochronal analysis (Luhman *et al*,2006; Saffel *et al*, 2005; Valenti and Fisher, 2005; and Wright *et al*, 2004). The effective temperature of the photosphere is about 5,062K, givin it the characteristics orange the of a K-type star.

Around January 16, 2002, a team of astronomers led by (Geoff Macy including Debra A. Fischer, Geoffrey W. Macy, R. Paul Butler, Steven S. Vogt and Ggregory W. Henry (2003, 2006)) announced the discovery of an extrasolar planet confirmed to be orbiting the star, between August 24 and September 18, 2006, two teams of astronomers announced the discovery to be direct imaging of a low mass substar companion "methane brown dwarf" companion called 54 Psc B or HD 3651 B (Luhman *et al*, 2006; and Mugrauer *et al*, 2006), with spectral type "T7.5V", located at 43 seconds arc away, it's mass is estimated to be at a minimum of 20 percent that of Jupiter (exoplanets.org; and Fischer *et al*, 2003) with radius of 0.082$R_\odot$ similar to Gliese 570 D. The planet moves around 54 Piscium at an average separation of 0.28 AUs Units (42.6 million kilometers, 26.5 million miles), or 73 percent of the distance of Mercury from the Sun. Its orbit around the star at a high eccentricity (e= 0.63 +/- 0.04) and takes about 62.2 days to complete making it a "hot Saturn". The luminosity of this sub-stellar object suggests that it has a mass of 0.051 $M_\odot$, have a surface temperature of about 810 K (537 °C), its low temperature makes it to be of low mass, therefore it is invisible to the eye, and can be seen only in the infrared part of the spectrum. The system thus shows more of the great variety of other planetary systems.

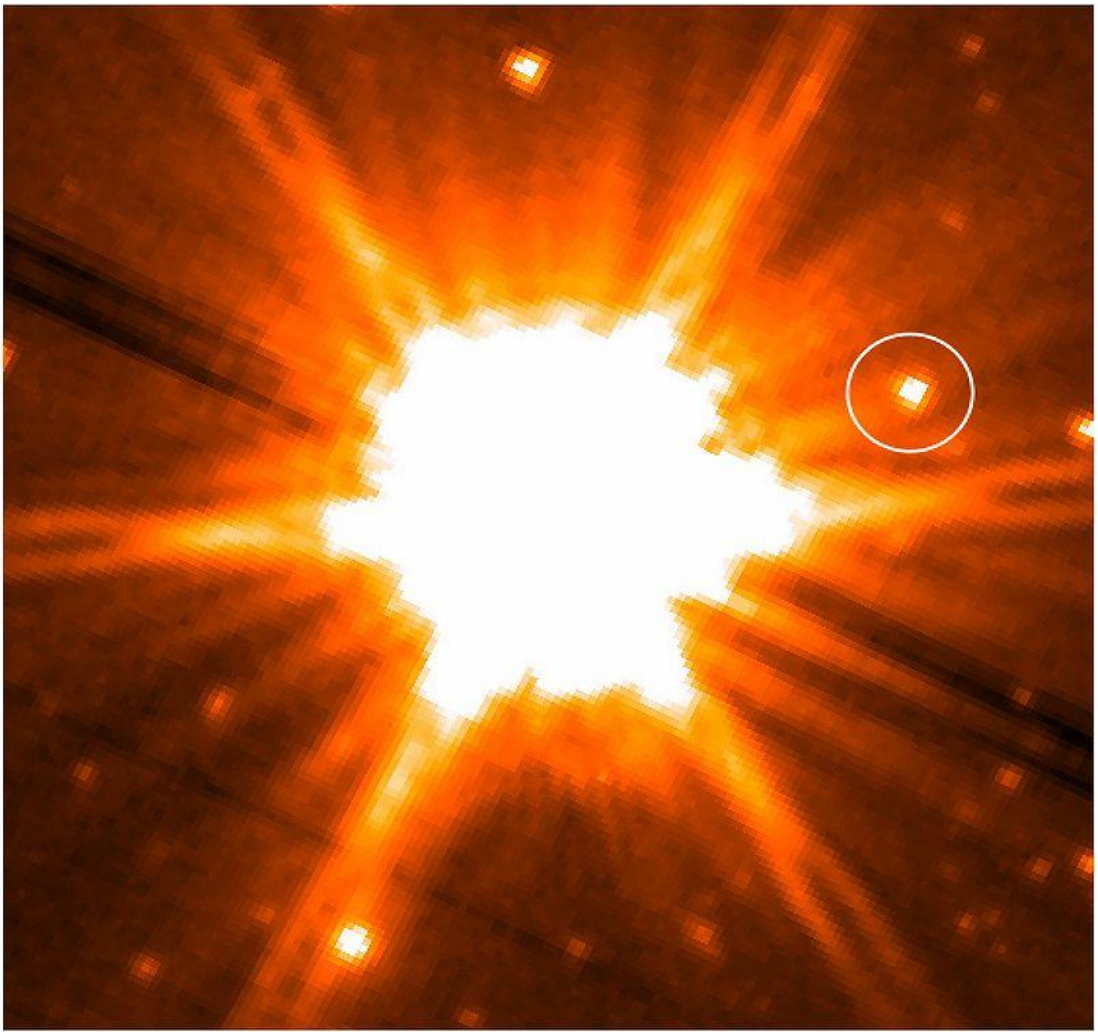


Figure 2. 5: Image showing 54 Piscium A and the brown dwarf 54 Piscium B (circled).

Table 2. 1: The 54 Piscium Planetray System

THE 54 PISCIUM PLANETRAY SYSTEM						
Companion	Mass	Semi-Major axis (AU)	Orbital Period	Eccentricity	Inclination	Radius
54 Piscium b	$> 0.227 \pm 0.023\ M_j$	0.296 ± 0.017	62.206 ± 0.021	$0.618 \pm 0.051 M_j$	-	-

CHAPTER THREE

LOCATION OF EQUILIBRIUM POINTS

3.1 INTRODUCTION

The photogravitational restricted three-body problem is another modification of the classical restricted three-body problem which in which the repulsive force of the radiation is considered in the potential function, it was initiated for three specific-body problem of Sun, planet, and a dust particle (Radzievskii 1950).Equations of motion of a test particle moving under the gravitational effect of two finite masses called the primaries, moving in elliptical orbits around their common center of mass/barycenter but does not influence the motion of the primaries is used in this study. Thus, the study of Singh and Umar (2012a) on elliptical photogravitational R3BP when the primary is an oblate spheroid and the secondary is luminous is extended, on the assumption that the primary is luminous and secondanry is an oblate spheroid up to J_4 (octupolar mass moment).

In this chapter three different aspect are considered, the first section gives a brief introduction about the chapter, the equation of motions of the problem in consideration are given in the second section, while the last section locates the position of equilibrium points. The last section is sub-divided into two as:- (i) position of collinear equilibrium points $L_1, L_2,$ and L_3 (ii) position of triangular points L_4 and L_5 for the problem in question.

3.2 Equation of Motion

The elliptic restricted three-body problem (ER3BP) considers the motion of a test particle (m_3) in the presence of the gravitational field of two primary masses moving in elliptic orbits about their common center of mass, under the assumption that the massive

body is luminous and the less is an oblate spheroid up to zonal harmonic J_4. The elliptic restricted three-body problem shares a common value of eccentricity, *e*. We assume that the coordinates of the test particle, the primary and secondary masses in a synodic-pulsating dimensionless frame are $(\xi,\ \eta)$, (ξ_1, η_1) and (ξ_2, η_2) respectively. Hence, the equations of motion for the test particle (m_3) in this system following (Singh and Umar 2012b)and (Singh and Taura 2014a) can be written as

$$\xi'' - 2\eta' = \Omega_\xi \ ;$$

$$\eta'' + 2\xi' = \Omega_\eta ;$$

$$\zeta'' = \Omega_\zeta \tag{3.1}$$

The force function is

$$\Omega = \frac{1}{(1-e^2)^{½}} \left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{A_2}{2r_2^3} - \frac{3\mu B_2}{8r_2^5}\right\}\right] \tag{3.2}$$

and the mean motion, n is given by

$$\boldsymbol{n^2} = \frac{(1+e^2)^{\frac{1}{2}}}{a\,(1-e^2)}\left(1 + \frac{3}{2}A_2 - \frac{15}{8}B_2\right) \tag{3.3}$$

The distances of the test particle from the primaries are

$$r_i^2 = (\xi - \xi_i)^2 + \eta^2 + \zeta^2 \quad \text{(i=1,2)} \tag{3.4}$$

Where; $\xi_1 = -\mu$,$\xi_2 = 1 - \mu$ (3.5)

3.3 Positions of Equilibrium Points

The equilibrium points of the problem are those points at which the velocity and acceleration of an infinitesimal mass are zero. Therefore, they are solutions of the equations;

$$\frac{\delta\Omega}{\delta\xi} = \frac{\delta\Omega}{\delta\eta} = \frac{\delta\Omega}{\delta\zeta} = 0 \tag{3.6}$$

where;

$$\Omega = \frac{1}{(1-e^2)^{½}} \left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} - \frac{3\mu B_2}{8r_2^5}\right\}\right] \tag{3.7}$$

3.3.1 Position of Collinear Libration Points

The Lagrangian point named after Joseph-Louis Lagrange in the eighteenth century is a location in space where combined gravitational forces of two large bodies, for example, Sun-Earth/Earth-Moon equal the centrifugal forces felt by the third body. Lagrange showed that five relative equilibrium solutions or libration points (three collinear and two triangular) exist for the CR3BP. The collinear libration points L_1, L_2 and L_3exist on a straight line joining the two primaries and all unstable points. Hence, the locations of the three (L_1, L_2, L_3) collinear libration points, are computed as shown below.

Recall that

$$\Omega = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} - \frac{3\mu B_2}{8r_2^5}\right\}\right] \tag{3.8}$$

Differentiating (3.2) to obtain w.r.t ξ, η, ζ respectively to obtain

$$\Omega_\xi = \frac{1}{(1-e^2)^{½}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1(\xi-\xi_1)}{r_1^3} + \frac{\mu(\xi-\xi_2)}{r_2^3} + \frac{3\mu A_2(\xi-\xi_2)}{2r_2^5} - \frac{15\mu B_2(\xi-\xi_2)}{8r_2^7}\right\}\right] = 0 \tag{3.9}$$

$$\Omega_\eta = \frac{1}{(1-e^2)^{½}}\left[\eta - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1\eta}{r_1^3} + \frac{\mu\,\eta}{r_2^3} + \frac{3\mu A_2\eta}{2r_2^5} - \frac{15\mu\, B_2\eta}{8r_2^7}\right\}\right] = 0 \tag{3.10}$$

$$\Omega_\zeta = \zeta\left[\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu\, B_2}{8r_2^7}\right] = 0 \tag{3.11}$$

where;

$$r_1^2 = (\xi - \xi_1)^2 + \eta^2 + \zeta^2$$

$$r_2^2 = (\xi - \xi_2)^2 + \eta^2 + \zeta^2 \tag{3.12}$$

To obtain the collinear equilibrium points, we put η=ζ= 0 in (3.9) above, since the collinear points lie on the line joining the two finite masses, thus we have;

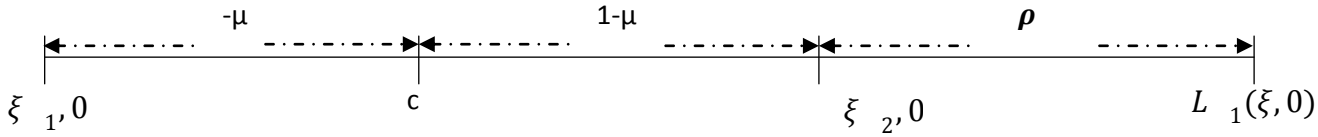
$$\frac{1}{(1-e^2)^{½}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1(\xi-\xi_1)}{r_1^3} + \frac{\mu(\xi-\xi_2)}{r_2^3} + \frac{3\mu A_2(\xi-\xi_2)}{2r_2^5} - \frac{15\mu B_2(\xi-\xi_2)}{8r_2^7}\right\}\right] = 0 \qquad (3.13)$$

$$n^2\xi - \left\{\frac{(1-\mu)q_1(\xi+\mu)}{(\xi+\mu)^3} + \frac{\mu(\xi-1+\mu)}{(\xi-1+\mu)^3} + \frac{3\mu A_2(\xi-1+\mu)}{2(\xi-1+\mu)^5} - \frac{15\mu B_2(\xi-1+\mu)}{8(\xi-1+\mu)^7}\right\} = 0 \qquad (3.14)$$

The collinear points on the ξ-axis are obtained by dividing the orbital plane into three parts: $\xi>\xi_2$, $\xi_1<\xi<\xi_2$ & $\xi_1>\xi$ with respect to their primaries.

POSITION OF COLLINEAR POINT L_1 ($\xi>\xi_2$)

Let the collinear point L_1 be on the right hand side of the smaller primary at a distance ρ from it on the ξ-axis. Then



$$n^2(\rho+1-\mu) - \left\{\frac{(1-\mu)q_1}{|\rho+1|^2} + \frac{\mu}{|\rho|^2} + \frac{3\mu A_2}{2|\rho|^4} - \frac{15\mu B_2}{8|\rho|^6}\right\} = 0 \qquad (3.15)$$

Multiply (3.15) through by $8(\rho+1)^2(\rho)^6$

$$\frac{8n^2(\rho+1)^2(\rho)^6(\rho+1-\mu) - 8(1-\mu)q_1(\rho)^6 - 8\mu(\rho+1)^2(\rho)^4 - 12\mu(\rho+1)^2(\rho)^2A_2 + 15\mu(\rho+1)^2B_2}{8(\rho+1)^2(\rho)^6} = 0 \qquad (3.16)$$

Clearing the fraction to obtain a linear equation

$$8n^2\rho^9 + 8n^2(3-\mu)\rho^8 + 8n^2[3-2\mu]\rho^7 + 8[n^2(1-\mu) - \mu - (1-\mu)q_1]\rho^6 - 16\mu\rho^5 - (8\mu + 12\mu A_2)\rho^4 - 24\mu\rho^3A_2 - (12\mu A_2 - 15\mu B_2)\rho^2 + 30\mu\rho B_2 + 15\mu B_2 = 0 \qquad (3.17)$$

For Verification, let n=1, q=1 and $A_2 = B_2 = 0$, the equation becomes

$$8\rho^9 + 8(3-\mu)\rho^8 + 8(3-2\mu)\rho^7 + 8(-\mu)\rho^6 - 16\mu\rho^5 - 8\mu\rho^4 = 0$$

$$\rho^9 + (3-\mu)\rho^8 + (3-2\mu)\rho^7 - \mu\rho^6 - 2\mu\rho^5 - \mu\rho^4 = 0 \qquad (3.18)$$

This is a ninth degree equation which verify the classical case, let assume that the solution of the classical case is Υ_1,when $e = A_2 = B_2 = 0$ and q_1=1.The introduction of eccentricity term(e), oblateness and octupolar mass moment (J_4) co-efficient and radiation factor q_1 will change the solution slightly by δ_1.

The new value of ρ can be defined as: -

The new value of ρ changes from Υ_1 in the classical case to $\Upsilon_1 + \delta_1$ can be defined as:-

$\rho = \Upsilon_1 + \delta_1$and q_1=1- β_1; where $\beta_1 \ll 1$

Substituting ρ= $\Upsilon_1 + \delta_1$into (3.17) above;

$$8n^2(\Upsilon_1 + \delta_1)^9 + 8n^2(3-\mu)(\Upsilon_1 + \delta_1)^8 + 8n^2[3-2\mu](\Upsilon_1 + \delta_1)^7 + 8[n^2(1-\mu) - \mu - (1-\mu)q_1](\Upsilon_1 + \delta_1)^6 - 16\mu(\Upsilon_1 + \delta_1)^5 - (8\mu + 12\mu A_2)(\Upsilon_1 + \delta_1)^4 - 24\mu(\Upsilon_1 + \delta_1)^3 A_2 - (12\mu A_2 - 15\mu B_2)(\Upsilon_1 + \delta_1)^2 + 30\mu(\Upsilon_1 + \delta_1)B_2 + 15\mu B_2 = 0 \quad (3.19)$$

Factor out $\delta_1$and re-written as;

$$\delta_1\left[72n^2{\Upsilon_1}^8 + 64n^2{\Upsilon_1}^7 + 56n^2(3-2\mu){\Upsilon_1}^6 + 6[8n^2(1-\mu) - 8]{\Upsilon_1}^5 - 80\mu{\Upsilon_1}^4 - 32\mu\Upsilon 13 + 481 - \mu\beta 1\Upsilon 15 - 24\mu\Upsilon 11 + 3\Upsilon 1 + 2\Upsilon 12\text{A2} + 30\mu(\Upsilon 1 + 1)\text{B2} = -8n2\Upsilon 19 - 8n23 - \mu\Upsilon 18 - 8n23 - 2\mu\Upsilon 17 - 8[n^2(1-\mu) - \mu]{\Upsilon_1}^6 + 16\mu{\Upsilon_1}^5 + 8\mu{\Upsilon_1}^4 - 8(1-\mu)\beta_1{\Upsilon_1}^6 + 12\mu{\Upsilon_1}^2(1+\Upsilon_1)^2 A_2 - 15\mu(1+\Upsilon_1)^2 B_2 \quad (3.20)$$

$$\therefore \delta_1[P_1 + Q_1\beta_1 + R_1A_2 + G_1B_2] = [L_1 + M_1\beta_1 + N_1A_2 + D_1B_2] \quad (3.21)$$

Where;

$$P_1 = 72n^2{\Upsilon_1}^8 + 64n^2{\Upsilon_1}^7 + 56n^2(3-2\mu){\Upsilon_1}^6 + 6[8n^2(1-\mu) - 8]{\Upsilon_1}^5 - 80\mu{\Upsilon_1}^4 - 32\mu{\Upsilon_1}^3$$

$$Q_1 = 48(1-\mu)\beta_1 {\Upsilon_1}^5;\ R_1 = -24\mu\Upsilon_1\left(1+3\Upsilon_1+2{\Upsilon_1}^2\right)\mathrm{A}_2;G_1 = 30\mu(\Upsilon_1+1)\mathrm{B}_2$$

$$L_1 = -8n^2{\Upsilon_1}^9 - 8n^2(3-\mu){\Upsilon_1}^8 - 8n^2(3-2\mu){\Upsilon_1}^7 - 8[n^2(1-\mu)-1]{\Upsilon_1}^6 + 16\mu{\Upsilon_1}^5 + 8\mu{\Upsilon_1}^4$$

$$M_1 = -8(1-\mu)\beta_1{\Upsilon_1}^6;N_1 = 12\mu{\Upsilon_1}^2(1+\Upsilon_1)^2\mathrm{A}_2;\ D_1 = -15\mu(1+\Upsilon_1)^2\mathrm{B}_2 \qquad (3.22)$$

From (3.21) above, we have

$$\delta_1 = \frac{1}{P_1}[L_1 + M_1\beta_1 + N_1A_2 + D_1B_2][1 - \frac{Q_1}{P_1}\beta_1 - \frac{R_1}{P_1}A_2 - \frac{G_1}{P_1}B_2]$$

Expanding $\delta_1$binomially and neglecting higher order terms;

$$\delta_1 = {P_1}^{-1}\left[L_1 + (M_1 - \frac{L_1Q_1}{P_1})\beta_1 + (N_1 - \frac{L_1R_1}{P_1})A_2 + (D_1 - \frac{L_1G_1}{P_1})B_2\right] \qquad (3.23)$$

Recall that the value of $\rho = \Upsilon_1 + \delta_1$, hence

$$\rho = \Upsilon_1 + {P_1}^{-1}\left[L_1 + (M_1 - \frac{L_1Q_1}{P_1})\beta_1 + (N_1 - \frac{L_1R_1}{P_1})A_2 + (D_1 - \frac{L_1G_1}{P_1})B_2\right] \qquad (3.24)$$

Where$\Upsilon_1$is the value of P_1

Recall that;

$$n^2 = \frac{(1+e^2)^{\frac{1}{2}}}{a(1-e^2)}[1 + \frac{3}{2}A_2 - \frac{15}{8}B_2]$$

Solving binomially and neglecting higher order terms.

$$n^2 = \frac{1}{a}[1 + \frac{3e^2}{2} + \frac{3}{2}A_2 - \frac{15}{8}B_2];\ n^2 = \frac{8+12e^2+12A_2-15B_2}{8a} \qquad (3.25)$$

From (3.22) above,

$$P_1 = 72n^2{\Upsilon_1}^8 + 64n^2(3-\mu){\Upsilon_1}^7 + 56n^2(3-2\mu){\Upsilon_1}^6 + 6[8n^2(1-\mu)-8]{\Upsilon_1}^5 - 80\mu{\Upsilon_1}^4 - 32\mu{\Upsilon_1}^3$$

Substituting (3.25) intoP_1;

$$P_1 = 72\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right]\Upsilon_1{}^8 + 64\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right](3-\mu)\Upsilon_1{}^7$$
$$+ 56\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right](3-2\mu)\Upsilon_1{}^6$$
$$+ 48\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right](1-\mu)\Upsilon_1{}^5 - 48\Upsilon_1{}^5 - 80\mu\Upsilon_1{}^4$$
$$- 32\mu\Upsilon_1{}^3$$

Now;

$$P_1 = \tfrac{72}{a}\Upsilon_1{}^8 + \tfrac{64(3-\mu)}{a}\Upsilon_1{}^7 + \tfrac{56(3-2\mu)}{a}\Upsilon_1{}^6 + \tfrac{48(1-\mu)}{a}\Upsilon_1{}^5 - 48\Upsilon_1{}^5 - 80\mu\Upsilon_1{}^4 -$$
$$32\mu\Upsilon_1{}^3 + e^2\left\{\tfrac{108(1-\mu)}{a}\Upsilon_1{}^8 + \tfrac{96(3-\mu)}{a}\Upsilon_1{}^7 + \tfrac{84(3-2\mu)}{a}\Upsilon_1{}^6 + \tfrac{72(1-\mu)}{a}\Upsilon_1{}^5\right\} +$$
$$A_2\left\{\tfrac{108(1-\mu)}{a}\Upsilon_1{}^8 + \tfrac{96(3-\mu)}{a}\Upsilon_1{}^7 + \tfrac{84(3-2\mu)}{a}\Upsilon_1{}^6 + \tfrac{72(1-\mu)}{a}A_2\Upsilon_1{}^5\right\} + B_2\{-\tfrac{135(1-\mu)}{a}\Upsilon_1{}^8 -$$
$$\tfrac{120(3-\mu)}{a}\Upsilon_1{}^7 - \tfrac{105(3-2\mu)}{a}\Upsilon_1{}^6 - \tfrac{90(1-\mu)}{a}\Upsilon_1{}^5\} \qquad (3.26)$$

Hence P_1 can written as

$$P_1 = a_1 + b_1e^2 + c_1A_2 + d_1B_2 \ ;$$

Where;

$$a_1 = \frac{72}{a}\Upsilon_1{}^8 + \frac{64(3-\mu)}{a}\Upsilon_1{}^7 + \frac{56(3-2\mu)}{a}\Upsilon_1{}^6 + \frac{48(1-\mu)}{a}\Upsilon_1{}^5 - 48\Upsilon_1{}^5 - 80\mu\Upsilon_1{}^4$$
$$- 32\mu\Upsilon_1{}^3$$

$$b_1 = c_1 = \frac{108(1-\mu)}{a}\Upsilon_1{}^8 + \frac{96(3-\mu)}{a}\Upsilon_1{}^7 + \frac{84(3-2\mu)}{a}\Upsilon_1{}^6 + \frac{72(1-\mu)}{a}\Upsilon_1{}^5$$

$$d_1 = -\frac{135(1-\mu)}{a}{\Upsilon_1}^8 - \frac{120(3-\mu)}{a}{\Upsilon_1}^7 - \frac{105(3-2\mu)}{a}{\Upsilon_1}^6 - \frac{90(1-\mu)}{a}{\Upsilon_1}^5$$

From (3.24) we have ${P_1}^{-1}$;

$${P_1}^{-1} = a^{-1}[1 - \frac{b_1}{a_1}e^2 - \frac{c_1}{a_1}A_2 - \frac{d_1}{a_1}B_2]$$

Where;

$E_1 = \frac{b_1}{a_1} \& F_1 = \frac{c_1}{a_1} \& S_1 = \frac{d_1}{a}$

So that;

$$b_1 = \ c_1 = \frac{108(1-\mu)}{a}{\Upsilon_1}^8 + \frac{96(3-\mu)}{a}{\Upsilon_1}^7 + \frac{84(3-2\mu)}{a}{\Upsilon_1}^6 + \frac{72(1-\mu)}{a}{\Upsilon_1}^5$$

$$S_1 = -\frac{135(1-\mu)}{a}{\Upsilon_1}^8 - \frac{120(3-\mu)}{a}{\Upsilon_1}^7 - \frac{105(3-2\mu)}{a}{\Upsilon_1}^6 - \frac{90(1-\mu)}{a}{\Upsilon_1}^5$$

$$\therefore a_1 = [\frac{72}{a}{\Upsilon_1}^8 + \frac{64(3-\mu)}{a}{\Upsilon_1}^7 + \frac{56(3-2\mu)}{a}{\Upsilon_1}^6 + \frac{48(1-\mu)}{a}{\Upsilon_1}^5 - 48{\Upsilon_1}^5 - 80\mu{\Upsilon_1}^4 - 32\mu{\Upsilon_1}^3]^{-1}$$

$$E_1 = \frac{\frac{108(1-\mu)}{a}{\Upsilon_1}^8 + \frac{96(3-\mu)}{a}{\Upsilon_1}^7 + \frac{84(3-2\mu)}{a}{\Upsilon_1}^6 + \frac{72(1-\mu)}{a}{\Upsilon_1}^5}{\frac{72}{a}{\Upsilon_1}^8 + \frac{64(3-\mu)}{a}{\Upsilon_1}^7 + \frac{56(3-2\mu)}{a}{\Upsilon_1}^6 + \frac{48(1-\mu)}{a}{\Upsilon_1}^5 - 48{\Upsilon_1}^5 - 80\mu{\Upsilon_1}^4 - 32\mu{\Upsilon_1}^3}$$

$$S_1 = \frac{-\frac{135(1-\mu)}{a}{\Upsilon_1}^8 - \frac{120(3-\mu)}{a}{\Upsilon_1}^7 - \frac{105(3-2\mu)}{a}{\Upsilon_1}^6 - \frac{90(1-\mu)}{a}{\Upsilon_1}^5}{\frac{72}{a}{\Upsilon_1}^8 + \frac{64(3-\mu)}{a}{\Upsilon_1}^7 + \frac{56(3-2\mu)}{a}{\Upsilon_1}^6 + \frac{48(1-\mu)}{a}{\Upsilon_1}^5 - 48{\Upsilon_1}^5 - 80\mu{\Upsilon_1}^4 - 32\mu{\Upsilon_1}^3}$$

$$\therefore P_1^{-1} = a_1 + E_1e^2 + F_1A_2 + S_1B_2 \qquad (3.27)$$

From (3.22);

$$L_1 = -8n^2\Upsilon_1{}^9 - 8n^2(3-\mu)\Upsilon_1{}^8 - 8n^2(3-2\mu)\Upsilon_1{}^7 - 8[n^2(1-\mu)-1]\Upsilon_1{}^6 + 16\mu\Upsilon_1{}^5 + 8\mu\Upsilon_1{}^4$$

Substituting (3.25) intoL_1;

$$L_1 = -8[\frac{8+12e^2+12A_2-15B_2}{8a}]\Upsilon_1{}^9 - 8[\frac{8+12e^2+12A_2-15B_2}{8a}](3-\mu)\Upsilon_1{}^8 - 8[\frac{8+12e^2+12A_2-15B_2}{8a}](3-2\mu)\Upsilon_1{}^7 - 8\left[\{\frac{8+12e^2+12A_2-15B_2}{8a}\}(1-\mu)-1\right]\Upsilon_1{}^6 + 16\mu\Upsilon_1{}^5 + 8\mu\Upsilon_1{}^4$$

$$L_1 = -8[\tfrac{1}{a}\Upsilon_1{}^9 + \tfrac{(3-\mu)}{a}\Upsilon_1{}^8 + \tfrac{(3-2\mu)}{a}\Upsilon_1{}^7 + \tfrac{(1-\mu)}{a}\Upsilon_1{}^6 - \Upsilon_1{}^6 - 2\mu\Upsilon_1{}^5 - \mu\Upsilon_1{}^4] - \tfrac{12}{a}e^2\{\Upsilon_1{}^9 + (3-\mu)\Upsilon_1{}^8 + (3-2\mu)\Upsilon_1{}^7 + (1-\mu)\Upsilon_1{}^6\} - \mathrm{A}_2\{\Upsilon_1{}^9 + (3-\mu)\Upsilon_1{}^8 + (3-2\mu)\Upsilon_1{}^7 + (1-\mu)\Upsilon_1{}^6\} + \tfrac{15}{a}\mathrm{B}_2\{\Upsilon_1{}^9 + (3-\mu)\Upsilon_1{}^8 + (3-2\mu)\Upsilon_1{}^7 + (1-\mu)\Upsilon_1{}^6\}$$

(3.28)

Where;

$$U_1 = -8[\frac{1}{a}\Upsilon_1{}^9 + \frac{(3-\mu)}{a}\Upsilon_1{}^8 + \frac{(3-2\mu)}{a}\Upsilon_1{}^7 + \frac{(1-\mu)}{a}\Upsilon_1{}^6 - \Upsilon_1{}^6 - 2\mu\Upsilon_1{}^5 - \mu\Upsilon_1{}^4$$

$$V_1 = -\frac{12}{a}\{\Upsilon_1{}^9 + (3-\mu)\Upsilon_1{}^8 + (3-2\mu)\Upsilon_1{}^7 + (1-\mu)\Upsilon_1{}^6\}$$

$$W_1 = \frac{15}{a}\mathrm{B}_2\{\Upsilon_1{}^9 + (3-\mu)\Upsilon_1{}^8 + (3-2\mu)\Upsilon_1{}^7 + (1-\mu)\Upsilon_1{}^6\}$$

Recall from (3.24)

$$L_1P_1^{-1} = [U_1 + V_1e^2 + V_1A_2 + W_1B_2][a_1 + E_1e^2 + F_1A_2 + S_1B_2]$$

$$L_1P_1^{-1} = U_1a_1 + e^2(U_1E_1 + V_1a_1) + A_2(U_1F_1 + V_1a_1) + B_2(U_1S_1 + W_1a_1) \qquad (3.29)$$

Now;$M_1 - \frac{Q_1 L_1}{P_1}$; $M_1 = -8(1-\mu)\Upsilon_1^6$

$$\left(M_1 - \frac{Q_1 L_1}{P_1}\right)\beta_1$$

$$= -8(1-\mu)\Upsilon_1^6 - 48(1-\mu)\Upsilon_1^5[U_1 a_1 + e^2(U_1 E_1 + V_1 a_1)$$

$$+ A_2(U_1 F_1 + V_1 a_1) + B_2(U_1 S_1 + W_1 a_1)]$$

$$\left(M_1 - \frac{Q_1 L_1}{P_1}\right)\beta_1 = -8(1-\mu)\Upsilon_1^5(\Upsilon_1 + 6U_1 a_1)\beta_1 \quad (3.30)$$

Similarly; $(N_1 - \frac{R_1 L_1}{P_1})A_2$; where $N_1 = 12\mu{\Upsilon_1}^2(1+\Upsilon_1)^2 A_2$

$$\left(N_1 - \frac{R_1 L_1}{P_1}\right)A_2$$

$$= 12\mu{\Upsilon_1}^2(1+\Upsilon_1)^2 A_2 - 24\mu\Upsilon_1(1 + 3\Upsilon_1 + 2{\Upsilon_1}^2)A_2[U_1 a_1$$

$$+ e^2(U_1 E_1 + V_1 a_1) + A_2(U_1 F_1 + V_1 a_1) + B_2(U_1 S_1 + W_1 a_1)]$$

$$\left(N_1 - \frac{R_1 L_1}{P_1}\right)A_2 = 12\mu\Upsilon_1[\Upsilon_1(1+\Upsilon_1)^2 - 2\mu(1 + 3\Upsilon_1 + 2{\Upsilon_1}^2)(U_1 a_1)]A_2 \quad (3.31)$$

So that; $D_1 - \frac{G_1 L_1}{P_1}$; where$D_1 = -15\mu(1+\Upsilon_1)^2 B_2$

$$(D_1 - \frac{G_1 L_1}{P_1})B_2$$

$$= -15\mu(1+\Upsilon_1)^2 B_2 + 30\mu(\Upsilon_1 + 1)B_2[U_1 a_1 + e^2(U_1 E_1 + V_1 a_1)$$

$$+ A_2(U_1 F_1 + V_1 a_1) + B_2(U_1 S_1 + W_1 a_1)]$$

$$(D_1 - \frac{G_1 L_1}{P_1})B_2 = -15\mu(\Upsilon_1 + 1)[(\Upsilon_1 + 1) - 2U_1 a_1]B_2 \quad (3.32)$$

Recall from (3.24)

$$\rho = \Upsilon_1 + {P_1}^{-1}\left[L_1 + (M_1 - \frac{L_1 Q_1}{P_1})\beta_1 + (N_1 - \frac{L_1 R_1}{P_1})A_2 + (D_1 - \frac{L_1 G_1}{P_1})B_2\right]; \rho = \Upsilon_1 + \delta_1$$

From (3.27);

$$\delta_1 = \frac{1}{P_1}[L_1 + (M_1 - \frac{L_1 Q_1}{P_1})\beta_1 + (N_1 - \frac{L_1 R_1}{P_1})A_2 + (D_1 - \frac{L_1 G_1}{P_1})B_2]$$

Also;$P_1^{-1}\left(M_1 - \frac{Q_1 L_1}{P_1}\right)\beta_1 = [a_1 + E_1 e^2 + F_1 A_2 + S_1 B_2][-8(1-\mu)\Upsilon_1^5(\Upsilon_1 + U_1 a_1)\beta_1]$

$$P_1^{-1}\left(M_1 - \frac{Q_1 L_1}{P_1}\right)\beta_1 = -8(1-\mu)\Upsilon_1^5 a_1(\Upsilon_1 + 6U_1 a_1)\beta_1 \quad (3.33)$$

Then;$P_1^{-1}\left(N_1 - \frac{R_1 L_1}{P_1}\right)A_2 = [a_1 + E_1 e^2 + F_1 A_2 + S_1 B_2]12\mu\Upsilon_1\left[\Upsilon_1(1+\Upsilon_1)^2 -\right.$

2μ1+3Υ1+2Υ12U1a1A2

$$P_1^{-1}\left(N_1 - \frac{R_1 L_1}{P_1}\right)A_2 = 12\mu a_1\Upsilon_1[\Upsilon_1(1+\Upsilon_1)^2 - 2\mu(1+3\Upsilon_1+2{\Upsilon_1}^2)(U_1 a_1)]A_2 \quad (3.34)$$

and;

$$P_1^{-1}\left(D_1 - \frac{G_1 L_1}{P_1}\right)B_2$$

$$= [a_1 + E_1 e^2 + F_1 A_2 + S_1 B_2] - 15\mu(\Upsilon_1 + 1)[(\Upsilon_1 + 1) - 2U_1 a_1]B_2$$

$$P_1^{-1}\left(D_1 - \frac{G_1 L_1}{P_1}\right)B_2 = -15\mu(\Upsilon_1 + 1)a_1[(\Upsilon_1 + 1) - 2U_1 a_1]B_2 \quad (3.35)$$

So;$\delta_1 = \frac{1}{P_1}[L_1 + (M_1 - \frac{L_1 Q_1}{P_1})\beta_1 + (N_1 - \frac{L_1 R_1}{P_1})A_2 + (D_1 - \frac{L_1 G_1}{P_1})B_2]$

Substituting (3.29), (3.35),(3.34) and (3.33) into δ_1; to obtain

$$\begin{aligned}\delta_1 = {} & U_1 a_1 + e^2(U_1 E_1 + V_1 a_1) - 8(1-\mu)\Upsilon_1^5 a_1(\Upsilon_1 + 6U_1 a_1)\beta_1 \\ & + \{(U_1 F_1 + V_1 a_1) \\ & + 12\mu a_1\Upsilon_1\left[\Upsilon_1(1+\Upsilon_1)^2 - 2\mu\left(1+3\Upsilon_1+2{\Upsilon_1}^2\right)(U_1 a_1)\right]\}A_2 \\ & + \{(U_1 S_1 + W_1 a_1) - 15\mu(\Upsilon_1 + 1)a_1[(\Upsilon_1 + 1) - 2U_1 a_1]\}B_2\end{aligned}$$

The value of ρ is define as;

$$\rho = \Upsilon_1 + P_1^{-1}\left[L_1 + (M_1 - \frac{L_1 Q_1}{P_1})\beta_1 + (N_1 - \frac{L_1 R_1}{P_1})A_2 + (D_1 - \frac{L_1 G_1}{P_1})B_2\right]$$

$$\rho = \Upsilon_1 + U_1 a_1 + e^2(U_1 E_1 + V_1 a_1) - 8(1-\mu)\Upsilon_1^5 a_1(\Upsilon_1 + 6U_1 a_1)\beta_1 + \{(U_1 F_1 +$$

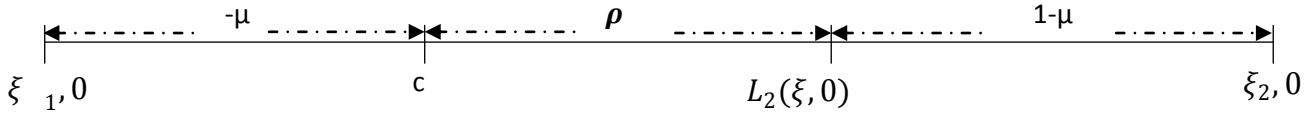
$$V1a1+12\mu a1\Upsilon 1\Upsilon 11+\Upsilon 12-2\mu 1+3\Upsilon 1+2\Upsilon 12U1a1A2+\{U1S1+W1a1-15\mu(\Upsilon 1+1)a$$

$$1[\Upsilon 1+1-2U1a1]\}B2 \qquad (3.36)$$

Thus $\rho = \Upsilon_1 + \delta_1$ is determine in this case;

POSITON OF COLLINEAR POINT L_2 ($\xi_1<\xi<\xi_2$)

Let the collinear point L_2 be on the left hand side of the smaller primary at a distance ρ from it on the ξ-axis.



$$n^2(1-\rho-\mu) - \left\{\frac{(1-\mu)q_1}{|1-\rho|^2} - \frac{\mu}{|\rho|^2} - \frac{3\mu A_2}{2|\rho|^4} - \frac{15\mu B_2}{8|\rho|^6}\right\} = 0$$

Multiply through by $8|\rho|^6|1-\rho|^2$

$$8n^2\rho^9 - 8n^2\{3-\mu\}\rho^8 + 8n^2\{3-2\mu\}\rho^7 - 8\{n^2(1-\mu) - (1-\mu)q_1 + \mu\}\rho^6 +$$

$$16\mu\rho^5 - 4\mu\{3A_2 + 2\}\rho^4 + 24\mu\rho^3 A_2 - 3\mu\{4A_2 - 5B_2\}\rho^2 - 30\mu\rho B_2 + 15\mu B_2 = 0$$

(3.37)

For verification, let $n = 1, q_1 = 1, A_2 = B_2 = 0$,we have;

$$8\rho^9 - 8\{3-\mu\}\rho^8 + 8\{3-2\mu\}\rho^7 - 8\mu\rho^6 + 16\mu\rho^5 - 8\mu\rho^4 = 0$$

$$\therefore \rho^9 - \{3-\mu\}\rho^8 + \{3-2\mu\}\rho^7 - \mu\rho^6 + 2\mu\rho^5 - \mu\rho^4 = 0$$

This is a ninth degree equation which verifies the classical case,

Let Υ_2 be the value ofρ, the distance betweenL_2,

i.e. $A_2 = 0$, $B_2 = 0$, $e = 0$, $a = 1$and $q_1 = 0$.The introduction of eccentricity term(e), oblateness and octupolar mass moment (J_4) co-efficient and radiation factor q_1 will change the solution slightly by δ_2.

