PERTURBED ROBE'S CIRCULAR RESTRICTED THREE-BODY PROBLEM UNDER AN OBLATE PRIMARY

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DECLARATION

I hereby declare that the work in this thesis entitled "**PERTURBED ROBE'S CIRCULAR RESTRICTED THREE-BODY PROBLEM UNDER AN OBLATE PRIMARY**" has been carried out by me in the Department of Mathematics under the supervision of Prof. Jagadish Singh and Dr. Abdul Mohammed. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other institution.

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Name of student	Signature	Date

CERTIFICATION

This thesis entitled "**PERTURBED ROBE'S CIRCULAR RESTRICTED THREE-BODY PROBLEM UNDER AN OBLATE PRIMARY**" by Veronica Ugbedeojo OMEBIJE (M.Sc./Sci/12914/2011-2012) meets the regulations governing the award of Master of Science of the Ahmadu Bello University, Zaria, and is approved for its contributions to knowledge and literary presentation.

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DEDICATION

This research work is dedicated to God The Almighty who is my all in all for seeing me through in this program and to my lovely husband Mr. Cyril Okeme for his love and encouragement.

ABSTRACT

The research work examines the existence and linear stability of equilibrium points in the perturbed Robe's circular restricted three-body problem under the assumption that the hydrostatic equilibrium figure of the first primary is an oblate spheroid. The problem is perturbed in the sense that small perturbations given to the Coriolis and centrifugal forces are being considered. Results of the analysis found two axial equilibrium points on the line joining the centre of both primaries. It is further observed that under certain conditions, points on the circle within the first primary are also equilibrium points. And a special case where the density of the fluid and that of the infinitesimal mass are equal ($D = 0$) is discussed. The linear stability of this configuration is examined; it is observed that the axial equilibrium points $(p_1, 0,0)$ and $(x_{11} + p_2, 0, 0)$ are conditionally stable, while the circular points $(1 + r\cos\theta, r\sin\theta, 0)$ are unstable.

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m_1 mass of the first primary……..……..……...29

m_2 mass of the second primary 29

m_3 mass of the infinitesimal body..... 29

ρ_1 density of the fluid 29

ρ_3 density of the infinitesimal body 29

α oblateness coefficient…….……30

n mean motion……. …….30

B potential due to the first primary…….……...…….. 30

B' potential due to the second primary……. ……30

V potential that explain the combine force upon the infinitesimal mass 30

ϕ perturbation in Coriolis force 30

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μ mass ratio 30

ξ, η, ζ, small displacements 48

λ constant 50

Δ discriminant…….59

CHAPTER 1

GENERAL INTRODUCTION

1.1 Introduction

In the scientific exploration of space by man, space dynamics is very important and this later rely on celestial mechanics- which is a branch of science devoted to the movement of artificial celestial bodies such as artificial satellite, while space dynamics is a branch of astronomy which deals with the nature and motions of stellar phenomena such as planets, asteroids and stars. The most celebrated problem of space dynamics is the problem of three bodies, known as the three-body problem (3BP). This is defined in terms of three bodies with arbitrary masses and free to move in space attracting one another according to Newton's law of gravitation. An example of 3BP is the Sun-Earth-Moon system, when they are considered as point masses; they form the main problem of the lunar theory. (Singh, J 2012)

The R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, which move in circular orbits around the center of mass on the account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The R3BPs have produced many significant results by well-known mathematicians and scientists in an attempt to understand and predict the motion of natural bodies. A system made up of the Sun and Jupiter as primaries and Trojan asteroid assuming the role of the infinitesimal mass in the Sun-Jupiter system is a typical example of the R3BP.

A special case of R3BP referred to as Robe's problem was introduced by Robe (1977), where the infinitesimal mass is embedded in the first primary which is filled with

homogenous incompressible fluid of known density, the second primary is a point mass located outside an orbiting shell. The infinitesimal mass is under the gravitational attraction of the primaries as well as buoyancy force due to the fluid. Robe's models provides an insight into the study of the oscillations of the Earth's inner core taking into cognizance the attraction of the Moon; and the motion of an artificial Earth satellite located inside another satellite. This Robe's problem has been varied by the introduction of perturbing forces by many researchers like Hallan and Rana (2001b), Hallan and Mangang (2007), Singh and Sandah (2012), Singh and Mohammed (2013) to mention a few.

1.2 Statement of the Problem

Hallan and Mangang (2007) studied the existence and linear stability of equilibrium points in the Robe's circular restricted three-body problem. They assumed one of the primaries of mass m_1 to be a rigid spherical shell filled with homogenous, incompressible fluid of density ρ_1 in the shape of an oblate spheroid, the second primary m_2 is a point mass located outside the shell and moving around the mass m_1 in a Keplerian orbit; the infinitesimal mass m_3 is a small solid sphere of density ρ_3 moving inside the shell and is subject to the attraction of m_2 and the buoyancy force due to the fluid of the first primary. It will be interesting to see what happens when perturbations are included in their study. In this present thesis, we examine the effect of small perturbations in Coriolis and centrifugal forces on the existence and linear stability of the equilibrium points. Thus our study will be a generalization of Hallan and Mangang (2007).

1.3 Significance / Justification of the Study

In reality, we found that several heavenly bodies are sufficiently oblate. Earth, Jupiter, Saturn and Ragulus are oblate. The participating bodies in the classical R3BP are assumed to be strictly spherical in shape. The minor planets and meteoroids have

irregular shapes. Large perturbations from the two-body orbit are caused by the lack of sphericity, oblateness or triaxiality of the celestial bodies; an example of this is the motion of artificial Earth satellites.

In recent time, many perturbing forces like oblateness, triaxiality, variation of masses for the primaries and of the infinitesimal mass etc, have been included in the study of the existence and linear stability of the equilibrium points in the Robe's circular restricted three-body problem.

To this end, we modify the model by considering the effect of small perturbations in the Coriolis and centrifugal forces on the Robe's circular restricted three-body when the first primary is an oblate spheroid. Taking into account the full buoyancy force and other forces acting on the infinitesimal body when the density parameter is not zero $(i.e., \rho_1 \neq \rho_3)$.

The Robe's model under consideration can be applied in space mission design; it also has enormous applications in various astronomical problems. Examples include small oscillations of the earth's core in the gravitational field of Earth-Moon system.

1.4 Research Methodology

We adopt a rotating coordinate system to derive the equations of motion of the infinitesimal body with respect to the primaries as given in chapter three.

1.5 Aim and Objectives of the Study

The research work is aimed at investigating the motion of an infinitesimal mass in the perturbed Robe's circular restricted three body problem under an oblate spheroid.

In view of the problem stated above, we therefore set the following objectives:

(a) To derive the equations of motion of the infinitesimal mass

(b) To determine the locations of the axial equilibrium points

(c) To examine the linear stability of the axial equilibrium points

(d) To investigate the model when the density parameter is zero $(i.e., \rho_1 = \rho_3)$

1.6 Preliminary Ideas

1.6.1 Circular restricted three-body problem

The circular restricted three-body problem (CR3BP) describes the motion of a body of infinitesimal mass under the gravitational influence of two massive bodies, called the primaries, which orbit about their common centre of mass in circular orbits on account of their mutual attraction. The usefulness of studying this problem was verified by the discovery of the Trojan asteroids at the stable Lagrange points of the Sun-Jupiter orbital system.

The classical restricted three-body problem possesses five equilibrium points: three collinear points $L_{1,2,3}$ and two triangular points $L_{4,5}$, where the gravitational and centrifugal forces just balance each other. The collinear points are unstable, while the triangular points are stable for the mass ratio $\mu < 0.03852\ldots$ (Szebehely, 1967a). Their stability occurs in spite that the potential energy has a maximum rather than a minimum at the latter points. The stability is actually achieved through the influence of the Coriolis force, because the coordinate system is rotating (Wintner, 1941; Contopoulus, 2012). These equilibrium points are of great astronomical importance. The Solar and Heliospheric observatory (SOHO) and WMAP launched by NASA are in operation at the Sun-Earth L_1 and L_2 respectively. The Laser Interferometer Space Antenna (LISA) pathfinder is planned to go to L_1.

1.6.2 Equations of motion of two-body problem

In the circular restricted three-body problem, the motion of the primaries is governed by the equations of motion of two-body problem with constant masses. Let $m_1 \; and \; m_2$ be the masses of the two primaries, 0 be a fixed point in the space of the motion. Let $\vec{r}_1 \, and$ $\vec{r}_2$ be their respective vectors from 0 to the respective masses and $\vec{r}$ be the radius vector between the bodies. Then, the force of attraction on the particle of mass m_1 due to another particle of mass m_2 by Newton's law of gravitation as in Singh and Sandah (2012) is given as

$$\vec{F}_1 = G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \tag{1.1}$$

and the force acting on the particle of mass m_2 due to the particle of mass m_1 is

$$\vec{F}_2 = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \tag{1.2}$$

Where G is the gravitational constant and r is the distance between the primaries.

By Newton's second law

$$\vec{F}_1 = \frac{d^2\vec{r}_1}{dt^2}m_1,$$

$$\vec{F}_2 = \frac{d^2\vec{r}_2}{dt^2}m_2. \tag{1.3}$$

Putting the equations of system (1.3) in Eq. (1.1) and Eq. (1.2), we get

$$\frac{d^2\vec{r}_1}{dt^2}m_1 = G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r}, \quad \frac{d^2\vec{r}_2}{dt^2}m_2 = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r}$$

$$\text{or} \qquad \ddot{\vec{r}_1} = G\frac{m_2}{r^2}\frac{\vec{r}}{r}, \quad \ddot{\vec{r}_2} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r} \qquad where \qquad \vec{r} = \vec{r}_2 - \vec{r}_1 \tag{1.4}$$

Equations of system (1.4) are the vectorial equations of motion of the bodies of masses $m_1 \; and \; m_2$.

1.6.3 Equations of relative motion

We assume the velocity of the bodies $m_1 \; and \; m_2$ to be $\vec{v}_1 \, and \, \vec{v}_2$ respectively, and $\vec{v}_{\mathrm{R}}$ be the velocity vector of the body $m_2 \, with \, respect \, to \, m_1$.

Then, $\vec{v}_{\mathrm{R}} = \vec{v}_2 - \vec{v}$ (1.5)

Now the vectorial equations of system (1.4) take the form

$$\frac{d\vec{v}_1}{dt} = G\frac{m_2}{r^2}\frac{\vec{r}}{r} \; , \tag{1.6}$$

$$\frac{d\vec{v}_2}{dt} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r}. \tag{1.7}$$

Subtracting equation (1.6) from equation (1.7), we get

$$\frac{d\vec{v}_2}{dt} - \frac{d\vec{v}_1}{dt} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r} - G\frac{m_2}{r^2}\frac{\vec{r}}{r} = -G\frac{(m_1 + m_2)}{r^2}\frac{\vec{r}}{r}$$

or $$\frac{d}{dt}(\vec{v}_2 - \vec{v}_1) = -G\frac{(m_1 + m_2)}{r^2}\frac{\vec{r}}{r}$$

Using equation (1.5), we have

$$\frac{d\vec{v}_R}{dt} = -G\frac{(m_1 + m_2)}{r^2}\frac{\vec{r}}{r} \tag{1.8}$$

$$Eq.(1.8) \; becomes \; \frac{d\vec{v}_R}{dt} = -\frac{\mu}{r^2}\frac{\vec{r}}{r} \; , \; where \quad \mu = G(m_1 + m_2)$$

And removing the suffix, we have

$$\frac{d\vec{v}}{dt} = -\frac{\mu}{r^2}\frac{\vec{r}}{r} \tag{1.9}$$

Equation (1.9) is the relative motion of the body of mass m_2 with respect to the body of mass m_1, where r is the distance between the bodies.

1.7 Theoretical Framework

The theoretical framework upon which the derivations and finding of this research work is built on, is given in this section

1.7.1 The buoyancy force

The force exerted on an object that is completely or partly immersed in a fluid is known as buoyancy force. It acts upward, generally opposite to the direction of the frame of reference acceleration and its magnitude is equal to the weight of the fluid displaced by the submerged object. This force is caused by the difference between the pressure at the top of the object, which pushes downward and the pressure at the bottom of the object which pushes upward; every submerged object feels upward buoyancy force, because the pressure at the bottom of the object is always greater than the pressure at the top.

Consider a small particle of density *d* and volume *V,* falling with a velocity $\bar{V}$in a fluid of density ρ_1, the force acting on this particle as in Singh and Sandah (2012) are given as

(i) The gravitational force F_A

(ii) The buoyancy force, given by

$$F_B = \rho_1 V \mathrm{g} \tag{1.10}$$

where, g is the gravity of the fluid.

The pressure distribution on a submerged particle is statistically equivalent to a single point force F_B then acts at the location called the center of pressure (center of the mass of

the fluid displaced). Equation (1.10) is derived by relating hydrostatic distribution to an equivalent force – the buoyancy force (F_B).

Now, consider a completely submerged cylinder of length l and radius R oriented vertically in the fluid basin with depth L (see figure 1.1 below)

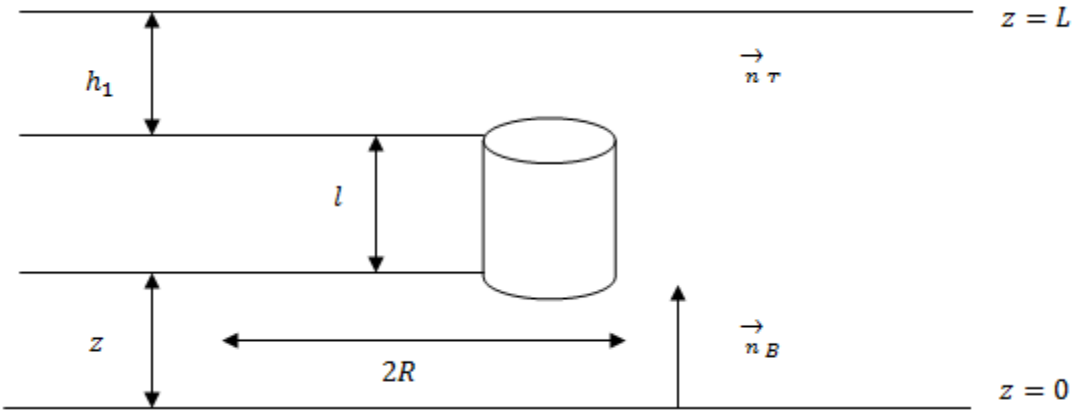


Figure 1.1 A cylinder of height h and radius R fully immersed in a fluid

The distance between the top surface of the cylinder and the below surface fluid is h_1, the thrust at the top and bottom of the cylinder are $\vec{n}_T$ and $\vec{n}_B$ respectively acting in opposite direction.

Now, from the relation

$$pressure = \frac{Force}{Area} \tag{1.11}$$

The force acting on the top of the surface due to hydrostatics is given by

$$\vec{F}_{top} = -\int_{AT} \vec{n}_T P dA \tag{1.12}$$

where A_T is the area of the top of the cylinder.

Now, the equation of pressure field in hydrostatics is given by

$$\frac{\partial p}{\partial z} = -\rho \tag{1.13}$$

here, p is a constant for an incompressible fluid, by integrating the last equation above with respect to *z*, we have

$$p = -\rho g z + c$$

At $z = L, \quad p = \rho_0 \quad so\ that \quad c = p_0 + \rho g L\ and\ \ P(z) = p_0 + \rho g(L - z).$

Thus

$$\int_1^z \partial p = -\int_1^z \rho g \partial z \tag{1.14}$$

Evaluating equation (1.14) at $z = L - h_1$, we immediately have

$$P = \rho_0 + \rho g(L - L + h_1)$$

Or $$P = \rho_0 + \rho g h_1 \tag{1.15}$$

Substituting equation (1.15) in (1.12), we have

$$\vec{F}_{top} = -\int_{AT} \vec{n}_T (P_0 + \rho g h_1) dA$$

or $$\vec{F}_{top} = -\vec{n}_T (P_0 + \rho g h_1) \int_{AT} dA$$

$Since \int_{AT} dA$ is actually the cross sectional area of the cylinder, therefore the force acting at the top of the solid is

$$\vec{F}_{top} = -\vec{n}_T \pi R^2 (P_0 + \rho g h_1) \tag{1.16}$$

Similarly, the force acting at the bottom of the cylinder is

$$\vec{F}_{bot} = -\int_{AB} \vec{n}_B P dA = -\vec{n}_B \pi R^2 \{P_0 + \rho g (h_1 + L)\} \tag{1.17}$$

while we have substituted $z = L - (h_1 + l)$ in equation (1.12).

Now, since $\vec{n}_T = \vec{n}_B = k$, where k is a unit vector along positive z direction and perpendicular to the $x - axis$, consequently, equation (1.17) may be written as

$$\vec{F}_{bot} = \vec{n}_T \pi R^2 \{P_0 + pg(h_1 + l)\} \tag{1.18}$$

Hence, the net vertical force (i.e., force acting downwards) on the cylinder can be obtained using equations (1.16) and (1.18) as

$$\vec{F}_{net} = \vec{F}_{top} + \vec{F}_{bot} = -\vec{n}_T \pi R^2 (P_0 + \rho g h_1) + \vec{n}_T \pi R^2 \{P_0 + \rho g (h_1 + l)\}$$

$$= \vec{n}_T \pi R^2 (-P_0 - \rho g h_1 + P_0 + \rho g h_1 + \rho g l)$$

$$\vec{F}_{net} = \vec{n}_T \pi R^2 (\rho g l)$$

or $\vec{F}_{net} = \vec{k} \rho g \pi R^2 l$ (1.19)

Since the volume of cylinder is $\pi R^2 l$, where R is radius and l is the height. The net force on the cylinder then takes the form

$$\vec{F}_{net} = \vec{k} \rho g V \tag{1.20}$$

therefore the buoyancy force is

$$\vec{F}_B = \rho g \vec{k} V_{cyl} \tag{1.21}$$

V_{cyl} here refers to the volume of the cylinder, ρ is the density of the fluid and g is the gravity of the fluid. Thus, the buoyancy force acting upward (positive z – direction) is equal to the weight of the displaced fluid.

Consider an arbitrary shape body and extend the above analysis using the projected area theorem. Let the submerged body in a fluid be separated into two parts, with the upper surface denoted by z_2 and the lower surface by z_1. Therefore, the force acting on the submerged body is given by

$$\vec{F} = -\int_{A_2} \overrightarrow{n_2} P(z_2) dA - \int_{A_1} \overrightarrow{n_1} P(z_1) dA \tag{1.22}$$

where $P(z_1) and\ P(z_2)$ are the pressure acting on the lower and upper surface of the body.

The component of this force in the $\vec{k}$ direction gives the buoyancy force. Taking the component of $\vec{F}\ in\ the\ \vec{k}$ direction, the buoyancy force is then

$$F_B = \vec{F}.\vec{k} = -\int_{A_2} \vec{n}_2.\vec{k} P(z_2) dA - \int_{A_1} \vec{n}_1.\vec{k} P(z_1) dA \tag{1.23}$$

From the projection theorem, we have

$$\vec{n}_2.\vec{k} dA = dA_z, \quad \vec{n}_1.\vec{k} = -dA_z \tag{1.24}$$

where dA_z is the projection of dA on the $xy - plane$.

The use of equation (1.24) in equation (1.23), we have the magnitude of the buoyancy force (the component of the force acting in the vertical direction)

$$F_B = -\int_{A_2} P(z_2)dA_z - \int_{A_1} P(z_1)dA_z \tag{1.25}$$

Now, since the hydrostatic pressure at any point on z in the fluid is given by equation (1.14), then

$$F_B = \int_{A_2} \rho g(z_2 - z_1)dA_z$$

or $$F_B = \rho g \int_{A_2} (z_2 - z_1)dA_z \tag{1.26}$$

From the principal surface of the integral, the equation (1.26) is just the volume of the solid, that is

$$\int_{A_2} (z_2 - z_1)dA_z = V_s$$

Therefore, the buoyancy force is

$$F_B = \rho g V_s \tag{1.27}$$

where V_s is the volume of the solid, this shows that the body is buoyed up by the weight of the displaced fluid.

Consider a solid spherical body of mass m_3 with center of mass M_3, density ρ_3 and radius b, so that

$$m_3 = \rho_3 \frac{4}{3}\pi b^3 \tag{1.28}$$

or $$b^3 = \frac{3m_3}{4\pi\rho_3} \tag{1.29}$$

Using equation (1.29) in (1.27) to get the buoyancy force (in magnitude) in a fluid of ρ_1

$$F_B = \rho_1 g \frac{4}{3}\pi \times \frac{3m_3}{4\pi\rho_3} \quad \text{or} \quad F_B = \rho_1 g \frac{m_3}{\rho_3}$$

this implies that $\vec{F}_B = -m_3 \frac{\rho_1}{\rho_3}\vec{g}$ (1.30)

The weight of the buoyed body is $\vec{F}_s = \rho_s \vec{g} V_s$ (1.31)

where $\vec{F}_s$ and ρ_s are the density and volume of the solid body respectively, while $\vec{g}$ is the gravity of the fluid ρ_1 acting at M_3. Equation (1.31) may be expressed as

$$\vec{F}_s = \rho_s \vec{g} V_s = \frac{G m_1 m_3 \overrightarrow{M_1 M_3}}{R_{13}^3} \quad (1.32)$$

Where M_1 is the center of the spherical shell having radius R filled with fluid ρ_1 and $\left|\overrightarrow{M_3 M_1}\right| = R_{31} = R_{13}$ and $m_3 = V_s \rho_s$

Putting equation (1.28) in equation (1.32), we have

$$\frac{G m_1 m_3 \vec{R}_{13}}{R_{13}^3} = \rho_3 g \frac{4}{3}\pi b^3 \quad (1.33)$$

using equation (1.29), we get $\frac{G m_1 m_3 \overrightarrow{M_1 M_3}}{R_{13}^3} = \rho_3 \frac{4}{3}\pi \times \frac{3m_3}{4\pi\rho_3}$

which implies

$$\vec{g} = \frac{G m_1 \overrightarrow{M_1 M_3}}{R_{13}^3}. \quad since \quad m_1 = \frac{4}{3}\pi R^3 \rho_1 \quad (1.34)$$

therefore,

$$\vec{g} = \frac{4}{3} G\pi R^3 \rho_1 \frac{\overrightarrow{M_3 M_1}}{R_{13}^3} \qquad but \quad R = R_{13} = R_{31} \left|\overrightarrow{M_3 M_1}\right|,$$

the above takes the value

$$\vec{g} = \frac{4}{3} G\pi R^3 \rho_1 \frac{\overrightarrow{M_3 M_1}}{R_{13}^3} \quad \text{or} \quad \vec{g} = \frac{4}{3} G\pi \rho_1 \overrightarrow{M_3 M_1} \tag{1.35}$$

Putting equation (1.35) in equation (1.30), we have the buoyant force as

$$\vec{F}_B = \frac{-4m_3\rho_1^2}{3\rho_3} G\pi \overrightarrow{M_3 M_1} \tag{1.36}$$

1.7.2 Robe's restricted three-body problem

Robe (1977) introduced a new kind of restricted three-body problem in which he regarded one of the primaries of mass m_1 as a rigid spherical shell filled with homogenous, incompressible fluid of density ρ_1; the second one is a point mass located outside the shell and moving around the mass m_1 in a Keplerian orbit; the infinitesimal mass m_3 is a small solid sphere of density ρ_3 moving inside the shell and is subject to the attraction of m_2 and the buoyancy force due to the fluid of the first primary. He discussed the linear stability of an equilibrium point obtained in two cases; in the first case, the orbit of m_2 around m_1 is circular and in the second case, the orbit is elliptic, while the shell is empty (without fluid) or the densities of m_1 and m_3 are equal.

