ROBE'S CIRCULAR RESTRICTED THREE-BODY PROBLEM WITH ZONAL HARMONICS AND A ROCHE ELLIPSOID-TRIAXIAL SYSTEM

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M.Sc/SCI/714/2011-2012

DEPARTMENT OF MATHEMATICS

AHMADU BELLO UNIVERSITY, ZARIA

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DECLARATION

I hereby declare that the work in this thesis entitled "Robe's circular restricted three-body problem with zonal harmonics and a Roche ellipsoid-triaxial system" has been performed by me in the Department of Mathematics under the supervision of Prof. J. Singh and Dr. B. Sani. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at any institution.

Achonu Joseph Omale

Name of student	Signature	Date

CERTIFICATION

This thesis entitled "ROBE'S CIRCULAR RESTRICTED THREE-BODY PROBLEM WITH ZONAL HARMONICS AND A ROCHE ELLIPSOID-TRIAXIAL SYSTEM" by Omale, Joseph Achonu (M.Sc/Sci/714/2011-2012) meets the regulations governing the award of Master of Science of Ahmadu Bello University, Zaria, and is approved for its contributions to knowledge and literary presentation.

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I return all glory to Almighty God for his grace, mercy and favour upon my life

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DEDICATION

This research work is dedicated to God Almighty for his grace and to all my siblings; Omale Enema, Akubo Faith Ojone, Omale Ebi-Ojo, Omale Ibe and Omale Ezekiel, also to my cousin Ahmodu Peter.

ABSTRACT

This research work analyzes the motion of an infinitesimal mass in the framework of Robe's circular restricted three-body problem in two cases: (*i*) when the hydrostatic equilibrium figure of the first primary is an oblate spheroid, the shape of the second primary is considered as an oblate spheroid with oblateness coefficients up to the second zonal harmonic, and (*ii*) when the primary bodies form a Roche ellipsoid-triaxial system. Without ignoring any component in both cases, a full treatment is given to the buoyancy force. The relevant equations of motion are established, and a special case where the density of the fluid and that of the infinitesimal mass are equal ($D = 0$) is discussed. It is observed in the first case that there are two axial libration points on the line joining the centers of the primaries, points on the circle within the first primary are also libration points under certain conditions. It is further found that the first axial point is stable, while the second one is conditionally stable, and the circular points are unstable. The location of the libration point and its stability when the infinitesimal mass is denser than the medium ($D > 0$), in the second case, are also studied and it is found that the origin (0, 0, 0) of the system is the only libration point, and this point is stable.

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LIST OF NOTATIONS

m_1	Mass of the first primary……..24
m_2	Mass of the second primary…….……...…….24
m_3	Mass of the infinitesimal body...…….…….....24
ρ_1	Density of the fluid of the first primary……..…….24
ρ_3	Density of the infinitesimal mass…….……..……..24
J_2	Expression for first zonal harmonic……25
J_4	Expression for second zonal harmonic……25
$\sigma_1 \ and \ \sigma_2$	Characterize triaxiality of the second primary……...……..…....25
B	Potential due to the fluid of the first primary...……..……..…26
B'	Potential due to the second oblate primary……..26
n	Mean motion……26
α_1	Oblateness coefficient of the first primary ….……...…….27
α_2	Oblateness coefficient of the second primary …….…...…….27
α_3	Oblateness coefficient of the second primary with zonal harmonics..…….27
$\alpha, \ \beta \ and \ \gamma$	Small displacement……..……....55
λ	Constant……..……..57

CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

The most celebrated problem of space dynamics is the problem of three bodies, known as the three-body problem (3BP). The problem is defined in terms of three bodies with arbitrary masses attracting one another according to Newtonian law of gravitation, and is free to move in space. A classical example of 3BP is the Sun-Earth-Moon system, when they are considered as point masses; they form the main problem of the lunar theory. Another approximate example of the 3BP is the Earth, the Moon and the space vehicle in the Earth-Moon space.

In the general 3BP, 18 first order, coupled, nonlinear differential equations govern the motion. However, only ten integrals of the motion are known to exist; they are derived from the conservation of linear momentum, angular momentum and energy. Thus, the equations of motion are not solvable analytically. In an attempt to solve the problem, Lanrange reduced the 3BP to the restricted three-body problem (R3BP), where one of the bodies is assumed to posses' infinitesimal mass. The R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of two finite masses, called primaries, which move in circular orbits around the center of mass on the account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries.

The difference between the general 3BP and the R3BP is first of all that in the latter the masses of only two particles are arbitrary; the third mass is much smaller than the other two. The general problem allows any sets of initial conditions for the three particles involved; the restricted problem requires circular orbits for the primaries and requires that

the motion of the infinitesimal body takes place in the plane defined by the motion of the primaries.

The R3BP is one of the most widely studied areas in space dynamics as well as celestial mechanics; and very significant results have been produced by well known mathematicians and scientists in an attempt to understand and predict the motion of natural bodies. The application of R3BP spans solar system dynamics, lunar theory, motion of space craft and stellar dynamics. A typical example of the R3BP is seen in a system made up of the Sun and Jupiter as primaries, and then a Trojan asteroid assuming the role of the infinitesimal mass in the Sun-Jupiter system.

The Robe's R3BP, referred to as Robe's problem, is a peculiar case of R3BP, where the infinitesimal mass is embedded in the first primary which is filled with homogenous incomparable fluid of known density, the second primary is a point mass located outside an orbiting shell. The infinitesimal mass is under the gravitational attraction of the primaries as well as buoyancy force due to the fluid. The study of the oscillations of the Earth's inner core taking into cognizance the attraction of the Moon; and the motion of an artificial Earth satellite located inside another satellite, constitute models for the Robe's problem. This Robe's problem has been varied by the introduction of perturbing forces such as oblatenes and triaxiality.

1.2 Statement of The Problem

The present problem describes the motion of a third body of an infinitesimal mass in the gravitational field of two massive bodies which move in a circular orbits around their center of mass on account of their mutual attraction. In the first problem, the first primary m_1 is a shell filled with an incompressible homogenous fluid of density ρ_1 such that, the hydrostatic equilibrium figure of the first primary is an oblate spheroid; the second primary

m_2 is also considered as an oblate spheroid with oblateness coefficients up to the second zonal harmonic, located outside the shell and describes a circular orbit around the first one and the infinitesimal mass m_3 is a solid sphere of density ρ_3, located inside the shell. In the second problem, we assume the first primary m_1 is a fluid of density ρ_1 in the shape of a Roche ellipsoid, the second primary m_2 be a triaxial rigid body. Our effort in this research work is to study the combined effect of the full buoyancy force of the fluid of the first primary and (*i*) the oblateness of the primaries, (*ii*) the triaxiality of the second primary; on their locations and linear stability of the libration points.

1.3 Aim and Objectives of Study

The aim of this study is to investigate the motion of an infinitesimal mass in the framework of Robe's circular restricted three-body problem in two cases: (*i*) when the hydrostatic equilibrium figure of the first primary is an oblate spheroid, the shape of the second primary is considered as an oblate spheroid with oblateness coefficients up to the second zonal harmonic, and (*ii*) when the primary bodies form a Roche ellipsoid-triaxial system.

The objectives of the study are to

i) Establish the equations of motion of the infinitesimal mass of the stated problems

ii) Examine possible libration points,

iii) Investigate the effect of zonal harmonic and triaxiality of the primaries on the possible libration points and their stability.

1.4 Significant / Justification of the Study

In the classical restricted three-body problem, the participating bodies are assumed to be spherical in shape, but we found that in reality, several bodies are sufficiently oblate or triaxial. Earth, Jupiter, Saturn and Ragulus are oblate while Moon, Pluto and Charon are

triaxial. The meteoroids and the minor planets have irregular shapes. The lack of spherisity oblateness, or triaxiality of the celestial bodies causes large perturbations from the two-body orbit. Examples of this are the motions of artificial Earth satellite. Many researchers like Subbarao and Sharma (1975), Singh and Ishwar (1999), Singh and Begha (2011), Singh and Leke (2012), Singh and Sandah (2012), Singh and Mohammed (2013) and Singh and Omale (2014) have included obaleteness and triaxiality in their study of R3BP.

It is reasonable to modify the model by considering first, the hydrostatic equilibrium figure of the first primary as an oblate spheroid, the shape of the second one as oblate spheroid too with oblateness coefficients up to the second zonal harmonic. The shape of the first primary in the second case is considered as Roche-ellipsoid and the second primary as a triaxial rigid body; and the full buoyancy of the fluid is taken into account as well as other forces acting on the infinitesimal mass when the density parameter is not zero $(i.e., \rho_1 \neq \rho_3)$.

This Robe's model under consideration will have many applications in various astronomical problems as it may provide insight into the problem of small oscillations of the earth's core in the gravitational field of Earth-Moon system, and the stability of the Earth centre as an libration point of the Robe's problem and the motion of the artificial satellites in the Earth-Moon vicinity. It can also be applicable in space mission design.

1.5 Research Methodology

The equations of motion of the infinitesimal body with respect to the primaries are established by adopting a synodic coordinate system.

1.6 Preliminary Ideas

1.6.1 Circular restricted three-body problem

In general, R3BP involves three particles of arbitrary masses moving freely under their mutual gravitational attraction. The particle with infinitesimal mass, sometimes called a massless particle, does not perturb the motions of the two massive bodies. In the circular problem, the two finite masses are fixed in a coordinate system rotating at the orbital angular velocity, with the origin (axis of rotation) at the centre of mass of the two bodies. Lagrange showed that in this rotating frame there were five libration points at which the infinitesimal body would remain fixed if placed there. Three of these points called collinear libration points were found by L. Euler (1767), there lie on the straight line connecting the two finite masses: one between the masses and one outside each of the masses. While the other two called the triangular libration points found by J. L Lagrange (1772), are located equidistant from the two finite masses at a distance equal to the finite mass separation. The two masses and the triangular libration points are thus located at the vertices of equilateral triangles in the plane of the circular orbit. These are illustrated in the figure below, by the use of Sun-Earth-Moon system. The velocity and acceleration are zero at these points.

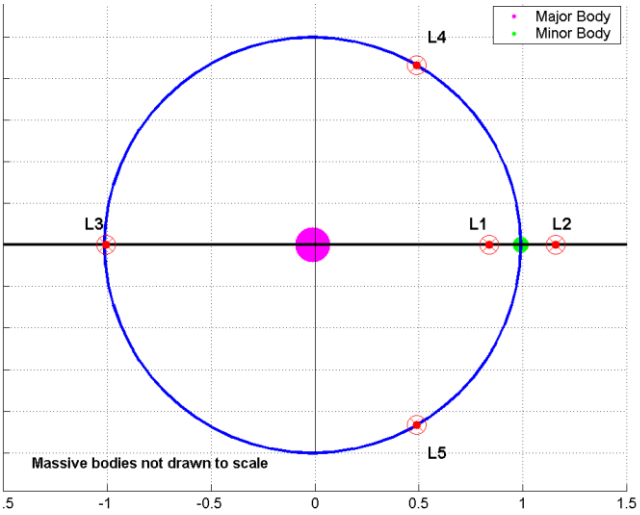


Fig. 1.1 The Five Lagrangian points in CR3BP (Sun-Earth-Moon system)

1.6.2 Equations of motion of two-body problem

Let the masses of the two primaries be m_1 *and* m_2 and their respective vectors from the fixed point 0 in the space of the motion to the respective masses be $\vec{r}_1$ *and* $\vec{r}_2$, and $\vec{r}$ be the radius vector between 4the bodies.

By Newton law of gravitation, the force of attraction on the particle of mass m_1 due to another particle of mass m_2 and the force acting on the particle of mass m_2 due to the particle of mass m_1 are

$$\vec{F}_1 = G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \quad and \quad \vec{F}_2 = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \text{ respectively} \tag{1.1}$$

where G is the gravitational constant and r is the distance between the primaries. By Newton's second law, we have

$$\vec{F}_1 = \frac{d^2\vec{r}_1}{dt^2}m_1 \ and \ \vec{F}_2 = \frac{d^2\vec{r}_2}{dt^2}m_2. \tag{1.2}$$

Equating the equations in system (1.1) and (1.2), we have

$$\frac{d^2\vec{r}_1}{dt^2}m_1 = G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r} \quad and \quad \frac{d^2\vec{r}_2}{dt^2}m_2 = -G\frac{m_1 m_2}{r^2}\frac{\vec{r}}{r}$$

Therefore $\ddot{\vec{r}_1} = G\frac{m_2}{r^2}\frac{\vec{r}}{r}, \quad \ddot{\vec{r}_2} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r},$ (1.3)

Equations of system (1.3) are the vectorial equations of motion of the bodies of arbitrary masses $m_1\ and\ m_2$.

1.6.3 Equations of relative motion

Let the velocity of the bodies $m_1\ and\ m_2$ be $\vec{v}_1\ and\ \vec{v}_2$ respectively, and $\vec{v}_R$ be the velocity vector of the body $m_2\ with\ resprct\ to\ m_1$.

Then $\vec{v}_R = \vec{v}_2 - \vec{v}_1$ (1.4)

Now the vectorial equations of system (1.3) recast to the form

$$\frac{d\vec{v}_1}{dt} = G\frac{m_2}{r^2}\frac{\vec{r}}{r} \ and \ \frac{d\vec{v}_2}{dt} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r}. \quad (1.5)$$

Subtracting the equations in system (1.5), we have

$$\frac{d\vec{v}_2}{dt} - \frac{d\vec{v}_1}{dt} = -G\frac{m_1}{r^2}\frac{\vec{r}}{r} - G\frac{m_2}{r^2}\frac{\vec{r}}{r} = -G\frac{(m_1 + m_2)}{r^2}\frac{\vec{r}}{r}$$

or $$\frac{d}{dt}(\vec{v}_2 - \vec{v}_1) = -G\frac{(m_1 + m_2)}{r^2}\frac{\vec{r}}{r}$$

Using Eq. (1.4), we have

$$\frac{d\vec{v}_R}{dt} = -G\frac{(m_1 + m_2)}{r^2}\frac{\vec{r}}{r}$$

Let $\mu = G(m_1 + m_2)$, we have

$\frac{d\vec{v}_R}{dt} = -\frac{\mu}{r^2}\frac{\vec{r}}{r}$ and dropping the suffix, we have

$$\frac{d\vec{v}}{dt} = -\frac{\mu}{r^2}\frac{\vec{r}}{r} \tag{1.6}$$

Equation (1.6) is refers to as the relative motion of the body of mass m_2 with respect to the body of mass m_1, where r is the distance between the bodies.

1.7 Theoretical Framework

The theoretical framework upon which the derivations and finding of this research work is built on is given in this section

1.7.1 Relative motion of Robe's restricted three-body

The volume of the fluid of density ρ_1 embedded in the heavier primary is given by

$$\mathrm{V} = \frac{4}{3}\pi \mathrm{R}^3, \qquad \text{where R is the radius of the sphere.}$$

$$\text{Density} = \frac{\text{Mass}}{\text{Volume}}, \qquad \text{which implies that Mass} = \text{Volume} \times \text{Density}$$

so the mass of the fluid with density ρ_1 can be written as

$$m_1 = \frac{4}{3}\pi R^3 \rho_1 \tag{1.7}$$

Therefore, the mass of the shell which is denoted by m_1^* is such that

$$m_1^* = m_1 + \ mass\ of\ the\ fluid$$

$$m_1^* = m_1 + m_F$$

$$or \qquad m_1^* = m_1 + \frac{4}{3}\pi R^3 \rho_1 \tag{1.8}$$

From Newton second law of motion,

$$\vec{F}_1 = \frac{d^2\vec{R}_1 m_1^*}{dt^2} \quad and \quad \vec{F}_2 = \frac{d^2\vec{R}_2 m_2}{dt^2} \tag{1.9}$$

Using the Newton second law of motion in system (1.9), we have

$$m_1^* \ddot{\vec{R}}_1 = \frac{Gm_1^* m_2 \vec{R}_{12}}{R_{12}^3} \quad and \quad m_2 \ddot{\vec{R}}_2 = \frac{-Gm_1^* m_2 \vec{R}_{12}}{R_{12}^3}$$

Where G and R_{12} are the gravitational constant and the distance between the primaries respectively.

$$\text{or} \quad \ddot{\vec{R}}_1 = \frac{Gm_2\vec{R}_{12}}{R_{12}^3} \quad and \quad \ddot{\vec{R}}_2 = \frac{-Gm_1^*\vec{R}_{12}}{R_{12}^3}, \quad \text{where} \quad \vec{R}_{12} = \vec{R}_2 - \vec{R}_1 \tag{1.10}$$

The vectorial equations of motion of the bodies of masses $m_1^* \ and \ m_2$ are the equations of system (1.10).

Subtracting the equations of system (1.10), we have

$$\ddot{\vec{R}}_2 - \ddot{\vec{R}}_1 = \frac{-Gm_1^*\vec{R}_{12}}{R_{12}^3} - \frac{Gm_2\vec{R}_{12}}{R_{12}^3} \quad or \quad \ddot{\vec{R}}_2 - \ddot{\vec{R}}_1 = \frac{-G\vec{R}_{12}(m_1^* + m_2)}{R_{12}^3} \tag{1.11}$$

Therefore, the relative motion of the mass m_2 about the spherical shell of mass m_1^* of density ρ_1.is given by equation (1.11)

1.7.2 The Robe's restricted three-body problem

Robe (1977) formulated a new kind of R3BP in which one of the primaries of mass m_1 is a rigid spherical shell, filled with homogenous, incompressible fluid of density ρ_1; the second one is a point mass m_2 outside the shell, and the third body of mass m_3 is a small solid sphere of density ρ_3 moving inside the shell, with assumption that the mass and radius of m_3 are infinitesimal; as illustrated in figure 1.3. He showed the centre of the first primary as a libration point and examined its linear stability in two cases. In the first case,

the orbit of m_2 around m_1 is circular and in the second case, the orbit is elliptic, but the shell is empty $(no\ fluid\ inside\ it)$, or the densities ρ_1 and ρ_3 are equal. In evaluating buoyancy force, he assumed that the pressure field of the fluid ρ_1 has spherical symmetry around the centre of the shell, which are due to the own gravitational field of the fluid ρ_1.

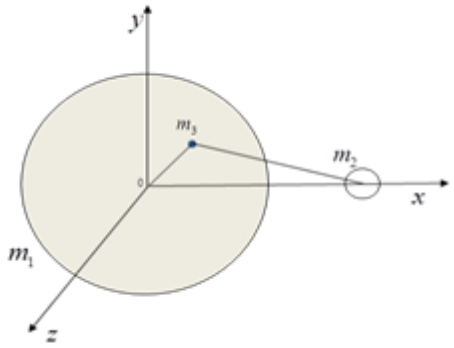


Figure 1.2 The Robe's restricted three-body problem

The equations of motion of the infinitesimal mass m_3 of Robe's circular R3BP are given in this section, since in our work; we shall consider a modified Robe's model. It can be written as

$$\ddot{x}_1 - 2\dot{x}_2 = U_{x_1},$$

$$\ddot{x}_2 + 2\dot{x}_1 = U_{x_2}, \tag{1.12}$$

$$\ddot{x}_3 \quad = U_{x_2},$$

where

$$U = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) + \frac{\mu}{(1-\mu-x_1^2)^2 + x_2^2 + x_3^2} - \frac{K}{2}[(1-\mu-x_1^2)^2 + x_2^2 + x_3^2]$$

$$\mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2} \quad and \quad K = \frac{4}{3}\pi\rho_1\left(1 - \frac{\rho_1}{\rho_3}\right)$$

1.7.3 The buoyancy force

The force exerted on an object that is wholly or partially immersed in a fluid is known as buoyancy force. It acts upward, generally opposite to the direction of the frame of reference acceleration and its magnitude is equal to the weight of the fluid displaced by the submerged object. This force is caused by the difference between the pressure at the top of the object, which pushes downward and the pressure at the bottom of the object which pushes upward; every submerged object feels upward buoyancy force, because the pressure at the bottom of the object is always greater than the pressure at the top.

Suppose the orbital plane of m_2 around m_1^* (the shell with fluid ρ_1) be taken as $xy-plane$, and also let the origin of the coordinate system be at the center of mass, 0, of the two finite masses. Then the forces acting on the third body are;

1) The attraction of m_2

2) The gravitation force $\vec{F}_A$ exerted by the fluid of density ρ_1 is given by

$$\vec{F}_A = \frac{-Gm_1m_3\vec{R}_{13}}{R_{13}^3}$$

By the use of equation (1.7), we have

$$\vec{F}_A = -G\frac{4\pi R^3 m_3\vec{R}_{13}}{3R_{13}^3} \tag{1.13}$$

Since the body of mass m_3 maintains a spherical symmetry about M_1 (center of m_1) due to the pressure of the fluid inside m_1, in the case, the radius R becomes $R = R_{13}$. Therefore, the gravitational force equation (1.13) is recast to the form

$$\vec{F}_A = -\left(\frac{4}{3}\right)\pi G\rho_1^2 m_3 \overline{M_1 M_3} \tag{1.14}$$

3) The buoyancy force $\vec{F}_B$, obtained from equation (1.14) is given as

$$\vec{F}_B = -\frac{4}{3}\pi G\rho_1^2 m_3 \frac{\overline{M_1 M_3}}{\rho_3} \tag{1.15}$$

1.8 Definition of Terms

1.8.1 An ellipsoid

An ellipsoid is a closed quadric surface that is a three-dimensional analogue of an ellipse

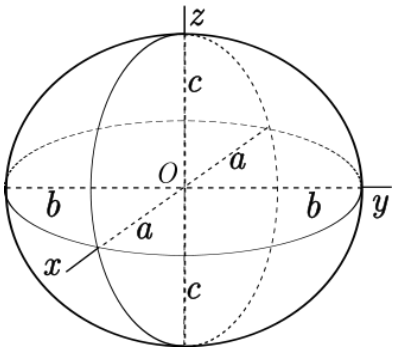


Figure 1.3 shape of an ellipsoid

1.8.2 Oblatenesss

In the above figure, if $a = b > c$, we have an oblate spheroid. In other words, an ellipsoid having a polar axis shorter than the diameter of the equatorial circle whose plane bisects it is known as an oblate spheroid, thus, its two moments of inertial are equal out of the three. It is the approximated shape of many planets and celestial bodies, including Saturn, Jupiter and to a lesser extent the Earth. It is therefore the most used geometric figure for defining reference ellipsoid, upon which cartographic and geodetic system are based.

1.8.3 Triaxiality

If $a > b > c$, we have a triaxial rigid body; that is, triaxiality is a situation that involves three axes. A rigid body that has three mutual perpendicular symmetric axes is called a

triaxial rigid body, for example, an ellipsoid. A triaxial body has all its moments of inertial to be distinct.

1.8.4 Roche ellipsoid

Roche ellipsoid is a general triaxial ellipsoid with all axes having different length.

1.8.5 Zonal hamomic

A harmonic function is a function f that satisfies the Laplace's equation $\nabla^2 f = 0$; the spherical harmonic is an infinite set of harmonic functions defined on a sphere. Zonal harmonic is a spherical harmonic projection of functions that have rotational symmetry around the axis.

1.8.6 Moment of Inertia

The moment of inertial of a system or particle with respect to a plane, point or line is defined as the sum of the products of the masses and the square of their respective perpendicular distances from the plane, point or line. If $\boldsymbol{m_1, m_2, m_3, \dots, m_n}$ are the masses of the particles with perpendicular distances $\boldsymbol{r_1, r_2, r_3, \dots, r_n}$, then the moment of inertia is given as

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2 = \sum_{i=1}^{n} m_i r_i^2 \qquad (1.16)$$

For a body, we have

$$I = \int_M r^2 dm \qquad (1.17)$$

where $dm = \rho dM$, ρ is the density of the body and subscript M shows that the integration is done over the whole mass of the body.