Hence, the new value of ρ can be defined as: $\rho = \Upsilon_2 + \delta_2$

Substituting $\rho = \Upsilon_2 + \delta_2$ into (3.37)

$$8n^2(\Upsilon_2 + \delta_2)^9 - 8n^2\{3-\mu\}(\Upsilon_2 + \delta_2)^8 + 8n^2\{3-2\mu\}(\Upsilon_2 + \delta_2)^7 - 8\{n^2(1-\mu) - (1-\mu)q_1 + \mu\}(\Upsilon_2 + \delta_2)^6 + 16\mu(\Upsilon_2 + \delta_2)^5 - 4\mu\{3A_2 + 2\}(\Upsilon_2 + \delta_2)^4 + 24\mu(\Upsilon_2 + \delta_2)^3 A_2 - 3\mu\{4A_2 - 5B_2\}(\Upsilon_2 + \delta_2)^2 - 30\mu(\Upsilon_2 + \delta_2)B_2 + 15\mu B_2 = 0$$

$$\delta_2\big[72n^2\Upsilon_2{}^8 - 64n^2\{3-\mu\}\Upsilon_2{}^7 + 56n^2\{3-2\mu\}\Upsilon_2{}^6 - 48n^2(1-\mu)\Upsilon_2{}^5 - 96\mu\Upsilon_2{}^5 + 48\Upsilon 25 + 80\mu\Upsilon 24 - 32\mu\Upsilon 23 - 48 1 - \mu\beta 1\Upsilon 25 - 24\mu\Upsilon 2(2\mu\Upsilon 22 - 3\Upsilon 2 + 1)A2 + 30\mu(\Upsilon 2 - 1)B2 + 8n2\Upsilon 29 - 8n23 - \mu\Upsilon 28 + 8n23 - 2\mu\Upsilon 27 - 8n21 - \mu\Upsilon 26 + -16\mu\Upsilon 26 + 8\Upsilon 26 + 16\mu\Upsilon 25 - 8\mu\Upsilon 24 - 8\beta 11 - \mu\Upsilon 26 - 12\mu\Upsilon 22\Upsilon 22 - 12A2 + 15\mu\Upsilon 2 - 12B2$$

(3.38)

Where;

$$\delta_2[P_2 + Q_2\beta_1 + R_2A_2 + G_2B_2] = [L_2 + M_2\beta_1 + N_2A_2 + D_2B_2] \quad (3.39)$$

$$P_2 = 72n^2\Upsilon_2{}^8 - 64n^2\{3-\mu\}\Upsilon_2{}^7 + 56n^2\{3-2\mu\}\Upsilon_2{}^6 - 48\{n^2(1-\mu) - 2\square + 1\}\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3$$

$$Q_2 = -48(1-\mu)\beta_1\Upsilon_2{}^5;\ R_2 = -24\mu\Upsilon_2(2\mu\Upsilon_2{}^2 - 3\Upsilon_2 + 1)A_2;\ G_2 = 30\mu(\Upsilon_2 - 1)B_2$$

$$L_2 = 8n^2\Upsilon_2{}^9 - 8n^2\{3-\mu\}\Upsilon_2{}^8 + 8n^2\{3-2\mu\}\Upsilon_2{}^7 - 8[n^2(1-\mu) + 2\text{□} - 1]\Upsilon_2{}^6$$
$$+ 16\mu\Upsilon_2{}^5 - 8\mu\Upsilon_2{}^4$$

$$M_2 = -8\beta_1(1-\mu)\Upsilon_2{}^6;\ N_2 = -12\mu\Upsilon_2{}^2\left(\Upsilon_2{}^2 - 1\right)^2 A_2;\ D_2 = 15\mu(\Upsilon_2 - 1)^2 B_2 \quad (3.40)$$

$$\delta_2 = \frac{1}{P_2}[L_2 + M_2\beta_1 + N_2A_2 + D_2B_2][1 - \frac{Q_2}{P_2}\beta_1 - \frac{R_2}{P_2}A_2 - \frac{G_2}{P_2}A_2]^{-1} \quad (3.41)$$

Expanding δ_2 binomially and neglecting higher order terms;

$$\delta_2 = P_2{}^{-1}[L_2 + (M_2 - \frac{Q_2L_2}{P_2})\beta_1 + (N_2 - \frac{R_2L_2}{P_2})A_2 + (D_2 - \frac{G_2L_2}{P_2})B_2] \quad (3.42)$$

Recall that $\rho = \Upsilon_2 + \delta_2$

$$\therefore \rho = \Upsilon_2 + P_2{}^{-1}[L_2 + (M_2 - \frac{Q_2L_2}{P_2})\beta_1 + (N_2 - \frac{R_2L_2}{P_2})A_2 + (D_2 - \frac{G_2L_2}{P_2})B_2] \quad (3.43)$$

recall $n^2 = \frac{8+12e^2+12A_2-15B_2}{8a}$

substituting (3.25) into (3.40) above

$$P_2 = 72\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right]\Upsilon_2{}^8 - 64\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right]\{3-\mu\}\Upsilon_2{}^7$$
$$+ 56\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right]\{3-2\mu\}\Upsilon_2{}^6$$
$$- 48\left\{\left[\frac{8+12e^2+12A_2-15B_2}{8a}\right](1-\mu) - 2\text{□} + 1\right\}\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4$$
$$- 32\mu\Upsilon_2{}^3$$

Now;

$$P_2 = \frac{72}{a}\Upsilon_2{}^8 - \frac{64}{a}(3-\mu)\Upsilon_2{}^7 + \frac{56}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{48}{a}(1-\mu)\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 +$$
$$80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3 + e^2\left\{\frac{108}{a}\Upsilon_2{}^8 - \frac{96}{a}(3-\mu)\Upsilon_2{}^7 + \frac{84}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{72}{a}(1-\right.$$

$$\mu)\Upsilon_2{}^5\Big\} + A_2\Big\{\frac{108}{a}\Upsilon_2{}^8 - \frac{96}{a}(3-\mu)\Upsilon_2{}^7 + \frac{84}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{72}{a}(1-\mu)A_2\Upsilon_2{}^5\Big\} +$$

$$B_2\{-\frac{135}{a}\Upsilon_2{}^8 + \frac{120}{a}(3-\mu)\Upsilon_2{}^7 - \frac{105}{a}(3-2\mu)\Upsilon_2{}^6 + \frac{90}{a}(1-\mu)\Upsilon_2{}^5\} \quad (3.44)$$

Where

$$a_2 = \frac{72}{a}\Upsilon_2{}^8 - \frac{64}{a}(3-\mu)\Upsilon_2{}^7 + \frac{56}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{48}{a}(1-\mu)\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3$$

$$b_2 = c_2 = \frac{108}{a}\Upsilon_2{}^8 - \frac{96}{a}(3-\mu)\Upsilon_2{}^7 + \frac{84}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{72}{a}(1-\mu)\Upsilon_2{}^5$$

$$d_2 = -\frac{135}{a}\Upsilon_2{}^8 + \frac{120}{a}(3-\mu)\Upsilon_2{}^7 - \frac{105}{a}(3-2\mu)B_2\Upsilon_2{}^6 + \frac{90}{a}(1-\mu)B_2\Upsilon_2{}^5$$

Thus P_2 can be re-written as

$$P_2 = [a_2 + b_2e^2 + C_2A_2 + d_2B_2]$$

Hence; $\frac{1}{P_2} \equiv P_2^{-1} = [a_2 + b_2e^2 + C_2A_2 + d_2B_2]^{-1}$

$$\therefore P_2^{-1} = a_2{}^{-1}[1 - \frac{b_2}{a_2}e^2 - \frac{c_2}{a_2}A_2 - \frac{d_2}{a_2}B_2]$$

So that;

$$P_2^{-1} = [\frac{72}{a}\Upsilon_2{}^8 - \frac{64}{a}(3-\mu)\Upsilon_2{}^7 + \frac{56}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{48}{a}(1-\mu)\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3]^{-1}\{1 -$$

$$\frac{\left[\frac{108}{a}\Upsilon_2{}^8 - \frac{96}{a}(3-\mu)\Upsilon_2{}^7 + \frac{84}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{72}{a}(1-\mu)\Upsilon_2{}^5\right]}{\left[\frac{72}{a}\Upsilon_2{}^8 - \frac{64}{a}(3-\mu)\Upsilon_2{}^7 + \frac{56}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{48}{a}(1-\mu)\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3\right]}e^2 -$$

$$\frac{\left[\frac{108}{a}\Upsilon_2{}^8 - \frac{96}{a}(3-\mu)\Upsilon_2{}^7 + \frac{84}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{72}{a}(1-\mu)\Upsilon_2{}^5\right]}{\left[\frac{72}{a}\Upsilon_2{}^8 - \frac{64}{a}(3-\mu)\Upsilon_2{}^7 + \frac{56}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{48}{a}(1-\mu)\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3\right]}A_2 -$$

$$\frac{\left[-\frac{135}{a}\Upsilon_2{}^8 + \frac{120}{a}(3-\mu)\Upsilon_2{}^7 - \frac{105}{a}(3-2\mu)B_2\Upsilon_2{}^6 + \frac{90}{a}(1-\mu)B_2\Upsilon_2{}^5\right]}{\left[\frac{72}{a}\Upsilon_2{}^8 - \frac{64}{a}(3-\mu)\Upsilon_2{}^7 + \frac{56}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{48}{a}(1-\mu)\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3\right]}B_2\} \quad (3.45)$$

Therefore;

$$a_2 = [\frac{72}{a}\Upsilon_2{}^8 - \frac{64}{a}(3-\mu)\Upsilon_2{}^7 + \frac{56}{a}(3-2\mu)\Upsilon_2{}^6 - \frac{48}{a}(1-\mu)\Upsilon_2{}^5 + 96\mu\Upsilon_2{}^5 - 48\Upsilon_2{}^5 + 80\mu\Upsilon_2{}^4 - 32\mu\Upsilon_2{}^3]^{-1}$$

Where;

$$E_2 = \frac{b_2}{a_2} \& F_2 = \frac{c_2}{a_2} = \frac{1}{a}(108\Upsilon_2{}^8 - 96(3-\mu)\Upsilon_2{}^7 + 84(3-2\mu)\Upsilon_2{}^6 - 72(1-\mu)\Upsilon_2{}^5)$$

$$S_2 = \frac{d_2}{a_2} = -\frac{1}{a}(135\Upsilon_2{}^8 - 120(3-\mu)\Upsilon_2{}^7 + 105(3-2\mu)B_2\Upsilon_2{}^6 - 90(1-\mu)B_2\Upsilon_2{}^5)$$

`

(3.46)

Also;

$$L_2 = 8\left[\frac{1}{a}\Upsilon_2{}^9 - \frac{(3-\mu)}{a}\Upsilon_2{}^8 + \frac{(3-2\mu)}{a}\Upsilon_2{}^7 - \frac{(1-\mu)}{a}\Upsilon_2{}^6 - 2\mu\Upsilon_2{}^6 + \Upsilon_2{}^6 + 2\mu\Upsilon_2{}^5 - \mu\Upsilon_2{}^4\right] + \frac{12}{a}e^2\{\Upsilon_2{}^9 - (3-\mu)\Upsilon_2{}^8 + (3-2\mu)\Upsilon_2{}^7 - (1-\mu)\Upsilon_2{}^6\} + \frac{12}{a}A_2\{\Upsilon_2{}^9 - (3-\mu)\Upsilon_2{}^8 + (3-2\mu)\Upsilon_2{}^7 - (1-\mu)\Upsilon_2{}^6\} - \frac{15}{a}B_2\{\Upsilon_2{}^9 - (3-\mu)\Upsilon_2{}^8 + (3-2\mu)\Upsilon_2{}^7 - (1-\mu)\Upsilon_2{}^6\}$$

(3.47)

Where;

$$U_2 = \frac{8}{a}(\Upsilon_2{}^9 - (3-\mu)\Upsilon_2{}^8 + (3-2\mu)\Upsilon_2{}^7 - (1-\mu)\Upsilon_2{}^6) - 16\mu\Upsilon_2{}^6 + 8\Upsilon_2{}^6 + 16\mu\Upsilon_2{}^5 - 8\mu\Upsilon_2{}^4$$

$$V_2 = \frac{12}{a}(\Upsilon_2{}^9 - (3-\mu)\Upsilon_2{}^8 + (3-2\mu)\Upsilon_2{}^7 - (1-\mu)\Upsilon_2{}^6)$$

$$W_2 = -\frac{15}{a}B_2(\Upsilon_2{}^9 - (3-\mu)\Upsilon_2{}^8 + (3-2\mu)\Upsilon_2{}^7 - (1-\mu)\Upsilon_2{}^6)$$

$$L_2 = U_2 + V_2 e^2 + V_2 A_2 + W_2 B_2 \quad (3.48)$$

From (3.43);

$$\rho = \Upsilon_2 + {P_2}^{-1}[L_2 + (M_2 - \frac{Q_2 L_2}{P_2})\beta_1 + (N_2 - \frac{R_2 L_2}{P_2})A_2 + (D_2 - \frac{G_2 L_2}{P_2})B_2]$$

Hence;

$$L_2 {P_2}^{-1} = [U_2 + V_2 e^2 + V_2 A_2 + W_2 B_2][a_2 + E_2 e^2 + F_2 A_2 + S_2 B_2]$$

$$L_2 {P_2}^{-1} = U_2 a_2 + e^2(U_2 E_2 + V_2 a_2) + A_2(U_2 F_2 + V_2 a_2) + A_2(U_2 S_2 + w_2 a_2) \quad (3.49)$$

For$(M_2 - \frac{Q_2 L_2}{P_2})\beta_1$; Where $M_2 = -8\,(1-\mu)\Upsilon_2^6\beta_1$

$\left(M_2 - \frac{Q_2 L_2}{P_2}\right)\beta_1 = -8\,(1-\mu)\Upsilon_2^6\beta_1 - 48\,(1-\mu)\Upsilon_2^5\beta_1 U_2 a_2 = -8\,(1-\mu)\Upsilon_2^5[\Upsilon_2 + 6U_2 a_2]\beta_1$ (3.50)

Similarly; $(N_2 - \frac{R_2 L_2}{P_2})A_2$, where $N_2 = -12\,\mu{\Upsilon_2}^2(\Upsilon_2 - 1)^2 A_2$

$$\left(N_2 - \frac{R_2 L_2}{P_2}\right)A_2 = -12\,\mu{\Upsilon_2}^2(\Upsilon_2 - 1)^2 A_2 - 24\,\mu\Upsilon_2(1 - 3\Upsilon_2 + 2{\Upsilon_2}^2)A_2 U_2 a_2$$

$= -12\,\mu\Upsilon_2[\Upsilon_2(\Upsilon_2 - 1)^2 + 2\mu(1 - 3\Upsilon_2 + 2{\Upsilon_2}^2)]U_2 a_2 A_2$ (3.51)

Also $(D_2 - \frac{G_2 L_2}{P_2})B_2$; $\quad D_2 = 15\,\mu(\Upsilon_2 - 1)^2 B_2$

Hence;

$\left(D_2 - \frac{G_2 L_2}{P_2}\right)B_2 = 15\,\mu(\Upsilon_2 - 1)^2 B_2 - 30\,\mu\Upsilon_2(\Upsilon_2 - 1)U_2 a_2 B_2 = 15\,\mu(\Upsilon_2 - 1)[(\Upsilon_2 - 1) - 2\mu\Upsilon_2 U_2 a_2]B_2$

(3.52)

Recall that the value of $\rho = \Upsilon_2 + \delta_2$

But; $\delta_2 = P_2^{-1}[L_2 + (M_2 - \frac{Q_2 L_2}{P_2})\beta_1 + (N_2 - \frac{R_2 L_2}{P_2})A_2 + (D_2 - \frac{G_2 L_2}{P_2})B_2]$

Now;

$$P_2^{-1}\left(M_2 - \frac{Q_2 L_2}{P_2}\right)\beta_1 = [a_2 + E_2 e^2 + F_2 A_2 + S_2 B_2][-8\,(1-\mu)\Upsilon_2^5[\Upsilon_2 + 6U_2 a_2]\beta_1]$$

$$= -8\,(1-\mu)\Upsilon_2^5 a_2[\Upsilon_2 + 6U_2 a_2]\beta_1 \tag{3.53}$$

$$P_2^{-1}\left(N_2 - \frac{R_2 L_2}{P_2}\right)A_2$$

$$= [a_2 + E_2 e^2 + F_2 A_2 + S_2 B_2][-12\,\mu\Upsilon_2[\Upsilon_2(\Upsilon_2 - 1)^2 + 2\mu(1 - 3\Upsilon_2 + 2{\Upsilon_2}^2)]U_2 a_2]A_2$$

$$= -12\mu a_2 \Upsilon_2[\Upsilon_2(\Upsilon_2 - 1)^2 + 2\mu(1 - 3\Upsilon_2 + 2{\Upsilon_2}^2)]U_2 a_2]A_2 \tag{3.54}$$

$$P_2^{-1}\left(D_2 - \frac{G_2 L_2}{P_2}\right)B_2$$

$$= [a_2 + E_2 e^2 + F_2 A_2 + S_2 B_2][15\,\mu(\Upsilon_2 - 1)[(\Upsilon_2 - 1) - 2\mu\Upsilon_2 U_2 a_2]B_2$$

$$= 15\mu a_2(\Upsilon_2 - 1)[(\Upsilon_2 - 1) - 2\mu\Upsilon_2 U_2 a_2]B_2 \tag{3.55}$$

$$\delta_2 = P_2^{-1}[L_2 + (M_2 - \tfrac{Q_2 L_2}{P_2})\beta_1 + (N_2 - \tfrac{R_2 L_2}{P_2})A_2 + (D_2 - \tfrac{G_2 L_2}{P_2})B_2]$$

Substituting (3.49), (3.53),(3.54), and (3.55) into δ_2;

$$\delta_2 = U_2 a_2 + e^2(U_2 E_2 + V_2 a_2) + A_2(U_2 F_2 + V_2 a_2) + B_2(U_2 S_2 + w_2 a_2) - 8\,(1 - \mu)\Upsilon_2^5 a_2[\Upsilon_2 + 6U_2 a_2]\beta_1 - 12\mu a_2 \Upsilon_2[\Upsilon_2(\Upsilon_2 - 1)^2 + 2\mu(1 - 3\Upsilon_2 + 2{\Upsilon_2}^2)]U_2 a_2]A_2 + 15\mu a_2(\Upsilon_2 - 1)[(\Upsilon_2 - 1) - 2\mu\Upsilon_2 U_2 a_2]B_2$$

$$\delta_2 = U_2 a_2 + e^2(U_2 E_2 + V_2 a_2) - 8\,(1-\mu)\Upsilon_2^5 a_2[\Upsilon_2 + 6U_2 a_2]\beta_1 + A_2\{(U_2 F_2 +$$

$$V2a2-12\mu a2\Upsilon 2[\Upsilon 2\Upsilon 2-12+2\mu(1-3\Upsilon 2+2\Upsilon 22)]U2a2]A2\}+B2\{U2S2+w2a2+15\mu a$$

$$2(\Upsilon 2-1)[\Upsilon 2-1-2\mu\Upsilon 2U2a2]\} \qquad (3.56)$$

The value of ρ is define as:

$$\rho = \Upsilon_2 + \delta_2$$

$$\rho = \Upsilon_2 + P_2^{-1}[L_2 + (M_2 - \frac{Q_2 L_2}{P_2})\beta_1 + (N_2 - \frac{R_2 L_2}{P_2})A_2 + (D_2 - \frac{G_2 L_2}{P_2})B_2]$$

$$\rho = \Upsilon_2 + U_2 a_2 + e^2(U_2 E_2 + V_2 a_2) - 8\,(1-\mu)\Upsilon_2^5 a_2[\Upsilon_2 + 6U_2 a_2]\beta_1 + A_2\{(U_2 F_2 +$$

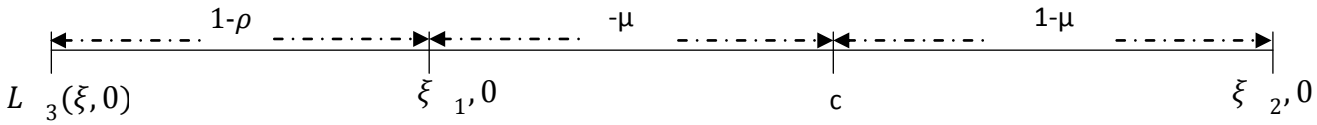
$$V2a2-12\mu a2\Upsilon 2[\Upsilon 2\Upsilon 2-12+2\mu(1-3\Upsilon 2+2\Upsilon 22)]U2a2]A2\}+B2\{U2S2+w2a2+15\mu a$$

$$2(\Upsilon 2-1)[\Upsilon 2-1-2\mu\Upsilon 2U2a2]\} \qquad (3.57)$$

Thus; $\rho = \Upsilon_2 + \delta_2$ is determined in this case.

POSITON OF COLLINEAR POINT L_3 ($\xi_1 > \xi$)

Let the collinear point L_3 be on the left hand side of the bigger primary at a distance $1-\rho$ from it on the ξ-axis.



$$n^2(\rho + 1 - \mu) - \left\{\frac{-(1-\mu)q_1}{|(1-\rho)|^2} - \frac{\mu}{|(2-\rho)|^2} - \frac{3\mu A_2}{2|(2-\rho)|^4} + \frac{15\mu B_2}{8|(2-\rho)|^6}\right\} = 0 \qquad (3.58)$$

Multiply (3.58) through by $8\,(2-\rho)^6(1-\rho)^2$

$$\frac{8n^2(2-\rho)^6(1-\rho)^2(\rho-1-\mu)+8(1-\mu)q_1(2-\rho)^6+8\mu(1-\rho)^2(2-\rho)^4+12\mu(1-\rho)^2(2-\rho)^2A_2-15\mu(\rho+1)^2B_2}{8\,(1-\rho)^2(2-\rho)^6} =$$

$$0 \qquad (3.59)$$

Clearing the fraction to obtain a linear equation

$$n^2\rho^9 + n^2[-15 - \square]\rho^8 + n^2[99 + 14\square]\rho^7 + n^2[-377 - 84\square + (1 - \mu)q_1]\rho^6 +$$

$$n^2[912 - 292\square - 10\square - 12(1 - \mu)q_1]\rho^5 + n^2\left[-1452 - 579\square + 60(1 - \mu)q_1 +\right.$$

$$32\ \mu A2\rho 4 + n21520 + 744\mu - 160\ 1 - \mu\ q1 - 9\mu A2\rho 3 + n2 - 1008 - 584\mu + 240\ 1 - \mu$$

$$q1 + 392\mu A2 - 158\mu B2\rho 2 + n2364 + 256\mu - 192 \qquad 1 - \qquad \mu$$

$$q1 - 18\mu A2 + 154\mu B2\rho + n2 - 64 - 48\mu + 64\ 1 - \mu\ q1 + 6\mu A2 - 158\mu B2 = 0 \quad (3.60)$$

For verification let $n = 1, q_1 = 1\ \&A_2 = B_2 = 0$

$$\rho^9 + [-15 - \square]\rho^8 + [99 + 14\square]\rho^7 + [-377 - 84\square + (1 - \mu)]\rho^6$$
$$+ [912 - 292\square - 10\square - 12(1 - \mu)]\rho^5$$
$$+ [-1452 - 579\square + 60(1 - \mu)]\rho^4$$
$$+ [1520 + 744\square - 160(1 - \mu)]\rho^3$$
$$+ [-1008 - 584\square + 240(1 - \mu)]\rho^2$$
$$+ [364 + 256\square - 192(1 - \mu)]\rho + [-64 - 48\square + 64(1 - \mu)] = 0$$

$$\rho^9 - [15 + \square]\rho^8 + [99 + 14\square]\rho^7 + [-376 - 83\square]\rho^6 + [900 + 290\square]\rho^5$$
$$- [-1392 - 519\square]\rho^4 + [1360 + 904\square]\rho^3 + [-768 - 824\square]\rho^2$$
$$+ [172 + 448\square]\rho + 112\square = 0$$

(3.61)

3.3.2 Positions of Triangular Points

Several reports have reported that French mathematician Lagrange, in 1772 showed that there are five libration points, which comprise of three collinear points $L_1, L_2 \& L_3$and two triangular points $L_4 \& L_5$. The triangular points L_4 and L_5, form the apex of the equilateral triangle that have large masses as their vertices, unlike other Lagrange

points, the triangular points are resistant to gravitational perturbations. Because of this stability, objects such as dust and asteroids tend to accumulate in these regions, as such at L_4 or L_5, a spacecraft is truly stable, like a ball in a large bowl. When gently pulled out of place, it orbits the Lagrange point without drifting away.

The equations of motion for the problem are of the form: -

$$\xi'' - 2\eta' = \frac{\delta\Omega}{\delta\xi}$$

$$\eta'' + 2\xi' = \frac{\delta\Omega}{\delta\eta}$$

$$\zeta'' = \frac{\delta\Omega}{\delta\zeta}$$

with;

$$\Omega = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} - \frac{3\mu B_2}{8r_2^5}\right\}\right] \quad (3.62)$$

Differentiating (3.62) with respect to ξ, η & ζ

with respect to ξ; $\Rightarrow \frac{1}{(1-e^2)^{½}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1(\xi-\xi_1)}{r_1^3} + \frac{\mu(\xi-\xi_2)}{r_2^3} + \frac{3\mu A_2(\xi-\xi_2)}{2r_2^5} - \right.\right.$

$15\mu\ \xi-\xi 2B28r27= 0$ (3.63)

With respect to η; $\Rightarrow \frac{1}{(1-e^2)^{½}}\left[\eta - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1\eta}{r_1^3} + \frac{\mu\eta}{r_2^3} + \frac{3\mu A_2\eta}{2r_2^5} - \frac{15\mu B_{2\eta}}{8r_2^7}\right\}\right] = 0$ (3.64)

and;

With respect to ζ; $\frac{-\zeta}{n^2(1-e^2)^{½}}\left[\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right] = 0$

or

$$\zeta\left[\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right] = 0 \quad (3.65)$$

Equation (3.65) implies that

$\zeta = 0$ or $\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7} = 0$, for the triangular points, $\eta \neq 0$ and $\zeta \neq 0$.

Recall; from equation (3.64) and $\eta \neq 0$, we have that

$$1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right\} = 0 \qquad (3.66)$$

Multiply equation (3.69) by $(\xi - \xi_1)$ and $(\xi - \xi_2)$ respectively to obtain

$$(\xi - \xi_1) - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1(\xi-\xi_1)}{r_1^3} + \frac{\mu(\xi-\xi_1)}{r_2^3} + \frac{3\mu A_2(\xi-\xi_1)}{2r_2^5} - \frac{15\mu B_2(\xi-\xi_1)}{8r_2^7}\right\} = 0 \qquad (3.67)$$

&

$$(\xi - \xi_2) - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1(\xi-\xi_2)}{r_1^3} + \frac{\mu(\xi-\xi_2)}{r_2^3} + \frac{3\mu A_2(\xi-\xi_2)}{2r_2^5} - \frac{15\mu B_2(\xi-\xi_2)}{8r_2^7}\right\} = 0 \qquad (3.68)$$

Subtracting equation (3.67) from (3.63);

$$\xi - \frac{(1-\mu)q_1(\xi-\xi_1)}{n^2r_1^3} - \frac{\mu(\xi-\xi_2)}{n^2r_2^3} - \frac{3\mu A_2(\xi-\xi_2)}{2n^2r_2^5} + \frac{15\mu B_2(\xi-\xi_{21})}{8n^2r_2^7}$$
$$- (\xi - \xi_1) + \frac{(1-\mu)q_1(\xi-\xi_1)}{n^2r_1^3} + \frac{\mu(\xi-\xi_1)}{n^2r_2^3} + \frac{3\mu A_2(\xi-\xi_1)}{2n^2r_2^5}$$
$$- \frac{15\mu B_2(\xi-\xi_1)}{8n^2r_2^7} = 0$$

To obtain;

$$\xi_1 + \frac{\mu(\xi_2-\xi_1)}{n^2r_2^3} + \frac{3\mu A_2(\xi_2-\xi_1)}{2n^2r_2^5} - \frac{15\mu B_2(\xi_2-\xi_1)}{8n^2r_2^7} = 0 \qquad (3.69)$$

Subtracting equation (3.67) from (3.63) above;

$$\xi - \frac{(1-\mu)q_1(\xi-\xi_1)}{n^2 r_1^3} - \frac{\mu(\xi-\xi_2)}{n^2 r_2^3} - \frac{3\mu A_2(\xi-\xi_2)}{2n^2 r_2^5} + \frac{15\mu B_2(\xi-\xi_2)}{8n^2 r_2^7}$$

$$-(\xi-\xi_2) + \frac{(1-\mu)q_1(\xi-\xi_2)}{n^2 r_1^3} + \frac{\mu(\xi-\xi_2)}{n^2 r_2^3} + \frac{3\mu A_2(\xi-\xi_2)}{2n^2 r_2^5}$$

$$-\frac{15\mu B_2(\xi-\xi_2)}{8n^2 r_2^7} = 0$$

To obtain;

$$\xi_2 - \frac{(1-\mu)q_1(\xi_2-\xi_1)}{n^2 r_1^3} = 0 \qquad (3.70)$$

The distance between the primaries $\xi_2 - \xi_1 = 1$ and from the definition of centre of mass

$$(1-\mu)\xi_1 + \mu\xi_2 = 0$$

$$\xi_1 - \mu(\xi_2 - \xi_1) = 0$$

$$\xi_1 + \mu = 0$$

$\xi_1 = -\mu \& \xi_2 = 1-\mu$

Substituting the values of $\xi_1 = -\mu$, $\xi_2 = 1-\mu \& (\xi_2 - \xi_1) = 1$ into equation (3.71) above as appropriate;

$-\mu + \frac{\mu}{n^2 r_2^3} + \frac{3\mu A_2}{2n^2 r_2^5} - \frac{15\mu B_2}{8n^2 r_2^7} = 0$ or $\mu\left[-1 + \frac{1}{n^2 r_2^3} + \frac{3A_2}{2n^2 r_2^5} - \frac{15B_2}{8n^2 r_2^7}\right] = 0$

since $\mu \neq 0$, then $-1 + \frac{1}{n^2 r_2^3} + \frac{3A_2}{2n^2 r_2^5} - \frac{15B_2}{8n^2 r_2^7} = 0$

$$n^2 = \frac{1}{r_2^3} + \frac{3A_2}{2r_2^5} - \frac{15B_2}{8r_2^7} \qquad (3.72)$$

Substituting $\xi_1 = -\mu$, $\xi_2 = 1-\mu \& (\xi_2 - \xi_1) = 1$ in (3.72) above;

$1 - \mu - \frac{(1-\mu)q_1}{n^2 r_1^3} = 0$; $1 - \mu\left[1 - \frac{q_1}{n^2 r_1^3}\right] = 0$

But $(1-\mu) \neq 0$

Hence,

$$1 - \frac{q_1}{n^2 r_1^3} = 0; \frac{q_1}{n^2 r_1^3} = 1; r_1^3 = \frac{q_1}{n^2} \tag{3.73}$$

When oblateness and zonal harmonics are absent; i.e. $A_2 = B_2 = 0$, equation (3.72) becomes;

$$n^2 = \frac{1}{r_2^3} \Rightarrow r_2 = \left(\frac{1}{n^2}\right)^{⅓} \tag{3.74}$$

When oblateness and Zonal harmonics are considered, the value of r_2 will change slightly byϵ,

So that;

$$r_2 = \epsilon + \left(\frac{1}{n^2}\right)^{⅓} \tag{3.75}$$

Recall that;

$$n^2 = \frac{1}{a}\left[1 + \frac{3}{2}e^2 + \frac{3}{2}A_2 - \frac{15}{8}B_2\right]$$

$$\therefore \frac{1}{n^2} = a\left[1 - \frac{3}{2}e^2 - \frac{3}{2}A_2 + \frac{15}{8}B_2\right] \tag{3.76}$$

Recall that (3.72) is $n^2 = \frac{1}{r_2^3} + \frac{3A_2}{2r_2^5} - \frac{15B_2}{8r_2^7}$

$$\Rightarrow \frac{1}{a}\left[1 + \frac{3}{2}e^2 + \frac{3}{2}A_2 - \frac{15}{8}B_2\right] = \frac{1}{r_2^3} + \frac{3A_2}{2r_2^5} - \frac{15B_2}{8r_2^7}$$

If $A_2 = B_2 = 0$

$$\frac{1}{a}\left[1 + \frac{3}{2}e^2\right] = \frac{1}{r_2^3}; r_2^3 = a\left[1 - \frac{3}{2}e^2\right]$$

$$\therefore r_2 = a^{⅓}\left(1 - \frac{e^2}{2}\right)$$

Then;

$$\frac{1}{n^2} = a\left[1 - \frac{3}{2}e^2 - \frac{3}{2}A_2 + \frac{15}{8}B_2\right]$$

$$\left(\frac{1}{n^2}\right)^{⅓} = a^{⅓}\left[1 - \frac{1}{2}e^2 - \frac{1}{2}A_2 + \frac{5}{8}B_2\right]$$

$\left(\frac{1}{n^2}\right)^{⅓} = a^{⅓}\left[1 - \frac{1}{2}e^2\right]$, If $A_2 = B_2 = 0$

$$\frac{1}{r_2^3} = \frac{1}{a^{⅓}\left(1-\frac{e^2}{2}\right)\left[1+\varepsilon a^{-⅓}\left(1-\frac{e^2}{2}\right)^{-1}\right]}; \frac{1}{r_2^3} = \frac{\left[1+\epsilon a^{-⅓}\left(1+\frac{e^2}{2}\right)\right]^{-1}}{a^{⅓}\left(1-\frac{e^2}{2}\right)}; \frac{1}{r_2^3} = \frac{\left[1-\epsilon a^{-⅓}\left(1+\frac{e^2}{2}\right)\right]}{a^{⅓}\left(1-\frac{e^2}{2}\right)}$$

$$\left(\frac{1}{r_2}\right)^3 = \left[\frac{\left[1 - \epsilon a^{-⅓}\left(1 + \frac{e^2}{2}\right)\right]}{a^{⅓}\left(1 - \frac{e^2}{2}\right)}\right]^3$$

Therefore;

$$\frac{1}{r_2^3} = \frac{1}{a}\left[1 + \frac{3}{2}e^2 - 3\epsilon a^{-⅓} - 6\epsilon e^2 a^{-⅓}\right] \tag{3.77}$$

Also;

$$\frac{1}{r_2^5} = \left[\frac{1-\epsilon a^{-⅓}\left(1+\frac{e^2}{2}\right)}{a^{⅓}\left(1-\frac{e^2}{2}\right)}\right]^5; \frac{1}{r_2^5} = \frac{1-5\epsilon a^{-⅓}\left(1+\frac{e^2}{2}\right)}{a^{\frac{5}{3}}\left(1-\frac{5e^2}{2}\right)}$$

$$\frac{1}{r_2^5} = \frac{1}{a^{\frac{5}{3}}}\left[1 + \frac{5}{2}e^2 - 5\epsilon a^{-⅓} - 15\epsilon e^2 a^{-⅓}\right] \tag{3.78}$$

and

$$\frac{1}{r_2^7} = \left[\frac{1-\epsilon a^{-⅓}\left(1+\frac{e^2}{2}\right)}{a^{⅓}\left(1-\frac{e^2}{2}\right)}\right]^7; \frac{1}{r_2^7} = \frac{1-7\epsilon a^{-⅓}\left(1+\frac{e^2}{2}\right)}{a^{\frac{7}{3}}\left(1-\frac{7e^2}{2}\right)}$$

$$\frac{1}{r_2^7} = \frac{1}{a^{\frac{7}{3}}}\left[1 + \frac{7}{2}e^2 - 7\epsilon a^{-⅓} - 28\epsilon e^2 a^{-⅓}\right] \tag{3.79}$$

Substituting $\frac{1}{r_2^3}, \frac{1}{r_2^5}, \frac{1}{r_2^7} \& n^2$ into equation (3.72) as appropriate;

$$n^2 = \frac{1}{r_2^3} + \frac{3A_2}{2r_2^5} - \frac{15B_2}{8r_2^7}$$

But; $n^2 = \frac{1}{a}\left[1 + \frac{3}{2}e^2 + \frac{3}{2}A_2 - \frac{15}{8}B_2\right]$

We obtain;

$$\frac{1}{a}\left[1 + \frac{3}{2}e^2 + \frac{3}{2}A_2 - \frac{15}{8}B_2\right]$$

$$= \frac{1}{a}\left[1 + \frac{3}{2}e^2 - 3\varepsilon a^{-⅓} - 6\varepsilon e^2 a^{-⅓}\right]$$

$$+ \frac{3A_2}{2a^{\frac{5}{3}}}\left[1 + \frac{5}{2}e^2 - 5\epsilon a^{-⅓} - 15\epsilon e^2 a^{-⅓}\right] - \frac{15B_2}{8a^{\frac{7}{3}}}[1 + \frac{7}{2}e^2 - 7\epsilon a^{-⅓}$$

$$- 28\epsilon e^2 a^{-⅓}\ ]$$

open up bracket

$$1 + \frac{3}{2}e^2 - 3\epsilon a^{-⅓} - 6\epsilon e^2 a^{-⅓} + \frac{3a^{\frac{-2}{3}}A_2}{2} + \frac{15e^2 a^{\frac{-2}{3}}A_2}{4} - \frac{15\epsilon a^{-1}A_2}{2} - \frac{45\epsilon e^2 a^{-1}A_2}{2}$$

$$- \frac{15a^{\frac{-4}{3}}B_2}{8} - \frac{105e^2 a^{\frac{-4}{3}}B_2}{16} + \frac{105\epsilon a^{\frac{-5}{3}}B_2}{8} + \frac{420\epsilon e^2 a^{\frac{-5}{3}}B_2}{8}$$

$$= 1 + \frac{3}{2}e^2 + \frac{3}{2}A_2 - \frac{15}{8}B_2$$

$$\epsilon = \frac{a^{-⅓}\left[\frac{A_2}{2}\left(a^{-⅓} + \frac{5e^2 a^{\frac{-1}{3}}}{2} - a^{⅓}\right) - \frac{5B_2}{8}\left(a^{-1} + \frac{7e^2 a^{-1}}{2} - a^{-⅓}\right)\right]}{a^{-⅓}\left[1 + 2e^2 + \frac{5a^{\frac{-2}{3}}A_2}{2} + \frac{15e^2 a^{\frac{-2}{3}}A_2}{2} - \frac{35a^{-\frac{4}{3}}B_2}{8} - \frac{35e^2 a^{-\frac{4}{3}}B_2}{2}\right]}$$

Neglecting Product of e^2A_2, e^2B_2 and higher powers of $A_2, B_2 \& e^2$ respectively

$$\epsilon = \frac{A_2}{2}(a^{-⅓} - a^{⅓}) - \frac{5B_2}{8}\left(a^{-1} - a^{⅓}\right) \qquad (3.80)$$

Substitutingϵ in equation (3.75) above;

i.e. ; $r_2 = \epsilon + (\frac{1}{n^2})^{⅓}$; But; $(\frac{1}{n})^{⅓} = a^{⅓}\,(1 - \frac{e^2}{2})$

$$r_2 = \frac{A_2}{2}(a^{-⅓} - a^{⅓}) - \frac{5B_2}{8}\,(a^{-1} - a^{⅓}) + a^{⅓}\,(1 - \frac{e^2}{2})$$

$$r_2{}^2 = [a^{⅓}\left(1 - \frac{e^2}{2}\right) + \varepsilon\,]^2 \qquad (3.81)$$

$$r_2{}^2 = a^{\frac{2}{3}}\,[1 - e^2 + A_2(a^{-⅓} - 1) - \frac{5B_2}{4}\left(a^{-\frac{4}{3}} - 1\right)] \qquad (3.82)$$

Recall that;

$$r_1{}^3 = \frac{q_1}{n^2};\ \Rightarrow r_1 = (\frac{q_1}{n^2})^{⅓} \Rightarrow r_1 = (q_1)^{⅓}(\frac{1}{n^2})^{⅓} \qquad (3.83)$$

$$r_1 = (aq_1)^{⅓}[\,1 - \frac{1}{2}e^2 - \frac{1}{2}A_2 + \frac{5}{4}B_2]$$

Hence;

$$r_1{}^2 = (aq_1)^{⅔}[\,1 - \frac{1}{2}e^2 - \frac{1}{2}A_2 + \frac{5}{4}B_2]^2$$

$$r_1{}^2 = (aq_1)^{⅔}[1 - e^2 - A_2 + \frac{5}{4}B_2] \qquad (3.84)$$

$r_1{}^2$ and $r_2{}^2$is defined as thus;

$$r_1{}^2 = (\xi - \xi_1)^2 + \eta^2;\ r_1{}^2 = [\xi - (-\mu)]^2 + \eta^2;\ r_1{}^2 = (\xi + \mu)^2 + \eta^2$$

and;

$$r_2{}^2 = (\xi - \xi_2)^2 + \eta^2;\ r_2{}^2 = [\xi - (1 - \mu)]^2 + \eta^2;\ r_2{}^2 = (\xi - 1 + \mu)^2 + \eta^2$$

So thatr_1^2&r_2^2 becomes; $r_1{}^2 - r_2{}^2 = 2\xi + 2\mu - 1 \qquad (3.85)$

Substituting $r_1{}^2$and $r_2{}^2$ into (3.85);

$$(aq_1)^{2/3}\left[1-\ e^2-\ A_2+\frac{5}{4}B_2\right]-\ a^{\frac{2}{3}}\left[1-\ e^2+\ A_2(a^{-2/3}\ -\ 1)-\frac{5B_2}{4}\left(a^{-\frac{4}{3}}-1\right)\right]$$

$$=\ 2\xi+2\mu-1$$

$$2\xi=\ 1-2\mu+\ (aq_1)^{2/3}\left[1-\ e^2-\ A_2+\frac{5}{4}B_2\right]$$

$$-\ a^{\frac{2}{3}}\left[1-\ e^2+\ A_2(a^{-2/3}\ -\ 1)-\frac{5B_2}{4}\left(a^{-\frac{4}{3}}-1\right)\right]$$

$$\xi=\frac{1}{2}-\ \mu+\frac{1}{2}\{(aq_1)^{2/3}\left[1-\ e^2-\ A_2+\frac{5}{4}B_2\right]-\ a^{\frac{2}{3}}\left[1-\ e^2+\ A_2(a^{-2/3}\ -\ 1)-$$

$$5B24a-43-1\ \}\qquad(3.86)$$

Adding μ, to both sides of (3.84);

$$\xi+\ \mu=\frac{1}{2}-\ \mu+\frac{1}{2}\Big\{(aq_1)^{2/3}\left[1-\ e^2-\ A_2+\frac{5}{4}B_2\right]-\ a^{\frac{2}{3}}\left[1-\ e^2+\ A_2(a^{-2/3}\ -\right.$$

$$1-5B24a-43-1+\mu\qquad(3.89)$$

Squaring both sides of (3.87);

$$(\xi+\ \mu)^2=\ [\frac{1}{2}-\ \mu$$

$$+\ \frac{1}{2}\Big\{(aq_1)^{2/3}\left[1-\ e^2-\ A_2+\frac{5}{4}B_2\right]$$

$$-\ a^{\frac{2}{3}}\left[1-\ e^2+\ A_2(a^{-2/3}\ -\ 1)-\frac{5B_2}{4}\left(a^{-\frac{4}{3}}-1\right)\right]\Big\}+\ \mu]^2$$

So that;

$$(\xi + \mu)^2 = [\frac{1}{2} + \frac{1}{2}\left\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \frac{5}{4}B_2\right] - a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-2/3} - 1) - \frac{5B_2}{4}\left(a^{-\frac{4}{3}} - 1\right)\right]\right\}]^2$$

$$(\xi + \mu)^2 = \frac{1}{4}[1 + 2\left\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \frac{5}{4}B_2\right] - a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-2/3} - 1 - 5B24a-43-1\right] \quad (3.88)$$

Also adding $-1 + \mu$ to (3.84), i.e.;

$$(\xi - 1 + \mu) = \frac{1}{2} - \mu + \frac{1}{2}\left\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \frac{5}{4}B_2\right] - a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-1/3} - 1) - \frac{5B_2}{4}\left(a^{-\frac{4}{3}} - 1\right)\right]\right\} - 1 + \mu$$

Therefore;

$$(\xi - 1 + \mu) = -\frac{1}{2} + \frac{1}{2}\left\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \frac{5}{4}B_2\right] - a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-1/3} - 1 - 5B24a-43-1\right] \quad (3.89)$$

squaring both sides;

$$(\xi - 1 + \mu)^2 = [-\frac{1}{2} + \frac{1}{2}\left\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \frac{5}{4}B_2\right] - a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-2/3} - 1) - \frac{5B_2}{4}\left(a^{-\frac{4}{3}} - 1\right)\right]\right\}]^2$$

$$(\xi - 1 + \mu)^2 = \frac{1}{4}\left[1 - 2\left\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \frac{5}{4}B_2\right] - a^{\frac{2}{3}}\left[1 - e^2 + A_2 a^{-2/3} - 1 - 5B_2 4a^{-4} 3^{-1}\right]\right.\right. \quad (3.90)$$

Hence,

$$r_1{}^2 = (\xi - \xi_1)^2 + \eta^2$$

therefore;

$$\eta^2 = r_1{}^2 - (\xi - \xi_1)^2; \eta^2 = r_1{}^2 - (\xi + \mu)^2 \qquad (3.91)$$

Substituting (3.84) and (3.88) into (3.91) as appropriate;

$$\eta^2 = (aq_1)^{2/3}[1 - e^2 - A_2 + \tfrac{5}{4}B_2] - \tfrac{1}{4}[1 + 2\left\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \tfrac{5}{4}B_2\right] -\right.$$

$$a231 - e2 + A2a^{-2/3} - 1 - 5B24a^{-43} - 1\ ] \qquad (3.92)$$

$$\eta^2 = a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-2/3} - 1) - \tfrac{5B_2}{4}\left(a^{-\frac{4}{3}} - 1\right)\right] - \tfrac{1}{4}\ [\ 1 + 2a^{\frac{2}{3}}\left[1 - e^2 +\right.$$

$$A2a^{-2/3} - 1 - 5B24a^{-43} - 1 - 2aq1^{2/3}1 - e2 - A2 + 54B2\ ] \quad (3.93)$$

Thus, the co-ordinates of triangular points $L_4$and L_5 are $\xi, \eta, 0$ and $\xi, -\eta, 0$ respectively.

Where;

$$\xi = \frac{1}{2} - \mu + \frac{1}{2}\{(aq_1)^{2/3}\left[1 - e^2 - A_2 + \frac{5}{4}B_2\right]$$

$$- a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-2/3} - 1) - \frac{5B_2}{4}\left(a^{-\frac{4}{3}} - 1\right)\right]\}$$

$$\eta = \{a^{\frac{2}{3}}\left[1 - e^2 + A_2(a^{-2/3} - 1) - \tfrac{5B_2}{4}\left(a^{-\frac{4}{3}} - 1\right)\right] - \tfrac{1}{4}\ [\ 1 + 2a^{\frac{2}{3}}\left[1 - e^2 +\right.$$

$$A2a^{-2/3} - 1 - 5B24a^{-43} - 1 - 2aq1^{2/3}1 - e2 - A2 + 54B2\ ]\}^{1/2} \quad (3.94)$$

CHAPTER FOUR

STABILITY OF LIBRATION POINTS

4.1 Introduction

When combined gravitational force of two massive bodies such as Earth and Sun or Earth and Moon equals the centrifugal force felt by a test particle, libration point is said to exist. The motion of a particle which remains in the small neighborhood of the equilibrium point after it is disturbed is termed as stable (Szebehely, 1967), while if the system moves away from the equilibrium after small disturbances, then the equilibrium is unstable.

This chapter is considered under three subsections, the first being the introductory aspects of the study, the variational equation and the characteristics equation are given in the second subsection. The third subsection is about the stability of equilibrium points, which is further sub-divided into two parts.

4.2 Variational and Characteristic Equations

The equations of motion for the elliptic photogravitational restricted three-body problem, when the bigger primary is a source of radiation and the smaller one an oblate spheroid with Zonal Harmonics are given as: -

$$\xi'' - 2\eta' = \frac{\delta\Omega}{\delta\xi}$$

$$\eta'' + 2\xi' = \frac{\delta\Omega}{\delta\eta}$$

$$\zeta'' = \frac{\delta\Omega}{\delta\zeta} \tag{4.1}$$

where;

$$\Omega = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} - \frac{3\mu B_2}{8r_2^5}\right\}\right]$$

Let $\alpha, \beta \; and \; \gamma$ be small displacement in the coordinates of the libration point (ξ_o, η_o, ζ_o) of the infinitesimal mass respectively. Then;

$\xi = \xi_o + \alpha;$

$$\eta = \eta_o + \beta$$

$$\zeta = \zeta_o + \gamma$$

Differentiating w.r.t to time; $\zeta_o = 0; \zeta = 0 + \gamma$

$\dot{\xi} = \dot{\xi_o} + \dot{\alpha},$ $\dot{\eta} = \dot{\eta_0} + \dot{\beta},$ & $\dot{\zeta} = \dot{\zeta_o} + \dot{\gamma}$

$\ddot{\xi} = \ddot{\xi_o} + \ddot{\alpha},$ $\ddot{\eta} = \ddot{\eta_o} + \ddot{\beta},$ $\&\ddot{\zeta} = \ddot{\zeta_o} +$

$\ddot{\gamma}$

Also $\dot{\xi_o}, \ddot{\xi_o}, \; \dot{\eta_0} \& \ddot{\eta_o}$ respectively are zero;

$\dot{\xi} = \dot{\alpha},$ $\dot{\eta} = \dot{\beta},$ $\dot{\zeta} = \dot{\zeta_o} + \dot{\gamma}$

$\ddot{\xi} = \ddot{\alpha}$ $\ddot{\eta} = \ddot{\beta}$ & $\ddot{\zeta} = \ddot{\gamma}$

Then;

$$\Omega_\xi = \Omega_\xi(\xi, \eta) = \Omega_\xi(\xi_o + \alpha, \eta_o + \beta)$$

Since $\alpha, \beta \& \gamma$ are small quantities, therefore their second and higher powers are very small quantities and hence they are negligible.