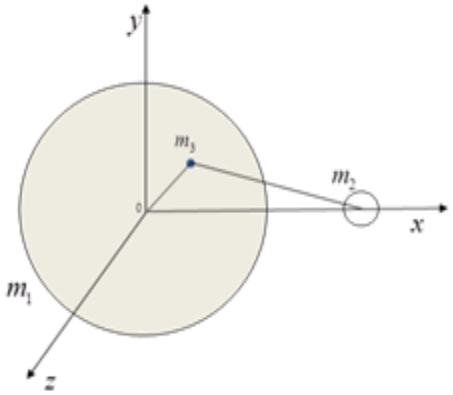


Figure 1.2 Robe's restricted three-body problem

Since we modified Robe's model in our work .The derivation of the equations of motion of Robe's problem (1977) is given in this section

Let the orbital plane of m_2 around m_1^* (the shell with fluid ρ_1) be taken as $xy-plane$, and also let the origin of the coordinate system be at the center of mass, 0, of the two finite masses. Then the forces acting on the third body are;

1) The attraction of m_2

2) The gravitational force $\vec{F}_A$ exerted by the fluid of density ρ_1 is given by

$$\vec{F}_A = \frac{Gm_1m_3\vec{R}_{13}}{R_{13}^3}$$

Substituting the mass of the fluid m_1 in the above gives

$$\vec{F}_A = -G\frac{4\pi R^3 m_1 m_3 \vec{R}_{13}}{3R_{13}^3} \tag{1.37}$$

Since the body of mass m_3 maintains a spherical symmetry about M_1 (center of m_1) due to the pressure of the fluid inside m_1, in this case, the radius R becomes $R = R_{13}$. Therefore, the gravitational force equation (1.37) is recast to the form

$$\vec{F}_A = -\left(\frac{4}{3}\right)\pi G\rho_1^2 m_3 \overline{M_1 M_3} \tag{1.38}$$

3) The buoyancy force $\vec{F}_B$, is obtained from equation (1.36) as

$$\vec{F}_B = -\frac{4}{3}\pi G\rho_1^2 m_3 \frac{\overline{M_1 M_3}}{\rho_3} \tag{1.39}$$

The equation of motion of the infinitesimal mass m_3 under the listed forces above, can be written as, Robe (1977).

$$\begin{aligned} \ddot{x} - 2\dot{y} &= U_x, \\ \ddot{y} + 2\dot{x} &= U_y, \\ \ddot{z} &= U_z, \end{aligned} \tag{1.40}$$

with

$$U = \frac{1}{2}(x^2 + y^2 + z^2) + \frac{\mu}{(1-\mu-x^2)^2 + y^2 + z^2} - \frac{K}{2}[(1-\mu-x^2)^2 + y^2 + z^2],$$

$$\mu = \frac{m_2}{m_1 + m_2} \leq 1 \quad and \quad K = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right).$$

System (1.40) is equations of motion for the Robe's CR3BP.

1.7.3 Perturbation

In astronomy, perturbations is the deviation in the motion of a celestial object caused either by the gravitational force of a passing object or by a collision with it. For example, predicting the Earth's orbit around the Sun would be rather straightforward if not for the slight perturbations in its orbital motion caused by the gravitational influence of the other planets.

1.7.4 Coriolis and centrifugal forces

1.7.4.1 Coriolis force: In classical mechanics, the Coriolis force is an inertial force described by the 19th-century French engineer-mathematician Gustave-Gaspard Coriolis in 1835. He described this force in terms of its effect. The effect of the Coriolis force is an apparent deflection of the path of an object that moves within a rotating coordinate system. The object does not actually deviate from its path, but it appears to do so because of the motion of the coordinate system.

The Coriolis effect has great significance in astrophysics and stellar dynamics, in which it is a controlling factor in the directions of rotation of sunspots. It is also an important consideration in ballistics, particularly in the launching and orbiting of space vehicles. (Encyclopedia Britainica Ultimate , 2014)

1.7.4.2 Centrifugal force: A fictitious force, peculiar to a particle moving on a circular path, that has the same magnitude and dimensions as the force that keeps the particle on its circular path (the centripetal force) but points in the opposite direction.

Although it is not a real force according to Newton's laws, the centrifugal-force concept is a useful one. For example, when analyzing the behaviour of the fluid in a cream separator or a centrifuge, it is convenient to study the fluid's behaviour relative to the rotating container rather than relative to the Earth; and, in order that Newton's laws be applicable in such a rotating frame of reference, an inertial force, or a fictitious force (the centrifugal force), equal and opposite to the centripetal force, must be included in the equations of motion. (Encyclopedia Britainica Ultimate 2014)

1.7.5 Oblate spheroid

An ellipsoid having a polar axis shorter than the diameter of the equatorial circle whose plane bisects it is known as an oblate spheroid, in other words an oblate spheroid is a three dimensional solid and is most easily described as a sphere that has been compressed from top to bottom, causing the equator to bulge out. Thus, its two moments of inertial are equal out of the three. It is the approximated shape of many planets and celestial bodies, including Saturn, Jupiter and to a lesser extent the Earth.

1.7.6 Moment of Inertia

The moment of inertia of a system or particle with respect to a plane, point or line is defined as the sum of the products of the masses and the square of their respective perpendicular distances from the plane, point or line. If $m_1, m_2, m_3, \dots, m_n$ are the masses of the particles with perpendicular distances $r_1, r_2, r_3, \dots, r_n$, then the moment of inertia is given as

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2 \qquad (1.42)$$

For a body, we have

$$I = \int_M r^2 dm \qquad (1.43)$$

where $dm = \rho dM$, ρ is the density of the body and subscript M shows that the integration is done over the whole mass of the body. In the case of I_x, I_y, I_z denote the moment of inertia with respect to the plane $x = y = z = 0\ respectively$.Using equation (1.42) in equation (1.43), we have

$$I_x = \sum_{i=1}^{n} m_1 x_i^2 = \int_M x^2$$

$$I_y = \sum_{i=1}^{n} m_1 y_i^2 = \int_M y^2 \qquad (1.44)$$

$$I_z = \sum_{i=1}^{n} m_1 z_i^2 = \int_M z^2$$

The moment of inertia with respect to $x, yand\ z$ axes are given respectively by

$$I_1 = \sum_{i=1}^{n} m_i (y_i^2 + z_i^2)^2 = \int_M (y^2 + z^2)\, dm$$

$$I_2 = \sum_{i=1}^{n} m_i (z_i^2 + x_i^2)^2 = \int_M (z^2 + x^2)\, dm \qquad (1.45)$$

$$I_3 = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2)^2 = \int_M (x^2 + y^2)\, dm$$

while the moment of inertia with respect to the origin is

$$I_0 = \sum_{i=1}^{n} m_i (x_i^2 + y_i^2 + z_i^2)^2 = \int_M (x^2 + y^2 + z^2)\, dm$$

$$= \int_M x^2\, dm + \int_M y^2\, dm + \int_M z^2\, dm \quad = I_x + I_y + I_z \ = \frac{1}{2}(I_1 + I_2 + I_3)$$

so that $\quad 2I_0 = I_1 + I_2 + I_3 \qquad (1.46)$

where I_0 is the moment of inertia about the origin and $I_1,\ I_2,\ I_3$ are the principal moment of inertia with respect to their principal axes.

The body is a spherical one if $I_1 = I_2 = I_3$, an oblate spheroid if $I_1 = I_2 \neq I_3$ and a triaxial rigid body if $I_1 \neq I_2 \neq I_3$.

The potential of the body with the origin at the center of the mass is given as

$$V = -\frac{GM}{r} - \frac{G}{r^3}(2I_0 - 3I) \tag{1.47}$$

where I is the moment of inertia about a line fixed from the center of the body to a point outside the body with distance r. Thus the potential between the infinitesimal mass and the primaries can be written as;

$$V = -GM\left[\frac{1}{r} + \frac{1}{2Mr^3}(2I_1 + I_3 - 3I) + \frac{1}{2Mr^3}(I_1' + I_2' + I_3' - 3I')\right] \tag{1.48}$$

where $I_1,\ I_2, I_3$ are the principal moments of inertia of the oblate spheroid (first primary) with $I_1 = I_2 \neq I_3$ and I_1', I_2', I_3' are the principal moment of inertia of the second primary so that $2I_0 = 2I_2 + I_3$ and I is the moment of inertia about the line joining the center of the primary and the infinitesimal body;

From Kepler's third law, the mean motion is expressed as

$$n^2 = \frac{G(m_1 + m_2)}{r^3} \tag{1.49}$$

When the first primary is an oblate spheroid, the second primary a point mass and the full buoyancy force of the fluid is considered, the mean motion of the primaries is given (McCuskey 1963) by

$$n^2 = \frac{G(m_1 + m_2)}{r^3}\left(1 + \frac{3}{2}\alpha\right). \tag{1.50}$$

1.8 Linear Stability

In ordinary differential equations, the stability of linear systems is determined by the eigen values of the coefficient matrix. The locations of the infinitesimal body would be displaced a little from the equilibrium point due to perturbations. If the resultant motion of the infinitesimal body is a rapid departure from the vicinity of the point, we call such location of equilibrium point an "unstable one" otherwise it is stable.

In order to examine the stability of the orbit in the vicinity of the equilibrium points, we apply this small displacement method by shifting the origin or coordinates of the infinitesimal mass and linearizing the equations of motion around the coordinates of the equilibrium. The variational equations of motion corresponding to the dynamical system are derived, which in turn through the trial solutions is transformed to a matrix form and a characteristic equation of the variational equations of the dynamical system is obtained. The stability of the solutions depends on the nature of the characteristic roots.

(i) For complex roots; the equilibrium point is asymptotically stable when all roots have negative real parts, and unstable when some or all roots have positive real part, while multiple complex roots can either be stable or unstable.

(ii) For pure imaginary roots; equilibrium point is stable, though not asymptotically stable. If there are multiple roots, the solution contains mixed terms (i.e., periodic and secular terms), the equilibrium point is unstable.

(iii) For real roots; the equilibrium point is stable if all the roots are both real and negative, but unstable if any of the roots are positive. This statement is also true for multiple roots.

CHAPTER TWO

LITERATURE REVIEW

The formulation of the R3BP originally suggested the approximate circular motion of the planet around the sun and the small masses of asteroids and the satellite of planets compared to the planet's masses. The restricted three body problem is of fundamental importance in mechanics, with significant applications to astrodynamics.

In the CR3BP, the effects of the gravitational attraction of the infinitesimal body and other perturbations have been ignored. Perturbations can as well arise from the causes such as the triaxiality, oblateness, perturbations in the Coriolis and centrifugal forces, variation of the masses, the atmospheric drag, and the solar wind. This enabled many researchers (Szebehely 1967b; Subbarao and Sharma 1975; Bhatnagar & Hallan 1978; AbdulRaheem & Singh 2006; Singh 2009) to study the R3BP by taking into account the effects of small perturbations in the Coriolis and the centrifugal forces, radiation pressures, oblateness, and triaxiality of the bodies.

Szebehely (1967b) studied the effect of small perturbation of the Coriolis force on the stability of equilibrium points by keeping the centrifugal force constant. He established that the collinear points remain unstable and obtained for the stability of the triangular points a relation between the critical value of the mass parameter μ_c and the change ε in the Coriolis force; $\mu_c = \mu_0 + \frac{16\epsilon}{3\sqrt{69}}$, and concluded that the Coriolis force is a stabilizing force. Subbarao & Sharma (1975) extended the work of Szebehely (1967b) by considering the primaries as oblate spheroids and its equatorial plane coinciding with the plane of motion. They showed that the oblateness of the primaries resulted in an increase in both the Coriolis and Centrifugal forces and further established that the range of linear

stability of the triangular points decreases, thereby concluding that the Coriolis force is not always a stabilizing force. Bhatnagar and Hallan (1978) considered the effect of perturbation in the Coriolis (ε) and centrifugal (ε') forces and discovered that the collinear points are not influenced by the perturbations and hence remain unstable. But for the triangular points they obtained the relation: $\mu_c = \mu_0 + \frac{4(36\varepsilon - 19\varepsilon')}{27\sqrt{69}}$ and then concluded that the range of stability increases or decreases depending on whether the point $(\varepsilon, \varepsilon')$lies on either of the two points in which the $(\varepsilon, \varepsilon')$ plane is divided by the line $36\varepsilon - 19\varepsilon' = 0$.

Singh and Ishwar (1984) investigated the effect of small perturbation in the Coriolis and the centrifugal forces on the location of equilibrium points in the R3BP with variable mass. They found that the triangular points form nearly equilateral triangles with the primaries while the collinear points lie on the line joining the primaries. Singh and Ishwar (1985) also considered the same problem and found that the range of stability of the triangular points increases or decreases depending on whether the perturbation point $(\varepsilon, \varepsilon')$ lies in either of the two parts in which $(\varepsilon, \varepsilon')$–plane is divided by the line $36\varepsilon - 19\varepsilon' = 0$

AbdulRaheem & Singh (2006) studied the combined effect of perturbations, radiation and oblateness on the stability of equilibrium points in the R3BP. This was studied under the influence of small perturbations in the Coriolis and centrifugal forces, together with the effects of oblateness and radiation pressure of the primaries. They found that the triangular points are stable for $0 < \mu < \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$, while the collinear points remain unstable. They showed further that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation and oblateness of the primaries

have destabilizing effect; the presence of any one or more of the latter makes weak the stabilizing ability of the former, consequently the overall effect is that the range of stability of the triangular point decreases. The non linear stability of the triangular points under small perturbations in the Coriolis and centrifugal forces together with the effect of oblateness and radiation pressures of the primaries was examined by Singh (2011). He found that the triangular points are stable for all mass ratios in the range of linear stability except for three mass ratios depending upon Coriolis and centrifugal forces, oblateness coefficients and mass reduction factors.

Singh and Begha (2011) studied the generalized three body problem in which the more massive primary is a triaxial rigid body, the less massive one is an oblate spheroid under the influence of small perturbation in the Coriolis and centrifugal forces. They ascertained that the position of the triangular and collinear equilibrium points was affected by the non-sphercity of the primaries and the change in the Centrifugal force. They also investigated the stability of the triangular points and found that they are stable for $0 < \mu \leq \mu_c$ and unstable for $\mu_c < \mu < \frac{1}{2}$, where μ_c is the critical value of the mass parameter which depends on the joint effect of perturbations, oblateness and triaxiality. The collinear points were found to be unstable in the linear approximation.

Singh (2013) also investigated the effects of small perturbations in the Coriolis and centrifugal forces, radiation pressures and triaxiality of the two stars (primaries) on the positions and stability of an infinitesimal mass (third body) in the framework of the planar CR3BP. He observed that the positions of the three collinear and triangular equilibrium points are affected by the radiation, triaxiality and a small perturbation in the centrifugal force, but are unaffected by that of the Coriolis force. Singh found the collinear points to remain unstable, while the triangular points are stable for $0 < \mu < \mu_c$

and unstable for $\mu_c < \mu < \frac{1}{2}$ where μ_c is the critical mass ratio influenced by small perturbations in the Coriolis and centrifugal forces, radiation and triaxiality. He also showed that the Coriolis force possesses a stabilizing behavior, while the centrifugal force possesses a destabilizing behavior. Conclusively, it can be seen that the size of the region of stability decreases with increase in the values of the parameters involved.

Robe (1977) introduced a new kind of R3BP in which he regarded one of the primaries of mass m_1 as a rigid spherical shell filled with homogenous, incompressible fluid of density ρ_1; the second one was a point mass located outside the shell and moving around the mass m_1 in a Keplerian orbit; the infinitesimal mass m_3 is a small solid sphere of density ρ_3 moving inside the shell and is subject to the attraction of m_2 and the buoyancy force due to the fluid of the first primary. He discussed the linear stability of an equilibrium point obtained in two cases; in the first case, the orbit of m_2 around m_1 is circular and in the second case, the orbit is elliptic, while the shell is empty (without fluid) or the densities of m_1 and m_3 are equal. In each case, the domain of stability has been investigated for the whole range of parameters occurring in the problem.

Shrivastava and Garain (1991) studied the effect of small perturbations in the Coriolis and centrifugal forces on the position of the equilibrium point in the Robe's CR3BP when the densities m_1 and m_3 are equal.

Hallan and Rana (2001a) studied the existence of all equilibrium points, their location and stability in Robe's (1977) CR3BP. It was found that, the centre of the first primary is always an equilibrium point, whatever be the values of the densities parameter K, eccentricity parameter e and mass parameter μ. The other equilibrium points exist only when $K \neq 0$ and $e = 0$, i.e when the second primary, a mass point moves around the first, a spherical shell filled with fluid in a circular orbit. They also showed that when

$K > 1$ there is one additional equilibrium point lying on the line joining the centre of the first primary and second primary. Also when $K + \mu = 1$ there are infinite number of equilibrium points in the $x - y$ plane lying on the circle of radius one and center as the second primary, provided the points are inside the spherical shell. When $K < 0$ and $K + \mu > 0$, there are two or more equilibrium points lying in the $x - z$ plane forming triangles with the center of the shell and the second primary. As regards the stability of the equilibrium point $(-\mu, 0,0)$, they discovered that the center of the primary are the same as those given by Robe (1977), whereas circular and triangular points are always unstable. The equilibrium point collinear with the center of the shell and the second primary was found to be stable provided $\mu \; and \; K$ satisfy the inequality $16\mu(K-1)^2 < (\mu + \sqrt{\mu(-4+4K+\mu)^3}$. Therefore, Robe's elliptic R3BP has only one equilibrium point for all values of the parameters $K \; and \; \mu$ while the Robe's CR3BP can have two, three or infinite number of equilibrium points depending upon the values of $K \; and \; \mu$.

Hallan and Rana (2001b) discussed the effect of small perturbations $\varepsilon \; and \; \varepsilon'$ in the Coriolis and centrifugal forces respectively on the location and stability of the equilibrium points in the Robe's CR3P as in the case studied by Shrivastava and Garain (1991) but with a different method, when the density of the first primary and that of the infinitesimal body is equal (i.e. the density parameter being zero). They showed that only one equilibrium point which lies on the right or left of the center of the shell on the line joining the center of the shell and the second primary accordingly as ε' is positive or negative and the change ε in the Coriolis force does not affect the location of the equilibrium point. They further found the point to be stable in the linear sense for $\mu_c < \mu < 1$ and unstable for $0 < \mu \leq \mu_c$ where $\mu_c = \frac{8}{9} + \frac{2}{3}\left[\left(\frac{45}{3}\varepsilon' - \frac{10}{3}\varepsilon\right)\right]$ and found that the range of stability increases or decreases depending upon whether the point $(\varepsilon, \varepsilon')$

lies in one or the other of the two regions in which $(\varepsilon, \varepsilon')$ plane is divided by the line $\frac{45}{3}\varepsilon' - \frac{10}{3}\varepsilon = 0$.

Shu and Lu (2005) investigated the locations and stability of the main libration points in the Robe's R3BP under perturbed Coriolis and centrifugal forces. Their efforts was aimed at improving the work of Shrivastava and Garain (1991) as they argued the assumption that the densities of the first primary and the infinitesimal mass are equal is hard to be fulfilled in practice. Hence their study regarded the density parameter as $K \geq 0$ $(\rho_1 \leq \rho_3)$, they obtained a unique libration point $(x_0, 0,0) = \left(-\mu + \frac{\mu\ \varepsilon'}{1+2\mu-K}, 0,0\right)$ when $\beta - K \geq 0$ and estimated the perturbation magnitude of their locations and their stability.

The existence and stability of equilibrium points in the Robe's (1977) CR3BP was investigated by Hallan and Mangang (2007) they considered the full buoyancy force as in Plastino and Plastino (1995) and assumed the hydrostatic equilibrium figure of the first primary as an oblate spheroid. The pertinent equations of motion were derived and they found that there is an equilibrium point near the center of the first primary. They further found that there can be one or more equilibrium point on the line joining the center of the first primary and the second primary, also an infinite number of equilibrium points lying on a circle in the orbital plane of the second primary. One more equilibrium point was found to be conditionally stable while the circular points are unstable.

Singh and Sandah (2012) investigated the existence and linear stability of an infinitesimal body around the equilibrium points in the framework of the Robe's CR3BP, with assumptions that the hydrostatic equilibrium figure of the first primary is an oblate spheroid and the second primary being an oblate body as well. They found that an equilibrium point exists near the center of the first primary, and there can be one more

equilibrium point on the line joining the centers of both primaries. They further found that the points on the circle within the first primary are also equilibrium points under certain conditions and the existence of two out of plane points. The linear stability of this configuration was examined and it was found that the points near the center of the first primary are conditionally stable, while the circular and out of plane equilibrium points are unstable. Singh and Omale (2014) extended the work of Singh and Sandah (2012) by including the oblateness coefficients of the second primary up to the second zonal harmonic. They observed that the two axial equilibrium points on the line joining the centers of the primaries are affected by the zonal harmonic. Points on the circle within the first primary are also equilibrium points under certain conditions. They further found that the first axial point is stable, while the second one is conditionally stable, and the circular points are unstable.

Singh and Leke (2013) extended the work of Hallan and Rana (2001b) by assuming that the masses of the primaries vary with time in accordance with unified Meshcherskii law and determined their motion by the GMP. Singh and Mohammed (2013) analyzed Robe's CR3BP when the hydrostatic equilibrium figure of the first primary is assumed to be an oblate spheroid, the shape of the second primary a triaxial rigid body, and the full buoyancy force of the fluid is taken into account. They found that there is an equilibrium point near the center of the first primary, another equilibrium point exists on the line joining the center of the primaries and there exist infinite number of equilibrium points on an ellipse in the orbital plane of the second primary. They further observed that all these equilibrium points can be stable under certain conditions. The most interesting and distinguishable results of their study was the existence of elliptical points and their stability.

CHAPTER THREE

EQUATIONS OF MOTION

3.1 Introduction

The motion of the infinitesimal mass has no impact on the motion of primaries but rather it moves under the gravitational influence of the two primaries which in turn moves in circular orbits around their centre of mass. The equations of motion of this problem described by this infinitesimal mass in the Robe's problem putting into consideration the full buoyancy force, oblateness of the first primary, small perturbations in the Coriolis and centrifugal forces are given in this chapter.

3.2. Mathematical Formulations of the Problems

Let the first primary m_1 be a fluid of density ρ_1 in the shape of an oblate spheroid as assumed by Hallan and Mangang (2007). Let the second primary $m_2 < m_1$ be a mass point, which describes a circular orbit around m_1. The infinitesimal mass m_3, whose density is $\rho_3 \neq \rho_1$, moves inside the first primary.