The potential of the body with the origin at the center of the mass is given as

$$V = -\frac{GM}{r} - \frac{G}{r^3}(2I_0 - 3I) \tag{1.18}$$

where I_0 is the moment of inertia about the origin while I is the moment of inertia about a line fixed from the center of the body to a point outside the body with distance r. Thus the potential between the infinitesimal mass and the primaries can be written as;

$$V = -GM\left[\frac{1}{r} + \frac{1}{2Mr^3}(2I_1 + I_3 - 3I) + \frac{1}{2Mr^3}(I_1 + I_2 + I_3 - 3I)\right] \tag{1.19}$$

Where I_1, I_2, I_3 are the principal moments of inertia of the oblate spheroid with $I_1 = I_2 \neq I_3$ and I_1, I_2, I_3 are the principal moment of inertia of the triaxial rigid body with $I_1 \neq I_2 \neq I_3$.

From Kepler's third law, the mean motion is expressed as

$$n^2 = \frac{G(m_1 + m_2)}{r^3} \tag{1.20}$$

In the first case where both primaries are oblate spheroid with oblateness coefficients up to the second zonal harmonic and the full buoyancy of the fluid is taken into account, the mean motion of the primaries is given (McCuskey 1963) by

$$n^2 = \frac{G(m_1 + m_2)}{r^3}\left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right) \tag{1.21}$$

while the second case where the primaries form a Roche ellipsoid-triaxial system, the mean motion of the primaries given also by McCuskey 1963 is

$$n^2 = \frac{G(m_1 + m_2)}{r^3}\left(1 + \frac{3}{2}(2\sigma_1 - \sigma_2)\right) \tag{1.22}$$

1.9 Linear Stability of the Solutions of Dynamical Systems

The stability of linear systems of ordinary differential equations is determined completely by the eigen values of the coefficient matrix. Due to perturbations, the position of the infinitesimal body would be displaced a little from the libration point. If the resultant motion of the infinitesimal body is a rapid departure from the vicinity of the point, we call such a position of libration point an “unstable one” if however the body merely oscillates about the libration point, it is said to be a “stable position” (in the sense of Lyapunov).

To investigate the stability of the orbit in the surrounding area of the libration points, the small displacement method is applied by shifting the coordinates or the origin of the infinitesimal mass, and then linearizing the equations of motion around the coordinates of the libration points to derive the variational equations of motion corresponding to the dynamical system. These equations are transformed to a matrix form through the trial solutions and a characteristic equation of the dynamical system is obtained.

The roots of the characteristic equation are obtained and the stability of the solutions depends on the nature of the characteristic roots; for complex roots, the libration point is asymptotically stable when all roots have negative real parts, and unstable when some or all roots have positive real part. For pure imaginary roots, the motion is oscillatory and the libration point is stable, though not asymptotically stable. If there are multiple roots, the solution contains mixed terms (periodic and secular terms) the libration point is unstable. If all the roots are both real and negative then the libration point is stable, but if any of the roots is positive, the libration point will be unstable.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

The R3BP describes the motion of an infinitesimal mass moving under the gravitational effects of two massive bodies, which move in circular orbits around their center of mass on the account that their mutual attraction and the infinitesimal mass does not influence the motion of the primaries. The approximate circular motion of planets around the Sun and the small masses of asteroids and the satellites of planets compared to the planet's masses originally suggested the formulation of the restricted problem. The R3BP is of fundamental importance in mechanics, with significant applications to astrodynamics. Since no general solution in the CR3BP is available, particular solutions are sought to obtain insight into the problem.

2.2 Libration Points

There are five libration points in the classical R3BP; two triangular and three collinear. L. Euler (1767) determined a set of three collinear libration points while J. L Lagrange (1772) discovered the triangular points.

Routh (1892) established the condition for linear stability of the triangular points. When this condition is satisfied all roots of the characteristic equation are pure imaginary, which leads to pure oscillatory solution. This solution was later referred to as "critical solution" by Lyapunov (1892). The collinear points were found to be unstable in the linear and nonlinear sense, as Routh original result shows, when interpreted in the light of Lyapunov's work.

2.3 Oblateness and Triaxiality

The bodies in the classical R3BP are strictly spherical in shape, but in the actual situation it is seen that the earth and several heavenly bodies such as Saturn and Jupiter are sufficiently oblate. The minor planets and meteoroids have irregular shapes. The lack of shpericity or the oblateness of the planets causes large perturbations from a two-body orbit. The motions of artificial Earth satellites are examples of this. Global studies of problem with oblateness or triaxiality have been carried out by many researchers to mention a few viz;

Szebehely (1967) studied the effect of a small perturbation of the Coriolis force on the stability of the libration points, keeping the centrifugal force constant. He maintained that the collinear points remain unstable and obtained for stability of the triangular points a relation between the critical value of the mass parameter μ_c and the change ϵ in the Coriolis force: $\mu_c = \mu_0 + \frac{16\epsilon}{3\sqrt{69}}$, and concluded that the Coriolis is a stabilizing one. Subbarao and Sharma (1975) investigate the same problem as Szebehely (1967) with the assumption that both primaries are oblate spheroids and its equatorial plane coinciding with the plane of motion. They showed that the oblateness of the primaries resulted in an increase in both the Coriolis and the centrifugal forces and further establish that the range of linear stability of the triangular decreases, thereby concluding that the Coriolis force is not always a stabilizing force. Sharma (1982) extended this work by considering the linear stability of the triangular libration points of the restricted three-body problem when the more massive primary is a source of the radiation and an oblate spheroid as well. He discovered that the eccentricity of the conditional retrograde elliptical periodic orbits around the triangular points at the critical mass μ_c increases with the increase in the oblateness coefficient and the radiation force; and becomes unity when μ_c is zero.

Singh and Ishwar (1999) generalized the R3BP by considering both primaries to be source of radiation and as well oblate spheroid. They observe that the equations of motion and locations of the libration are affected by the radiation pressure forces and oblateness of the primaries. They discovered that the triangular points are stable for $0 < \mu \ll \mu_c$ and unstable for $\mu_c < \mu \ll \frac{1}{2}$, and further established that the range of stability depends upon the radiating and oblateness coefficients.

AbdulRaheem and Singh (2006) studied the combined effect of perturbations, radiation and oblateness on the stability of the libration points in the R3BP. The model was studied under the influence of small perturbations in the Coriolis and centrifugal forces, together with the effects of oblateness and radiation pressure of the primaries. They found that the collinear points remain unstable, while the triangular points are stable for $0 < \mu \ll \mu_c$ and unstable for $\mu_c < \mu \ll \frac{1}{2}$. They observed further that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation and oblateness of the primaries have destabilizing effects; the presence of any one or more of the latter makes week stabilizing ability of the former, consequently the overall effect is that the range of stability of the triangular points decreases.

Singh and Jessica (2011) contributions regarding the considerations of the non-sphericity of the bodies in R3BP is an interesting one, the problem was generalized to include more massive primary as a triaxial rigid body while the less massive one is an oblate spheroid under the influence of small perturbations in the Coriolis and centrifugal forces; They asserted that the location of the triangular and collinear points was affected by the non-sphericity of the primaries and the change in the centrifugal force. The stabilities were investigated and found triangular points are stable for $0 < \mu \ll \mu_c$ and unstable for

$\mu_c < \mu \ll \frac{1}{2}$, where $\mu_c$is the critical value of the mass parameter which depends on the joint effect of perturbations, oblateness and triaxialty. But the collinear points are unstable.

2.4 Robe's Problem

Robe (1977) considered a new kind of R3BP, in which one of the primaries of mass m_1, is a rigid spherical shell, filled with homogenous, incompressible fluid of density ρ_1, with the second mass m_2 as a small point outside the shell and moving around the heavier mass in a Keplerian orbit; and the infinitesimal mass m_3 as a small sphere of density ρ_3, moving inside the shell and is subject to attraction of m_2 and the buoyancy force due to the fluid and that the radius of m_3 is assumed to be infinitesimal. Further, he discussed the linear stability of an libration point and obtained two cases. In the first case the orbit of m_2 around m_1 is circular and in the second case, the orbit is elliptic, but the shell is empty (there is no fluid inside it) or densities of m_1 and m_3 are equal.

Plastino and Plastino (1995) reanalyzed the Robe's problem with a new assumption by considering the first primary body as a Roche spheroid in an equilibrium state of a rotational fluid, called it the Robe-Roche model as a new type of restricted three-body problem. They explored its libration points and found that when the density parameter is taken as zero, every point inside the shell is a libration point and further analyzed the connection between the effect of the buoyancy forces and a perturbation of the Coriolis force. They also showed that the center of the spheroid is a libration point of the Robe-Roche restricted three-body problem and it is linearly stable. However, their approximation of the potential due to the second primary was only up to the second degree terms only in the variable involved.

Hallan and Mangang (2007) examined the existence and stability of libration points in the Robe's (1977) R3BP by considering the full buoyancy force as in Plastino and Plastino

(1995) and assuming the hydrostatic equilibrium figure of the first primary as an oblate spheroid. They derived the pertinent equations of motion and found that there is a libration point near the center of the first primary. They further found that there can be one more libration point on the line joining the center of the first primary and the second one, and an infinite number of libration points lying on a circle in the orbital plane of the second primary. The one more libration point was found to be conditionally stable while the circular points are unstable. Singh and Sandah (2012) extended the work of Hallan and Mangang (2007) by assuming that the second primary is an oblate spheroid as well, and took into account the first zonal harmonic only. They also found that libration point exists near the center of the first primary, and there can be one more libration point on the line joining the center of both primaries. Circular points within the first primary are also libration under certain conditions and the existence of two out-of-plane points is observed. The linear stability is examined and found that the points near the center of the first primary are conditionally stable, while the circular and the out-of-plane points are unstable.

Singh and Mohammed (2013) analyze Robe's CR3BP when the hydrostatic equilibrium figure of the first primary is assumed to be an oblate spheroid, the shape of the second primary is considered as a triaxial rigid body, and a full buoyancy force of the fluid is taken into account. They found that there is a libration point near the center of the primary, another libration point exists on the line joining the center of the primaries and there exist infinite number of libration points on an ellipse in the orbital plane of the second primary. They also observed that under certain conditions, all these libration points can be stable. The most interesting and distinguishable result of their study are the existence of elliptical points and their stabilities. Singh and Omale (2014b) extended the work of Singh and Mohammed (2013) by investigating the effect of small perturbations in Coriolis and

centrifugal forces on the axial libration points and their stability. They observed that the locations of the axial libration points are only influenced by a small change in the centrifugal force due to the force function Ω being dependent of it. They also found that magnitude of the centrifugal force shift the first libration point $(p_1, 0, 0)$ towards the positive side of the $x-axis$, while second axial libration $(x_{11}+p_2, 0,0)$ is shifted away from the origin towards the negative part of the $x-axis$ as a result of the increase in magnitude of the triaxiality and centrifugal force. The range of stability is established and is found that the first axial libration point is stable for $p_1 < 0$, conditionally stable for $0 < p_1 < \frac{c}{2}$ and unstable for $0 < \frac{c}{2} < p_1$, while the second axial libration point is stable whenever $x_{11} < 0$ and if $x_{11} > 0$, then it is conditionally stable.

Kaur and Aggarwal (2013) studied Robe's restricted problem by considering one of the primaries of mass m_1 as Plastino and Plastino (1995) and the smaller primary of mass m_2 as an oblate spheroid outside the ellipsoid, the third and fourth bodies of mass (m_3 and m_4 respectively) inside the ellipsoid. They assume that the mass and radius of the third and fourth body are infinitesimal and analyze the libration solutions of $m_3\ and\ m_4$ and their linear stabilities, they proved that the system has only six libration solutions lying within the Roche ellipsoid. When the primaries are considered as Earth-Moon and m_3 and m_4 are submarines; the libration solutions of m_3 and m_4 when displaced in the direction of x_1-axis or x_2-axis are unstable.

Singh *et al.* (2014) investigated Robe's CR3BP for two cases, (*i*) when the primaries form a Roche ellipsoid-triaxial system and (*ii*) when the primaries form a Roche ellipsoid-oblate system. Without ignoring any component in both problems, a full treatment is given to the buoyancy force. They establish the relevant equations of motion and discuss the special case where the density of the fluid and that of the infinitesimal mass are equal (*D = 0*).

They also studied the location of the libration point and its stability when the infinitesimal mass is denser than the medium ($D > 0$), and found that the point (0, 0, 0) is the only libration point and this point is stable.

CHAPTER THREE

EQUATIONS OF MOTION

3.1 Introduction

Robe (1977) formulated a new type of restricted three-body problem in which one of the primaries is a rigid spherical shell filled with homogeneous incompressible fluid. The second one is a point mass outside the shell, and the third one is a small solid sphere of infinitesimal mass moving inside the shell. Singh and Sandah (2012) studied the Robe's problem under the assumption that both primaries are oblate spheroid and took into account the first zonal harmonic only. The present research intends to first, investigate the effects of the above oblateness coefficients up to the second zonal harmonic and to study Robe's circular restricted three-body problem with a Roche ellipsoid-triaxial system in the second case.

3.2 Mathematical Model

Let us suppose m_1 and m_2 be the masses of the first and second primary respectively and the density of the homogenous incompressible fluid to be ρ_3. Let R be the radius of the sphere m_1, and $M_1, M_2 and\ M_3$ be the centers of $m_1, m_2\ and\ m_3$ respectively.

Let $\vec{R}_{12}$, $\vec{R}_{13}$ and $\vec{R}_{23}$ be the position vectors between the center of the mass of the body m_1 & m_2, m_1 & m_3 $and\ m_2$ & m_3 respectively. Let 0 be the center of the mass of the primaries. We assume that the first primary m_1 be a fluid of density ρ_1 in the shape of an oblate spheroid. Let the second primary m_2 be an oblate spheroid as Sharma and Subba Rao (1976) assumed, which describes a circular orbit around m_1. The infinitesimal mass m_3 moves inside the first primary. We consider a uniformly rotating coordinate system

$0x_1x_2x_3$ with origin at the center of mass m_1, $0x_1$ points towards m_2, and $0x_1x_2$ being the orbital plane of m_2 coinciding with the equatorial plane of m_1. Then, the equations of motion of the infinitesimal body of density ρ_3 in this coordinate system, as in Singh and Sandah (2012) and Plastino and Plastino (1995) take the form;

$$\ddot{x}_1 - 2n\dot{x}_2 = U_{x_1},$$

$$\ddot{x}_2 + 2n\dot{x}_1 = U_{x_2}, \qquad (3.1)$$

$$\ddot{x}_3 = U_{x_3}$$

where

$$U = V + \frac{1}{2}n^2\left[\left(x_1 - \frac{m_2R}{m_1+m_2}\right)^2 + x_2^2\right],$$

$$V = B + B' - \frac{\rho_1}{\rho_3}\left[B + B' + \frac{1}{2}n^2\left\{\left(x_1 - \frac{m_2R}{m_1+m_2}\right)^2 + x_2^2\right\}\right], \qquad (3.2)$$

Case 1

$B = \pi G\rho_1[I - A_1x_1^2 - A_1x_2^2 - A_3x_3^2]$.

$$B' = \frac{Gm_2}{[(R-x_1)^2 + x_2^2 + x_3^2\,]^{1/2}} + \frac{Gm_2\alpha_2}{2[(R-x_1)^2 + x_2^2 + x_3^2\,]^{3/2}}$$

$$- \frac{3Gm_2\alpha_3}{8[(R-x_1)^2 + x_2^2 + x_3^2\,]^{5/2}}$$

$I = 2a_1^2A_1 + a_2^2A_2$,

$$A_1 = a_1^2a_2\int_0^\infty \frac{du}{\Delta(a_1^2+u)}, \quad A_2 = a_1^2a_2\int_0^\infty \frac{du}{\Delta(a_2^2+u)}, \quad \Delta^2 = (a_1^2+u)^2(a_2^2+u)$$

$$n^2 = \frac{G(m_1 + m_2)}{R^2}\left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right),$$

$$\alpha_1 = \frac{a_1^2 - a_3^2}{5R^2}, \qquad \alpha_2 = J_2 R_2^2, \qquad \alpha_3 = J_4 R_2^2$$

Case 2

$$B = \pi G\rho_1[I - A_1 x_1^2 - A_2 x_2^2 - A_3 x_3^2].$$

$$B = \mu R^2 + \mu R x_1 + \mu\left(x_1^2 - \tfrac{1}{2}x_2^2 - \tfrac{1}{2}x_3^2\right) + \tfrac{\mu}{2}(2\sigma_1 - \sigma_2)\left(6x_1^2 - \tfrac{3}{2}x_2^2 - \tfrac{3}{2}x_3^2\right)$$
$$+ \frac{\mu R^2}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu R\,(2\sigma_1 - \sigma_2)x_1 - \frac{3}{2}\mu(\sigma_1 - \sigma_2)x_2^2 - \frac{3}{2}\mu\sigma_1 x_3^2,$$

$$\mu = \frac{Gm_2}{R^3}, I = a_1^2 A_1 + a_2^2 A_2 + a_3^2 A_3, \;\; A_i = a_1 a_2 a_3 \int_0^\infty \frac{du}{\Delta(a_i^2 + u)}, where\ i = 1,2,3$$

$$\Delta^2 = (a_1^2 + u)\,(a_2^2 + u)(a_3^2 + u),$$

$$n^2 = \frac{G(m_1 + m_2)}{R^3}\left(1 + \frac{3}{2}(2\sigma_1 - \sigma_2)\right), \;\; \sigma_1 = \frac{a^2 - c^2}{5R^2}, \sigma_2 = \frac{b^2 - c^2}{5R^2},$$

$$\sigma_i \ll 1\ (i = 1,2)$$

Here, V is the potential that explains the combined forces upon the infinitesimal mass, B denotes the gravitational potential due to the fluid mass, R is the distance between the primaries and G is the gravitational constant while *n* is the mean motion. I stands for the polar moment of inertia, while $A_i (i = 1,2)$ are the index symbols.

Case 1

B' denotes the potential due to the second oblate primary, R_2 is the mean radius of m_2, $J_2 and\ J_4$ are the zonal harmonic coefficients that characterize the size of the non spherical

components of the potential. $A_i(i = 1,2)$ are the index symbols. α_1 and α_2 are the oblateness coefficients of the first and second primaries respectively.

Case 2

B' stands for the potential due the second triaxial primary, a_1, $a_{2,}$ and a_3 are the semiaxes of the ellipsoidal figure and parallel respectively to the coordinate axes; σ_1, σ_2 characterize triaxiality of the second primary with semiaxes a, b, c.

Substituting the expression for V in (3.2) into U, we have

$$U = B + B' - \frac{\rho_1}{\rho_3}\left[B + B' + \frac{1}{2}n^2\left\{\left(x_1 - \frac{m_2R}{m_1 + m_2}\right)^2 + x_2^2\right\}\right] + \frac{1}{2}n^2\left[\left(x_1 - \frac{m_2R}{m_1 + m_2}\right)^2 + x_2^2\right]$$

or

$$U = B + B' - \frac{\rho_1}{\rho_3}B - \frac{\rho_1}{\rho_3}B' - \frac{\rho_1}{\rho_3}\left[\frac{1}{2}n^2\left\{\left(x_1 - \frac{m_2R}{m_1 + m_2}\right)^2 + x_2^2\right\}\right] + \frac{1}{2}n^2\left[\left(x_1 - \frac{m_2R}{m_1 + m_2}\right)^2 + x_2^2\right]$$

or

$$U = B\left(1 - \frac{\rho_1}{\rho_3}\right) + B\left(1 - \frac{\rho_1}{\rho_3}\right) + \left(1 - \frac{\rho_1}{\rho_3}\right)\left\{\frac{1}{2}n^2\left[\left(x_1 - \frac{m_2R}{m_1 + m_2}\right)^2 + x_2^2\right]\right\}$$

or

$$U = \left(1 - \frac{\rho_1}{\rho_3}\right)\left[B + B' + \frac{1}{2}n^2\left[\left(x_1 - \frac{m_2R}{m_1 + m_2}\right)^2 + x_2^2\right]\right].$$

Now, substituting the expression for the potentials B and B, the equations of motion in system (3.1) take the form;

$$\ddot{x}_1 - 2n\dot{x}_2 = U_{x_1},$$

$$\ddot{x}_2 + 2n\dot{x}_1 = U_{x_2}, \tag{3.3}$$

$$\ddot{x}_3 = U_{x_3},$$

where in case 1

$$U = \left(1 - \frac{\rho_1}{\rho_3}\right)\left[\pi G\rho_1[I - A_1x_1^2 - A_2x_2^2 - A_3x_3^2] + \frac{Gm_2}{[(R-x_1)^2 + x_2^2 + x_3^2]^{1/2}} + \frac{Gm_2\alpha_2}{2[(R-x_1)^2 + x_2^2 + x_3^2]^{3/2}} - \frac{3Gm_2\alpha_3}{8[(R-x_1)^2 + x_2^2 + x_3^2]^{5/2}} + \frac{1}{2}n^2\left\{\left(x_1 - \frac{m_2R}{m_1+m_2}\right)^2 + x_2^2\right\}\right]$$

or

$$U = D\left[\pi G\rho_1[I - A_1x_1^2 - A_2x_2^2 - A_3x_3^2] + \frac{Gm_2}{[(R-x_1)^2 + x_2^2 + x_3^2]^{1/2}} + \frac{Gm_2\alpha_2}{2[(R-x_1)^2 + x_2^2 + x_3^2]^{3/2}} - \frac{3Gm_2\alpha_3}{8[(R-x_1)^2 + x_2^2 + x_3^2]^{5/2}} + \frac{1}{2}n^2\{(x_1 - \mu R)^2 + x_2^2\}\right].$$

Case 2

$$U = \left(1 - \frac{\rho_1}{\rho_3}\right)\left[\pi G\rho_1[I - A_1x_1^2 - A_2x_2^2 - A_3x_3^2] + \mu R^2 + \mu R x_1 + \mu\left(x_1^2 - \tfrac{1}{2}x_2^2 - \tfrac{1}{2}x_3^2\right)\right.$$
$$+ \tfrac{\mu}{2}(2\sigma_1 - \sigma_2)\left(6x_1^2 - \tfrac{3}{2}x_2^2 - \tfrac{3}{2}x_3^2\right) + \frac{\mu R^2}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu R\,(2\sigma_1 - \sigma_2)x_1$$
$$\left. - \frac{3}{2}\mu(\sigma_1 - \sigma_2)x_2^2 - \frac{3}{2}\mu\sigma_1 x_3^2 + \frac{1}{2}n^2\left\{\left(x_1 - \frac{m_2R}{m_1 + m_2}\right)^2 + x_2^2\right\}\right]$$

or

$$U = D\left[\pi G\rho_1[I - A_1x_1^2 - A_2x_2^2 - A_3x_3^2] + \mu R^2 + \mu R x_1 + \mu\left(x_1^2 - \tfrac{1}{2}x_2^2 - \tfrac{1}{2}x_3^2\right)\right.$$
$$+ \tfrac{\mu}{2}(2\sigma_1 - \sigma_2)\left(6x_1^2 - \tfrac{3}{2}x_2^2 - \tfrac{3}{2}x_3^2\right) + \frac{\mu R^2}{2}(2\sigma_1 - \sigma_2) + \frac{3}{2}\mu R\,(2\sigma_1 - \sigma_2)x_1$$
$$\left. - \frac{3}{2}\mu(\sigma_1 - \sigma_2)x_2^2 - \frac{3}{2}\mu\sigma_1 x_3^2 + \frac{1}{2}n^2[(x_1 - \mu R)^2 + x_2^2]\right]$$

where

$$\mu = \frac{m_2R}{m_1 + m_2} \; and \; D = \left(1 - \frac{\rho_1}{\rho_3}\right).$$