Expanding the above using Taylor's expansion, we have;

$$\Omega_\xi = \Omega_\xi^o + \alpha\Omega_{\xi\xi}^o + \beta\Omega_{\xi\eta}^o + \gamma\Omega_{\xi\zeta}^o; \Omega_\eta = \Omega_\eta^o + \alpha\Omega_{\eta\xi}^o + \beta\Omega_{\eta\eta}^o + \gamma\Omega_{\eta\zeta}^o \quad (4.2)$$

where

$\Omega_{\xi}^{o} = \Omega_{\xi}^{o}(\xi_o, \eta_o, \zeta_0)$; $\Omega_{\eta}^{o} = \Omega_{\eta}^{o}(\xi_o, \eta_o, \zeta_0)$

Also;

$\Omega_{\xi\xi}^{o}, \Omega_{\xi\eta}^{o}, \Omega_{\eta}^{o}, \Omega_{\eta\xi}^{o}, \Omega_{\eta\eta}^{o}$, are the values of $\Omega_{\xi\xi}$, $\Omega_{\xi\eta}, \Omega_{\eta}, \Omega_{\eta\xi}$ &$\Omega_{\eta\eta}$ respectively at the point (ξ_o, η_o),

At the equilibrium points (ξ_o, η_o);

$$\Omega_{\xi}^{o} = \Omega_{\eta}^{o} = \Omega_{\zeta}^{o} = 0$$

From equation 4.3.1 above;

$\Omega_{\xi} = \alpha\Omega_{\xi\xi}^{o} + \beta\Omega_{\xi\eta}^{o} + \gamma\Omega_{\xi\zeta}^{o}$;$\Omega_{\eta} = \alpha\Omega_{\eta\xi}^{o} + \beta\Omega_{\eta\eta}^{o} + \gamma\Omega_{\eta\zeta}^{o}$

$\Omega_{\zeta} = \alpha\Omega_{\zeta\xi}^{o} + \beta\Omega_{\zeta\eta}^{o} + \gamma\Omega_{\zeta\zeta}^{o}$ but $\zeta = 0$ (4.3)

Hence the variational equations of motion are

$$\xi'' - 2\eta' = \alpha\Omega_{\xi\xi}^{o} + \beta\Omega_{\xi\eta}^{o}$$

$\eta'' + 2\xi' = \alpha\Omega_{\eta\xi}^{o} + \beta\Omega_{\eta\eta}^{o}$ (4.4)

Now let;

$\alpha = Ae^{\lambda t}$, $\beta = Be^{\lambda t}$, & , $\gamma = Ce^{\lambda t}$

Differentiating;

$\dot{\alpha} = \lambda Ae^{\lambda t}$ $\dot{\beta} = \lambda Be^{\lambda t}\dot{\gamma} = \lambda Ce^{\lambda t}$

$\ddot{\alpha} = A\lambda^2 e^{\lambda t}$ $\ddot{\beta} = B\lambda^2 e^{\lambda t}$ &$\ddot{\gamma} = \lambda^2 Ce^{\lambda t}$

where $A, B, C$and $\lambda$is a parameter, substituting these into equation (4.4) we have;

$$A\lambda^2 e^{\lambda t} - 2\lambda Be^{\lambda t} = Ae^{\lambda t}\Omega_{\xi\xi}^{*o} + Be^{\lambda t}\Omega_{\xi\eta}^{*0}$$

$$B\lambda^2 e^{\lambda t} + 2Ae^{\lambda t} = Ae^{\lambda t}\Omega_{\eta\xi}^{*o} + Be^{\lambda t}\Omega_{\eta\eta}^{*0} \tag{4.5}$$

$$\begin{vmatrix} \lambda^2 - \Omega_{\xi\xi}^{*o} & -\Omega_{\xi\eta}^{*0} - 2\lambda \\ 2\lambda - \Omega_{\eta\xi}^{*o} & \lambda^2 - \Omega_{\eta\eta}^{*0} \end{vmatrix} = 0 \tag{4.6}$$

Therefore, the characteristics equation is;

$$\lambda^4 + \lambda^2\,(4 - \Omega_{\eta\eta}^{o} - \Omega_{\xi\xi}^{o}) + \Omega_{\xi\xi}^{o}\Omega_{\eta\eta}^{0} - (\Omega_{\xi\eta}^{0})^2 = 0 \tag{4.7}$$

4.3.1 Stability of Collinear Points

The equations of motion for the problem are of the form: -

$$\xi'' - 2\eta' = \Omega_{\xi}$$

$$\eta'' + 2\xi' = \Omega_{\eta}$$

$$\&\zeta'' = \Omega_{\zeta}$$

$$\Omega = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{2}(\xi^2+\eta^2) + \frac{1}{n^2}\left\{\frac{(1-\square)q_1}{r_1} + \frac{\square}{r_2} + \frac{\square A_2}{2r_2^3} - \frac{3\square B_2}{8r_2^5}\right\}\right]$$

Now the first partial derivative of ξ, η and ζ respectively

$$\Omega_{\xi} = \frac{1}{(1-e^2)^{½}}\Bigg[\xi - \frac{1}{n^2}\left\{\frac{(1-\square)q_1}{r_1^3}(\xi - \xi_1) + \frac{\square}{r_2^3}(\xi - \xi_2) + \frac{3\square A_2}{2r_2^5}(\xi - \xi_2) - \frac{15\square B_2}{8r_2^7}(\xi - \xi_2)\right\}\Bigg]$$

$$\Omega_{\eta} = \frac{1}{(1-e^2)^{½}}\left[\eta - \frac{\eta}{n^2}\left\{\frac{(1-\square)q_1}{r_1^3} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right]$$

$$\Omega_{\zeta} = \frac{1}{(1-e^2)^{½}}\left[\frac{(1-\square)q_1}{r_1^3} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right]\zeta \tag{4.8}$$

Now the partial derivative to the 2nd order

$$\Omega_{\xi\xi} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-□)q_1}{r_1^3} + \frac{□}{r_2^3} + \frac{3□A_2}{2r_2^5} - \frac{15□B_2}{8r_2^7}\right\}\right.$$

$$+ \frac{1}{n^2}\left\{\frac{3(1-□)q_1}{r_1^5}(\xi - \xi_1)^2 + \frac{3□}{r_2^5}(\xi - \xi_2)^2 + \frac{15□A_2}{2r_2^7}(\xi - \xi_2)^2\right.$$

$$\left.\left. - \frac{105□B_2}{8r_2^9}(\xi - \xi_2)^2\right\}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-□)q_1}{r_1^3} + \frac{□}{r_2^3} + \frac{3□A_2}{2r_2^5} - \frac{15□B_2}{8r_2^7}\right\}\right.$$

$$\left. + \frac{\eta^2}{n^2}\left\{\frac{3(1-□)q_1}{r_1^5} + \frac{3□}{r_2^5} + \frac{15□A_2}{2r_2^7} - \frac{105□B_2}{8r_2^9}\right\}\right]$$

$$\Omega_{\xi\eta} = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{n^2}\left\{\frac{3(1-□)q_1}{r_1^5}(\xi - \xi_1)^2 + \frac{3□}{r_2^5}(\xi - \xi_2)^2 + \frac{15□A_2}{2r_2^7}(\xi - \xi_2)^2 -\right.\right.$$

$$105□B28r29\xi - \xi22\eta$$

(4.9)

Since the collinear points lie on the $\xi - axis$, $\eta = \zeta = 0$, from the first interval$(-1 -$ □, −□) i.e for L_1, $r_1 = -(\xi + □)$ and $r_2 = -(\xi + □ - 1)$

From Ω_ξ, i.e

$$\Omega_\xi = \frac{1}{(1-e^2)^{½}}\left[\xi\right.$$

$$- \frac{1}{n^2}\left\{\frac{(1-□)q_1}{r_1^3}(\xi - \xi_1) + \frac{□}{r_2^3}(\xi - \xi_2) + \frac{3□A_2}{2r_2^5}(\xi - \xi_2)\right.$$

$$\left.\left. - \frac{15□B_2}{8r_2^7}(\xi - \xi_2)\right\}\right]$$

$$\xi n^2 - \frac{(1-□)q_1}{r_1^3}(\xi - \xi_1) - \frac{□}{r_2^3}(\xi - \xi_2) - \frac{3□A_2}{2r_2^5}(\xi - \xi_2) + \frac{15□B_2}{8r_2^7}(\xi - \xi_2) = 0$$

But $\xi - \xi_1 = \xi + \square$ and $r_1 = -(\xi + \square)$, also $\xi - \xi_2 = \xi + \square - 1$ and $r_2 = -(\xi + \square - 1)$

We have,

$$\xi n^2 - \frac{(1-\square)q_1}{r_1^3}(\xi + \square) - \frac{\square}{r_2^3}(\xi + \square - 1) - \frac{3\square A_2}{2r_2^5}(\xi + \square - 1) + \frac{15\square B_2}{8r_2^7}(\xi + \square - 1) = 0$$

$$\xi n^2 - \frac{(1-\square)q_1}{|-(\xi + \square)^3|}(\xi + \square) - \frac{\square}{|-(\xi + \square - 1)^3|}(\xi + \square - 1) - \frac{3\square A_2}{2|-(\xi + \square - 1)^5|}(\xi + \square - 1) + \frac{15\square B_2}{8|-(\xi + \square - 1)^7|}(\xi + \square - 1) = 0$$

$$\xi n^2 + \frac{(1-\square)q_1}{|\xi+\square|^2} + \frac{\square}{|\xi+\square-1|^2} + \frac{3\square A_2}{2|\xi+\square-1|^5} - \frac{15\square B_2}{8|\xi+\square-1|^6} = 0 \quad (4.10)$$

$$\frac{(1-\square)q_1}{r_1^2} = -\xi n^2 - \frac{\square}{r_2^2} - \frac{3\square A_2}{2r_2^4} + \frac{15\square B_2}{8r_2^6} \quad \text{or} \quad \frac{(1-\square)q_1}{|\xi+\square|^2} = -\xi n^2 - \frac{\square}{|\xi+\square-1|^2} - \frac{3\square A_2}{2|\xi+\square-1|^4} + \frac{15\square B_2}{8|\xi+\square-1|^6} \quad (4.11)$$

Substituting (4.11) into $\Omega_{\eta\eta}$, recall that

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\square)q_1}{r_1^3} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\} + \frac{\eta^2}{n^2}\left\{\frac{3(1-\square)q_1}{r_1^3} + \frac{3\square}{r_2^5} + \frac{15\square A_2}{2r_2^7} - \frac{105\square B_2}{8r_2^9}\right\}\right]$$

The collinear points lies on the $\xi - axis$ and $\eta = \zeta = 0$. The second part of $\Omega_{\eta\eta} = 0$

That is; $\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\square)q_1}{r_1^3} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right]$

$$\Rightarrow \Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r_1}\left(\frac{(1-\square)q_1}{r_1^2}\right) + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right] \quad (4.12)$$

Now, substituting (4.11) into (4.12) as appropriate

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r_1}\{-\xi n^2 - \frac{□}{r_2^2} - \frac{3□A_2}{2r_2^4} + \frac{15□B_2}{8r_2^6}\} + \frac{□}{r_2^3} + \frac{3□A_2}{2r_2^5} - \frac{15□B_2}{8r_2^7}\right\}\right]$$

Open up bracket

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\{-\frac{\xi n^2}{r_1} - \frac{□}{r_1 r_2^2} - \frac{3□A_2}{2r_1 r_2^4} + \frac{15□B_2}{8r_1 r_2^6} + \frac{□}{r_2^3} + \frac{3□A_2}{2r_2^5} - \frac{15□B_2}{8r_2^7}\}\right]$$

Recall that, $r_1 = -(\xi + □) \Rightarrow r_1 = -\xi - □ \Rightarrow -\xi = r_1 + □$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\{\frac{(r_1 + □)n^2}{r_1} - \frac{□}{r_1 r_2^2} - \frac{3□A_2}{2r_1 r_2^4} + \frac{15□B_2}{8r_1 r_2^6} + \frac{□}{r_2^3} + \frac{3□A_2}{2r_2^5} - \frac{15□B_2}{8r_2^7}\}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{n^2}{n^2} - \frac{□n^2}{n^2 r_1} + \frac{□}{n^2 r_1 r_2^2} + \frac{3□A_2}{2n^2 r_1 r_2^4} - \frac{15□B_2}{8n^2 r_1 r_2^6} - \frac{□}{n^2 r_2^3} - \frac{3□A_2}{2n^2 r_2^5} + \frac{15□B_2}{8n^2 r_2^7}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[-\frac{□}{r_1} + \frac{1}{n^2}\{\frac{□}{r_2^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{3□A_2}{2r_2^4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{15□B_2}{8r_2^6}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\}\right] \quad (4.13)$$

Thus, in the first interval $(-1 - □, -□)$, $r_1 > 1$ and $r_2 < 1$

Hence, $\frac{□}{r_2^2} < 0$, $\frac{3□A_2}{2r_2^4} < 0$ and $-\frac{15□B_2}{8r_2^6} < 0 \Rightarrow \Omega_{\eta\eta}^{o} < 0$

Now for the second case L_2, the interval is $(-□, 1 - □)$, we have that

$r_1 = \xi - \xi_1$ and $r_1 = \xi - \xi_2$; where $\xi_1 = -□$ and $\xi_2 = 1 - □$

Hence;

$r_1 = |\xi + \square|$ and $r_2 = -(\xi + \square - 1)$

Recall Ω_ξ, hence,

$$\Rightarrow \xi n^2 - \frac{(1-\square)q_1}{r_1^3}(\xi + \square) - \frac{\square}{r_2^3}(\xi + \square - 1) - \frac{3\square A_2}{2r_2^5}(\xi + \square - 1) + \frac{15\square B_2}{8r_2^7}(\xi + \square - 1) = 0$$

$$\xi n^2 - \frac{(1-\square)q_1}{|\xi + \square|^3}(\xi + \square) - \frac{\square}{|-(\xi + \square - 1)|^3}(\xi + \square - 1) - \frac{3\square A_2}{2|-(\xi + \square - 1)|^5}(\xi + \square - 1) + \frac{15\square B_2}{8|-(\xi + \square - 1)|^7}(\xi + \square - 1) = 0$$

$$\xi n^2 - \frac{(1-\square)q_1}{|\xi + \square|^2} + \frac{\square}{|(\xi + \square - 1)|^2} + \frac{3\square A_2}{2|(\xi + \square - 1)|^4} - \frac{15\square B_2}{8|(\xi + \square - 1)|^6} = 0$$

or

$$\xi n^2 - \frac{(1-\square)q_1}{r_1^2} + \frac{\square}{r_2^2} + \frac{3\square A_2}{2r_2^4} - \frac{15\square B_2}{8r_2^6} = 0 \quad \Rightarrow \quad -\frac{(1-\square)q_1}{r_1^2} = -\xi n^2 - \frac{\square}{r_2^2} - \frac{3\square A_2}{2r_2^4} + \frac{15\square B_2}{8r_2^6} \tag{4.16}$$

Substituting (4.16) into (4.12);

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r_1}\left(\frac{(1-\square)q_1}{r_1^2}\right) + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right] \tag{4.17}$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r_1}\left\{\xi n^2 + \frac{\square}{r_2^2} + \frac{3\square A_2}{2r_2^4} - \frac{15\square B_2}{8r_2^6}\right\} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right]$$

Open up

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{\xi n^2}{r_1} + \frac{\square}{r_1 r_2^2} + \frac{3\square A_2}{2r_1 r_2^4} - \frac{15\square B_2}{8r_1 r_2^6} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right]$$

Recall that, $r_1 = \xi + \square \Rightarrow \xi = r_1 - \square$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{(r_1 - \square)n^2}{r_1} - \frac{\square}{r_1 r_2^2} - \frac{3\square A_2}{2r_1 r_2^4} + \frac{15\square B_2}{8r_1 r_2^6} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{n^2}{n^2} + \frac{\square n^2}{n^2 r_1} + \frac{\square}{n^2 r_1 r_2^2} + \frac{3\square A_2}{2n^2 r_1 r_2^4} - \frac{15\square B_2}{8n^2 r_1 r_2^6} - \frac{\square}{n^2 r_2^3} - \frac{3\square A_2}{2n^2 r_2^5} + \frac{15\square B_2}{8n^2 r_2^7}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[\frac{\square}{r_1} + \frac{1}{n^2}\left\{\frac{\square}{r_2^2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) + \frac{3\square A_2}{2r_2^4}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{15\square B_2}{8r_2^6}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)\right\}\right] \quad (4.18)$$

In the second interval $(-\square, 1 - \square,)$, and $0 < r_1 \ll 1$

That is $\frac{\square}{r_2^2} < 0$, $\frac{3\square A_2}{2r_2^4} < 0$ and $-\frac{15\square B_2}{8r_2^6} < 0 \Rightarrow \Omega_{\eta\eta}^{o} < 0$

In the third case L_3, we have the interval $(1 - \square, 2 - \square)$

$r_1 = \xi - \xi_1$ and $r_2 = \xi - \xi_2$, where $\xi_1 = -\square$, $\xi_2 = 1 - \square$

$\therefore \quad r_1 = \xi + \square \quad r_2 = \xi + \square - 1$

So, $r_1 = \xi + \square \quad \Rightarrow \quad \xi = r_1 - \square$

Recall that,

$$\xi n^2 - \frac{(1-\square)q_1}{r_1^3}(\xi + \square) - \frac{\square}{r_2^3}(\xi + \square - 1) - \frac{3\square A_2}{2r_2^5}(\xi + \square - 1) + \frac{15\square B_2}{8r_2^7}(\xi + \square - 1) = 0$$

$$\Rightarrow \xi n^2 - \frac{(1-\square)q_1}{r_1^2} - \frac{\square}{r_2^2} - \frac{3\square A_2}{2r_2^4} + \frac{15\square B_2}{8r_2^7} = 0 \qquad \Rightarrow \frac{(1-\square)q_1}{r_1^2} = \xi n^2 - \frac{\square}{r_2^2} - \frac{3\square A_2}{2r_2^4} + \frac{15\square B_2}{8r_2^6} \tag{4.19}$$

Substituting (4.18) into (4.12)

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r_1}\left(\frac{(1-\square)q_1}{r_1^2}\right) + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right] \tag{4.20}$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{1}{r_1}\left\{\xi n^2 - \frac{\square}{r_2^2} - \frac{3\square A_2}{2r_2^4} + \frac{15\square B_2}{8r_2^6}\right\} + \frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right\}\right]$$

but, $r_1 = \xi + \square$; $\Rightarrow \quad \xi = r_1 - \square$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{r_1 n^2}\left\{(r_1 - \square)n^2 - \frac{\square}{r_2^2} - \frac{3\square A_2}{2r_2^4} + \frac{15\square B_2}{8r_2^6} - \frac{1}{n^2}\left(\frac{\square}{r_2^3} + \frac{3\square A_2}{2r_2^5} - \frac{15\square B_2}{8r_2^7}\right)\right\}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{r_1 n^2}{r_1 n^2} + \frac{\square n^2}{r_1 n^2} + \frac{\square}{r_1 r_2^2} + \frac{3\square A_2}{2r_1 r_2^4} - \frac{15\square B_2}{8r_1 r_2^6} - \frac{\square}{n^2 r_2^3} - \frac{3\square A_2}{2n^2 r_2^5} + \frac{15\square B_2}{8n^2 r_2^6}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[\frac{\square}{r_1} + \frac{1}{n^2}\left\{\frac{\square}{r_2^2}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] + \frac{3\square A_2}{2r_2^4}\left[\frac{1}{r_1} - \frac{1}{r_2}\right] - \frac{15\square B_2}{8r_2^6}\left[\frac{1}{r_1} - \frac{1}{r_2}\right]\right\}\right] \tag{4.21}$$

Since $\square \leq ½$, as in 1st and 2nd interval, with $r_1 > 1$, $r_2 > 1$, $\frac{3\square A_2}{2r_2^4} < 0$ and $B_2 < 0$, we conclude that $\Omega_{\eta\eta}^o < 0$. Hence, from the characteristics equation $\lambda^4 - \left(\Omega_{\xi\xi}^o + \Omega_{\eta\eta}^o - 4\right)\lambda^2 + \Omega_{\xi\xi}^o \Omega_{\eta\eta}^o - \left(\Omega_{\xi\eta}^o\right)^2 = 0$, we observe that $\Omega_{\xi\xi}^o \Omega_{\eta\eta}^o - (\Omega_{\xi\eta}^o)^2 < 0$, the discriminant

of the characteristics equation is positive and have four real and complex conjugate roots with real parts $(\lambda_{1,2}, \lambda_{3,4})$ obtained from the software package Mathematica:-

where;

$$\lambda_{1,2} = \pm \frac{\sqrt{-4 + \Omega^0_{\eta\eta} + \Omega^0_{\xi\xi} - \sqrt{(4 - \Omega^0_{\eta\eta} - \Omega^0_{\xi\xi})^2 - 4(-\Omega^0_{\xi\eta}{}^2 + \Omega^0_{\eta\eta}\Omega^0_{\xi\xi})}}}{\sqrt{2}}$$

$$\lambda_{3,4} = \pm\sqrt{(-2 + \frac{\Omega^0_{\eta\eta}}{2} + \frac{\Omega^0_{\xi\xi}}{2} + \frac{1}{2}\sqrt{(4 - \Omega^0_{\eta\eta} - \Omega^0_{\xi\xi})^2 - 4(-\Omega^0_{\xi\eta}{}^2 + \Omega^0_{\eta\eta}\Omega^0_{\xi\xi})})} \qquad (4.22)$$

Thus; $\Omega^o_{\xi\xi} > 0$, $\Omega^o_{\eta\eta} < 0$, $\Omega^o_{\xi\eta} = 0$ then $\Omega^o_{\xi\xi}\Omega^o_{\eta\eta} - (\Omega^o_{\xi\eta})^2 < 0$ is positive, results to the instability of the collinear points due to unboundedness of motion in this neighbourhood.

4.3.2 Stability of Triangular Points

The characteristics equation obtained above is thus;

$$\lambda^4 + \lambda^2\,(4 - \Omega^o_{\eta\eta} - \Omega^o_{\xi\xi}) + \Omega^o_{\xi\xi}\Omega^o_{\eta\eta} - (\Omega^o_{\xi\eta})^2 = 0$$

Recall that;

$$\Omega = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{2}(\xi^2 + \eta^2) + \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1} + \frac{\mu}{r_2} + \frac{\mu A_2}{2r_2^3} - \frac{3\mu B_2}{8r_2^5}\right\}\right]$$

Where;

$r_1{}^2 = (\xi - \xi_1)^2 + \eta^2 + \rho^2$

$$r_1{}^2 = (\xi - \xi_1)^2 + \eta^2 + \rho^2$$

$\xi_1 = -\mu, \qquad \xi_2 = 1 - \mu \qquad\qquad \&n^2 = \frac{(1-e^2)^{½}}{a(1-e^2)}\,[\,1 + \frac{3}{2}A_2 - \frac{15}{8}B_2]$

Recall that;

$$\Omega_{\xi} = \frac{1}{(1-e^2)^{½}}\left[\xi - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1(\xi-\xi_1)}{r_1^3} + \frac{\mu(\xi-\xi_2)}{r_2^3} + \frac{3\mu A_2(\xi-\xi_2)}{2r_2^5} - \frac{15\mu(\xi-\xi_2)B_2}{8r_2^7}\right\}\right]$$

$$\Omega_{\eta} = \frac{1}{(1-e^2)^{½}}\left[\eta - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1\eta}{r_1^3} + \frac{\mu\eta}{r_2^3} + \frac{3\mu A_2\eta}{2r_2^5} - \frac{15\mu B_2\eta}{8r_2^7}\right\}\right]$$

Now;

$$\Omega_{\xi\xi} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right\} + \frac{1}{n^2}\left\{\frac{3(1-\mu)q_1(\xi-\xi_1)^2}{r_1^5} + \frac{3\mu(\xi-\xi_2)^2}{r_2^5} + \frac{15\mu A_2(\xi-\xi_2)^2}{2r_2^7} - \frac{105\mu B_2(\xi-\xi_2)^2}{8r_2^9}\right\}\right]$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right\} + \frac{\eta^2}{n^2}\left\{\frac{3(1-\mu)q_1}{r_1^5} + \frac{3\mu}{r_2^5} + \frac{15\mu A_2}{2r_2^7} - \frac{105\mu B_2}{8r_2^9}\right\}\right]$$

$$\Omega_{\xi\eta} = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{n^2}\left\{\frac{3(1-\mu)q_1(\xi-\xi_1)}{r_1^5} + \frac{3\mu(\xi-\xi_2)}{r_2^5} + \frac{15\mu A_2(\xi-\xi_2)}{2r_2^7} - \frac{105\mu(\xi-\xi_2)B_2}{8r_2^9}\right\}\eta\right]$$

$$\Omega_{\eta\zeta} = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{n^2}\left\{\frac{3(1-\mu)q_1\eta}{r_1^5} + \frac{3\mu\eta}{r_2^5} + \frac{15\mu A_2\eta}{2r_2^7} - \frac{105\mu B_2\eta}{8r_2^9}\right\}\zeta\right]$$

$$\Omega_{\xi\zeta} = \frac{1}{(1-e^2)^{½}}\left[\frac{1}{n^2}\left\{\frac{3(1-\mu)q_1(\xi-\xi_1)}{r_1^5} + \frac{3\mu(\xi-\xi_2)}{r_2^5} + \frac{15\mu A_2(\xi-\xi_2)}{2r_2^7} - \frac{105\mu(\xi-\xi_2)B_2}{8r_2^9}\right\}\zeta\right]$$

$$\Omega_{\zeta} = \frac{-\zeta}{n^2(1-e^2)^{½}}\left[\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right] = 0$$

so,

$$\Omega_{\zeta\zeta} = \frac{1}{(1-e^2)^{½}}\left[-\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right\} + \frac{\rho^2}{n^2}\left\{\frac{3(1-\mu)q_1}{r_1^5} + \frac{3\mu}{r_2^5} + \frac{15\mu A_2}{2r_2^7} - 105\mu B28r29\right.\right. \quad (4.23)$$

Recall back from equation above, we have: -

$$r_2 = a^{⅓}\left(1-\frac{e^2}{2}\right)\left[1 + \left\{-\frac{A_2}{2}a^{⅓}(1-a^{-⅔}) - \frac{5}{8}B_2a^{⅓}\left(a^{\frac{-4}{3}}-1\right)\right\}a^{\frac{-1}{3}}\left(1-\frac{e^2}{2}\right)^{-1}\right]$$

$$r_2^3 = [a^{⅓}\left(1-\frac{e^2}{2}\right)[1-\frac{A_2}{2}(1-a^{-⅔}) - \frac{5}{8}B_2\left(a^{\frac{-4}{3}}-1\right)]]^3$$

$$r_2^3 = a\left(1-\frac{3e^2}{2}\right)[1-\frac{3A_2}{2}(1-a^{-⅔}) - \frac{15}{8}B_2\left(a^{\frac{-4}{3}}-1\right)] \quad (4.24)$$

Therefore;

$$r_2^3 = a[1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{3A_2a^{-⅔}}{2}+\frac{15}{8}B_2-\frac{15}{8}B_2a^{\frac{-4}{3}}] \quad (4.25)$$

and

$$\frac{1}{r_2^3} = \frac{1}{a}[1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A_2a^{-⅔}}{2}-\frac{15}{8}B_2+\frac{15}{8}B_2a^{\frac{-4}{3}}] \quad (4.26)$$

$$\frac{1}{r_2^5} = \frac{1}{(r_2^3)^{\frac{5}{3}}}; \frac{1}{r_2^5} = \frac{1}{a^{\frac{5}{3}}}[1 + \frac{5e^2}{2} + \frac{5A_2}{2} - \frac{5A_2 a^{-⅔}}{2} - \frac{25}{8}B_2 + \frac{25}{8}B_2 a^{\frac{-4}{3}}] \quad (4.27)$$

also,

$$\frac{1}{r_2^7} = \frac{1}{(r_2^3)^{\frac{7}{3}}}; \frac{1}{r_2^7} = \frac{1}{a^{\frac{7}{3}}}[1 + \frac{7e^2}{2} + \frac{7A_2}{2} - \frac{7A_2 a^{-⅔}}{2} - \frac{35}{8}B_2 + \frac{35}{8}B_2 a^{\frac{-4}{3}}] \text{ and} \quad (4.28)$$

$$\frac{1}{r_2^9} = \frac{1}{(r_2^3)^{\frac{9}{3}}}; \frac{1}{r_2^9} = \frac{1}{a^{\frac{9}{3}}}[1 + \frac{9e^2}{2} + \frac{9A_2}{2} - \frac{9A_2 a^{-⅔}}{2} - \frac{45}{8}B_2 + \frac{45}{8}B_2 a^{\frac{-4}{3}}] \quad (4.29)$$

Recall that;

$r_1 = (q_1)^{⅓}\,(\frac{1}{n^2})^{⅓}$, thus;$r_1 = (aq_1)^{⅓}\,(1 - \frac{e^2}{2} - \frac{A_2}{2} + \frac{5B_2}{8})$

$$r_1^3 = [(aq_1)^{⅓}\,(1 - \frac{e^2}{2} - \frac{A_2}{2} + \frac{5B_2}{8})]^3 ; r_1^3 = (aq_1)^{⅓}\,(1 - \frac{3e^2}{2} - \frac{3A_2}{2} - \frac{15B_2}{8}] \quad (4.30)$$

$$\frac{1}{r_1^3} = \frac{1}{(aq_1)}\,(1 + \frac{3e^2}{2} + \frac{3A_2}{2} - \frac{15B_2}{8}) \quad (4.31)$$

$$\frac{1}{r_1^5} = \frac{1}{(aq_1)^{\frac{5}{3}}}\,(1 + \frac{3e^2}{2} + \frac{3A_2}{2} - \frac{15B_2}{8})^{\frac{5}{3}}; \frac{1}{r_1^5} = \frac{1}{(aq_1)^{\frac{5}{3}}}\,(1 + \frac{5e^2}{2} + \frac{5A_2}{2} - \frac{25B_2}{8}) \quad (4.32)$$

$$\frac{1}{r_1^7} = \frac{1}{(aq_1)^{\frac{7}{3}}}\,(1 + \frac{3e^2}{2} + \frac{3A_2}{2} - \frac{15B_2}{8})^{\frac{7}{3}}; \frac{1}{r_1^7} = \frac{1}{(aq_1)^{\frac{7}{3}}}\,(1 + \frac{7e^2}{2} + \frac{7A_2}{2} - \frac{35B_2}{8}) \quad (4.33)$$

$$\frac{1}{r_1^9} = \frac{1}{(aq_1)^{\frac{9}{3}}}\,(1 + \frac{3e^2}{2} + \frac{3A_2}{2} - \frac{15B_2}{8})^{\frac{9}{3}}; \frac{1}{r_1^9} = \frac{1}{(aq_1)^{\frac{9}{3}}}\,(1 + \frac{9e^2}{2} + \frac{9A_2}{2} - \frac{45B_2}{8}) \quad (4.34)$$

The values obtained above are used to evaluate the partial derivative of $\Omega_{\xi\xi}$, $\Omega_{\eta\eta}$ &$\Omega_{\xi\eta}$ at the point $\boldsymbol{\xi}_\circ$ &$\boldsymbol{\eta}_\circ$

Recall that

$$\Omega_{\xi\xi} = \frac{1}{(1-e^2)^{½}}\,[1 - \frac{1}{n^2}\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\} + \frac{1}{n^2}\{\frac{3(1-\mu)(\xi-\xi_2)^2 q_1}{r_1^5} + \frac{3\mu(\xi-\xi_2)^2}{r_2^5} + \frac{15\mu(\xi-\xi_2)^2 A_2}{2r_2^7} - \frac{105\mu(\xi-\xi_2)^2 B_2}{8r_2^9}\}] \quad (4.35)$$

Calculating $\Omega_{\xi\xi}$; Taking the first part of $\Omega_{\xi\xi}$;

$$\frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3\mu A_2}{2r_2^5}-\frac{15\mu B_2}{8r_2^7}\right]$$

substituting $r_1^3, r_2^3, 2r_2^5 \& 8r_2^7$ respectively

$$\frac{(1-\mu)q_1}{(aq_1)}\left[1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{15B_2}{8}\right]$$

$$+\frac{\mu}{a}\left[1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{3A_2a^{-2/3}}{2}-\frac{15}{8}B_2+\frac{15}{8}B_2a^{\frac{-4}{3}}\right]$$

$$+\frac{3\mu}{2a^{\frac{5}{3}}}\left[1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{5A_2a^{-2/3}}{2}-\frac{25}{8}B_2+\frac{25}{8}B_2a^{\frac{-4}{3}}\right]-\frac{15\mu}{8a^{\frac{7}{3}}}\left[1\right.$$

$$\left.+\frac{7e^2}{2}+\frac{7A_2}{2}-\frac{7A_2a^{-2/3}}{2}-\frac{35}{8}B_2+\frac{35}{8}B_2a^{\frac{-4}{3}}\right]$$

Neglecting the product of $A_2e^2, B_2e^2, and higher powers of A_2, B_2 \& e^2$ respectively to obtain;

$$\frac{1}{a}\left[1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{15B_2}{8}\right]=\frac{(1-\mu)q_1}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3\mu A_2}{2r_2^5}-\frac{15\mu B_2}{8r_2^7}$$

but,

$$\frac{1}{n^2}=a\left[1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right]$$

Hence,

$$\frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3\mu A_2}{2r_2^5}-\frac{15\mu B_2}{8r_2^7}\right]$$

$$=a\left[1-\frac{3e^2}{2}-\frac{3A_2}{2}-\frac{15B_2}{8}\right]\frac{1}{a}\left[1+\frac{3e^2}{2}+\frac{3A_2}{2}-\frac{15B_2}{8}\right]$$

which gives;

$$\frac{1}{n^2}\left[\frac{(1-\mu)q_1}{r_1^3}+\frac{\mu}{r_2^3}+\frac{3\mu A_2}{2r_2^5}-\frac{15\mu B_2}{8r_2^7}\right]=1 \tag{4.36}$$

but

$$(\xi-\xi_1)^2 = (\xi+\mu)^2$$

$$= \frac{1}{4}\,[\,1$$

$$+2\{\left(aq_1\right)^{2/3}\left[1-e^2-A_2+\frac{5B_2}{4}\right]$$

$$-a^{2/3}\left[1-e^2+A_2(a^{-2/3}-1)-\frac{5B_2}{4}\left(a^{\frac{-4}{3}}-1\right)\right]\}]$$

$$(\xi+\mu)^2 = \frac{1}{4}\,[\,1+2\,(aq_1)^{2/3}-2\,(aq_1)^{2/3}e^2-2\,(aq_1)^{2/3}A_2+\frac{5B_2}{2}(aq_1)^{2/3}-2a^{2/3}+$$

$$2a^{2/3}e^2+2a^{2/3}A_2-2A_2+\frac{5B_2}{2}a^{-2/3}-\frac{5B_2}{2}a^{2/3}\,] \tag{4.37}$$

and

$$(\xi-\xi_2)^2 = (\xi-1+\mu)^2$$

$$= \frac{1}{4}\,[\,1$$

$$-2\{\left(aq_1\right)^{2/3}\left[1-e^2-A_2+\frac{5B_2}{4}\right]$$

$$-a^{2/3}\left[1-e^2+A_2(a^{-2/3}-1)-\frac{5B_2}{4}\left(a^{\frac{-4}{3}}-1\right)\right]\}]$$

$$(\xi-1+\mu)^2 = \frac{1}{4}\,[\,1-2\,(aq_1)^{2/3}+2\,(aq_1)^{2/3}e^2+2\,(aq_1)^{2/3}A_2-\frac{5B_2}{2}(aq_1)^{2/3}+$$

$$2a^{2/3}-2a^{2/3}e^2-2a^{2/3}A_2+2A_2-\frac{5B_2}{2}a^{-2/3}+\frac{5B_2}{2}a^{2/3}\,] \tag{4.38}$$

Now for the second part of $\Omega_{\xi\xi}$,

i.e. $\frac{1}{n^2}\left\{\frac{3(1-\mu)(\xi-\xi_2)^2 q_1}{r_1^5}+\frac{3\mu(\xi-\xi_2)^2}{r_2^5}+\frac{15\mu(\xi-\xi_2)^2 A_2}{2r_2^7}-\frac{105\mu(\xi-\xi_2)^2 B_2}{8r_2^9}\right\}$

substituting $(\xi-\xi_1)^2, r_1^5, \& \frac{1}{n^2}$ respectively into $\frac{\mathbf{3(1-\mu)(\xi-\xi_2)^2 q_1}}{\boldsymbol{r_1^5}}$;

$$3(1-\mu)\left[q_1\left\{a\left[1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right]\frac{1}{(aq_1)^{\frac{5}{3}}}\left[1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{15B_2}{8}\right]\right\}\right]$$

$$\frac{3(1-\mu)}{(aq_1)^{2/3}}\left[1+e^2+A_2-\frac{5B_2}{4}\right](\xi-\xi_1)^2$$

$$\left\{\left[\frac{3(1-\mu)q_1}{4(aq_1)^{\frac{5}{3}}}\right.\right.\left[1+2(aq_1)^{2/3}-2(aq_1)^{2/3}e^2-2(aq_1)^{2/3}A_2\right.$$

$$\left.+\frac{5B_2}{2}\left(aq_1\right)^{2/3}-2a^{2/3}+2a^{2/3}e^2+2a^{2/3}A_2-2A_2+\frac{5B_2}{2}a^{-2/3}\right.$$

$$\left.\left.-\frac{5B_2}{2}a^{2/3}\right]\right\}$$

$$\frac{3(1-\mu)}{(aq_1)^{2/3}}\left[\frac{1}{4}+\frac{1}{4}e^2+\frac{1}{4}A_2-\frac{5}{16}B_2+\frac{(aq_1)^{2/3}}{2}+\frac{(aq_1)^{2/3}}{2}e^2+\frac{(aq_1)^{2/3}}{2}A_2\right.$$

$$-\frac{5(aq_1)^{2/3}}{8}B_2-\frac{(aq_1)^{2/3}}{2}e^2-\frac{(aq_1)^{2/3}}{2}A_2+\frac{5(aq_1)^{2/3}}{8}B_2-\frac{a^{2/3}}{2}$$

$$-\frac{a^{2/3}e^2}{2}-\frac{a^{2/3}A_2}{2}+\frac{5B_2}{8}a^{2/3}+\frac{a^{2/3}e^2}{2}-\frac{A_2}{2}+\frac{a^{2/3}A_2}{2}-\frac{5B_2}{2}a^{2/3}$$

$$\left.+\frac{5B_2}{2}a^{-2/3}\right]$$

$$\frac{3(1-\mu)}{(aq_1)^{2/3}}\left[\frac{1}{4}+\frac{1}{4}e^2-\frac{1}{4}A_2-\frac{5}{16}B_2+\frac{(aq_1)^{2/3}}{2}-\frac{a^{2/3}}{2}+\frac{5B_2}{2}a^{-2/3}\right]$$

$$\frac{3(1-\mu)}{4(aq_1)^{2/3}}+\frac{3(1-\mu)}{4(aq_1)^{2/3}}e^2-\frac{3(1-\mu)}{4(aq_1)^{2/3}}A_2-\frac{15(1-\mu)}{16(aq_1)^{2/3}}B_2+\frac{3(1-\mu)}{2}-\frac{3(1-\mu)}{2q_1{}^{2/3}}+\frac{15(1-\mu)B_2}{8a^{\frac{4}{3}}q_1{}^{2/3}} \quad (4.39)$$

for , $\frac{3\mu(\xi-\xi_2)^2}{n^2 r_2^5}$;

$$3\mu(\xi-\xi_2)^2\left\{a\left[\ 1-\ \frac{3e^2}{2}-\frac{3A_2}{2}+\ \frac{15B_2}{8}\right]\frac{1}{a^{\frac{5}{3}}}\left[\ 1+\ \frac{5e^2}{2}+\ \frac{5A_2}{2}-\ \frac{5A_2}{2}a^{-2/3}-\frac{25B_2}{8}\right.\right.$$

$$\left.\left.+\frac{25B_2}{8}a^{\frac{-4}{3}}\right]\right\}$$

$\frac{3\mu}{a^{2/3}}(\xi-\xi_2)^2\ [\ 1+\ e^2+A_2+-\ \frac{5A_2}{2}a^{-2/3}-\frac{5B_2}{4}+\frac{25B_2}{8}a^{\frac{-4}{3}}\ ]$; but

$$(\xi-\xi_2)^2=\frac{1}{4}-\frac{1}{2}\ (aq_1)^{2/3}\ +\ \frac{1}{2}\ (aq_1)^{2/3}e^2+\frac{1}{2}\ (aq_1)^{2/3}A_2-\ \frac{5B_2}{8}(aq_1)^{2/3}\ +\frac{1}{2}a^{2/3}$$

$$-\ \frac{1}{2}a^{2/3}e^2-\ \frac{1}{2}a^{2/3}A_2+\frac{1}{2}A_2-\ \frac{5B_2}{8}a^{-2/3}+\ \frac{5B_2}{8}a^{2/3}$$

$$\frac{3\mu}{a^{2/3}}\left[\ 1+\ e^2+A_2+-\ \frac{5A_2}{2}a^{-2/3}-\frac{5B_2}{4}+\frac{25B_2}{8}a^{\frac{-4}{3}}\right][\frac{1}{4}-\frac{1}{2}\ (aq_1)^{2/3}\ +\ \frac{1}{2}\ (aq_1)^{2/3}e^2$$

$$+\frac{1}{2}\ (aq_1)^{2/3}A_2-\ \frac{5B_2}{8}(aq_1)^{2/3}\ +\frac{1}{2}a^{2/3}-\ \frac{1}{2}a^{2/3}e^2-\ \frac{1}{2}a^{2/3}A_2+\frac{1}{2}A_2$$

$$-\ \frac{5B_2}{8}a^{-2/3}+\ \frac{5B_2}{8}a^{2/3}]$$

$$\frac{3\mu}{a^{2/3}}\ [\frac{1}{4}\ +\frac{1}{4}e^2+\ \frac{1}{4}A_2\ -\ \frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{16}-\ \frac{25B_2}{8}a^{\frac{-4}{3}}-\frac{1}{2}\ (aq_1)^{2/3}\ -\ \frac{1}{2}\ (aq_1)^{2/3}e^2$$

$$-\ \frac{1}{2}\ (aq_1)^{2/3}A_2+\ \frac{5}{4}q_1^{2/3}A_2+\frac{5B_2}{8}(aq_1)^{2/3}\ -\frac{25B_2}{8}a^{\frac{-4}{3}}q_1^{2/3}$$

$$+\ \frac{1}{2}\ (aq_1)^{2/3}e^2+\frac{1}{2}\ (aq_1)^{2/3}A_2-\ \frac{5B_2}{8}(aq_1)^{2/3}\ +\frac{1}{2}a^{2/3}+\ \frac{1}{2}a^{2/3}e^2$$

$$+\frac{1}{2}a^{2/3}A_2-\ \frac{5}{4}A_2-\frac{5}{8}a^{2/3}B_2+\ \frac{25B_2}{16}a^{-2/3}\ +\frac{1}{2}A_2-\ \frac{1}{2}a^{2/3}e^2-\ \frac{1}{2}a^{2/3}A_2$$

$$-\ \frac{5B_2}{8}a^{-2/3}+\ \frac{5B_2}{8}a^{2/3}]$$

$$\frac{3\mu}{a^{2/3}}\ [\frac{1}{4}\ +\frac{1}{4}e^2-\ \frac{1}{2}A_2\ -\ \frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{16}+\ \frac{25B_2}{32}a^{\frac{-4}{3}}-\frac{1}{2}\ (aq_1)^{2/3}\ +\ \frac{5}{4}q_1^{2/3}A_2$$

$$+\frac{25q_1^{2/3}}{16a^{2/3}}B_2\ +\frac{1}{2}a^{2/3}+\frac{5B_2}{16}a^{-2/3}]$$

$$\frac{3\mu}{4a^{2/3}} + \frac{3\mu}{4a^{2/3}}e^2 - \frac{3\mu}{2a^{2/3}}A_2 - \frac{15\mu}{8a^{\frac{4}{3}}}B_2 - \frac{15\mu B_2}{16a^{2/3}} + \frac{75\mu B_2}{32a^2} - \frac{3\mu}{2}q_1^{2/3} + \frac{15\mu}{4a^{2/3}}q_1^{2/3}A_2 - \frac{75\mu q_1^{2/3}}{16a^{\frac{4}{3}}}B_2 + \frac{3\mu}{2} + \frac{45\mu}{8a^{\frac{4}{3}}}B_2 \quad (4.40)$$

Now, $\frac{15\mu(\xi-\xi_2)^2A_2}{2n^2r_2^7}$

$$\frac{15\mu(\xi-\xi_2)^2A_2}{2}\{a\left[1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right]\frac{1}{a^{\frac{7}{3}}}\left[1+\frac{7e^2}{2}+\frac{7A_2}{2}-\frac{7A_2a^{-2/3}}{2}-\frac{35}{8}B_2+\frac{35}{8}B_2a^{\frac{-4}{3}}\right]\}$$

$$\frac{15\mu(\xi-\xi_2)^2A_2}{2a^{\frac{4}{3}}}\left[1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}+\frac{7e^2}{2}+\frac{7A_2}{2}-\frac{7A_2a^{-2/3}}{2}-\frac{35}{8}B_2+\frac{35}{8}B_2a^{\frac{-4}{3}}\right]$$

$$\frac{15\mu A_2}{2a^{\frac{4}{3}}}\left[1+e^2+A_2-\frac{7A_2a^{-2/3}}{2}-\frac{5}{2}B_2+\frac{35}{8}B_2a^{\frac{-4}{3}}\right][\frac{1}{4}-\frac{1}{2}(aq_1)^{2/3}+\frac{1}{2}(aq_1)^{2/3}e^2+\frac{1}{2}(aq_1)^{2/3}A_2-\frac{5B_2}{8}(aq_1)^{2/3}+\frac{1}{2}a^{2/3}-\frac{1}{2}a^{2/3}e^2-\frac{1}{2}a^{2/3}A_2+\frac{1}{2}A_2-\frac{5B_2}{8}a^{-2/3}+\frac{5B_2}{8}a^{2/3}]\frac{15\mu A_2}{2a^{\frac{4}{3}}}[\frac{1}{4}+\frac{e^2}{2}+\frac{A_2}{2}-\frac{5}{2}B_2-\frac{7A_2a^{-2/3}}{2}+\frac{35}{32}B_2a^{\frac{-4}{3}}-\frac{1}{2}(aq_1)^{2/3}-(aq_1)^{2/3}e^2-(aq_1)^{2/3}A_2+\frac{5}{4}(aq_1)^{2/3}B_2+\frac{7A_2q_1^{2/3}}{4}-\frac{35B_2}{16}a^{\frac{-4}{3}}q_1^{2/3}+\frac{1}{2}(aq_1)^{2/3}e^2+\frac{1}{2}(aq_1)^{2/3}A_2-\frac{5B_2}{8}(aq_1)^{2/3}+\frac{1}{2}a^{2/3}+a^{2/3}e^2+a^{2/3}A_2-\frac{5B_2}{4}a^{2/3}-\frac{7}{4}A_2+\frac{35B_2}{16}a^{-2/3}-\frac{1}{2}a^{2/3}e^2-\frac{1}{2}a^{2/3}A_2+\frac{1}{2}A_2-\frac{5B_2}{8}a^{-2/3}+\frac{5B_2}{8}a^{2/3}]$$

$$\frac{15\mu A_2}{2a^{\frac{4}{3}}}\left[\frac{1}{4}-\frac{1}{2}\,(aq_1)^{2/3}+\frac{1}{2}a^{2/3}\right]$$

$$\frac{15\mu A_2}{8a^{\frac{4}{3}}}-\frac{15\mu A_2}{4a^{2/3}}q_1^{2/3}+\frac{15\mu A_2}{4a^{2/3}} \tag{4.41}$$

and, $\frac{-105\mu(\xi-\xi_2)^2A_2}{8n^2r_2^9}$

$$\frac{-105\mu(\xi-\xi_2)^2A_2}{8}\left\{a\left[1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right]\frac{1}{a^{\frac{9}{3}}}\left[1+\frac{9e^2}{2}+\frac{9A_2}{2}-\frac{9A_2a^{-2/3}}{2}-\frac{45}{8}B_2+\frac{45}{8}B_2a^{\frac{-4}{3}}\right]\right\}$$

$$\frac{-105\mu(\xi-\xi_2)^2A_2}{8a^2}\left[1+3e^2+3A_2-\frac{15B_2}{4}-\frac{9A_2a^{-2/3}}{2}+\frac{45}{8}B_2a^{\frac{-4}{3}}\right]$$

$$\frac{-105\mu A_2}{8a^2}\left[1+3e^2+3A_2-\frac{15B_2}{4}-\frac{9A_2a^{-2/3}}{2}+\frac{45}{8}B_2a^{\frac{-4}{3}}\right]\left[\frac{1}{4}-\frac{1}{2}\,(aq_1)^{2/3}+\frac{1}{2}\,(aq_1)^{2/3}e^2+\frac{1}{2}\,(aq_1)^{2/3}A_2-\frac{5B_2}{8}(aq_1)^{2/3}+\frac{1}{2}a^{2/3}-\frac{1}{2}a^{2/3}e^2-\frac{1}{2}a^{2/3}A_2+\frac{1}{2}A_2-\frac{5B_2}{8}a^{-2/3}+\frac{5B_2}{8}a^{2/3}\right]$$

$$\frac{-105\mu A_2}{8a^2}\left[\frac{1}{4}+\frac{3}{4}e^2+\frac{3}{4}A_2-\frac{15B_2}{16}-\frac{9A_2a^{-2/3}}{8}+\frac{45}{32}B_2a^{\frac{-4}{3}}-\frac{1}{2}\,(aq_1)^{2/3}-\frac{3}{2}\,(aq_1)^{2/3}e^2-\frac{3}{2}\,(aq_1)^{2/3}A_2+\frac{15B_2}{8}(aq_1)^{2/3}+\frac{9A_2q_1^{2/3}}{4}-\frac{45B_2}{16}a^{\frac{-4}{3}}q_1^{2/3}+\frac{1}{2}\,(aq_1)^{2/3}e^2+\frac{1}{2}\,(aq_1)^{2/3}A_2-\frac{5B_2}{8}(aq_1)^{2/3}+\frac{1}{2}a^{2/3}+\frac{3}{2}a^{2/3}e^2+\frac{3}{2}a^{2/3}A_2-\frac{15B_2}{8}a^{2/3}-\frac{9}{4}A_2+\frac{45B_2}{16}a^{-2/3}-\frac{1}{2}a^{2/3}e^2-\frac{1}{2}a^{2/3}A_2+\frac{1}{2}A_2-\frac{5B_2}{8}a^{-2/3}+\frac{5B_2}{8}a^{2/3}\right] \tag{4.42}$$