A uniformly rotating coordinate system $oxyz$ with origin at the centre of mass m_1, ox pointing towards m_2, and oxy is considered the orbital plane of m_2 coinciding with the equatorial plane of m_1. Using dimensionless variables, the equations of motion of the infinitesimal mass in this coordinate system as in Hallan and Mangang (2007) and Bhatnagar and Hallan (1978) can be written as:

$$\begin{aligned} \ddot{x} - 2\phi n\dot{y} &= U_x \,, \\ \ddot{y} + 2\phi n\dot{x} &= U_y, \\ \ddot{z} &= U_z, \end{aligned} \tag{3.1}$$

where

$$U = V + \frac{n^2\psi}{2}\left[\left(x - \frac{m_2 R}{m_1 + m_2}\right)^2 + y^2\right]$$

$$V = B + B' - \frac{\rho_1}{\rho_3}\left[B + B^{'} + \frac{n^2\psi}{2}\left\{\left(x - \frac{m_2 R}{m_1 + m_2}\right)^2 + y^2\right\}\right]$$

$$n^2 = G\frac{(m_1 + m_2)}{R^3}\left(1 + \frac{3\alpha}{2}\right). 0 < \alpha = \frac{a^2 - c^2}{5R^2} < 1$$

$$B = \pi G\rho_1\ [\mathrm{I} - \mathrm{A}_1 x^2 - \mathrm{A}_1 \mathrm{y}^2 - \mathrm{A}_2\ \mathrm{z}^2]$$

$$B^{'} = \frac{Gm_2}{[(R - x)^2 + \mathrm{y}^2 + \ \mathrm{z}^2]^{\frac{1}{2}}}$$

$$\mathrm{A_i} = \ \mathrm{a}_1^2\ \mathrm{a}_2 \int_0^{\infty} \frac{\mathrm{du}}{\Delta\ (\mathrm{a}_\mathrm{i}^2 + \ \mathrm{u})}, (\mathrm{i} = 1{,}2)$$

$$\Delta^2 = (\mathrm{a}_1^2 + \ \mathrm{u})^2\ (\mathrm{a}_2^2 + \ \mathrm{u}),\ \ \mathrm{I} = 2\mathrm{a}_1^2\ \mathrm{A}_1 + \mathrm{a}_2^2\mathrm{A}_2, \mu = \frac{Gm_2}{R^3}.$$

Where V is the potential that explains the combined forces upon the infinitesimal mass, B denotes the gravitational potential due to the fluid mass. B' stands for the potential due to mass point. R is for the distance between the primaries. G is the gravitational constant, $\mathrm{a}_1$and a_2 are the equatorial and polar radii of the primary. I stand for the polar moment of inertia. α is the oblateness coefficient of the first primary, while $\mathrm{A_i}(\mathrm{i} = 1{,}2)$ are the index symbols. $\phi = 1 + \varepsilon,\ |\varepsilon| \ll 1\ and\ \psi = 1 + \varepsilon',\ |\varepsilon'| \ll 1$ are parameters for the Coriolis and centrifugal forces respectively, to which small perturbation $\boldsymbol{\varepsilon}$ and ε' are given. ⍰ is the mass ratio. *D>0* states that the infinitesimal body is denser than the surrounding fluid.

$$U = B + B' - \frac{\rho_1}{\rho_3}\left[B + B' + \frac{n^2\psi\left\{\left(x - \frac{m_2R}{m_1+m_2}\right)^2 + y^2\right\}}{2}\right] + \frac{n^2\psi\left\{\left(x - \frac{m_2R}{m_1+m_2}\right)^2 + y^2\right\}}{2}$$

$$= B + B' - \frac{\rho_1}{\rho_3}B - \frac{\rho_1}{\rho_3}B' - \frac{\rho_1}{\rho_3}\left[\frac{n^2\psi\left\{\left(x - \frac{m_2R}{m_1+m_2}\right)^2 + y^2\right\}}{2}\right] + \frac{n^2\psi\left\{\left(x - \frac{m_2R}{m_1+m_2}\right)^2 + y^2\right\}}{2}$$

$$= B\left(1 - \frac{\rho_1}{\rho_3}\right) + B'\left(1 - \frac{\rho_1}{\rho_3}\right) + \left(1 - \frac{\rho_1}{\rho_3}\right)\left[\frac{n^2\psi\left\{\left(x - \frac{m_2R}{m_1+m_2}\right)^2 + y^2\right\}}{2}\right]$$

$$= \left(1 - \frac{\rho_1}{\rho_3}\right)\left[B + B' + \frac{n^2\psi\left\{\left(x - \frac{m_2R}{m_1+m_2}\right)^2 + y^2\right\}}{2}\right]$$

A substitution of the expression for the potential B due to the fluid in the first primary and the potential B' due to the second primary, and taking the units such that the sum of the masses of the primaries and their distance apart are unity. The unit of time is selected such that G=1. Thus $0 < \mu = \frac{m_2}{m_1+m_2} < 1, R = 1,$ then the equations of motion for the infinitesimal mass in the dimensionless variables can be written as

$$\ddot{x} - 2\phi n\dot{y} = U_x\,,$$
$$\ddot{y} + 2\phi n\dot{x} = U_y, \qquad (3.2)$$
$$\ddot{z} = U_z,$$

where

$$U = D\left[\pi\rho_1\,[\mathrm{I} - \mathrm{A}_1(x^2 + \mathrm{y}^2) - \mathrm{A}_2\mathrm{z}^2] + \frac{\mu}{[(1-x)^2 + \mathrm{y}^2 + \mathrm{z}^2]^{\frac{1}{2}}} + \frac{\mathrm{n}^2\,\psi\{(x - \square)^2 + \mathrm{y}^2\}}{2}\right] \quad , D = 1 - \frac{\rho_1}{\rho_3}$$

3.3 The Case $(\boldsymbol{\rho_1 = \rho_3})$

We consider the case where the infinitesimal mass and the fluid have equal density. This reduces the equations of motion in system (3.2) to

$$\ddot{x} - 2n\phi\dot{y} = 0, \tag{3.3}$$

$$\ddot{y} - 2n\phi\dot{x} = 0, \tag{3.4}$$

$$\ddot{z} = 0, \tag{3.5}$$

In solving the equations above we let

$$\dot{x} = U \; and \; \dot{y} = V$$

$$\ddot{x} = \dot{U} \; and \; \ddot{y} = \dot{V} \tag{3.6}$$

Substituting system (3.6) in equations (3.3) and (3.4), we have

$$\dot{U} - 2\phi nV = 0 \tag{3.7}$$

$$\dot{V} + 2\phi nU = 0 \tag{3.8}$$

Now differentiating (3.7) with respect to t, we have

$$\ddot{U} - 2\phi n\dot{V} = 0 \tag{3.9}$$

Using equation (3.8) in equation (3.9), we have

$$\ddot{U} - 2\phi n(-2\phi nU) = 0 \quad \text{or} \quad \ddot{U} + 4\phi^2 n^2 U = 0 \tag{3.10}$$

The auxiliary equation to equation (3.10) is

$$m^2 + 4\phi^2 n^2 = 0 \quad ,$$

$$m = \pm 2\phi n\mathrm{i}$$

$$\therefore \quad \mathrm{U} = \mathrm{C}_1 \, \mathrm{Cos}\, 2\phi nt + \mathrm{C}_2 \, \mathrm{Sin}\, 2\phi nt \tag{3.11}$$

Now using equation (3.11) in equation (3.8) we have

$$\dot{V} + 2n(\mathrm{C}_1 \mathrm{Cos}\, 2\phi nt + \mathrm{C}_2 \mathrm{Sin}\, 2\phi nt) = 0 \quad \text{or} \quad \dot{\mathrm{V}} = -2n\mathrm{C}_1 \mathrm{Cos} 2\phi nt - 2n\mathrm{C}_2 \mathrm{Sin} 2\phi nt$$

Integrating, we have

$$V = C_1 \text{Sin} 2\phi nt + C_2 \text{Cos}\ 2\phi nt \tag{3.12}$$

Equation (3.11) can be written as

$$U = \sqrt{C_1^2 + C_2^2}\left[\frac{C_1}{\sqrt{C_1^2 + C_2^2}}\ \text{Cos} 2\phi nt + \frac{C_2}{\sqrt{C_1^2 + C_2^2}} \text{Sin} 2\phi nt\right] \tag{3.13}$$

$$Put\ \frac{C_2}{\sqrt{C_1^2 + C_2^2}} = \text{Cos}\ \theta, \qquad \therefore\ \text{Sin}\ \theta = \sqrt{1 - \text{Cos}\ \theta} = \frac{C_1}{\sqrt{C_1^2 + C_2^2}}\ also\ \sqrt{C_1^2 + C_2^2} = m$$

∴ Equation (3.13) becomes

$$U = m[\text{Sin}\ 2\phi nt\ \text{Cos}\ \theta + \text{Cos}\ 2\phi nt\ \text{Sin}\ \theta]$$

$$\text{or}\ \ U = m\ \text{Sin}\ (2n\phi t + \theta) \tag{3.14}$$

Using equation (3.14) in first equation of system (3.6), we have

$$\dot{x} = m\ \text{Sin}\ (2n\phi t + \theta)$$

$$\text{or}\quad x = \frac{-m}{2n}\ \text{Cos}\ (2n\phi t + \theta) + m_1 \tag{3.15}$$

Similarly from equation (3.12) we have

$$V = m[\text{Sin}\ 2\phi nt\ \text{Sin}\ \theta - \text{Cos}\ 2\phi nt\ \text{Cos}\ \theta]$$

$$= -m[\text{Cos}\ 2\phi nt\ \text{Cos}\ \theta - \text{Sin}\ 2\phi nt\ \text{Sin}\ \theta]$$

$$= m\ \text{Cos}\ (2n\phi t + \theta) \tag{3.16}$$

Using equation (3.16) in 2nd equation of system (3.6), we have

$$\dot{y} = m\ \text{Cos}(2n\phi t + \theta)$$

$$y = \frac{-m}{2n} \text{Sin}\,(2n\phi t + \theta) + m_2 \tag{3.17}$$

Therefore the solutions of equations (3.5), (3.6) and (3.7) can be written as

$x = \theta_1 \text{Cos}\,(2n\phi t + \theta) + \theta_2,$

$y = -\theta_1 \text{Sin}\,(2n\phi t + \theta) + \theta_3,$

$z = \theta_4 t + \theta_5,$

where $\theta_i (i = 1,2, \ldots 5)$ are constants. Since the fluid is assumed to be in the hydrostatic equilibrium in the rotating frame, all the elements of the fluid are at rest in this system. Also the density of the infinitesimal mass is the same as that of the fluid; it is therefore impossible to differentiate between any of its elements. Thus, every point inside the fluid is an equilibrium point which confirms the result of Plastino and Plastino (1995).

3.3 Conclusion

System (3.2) consists of the equations of motion of the perturbed Robe's circular restricted three-body problem under an oblate spheroid by considering the full buoyancy force of the fluid and the gravitational attraction of the primary bodies.

When the perturbations in the Coriolis and centrifugal forces are absent (i.e., $\phi = 1$ and $\psi = 1$), the equations of motion are the same with those of Hallan and Mangang (2007).

CHAPTER FOUR

LOCATIONS AND LINEAR STABILITY OF EQUILIBRIUM POINTS

4.1 Introduction

The classical restricted three-body problem possesses five equilibrium points, three collinear and two triangular points. Euler (1767) found the collinear points, while Lagrange (1772) found the triangular points. For example, let's assume two massive orbiting system, such as the Sun and the earth, five equilibrium points Li (i=1-5) called the Lagrangian points or libration points or equilibrium points exist. The gravitational pull at these equilibrium points are balanced in such a way that any infinitesimal body in the vicinity of the equilibrium points would be held there without pulling closer to either of the massive bodies. That is, provided there are no perturbations if a celestial body or satellite is located at one of these points with zero initial velocity relative to the rotating coordinate system, it will remain there permanently.

Our aim in this chapter is to investigate the locations and stability of the equilibrium points in the Robe's (1977) CR3BP with small perturbations in the Coriolis and centrifugal forces as in Hallan and Mangang (2007).

4.2 The Equilibrium Points

The equilibrium points are those points where the infinitesimal mass has zero velocity and zero acceleration in the rotating frame. They are therefore the solution of the equations

$$U_x = U_y = U_z = 0, \tag{4.1}$$

That is

$$U_x = D\left[-2\pi\rho_1 \mathrm{A}_1 x + \frac{\mu(1-x)}{[(1-x)^2 + y^2 + z^2]^{\frac{3}{2}}} + \mathrm{n}^2\,\psi(x-\mu)\right] = 0$$

$$U_y = D\left[-2\pi\rho_1 \mathrm{A}_1 y - \frac{\mu y}{[(1-x)^2 + y^2 + z^2]^{\frac{3}{2}}} + \mathrm{n}^2\,\psi y\right] = 0 \qquad (4.2)$$

and

$$U_z = D\left[-2\pi\rho_1 \mathrm{A}_2 z - \frac{\mu \mathrm{z}}{[(1-x)^2 + y^2 + z^2]^{\frac{3}{2}}}\right] = 0$$

4.2.1 Axial equilibrium points

The locations of the axial equilibrium points are the solutions of (4.2) when $U_x = 0$, $x \neq 0,\ y = 0,\ z = 0, D \neq 0$. It implies they lie on the x -axis. Therefore, the first equation of system (4.2) can be written as

$$-2\pi\rho_1 \mathrm{A}_1 x + \frac{\mu(1-x)}{|1-x|^3} + \mathrm{n}^2\,\psi(x-\mu) = 0 \qquad (4.3)$$

Substituting $\mathrm{n}^2 = 1 + \frac{3\alpha}{2}$ and $\psi = 1 + \varepsilon'$,

the $x-$corodinate of the equilibrium points are then the roots of the of equation

$$-2\pi\rho_1 \mathrm{A}_1 x + \frac{\mu(1-x)}{|1-x|^3} + \left(1 + \frac{3\alpha}{2}\right)(x-\mu)(1+\varepsilon') = 0. \qquad (4.4)$$

First we determine the roots of equation (4.4), after neglecting the oblateness term in the absence of small perturbations in the centrifugal force $(i.e\ \alpha = 0\ and\ \varepsilon' = 0)$, then equation (4.4) becomes

$$-2\pi\rho_1 \mathrm{A}_1 x + \frac{\mu(1-x)}{|1-x|^3} + (x-\mu) = 0 \qquad (4.5)$$

or $\quad -2\pi\rho_1 \mathrm{A}_1 x(1-x)^2 + \mu + (x-\mu)(1-x)^2 = 0$

or $\quad -2\pi\rho_1 \mathrm{A}_1 x(1-x)^2 + \mu + x(1-x)^2 - \mu(1-x)^2 = 0$

or $\quad x(1-x)^2[1-2\pi\rho_1 \mathrm{A}_1] + \mu\,[1-(1-x)^2] = 0$

or $\quad x(1-x)^2[1-2\pi\rho_1 \mathrm{A}_1] + \mu x(2-x) = 0 \qquad (4.6)$

This implies that $x = 0$, satisfies equation (4.5). Therefore, when $\alpha = 0 \; and \; \varepsilon' = 0$. The origin which is the center of the first primary is an equilibrium point for all values of the parameters $\mu, \mathrm{A}_1, \mathrm{A}_2, \rho_1 \; and \; D$. We now determine the other equilibrium points satisfying the condition $0 < |x| < 1$, from equation (4.5) we have

$$-2\pi\rho_1 \mathrm{A}_1 x + \frac{\mu}{(1-x)^2} + (x-\mu) = 0$$

Multiplying this equation throughout by $(1-x)^2$, we have

$$-2\pi\rho_1 \mathrm{A}_1 x(1-x)^2 + \mu + (x-\mu)(1-x)^2 = 0$$

Expanding, we have

$$-2\pi\rho_1 \mathrm{A}_1 x(1-2x+x^2) + \mu + (x-\mu)(1-2x+x^2) = 0$$

or $\quad -2\pi\rho_1 \mathrm{A}_1 x + 4\pi\rho_1 \mathrm{A}_1 x^2 - 2\pi\rho_1 \mathrm{A}_1 x^3 + \mu + x - 2x^2 + x^3 - \mu + 2\mu x - \mu x^2 = 0$

or $\quad x[-2\pi\rho_1 \mathrm{A}_1 + 4\pi\rho_1 \mathrm{A}_1 x - 2\pi\rho_1 \mathrm{A}_1 x^2 + 1 - 2x + x^2 + 2\mu - \mu x] = 0 \qquad (4.7)$

Since $x \neq 0$, then

$$-2\pi\rho_1 \mathrm{A}_1 + 4\pi\rho_1 \mathrm{A}_1 x - 2\pi\rho_1 \mathrm{A}_1 x^2 + 1 - 2x + x^2 + 2\mu - \mu x = 0$$

or $\quad x^2 - 2\pi\rho_1 \mathrm{A}_1 x^2 + 4\pi\rho_1 \mathrm{A}_1 x - 2x - \mu x + 1 + 2\mu - 2\pi\rho_1 \mathrm{A}_1 = 0$

or $\quad x^2(1-2\pi\rho_1 \mathrm{A}_1) + (4\pi\rho_1 \mathrm{A}_1 - 2 - \mu)x + 1 + 2\mu - 2\pi\rho_1 \mathrm{A}_1 = 0 \qquad (4.8)$

The roots of equation (4.8) are such that using the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

where $a = (1-2\pi\rho_1 \mathrm{A}_1), \; \mathrm{b} = (4\pi\rho_1 \mathrm{A}_1 - 2 - \mu), \; \text{and c} = 1 + 2\mu - 2\pi\rho_1 \mathrm{A}_1$

$$x = \frac{-(4\pi\rho_1 \mathrm{A}_1 - 2 - \mu) \pm \sqrt{(4\pi\rho_1 \mathrm{A}_1 - 2 - \mu)^2 - 4(1-2\pi\rho_1 \mathrm{A}_1)(1+2\mu-2\pi\rho_1 \mathrm{A}_1)}}{2(1-2\pi\rho_1 \mathrm{A}_1)}$$

$$= \frac{-(4\pi\rho_1 \mathrm{A}_1 - 2 - \mu) \pm \sqrt{\begin{matrix} 16\pi^2 {\rho_1}^2 {\mathrm{A}_1}^2 - 16\pi\rho_1 \mathrm{A}_1 - 8\pi\rho_1 \mathrm{A}_1 \mu + 4 - 4\mu + \mu^2 - 4 \\ +16\pi\rho_1 \mathrm{A}_1 + 16\mu\pi\rho_1 \mathrm{A}_1 - 16\pi^2 {\rho_1}^2 {\mathrm{A}_1}^2 \end{matrix}}}{2(1-2\pi\rho_1 \mathrm{A}_1)}$$

$$= \frac{-(4\pi\rho_1 \mathrm{A}_1 - 2 - \mu) \pm \sqrt{8\pi\rho_1 \mathrm{A}_1 \mu - 4\mu + \mu^2}}{2(1 - 2\pi\rho_1 \mathrm{A}_1)}$$

$$= \frac{2 - 4\pi\rho_1 \mathrm{A}_1 + \mu \pm \sqrt{8\pi\rho_1 \mathrm{A}_1 \mu - 4\mu + \mu^2}}{2(1 - 2\pi\rho_1 \mathrm{A}_1)}$$

$$= \frac{2(1 - 2\pi\rho_1 \mathrm{A}_1) + \mu \pm \sqrt{\mu^2 + 8\mu\pi\rho_1 \mathrm{A}_1 - 4\mu}}{2(1 - 2\pi\rho_1 \mathrm{A}_1)}$$

$$= \frac{2(1 - 2\pi\rho_1 \mathrm{A}_1)}{2(1 - 2\pi\rho_1 \mathrm{A}_1)} + \frac{\mu \pm \sqrt{\mu^2 + 8\mu\pi\rho_1 \mathrm{A}_1 - 4\mu}}{2(1 - 2\pi\rho_1 \mathrm{A}_1)}$$

This implies that

$$x = \frac{1 + \mu \pm \sqrt{\mu^2 + 8\mu\pi\rho_1 \mathrm{A}_1 - 4\mu}}{2(1 - 2\pi\rho_1 \mathrm{A}_1)}$$

Therefore, the roots of equation (4.8) are

$$x_{11} = \frac{1 + \mu + \sqrt{\mu^2 + 8\mu\pi\rho_1 \mathrm{A}_1 - 4\mu}}{2(1 - 2\pi\rho_1 \mathrm{A}_1)} \tag{4.9}$$

$$x_{12} = \frac{1 + \mu - \sqrt{\mu^2 + 8\mu\pi\rho_1 \mathrm{A}_1 - 4\mu}}{2(1 - 2\pi\rho_1 \mathrm{A}_1)}$$

If the discriminant is non-negative, then these roots will be real, that is if

$$\mu^2 + 8\mu\pi\rho_1 \mathrm{A}_1 - 4\mu \geq 0$$

This yield

$$\mu + 8\pi\rho_1 \mathrm{A}_1 - 4 \geq 0 \tag{4.10}$$

Dividing the above equation throughout by 4, we get

$$\frac{\mu}{4} + 2\,\pi\rho_1 \mathrm{A}_1 - 1 \geq 0 \tag{4.11}$$

When $\frac{\mu}{4} \geq 1 - 2\,\pi\rho_1 \mathrm{A}_1 > 0$, both roots are greater than unity and we neglect them because they lie outside the first primary.

Now, if $1 - 2\,\pi\rho_1 \mathrm{A}_1 > 0$, we have $x_{12} > 1\ and\ x_{11} < 1$. Further, we see that $x_{11} > -1$ when $1 - 2\,\pi\rho_1 \mathrm{A}_1 < -\frac{3}{4}\mu$.

Therefore, in the case when $\varepsilon' < -\frac{3}{4}\mu$, the point $(x_{11}, 0,0)$ lies within the oblate spheroid if $|x_{11}| < a_1$. When $1 - 2\pi\rho_1 A_1 < -\frac{3}{4}\mu$, $|x_{11}| < a_1$. x_{11} is a root of equation (4.9).

When $1 - 2\pi\rho_1 A_1 = 0$, the only root is $x_{11} = 2$ which lie outside the first primary and we reject it, for $\alpha \neq 0, and\ \varepsilon' \neq 0, x_{11} = 0$ is always a root

The roots of equation (4.4) when oblateness of the first primary and the effect of perturbation are considered (i.e $\alpha \neq 0, \varepsilon' \neq 0$) are:

$$x = 0 + p_1, \qquad |p_1| \ll 1 \tag{4.12}$$

$$x = x_{11} + p_2 \qquad |p_2| \ll 1$$

Substituting these values of x in equation (4.4), we have

$$-2\pi\rho_1 A_1 p_1 + \frac{\mu}{(1-p_1)^2} + n^2\psi(p_1 - \mu) = 0$$

Multiplying through by $(1 - p_1)^2$ we get

$$(1-p_1)^2\left[-2\pi\rho_1 A_1 p_1 + n^2\psi(p_1 - \mu)\right] + \mu = 0$$

$$(1-2p_1)\left[-2\pi\rho_1 A_1 p_1 + n^2\psi(p_1 - \mu)\right] + \mu = 0$$

Expanding and neglecting 2^{nd} powers of p_1 we get

$$-2\pi\rho_1 A_1 p_1 + n^2\psi(p_1 - \mu) + 2p_1 n^2\psi\mu + \mu = 0$$

Factoring p_1 we get

$$p_1\left[n^2\psi - 2\pi\rho_1 A_1 + 2n^2\psi\mu\right] = n^2\psi\mu - \mu$$

$$\therefore\ p_1 = \frac{\mu(n^2\psi - 1)}{n^2\psi(1+2\mu) - 2\pi\rho_1 A_1}$$

Putting the value of $n^2 = 1 + \frac{3\alpha}{2}$, we get

$$p_1 = \frac{\mu\left[\left(1+\frac{3\alpha}{2}\right)\psi - 1\right]}{\left(1+\frac{3\alpha}{2}\right)\psi(1+2\mu) - 2\pi\rho_1 A_1}$$

$$= \mu\left[\frac{\psi + \frac{3\alpha}{2}\psi - 1}{\left(1+\frac{3\alpha}{2}\right)\psi(1+2\mu) - 2\pi\rho_1 A_1}\right]$$

$$= \frac{-\mu(\psi - 1) + \frac{3\alpha\psi}{2}}{2\pi\rho_1 A_1 - (1+2\mu)(1+\frac{3\alpha}{2})\psi}$$

Putting the value of $\psi = 1 + \varepsilon'$, we get

$$p_1 = \frac{-\mu\left[(1+\varepsilon'-1) + \frac{3\alpha}{2}(1+\varepsilon')\right]}{2\pi\rho_1 A_1 - (1+2\mu)(1+\frac{3\alpha}{2})(1+\varepsilon')}$$

Now, neglecting the product of ε' and α, we get

$$p_1 = \frac{-\mu\left[\varepsilon' + \frac{3\alpha}{2}\right]}{2\pi\rho_1 A_1 - (1+2\mu)\left(1+\frac{3\alpha}{2}+\varepsilon'\right)}$$

$$= \frac{-\mu\left[\varepsilon' + \frac{3\alpha}{2}\right]}{2\pi\rho_1 A_1 - (1+2\mu) - (1+2\mu)\left(\frac{3\alpha}{2}+\varepsilon'\right)}$$

Using binominal expansion in the denominator we have

$$p_1 = \frac{-\mu\left[\varepsilon' + \frac{3\alpha}{2}\right]}{2\pi\rho_1 A_1 - (1+2\mu)\left[1 - \frac{(1+2\mu)\frac{3\alpha}{2} + \varepsilon'}{2\pi\rho_1 A_1 - (1+2\mu)}\right]}$$

$$or \;\; p_1 = \frac{-\mu\left(\varepsilon' + \frac{3\alpha}{2}\right)}{2\pi\rho_1 A_1 - (1+2\mu)}$$