We take the units such that the sum of the masses and the distance between the primaries are unity. Thus, we let $m_2 = \mu$, $0 < \mu = \frac{m_2}{(m_1 + m_2)} < 1$ and R = 1. For the unit of time, we recall that the time which the mass m_2 takes to describe one radian of an arc about m_1. In this unit the time period of m_2 about m_1 will consist of $2\pi\ units$. consequently T = 2π. Also, from Kepler's third law of planetary motion, we have

$$\frac{a^3}{T^2} = \frac{G(m_1 + m_2)}{4\pi^2}, \qquad here\ a = R = m_1 + m_2 = 1, T = 2\pi.$$

Therefore,

$$\frac{1}{4\pi^2} = \frac{G}{4\pi^2}, \quad so\ that\ G = 1.$$

With these units, the equations of motion (3.3) are recast in the form;

$$\ddot{x}_1 - 2n\dot{x}_2 = U_{x_1},$$

$$\ddot{x}_2 + 2n\dot{x}_1 = U_{x_2}, \tag{3.4}$$

$$\ddot{x}_3 = U_{x_3},$$

where **Case 1**

$$U = D\left[\pi\rho_1\{\mathrm{I} - A_1(x_1^2 + x_2^2) - A_2x_3^2\} + \frac{\mu}{r_1} + \frac{\mu\alpha_2}{2r_1^3} - \frac{3\mu\alpha_3}{8r_1^5} + \frac{n^2\{(x_1 - \mu)^2 + x_2^2\}}{2}\right]$$

$$n^2 = \left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right), \quad r_1 = [(1 - x_1)^2 + x_2^2 + x_3^2]^{1/2}. \tag{3.5}$$

Case 2

$$U = D\left[\pi\rho_1(I - A_1x_1^2 - A_2x_2^2 - A_3x_3^2) + \mu\left(x_1^2 - \tfrac{1}{2}x_2^2 - \tfrac{1}{2}x_3^2\right) + \tfrac{n^2}{2}(x_1^2 + x_2^2)\right.$$
$$+ \tfrac{\mu}{2}(2\sigma_1 - \sigma_2)\left(6x_1^2 - \tfrac{3}{2}x_2^2 - \tfrac{3}{2}x_3^2\right) - \tfrac{3\mu}{2}(\sigma_1 - \sigma_2)x_2^2 - \tfrac{3\mu}{2}\sigma_1x_2^2$$
$$\left. + \frac{\mu}{2}[\mu n^2 + (2\sigma_1 - \sigma_2) + 2]\right], \tag{3.6}$$

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2), \quad 0 < \mu < 1, \qquad \sigma_1, \qquad \sigma_2 \ll 1.$$

The equations of motion in system (3.4) with (3.6) can be recast in the form

$$\ddot{x}_1 - 2n\dot{x}_2 = D\left[1 + 2\mu - C_1 + \tfrac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2)\right]x_1,$$

$$\ddot{x}_2 + 2n\dot{x}_1 = D\left[1 - \mu - C_2 + \frac{3}{2}(1-2\mu)(2\sigma_1 - \sigma_2) + \frac{3\mu\sigma_2}{2}\right]x_2 \ , \qquad (3.7)$$

$$\ddot{x}_3 = -D\left[\mu + C_3 + \frac{3\mu}{2}(4\sigma_1 - \sigma_2)\right]x_3,$$

where

$C_i = 2\pi\rho_1 A_i, \quad i = 1, 2, 3.$

3.2 The Case $\boldsymbol{\rho_1 = \rho_3}$

We consider a special case here where the density of the infinitesimal mass and the fluid are the same. This reduces the equations (3.4) and (3.7) of motion to the form

$$\ddot{x}_1 - 2n\dot{x}_2 = 0$$

$$\ddot{x}_2 + 2n\dot{x}_1 = 0, \qquad (3.8)$$

$$\ddot{x}_3 = 0.$$

The solution of these equations can be written as

$x_1 = k\cos(2nt + \theta) + k_1,$

$x_2 = -k\sin(2nt + \theta) + k_2,$

$x_3 = k_3 t + k_4,$

where $k,\ k_1,\ k_2,\ k_3,\ k_4\ and\ \theta$ are constants. We see that each point $(x_1,\ x_2,\ x_3)$ is a libration point of the equations (3.8). This happens because the fluid is assumed to be in the hydrostatic libration in the rotating frame. Therefore all the elements of the fluid are at rest in this system. If the density of the infinitesimal mass is the same as that of the fluid

($i.e.,\ \rho_1 = \rho_3$), it is impossible to differentiate between any of its elements. Thus, it will be in libration everywhere.

3.3 Conclusion

The equations of motion of the infinitesimal mass m_3 under the framework of the Robe's restricted problem have been obtained by taking into account the influences of the full buoyancy force of the fluid, oblateness, triaxiality and gravitational attractions of the primary bodies. The equations of motion (3.4) with (3.6) are different from those of Hallan and Mangang (2007) due to the presence of the first zonal harmonic coefficient of the second primary and from those of Singh and Sandah (2012) on account of the appearance of the second zonal harmonic coefficients of the second primary. If the second primary is spherical (i.e., $\alpha_2 = \alpha_3 = 0$), then the equations fully coincide with those of Hallan and Mangang (2007). If the second primary has not the second zonal harmonic coefficient (i.e. $\alpha_3 = 0$), then the equations fully coincide with those of Singh and Sandah (2012). The equations of motion in system (3.7) differ from those of Plastino and Plastino (1995) due to the appearance of triaxial terms, and it shows that the triaxiality of the second primary has the significant effect on the equations of motion.

CHAPTER FOUR

LOCATIONS AND LINEAR STABILITY OF LIBRATION POINTS

4.1 Introduction

The gravitational pull at these points are in balance, so that the infinitesimal body in the vicinity of the libration points would be held there without being pulled closer to either of the massive bodies, in other words, if a satellite or celestial body is located at one of these points with zero initial velocity relative to the rotating coordinate system, it will remain there permanently provided there are no perturbations. These points are also called stationary points. There are five libration points in the circular restricted three-body problem; two triangular and three collinear. In any assumed isolated two-massive orbiting system, such as the sun and the earth, these five points denoted by L_i $(i = 1,2 \dots 5)$, and called Lagrangain points or libration points.

The stability of the dynamical system implies that there is a class of models or initial conditions for which the trajectories would be equivalent, the operation for comparing orbits to establish their equivalence changes with the different notions of stability. In general, the stability of differential equations refers to the global effects of local perturbations.

A.M Lyapunov (1857-1918) focused on the stability of libration points, the motion of the mechanical system and stability of a uniformly rotating fluid and developed approximation method used to determine the stability of sets of ordinary differential equations.

Our aim in this chapter is to find the location and study the stability of the libration points in the Robe's restricted three-body problem in both cases.

4.2 Stability of Libration Points

The stability constant coefficients of linear dynamical systems can be determined from eigen values of the coefficient matrix. If any eigen value has a positive real part; the system will tend to move away from the fixed point. Let the location of the infinitesimal body be displaced a little from the libration point. If the resultant motion is a rapid departure from the vicinity of the point, we say the libration point is unstable, and is stable if the body merely oscillates about the libration point.

4.3 Variational and Characteristic Equations

To study the stability of an orbit in the vicinity of any libration point, we adopt the small displacement method by displacing the coordinates of the infinitesimal body and linearizing the equations of motion around the libration point with coordinates (x_{10}, x_{20}, x_{30}).

Let $\alpha, \beta \; and \; \gamma$ are small displacement in the coordinates of the libration point (x_{10}, x_{20}, x_{30}) of the infinitesimal mass respectively. We displace the third body to the position $(x_{10} + \alpha,\ x_{20} + \beta,\ \ x_{30} + \gamma)$, so that

$$\begin{array}{llll} x_1 = x_{10} + \alpha, & x_2 = x_{20} + \beta, & x_3 = x_{30} + \gamma & position \\ \dot{x}_1 = \dot{\alpha}, & \dot{x}_2 = \dot{\beta}, & \dot{x}_3 = \dot{\gamma} & velosity \\ \ddot{x}_1 = \ddot{\alpha} & \ddot{x}_2 = \ddot{\beta} & \ddot{x}_3 = \ddot{\gamma} & acceleration \end{array} \qquad (4.1)$$

Now, from the system of equations of motion (3.3) and (3.4) using Taylor's theorem, the function U can be expanded around the libration point (x_{10}, x_{20}, x_{30}) to give

$$U(x_1, x_2, x_3) = U(x_{10} + \alpha, \quad x_{20} + \beta, \quad x_{30} + \gamma)$$

$$= U(x_{10}, x_{20}, x_{30}) + U_{x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_3}(x_{10}, x_{20}, x_{30})\gamma$$

$$+ \frac{1}{2!}U_{x_1x_1}(x_{10}, x_{20}, x_{30})\alpha^2 + U_{x_1x_2}(x_{10}, x_{20}, x_{30})\alpha\beta$$

$$+ U_{x_1x_2}(x_{10}, x_{20}, x_{30})\alpha\beta + \frac{1}{2!}U_{x_2x_2}(x_{10}, x_{20}, x_{30})\beta^2$$

$$+ \frac{1}{2!}U_{x_2x_2}(x_{10}, x_{20}, x_{30})\beta^2 + \frac{1}{2!}U_{x_3x_3}(x_{10}, x_{20}, x_{30})\gamma^2$$

$$+ U_{x_1x_3}(x_{10}, x_{20}, x_{30})\alpha\gamma + U_{x_2x_3}(x_{10}, x_{20}, x_{30})\beta\gamma + \cdots \qquad (4.2)$$

Expanding the right hand side of systems (3.3) and (3.4) near the libration point (x_{10}, x_{20}, x_{30}) using the fact that at libration point

$U_{x_1}(x_{10}, x_{20}, x_{30}) = U_{x_2}(x_{10}, x_{20}, x_{30}) = U_{x_3}(x_{10}, x_{20}, x_{30}) = 0,$

we have

$$U_{x_1} = U_{x_1x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_1x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_1x_3}(x_{10}, x_{20}, x_{30})\gamma + 0(2)$$

$$+ 0(3)$$

$$U_{x_2} = U_{x_2x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_2x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_2x_3}(x_{10}, x_{20}, x_{30})\gamma + 0(2)$$

$$+ 0(3)$$

$$U_{x_3} = U_{x_3x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_3x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_3x_3}(x_{10}, x_{20}, x_{30})\gamma + 0(2)$$

$$+ 0(3)$$

where $0(2)\ and\ 0(3)$ denote second, third and higher terms of $\alpha, \beta\ and\ \gamma$ respectively, and they can be ignored since $\alpha, \beta\ and\ \gamma$ are small quantities, we have

$$U_{x_1} = U_{x_1x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_1x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_1x_3}(x_{10}, x_{20}, x_{30})\gamma$$

$$U_{x_2} = U_{x_2x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_2x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_2x_3}(x_{10}, x_{20}, x_{30})\gamma \qquad (4.3)$$

$$U_{x_3} = U_{x_3x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_3x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_3x_3}(x_{10}, x_{20}, x_{30})\gamma.$$

Making use of (4.3) in (3.3) and (3.4), we have

$$\ddot{\alpha} - 2n\dot{\beta} = U_{x_1x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_1x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_1x_3}(x_{10}, x_{20}, x_{30})\gamma,$$

$$\ddot{\beta} + 2n\dot{\alpha} = U_{x_2x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_2x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_2x_3}(x_{10}, x_{20}, x_{30})\gamma, \quad (4.4)$$

$$\ddot{\gamma} = U_{x_3x_1}(x_{10}, x_{20}, x_{30})\alpha + U_{x_3x_2}(x_{10}, x_{20}, x_{30})\beta + U_{x_3x_3}(x_{10}, x_{20}, x_{30})\gamma,$$

The equations in system (4.4) can be written as

$$\ddot{\alpha} - 2n\dot{\beta} = \left(U^0_{x_1x_1}\right)\alpha + \left(U^0_{x_1x_2}\right)\beta + \left(U^0_{x_1x_3}\right)\gamma,$$

$$\ddot{\beta} + 2n\dot{\alpha} = \left(U^0_{x_1x_2}\right)\alpha + \left(U^0_{x_2x_2}\right)\beta + \left(U^0_{x_2x_3}\right)\gamma, \quad (4.5)$$

$$\ddot{\gamma} = \left(U^0_{x_1x_3}\right)\alpha + \left(U^0_{x_2x_3}\right)\beta + \left(U^0_{x_3x_3}\right)\gamma.$$

The system (4.5) is the variational equations corresponding to the equations of motion (3.3) and (3.4). Only the linear terms in $\alpha, \beta \; and \; \gamma$ are taken, the second order partial derivatives of *U* are denoted by the subscripts and the superscript 0 denotes that the partial derivatives are to be evaluated at the libration point under consideration.

In order to consider the motion near any libration point in the x, y-plane, we let solutions of the first two equations of (4.5) be

$$\alpha = Aexp(\lambda t) \; and \; \beta = Bexp(\lambda t)$$

so that $\dot{\alpha} = A\lambda exp(\lambda t), \quad \ddot{\alpha} = A\lambda^2 exp(\lambda t)$ (4.6)

$$\dot{\beta} = B\lambda exp(\lambda t), \quad \ddot{\beta} = B\lambda^2 exp(\lambda t)$$

where A, B and λ are constants.

Substituting the equations in system (4.6) into the first two of system (4.5), we have

$$\lambda^2 Aexp(\lambda t) - 2n\lambda Bexp(\lambda t) = Aexp(\lambda t)U^0_{x_1x_1} + Bexp(\lambda t)U^0_{x_1x_2},$$

$$\lambda^2 Bexp(\lambda t) + 2n\lambda Aexp(\lambda t) = Aexp(\lambda t)U^0_{x_1x_2} + Bexp(\lambda t)U^0_{x_2x_2}.$$

Dividing throughout by $exp(\lambda t)$, we have

$$\lambda^2 A - 2n\lambda B = U^0_{x_1x_1}A + U^0_{x_1x_2}B,$$

$$\lambda^2 B + 2n\lambda A = U^0_{x_1x_2}A + U^0_{x_2x_2}B$$

or

$$\lambda^2 A - 2n\lambda B - U^0_{x_1x_1}A - U^0_{x_1x_2}B = 0,$$

$$\lambda^2 B + 2n\lambda A - U^0_{x_1x_2}A - U^0_{x_2x_2}B = 0$$

or

$$\left[\lambda^2 - U^0_{x_1x_1}\right]A - \left[2n\lambda + U^0_{x_1x_2}\right]B = 0,$$

$$\left[\lambda^2 - U^0_{x_2x_2}\right]B + \left[2n\lambda - U^0_{x_1x_2}\right]A = 0.$$

The system can be written in matrix notation as

$$\begin{pmatrix} \left(\lambda^2 - U^0_{x_1x_1}\right) & -\left(2n\lambda + U^0_{x_1x_2}\right) \\ \left(2n\lambda - U^0_{x_1x_2}\right) & \left(\lambda^2 - U^0_{x_2x_2}\right) \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

This has a non zero solution if

$$\begin{vmatrix} \left(\lambda^2 - U^0_{x_1x_1}\right) & -\left(2n\lambda + U^0_{x_1x_2}\right) \\ \left(2n\lambda - U^0_{x_1x_2}\right) & \left(\lambda^2 - U^0_{x_2x_2}\right) \end{vmatrix} = 0$$

Expanding the determinant, we have

$$\left[\lambda^2 - U^0_{x_1x_1}\right]\left[\lambda^2 - U^0_{x_2x_2}\right] + \left[2n\lambda + U^0_{x_1x_2}\right]\left[2n\lambda - U^0_{x_1x_2}\right] = 0$$

or

$$\lambda^4 - \lambda^2 U^0_{x_2x_2} - \lambda^2 U^0_{x_1x_1} + U^0_{x_1x_1}U^0_{x_2x_2} + 4n^2\lambda^2 - 2n\lambda U^0_{x_1x_2} + 2n\lambda U^0_{x_1x_2} - \left(U^0_{x_1x_2}\right)^2 = 0$$

Hence,

$$\lambda^4 - \left(U^0_{x_1x_1} + U^0_{x_2x_2} - 4n^2\right)\lambda^2 + U^0_{x_1x_1}U^0_{x_2x_2} - \left(U^0_{x_1x_2}\right)^2 = 0 \qquad (4.7)$$

Equation (4.7) is the characteristic equation corresponding to the variational equations (4.5) in the case when motion is considered in the x, y plane.

4.4 Jacobi Integral

Multiply the first, second and third equation of system (3.4) by $2\dot{x}_1$, $2\dot{x}_2$ and $2\dot{x}_3$ respectively, we have

$$2\ddot{x}_1\dot{x}_1 - 4n\dot{x}_1\dot{x}_2 = 2\dot{x}_1 U_{x_1},$$

$$2\ddot{x}_2\dot{x}_2 + 4n\dot{x}_1\dot{x}_2 = 2\dot{x}_2 U_{x_2},$$

$$2\ddot{x}_3\dot{x}_3 = 2\dot{x}_3 U_{x_3},$$

adding the equations above, we have

$$2\ddot{x}_1\dot{x}_1 + 2\ddot{x}_2\dot{x}_2 + 2\ddot{x}_3\dot{x}_3 = 2\dot{x}_1 U_{x_1} + 2\dot{x}_2 U_{x_2} + 2\dot{x}_3 U_{x_3} \qquad (4.8)$$

Recall that $\frac{d\dot{x}_1^2}{dt} = 2\ddot{x}_1\dot{x}_1$, $\frac{d\dot{x}_2^2}{dt} = 2\ddot{x}_2\dot{x}_2$ *and* $\frac{d\dot{x}_3^2}{dt} = 2\ddot{x}_3\dot{x}_3$ *and unsing them*

in (4.8), we have

$$\frac{d\dot{x}_1^2}{dt} + \frac{d\dot{x}_2^2}{dt} + \frac{d\dot{x}_3^2}{dt} = \frac{d2U}{dt} \qquad or \qquad \frac{d}{dx}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - 2U) = 0$$

Integrating both side, we have

$$\int \frac{d}{dx}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - 2U) = \int o$$

or $\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - 2U + C = 0$, where C is the Jacobian constant.

Hence, the equations of motion in both problems admit the Jacobi integral.

4.5 The Libration Points (Case 1)

The libration points are the singularity of the manifold

$$F(x_1, x_2, x_{2,}\dot{x}_1, \dot{x}_2, \dot{x}_3) = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - 2U + C = 0,$$

that is, they are the solutions of the equations

$$U_{x_1} = U_{x_2} = U_{x_3} = 0 \tag{4.9}$$

$$U_{x_1} = D\left[-2\pi\rho_1 A_1 x_1 + \frac{\mu(1-x_1)}{r_2^3} + \frac{3\mu\alpha_2(1-x_1)}{5r_2^5} - \frac{15\mu\alpha_3(1-x_1)}{8r_2^7} + n^2(x_1-\mu)\right]$$

$$U_{x_2} = Dx_2\left[-2\pi\rho_1 A_1 - \frac{\mu}{r_2^3} - \frac{3\mu\alpha_2}{2r_2^5} + \frac{15\mu\alpha_3}{8r_2^7} + n^2\right] \tag{4.10}$$

$$U_{x_3} = Dx_3\left[-2\pi\rho_1 A_2 - \frac{\mu}{r_2^3} - \frac{3\mu\alpha_2}{2r_2^5} + \frac{15\mu\alpha_3}{8r_2^7}\right]$$

$r_2^2 = (1-x_1)^2 + x_2^2 + x_3^2.$

4.5.1 Axial libration points

The locations of the axial libration points are the solutions of the system (4.10) when $x_1 \neq 0, x_2 = x_3 = 0,\ D \neq 0$ these points lie on the $x_1 - axis$. Now, from system (4.10), we have

$$U_{x_1} = D\left[-2\pi\rho_1 A_1 x_1 - \frac{\mu(1-x_1)}{r_2^3} - \frac{3\mu\alpha_2(1-x_1)}{5r_2^5} + \frac{15\mu\alpha_3(1-x_1)}{8r_2^7} + n^2(x_1-\mu)\right]$$

$$= 0$$

or

$$D\left[-2\pi\rho_1 A_1 x_1 + \frac{\mu(1-x_1)}{|1-x_1|^3} + \frac{3\mu\alpha_2(1-x_1)}{2|1-x_1|^5} - \frac{15\mu\alpha_3(1-x_1)}{8|1-x_1|^7} + n^2(x_1-\mu)\right] = 0$$

Substituting the expression for n^2, we have

$$D\left[-2\pi\rho_1 A_1 x_1 + \frac{\mu}{|1-x_1|^3} + \frac{3\mu\alpha_2}{2|1-x_1|^4} - \frac{15\mu\alpha_3}{8|1-x_1|^6} + \left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right)(x_1-\mu)\right] = 0$$

and since $D \neq 0$, we have

$$-2\pi\rho_1 A_1 x_1 + \frac{\mu}{|1-x_1|^2} + \frac{3\mu\alpha_2}{2|1-x_1|^4} - \frac{15\mu\alpha_3}{8|1-x_1|^6} + \left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right)(x_1-\mu) = 0. \quad (4.11)$$

First, we find the roots of the equation (4.11) when the primaries are spherical (i.e., $\alpha_i = 0\,, i = 1,2,3$). In this case, we have

$$-2\pi\rho_1 A_1 x_1 + \frac{\mu}{|1-x_1|^2} + (x_1-\mu) = 0. \quad (4.12)$$

We see that $x_1 = 0$ satisfies the equation (4.12) above. Therefore, when $\alpha_i = 0\,, (i = 1,2,3)$, the centre of the first primary is a libration point for all values of the parameters $\mu, A_1, A_2, \rho_1 \; and \; D$. Now we find the other roots of equation (4.12) satisfying the condition $0 < |x_1| < 1$. Hence, from equation (4.12), we have

$$(1-x_1)^2[-2\pi\rho_1 A_1 x_1 + (x_1-\mu)] + \mu = 0$$

or $\quad (1-2x_1+x_1^2)[(1-2\pi\rho_1 A_1)x_1 - \mu] + \mu = 0$

or $\quad (1-2\pi\rho_1 A_1)x_1 - \mu + (4\pi\rho_1 A_1 - 2)x_1^2 + 2\mu x_1 + (1-2\pi\rho_1 A_1)x_1^3 - \mu x_1^2 + \mu = 0$

or $\quad (1-2\pi\rho_1 A_1)x_1^3 + [4\pi\rho_1 A_1 - (2+\mu)]x_1^2 + (1+2\mu-2\pi\rho_1 A_1)x_1 = 0$

or $\quad (1-2\pi\rho_1 A_1)x_1^2 + [4\pi\rho_1 A_1 - (2+\mu)]x_1 + (1+2\mu-2\pi\rho_1 A_1) = 0 \quad (4.13)$

the roots of equation (4.13) are

x_{11}, x_{12}

$$= \frac{4\pi\rho_1 A_1 - (2+\mu) \pm \left[\left(4\pi\rho_1 A_1 - (2+\mu)\right)^2 - 4(1-2\pi\rho_1 A_1)(1+2\mu-2\pi\rho_1 A_1)\right]^{\frac{1}{2}}}{2(1-2\pi\rho_1 A_1)}$$

$$= \frac{4\pi\rho_1 A_1 - (2+\mu) \pm \left[\begin{matrix}16(\pi\rho_1 A_1)^2 - 16\pi\rho_1 A_1 - 8\mu\pi\rho_1 A_1 + 4\mu + 4 + \mu^2 - 4 - 8\mu + \\ +16\pi\rho_1 A_1 + 16\mu\pi\rho_1 A_1 - 16(\pi\rho_1 A_1)^2\end{matrix}\right]^{\frac{1}{2}}}{2(1-2\pi\rho_1 A_1)}$$

$$= \frac{2(1-2\pi\rho_1 A_1) + \mu \pm [\mu^2 + 8\mu\pi\rho_1 A_1 - 4\mu]^{\frac{1}{2}}}{2(1-2\pi\rho_1 A_1)} = 1 + \frac{\mu \pm [\mu^2 + 8\mu\pi\rho_1 A_1 - 4\mu]^{\frac{1}{2}}}{2(1-2\pi\rho_1 A_1)}$$

Therefore, the roots of equation (4.12) are

$$x_{11} = 1 + \frac{\mu + [\mu^2 + 8\mu\pi\rho_1 A_1 - 4\mu]^{\frac{1}{2}}}{2(1-2\pi\rho_1 A_1)}, \quad x_{12} = 1 + \frac{\mu - [\mu^2 + 8\mu\pi\rho_1 A_1 - 4\mu]^{\frac{1}{2}}}{2(1-2\pi\rho_1 A_1)}. \qquad (4.14)$$

The term A_1 which appears in the equation of system (4.10) is due to the fluid mass effect, therefore these roots will be real if the discriminant is non-negative, that is if

$$\mu^2 + 8\mu\pi\rho_1 A_1 - 4\mu \geq 0$$

this gives

$$\mu + 8\pi\rho_1 A_1 - 4 \geq 0 \qquad (4.15)$$

or $$\frac{1}{4}\mu + 2\pi\rho_1 A_1 - 1 \geq 0. \qquad (4.16)$$

when $\frac{1}{4}\mu \geq 1 - 2\pi\rho_1 A_1 > 0$, both roots are greater than unity and we ignore them because they lie outside the first primary. Now if $1 - 2\pi\rho_1 A_1 < 0$, we have $x_{11} < 1 \; and \; x_{12} > 1$. we further see that $x_{11} \geq -1$ when $1 - 2\pi\rho_1 A_1 < -\frac{3}{4}\mu$, the point $(x_{11}, 0,0)$ lies within the first primary if $|x_{11}| < a_1$. When $1 - 2\pi\rho_1 A_1 < -\frac{3}{4}\mu, |x_{11}| < a_1$; x_{11} is a root of equation (4.12). Hence, for $1 - 2\pi\rho_1 A_1 = 0$ the only root is $x_{11} = 2$ which lies outside

the first primary and we ignore it, thus $\alpha_1 = \alpha_2 = \alpha_3 = x_1 = 0$ is always a root of equation (4.12), also $x_1 = x_{11}$ is a root provided $1 - 2\pi\rho_1 A_1 < -\frac{3}{4}\mu$, $|x_{11}| < a_1$.