Substituting (4.36), (4.39) – (4.42) into (4.35);

$$\Omega_{\xi\xi} = \frac{1}{(1-e^2)^{1/2}}$$

$$= [1 - 1 + \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{4(aq_1)^{2/3}}e^2 - \frac{3(1-\mu)}{4(aq_1)^{2/3}}A_2 - \frac{15(1-\mu)}{16(aq_1)^{2/3}}B_2$$

$$+ \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{15(1-\mu)B_2}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{4a^{2/3}}e^2 - \frac{3\mu}{2a^{2/3}}A_2$$

$$- \frac{15\mu}{8a^{\frac{4}{3}}}B_2 - \frac{15\mu B_2}{16a^{2/3}} + \frac{75\mu B_2}{32a^2} - \frac{3\mu}{2}q_1^{2/3} + \frac{15\mu}{4a^{2/3}}q_1^{2/3}A_2 - \frac{75\mu q_1^{2/3}}{16a^{\frac{4}{3}}}B_2$$

$$+ \frac{3\mu}{2} + \frac{45\mu}{8a^{\frac{4}{3}}}B_2 + \frac{15\mu A_2}{8a^{\frac{4}{3}}} - \frac{15\mu A_2}{4a^{2/3}}q_1^{2/3} + \frac{15\mu A_2}{4a^{2/3}} - \frac{-105\mu B_2}{32a^2}$$

$$+ \frac{105\mu B_2}{16a^{\frac{4}{3}}}q_1^{2/3} - \frac{105\mu B_2}{16a^{\frac{4}{3}}}]$$

$$\Omega_{\xi\xi} = \frac{1}{(1-e^2)^{1/2}}[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3} + \frac{3\mu}{2} + e^2\left\{\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \right.$$

$3\mu 4a^{2/3} + A29\mu 4a^{2/3} - 3\ 1-\mu 4(aq1)^{2/3} - B2 - 15\ 1-\mu 16(aq1)^{2/3} + 15$

$1-\mu 8a^{2/3}q1^{2/3} - 15\mu 4a^{2/3} + 15\mu q1^{2/3}8a^{2/3} - 15\mu 16a2 - 15\mu 4a43]$

(4.43)

Calculating $\Omega_{\eta\eta}$;

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{1/2}}\left[1 - \frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right\} + \frac{\eta_2}{n^2}\left\{\frac{3(1-\mu)q_1}{r_1^5} + \frac{3\mu}{r_2^5} + \frac{15\mu A_2}{2r_2^7} - \right.\right.$$

$105\mu B28r29]$ (4.44)

Recall from (4.36) above;

$$\frac{1}{n^2}\left\{\frac{(1-\mu)q_1}{r_1^3} + \frac{\mu}{r_2^3} + \frac{3\mu A_2}{2r_2^5} - \frac{15\mu B_2}{8r_2^7}\right\} = 1 \quad (4.45)$$

Now for the second part;

$$\frac{\eta_2}{n^2}\left\{\frac{3(1-\mu)q_1}{r_1^5} + \frac{3\mu}{r_2^5} + \frac{15\mu A_2}{2r_2^7} - \frac{105\mu B_2}{8r_2^9}\right\}$$

Substituting (4.3`1) into $\frac{3(1-\mu)q_1}{r_1^5}\eta_2$

$$3(1-\mu)\eta_2 = [\frac{(aq_1)}{(aq_1)^{\frac{5}{3}}}\left\{1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{25B_2}{8}-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right\}]$$

$$\frac{3(1-\mu)\eta_2}{(aq_1)}\ [1+e^2+A_2-\frac{5B_2}{4}]$$

$$\eta_2\ [\frac{3(1-\mu)}{(aq_1)^{2/3}}+\frac{3(1-\mu)}{(aq_1)^{2/3}}e^2+\frac{3(1-\mu)}{(aq_1)^{2/3}}A_2-\frac{15(1-\mu)B_2}{4(aq_1)^{2/3}}] \quad (4.46)$$

Substituting (4.32) into $\frac{3\mu\eta_2}{n^2r_2^5}$;

$$3\mu\eta_2\ \ [\frac{1}{a^{\frac{5}{3}}}\left\{1+\frac{5e^2}{2}+\frac{5A_2}{2}-\frac{5a^{-2/3}}{2}-\frac{25B_2}{8}+\frac{25B_2}{8}a^{\frac{-4}{3}}\right\}a\left\{1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right\}]$$

$$\frac{3\mu\eta_2}{a^{2/3}}\ [1+e^2+A_2-\frac{5B_2}{4}-\frac{5A_2a^{-2/3}}{2}+\frac{25B_2}{8}a^{\frac{-4}{3}}]$$

$$\eta_2\ [\frac{3\mu}{a^{2/3}}+\frac{3\mu}{a^{2/3}}e^2+\frac{3\mu}{a^{2/3}}A_2-\frac{15\mu B_2}{4a^{2/3}}-\frac{15\mu}{2a^{\frac{4}{3}}}A_2+\frac{75\mu B_2}{8a^2}] \quad (4.47)$$

Substituting (4.33) into $\frac{15\mu\eta_2 A_2}{2n^2r_2^7}$;

$$\frac{15\mu\eta_2 A_2}{2}\ [\frac{1}{a^{\frac{7}{3}}}\left\{1+\frac{7e^2}{2}+\frac{7A_2}{2}-\frac{7a^{-2/3}}{2}A_2-\frac{35B_2}{8}+\frac{35B_2}{8}a^{\frac{-4}{3}}\right\}a\left\{1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right\}]$$

$$\frac{15\mu\eta_2 A_2}{2a^{\frac{4}{3}}}\ [1+\ 2e^2+2A_2-\frac{7a^{-2/3}}{2}-\frac{5B_2}{2}+\frac{35B_2}{8}a^{\frac{-4}{3}}\ ]$$

$$\frac{15\mu\eta_2 A_2}{2a^{\frac{4}{3}}} \quad (4.48)$$

and substituting (4.34) into $-\frac{105\mu\eta_2 B_2}{8r_2^9}$;

$$-\frac{105\mu\eta_2 B_2}{8r_2^9}\left[\frac{1}{a^{\frac{9}{3}}}\left\{1+\frac{9e^2}{2}+\frac{9A_2}{2}-\frac{9a^{-2/3}A_2}{2}-\frac{45B_2}{8}+\frac{45B_2}{8}a^{\frac{-4}{3}}\right\}a\left\{1-\frac{3e^2}{2}-\frac{3A_2}{2}+\frac{15B_2}{8}\right\}\right]$$

$$\frac{-105\mu\eta_2 B_2}{8a^2} \tag{4.49}$$

Substituting (4.36), (4.46) – (4.49) into (4.44);

$$\Omega_{\eta\eta}=\frac{\eta_2}{(1-e^2)^{1/2}}\left[1-1+\frac{3(1-\mu)}{(aq_1)^{2/3}}+\frac{3(1-\mu)}{(aq_1)^{2/3}}e^2+\frac{3(1-\mu)}{(aq_1)^{2/3}}A_2-\frac{15(1-\mu)}{4(aq_1)^{2/3}}B_2+\frac{3\mu}{a^{2/3}}+\frac{3\mu}{a^{2/3}}e^2-\frac{3\mu}{a^{2/3}}A_2-\frac{15\mu B_2}{4a^{2/3}}-\frac{15\mu}{2a^{\frac{4}{3}}}A_2+\frac{75\mu B_2}{8a^2}+\frac{15\mu\eta_2 A_2}{2a^{\frac{4}{3}}}\frac{-105\mu\eta_2 B_2}{8a^2}\right] \tag{4.50}$$

$$\Omega_{\eta\eta}=\frac{\eta_2}{(1-e^2)^{1/2}}\left[\frac{3(1-\mu)}{(aq_1)^{2/3}}+\frac{3(1-\mu)}{(aq_1)^{2/3}}e^2+\frac{3(1-\mu)}{(aq_1)^{2/3}}A_2-\frac{15(1-\mu)}{4(aq_1)^{2/3}}+\frac{3\mu}{a^{2/3}}+\frac{3\mu}{a^{2/3}}e^2-\frac{3\mu}{a^{2/3}}A_2-\frac{15\mu B_2}{4a^{2/3}}\frac{-105\mu\eta_2 B_2}{8a^2}\right] \tag{4.51}$$

But; $$\eta_2=\frac{a^{2/3}}{2}-\frac{a^{2/3}}{2}e^2+\frac{A_2}{2}-\frac{a^{2/3}}{2}A_2-\frac{5a^{\frac{-2}{3}}}{8}B_2+\frac{5a^{2/3}}{8}B_2+\frac{(aq_1)^{2/3}}{2}-\frac{(aq_1)^{2/3}}{2}A_2+\frac{5(aq_1)^{2/3}}{8}B_2-\frac{1}{4} \tag{4.52}$$

$$\Omega_{\eta\eta} = \frac{1}{(1-e^2)^{1/2}}\left[1 - 1 - \frac{3(1-\mu)}{2q_1^{2/3}}e^2 - \frac{3\mu}{2}e^2 + \frac{3(1-\mu)}{2(aq_1)^{2/3}}A_2 + \frac{3\mu}{2}a^{2/3}A_2\right.$$

$$- \frac{3(1-\mu)}{2q_1^{2/3}}A_2 - \frac{3\mu}{2}A_2 - \frac{15(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}}A_2 - \frac{15\mu}{8a^{\frac{4}{3}}}B_2 + \frac{15(1-\mu)}{8q_1^{2/3}}B_2$$

$$+ \frac{15\mu}{8}B_2 + \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3(1-\mu)}{2q_1^{2/3}}e^2 + \frac{3(1-\mu)}{2q_1^{2/3}}A_2$$

$$- \frac{15(1-\mu)}{4q_1^{2/3}}B_2 + \frac{3\mu}{2} + \frac{3\mu}{2}e^2 + \frac{3\mu}{2}A_2 - \frac{15\mu}{8}B_2 - \frac{15\mu}{8a^2}B_2$$

$$+ \frac{3(1-\mu)}{2} + \frac{3(1-\mu)}{2}e^2 + \frac{3(1-\mu)}{2}A_2 - \frac{15(1-\mu)}{8}B_2$$

$$+ \frac{3\mu}{2q_1^{2/3}} + \frac{3\mu}{2q_1^{2/3}}e^2 + \frac{3\mu}{2q_1^{2/3}}A_2 - \frac{15\mu}{8q_1^{2/3}}B_2 - \frac{15\mu}{8a^{\frac{4}{3}}q_1^{2/3}}B_2$$

$$- \frac{3(1-\mu)}{2}e^2 - \frac{3\mu}{2q_1^{2/3}}e^2 - \frac{3(1-\mu)}{2}A_2 - \frac{3\mu}{2q_1^{2/3}}A_2$$

$$+ \frac{15(1-\mu)}{8}B_2 + \frac{15\mu}{8q_1^{2/3}}B_2 - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3(1-\mu)}{4(aq_1)^{2/3}}e^2$$

$$- \frac{3(1-\mu)}{4(aq_1)^{2/3}}A_2 + \frac{15(1-\mu)}{16(aq_1)^{2/3}}A_2 - \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{4a^{2/3}}e^2 - \frac{3\mu}{4a^{2/3}}B_2$$

$$\left. + \frac{15\mu}{16a^{2/3}}B_2 + \frac{15\mu}{8a^2}B_2\right]$$

$\Omega_{\eta\eta} =$

$\frac{1}{(1-e^2)^{1/2}}\left[\frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{2q_1^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2} + e^2\right\{ - \frac{3(1-\mu)}{4(aq_1)^{2/3}} -$

$3\mu 4a^{2/3} + A23\ 1-\mu 4aq1^{2/3} + 3\mu 4a^{2/3} + B2 - 15\ 1-\mu 8a43q1^{2/3} + 15\mu 16a2 -$

$15\mu 4a43 - 15\mu 8a43q1^{2/3} + 15\ 1-\mu 16(aq1)^{2/3} + 15\mu 16a^{2/3}]$

(4.54)

Substituting (4.43) and (4.53) into $\Omega_{\xi\xi} + \Omega_{\eta\eta}$;

$$\Omega_{\xi\xi} + \Omega_{\eta\eta} = \frac{1}{(1-e^2)^{1/2}}\left\{\left[\left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3}\right]\right.\right.$$

$$+ \left[\frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{2q_1^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2}\right] + \frac{3\mu}{2}$$

$$+ e^2\left\{\left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4a^{2/3}}\right] + \left[-\frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}}\right]\right\}$$

$$+ A_2\left\{\left[\frac{9\mu}{4a^{2/3}} - \frac{3(1-\mu)}{4(aq_1)^{2/3}}\right] + \left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4a^{2/3}}\right]\right\}$$

$$- B_2\left\{\left[-\frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^{2/3}q_1^{2/3}} - \frac{15\mu}{4a^{2/3}} + \frac{15\mu q_1^{2/3}}{8a^{2/3}} - \frac{15\mu}{16a^2}\right.\right.$$

$$\left.\left. - \frac{15\mu}{4a^{\frac{4}{3}}}\right]\right.$$

$$+ \left[-\frac{15(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{\frac{4}{3}}} - \frac{15\mu}{8a^{\frac{4}{3}}}q_1^{2/3} + \frac{15(1-\mu)}{16(aq_1)^{2/3}}\right.$$

$$\left.\left.\left.\left. + \frac{15\mu}{16a^{2/3}}\right]\right\}\right]\right\}$$

$$\Omega_{\xi\xi} + \Omega_{\eta\eta} = \frac{1}{(1-e^2)^{1/2}}\left[3(1-\mu) + 3\mu + A_2\left\{\frac{3\mu}{a^{2/3}}\right\} + B_2\left\{\frac{-15\mu}{2a^{\frac{4}{3}}}\right\}\right]$$

$$\Omega_{\xi\xi} + \Omega_{\eta\eta} = \frac{1}{(1-e^2)^{1/2}}\left[3 + \frac{3\mu}{a^{2/3}}A_2 - \frac{15\mu}{2a^{\frac{4}{3}}}B_2\right] \tag{4.55}$$

Calculating $\Omega_{\xi\eta}$;

$$\Omega_{\xi\eta} = \frac{\eta}{(1-e^2)^{1/2}}\left[\frac{1}{n^2}\left\{\frac{3(1-\mu)(\xi-\xi_2)^2 q_1}{r_1^5} + \frac{3\mu(\xi-\xi_2)^2}{r_2^5} + \frac{15\mu(\xi-\xi_2)^2 A_2}{2r_2^7}\right.\right.$$

$$\left.\left. - \frac{105\mu(\xi-\xi_2)^2 B_2}{8r_2^9}\right\}\right]$$

(4.56)

$$\xi+\mu=\frac{1}{2}+\frac{1}{2}\{(aq_1)^{2/3}\left[1-e^2-A_2+\frac{5B_2}{4}\right]$$

$$-a^{2/3}\left[1-e^2+A_2(a^{-2/3}-1)-\frac{5B_2}{4}\left(a^{\frac{-4}{3}}-1\right)\right]\}$$

$$\xi+\mu=\frac{1}{2}+\frac{1}{2}(aq_1)^{2/3}-\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}(aq_1)^{2/3}A_2+\frac{5B_2}{8}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}$$

$$+\frac{1}{2}a^{2/3}e^2+\frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{8}a^{2/3}$$

Therefore;

$$\frac{3(1-\mu)(\xi-\xi_2)^2q_1}{n^2r_1^5}$$

$$=\frac{3(1-\mu)q_1\eta}{(aq_1)^{2/3}}\left[\left\{1-e^2-A_2+\frac{5B_2}{4}\right\}\{\frac{1}{2}+\frac{1}{2}(aq_1)^{2/3}\right.$$

$$-\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}(aq_1)^{2/3}A_2+\frac{5B_2}{8}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}+\frac{1}{2}a^{2/3}e^2$$

$$\left.+\frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{8}a^{2/3}\}\right]$$

$$\frac{3(1-\mu)q_1\eta}{(aq_1)^{2/3}}\left[\frac{1}{2}+\frac{1}{2}(aq_1)^{2/3}-\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}(aq_1)^{2/3}A_2+\frac{5B_2}{8}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}\right.$$

$$+\frac{1}{2}a^{2/3}e^2+\frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{8}a^{2/3}+\frac{1}{2}e^2$$

$$+\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}a^{2/3}e^2+\frac{1}{2}A_2+\frac{1}{2}(aq_1)^{2/3}A_2-\frac{1}{2}a^{2/3}A_2-\frac{5B_2}{8}$$

$$\left.+\frac{5B_2}{8}(aq_1)^{2/3}+\frac{5B_2}{8}a^{2/3}A_2\right]$$

$$\frac{3(1-\mu)q_1\eta}{(aq_1)^{2/3}}\left[\frac{1}{2}+\frac{1}{2}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}-\frac{5B_2}{8}+\frac{1}{2}e^2+\frac{5B_2}{8}a^{-2/3}\right]$$

$$\eta\left[\frac{3(1-\mu)}{2(aq_1)^{2/3}}+\frac{3(1-\mu)}{2}-\frac{3(1-\mu)}{2q_1^{2/3}}-\frac{15(1-\mu)B_2}{8(aq_1)^{2/3}}+\frac{3(1-\mu)}{2(aq_1)^{2/3}}e^2+\frac{15(1-\mu)B_2}{8a^{\frac{4}{3}}q_1^{2/3}}\right] \quad (4.57)$$

$$(\xi-\xi_2)^2 = -\frac{1}{2}+\frac{1}{2}\{(aq_1)^{2/3}\left[1-e^2-A_2+\frac{5B_2}{4}\right]$$

$$-a^{2/3}\left[1-e^2+A_2(a^{-2/3}-1)-\frac{5B_2}{4}\left(a^{\frac{-4}{3}}-1\right)\right]\}$$

$$(\xi-\xi_2)^2 = -\frac{1}{2}+\frac{1}{2}(aq_1)^{2/3}-\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}(aq_1)^{2/3}A_2+\frac{5B_2}{8}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}$$

$$+\frac{1}{2}a^{2/3}e^2+\frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{8}a^{2/3}\frac{3\mu(\xi-\xi_2)^2\eta}{n^2r_2^5}$$

$$=\frac{3\mu}{a^{2/3}}\eta\,[\{1+e^2+A_2-\frac{5a^{-2/3}A_2}{4}-\frac{5B_2}{4}+\frac{25a^{\frac{-4}{3}}B_2}{4}\}\{-\frac{1}{2}$$

$$+\frac{1}{2}(aq_1)^{2/3}-\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}(aq_1)^{2/3}A_2+\frac{5B_2}{8}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}$$

$$+\frac{1}{2}a^{2/3}e^2+\frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{8}a^{2/3}\}]$$

$$\frac{3\mu}{a^{2/3}}\eta\,[-\frac{1}{2}+\frac{1}{2}(aq_1)^{2/3}-\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}(aq_1)^{2/3}A_2+\frac{5B_2}{8}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}$$

$$+\frac{1}{2}a^{2/3}e^2+\frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\frac{5B_2}{8}a^{-2/3}-\frac{5B_2}{8}a^{2/3}-\frac{1}{2}e^2$$

$$+\frac{1}{2}(aq_1)^{2/3}e^2-\frac{1}{2}a^{2/3}e^2-\frac{1}{2}A_2+\frac{1}{2}(aq_1)^{2/3}A_2-\frac{1}{2}a^{2/3}A_2+\frac{5a^{-2/3}A_2}{4}$$

$$-\frac{5A_2}{4}q_1^{2/3}+\frac{5A_2}{4}+\frac{5B_2}{8}-\frac{5B_2}{8}(aq_1)^{2/3}+\frac{5a^{2/3}B_2}{8}-\frac{25a^{\frac{-4}{3}}B_2}{16}$$

$$+\frac{25q_1^{2/3}B_2}{16a^{2/3}}-\frac{25B_2}{16a^{2/3}}]$$

$$\frac{3\mu}{a^{2/3}}\eta\,[-\frac{1}{2}+\frac{1}{2}(aq_1)^{2/3}-\frac{1}{2}a^{2/3}+\frac{1}{2}A_2-\frac{15B_2}{16a^{2/3}}-\frac{1}{2}e^2+\frac{5a^{-2/3}A_2}{4}-\frac{5A_2}{4}q_1^{2/3}$$

$$+\frac{5B_2}{8}+\frac{25a^{\frac{-4}{3}}B_2}{16}+\frac{25q_1^{2/3}B_2}{16a^{2/3}}]$$

$$\eta\left[-\frac{3\mu}{2a^{2/3}}+\frac{3\mu}{2}q_1^{2/3}-\frac{3\mu}{2}+\frac{3\mu}{4a^{2/3}}A_2-\frac{45\mu B_2}{16\,a^{\frac{4}{3}}}-\frac{3\mu}{2a^{2/3}}e^2+\frac{45\mu A_2}{4\,a^{\frac{4}{3}}}-\frac{15q_1^{2/3}B_2}{4a^{2/3}}+\frac{15\mu B_2}{8a^{2/3}}-\right.$$

$$\left.\frac{75B_2}{16a^2}+\frac{75\mu q_1^{2/3}B_2}{16\,a^{\frac{4}{3}}}\right] \qquad (4.58)$$

Also; $\frac{15\mu(\xi-\xi_2)^2A_2}{2n^2r_2^7}$;

$$\frac{15\mu A_2\eta}{2\,a^{\frac{4}{3}}} = [\{1+\ 2e^2+\ 2A_2-\ \frac{7a^{-2/3}A_2}{2}-\ \frac{5B_2}{2}+\frac{35a^{\frac{-4}{3}}B_2}{4}\}\{-\ \frac{1}{2}+\frac{1}{2}\ (aq_1)^{2/3}$$
$$-\ \frac{1}{2}\ (aq_1)^{2/3}\,e^2-\frac{1}{2}\ (aq_1)^{2/3}\,A_2+\ \frac{5B_2}{8}(aq_1)^{2/3}\ -\frac{1}{2}a^{2/3}\ +\ \frac{1}{2}a^{2/3}e^2$$
$$+\ \frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\ \frac{5B_2}{8}a^{-2/3}-\ \frac{5B_2}{8}a^{2/3}\}]$$

$\frac{15\mu A_2\eta}{2\,a^{\frac{4}{3}}}[-\ \frac{1}{2}\ +\frac{1}{2}\ (aq_1)^{2/3}\ -\frac{1}{2}a^{2/3}]$

$\eta[-\frac{15\mu A_2}{4\,a^{\frac{4}{3}}}\ +\frac{15\mu A_2}{4\,a^{\frac{4}{3}}}q_1{}^{2/3}-\frac{15\mu A_2}{4\,a^{2/3}}]$ (4.59)

and $-\ \frac{105\mu(\xi-\xi_2)^2B_2}{8r_2^9}$;

$$-\frac{105\mu B_2}{8a^2}\eta\ [\{1+\ 3e^2+\ 3A_2-\ \frac{9a^{-2/3}A_2}{2}-\ \frac{15B_2}{4}+\frac{45a^{\frac{-4}{3}}B_2}{8}\}\{-\ \frac{1}{2}+\frac{1}{2}\ (aq_1)^{2/3}$$
$$-\ \frac{1}{2}\ (aq_1)^{2/3}\,e^2-\frac{1}{2}\ (aq_1)^{2/3}\,A_2+\ \frac{5B_2}{8}(aq_1)^{2/3}\ -\frac{1}{2}a^{2/3}\ +\ \frac{1}{2}a^{2/3}e^2$$
$$+\ \frac{1}{2}a^{2/3}A_2-\frac{1}{2}A_2+\ \frac{5B_2}{8}a^{-2/3}-\ \frac{5B_2}{8}a^{2/3}\}]$$

$\frac{105\mu B_2}{8a^2}\eta[-\ \frac{1}{2}\ +\frac{1}{2}\ (aq_1)^{2/3}\ -\frac{1}{2}a^{2/3}]$

$\eta[\frac{105\mu B_2}{16a^2}-\frac{105\mu B_2}{16\,a^{\frac{4}{3}}}q_1{}^{2/3}+\frac{105\mu B_2}{16\,a^{\frac{4}{3}}}]$ (4.60)

Substituting (4.57) – (4.50) into (4.56) we have;

$$\Omega_{\xi\eta} = \frac{\eta}{(1-e^2)^{1/2}}\left[\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{15(1-\mu)B_2}{8(aq_1)^{2/3}}\right.$$

$$+ \frac{3(1-\mu)}{2(aq_1)^{2/3}}e^2 + \frac{15(1-\mu)B_2}{8a^{\frac{4}{3}}q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2}q_1^{2/3} - \frac{3\mu}{2} + \frac{3\mu}{4a^{2/3}}A_2$$

$$- \frac{45\mu B_2}{16\,a^{\frac{4}{3}}} - \frac{3\mu}{2a^{2/3}}e^2 + \frac{45\mu A_2}{4\,a^{\frac{4}{3}}} - \frac{15q_1^{2/3}B_2}{4a^{2/3}} + \frac{15\mu B_2}{8a^{2/3}} - \frac{75B_2}{16a^2}$$

$$+ \frac{75\mu q_1^{2/3}B_2}{16\,a^{\frac{4}{3}}} - \frac{15\mu A_2}{4\,a^{\frac{4}{3}}} + \frac{15\mu A_2}{4\,a^{\frac{4}{3}}}q_1^{2/3} - \frac{15\mu A_2}{4\,a^{2/3}} + \frac{105\mu B_2}{16a^2}$$

$$\left. - \frac{105\mu B_2}{16\,a^{\frac{4}{3}}}q_1^{2/3} + \frac{105\mu B_2}{16\,a^{\frac{4}{3}}}\right]$$

to obtain

$$\Omega_{\xi\eta} = \frac{\eta}{(1-e^2)^{1/2}}\left[\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2}q_1^{2/3} - \frac{3\mu}{2} + e^2\left\{\frac{3(1-\mu)}{2(aq_1)^{2/3}} - \right.\right.$$

$$3\mu 2a^{2/3} + A2 - 3\mu a^{2/3} + B2\{-15\,1-\mu B28(aq1)^{2/3} + 15\,1-\mu 8a43q1^{2/3} + 15\mu 4\,a43 +$$

$$\left.\left.\frac{15\mu}{8\,a^{2/3}} + \frac{15\mu}{8a^2} - + \frac{15\mu}{8\,a^{\frac{4}{3}}}q_1^{2/3}\right\}\right] \tag{4.61}$$

Now, calculating $\left(\Omega_{\xi\eta}\right)^2$, i.e.

$$\left(\Omega_{\xi\eta}\right)^2 = \left\{\frac{\eta}{(1-e^2)^{1/2}}\left[\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2}q_1^{2/3} - \frac{3\mu}{2} + \right.\right.$$

$$e23\;1-\mu 2(aq1)^{2/3} - 3\mu 2a^{2/3} + A2 - 3\mu a^{2/3} + B2\{-15\;1-\mu B28(aq1)^{2/3} + 15$$

$$\left.\left.1-\mu 8a43q1^{2/3} + 15\mu 4\,a43 + 15\mu 8\,a^{2/3} + 15\mu 8a2 - 15\mu 8\,a43q1^{2/3}\right\}\right]\}2$$

(4.62)

this can be re –written as;

$$\left(\Omega_{\xi\eta}\right)^2 = \frac{\eta^2}{(1-e^2)}[L + Me^2 + NA_2 + OB_2]^2 \tag{4.63}$$

where;

$$L = \frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2}q_1^{2/3} - \frac{3\mu}{2}$$

$$M = \frac{3(1-\mu)}{2(aq_1)^{2/3}} - \frac{3\mu}{2a^{2/3}}$$

$$N = -\frac{3\mu}{a^{2/3}}$$

$$O = -\frac{15(1-\mu)B_2}{8(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{15\mu}{4\,a^{\frac{4}{3}}} + \frac{15\mu}{8\,a^{2/3}} + \frac{15\mu}{8a^2} - + \frac{15\mu}{8\,a^{\frac{4}{3}}}q_1^{2/3}$$

$$\left(\Omega_{\xi\eta}\right)^2 = \frac{\eta^2}{(1-e^2)}[L^2 + LMe^2 + LMe^2 + LNA_2 + LNA_2 + LOB_2 + LOB_2]$$

$$\left(\Omega_{\xi\eta}\right)^2 = \frac{\eta^2}{(1-e^2)}[L^2 + 2LMe^2 + 2LNA_2 + 2LOB_2] \tag{4.64}$$

also η^2;

$$\eta^2 = \frac{a^{2/3}}{2} - \frac{a^{2/3}}{2}e^2 + \frac{A_2}{2} - \frac{a^{2/3}}{2}A_2 - \frac{5a^{-2/3}}{8}B_2 + \frac{5a^{2/3}}{2}B_2 + \frac{(aq_1)^{2/3}}{8} - \frac{(aq_1)^{2/3}}{2}e^2 - \frac{(aq_1)^{2/3}}{2}A_2 + \frac{5(aq_1)^{2/3}}{8}B_2 - \frac{1}{4}$$

factor out $\frac{1}{2}$ to obtain;

$$\eta^2 = \frac{1}{2}[\,a^{2/3} + (aq_1)^{2/3} - \frac{1}{2} - a^{2/3}e^2 - (aq_1)^{2/3}e^2 + A_2 - a^{2/3}A_2 - (aq_1)^{2/3}A_2 - \frac{5a^{-2/3}}{4}B_2 + \frac{5a^{2/3}}{4}B_2 + \frac{5(aq_1)^{2/3}}{4}B_2$$

$$\eta^2 = \frac{1}{2}[\,a^{2/3} + (aq_1)^{2/3} - \frac{1}{2} - e^2\{a^{2/3} + (aq_1)^{2/3}\} + A_2\{1 - a^{2/3} - (aq_1)^{2/3}\} - B_2\left\{\frac{5a^{-2/3}}{4} - \frac{5a^{2/3}}{4} - \frac{5(aq_1)^{2/3}}{4}\right\}] \tag{4.65}$$

which can be re – written as;

$$\eta^2 = \frac{1}{2}[\, Q - \, Se^2 + \, RA_2 \, - \, TB_2]$$

where;

$$Q = a^{2/3} + (aq_1)^{2/3} - \frac{1}{2}$$

$$S = \, a^{2/3} + \, \, (aq_1)^{2/3}$$

$$R = 1 - \, \, a^{2/3} \, - \, (aq_1)^{2/3}$$

$$T = \frac{5a^{-2/3}}{4} - \frac{5a^{2/3}}{4} - \frac{5(aq_1)^{2/3}}{4}$$

$$\left(\Omega_{\xi\eta}\right)^2 = \frac{1}{2(1 - \, e^2)}[Q - \, Se^2 + \, RA_2 \, - \, TB_2][L^2 \, + \, 2LMe^2 + 2LNA_2 + 2LOB_2]$$

$$\left(\Omega_{\xi\eta}\right)^2 = \frac{1}{2(1-e^2)}\, [\, QL^2 + \, e^2\{2LMQ - \, SL^2\} + \, A_2\,\{2LNQ + \, RL^2\} + B_2\,\{2LOQ -$$

$\mathcal{I}L2]$ (4.66)

Calculating L^2

$$L^2 = [\frac{3(\,1-\mu)}{2(aq_1)^{2/3}} \, + \frac{3(\,1-\mu)}{2} - \frac{3(\,1-\mu)}{2q_1{}^{2/3}} - \frac{3\mu}{2a^{2/3}} \, + \frac{3\mu}{2} q_1{}^{2/3} \, - \frac{3\mu}{2}\,][\frac{3(\,1-\mu)}{2(aq_1)^{2/3}}$$

$$+ \frac{3(\,1-\mu)}{2} - \frac{3(\,1-\mu)}{2q_1{}^{2/3}} - \frac{3\mu}{2a^{2/3}} \, + \frac{3\mu}{2} q_1{}^{2/3} \, - \frac{3\mu}{2}\,]$$

$$L^2 = \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}}$$

$$- \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{4} - \frac{9(1-\mu)^2}{4q_1^{2/3}} - \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{4}q_1^{2/3} - \frac{9\mu(1-\mu)}{4} - \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{4q_1^{2/3}}$$

$$+ \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{4} + \frac{9\mu(1-\mu)}{4q_1^{2/3}}$$

$$- \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{9\mu^2}{4a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu^2}{4a^{\frac{2}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4}q_1^{2/3} - \frac{9\mu(1-\mu)}{4} - \frac{9\mu^2}{4a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu^2}{4}q_1^{\frac{4}{3}}$$

$$- \frac{9\mu^2}{4}q_1^{2/3} - \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{4} + \frac{9\mu(1-\mu)}{4q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{2}{3}}}$$

$$- \frac{9\mu^2}{4}q_1^{2/3} + \frac{9\mu^2}{4a^{\frac{2}{3}}}$$

$L^2 =$

$$\frac{9(1-\mu)^2}{4} - 9\mu(1-\mu) + \frac{9\mu^2}{4} + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}} -$$

$$\frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{2q_1^{2/3}} + \frac{9\mu(1-\mu)}{2}q_1^{2/3} + \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} - \frac{9\mu^2}{2a^{\frac{2}{3}}}q_1^{2/3} +$$

$$\frac{9\mu^2}{4}q_1^{\frac{4}{3}} - \frac{9\mu^2}{2}q_1^{2/3} \qquad (4.67)$$

Calculating QL^2;

$$QL^2 = \left[\frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}}\right.$$

$$+ \frac{9(1-\mu)^2}{4} - \frac{9(1-\mu)^2}{2q_1^{2/3}} + \frac{9\mu(1-\mu)}{4}q_1^{2/3} - 9\mu(1-\mu)$$

$$+ \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{9\mu^2}{2a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9\mu^2}{4}q_1^{\frac{4}{3}}$$

$$\left. - \frac{9\mu^2}{2}q_1^{2/3} + \frac{9\mu^2}{4}\right][a^{2/3} + (aq_1)^{2/3} - \frac{1}{2}]$$

$$QL^2 = \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4} + \frac{9(1-\mu)^2 (aq_1)^{\frac{2}{3}}}{4} - \frac{9(1-\mu)^2}{8} - 9\mu(1-\mu)a^{\frac{2}{3}}$$

$$- 9\mu(1-\mu)(aq_1)^{\frac{2}{3}} - \frac{9\mu(1-\mu)}{2} + \frac{9\mu^2 a^{\frac{2}{3}}}{4} + \frac{9\mu^2 (aq_1)^{\frac{2}{3}}}{4} - \frac{9\mu^2}{8}$$

$$+ \frac{9\mu^2}{2} + \frac{9\mu^2 q_1^{2/3}}{2} + \frac{9\mu^2}{4a^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{4a^{\frac{2}{3}} q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}}$$

$$+ \frac{9(1-\mu)^2}{2q_1^{2/3}} + \frac{9(1-\mu)^2}{2} - \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{2q_1^{\frac{2}{3}}}$$

$$+ \frac{9(1-\mu)^2}{4a^{\frac{2}{3}} q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}} q_1^{\frac{2}{3}}} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{2q_1^{2/3}} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{2} + \frac{9(1-\mu)^2}{4q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{2}{3}}}$$

$$+ \frac{9\mu^2}{4a^{\frac{2}{3}}} q_1^{2/3} - \frac{9\mu^2}{8a^{\frac{4}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} - \frac{9\mu(1-\mu)}{4q_1^{\frac{2}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{2}(aq_1)^{\frac{2}{3}} - \frac{9\mu(1-\mu)}{2} a^{\frac{2}{3}} q_1^{\frac{4}{3}} - \frac{9\mu(1-\mu)}{4} q_1^{2/3}$$

$$+ \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8q_1^{\frac{4}{3}}} - \frac{9\mu^2}{2} q_1^{2/3} - \frac{9\mu^2}{2} q_1^{\frac{4}{3}}$$

$$+ \frac{9\mu^2}{4a^{\frac{2}{3}}} q_1^{2/3} + \frac{9\mu^2}{4} a^{\frac{2}{3}} q_1^{\frac{4}{3}} + \frac{9\mu^2}{4} a^{\frac{2}{3}} q_1^2 - \frac{9\mu^2}{8} q_1^{\frac{4}{3}} - \frac{9\mu^2}{2}(aq_1)^{\frac{2}{3}} - \frac{9\mu^2}{2} a^{\frac{2}{3}} q_1^{\frac{4}{3}}$$

$$+ \frac{9\mu^2}{4} q_1^{\frac{4}{3}}$$

$$QL^2 = -\frac{9\mu(1-\mu)}{4}a^{2/3} + \frac{9(1-\mu)^2(aq_1)^{\frac{2}{3}}}{4} + \frac{27(1-\mu)^2}{8} + \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}$$

$$-\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4q_1^{2/3}} - \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2a^{2/3}}{4q_1^{2/3}}$$

$$+\frac{9(1-\mu)^2a^{2/3}}{4q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} - \frac{9\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{2} + \frac{9\mu(1-\mu)}{2}$$

$$-\frac{9\mu(1-\mu)}{2(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}}$$

$$-\frac{9\mu(1-\mu)}{4q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} - \frac{9\mu(1-\mu)q_1^{2/3}}{4} + \frac{9\mu^2a^{\frac{2}{3}}}{4}$$

$$-\frac{9\mu^2(aq_1)^{\frac{2}{3}}}{4} + \frac{27\mu^2}{8} + \frac{9\mu^2q_1^{2/3}}{4} + \frac{9\mu^2q_1^{2/3}}{2a^{\frac{2}{3}}} - \frac{9\mu^2}{8a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{4}{3}}}{8}$$

$$-\frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} + \frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4} \qquad (4.68)$$

Calculating $e^2\{2LMQ - SL^2\}$

$$LM = [\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2}q_1^{2/3} - \frac{3\mu}{2}][\frac{3(1-\mu)}{2(aq_1)^{2/3}}$$

$$-\frac{3\mu}{2a^{2/3}}]$$

$$LM = \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{2/3}}$$

$$- \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{4a^{2/3}} + \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{9\mu^2}{4a^{\frac{4}{3}}}$$

$$- \frac{9\mu^2 q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}} + \frac{9\mu^2}{4a^{\frac{2}{3}}}$$

$$LM = \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}}$$

$$+ \frac{9\mu^2}{4a^{\frac{2}{3}}}$$

$$2LM = \frac{9(1-\mu)^2}{2(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu^2}{2a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}}$$

$$+ \frac{9\mu^2}{2a^{\frac{2}{3}}}$$

then $2LMQ$

$$2LMQ = \left[\frac{9(1-\mu)^2}{2(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu^2}{2a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}}\right.$$

$$\left. + \frac{9\mu^2}{2a^{\frac{2}{3}}}\right]\left[a^{2/3} + (aq_1)^{2/3} - \frac{1}{2}\right]$$

$$2LMQ = \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2q_1^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{(aq_1)^{2/3}} + \frac{9\mu^2}{2a^{\frac{2}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{2}$$

$$+ \frac{9\mu^2}{2} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{2} - \frac{9(1-\mu)^2}{2q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{a^{\frac{2}{3}}}$$

$$+ \frac{9\mu^2 q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}} - \frac{9\mu^2 q_1^{\frac{4}{3}}}{2} + \frac{9\mu^2 q_1^{\frac{2}{3}}}{2} - \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{4(aq_1)^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu^2}{4a^{\frac{4}{3}}} + \frac{9\mu^2 q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}} - \frac{9\mu^2}{4a^{\frac{2}{3}}}$$

$$2LMQ = \frac{27(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{(aq_1)^{2/3}} + \frac{9\mu^2}{4a^{\frac{2}{3}}} + \frac{9\mu^2}{2} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{2} -$$

$$\frac{9\mu(1-\mu)}{a^{\frac{2}{3}}} + \frac{27\mu^2 q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}} - \frac{9\mu^2 q_1^{\frac{4}{3}}}{2} - \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu^2}{4a^{\frac{4}{3}}} \qquad (4.69)$$

Now SL^2;

$$SL^2 = \left[\frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}}\right.$$

$$+ \frac{9(1-\mu)^2}{4} - \frac{9(1-\mu)^2}{2q_1^{2/3}} + \frac{9\mu(1-\mu)}{4}q_1^{2/3} - 9\mu(1-\mu)$$

$$\left. + \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{9\mu^2}{2a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9\mu^2}{4}q_1^{\frac{4}{3}}\right.$$

$$\left. - \frac{9\mu^2}{2}q_1^{2/3} + \frac{9\mu^2}{4}\right][a^{2/3} + (aq_1)^{2/3}]$$

$$SL^2 = \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2q_1^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} + \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4}$$

$$- \frac{9(1-\mu)^2a^{\frac{2}{3}}}{2q_1^{2/3}} + \frac{9\mu(1-\mu)}{4}(aq_1)^{2/3} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{1} + \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}}$$

$$+ \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{2}{3}}} - \frac{9\mu^2}{2}q_1^{2/3} + \frac{9\mu^2}{2} + \frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2}{2}(aq_1)^{2/3}$$

$$+ \frac{9\mu^2a^{\frac{2}{3}}}{4} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{2} - \frac{9(1-\mu)^2}{2q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}}$$

$$+ \frac{9(1-\mu)^2(aq_1)^{2/3}}{4} - \frac{9(1-\mu)^2a^{\frac{2}{3}}}{2} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2}$$

$$- \frac{9\mu(1-\mu)(aq_1)^{2/3}}{1} + \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{2/3}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} + \frac{9\mu^2q_1^{2/3}}{4a^{\frac{2}{3}}}$$

$$- \frac{9\mu^2q_1^{\frac{4}{3}}}{2} + \frac{9\mu^2q_1^{2/3}}{2} + \frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4} - \frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} + \frac{9\mu^2(aq_1)^{2/3}}{4}$$

$$SL^2 = \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} - \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4} - \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{2/3}} - \frac{9\mu(1-\mu)(aq_1)^{2/3}}{2} -$$

$$\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} + \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{2}{3}}} + \frac{9\mu^2}{2} - \frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2}{4}(aq_1)^{2/3} + \frac{9\mu^2a^{\frac{2}{3}}}{4} +$$

$$\frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{2} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9(1-\mu)^2(aq_1)^{2/3}}{4} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} + \frac{9\mu^2q_1^{2/3}}{4a^{\frac{2}{3}}} - \frac{9\mu^2}{2}q_1^{\frac{4}{3}} +$$

$$\frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4} \qquad (4.70)$$

Hence $e^2\{2LMQ - SL^2\}$;

$$e^2\{2LMQ - SL^2\}$$

$$= e^2\{\frac{27(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{(aq_1)^{2/3}} + \frac{9\mu^2}{4a^{\frac{2}{3}}} + \frac{9\mu^2}{2}$$

$$+ \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{2} - \frac{9\mu(1-\mu)}{a^{\frac{2}{3}}} + \frac{27\mu^2 q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}} - \frac{9\mu^2 q_1^{\frac{4}{3}}}{2}$$

$$- \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu^2}{4a^{\frac{4}{3}}} - [\frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}}$$

$$- \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{2/3}} - \frac{9\mu(1-\mu)(aq_1)^{2/3}}{2}$$

$$- \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} + \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{2}{3}}} + \frac{9\mu^2}{2}$$

$$- \frac{9\mu^2 a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2}{4}(aq_1)^{2/3} + \frac{9\mu^2 a^{\frac{2}{3}}}{4} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{2}$$

$$- \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9(1-\mu)^2(aq_1)^{2/3}}{4} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} + \frac{9\mu^2 q_1^{2/3}}{4a^{\frac{2}{3}}}$$

$$- \frac{9\mu^2}{2}q_1^{\frac{4}{3}} + \frac{9\mu^2 a^{\frac{2}{3}}q_1^2}{4}]\}$$

$$e^2\{2LMQ - SL^2\} = e^2\{\frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} + \frac{9\mu^2 q_1^{2/3}}{2a^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu^2}{4a^{\frac{4}{3}}} +$$

$$\frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4} + \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)(aq_1)^{2/3}}{2} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{9\mu^2 a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} +$$

$$\frac{9\mu^2}{4}(aq_1)^{2/3} + \frac{9\mu^2}{4}a^{\frac{2}{3}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} - \frac{9(1-\mu)^2(aq_1)^{2/3}}{4} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} - \frac{9\mu^2 a^{\frac{2}{3}}q_1^2}{4} +$$

$$\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}}\} \qquad (4.71)$$

Calculating $A_2\{2LNQ + RL^2\}$

Now $LN = \left[\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2}q_1^{2/3} - \frac{3\mu}{2}\right]\left[-\frac{3\mu}{a^{2/3}}\right]$

$$LN = -\frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} + \frac{9\mu^2}{2a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}} + \frac{9\mu^2}{2a^{\frac{2}{3}}}$$

$$2LN = -\frac{9\mu(1-\mu)}{a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{a^{\frac{2}{3}}} + \frac{9\mu^2}{a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{(aq_1)^{2/3}} + \frac{9\mu^2}{a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{a^{\frac{2}{3}}}$$

now,

$$2LNQ = \left[-\frac{9\mu(1-\mu)}{a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{a^{\frac{2}{3}}} + \frac{9\mu^2}{a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{(aq_1)^{2/3}} + \frac{9\mu^2}{a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{a^{\frac{2}{3}}}\right]\left[a^{2/3} + (aq_1)^{2/3} - \frac{1}{2}\right]$$

$$2LNQ = -\frac{9\mu(1-\mu)}{(aq_1)^{2/3}} - \frac{9\mu(1-\mu)}{a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - 9\mu(1-\mu) - 9\mu(1-\mu)q_1^{\frac{2}{3}} + \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + 9\mu^2 + 9\mu^2 q_1^{\frac{2}{3}} - \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{q_1^{\frac{2}{3}}} + 9\mu(1-\mu) - \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} + \frac{9\mu^2}{a^{\frac{2}{3}}} + \frac{9\mu^2 q_1^{\frac{2}{3}}}{a^{\frac{2}{3}}} - \frac{9\mu^2}{2a^{\frac{4}{3}}} - 9\mu^2 q_1^{\frac{2}{3}} - 9\mu^2 q_1^{\frac{4}{3}} + \frac{9\mu^2 q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}}$$

$$2LNQ = -\frac{27\mu(1-\mu)}{2(aq_1)^{2/3}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - 9\mu(1-\mu)q_1^{\frac{2}{3}} + 9\mu^2 + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{q_1^{\frac{2}{3}}} + \frac{27\mu^2 q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}} - \frac{9\mu^2}{2a^{\frac{4}{3}}} - 9\mu^2 q_1^{\frac{4}{3}} \quad (4.72)$$

also RL^2

$$RL^2 = \left[\frac{9(1-\mu)^2}{4} - 9\mu(1-\mu) + \frac{9\mu^2}{4} + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}}\right.$$

$$- \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{2q_1^{2/3}}$$

$$\left. + \frac{9\mu(1-\mu)}{2}q_1^{2/3} + \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} - \frac{9\mu^2}{2a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu^2}{4}q_1^{\frac{4}{3}} - \frac{9\mu^2}{2}q_1^{2/3}\right][1$$

$$- a^{2/3} - (aq_1)^{2/3}]$$

$$RL^2 = \frac{9(1-\mu)^2}{4} - \frac{9(1-\mu)^2 a^{2/3}}{4} - \frac{9(1-\mu)^2 (aq_1)^{2/3}}{4} - 9\mu(1-\mu)$$

$$+ 9\mu(1-\mu)a^{2/3} + 9\mu(1-\mu)(aq_1)^{2/3} + \frac{9\mu^2}{4} - \frac{9\mu^2 a^{2/3}}{4}$$

$$- \frac{9\mu^2}{4}(aq_1)^{2/3} + \frac{9\mu^2}{2a^{\frac{2}{3}}} - \frac{9\mu^2}{2} - \frac{9\mu^2 q_1^{2/3}}{2} + \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}}$$

$$- \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{2(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{2q_1^{2/3}} - \frac{9(1-\mu)^2}{2} - \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}$$

$$+ \frac{9(1-\mu)^2}{2q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2q_1^{2/3}} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}}$$

$$+ \frac{9\mu(1-\mu)}{2a^{2/3}} - \frac{9(1-\mu)^2}{2q_1^{2/3}} + \frac{9(1-\mu)^2 a^{2/3}}{2q_1^{2/3}} + \frac{9(1-\mu)^2 a^{2/3}}{2} + \frac{9\mu^2}{4a^{\frac{4}{3}}}$$

$$- \frac{9\mu^2}{4a^{\frac{2}{3}}} - \frac{9\mu^2}{4a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu(1-\mu)}{2q_1^{2/3}} - \frac{9\mu(1-\mu)a^{2/3}}{2q_1^{2/3}} - \frac{9\mu(1-\mu)a^{2/3}}{2}$$

$$+ \frac{9\mu(1-\mu)}{2}q_1^{2/3} - \frac{9\mu(1-\mu)}{2}(aq_1)^{2/3} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2}$$

$$+ \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{2/3}} - \frac{9\mu^2}{2a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu^2}{2}q_1^{2/3}$$

$$+ \frac{9\mu^2 q_1^{\frac{4}{3}}}{2} + \frac{9\mu^2 q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2 a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2 a^{\frac{2}{3}}q_1^2}{4} - \frac{9\mu^2}{2}q_1^{2/3} + \frac{9\mu^2}{2}(aq_1)^{2/3}$$

$$+ \frac{9\mu^2 a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2}$$

$$RL^2 = -\frac{9(1-\mu)^2}{4} + \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4} - \frac{9(1-\mu)^2 (aq_1)^{2/3}}{4} + \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} - \frac{27(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} +$$

$$\frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{2}{3}}} - 9\mu(1-\mu) + \frac{27(1-\mu)^2}{4q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} + \frac{9\mu(1-\mu)(aq_1)^{2/3}}{2} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} +$$

$$\frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} + \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{2q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)q_1^{\frac{2}{3}}}{2} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} - \frac{9\mu^2}{4} - \frac{9\mu^2a^{\frac{2}{3}}}{4} +$$

$$\frac{9\mu^2(aq_1)^{2/3}}{4} + \frac{9\mu^2}{4a^{\frac{2}{3}}} - \frac{9\mu^2q_1^{\frac{2}{3}}}{2} + \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{27\mu^2q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}} - \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{27\mu^2q_1^{\frac{4}{3}}}{4} + \frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4} -$$

$$\frac{9(1-\mu)^2}{2q_1^{\frac{2}{3}}} \qquad (4.73)$$

Hence $A_2\{2LNQ + RL^2\}$ is ;