Similarly, putting $x = x_{11} + p_2$ in equation (4.4), we get

$$2\pi\rho_1 A_1(x_{11} + p_2) + \frac{\mu}{(1 - x_{11} - p_2)^2} + n^2\psi(x_{11} + p_2 - \mu) = 0$$

Multiplying through by $(1 - x_{11} - p_2)^2$ we get

$$(1 - x_{11} - p_2)^2\left[-2\pi\rho_1 A_1(x_{11} + p_2) + n^2\psi(x_{11} + p_2 - \mu)\right] + \mu = 0$$

Expanding and neglecting 2nd and higher powers of p_2 as they are very small quantities we have

$$(1 - x_{11})^2 - 2p_2(1 - x_{11})\left[-2\pi\rho_1 A_1 x_{11} - 2\pi\rho_1 A_1 p_2 + n^2\psi p_2 + n^2\psi(x_{11} - \mu)\right] + \mu = 0$$

$$\text{or} \quad [(1 - x_{11})^2 - 2p_2(1 - x_{11})]\left[-2\pi\rho_1 A_1 x_{11} + n^2\psi(x_{11} - \mu) + p_2\left(-2\pi\rho_1 A_1 + n^2\psi\right)\right] + \mu = 0$$

$$\text{or} \quad (1 - x_{11})^2\left[-2\pi\rho_1 A_1 x_{11} + n^2\psi(x_{11} - \mu)\right] + p_2(1 - x_{11})^2\left(-2\pi\rho_1 A_1 + n^2\psi\right) - 2p_2(1 - x_{11})\left[-2\pi\rho_1 A_1 x_{11} + n^2\psi(x_{11} - \mu)\right] + \mu = 0$$

or $(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+n^2\psi(x_{11}-\mu)\right]$

$+p_2(1-x_{11})\left[(1-x_{11})\left(-2\pi\rho_1 A_1+n^2\psi\right)\right.$

$\left.-2\left(-2\pi\rho_1 A_1+n^2\psi\right)(x_{11}-\mu)\right]+\mu=0$

Expanding, we have

$(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+n^2\psi(x_{11}-\mu)\right]$

$+p_2(1-x_{11})\left[-2\pi\rho_1 A_1+n^2\psi+2\pi\rho_1 A_1 x_{11}-n^2\psi x_{11}+4\pi\rho_1 A_1 x_{11}\right.$

$\left.-2n^2\psi x_{11}+2n^2\psi\mu\right]+\mu=0$

or $(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+n^2\psi(x_{11}-\mu)\right]$

$+p_2(1-x_{11})\left[-2\pi\rho_1 A_1+6\pi\rho_1 A_1 x_{11}-3n^2\psi x_{11}+n^2\psi+2n^2\psi\mu\right]+\mu$

$=0$

or $(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+n^2\psi(x_{11}-\mu)\right]$

$+p_2(1-x_{11})\left[-2\pi\rho_1 A_1(1-3x_{11})+n^2\psi(1-3x_{11})+2n^2\psi\mu\right]+\mu=0$

or $(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+n^2\psi(x_{11}-\mu)\right]$

$+p_2(1-x_{11})\left[(1-3x_{11})\left(n^2\psi-2\pi\rho_1 A_1\right)+2n^2\psi\mu\right]+\mu=0$

Now, Substituting $n^2=1+\frac{3\alpha}{2}$, we get

$$(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}\left(1+\frac{3\alpha}{2}\right)\psi(x_{11}-\mu)\right]$$

$$+p_2(1-x_{11})\left[(1-3x_{11})\left\{\left(1+\frac{3\alpha}{2}\right)\psi-2\pi\rho_1 A_1\right\}+2\psi\mu\left(1+\frac{3\alpha}{2}\right)\right]+\mu$$

$$=0$$

or $$(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+\psi(x_{11}-\mu)+\left(1+\frac{3\alpha}{2}\right)(x_{11}-\mu)\right]$$

$$+p_2(1-x_{11})\left[(1-3x_{11})\left\{\psi\left(1+\frac{3\alpha}{2}\right)-2\pi\rho_1 A_1\right\}+2\psi\mu\left(1+\frac{3\alpha}{2}\right)\right]+\mu$$

$$=0$$

or $$(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+\psi(x_{11}-\mu)\right]+\mu+\frac{3\alpha}{2}(x_{11}-\mu)(1-x_{11})$$

$$+p_2(1-x_{11})\left[(1-3x_{11})\left\{\psi\left(1+\frac{3\alpha}{2}\right)-2\pi\rho_1 A_1\right\}+2\psi\mu\left(1+\frac{3\alpha}{2}\right)\right]=0$$

or $$(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+(1+\varepsilon')(x_{11}-\mu)\right]+\mu+\frac{3\alpha\psi}{2}(x_{11}-\mu)(1-x_{11})^2$$

$$+p_2(1-x_{11})\left[(1-3x_{11})\left\{\psi\left(1+\frac{3\alpha}{2}\right)-2\pi\rho_1 A_1\right\}+2\psi\mu\left(1+\frac{3\alpha}{2}\right)\right]=0$$

or $$(1-x_{11})^2\left[-2\pi\rho_1 A_1 x_{11}+(x_{11}-\mu)\right]+\mu+\varepsilon'(1-x_{11})^2(x_{11}-\mu)$$

$$+\frac{3\alpha\psi}{2}(x_{11}-\mu)(1-x_{11})^2$$

$$+p_2(1-x_{11})\left[(1-3x_{11})\left\{\psi\left(1+\frac{3\alpha}{2}\right)-2\pi\rho_1 A_1\right\}+2\psi\mu\left(1+\frac{3\alpha}{2}\right)\right]=0$$

From equation (4.5),we have that

$$-2\pi\rho_1 A_1 x_{11} + \frac{\mu}{(1-x_{11})^2} + (x_{11}-\mu) = 0$$

Multiplying throughout by $(1-x_{11})^2$, we get

$$-2\pi\rho_1 A_1 x_{11}(1-x_{11})^2 + \mu + (x_{11}-\mu)(1-x_{11})^2 = 0$$

or $\quad (1-x_{11})^2\left[(x_{11}-\mu) - 2\pi\rho_1 A_1 x_{11}\right] + \mu = 0$

That implies that

$$p_2(1-x_{11})\left[(1-3x_{11})\left\{\psi\left(1+\frac{3\alpha}{2}\right) - 2\pi\rho_1 A_1\right\} + 2\psi\mu\left(1+\frac{3\alpha}{2}\right)\right]$$

$$= \frac{3\alpha\psi}{2}(x_{11}-\mu)(1-x_{11})^2 - \varepsilon'(1-x_{11})^2(x_{11}-\mu)$$

$$or \qquad p_2 = \frac{-(x_{11}-\mu)(1-x_{11})\left(\frac{3\alpha\psi}{2}+\varepsilon'\right)}{\left[(1-3x_{11})\left\{\left(1+\frac{3\alpha}{2}\right)\psi - 2\pi\rho_1 A_1\right\} + 2\psi\mu\left(1+\frac{3\alpha}{2}\right)\right]}$$

Putting the value of $\psi = 1+\varepsilon'$ we get

$$p_2 = \frac{-(x_{11}-\mu)(1-x_{11})\left(\frac{3\alpha(1+\varepsilon')}{2}+\varepsilon'\right)}{\left[(1-3x_{11})\left\{\left(1+\frac{3\alpha}{2}\right)(1+\varepsilon') - 2\pi\rho_1 A_1\right\} + 2\mu(1+\varepsilon')\left(1+\frac{3\alpha}{2}\right)\right]}$$

Neglecting the product of $\alpha\ and\ \varepsilon'$ as they are very small quantities, we have

$$p_2 = \frac{-(x_{11}-\mu)(1-x_{11})\left(\frac{3\alpha}{2}+\varepsilon'\right)}{\left[(1-3x_{11})\left[1+\frac{3\alpha}{2}+\varepsilon' - 2\pi\rho_1 A_1\right] + 2\mu\left(1+\frac{3\alpha}{2}+\varepsilon'\right)\right]}$$

$$or \quad p_2 = \frac{-(x_{11}-\mu)(1-x_{11})\left(\frac{3\alpha}{2}+\varepsilon'\right)}{\left[(1-3x_{11})(1-2\pi\rho_1 A_1) + 2\mu + (1-3x_{11})\left(1+\frac{3\alpha}{2}+\varepsilon'\right)\right]}$$

$$or \qquad p_2 = \frac{-(x_{11}-\mu)(1-x_{11})\left(\frac{3\alpha}{2}+\varepsilon'\right)}{(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu} \tag{4.13}$$

Putting equation (4.13) in the second equation of (4.12) at once gives the position of the other axial equilibrium point near the center of the first primary.

4.2.2 Locations of circular points

The locations of circular points are evaluated using the first two equations of system (4.2) with the conditions $x \neq 0, y \neq 0, z = 0$ that is, these points lie in the xy-plane and their x, y coordinates are the solutions of equations

$$x\left[-2\pi\rho_1 \mathrm{A}_1 - \frac{\mu}{[(1-x)^2 + y^2]^{\frac{3}{2}}} + n^2\psi\right] + \frac{\mu}{[(1-x)^2 + y^2]^{\frac{3}{2}}} - n^2\psi\mu = 0 \tag{4.14}$$

$$and \qquad -2\pi\rho_1 \mathrm{A}_1 - \frac{\mu}{[(1-x)^2 + y^2]^{\frac{3}{2}}} + n^2\psi = 0$$

Solving the above equations and knowing that $\mu \neq 0$, we get

$$\frac{1}{[(1-x)^2 + y^2]^{\frac{3}{2}}} - n^2\psi = 0 \tag{4.15}$$

If we let

$$(1-x)^2 + y^2 = r^2 \tag{4.16}$$

Substituting equation (4.16) in equation (4.15) gives

$$\frac{1}{[r^2]^{\frac{3}{2}}} - n^2\psi = 0 \quad \text{or} \quad r^3 = n^2\psi \quad or \quad r^3 = \frac{1}{n^2\psi} \tag{4.17}$$

Now substituting $n^2 = 1 + \frac{3\alpha}{2}$ and $\psi = 1 + \varepsilon'$ in equation (4.17) we have

$$r = \frac{1}{\left(1+\frac{3\alpha}{2}\right)^{\frac{1}{3}}(1+\varepsilon')^{\frac{1}{3}}}$$

Using Binomial expansion we get

$$r = \frac{1}{\left(1+\frac{\alpha}{2}\right)\left(1+\frac{\varepsilon'}{3}\right)} \quad \text{or} \quad r = 1-\frac{\alpha}{2}-\frac{\varepsilon'}{3} \tag{4.18}$$

A substitution of equation (4.15) in the 2nd equation of system (4.14) gives

$$-2\pi\rho_1 A_1 - n^2\psi\mu + n^2\psi = 0$$

$$-2\pi\rho_1 A_1 + n^2\psi(1-\mu) = 0 \tag{4.19}$$

Therefore, when $2\pi\rho_1 A_1 = n^2\psi(1-\mu)$, the points on the circle given by equation (4.17) with $z = 0$ and $r = 1-\frac{\alpha}{2}-\frac{\varepsilon'}{3}$ lying within the first primary are also equilibrium points and are called circular points. These points are affected by both the oblateness of the first primary and small perturbation in the centrifugal force.

The general coordinates of circular points are given by $(1 + r\cos\theta, r\sin\theta, 0)$, where θ is a parameter.

4.3 Linear Stability of The Equilibrium Points

When a slight displacement of a particle away from an equilibrium point will not produce unbounded motion, but rather an oscillation (libration) about the point, then that point is said to be stable otherwise unstable.

In order to investigate the linear stability of the equilibrium points, the lyapunov theory is applied to the roots of the characteristic equation, after the variational equation of each equilibrium point is found.

4.3.1 Variational equations

In order to study the linear stability of an orbit in the vicinity of any equilibrium point, we apply the small displacement method by shifting the coordinates of the infinitesimal body and linearizing the equations of motion around the equilibrium point with coordinates(x_0, y_0, z_0).

Let $\xi, \eta \; and \; \zeta$ be small displacement in the coordinates of the equilibruim point (x_0, y_0, z_0) of the infinitesimal mass respectively. Then the third body will be displaced to the point $(x_0 + \xi,\ y_0 + \eta,\ z_0 + \zeta)$ so that

$$x = x_0 + \xi, \qquad y = y_0 + \eta, \qquad z = z_0 + \zeta \tag{4.20}$$

Using Taylor's theorem in system (3.3), the function U can be expanded around the equilibrium point (x_0, y_0, z_0) to give

$$U(x, y, z) = U(x_0 + \xi, \quad y_0 + \eta, \quad z_0 + \zeta)$$

$$= U(x_0, y_0, z_0) + U_x(x_0, y_0, z_0)\xi + U_y(x_0, y_0, z_0)\eta + U_z(x_0, y_0, z_0)\zeta$$

$$+ \frac{1}{2!}U_{xx}(x_0, y_0, z_0)\xi^2 + U_{xy}(x_0, y_0, z_0)\xi\eta + \frac{1}{2!}U_{yy}(x_{10}, x_{20}, x_{30})\eta^2$$

$$+ \frac{1}{2!}U_{zz}(x_0, y_0, z_0)\zeta^2 + + \frac{1}{2!}U_{x_3x_3}(x_0, y_0, z_0)\zeta^2$$

$$+ U_{xz}(x_0, y_0, z_0)\xi\zeta + U_{yz}(x_0, y_0, z_0)\eta\zeta + \cdots \tag{4.21}$$

From system (4.20), we have

$$\dot{x} = \dot{\xi} \qquad \dot{y} = \dot{\eta} \qquad \dot{z} = \dot{\zeta}$$

$$\ddot{x} = \ddot{\xi} \qquad \ddot{y} = \ddot{\eta} \qquad \ddot{z} = \ddot{\zeta} \tag{4.22}$$

Now expanding the right hand side of system (3.3) near the equilibrium point (x_0, y_0, z_0) using the fact that at equilibrium point

$U_x(x_0, y_0, z_0) = U_y(x_0, y_0, z_0) = U_z(x_0, y_0, z_0) = 0,$

We have

$$U_x = U_{xx}(x_0, y_0, z_0)\xi + U_{xy}(x_0, y_0, z_0)\eta + U_{xz}(x_0, y_0, z_0)\zeta + \phi(2) + \phi(3)$$

$$U_y = U_{xy}(x_0, y_0, z_0)\xi + U_{yy}(x_0, y_0, z_0)\eta + U_{yz}(x_0, y_0, z_0)\zeta + \phi(2) + \phi(3) \qquad (4.23)$$

$$U_z = U_{xz}(x_0, y_0, z_0)\xi + U_{yz}(x_0, y_0, z_0)\eta + U_{zz}(x_0, y_0, z_0)\zeta + \phi(2) + \phi(3)$$

here $\phi(2)\ and\ \phi(3)$ denotes second, third and higher terms of $\xi, \eta\ and\ \zeta$ respectively, they can be ignored, making use of equations (4.22) and (4.23) in system (3.3) , we get

$$\ddot{\xi} - 2n\phi\dot{\eta} = (U_{xx}^0)\xi + \left(U_{xy}^0\right)\eta + (U_{xz}^0)\zeta,$$

$$\ddot{\eta} + 2n\phi\dot{\xi} = \left(U_{yx}^0\right)\xi + \left(U_{yy}^0\right)\eta + \left(U_{yz}^0\right)\zeta, \qquad (4.24)$$

$$\ddot{\zeta} = (U_{zx}^0)\xi + \left(U_{zy}^0\right)\eta + (U_{zz}^0)\zeta,$$

These are called the variational equations of motion for system (3.3). Here only the linear terms in $\xi, \eta\ and\ \zeta$ have been taken, the second order partial derivatives of U are denoted by the subscripts and the superscript 0 indicates that the derivatives are to be evaluated at the equilibrium point under consideration.

In order to consider the motion near any equilibrium point in the x, y-plane, we let solutions of the first two equations of system (4.24) be

$$\xi = Lexp(\lambda t),\ \ \eta = Mexp(\lambda t)\ , \qquad (4.25)$$

where $L, M\ and\ \lambda$ are constants

then

$$\dot{\xi} = L\lambda exp(\lambda t), \quad \ddot{\xi} = L\lambda^2 exp(\lambda t)$$

$$\dot{\eta} = M\lambda exp(\lambda t), \quad \ddot{\eta} = M\lambda^2 exp(\lambda t) \qquad (4.26)$$

Now, substituting system (4.26) into the first two equations of system (4.24), we have

$$\lambda^2 Le^{\lambda t} - 2n\phi\lambda Me^{\lambda t} = Le^{\lambda t}U_{xx}^0 + Me^{\lambda t}U_{xy}^0,$$

$$\lambda^2 M e^{\lambda t} + 2n\phi\lambda L e^{\lambda t} = L e^{\lambda t} U_{xy}^0 + M e^{\lambda t} U_{yy}^0.$$

Dividing through by $e^{\lambda t}$, we have

$$\lambda^2 L - 2n\phi\lambda M = L U_{xx}^0 + M U_{xy}^0,$$

$$\lambda^2 M + 2n\phi\lambda L = L U_{xy}^0 + M U_{yy}^0.$$

or

$$[\lambda^2 - U_{xx}^0]L - [2\phi n\lambda + U_{xy}^0]M = 0$$

$$[\lambda^2 - U_{yy}^0]M + [2\phi n\lambda - U_{xy}^0]L = 0 \qquad (4.27)$$

Systems (4.27) can be written in matrix notation as

$$\begin{pmatrix} (\lambda^2 - U_{xx}^0) & -(2\phi n\lambda + U_{xy}^0) \\ (2\phi n\lambda - U_{xy}^0) & (\lambda^2 - U_{yy}^0) \end{pmatrix} \begin{pmatrix} L \\ M \end{pmatrix} = 0$$

They have non-zero solution if

$$\begin{vmatrix} (\lambda^2 - U_{xx}^0) & -(2\phi n\lambda + U_{xy}^0) \\ (2\phi n\lambda - U_{xy}^0) & (\lambda^2 - U_{yy}^0) \end{vmatrix} = 0 \qquad (4.28)$$

Expanding system (4.28), we have

$$[\lambda^2 - U_{xx}^0][\lambda^2 - U_{yy}^0] + [2\phi n\lambda + U_{xy}^0][2\phi n\lambda - U_{xy}^0] = 0$$

or

$$\lambda^4 - \lambda^2 U_{yy}^0 - \lambda^2 U_{xx}^0 + U_{xx}^0 U_{yy}^0 + 4\phi^2 n^2 \lambda^2 - 2\phi n\lambda U_{xy}^0 + 2\phi n\lambda U_{xy}^0 - (U_{xy}^0)^2 = 0$$

or

$$\lambda^4 - (U_{xx}^0 + U_{yy}^0 - 4\phi^2 n^2)\lambda^2 + U_{xx}^0 U_{yy}^0 - (U_{xy}^0)^2 = 0 \qquad (4.29)$$

Equation (4.29) is the characteristic equation corresponding to the variational equation in system (4.24) when motion is considered in the x, y plane.

4.3.2 Stability of axial equilibrium point $(p_1, 0, 0)$

The second order partial derivatives at the equilibrium point $(p_1, 0, 0)$, that is $x = p_1$, $y = 0$ and $z = 0$ are as follows;

From equation (4.2), the second order partial derivatives are

$$U_{xx} = D\left[-2\pi\rho_1 A_1 + \mu(-1)[(1-x)^2 + y^2 + z^2]^{-\frac{3}{2}} - \frac{3}{2}\mu(1-x)[(1-x)^2 + y^2 + z^2]^{-\frac{5}{2}}.2\,(1-x)(-1) + n^2\psi\right]$$

$$= D\left[-2\pi\rho_1 A_1 + \mu\left\{-[(1-x)^2 + y^2 + z^2]^{-\frac{3}{2}} + 3(1-x)^2[(1-x)^2 + y^2 + z^2]^{-\frac{5}{2}}.\right\} + n^2\psi\right]$$

$$= D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{-(1-x)^2 + y^2 + z^2 + 3(1-x)^2}{[(1-x)^2 + y^2 + z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{2(1-x)^2 + y^2 + z^2}{[(1-x)^2 + y^2 + z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right] \quad (4.30)$$

Recall from equation (4.4) that

$$-2\pi\rho_1 A_1 x + \frac{\mu}{(1-x)^2} + n^2\psi(x-\mu) = 0$$

Or

$$-2\pi\rho_1 A_1 = -\frac{\mu}{x(1-x)^2} - \frac{n^2\psi(x-\mu)}{x} \quad (4.31)$$

Substituting equation (5.12) in equation (5.11), we have

$$U_{xx} = D\left[-\frac{\mu}{x(1-x)^2} - \frac{n^2\psi(x-\mu)}{x} + \mu\left\{\frac{2(1-x)^2+y^2+z^2}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

At $(p_1, 0, 0)$ that is $x = p_1,\ y = 0,\ z = 0$, we get

$$U_{xx}^0 = D\left[-\frac{\mu}{p_1(1-p_1)^2} - \frac{n^2\psi(p_1-\mu)}{p_1} + \frac{2\mu}{(1-p_1)^3} + n^2\psi\right]$$

$$= D\left[-\frac{\mu}{p_1(1-p_1)^2} + \frac{2\mu}{(1-p_1)^3} + n^2\psi\left\{\frac{p_1-p_1+\mu}{p_1}\right\}\right]$$

$$= D\mu\left[-\frac{1}{p_1(1-p_1)^2} + \frac{2}{(1-p_1)^3} + \frac{n^2\psi}{p_1}\right]$$

Substituting the value of $p_1 = q\left(\varepsilon' + \frac{3\alpha}{2}\right)$, we have

$$U_{xx}^0 = D\mu\left[-\frac{1}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2} + \frac{2}{\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^3} + \frac{n^2\psi}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)}\right]$$

$$U_{xx}^0 = D\mu\left[\frac{-1+n^2\psi\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2} + \frac{2}{\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^3}\right]$$

Putting the value of $n^2 = 1 + \frac{3\alpha}{2},\ \psi = 1 + \varepsilon'$ *and neglecting the second and* higher power of α, ε' and their products, we have

$$U_{xx}^0 = D\mu\left[\frac{-1+\left(1+\frac{3\alpha}{2}+\varepsilon'\right)\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2} + \frac{2}{\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^3}\right]$$

Using binomial expansion and neglecting the second and higher power of ε' *and* α, we have

$$U_{xx}^0 = D\mu\left[\frac{-1+\left(1+\frac{3\alpha}{2}+\varepsilon'\right)\left[1-2q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1-2q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]} + \frac{2}{\left[1-3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{-1+\left(1+\frac{3\alpha}{2}+\varepsilon'\right)-\left(1+\frac{3\alpha}{2}+\varepsilon'\right)2q\left(\varepsilon' + \frac{3\alpha}{2}\right)}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1-2q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]} + \frac{2}{\left[1-3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

Expanding and neglecting the products of $\varepsilon^{'} \; and \; \alpha$, we have

$$U_{xx}^{0} = D\mu\left[\frac{-1+1+\frac{3\alpha}{2}+\varepsilon^{'}-2q\varepsilon^{'}-3q\alpha}{q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)}+\frac{2}{\left[1-3q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{\frac{3\alpha}{2}+\varepsilon^{'}-2q\varepsilon^{'}-3q\alpha}{q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)}+\frac{2}{\left[1-3q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{(1-2q)\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)}{q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)}+\frac{2}{\left[1-3q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{(1-2q)}{q}+\frac{2}{\left[1-3q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{(1-2q)}{q}+2\left[1+3q\left(\varepsilon^{'}+\frac{3\alpha}{2}\right)\right]\right]$$

$$= D\mu\left[\frac{(1-2q)}{q}+2+6q\varepsilon^{'}+9q\alpha\right] \qquad (4.32)$$

