STUDY OF FUZZY MULTISETS AND THEIR ALGEBRA

By

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DECLARATION

I declare that the work in this dissertation entitled " STUDY OF FUZZY MULTI-SETS AND THEIR ALGEBRA" has been performed by me in the Department of Mathematics under the supervision of Professor D. Singh, Dr. A. M. Ibrahim and Professor G. U. Garba.

The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at this or any other Institution.

Alhaji Jibril ALKALI

Name of student	Signature	Date

CERTIFICATION

This dissertation entitled " STUDY OF FUZZY MULTISETS AND THEIR ALGEBRA" by Alhaji Jibril ALKALI meets the regulations governing the award of the degree of Doctor of Philosophy of the Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This dissertation is dedicated to my lovely Family

.

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ABSTRACT

In this dissertation, fundamentals of the theory of fuzzy sets, multisets and fuzzy multisets were studied, and some properties of α-cuts, specifically two decomposition theorems for fuzzy sets, were extended to fuzzy multisets. Also, properties of inverse α-cuts for fuzzy sets were strengthened and extended to fuzzy multisets. Moreover, some new operations were introduced which helped developing certain monoids of multiset partitions and fuzzy partitions. Finally, fuzzy groups, multigroups and fuzzy multigroups were studied and abelian fuzzy multigroup was introduced.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Introduction

Essentially, mathematics studies *structures*. A structure is a collection (*Bereich*) of objects (*events, states,* etc.) with certain operations and relations defined on these objects or their n-tuples. As such, the notion of a *set* is not only fundamental to mathematics but also to all sciences, including natural languages in sofar as they are required to possess a structure.

In fact, the classical (Cantorian) set theory, despite embodying severe limitations on the nature of objects to form a set, has provided a firm foundation for *bivalent* logic based mathematics in which all variables as well as relations between variables are crisp. However, if mathematics is required to model the concept of a class of objects and relations between them which may not possess crisply defined criteria, the classical paradigm based on two-valued logic is found *incapacitated* and even *inapplicable* at times. For example, *the class of all real numbers which are much larger than ten*, *the class of all handsome living statesmen across the world*, etc., cannot be characterized extensionally because their defining predicates involve *vagueness* and *uncertainty*, and admit *gradations* in values of membership beyond zero and one.

Yet, the fact remains that such imprecisely defined classes play an important role in human thinking, particularly in the domain of pattern recognition, communication of information and abstraction (Zadeh, 1965).

Regular mathematics is like the traffic camera that uses a logic that has rigid limits.

Sometimes, it does not work for the purposes you need, especially when something approximating human judgment is needed (Sowell, 2008).

In order to adequately address the concept of vagueness and uncertainty, the concept of fuzzy sets got introduced. Fuzzy set theory is a mathematical theory to model a kind of uncertainty.

A fuzzy set is characterized by its elements occurring with a continuum of *degrees* (*grades*) of membership. The members of a fuzzy set are always assumed to be sets. Moreover, the grade of membership (full or partial) and non-membership (full or partial) of an element in a fuzzy set is required to be commensurate with the concept represented by the fuzzy set. A perceptive advantage can be observed in figure 1.1(a) and figure 1.1(b) in representing vague concepts such as *low*, *medium* and *high* by fuzzy sets respectively.

It is instructive to note that an appropriate representation of a fuzzy set intended