$A_2\{2LNQ + RL^2\}$

$$= A_2\{-\frac{27\mu(1-\mu)}{2(aq_1)^{2/3}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - 9\mu(1-\mu)q_1^{\frac{2}{3}}$$

$$+ 9\mu^2 + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{q_1^{\frac{2}{3}}} + \frac{27\mu^2 q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}} - \frac{9\mu^2}{2a^{\frac{4}{3}}} - 9\mu^2 q_1^{\frac{4}{3}}$$

$$- [-\frac{9(1-\mu)^2}{4} + \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4} - \frac{9(1-\mu)^2(aq_1)^{2/3}}{4} + \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}}$$

$$- \frac{27(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{2}{3}}} - 9\mu(1-\mu)$$

$$+ \frac{27(1-\mu)^2}{4q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} + \frac{9\mu(1-\mu)(aq_1)^{2/3}}{2} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} + \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{2q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}}$$

$$+ \frac{9\mu(1-\mu)q_1^{\frac{2}{3}}}{2} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} - \frac{9\mu^2}{4} - \frac{9\mu^2 a^{\frac{2}{3}}}{4} + \frac{9\mu^2(aq_1)^{2/3}}{4}$$

$$+ \frac{9\mu^2}{4a^{\frac{2}{3}}} - \frac{9\mu^2 q_1^{\frac{2}{3}}}{2} + \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{27\mu^2 q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}} - \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{27\mu^2 q_1^{\frac{4}{3}}}{4}$$

$$+ \frac{9\mu^2 a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2 a^{\frac{2}{3}}q_1^2}{4} - \frac{9(1-\mu)^2}{2q_1^{\frac{2}{3}}}]\}$$

$$A_2\{2LNQ + RL^2\} = A_2\{-\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{(aq_1)^{2/3}} - \frac{9\mu(1-\mu)q_1^{\frac{2}{3}}}{2} + \frac{27\mu(1-\mu)}{2\,q_1^{\frac{2}{3}}} - 9\mu(1-\mu) +$$

$$\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} + \frac{9\mu(1-\mu)(aq_1)^{2/3}}{2} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2\,q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} - \frac{9(1-\mu)^2}{4} + \frac{9(1-\mu)^2 a^{\frac{2}{3}}}{4} -$$

$$\frac{9(1-\mu)^2(aq_1)^{2/3}}{4} - \frac{9(1-\mu)^2}{2\,q_1^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} - \frac{27(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4\,q_1^{\frac{2}{3}}} + \frac{27(1-\mu)^2}{4q_1^{\frac{4}{3}}} -$$

$$\frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{27\mu^2}{4} + \frac{27\mu^2}{4a^{\frac{2}{3}}} - \frac{9\mu^2a^{\frac{2}{3}}}{4} + \frac{9\mu^2(aq_1)^{2/3}}{4} + \frac{27\mu^2q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}} - \frac{9\mu^2q_1^{\frac{2}{3}}}{4} - \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{9\mu^2q_1^{\frac{4}{3}}}{4} +$$

$$\frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} - \frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4}\} \qquad (4.74)$$

Calculating $B_2\{2LOQ - TL^2\}$

$$LO = \left[\frac{3(1-\mu)}{2(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} - \frac{3\mu}{2a^{2/3}} + \frac{3\mu}{2}q_1^{2/3}\right.$$

$$\left. - \frac{3\mu}{2}\right]\left[- \frac{15(1-\mu)B_2}{8(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{15\mu}{4\,a^{\frac{4}{3}}} + \frac{15\mu}{8\,a^{2/3}} + \frac{15\mu}{8a^2}\right.$$

$$\left. - \frac{15\mu}{8\,a^{\frac{4}{3}}}q_1^{2/3}\right]$$

$$LO = -\frac{45(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{8}{3}}q_1^{\frac{2}{3}}}$$

$$- \frac{45\mu(1-\mu)}{16a^2} - \frac{45(1-\mu)^2}{16(aq_1)^{2/3}} + \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}}$$

$$+ \frac{45\mu(1-\mu)}{16a^{2/3}} + \frac{45\mu(1-\mu)}{16a^2} - \frac{45\mu(1-\mu)q_1^{2/3}}{16a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{16a^{2/3}q_1^{\frac{4}{3}}}$$

$$- \frac{45(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{16(aq_1)^{2/3}} - \frac{45\mu(1-\mu)}{16a^2q_1^{2/3}}$$

$$+ \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} - \frac{45\mu(1-\mu)}{16a^2q_1^{2/3}} - \frac{45\mu^2}{8a^2} - \frac{45\mu^2}{16a^{\frac{4}{3}}} - \frac{45\mu^2}{16a^{\frac{8}{3}}}$$

$$+ \frac{45\mu^2q_1^{2/3}}{16a^2} - \frac{45\mu(1-\mu)}{16a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{2/3}}{8a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{2/3}}{16a^{2/3}}$$

$$+ \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^2} - \frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{16(aq_1)^{2/3}} - \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} - \frac{45\mu^2}{8a^{\frac{4}{3}}} - \frac{45\mu^2}{16a^{2/3}}$$

$$- \frac{45\mu^2}{16a^2} + \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}}$$

$$2LO = -\frac{45(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8a^2q_1^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^{\frac{8}{3}}q_1^{2/3}} - \frac{45(1-\mu)^2}{8(aq_1)^{2/3}}$$

$$+ \frac{45(1-\mu)^2}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{2a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)q_1^{2/3}}{8a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}}$$

$$- \frac{135\mu^2}{8a^2} - \frac{135\mu^2}{8a^{\frac{4}{3}}} - \frac{45\mu^2}{8a^{\frac{8}{3}}} + \frac{45\mu^2q_1^{\frac{2}{3}}}{4a^2} + \frac{135\mu^2q_1^{2/3}}{8a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^{2/3}}$$

$$- \frac{45\mu^2q_1^{\frac{4}{3}}}{8a^{\frac{4}{3}}} - \frac{45\mu^2}{8a^{2/3}}$$

Now; $2LOQ$

$$2LOQ = -\frac{45(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8a^2 q_1^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}} q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^{\frac{8}{3}} q_1^{2/3}}$$

$$- \frac{45(1-\mu)^2}{8(aq_1)^{2/3}} + \frac{45(1-\mu)^2}{8a^{\frac{4}{3}} q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{2a^{\frac{4}{3}}} - \frac{45\mu(1-\mu) q_1^{2/3}}{8a^{\frac{4}{3}}}$$

$$+ \frac{45(1-\mu)^2}{8a^{2/3} q_1^{\frac{4}{3}}} - \frac{135\mu^2}{8a^2} - \frac{135\mu^2}{8a^{\frac{4}{3}}} - \frac{45\mu^2}{8a^{\frac{8}{3}}} + \frac{45\mu^2 q_1^{\frac{2}{3}}}{4a^2} + \frac{135\mu^2 q_1^{2/3}}{8a^{\frac{4}{3}}}$$

$$+ \frac{45\mu^2 q_1^{\frac{2}{3}}}{8a^{2/3}} - \frac{45\mu^2 q_1^{\frac{4}{3}}}{8a^{\frac{4}{3}}} - \frac{45\mu^2}{8a^{2/3}}][\, a^{\frac{2}{3}} + (aq_1)^{\frac{2}{3}} - \frac{1}{2}\,]$$

$$2LOQ = [-\frac{45(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{4(aq_1)^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}}$$

$$+ \frac{45(1-\mu)^2}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{8a^{2/3}}$$

$$+ \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^2} - \frac{45\mu(1-\mu)}{16a^{\frac{8}{3}}q_1^{\frac{2}{3}}}$$

$$- \frac{45(1-\mu)^2}{8q_1^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{8} + \frac{45(1-\mu)^2}{16(aq_1)^{2/3}} + \frac{45(1-\mu)^2}{8(aq_1)^{2/3}}$$

$$+ \frac{45(1-\mu)^2}{8a^{2/3}} - \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{2a^{\frac{2}{3}}}$$

$$- \frac{45\mu(1-\mu)}{4a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{8a^{\frac{2}{3}}} - \frac{45\mu(1-\mu)q_1^{\frac{4}{3}}}{8a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}}$$

$$+ \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8q_1^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{16a^{2/3}q_1^{\frac{4}{3}}} - \frac{135\mu^2}{8a^{\frac{4}{3}}} - \frac{135\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}}$$

$$+ \frac{135\mu^2}{16a^2} - \frac{135\mu^2}{8a^{\frac{2}{3}}} - \frac{135\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{2}{3}}} + \frac{135\mu^2}{16a^{\frac{4}{3}}} - \frac{45\mu^2}{8a^2} - \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^2} + \frac{45\mu^2}{16a^{\frac{8}{3}}}$$

$$+ \frac{45\mu^2q_1^{\frac{2}{3}}}{4a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{4a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^2} + \frac{135\mu^2q_1^{\frac{2}{3}}}{8a^{2/3}} + \frac{135\mu^2q_1^{\frac{4}{3}}}{8a^{2/3}} - \frac{135\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}}$$

$$+ \frac{45\mu^2q_1^{\frac{2}{3}}}{8} + \frac{45\mu^2q_1^{\frac{4}{3}}}{8} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}} - \frac{45\mu^2q_1^{\frac{4}{3}}}{8a^{\frac{2}{3}}} - \frac{45\mu^2q_1^2}{8a^{\frac{2}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}}$$

$$- \frac{45\mu^2}{8} - \frac{45\mu^2q_1^{\frac{2}{3}}}{8} + \frac{45\mu^2}{16a^{\frac{2}{3}}}$$

$$2LOQ = -\frac{225\mu(1-\mu)}{16a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16(aq_1)^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} +$$

$$\frac{135\mu(1-\mu)}{8a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^2} - \frac{45\mu(1-\mu)}{16a^{\frac{8}{3}}q_1^{\frac{2}{3}}} + \frac{135\mu(1-\mu)q_1^{\frac{2}{3}}}{8a^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{8} +$$

$$\frac{45(1-\mu)^2}{8a^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{4a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)q_1^{\frac{4}{3}}}{8a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} - \frac{135\mu^2}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}} +$$

$$\frac{45\mu^2}{16a^2} - \frac{225\mu^2}{16a^{\frac{2}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{4a^2} + \frac{45\mu^2}{16a^{\frac{8}{3}}} + \frac{225\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{4a^{\frac{2}{3}}} - \frac{135\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{8} -$$

$$\frac{45\mu^2q_1^2}{8a^{\frac{2}{3}}} - \frac{45\mu^2}{8} \qquad (4.75)$$

Calculating TL^2;

$$TL^2 = \left[\frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{2(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{4}\right.$$

$$- \frac{9(1-\mu)^2}{2q_1^{2/3}} + \frac{9\mu(1-\mu)}{2}q_1^{2/3} - 9\mu(1-\mu) + \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{2q_1^{2/3}} + \frac{9\mu^2}{4a^{\frac{4}{3}}} - \frac{9\mu^2}{2a^{\frac{2}{3}}}q_1^{2/3} + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9\mu^2}{4}q_1^{\frac{4}{3}} - \frac{9\mu^2}{2}q_1^{2/3}$$

$$\left. + \frac{9\mu^2}{4}\right]\left[\frac{5}{4a^{\frac{2}{3}}} - \frac{5a^{2/3}}{4} - \frac{5(aq_1)^{2/3}}{4}\right]$$

$$TL^2 = \frac{45(1-\mu)^2}{16a^2 q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16a^{\frac{2}{3}} q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16(aq_1)^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{8a^{\frac{4}{3}} q_1^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{8q_1^{\frac{2}{3}}}$$

$$- \frac{45(1-\mu)^2}{8} - \frac{45(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8q_1^{\frac{2}{3}}}$$

$$- \frac{45\mu(1-\mu)}{8a^2 q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{16a^{\frac{2}{3}}}$$

$$- \frac{45(1-\mu)^2 a^{\frac{2}{3}}}{16} - \frac{45(1-\mu)^2 (aq_1)^{\frac{2}{3}}}{16} - \frac{45(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}}$$

$$+ \frac{45(1-\mu)^2 a^{\frac{2}{3}}}{8q_1^{\frac{2}{3}}} + \frac{45(1-\mu)^2 a^{\frac{2}{3}}}{8} + \frac{45\mu(1-\mu) q_1^{\frac{2}{3}}}{8a^{\frac{2}{3}}}$$

$$- \frac{45\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{8} - \frac{45\mu(1-\mu)}{8} a^{\frac{2}{3}} q_1^{\frac{4}{3}} - \frac{45\mu(1-\mu)}{4a^{\frac{2}{3}}}$$

$$+ \frac{45\mu(1-\mu)}{4} a^{\frac{2}{3}} + \frac{45\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{4} + \frac{45(1-\mu)^2}{16a^{\frac{2}{3}} q_1^{\frac{4}{3}}}$$

$$- \frac{45(1-\mu)^2}{16q_1^{\frac{4}{3}}} a^{\frac{2}{3}} - \frac{45(1-\mu)^2}{16q_1^{\frac{2}{3}}} a^{\frac{2}{3}} + \frac{45\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}}$$

$$- \frac{45\mu(1-\mu)}{8q_1^{\frac{2}{3}}} a^{\frac{2}{3}} - \frac{45\mu(1-\mu)}{8} a^{\frac{2}{3}} + \frac{45\mu^2}{16a^2} - \frac{45\mu^2}{16a^{\frac{2}{3}}} - \frac{45\mu^2 q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}}$$

$$- \frac{45\mu^2 q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}} + \frac{45\mu^2 q_1^{\frac{2}{3}}}{8} + \frac{45\mu^2 q_1^{\frac{4}{3}}}{8} + \frac{45\mu^2}{8a^{\frac{4}{3}}} - \frac{45\mu^2}{8} - \frac{45\mu^2 q_1^{\frac{2}{3}}}{8}$$

$$+ \frac{45\mu^2 q_1^{\frac{4}{3}}}{16a^{\frac{2}{3}}} - \frac{45\mu^2}{16} a^{\frac{2}{3}} q_1^{\frac{4}{3}} - \frac{45\mu^2}{16} a^{\frac{2}{3}} q_1^2 - \frac{45\mu^2 q_1^{\frac{2}{3}}}{8a^{\frac{2}{3}}} + \frac{45\mu^2}{8}(aq_1)^{\frac{2}{3}}$$

$$+ \frac{45\mu^2}{8} a^{\frac{2}{3}} q_1^{\frac{4}{3}} + \frac{45\mu^2}{16a^{\frac{2}{3}}} - \frac{45\mu^2}{16} a^{\frac{2}{3}} - \frac{45\mu^2}{16}(aq_1)^{\frac{2}{3}}$$

Now $(-TL^2)$;

$$-TL^2 = -\frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}} + \frac{135(1-\mu)^2}{16(aq_1)^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{8} + \frac{45(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} +$$

$$\frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{16a^{\frac{2}{3}}} - \frac{45(1-\mu)^2a^{\frac{2}{3}}}{16} + \frac{45(1-\mu)^2(aq_1)^{\frac{2}{3}}}{16} -$$

$$\frac{45(1-\mu)^2a^{\frac{2}{3}}}{16q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}}q_1^{\frac{2}{3}} - \frac{45\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{8} + \frac{45\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{8} - \frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8} +$$

$$\frac{45(1-\mu)^2a^{\frac{2}{3}}}{16\,q_1^{\frac{4}{3}}} + \frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8q_1^{\frac{2}{3}}} + \frac{135\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}} + \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{4}{3}}}{8} + \frac{45\mu^2}{8} - \frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{2}{3}}} - \frac{45\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{16} +$$

$$\frac{45\mu^2a^{\frac{2}{3}}q_1^2}{16} - \frac{45\mu^2(aq_1)^{\frac{2}{3}}}{16} + \frac{45\mu^2a^{\frac{2}{3}}}{16} - \frac{45\mu^2}{16a^2} - \frac{45\mu^2}{8a^{\frac{4}{3}}} \qquad (4.76)$$

Hence $B_2\{2LOQ - TL^2\}$;

$B_2\,[2LOQ - TL^2]$

$$= B_2\,[-\frac{225\mu(1-\mu)}{16a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16(aq_1)^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}}$$

$$- \frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} + \frac{135\mu(1-\mu)}{8a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}}$$

$$+ \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^2} - \frac{45\mu(1-\mu)}{16a^{\frac{8}{3}}q_1^{\frac{2}{3}}} + \frac{135\mu(1-\mu)q_1^{\frac{2}{3}}}{8a^{\frac{2}{3}}}$$

$$- \frac{45(1-\mu)^2}{8} + \frac{45(1-\mu)^2}{8a^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{4a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)q_1^{\frac{4}{3}}}{8a^{\frac{2}{3}}}$$

$$+ \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} - \frac{135\mu^2}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}} + \frac{45\mu^2}{16a^2}$$

$$- \frac{225\mu^2}{16a^{\frac{2}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{4a^2} + \frac{45\mu^2}{16a^{\frac{8}{3}}} + \frac{225\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{4a^{\frac{2}{3}}}$$

$$- \frac{135\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{8} - \frac{45\mu^2q_1^2}{8a^{\frac{2}{3}}} - \frac{45\mu^2}{8} - \frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}}$$

$$+ \frac{135(1-\mu)^2}{16(aq_1)^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{8} + \frac{45(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}}$$

$$- \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}}$$

$$- \frac{45(1-\mu)^2}{16a^{\frac{2}{3}}} - \frac{45(1-\mu)^2a^{\frac{2}{3}}}{16} + \frac{45(1-\mu)^2(aq_1)^{\frac{2}{3}}}{16}$$

$$- \frac{45(1-\mu)^2a^{\frac{2}{3}}}{16q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}}q_1^{\frac{2}{3}} - \frac{45\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{8}$$

$$+\frac{45\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{8}-\frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8}+\frac{45(1-\mu)^2a^{\frac{2}{3}}}{16\,q_1^{\frac{4}{3}}}$$

$$+\frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8q_1^{\frac{2}{3}}}+\frac{135\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}}+\frac{45\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}}-\frac{45\mu^2q_1^{\frac{4}{3}}}{8}+\frac{45\mu^2}{8}$$

$$-\frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{2}{3}}}-\frac{45\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{16}+\frac{45\mu^2a^{\frac{2}{3}}q_1^2}{16}-\frac{45\mu^2(aq_1)^{\frac{2}{3}}}{16}+\frac{45\mu^2a^{\frac{2}{3}}}{16}-\frac{45\mu^2}{16a^2}$$

$$-\frac{45\mu^2}{8a^{\frac{4}{3}}}]$$

Therefore $B_2\,[2LOQ-\ TL^2]$ is thus;

$$B_2\,[2LOQ-\ TL^2]=\ B_2\,[-\frac{225\mu(1-\mu)}{16a^{\frac{2}{3}}q_1^{\frac{4}{3}}}+\frac{45(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}}+\frac{135(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}}-\frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}}-$$

$$\frac{45(1-\mu)^2}{8a^2q_1^{\frac{4}{3}}}-\frac{135\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{2a^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{4a^2q_1^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{8a^2}-\frac{45\mu(1-\mu)}{8a^{\frac{8}{3}}q_1^{\frac{2}{3}}}+$$

$$\frac{45\mu(1-\mu)}{4a^{\frac{2}{3}}}q_1^{\frac{2}{3}}+\frac{45(1-\mu)^2}{16a^{\frac{2}{3}}}-\frac{45\mu(1-\mu)}{4a^{\frac{4}{3}}}-\frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}}q_1^{\frac{4}{3}}+\frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}}q_1^{\frac{2}{3}}-\frac{225\mu^2}{16a^{\frac{4}{3}}}+\frac{45\mu^2}{8a^{\frac{2}{3}}}q_1^{\frac{2}{3}}-$$

$$\frac{225\mu^2}{16a^{\frac{2}{3}}}-\frac{45\mu^2}{4a^2}q_1^{\frac{2}{3}}+\frac{45\mu^2}{16a^{\frac{8}{3}}}+\frac{225\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}}+\frac{135\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{2}{3}}}-\frac{135\mu^2}{16a^{\frac{4}{3}}}q_1^{\frac{2}{3}}-\frac{45\mu^2q_1^2}{8a^{\frac{2}{3}}}-\frac{45(1-\mu)^2a^{\frac{2}{3}}}{16}+$$

$$\frac{45(1-\mu)^2(aq_1)^{\frac{2}{3}}}{16}-\frac{45(1-\mu)^2a^{\frac{2}{3}}}{16q_1^{\frac{2}{3}}}-\frac{45\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{8}+\frac{45\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{8}-\frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8}+$$

$$\frac{45(1-\mu)^2a^{\frac{2}{3}}}{16q_1^{\frac{4}{3}}}+\frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8q_1^{\frac{2}{3}}}-\frac{45\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{16}+\frac{45\mu^2a^{\frac{2}{3}}q_1^2}{16}-\frac{45\mu^2(aq_1)^{\frac{2}{3}}}{16}+\frac{45\mu^2a^{\frac{2}{3}}}{16}]$$

(4.77)

Substituting (4.68), (4.71), (4.74), and (4.77) into (4.66) i.e.;

$$(\Omega_{\xi\eta})^2 = \frac{1}{2(1-e^2)}\,[\,QL^2 + e^2\{2LMQ - SL^2\} + A_2\{2LNQ + RL^2\} + B_2\{2LOQ - TL2]$$

$$(\Omega_{\xi\eta}^{\circ})^2 = \frac{1}{2(1-e^2)}\Big[-\frac{9\mu(1-\mu)}{4}a^{2/3} + 3 + \frac{27(1-\mu)^2}{8} + \frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}$$

$$-\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} + \frac{9(1-\mu)^2}{4q_1^{2/3}} - \frac{45(1-\mu)^2}{8q_1^{\frac{4}{3}}} - \frac{9(1-\mu)^2a^{2/3}}{4q_1^{2/3}}$$

$$+\frac{9(1-\mu)^2a^{2/3}}{4q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} - \frac{9\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{2} + \frac{9\mu(1-\mu)}{2}$$

$$-\frac{9\mu(1-\mu)}{2(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}}$$

$$-\frac{9\mu(1-\mu)}{4q_1^{\frac{2}{3}}} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} - \frac{9\mu(1-\mu)q_1^{2/3}}{4} + \frac{9\mu^2a^{\frac{2}{3}}}{4}$$

$$-\frac{9\mu^2(aq_1)^{\frac{2}{3}}}{4} + \frac{27\mu^2}{8} + \frac{9\mu^2q_1^{2/3}}{4} + \frac{9\mu^2q_1^{2/3}}{2a^{\frac{2}{3}}} - \frac{9\mu^2}{8a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{4}{3}}}{8}$$

$$-\frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4} + \frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4}$$

$$+e^2\Bigg\{\frac{9(1-\mu)^2}{2a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{2(aq_1)^{2/3}} + \frac{9\mu^2q_1^{2/3}}{2a^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}}$$

$$+\frac{9\mu(1-\mu)}{2a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{9\mu^2}{4a^{\frac{4}{3}}} + \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4} + \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{\frac{2}{3}}}$$

$$+\frac{9\mu(1-\mu)(aq_1)^{2/3}}{2} + \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2} - \frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}} + \frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4}$$

$$+\frac{9\mu^2}{4}(aq_1)^{2/3} - \frac{9\mu^2}{4}a^{\frac{2}{3}} + \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}} - \frac{9(1-\mu)^2(aq_1)^{2/3}}{4}$$

$$-\frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2} - \frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4} - \frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2q_1^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{2a^{\frac{2}{3}}}\Bigg\}$$

$$+\ A_2\left\{-\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{(aq_1)^{2/3}}-\frac{9\mu(1-\mu)q_1^{\frac{2}{3}}}{2}+\frac{27\mu(1-\mu)}{2\,q_1^{\frac{2}{3}}}-9\mu(1-\mu)\right.$$

$$+\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2}+\frac{9\mu(1-\mu)(aq_1)^{2/3}}{2}-\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2\,q_1^{\frac{2}{3}}}$$

$$-\frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{2}-\frac{9(1-\mu)^2}{4}+\frac{9(1-\mu)^2a^{\frac{2}{3}}}{4}$$

$$-\frac{9(1-\mu)^2(aq_1)^{2/3}}{4}-\frac{9(1-\mu)^2}{2\,q_1^{\frac{2}{3}}}+\frac{9(1-\mu)^2}{4(aq_1)^{\frac{4}{3}}}-\frac{27(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}}$$

$$+\frac{9(1-\mu)^2}{4(aq_1)^{2/3}}+\frac{9(1-\mu)^2a^{\frac{2}{3}}}{4\,q_1^{\frac{2}{3}}}+\frac{27(1-\mu)^2}{4q_1^{\frac{4}{3}}}-\frac{9(1-\mu)^2a^{\frac{2}{3}}}{4q_1^{\frac{4}{3}}}+\frac{27\mu^2}{4}$$

$$+\frac{27\mu^2}{4a^{\frac{2}{3}}}-\frac{9\mu^2a^{\frac{2}{3}}}{4}+\frac{9\mu^2(aq_1)^{2/3}}{4}+\frac{27\mu^2q_1^{\frac{2}{3}}}{4a^{\frac{2}{3}}}-\frac{9\mu^2q_1^{\frac{2}{3}}}{4}-\frac{9\mu^2}{4a^{\frac{4}{3}}}$$

$$\left.-\frac{9\mu^2q_1^{\frac{4}{3}}}{4}+\frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4}-\frac{9\mu^2a^{\frac{2}{3}}q_1^2}{4}\right\}$$

$$+\ B_2\left\{-\frac{225\mu(1-\mu)}{16a^{\frac{2}{3}}q_1^{\frac{4}{3}}}+\frac{45(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}}+\frac{135(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}}\right.$$

$$-\frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}}-\frac{45(1-\mu)^2}{8a^2q_1^{\frac{4}{3}}}-\frac{135\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{2a^{\frac{2}{3}}}$$

$$+\frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{4a^2q_1^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{8a^2}-\frac{45\mu(1-\mu)}{8a^{\frac{8}{3}}q_1^{\frac{2}{3}}}$$

$$+\frac{45\mu(1-\mu)}{4a^{\frac{2}{3}}}q_1^{\frac{2}{3}}+\frac{45(1-\mu)^2}{16a^{\frac{2}{3}}}-\frac{45\mu(1-\mu)}{4a^{\frac{4}{3}}}-\frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}}q_1^{\frac{4}{3}}$$

$$+\frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}}q_1^{\frac{2}{3}}-\frac{225\mu^2}{16a^{\frac{4}{3}}}+\frac{45\mu^2}{8a^{\frac{2}{3}}}q_1^{\frac{2}{3}}-\frac{225\mu^2}{16a^{\frac{2}{3}}}-\frac{45\mu^2}{4a^2}q_1^{\frac{2}{3}}+\frac{45\mu^2}{16a^{\frac{8}{3}}}$$

$$+\frac{225\mu^2 q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}}+\frac{135\mu^2 q_1^{\frac{4}{3}}}{16a^{\frac{2}{3}}}-\frac{135\mu^2}{16a^{\frac{4}{3}}}q_1^{\frac{2}{3}}-\frac{45\mu^2 q_1^2}{8a^{\frac{2}{3}}}-\frac{45(1-\mu)^2 a^{\frac{2}{3}}}{16}$$

$$+\frac{45(1-\mu)^2(aq_1)^{\frac{2}{3}}}{16}-\frac{45(1-\mu)^2 a^{\frac{2}{3}}}{16q_1^{\frac{2}{3}}}-\frac{45\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{8}$$

$$+\frac{45\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{8}-\frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8}+\frac{45(1-\mu)^2 a^{\frac{2}{3}}}{16q_1^{\frac{4}{3}}}$$

$$+\frac{45\mu(1-\mu)a^{\frac{2}{3}}}{8q_1^{\frac{2}{3}}}-\frac{45\mu^2 a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{16}+\frac{45\mu^2 a^{\frac{2}{3}}q_1^2}{16}-\frac{45\mu^2(aq_1)^{\frac{2}{3}}}{16}$$

$$+\frac{45\mu^2 a^{\frac{2}{3}}}{16}\Bigg\}]$$

Hence $(\Omega_{\xi\eta})^2$ is re-written as

$$(\Omega_{\xi\eta})^2=\frac{1}{(1-e^2)}\left[\frac{QL^2}{2}+e^2\left\{\frac{2LMQ-SL^2}{2}\right\}\right.$$

$$\left.+A_2\left\{\frac{2LNQ+RL^2}{2}\right\}+B_2\left\{\frac{2LOQ-TL^2}{2}\right\}\right]$$

$$(\Omega_{\xi\eta})^2=\frac{1}{(1-e^2)}\left[\frac{9(1-\mu)^2}{4a^{2/3}q_1^{\frac{4}{3}}}+\frac{9(1-\mu)^2}{8q_1^{\frac{2}{3}}}-\frac{45(1-\mu)^2}{16q_1^{\frac{4}{3}}}-\frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}}-\frac{9(1-\mu)^2 a^{2/3}}{8}-\right.$$

$$\frac{9\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{4}-\frac{9\mu(1-\mu)a^{2/3}}{4}+\frac{9(1-\mu)^2 a^{2/3}}{8q_1^{\frac{4}{3}}}+\frac{9\mu(1-\mu)a^{2/3}}{4q_1^{\frac{2}{3}}}+\frac{9\mu^2 q_1^{\frac{2}{3}}}{8}+\frac{27\mu^2}{16}-\frac{9\mu^2 a^{2/3}q_1^{\frac{4}{3}}}{8}-$$

$$\left.\frac{9\mu^2(aq_1)^{\frac{2}{3}}}{8}+\frac{9\mu^2 a^{2/3}}{8}+\frac{27(1-\mu)^2}{16}-\frac{9\mu(1-\mu)}{4a^{2/3}}+\frac{9(1-\mu)^2(aq_1)^{\frac{2}{3}}}{8}+\frac{9\mu(1-\mu)a^{2/3}q_1^{\frac{4}{3}}}{4}-\right.$$

$$\frac{9(1-\mu)^2a^{2/3}}{8q_1^{\frac{2}{3}}}+\frac{9\mu^2q_1^{\frac{2}{3}}}{4a^{2/3}}-\frac{45\mu^2q_1^{\frac{4}{3}}}{16}+\frac{9\mu^2a^{2/3}q_1^2}{4}-\frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}}+\frac{9\mu(1-\mu)}{8\,a^{\frac{4}{3}}q_1^{\frac{2}{3}}}-\frac{9\mu(1-\mu)q_1^{\frac{2}{3}}}{8}+$$

$$\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{8q_1^{\frac{2}{3}}}-\frac{9\mu^2}{16a^{\frac{4}{3}}}+e^2\left\{\frac{9(1-\mu)^2}{4a^{2/3}q_1^{\frac{4}{3}}}-\frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}}+\frac{9\mu^2q_1^{\frac{2}{3}}}{4a^{2/3}}-\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}}+\right.$$

$$\frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{\frac{2}{3}}}-\frac{9\mu^2}{8a^{\frac{4}{3}}}+\frac{9(1-\mu)^2a^{2/3}}{8}+\frac{9(1-\mu)^2a^{2/3}}{8q_1^{\frac{2}{3}}}+\frac{9\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{4}+\frac{9\mu(1-\mu)a^{2/3}}{4}-$$

$$\frac{9(1-\mu)^2a^{2/3}}{8q_1^{\frac{4}{3}}}+\frac{9\mu^2a^{2/3}q_1^{\frac{4}{3}}}{8}+\frac{9\mu^2(aq_1)^{\frac{2}{3}}}{8}-\frac{9\mu^2a^{2/3}}{8}+\frac{9\mu(1-\mu)}{4a^{2/3}}-\frac{9(1-\mu)^2(aq_1)^{\frac{2}{3}}}{8}-$$

$$\left.\frac{9\mu(1-\mu)a^{2/3}q_1^{\frac{4}{3}}}{4}-\frac{9\mu^2a^{2/3}q_1^2}{8}-\frac{9\mu(1-\mu)a^{2/3}}{4q_1^{\frac{2}{3}}}-\frac{9\mu(1-\mu)}{2a^{2/3}}\right\}+A_2\left\{-\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{2(aq_1)^{2/3}}-\frac{9\mu(1-\mu)q_1^{\frac{2}{3}}}{4}+\right.$$

$$\frac{27\mu(1-\mu)}{4\,q_1^{\frac{2}{3}}}-\frac{9\mu(1-\mu)}{2}+\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{4}+\frac{9\mu(1-\mu)(aq_1)^{2/3}}{4}-\frac{9\mu(1-\mu)a^{\frac{2}{3}}}{4\,q_1^{\frac{2}{3}}}-\frac{9\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{4}-$$

$$\frac{9(1-\mu)^2}{8}+\frac{9(1-\mu)^2a^{\frac{2}{3}}}{8}-\frac{9(1-\mu)^2(aq_1)^{2/3}}{8}-\frac{9(1-\mu)^2}{4\,q_1^{\frac{2}{3}}}+\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}}-\frac{27(1-\mu)^2}{8a^{\frac{2}{3}}q_1^{\frac{4}{3}}}+\frac{9(1-\mu)^2}{8(aq_1)^{2/3}}+$$

$$\frac{9(1-\mu)^2a^{\frac{2}{3}}}{8\,q_1^{\frac{2}{3}}}+\frac{27(1-\mu)^2}{8q_1^{\frac{4}{3}}}-\frac{9(1-\mu)^2a^{\frac{2}{3}}}{8q_1^{\frac{4}{3}}}+\frac{27\mu^2}{8}+\frac{27\mu^2}{8a^{\frac{2}{3}}}-\frac{9\mu^2a^{\frac{2}{3}}}{8}+\frac{9\mu^2(aq_1)^{2/3}}{8}+\frac{27\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{2}{3}}}-$$

$$\left.\frac{9\mu^2q_1^{\frac{2}{3}}}{4}-\frac{9\mu^2}{8a^{\frac{4}{3}}}-\frac{9\mu^2q_1^{\frac{4}{3}}}{8}+\frac{9\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{8}-\frac{9\mu^2a^{\frac{2}{3}}q_1^2}{8}\right\}+B_2\left\{-\frac{225\mu(1-\mu)}{32a^{\frac{2}{3}}q_1^{\frac{4}{3}}}+\frac{45(1-\mu)^2}{16(aq_1)^{\frac{2}{3}}}+\right.$$

$$\frac{135(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}}-\frac{45(1-\mu)^2}{32a^{\frac{4}{3}}q_1^{\frac{2}{3}}}-\frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}}-\frac{135\mu(1-\mu)}{16(aq_1)^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{4a^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{32a^{\frac{4}{3}}q_1^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}}+$$

$$\frac{45\mu(1-\mu)}{16a^2}-\frac{45\mu(1-\mu)}{16a^{\frac{8}{3}}q_1^{\frac{2}{3}}}+\frac{45\mu(1-\mu)}{8a^{\frac{2}{3}}}q_1^{\frac{2}{3}}+\frac{45(1-\mu)^2}{32a^{\frac{2}{3}}}-\frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}}-\frac{45\mu(1-\mu)}{16a^{\frac{2}{3}}}q_1^{\frac{4}{3}}+$$

$$\frac{45\mu(1-\mu)}{32a^{\frac{4}{3}}}q_1^{\frac{2}{3}}-\frac{225\mu^2}{32a^{\frac{4}{3}}}+\frac{45\mu^2}{16a^{\frac{2}{3}}}q_1^{\frac{2}{3}}-\frac{225\mu^2}{32a^{\frac{2}{3}}}-\frac{45\mu^2}{8a^2}q_1^{\frac{2}{3}}+\frac{45\mu^2}{32a^{\frac{8}{3}}}+\frac{225\mu^2q_1^{\frac{2}{3}}}{32a^{\frac{4}{3}}}+\frac{135\mu^2q_1^{\frac{4}{3}}}{32a^{\frac{2}{3}}}-$$

$$\frac{135\mu^2}{32a^{\frac{4}{3}}}q_1^{\frac{2}{3}}-\frac{45\mu^2q_1^2}{16a^{\frac{2}{3}}}-\frac{45(1-\mu)^2a^{\frac{2}{3}}}{32}+\frac{45(1-\mu)^2(aq_1)^{\frac{2}{3}}}{32}-\frac{45(1-\mu)^2a^{\frac{2}{3}}}{32q_1^{\frac{2}{3}}}-\frac{45\mu(1-\mu)(aq_1)^{\frac{2}{3}}}{16}+$$

$$\frac{45\mu(1-\mu)a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{16} - \frac{45\mu(1-\mu)a^{\frac{2}{3}}}{16} + \frac{45(1-\mu)^2a^{\frac{2}{3}}}{32q_1^{\frac{4}{3}}} + \frac{45\mu(1-\mu)a^{\frac{2}{3}}}{16q_1^{\frac{2}{3}}} - \frac{45\mu^2a^{\frac{2}{3}}q_1^{\frac{4}{3}}}{32} + \frac{45\mu^2a^{\frac{2}{3}}q_1^2}{32} - \frac{45\mu^2(aq_1)^{\frac{2}{3}}}{32} + \frac{45\mu^2a^{\frac{2}{3}}}{32}\Bigg\}]$$

(4.78)

Calculating $\Omega_{\xi\xi}^{\circ}\Omega_{\eta\eta}^{\circ}$, recall from (4.43) and (4.54) i.e.

$$(\Omega_{\xi\xi}^{\circ})(\Omega_{\eta\eta}^{\circ}) = [\frac{1}{(1-e^2)^{1/2}}]^2\{[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3} + \frac{3\mu}{2} + e^2\left\{\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4a^{2/3}}\right\} + A_2\left\{\frac{9\mu}{4a^{2/3}} - \frac{3(1-\mu)}{4(aq_1)^{2/3}}\right\} - B_2\left\{-\frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^{2/3}q_1^{2/3}} - \frac{15\mu}{4a^{2/3}} + 15\mu q1^{2/3}8a^{2/3} - 15\mu 16a2 - 15\mu 4a43][3\ 1-\mu 2q1^{2/3} + 3\mu 2q1^{2/3} + 3\ 1-\mu 2 - 3\ 1-\mu 4aq1^{2/3} - 3\mu 4a^{2/3} + 3\mu 2 + e2 - 3\ 1-\mu 4aq1^{2/3} - 3\mu 4a^{2/3} + A23\ 1-\mu 4aq1^{2/3} + 3\mu 4a^{2/3} + B2 - 15\ 1-\mu 8a43q1^{2/3} + 15\mu 16a2 - 15\mu 4a43 - 15\mu 8a43q1^{2/3} + 15\ 1-\mu 16(aq1)^{2/3} + 15\mu 16a^{2/3}]\}$$ (4.79)

which can be re – written as

$$(\Omega_{\xi\xi}^{\circ})(\Omega_{\eta\eta}^{\circ}) = \frac{1}{(1-e^2)}\{[F + Ge^2 + HA_2 + RB_2][J + Ke^2 + TA_2 + SB_2]\}$$

$$(\Omega_{\xi\xi}^{\circ})(\Omega_{\eta\eta}^{\circ}) = \frac{1}{(1-e^2)}\{FJ + FKe^2 + GJe^2 + FTA_2 + HJA_2 + FSB_2 + RJB_2\}$$

$$(\Omega_{\xi\xi}^{\circ})(\Omega_{\eta\eta}^{\circ}) = \frac{1}{(1-e^2)}\{FJ + e^2(FK + GJ) + A_2(FT + HJ) + B_2(FS + RJ)\}$$

(4.80)

where;

$$F = \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3} + \frac{3\mu}{2}$$

$$G = \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4a^{2/3}}$$

$$H = \frac{9\mu}{4a^{2/3}} - \frac{3(1-\mu)}{4(aq_1)^{2/3}}$$

$$R = -\frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^{2/3}q_1^{2/3}} - \frac{15\mu}{16a^{2/3}} + \frac{15\mu q_1^{2/3}}{8a^{2/3}} - \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{\frac{4}{3}}}$$

$$J = \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu q_1^{2/3}}{2} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2}$$

$$K = -\frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}}$$

$$T = \frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3\mu}{4a^{2/3}}$$

$$S = -\frac{15(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{\frac{4}{3}}} - \frac{15\mu}{8a^{\frac{4}{3}}}q_1^{2/3} + \frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15\mu}{16a^{2/3}}$$

Now, calculating FJ

$$FJ = \left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3} + \frac{3\mu}{2}\right]\left[\frac{3(1-\mu)}{2q_1^{2/3}}\right.$$
$$\left. + \frac{3\mu q_1^{2/3}}{2} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2}\right]$$

$$FJ = \frac{9(1-\mu)^2}{8a^{2/3}q_1^{2/3}} + \frac{9\mu(1-\mu)}{8a^{2/3}} + \frac{9(1-\mu)^2}{8(aq_1)^{2/3}} - \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}}$$

$$+ \frac{9\mu(1-\mu)}{8(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{4q_1^{2/3}} + \frac{9\mu(1-\mu)q_1^{2/3}}{4} + \frac{9(1-\mu)^2}{4}$$

$$- \frac{9(1-\mu)^2}{8(aq_1)^{2/3}} - \frac{9\mu(1-\mu)}{8a^{2/3}} + \frac{9\mu(1-\mu)}{4} - \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{4}$$

$$- \frac{9(1-\mu)^2}{4q_1^{2/3}} + \frac{9(1-\mu)^2}{8a^{2/3}q_1^{2/3}} + \frac{9\mu(1-\mu)}{8(aq_1)^{2/3}} - \frac{9\mu(1-\mu)}{8q_1^{2/3}} + \frac{9\mu(1-\mu)}{8(aq_1)^{2/3}}$$

$$+ \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} + \frac{9\mu(1-\mu)}{8a^{2/3}} - \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu^2}{8a^{2/3}} - \frac{9\mu(1-\mu)}{4}$$

$$- \frac{9\mu^2 q_1^{\frac{4}{3}}}{4} - \frac{9\mu(1-\mu)q_1^{2/3}}{4} + \frac{9\mu(1-\mu)}{8a^{2/3}} + \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} - \frac{9\mu^2 q_1^{2/3}}{4}$$

$$+ \frac{9\mu(1-\mu)}{4q_1^{2/3}} + \frac{9\mu^2 q_1^{2/3}}{4} + \frac{9\mu(1-\mu)}{4} - \frac{9\mu(1-\mu)}{8(aq_1)^{2/3}} - \frac{9\mu^2}{8a^{2/3}} + \frac{9\mu^2}{4}$$

$$FJ = \frac{9(1-\mu)^2}{4a^{2/3}q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8a^{2/3}} - \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{9\mu(1-\mu)}{4(aq_1)^{2/3}} + \frac{9(1-\mu)^2}{4} - \frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} +$$

$$\frac{9\mu^2 q_1^{2/3}}{4a^{2/3}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{\frac{4}{3}}}{4} + \frac{9\mu^2}{4} \qquad (4.81)$$