Similarly,

$$U_{xy} = D\left[-\mu(1-x)\times\frac{-3}{2}[(1-x)^2+y^2+z^2]^{\frac{-5}{2}}\times 2y\right]$$

$$= D\left[\frac{3\mu(1-x)y}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right]$$

At $(p_1, 0, 0)$ that is $x = p_1,\ y = 0,\ \ z = 0$, we get

$$U_{xy}^{0} = 0 \qquad (4.33)$$

Also,

$$U_{xz} = D\left[-\mu(1-x)\times\frac{-3}{2}[(1-x)^2+y^2+z^2]^{\frac{-5}{2}}\times 2z+0\right]$$

$$= D\left[\frac{3\mu(1-x)z}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right]$$

At $(p_1, 0, 0)$ that is $x = p_1,\ y = 0,\ \ z = 0$, we get

$$U_{xz}^{0} = 0 \tag{4.34}$$

Similarly,

$$U_{yy} = D\left[-2\pi\rho_1 A_1 - \mu\left\{[(1-x)^2+y^2+z^2]^{-\frac{3}{2}} + \frac{-3}{2}y[(1-x)^2+y^2+z^2]^{-\frac{5}{2}}.2y\right\} + n^2\psi\right]$$

$$= D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{(1-x)^2+y^2+z^2-3y^2}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{(1-x)^2-2y^2+z^2}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right] \tag{4.35}$$

Substituting equation (4.31) in equation (4.35), we get

$$= D\left[-\frac{\mu}{x(1-x)^2} - \frac{n^2\psi(x-\mu)}{x} - \mu\left\{\frac{(1-x)^2-2y^2+z^2}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[-\frac{\mu}{x(1-x)^2} - n^2\psi + \frac{n^2\psi\mu}{x} - \mu\left\{\frac{(1-x)^2-2y^2+z^2}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[-\frac{\mu}{x(1-x)^2} + \frac{n^2\psi\mu}{x} - \mu\left\{\frac{(1-x)^2-2y^2+z^2}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right\}\right]$$

At $x = p_1,\ y = 0,\ z = 0$, we have

$$= D\left[-\frac{\mu}{p_1(1-p_1)^2} + \frac{n^2\psi\mu}{p_1} - \frac{\mu}{(1-p_1)^3}\right]$$

Putting the value of $p_1, we\ have$

$$U_{yy}^{0} = D\left[-\frac{\mu}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2} - n^2\psi + \frac{n^2\psi\mu}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)} - \frac{\mu}{\left[1-q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^3} + n^2\psi\right]$$

$$= D\left[-\frac{\mu}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1 - q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2} + \frac{n^2\psi\mu}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)} - \frac{\mu}{\left[1 - q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^3}\right]$$

$$= D\left[-\frac{\mu + n^2\psi\mu\left[1 - q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1 - q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^2} - \frac{\mu}{\left[1 - q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^3}\right]$$

Using binomial expansion and neglecting the second and higher power of $\varepsilon' and\ \alpha$, we have

$$U_{yy}^0 = D\left[-\frac{\mu + n^2\psi\mu\left[1 - 2q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1 - 2q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]} - \frac{\mu}{\left[1 - 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

Putting the values of $n^2 = 1 + \frac{3\alpha}{2}$, $\psi = 1 + \varepsilon'$ expanding and neglecting the second and higher powers of α, ε' and their products, we have

$$U_{yy}^0 = D\left[\frac{-\mu + \mu\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)\left[1 - 2q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)\left[1 - 2q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]} - \frac{\mu}{\left[1 - 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{-1 + 1 + \frac{3\alpha}{2} + \varepsilon' - 2q\varepsilon' - 3q\alpha}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)} - \frac{\mu}{\left[1 - 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{\frac{3\alpha}{2} + \varepsilon' - 2q\varepsilon' - 3q\alpha}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)} - \frac{1}{\left[1 - 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{(1 - 2q)\left(\varepsilon' + \frac{3\alpha}{2}\right)}{q\left(\varepsilon' + \frac{3\alpha}{2}\right)} - \frac{1}{\left[1 - 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{(1 - 2q)}{q} - \frac{1}{\left[1 - 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\mu\left[\frac{(1 - 2q)}{q} - \left[1 + 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]\right]$$

$$= D\mu\left[\frac{(1-2q)}{q} - 1 - 3q\varepsilon' - \frac{9\alpha q}{2}\right] \qquad (4.36)$$

$$U_{yz} = D\left[-\mu y \times \frac{-3}{2}[(1-x)^2 + y^2 + z^2]^{-\frac{5}{2}}.2z + 0\right]$$

$$= D\left[\frac{3\mu yz}{[(1-x)^2 + y^2 + z^2]^{\frac{5}{2}}}\right]$$

At $x = p_1,\ y = 0,\ z = 0$, we have

$$U_{yz}^0 = 0 \qquad (4.37)$$

$$U_{zz} = D\left[-2\pi\rho_1 A_2 - \mu\left\{[(1-x)^2 + y^2 + z^2]^{-\frac{3}{2}} + z.\frac{-3}{2}[(1-x)^2 + y^2 + z^2]^{-\frac{5}{2}}.2z\right\}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \frac{\mu\{(1-x)^2 + y^2 + z^2 - 3z^2\}}{[(1-x)^2 + y^2 + z^2]^{\frac{5}{2}}}\right]$$

At $x = p_1,\ y = 0,\ z = 0$, we have

$$U_{zz}^0 = D\left[-2\pi\rho_1 A_2 - \frac{\mu}{(1-p_1)^3}\right]$$

Putting the value of $p_1 = q\left(\varepsilon' + \frac{3\alpha}{2}\right)$, we have

$$U_{zz}^0 = D\left[-2\pi\rho_1 A_2 - \frac{\mu}{\left[1 - q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]^3}\right]$$

Using binomial expansion and neglecting the second and higher power of ε' *and* α , we have

$$U_{zz}^0 = D\left[-2\pi\rho_1 A_2 - \frac{\mu}{\left[1 - 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \mu\left[1 + 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]\right]$$

$$= -D\left[2\pi\rho_1 A_2 + \mu\left[1 + 3q\left(\varepsilon' + \frac{3\alpha}{2}\right)\right]\right]$$

$$= -D\left[2\pi\rho_1 A_2 + \mu + 3\mu q\varepsilon^{'} + \frac{9\mu\alpha}{2}\right] \tag{4.38}$$

Using equations (4.32), (4.33), (4.34), (4.36), (4.37) and (4.38) in the variational equations of system (4.24), we have

$$\ddot{\xi} - 2n\phi\dot{\eta} = D\mu\left[\frac{(1-2q)}{q} + 2 + 6q\varepsilon^{'} + 9q\alpha\right]\xi$$

$$\ddot{\eta} + 2n\phi\dot{\xi} = D\mu\left[\frac{(1-2q)}{q} - 1 - 3q\varepsilon^{'} - \frac{9\alpha q}{2}\right]\eta \tag{4.39}$$

$$\ddot{\zeta} = -D\left[2\pi\rho_1 A_2 + \mu + 3\mu q\varepsilon^{'} + \frac{9\mu\alpha}{2}\right]\zeta \tag{4.40}$$

Equation (4.40) is independent of system (4.39), the solution being a bounded periodic function, hence, the motion of the infinitesimal mass along the z- direction is stable.

The characteristic equation corresponding to system (4.39) can be written as:

$$\lambda^4 - k_1\lambda^2 + k_2 = 0\,, \tag{4.41}$$

where

$$k_1 = \left[U_{xx}^0 + U_{yy}^0 - 4 - 8\varepsilon - 6\alpha\right]$$

$$= \left\{D\mu\left[\frac{(1-2q)}{q} + 2 + 6q\varepsilon^{'} + 9q\alpha\right] + D\mu\left[\frac{(1-2q)}{q} - 1 - 3q\varepsilon^{'} - \frac{9\alpha q}{2}\right] - 4 - 8\varepsilon - 6\alpha\right\}$$

$$= \left\{D\mu\left[\frac{2(1-2q)}{q} + 1 + 3q\varepsilon^{'} + \frac{9\alpha q}{2}\right] - 4 - 8\varepsilon - 6\alpha\right\}$$

$$= \left\{D\mu\left[\frac{2(2-4q) + 2q + 6q^2\varepsilon^{'} + 9\alpha q^2}{2q}\right] - 4 - 8\varepsilon - 6\alpha\right\}$$

$$= \left\{ D\mu \left[\frac{4 - 6q + 6q^2\varepsilon' + 9\alpha q^2}{2q} \right] - 4 - 8\varepsilon - 6\alpha \right\}$$

$$= \frac{D\mu}{2q} [4 - 6q + 6q^2\varepsilon' + 9\alpha q^2] - 4 - 8\varepsilon - 6\alpha \qquad (4.42)$$

$$k_2 = \left[U_{xx}^0 U_{yy}^0 \right]$$

$$= D\mu \left[\frac{(1-2q)}{q} + 2 + \ 6q\varepsilon' + 9q\alpha \right] \times D\mu \left[\frac{(1-2q)}{q} - 1 - 3q\varepsilon' - \frac{9\alpha q}{2} \right]$$

$$= \quad D^2\mu^2 \left[\left(\frac{1-2q}{q} \right)^2 - \frac{(1-2q)}{q} - \frac{3q\varepsilon'(1-2q)}{q} - \frac{9q\alpha(1-2q)}{q} + \frac{2(1-2q)}{q} - 2 - 6q\varepsilon' - 9q\alpha + \right.$$

$$\left. \frac{6q\varepsilon'(1-2q)}{q} - 6q\varepsilon' - 18q^2\varepsilon'^2 - 27q^2\varepsilon'\alpha + \frac{9q\alpha(1-2q)}{q} - 9q\alpha - 27q^2\varepsilon'\alpha - \frac{81}{2}q^2\alpha^2 \right]$$

Neglecting the second powers and products of ε' *and* α , we have

$$= D^2\mu^2 \left[\frac{(1-4q+4q^2)}{q^2} - \frac{(1-2q)}{q} + \frac{3q\varepsilon'(1-2q)}{q} + \frac{9q\alpha(1-2q)}{2q} + \frac{2(1-2q)}{q} - 2 - 12q\varepsilon' - 18q\alpha \right]$$

$$= D^2\mu^2 \left[\frac{(1-4q+4q^2)}{q^2} + \frac{(1-2q)}{q} + \frac{3q\varepsilon' - 6q^2\varepsilon'}{q} + \frac{9q\alpha - 18q^2\alpha}{2q} + -2 - 12q\varepsilon' - 18q\alpha \right]$$

Taking L.C.M we have

$$= \ D^2\mu^2 \left[\frac{2 - 8q + 8q^2 + 2q - 4q^2 + 6q^2\varepsilon' - 12q^3\varepsilon' + 9q^2\alpha - 18q^3\alpha \quad -4q^2 - 24q^3\varepsilon' - 36q^3\alpha}{2q^2} \right]$$

$$= \ D^2\mu^2 \left[\frac{2 - 6q + 6q^2\varepsilon' - 36q^3\varepsilon' + 9q^2\alpha - 54q^3\alpha}{2q^2} \right]$$

$$= \ D^2\mu^2 \left[\frac{2 - 6q + 6q^2\varepsilon'(1-6q) + 9q^2\alpha(1-6q)}{2q^2} \right]$$

$$= \frac{D^2\mu^2}{2q^2}[2 - 6q + (1 - 6q)(6q^2\varepsilon' + 9q^2\alpha)] \quad (4.43)$$

The roots of Eq.(4.43) becomes

$$\lambda_{1,2}^2 = \frac{-K_1 \pm \sqrt{\Delta}}{2}$$

Or

$$\lambda_{1,2}^2 = \frac{-\left[\frac{D\mu}{2q}[4 - 6q + 6q^2\varepsilon' + 9\alpha q^2] - 4 - 8\varepsilon - 6\alpha\right] \pm \sqrt{\Delta}}{2}, \quad (4.44)$$

where

The discriminant $\Delta = K_1^2 - 4K_2$

$$K_1^2 = \left[\frac{D\mu}{2q}[4 - 6q + 6q^2\varepsilon' + 9\alpha q^2] - [4 + 8\varepsilon + 6\alpha]\right]^2$$

Expanding term by term, we have

$$K_1^2 = \left[\frac{D\mu}{2q}[4 - 6q + 6q^2\varepsilon' + 9\alpha q^2] - [4 + 8\varepsilon + 6\alpha]\right]\left[\frac{D\mu}{2q}[4 - 6q + 6q^2\varepsilon' + 9\alpha q^2] - [4 + 8\varepsilon + 6\alpha]\right]$$

$$= \left[\frac{D\mu}{2q}[4 - 6q + 6q^2\varepsilon' + 9\alpha q^2]\right]^2 - 2\left[\frac{D\mu}{2q}[4 - 6q + 6q^2\varepsilon' + 9\alpha q^2][4 + 8\varepsilon + 6\alpha]\right] + [4 + 8\varepsilon + 6\alpha]^2$$

$$= \frac{D^2\mu^2}{4q^2}[16 - 48q + 36q^2 + 48q^2\varepsilon' + 72\alpha q^2 - 72q^3\varepsilon' - 108q^3\alpha]$$

$$- \left[\frac{D\mu}{q}(16 + 32\varepsilon + 24\alpha - 24q - 48q\varepsilon - 36q\alpha + 24q^2\varepsilon' + 36\alpha q^2)\right]$$

$$+ 16 + 64\varepsilon + 48\alpha$$

$$= D^2\mu^2\left[\frac{16}{4q^2} - \frac{48q}{4q^2} + \frac{36q^2}{4q^2} + \frac{48q^2\varepsilon'}{4q^2} + \frac{72\alpha q^2}{4q^2} - \frac{72q^3\varepsilon'}{4q^2} - \frac{108q^3\alpha}{4q^2}\right]$$

$$+ D\mu\left[\frac{-16}{q} - \frac{32\varepsilon}{q} - \frac{24\alpha}{q} + 24 + 48\varepsilon + 36\alpha - 24q\varepsilon' - 36q\alpha\right] + 16$$

$$+ 64\varepsilon + 48\alpha$$

$$= D^2\mu^2\left[\frac{4}{q^2} - \frac{12}{q} + 9 + 12\varepsilon' + 18\alpha - 18q\varepsilon' - 27q\alpha\right]$$

$$- \left[\frac{D\mu}{q}(16 + 32\varepsilon + 24\alpha - 24q - 48q\varepsilon - 36q\alpha + 24q^2\varepsilon' + 36\alpha q^2)\right]$$

$$+ 16 + 64\varepsilon + 48\alpha \qquad (4.45)$$

$$-4K_2 = -4\,D^2\mu^2\left[\frac{2 - 6q + 6q^2\varepsilon' - 36q^3\varepsilon' - 9q^2\alpha - 54q^3\alpha}{2q^2}\right]$$

$$= D^2\mu^2\left[\frac{-4}{q^2} + \frac{12}{q} - 12\varepsilon' + 72q\varepsilon' - 18\alpha + 108q\alpha\right] \qquad (4.46)$$

Using equations (4.45) and (4.46), we have

$$\Delta = D^2\mu^2\left[\frac{4}{q^2} - \frac{12}{q} + 9 + 12\varepsilon' + 18\alpha - 18q\varepsilon' - 27q\alpha\right]$$

$$- \left[\frac{D\mu}{q}(16 + 32\varepsilon + 24\alpha - 24q - 48q\varepsilon - 36q\alpha + 24q^2\varepsilon' + 36\alpha q^2)\right]$$

$$+ 16 + 64\varepsilon + 48\alpha + D^2\mu^2\left[\frac{-4}{q^2} + \frac{12}{q} - 12\varepsilon' + 72q\varepsilon' - 18\alpha + 108q\alpha\right]$$

$$= D^2\mu^2[+9 + 54q\varepsilon' + 81q\alpha]D\mu\left[-\frac{16}{q} - \frac{32\varepsilon}{q} - \frac{24\alpha}{q} + 24 + 48\varepsilon + 36\alpha - 24q\varepsilon'\right.$$

$$\left. - 36q\alpha\right] + 16 + 64\varepsilon + 48\alpha$$

$$= 9\,D^2\mu^2 + 27\,D^2\mu^2 q[2\varepsilon' + 3\alpha] - \frac{16}{q}\,D\mu\left[1 + 2\varepsilon + \frac{3\alpha}{2}\right] - 12\,D\mu q[2\varepsilon' + 3\alpha]$$

$$+ 24D\mu\left[1 + 2\varepsilon + \frac{3\alpha}{2}\right] + 16 + 64\varepsilon + 48\alpha$$

$$= 9\,D^2\mu^2 + [2\varepsilon' + 3\alpha][27\,D^2\mu^2 q - 12\,D\mu q] + \left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)\left(24D\mu - \frac{16}{q}\,D\mu\right) + 16$$

$$+ 64\varepsilon + 48\alpha$$

$$= 9\,D^2\mu^2 + \mu q(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D) + \left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)\left(24D\mu - \frac{16}{q}\,D\mu\right) + 16$$

$$+ 64\varepsilon + 48\alpha$$

Substituting the value of

$q = \dfrac{-\mu}{2\pi\rho_1 A_1 - (1 + 2\mu)}$, we have

$$\Delta = 9\,D^2\mu^2 + (2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)\frac{-\mu^2}{2\pi\rho_1 A_1 - (1+2\mu)}$$
$$+ \left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)\left(24D\mu - 16\,D\mu \times \frac{-[2\pi\rho_1 A_1 - (1+2\mu)]}{\mu}\right) + 16 + 64\varepsilon$$
$$+ 48\alpha$$

$$= 9\,D^2\mu^2 + \frac{-\mu^2(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)}$$
$$+ \left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)(24D\mu + 16\,D[2\pi\rho_1 A_1 - (1+2\mu)]) + 16 + 64\varepsilon$$
$$+ 48\alpha$$

$$= 9\,D^2\mu^2 + \frac{-\mu^2(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)} + 24D\mu\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)$$
$$+ \left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)(16\,D[2\pi\rho_1 A_1 - (1+2\mu)]) + 16 + 64\varepsilon + 48\alpha$$

$$= 9\,D^2\mu^2 - \frac{\mu^2(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)} + 24D\mu\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)$$
$$+ 16\,D(2\pi\rho_1 A_1 - 1)\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) - 32D\mu\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) + 16 + 64\varepsilon$$
$$+ 48\alpha$$

$$= 9\,D^2\mu^2 - \frac{\mu^2(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)} - 8D\mu\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right)$$
$$+ 16\,D(2\pi\rho_1 A_1 - 1)\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) + 16 + 64\varepsilon + 48\alpha$$

$$= 9\,D^2\mu^2 - \frac{\mu^2(2\varepsilon^{'} + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)} - 8D\mu - 8D\mu\left(2\varepsilon + \frac{3\alpha}{2}\right)$$
$$+ 16\,D(2\pi\rho_1 A_1)\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) - 16\,D\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) + 16 + 64\varepsilon$$
$$+ 48\alpha$$

$$= 9\,D^2\mu^2 - 8D\mu - 8D\mu\left(2\varepsilon + \frac{3\alpha}{2}\right) - \frac{\mu^2(2\varepsilon^{'} + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)}$$
$$+ 16\,D(2\pi\rho_1 A_1)\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) - 16\,D\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) + 16 + 64\varepsilon$$
$$+ 48\alpha$$

$$= D\mu(9D\mu - 8) - 8D\mu\left(2\varepsilon + \frac{3\alpha}{2}\right) - \frac{\mu^2(2\varepsilon^{'} + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)}$$
$$+ 16\,D(2\pi\rho_1 A_1)\left(1 + 2\varepsilon + \frac{3\alpha}{2}\right) - 16\,D - 16\,D\left(2\varepsilon + \frac{3\alpha}{2}\right) + 16 + 64\varepsilon$$
$$+ 48\alpha$$

$$= D\mu(9D\mu - 8) - \left(2\varepsilon + \frac{3\alpha}{2}\right)(8D\mu + 16D) - \frac{\mu^2(2\varepsilon^{'} + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)}$$
$$+ 16\,D(2\pi\rho_1 A_1) + 16\,D(2\pi\rho_1 A_1)\left(2\varepsilon + \frac{3\alpha}{2}\right) - 16\,(D-1) + 64\varepsilon + 48\alpha$$

$$= -16\,(D-1) + D\mu(9D\mu - 8) - \left(2\varepsilon + \frac{3\alpha}{2}\right)(8D\mu + 16D) + 16\,D(2\pi\rho_1 A_1)\left(2\varepsilon + \frac{3\alpha}{2}\right)$$
$$+ 32\left(2\varepsilon + \frac{3\alpha}{2}\right) - \frac{\mu^2(2\varepsilon^{'} + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)} + 16\,D(2\pi\rho_1 A_1)$$

$$= -16\,(D-1) + D\mu(9D\mu - 8) + \left(2\varepsilon + \frac{3\alpha}{2}\right)[-8D\mu - 16D + 32 + 32D\pi\rho_1 A_1]$$

$$- \frac{\mu^2(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)} + 32D\pi\rho_1 A_1$$

$$= -16\,(D-1) + D\mu(9D\mu - 8) + 2\varepsilon[-8D\mu - 16D + 32 + 32D\pi\rho_1 A_1]$$

$$+ \frac{3\alpha}{2}[-8D\mu - 16D + 32 + 32D\pi\rho_1 A_1] - \frac{\mu^2(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)}$$

$$+ 32D\pi\rho_1 A_1$$

$$= -16\,(D-1) + D\mu(9D\mu - 8) + 16\varepsilon[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1]$$

$$+ 12\alpha[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{\mu^2(2\varepsilon' + 3\alpha)(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)}$$

$$+ 32D\pi\rho_1 A_1$$

$$= -16\,(D-1) + D\mu(9D\mu - 8) + 16\varepsilon[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1]$$

$$+ 12\alpha[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1] + 32D\pi\rho_1 A_1$$

$$- \frac{\mu^2(2\varepsilon')(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)} - \frac{\mu^2 3\alpha(27\,D^2\mu - 12\,D)}{2\pi\rho_1 A_1 - (1+2\mu)}$$

$$= -16\,(D-1) + D\mu(9D\mu - 8) + 32D\pi\rho_1 A_1 + 16\varepsilon[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1]$$

$$+ 12\alpha[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{2\mu^2 D\varepsilon'(27D\mu - 12\,)}{2\pi\rho_1 A_1 - (1+2\mu)}$$

$$- \frac{3\mu^2 D\alpha(27D\mu - 12)}{2\pi\rho_1 A_1 - (1+2\mu)}$$

$$= -16\,(D-1) + D\mu(9D\mu - 8) + 32D\pi\rho_1 A_1 + 16\varepsilon[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1]$$

$$-\frac{2\mu^2 D\varepsilon'(27D\mu - 12\,)}{2\pi\rho_1 A_1 - (1+2\mu)}$$

$$+\alpha\left\{12[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3\mu^2 D(27D\mu - 12)}{2\pi\rho_1 A_1 - (1+2\mu)}\right\}$$

$$\Delta = -16\,(D-1) + D\mu(9D\mu - 8) + 32D\pi\rho_1 A_1 + 16\varepsilon[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1]$$

$$-\frac{2\mu^2 D\varepsilon'(27D\mu - 12\,)}{2\pi\rho_1 A_1 - (1+2\mu)} + \alpha\left\{\frac{12[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1]\times [2\pi\rho_1 A_1 - (1+2\mu)] - 3\mu^2 D(27D\mu - 12)}{2\pi\rho_1 A_1 - (1+2\mu)}\right\} \quad (4.47)$$

When $\mu = 0$, the discriminant Δ becomes

$$\Delta = -16\,(D-1) + 32D\pi\rho_1 A_1 + 16\varepsilon[-2D + 4 + 4D\pi\rho_1 A_1]$$

$$+\alpha\{12[-D\mu - 2D + 4 + 4D\pi\rho_1 A_1]\} \quad > 0$$

$$\because\ |\varepsilon|, |\varepsilon'| and |\alpha| \ll 1$$

For $\mu = 1$, the discriminant Δ reduces to

$$\Delta = -16\,(D-1) + D(9D-8) + 32D\pi\rho_1 A_1 - 16\varepsilon[D + 2D - 4 - 4D\pi\rho_1 A_1]$$

$$-\frac{2D\varepsilon'(27D-12)}{2\pi\rho_1 A_1 - 3}$$

$$-\alpha\left\{\frac{12[-D+2D-4-4D\pi\rho_1 A_1][2\pi\rho_1 A_1+3] + 3D(27D\mu-12)}{2\pi\rho_1 A_1 - 3}\right\}$$

$$< 0 \qquad \because\ |\varepsilon|, |\varepsilon'| and |\alpha| \ll 1$$

Since Δ is a monotonic function of μ in the interval (0,1) and has values opposite in signs at the end points, there is only one value of μ, call the critical value of the mass parameter and we denote it by μ_c in the open interval $0 < \mu < 1$ for which the discriminant vanishes

4.3.2.1 Critical Mass

The critical value of the mass parameter is the value of the mass ratio μ for which the discriminant Δ given in equation (4.47) vanishes. The value of the critical mass parameter is obtained by solving $\Delta = 0$ from equation (4.47).