Now, we find the roots of equation (4.12) when oblateness of the primaries is considered, (i.e., $\alpha_1 \neq 0,\ \alpha_2 \neq 0\ and\ \alpha_3 \neq 0$). In this case we let them be

$$x_1 = 0 + p_1, \qquad |p_1| \ll 1, \tag{4.17}$$

$$x_1 = x_{11} + p_2 \qquad |p_2| \ll 1.$$

Substituting the value of $x_1 = 0 + p_1$ into equation (4.11), we have

$$-2\pi\rho_1 A_1 p_1 + \frac{\mu}{(1-p_1)^2} + \frac{3\mu\alpha_2}{2(1-p_1)^4} - \frac{15\mu\alpha_3}{8(1-p_1)^6} + \left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right)(p_1 - \mu) = 0$$

or

$$-2\pi\rho_1 A_1 p_1 + \mu(1-p_1)^{-2} + \frac{3}{2}\mu\alpha_2(1-p_1)^{-4} - \frac{15}{8}\mu\alpha_3(1-p_1)^{-6} + \left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right)(p_1 - \mu) = 0.$$

Expanding and ignoring second and higher powers of p_1, as is are very small quantities, we obtain

$$-2\pi\rho_1 A_1 p_1 + \mu(1+2p_1) + \frac{3}{2}\mu\alpha_2(1+4p_1) - \frac{15}{8}\mu\alpha_3(1+6p_1) + \left(1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right)(p_1 - \mu) = 0,$$

Expanding and ignoring the product of $\alpha_i (i = 1,2,3)\ and\ p_1$, we have

$$-2\pi\rho_1 A_1 p_1 + \mu + 2\mu p_1 + \tfrac{3}{2}\mu\alpha_2 - \tfrac{15}{8}\mu\alpha_3 + p_1 - \mu - \tfrac{3}{2}\mu\alpha_1 - \tfrac{3}{2}\mu\alpha_2 - \tfrac{15}{8}\mu\alpha_3 = 0$$

$$or \quad -2\pi\rho_1 A_1 p_1 + 2\mu p_1 + p_1 - \frac{3}{2}\mu\alpha_1 = 0$$

$$or \quad p_1[-2\pi\rho_1 A_1 + (1+2\mu)] = \frac{3}{2}\mu\alpha_1$$

$$or \quad -p_1[2\pi\rho_1 A_1 - (1+2\mu)] = \frac{3}{2}\mu\alpha_1$$

$$p_1 = -\frac{3\alpha_1}{2}\frac{\mu}{[2\pi\rho_1 A_1 - (1+2\mu)]} \tag{4.18}$$

Similarly, substituting the value of $x_1 = x_{11} + p_2$ into equation (4.11), we have

$$-2\pi\rho_1 A_1(x_{11}+p_2) + \frac{\mu}{(1-x_{11}-p_2)^2} + \frac{3\mu\alpha_2}{2(1-x_{11}-p_2)^4} - \frac{15\mu\alpha_3}{8(1-x_{11}-p_2)^6}$$
$$+\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}+p_2-\mu) = 0$$

or

$$(1-x_{11}-p_2)^6\left[-2\pi\rho_1 A_1(x_{11}+p_2)+\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}+p_2-\mu)\right]$$
$$+\mu(1-x_{11}-p_2)^4+\frac{3}{2}\mu\alpha_2(1-x_{11}-p_2)^2-\frac{15\mu\alpha_3}{8}=0$$

Expanding and ignoring the second and higher power of p_2, we have

$$[(1-x_{11})^6 - 6\,p_2(1-x_{11})^5]\left[-2\pi\rho_1 A_1 x_{11} - 2\pi\rho_1 A_1\,p_2 + p_2\right.$$
$$\left.+\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right] + \mu[(1-x_{11})^4 - 4(1-x_{11})^3]$$
$$+\frac{3}{2}\mu\alpha_2[(1-x_{11})^2 - 2(1-x_{11})] - \frac{15\mu\alpha_3}{8} = 0$$

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]^2\Big\{[(1-x_{11})^2$$

$$-2\,p_2(1-x_{11})]\Big[-2\pi\rho_1 A_1 x_{11}+p_2(1-2\pi\rho_1 A_1)$$

$$+\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\Big]+\mu\Big\}$$

$$+\frac{3}{2}\mu\alpha_2[(1-x_{11})^2-2(1-x_{11})]=\frac{15\mu\alpha_3}{8}$$

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]^2\Big\{(1-x_{11})^2\Big[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)$$

$$-2\pi\rho_1 A_1 x_{11}\Big]+p_2(1-x_{11})^2(1-2\pi\rho_1 A_1)$$

$$-2\,p_2(1-x_{11})[x_{11}-\mu-2\pi\rho_1 A_1 x_{11}]+\mu\Big\}+\frac{3}{2}\mu\alpha_2(1-x_{11})^2=\frac{15\mu\alpha_3}{8}$$

(Ignoring the product of $\alpha_i(i=1,2,3)$ and p_2)

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]^2\Big\{(1-x_{11})^2\Big[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)$$

$$-2\pi\rho_1 A_1 x_{11}\Big]+p_2(1-x_{11})[(1-x_{11})(1-2\pi\rho_1 A_1)]$$

$$+p_2(1-x_{11})[-2x_{11}+2\mu+4\pi\rho_1 A_1 x_{11}]+\mu\Big\}$$

$$=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]^2\left\{(1-x_{11})^2\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right.\right.$$
$$\left.-2\pi\rho_1A_1x_{11}\right]$$
$$+p_2(1-x_{11})[1-2\pi\rho_1A_1-x_{11}+2\pi\rho_1A_1x_{11}-2x_{11}+2\mu$$
$$\left.+4\pi\rho_1A_1x_{11}]+\mu\right\}=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]^2\left\{(1-x_{11})^2\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right.\right.$$
$$\left.-2\pi\rho_1A_1x_{11}\right]+p_2(1-x_{11})[1-3x_{11}+2\mu-2\pi\rho_1A_1+6\pi\rho_1A_1x_{11}]$$
$$\left.+\mu\right\}=\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]^2\left\{(1-x_{11})^2\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right.\right.$$
$$\left.-2\pi\rho_1A_1x_{11}\right]+p_2(1-x_{11})[(1-3x_{11})-2\pi\rho_1A_1(1-3x_{11})+2\mu]$$
$$\left.+\mu\right\}=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]^2\left\{(1-x_{11})^2\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right.\right.$$
$$\left.-2\pi\rho_1A_1x_{11}\right]+p_2(1-x_{11})[(1-3x_{11})(1-2\pi\rho_1A_1)+2\mu]+\mu\Big\}$$
$$=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]\Bigg[[(1-x_{11})^2$$

$$-2\,p_2(1-x_{11})]\Big\{(1-x_{11})^2\Big[\Big(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\Big)(x_{11}-\mu)$$

$$-2\pi\rho_1 A_1 x_{11}\Big]+p_2(1-x_{11})[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]+\mu\Big\}\Bigg]$$

$$=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

expanding and ignoring the second and higher power of p_2, we have

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]\Big\{(1-x_{11})^4\Big[\Big(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\Big)(x_{11}-\mu)$$

$$-2\pi\rho_1 A_1 x_{11}\Big]+p_2(1-x_{11})^3[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]$$

$$+\mu(1-x_{11})^2$$

$$-2p_2(1-x_{11})^3\Big[\Big(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\Big)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\Big]$$

$$-2\mu p_2(1-x_{11})\Big\}=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

expanding and ignoring the product of p_2 and $\alpha_1(i=1,2,3)$, we have

$$[(1-x_{11})^2-2\,p_2(1-x_{11})]\Big\{(1-x_{11})^4\Big[\Big(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\Big)(x_{11}-\mu)$$

$$-2\pi\rho_1 A_1 x_{11}\Big]+p_2(1-x_{11})^3[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]$$

$$+\mu(1-x_{11})^2-2p_2(1-x_{11})^3[(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}]$$

$$-2\mu p_2(1-x_{11})\Big\}=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

expanding and neglecting the second and higher power of p_2, we have

$$(1-x_{11})^6\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]$$
$$+p_2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]+\mu(1-x_{11})^4$$
$$-2p_2(1-x_{11})^5[(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}]-2\mu p_2(1-x_{11})^3$$
$$-2p_2(1-x_{11})^5\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]$$
$$-2\mu p_2(1-x_{11})^3=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

expanding and ignoring the product of p_2 and $\alpha_1(i=1,2,3)$, we have

$$(1-x_{11})^6\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]$$
$$+p_2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]+\mu(1-x_{11})^4$$
$$-2p_2(1-x_{11})^5[(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}]--2\mu p_2(1-x_{11})^3$$
$$-2p_2(1-x_{11})^5[(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}]-2\mu p_2(1-x_{11})^3$$
$$=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$(1-x_{11})^6\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]$$
$$+p_2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]+\mu(1-x_{11})^4$$
$$-4p_2(1-x_{11})^5[(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}]-4\mu p_2(1-x_{11})^3$$
$$=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$(1-x_{11})^6\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]$$
$$+p_2(1-x_{11})^5[(1--3x_{11})(1-2\pi\rho_1 A_1)+2\mu]+\mu(1-x_{11})^4$$
$$-4p_2(1-x_{11})^3[(1-x_{11})^2(x_{11}-\mu--2\pi\rho_1 A_1 x_{11})+\mu]$$
$$=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}. \qquad (4.19)$$

From equation (4.12), we have

$$(1-x_1)^2[-2\pi\rho_1 A_1 x_1+(x_1-\mu)]+\mu=0,$$

but x_1 is a root of equation (4.12), so putting $x_1=x_{11}$, we have

$$(1-x_{11})^2(x_{11}-\mu-2\pi\rho_1 A_1 x_{11})+\mu=0. \qquad (4.20)$$

Now using (4.20) in (4.19), we have equation (4.19) reduced to

$$(1-x_{11})^6\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]$$
$$+p_2(1-x_{11})^5[(1—3x_{11})(1-2\pi\rho_1 A_1)+2\mu]+\mu(1-x_{11})^4$$
$$=-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$p_2(1-x_{11})^5[(1—3x_{11})(1-2\pi\rho_1 A_1)+2\mu]$$
$$=-(1-x_{11})^6\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]$$
$$-\mu(1-x_{11})^4-\frac{3}{2}\mu\alpha_2(1-x_{11})^2+\frac{15\mu\alpha_3}{8}$$

or

$$p_2 = \frac{-(1-x_{11})^6\left[\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]-\mu(1-x_{11})^4}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2(1-x_{11})^2}{2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

or

$$p_2 = \frac{\begin{array}{c}-(1-x_{11})^6\left[(x_{11}-\mu)+\left(\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}\right]-\\ -\mu(1-x_{11})^4\end{array}}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2(1-x_{11})^2}{2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

or

$$p_2 = \frac{\begin{array}{c}-(1-x_{11})^6\left[\left(\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right]-\\ -(1-x_{11})^6[(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}]-\mu(1-x_{11})^4\end{array}}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2(1-x_{11})^2}{2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

or

$$p_2 = \frac{\begin{array}{c} -(1-x_{11})^6\left[\left(\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right]- \\ -(1-x_{11})^4\{[(1-x_{11})^2(x_{11}-\mu)-2\pi\rho_1 A_1 x_{11}]+\mu\} \end{array}}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2(1-x_{11})^2}{2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]} \qquad (4.21)$$

From equation (4.20), we have

$$-2\pi\rho_1 A_1 x_{11} = \frac{-\mu}{(1-x_{11})^2} - (x_{11}-\mu) \qquad (4.22)$$

Now, using (4.22) in (4.21), we have

$$p_2 = \frac{\begin{array}{c} -(1-x_{11})^6\left[\left(\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right]- \\ -(1-x_{11})^4\left\{\left[(1-x_{11})^2(x_{11}-\mu)-\frac{-\mu}{(1-x_{11})^2}-(x_{11}-\mu)\right]+\mu\right\} \end{array}}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2(1-x_{11})^2}{2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

or

$$p_2 = \frac{-(1-x_{11})^6\left[\left(\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right]-(1-x_{11})^4[-\mu+\mu]}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2(1-x_{11})^2}{2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

or

$$p_2 = \frac{-(1-x_{11})^6\left[\left(\frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right]}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2(1-x_{11})^2}{2(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

or

$$p_2 = \frac{-(1-x_{11})^6\left[\left(\frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3\right)(x_{11}-\mu)\right]}{(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$-\frac{3\mu\alpha_2}{2(1-x_{11})^3[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]}$$

$$+\frac{15\mu\alpha_3}{8(1-x_{11})^5[(1-3x_{11})(1-2\pi\rho_1 A_1)+2\mu]} \qquad (4.23)$$

A substitution of equation (4.21) in the second equation of system (4.17) gives the position of the other libration point near the centre of the first primary.

4.5.2 Locations of circular points

The locations of the circular points are obtained by using the first two equations of system (4.10) with the conditions $x_1 \neq 0, x_2 \neq 0,\ U_{x_1} = U_{x_2} = 0$.

Now from first two equations of system (4.10), we have

$$D\left[-2\pi\rho_1 A_1 x_1 + \frac{\mu(1-x_1)}{[(1-x_1)^2 + x_2^2]^{3/2}} + \frac{3\mu\alpha_2(1-x_1)}{2[(1-x_1)^2 + x_2^2]^{5/2}} - \frac{15\mu\alpha_3(1-x_1)}{8[(1-x_1)^2 + x_2^2]^{7/2}}\right.$$

$$\left. + n^2(x_1-\mu)\right] = 0$$

and

$$Dx_2\left[-2\pi\rho_1 A_1 - \frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} - \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} + \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} + n^2\right] = 0.$$

Since $D \neq 0$, we have

$$-2\pi\rho_1 A_1 x_1 + \frac{\mu(1-x_1)}{[(1-x_1)^2 + x_2^2]^{3/2}} + \frac{3\mu\alpha_2(1-x_1)}{2[(1-x_1)^2 + x_2^2]^{5/2}} - \frac{15\mu\alpha_3(1-x_1)}{8[(1-x_1)^2 + x_2^2]^{7/2}} + n^2(x_1 - \mu) = 0$$

and

$$x_2\left[-2\pi\rho_1 A_1 - \frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} - \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} + \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} + n^2\right] = 0$$

or

$$x_1\left[-2\pi\rho_1 A_1 - \frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} - \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} + \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} + n^2\right] + \frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} + \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} - \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} - \mu n^2 = 0 \qquad (4.24)$$

and

$$x_2\left[-2\pi\rho_1 A_1 - \frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} - \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} + \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} + n^2\right] = 0. \qquad (4.25)$$

Since $x_2 \neq 0$, we have from equation (4.25)

$$-2\pi\rho_1 A_1 - \frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} - \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} + \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} + n^2 = 0. \quad (4.26)$$

Using (4.26) in (4.24), we have

$$\frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} + \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} - \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} - \mu n^2 = 0 \quad (4.27)$$

or

$$\mu\left[\frac{1}{[(1-x_1)^2 + x_2^2]^{3/2}} + \frac{3\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} - \frac{15\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} - n^2\right] = 0$$

Since $\mu \neq 0$, we have

$$\frac{1}{[(1-x_1)^2 + x_2^2]^{3/2}} + \frac{3\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} - \frac{15\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} - n^2 = 0. \quad (4.28)$$

writing $(1-x_1)^2 + x_2^2 = r^2$, we have

$$\frac{1}{r^3} + \frac{3\alpha_2}{2r^5} - \frac{15\alpha_3}{8r^7} - n^2 = 0$$

or

$$n^2 r^7 - r^4 - \frac{3\alpha_2 r^2}{2} + \frac{15\alpha_3}{8} = 0. \quad (4.29)$$

Now, let

$$r = 1 + \varepsilon, \qquad \varepsilon \ll 1 \quad (4.30)$$

Substituting (4.30) in (4.29), we have

$$n^2(1+\varepsilon)^7-(1+\varepsilon)^4-\frac{3\alpha_2(1+\varepsilon)^2}{2}+\frac{15\alpha_3}{8}=0.$$

Expanding and neglecting the second and higher power of ε, we have

$$n^2(1+7\varepsilon)-(1+4\varepsilon)-\frac{3\alpha_2}{2}(1+2\varepsilon)+\frac{15\alpha_3}{8}=0$$

or

$$\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(1+7\varepsilon)-(1+4\varepsilon)-\frac{3\alpha_2}{2}(1+2\varepsilon)+\frac{15\alpha_3}{8}=0$$

neglecting the product of ε and α_i $(i=1,2,3)$ we have

$$1+7\varepsilon+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3-1-4\varepsilon-\frac{3\alpha_2}{2}+\frac{15\alpha_3}{8}=0$$

$$or \qquad 7\varepsilon-4\varepsilon+\frac{3}{2}\alpha_1=0$$

$$or \qquad 3\varepsilon+\frac{3}{2}\alpha_1=0$$

$$\varepsilon=-\frac{1}{2}\alpha_1. \tag{4.31}$$

Now, using (4.31) in (4.30), we have

$$r=1-\frac{1}{2}\alpha_1 \tag{4.32}$$

so that

$$r^2=\left(1-\frac{1}{2}\alpha_1\right)^2,\quad r^3=\left(1-\frac{1}{2}\alpha_1\right)^3,\quad r^5=\left(1-\frac{1}{2}\alpha_1\right)^5 \tag{4.33}$$

also

$$r^{-3} = \left(1 - \frac{1}{2}\alpha_1\right)^{-3} = 1 + \frac{3}{2}\alpha_1 ,$$

$$r^{-5} = \left(1 - \frac{1}{2}\alpha_1\right)^{-5} = 1 + \frac{5}{2}\alpha_1, \qquad (4.34)$$

$$r^{-7} = \left(1 - \frac{1}{2}\alpha_1\right)^{-7} = 1 + \frac{7}{2}\alpha_1.$$

Now, from equation (4.27), we have

$$\frac{\mu}{[(1-x_1)^2 + x_2^2]^{3/2}} + \frac{3\mu\alpha_2}{2[(1-x_1)^2 + x_2^2]^{5/2}} - \frac{15\mu\alpha_3}{8[(1-x_1)^2 + x_2^2]^{7/2}} = \mu n^2 \qquad (4.35)$$

Substituting equation (4.35) in equation (4.26), we have

$$-2\pi\rho_1 A_1 - \mu n^2 + n^2 = 0$$

or $$2\pi\rho_1 A_1 + \mu n^2 - n^2 = 0$$

or $$2\pi\rho_1 A_1 = n^2(1-\mu).$$

When $2\pi\rho_1 A_1 = n^2(1-\mu)$ the points on the circle given by $r^2 = (1-x_1)^2 + x_2^2$ with $x_3 = 0$ and $r = 1 - \frac{1}{2}\alpha_1$ lying within the first primary are also libration points, the general coordinates of these circular points are given by $(1 + r\cos\theta,\ r\sin\theta)$ where θ is a parameter.

4.6 Location of Libration Point When $D > 0$ (Case 2)

Those points at which the velocity and acceleration of the infinitesimal mass are zero are the libration points. These points are the solutions of the equations

$U_{x_1} = U_{x_2} = U_{x_3} = 0$, that is

$$U_{x_1} = D\left[1 + 2\mu - C_1 + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2)\right]x_1 = 0,$$

$$U_{x_2} = D\left[1 - \mu - C_2 + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3\mu\sigma_2}{2}\right]x_2 = 0, \qquad (4.36)$$

$$U_{x_3} = -D\left[\mu + C_3 + \frac{3\mu}{2}(4\sigma_1 - \sigma_2)\right]x_3 = 0.$$

Since $D \neq 0$, it implies that

$$\left[1 + 2\mu - C_1 + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2)\right]x_1 = 0,$$

$$\left[1 - \mu - C_2 + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3\mu\sigma_2}{2}\right]x_2 = 0, \qquad (4.37)$$

$$-D\left[\mu + C_3 + \frac{3\mu}{2}(4\sigma_1 - \sigma_2)\right]x_3 = 0.$$

Since $(1 + 2\mu) < C_1$ and $(1 - \mu) < C_2$ by Plastino and Plastino (1995), $0 < \mu < 1$, and $0 < \sigma_1, \sigma_2 \ll 1$; the coefficients of $x_1,\ x_2\ and\ x_3$ in (4.37) are non zero. This implies that the origin (0, 0, 0) is the only libration point.

4.7 Stability of Axial Libration Points (Case 1)

The second order partial derivatives are to be evaluated at the point $(x_l, 0, 0)$.