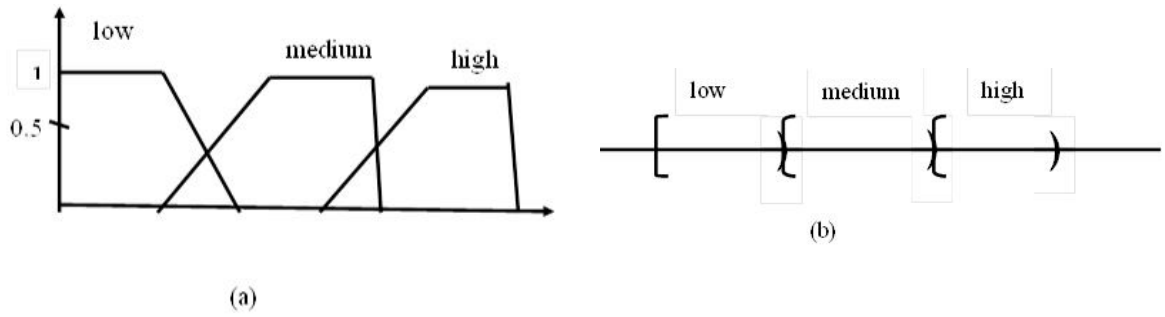


Figure 1.1.1: Representation of *low, medium* and *high* by fuzzy sets

to model a vague concept must account for the *context* as well. For example, the concept of *high temperature* in the context of weather and in the context of nuclear reactor would require different fuzzy sets for their representations. Moreover, even for similar contexts (e.g., weather in different seasons in the same climate) representing the same concept (e.g., high temperature) distinct fuzzy sets would be required.

For example, the graphs Figure 1.2(a) and Figure 1.2(b) represent the same concept viz., *the class of real numbers close to ten.*

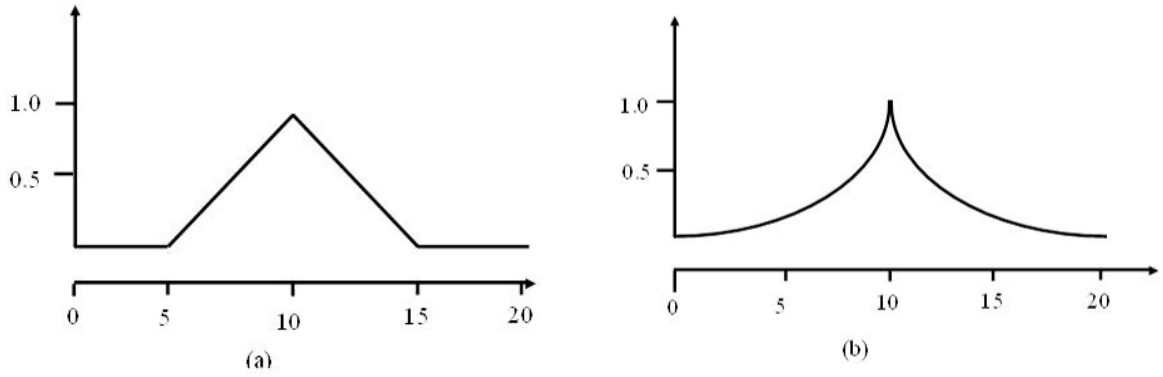


Figure 1.1.2: Class of real numbers close to ten

Some other variations in the shapes of the graph of the same concept are possible depending on the context of a particular application ((Palaniappan, 2005) contains some details). In most of the applications, especially in order to achieve computational efficiency, the shapes for the membership functions of fuzzy variables are *triangular*, *bell* shaped*, trapezoidal or Gaussian functions* whereas, *right open intervals* of real numbers are used in dealing with functions of crisp variables. The triangular representation is known to be coarser than others.

Undoubtedly, the extensional representations of a vague concept by ordinary sets are mathematically sound. However, it is inherently far from reality, simply because the decision to include the points falling into a close neighborhood of each sharply defined boundary between states of a crisp variable in only one of the states involves inevitable uncertainty which is autocratically ignored. For example, crisply defined ranges to represent body temperatures of human beings viz., *not feverish*, *feverish*, and *very highly feverish,* are incongruent with *reality* which could best be brought out by some suitable nonextensional approach.

Zadeh (1965) in order to study the control problem of complex systems and to deal with fuzzy information developed fuzzy set theory. A *fuzzy set (class)* $\tilde{A}$ in a space X of points (objects) is characterized by a *membership (generalized characteristic) function*, usually denoted $\mu_{\tilde{A}}(x)$ or $\tilde{A}(x)$, which associates with each point x in X a real number in the interval $[0,\ 1]$, with the values of $\mu_{\tilde{A}}(x)$ at x representing the *grade of membership* of x in $\tilde{A}$. This approach for formulating the theory of fuzzy sets can be seen straightforwardly related to an infinite-valued logic by interpreting the membership grades as *the truth degrees* with $[0,\ 1]$ as its *truth degree set*.