In the absence of perturbations of the Coriolis and centrifugal forces and oblateness of the first primary, i.e., $\varepsilon = \varepsilon' = \alpha = 0$, and solving for $\Delta = 0$, then μ_c reduces to μ_0 as follows

$$\Delta = -16\,(D-1) + D\mu(9D\mu - 8) + 32D\pi\rho_1 A_1 = 0$$

$$= 9\,D^2\mu^2 - 8D\mu - 16\,(D-1) + 32D\pi\rho_1 A_1 = 0$$

Using quadratic equation, we get

$$\mu = \frac{8D \pm \{(8D)^2 - 4 \times 9D^2[-16\,(D-1) + 32D\pi\rho_1 A_1]\}^{\frac{1}{2}}}{2 \times 9\,D^2}$$

$$= \frac{8D \pm \{64D^2 - 36D^2[-16\,D + 16 + 32D\pi\rho_1 A_1]\}^{\frac{1}{2}}}{18\,D^2}$$

$$= \frac{8D \pm \{64D^2 + 576D^3 - 576D^2 + 32D\pi\rho_1 A_1\}^{\frac{1}{2}}}{18\,D^2}$$

$$= \frac{8D \pm \{576D^3 - 512D^2 + 32D\pi\rho_1 A_1\}^{\frac{1}{2}}}{18\,D^2}$$

$$= \frac{8D \pm \{64D^2(9D - 8) + 32D\pi\rho_1 A_1\}^{\frac{1}{2}}}{18\,D^2}$$

$$= \frac{8D \pm \sqrt{64D^2}\left\{(9D - 8) + \frac{\pi\rho_1 A_1}{2D}\right\}^{\frac{1}{2}}}{18\,D^2}$$

$$= \frac{8D \pm 8D\left\{(9D - 8) + \frac{\pi\rho_1 A_1}{2D}\right\}^{\frac{1}{2}}}{18\,D^2}$$

$$= \frac{8D}{18\,D^2} \pm \frac{8D\left\{(9D - 8) + \frac{\pi\rho_1 A_1}{2D}\right\}^{\frac{1}{2}}}{18\,D^2}$$

$$= \frac{4}{9D} \pm \frac{4\left\{(9D - 8) + \frac{\pi\rho_1 A_1}{2D}\right\}^{\frac{1}{2}}}{9D}$$

Therefore,

$$\mu_0 = \frac{4}{9D} + \frac{4\sqrt{a}}{9D}, \qquad (4.48)$$

where $a = 9D - 8 + \frac{\pi\rho_1 A_1}{2D}$.

When considering perturbations and oblateness, we put $\mu = \mu_0 + \delta = \mu_c$ in equation (4.47) and then equate Δ to zero, where $\delta \ll 1$, we have

$$0 = -16\,(D-1) + D(\mu_0+\delta)(9D(\mu_0+\delta)-8) + 32D\pi\rho_1 A_1$$
$$+ 16\varepsilon[-D(\mu_0+\delta) - 2D + 4 + 4D\pi\rho_1 A_1]$$
$$- \frac{2(\mu_0+\delta)^2 D\varepsilon'(27D(\mu_0+\delta)-12)}{2\pi\rho_1 A_1 - (1+2(\mu_0+\delta))}$$
$$+ \alpha\left\{12[-D(\mu_0+\delta) - 2D + 4 + 4D\pi\rho_1 A_1]\right.$$
$$\left. - \frac{3D(\mu_0+\delta)^2(27D(\mu_0+\delta)-12)}{2\pi\rho_1 A_1 - (1+2(\mu_0+\delta))}\right\}$$

$$= -16\,(D-1) + D\mu_0(9D\mu_0 + 9D\delta - 8) + Dr(9D\mu_0 + 9D\delta - 8) + 32D\pi\rho_1 A_1$$
$$+ 16\varepsilon[-D\mu_0 + D\delta - 2D + 4 + 4D\pi\rho_1 A_1]$$
$$- \frac{2({\mu_0}^2 + 2\mu_0\delta + \delta^2)D\varepsilon'(27D\mu_0 + 27D\delta - 12)}{2\pi\rho_1 A_1 - (1+2\mu_0+2\delta)}$$
$$+ \alpha\left\{12[-D\mu_0 + D\delta - 2D + 4 + 4D\pi\rho_1 A_1]\right.$$
$$\left. - \frac{3D({\mu_0}^2 + 2\mu_0\delta + \delta^2)(27D\mu_0 + 27\delta - 12)}{2\pi\rho_1 A_1 - (1+2\mu_0+2\delta)}\right\} = 0$$

Neglecting the second and higher power of δ and its product with $\alpha,\ \varepsilon\ and\ \varepsilon'$ as they are very small quantity, we get

$$0 = -16\,(D-1) + D\mu_0(9D\mu_0 + 9D\delta - 8) + D\delta(9D\mu_0 - 8) + 32D\pi\rho_1 A_1$$
$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{2(\mu_0+\delta)^2 D\varepsilon'(27D\mu_0 - 12)}{2\pi\rho_1 A_1 - (1 + 2\mu_0 + 2\delta)}$$
$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3D(\mu_0+\delta)^2(27D\mu_0 - 12)}{2\pi\rho_1 A_1 - (1 + 2\mu_0 + 2\delta)}\right\}$$

$$= -16\,(D-1) + D\mu_0(9D\mu_0 + 9D\delta - 8) + D\delta(9D\mu_0 - 8) + 32D\pi\rho_1 A_1$$
$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1]$$
$$- \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12)\left[1 - \frac{2\delta}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right]^{-1}}{[2\pi\rho_1 A_1 - (1 + 2\mu_0)]}$$
$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1]\right.$$
$$\left. - \frac{3D{\mu_0}^2(27D\mu_0 - 12)\left[1 - \frac{2\delta}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right]^{-1}}{[2\pi\rho_1 A_1 - (1 + 2\mu_0)]}\right\}$$

Now, using binomial expansion, we get

$$= -16\,(D-1) + D\mu_0(9D\mu_0 + 9D\delta - 8) + D\delta(9D\mu_0 - 8) + 32D\pi\rho_1 A_1$$
$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1]$$
$$- \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12)\left[1 + \frac{2\delta}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right]}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$
$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1]\right.$$
$$\left. - \frac{3D\mu^2(27D\mu_0 - 12)\left[1 + \frac{2\delta}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right]}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\}$$

Neglecting the products of ε', α and δ, we get

$$0 = -16\,(D-1) + D\mu_0(9D\mu_0 + 9D\delta - 8) + D\delta(9D\mu_0 - 8) + 32D\pi\rho_1 A_1$$
$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$
$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3D{\mu_0}^2(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\}$$

$$= -16\,(D-1) + 9D^2{\mu_0}^2 + 9D^2\mu_0\delta - 8D\mu_0 + 9D^2\mu_0\delta - 8D\delta + 32D\pi\rho_1 A_1$$
$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$
$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3D{\mu_0}^2(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\}$$

$$= -16\,(D-1) + 9D^2{\mu_0}^2 + 18D^2\mu_0\delta - 8D\mu_0 - 8D\delta + 32D\pi\rho_1 A_1$$

$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$

$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3D{\mu_0}^2(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\}$$

$$= -16\,(D-1) - 8D\mu_0\left(1 - \frac{9D\mu_0}{8}\right) - 8D\delta\left(1 - \frac{9D\mu_0}{4}\right) + 32D\pi\rho_1 A_1$$

$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$

$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3D{\mu_0}^2(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\}$$

or

$$-16\,(D-1) - 8D\mu_0\left(1 - \frac{9D\mu_0}{8}\right) + 32D\pi\rho_1 A_1 + 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1]$$

$$- \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$

$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3D{\mu_0}^2(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\}$$

$$= 8D\delta\left(1 - \frac{9D\mu_0}{4}\right)$$

or

$$\delta = \frac{1}{8D\left(1-\frac{9D\mu_0}{4}\right)}\Bigg\{-16\,(D-1) - 8D\mu_0\left(1-\frac{9D\mu_0}{8}\right) + 32D\pi\rho_1 A_1$$

$$+ 16\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{2{\mu_0}^2 D\varepsilon'(27D\mu_0 - 12\,)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$

$$+ \alpha\left\{12[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1] - \frac{3D{\mu_0}^2(27D\mu_0 - 12)}{[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\}\Bigg\}$$

$$\delta = \frac{-2\,(D-1)}{D\left(1-\frac{9D\mu_0}{4}\right)} - \frac{\mu_0\left(1-\frac{9D\mu_0}{8}\right)}{\left(1-\frac{9D\mu_0}{4}\right)} + \frac{4\pi\rho_1 A_1}{\left(1-\frac{9D\mu_0}{4}\right)} + \frac{2\varepsilon[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1]}{D\left(1-\frac{9D\mu_0}{4}\right)}$$

$$- \frac{{\mu_0}^2\varepsilon'(27D\mu_0 - 12\,)}{4\left(1-\frac{9D\mu_0}{4}\right)[2\pi\rho_1 A_1 - (1+2\mu_0)]}$$

$$+ \alpha\left\{\frac{3[-D\mu_0 - 2D + 4 + 4D\pi\rho_1 A_1]}{2\left(1-\frac{9D\mu_0}{4}\right)}\right.$$

$$\left. - \frac{3{\mu_0}^2(27D\mu_0 - 12)}{8\left(1-\frac{9D\mu_0}{4}\right)[2\pi\rho_1 A_1 - (1+2\mu_0)]}\right\} \qquad (4.49)$$

$\mu_c = \mu_0 + \delta$. Therefore, from equations (4.48) and (4.49), we get

$$\mu_c = \frac{4}{9D} + \frac{4\sqrt{a}}{9D} - \frac{2\,(D-1)}{D\left[1-\frac{9D}{4}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]} - \frac{\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\left[1-\frac{9D}{8}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]}{\left[1-\frac{9D}{4}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]}$$

$$+ \frac{4\pi\rho_1 A_1}{\left[1-\frac{9D}{4}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]} + \frac{2\varepsilon\left[-D\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)-2D+4+4D\pi\rho_1 A_1\right]}{D\left[1-\frac{9D}{4}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]}$$

$$- \frac{\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)^2 \varepsilon'\left(27D\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)-12\right)}{4\left[1-\frac{9D}{4}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]\left[2\pi\rho_1 A_1-\left(1+2\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right)\right]}$$

$$+ \alpha\left\{\frac{3\left[-D\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)-2D+4+4D\pi\rho_1 A_1\right]}{2\left[1-\frac{9D}{4}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]}\right.$$

$$\left. - \frac{3\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)^2\left(27D\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)-12\right)}{8\left[1-\frac{9D}{4}\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right]\left[2\pi\rho_1 A_1-\left(1+2\left(\frac{4}{9D}+\frac{4\sqrt{a}}{9D}\right)\right)\right]}\right\}$$

$$= \frac{4}{9D} + \frac{4\sqrt{a}}{9D} - \frac{2\,(D-1)}{D(-\sqrt{a})} - \frac{\left(\frac{4}{9D} + \frac{4\sqrt{a}}{9D}\right)\left[1 - \frac{1}{2} - \frac{\sqrt{a}}{2}\right]}{(-\sqrt{a})} + \frac{4\pi\rho_1 A_1}{(-\sqrt{a})}$$

$$+ \frac{2\varepsilon\left[-\frac{4}{9} - \frac{4\sqrt{a}}{9} - 2D + 4 + 4D\pi\rho_1 A_1\right]}{D(-\sqrt{a})}$$

$$- \frac{\left(\frac{4}{9D} + \frac{4\sqrt{a}}{9D}\right)^2 (12 + 12\sqrt{a} - 12\,)\varepsilon'}{4(-\sqrt{a})\left[2\pi\rho_1 A_1 - 1 - \frac{8}{9D} - \frac{8\sqrt{a}}{9D}\right]}$$

$$+ \alpha\left\{\frac{3\left[-\frac{4}{9} - \frac{4\sqrt{a}}{9} - 2D + 4 + 4D\pi\rho_1 A_1\right]}{2(-\sqrt{a})}\right.$$

$$\left. - \frac{3\left(\frac{4}{9D} + \frac{4\sqrt{a}}{9D}\right)^2 (12 + 12\sqrt{a} - 12\,)}{8(-\sqrt{a})\left[2\pi\rho_1 A_1 - 1 - \frac{8}{9D} - \frac{8\sqrt{a}}{9D}\right]}\right\}$$

$$= \frac{4}{9D} + \frac{4\sqrt{a}}{9D} + \frac{2\,(D-1)}{D\sqrt{a}} + \frac{1}{\sqrt{a}}\left(\frac{4}{9D} + \frac{4\sqrt{a}}{9D}\right)\left[\frac{1}{2} - \frac{\sqrt{a}}{2}\right] - \frac{4\pi\rho_1 A_1}{\sqrt{a}}$$

$$- \frac{2\varepsilon}{D\sqrt{a}}\left[\frac{32}{9} - \frac{4\sqrt{a}}{9} - 2D + 4D\pi\rho_1 A_1\right] + \frac{\left(\frac{4}{9D} + \frac{4\sqrt{a}}{9D}\right)^2 (12\sqrt{a}\,)\varepsilon'}{4\sqrt{a}\left[2\pi\rho_1 A_1 - 1 - \frac{8}{9D} - \frac{8\sqrt{a}}{9D}\right]}$$

$$+ \alpha\left\{-\frac{3}{2\sqrt{a}}\left[\frac{32}{9} - \frac{4\sqrt{a}}{9} - 2D + 4D\pi\rho_1 A_1\right]\right.$$

$$\left. + \frac{3\left(\frac{4}{9D} + \frac{4\sqrt{a}}{9D}\right)^2 (12\sqrt{a}\,)}{8\sqrt{a}\left[2\pi\rho_1 A_1 - 1 - \frac{8}{9D} - \frac{8\sqrt{a}}{9D}\right]}\right\}$$

$$= \frac{4}{9D} + \frac{4\sqrt{a}}{9D} + \frac{2}{\sqrt{a}} - \frac{2}{D\sqrt{a}} + \frac{2}{9D\sqrt{a}} - \frac{2}{9D} + \frac{2}{9D} - \frac{2\sqrt{a}}{9D} - \frac{4\pi\rho_1 A_1}{\sqrt{a}}$$

$$- \frac{2\varepsilon}{D\sqrt{a}}\left[\frac{4}{9}\left(8-\sqrt{a}\right) + 2D(2\pi\rho_1 A_1 - 1)\right] + \frac{3\left(\frac{16}{81D^2} + \frac{32\sqrt{a}}{81D^2} + \frac{16a}{81D^2}\right)\varepsilon'}{\left[2\pi\rho_1 A_1 - 1 - \frac{8}{9D} - \frac{8\sqrt{a}}{9D}\right]}$$

$$+ \alpha\left\{-\frac{3}{2D\sqrt{a}}\left[\frac{4}{9}\left(8-\sqrt{a}\right) + 2D(2\pi\rho_1 A_1 - 1)\right]\right.$$

$$\left. + \frac{9\left(\frac{16}{81D^2} + \frac{32\sqrt{a}}{81D^2} + \frac{16a}{81D^2}\right)}{2\left[2\pi\rho_1 A_1 - 1 - \frac{8}{9D} - \frac{8\sqrt{a}}{9D}\right]}\right\}$$

$$= \frac{4}{9D} + \frac{4\sqrt{a}}{9D} - \frac{16}{9D\sqrt{a}} + \frac{2}{\sqrt{a}}\left(1 - \frac{a}{9D}\right) - \frac{4\pi\rho_1 A_1}{\sqrt{a}}$$

$$- \frac{8\varepsilon}{9D\sqrt{a}}\left[\left(8-\sqrt{a}\right) + \frac{9D}{2}(2\pi\rho_1 A_1 - 1)\right]$$

$$+ \frac{\frac{48}{81D^2}\left(1 + 2\sqrt{a} + a\right)\varepsilon'}{\frac{8}{9D}\left[\frac{9D}{8}(2\pi\rho_1 A_1 - 1) - \left(1+\sqrt{a}\right)\right]}$$

$$+ \alpha\left\{-\frac{12}{18D\sqrt{a}}\left[\left(8-\sqrt{a}\right) + \frac{9D}{2}(2\pi\rho_1 A_1 - 1)\right]\right.$$

$$\left. + \frac{\frac{144}{81D}\left(1 + 2\sqrt{a} + a\right)}{\frac{16}{9D}\left[\frac{9D}{8}(2\pi\rho_1 A_1 - 1) - \left(1+\sqrt{a}\right)\right]}\right\}$$

$$= \frac{4}{9D} + \frac{4\sqrt{a}}{9D} - \frac{16}{9D\sqrt{a}} + \frac{2}{\sqrt{a}}\left(1 - \frac{a}{9D}\right) - \frac{4\pi\rho_1 A_1}{\sqrt{a}}$$

$$- \frac{8\varepsilon}{9D\sqrt{a}}\left[\left(8 - \sqrt{a}\right) + \frac{9D}{2}(2\pi\rho_1 A_1 - 1)\right]$$

$$+ \left\{\frac{3}{8D} \times \frac{1}{4}\frac{\left(1 + 2\sqrt{a} + a\right)\varepsilon'}{\left[\frac{9D}{8}(2\pi\rho_1 A_1 - 1) - \left(1 + \sqrt{a}\right)\right]}\right\}$$

$$+ \alpha\left\{-\frac{2}{3D\sqrt{a}}\left[\left(8 - \sqrt{a}\right) + \frac{9D}{2}(2\pi\rho_1 A_1 - 1)\right]\right.$$

$$\left.+ \left\{\frac{144}{81D} \times \frac{9D}{16}\frac{\left(1 + 2\sqrt{a} + a\right)}{\left[\frac{9D}{8}(2\pi\rho_1 A_1 - 1) - \left(1 + \sqrt{a}\right)\right]}\right\}\right\}$$

$$= \frac{4}{9D} + \frac{4\sqrt{a}}{9D} - \frac{16}{9D\sqrt{a}} + \frac{2}{\sqrt{a}}\left(1 - \frac{a}{9D}\right) - \frac{4\pi\rho_1 A_1}{\sqrt{a}}$$

$$- \frac{8\varepsilon}{9D\sqrt{a}}\left[\left(8 - \sqrt{a}\right) + \frac{9D}{2}(2\pi\rho_1 A_1 - 1)\right]$$

$$+ \left\{\frac{3}{4D}\frac{\left(1 + 2\sqrt{a} + a\right)\varepsilon'}{\left[\frac{9D}{8}(2\pi\rho_1 A_1 - 1) - \left(1 + \sqrt{a}\right)\right]}\right\}$$

$$+ \alpha\left\{-\frac{2}{3D\sqrt{a}}\left[\left(8 - \sqrt{a}\right) + \frac{9D}{2}(2\pi\rho_1 A_1 - 1)\right]\right.$$

$$\left.+ \frac{\left(1 + 2\sqrt{a} + a\right)}{\left[\frac{9D}{8}(2\pi\rho_1 A_1 - 1) - \left(1 + \sqrt{a}\right)\right]}\right\}$$

$$= \frac{4}{9D}\left[1+\sqrt{a}-\frac{4}{\sqrt{a}}+\frac{9D}{2\sqrt{a}}\left(1-\frac{a}{9D}\right)-\frac{9D\pi\rho_1 A_1}{\sqrt{a}}-\frac{\varepsilon}{\sqrt{a}}\left[16-2\sqrt{a}+18D\pi\rho_1 A_1-9D\right]\right.$$

$$+\frac{6\left(1+2\sqrt{a}+a\right)\varepsilon'}{\frac{1}{8}\left[18D\pi\rho_1 A_1-9D-8-8\sqrt{a}\right]}$$

$$+\alpha\left\{-\frac{3}{4\sqrt{a}}\left[16-2\sqrt{a}+18D\pi\rho_1 A_1-9D\right]\right.$$

$$\left.\left.+\frac{\left(1+2\sqrt{a}+a\right)}{\frac{1}{8}\left[18D\pi\rho_1 A_1-9D-8-8\sqrt{a}\right]}\right\}\right]$$

$$= \frac{4}{9D}\left[1+\sqrt{a}-\frac{4}{\sqrt{a}}+\frac{9D}{2\sqrt{a}}\left(1-\frac{a}{9D}\right)-\frac{9D\pi\rho_1 A_1}{\sqrt{a}}-\frac{\varepsilon}{\sqrt{a}}\left[16-2\sqrt{a}+18D\pi\rho_1 A_1-9D\right]\right.$$

$$+\frac{48\left(1+2\sqrt{a}+a\right)\varepsilon'}{\left[18D\pi\rho_1 A_1-9D-8-8\sqrt{a}\right]}$$

$$+\alpha\left\{-\frac{3}{4\sqrt{a}}\left[16-2\sqrt{a}+18D\pi\rho_1 A_1-9D\right]\right.$$

$$\left.\left.+\frac{8\left(1+2\sqrt{a}+a\right)}{\left[18D\pi\rho_1 A_1-9D-8-8\sqrt{a}\right]}\right\}\right]$$

Therefore,

μ_c

$$= \frac{4}{9D}\left[\begin{array}{l} 1+\sqrt{a}-\frac{4}{\sqrt{a}}+\frac{9D}{2\sqrt{a}}\left(1-\frac{a}{9D}\right)-\frac{9D\pi\rho_1 A_1}{\sqrt{a}}--\frac{\varepsilon}{\sqrt{a}}\left[16-2\sqrt{a}+b\right] \\ +\frac{48\left(1+2\sqrt{a}+a\right)\varepsilon'}{\left[b-8-8\sqrt{a}\right]}+\alpha\left\{-\frac{3}{4\sqrt{a}}\left[16-2\sqrt{a}+b\right]+\frac{8\left(1+2\sqrt{a}+a\right)}{\left[b-8-8\sqrt{a}\right]}\right\} \end{array}\right] \quad (4.50)$$

where $b = 18D\pi\rho_1 A_1 - 9D$.

Therefore, equation (4.50) represents the combined effects of perturbations in the Coriolis and centrifugal forces and oblateness of the first primary on the critical mass value.

The nature of the roots in equation (4.44) depends on the discriminant, small perturbations in the Coriolis and centrifugal forces, the oblateness of the first primary and the density parameter D. Therefore we consider three cases of Δ together with changes in K_1.

When $\mu < \mu_c$, Δ is negative for $K_1 > 0$ or $K_1 < 0$, so the real part of two of the roots are positive and equal in both cases, therefore the equilibrium points are unstable.

When $\mu = \mu_c$ and $K_1 < 0$, two of the roots are real and equal, while the remaining two are negative and equal as well. Hence the equilibrium points are unstable. In the situation when $K_1 = 0$ all the roots are zero and the points are unstable. Also when $K_1 > 0$, all the four roots are imaginary with two positive and equal and the other two negative and equal. Therefore the equilibrium points have a positive stable resonance.