Now $e^2(FK + GJ)$

$$FK = \left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3} + \frac{3\mu}{2}\right]\left[-\frac{3(1-\mu)}{4(aq_1)^{2/3}}\right.$$

$$\left. - \frac{3\mu}{4a^{2/3}}\right]$$

$$FK = -\frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} - \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{8a^{2/3}} + \frac{9(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}}$$

$$+ \frac{9\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8a^{2/3}} + \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}}$$

$$- \frac{9\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} - \frac{9\mu^2}{8a^{2/3}}$$

$$FK = -\frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} - \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} - \frac{9\mu^2}{8a^{2/3}} \quad (4.82)$$

then GJ

$$GJ = \left[\frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu q_1^{2/3}}{2} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2}\right]\left[\frac{3(1-\mu)}{4(aq_1)^{2/3}}\right.$$

$$\left. + \frac{3\mu}{4a^{2/3}}\right]$$

$$GJ = \frac{9(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{8a^{\frac{2}{3}}} + \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} + \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{8a^{\frac{2}{3}}}$$

$$- \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} - \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}}$$

$$+ \frac{9\mu^2}{8a^{2/3}}$$

$$GJ = \frac{9(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} + \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} + \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu^2}{8a^{2/3}} +$$

$$\frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} \quad (4.83)$$

also $e^2(FK + GJ)$

$$e^2(FK + GJ) = e^2[-\frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} - \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} + \frac{9(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}}$$

$$- \frac{9\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} - \frac{9\mu^2}{8a^{2/3}}$$

$$+\left\{\frac{9(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} + \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} + \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}}\right.$$

$$\left. - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu^2}{8a^{2/3}} + \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}}\right\}]$$

$$e^2(FK + GJ) = e^2[-\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{4a^{2/3}q_1^{\frac{4}{3}}} - \frac{9\mu^2}{8a^{\frac{4}{3}}} + \frac{9\mu^2 q_1^{2/3}}{4a^{2/3}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} +$$

$$\frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}}] \quad (4.84)$$

Calculating $A_2(FT + HJ)$

$$FT = \left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3} + \frac{3\mu}{2}\right][\frac{3(1-\mu)}{4(aq_1)^{2/3}}$$

$$+ \frac{3\mu}{4a^{2/3}}]$$

$$FT = \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{8a^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8a^{2/3}q_1^{\frac{4}{3}}}$$

$$- \frac{9\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} + \frac{9\mu^2}{8a^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{2}{3}}} - \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}}$$

$$+ \frac{9\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} + \frac{9\mu^2}{8a^{\frac{2}{3}}}$$

$$FT = \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{9\mu^2}{16a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} + \frac{9\mu^2}{8a^{\frac{2}{3}}} \quad (4.85)$$

$$HJ = \left[\frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu q_1^{2/3}}{2} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2}\right]\left[\frac{9\mu}{4a^{2/3}} - \frac{3(1-\mu)}{4(aq_1)^{2/3}}\right]$$

$$HJ = \frac{27\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{27\mu^2 q_1^{2/3}}{8a^{2/3}} - \frac{9\mu(1-\mu)}{8a^{\frac{2}{3}}} + \frac{27\mu(1-\mu)}{8a^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{27\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{27\mu^2}{16a^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{27\mu^2}{8a^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{8(aq_1)^{\frac{2}{3}}}$$

$$HJ = \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{27\mu^2 q_1^{2/3}}{8a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{27\mu^2}{16a^{\frac{4}{3}}} + \frac{27\mu^2}{8a^{\frac{2}{3}}} \quad (4.86)$$

Hence, $A_2(FT + HJ)$;

$$A_2\left[\frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{9\mu^2}{16a^{\frac{4}{3}}} - \frac{9\mu^2 q_1^{2/3}}{8a^{2/3}} + \frac{9\mu^2}{8a^{\frac{2}{3}}} + \left\{\frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{27\mu^2 q_1^{2/3}}{8a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} - \frac{9(1-\mu)^2}{8(aq_1)^{\frac{2}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{27\mu^2}{16a^{\frac{4}{3}}} + \frac{27\mu^2}{8a^{\frac{2}{3}}}\right\}\right]$$

$$A_2(FT + HJ) = A_2\left[\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu^2}{8a^{\frac{4}{3}}} + \frac{9\mu^2 q_1^{2/3}}{4a^{2/3}} + \frac{9\mu^2}{2a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}}\right] \quad (4.87)$$

Calculating $B_2(FS + RJ)$

So, $FS = \left[\frac{3(1-\mu)}{4(aq_1)^{2/3}} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{2q_1^{2/3}} + \frac{3\mu}{4a^{2/3}} - \frac{3\mu}{2}q_1^{2/3} + \frac{3\mu}{2}\right]\left[-\frac{15(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{\frac{4}{3}}} + \frac{15(1-\mu)}{16(aq_1)^{2/3}} - \frac{15\mu}{8a^{\frac{4}{3}}}q_1^{2/3} + \frac{15\mu}{16a^{2/3}}\right]$

$$\text{FS} = -\frac{45(1-\mu)^2}{32a^2q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{32a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}}$$

$$- \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{64a^{\frac{8}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{32a^2} - \frac{45\mu(1-\mu)}{32a^2q_1^{\frac{2}{3}}} + \frac{45\mu^2}{64a^{\frac{8}{3}}}$$

$$- \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^2} + \frac{45\mu^2}{32a^2} - \frac{45\mu(1-\mu)}{16a^2q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}}$$

$$- \frac{45\mu^2}{16a^2} + \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}} - \frac{45\mu^2}{8a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{64(aq_1)^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{32(aq_1)^{\frac{2}{3}}}$$

$$- \frac{45(1-\mu)^2}{32a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{64a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{32a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{32(aq_1)^{\frac{2}{3}}}$$

$$- \frac{45\mu(1-\mu)}{32a^2} - \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^2}$$

$$+ \frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{64a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{32a^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{32(aq_1)^{\frac{2}{3}}}$$

$$+ \frac{45\mu^2}{64a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^{\frac{2}{3}}} + \frac{45\mu^2}{32a^{\frac{2}{3}}}$$

$$FS = -\frac{45(1-\mu)^2}{32a^2q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{225(1-\mu)^2}{64(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{135\mu(1-\mu)}{32a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{64a^{\frac{8}{3}}q_1^{\frac{2}{3}}} +$$

$$\frac{45\mu^2}{64a^{\frac{8}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^2} - \frac{45\mu^2}{32a^2} + \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} - \frac{315\mu^2}{64a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{32(aq_1)^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{32a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} +$$

$$\frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^{\frac{2}{3}}} + \frac{45\mu^2}{32a^{\frac{2}{3}}} \qquad (4.88)$$

Now, RJ

$$RJ = \left[-\frac{15(1-\mu)}{16(aq_1)^{2/3}} + \frac{15(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} - \frac{15\mu}{16a^{2/3}} + \frac{15\mu q_1^{2/3}}{8a^{\frac{4}{3}}} - \frac{15\mu}{16a^2} - \frac{15\mu}{4a^{\frac{4}{3}}}\right]\left[\frac{3(1-\mu)}{2q_1^{2/3}}\right.$$

$$\left. + \frac{3\mu q_1^{2/3}}{2} + \frac{3(1-\mu)}{2} - \frac{3(1-\mu)}{4(aq_1)^{2/3}} - \frac{3\mu}{4a^{2/3}} + \frac{3\mu}{2}\right]$$

$$RJ = -\frac{45(1-\mu)^2}{32a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{32(aq_1)^{2/3}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{32a^2q_1^{\frac{2}{3}}}$$

$$- \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{32a^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^{\frac{2}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}}$$

$$- \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^2} - \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{32(aq_1)^{2/3}} + \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{32a^{\frac{2}{3}}}$$

$$+ \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{32a^2} - \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{64(aq_1)^{\frac{4}{3}}}$$

$$- \frac{45(1-\mu)^2}{32a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{64a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{32a^2} + \frac{45\mu(1-\mu)}{64a^{\frac{8}{3}}q_1^{\frac{2}{3}}}$$

$$+ \frac{45\mu(1-\mu)}{16a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{64a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{32a^2q_1^{\frac{2}{3}}} + \frac{45\mu^2}{64a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^2}$$

$$+ \frac{45\mu^2}{64a^{\frac{8}{3}}} + \frac{45\mu^2}{16a^2} - \frac{45\mu(1-\mu)}{32(aq_1)^{2/3}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu^2}{32a^{\frac{2}{3}}} + \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}}$$

$$- \frac{45\mu^2}{32a^2} - \frac{45\mu^2}{8a^{\frac{4}{3}}}$$

$$RJ = -\frac{45(1-\mu)^2}{32a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{225(1-\mu)^2}{64(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{16(aq_1)^{2/3}} - \frac{45\mu(1-\mu)}{32a^{\frac{4}{3}}q_1^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{16a^{\frac{2}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^{\frac{2}{3}}} +$$

$$\frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^2} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{32(aq_1)^{2/3}} + \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{16a^2} -$$

$$\frac{45(1-\mu)^2}{32a^2q_1^{\frac{4}{3}}} + \frac{45\mu(1-\mu)}{64a^{\frac{8}{3}}q_1^{\frac{2}{3}}} + \frac{315\mu^2}{64a^{\frac{4}{3}}} + \frac{45\mu^2}{64a^{\frac{8}{3}}} + \frac{45\mu^2}{32a^2} - \frac{45\mu^2}{32a^{\frac{2}{3}}} \qquad (4.89)$$

Now; $B_2(FS + RJ)$

$$B_2(FS+RJ) = B_2[-\frac{45(1-\mu)^2}{32a^2q_1^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{225(1-\mu)^2}{64(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{135\mu(1-\mu)}{32a^{\frac{4}{3}}q_1^{\frac{2}{3}}} +$$

$$\frac{45\mu(1-\mu)}{64a^{\frac{8}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu^2}{64a^{\frac{8}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^2} - \frac{45\mu^2}{32a^2} + \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} - \frac{315\mu^2}{64a^{\frac{4}{3}}} + \frac{45(1-\mu)^2}{32(aq_1)^{\frac{2}{3}}} - \frac{45(1-\mu)^2}{32a^{\frac{2}{3}}q_1^{\frac{4}{3}}} -$$

$$\frac{45\mu(1-\mu)q_1^{\frac{2}{3}}}{16a^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{16a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{32a^{\frac{2}{3}}} + \frac{45\mu^2}{32a^{\frac{2}{3}}} + \left\{-\frac{45(1-\mu)^2}{32a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{225(1-\mu)^2}{64(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{16(aq_1)^{2/3}} -\right.$$

$$45\mu\ 1-\mu32a43q123- \quad 45\mu\ 1-\mu16a23- \ 45\mu2q12332a23+ \ 45\mu2q14316a43-$$

$$45\mu2q12316a2- \ 45\mu2q12316a43-45\ 1-\mu232(aq1)^{2/3} + 45\ 1-\mu216a43q123+$$

$$45\mu \quad 1-\mu q12316a43- \quad 45\mu \quad 1-\mu16a2- \quad 45 \quad 1-\mu232a2q143+ \quad 45\mu$$

$$1-\mu64a83q123+315\mu264a43+ 45\mu264a83 + \ 45\mu232a2- 45\mu232a23]$$

(4.90)

$$B_2(FS+RJ) = B_2[-\frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}} + \frac{225(1-\mu)^2}{32(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{32a^{\frac{4}{3}}} +$$

$$\frac{45\mu^2}{32a^{\frac{8}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{8a^2} - \frac{315\mu^2}{32a^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{45\mu^2q_1^{\frac{4}{3}}}{8a^{\frac{4}{3}}} - \frac{45\mu^2q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{16a^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{16a^2} -$$

$$\frac{45\mu(1-\mu)}{16(aq_1)^{\frac{2}{3}}}] \qquad (4.91)$$

Recall from (4.79), then Substituting (4.81), (4.84), (4.87) and (4.91) into it;

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) = \frac{1}{(1-e^2)}\left[\frac{9(1-\mu)^2}{4a^{2/3}q_1^{\frac{4}{3}}} + \frac{9\mu(1-\mu)}{8a^{2/3}} - \frac{9(1-\mu)^2}{16(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{8a^{\frac{4}{3}}q_1^{2/3}} + \frac{9\mu(1-\mu)}{4(aq_1)^{2/3}}\frac{9(1-\mu)^2}{4} -\right.$$

$$\frac{9(1-\mu)^2}{4q_1^{\frac{4}{3}}} + \frac{9\mu^2q_1^{2/3}}{4a^{2/3}} - \frac{9\mu^2}{16a^{\frac{4}{3}}} - \frac{9\mu^2q_1^{\frac{4}{3}}}{4} + \frac{9\mu^2}{4} + e^2\left\{-\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} - \frac{9\mu(1-\mu)}{4a^{\frac{4}{3}}q_1^{2/3}} + \frac{9(1-\mu)^2}{4a^{2/3}q_1^{\frac{4}{3}}} -\right.$$

$$\frac{9\mu^2}{8a^{\frac{4}{3}}} + \frac{9\mu^2q_1^{2/3}}{4a^{2/3}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}}\Big\} + A_2\left\{\frac{9(1-\mu)^2}{8(aq_1)^{\frac{4}{3}}} - \frac{9(1-\mu)^2}{4a^{\frac{2}{3}}q_1^{\frac{4}{3}}} - \frac{9\mu^2}{8a^{\frac{4}{3}}} + \frac{9\mu^2q_1^{2/3}}{4a^{2/3}} + \frac{9\mu^2}{2a^{\frac{2}{3}}} +\right.$$

$$\frac{9\mu(1-\mu)}{4(aq_1)^{\frac{2}{3}}} + \frac{9\mu(1-\mu)}{4a^{\frac{2}{3}}}\Big\} + B_2\{-\frac{45(1-\mu)^2}{16a^2q_1^{\frac{4}{3}}} + \frac{225(1-\mu)^2}{32(aq_1)^{\frac{4}{3}}} - \frac{45\mu(1-\mu)}{8a^2q_1^{\frac{2}{3}}} + \frac{45\mu(1-\mu)}{16a^{\frac{4}{3}}q_1^{\frac{2}{3}}} +$$

$$\frac{45\mu(1-\mu)}{32a^{\frac{4}{3}}} + \frac{45\mu^2}{32a^{\frac{8}{3}}} - \frac{45\mu^2 q_1^{\frac{2}{3}}}{8a^2} - \frac{315\mu^2}{32a^{\frac{4}{3}}} - \frac{45(1-\mu)^2}{16a^{\frac{2}{3}}q_1^{\frac{4}{3}}} + \frac{45\mu^2 q_1^{\frac{4}{3}}}{8a^{\frac{4}{3}}} - \frac{45\mu^2 q_1^{\frac{2}{3}}}{16a^{\frac{2}{3}}} - \frac{45\mu(1-\mu)}{16a^{\frac{2}{3}}} -$$

$$\frac{45\mu(1-\mu)}{16a^2} - \frac{45\mu(1-\mu)}{16(aq_1)^{\frac{2}{3}}}\}] \qquad (4.92)$$

For simplication, let $a = 1 - \alpha$ and $q = 1 - \beta_1$ and substituting into $(\Omega_{\xi}^{\circ})(\Omega_{\eta\eta}^{\circ})\&(\Omega_{\xi\eta}^{\circ})^{\mathbf{2}}$, considering only the linear terms, neglecting higher other terms and products. Thus, we have:

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) = \frac{1}{(1-e^2)}\Big[\frac{9(1-\mu)^2}{4}\left(1+\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right) + \frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha\right)$$

$$-\frac{9(1-\mu)^2}{16}\left(1+\frac{4}{3}\alpha+\frac{4}{3}\beta_1\right) - \frac{9\mu(1-\mu)}{8}\left(1+\frac{4}{3}\alpha+\frac{2}{3}\beta_{1)}\right.$$

$$+\frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right) + \frac{9(1-\mu)^2}{4} - \frac{9(1-\mu)^2}{4}(1$$

$$+\frac{4}{3}\beta_{1)} + \frac{9\mu^2}{4}\left(1+\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right) - \frac{9\mu^2}{16}\left(1+\frac{4}{3}\alpha\right) - \frac{9\mu^2}{4}(1$$

$$-\frac{4}{3}\beta_1) + \frac{9\mu^2}{4}$$

$$+e^2\left\{-\frac{9(1-\mu)^2}{8}\left(1+\frac{4}{3}\alpha+\frac{4}{3}\beta_1\right) - \frac{9\mu(1-\mu)}{4}\left(1+\frac{4}{3}\alpha+\frac{2}{3}\beta_1\right)\right.$$

$$+\frac{9(1-\mu)^2}{4}\left(1+\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right) - \frac{9\mu^2}{8}\left(1+\frac{4}{3}\alpha\right)$$

$$+\frac{9\mu^2}{4}\left(1+\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right) + \frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha\right) + \frac{9\mu(1-\mu)}{4}(1$$

$$\left.+\frac{2}{3}\alpha+\frac{2}{3}\beta_1)\right\}$$

$$+A_2\left\{\frac{9(1-\mu)^2}{8}\left(1+\frac{4}{3}\alpha+\frac{4}{3}\beta_1\right) - \frac{9(1-\mu)^2}{4}\left(1+\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)\right.$$

$$-\frac{9\mu^2}{8}\left(1+\frac{4}{3}\alpha\right) + \frac{9\mu^2}{4}\left(1+\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right) + \frac{9\mu^2}{2}\left(1+\frac{2}{3}\alpha\right)$$

$$\left.+\frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right) + \frac{9\mu(1-\mu)}{4}\left(1+\frac{4}{3}\alpha\right)\right\}$$

$$+B_2\left\{-\frac{45(1-\mu)^2}{16}(1+2\alpha+\frac{4}{3}\beta_1) + \frac{225(1-\mu)^2}{32}\left(1+\frac{4}{3}\alpha\right.\right.$$

$$+\frac{4}{3}\beta_1) - \frac{45\mu(1-\mu)}{8}(1+2\alpha+\frac{2}{3}\beta_1) + \frac{45\mu(1-\mu)}{16}\left(1+\frac{4}{3}\alpha\right.$$

$$+\frac{2}{3}\beta_1) + \frac{45\mu(1-\mu)}{32}\left(1+\frac{4}{3}\alpha\right) + \frac{45\mu^2}{32}\left(1+\frac{8}{3}\alpha\right) - \frac{45\mu^2}{8}(1$$

$$+\ 2\alpha - \frac{2}{3}\beta_1\ ) - \ \frac{315\mu^2}{32}\ (1 + \frac{4}{3}\alpha\ )\ -\ \frac{45(\ 1-\mu)^2}{16}\ (1 + \frac{2}{3}\alpha$$

$$+ \frac{4}{3}\beta_1\ )\ +\ \frac{45\mu^2}{8}\ (1\ \frac{4}{3}\alpha - \frac{4}{3}\beta_1\ ) - \ \frac{45\mu^2}{16}\ (1 + \frac{2}{3}\alpha - \frac{2}{3}\beta_1\ )$$

$$- \frac{45\mu(\ 1-\mu)}{16}\ (1 + \frac{2}{3}\alpha\ )\ -\ \frac{45\mu(\ 1-\mu)}{16}(1 +\ 2\alpha\ )$$

$$- \frac{45\mu(\ 1-\mu)}{16}\ \ (1 + \frac{2}{3}\alpha\ + \frac{2}{3}\beta_1\ )\Big\}\Big]$$

Expanding the inner bracket and neglecting higher order terms;

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) = \frac{1}{(1-e^2)}\left[\frac{9(1-\mu)^2}{4} + \frac{9(1-\mu)^2}{6}\alpha + \frac{9(1-\mu)^2}{3}\beta_1\right)$$

$$+\frac{9\mu(1-\mu)}{4}+\frac{9\mu(1-\mu)}{6}\alpha - \frac{9(1-\mu)^2}{16} + \frac{9(1-\mu)^2}{12}\alpha$$

$$+ \frac{9(1-\mu)^2}{12}\beta_1 - \frac{9\mu(1-\mu)}{8} + \frac{9\mu(1-\mu)}{6}\alpha + \frac{9\mu(1-\mu)}{12}\beta_1$$

$$+\frac{9\mu(1-\mu)}{4} + \frac{9\mu(1-\mu)}{6}\alpha + \frac{9\mu(1-\mu)}{6}\beta_1 + \frac{9(1-\mu)^2}{4}$$

$$-\frac{9(1-\mu)^2}{4} - \frac{9(1-\mu)^2}{3}\beta_1 + \frac{9\mu^2}{4} + \frac{9\mu^2}{6}\alpha - \frac{9\mu^2}{6}\beta_1 - \frac{9\mu^2}{16}$$

$$-\frac{9\mu^2}{12}\alpha - \frac{9\mu^2}{4} + \frac{9\mu^2}{3}\beta_1 + \frac{9\mu^2}{4}$$

$$+ e^2\left\{-\frac{9(1-\mu)^2}{8} - \frac{9\mu(1-\mu)}{4} + \frac{9(1-\mu)^2}{4} - \frac{9\mu^2}{8} + \frac{9\mu^2}{4}\right.$$

$$\left.+\frac{9\mu(1-\mu)}{4} + \frac{9\mu(1-\mu)}{4}\right\}$$

$$+ A_2\left\{\frac{9(1-\mu)^2}{8} - \frac{9(1-\mu)^2}{4} - \frac{9\mu^2}{8} + \frac{9\mu^2}{4} + \frac{9\mu^2}{2} + \frac{9\mu(1-\mu)}{4}\right.$$

$$\left.+\frac{9\mu(1-\mu)}{4}\right\}$$

$$+ B_2\left\{-\frac{45(1-\mu)^2}{16} + \frac{225(1-\mu)^2}{32} - \frac{45\mu(1-\mu)}{8} + \frac{45\mu(1-\mu)}{16}\right.$$

$$+\frac{45\mu(1-\mu)}{32} + \frac{45\mu^2}{32} - \frac{45\mu^2}{8} - \frac{315\mu^2}{32} - \frac{45(1-\mu)^2}{16}$$

$$\left.+\frac{45\mu^2}{8} - \frac{45\mu^2}{16} - \frac{45\mu(1-\mu)}{16} - \frac{45\mu(1-\mu)}{16} - \frac{45\mu(1-\mu)}{16}\right\}]$$

Simplifying to become

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) = (1+e^2)\ [\frac{27(1-\mu)^2}{16} + \frac{27\mu(1-\mu)}{8} + \frac{27\mu^2}{16} + \frac{3(1-\mu)^2}{4}\alpha + \frac{3\mu(1-\mu)}{2}\alpha +$$

$$\frac{3\mu^2}{4}\alpha - \frac{3(1-\mu)^2}{4}\beta_1 + \frac{3\mu^2}{2}\beta_1 + \frac{3\mu(1-\mu)}{2}\beta_1 + e^2\left\{\frac{9(1-\mu)^2}{8} + \frac{9\mu(1-\mu)}{4} + \frac{9\mu^2}{8}\right\} +$$

$$A_2\left\{-\frac{9(1-\mu)^2}{8} + \frac{45\mu^2}{8} + \frac{9\mu(1-\mu)}{2}\right\} + B_2\left\{\frac{45(1-\mu)^2}{32} - \frac{315\mu(1-\mu)}{32} - \frac{45\mu^2}{4}\right\}] \quad (4.93)$$

Multiply through by $(1+e^2)$

$$\begin{aligned}(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) = \ & [\frac{27(1-\mu)^2}{16} + \frac{27(1-\mu)^2}{16}e^2 + \frac{27\mu(1-\mu)}{8} + \frac{27\mu(1-\mu)}{8}e^2 \\ & + \frac{27\mu^2}{16} + \frac{27\mu^2}{16}e^2 + \frac{3(1-\mu)^2}{4}\alpha + \frac{3(1-\mu)^2}{4}\alpha e^2 + \frac{3\mu(1-\mu)}{2}\alpha \\ & + \frac{3\mu(1-\mu)}{2}\alpha e^2 + \frac{3\mu^2}{4}\alpha + \frac{3\mu^2}{4}\alpha e^2 - \frac{3(1-\mu)^2}{4}\beta_1 \\ & - \frac{3(1-\mu)^2}{4}\beta_1 e^2 + \frac{3\mu(1-\mu)}{2}\beta_1 + \frac{3\mu(1-\mu)}{2}\beta_1 e^2 + \frac{3\mu^2}{2}\beta_1 \\ & + \frac{3\mu^2}{2}\beta_1 e^2 + e^2\left\{\frac{9(1-\mu)^2}{8} + \frac{9\mu(1-\mu)}{4} + \frac{9\mu^2}{8}\right\} \\ & + e^4\left\{\frac{9(1-\mu)^2}{8} + \frac{9\mu(1-\mu)}{4} + \frac{9\mu^2}{8}\right\} \\ & + A_2\left\{-\frac{9(1-\mu)^2}{8} + \frac{45\mu^2}{8} + \frac{9\mu(1-\mu)}{2}\right\} \\ & + A_2e^2\left\{-\frac{9(1-\mu)^2}{8} + \frac{45\mu^2}{8} + \frac{9\mu(1-\mu)}{2}\right\} \\ & + B_2\left\{\frac{45(1-\mu)^2}{32} - \frac{315\mu(1-\mu)}{32} - \frac{45\mu^2}{4}\right\} \\ & + B_2e^2\left\{\frac{45(1-\mu)^2}{32} - \frac{315\mu(1-\mu)}{32} - \frac{45\mu^2}{4}\right\}]\end{aligned}$$

(4.94)

Neglecting higher powers of e^2 and the product of A_2e^2, β_1e^2, etc.

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) = \left[\frac{27(1-\mu)^2}{16} + \frac{27(1-\mu)^2}{16}e^2 + \frac{27\mu(1-\mu)}{8} + \frac{27\mu(1-\mu)}{8}e^2 + \frac{27\mu^2}{16} + \frac{27\mu^2}{16}e^2 + \frac{3(1-\mu)^2}{4}\alpha + \frac{3\mu(1-\mu)}{2}\alpha + \frac{3\mu^2}{4}\alpha - \frac{3(1-\mu)^2}{4}\beta_1 + \frac{3\mu(1-\mu)}{2}\beta_1 + \frac{3\mu^2}{2}\beta_1 + e^2\left\{\frac{9(1-\mu)^2}{8} + \frac{9\mu(1-\mu)}{4} + \frac{9\mu^2}{8}\right\} + A_2\left\{-\frac{9(1-\mu)^2}{8} + \frac{45\mu^2}{8} + \frac{9\mu(1-\mu)}{2}\right\} + B_2\left\{\frac{45(1-\mu)^2}{32} - \frac{315\mu(1-\mu)}{32} - \frac{45\mu^2}{4}\right\}\right] \quad (4.95)$$

For$a = 1 - \alpha$ and $q = 1 - \beta_1$ and substituting into $(\Omega^{\circ}_{\xi})(\Omega^{\circ}_{\eta\eta})\&(\Omega^{\circ}_{\xi\eta})^{2}$, considering only the linear terms, neglecting higher other terms and productsinto (4.78);

$$(\Omega^{\circ}_{\xi\eta})^2 = \frac{1}{(1-e^2)}\Big[\frac{9(1-\mu)^2}{4}\left(1+\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)+\frac{9(1-\mu)^2}{8}\left(1+\frac{2}{3}\beta_1\right)$$

$$-\frac{45(1-\mu)^2}{16}\left(1+\frac{4}{3}\beta_1\right)-\frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)$$

$$-\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha\right)-\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha\right.$$

$$\left.-\frac{2}{3}\beta_1\right)-\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha\right)+\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)$$

$$+\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)+\frac{9\mu^2}{8}\left(1-\frac{2}{3}\beta_1\right)+\frac{27\mu^2}{16}-\frac{9\mu^2}{8}(1$$

$$-\frac{2}{3}\alpha-\frac{4}{3}\beta_1)-\frac{9\mu^2}{8}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)+\frac{9\mu^2}{8}\left(1-\frac{2}{3}\right)$$

$$+\frac{27(1-\mu)^2}{16}-\frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\right)+\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)$$

$$+\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha-\frac{4}{3}\beta_1\right)-\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)$$

$$+\frac{9\mu^2}{4}\left(1+\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)-\frac{45\mu^2}{16}\left(1-\frac{4}{3}\beta_1\right)+\frac{9\mu^2}{4}\left(1-\frac{2}{3}\alpha-2\beta_1\right)$$

$$-\frac{9(1-\mu)^2}{16}\left(1+\frac{4}{3}\alpha+\frac{4}{3}\beta_1\right)+\frac{9\mu(1-\mu)}{8}\left(1+\frac{4}{3}\alpha+\frac{2}{3}\beta_1\right)$$

$$-\frac{9\mu(1-\mu)}{8}\left(1-\frac{2}{3}\beta_1\right)+\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{8}\left(1-\frac{2}{3}\beta_1\right)$$

$$-\frac{9\mu^2}{16}\left(1+\frac{4}{3}\alpha\right)$$

$$+e^2\left\{\frac{9(1-\mu)^2}{4}\left(1+\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)-\frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)\right.$$

$$+\frac{9\mu^2}{4}\left(1+\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)-\frac{9(1-\mu)^2}{8}\left(1+\frac{4}{3}\alpha+\frac{4}{3}\beta_1\right)$$

$$+\frac{9\mu(1-\mu)}{4}\left(1+\frac{4}{3}\alpha+\frac{4}{3}\beta_1\right)-\frac{9\mu^2}{8}\left(1+\frac{4}{3}\alpha\right)$$

$$+\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha\right)+\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)$$

$$+\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)+\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha\right)$$

$$-\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)+\frac{9\mu^2}{8}\left(1-\frac{2}{3}\alpha-\frac{4}{3}\beta_1\right)$$

$$+\frac{9\mu^2}{8}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)-\frac{9\mu^2}{8}\left(1-\frac{2}{3}\alpha\right)+\frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha\right)$$

$$-\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)-\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha-\frac{4}{3}\beta_1\right)$$

$$-\frac{9\mu^2}{8}\left(1-\frac{2}{3}\alpha-2\beta_1\right)-\frac{9\mu(1-\mu)}{4}\left(1+\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)$$

$$-\frac{9\mu(1-\mu)}{2}(1+\frac{2}{3}\alpha)\Big\}$$

$$+A_2\left\{-\frac{9\mu(1-\mu)}{2}\left(1+\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)-\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\beta_1\right)\right.$$

$$+\frac{27\mu(1-\mu)}{4}\left(1+\frac{2}{3}\beta_1\right)-\frac{9\mu(1-\mu)}{2}+\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha\right)$$

$$+\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)-\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)$$

$$-\frac{9\mu(1-\mu)}{4}\left(1-\frac{2}{3}\alpha-\frac{4}{3}\beta_1\right)-\frac{9(1-\mu)^2}{8}$$

$$+\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha_1\right)-\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha_1-\frac{2}{3}\beta_1\right)$$

$$-\frac{9(1-\mu)^2}{4}\left(1+\frac{2}{3}\beta_1\right)+\frac{9(1-\mu)^2}{8}\left(1+\frac{4}{3}\alpha+\frac{4}{3}\beta_1\right)$$

$$-\frac{27(1-\mu)^2}{8}\left(1+\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)+\frac{9(1-\mu)^2}{8}\left(1+\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)$$

$$+\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)+\frac{27(1-\mu)^2}{8}\left(1+\frac{4}{3}\beta_1\right)$$

$$-\frac{9(1-\mu)^2}{8}\left(1-\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)+\frac{27\mu^2}{8}+\frac{27\mu^2}{8}\left(1+\frac{2}{3}\alpha\right)$$

$$-\frac{9\mu^2}{8}\left(1-\frac{2}{3}\alpha\right)+\frac{9\mu^2}{8}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)+\frac{27\mu^2}{8}\left(1+\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)$$

$$
\begin{aligned}
&- \frac{9\mu^2}{4}\left(1 - \frac{2}{3}\beta_1\right) - \frac{9\mu^2}{8}\left(1 + \frac{4}{3}\alpha\right) - \frac{9\mu^2}{8}\left(1 - \frac{4}{3}\beta_1\right) \\
&+ \frac{9\mu^2}{8}\left(1 - \frac{2}{3}\alpha - \frac{4}{3}\beta_1\right) - \frac{9\mu^2}{8}(1 - \frac{2}{3}\alpha - 2\beta_1)\Big\} \\
&+ B_2\left\{- \frac{225\mu(1-\mu)}{32}\left(1 + \frac{2}{3}\alpha + \frac{4}{3}\beta_1\right)\right. \\
&+ \frac{45(1-\mu)^2}{16}\left(1 + \frac{2}{3}\alpha + \frac{2}{3}\beta_1\right) + \frac{135(1-\mu)^2}{16}\left(1 + \frac{4}{3}\alpha + \frac{4}{3}\beta_1\right) \\
&- \frac{45(1-\mu)^2}{32}\left(1 + \frac{4}{3}\alpha + \frac{2}{3}\beta_1\right) - \frac{45(1-\mu)^2}{16}\left(1 + 2\alpha + \frac{4}{3}\beta_1\right) \\
&- \frac{135\mu(1-\mu)}{16}\left(1 + \frac{2}{3}\alpha + \frac{2}{3}\beta_1\right) + \frac{45\mu(1-\mu)}{4}\left(1 + \frac{2}{3}\alpha\right) \\
&+ \frac{45\mu(1-\mu)}{32}\left(1 + \frac{4}{3}\alpha + \frac{2}{3}\beta_1\right) + \frac{45\mu(1-\mu)}{8}\left(1 + 2\alpha + \frac{2}{3}\beta_1\right) \\
&+ \frac{45\mu(1-\mu)}{16}(1 + 2\alpha) - \frac{45\mu(1-\mu)}{16}\left(1 + \frac{8}{3}\alpha + \frac{2}{3}\beta_1\right) \\
&+ \frac{45\mu(1-\mu)}{8}\left(1 + \frac{2}{3}\alpha - \frac{2}{3}\beta_1\right) + \frac{45(1-\mu)^2}{32}\left(1 + \frac{2}{3}\alpha\right) \\
&- \frac{45\mu(1-\mu)}{8a^{\frac{4}{3}}}\left(1 + \frac{4}{3}\alpha\right) - \frac{45\mu(1-\mu)}{16}\left(1 + \frac{2}{3}\alpha - \frac{4}{3}\beta_1\right) \\
&+ \frac{45\mu(1-\mu)}{32}\left(1 + \frac{4}{3}\alpha - \frac{2}{3}\beta_1\right) - \frac{225\mu^2}{32}\left(1 + \frac{4}{3}\alpha\right) \\
&+ \frac{45\mu^2}{16}\left(1 + \frac{2}{3}\alpha - \frac{2}{3}\beta_1\right) - \frac{225\mu^2}{32}\left(1 + \frac{2}{3}\alpha\right) \\
&- \frac{45\mu^2}{8}\left(1 + 2\alpha - \frac{2}{3}\beta_1\right) + \frac{45\mu^2}{32}\left(1 + \frac{8}{3}\alpha\right) \\
&+ \frac{225\mu^2}{32}\left(1 + \frac{4}{3}\alpha - \frac{2}{3}\beta_1\right) + \frac{135\mu^2}{32}\left(1 + \frac{2}{3}\alpha - \frac{4}{3}\beta_1\right) \\
&- \frac{135\mu^2}{32}\left(1 + \frac{4}{3}\alpha - \frac{2}{3}\beta_1\right) - \frac{45\mu^2}{16}\left(1 + \frac{2}{3}\alpha - 2\beta_1\right) \\
&- \frac{45(1-\mu)^2}{32}\left(1 - \frac{2}{3}\alpha\right) + \frac{45(1-\mu)^2}{32}\left(1 - \frac{2}{3}\alpha - \frac{2}{3}\beta_1\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{45(1-\mu)^2}{32}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)-\frac{45\mu(1-\mu)}{16}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)\\
&+\frac{45\mu(1-\mu)}{16}\left(1-\frac{2}{3}\alpha-\frac{4}{3}\beta_1\right)-\frac{45\mu(1-\mu)}{16}\left(1+\frac{2}{3}\alpha\right)\\
&+\frac{45(1-\mu)^2}{32}\left(1-\frac{2}{3}\alpha+\frac{4}{3}\beta_1\right)+\frac{45\mu(1-\mu)}{16}\left(1-\frac{2}{3}\alpha+\frac{2}{3}\beta_1\right)\\
&-\frac{45\mu^2}{32}\left(1-\frac{2}{3}\alpha-\frac{4}{3}\beta_1\right)+\frac{45\mu^2}{32}\left(1-\frac{2}{3}\alpha-2\beta_1\right)\\
&-\frac{45\mu^2}{32}\left(1-\frac{2}{3}\alpha-\frac{2}{3}\beta_1\right)+\frac{45\mu^2}{32}\left(1-\frac{2}{3}\alpha\right)\Big\}]
\end{aligned}
$$

Expanding the inner bracket and neglecting higher order terms

$$(\Omega^{\circ}_{\xi\eta})^2 = \frac{1}{(1-e^2)}\Big[\frac{9(1-\mu)^2}{4}+\frac{9(1-\mu)^2}{6}\,\alpha+\frac{9(1-\mu)^2}{3}\beta_1+\frac{9(1-\mu)^2}{8}$$

$$+\frac{9(1-\mu)^2}{12}\beta_1-\frac{45(1-\mu)^2}{16}$$

$$-\frac{45(1-\mu)^2}{12}\beta_1-\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{6}\,\alpha-\frac{9\mu(1-\mu)}{6}\beta_1$$

$$-\frac{9(1-\mu)^2}{8}+\frac{9(1-\mu)^2}{12}\,\alpha-\frac{9\mu(1-\mu)}{4}+\frac{9\mu(1-\mu)}{6}\alpha$$

$$+\frac{9\mu(1-\mu)}{6}\beta_1-\frac{9\mu(1-\mu)}{4}+\frac{9\mu(1-\mu)}{6}\,\alpha+\frac{9(1-\mu)^2}{8}$$

$$-\frac{9(1-\mu)^2}{12}\alpha+\frac{9(1-\mu)^2}{6}\beta_1+\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{6}\alpha$$

$$+\frac{9\mu(1-\mu)}{6}\beta_1+\frac{9\mu^2}{8}-\frac{9\mu^2}{12}\beta_1+\frac{27\mu^2}{16}-\frac{9\mu^2}{8}+\frac{9\mu^2}{12}\alpha+\frac{9\mu^2}{6}\beta_1$$

$$-\frac{9\mu^2}{8}+\frac{9\mu^2}{12}\alpha-\frac{9\mu^2}{12}\beta_1+\frac{9\mu^2}{8}+\frac{9\mu^2}{12}\,\alpha+\frac{27(1-\mu)^2}{16}$$

$$-\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{6}\,\alpha+\frac{9(1-\mu)^2}{8}-\frac{9(1-\mu)^2}{12}\alpha$$

$$-\frac{9(1-\mu)^2}{12}\beta_1+\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{6}\,\alpha-\frac{9\mu(1-\mu)}{3}\beta_1$$

$$-\frac{9(1-\mu)^2}{8}+\frac{9(1-\mu)^2}{12}\,\alpha+\frac{9(1-\mu)^2}{12}\beta_1+\frac{9\mu^2}{4}+\frac{9\mu^2}{6}\alpha$$

$$+\frac{9\mu^2}{3}\beta_1-\frac{45\mu^2}{16}+\frac{45\mu^2}{12}\beta_1+\frac{9\mu^2}{4}-\frac{9\mu^2}{6}\alpha-\frac{9\mu^2}{2}\beta_1$$

$$-\frac{9(1-\mu)^2}{16}-\frac{9(1-\mu)^2}{12}\alpha-\frac{9(1-\mu)^2}{12}\beta_1+\frac{9\mu(1-\mu)}{8}$$

$$+\frac{9\mu(1-\mu)}{6}\,\alpha+\frac{9\mu(1-\mu)}{12}\beta_1-\frac{9\mu(1-\mu)}{8}+\frac{9\mu(1-\mu)}{12}\beta_1$$

$$+\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{8}+\frac{9\mu(1-\mu)}{12}\beta_1-\frac{9\mu^2}{16}-\frac{9\mu^2}{12}\,\alpha$$

$$+e^2\Big\{\frac{9(1-\mu)^2}{4}-\frac{9\mu(1-\mu)}{4}+\frac{9\mu^2}{4}-\frac{9(1-\mu)^2}{8}+\frac{9\mu(1-\mu)}{4}$$

$$-\frac{9\mu^2}{8}+\frac{9(1-\mu)^2}{8}+\frac{9(1-\mu)^2}{8}+\frac{9\mu(1-\mu)}{4}+\frac{9\mu(1-\mu)}{4}$$

$$-\frac{9(1-\mu)^2}{8}+\frac{9\mu^2}{8}+\frac{9\mu^2}{8}-\frac{9\mu^2}{8}+\frac{9\mu(1-\mu)}{4a^{2/3}}-\frac{9(1-\mu)^2}{8}$$

$$-\frac{9\mu(1-\mu)}{4}-\frac{9\mu^2}{8}-\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{2}\Bigg\}$$

$$+A_2\left\{-\frac{9\mu(1-\mu)}{2}-\frac{9\mu(1-\mu)}{4}+\frac{27\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{2}\right.$$

$$+\frac{9\mu(1-\mu)}{4}+\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{4}-\frac{9\mu(1-\mu)}{4}$$

$$-\frac{9(1-\mu)^2}{8}+\frac{9(1-\mu)^2}{8}-\frac{9(1-\mu)^2}{8}-\frac{9(1-\mu)^2}{4}+\frac{9(1-\mu)^2}{8}$$

$$-\frac{27(1-\mu)^2}{8}+\frac{9(1-\mu)^2}{8}+\frac{9(1-\mu)^2}{8}+\frac{27(1-\mu)^2}{8}$$

$$-\frac{9(1-\mu)^2}{8}+\frac{27\mu^2}{8}+\frac{27\mu^2}{8}-\frac{9\mu^2}{8}+\frac{9\mu^2}{8}+\frac{27\mu^2}{8}-\frac{9\mu^2}{4}-\frac{9\mu^2}{8}$$

$$-\frac{9\mu^2}{8}+\frac{9\mu^2}{8}-\frac{9\mu^2}{8}\Bigg\}$$

$$+B_2\left\{-\frac{225\mu(1-\mu)}{32}+\frac{45(1-\mu)^2}{16}+\frac{135(1-\mu)^2}{16}\right.$$

$$-\frac{45(1-\mu)^2}{32}-\frac{45(1-\mu)^2}{16}-\frac{135\mu(1-\mu)}{16}+\frac{45\mu(1-\mu)}{4}$$

$$+\frac{45\mu(1-\mu)}{32}+\frac{45\mu(1-\mu)}{8}+\frac{45\mu(1-\mu)}{16}-\frac{45\mu(1-\mu)}{16}$$

$$+\frac{45\mu(1-\mu)}{8}+\frac{45(1-\mu)^2}{32}-\frac{45\mu(1-\mu)}{8}-\frac{45\mu(1-\mu)}{16}$$

$$+\frac{45\mu(1-\mu)}{32}-\frac{225\mu^2}{32}+\frac{45\mu^2}{16}-\frac{225\mu^2}{32}-\frac{45\mu^2}{8}+\frac{45\mu^2}{32}$$

$$+\frac{225\mu^2}{32}+\frac{135\mu^2}{32}-\frac{135\mu^2}{32}-\frac{45\mu^2}{16}-\frac{45(1-\mu)^2}{32}+\frac{45(1-\mu)^2}{32}$$

$$-\frac{45(1-\mu)^2}{32}-\frac{45\mu(1-\mu)}{16}+\frac{45\mu(1-\mu)}{16}-\frac{45\mu(1-\mu)}{16}$$

$$+\frac{45(1-\mu)^2}{32}+\frac{45\mu(1-\mu)}{16}-\frac{45\mu^2}{32}+\frac{45\mu^2}{32}-\frac{45\mu^2}{32}+\frac{45\mu^2}{32}\Big\}]$$

Simplifying

$$(\Omega^{\circ}_{\xi\eta})^2=(1+e^2)\,[\,\frac{27(1-\mu)^2}{16}-\frac{27\mu(1-\mu)}{8}+\frac{27\mu^2}{16}+\frac{3(1-\mu)^2}{4}\alpha-\frac{3\mu(1-\mu)}{2}\alpha+$$

$$\frac{3\mu^2}{4}\alpha-\frac{3(1-\mu)^2}{4}\beta_1--\frac{3\mu(1-\mu)}{2}\beta_1+\frac{3\mu^2}{4}\beta_1+e^2\Big\{\frac{9(1-\mu)^2}{8}-\frac{9\mu(1-\mu)}{4}+\frac{9\mu^2}{8}\Big\}+$$

$$A_2\Big\{-\frac{9(1-\mu)^2}{2}-\frac{9\mu(1-\mu)}{8}+\frac{9\mu^2}{8}\Big\}+B_2\Big\{\frac{45(1-\mu)^2}{32}+\frac{315\mu(1-\mu)}{32}-\frac{45\mu^2}{4}\Big\}]\quad(4.96)$$

Multiply through by $(1+e^2)$

$$(\Omega^{\circ}_{\xi\eta})^2=[\,\frac{27(1-\mu)^2}{16}+\frac{27(1-\mu)^2}{16}e^2-\frac{27\mu(1-\mu)}{8}-\frac{27\mu(1-\mu)}{8}e^2$$

$$+\frac{27\mu^2}{16}+\frac{27\mu^2}{16}e^2+\frac{3(1-\mu)^2}{4}\alpha+\frac{3(1-\mu)^2}{4}\alpha e^2-\frac{3\mu(1-\mu)}{2}\alpha$$

$$-\frac{3\mu(1-\mu)}{2}\alpha e^2+\frac{3\mu^2}{4}\alpha+\frac{3\mu^2}{4}\alpha e^2-\frac{3(1-\mu)^2}{4}\beta_1$$

$$-\frac{3(1-\mu)^2}{4}\beta_1e^2-\frac{3\mu(1-\mu)}{2}\beta_1-\frac{3\mu(1-\mu)}{2}\beta_1e^2+\frac{3\mu^2}{4}\beta_1$$

$$+\frac{3\mu^2}{4}\beta_1e^2+e^2\Big\{\frac{9(1-\mu)^2}{8}-\frac{9\mu(1-\mu)}{4}+\frac{9\mu^2}{8}\Big\}$$

$$+e^4\Big\{\frac{9(1-\mu)^2}{8}-\frac{9\mu(1-\mu)}{4}+\frac{9\mu^2}{8}\Big\}$$

$$+A_2\Big\{-\frac{9(1-\mu)^2}{2}-\frac{9\mu(1-\mu)}{8}+\frac{9\mu^2}{8}\Big\}$$

$$+A_2e^2\Big\{-\frac{9(1-\mu)^2}{2}-\frac{9\mu(1-\mu)}{8}+\frac{9\mu^2}{8}\Big\}$$

$$+B_2\Big\{\frac{45(1-\mu)^2}{32}+\frac{315\mu(1-\mu)}{32}-\frac{45\mu^2}{4}\Big\}$$

$$+B_2e^2\Big\{\frac{45(1-\mu)^2}{32}+\frac{315\mu(1-\mu)}{32}-\frac{45\mu^2}{4}\Big\}]$$

(4.97)

Neglecting higher order terms of e^2 and the product of $\alpha_1 e^2, A_2 e^2$ e.t.c

$$(\Omega^{\circ}_{\xi\eta})^2 = [\ \frac{27(1-\mu)^2}{16} + \frac{27(1-\mu)^2}{16}e^2 - \frac{27\mu(1-\mu)}{8} - \frac{27\mu(1-\mu)}{8}e^2 + \frac{27\mu^2}{16} + \frac{27\mu^2}{16}e^2 +$$

$$\frac{3(1-\mu)^2}{4}\alpha - \frac{3\mu(1-\mu)}{2}\alpha + \frac{3\mu^2}{4}\alpha - \frac{3(1-\mu)^2}{4}\beta_1 - \frac{3\mu(1-\mu)}{2}\beta_1 + \frac{3\mu^2}{4}\beta_1 +$$

$$e^2\left\{\frac{9(1-\mu)^2}{8} - \frac{9\mu(1-\mu)}{4} + \frac{9\mu^2}{8}\right\} + A_2\left\{-\frac{9(1-\mu)^2}{2} - \frac{9\mu(1-\mu)}{8} + \frac{9\mu^2}{8}\right\} + B_2\left\{\frac{45(1-\mu)^2}{32} +\right.$$

$$315\mu 1-\mu 32 - 45\mu 24 \qquad (4.98)$$

Subtracting (4.98) from (4.95);