For example, let X be the real line $\mathbb{R}$ and let $\tilde{A}$ be a fuzzy set of numbers that are considerably larger than ten. Then $\tilde{A}$ can be precisely, although subjectively, characterized by a membership function $\mu_{\tilde{A}}(x)$ on $\mathbb{R}$, where $\mu_{\tilde{A}}(10) = 0$, $\mu_{\tilde{A}}(15) = 0$, $\mu_{\tilde{A}}(50) = 0.02$, $\mu_{\tilde{A}}(100) = 0.06$, $\mu_{\tilde{A}}(500) = 0.6$, $\mu_{\tilde{A}}(1000) = 1$, etc., can be taken as its representative values. Customarily, while listing the elements of a fuzzy set, elements with *zero degree* of membership are usually omitted.

Schematically, a fuzzy set $\tilde{A}$ in X can be represented as

$$\mu_{\tilde{A}} : X \longrightarrow [0,1],$$

where $\mu_{\tilde{A}}$ is the membership function.

Also, $\tilde{A}$ is a fuzzy set in X is usually represented as

$\tilde{A} :\ X \longrightarrow [0,1]$, or $\tilde{A} = \{(x, \mu_{\tilde{A}}(x))|\ x \in X\}$, a set of ordered pairs.

Example 1.1.1 (See Figure 1.3 and 1.4)shows the difference between the crisp and fuzzy sets represented by membership functions, respectively.

Theory of fuzzy sets has been progressing steadily and a wealth of mathematics has been developed.

Similar to the concept of fuzziness, the concept of *multiple-membership collections*

has a long and tortuous history. Knuth (1969) noted that despite frequent occurrences of multi-set like structures in mathematics, there is currently no structured way to deal with multisets. The term *multiset,* as Knuth notes, was first suggested by N. G. De Bruijn in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz., *heap*, *bunch*, *bag*, *sample*, *weighted set*, *occurrence set*, *fire set*, etc.

As mentioned earlier, this demand of cantorian set theory did not go hand in hand with the nature of a large number of problems arising in mathematics as well as in other *hard* and *soft* science. For example, consideration of repeated roots of a polynomial equation, repeated observations in a statistical sample, repeated hydrogen atoms in a water molecule H_2O, etc., do play a significant role.

Relatively recently, in view of the aforesaid two very dominant generalizations of the ordinary set theory, efforts to develop mathematical structures characterizing classes of more complex objects, possessing both fuzziness and multiplicity, have been made. The concept in view is that of a *fuzzy multiset.* In order to distinguish it, an ordinary nonfuzzy multiset is called a *crisp (pure) multiset.* For example,

$$A = \{(x,1)\,,\ (x,0.6)\,,\ (x,0.3)\,,(y,0.8)\,,\ (y,0.7)\}\,,$$

equivalently represented as

$$A = \{\ \{1,0.6,\ 0.3\}/x\ ,\{0.8,0.7\}/y\}\},$$

is a fuzzy multiset in $X = \{x,y,z\}$.