When $\mu_c < \mu$, Δ is positive and if $K_1 > 0$, then the roots are distinct and imaginary and the equilibrium points are stable. In the case $K_1 < 0$, the roots are real and distinct and due to a positive root the equilibrium points are unstable. Therefore we conclude that the equilibrium points of the axial point $(p_1, 0,0)$ with small perturbations in the centrifugal force and the density parameter D are unstable for $0 < \mu \leq \mu_c$ and stable for $\mu_c \leq \mu < 1$ depending on the density parameter D, the mass ratio μ, small perturbations given to the Coriolis and centrifugal forces and oblateness of the first primary.

4.3.2.2 Range of stability

In order to get the value of the range of stability we differentiate the discriminant in equation (4.47) with respect to μ i.e.,

$$-16\,(D-1)+D\mu(9D\mu-8)+32D\pi\rho_1A_1+16\varepsilon[-D\mu-2D+4+4D\pi\rho_1A_1]$$
$$-\frac{2\mu^2D\varepsilon'(27D\mu-12\,)}{2\pi\rho_1A_1-(1+2\mu)}$$
$$+\alpha\left\{\frac{12[-D\mu-2D+4+4D\pi\rho_1A_1][2\pi\rho_1A_1-(1+2\mu)]-3\mu^2D(27D\mu-12)}{2\pi\rho_1A_1-(1+2\mu)}\right\}=\Delta$$

$$=-16\,(D-1)+D\mu(9D\mu-8)+32D\pi\rho_1A_1+16\varepsilon[-D\mu-2D+4+4D\pi\rho_1A_1]$$
$$-2\mu^2D\varepsilon'(27D\mu-12\,)[2\pi\rho_1A_1-(1+2\mu)]^{-1}$$
$$+\alpha\{12[-D\mu-2D+4+4D\pi\rho_1A_1][2\pi\rho_1A_1-(1+2\mu)]$$
$$-3\mu^2D(27D\mu-12)\}[2\pi\rho_1A_1-(1+2\mu)]^{-1}$$

$$=-16\,(D-1)+9D^2\mu^2-8D\mu+32D\pi\rho_1A_1+16\varepsilon[-D\mu-2D+4+4D\pi\rho_1A_1]$$
$$-\varepsilon'\{[54D^2\mu^3-24D\mu^2][2\pi\rho_1A_1-(1+2\mu)]^{-1}\}$$
$$+\alpha\{12[-D\mu-2D+4+4D\pi\rho_1A_1][2\pi\rho_1A_1-(1+2\mu)]-81D^2\mu^3$$
$$-36D\mu^2\}[2\pi\rho_1A_1-(1+2\mu)]^{-1}$$

$$\frac{d\Delta}{d\mu} = 18D^2\mu - 8D - 16D\varepsilon$$

$$- \varepsilon'\{2[54\mu^3D^2 - 24\mu^2D][2\pi\rho_1A_1 - (1+2\mu)]^{-2}$$

$$+ [2\pi\rho_1A_1 - (1+2\mu)]^{-1}[162D^2\mu^2 - 48D\mu]\}$$

$$+ \alpha[\{12[-D\mu - 2D + 4 + 4D\pi\rho_1A_1][2\pi\rho_1A_1 - (1+2\mu)] - 81D^2\mu^3$$

$$- 36D\mu^2\} \times 2[2\pi\rho_1A_1 - (1+2\mu)]^{-2}$$

$$+ [2\pi\rho_1A_1 - (1+2\mu)]^{-1}\{-24[-D\mu - 2D + 4 + 4D\pi\rho_1A_1]$$

$$- 12D[2\pi\rho_1A_1 - (1+2\mu)] - 243\mu^2D^2 - 72D\mu\} +]$$

$$= 2D(9D\mu - 4) - 16\varepsilon - \varepsilon'\left\{\frac{2[54\mu^3D^2 - 24\mu^2D]}{[2\pi\rho_1A_1 - (1+2\mu)]^2} + \frac{[162D^2\mu^2 - 48D\mu]}{[2\pi\rho_1A_1 - (1+2\mu)]}\right\}$$

$$+ \alpha\left[\frac{2\{12[-D\mu - 2D + 4 + 4D\pi\rho_1A_1][2\pi\rho_1A_1 - (1+2\mu)] - 18D\mu^2(9D\mu + 4)\}}{[2\pi\rho_1A_1 - (1+2\mu)]^2}\right.$$

$$\left. + \frac{\{-24[-D\mu - 2D + 4 + 4D\pi\rho_1A_1] - 12D[2\pi\rho_1A_1 - (1+2\mu)] - 9D\mu(27D\mu + 8)\}}{[2\pi\rho_1A_1 - (1+2\mu)]}\right]$$

$$= 2D(9D\mu - 4) - 16\varepsilon - \varepsilon'\left\{\frac{2[54\mu^3D^2 - 24\mu^2D]}{[2\pi\rho_1A_1 - (1+2\mu)]^2} + \frac{[162D^2\mu^2 - 48D\mu]}{[2\pi\rho_1A_1 - (1+2\mu)]}\right\}$$

$$+ \alpha\left[\frac{24[-D\mu - 2D + 4 + 4D\pi\rho_1A_1][2\pi\rho_1A_1 - (1+2\mu)]}{[2\pi\rho_1A_1 - (1+2\mu)]^2}\right.$$

$$- \frac{36D\mu^2(9D\mu + 4)}{[2\pi\rho_1A_1 - (1+2\mu)]^2} - \frac{24[-D\mu - 2D + 4 + 4D\pi\rho_1A_1]}{[2\pi\rho_1A_1 - (1+2\mu)]}$$

$$\left. - \frac{-12D[2\pi\rho_1A_1 - (1+2\mu)]}{[2\pi\rho_1A_1 - (1+2\mu)]} - \frac{9D\mu(27D\mu + 8)}{[2\pi\rho_1A_1 - (1+2\mu)]}\right]$$

$$= 2D(9D\mu - 4) - 16\varepsilon$$

$$- \varepsilon'\left\{\frac{2[54\mu^3D^2 - 24\mu^2D] + [162D^2\mu^2 - 48D\mu][2\pi\rho_1A_1 - (1+2\mu)]}{[2\pi\rho_1A_1 - (1+2\mu)]^2}\right\}$$

$$+ \alpha\left[\frac{24[-D\mu - 2D + 4 + 4D\pi\rho_1A_1][2\pi\rho_1A_1 - (1+2\mu)]}{[2\pi\rho_1A_1 - (1+2\mu)]^2}\right.$$

$$- \frac{36D\mu^2(9D\mu + 4) - 9D\mu(27D\mu + 8)[2\pi\rho_1A_1 - (1+2\mu)]}{[2\pi\rho_1A_1 - (1+2\mu)]^2}$$

$$\left. - \frac{24[-D\mu - 2D + 4 + 4D\pi\rho_1A_1] - 12D[2\pi\rho_1A_1 - (1+2\mu)]}{[2\pi\rho_1A_1 - (1+2\mu)]}\right]$$

$$\cong 2D(9D\mu - 4) \qquad (since\ |\varepsilon|, |\varepsilon'|, |\propto| \ll 1)$$

$\frac{d\Delta}{d\mu} > 0 \;\; if \;\; \mu > \frac{4}{9D} \;\; and \;\; \frac{d\Delta}{d\mu} < 0 \;\; if \;\; \mu < \frac{4}{9D}$Therefore, Δ is strictly increasing for $\mu > \frac{4}{9D}$ and strictly decreasing for $\mu < \frac{4}{9D}$. Hence, Δ decreases from 0 to $-\frac{16}{9D}$ approximately as $\mu\ increases$ from 0 to $\frac{4}{9D}$. Further, as μ increases from $\frac{4}{9D}\ to\ \mu_{c.}$. Δ increases from $-\frac{16}{9D}\ to\ 0$.

Table 1: numerical computation of the critical mass $\boldsymbol{\mu_c}$

D	$\varepsilon > 0, \varepsilon' > 0, \alpha > 0$	$\varepsilon > 0, \varepsilon' < 0, \alpha > 0$	$\varepsilon < 0, \varepsilon' > 0, \alpha > 0$	$\varepsilon < 0, \varepsilon' < 0, \alpha > 0$
0.1	6.54866 +14.0593*i*	6.66769 +14.3108i	6.53088 +13.9921i	6.64991 +14.2437i
0.15	3.94983 +9.23492i	4.00284 +9.35176i	3.93798 +9.19204i	3.99099 +9.30887i
0.2	2.7947 +6.89476i	2.82458 +6.96036i	2.78581 +6.8625i	2.81569 +6.92811i
0.25	2.15239 +5.52468i	2.17155 +5.56584i	2.14527 +5.49833i	2.16444 +5.5395 i
0.3	1.74608 +4.63347i	1.75941 +4.6612i	1.74015 +4.61082i	1.75349 +4.63855i
0.35	1.46671 +4.01508i	1.47653 +4.03469i	1.46163 +3.99488i	1.47145 +4.01449i
0.4	1.2631 +3.56881i	1.27064 +3.58315i	1.25866 +3.55029i	1.26619 +3.56464i
0.45	1.10816 +3.2406 i	1.11412 +3.25135i	1.10421 +3.22322i	1.11017 +3.23397i
0.5	0.986233 +3.00017i	0.991075 +3.00835i	0.982677 +2.98352i	0.98752 +2.9917i
0.55	0.887658 +2.83133i	0.891669 +2.83762i	0.884426 +2.81503i	0.888437 +2.82132i
0.6	0.806109 +2.72834i	0.809486 +2.73319i	0.803146 +2.71203i	0.806523 +2.71688i
0.65	0.73722 +2.69676i	0.740104 +2.70047i	0.734485 +2.68i	0.737369 +2.68371i
0.7	0.677787 +2.76153i	0.680279 +2.76431i	0.675247 +2.74364i	0.677739 +2.74643i
0.75	0.62519 +2.99919i	0.627366 +3.00119i	0.62282 +2.97892i	0.624995 +2.98093i
0.8	0.576761 +3.71633i	0.578678 +3.7176i	0.574539 +3.69011i	0.576456 +3.69138i
0.85	0.528251 +10.9424i	0.529953 +10.9428i	0.52616 +10.8619i	0.527862 +10.8622i
0.86	-10.5874	-10.5855	-10.5071	-10.5051
0.87	-5.70779	-5.7056	-5.66334	-5.66115
0.89	-3.42694	-3.42456	-3.39928	-3.39691
0.9	-2.92166	-2.91923	-2.89779	-2.89536
0.95	-2.27456	-2.272	-2.25959	-2.25704
0.99	-0.817355	-0.814785	-0.805601	-0.80303

We have used the software package Mathematica to compute the critical mass values μ_c and it is presented in the table 1 above.Table 1 gives the critical mass values for$0 < D <$

1, and $|\varepsilon| = 0.001, |\varepsilon'| = 0.002, |\alpha| = 0.01$ when the perturbations and oblateness are considered as positive or negative. We can see that when $a \leq 0$ the critical mass does not exist whether $\varepsilon, \varepsilon' and\ \alpha$ is positive or negative but exists for $a > 0$. Therefore, the values of μ_c increases with increase in D and whether the perturbations and oblateness are positive or negative. The presence of the complex and negative number in the table shows that the equilibrium point is unstable.

4.3.3 Stability of the axial equilibrium point $(x_{11} + p_2\ , 0, 0)$

The stability behaviors of the second axial equilibrium point $(x_{11} + p_2\ ,0,0)$ are examined in this section. The second order partial derivatives are as follows

$$U_{xx}^0 = U_{xy}^0 = U_{yz}^0 = 0$$

$$U_{xx} = D\left[-\frac{\mu}{x(1-x)^2} - \frac{n^2\psi(x-\mu)}{x} + \mu\left\{\frac{2(1-x)^2 + y^2 + z^2}{[(1-x)^2 + y^2 + z^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$U_{xx}^0 = D\left[-\frac{\mu}{x(1-x)^2} - \frac{n^2\psi(x-\mu)}{x} + \frac{2\mu}{(1-x)^3} + n^2\psi\right] \quad (i.e., y = z = o)$$

$$= D\left[-\frac{\mu}{x(1-x)^2} - n^2\psi + \frac{n^2\psi\mu}{x} + \frac{2\mu}{(1-x)^3} + n^2\psi\right]$$

$$= D\left[-\frac{\mu}{x(1-x)^2} + \frac{n^2\psi\mu}{x} + \frac{2\mu}{(1-x)^3}\right]$$

$$= D\left[\frac{-\mu(1-x) + n^2\psi\mu(1-x)^3 + 2x\mu}{x(1-x)^3}\right]$$

$$= D\mu\left[\frac{-(1-x) + n^2\psi(1-x)^3 + 2x}{x(1-x)^3}\right]$$

$$= D\mu\left[\frac{-1 + x + n^2\psi(1-x)^3 + 2x}{x(1-x)^3}\right]$$

$$= D\mu\left[\frac{3x - 1 + n^2\psi(1-x)^3}{x(1-x)^3}\right]$$

At $(x_{11} + p_2\ ,0,0)$ we get

$$U_{xx}^0 = D\mu\left[\frac{3(x_{11} + p_2)\ - 1 + n^2\psi(1 - x_{11} - p_2\ )^3}{(x_{11} + p_2\ )(1 - x_{11}p_2\ )^3}\right]$$

$$= D\mu\left[\frac{3(x_{11}+p_2)\ -1+n^2\psi(1-x_{11})^3\left(1-\frac{p_2}{(1-x_{11})}\right)^3}{x_{11}\left(1+\frac{p_2}{x_{11}}\right)(1-x_{11})^3\left(1-\frac{p_2}{(1-x_{11})}\right)^3}\right]$$

Using binomial expansion and neglecting the second and higher power of p_2, we get

$$U_{xx}^0 = D\mu\left[\frac{3(x_{11}+p_2)\ -1+n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)}{x_{11}(1-x_{11})^3\left(1+\frac{p_2}{x_{11}}\right)\left(1-\frac{3p_2}{(1-x_{11})}\right)}\right]$$

$$= D\mu\left[\frac{3(x_{11}+p_2)\ -1+n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)}{x_{11}(1-x_{11})^3\left(1+\frac{p_2}{x_{11}}\ -\frac{3p_2}{(1-x_{11})}\right)}\right]$$

$$= D\mu\left[\frac{\left\{3(x_{11}+p_2)\ -1+n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)\right\}\left(1+\frac{p_2}{x_{11}}\ -\frac{3p_2}{(1-x_{11})}\right)^{-1}}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{\left\{3(x_{11}+p_2)\ -1+n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)\right\}\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\begin{array}{c}\frac{3(x_{11}+p_2)\ \left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)-\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)}{+n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)} \\ x_{11}(1-x_{11})^3 \\ \end{array}\right]$$

Expanding and neglecting the product of $p_2, we\ get$

$$U_{xx}^0 = D\mu\left[\begin{array}{c}3x_{11}-3p_2+\frac{9p_2x_{11}}{(1-x_{11})}+3p_2-1+\frac{p_2}{x_{11}}-\frac{3p_2}{(1-x_{11})} \\ +n^2\psi(1-x_{11})^3\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}-\frac{3p_2}{(1-x_{11})}\right) \\ x_{11}(1-x_{11})^3 \\ \end{array}\right]$$

$$= D\mu\left[\frac{3x_{11} + \frac{9p_2x_{11}}{(1-x_{11})} - 1 + \frac{p_2}{x_{11}} - \frac{3p_2}{(1-x_{11})} + n^2\psi(1-x_{11})^3\left(1-\frac{p_2}{x_{11}}\right)}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{-(1-3x_{11}) + n^2\psi(1-x_{11})^3 + \frac{p_2}{x_{11}} - \frac{3p_2(1-3x_{11})}{(1-x_{11})} - \frac{p_2}{x_{11}}n^2\psi(1-x_{11})^3}{x_{11}(1-x_{11})^3}\right]$$

The last term of the numerator becomes $-\frac{p_2}{x_{11}}(1-x_{11})^3$, since we are neglecting the product of $\varepsilon,\ \varepsilon',\ \alpha\ and\ p_2$, therefore we get

$$U_{xx}^0 = D\mu\left[\frac{-(1-3x_{11}) + n^2\psi(1-x_{11})^3 + \frac{p_2}{x_{11}} - \frac{3p_2(1-3x_{11})}{(1-x_{11})} - \frac{p_2}{x_{11}}(1-x_{11})^3}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{-(1-3x_{11}) + n^2\psi(1-x_{11})^3 + p_2\left[\frac{1}{x_{11}} - \frac{3(1-3x_{11})}{(1-x_{11})} - \frac{(1-x_{11})^3}{x_{11}}\right]}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi(1-x_{11})^3}{x_{11}(1-x_{11})^3}\right.$$

$$\left. + p_2\left[\frac{1}{x_{11}^2(1-x_{11})^3} - \frac{3(1-3x_{11})}{x_{11}(1-x_{11})^4} - \frac{(1-x_{11})^3}{x^2{}_{11}(1-x_{11})^4}\right]\right]$$

$$= D\mu\left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} + p_2\left[\frac{(1-x_{11}) - 3x_{11}(1-3x_{11}) - (1-x_{11})}{x_{11}^2(1-x_{11})^4}\right]\right]$$

$$U_{xx}^0 = D\mu\left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2}{x_{11}}\left[\frac{(1-3x_{11})}{(1-x_{11})^4}\right]\right] \qquad (4.51)$$

Similarly,

$$U_{yy} = D\left[-\frac{\mu}{x(1-x)^2}+\frac{n^2\psi\mu}{x}-\mu\left\{\frac{(1-x)^2-2y^2+z^2}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right\}\right]$$

$$U_{yy}^0 = D\left[-\frac{\mu}{x(1-x)^2}+\frac{n^2\psi\mu}{x}-\frac{\mu}{(1-x)^3}\right] \quad (i.e., y=z=0)$$

$$= D\mu\left[-\frac{1}{x(1-x)^2}+\frac{n^2\psi}{x}-\frac{1}{(1-x)^3}\right]$$

$$= D\mu\left[\frac{-(1-x)+n^2\psi(1-x)^3-x}{x(1-x)^3}\right]$$

$$= D\mu\left[\frac{-1+x+n^2\psi(1-x)^3-x}{x(1-x)^3}\right]$$

$$= D\mu\left[\frac{n^2\psi(1-x)^3-1}{x(1-x)^3}\right]$$

At the point $(x_{11}+p_2=x)$, we get

$$U_{yy}^0 = D\mu\left[\frac{n^2\psi(1-x_{11}-p_2)^3-1}{(x_{11}+p_2)(1-x_{11}-p_2)^3}\right]$$

$$= D\mu\left[\frac{n^2\psi(1-x_{11})^3\left(1-\frac{p_2}{(1-x_{11})}\right)^3-1}{x_{11}\left(1+\frac{p_2}{x_{11}}\right)(1-x_{11})^3\left(1-\frac{p_2}{(1-x_{11})}\right)^3}\right]$$

$$= D\mu\left[\frac{n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)-1}{x_{11}(1-x_{11})^3\left(1+\frac{p_2}{x_{11}}-\frac{3p_2}{(1-x_{11})}\right)}\right]$$

$$= D\mu\left[\frac{\left\{n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)-1\right\}\left(1+\frac{p_2}{x_{11}}-\frac{3p_2}{(1-x_{11})}\right)^{-1}}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{\left\{n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)-1\right\}\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{n^2\psi(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)-\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{n^2\psi(1-x_{11})^3\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}-\frac{3p_2}{(1-x_{11})}\right)-\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{n^2\psi(1-x_{11})^3\left(1-\frac{p_2}{x_{11}}\right)-\left(1-\frac{p_2}{x_{11}}+\frac{3p_2}{(1-x_{11})}\right)}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{n^2\psi(1-x_{11})^3-\frac{p_2n^2\psi(1-x_{11})^3}{x_{11}}-1+\frac{p_2}{x_{11}}-\frac{3p_2}{(1-x_{11})}}{x_{11}(1-x_{11})^3}\right]$$

Neglecting the product of $\varepsilon,\ \varepsilon'\ \alpha\ and\ p_2\ in\ n^2\psi$, therefore we get

$$U_{yy}^0 = D\mu\left[\frac{n^2\psi(1-x_{11})^3-\frac{p_2}{x_{11}}(1-x_{11})^3-1+\frac{p_2}{x_{11}}-\frac{3p_2}{(1-x_{11})}}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{-1+n^2\psi(1-x_{11})^3-\frac{p_2(1-x_{11})^4+p_2(1-x_{11})-3p_2x_{11}}{x_{11}(1-x_{11})}}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{-1+n^2\psi(1-x_{11})^3-\frac{p_2(1-x_{11})^4+p_2-p_2x_{11}-3p_2x_{11}}{x_{11}(1-x_{11})}}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{-1+n^2\psi(1-x_{11})^3-\frac{p_2(1-x_{11})^4+p_2(1-4x_{11})}{x_{11}(1-x_{11})}}{x_{11}(1-x_{11})^3}\right]$$

$$= D\mu\left[\frac{-1}{x_{11}(1-x_{11})^3}+\frac{n^2\psi(1-x_{11})^3}{x_{11}(1-x_{11})^3}-\frac{p_2(1-x_{11})^4+p_2(1-4x_{11})}{x_{11}^2(1-x_{11})^4}\right]$$

$$= D\mu\left[\frac{-1}{x_{11}(1-x_{11})^3}+\frac{n^2\psi}{x_{11}}-\frac{p_2}{x_{11}^2}+\frac{p_2(1-4x_{11})}{x_{11}^2(1-x_{11})^4}\right]$$

$$U_{yy}^0 = D\mu\left[\frac{-1}{x_{11}(1-x_{11})^3}+\frac{n^2\psi}{x_{11}}-\frac{p_2}{x_{11}^2}\left\{1-\frac{(1-4x_{11})}{(1-x_{11})^4}\right\}\right] \qquad (4.52)$$

Similarly,

$$U_{zz} = D\left[-2\pi\rho_1 A_2-\frac{\mu\{(1-x)^2+y^2+z^2-3z^2\}}{[(1-x)^2+y^2+z^2]^{\frac{5}{2}}}\right]$$

$$= D\left[-2\pi\rho_1 A_2-\frac{\mu}{(1-x)^3}\right] \quad i.e., \qquad y=z=0$$

Now at $x = x_{11}+p_2$, we get

$$U_{zz}^0 = D\left[-2\pi\rho_1 A_2-\frac{\mu}{(1-x_{11}-p_2)^3}\right]$$

$$= D\left[-2\pi\rho_1 A_2-\frac{\mu}{(1-x_{11})^3\left(1-\frac{p_2}{(1-x_{11})}\right)^3}\right]$$

$$= D\left[-2\pi\rho_1 A_2-\frac{\mu}{(1-x_{11})^3\left(1-\frac{3p_2}{(1-x_{11})}\right)}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \frac{\mu\left(1-\frac{3p_2}{(1-x_{11})}\right)^{-1}}{(1-x_{11})^3}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \frac{\mu\left(1+\frac{3p_2}{(1-x_{11})}\right)}{(1-x_{11})^3}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \frac{\mu}{(1-x_{11})^3} - \frac{3\mu p_2}{(1-x_{11})^4}\right] \tag{4.53}$$

Substituting equations (4.51), (4.52) and (4.53) in variational equations of system (4.24), we get

$$\ddot{\xi} - 2n\phi\dot{\eta} = D\mu\left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2}{x_{11}}\left[\frac{(1-3x_{11})}{(1-x_{11})^4}\right]\right]\xi$$

$$\ddot{\eta} + 2n\phi\dot{\xi} = D\mu\left[\frac{-1}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2}\left\{1 - \frac{(1-4x_{11})}{(1-x_{11})^4}\right\}\right]\eta \tag{4.54}$$

$$\ddot{\xi} = -D\left[2\pi\rho_1 A_2 + \frac{\mu}{(1-x_{11})^3} + \frac{3\mu p_2}{(1-x_{11})^4}\right]\xi \tag{4.55}$$

Equation (4.55) is independent of system (4.54), the solution being a bounded periodic function, hence, the motion of the infinitesimal mass along the z- direction is stable.