Now from the first equation of system (4.10), we have

$$U_{x_1} = D\left[-2\pi\rho_1 A_1 x_1 + \frac{\mu(1 - x_1)}{r_2^3} + \frac{3\mu\alpha_2(1 - x_1)}{5r_2^5} - \frac{15\mu\alpha_3(1 - x_1)}{8r_2^7} + n^2(x_1 - \mu)\right]$$

The second derivative with respect to x_1 gives

$$U_{x_1x_1} = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{-1}{r_2^3} + \frac{3(1-x_1)^2}{r_2^5}\right\} + \frac{3\mu\alpha_2}{2}\left\{\frac{-1}{r_2^5} + \frac{5(1-x_1)^2}{r_2^7}\right\}\right.$$

$$\left. - \frac{15\mu\alpha_3}{8}\left\{\frac{-1}{r_2^7} + \frac{7(1-x_1)^2}{r_2^9}\right\} + n^2\right] \qquad (4.38)$$

similarly, the second derivative with respect to x_2 gives

$$U_{x_1x_2} = D\left[\frac{-3\mu(1-x_1)x_2}{r_2^5} - \frac{15\mu\alpha_2(1-x_1)x_2}{2r_2^7} + \frac{105\mu\alpha_3(1-x_1)x_2}{8r_2^9}\right]$$

$$or \qquad U_{x_1x_2} = D3\mu(1-x_1)x_2\left[\frac{-1}{r_2^5} - \frac{5\alpha_2}{2r_2^7} + \frac{35\alpha_3}{8r_2^9}\right] \qquad (4.39)$$

also, the second derivative with respect to x_3 gives

$$U_{x_1x_3} = D\left[\frac{-3\mu(1-x_1)x_3}{r_2^5} - \frac{15\mu\alpha_2(1-x_1)x_3}{2r_2^7} + \frac{105\mu\alpha_3(1-x_1)x_3}{8r_2^9}\right]$$

$$or \qquad U_{x_1x_2} = D3\mu(1-x_1)x_3\left[\frac{-1}{r_2^5} - \frac{5\alpha_2}{2r_2^7} + \frac{35\alpha_3}{8r_2^9}\right]. \qquad (5.40)$$

In the same vein, the second equation of system (4.10), gives

$$U_{x_2} = D\left[-2\pi\rho_1 A_1 x_2 - \frac{\mu x_2}{r_2^3} - \frac{3\mu\alpha_2 x_2}{2r_2^5} + \frac{15\mu\alpha_3 x_2}{8r_2^7} + n^2 x_2\right]$$

Differentiating $U_{x_2}$with respect to x_2, we have

$$U_{x_2x_2} = D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{1}{r_2^3} - \frac{3x_2^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{1}{r_2^5} - \frac{5x_2^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{1}{r_2^7} - \frac{7x_2^2}{r_2^9}\right\}\right.$$

$$\left. + n^2\right]. \qquad (5.41)$$

Differentiating $U_{x_2}$with respect to x_3, we have

$$U_{x_2x_3} = D\left[\frac{3\mu x_2 x_3}{r_2^5} + \frac{15\mu\alpha_2 x_2 x_3}{2r_2^7} - \frac{105\mu\alpha_3 x_2 x_3}{8r_2^9}\right]$$

$$or \qquad U_{x_2x_3} = 3D\mu x_2 x_3\left[\frac{1}{r_2^5} + \frac{5\alpha_2}{2r_2^7} - \frac{35\alpha_3}{8r_2^9}\right]. \tag{4.42}$$

From the third equation of system (4.10), we have

$$U_{x_3} = D\left[-2\pi\rho_1 A_2 x_3 - \frac{\mu x_3}{r_2^3} - \frac{3\mu\alpha_2 x_3}{2r_2^5} + \frac{15\mu\alpha_3 x_3}{8r_2^7}\right]$$

The second derivatives with respect to x_3 give

$$U_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{1}{r_2^3} - \frac{3x_3^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{1}{r_2^5} - \frac{5x_3^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{1}{r_2^7} - \frac{7x_3^2}{r_2^9}\right\}\right] \tag{4.43}$$

$$where \qquad r_2^2 = (1-x_1)^2 x_2^2 + x_3^2$$

Now, evaluating the values of the second order partial derivatives at the libration point $(x_l, 0, 0)$, where $x_l = p_1$ and $x_l = x_{11} + p_1$ stands for the first and second axial libration points respectively.

Substituting $x_1 = x_l, x_2 = 0\ and\ x_3 = 0$ in equation (4.38), we have

$$U_{x_1x_1}^0 = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{-1}{(1-x_l)^3} + \frac{3(1-x_l)^2}{(1-x_l)^5}\right\} + \frac{3\mu\alpha_2}{2}\left\{\frac{-1}{(1-x_l)^5} + \frac{5(1-x_l)^2}{(1-x_l)^7}\right\}\right.$$
$$\left. - \frac{15\mu\alpha_3}{8}\left\{\frac{-1}{(1-x_l)^7} + \frac{7(1-x_l)^2}{(1-x_l)^9}\right\} + n^2\right]$$

or

$$U_{x_1x_1}^0 = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{-1}{(1-x_l)^3} + \frac{3}{(1-x_l)^3}\right\} + \frac{3\mu\alpha_2}{2}\left\{\frac{-1}{(1-x_l)^5} + \frac{5}{(1-x_l)^5}\right\}\right.$$
$$\left. - \frac{15\mu\alpha_3}{8}\left\{\frac{-1}{(1-x_l)^7} + \frac{7}{(1-x_l)^7}\right\} + n^2\right]$$

or

$$U^0_{x_1x_1} = D\left[-2\pi\rho_1 A_1 + \frac{2\mu}{(1-x_l)^3} + \frac{3\mu\alpha_2}{2}\left\{\frac{4}{(1-x_l)^5}\right\} - \frac{15\mu\alpha_3}{8}\left\{\frac{6}{(1-x_l)^7}\right\} + n^2\right]$$

or

$$U^0_{x_1x_1} = D\left[-2\pi\rho_1 A_1 + \frac{2\mu}{(1-x_l)^3} + \frac{6\mu\alpha_2}{(1-x_l)^5} - \frac{45\mu\alpha_3}{4(1-x_l)^7} + n^2\right] \qquad (4.44)$$

Putting $x_1 = x_l$ in equation (4.11), we have

$$-2\pi\rho_1 A_1 x_l + \frac{\mu}{(1-x_l)^2} + \frac{3\mu\alpha_2}{2(1-x_l)^4} - \frac{15\mu\alpha_3}{8(1-x_l)^6} + n^2(x_l - \mu) = 0$$

or

$$-2\pi\rho_1 A_1 x_l = -\frac{\mu}{(1-x_l)^2} - \frac{3\mu\alpha_2}{2(1-x_l)^4} + \frac{15\mu\alpha_3}{8(1-x_l)^6} - n^2(x_l - \mu) = 0$$

$$-2\pi\rho_1 A_1 = -\frac{\mu}{x_l(1-x_l)^2} - \frac{3\mu\alpha_2}{2x_l(1-x_l)^4} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6} - \frac{n^2}{x_l}(x_l - \mu) \qquad (4.45)$$

Substituting equation (4.45) into equation (4.44), we have

$$U^0_{x_1x_1} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - \frac{3\mu\alpha_2}{2x_l(1-x_l)^4} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6} - \frac{n^2}{x_l}(x_l - \mu) + \frac{2\mu}{(1-x_l)^3} + \frac{6\mu\alpha_2}{(1-x_l)^5} - \frac{45\mu\alpha_3}{4(1-x_l)^7} + n^2\right]$$

or

$$U^0_{x_1x_1} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - + \frac{2\mu}{(1-x_l)^3}\frac{3\mu\alpha_2}{2x_l(1-x_l)^4} + \frac{6\mu\alpha_2}{(1-x_l)^5} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6} - \frac{45\mu\alpha_3}{4(1-x_l)^7} + n^2 - \frac{n^2}{x_l}(x_l - \mu)\right]$$

or

$$U^0_{x_1x_1} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - + \frac{2\mu}{(1-x_l)^3}\frac{3\mu\alpha_2}{2x_l(1-x_l)^4} + \frac{6\mu\alpha_2}{(1-x_l)^5} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6}\right.$$
$$\left. - \frac{45\mu\alpha_3}{4(1-x_l)^7} + n^2\left\{1 - - \frac{x_l+\mu}{x_l}\right\}\right]$$

or

$$U^0_{x_1x_1} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - + \frac{2\mu}{(1-x_l)^3}\frac{3\mu\alpha_2}{2x_l(1-x_l)^4} + \frac{6\mu\alpha_2}{(1-x_l)^5} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6}\right.$$
$$\left. - \frac{45\mu\alpha_3}{4(1-x_l)^7} + n^2\left\{\frac{x_l - x_l + \mu}{x_l}\right\}\right]$$

or

$$U^0_{x_1x_1} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - + \frac{2\mu}{(1-x_l)^3}\frac{3\mu\alpha_2}{2x_l(1-x_l)^4} + \frac{6\mu\alpha_2}{(1-x_l)^5} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6}\right.$$
$$\left. - \frac{45\mu\alpha_3}{4(1-x_l)^7} + \frac{n^2\mu}{x_l}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[-\frac{1}{x_l(1-x_l)^2} - + \frac{2}{(1-x_l)^3}\frac{3\alpha_2}{2x_l(1-x_l)^4} + \frac{6\alpha_2}{(1-x_l)^5} + \frac{n^2}{x_l} + \frac{15\alpha_3}{8x_l(1-x_l)^6}\right.$$
$$\left. - \frac{45\alpha_3}{4(1-x_l)^7}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[\frac{\begin{array}{c}-(1-x_l)^3 + 2x_l(1-x_l)^2 - \left(3/2\right)\alpha_2(1-x_l) + \\ + 6\alpha_2 x_l + n^2(1-x_l)^5\end{array}}{x_l(1-x_l)^5} + \frac{15\alpha_3(1-x_l) - 90x_l\alpha_3}{8x_l(1-x_l)^7}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[\frac{\begin{matrix}-(1-x_l)^3 \ + \ 2x_l(1-x_l)^2 - (3/2)\alpha_2(1-x_l) + \\ + 6\alpha_2 x_l \ + \ n^2(1-x_l)^5\end{matrix}}{x_l(1-x_l)^5} + \frac{15\alpha_3\{1-x_l-6x_l\}}{8x_l(1-x_l)^7}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[\frac{\begin{matrix}-(1-x_l)^3 \ + \ 2x_l(1-x_l)^2 - (3/2)\alpha_2(1-x_l) + \\ + 6\alpha_2 x_l \ + \ n^2(1-x_l)^5\end{matrix}}{x_l(1-x_l)^5} + \frac{15\alpha_3(1-7x_l)}{8x_l(1-x_l)^7}\right] \quad (4.46)$$

Substituting $x_1 = x_l, x_2 = 0 \ and \ x_3 = 0$ in equation (4.39), (4.40) and (4.42), we have

$$U^0_{x_1x_2} = U^0_{x_1x_3} = U^0_{x_2x_3} = 0 \quad (4.47)$$

From equation (4.41), we have

$$U_{x_2x_2} = D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{1}{r_2^3} - \frac{3x_2^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{1}{r_2^5} - \frac{5x_2^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{1}{r_2^7} - \frac{7x_2^2}{r_2^9}\right\} + n^2\right],$$

again, substituting $x_1 = x_l, x_2 = 0 \ and \ x_3 = 0$, we have

$$U_{x_2x_2} = D\left[-2\pi\rho_1 A_1 - \frac{\mu}{(1-x_l)^3} - \frac{3\mu\alpha_2}{2(1-x_l)^5} + \frac{15\mu\alpha_3}{8(1-x_l)^7} + n^2\right]. \quad (4.48)$$

Using equation (4.45) in equation (4.48), we have

$$U_{x_2x_2} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - \frac{3\mu\alpha_2}{2x_l(1-x_l)^4} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6} - \frac{n^2}{x_l}(x_l-\mu) - \frac{\mu}{(1-x_l)^3}\right.$$
$$\left. - \frac{3\mu\alpha_2}{2(1-x_l)^5} + \frac{15\mu\alpha_3}{8(1-x_l)^7} + n^2\right]$$

or

$$U_{x_2x_2} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - \frac{\mu}{(1-x_l)^3} - \frac{3\mu\alpha_2}{2x_l(1-x_l)^4} - \frac{3\mu\alpha_2}{2(1-x_l)^5} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6} + \frac{15\mu\alpha_3}{8(1-x_l)^7} + n^2 - \frac{n^2}{x_l}(x_l-\mu)\right]$$

or

$$U_{x_2x_2} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - \frac{\mu}{(1-x_l)^3} - \frac{3\mu\alpha_2}{2x_l(1-x_l)^4} - \frac{3\mu\alpha_2}{2(1-x_l)^5} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6} + \frac{15\mu\alpha_3}{8(1-x_l)^7} + n^2\left\{\frac{x_l - x_l + \mu}{x_l}\right\}\right]$$

or

$$U_{x_2x_2} = D\left[-\frac{\mu}{x_l(1-x_l)^2} - \frac{\mu}{(1-x_l)^3} - \frac{3\mu\alpha_2}{2x_l(1-x_l)^4} - \frac{3\mu\alpha_2}{2(1-x_l)^5} + \frac{15\mu\alpha_3}{8x_l(1-x_l)^6} + \frac{15\mu\alpha_3}{8(1-x_l)^7} + \frac{n^2\mu}{x_l}\right]$$

or

$$U_{x_2x_2} = D\mu\left[-\frac{1}{x_l(1-x_l)^2} - \frac{1}{(1-x_l)^3} - \frac{3\alpha_2}{2x_l(1-x_l)^4} - \frac{3\alpha_2}{2(1-x_l)^5} + \frac{15\alpha_3}{8x_l(1-x_l)^6} + \frac{15\alpha_3}{8(1-x_l)^7} + \frac{n^2}{x_l}\right]$$

or

$$U^0_{x_2x_2} = D\mu\left[\frac{\begin{matrix}-(1-x_l)^3 - x_l(1-x_l)^2 - (3/2)\alpha_2(1-x_l) \\ -(3/2)\alpha_2 x_l \; + \; n^2(1-x_l)^5\end{matrix}}{x_l(1-x_l)^5} + \frac{15\alpha_3(1-x_l) + 15\alpha_3 x_l}{8x_l(1-x_l)^7}\right]$$

or

$$U^0_{x_2x_2} = D\mu\left[\frac{\begin{matrix}-(1-x_l)^3 - x_l(1-x_l)^2 - (3/2)\alpha_2(1-x_l) - \\ -(3/2)\alpha_2 x_l + n^2(1-x_l)^5\end{matrix}}{x_l(1-x_l)^5} + \frac{15\alpha_3(1-x_l+x_l)}{8x_l(1-x_l)^7}\right]$$

or

$$U^0_{x_2x_2} = D\mu\left[\frac{\begin{matrix}-(1-x_l)^3 - x_l(1-x_l)^2 - (3/2)\alpha_2(1-x_l) - \\ -(3/2)\alpha_2 x_l + n^2(1-x_l)^5\end{matrix}}{x_l(1-x_l)^5} + \frac{15\alpha_3}{8x_l(1-x_l)^7}\right]. \qquad (4.49)$$

Again, from equation (4.43), we have

$$U_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{1}{r_2^3} - \frac{3x_3^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{1}{r_2^5} - \frac{5x_3^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{1}{r_2^7} - \frac{7x_3^2}{r_2^9}\right\}\right],$$

also, putting $x_1 = x_l, x_2 = 0\ and\ x_3 = 0$ in the above, we have

$$U^0_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \frac{\mu}{(1-x_l)^3} - \frac{3\mu\alpha_2}{2(1-x_l)^5} + \frac{15\mu\alpha_3}{8(1-x_l)^7}.\right] \qquad (4.50)$$

Therefore, the second order partial derivatives evaluated at the point $(x_l, 0, 0)$ are

$$U^0_{x_1x_1} = D\mu\left[\frac{\begin{matrix}-(1-x_l)^3 + 2x_l(1-x_l)^2 - (3/2)\alpha_2(1-x_l) + \\ +6\alpha_2 x_l + n^2(1-x_l)^5\end{matrix}}{x_l(1-x_l)^5} + \frac{15\alpha_3(1-7x_l)}{8x_l(1-x_l)^7}\right]$$

$$U^0_{x_2x_2} = D\mu\left[\frac{\begin{matrix}-(1-x_l)^3 - (3/2)\alpha_2(1-x_l) - x_l(1-x_l)^2 - \\ -(3/2)\alpha_2 x_l + + n^2(1-x_l)^5\end{matrix}}{x_l(1-x_l)^5} + \frac{15\alpha_3}{8x_l(1-x_l)^7}\right] \qquad (4.51)$$

$$U^0_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \frac{\mu}{(1-x_l)^3} - \frac{3\mu\alpha_2}{2(1-x_l)^5} + \frac{15\mu\alpha_3}{8(1-x_l)^7}\right],$$

$$U^0_{x_1x_2} = U^0_{x_1x_3} = U^0_{x_2x_3} = 0.$$

superscript 0 denotes that the partial derivatives are to be evaluated at the point $(x_l, 0, 0)$. Substituting the equations (4.51) in the variational equation (4.5), we have

$$\ddot{\alpha} - 2n\dot{\beta} = \left(U^0_{x_1x_1}\right)\alpha,$$

$$\ddot{\beta} + 2n\dot{\alpha} = \left(U^0_{x_2x_2}\right)\beta \tag{4.52}$$

$$\ddot{\gamma} = -D\left[2\pi\rho_1 A_2 + \frac{\mu}{(1-x_l)^3} + \frac{3\mu\alpha_2}{2(1-x_l)^5} - \frac{15\mu\alpha_3}{8(1-x_l)^7}\right]\gamma \tag{4.53}$$

Equation (4.53) is independent of system (4.52), and its solution being periodic is bounded and therefore, the motion of the infinitesimal body in the $x_3 -$ direction is stable.

The characteristic equation corresponding to system (4.52) can be written as

$$\lambda^4 - \left(U^0_{x_1x_1} + U^0_{x_2x_2} - 4n^2\right)\lambda^2 + U^0_{x_1x_1}U^0_{x_2x_2} = 0 \tag{4.54}$$

4.7.1 Stability of the axial libration point $(\boldsymbol{p_1}, \mathbf{0}, \mathbf{0})$

The values of the second order partial derivatives in system (4.51) are evaluated at the axial point $(p_1, 0, 0)$,

$$wherer \quad p_1 = -\frac{3\alpha_1}{2}\frac{\mu}{[2\pi\rho_1 A_1 - (1+2\mu)]}.$$

$$Let\ q = \frac{-\mu}{[2\pi\rho_1 A_1 - (1+2\mu)]}, \tag{4.55}$$

so that.

$$p_1 = -\frac{3\alpha_1 q}{2}$$

Now substituting $x_l = -\frac{3\alpha_1 q}{2}$ in the first equations of system (4.51), we have

$$U^0_{x_1x_1} = D\mu\left[\frac{-\left(1-\frac{3\alpha_1 q}{2}\right)^3 + 2.\frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^2 - \frac{-(3/2)\alpha_2\left(1-\frac{3\alpha_1 q}{2}\right) + 6\alpha_2.\frac{3\alpha_1 q}{2} + n^2\left(1-\frac{3\alpha_1 q}{2}\right)^5}{\frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^5} + \frac{15\alpha_3\left(1-7.\frac{3\alpha_1 q}{2}\right)}{8\frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^7}\right]$$

$$U^0_{x_1x_1} = D\mu\left[\frac{-\left(1-\frac{9\alpha_1 q}{2}\right) + 3\alpha_1 q(1-3\alpha_1 q)^2 - (3/2)\alpha_2\left(1-\frac{3\alpha_1 q}{2}\right) + + 9\alpha_1\alpha_2 q + \left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)\left(1-\frac{15\alpha_1 q}{2}\right)}{\frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^5} + \frac{15\alpha_3\left(1-\frac{21\alpha_1 q}{2}\right)}{12\alpha_1 q\left(1-\frac{21\alpha_1 q}{2}\right)}\right]$$

$$or \quad U^0_{x_1x_1} = D\mu\left[\frac{-\left(1-\frac{9\alpha_1 q}{2}\right) + 3\alpha_1 q(1-3\alpha_1 q)^2 - (3/2)\alpha_2\left(1-\frac{3\alpha_1 q}{2}\right) + + 9\alpha_1\alpha_2 q + \left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)\left(1-\frac{15\alpha_1 q}{2}\right)}{\frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^5} + \frac{5\alpha_3}{4\alpha_1 q}\right]$$

Now, expanding and ignoring the products of α_i $(i = 1,2,3)$, we get

$$U^0_{x_1x_1} = D\mu\left[\frac{-1 + \frac{9\alpha_1 q}{2} + 3\alpha_1 q - \frac{3}{2}\alpha_2 + 1 + \frac{3}{2}\alpha_1 + \frac{3}{2}\alpha_2 - \frac{15}{8}\alpha_3 - \frac{15\alpha_1 q}{2}}{\frac{3\alpha_1 q}{2}} + \frac{5\alpha_3}{4\alpha_1 q}\right]$$

$$or \qquad U^0_{x_1x_1} = D\mu\left[\frac{\frac{3}{2}\alpha_1 - \frac{15}{8}\alpha_3}{\frac{3\alpha_1 q}{2}} + \frac{5\alpha_3}{4\alpha_1 q}\right]$$

$$or \qquad U^0_{x_1x_1} = D\mu\left[\frac{3\alpha_1}{3\alpha_1 q} - \frac{30\alpha_3}{24\alpha_1 q} + \frac{5\alpha_3}{4\alpha_1 q}\right]$$

$$\text{or} \qquad U^0_{x_1x_1} = D\mu\left[\frac{3\alpha_1}{3\alpha_1 q}\right]$$

$$or \qquad U^0_{x_1x_1} = \left[\frac{D\mu}{q}\right] \tag{4.56}$$

now, using (4.55) in (4.56), we have

$$U^0_{x_1x_1} = D\mu.\frac{1}{q} = \frac{-D\mu[2\pi\rho_1 A_1 - (1+2\mu)]}{\mu}$$

$$or \qquad U^0_{x_1x_1} = -D[2\pi\rho_1 A_1 - (1+2\mu)] \tag{4.57}$$

Also substituting $x_l = -\frac{3\alpha_1 q}{2}$ in the second equations of system (4.51), we have

$$U^0_{x_2x_2} = D\mu\left[\frac{-\left(1-\frac{3\alpha_1 q}{2}\right)^3 - \frac{3\alpha_2}{2}\left(1-\frac{3\alpha_1 q}{2}\right) - \frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^2 - \frac{3\alpha_2}{2}.\frac{3\alpha_1 q}{2} + n^2\left(1-\frac{3\alpha_1 q}{2}\right)^5}{\frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^5} + \frac{15\alpha_3}{8.\frac{3\alpha_1 q}{2}\left(1-\frac{3\alpha_1 q}{2}\right)^7}\right]$$

Expanding using binomial expansion and ignoring the second and higher power of α_1, we have

$$U^0_{x_2x_2} = D\mu\left[\frac{-\left(1-\frac{9\alpha_1 q}{2}\right)-\frac{3\alpha_2}{2}\left(1-\frac{3\alpha_1 q}{2}\right)-\frac{3\alpha_1 q}{2}(1-3\alpha_1 q)-\frac{9\alpha_1\alpha_2 q}{4}+ + \left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)\left(1-\frac{15\alpha_1 q}{2}\right)}{\frac{3\alpha_1 q}{2}\left(1-\frac{15\alpha_1 q}{2}\right)}\right.$$

$$\left. + \frac{15\alpha_3}{12\alpha_1 q\left(1-\frac{21\alpha_1 q}{2}\right)}\right]$$

now, expanding and ignoring the products of α_i $(i = 1,2,3)$, we have

$$U^0_{x_2x_2} == D\mu\left[\frac{-1-\frac{9\alpha_1 q}{2}-\frac{3\alpha_2}{2}-\frac{3\alpha_1 q}{2}+1+\frac{3\alpha_1}{2}+\frac{3\alpha_2}{2}-\frac{15\alpha_3}{8}-\frac{15\alpha_1 q}{2}}{\frac{3\alpha_1 q}{2}}+\frac{15\alpha_3}{12\alpha_1 q}\right]$$

$$or \quad U^0_{x_2x_2} = D\mu\left[\frac{-\frac{9\alpha_1 q}{2}+\frac{3\alpha_1}{2}-\frac{15\alpha_3}{8}}{\frac{3\alpha_1 q}{2}}+\frac{15\alpha_3}{12\alpha_1 q}\right]$$

$$or \quad U^0_{x_2x_2} = D\mu\left[\left(-\frac{9\alpha_1 q}{2}+\frac{3\alpha_1}{2}-\frac{15\alpha_3}{8}\right)\left(\frac{2}{3\alpha_1 q}\right)+\frac{15\alpha_3}{12\alpha_1 q}\right]$$

$$or \quad U^0_{x_2x_2} = D\mu\left[-\frac{9\alpha_1 q}{3\alpha_1 q}+\frac{3\alpha_1}{3\alpha_1 q}-\frac{15\alpha_3}{12\alpha_1 q}+\frac{15\alpha_3}{12\alpha_1 q}\right]$$

$$or \quad U^0_{x_2x_2} = D\mu\left[-3+\frac{1}{q}\right] \tag{4.58}$$

now, using (4.55) in (4.58), we have

$$U^0_{x_2x_2} = D\mu\left[-3+\frac{-[2\pi\rho_1 A_1-(1+2\mu)]}{\mu}\right]$$

$$or \quad U^0_{x_2x_2} = \left[-3D\mu - D[2\pi\rho_1 A_1-(1+2\mu)]\right] \tag{4.59}$$