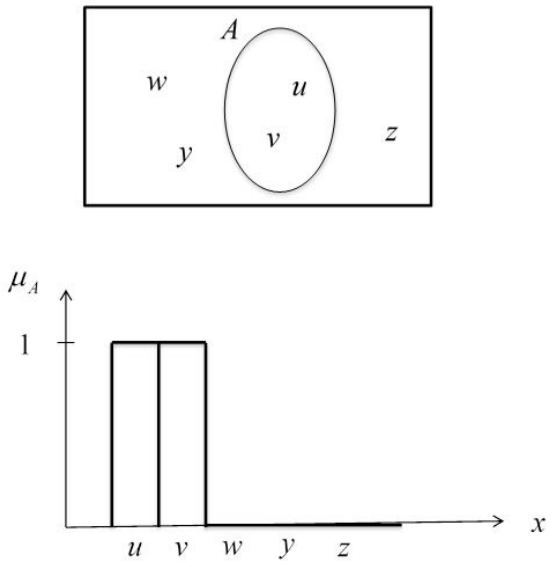


Figure 1.1.3: Graphical representation of crisp set

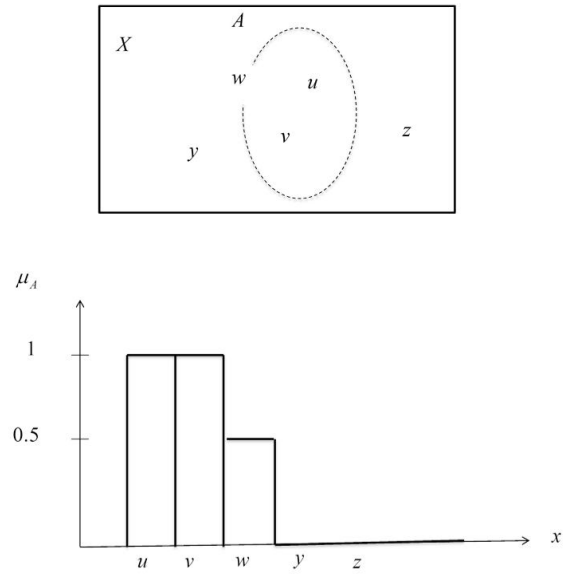


Figure 1.1.4: Graphical representation of fuzzy set

1.1.1 The Concept of Fuzziness

Since the 16th century, probability theory has been studying a kind of uncertainty in dealing with occurrences of an event. In probability theory, an event $u \in U$ either happens or does not happen and its probability $p(u)$ represents the chance for the event to happen; that is, the chances that a random variable x takes a value u. Note, however, that event itself is certain. A probability $p(u)$ can be represented by the ratio between the number of experiments of a series when u happens and the total number of experiments performed. For example, the probability that it will rain on August 20 is 0.69, because 69 out of 100 days on this date were recorded for the last 100 years as rainy days. The probability density function $P(x)$ gives the probability of each of the possible values of a random variable x (events $u \in U$). However, a particular August 20 comes and it is not quite clear whether it is a *rainy* day or not. The notion of *rainy* can be represented as a fuzzy set $\tilde{A}$ on the universe of the *rainfall* and represented by its membership function $\mu_{\tilde{A}}$. In general, a membership coefficient $\mu_{\tilde{A}}(u)$ measures the degree to which an event u, which has already happened, is a member of the concept labeled as $\tilde{A}$. The membership function represents the degree to which the total set U (every element from U) is contained in the subset $\tilde{A}$. For example, August 20 was a rainy day to a degree of 0.7. So the probability density function $P(x)$ is a completely different concept from the membership function of a fuzzy set $\tilde{A}$. Another classical example is given below: Consider the experiment of throwing a single die once Figure 1.5 shows in a tabular and graphical form the probability distribution table of the outcome (X) obtained after the throw and the membership function that the number is *small*.

Outcome X	1	2	3	4	5	6
Probability distribution of X	1/6	1/6	1/6	1/6	1/6	1/6
$\mu_{\tilde{A}}$ Small number	1	0.5	0	0	0	0

Figure 1.1.5: Probability distribution graph for throwing a single die and the membership function of the concepts *Small, Medium, Big.*

In the above example two events can be distinguished-a crisp event, achieving a number of 2, and a fuzzy event, achieving a *small* number. One can calculate the probability for a crisp event, $p(x = 2) = 1/6$, but it is also possible to calculate the probability $p(\tilde{A})$ of a fuzzy event, the probability that a small number will be achieved, $p(x = small)$.
In general, a probability of a fuzzy event is calculated as:

$$p(\tilde{A}) = \sum \mu_{\tilde{A}}(u)/n$$

for uniformly distributed variable x having n discrete values, and

$$p(\tilde{A}) = \sum \mu_{\tilde{A}}(u) \cdot p(x = u)$$

for the general case.

Example 1.1.2 For the membership functions shown in Figure (1.1.5), we calculate

$$p(x = small) = 1.5/6;\ p(x = medium) = 2/6;\ p(x = big) = 2.5/6.$$

1.2 Statement of Problem

A number of fundamentals notions of fuzzy set theory and some important algebraic structures dealing with fuzzy sets have not been studied in fuzzy multisets theory.

1.3 Justification

Fuzzy set theory has found diverse applications, especially in information control system, artificial intelligence, decision theory, etc. On the other hand, studies on multisets and their applications abound in mathematics, computer science, Linguistics, etc. Currently, the study of fuzzy multisets and applications is being undertaken.