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) - \left(\Omega^{\circ}_{\xi\eta}\right)^2$$

$$= \left[\frac{27(1-\mu)^2}{16} + \frac{27(1-\mu)^2}{16}e^2 + \frac{27\mu(1-\mu)}{8} + \frac{27\mu(1-\mu)}{8}e^2\right.$$

$$+ \frac{27\mu^2}{16} + \frac{27\mu^2}{16}e^2 + \frac{3(1-\mu)^2}{4}\alpha + \frac{3\mu(1-\mu)}{2}\alpha + \frac{3\mu^2}{4}\alpha$$

$$- \frac{3(1-\mu)^2}{4}\beta_1 + \frac{3\mu(1-\mu)}{2}\beta_1 + \frac{3\mu^2}{2}\beta_1 - \left[\frac{27(1-\mu)^2}{16}\right.$$

$$+ \frac{27(1-\mu)^2}{16}e^2 - \frac{27\mu(1-\mu)}{8} - \frac{27\mu(1-\mu)}{8}e^2 + \frac{27\mu^2}{16}$$

$$+ \frac{27\mu^2}{16}e^2 + \frac{3(1-\mu)^2}{4}\alpha - \frac{3\mu(1-\mu)}{2}\alpha + \frac{3\mu^2}{4}\alpha - \frac{3(1-\mu)^2}{4}\beta_1$$

$$\left.\left.- \frac{3\mu(1-\mu)}{2}\beta_1 + \frac{3\mu^2}{4}\beta_1\right]\right.$$

$$+ e^2\left\{\frac{9(1-\mu)^2}{8} + \frac{9\mu(1-\mu)}{4} + \frac{9\mu^2}{8} - \left[\frac{9(1-\mu)^2}{8}\right.\right.$$

$$\left.\left.- \frac{9\mu(1-\mu)}{4} + \frac{9\mu^2}{8}\right]\right\}$$

$$+ A_2\left\{-\frac{9(1-\mu)^2}{8} + \frac{45\mu^2}{8} + \frac{9\mu(1-\mu)}{2} - \left[-\frac{9(1-\mu)^2}{2}\right.\right.$$

$$\left.\left.- \frac{9\mu(1-\mu)}{8} + \frac{9\mu^2}{8}\right]\right\}$$

$$+ B_2\left\{\frac{45(1-\mu)^2}{32} - \frac{315\mu(1-\mu)}{32} - \frac{45\mu^2}{4} - \left[\frac{45(1-\mu)^2}{32}\right.\right.$$

$$\left.\left.+ \frac{315\mu(1-\mu)}{32} - \frac{45\mu^2}{4}\right]\right\}\Big]$$

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) - \left(\Omega^{\circ}_{\xi\eta}\right)^2 = \frac{27\mu(1-\mu)}{4} + 3\mu(1-\mu)\alpha + \frac{3\mu(1-\mu)}{2}\beta_1 + 9\mu(1-\mu)A_2 +$$

$$\frac{45\mu(1-\mu)}{4}e^2 - \frac{315\mu(1-\mu)}{16}B_2 \qquad (4.99)$$

Let

$$\varepsilon_2 = 3\mu(1-\mu)\alpha + \frac{3\mu(1-\mu)}{2}\beta_1 + 9\mu(1-\mu)A_2 + \frac{45\mu(1-\mu)}{4}e^2 + +9\mu(1-\mu)A_2 - \frac{315\mu(1-\mu)}{16}B_2 \quad (4.100)$$

Hence

$$(\Omega^{\circ}_{\xi\xi})(\Omega^{\circ}_{\eta\eta}) - (\Omega^{\circ}_{\xi\eta})^2 = \frac{27\mu(1-\mu)}{4} + \varepsilon_2 \quad (4.101)$$

Recall from (4.55), i.e.;

$$\Omega^{\circ}_{\xi\xi} + \Omega^{\circ}_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[3 + \frac{3\mu}{a^{⅔}}A_2 - \frac{15\mu}{2a^{\frac{4}{3}}}B_2\right]$$

$$\Omega^{\circ}_{\xi\xi} + \Omega^{\circ}_{\eta\eta} = \frac{1}{(1-e^2)^{½}}\left[3 + 3\mu\left(1+\frac{2}{3}\alpha\right)A_2 - \frac{15\mu}{2}\left(1+\frac{4}{3}\alpha\right)B_2\right]$$

$$\Omega^{\circ}_{\xi\xi} + \Omega^{\circ}_{\eta\eta} = \frac{3}{(1-e^2)^{½}}\left[1 + \mu\left(1+\frac{2}{3}\alpha\right)A_2 - \frac{5\mu}{2}\left(1+\frac{4}{3}\alpha\right)B_2\right] \quad (4.102)$$

where

$$\varepsilon_1 = \frac{1}{(1-e^2)^{½}}\left[1 + \mu\left(1+\frac{2}{3}\alpha\right)A_2 - \frac{5\mu}{2}\left(1+\frac{4}{3}\alpha\right)B_2\right] \quad (4.103)$$

Therefore;

$$\Omega^{\circ}_{\xi\xi} + \Omega^{\circ}_{\eta\eta} = 3\varepsilon_1 \quad (4.104)$$

Substituting (4.104) and (4.101) into the characteristics equation (4.7)

$$\lambda^4 - (3\varepsilon_1 - 4)\lambda^2 + \frac{27\mu(1-\mu)}{4} + \varepsilon_2 = 0$$

this can be re – written as a quadratic equation of the form(λ^2)

$$(\lambda^2)^2 + (4 - 3\varepsilon_1)\lambda^2 + \frac{27\mu(1-\mu)}{4} + \varepsilon_2 = 0$$

$$4(\lambda^2)^2 + 4(4 - 3\varepsilon_1)\lambda^2 + 27\mu(1 - \mu) + 4\varepsilon_2 = 0 \qquad (4.105)$$

This is in form of a quadratic equation. Thus using the quadratic formula i.e. $\lambda^2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, where

$$a = 4, b = 4(4 + 3\varepsilon_1) \& c = 27\mu(1 - \mu) + 4\varepsilon_2$$

Substituting $a, b \& c$ into the quadratic formula

$$\lambda^2 = \frac{-4(4-3\varepsilon_1) \pm \sqrt{\{4(4-3\varepsilon_1)^2 - 4\{4.[27\mu(1-\mu) + 4\varepsilon_2]\}}}{2(4)};$$

$$\lambda^2 = \frac{-4(4 - 3\varepsilon_1) \pm \{16(4 - 3\varepsilon_1)^2 - 16[27\mu(1 - \mu) + 4\varepsilon_2]\}^{1/2}}{2(4)}$$

$$\lambda^2 = \frac{-(4-3\varepsilon_1) \pm \{(4-3\varepsilon_1)^2 - [27\mu(1-\mu) + 4\varepsilon_2]\}^{1/2}}{2} \qquad (4.106)$$

It can be concluded from (4.106) that it is bounded and purely imaginary. The roots are given as

$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}$; where the discriminant $\Delta = (4 - 3\varepsilon_1)^2 - [27\mu(1 - \mu) + 4\varepsilon_2]$

For the stability of motion, we have that

$3\varepsilon_1 - 4 \leq 0$ and $(4 - 3\varepsilon_1)^2 - [27\mu(1 - \mu) + 4\varepsilon_2] \geq 0$

Now taking $3\varepsilon_1 - 4 \leq 0$, but $\varepsilon_1 = \frac{1}{(1-e^2)^{1/2}} [1 + \mu(1 + \frac{2}{3}\alpha) A_2 - \frac{5\mu}{2}(1 + \frac{4}{3}\alpha) B_2]$

$\frac{3}{(1-e^2)^{1/2}} [1 + \mu(1 + \frac{2}{3}\alpha) A_2 - \frac{5\mu}{2}(1 + \frac{4}{3}\alpha) B_2] - 4 < 0;$

$\frac{9}{(1-e^2)}\left[1 + \mu\left(1 + \frac{2}{3}\alpha\right) A_2 - \frac{5\mu}{2}\left(1 + \frac{4}{3}\alpha\right) B_2\right] < 16;$

$$9\left[1+\ \mu\left(1+\frac{2}{3}\alpha\right)A_2-\frac{5\mu}{2}\left(1+\frac{4}{3}\alpha\right)B_2\right]^2<16\ (1-\ e^2);$$

$$\frac{9}{16}\left[1+\ \mu\left(1+\frac{2}{3}\alpha\right)A_2-\frac{5\mu}{2}\left(1+\frac{4}{3}\alpha\right)B_2\right]^2<(1-\ e^2);$$

$$e^2<[1-\frac{9}{16}\{1+\ \mu\left(1+\frac{2}{3}\alpha\right)A_2-\frac{5\mu}{2}\left(1+\frac{4}{3}\alpha\right)B_2\}^2];$$

$$0\ \leq\ e\leq[1-\frac{9}{16}\{1+\ \mu\left(1+\frac{2}{3}\alpha\right)A_2-\frac{5\mu}{2}\left(1+\frac{4}{3}\alpha\right)B_2\}^2]^{½};$$

let $A_2=0, B_2=0$ (4.107)

$$0\ \leq\ e\leq[1-\frac{9}{16}]^{½}\ ;\ 0\ \leq\ e\leq\ 0.6614378278$$

It is evident that (4.107) satisfy the condition of β, A_2 and B_2 respectively, hence the stability of the equilibrium point, but if eccentricity does not satisfy (4.107), the characteristics equation can either be real and complex conjugates roots.

Now for $(4-3\varepsilon_1)^2-[27\mu(1-\mu)+\ 4\varepsilon_2]\geq 0$, we have that

$$16-24\varepsilon_1+\ 9\varepsilon_1^2-27\mu+\ 27\mu^2-4\varepsilon_2\geq 0$$

$$27\mu^2-27\mu\ +\ 9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\geq 0 \qquad (4.108)$$

Equation (4.108) is a quadratic equation in form of μ, thus it can be written as: -

$$□=\frac{27\pm[(27)^2+4.27\{9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\}^{½}]}{54} \qquad (4.109)$$

where, $a=27$, $b=\ -27$ and $c=\ 9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16$ and in inequality form

$$\left\{ \square - \frac{27-\left[(27)^2+4.27\left\{9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\right\}^{½}\right]}{54} \right\} \left\{ \square - \frac{27+\left[(27)^2+4.27\left\{9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\right\}^{½}\right]}{54} \right\} \geq 0$$

(4.110)

When both factors are positive, we have it as

$$\mu > \frac{27+[(27)^2+108\ \{9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\}^{1/2}]}{54} \quad (4.111)$$

$$\mu > \frac{27-[(27)^2+108\ \{9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\}^{1/2}]}{54} \quad (4.112)$$

and when they are both negative it will be as

$$\mu < \frac{27+[(27)^2+108\ \{9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\}^{1/2}]}{54} \quad (4.113)$$

$$\mu < \frac{27-[(27)^2-108\ \{9\varepsilon_1^2-24\varepsilon_1-4\varepsilon_2+16\}^{1/2}]}{54} \quad (4.114)$$

To verify this study with the classical case, we let $\alpha_1 = \beta_2 = e = A_2 = B_2 = 0$. So $\varepsilon_1 = 1$ and $\varepsilon_2 = 0$ in (4.114).

$$\mu < \frac{27-[(27)^2-108\ \{9-24+16\}^{1/2}]}{54}; \qquad \mu < \frac{27-[729-108]}{54}; \qquad \mu < \frac{27-[729-108]^{1/2}}{54};$$

$$\mu < \frac{27-[621]^{1/2}}{54}; \qquad \mu < \frac{27-24.92}{54}; \qquad \mu < \frac{2.08}{54}; \qquad \mu < 0.03851 \quad (4.115)$$

which agrees with McCuskey (1963). Thus equation (4.114) is the only equation that satisfies the stability of the triangular points, when $\mu < 1/2$.The value of μ obtained in (4.114) would be the critical mass parameter value denoted as μ_c.

Taking, $\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}$; where the discriminant $\Delta = (4-3\varepsilon_1)^2 - [27\mu(1-\mu) + 4\varepsilon_2]$

Taking the discriminant $\Delta = b^2 - 4ac$

$$b^2 - 4ac = (4-3\varepsilon_1)^2 - 27\mu + 27\mu^2 - 4\varepsilon_2; \quad (4.116)$$

therefore;

$$(4 - 3\varepsilon_1)^2 = 16 - 24\varepsilon_1 + 9{\varepsilon_1}^2 ; \quad (4.117)$$

Recall that $\varepsilon_1 = \frac{1}{(1-e^2)^{½}} [1 + 1\mu A_2 - \frac{5\mu}{2} B_2]$, solve binomially and neglecting higher order terms, so ε_1, becomes

$\varepsilon_1 = \left(1 + \frac{e^2}{2}\right)(1 + \mu A_2 - \frac{5\mu}{2} B_2)$, open up and neglecting higher order terms,

$$\varepsilon_1 = 1 + \frac{e^2}{2} + \mu A_2 - \frac{5\mu}{2} B_2 \quad (4.118)$$

$$\therefore {\varepsilon_1}^2 = [1 + \frac{e^2}{2} + \mu A_2 - \frac{5\mu}{2} B_2]^2$$

$${\varepsilon_1}^2 = 1 + e^2 + 2\mu A_2 - 5\mu B_2 \quad (4.119)$$

Now, substituting (4.118) and (4.119) into (4.117);

$$16 - 24\varepsilon_1 + 9{\varepsilon_1}^2 = 16 - 24\left(1 + \frac{e^2}{2} + \mu A_2 - \frac{5\mu}{2} B_2\right) + 9(1 + e^2 + 2\mu A_2 - 5\mu B_2)$$

$$16 - 24\varepsilon_1 + 9{\varepsilon_1}^2 = 1 - 6\mu A_2 + 15\mu B_2 - 3e^2 \quad (4.120)$$

Now, $-4\varepsilon_2$

But; $\varepsilon_2 = 3\mu(1-\mu)\alpha + \frac{3\mu(1-\mu)}{2}\beta_1 + 9\mu(1-\mu)A_2 + \frac{45\mu(1-\mu)}{4}e^2 + +9\mu(1-$

$\mu A2 - 315\mu 1 - \mu 16B2$

Hence $-4\varepsilon_2$

$$-4\varepsilon_2 = -4\{3\mu(1-\mu)\alpha + \frac{3\mu(1-\mu)}{2}\beta_1 + 9\mu(1-\mu)A_2 + \frac{45\mu(1-\mu)}{4}e^2$$

$$+ +9\mu(1-\mu)A_2 - \frac{315\mu(1-\mu)}{16}B_2\}B_2$$

$$-4\varepsilon_2 = 12\mu^2\alpha \ + 6\mu^2\beta_1 + 45\mu^2e^2 + 36\mu^2A_2 - \frac{315\mu^2}{4}B_2 - 12\mu\alpha - 6\mu\beta_1 -$$

$$36\mu A_2 - 45\mu e^2 \ + \frac{315\mu}{4}B_2 \qquad (4.121)$$

Substituting (4.119) and (4.121) into (4.116);

$$(4 - 3\varepsilon_1)^2 - 27\mu + 27\mu^2 - 4\varepsilon_2$$

$$= 1 - 6\mu A_2 + 15\mu B_2 - 3e^2 - 27\mu + 27\mu^2 - 12\mu\alpha + 12\mu^2\alpha$$

$$- 6\mu\beta_1 + 6\mu^2\beta_1 - 36\mu A_2 + 36\mu^2A_2 - 45\mu e^2 + 45\mu^2e^2 + \frac{315\mu}{4}B_2$$

$$- \frac{315\mu^2}{4}B_2$$

$$(4 - 3\varepsilon_1)^2 - 27\mu + 27\mu^2 - 4\varepsilon_2]$$

$$= 27\mu^2 + 12\mu^2\alpha + 6\mu^2\beta_1 + 45\mu^2e^2 + 36\mu^2A_2 - \frac{315\mu^2}{4}B_2 - 27\mu$$

$$- 12\mu\alpha - 6\mu\beta_1 - 45\mu e^2 - 42\mu A_2 + \frac{375\mu}{4}B_2 + 1 - 3e^2$$

A quadratic equation in terms of μ is obtained;

$$\mu^2\left(27 + 12\alpha + 6\mu^2 + 45\mu^2 + 36A_2 - \frac{315}{4}B_2\right) - \mu\,(27 + 12\alpha + 6\beta_1 + 42A_2 +$$

$$45e^2 + \frac{375}{4}B_2) + (1 - 3e^2) \qquad (4.122)$$

Using, $\mu = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, let

$$a = 27 + 12\alpha + 6\beta + 45e^2 + 36A_2 - \frac{315}{4}B_2$$

$$\text{b} = -27 - 12\alpha - 6\beta_1 - 42A_2 - 45e^2 + \frac{375}{4}B_2;\ \text{c} = 1 - 3e^2 \qquad (4.123)$$

Hence from (4.123);

$$-\mathrm{b} = 27 + 12\alpha + 6\beta_1 + 42A_2 + 45e^2 - \frac{375}{4}B_2 \quad (4.124)$$

$$\mathrm{b}^2 = \left(27 + 12\alpha + 6\beta_1 + 42A_2 + 45e^2 - \frac{375}{4}B_2\right)^2 = 729 + 648\alpha + 324\beta_1 + 2430e^2 + 2268A_2 - \frac{10125}{2}B_2 \quad (4.125)$$

$$\mathrm{ac} = (27 + 12\alpha + 6\beta + 45e^2 + 36A_2 - \frac{315}{4}B_2)(1 - 3e^2)$$

$$\mathrm{ac} = 27 + 12\alpha + 6\beta - 36e^2 + 36A_2 - \frac{315}{4}B_2$$

$$-4\mathrm{ac} = -108 - 48\alpha - 24\beta + 144e^2 - 144A_2 + 315B_2 \quad (4.126)$$

Substituting (4.125) and (4.126) into $\mathrm{b}^2 - 4ac$

$$\mathrm{b}^2 - 4ac = 729 + 648\alpha + 324\beta_1 + 2430e^2 + 2268A_2 - \frac{10125}{2}B_2 - 108 - 48\alpha - 24\beta + 144e^2 - 144A_2 + 315B_2$$

$$\mathrm{b}^2 - 4ac = 621 + 600\alpha + 300\beta_1 + 2574e^2 + 2124A_2 - \frac{9495}{2}B_2 \quad (4.127)$$

$$2\mathrm{a} = 2(27 + 12\alpha + 6\beta + 45e^2 + 36A_2 - \frac{315}{4}B_2) = 54 + 24\alpha + 12\beta + 90e^2 + 72A_2 - \frac{315}{2}B_2 \quad (4.128)$$

substituting (4.124), (4.127) and (4.128) into μ;

$$\mu = \frac{27 + 12\alpha + 6\beta_1 + 42A_2 + 45e^2 - \frac{375}{4}B_2 \pm \sqrt{621 + 600\alpha + 300\beta_1 + 2574e^2 + 2124A_2 - \frac{949}{2}}}{54 + 24\alpha + 12\beta + 90e^2 + 72A_2 - \frac{315}{2}B_2}$$

Taking, $\frac{-\mathrm{b}}{2a}$, i.e.

$$\frac{-\mathrm{b}}{2a} = \frac{27+12\alpha+6\beta_1+42A_2+45e^2-\frac{375}{4}B_2}{54+24\alpha+12\beta+90e^2+72A_2-\frac{315}{2}B_2}; \frac{-\mathrm{b}}{2a} = \frac{27\,[\,1+\frac{1}{27}\{12\alpha+6\beta_1+42A_2+45e^2-\frac{375}{4}B_2\}]}{54\,[\,1+\frac{1}{54}\{24\alpha+12\beta+90e^2+72A_2-\frac{315}{2}B_2\}]}$$

$$\frac{-\mathrm{b}}{2a} = \frac{27}{54}\,[1+\frac{4}{9}\alpha+\frac{2}{9}\beta_1+\frac{5}{3}e^2+\frac{14}{9}A_2 - \frac{125}{36}B_2][1+\frac{4}{9}\alpha+\frac{2}{9}\beta_1+\frac{5}{3}e^2+\frac{4}{3}A_2-\frac{35}{12}B_2]^{-1}$$

expanding, solving binomially, neglecting higher order terms and product $\alpha e^2, A_2 e^2, e^4$e.t.c.

$$\frac{-\mathrm{b}}{2a} = \frac{1}{2}\left[1 + \frac{4}{9}\alpha - \frac{4}{9}\alpha + \frac{2}{9}\beta_1 - \frac{2}{9}\beta_1 + \frac{5}{3}e^2 - \frac{5}{3}e^2 + \frac{14}{9}A_2 - \frac{4}{3}A_2 - \frac{125}{36}B_2 + \frac{35}{12}B_2\right]$$

$$\frac{-\mathrm{b}}{2a} = \frac{1}{2}\left[1 + \frac{2}{9}A_2 - \frac{5}{9}B_2\right] \qquad (4.129)$$

For $\frac{\sqrt{\mathrm{b}^2-4ac}}{2a}$;

$$\frac{\sqrt{\mathrm{b}^2-4ac}}{2a} = \frac{\sqrt{621 + 600\alpha + 300\beta_1 + 2574e^2 + 2124A_2 - \frac{9495}{2}B_2}}{54 + 24\alpha + 12\beta + 90e^2 + 72A_2 - \frac{315}{2}B_2}$$

$\frac{\sqrt{\mathrm{b}^2-4ac}}{2a} = \frac{\sqrt{621}\left[1+\frac{1}{621}\left\{600\alpha+300\beta_1+2574e^2+2124A_2-\frac{9495}{2}B_2\right\}\right]^{1/2}}{54\left[1+\frac{1}{54}\left\{24\alpha+12\beta+90e^2+72A_2-\frac{315}{2}B_2\right\}\right]}$; expanding binomially;

$$\frac{\sqrt{\mathrm{b}^2-4ac}}{2a} = \frac{\sqrt{621}}{54}\left[1 + \frac{300}{621}\alpha + \frac{150}{621}\beta_1 + \frac{1287}{621}e^2 + \frac{1062}{621}A_2 - \frac{1055}{621}B_2\right][1 - \frac{4}{9}\alpha - \frac{2}{9}\beta_1 - \frac{5}{3}e^2 - \frac{4}{3}A_2 + \frac{35}{12}B_2]$$

open up bracket, neglecting higher order terms and product $\alpha e^2, A_2 e^2, e^4$, e.t.c.

$$\frac{\sqrt{\mathrm{b}^2-4ac}}{2a} = \frac{\sqrt{69}}{18}\left[1 + \left\{\frac{300}{621} - \frac{4}{9}\right\}\alpha + \left\{\frac{150}{621} - \frac{2}{9}\right\}\beta_1 + \left\{\frac{1287}{621} - \frac{5}{3}\right\}e^2 + \left\{\frac{1062}{621} - \frac{4}{3}\right\}A_2 - \left\{\frac{1055}{621} + \frac{35}{12}\right\}B_2\right]$$

$$\frac{\sqrt{\mathrm{b}^2-4ac}}{2a} = \frac{\sqrt{69}}{18}\left[1 + \left\{\frac{8}{207}\right\}\alpha + \left\{\frac{4}{207}\right\}\beta_1 + \left\{\frac{28}{69}\right\}e^2 + \left\{\frac{26}{29}\right\}A_2 + \left\{-\frac{125}{138}\right\}B_2\right] \qquad (4.130)$$

Substituting (4.129) and (4.130), $\mu = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$\mu = \frac{1}{2}\left[1 + \frac{2}{9}A_2 - \frac{5}{9}B_2\right] + \frac{1}{2}\{\frac{\sqrt{69}}{9}\left[1 + \left\{\frac{8}{207}\right\}\alpha + \left\{\frac{4}{207}\right\}\beta_1 + \left\{\frac{28}{69}\right\}e^2 + \left\{\frac{26}{29}\right\}A_2 +$$

{−125138}B2}

(4.131)

OR

$$\mu = \frac{1}{2}\left[1 + \frac{2}{9}A_2 - \frac{5}{9}B_2\right] - \frac{1}{2}\{\frac{\sqrt{69}}{9}\left[1 + \left\{\frac{8}{207}\right\}\alpha + \left\{\frac{4}{207}\right\}\beta_1 + \left\{\frac{28}{69}\right\}e^2 + \left\{\frac{26}{29}\right\}A_2 +$$

{−125138}B2}

(4.132)

$$\therefore \mu = \frac{1}{2}\left[1 - \sqrt{\frac{23}{27}}\right] - [\frac{4\sqrt{69}}{27 * 69}]\alpha - [\frac{2\sqrt{69}}{27 * 69}]\beta_1 - [\frac{14\sqrt{69}}{9 * 69}]e^2 + \frac{2}{18}[1 - \frac{13\sqrt{69}}{96}]A_2 - \frac{5}{18}[1 - \frac{25\sqrt{69}}{138}]B_2$$

$$\mu = \frac{1}{2}\left[1 - \sqrt{\frac{23}{27}}\right] - [\frac{4\sqrt{69}}{27(\sqrt{69})^2}]\alpha - [\frac{2\sqrt{69}}{27(\sqrt{6}9)^2}]\beta_1 - [\frac{14\sqrt{69}}{9(\sqrt{6}9)^2}]e^2 + \frac{2}{18}[1 - \frac{13\sqrt{69}}{(\sqrt{6}9)^2}]A_2 - \frac{5}{18}[1 - \frac{25\sqrt{69}}{2(\sqrt{6}9)^2}]B_2$$

$$\mu_c = \frac{1}{2}\left[1 - \sqrt{\frac{23}{27}}\right] - [\frac{4}{27\sqrt{69}}]\alpha - [\frac{2}{27\sqrt{69}}]\beta_1 - [\frac{14}{9\sqrt{69}}]e^2 + \frac{2}{18}[1 - \frac{13}{\sqrt{69}}]A_2 - \frac{5}{18}[1 - \frac{25}{2\sqrt{69}}]B_2 \quad (4.133)$$

$$\mu_c = \mu_o - [\frac{4}{27\sqrt{69}}]\alpha - [\frac{2}{27\sqrt{69}}]\beta_1 - [\frac{14}{9\sqrt{69}}]e^2 + \frac{2}{18}[1 - \frac{13}{\sqrt{69}}]A_2 - \frac{5}{18}[1 - \frac{25}{2\sqrt{69}}]B_2$$

where;

$$\mu_o = \frac{1}{2}\left[1 - \sqrt{\frac{23}{27}}\right] = 0.0385208965$$

Again to verify the classical case from (4.133), we let $\alpha_1 = \beta_2 = e = A_2 = B_2 = 0$ to obtain $\mu_c = 0.0385208965$ which agrees with Mc Cuskkey (1963). Therefore, resulting in the boundary of stability given as $0 < \square < \square_c$.

Since $\Delta > 0$, in the interval $0 < \mu < \mu_c$, this implies that the roots of equation $\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}$….. are pureimaginary numbers, hence thetriangular libration points are stable in this region. In$\mu_c < \mu < ½, \Delta < 0$, the real parts of the two roots are $\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}$….. are positive, therefore the triangular points are unstable. If$\mu = \mu_c$, $\Delta = 0$, the roots in $\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2}$….. are double roots and hencethe triangular points are unstable.

Hence, the triangular points are stable for $0 < \mu < \mu_c$ where the critical mass parameterμ_c depends on the radiation pressure factor, oblateness and the quadruple and octupolar mass moment of the smaller primary, the semi-major axis and eccentricity of the orbits on the critical mass value.

CHAPTER FIVE

RESULTS AND DISCUSSION

5.1 INTRODUCTION

The problem studied in this research "the photogravitational restricted three-body problem in which two primaries have elliptical orbit and the third mass is very-very small that it does not affect the motion of the two primaries" it's primary is a source of radiation and the secondary is an oblate spheroid. The equations of motion of the system are solved using analytical method, the analytical solutions are then transformed into numerical solutions using different binary systems with the help of the Software Mathematica.

Tables 5.1 and 5.2 show the numerical data of the binary system used in this research.

Table 5. 1: Numerical Data_(NASA ADS)_

S/No.	Binary System	Mass Ratio (μ)	Radation Presure (q)	Eccentricty (e)	Masses (M_{sun})		Luminosity (L_{sun})	Semi-Major Axis (AU)	Spectral Type
					M_1	M_2			
1	Zeta Cygni	0.16384	0.960868	0.22	3.05	0.60	112	0.19	G8III
2	Procyon A/B	0.28653	0.995	0.4	1.499	0.602	6.93	0.6	F5 II-V/D
3	54 Piscium	0.062885	0.999	0.3	0.76	0.051	0.52	0.7	KOV/T7

Table 5. 2: Numerical data_(NASA ADS)_

Binary System	M_1	M_2	μ
NLTT 11748	0.15	0.71	0.1744
LP400-22	0.19	>0.48	0.2836
J1257+5428	0.20	>0.95(0.98)	0.16949

Zeta Cygni (ζ Cyg) belongs to the northern constellation of Cygnus and is the brightest member of the constellation, with an apparent visual magnitude of 3.26. The primary component is a giant star and the secondary component has a 12^{th} magnitude companion believed to be a white dwarf (Cygnus Constellation 2016).Procyon (Alpha Canis Minoris) is a binary star, consisting of a white main sequence star, a yellowish star brighter than our sun belonging to spectra type of F5IV called Procyon A and a faint white dwarf companion of spectra type DA. It is the eighth brightest star in the night sky, and has an apparent magnitude of 0.4 and absolute magnitude of 2.68 (Fred 2011). While, 54 Piscium is an orange dwarf star of the sixth magnitude belonging to the Pisces constellation, with an apparent visual magnitude of 5.88 (Jim 2013). We assume an eccentricity 0.3 and semi major axis 0.7 in our computation. The numerical data obtain from NASA ADS is given in table 5.1.

5.2 Numerical Results of Collinear Points

Using data in table (5.1), the numerical values obtained for collinear points are presented in tables (5.3) – (5.11) while the graphically representations are shown in figures (5.1) – (5.6). In tables (5.3) – (5.6), the first two row of each tables indicate its classical (circular) and elliptic, this implies that for the classical, pertubing agents in the system are ignored as well as the eccentricity (e), i.e. $e = 0, a = 1, A_2 = B_2 = 0$, while for the elliptic case they are ignored but a value is assigned to the eccentricity, i.e.$e \neq 0$, and $B_1 = B_2 = 0$.

Table 5. 3: Effect of J_2 on the roots of the collinear points of Zeta Cygni, Procyon A/B and 54 Piscium

Oblateness		ROOTS								
A_2	B_2	ρ_{L_1}			ρ_{L_2}			ρ_{L_3}		
		1	2	3	1	2	3	1	2	3
Classical		0.432848	0.545731	0.302411	0.345014	0.40807	0.252161	0.108407	0.170135	0.0370214
Elliptic		0.0787916	0.304545	0.181032	0.640277	0.448669	0.321565	0.745067	0.482081	0.278128
0.001	-0.000001	0.0869503	0.306412	0.184403	0.640423	0.44931	0.322452	0.745204	0.482318	0.278455
0.01	"	0.113267	0.320494	0.204871	0.641725	0.45476	0.329728	0.746421	0.484428	0.281368
0.1	"	0.172267	0.385526	0.271471	0.653505	0.492844	0.375097	0.757565	0.503951	0.308185
0.2	"	0.19476	0.417209	0.298968	0.664478	0.519397	0.404931	0.768145	0.522815	0.333878

A_2	B_2	ρ_{L_1}			ρ_{L_2}			ρ_{L_3}		
		1	2	3	1	2	3	1	2	3
Classical		0.432848	0.545731	0.302411	0.345014	0.40807	0.252161	0.108407	0.170135	0.0370214
Elliptic		0.091751	0.304545	0.181032	0.611522	0.444369	0.321565	0.713823	0.482081	0.278128
0.002	-0.0000001	0.103123	0.320931	0.187255	0.611866	0.445707	0.323306	0.713132	0.4646	0.27878
"	-0.000001	0.103582	0.320951	0.187349	0.611866	0.44571	0.323313	0.713132	0.4646	0.278781
"	-0.00001	0.107443	0.321148	0.188273	0.611869	0.445745	0.323391	0.713134	0.464603	0.278784
"	-0.0001	0.125422	0.320931	0.195835	0.611894	0.446089	0.324151	0.713151	0.464633	0.278821

Table 5. 4: Effect of J_4 on the roots of the collinear points of Zeta Cygni, Procyon A/B and 54 Piscium

Table 5. 5: Effect of J_2 on the locations of the collinear points of Zeta Cygni, Procyon A/B and 54 Piscium

Oblateness		LOCATIONS								
A_2	B_2	ξ_{L_1}			ξ_{L_2}			ξ_{L_3}		
		1	2	3	1	2	3	1	2	3
Classical		1.268464	1.259201	1.239526	0.491146	0.3054	0.684954	-1.05543	-1.1164	-1.0258636
Elliptic		0.914408	1.018015	1.118147	0.195883	0.264801	0.61555	-0.41877	-0.80445	-0.784757
0.001	-0.000001	0.922566	1.019882	1.21518	0.195737	0.26416	0.614663	-0.41864	-0.80421	-0.78411
0.01	"	0.948883	1.033964	1.141986	0.194435	0.25871	0.607387	-0.41742	-0.8021	-0.7841
0.1	"	1.007883	1.098996	1.208586	0.182655	0.220626	0.562018	-0.40626	-0.78258	-0.7841
0.2	""	1.030376	1.130679	1.236083	0.171682	0.194073	0.532184	-0.3957	-0.76372	-0.78406

Table 5. 6: Effect of J_4 on the locations of the collinear points of Zeta Cygni, Procyon A/B and 54 Piscium

Oblateness		Locations								
A_2	B_2	ξ_{L_1}			ξ_{L_2}			ξ_{L_3}		
		1	2	3	1	2	3	1	2	3
Classical		1.268464	1.259201	1.239526	0.490602	0.3054	0.684954	-1.05598	-1.1164	-1.02586
Elliptic		0.927373	1.018015	1.118147	0.224094	0.269101	0.61555	-0.415156	-0.80445	-0.78476
0.002	-0.0000001	0.938739	1.034401	1.12437	0.22375	0.267763	0.613809	-0.45125	-0.82193	-0.78422
"	-0.000001	0.939198	1.034421	1.124464	0.22375	0.26776	0.613802	-0.45125	-0.82193	-0.7841
"	-0.00001	0.943059	1.034618	1.125388	0.223747	0.267725	0.613724	-0.45125	-0.82193	-0.7841
"	-0.0001	0.961038	1.034401	1.13295	0.223722	0.267381	0.612964	-0.45123	-0.8219	-0.78406

Table 5. 6: Zeta Cygni showing effect of Semi-Major axis A_2 =0.001 and B_2=-0.000001

Oblateness		Semi-Major axis	Roots			Locations		
A_2	B_2	a	ρ			ξ		
			L_1	L_2	L_3	L_1	L_2	L_3
0.001	-0.000001	0.1	0.0544628	0.733601	0.847024	0.890623	0.102559	-0.31682
"	"	0.2	0.0905292	0.631356	0.736015	0.926689	0.204804	-0.42783
"	"	0.3	0.128551	0.549743	0.64089	0.964711	0.286417	-0.52295
"	"	0.4	0.167706	0.486358	0.555092	1.003866	0.349802	-0.60875
"	"	0.5	0.207696	0.439594	0.475697	1.043856	0.396566	-0.68814
"	"	0.6	0.24843	0.406602	0.401089	1.08459	0.429558	-0.76275
"	"	0.7	0.28986	0.383754	0.330253	1.12602	0.452406	-0.83359
"	"	0.8	0.331933	0.367835	0.262502	1.168093	0.468325	-0.90134
"	"	0.9	0.374581	0.356522	0.197344	1.210741	0.479638	-0.9665
"	"	0.99	0.413391	0.348998	0.140612	1.249551	0.487162	-1.02308
"	"	1.0	0.417726	0.348281	0.134411	1.253886	0.487879	-1.02943

Table 5. 7: Zeta Cygni showing effect of Eccentricity A_2=0.01 and B_2=-0.0000001

Oblateness		Ecce.	Roots			Locations		
A_2	B_2	e		ρ			ξ	
			L_1	L_2	L_3	L_1	L_2	L_3
0.01	-0.0000001	0.1	0.114947	0.637059	0.740859	0.951107	0.199101	-0.42298
"	"	0.2	0.113385	0.640862	0.745091	0.949545	0.195298	-0.41875
"	"	0.3	0.110978	0.646765	0.751636	0.947138	0.189395	-0.4122
"	"	0.4	0.10796	0.65424	0.759888	0.94412	0.18192	-0.40395
"	"	0.5	0.104567	0.662735	0.769219	0.940727	0.173425	-0.39462
"	"	0.6	0.101002	0.671755	0.779082	0.937162	0.164405	-0.38476
"	"	0.7	0.0974163	0.680909	0.789053	0.933576	0.155251	-0.37479
"	"	0.8	0.093915	0.689916	0.798836	0.930075	0.146244	-0.365
"	"	0.9	0.0905612	0.698593	0.808242	0.926721	0.137567	-03556

Oblateness		Semi-Major axis	Roots			Locations		
A_2	B_2	a		ρ			ξ	
			L_1	L_2	L_3	L_1	L_2	L_3
0.001	-0.000001	0.1	0.0657045	0.667082	0.881546	0.779175	0.046338	-0.61945
"	"	0.2	0.113479	0.597922	0.786036	0.826949	0.115548	-0.68861
"	"	0.3	0.162125	0.540801	0.701784	0.875595	0.172669	-0.74573
"	"	0.4	0.210588	0.498728	0.624655	0.924058	0.214742	-0.7878
"	"	0.5	0.258684	0.469392	0.5526011	0.972154	0.244078	-0.81714
"	"	0.6	0.306412	0.44931	0.484428	1.019882	0.26416	-0.83722
"	"	0.7	0.353793	0.43546	0.41936	1.067263	0.27801	-0.85107
"	"	0.8	0.400863	0.425708	0.356865	1.114333	0.287762	-0.86082
"	"	0.9	0.447615	0.418667	0.29655	1.161085	0.294803	-0.86786
"	"	0.99	0.489425	0.413911	0.243887	1.202895	0.299559	-0.87262
"	"	1.0	0.494055	0.413454	0.238122	1.207525	0.300016	-0.87308

Table 5. 8: Procyon A/B showing effect of Semi-Major axis A_2=0.001 and B_2=-0.000001

Table 5. 9: Procyon A/B showing effect of Eccentricity A_2=0.01 and B_2=-0.0000001

Oblateness		Ecce.	Roots			Locations		
A_2	B_2	e		ρ			ξ	
			L_1	L_2	L_3	L_1	L_2	L_3
0.01	-0.0000001	0.1	0.348627	0.446071	0.443564	1.062097	0.267399	-0.84297
"	"	0.2	0.342295	0.447853	0.452656	1.055765	0.265617	-0.83387
"	"	0.3	0.332581	0.450773	0.4667215	1.046051	0.262697	-0.81982
"	"	0.4	0.320477	0.454757	0.484427	1.033947	0.258713	-0.8021
"	"	0.5	0.306983	0.459708	0.504443	1.020453	0.253762	-0.78209
"	"	0.6	0.292946	0.465503	0.525585	1.006416	0.247967	-0.76095
"	"	0.7	0.278997	0.472006	0.546943	0.992467	0.241467	-0.73959
"	"	0.8	0.26555	0.479069	0.567883	0.97902	0.234401	-0.71865
"	"	0.9	0.252848	0.486546	0.588001	0.966318	0.226924	-0.69853

Table 5. 10: 54 Piscium showing effect of Semi-Major axis B_1=0.001 and B_2=-0.000001

Oblateness		Semi-Major axis	Roots			Locations		
A_2	B_2	a		ρ			ξ	
			L_1	L_2	L_3	L_1	L_2	L_3
0.001	-0.000001	0.1	0.0394972	0.775133	0.817655	0.976612	0.161982	-0.24523

“	“	0.2	0.0598353	0.658951	0.699933	0.99695	0.278164	-0.36295
“	“	0.3	0.0813569	0.563596	0.60074	1.018472	0.373519	-0.46215
“	“	0.4	0.104557	0.482745	0.51199	1.041672	0.45437	-0.5509
“	“	0.5	0.129436	0.415402	0.430283	1.066551	0.521713	-0.6326
“	“	0.6	0.156027	0.362068	0.353786	1.093142	0.575032	-0.7091
“	“	0.7	0.184403	0.322452	0.281368	1.121518	0.614663	-0.78152
“	“	0.8	0.214634	0.294339	0.212266	1.151749	0.642776	-0.85062
“	“	0.9	0.246752	0.274701	0.145941	1.183867	0.662414	-0.91694
“	“	0.99	0.277254	0.262035	0.0882869	1.214369	0.67508	-0.9746
“	“	1.0	0.280734	0.260849	0.081989	1.217849	0.676266	-0.9809

Table 5. 11: 54 Piscium showing effect of Eccentricity A_2=0.01 and B_2=-0.0000001

Oblateness		Ecce.	Roots			Locations		
A_2	$\mathbf{B_2}$	$\boldsymbol{e}$	$\boldsymbol{\rho}$			$\boldsymbol{\xi}$		
			$\boldsymbol{L_1}$	$\boldsymbol{L_2}$	$\boldsymbol{L_3}$	$\boldsymbol{L_1}$	$\boldsymbol{L_2}$	$\boldsymbol{L_3}$
0.01	-0.0000001	0.1	0.215601	0.318212	0.253603	1.152716	0.618903	-0.80928
“	“	0.2	0.211314	0.32255	0.264499	1.148429	0.614565	-0.79839
“	“	0.3	0.204821	0.329722	0.281367	1.141936	0.607393	-0.78152
“	“	0.4	0.196864	0.3396	0.302659	1.134001	0.597515	-0.76023
“	“	0.5	0.188161	0.351947	0.326776	1.125276	0.585168	-0.73611
“	“	0.6	0.179286	0.366394	0.352316	1.116401	0.570721	-0.71057
“	“	0.7	0.170632	0.382463	0.378192	1.107747	0.554652	-0.68469
“	“	0.8	0.162433	0.39962	0.40364	1.099548	0.537492	-0.65925
“	“	0.9	0.154806	0.417345	0.428168	1.091915	0.51977	-0.63472