Using equation (4.57) and (4.59) in the characteristic equation (4.54), we have

$$\lambda^4 - (-D[2\pi\rho_1 A_1 - (1+2\mu)] - 3D\mu - D[2\pi\rho_1 A_1 - (1+2\mu)] - 4n^2)\lambda^2 +$$

$$\{-D[2\pi\rho_1 A_1 - (1+2\mu)]\}\{-3D\mu - D[2\pi\rho_1 A_1 - (1+2\mu)]\} = 0$$

or

$$\lambda^4 - (-2D[2\pi\rho_1 A_1 - (1+2\mu)] - 3D\mu + 4n^2)\lambda^2 + D^2[2\pi\rho_1 A_1 - (1+2\mu)]^2 +$$

$$3\mu D^2[2\pi\rho_1 A_1 - (1+2\mu)] = 0$$

or

$$\lambda^4 - \left(-D2[2\pi\rho_1 A_1 - 1] + 4D\mu - 3D\mu + 4n^2\right)\lambda^2 +$$

$$D^2[2\pi\rho_1 A_1 - (1+2\mu)]\{2\pi\rho_1 A_1 - (1+2\mu) + 3\mu\} = 0$$

or

$$\lambda^4 - \left(-D2[2\pi\rho_1 A_1 - 1] + D\mu - 4n^2\right)\lambda^2 + D^2[2\pi\rho_1 A_1 - (1+2\mu)]\{2\pi\rho_1 A_1 - 1 +$$

$\mu=0$

or

$$\lambda^4 - \left(-D2\left[2\pi\rho_1 A_1 - 1 - \frac{\mu}{2}\right] - 4n^2\right)\lambda^2 + D^2[2\pi\rho_1 A_1 - (1+2\mu)]\{2\pi\rho_1 A_1 -$$

$1-\mu=0$

or $$\lambda^4 - \left(-D2\left[2\pi\rho_1 A_1 - \left(1+\frac{\mu}{2}\right)\right] - 4n^2\right)\lambda^2 +$$

$$+ \quad D^2[2\pi\rho_1 A_1 - (1+2\mu)]\{2\pi\rho_1 A_1 - (1-\mu)\} = 0 \qquad (4.60)$$

If λ_1^2 *and* λ_2^2 are the two roots of equation (4.60) satisfying

$$\lambda_1^2 + \lambda_2^2 = U_{x_1x_1}^0 + U_{x_2x_2}^0 - 4n^2 = -2D\left[2\pi\rho_1 A_1 - (1+\frac{\mu}{2})\right] - 4n^2 < 0, \qquad (4.61)$$

and

$$\lambda_1^2\,\lambda_2^2 = U^0_{x_1x_1}U^0_{x_2x_2} = D^2[2\pi\rho_1 A_1 - (1+2\mu)][2\pi\rho_1 A_1 - (1-\mu)] > 0, \qquad (4.62)$$

then the axial libration point is stable.

Since $2\pi\rho_1 A_1 > (1+2\mu) > \left(1+\frac{\mu}{2}\right) > (1-\mu) by Plastino\ and\ Plastino\ (1995)$, it shows that both conditions (4.61) and (4.62) are satisfied. Therefore, the axial libration point $(p_1\ ,0,0)$ is stable.

4.7.2 Stability of the axial libration point $(x_{11}+p_2\ ,0,0)$

The values of the second order partial derivatives in system (4.51) are evaluated at the axial point $(x_{11}+p_2\ ,0,0)$,

Now, putting $x_l = x_{11}+p_2$ in the first equations of system (5.20), we have

$$U^0_{x_1x_1} = D\mu\left[\frac{\begin{matrix}-(1-x_{11}-p_2)^3 + 2x_{11}+2p_2(1-x_{11}-p_2)^2 + 6\alpha_2(x_{11}+p_2)\\ -\left({}^3/_2\right)\alpha_2(1-x_{11}-p_2) + n^2(1-x_{11}-p_2)^5\end{matrix}}{(x_{11}+p_2)(1-x_{11}-p_2)^5}\right.$$

$$\left. + \frac{15\alpha_3(1-7x_{11}-7p_2)}{8(x_{11}+p_2)(1-x_{11}-p_2)^7}\right] \qquad (4.63)$$

We now expand the first two terms of equation (4.63), that is

$$\frac{-(1-x_{11})^3\left(1-\frac{3p_2}{1-x_{11}}\right)+(2x_{11}+2p_2)(1-x_{11})^2\left(1-\frac{2p_2}{1-x_{11}}\right)}{x_{11}\left(1+\frac{p_2}{x_{11}}\right)(1-x_{11})^5\left(1-\frac{5p_2}{1-x_{11}}\right)}$$

$$\frac{-(1-x_{11}-p_2)^3 + 2(x_{11}+p_2)(1-x_{11}-p_2)^2}{(x_{11}+p_2)(1-x_{11}-p_2)^5} =$$

$$= \frac{-(1-x_{11})^3\left[1-\frac{p_2}{1-x_{11}}\right]^3 + (\,2x_{11}+2p_2)(1-x_{11})^2\left[1-\frac{p_2}{1-x_{11}}\right]^2}{x_{11}\left[1+\frac{p_2}{x_{11}}\right](1-x_{11})^5\left[1-\frac{p_2}{1-x_{11}}\right]^5}$$

expanding using binomial expansion and ignoring the second and higher power of p_2,we have

$$\frac{-(1-x_{11})^3\left(1-\frac{3p_2}{1-x_{11}}\right) + (\,2x_{11}+2p_2)(1-x_{11})^2\left(1-\frac{2p_2}{1-x_{11}}\right)}{x_{11}\left(1+\frac{p_2}{x_{11}}\right)(1-x_{11})^5\left(1-\frac{5p_2}{1-x_{11}}\right)}$$

or

$$\frac{-(1-x_{11})^3\left(1-\frac{3p_2}{1-x_{11}}\right) + [\,2x_{11}(1-x_{11})^2 + 2p_2(1-x_{11})^2]\left(1-\frac{2p_2}{1-x_{11}}\right)}{x_{11}(1-x_{11})^5\left(1+\frac{p_2}{x_{11}}\right)\left(1-\frac{5p_2}{1-x_{11}}\right)}$$

now, expanding and ignoring the products of p_2, we have

$$\frac{-(1-x_{11})^3 + 3p_2(1-x_{11})^2 + 2x_{11}(1-x_{11})^2 - 4x_{11}p_2(1-x_{11}) + 2p_2(1-x_{11})^2}{x_{11}(1-x_{11})^5\left(1+\frac{p_2}{x_{11}}-\frac{5p_2}{1-x_{11}}\right)}$$

or

$$\frac{-(1-x_{11})^2[(1-x_{11})-2x_{11}] + 5p_2(1-x_{11})^2 - 4x_{11}p_2(1-x_{11})}{x_{11}(1-x_{11})^5\left(1+\frac{p_2}{1-x_{11}}-\frac{5p_2}{1-x_{11}}\right)}$$

or

$$\frac{\{-(1-x_{11})^2[(1-x_{11})-2x_{11}] + 5p_2(1-x_{11})^2 - 4x_{11}p_2(1-x_{11})\}\left\{\left(1+\frac{p_2}{1-x_{11}}-\frac{5p_2}{1-x_{11}}\right)\right\}^{-1}}{x_{11}(1-x_{11})^5}$$

or

$$\frac{\{-(1-x_{11})^2(1-3x_{11})+5p_2(1-x_{11})^2-4x_{11}p_2(1-x_{11})\}\left(1-\frac{p_2}{1-x_{11}}+\frac{5p_2}{1-x_{11}}\right)}{x_{11}(1-x_{11})^5}$$

also expanding and ignoring the products of p_2, we have

$$\frac{-(1-x_{11})^2(1-3x_{11})+5p_2(1-x_{11})^2-4x_{11}p_2(1-x_{11})+\frac{p_2}{x_{11}}(1-x_{11})^2(1-3x_{11})-\\-5p_2(1-x_{11})(1-3x_{11})}{x_{11}(1-x_{11})^5}$$

or

$$\frac{-(1-x_{11})^2(1-3x_{11})+5p_2(1-x_{11})(2x_{11})-4x_{11}p_2(1-x_{11})+\frac{p_2}{x_{11}}(1-x_{11})^2(1-3x_{11})}{x_{11}(1-x_{11})^5}$$

or

$$\frac{-(1-x_{11})^2(1-3x_{11})+10p_2(1-x_{11})-4x_{11}p_2(1-x_{11})+\frac{p_2}{x_{11}}(1-x_{11})^2(1-3x_{11})}{x_{11}(1-x_{11})^5}$$

or

$$\frac{-(1-x_{11})^2(1-3x_{11})+\ 6p_2(1-x_{11})+\ \frac{p_2}{x_{11}}(1-x_{11})^2(1-3x_{11})}{x_{11}(1-x_{11})^5}\ . \qquad (4.64)$$

We now expand the third, fourth and fifth terms of equation (4.63), that is

$$\frac{-\ \frac{3\alpha_2}{2}(1-x_{11}-p_2)\ +\ 6\alpha_2(x_{11}+p_2)+\ n^2(1-x_{11}-p_2)^5}{(x_{11}+p_2)(1-x_{11}-p_2)^5}$$

or

$$\frac{-\ \frac{3\alpha_2}{2}(1-x_{11})+\frac{3\alpha_2p_2}{2}+\ 6\alpha_2x_{11}+\ 6\alpha_2p_2+\ n^2(1-x_{11})^5\left(1-\frac{5p_2}{1-x_{11}}\right)}{x_{11}(1-x_{11})^5\left(1-\frac{p_2}{x_{11}}\right)\left(1-\frac{5p_2}{1-x_{11}}\right)}$$

$$\frac{-\frac{3\alpha_2}{2}(1-x_{11})+\frac{3\alpha_2 p_2}{2}+\ 6\alpha_2 x_{11}+\ 6\alpha_2 p_2+\ n^2(1-x_{11})^5\left(1-\frac{5p_2}{1-x_{11}}\right)}{x_{11}(1-x_{11})^5\left(1+\frac{p_2}{1-x_{11}}-\frac{5p_2}{1-x_{11}}\right)}$$

or

$$\frac{\left\{-\frac{3\alpha_2}{2}(1-x_{11})+\frac{3\alpha_2 p_2}{2}+\ 6\alpha_2 x_{11}+\ 6\alpha_2 p_2+n^2(1-x_{11})^5\left(1-\frac{5p_2}{1-x_{11}}\right)\right\}\times}{x_{11}(1-x_{11})^5}$$

$$\times\left(1+\frac{p_2}{x_{11}}-\frac{5p_2}{1-x_{11}}\right)^{-1}$$

or

$$\frac{\left\{-\frac{3\alpha_2}{2}(1-x_{11})+\frac{3\alpha_2 p_2}{2}+\ 6\alpha_2 x_{11}+\ 6\alpha_2 p_2+\ n^2(1-x_{11})^5\left(1-\frac{5p_2}{1-x_{11}}\right)\right\}\times}{x_{11}(1-x_{11})^5}$$

$$\times\left(1-\frac{p_2}{x_{11}}+\frac{5p_2}{1-x_{11}}\right)$$

ignoring the products of $\alpha_2\ and\ p_2$, we have

$$\frac{\left\{-\frac{3\alpha_2}{2}(1-x_{11})+\ 6\alpha_2 x_{11}+\ n^2(1-x_{11})^5-\ 5n^2p_2(1-x_{11})^4\right\}\left(1-\frac{p_2}{x_{11}}+\frac{5p_2}{1-x_{11}}\right)}{x_{11}(1-x_{11})^5}$$

expanding and ignoring the products of $\alpha_2\ and\ p_2$, we have

$$\frac{-\frac{3\alpha_2}{2}(1-x_{11})+\ 6\alpha_2 x_{11}+\ n^2(1-x_{11})^5-\ 5n^2p_2(1-x_{11})^4-\frac{p_2}{x_{11}}n^2(1-x_{11})^5+}{x_{11}(1-x_{11})^5}$$

$$+5p_2(1-x_{11})^4$$

or

$$\frac{-\frac{3\alpha_2}{2}(1-x_{11}-4x_{11})\ +\ n^2(1-x_{11})^5\ -\frac{p_2}{x_{11}}n^2(1-x_{11})^5}{x_{11}(1-x_{11})^5}$$

or

$$\frac{-\frac{3\alpha_2}{2}(1-5x_{11})\ +\ n^2(1-x_{11})^5\ -\frac{p_2}{x_{11}}n^2(1-x_{11})^5}{x_{11}(1-x_{11})^5}$$

or

$$\frac{-\frac{3\alpha_2}{2}(1-5x_{11})\ +\ n^2(1-x_{11})^5\ -\frac{p_2}{x_{11}}\left(1+\frac{3}{2}\alpha_1+\frac{3}{2}\alpha_2-\frac{15}{8}\alpha_3\right)(1-x_{11})^5}{x_{11}(1-x_{11})^5}$$

expanding and ignoring the products of $\alpha_2\ and\ p_2$, we have

$$\frac{-\frac{3\alpha_2}{2}(1-5x_{11})\ +\ n^2(1-x_{11})^5\ -\frac{p_2}{x_{11}}(1-x_{11})^5}{x_{11}(1-x_{11})^5} \tag{4.65}$$

we now expand the last term of equation (4.63), that is

$$\frac{15\alpha_3(1-7x_{11}-7p_2)}{8(x_{11}+p_2)(1-x_{11}-p_2)^7}=\frac{15\alpha_3(1-7x_{11}-7p_2)}{8x_{11}\left(1+\frac{p_2}{x_{11}}\right)(1-x_{11})^7\left[1-\frac{p_2}{1-x_{11}}\right]^7}$$

$$=\frac{15\alpha_3(1-7x_{11}-7p_2)}{8x_{11}(1-x_{11})^7\left(1+\frac{p_2}{x_{11}}\right)\left[1-\frac{7p_2}{1-x_{11}}\right]}$$

expanding and ignoring the product of p_2, we have

$$\frac{15\alpha_3(1-7x_{11}-7p_2)}{8x_{11}(1-x_{11})^7\left[1\ +\ \frac{p_2}{x_{11}}-\ \frac{7p_2}{1-x_{11}}\right]}=\frac{15\alpha_3(1-7x_{11}-7p_2)\left[1\ +\ \frac{p_2}{x_{11}}-\ \frac{7p_2}{1-x_{11}}\right]^{-1}}{8x_{11}(1-x_{11})^7}$$

$$=\frac{15\alpha_3(1-7x_{11}-7p_2)\left[1-\frac{p_2}{x_{11}}+\ \frac{7p_2}{1-x_{11}}\right]}{8x_{11}(1-x_{11})^7}$$

expanding and ignoring the products of $\alpha_3\ and\ p_2$, we have

$$\frac{15\alpha_3(1-7x_{11})}{8x_{11}(1-x_{11})^7}. \qquad (4.66)$$

Now, combining equation (4.64), (4.65) and (4.66), we have

$$U^0_{x_1x_1} = D\mu\left[\frac{-(1-x_{11})^2(1-3x_{11}) + 6p_2(1-x_{11}) + \frac{p_2}{x_{11}}(1-x_{11})^2(1-3x_{11}) - -\frac{3\alpha_2}{2}(1-5x_{11}) + n^2(1-x_{11})^5 - \frac{p_2}{x_{11}}(1-x_{11})^5}{x_{11}(1-x_{11})^5} + \frac{15\alpha_3(1-7x_{11})}{8x_{11}(1-x_{11})^7}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[\frac{8n^2(1-x_{11})^7 - 8(1-x_{11})^4(1-3x_{11}) - 12\alpha_2(1-x_{11})^2(1-5x_{11}) + +15\alpha_3(1-7x_{11})}{8x_{11}(1-x_{11})^7}\right] + D\mu\left[\frac{p_2\{6x_{11}^2 + (1-x_{11})(1-3x_{11}) - (1-x_{11})^4\}}{x_{11}^2(1-x_{11})^4}\right] = U_{11} \qquad (4.67)$$

Now, putting $x_l = x_{11} + p_2$ in the second equations of system (4.51), we have

$$U^0_{x_2x_2} = D\mu\left[\frac{\begin{array}{c}-(1-x_{11}-p_2)^3 - (x_{11}+p_2)(1-x_{11}-p_2)^2 - \\ -\frac{3\alpha_2}{2}(1-x_{11}-p_2) - \frac{3\alpha_2}{2}(x_{11}+p_2) + n^2(1-x_{11}-p_2)^5\end{array}}{(x_{11}+p_2)(1-x_{11}-p_2)^5}\right.$$

$$\left. + \frac{15\alpha_3}{8(x_{11}+p_2)(1-x_{11}-p_2)^7}\right]. \qquad (4.68)$$

We now expand the first two terms of equation (4.68), that is

$$\frac{-(1-x_{11}-p_2)^3 + (x_{11}+p_2)(1-x_{11}-p_2)^2}{(x_{11}+p_2)(1-x_{11}-p_2)^5}$$

Expanding using binomial expansion and ignoring the second and higher power of p_2,we have

$$\frac{-(1-x_{11})^3\left(1-\frac{3p_2}{1-x_{11}}\right) - (x_{11}+p_2)(1-x_{11})^2\left(1-\frac{2p_2}{1-x_{11}}\right)}{x_{11}(1-x_{11})^5\left[1+\frac{p_2}{x_{11}}\right]\left[1-\frac{5p_2}{1-x_{11}}\right]} =$$

$$= \frac{-(1-x_{11})^3\left(1-\frac{3p_2}{1-x_{11}}\right) - (x_{11}+p_2)(1-x_{11})^2\left(1-\frac{2p_2}{1-x_{11}}\right)}{x_{11}(1-x_{11})^5\left[1+\frac{p_2}{x_{11}}-\frac{5p_2}{1-x_{11}}\right]}$$

Expanding and ignoring the product of p_2, we have

$$\frac{-(1-x_{11})^3 + 3p_2(1-x_{11})^2 - x_{11}(1-x_{11})^2 - p_2(1-x_{11})^2 + 2x_{11}p_2(1-x_{11})}{x_{11}(1-x_{11})^5\left[1+\frac{p_2}{x_{11}}-\frac{5p_2}{1-x_{11}}\right]}$$

or

$$\frac{[-(1-x_{11})^3 + 3p_2(1-x_{11})^2 - x_{11}(1-x_{11})^2 - p_2(1-x_{11})^2 + 2x_{11}p_2(1-x_{11})] \times \left[1+\frac{p_2}{x_{11}}-\frac{5p_2}{1-x_{11}}\right]^{-1}}{x_{11}(1-x_{11})^5}$$

or

$$\frac{[-(1-x_{11})^3 + 3p_2(1-x_{11})^2 - x_{11}(1-x_{11})^2 - p_2(1-x_{11})^2 + 2x_{11}p_2(1-x_{11})] \times \left[1-\frac{p_2}{x_{11}}+\frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

or

$$\frac{[-(1-x_{11})^3 - x_{11}(1-x_{11})^2 + 2p_2(1-x_{11})^2 + 2x_{11}p_2(1-x_{11})]\left[1-\frac{p_2}{x_{11}}+\frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

or

$$\frac{[-(1-x_{11})^2\{1-x_{11}-x_{11}\} + 2p_2(1-x_{11})\{1-x_{11}+x_{11}\}]\left[1-\frac{p_2}{x_{11}}+\frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

or

$$\frac{[-(1-x_{11})^2(1-2x_{11}) + 2p_2(1-x_{11})]\left[1-\frac{p_2}{x_{11}}+\frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

also expanding and ignoring the product of p_2, we have

$$\frac{-(1-x_{11})^2(1-2x_{11}) + 2p_2(1-x_{11}) + \frac{p_2}{x_{11}}(1-x_{11})^2(1-2x_{11}) - -5p_2(1-x_{11})(1-2x_{11})}{x_{11}(1-x_{11})^5}$$

or

$$\frac{-(1-x_{11})^2(1-2x_{11}) \; + \; p_2(1-x_{11})[2-5(1-2x_{11})] \; + \; \frac{p_2}{x_{11}}(1-x_{11})^2(1-2x_{11})}{x_{11}(1-x_{11})^5}$$

or

$$\frac{-(1-x_{11})^2(1-2x_{11}) - \; p_2(1-x_{11})(3-10x_{11}) \; + \; \frac{p_2}{x_{11}}(1-x_{11})^2(1-2x_{11})}{x_{11}(1-x_{11})^5}. \quad (4.69)$$

We now expand the third, fourth and fifth terms of equation (4.68), that is

$$\frac{-\frac{3\alpha_2}{2}(1-x_{11}-p_2) - \frac{3\alpha_2}{2}(x_{11}+p_2) \; + \; n^2(1-x_{11}-p_2)^5}{(x_{11}+p_2)(1-x_{11}-p_2)^5} =$$

$$= \frac{-\frac{3\alpha_2}{2}(1-x_{11}-p_2) - \frac{3\alpha_2}{2}(x_{11}+p_2) \; + \; n^2(1-x_{11})^5\left[1-\frac{p_2}{1-x_{11}}\right]^5}{x_{11}\left[1+\frac{p_2}{x_{11}}\right](1-x_{11})^5\left[1-\frac{p_2}{1-x_{11}}\right]^5}$$

Expanding using binomial expansion and ignoring the second and higher power of p_2,we have

$$\frac{-\frac{3\alpha_2}{2}(1-x_{11}-p_2) - \frac{3\alpha_2}{2}(x_{11}+p_2) \; + \; n^2(1-x_{11})^5\left[1-\frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5\left[1+\frac{p_2}{x_{11}}\right]\left[1-\frac{5p_2}{1-x_{11}}\right]}$$

Expanding and ignoring the product of p_2, we have

$$\frac{-\frac{3\alpha_2}{2}(1-x_{11}-p_2) - \frac{3\alpha_2}{2}(x_{11}+p_2) \; + \; n^2(1-x_{11})^5\left[1-\frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5\left[1+\frac{p_2}{x_{11}}-\frac{5p_2}{1-x_{11}}\right]}$$