1.4 Aim and Objectives

The aim of this Dissertation is to study Fuzzy Sets, Multisets, Fuzzy Multisets and develop some algebraic structures.

The objectives are as follows:

i To comprehensively study the origin, structure and development of fuzzy set theory.

ii To critically study the fundamentals of fuzzy sets, multisets, and fuzzy multisets.

iii To extend the properties of α- cuts for fuzzy sets to fuzzy multisets.

iv To strengthen properties of inverse α-cuts for fuzzy sets and extend them to fuzzy multisets.

v To formulate certain monoids of multiset partitions and that of fuzzy partitions.

vi To study fuzzy groups, multigroups, and fuzzy multigroups for further extensions.

1.5 Methodology

In order to achieve the objectives outlined above, we propose to conduct an up-to-date critical review of literature dealing with various fundamental aspects of fuzzy sets, multisets, fuzzy multisets, and define some new operations which would help formulating new algebraic structures.

1.6 Organization of the Dissertation

The Dissertation consists of six chapters. Besides chapter one, a comprehensive and critical literature survey will be presented in chapter two. In Chapter three, the fundamentals of fuzzy sets, multisets, and fuzzy multisets will be presented. In chapter four, α-cuts, inverse α-cuts and their properties for fuzzy sets, and their extensions to fuzzy multisets will be presented. In chapter five, certain monoids of multisets and fuzzy sets will be presented. In chapter six, summary, conclusion and recommendations will be presented.

CHAPTER TWO

LITERATURE REVIEW

The queerness of the concept of vagueness has long been drawing the attention of philosophers,linguists, logicians, and mathematicians. As noted in (Magnus, 1997), Nietsche was the first to recognize the notion of vagueness. In course of time, various other closely related notions such as *loose concepts*, *haziness*, *borderline cases*, *fluent boundaries*, *case by grades*, etc., appeared. As mentioned earlier, Gottlob Frege (1848–1925) was the first to provide a mathematical definition of vagueness in terms of having an *unsharp boundary*. A seminal contribution towards investigating the concept of *vagueness* was made by (Black, 1937). The epicentre of Black's explication can be seen as a unifying thread between Bertrand Russell's and C. S. Peirce's approach.

Menger (1979) argued that the notion of *probability* could adequately deal with loose concepts. He also introduced the notion of *hazy set*. However, it was not explicit until the formulation of the theory of fuzzy sets (Zadeh, 1965) that the notion of probability could not deal with vagueness and other loose concepts if the meaning of these concepts is the *absence of sharp boundaries*. The concept of fuzziness essentially refers to the semantic feature of the vagueness of a phenomenon rather than its stochastic explication which is devoid of it.

A distinctive feature of the concept of *fuzziness* can be seen summarized in the following: *In contrast to the stochastic uncertainty-type vagueness, the vagueness concerning the description of the semantic meaning of events, phenomena or statements is called fuzziness* [(Moreno-Armella and Waldegg, 1993). Kaushal et al. (2010) and Seising (2005) provided a good deal of illustrations to describe the relevancy of fuzzy concept in mathematics.]

In view of the pervasive role played by set-theoretic foundation, it was seemingly natural to look for a set theory-like framework to model the class of problems in which the source of vagueness is not the presence of random variables rather the absence of precisely defined criteria of class membership. Fortunately, it was found forthcoming by way of relaxing the restriction of *definiteness* imposed on objects to form a Cantorian set. L. A. Zadeh was the first who formulated a set-theoretic model in (Zadeh, 1965) and titled it *fuzzy set theory* in contrast to *crisp set theory*. Fuzzy set theory is a mathematical theory to model vagueness and other loose concepts. It deals with fuzzy variables and fuzzy relations.

A fuzzy set is represented by a *generalized* characteristic function, first introduced in (Whitney, 1933), in contrast to the characteristic function taking values in $\{0,\ 1\}$ in the case of an ordinary set. It follows that fuzzy models subsume the role of classical models and, in this sense, fuzzy set theory is a generalization of the classical set theory. As indicated in (Zadeh, 1965), other ranges of the membership function beyond the unit interval (for example, a partially ordered set) could be gainfully considered. Goguen (1967) formulated theory of the *lattice fuzzy* (L - *Fuzzy*) sets by taking a complete lattice as the range of membership function. In view of these generalizations, if the degrees of membership of a fuzzy set are real from the unit interval, it is called *standard* or *ordinary* or *first order*. It is interesting to note that the range of membership function in the case of a classical set viz., $\{0,\ 1\}$ forms a two-element Boolean algebra, a partially ordered structure.