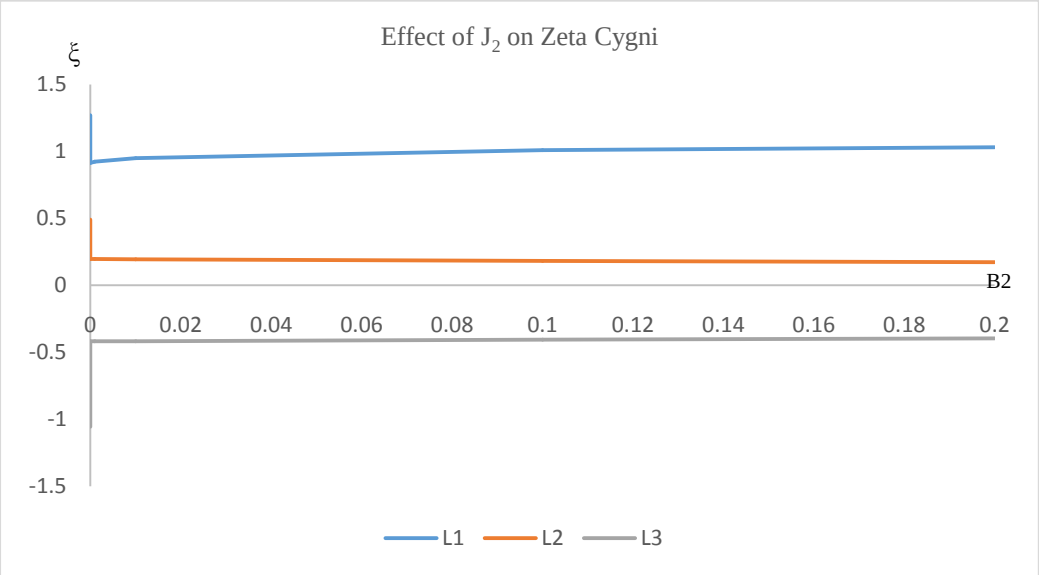


Figure 5. 1: Effect of J_2 on Zeta Cygni

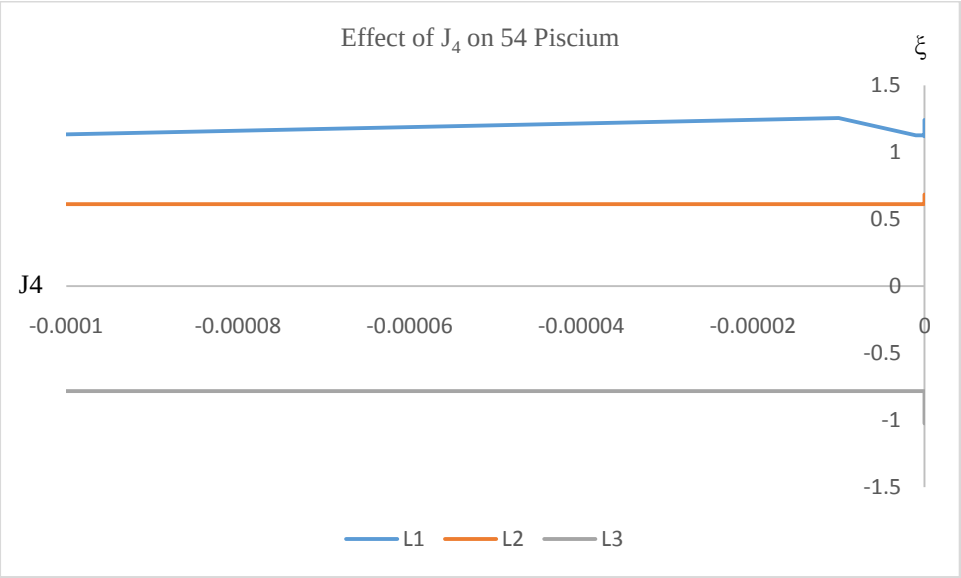


Figure 5. 2: Effect of J_4 on 54 Piscium

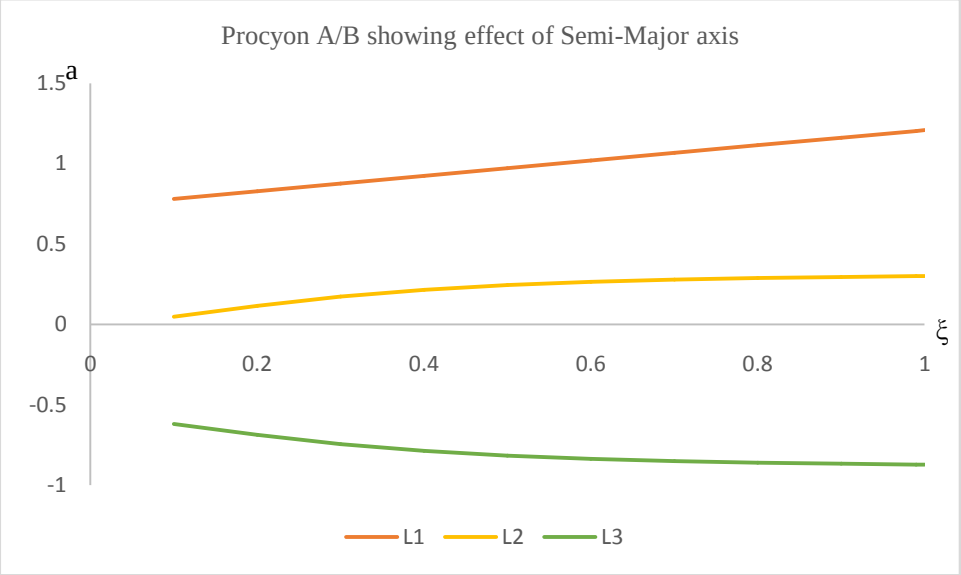


Figure 5. 3: Effect of Semi-Major Axis on Procyon A/B

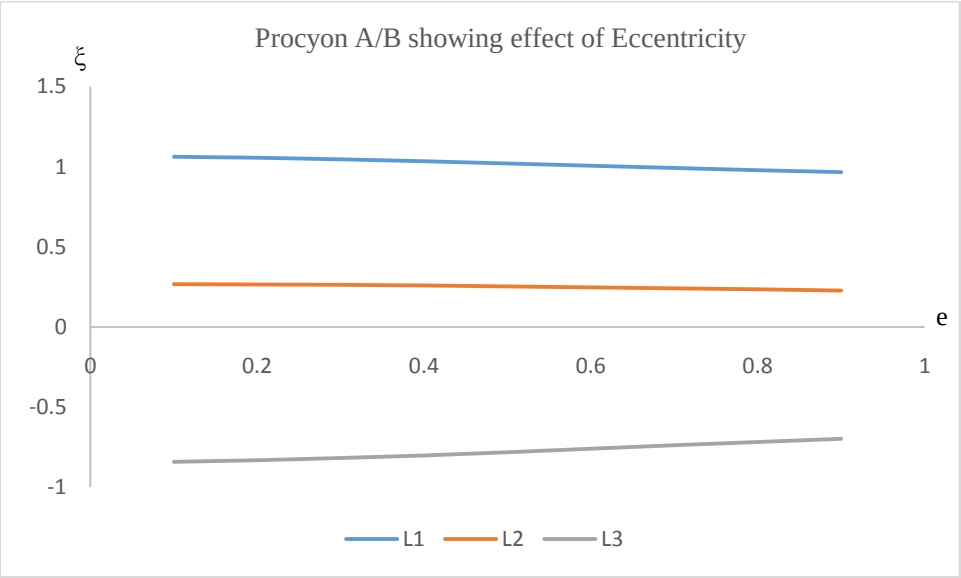


Figure 5. 4: Effect of Eccentricity on Procyon A/B

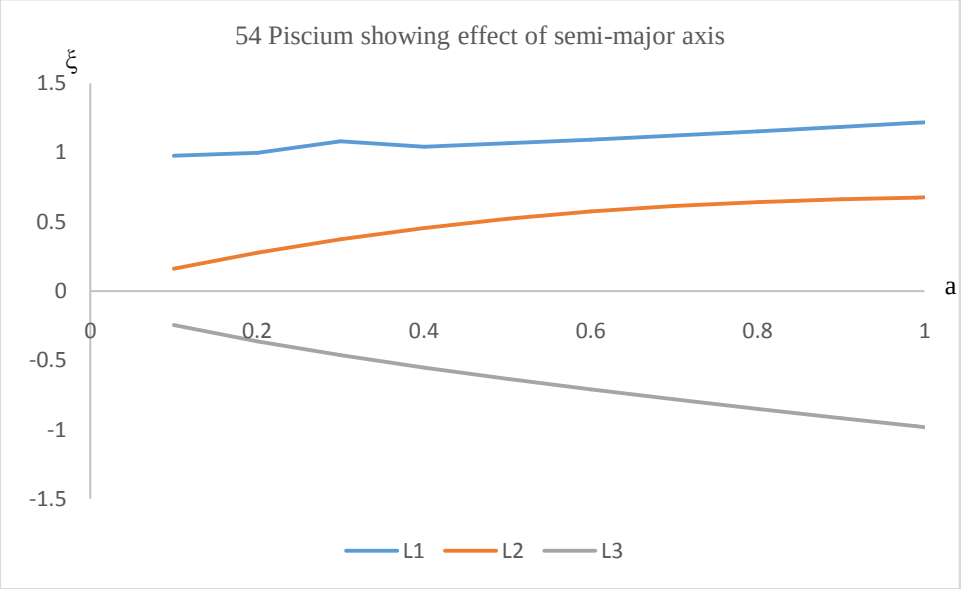


Figure 5. 5: Effect of Semi-Major Axis on 54 Piscium

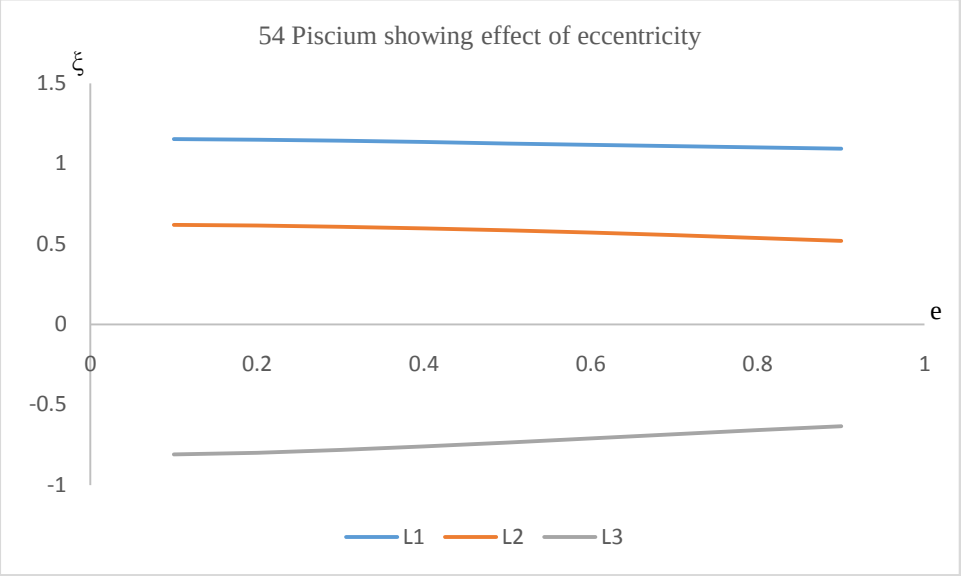


Figure 5. 6: Effect of Eccentricity on 54 Piscium

5.3 Numerical Results of Triangular Points

The numerical values obtained for triangular points are presented in tables (5.12) – (5.21) while the graphically representations are shown in figures (5.7) – (5.17). Also in table (5.12) the first two row of each binary system indicate its classical (circular) and elliptic, this implies that for the classical, pertubing agents in the system are ignored as well as the eccentricity (e), i.e. $e = 0, a = 1, A_2 = B_2 = 0$, while for the elliptic case they are ignored but a value is assigned to the eccentricity, i.e. $e \neq 0$, and $B_1 = B_2 = 0$

Table 5. 12: Locations of Triangular Points of Zeta Cygni, Procyon A/B and 54 Piscium

Binary System	Mass Ratio (μ)	Radiation Pressure Factor	Oblateness		Locations Of Triangular points	
	μ	q_1	A_2	B_2	ξ	$\pm\eta$
Zeta Cygni	0.164384	0.960868	Circular		0.322485	0.858411
		Elliptic	0	0	0.331486	0.245708
			0.001	-0.0000001	0.331661	0.246061
			0.01	-0.000001	0.33323	0.24921
			0.1	-0.00001	0.34892	0.278754
Procyon A/B	0.28653	0.995	Circular		0.211862	0.86062
		Elliptic	0	0	0.212473	0.588695
			0.001	-0.0000001	0.212263	0.588516
			0.01	-0.000001	0.210371	0.586906
			0.1	-0.00001	0.191456	0.570554

54 Piscium	0.062885	0.999	Circular		0.436782	0.865833
		Elliptic	0	0	0.436876	0.683506
			0.001	-0.0000001	0.436588	0.683295
			0.01	-0.000001	0.433995	0.681395
			0.1	-0.00001	0.408064	0.662088

Table 5. 13: Effect of eccentricity on $L_{4,5}$ of Zeta Cygni for A_2= 0.1, B_2 = -0.00001

Eccentricity	Locations of Triangular Points	
e	ξ	$\pm\eta$
0.1	0.348753	0.300380
0.2	0.348883	0.283625
0.3	0.349100	0.253250
0.4	0.349404	0.203235
0.5	0.349794	0.109316

Table 5. 14: Effect of semi-major axis on $L_{4,5}$ of Zeta Cygni for A_2= 0.1, B_2 = -0.00001 and e=0.2

Semi-major axis	Locations of Triangular Points	
a	ξ	$\pm\eta$
0.2	0.347600	0.300341
0.3	0.335764	0.424632
0.4	0.325209	0.510618
0.5	0.315512	0.578480
0.6	0.306447	0.635407
0.7	0.297875	0.684902
0.8	0.289704	0.728963
0.9	0.281868	0.768852
0.99	0.275061	0.801894

Table 5. 15: Effects of A_2 and B_2 for e=0.6, a=0.8; B_2=-0.00001 and A_2=0.1on LP400-22.

A_2	$\xi(A_2)$	$\pm\eta(A_2)$	B_2	$\xi(B_2)$	$\pm\eta(B_2)$
0.001	0.215893	0.548790	-0.000001	0.166399	0.515129
0.01	0.211393	0.545815	-0.00001	0.166393	0.515126
0.1	0.166393	0.515126	-0.0001	0.166327	0.515095
0.2	0.116393	0.478724	-0.001	0.165675	0.514787
0.3	0.0663927	0.439317	-0.01	0.159148	0.511701

Table 5. 16: Effects of eccentricity and semi-major axis for e=0.6; a=0.8; A_2=0.01 and B_2=-0.00001on LP400-22.

e/a	$\xi(e)$	$\pm\eta(e)$	$\xi(a)$	$\pm\eta(a)$
0.99	0.211393	0.486284i	0.211394	0.617079
0.9	0.211393	0.299807i	0.211393	0.585032
0.8	0.211393	0.237944	0.211393	0.545815
0.7	0.211393	0.431142	0.211392	0.501671
0.6	0.211393	0.545815	0.211391	0.450742
0.5	0.211393	0.626665	0.211390	0.389714
0.4	0.211393	0.685761	0.211388	0.311482
0.3	0.211393	0.728418	0.211386	0.193227
0.2	0.211393	0.757418	0.211382	0.17184i
0.1	0.211393	0.774296	0.211371	0.330521i

Table 5. 17: Effects of A_2 and B_2 for e=0.8, a=0.7; B_1=0.1 and B_2=-0.00001 on J1257+5428

A_2	$\xi(A_2)$	$\pm\eta(A_2)$	B_2	$\xi(B_2)$	$\pm\eta(B_2)$
0.001	0.330002	0.183096	-0.000001	0.280509	0.0705473
0.01	0.325502	0.175866	-0.00001	0.280502	0.0705350
0.1	0.280502	0.070535	-0.0001	0.280431	0.0704120
0.2	0.230502	0.154474i	-0.001	0.279717	0.0691695
0.3	0.180502	0.229564i	-0.01	0.272582	0.0552282

Table 5. 18: Effects of eccentricity and semi-major axis for e=0.8; a=0.7; A_2=0.01 and B_2=-0.00001 on J1257+5428

e/a	$\xi(e)$	$\pm\eta(e)$	$\xi(a)$	$\pm\eta(a)$
0.99	0.325502	0.487029i	0.325504	0.320401
0.9	0.325502	0.321084i	0.325503	0.285052
0.8	0.325502	0.175866	0.325503	0.237944
0.7	0.325502	0.386245	0.325502	0.175866
0.6	0.325502	0.501671	0.325501	0.0631065
0.5	0.325502	0.581717	0.325500	0.156562i
0.4	0.325502	0.639803	0.325498	0.234491i
0.3	0.325502	0.681567	0.325496	0.296888i
0.2	0.325502	0.709896	0.325492	0.35396i
0.1	0.325502	0.726364	0.325481	0.411787i

Table 5. 19: Effects of B_1 and B_2 for e=0.3, a=0.7; A_2=0.1 and B_2=-0.00001 on NLTT 11748.

A_2	$\xi(A_2)$	$\pm\eta(A_2)$	B_2	$\xi(B_2)$	$\pm\eta(B_2)$
0.001	0.3256	$\frac{\sqrt{3}}{2}$	-0.000001	0.320599	0.681569
0.01	0.325092	0.683469	-0.00001	0.320592	0.681567
0.1	0.320592	0.681567	-0.0001	0.325021	0.681555
0.2	0.275592	0.662254	-0.001	0.319807	0.681428
0.3	0.225592	0.640112	-0.01	0.312672	0.680154

Table 5. 20: Effects of eccentricity and semi-major axis for e=0.3; a=0.8; A_2=0.01 and B_2=-0.00001 on NLTT11748.

e/a	$\xi(e)$	$\pm\eta(e)$	$\xi(a)$	$\pm\eta(a)$
0.99	0.320593	0.486284i	0.320594	0.805595
0.9	0.320593	0.299807i	0.320593	0.77068
0.8	0.320593	0.237944	0.320593	0.728418
0.7	0.320593	0.431142	0.320592	0.681567
0.6	0.320593	0.545815	0.320591	0.628682
0.5	0.320593	0.626665	0.320590	0.567421
0.4	0.320593	0.685761	0.320588	0.493558
0.3	0.320593	0.728418	0.320586	0.397913
0.2	0.320593	0.757418	0.320582	0.250619
0.1	0.320593	0.774296	0.320571	0.225997i

Table 5. 21: Effect of radiation (q) on Ω_c varying eccentricity

q	Ω_c		
	$e = 0.2$	$e = 0.25$	$e = 0.30$
0.999	0.0268586	0.0172946	0.0121448
0.99	0.0214278	0.0172143	0.0120645
0.9	0.0206253	0.0164118	0.0112619
0.85	0.0201794	0.0159659	0.0108161
0.8	0.0197335	0.01552	0.0103702
0.75	0.0192877	0.0150742	0.00992431
0.7	0.0197335	0.0146283	0.00947844
0.65	0.0183959	0.0141824	0.00769494

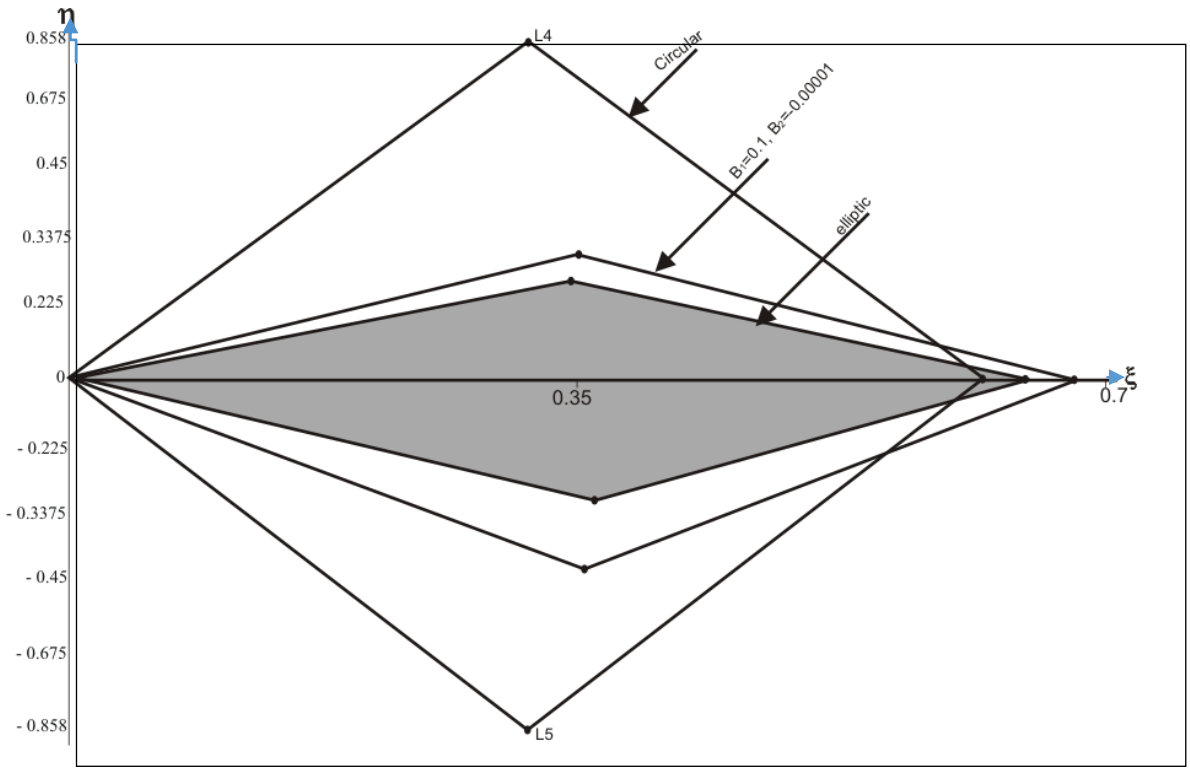


Figure 5. 7: Effect of A_2 and B_2 on $L_{4,5}$ of Zeta Cygni

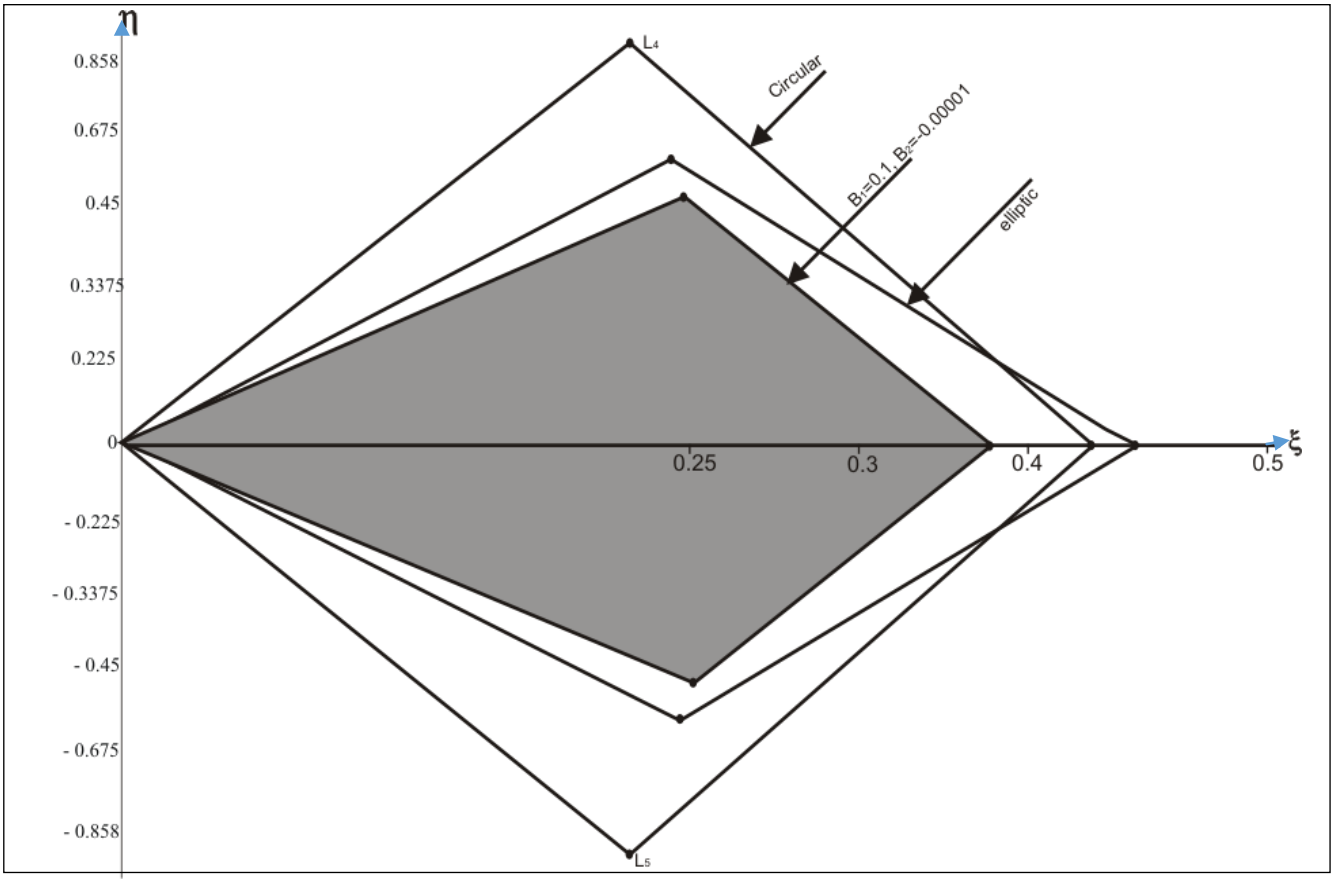


Figure 5. 8: Effect of A_2 and B_2 on $L_{4,5}$ Procyon A/B

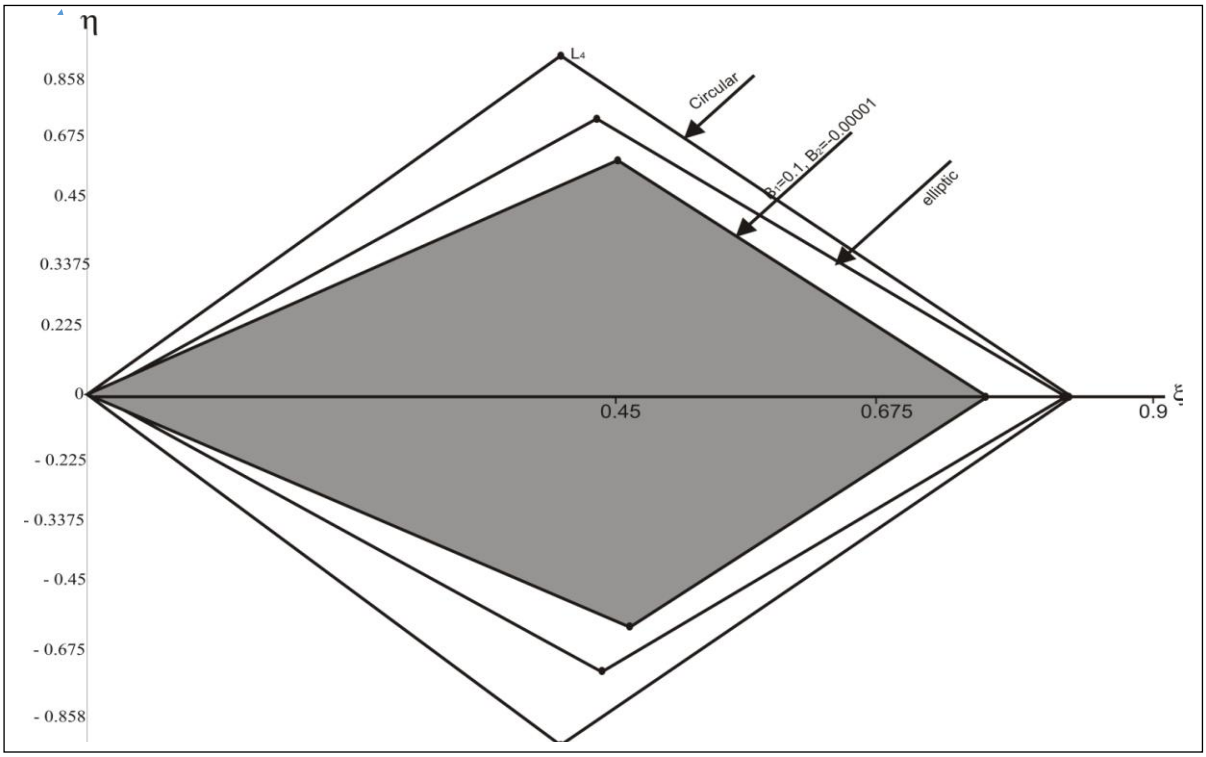


Figure 5. 9: Effect of A_2 and B_2 on L4,5 for 54 Piscium

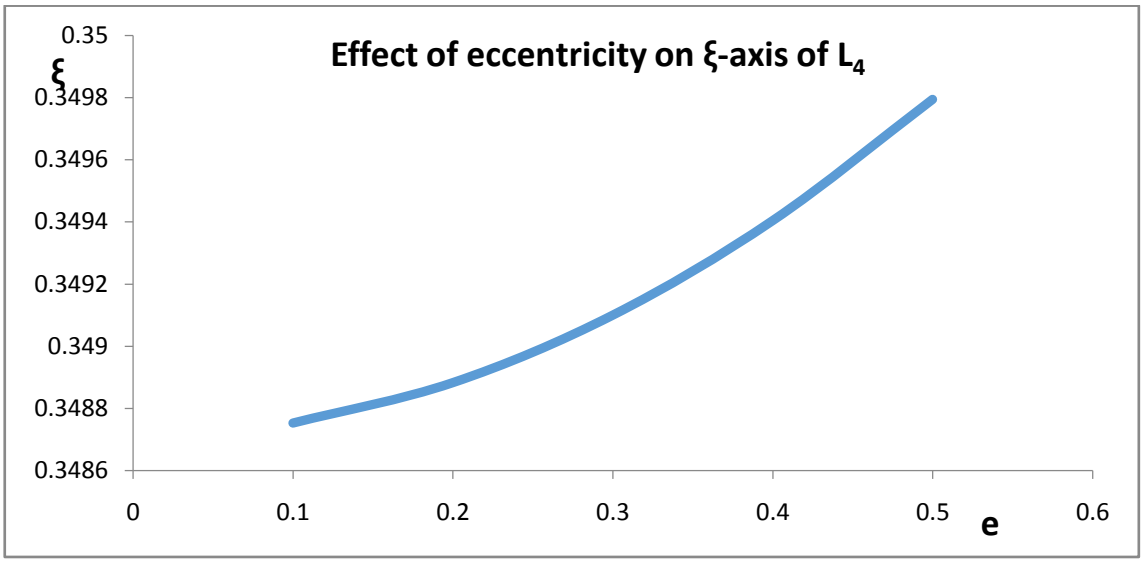


Figure 5. 10: Effect of eccentricity on ξ- co-ordinates of Zeta Cygni on L_4 for A_2= 0.1, B_2 = -0.00001

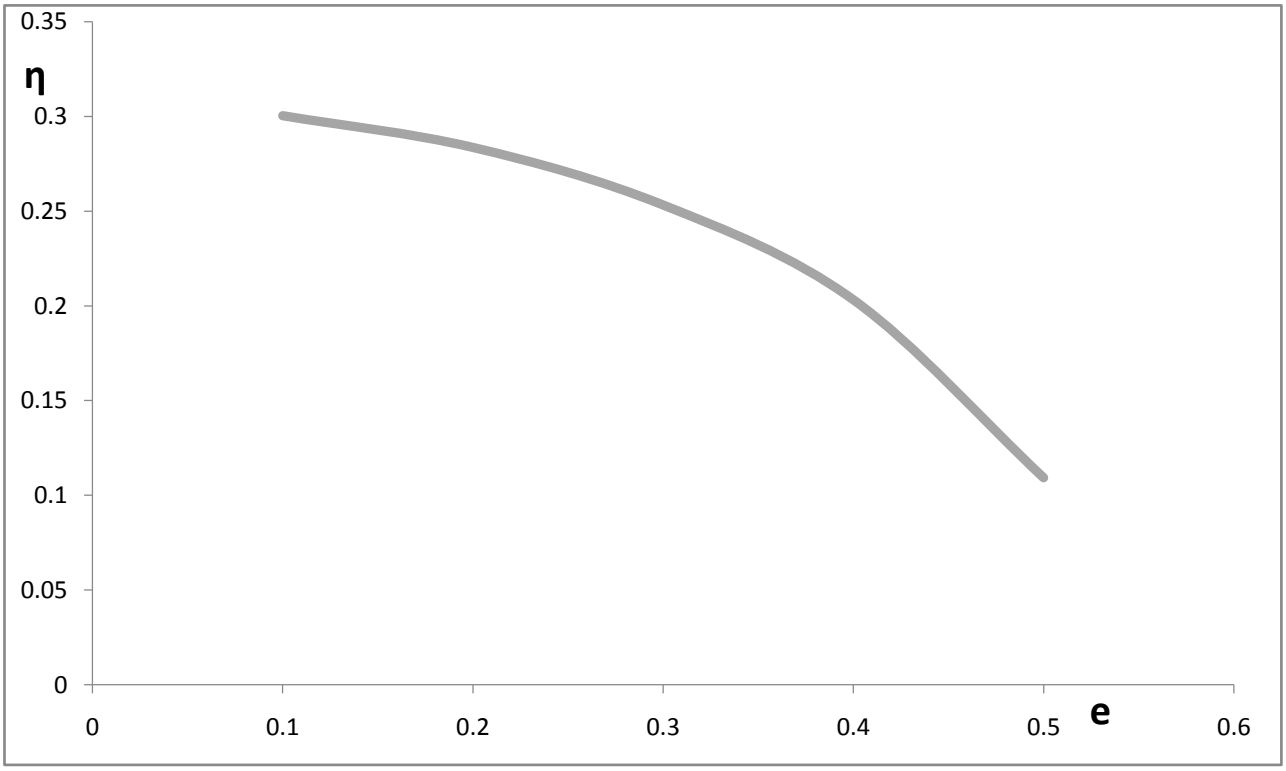


Figure 5. 11: Effect eccentricity on η – co-ordinates of Zeta-Cygni on L_4 for A_2= 0.1, B_2 = -0.00001

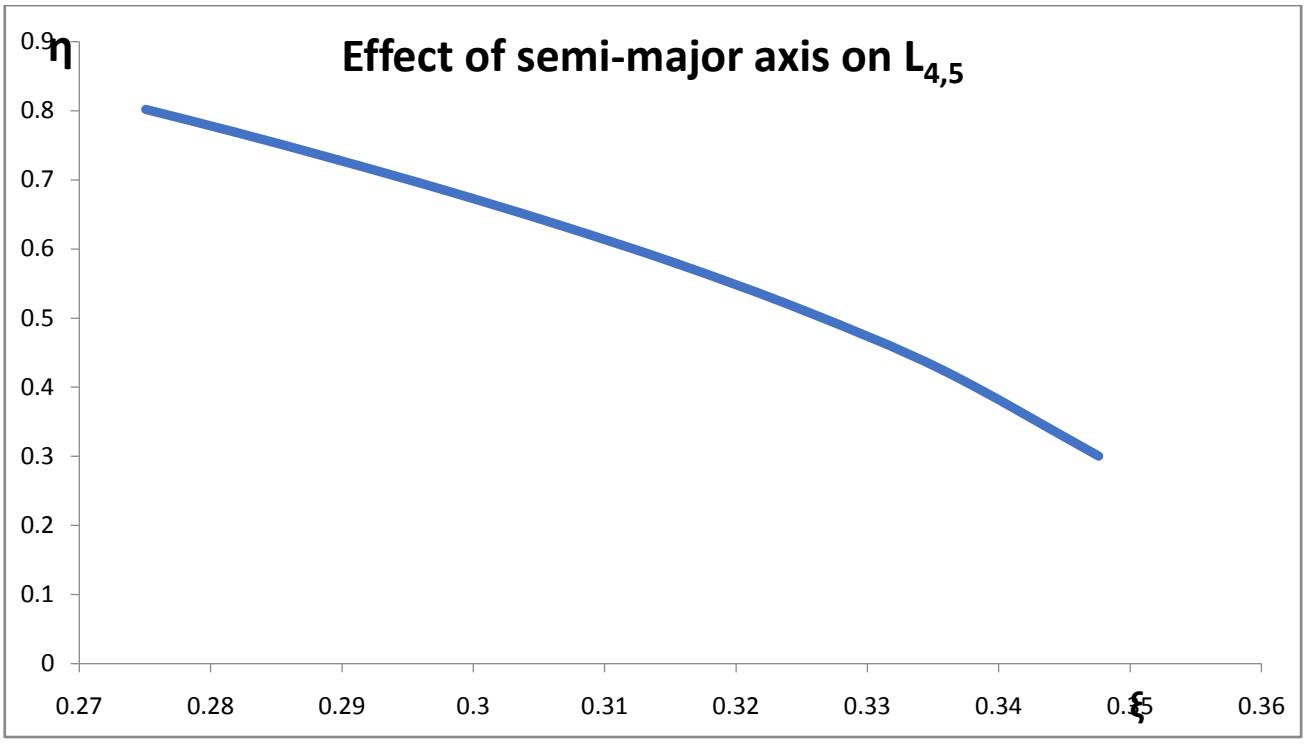


Figure 5. 12: Effect of Semi-major axis (a) on $L_{4,5}$ of Zeta Cygni for A_2= 0.1, B_2 = -0.00001 and e=0.2

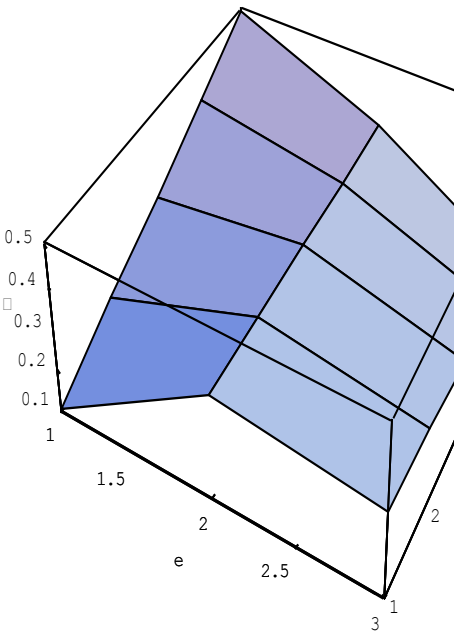


Figure 5. 13: Surface representation of the effect of eccentricity on L_4 of Zeta Cygni

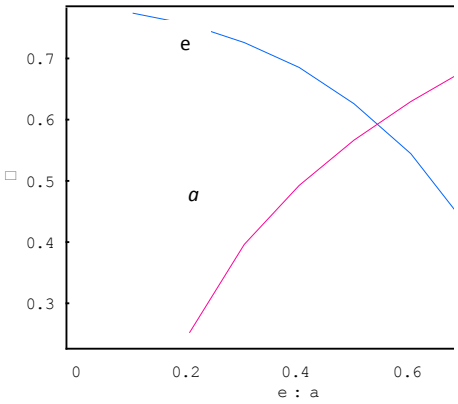


Figure 5. 14: Effects of eccentricity and Semi-major axis on the triangular points ofNLTT11748

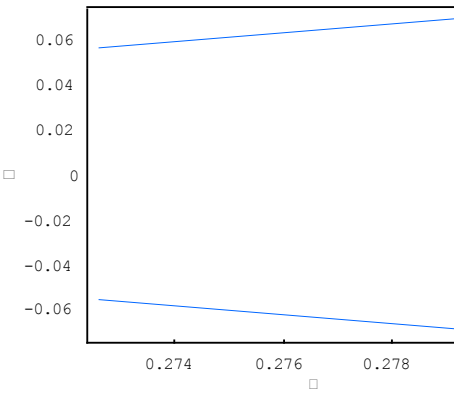


Figure 5. 15: Effect of B_2 on $L_{4,5}$ of J1257+1548 for A_2=0.1, e=0.8 and a=0.7.

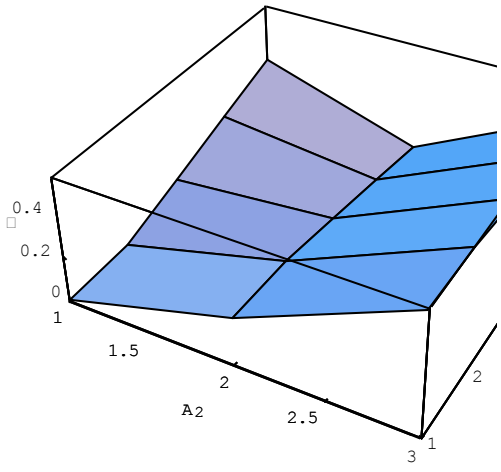


Figure 5. 16: Surface representation of the effect of the quadruple mass moment for e=0.6, a=0.8 and B_2=-0.00001 of LP400-22

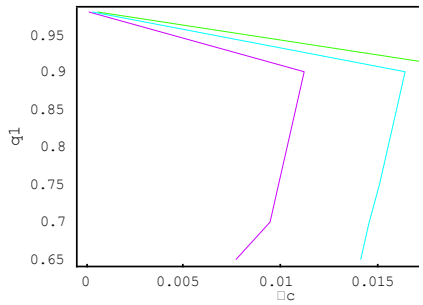


Figure 5. 17: Effect of Radiation Pressure Force (q) on Critical Mass (μ_c) varying eccentricity

5.4 CONCLUSION

The position and stability of the collinear equilibrium points have been obtained in this study, which are affected by all the perturbing agents present in this study using the three binary system Zeta Cygni, Procyon A/B and 54 Piscium. The effect of increasing J_2 (tables (5.3) and (5.5) and figure (5.1)) shows that there is a shift away from the origin in L_1 and towards the origin for $L_{2,3}$. While, there is a general shift towards the origin for all the points as J_4 increases tables (5.4) and (5.6) and figure (5.2). Tables (5.9), (5.10), (5.11) & (5.12) show that as the semi-major axis and eccentricity increase, the locations of $L_{1,2,3}$ of Procyon A/B and 54 Piscium move away and towards the origin respectively.

Our results confirm those of Singh and Umar (2013) in the absence of radiation of the secondary (i.e. $q_2 = 1$) and oblateness of the primary ($A_2=0$) in their study and the second zonal harmonics $J_4=0$ in ours. In the circular case, under the same conditions with the absence of small perturbations in the Coriolis and centrifugal forces in their paper, our results coincide with AbdulRaheem and Singh 2006. In all cases, despite the introduction of the second zonal harmonics, the collinear points remain unstable in the Lyopunov sense.

The position and linear stability of the triangular libration points have also been obtained they are affected by the perturbing agents–*oblateness* (*up to J_4*), *eccentricity*, *semi-majoraxis* and *radiation pressure* – and are shown numerically and graphically above. Figures (5.7), (5.8) and (5.9), table (5.13) show the effects of Zeta Cygni, Procyon A/B and 54 Piscium, from tables (5.14) and (5.16) it is seen clearly that as eccentricity and semi-major axis increases the $\xi_{\text{co-ordinate}}$shifts away from and towards the origin respectively, while $\eta_{\text{co-ordiante}}$shifts towards and away from the lines joining the primaries respectively. These are shown graphically in figures 16-19, the latter agreeing with Singh and Umar (2015). The size of the region of stability is seen to decrease with increase in eccentricity (e =0.3, 0.25 and 0.2) and radiation pressure in figure (5.17) also confirming the results of Singh and Umar $2012_{a,\ b}$. The combined effect of Oblateness (J_2 and J_4), eccentricity (e) and radiation pressure (q_1) on the size of the region of stability is a general reduction as given in table (5.13).

Our result in the circular case confirms those of Sharma (1987) and Ishwar and Elipe (2001) with $J_4=0$ in ours. They also agree with those of Singh and Ishwar (1999) and AbdulRaheem and Singh (2006) when the primaries are spherical with a dark degenerate secondary together with the absence of small perturbations in Coriolis and Centrifugal forces in the latter case. Under the same condition in the elliptic case when the octupolar mass moment $J_4=0$, it verifies the result of Singh and Umar (2012).

In the absence of radiation pressure force, the position and linear stability of the triangular libration point obtained using other binaries (table (5.2), are shown in tables (5.16 –5.21) and presented graphically in figures (5.14 – 5.16) as affected by the perturbing agents– oblateness (up to J_4), eccentricity, semi-major axis – in the absence of radiation pressure (q=1). It is observed that from (5.14-5.21) and figures (5.14-5.16)

that eccentricity does not affect the ξ coordinateand cease to exist in the quasi-parabolic case. While a shift away from the line joining the primaries is seen with increase in eccentricity on theη coordinate, the reverse is the case with increase in the semi-major axis (tables 5.17, 5.19 and 5.21). The triangular points also cease to exist as the semi-major axis approaches unity. The quadruple and octupolar mass moment on the other hand, cause a shift towards the origin and away from the line joining the primaries respectively on theξ and η coordinates. Our results in the circular case confirm those of Sharma (1987) and Ishwar and Elipe (2001) in the absence of radiation in their problem and with J_4=0 in ours. They also agree with those of Singh and Ishwar (1999) and AbdulRaheem and Singh (2006) when the primaries are spherical and radiation is absent together with the absence of small perturbations in Coriolis and centrifugal forces in the latter cases and that of Singh and Taura (2013) without considering oblateness of the bigger primary as well as neglecting the gravitational potential from the circular cluster of material points. Under the same conditions in the elliptic case when the octupolar mass moment J_4 is taken as zero, it verifies the result of Singh and Umar (2012a). The triangular points are stable in the interval$0 < \mu < \mu_c$. The mass ratios of the double white dwarf binaries (table 5.2) are outside this range and are as such unstable.

CHAPTER SIX

SUMMARY, CONCLUSION AND RECONMMENRATIONS

6.1 SUMMARY

This research work investigates the motion of a test particle of infinitesimal mass called the third body, under the influence of two massive primaries moving in elliptic orbits around their centre of mass by considering the primary to be source of radiation and the secondary an oblate spheroid up to the second zonal harmonics (J_4). This research work has been conducted under six different chapters. Chapter one gives general introduction, chapter two is about the literature review, chapter three deals with the locations of equilibrium points, by obtaining the location of both the collinear and triangular points, the stability of equilibrium points is given in chapter four. Chapter five and six encompasses results and discussion and summary, conclusion and recommendations.

Three different cases of the line joining the primaries (collinear equilibrium points) arise. A ninth degree equation was obtained for the collinear libration points, corresponding to $\xi>\xi_2$, $\xi_1<\xi<\xi_2$ and $\xi_1>\xi$ respectively with respect to the primaries. The presence of the perturbing forces and additional parameters δ_1 and δ_2 made the three points L_1, L_2 and L_3 move away and towards the origin respectively. It was found that the points remain unstable just as in the classical case.

The triangular points $L_4 \& L_5$ have a slight change in the shape of triangles formed different from the equilateral triangles which are formed in the classical case, dueto the perturbing agents present in the system.

The coordinates of triangular points correspond to the classical (Mc Cuskey 1963) case in the absence of the perturbing forces and in the absence of the zonal harmonics (J_4).

Theyagree with Singh and Umar (2012a), when the primary is non-luminous, oblate and secondary is luminous. The condition for stability $0 < \square < \square_c$was also obtained for the triangular libration points with the presence of radiation pressure force, eccentricity, oblateness up to J_4. In the absence of *radiation*, *eccentricity*, *semi-major axis* and *oblateness up to* J_4 our value of μ_c is the same as that of the classical (Mc Cuskey 1963) case.

6.2 Conclusion

The motion of the infinitesimal mass has been considered in the frame work of photogravitational elliptic restricted three-body problem with Zonal Harmonics of the secondary.

Equations (3.1), (3.2) and (3.3) are the equations of motion of our problem. The perturbing agents (eccentricity e, radiation pressure q_1 and oblateness up to J_4) influence the equations of motion as well as the mean motion equation (3.3).

Some binary systems (Zeta Cygni, Procyon A/B and 54 Piscium) were used to obtain the position of the collinear equilibrium points (Equations (3.22), (3.55), (3.91) and figures (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6)). It was discovered that their stability behavior does not change despite the introduction of *eccentricitye*, *radiation pressure* q_1 and *oblateness up to*J_4, they still remain unstable.

The linear stability of the triangular points has been obtained in the presence of *eccentricity* (e), *semi-major axis* (a), *radiation pressure* (q_1) of the primary and *oblateness up to* $J_4$of the secondary. The locations of the triangular points were obtained (Table (5.13) and figures (5.7), (5.8), and (5.9)) using various binary systems. For the first system (Zeta Cygni) as A_2 and B_2 increases the coordinates moves away from the

origin, for increasing A_2 and B_2 in Procyon A/B they seem to move towards the origin, likewise 54 Piscium. It was also observed that for increasing $A_2$and B_2, the critical mass ratio comes closer to the μ_o. From table 5.14 and figures(5.10 and 5.11) with increase in eccentricity (e) value and fixed semi-major axis (a), as well as A_2 and B_2, it is observed that the ξ co-ordinatemoves slightly away from the origin whilethe η co-ordinate moves closer to the origin, while for table (5.15) and figure (5.12) with increase in semi-major axis (a) and fixed eccentricity (e) as well as A_2 and B_2, ξ co-ordinate tends to move to move closer to the origin and η co-ordinate moves away from the origin.

It was found that the triangular points are stable for$0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq$ ½. Some white dwarf binary systems (NLTT 1178, LP400-22 and J1257+5428) were also used to obtain the stability of triangular points in the absence of radiation pressure, from table (5.16) at increasing A_2 and decreasing B_2we observed that the co-ordinates shift towards the origin. Therefore, from table (5.19) with increase in both eccentricity (e) and semi-major axis (a) at fixed A_2 and B_2, it is found that as semi major axis and (A_2 and B_2) are fixed, the ξ co-ordinateisfixed while η co-ordinateattains complex roots, thereafter start increasing, then shifts away from the origin, also at fixed eccentricity (e) and (A_2 and B_2) ξ co-ordinatemoves slightly towards the origin and the η co-ordinate shifts towards the origin and then attains complex roots. Thus it was discovered that the triangular points are unstable because the mass ratios of the system is outside the range.

6.3 Recommendations/Future Direction

The study of space dyanamics in our institutions will boost research work in mathematics as well as assist (Nigeria Airspace Management Agency) in improving research development of space reaserch in their agency. Research work is a continuous

process,we examine some aspets that have not been examined in this dissertation , the out-of-plane points and periodic points around the collinear points and triangular points,will be carried out in future study.

6.4 Contribution to Knowledge

This dissertation examines the lagrangian equilibrium points in the photogravitational ER3BP with zonal harmonics of the secondary under the assumption that the primary is an intense emitter of radiation and the secondary has the shape of an oblate spheroid up to J_4 (octupolar mass moment).

1. We established that the mass ratios of the binaries are outside the range $0 < \mu < \mu_c$ and the quadruple and octupolar mass moment causes a shift towards the origin and away from the line joining the primaries on both co-ordinates for the white-dwarf binary systems (*NLTT 1178, LP400-22 and J127+5428*) in the absence of one perturbing agents (radiation), also eccentricity does not affect the $\xi_{\text{co-ordinate}}$ and ceases to exist in the quasi-parabolic case while there is a shift away from the primaries in the $\eta_{\text{co-ordinate}}$, for the semi-major axis reverse is the case for both co-ordiantes.
2. The binary systems (*Zeta Cygni, Procyon A/B and 54 Piscium*) were used to analyse the positions and stability of the triangular points with the presence of all the perturbing agents, we established that these points exist and form scalene triangles with the line joining the primaries.

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