Expanding and ignoring the products of $\alpha_i \; p_2$, we have

$$\frac{-\frac{3\alpha_2}{2}(1-x_{11}) - \frac{3\alpha_2}{2}(x_{11}) \; + \; n^2(1-x_{11})^5 - 5\,n^2p_2(1-x_{11})^4}{x_{11}(1-x_{11})^5\left[1+\frac{p_2}{x_{11}}-\frac{5p_2}{1-x_{11}}\right]}$$

$$or \quad \frac{\left[-\frac{3\alpha_2}{2}(1-x_{11}) - \frac{3\alpha_2}{2}(x_{11}) + n^2(1-x_{11})^5 - 5\,n^2 p_2(1-x_{11})^4\right]\left[1+\frac{p_2}{x_{11}} - \frac{5p_2}{1-x_{11}}\right]^{-1}}{x_{11}(1-x_{11})^5}$$

$$or \quad \frac{\left[-\frac{3\alpha_2}{2}(1-x_{11}) - \frac{3\alpha_2}{2}(x_{11}) + n^2(1-x_{11})^5 - 5\,n^2 p_2(1-x_{11})^4\right]\left[1-\frac{p_2}{x_{11}} + \frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

$$or \quad \frac{\left[-\frac{3\alpha_2}{2}(1-x_{11}+x_{11}) + n^2(1-x_{11})^5 - 5\,n^2 p_2(1-x_{11})^4\right]\left[1-\frac{p_2}{x_{11}} + \frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

$$or \quad \frac{\left[-\frac{3\alpha_2}{2} + n^2(1-x_{11})^5 - 5\,n^2 p_2(1-x_{11})^4\right]\left[1-\frac{p_2}{x_{11}} + \frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

Expanding and ignoring the product of $p_2 \; and \; \alpha_i \; (i = 1, 2, 3) \; in \; n^2$, we have

$$\frac{\left[-\frac{3\alpha_2}{2} + n^2(1-x_{11})^5 - 5\,p_2(1-x_{11})^4\right]\left[1-\frac{p_2}{x_{11}} + \frac{5p_2}{1-x_{11}}\right]}{x_{11}(1-x_{11})^5}$$

Expanding and ignoring the products of $\alpha_i \;\; p_2$, we have

$$\frac{-\frac{3\alpha_2}{2} + n^2(1-x_{11})^5 - 5\,p_2(1-x_{11})^4 - \frac{p_2}{x_{11}}n^2(1-x_{11})^5 + 5p_2 n^2(1-x_{11})^4}{x_{11}(1-x_{11})^5}$$

Expanding and ignoring the product of $p_2 \; and \; \alpha_i \; (i = 1, 2, 3) \; in \; n^2$, we have

$$\frac{-\frac{3\alpha_2}{2} + n^2(1-x_{11})^5 - 5\,p_2(1-x_{11})^4 - \frac{p_2}{x_{11}}(1-x_{11})^5 + 5p_2(1-x_{11})^4}{x_{11}(1-x_{11})^5}$$

$$or \quad \frac{-\frac{3\alpha_2}{2} + n^2(1-x_{11})^5 - \frac{p_2}{x_{11}}(1-x_{11})^5}{x_{11}(1-x_{11})^5}. \qquad (4.70)$$

We now expand the last term of equation (4.68), that is

$$\frac{15\alpha_3}{8(x_{11}+p_2)(1-x_{11}-p_2)^7} = \frac{15\alpha_3}{8x_{11}\left(1+\frac{p_2}{x_{11}}\right)(1-x_{11})^7\left[1-\frac{p_2}{1-x_{11}}\right]^7}$$

Expanding using binomial and ignoring the second and higher power of p_2, we have

$$\frac{15\alpha_3}{x_{11}(1-x_{11})^7\left(1+\frac{p_2}{x_{11}}\right)\left[1-\frac{7p_2}{1-x_{11}}\right]}$$

Expanding and ignoring the products of p_2, we have

$$\frac{15\alpha_3}{8x_{11}(1-x_{11})^7\left[1+\frac{p_2}{x_{11}}-\frac{7p_2}{1-x_{11}}\right]} =$$

$$\frac{15\alpha_3\left[1+\frac{p_2}{x_{11}}-\frac{7p_2}{1-x_{11}}\right]^{-1}}{8x_{11}(1-x_{11})^7} = \frac{15\alpha_3\left[1-\frac{p_2}{x_{11}}+\frac{7p_2}{1-x_{11}}\right]}{8x_{11}(1-x_{11})^7}$$

Expanding and ignoring the product of $p_2 \; and \; \alpha_3$, we have

$$\frac{15\alpha_3}{8x_{11}(1-x_{11})^7} \tag{4.71}$$

Now, combining equation (4.69), (4.70) and (4.71), we have

$$U^0_{x_2x_2} = D\mu\left[\frac{\begin{array}{c}-(1-x_{11})^2(1-2x_{11}) - p_2(1-x_{11})(3-10x_{11}) - \frac{3\alpha_2}{2} \\ +\frac{p_2}{x_{11}}(1-x_{11})^2(1-2x_{11}) + n^2(1-x_{11})^5 - \frac{p_2}{x_{11}}(1-x_{11})^5\end{array}}{x_{11}(1-x_{11})^5}\right.$$

$$\left. + \frac{15\alpha_3}{8x_{11}(1-x_{11})^7}\right]$$

or

$$U^0_{x_2x_2} = D\mu\left[\frac{8n^2(1-x_{11})^7 - 8(1-x_{11})^4(1-2x_{11}) - 12\alpha_2(1-x_{11})^2 + 15\alpha_3}{8x_{11}(1-x_{11})^7}\right]$$

$$-D\mu\left[\frac{p_2\{x_{11}(3-10x_{11}) - (1-x_{11})(1-2x_{11}) - (1-x_{11})^4\}}{x_{11}^2(1-x_{11})^4}\right] = U_{22} \qquad (4.72)$$

Using equation (4.67) and (4.72) in the characteristic equation (4.54), we have

$$\lambda^4 - (U_{11} + U_{22} - 4n^2)\lambda^2 + U_{11}U_{22} = 0. \qquad (4.73)$$

If $\lambda_1^2 and\ \lambda_2^2$ are the two roots of equation (4.73) satisfying the following conditions

$$\Lambda_1^2 + \lambda_2^2 = U_{11} + U_{22} - 4n^2 < 0, \qquad (4.74)$$

and

$$\lambda_1^2\,\lambda_2^2 = U_{11}U_{22} > 0, \qquad (4.75)$$

then, the axial libration point is stable.

4.7.3 Stability of circular points

At the point $(1 + r\cos\theta\,, r\sin\theta\,, 0)$, the values of the second order partial derivatives are evaluated. Now from equation (4.38), we have

$$U_{x_1x_1} = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{-1}{r_2^3} + \frac{3(1-x_1)^2}{r_2^5}\right\} + \frac{3\mu\alpha_2}{2}\left\{\frac{-1}{r_2^5} + \frac{5(1-x_1)^2}{r_2^7}\right\}\right.$$
$$\left. - \frac{15\mu\alpha_3}{8}\left\{\frac{-1}{r_2^7} + \frac{7(1-x_1)^2}{r_2^9}\right\} + n^2\right]$$

$$U_{x_1x_1} = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{-r_2^2 + 3(1-x_1)^2}{r_2^5}\right\} + \frac{3\mu\alpha_2}{2}\left\{\frac{-r_2^2 + 5(1-x_1)^2}{r_2^7}\right\}\right.$$
$$\left. - \frac{15\mu\alpha_3}{8}\left\{\frac{-r_2^2 + 7(1-x_1)^2}{r_2^9}\right\} + n^2\right]$$

Using the values of $r_2^2 = (1-x_1)^2 + x_2^2 + x_3^2$, we have

$$U_{x_1x_1} = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{-(1-x_1)^2 - x_2^2 - x_3^2 + 3(1-x_1)^2}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right\}\right.$$

$$+\frac{3\mu\alpha_2}{2}\left\{\frac{-(1-x_1)^2 - x_2^2 - x_3^2 + 5(1-x_1)^2}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right\}$$

$$\left.-\frac{15\mu\alpha_3}{8}\left\{\frac{-(1-x_1)^2 - x_2^2 - x_3^2 + 7(1-x_1)^2}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right\} + n^2\right]$$

$$U_{x_1x_1} = D\left[-2\pi\rho_1 A_1 + \mu\left\{\frac{2(1-x_1)^2 - x_2^2 - x_3^2}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right\} + \frac{3\mu\alpha_2}{2}\left\{\frac{4(1-x_1)^2 - x_2^2 - x_3^2}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right\}\right.$$

$$\left.-\frac{15\mu\alpha_3}{8}\left\{\frac{6(1-x_1)^2 - x_2^2 - x_3^2}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right\} + n^2\right] \tag{4.76}$$

Similarly, from equation (4.39), we have

$$U_{x_1x_2} = D\left[\frac{-3\mu(1-x_1)x_2}{r_2^5} - \frac{15\mu\alpha_2(1-x_1)x_2}{2r_2^7} + \frac{105\mu\alpha_3(1-x_1)x_2}{8r_2^9}\right]$$

or

$$U_{x_1x_2} = D\left[\frac{-3\mu(1-x_1)x_2}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}} - \frac{15\mu\alpha_2(1-x_1)x_2}{2[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right.$$

$$\left.+\frac{105\mu\alpha_3(1-x_1)x_2}{8[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right] \tag{4.77}$$

Also, from equation (4.40), we have

$$U_{x_1x_3} = D\left[\frac{-3\mu(1-x_1)x_3}{r_2^5} - \frac{15\mu\alpha_2(1-x_1)x_3}{2r_2^7} + \frac{105\mu\alpha_3(1-x_1)x_3}{8r_2^9}\right]$$

or

$$U_{x_1x_3} = D\left[\frac{-3\mu(1-x_1)x_3}{[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}} - \frac{15\mu\alpha_2(1-x_1)x_3}{2[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right.$$

$$\left.+\frac{105\mu\alpha_3(1-x_1)x_3}{8[(1-x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right]. \tag{4.78}$$

Again, from equation (4.41), we have

$$U_{x_2x_2} = D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{1}{r_2^3} - \frac{3x_2^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{1}{r_2^5} - \frac{5x_2^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{1}{r_2^7} - \frac{7x_2^2}{r_2^9}\right\} + n^2\right]$$

or

$$U_{x_2x_2} = D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{r_2^2 - 3x_2^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{r_2^2 - 5x_2^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{r_2^2 - 7x_2^2}{r_2^9}\right\} + n^2\right]$$

Now, using the values of $r_2^2 = (1 - x_1)^2 + x_2^2 + x_3^2$, we have

$$U_{x_2x_2} = D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{(1 - x_1)^2 + x_2^2 + x_3^2 - 3x_2^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right\}\right.$$

$$- \frac{3\mu\alpha_2}{2}\left\{\frac{(1 - x_1)^2 + x_2^2 + x_3^2 - 5x_2^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right\}$$

$$\left. + \frac{15\mu\alpha_3}{8}\left\{\frac{(1 - x_1)^2 + x_2^2 + x_3^2 - 7x_2^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right\} + n^2\right]$$

or

$$U_{x_2x_2} = D\left[-2\pi\rho_1 A_1 - \mu\left\{\frac{(1 - x_1)^2 - 2x_2^2 + x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{(1 - x_1)^2 - 4x_2^2 + x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right\}\right.$$

$$\left. + \frac{15\mu\alpha_3}{8}\left\{\frac{(1 - x_1)^2 - 6x_2^2 + x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right\} + n^2\right] \tag{4.79}$$

Similarly, from equation (4.42), we have

$$U_{x_2x_3} = D\left[\frac{3\mu x_2 x_3}{r_2^5} + \frac{15\mu\alpha_2 x_2 x_3}{2r_2^7} - \frac{105\mu\alpha_3 x_2 x_3}{8r_2^9}\right]$$

or

$$U_{x_2x_3} = D\left[\frac{3\mu x_2 x_3}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}} + \frac{15\mu\alpha_2 x_2 x_3}{2[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right.$$

$$\left. - \frac{105\mu\alpha_3 x_2 x_3}{8[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right] \tag{4.80}$$

In the same vein, from equation (4.43), we have

$$U_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{1}{r_2^3} - \frac{3x_3^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{1}{r_2^5} - \frac{5x_3^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{1}{r_2^7} - \frac{7x_3^2}{r_2^9}\right\}\right]$$

or

$$U_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{r_2^2 - 3x_3^2}{r_2^5}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{r_2^2 - 5x_3^2}{r_2^7}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{r_2^2 - 7x_3^2}{r_2^9}\right\}\right]$$

Also, using the values of $r_2^2 = (1 - x_1)^2 + x_2^2 + x_3^2$, we have

$$U_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{(1 - x_1)^2 + x_2^2 + x_3^2 - 3x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right\}\right.$$
$$- \frac{3\mu\alpha_2}{2}\left\{\frac{(1 - x_1)^2 + x_2^2 + x_3^2 - 5x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right\}$$
$$\left. + \frac{15\mu\alpha_3}{8}\left\{\frac{(1 - x_1)^2 + x_2^2 + x_3^2 - 7x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right\}\right]$$

or

$$U_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{(1 - x_1)^2 + x_2^2 - 2x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{5}{2}}}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{(1 - x_1)^2 + x_2^2 - 4x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{7}{2}}}\right\}\right.$$
$$\left. + \frac{15\mu\alpha_3}{8}\left\{\frac{(1 - x_1)^2 + x_2^2 - 6x_3^2}{[(1 - x_1)^2 + x_2^2 + x_3^2]^{\frac{9}{2}}}\right\}\right] \qquad (4.81)$$

Substituting $x_1 = 1 + r\cos\theta$, $x_2 = \mathrm{r}\sin\theta$, $x_3 = 0$ and $2\pi\rho_1 A_1 = n^2(1 - \mu)$ in equation (4.76), we have

$$U^0_{x_1x_1} = D\left[-n^2(1 - \mu) + \mu\left\{\frac{2(1 - 1 - r\cos\theta)^2 - (\mathrm{r}\sin\theta)^2}{[(1 - 1 - r\cos\theta)^2 + (\mathrm{r}\sin\theta)^2]^{\frac{5}{2}}}\right\}\right.$$
$$+ \frac{3\mu\alpha_2}{2}\left\{\frac{4(1 - 1 - r\cos\theta)^2 - (\mathrm{r}\sin\theta)^2}{[(1 - 1 - r\cos\theta)^2 + (\mathrm{r}\sin\theta)^2]^{\frac{7}{2}}}\right\}$$
$$\left. - \frac{15\mu\alpha_3}{8}\left\{\frac{6(1 - 1 - r\cos\theta)^2 - (\mathrm{r}\sin\theta)^2}{[(1 - 1 - r\cos\theta)^2 + (\mathrm{r}\sin\theta)^2]^{\frac{9}{2}}}\right\} + n^2\right]$$

or

$$U^0_{x_1x_1} = D\left[-n^2 + n^2\mu + n^2 + \mu\left\{\frac{2r^2\cos^2\theta - r^2\sin^2\theta}{[r^2\cos^2\theta + r^2\sin^2\theta]^{\frac{5}{2}}}\right\}\right.$$

$$\left. + \frac{3\mu\alpha_2}{2}\left\{\frac{4r^2\cos^2\theta - r^2\sin^2\theta}{[r^2\cos^2\theta + r^2\sin^2\theta]^{\frac{7}{2}}}\right\} - \frac{15\mu\alpha_3}{8}\left\{\frac{6r^2\cos^2\theta - r^2\sin^2\theta}{[r^2\cos^2\theta + r^2\sin^2\theta]^{\frac{9}{2}}}\right\}\right]$$

or

$$U^0_{x_1x_1} = D\left[n^2\mu + \mu r^2\left\{\frac{2\cos^2\theta - \sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}}\right\} + \frac{3\mu\alpha_2 r^2}{2}\left\{\frac{4\cos^2\theta - \sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{7}{2}}}\right\}\right.$$

$$\left. - \frac{15\mu\alpha_3 r^2}{8}\left\{\frac{6\cos^2\theta - \sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{9}{2}}}\right\}\right]$$

Since $\cos^2\theta + \sin^2\theta = 1$, we have

$$U^0_{x_1x_1} = D\left[n^2\mu + \frac{\mu r^2}{r^5}\{2\cos^2\theta - \sin^2\theta\} + \frac{3\mu\alpha_2 r^2}{2r^7}\{4\cos^2\theta - \sin^2\theta\}\right.$$

$$\left. - \frac{15\mu\alpha_3 r^2}{8r^9}\{6\cos^2\theta - \sin^2\theta\}\right]$$

or

$$U^0_{x_1x_1} = D\left[n^2\mu + \frac{\mu}{r^3}\{2\cos^2\theta - \sin^2\theta\} + \frac{3\mu\alpha_2}{2r^5}\{4\cos^2\theta - \sin^2\theta\}\right.$$

$$\left. - \frac{15\mu\alpha_3}{8r^7}\{6\cos^2\theta - \sin^2\theta\}\right]$$

using $\sin^2\theta = 1 - \cos^2\theta$, we have

$$U^0_{x_1x_1} = D\left[n^2\mu + \frac{\mu}{r^3}\{2\cos^2\theta - (1 - \cos^2\theta)\} + \frac{3\mu\alpha_2}{2r^5}\{4\cos^2\theta - (1 - \cos^2\theta)\}\right.$$

$$\left. - \frac{15\mu\alpha_3}{8r^7}\{6\cos^2\theta - (1 - \cos^2\theta)\}\right]$$

or

$$U^0_{x_1x_1} = D\left[n^2\mu + \frac{\mu}{r^3}\{2\cos^2\theta - 1 + \cos^2\theta\} + \frac{3\mu\alpha_2}{2r^5}\{4\cos^2\theta - 1 + \cos^2\theta\}\right.$$

$$\left. - \frac{15\mu\alpha_3}{8r^7}\{6\cos^2\theta - 1 + \cos^2\theta\}\right]$$

or

$$U^0_{x_1x_1} = D\left[n^2\mu + \frac{\mu}{\mathrm{r}^3}\{3\cos^2\theta - 1\} + \frac{3\mu\alpha_2}{2\mathrm{r}^5}\{5\cos^2\theta - 1\} - \frac{15\mu\alpha_3}{8\mathrm{r}^7}\{7\cos^2\theta - 1\}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[n^2 + \frac{1}{\mathrm{r}^3}\{3\cos^2\theta - 1\} + \frac{3\alpha_2}{2\mathrm{r}^5}\{5\cos^2\theta - 1\} - \frac{15\alpha_3}{8\mathrm{r}^7}\{7\cos^2\theta - 1\}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[n^2 + \frac{3\cos^2\theta}{\mathrm{r}^3} - \frac{1}{\mathrm{r}^3} + \frac{15\alpha_2\cos^2\theta}{2\mathrm{r}^5} - \frac{3\alpha_2}{2\mathrm{r}^5} - \frac{105\alpha_3\cos^2\theta}{8\mathrm{r}^7} + \frac{15\alpha_3}{8\mathrm{r}^7}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[n^2 + 3\cos^2\theta\left\{\frac{1}{\mathrm{r}^3} + \frac{5\alpha_2}{2\mathrm{r}^5} - \frac{35\alpha_3}{8\mathrm{r}^7}\right\} - \left\{\frac{1}{\mathrm{r}^3} + \frac{3\alpha_2}{2\mathrm{r}^5} - \frac{15\alpha_3}{8\mathrm{r}^7}\right\}\right]. \qquad (4.82)$$

But from equation (4.29), we have

$$\frac{1}{r^3} + \frac{3\alpha_2}{2r^5} - \frac{15\alpha_3}{8r^7} - n^2 = 0$$

$$or \quad \frac{1}{r^3} = -\frac{3\alpha_2}{2r^5} + \frac{15\alpha_3}{8r^7} + n^2 \qquad (4.83).$$

Now, using (4.83) in (4.82), we have

$$U^0_{x_1x_1} = D\mu\left[n^2 + 3\cos^2\theta\left\{-\frac{3\alpha_2}{2r^5} + \frac{15\alpha_3}{8r^7} + n^2 + \frac{5\alpha_2}{2r^5} - \frac{35\alpha_3}{8\mathrm{r}^7}\right\} - \left\{-\frac{3\alpha_2}{2r^5} + \frac{15\alpha_3}{8r^7} + n^2 + \frac{3\alpha_2}{2\mathrm{r}^5} - \frac{15\alpha_3}{8\mathrm{r}^7}\right\}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[n^2 - n^2 + 3\cos^2\theta\left\{n^2 + \frac{\alpha_2}{r^5} - \frac{5\alpha_3}{2r^7}\right\}\right]$$

or

$$U^0_{x_1x_1} = D\mu\left[n^2 - n^2 + 3\cos^2\theta\left\{n^2 + \alpha_2 r^{-5} - \frac{5}{2}\alpha_3 r^{-7}\right\}\right]$$

By the use of the expression in system (4.34), we have

$$U^0_{x_1x_1} = D\mu\left[n^2 - n^2 + 3\cos^2\theta\left\{n^2 + \alpha_2\left(1 + \frac{5}{2}\alpha_1\right) - \frac{5}{2}\alpha_3\left(1 + \frac{7}{2}\alpha_1\right)\right\}\right]$$

Expanding and neglecting the products of $\alpha_i \quad (i = 1, 2, 3)$, we have

$$U^0_{x_1x_1} = D\mu\left[n^2 - n^2 + 3\cos^2\theta\left\{n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right\}\right]$$

or

$$U^0_{x_1x_1} = 3\,D\mu\cos^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) \tag{4.84}$$

Similarly, Substituting $x_1 = 1 + r\cos\theta, \quad x_2 = \mathrm{r}\sin\theta, \; x_3 = 0$ and $2\pi\rho_1 A_1 = n^2(1-\mu)$ in equation (4.77), we have

$$U^0_{x_1x_2} = D\left[\frac{-3\mu(1-1-r\cos\theta)r\sin\theta}{[(1-1-r\cos\theta)^2 + (r\sin\theta)^2]^{\frac{5}{2}}} - \frac{15\mu\alpha_2(1-1-r\cos\theta)r\sin\theta}{2[(1-1-r\cos\theta)^2 + (r\sin\theta)^2]^{\frac{7}{2}}} + \frac{105\mu\alpha_3(1-1-r\cos\theta)r\sin\theta}{8[(1-1-r\cos\theta)^2 + (r\sin\theta)^2]^{\frac{9}{2}}}\right]$$

or

$$U^0_{x_1x_2} = D\left[\frac{3\mu r^2\cos\theta\sin\theta}{[r^2\cos^2\theta + \sin^2\theta]^{\frac{5}{2}}} + \frac{15\mu\alpha_2 r^2\cos\theta\sin\theta}{2[r^2\cos^2\theta + \sin^2\theta]^{\frac{7}{2}}} - \frac{105\mu\alpha_3 r^2\cos\theta\sin\theta}{8[r^2\cos^2\theta + \sin^2\theta]^{\frac{9}{2}}}\right]$$

or

$$U^0_{x_1x_2} = D\left[\frac{3\mu r^2\cos\theta\sin\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}} + \frac{15\mu\alpha_2 r^2\cos\theta\sin\theta}{2[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{7}{2}}} - \frac{105\mu\alpha_3 r^2\cos\theta\sin\theta}{8[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{9}{2}}}\right]$$

or

$$U^0_{x_1x_2} = D\left[\frac{3\mu r^2\cos\theta\sin\theta}{r^5} + \frac{15\mu\alpha_2 r^2\cos\theta\sin\theta}{2r^7} - \frac{105\mu\alpha_3 r^2\cos\theta\sin\theta}{8r^9}\right]$$

or

$$U^0_{x_1x_2} = 3D\mu\cos\theta\sin\theta\left[\frac{1}{r^3} + \frac{5\alpha_2}{2r^5} - \frac{35\alpha_3}{8r^7}\right]$$

Using the expression in equation (4.83), we have

$$U^0_{x_1x_2} = 3D\mu\cos\theta\sin\theta\left[-\frac{3\alpha_2}{2r^5} + \frac{15\alpha_3}{8r^7} + n^2 + \frac{5\alpha_2}{2r^5} - \frac{35\alpha_3}{8r^7}\right]$$

$$or \qquad U^0_{x_1x_2} = 3D\mu\cos\theta\sin\theta\left[n^2 + \frac{\alpha_2}{r^5} - \frac{5\alpha_3}{2r^7}\right]$$

using of the expression in system (4.43), we have

$$U^0_{x_1x_2} = 3D\mu\cos\theta\sin\theta\left[n^2 + \alpha_2\left(1+\frac{5}{2}\alpha_1\right) - \frac{5}{2}\alpha_3\left(1+\frac{7}{2}\alpha_1\right)\right]$$