The characteristic equation corresponding to system (4.54) can be written as:

$$\lambda^4 - \left\{ D\mu \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right] \right.$$
$$+ D\mu \left[\frac{-1}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right]$$
$$\left. - 4n^2\phi^2 \right\} \lambda^2 + D^2\mu^2 \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right] \left[\frac{-1}{x_{11}(1-x_{11})^3} \right.$$
$$\left. + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right] = 0. \qquad (4.56)$$

or

$$\lambda^4 - \left\{ D\mu \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} - \frac{1}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} + \frac{n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right. \right.$$
$$\left. \left. - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right] - 4n^2\phi^2 \right\} \lambda^2 + D^2\mu^2 \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} \right.$$
$$\left. - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right] \left[\frac{-1}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right] = 0$$

or

$$\lambda^4 - \left\{ D\mu \left[\frac{-2+3x_{11}}{x_{11}(1-x_{11})^3} + \frac{2n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} - \frac{p_2}{x_{11}^2} + \frac{p_2(1-4x_{11})}{x_{11}^2(1-x_{11})^4} \right] \right.$$
$$\left. - 4n^2\phi^2 \right\} \lambda^2 + D^2\mu^2 \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right] \left[\frac{-1}{x_{11}(1-x_{11})^3} \right.$$
$$\left. + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right] = 0$$

or

$$\lambda^4 - \left\{ D\mu \left[\frac{-2+3x_{11}}{x_{11}(1-x_{11})^3} + \frac{2n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ \frac{3(1-3x_{11})}{(1-x_{11})^4} - \frac{1}{x_{11}} + \frac{(1-4x_{11})}{x_{11}(1-x_{11})^4} \right\} \right] \right.$$
$$\left. - 4n^2\phi^2 \right\} \lambda^2 + D^2\mu^2 \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right] \left[\frac{-1}{x_{11}(1-x_{11})^3} \right.$$
$$\left. + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right] = 0$$

or

$$\lambda^4 - \left\{ D\mu \left[\frac{-2+3x_{11}}{x_{11}(1-x_{11})^3} + \frac{2n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ \frac{3(1-3x_{11})}{(1-x_{11})^4} - \frac{1}{x_{11}} + \frac{(1-4x_{11})}{x_{11}(1-x_{11})^4} \right\} \right] \right.$$
$$\left. - 4n^2\phi^2 \right\} \lambda^2 + D^2\mu^2 \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right] \left[\frac{-1}{x_{11}(1-x_{11})^3} \right.$$
$$\left. + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right] = 0$$

$$\lambda^4 - \left\{ D\mu \left[\frac{-2+3x_{11}}{x_{11}(1-x_{11})^3} + \frac{2n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ \frac{1-x_{11}(1+9x_{11})}{x_{11}(1-x_{11})^4} - \frac{1}{x_{11}} \right\} \right] \right.$$
$$\left. - 4n^2\phi^2 \right\} \lambda^2 + D^2\mu^2 \left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4} \right] \left[\frac{-1}{x_{11}(1-x_{11})^3} \right.$$
$$\left. + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2} \left\{ 1 - \frac{(1-4x_{11})}{(1-x_{11})^4} \right\} \right] = 0$$

or

$$\lambda^4 - \left\{D\mu\left[\frac{-2+3x_{11}}{x_{11}(1-x_{11})^3} + \frac{2+3\alpha+2\varepsilon'}{2x_{11}} - \frac{p_2}{x_{11}^2}\left\{\frac{1-x_{11}(1+9x_{11})}{x_{11}(1-x_{11})^4} - \frac{1}{x_{11}}\right\}\right] - 4 - 6\alpha - 8\varepsilon\right\}\lambda^2 + D^2\mu^2\left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{1+\frac{3\alpha}{2}+\varepsilon'}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4}\right]\left[\frac{-1}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2}\left\{1 - \frac{(1-4x_{11})}{(1-x_{11})^4}\right\}\right] = 0 \quad (4.57)$$

If $\lambda_1^2 and\, \lambda_2^2$ are the two roots of Eq. (4.57) satisfying the following conditions

$$\lambda_1^2 + \lambda_2^2 = D\mu\left[\frac{-2+3x_{11}}{x_{11}(1-x_{11})^3} + \frac{2+3\alpha+2\varepsilon'}{x_{11}} - \frac{p_2}{x_{11}^2}\left\{\frac{1-x_{11}(1+9x_{11})}{x_{11}(1-x_{11})^4} - \frac{1}{x_{11}}\right\}\right] - 4 - 6\alpha - 8\varepsilon < 0 \quad (4.58)$$

and

$$\lambda_1^2\,\lambda_2^2 = D^2\mu^2\left[\frac{-(1-3x_{11})}{x_{11}(1-x_{11})^3} + \frac{1+\frac{3\alpha}{2}+\varepsilon'}{x_{11}} - \frac{3p_2(1-3x_{11})}{x_{11}(1-x_{11})^4}\right]\left[\frac{-1}{x_{11}(1-x_{11})^3} + \frac{n^2\psi}{x_{11}} - \frac{p_2}{x_{11}^2}\left\{1 - \frac{(1-4x_{11})}{(1-x_{11})^4}\right\}\right] > 0 \quad (4.59)$$

then the axial equilibrium point is stable. Therefore, the axial equilibrium point is conditionally stable.

4.3.4 Stability of the circular equilibrium point

The general coordinates of a circular point is given by $(1 + r\cos\theta, r\sin\theta, 0)$, where θ is a parameter. Evaluating the second order partial derivative at the point $(1 + r\cos\theta, r\sin\theta, 0)$, we get

$$U_{xx}^{0} = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{2(1-1-r\cos\theta)^2 - (r\sin\theta)^2}{[(1-1-r\cos\theta)^2 + (r\sin\theta)^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

Recall from equation (4.19) $2\pi\rho_1 A_1 = n^2\psi(1-\mu)$, we get

$$U_{xx}^{0} = D\left[-n^2\psi(1-\mu) + \mu\left\{\frac{2(-r\cos\theta)^2 - (r\sin\theta)^2}{[(-r\cos\theta)^2 + (r\sin\theta)^2]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[-n^2\psi + n^2\psi\mu + \mu\left\{\frac{2r^2\cos^2\theta - r^2\sin^2\theta}{[r^2\cos^2\theta + r^2\sin^2\theta]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[n^2\psi\mu + \mu\left\{\frac{2r^2\cos^2\theta - r^2\sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[n^2\psi\mu + \frac{\mu r^2}{r^5}[2\cos^2\theta - \sin^2\theta]\right]$$

$$= D\left[n^2\psi\mu + \frac{\mu}{r^3}[2\cos^2\theta - \sin^2\theta]\right]$$

$$= D\left[n^2\psi\mu + \frac{\mu}{r^3}[2\cos^2\theta - 1 + \cos^2\theta]\right] \quad since \ \sin^2\theta = 1 - \cos^2\theta$$

$$= D\mu\left[n^2\psi + \frac{1}{r^3}[3\cos^2\theta - 1]\right]$$

But $\frac{1}{r^3} = n^2\psi$

$$U_{xx}^{0} = D\mu\left[n^2\psi + n^2\psi[3\cos^2\theta - 1]\right]$$

$$= D\mu[n^2\psi + 3n^2\psi\cos^2\theta - n^2\psi]$$

$$= D\mu[3n^2\psi\cos^2\theta]$$

$$= 3D\mu n^2\psi\cos^2\theta$$

$$= 3D\mu\left(1 + \frac{3\alpha}{2}\right)(1+\varepsilon')\cos^2\theta$$

$$U_{xx}^{0} = 3D\mu\cos^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right) \qquad (4.60)$$

Similarly,

$$U_{yy} = D\left[-n^2\psi(1-\mu) - \mu\left\{\frac{(1-1-r\cos\theta)^2 - 2r^2\sin^2\theta}{[(1-1-r\cos\theta)^2 + r^2\sin^2\theta]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[-n^2\psi(1-\mu) - \mu\left\{\frac{(-r\cos\theta)^2 - 2r^2\sin^2\theta}{[-r\cos\theta + r^2\sin^2\theta]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[-n^2\psi + n^2\psi\mu - \mu\left\{\frac{r^2\cos^2\theta - 2r^2\sin^2\theta}{[r^2\cos^2\theta + r^2\sin^2\theta]^{\frac{5}{2}}}\right\} + n^2\psi\right]$$

$$= D\left[n^2\psi\mu - \mu r^2\left\{\frac{\cos^2\theta - 2\sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[n^2\psi\mu - \mu r^2\left\{\frac{1-\sin^2\theta - 2\sin^2\theta}{[r^2]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[n^2\psi\mu - \frac{\mu r^2}{r^5}[1 - 3\sin^2\theta]\right]$$

$$= D\left[n^2\psi\mu - \frac{\mu}{r^3}[1 - 3\sin^2\theta]\right]$$

$$= D\mu\left[n^2\psi - \frac{1}{r^3}[1 - 3\sin^2\theta]\right]$$

Using $\frac{1}{r^3} = n^2\psi, we\ have$

$$U_{yy} = D\mu\left[n^2\psi - n^2\psi[1 - 3\sin^2\theta]\right]$$

$$= D\mu[n^2\psi - n^2\psi + n^2\psi\, 3\sin^2\theta]$$

$$= 3D\mu\sin^2\theta\left(1 + \frac{3\alpha}{2}\right)(1+\varepsilon')$$

$$= 3D\mu\sin^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right) \qquad (4.61)$$

Similarly,

$$U^0_{xy} = D\left[\frac{-3\mu(1-1-r\cos\theta)r\sin\theta}{[(1-1-r\cos\theta)^2 + (r\sin\theta)^2]^{\frac{5}{2}}}\right]$$

$$= D\left[\frac{3\mu r^2 \cos\theta \sin\theta}{[(-r\cos\theta)^2 + (r\sin\theta)^2]^{\frac{5}{2}}}\right]$$

$$= D\left[\frac{3\mu r^2 \cos\theta \sin\theta}{[r^2\cos^2\theta + r^2\sin^2\theta]^{\frac{5}{2}}}\right]$$

$$= D\left[\frac{3\mu r^2 \cos\theta \sin\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}}\right]$$

$$= D\left[\frac{3\mu r^2 \cos\theta \sin\theta}{[r^2]^{\frac{5}{2}}}\right]$$

$$= D\left[\frac{3\mu \cos\theta \sin\theta}{r^3}\right]$$

Using $\frac{1}{r^3} = n^2\psi, we\ have$

$$\mathrm{U}_{xy}^{0} = Dn^2\psi 3\mu\cos\theta\sin\theta$$

$$= 3D\mu\cos\theta\sin\theta\left(1 + \frac{3\alpha}{2}\right)(1+\varepsilon')$$

$$= 3D\mu\cos\theta\sin\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right) \qquad (4.62)$$

Similarly,

$$U_{zz}^{0} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{(1-1-r\cos\theta)^2 + r^2\sin^2\theta}{[(1-1-r\cos\theta)^2 + r^2\sin^2\theta]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{(-r\cos\theta)^2 + r^2\sin^2\theta}{[(-r\cos\theta)^2 + r^2\sin^2\theta]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{r^2\cos^2\theta + r^2\sin^2\theta}{[r^2\cos^2\theta + r^2\sin^2\theta]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{r^2(\cos^2\theta + \sin^2\theta)}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{r^2}{[r^2]^{\frac{5}{2}}}\right\}\right]$$

$$= D\left[-2\pi\rho_1 A_2 - \frac{\mu}{r^3}\right]$$

$$= -D\left[2\pi\rho_1 A_2 + \frac{\mu}{r^3}\right]$$

Using $\frac{1}{r^3} = \mu n^2 \psi, we\ have$

$$= -D[2\pi\rho_1 A_2 + \mu n^2 \psi]$$

$$= -D\left[2\pi\rho_1 A_2 + \mu\left(1 + \frac{3\alpha}{2}\right)(1 + \varepsilon')\right]$$

$$= -D\left[2\pi\rho_1 A_2 + \mu\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)\right] \qquad (4.63)$$

Similarly,

$$U_{xz} = D\left[\frac{-3\mu(1 - x_1)x_3}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right]$$

$$U_{xz}^0 = 0 \qquad (4.64)$$

$$U_{yz} = D\left[\frac{3\mu x_2 x_3}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right]$$

$$U_{yz}^0 = 0 \qquad (4.65)$$

Now, using equations (4.60), (4.61), (4.62), (4.63), (4.64) and (4.65) in the characteristic equation (4.29), we get

$$\lambda^4 - \left[3D\mu\cos^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right) + 3D\mu\sin^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right) - 4n^2\phi^2\right]\lambda^2$$
$$+ \left[3D\mu\cos^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)\right]\left[3D\mu\sin^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)\right]$$
$$- \left[3D\mu\sin\theta\cos\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)\right]^2 = 0$$

Or

$$\lambda^4 - \left[3D\mu\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)(\sin^2\theta + \cos^2\theta) - 4n^2\phi^2\right]\lambda^2$$

$$+ 9D^2\mu^2\cos^2\theta\sin^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)^2$$

$$- 9D^2\mu^2\cos^2\theta\sin^2\theta\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)^2 = 0$$

Or

$$\lambda^4 - \left[3D\mu\left(1 + \frac{3\alpha}{2} + \varepsilon'\right)(\sin^2\theta + \cos^2\theta) - 4n^2\phi^2\right]\lambda^2 = 0$$

Or

$$\lambda^4 - \left[3D\mu\left(1 + \frac{3\alpha}{2} + \varepsilon'\right) - 4\left(1 + \frac{3\alpha}{2}\right)(1+\varepsilon)^2\right]\lambda^2 = 0$$

Or

$$\lambda^4 - \left[3D\mu\left(1 + \frac{3\alpha}{2} + \varepsilon'\right) - 4\left(1 + \frac{3\alpha}{2} + 2\varepsilon\right)\right]\lambda^2 = 0$$

Or

$$\lambda^4 - \left[3D\mu + \frac{9D\mu\alpha}{2} + 3D\mu\varepsilon' - 4 - 6\alpha - 8\varepsilon\right]\lambda^2 = 0$$

Or

$$\lambda^4 - \left[3D\mu - 4 + \frac{3\alpha}{2}(3D\mu - 4) + 3D\mu\varepsilon' - 8\varepsilon\right]\lambda^2 = 0$$

Or

$$\lambda^4 - \left[(3D\mu - 4)\left(1 + \frac{3\alpha}{2}\right) + 3D\mu\varepsilon' - 8\varepsilon\right]\lambda^2 = 0 \qquad (4.66)$$

Let $\Lambda = \lambda^2$, the characteristic equation (4.66) becomes

$$\Lambda^2 - \left[(3D\mu - 4)\left(1 + \frac{3\alpha}{2}\right) + 3D\mu\varepsilon' - 8\varepsilon\right]\Lambda = 0$$

Or

$$\Lambda\left\{\Lambda - \left[(3D\mu - 4)\left(1 + \frac{3\alpha}{2}\right) + 3D\mu\varepsilon' - 8\varepsilon\right]\right\} = 0$$

Therefore,

$$\Lambda = 0 \; or \; \Lambda = (3D\mu - 4)\left(1 + \frac{3\alpha}{2}\right) + 3D\mu\varepsilon' - 8\varepsilon$$

Thus, the roots of the characteristic equation (4.66) are

$$\lambda = 0 \; twice \; or \; \lambda = \pm\left[(3D\mu - 4)\left(1 + \tfrac{3\alpha}{2}\right) + 3D\mu\varepsilon' - 8\varepsilon\right]^{1/2}$$

Therefore, circular points are unstable due to the presence of multiple roots.

4.4 Conclusion

We have seen that when the first primary is an oblate spheroid and small perturbations in the centrifugal force are present, the point (p_1,0,0) is always an equilibrium point, and another equilibrium point ($x_{11} + p_2$, 0,0) exists when $1 - 2\pi\rho_1 A_1 < -\frac{3}{4}\mu, |x_{11}| < a_1$ .It was found that these points are different from those of Hallan and Mangang (2007) due to small perturbations in the centrifugal force. Further, points on the circle $(1-x)^2 + y^2 = r^2, z = 0$ lying within the first primary are also equilibrium points. These points, which are called circular points are also affected by oblateness of the first primary and small perturbations in the centrifugal force.

The linear stability of the equilibrium points which we have found in this research work has been discussed. The equations of motion have been linearized, we have used trial solutions and the variational equations are solved to get the characteristic equation of the different equilibrium points.

The stability of the first axial equilibrium point is verified using numerical method; and we found this point to be unstable due to the presence of complex and negative quantity. The increase, decrease or unchanged range of stability depends on the density parameter

D. The characteristic equation is used to analyze the second axial equilibrium point and it found to be conditionally stable, while the circular points are unstable.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

The summary, conclusion and recommendations as regards this study are given in this chapter.

5.2 Summary

To summarize the work in this thesis, we note that; the general introduction and the literature review are given in chapter 1 and 2 respectively. In chapter 3, equation (3.2) gives the equations of motion of the Robe's CR3BP with small perturbations in the Coriolis and centrifugal forces. They differ from those of Hallan and Mangang (2007) due to the presence of perturbations in the Coriolis and centrifugal forces.

In chapter 4, the equilibrium points are investigated by equating the right hand side of system (4.2) to zero. By doing this we have found that the axial equilibrium points $(p_1,0,0)$ and $(x_{11}+p_2,0,0)$ near the centre of the first primary are all equilibrium points. The locations of these axial equilibrium points differ from those of Hallan and Mangang (2007) due to a small perturbation in the centrifugal force. We also found that the circular points lying within the first primary under certain conditions are also equilibrium points. These points are influenced by the oblateness of the first primary and the perturbation of the centrifugal force. The linear stability of the equilibrium points are examined. We have seen that the axial equilibrium points are conditionally stable while the circular points are unstable.

5.3 Conclusion

By considering the hydrostatic equilibrium figure of the first primary as an oblate spheroid, the full buoyancy force of the fluid acting on the infinitesimal mass and the presence of perturbations in the Coriolis and centrifugal forces, the existence of an axial equilibrium point $(p_{1,}0,0)$ is shown. Another axial equilibrium point $(x_{11}+p_2,0,0)$ is found within the first primary on the line joining the centres of the primaries when $1-2\pi\rho_1 A_1 < -\frac{3\mu}{4}$ and $|x_{11}| < a_1$. We have also discussed a special case in which the fluid and the infinitesimal mass have equal densities; it is found that all the points inside the fluid in this case are in equilibrium. When $2\pi\rho_1 A_1 = n^2\ \psi(1-\mu)$, points on the circle $\mathrm{r}^2 = (1-x)^2 + \mathrm{y}^2$,$z=0$ lying within the first primary are also equilibrium points – these are also known as circular points. It is found that the axial equilibrium points are conditionally stable and the circular points are unstable.

All the results fully coincide with those of Hallan and Mangang (2007) if there are no perturbations in the Coriolis and centrifugal forces.

5.4 Recommendations

The research work can be extended by considering

1. The non linear stability of the equilibrium points .
2. The Robe's CR3BP with variable mass.

REFERENCES

AbdulRaheem, A. and Singh J. (2006): Combined effects of perturbations, radiation and oblateness on the stability of equilibrium points in the R3BP, *Astronomical journal*, Vol. 131, 1880-1885.

Bhatnagar, K. B. and Hallan, P.P. (1978): Effect of perturbation in the Coriolis and centrifugal forces on the stability of equilibrium points in the R3BP, *Celestial mechanics and dynamical astronomy*, Vol.18, 105 – 112.

Contopoulus G. (2012): *Order and chaos in dynamical Astronomy*, Springer, Berlin.

Encyclopedia Britainica Ultimate. (2014): *Centrifugal force*, Reference Suite. Chicago

Encyclopedia Britainica Ultimate. (2014): *Coriolis force*, Reference Suite. Chicago

Euler, L., (1767): *The motion in the rectilinear three-body problem*, Nov. Comm. Petrop. Vol.11, 144-149

Hallan, P.P. and Rana, N. (2001a): Existence and stability of equilibrium points in the Robe's restricted problem, *Celestial mechanics and dyna mical astronomy*, Vol.79, 145-155

Hallan, P.P. and Rana, N. (2001b): Effect of pertubations in the Coriolis and centrifugal forces on the location and stability of equilibrium point in Robe's circular restricted three-body problem, *Planetary and space science*, Vol. 49 957-960

Hallan, P.P. and Mangang K.B. (2007), Existence and linear stability of equilibrium in the Robe's restricted three-body problem when the first primary is an oblate spheroid, *planetary and space science*, Vol.55, 512-516.

Lagrange,J.,(1772):Collected works of Lagrange, Vol .9.panekoucke,Paris

McCuskey, S. W. (1963): *Introduction to Celestial Mechanics*. Addison Wesley, New York.

Plastino, A.R. and Plastino, A. (1995): Robe's restricted three-body problem revisited, *Celestial mechanics and dynamical astronomy,* Vol.61, 197-206.

Robe, H.A.G., (1977): A new kind of three-problem, *Celestial Mechanics*, Vol.16, 343-349.

Shrivastava, A.K. and Garain, D. (1991): "Effect of perturbation on the location of libration point in the Robe's restricted problem of three bodies", *Celestial mechanics and dynamical astronomy,* Vol.51, 67- 71

Shu, S. and Lu, B. (2005): "Effect of perturbation of the Coriolis and centrifugal forces on the location and linear stability of the libration points in Robe's problem," *Chinese Astronomy and Astrophysics Mechanics*, Vol. 29, 421-429

Singh, J. (2012): Collected notes on space dynamics, ABU Zaria.

Singh, J. (2013): The equilibrium points in the perturbed R3BP with triaxial and luminous primaries, *Astrophysics and Space Science*,Vol.346, 41 – 49.

Singh, J. and Ishwar, B. (1984): Effects of small perturbations in the Coriolis and centrifugal forces on the locations of points in the restricted problem of three bodies with variable mass, *Celestial mechanics and dynamical astronomy*, Vol.32, 297 – 305.

Singh, J. and Ishwar, B. (1985): Effects of small perturbations in the Coriolis and centrifugal forces on the stability of triangular points in the restricted problem of three bodies with variable mass, *Celestial mechanics and dynamical astronomy*, Vol. 35, 201 – 207.

Singh, J. (2009): Combined effects of oblateness and radiation on the nonlinear stability of L_4 in the restricted three-body problem. *Astronomical Journal*, Vol.137, 3286-3293.

Singh, J. (2011): Nonlinear stability of equilibrium points in the restricted three-body problem with oblate and variable mass, *Astrophysics and Space Science*, Vol. 333, 69-71.

Singh, J. and Begha, J.M. (2011): Stability of equilibrium points in the generalized perturbed restricted three-body problem, *Astrophysics and Space Science*, Vol.331, 511-519.

Singh, J. and Leke, O., (2013): Equilibrium points and stability in the restricted three-body problem with variable masses, *Astrophysics and Space Science*, Vol.2013, 8-15

Singh, J. and Mohammed H. L. (2013): Robe's circular restricted three-body problem under oblate and triaxial primaries, *Earth, Moon and Planets, Vol. 109, 1-11.*

Singh, J. and Omale, J. A. (2014): Robe's circular restricted three-body problem with zonal harmonics. *Astrophysics and Space Science*, Vol.353, 89–96.

Singh, J. and Sandah A.U. (2012): Existence and linear stability of equilibrium points in the Robe's restricted three-body problem with oblateness, *Advances in Mathematical Physics*, Vol.2012, 1-13.

SubbaRao, P.V. and Sharma, R.K. (1975): A note on the stability of points in the restricted three – body problem, *Astronomical Astrophysics*, Vol.43, 381 – 383.

Szebehely,V.V. (1967a): *Stability of points in the restricted problem*, Astronomical Journal, Vol.72 ,7 – 9.

Szebehely,V.V. (1967b): Theory of orbits; The restricted problem of three bodies. *Academic press*, New York

Wintner, A. (1941): The Analytical Foundation of Celestial Mechanics. *Princeton university press, Princeton New Jersey*, 372-373