Expanding and neglecting the products of $\alpha_i \quad (i = 1,2,3)$, we have

$$U^0_{x_1x_2} = 3D\mu\cos\theta\sin\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) \tag{4.85}$$

Also, Substituting $x_1 = 1 + r\cos\theta, \quad x_2 = \mathrm{r}\sin\theta, \; x_3 = 0$ and $2\pi\rho_1 A_1 = n^2(1-\mu)$ in equation (4.78), we have

$$U^0_{x_1x_3} = 0 \tag{4.86}$$

Substitution of $x_1 = 1 + r\cos\theta, \quad x_2 = \mathrm{r}\sin\theta, \; x_3 = 0$ and $2\pi\rho_1 A_1 = n^2(1-\mu)$ in equation (4.79), gives

$$U_{x_2x_2} = D\left[-n^2(1-\mu) - \mu\left\{\frac{(1-1-\mathrm{r}\cos\theta)^2 - 2\mathrm{r}^2\sin^2\theta}{[1-1-\mathrm{r}\cos\theta + \mathrm{r}^2\sin^2\theta]^{\frac{5}{2}}}\right\}\right.$$
$$- \frac{3\mu\alpha_2}{2}\left\{\frac{(1-1-\mathrm{r}\cos\theta)^2 - 4\mathrm{r}^2\sin^2\theta}{[(1-1-\mathrm{r}\cos\theta)^2 + \mathrm{r}^2\sin^2\theta]^{\frac{7}{2}}}\right\}$$
$$\left. + \frac{15\mu\alpha_3}{8}\left\{\frac{(1-1-\mathrm{r}\cos\theta)^2 - 6\mathrm{r}^2\sin^2\theta}{[(1-1-\mathrm{r}\cos\theta)^2 + \mathrm{r}^2\sin^2\theta]^{\frac{9}{2}}}\right\} + n^2\right]$$

or

$$U_{x_2x_2} = D\left[-n^2 + n^2\mu + n^2 - \mu\left\{\frac{\mathrm{r}^2\cos^2\theta - 2\mathrm{r}^2\sin^2\theta}{[\mathrm{r}^2\cos^2\theta + \mathrm{r}^2\sin^2\theta]^{\frac{5}{2}}}\right\}\right.$$
$$\left. - \frac{3\mu\alpha_2}{2}\left\{\frac{\mathrm{r}^2\cos^2\theta - 4\mathrm{r}^2\sin^2\theta}{[\mathrm{r}^2\cos^2\theta + \mathrm{r}^2\sin^2\theta]^{\frac{7}{2}}}\right\} + \frac{15\mu\alpha_3}{8}\left\{\frac{\mathrm{r}^2\cos^2\theta - 6\mathrm{r}^2\sin^2\theta}{[\mathrm{r}^2\cos^2\theta + \mathrm{r}^2\sin^2\theta]^{\frac{9}{2}}}\right\}\right]$$

or

$$U_{x_2x_2} = D\left[n^2\mu - \mu r^2\left\{\frac{\cos^2\theta - 2\sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}}\right\} - \frac{3\mu\alpha_2 r^2}{2}\left\{\frac{\cos^2\theta - 4\sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{7}{2}}}\right\}\right.$$

$$\left. + \frac{15\mu\alpha_3 r^2}{8}\left\{\frac{\cos^2\theta - 6\sin^2\theta}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{9}{2}}}\right\}\right]$$

or

$$U_{x_2x_2} = D\left[n^2\mu - \frac{\mu r^2}{r^5}\{\cos^2\theta - 2\sin^2\theta\} - \frac{3\mu\alpha_2 r^2}{2r^7}\{\cos^2\theta - 4\sin^2\theta\}\right.$$

$$\left. + \frac{15\mu\alpha_3 r^2}{8r^9}\{\cos^2\theta - 6\sin^2\theta\}\right]$$

Putting $\cos^2\theta = 1 - \sin^2\theta$, we have

$$U_{x_2x_2} = D\left[n^2\mu - \frac{\mu}{r^3}\{1 - \sin^2\theta - 2\sin^2\theta\} - \frac{3\mu\alpha_2}{2r^5}\{1 - \sin^2\theta - 4\sin^2\theta\}\right.$$

$$\left. + \frac{15\mu\alpha_3}{8r^7}\{1 - \sin^2\theta - 6\sin^2\theta\}\right]$$

or

$$U_{x_2x_2} = D\left[n^2\mu - \frac{\mu}{r^3}\{1 - 3\sin^2\theta\} - \frac{3\mu\alpha_2}{2r^5}\{1 - 5\sin^2\theta\} + \frac{15\mu\alpha_3}{8r^7}\{1 - 7\sin^2\theta\}\right]$$

or

$$U_{x_2x_2} = D\mu\left[n^2 - \frac{1}{r^3}\{1 - 3\sin^2\theta\} - \frac{3\alpha_2}{2r^5}\{1 - 5\sin^2\theta\} + \frac{15\alpha_3}{8r^7}\{1 - 7\sin^2\theta\}\right]$$

or

$$U_{x_2x_2} = D\mu\left[n^2 - \frac{1}{r^3} + \frac{3\sin^2\theta}{r^3} - \frac{3\alpha_2}{2r^5} + \frac{15\alpha_2\sin^2\theta}{2r^5} + \frac{15\alpha_3}{8r^7} - \frac{105\sin^2\theta}{8r^7}\right]$$

or

$$U_{x_2x_2} = D\mu\left[n^2 - \left\{\frac{1}{r^3} + \frac{3\alpha_2}{2r^5} - \frac{15\alpha_3}{8r^7}\right\} + 3\sin^2\theta\left\{\frac{1}{r^3} + \frac{5\alpha_2}{2r^5} - \frac{35\alpha_3}{8r^7}\right\}\right]$$

Using the expression in equation (4.83), we have

$$U_{x_2x_2} = D\mu\left[n^2 - \left\{-\frac{3\alpha_2}{2r^5} + \frac{15\alpha_3}{8r^7} + n^2 + \frac{3\alpha_2}{2r^5} - \frac{15\alpha_3}{8r^7}\right\}\right.$$

$$\left. + 3\sin^2\theta\left\{-\frac{3\alpha_2}{2r^5} + \frac{15\alpha_3}{8r^7} + n^2 + \frac{5\alpha_2}{2r^5} - \frac{35\alpha_3}{8r^7}\right\}\right]$$

or

$$U_{x_2x_2} = D\mu\left[n^2 - n^2 + 3\sin^2\theta\left\{n^2 + \frac{\alpha_2}{r^5} - \frac{5\alpha_3}{2r^7}\right\}\right]$$

or

$$U_{x_2x_2} = D\mu\left[3\sin^2\theta\left\{n^2 + \alpha_2 r^{-5} - \frac{5}{2}\alpha_3 r^{-7}\right\}\right]$$

using of the expression in system (4.34), we have

$$U_{x_2x_2} = D\mu\left[3\sin^2\theta\left\{n^2 + \alpha_2\left(1 + \frac{5}{2}\alpha_1\right) - \frac{5}{2}\alpha_3\left(1 + \frac{7}{2}\alpha_1\right)\right\}\right]$$

Expanding and neglecting the products of $\alpha_i \quad (i = 1, 2, 3)$, we have

$$U_{x_2x_2} = 3D\mu\sin^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) \tag{4.87}$$

Putting $x_1 = 1 + r\cos\theta$, $x_2 = \mathrm{r}\sin\theta$, $x_3 = 0$ and $2\pi\rho_1 A_1 = n^2(1-\mu)$ in equation (4.80), we have

$$U^0_{x_2x_3} = 0 \tag{4.88}$$

Finally, Putting $x_1 = 1 + r\cos\theta$, $x_2 = \mathrm{r}\sin\theta$, $x_3 = 0$ and $2\pi\rho_1 A_1 = n^2(1-\mu)$ in equation (4.81), we have

$$U^0_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{(1-1-r\cos\theta)^2 + r^2\sin^2\theta}{[(1-1-r\cos\theta)^2 + r^2\sin^2\theta]^{\frac{5}{2}}}\right\}\right.$$

$$-\frac{3\mu\alpha_2}{2}\left\{\frac{(1-1-r\cos\theta)^2 + r^2\sin^2\theta}{[(1-1-r\cos\theta)^2 + r^2\sin^2\theta]^{\frac{7}{2}}}\right\}$$

$$\left. + \frac{15\mu\alpha_3}{8}\left\{\frac{(1-1-r\cos\theta)^2 + r^2\sin^2\theta}{[(1-1-r\cos\theta)^2 + r^2\sin^2\theta]^{\frac{9}{2}}}\right\}\right]$$

or

$$U^0_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{r^2(\cos^2\theta + \sin^2\theta)}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{5}{2}}}\right\} - \frac{3\mu\alpha_2}{2}\left\{\frac{r^2(\cos^2\theta + \sin^2\theta)}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{7}{2}}}\right\}\right.$$

$$\left. + \frac{15\mu\alpha_3}{8}\left\{\frac{r^2(\cos^2\theta + \sin^2\theta)}{[r^2(\cos^2\theta + \sin^2\theta)]^{\frac{9}{2}}}\right\}\right]$$

or

$$U^0_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \frac{\mu r^2}{r^5} - \frac{3\mu\alpha_2 r^2}{2r^7} + \frac{15\mu\alpha_3 r^2}{8r^9}\right]$$

or

$$U^0_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \frac{\mu}{r^3} - \frac{3\mu\alpha_2}{2r^5} + \frac{15\mu\alpha_3}{8r^7}\right]$$

or

$$U^0_{x_3x_3} = D\left[-2\pi\rho_1 A_2 - \mu\left\{\frac{1}{r^3} + \frac{3\alpha_2}{2r^5} - \frac{15\alpha_3}{8r^7}\right\}\right]$$

or

$$U^0_{x_3x_3} = -D\left[2\pi\rho_1 A_2 + \mu\left\{\frac{1}{r^3} + \frac{3\alpha_2}{2r^5} - \frac{15\alpha_3}{8r^7}\right\}\right]$$

Using the expression in equation (4.83), we have

$$U^0_{x_3x_3} = -D\left[2\pi\rho_1 A_2 + \mu\left\{-\frac{3\alpha_2}{2r^5} + \frac{15\alpha_3}{8r^7} + n^2 + \frac{3\alpha_2}{2r^5} - \frac{15\alpha_3}{8r^7}\right\}\right]$$

or

$$U^0_{x_3x_3} = -D[2\pi\rho_1 A_2 + \mu n^2] \qquad (4.89)$$

Therefore, the values of the second partial derivatives evaluated at the point $(1 + r\cos\theta, r\sin\theta, 0$ are

$$U^0_{x_1x_1} = 3D\mu\cos^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right),$$

$$U^0_{x_1x_2} = 3D\mu\sin\theta\cos\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right),$$

$$U^0_{x_2x_2} = 3D\mu \sin^2\theta \left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right), \tag{4.90}$$

$$U^0_{x_3x_3} = -D(2\pi\rho_1 A_2 + \mu n^2),$$

$$U^0_{x_1x_3} = U^0_{x_2x_3} = 0.$$

Substituting these values of system (4.90) in the variational equation (4.5), we obtain

$$\begin{aligned} \ddot{\alpha} - 2n\dot{\beta} &= 3D\mu\cos^2\theta\left(n^2 + \alpha_2 - \tfrac{5}{2}\alpha_3\right)\alpha + 3D\mu\sin\theta\cos\theta\left(n^2 + \alpha_2 - \tfrac{5}{2}\alpha_3\right)\beta \\ \ddot{\beta} + 2n\dot{\alpha} &= 3D\mu\sin\theta\cos\theta\left(n^2 + \alpha_2 - \tfrac{5}{2}\alpha_3\right)\alpha + 3D\mu\sin^2\theta\left(n^2 + \alpha_2 - \tfrac{5}{2}\alpha_3\right) \end{aligned} \tag{4.91}$$

$$\ddot{\gamma} = -D(2\pi\rho_1 A_2 + \mu n^2)\gamma \tag{4.92}$$

Equation (4.92) is independent of system (4.91) and shows that the motion of the infinitesimal mass along x_3 −direction is stable.

Using the second order partial derivatives of system (4.90) in the characteristic equation (4.7), we have

$$\begin{aligned} \lambda^4 - \left[3D\mu\cos^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) + 3D\mu\sin^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) - 4n^2\right]\lambda^2 \\ + \left[3D\mu\cos^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right)\right]\left[3D\mu\sin^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right)\right] \\ - \left[3D\mu\sin\theta\cos\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right)\right]^2 = 0 \end{aligned}$$

or

$$\lambda^4 - \left[3D\mu\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right)(\sin^2\theta + \cos^2\theta) - 4n^2\right]\lambda^2$$

$$+ 9D^2\mu^2\cos^2\theta\sin^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right)^2$$

$$- 9D^2\mu^2\cos^2\theta\sin^2\theta\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right)^2 = 0$$

or

$$\lambda^4 - \left[3D\mu\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) - 4n^2\right]\lambda^2 = 0 \qquad (4.93)$$

Equation (4.93) is the characteristic equation corresponding to the system (4.91).

Let $\lambda^2 = \Lambda$, then equation (4.93) above reduces to

$$\Lambda^2 - \left[3D\mu\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) - 4n^2\right]\Lambda = 0$$

$$\Lambda\left[\Lambda - \left\{3D\mu\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) - 4n^2\right\}\right] = 0$$

This shows that either $\Lambda = 0 \quad or \quad \Lambda = 3D\mu\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) - 4n^2$

Therefore, $\lambda = 0 \quad twice \quad \lambda = \pm\left[3D\mu\left(n^2 + \alpha_2 - \frac{5}{2}\alpha_3\right) - 4n^2\right]^{\frac{1}{2}}$ (4.94)

Equation (4.94) gives the roots of the characteristic equation of equation (4.93). The presence of the multiple roots shows that the circular points are unstable.

4.8 Stability of Axial Libration Points (Case 2)

The second order partial derivatives at this case are

$$U^0_{x_1x_1} = D\left[1 + 2\mu - C_1 + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2)\right],$$

$$U^0_{x_2x_2} = D\left[1 - \mu - C_2 + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3\mu\sigma_2}{2}\right], \qquad (4.95)$$

$$U^0_{x_3x_3} = D\left[-\mu - C_3 - \tfrac{3\mu}{2}(4\sigma_1 - \sigma_2)\right],$$

$$U^0_{x_1x_2} = U^0_{x_1x_3} = U^0_{x_2x_3} = 0.$$

The superscript 0 denotes that the partial derivatives are to be evaluated at the point $(0, 0, 0)$. Substituting the values of the partial derivatives from (4.76) in the variational equation (4.5), we have

$$\ddot{\alpha} - 2n\dot{\beta} = D\left[1 + 2\mu - C_1 + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2)\right]\alpha,$$

$$\ddot{\beta} + 2n\dot{\alpha} = D\left[1 - \mu - C_2 + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3\mu\sigma_2}{2}\right]\beta, \tag{4.96}$$

$$\ddot{\gamma} = D\left[-\mu - C_3 - \tfrac{3\mu}{2}(4\sigma_1 - \sigma_2)\right]\gamma. \tag{4.97}$$

Equation (4.97) is independent of system (4.96), and its solutions being periodic are bounded and therefore, the motion of the infinitesimal body in both cases in the x_3- direction is stable.

Using the equations in system (4.76) in the characteristic equation (4.5), we have

$$\begin{aligned}\lambda^4 - \Big\{-4n^2 + D\left[1 + 2\mu - C_1 + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2)\right] \\ + D\left[1 - \mu - C_2 + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3\mu\sigma_2}{2}\right]\Big\}\lambda^2 \\ + D\left[1 + 2\mu - C_1 + \frac{3}{2}(1 + 4\mu)(2\sigma_1 - \sigma_2)\right] \\ \times D\left[1 - \mu - C_2 + \frac{3}{2}(1 - 2\mu)(2\sigma_1 - \sigma_2) + \frac{3\mu\sigma_2}{2}\right] = 0\end{aligned}$$

or

$$\lambda^4 - \left\{-4\left(1+\frac{3}{2}(2\sigma_1-\sigma_2)\right) + D\left[1+2\mu-C_1+\frac{3}{2}(1+4\mu)(2\sigma_1-\sigma_2)\right]\right.$$
$$\left. + D\left[1-\mu-C_2+\frac{3}{2}(1-2\mu)(2\sigma_1-\sigma_2)+\frac{3\mu\sigma_2}{2}\right]\right\}\lambda^2$$
$$+ D^2\left[1+2\mu-C_1+\frac{3}{2}(1+4\mu)(2\sigma_1-\sigma_2)\right]\left[1-\mu-C_2\right.$$
$$\left. +\frac{3}{2}(1-2\mu)(2\sigma_1-\sigma_2)+\frac{3\mu\sigma_2}{2}\right] = 0$$

or

$$\lambda^4 - \left\{-4-6(2\sigma_1-\sigma_2) + D\left[2+\mu-C_1-C_2+\frac{3}{2}(1+\mu)(4\sigma_1-\sigma_2)-\frac{3\mu\sigma_2}{2}\right]\right\}\lambda^2$$
$$+ D^2\left[1+2\mu-C_1+\frac{3}{2}(1+4\mu)(2\sigma_1-\sigma_2)\right]\left[1-\mu-C_2\right.$$
$$\left. +\frac{3}{2}(1-2\mu)(2\sigma_1-\sigma_2)+\frac{3\mu\sigma_2}{2}\right] = 0. \qquad (4.98)$$

Equation (4.98) is the characteristic equation corresponding to systems (4.96)

If λ_1^2 & λ_2^2 are the roots of the above characteristic equation (4.98), such that

$$\lambda_1^2+\lambda_2^2 = -4-6(2\sigma_1-\sigma_2)+D\left[2+\mu-C_1-C_2+\frac{3}{2}(1+\mu)(4\sigma_1-\sigma_2)-\frac{3\mu\sigma_2}{2}\right] < 0,$$

and

$$\lambda_1^2\lambda_2^2 = D^2\left[1+2\mu-C_1+\frac{3}{2}(1+4\mu)(2\sigma_1-\sigma_2)\right]\left[1-\mu-C_2+\frac{3}{2}(1-2\mu)(2\sigma_1-\sigma_2)\right.$$
$$\left. +\frac{3\mu\sigma_2}{2}\right] > 0,$$

then the libration point is stable; otherwise unstable.

Since $C_1 > (1 + 2\mu)$, $C_2 > (1 - \mu)$, $\sigma_i \ll 1$. It shows that all the conditions above are satisfied. Therefore, the libration point (0, 0, 0) is stable. We also see that the conditions for the stability are influenced by the triaxiality of the second primary.

4.9 Conclusion

In case 1, we see that when both primaries are oblate spheroids, the points $(p_1, 0, 0)$ is always a libration point, and another libration point $(x_{11} + p_2, 0, 0)$ exists when $1 - 2\pi\rho_1 A_1 < -\frac{3}{4}\mu$, $|x_{11}| < a_1$. The later point is different from that of Hallan and Mangang (2007) and Singh and Sandah (2012). The points on the circle $(1 - x_1)^2 + x_2^2 = r^2$ lying within the first primary and satisfying the condition (4.29) are libration points. All these points are affected by oblateness of both primaries and the zonal harmonic. When $x_2 = 0$, the circular points coalesce to those lying on the line joining the primaries. In case 2, we observe that the triaxiality of the second primary does not affect the location of the libration point.

The investigation of the linear stability is according to Lyapunov's theorem. In the first case, we have retained the linear terms in the oblateness and zonal harmonic coefficients of the primaries. The stability in the first approximation of this configuration shows that the first axial point near the first primary is stable; the second axial point is conditionally stable, while the circular points are unstable. This confirm the result of Hallan and Mangang (2007), Hallan and Rana (2001) and Singh and Sandah (2012). In the second case, we found that the origin (0, 0, 0) is the only libration point. It is also noticed that the triaxiality of the second primary does not affect the location of the libration point; it is stable. The stability behaviour is the same as that of Plastino and Plastino (1995).

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

The summary, conclusion and recommendations as regard to this study are given in this chapter.

5.2 Summary

The research work is based on investigating the motion of an infinitesimal mass in the framework of Robe's CR3BP in two different cases. (*i*) when the hydrostatic equilibrium figure of the first primary is an oblate spheroid, the shape of the second primary is considered as an oblate spheroid with oblateness coefficients up to the second zonal harmonic. (*ii*) when the primary bodies form a Roche ellipsoid-triaxial system.

The equations of motion of Robe's CR3BP under oblate primaries with oblateness coefficients up to the second zonal harmonic are established, and are given in system (3.4), they are different from those of Hallan and Mangang (2007) due to the oblateness of the second primary and those of Singh and Sandah (2012) on the account of the zonal harmonic of the second primary. While system (3.7) gives the equations of motion when the first primary is a Roche ellipsoid and the second one is a triaxial rigid body, and they are different from those of Plastino and Plastino (1995) due to the triaxiality terms.

The examination of the existence of libration points in both cases is carried out in chapter four. It is observed in the first case that two axial libration points $(p_1, 0, 0)$ and $(x_{11} + p_2, 0, 0)$ are found near the centre of the first primary. The points on the circle are also libration points. The origin (0, 0, 0) is the only libration in the second case and this fully coincides with that of Plastino and Plastino (1995).

In chapter five, the investigation of the linear stability of the libration solutions is done by the use of characteristics roots. The stability in the first approximation of the system considered shows that the first axial point is stable and the second one is conditionally stable, while the circular points are unstable. This confirms the earlier results of Hallan and Mangang (2007) and Singh and Sandah (2012). The stability behaviour of the point (0,0,0) is as that of Plastino and Plastino (1995)

5.3 Conclusions

The main problem of dynamics is to investigate all the possible motions of a system. By considering the hydrostatic equilibrium figure of the fluid of the first primary as an oblate spheroid in the first case, the full buoyancy force of the fluid acting on the infinitesimal mass, and the shape of the second primary as an oblate spheroid with zonal harmonics; we have established the pertinent equations of motion and investigated the existence of the libration points and have discussed their linear stability. It is found that, the existence and stability of the first axial libration point $(p_1, 0\ ,0)$ is as in Hallan and Mangang (2007) and Singh and Sandah (2012). The second axial libration point $(x_{11} + p_2, 0\ ,0)$ is different from those of Hallan and Mangang (2007) and Singh and Sandah (2012) due to the appearance of zonal hamonic, and it is found within the first primary on the line joining the centres of the primaries when $1-2\pi\rho_1 A_1 < -{}^{3\mu}/_{4}$ and $|x_{11}| < a_1$. When $2\pi\rho_1 A_1 = n^2(1-\mu)$, points on the circle $(1-x_1)^2 + x_2^2 = r^2$, $x_3 = 0$ lying within the first primary are also libration points. We call them circular points. It is further found that the first axial point is stable, while the second one is conditionally stable, and the circular points are unstable.

In the second case, we have assumed the shape of the first primary as a Roche ellipsoid and the secondary is a triaxial rigid body. We have also established the pertinent equations of motion, investigated the existence and stability of the libration points. It observed that the

systems have only one libration point (0, 0, 0) when D > 0, and this point is stable. In both cases, we have discussed a special case in which the fluid and the infinitesimal mass have equal densities; it is found that all the points inside the fluid in this case are in libration points.

5.4 Recommendations

The research work can be extended in the following ways

(i) The investigation of nonlinear stability of these problems

(ii) The shape of the first primary can be considered as a Roche ellipsoid in the first case

(iii) The out-of-plane points of Robe's circular restricted three-body problem.

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