The notion of fuzzy sets is essentially *nonstatistical*. Although the membership function of a fuzzy set does possess some resemblance to a probability function where X is countable (or a probability density function when X is a continuum), Probability theory is not an appropriate vehicle to deal with the kind of uncertainty that

appears especially in *pattern classification* and *information processing.* This type of uncertainty seems to be more of an *ambiguity* than a statistical variation ((Zadeh, 1965) and (Goguen, 1967)). Of course, probability theories can help solving problems involving stochastic- type vaguenesses.

Following the formulation of fuzzy set theory, other approaches addressing the problem of how to understand and manipulate imperfect knowledge got underway: *rough set theory* (Pawlak, 1982) and *soft set theory* (Molodtsov, 1999) are two prominent directions. It is but natural that both the approaches share a lot many commonalities with the fuzzy set theory, yet being different. As Pawlak notes, the most successful one is, no doubt, the fuzzy set theory.

During the last three decades or so, many formulations and interpretations of the theories of fuzzy sets have appeared. *Infinite valued logic* based approaches, *set theocratic* approaches, *model theoretic* approaches and *category theoretic* approaches have been the major directions for developing theories of fuzzy sets (Gottwald (2005) is a crisp and excellent survey).

As to the limitations of fuzzy set theory, the only difficulty, especially raised by soft set theoreticians, is the nature of membership function being extremely individual. On our view, it is actually the most competitive feature that makes fuzzy set theory more comprehensive and expressive than other approaches known so far to deal with vagueness and uncertainty as closely as possible to human thinking. In fact, the subjectivity involved is not bald. It is guided by taking account of both the *concept* and *context* in a harmonious framework. However, constructing appropriate membership function for a wide variety of concepts in various contexts is quite an involved task.

Similar to the concept of fuzziness, the concept of *multiple-membership collections* has a long and tortuous history. Knuth (1969) noted that despite frequent occurrences of multi-set like structures in mathematics, there is currently no structured way to deal with multisets. The term *multiset,* as Knuth noted, was first suggested

by N. G. De Bruijn in a private communication to him. Owing to its aptness, it has replaced a variety of terms viz., *heap*, *bunch*, *bag*, *sample*, *weighted set*, *occurrence set*, *fire set*, etc. In fact, prior to coinage of the term *multiset*, the term *bag* was in frequent use. Currently, multiset and bag are being used interchangeably. Knuth (1969), along with providing a concise historical perspective of the concept of multiset, presented many results on multiset operations and their applications in mathematics, computer science, and other sciences.

The challenging task of formulating sufficiently rich mathematics of multisets started receiving serious attention from beginning of the 1970s. An updated exposition on both historical and mathematical perspective of the development of the theory of multisets can be found in, (Blizard, 1991; Singh et al., 2007). The underlying idea was to develop a generalization of the ordinary set theory by way of relaxing the restriction of *distinctness* on the nature of the objects forming a set. As mentioned earlier, this demand of cantorian set theory did not go hand in hand with the nature of a large number of problems arising in mathematics as well as in other *hard* and *soft* sciences. For example, consideration of repeated roots of a polynomial equation, repeated observations in a statistical sample, repeated hydrogen atoms in a water molecule H_2O, etc., do play a significant role. The *principle of indistinguishability* attributed to Henri Poincare (Seising, 2005) and elaborately described by Parker–Rhodes (Singh et al., 2007) contemplated that objects may be *identical*, *distinct* or *twins*. For example in H_2O, the two hydrogen atoms H^1 and H^2 are the same but separate, H^1 and O (or H^2 and O) are obviously distinct, while H^1 and H^1 (or H^2 and H^2) are coinciding and identical.

A *multiset* (mset for short) or a *bag* is an unordered collection of objects in which, unlike an ordinary set, objects are allowed to repeat. Each individual occurrence of an object in an mset is called its *element*. All duplicates of an object in an mset are

treated indistinguishably. The objects of an mset are its distinguishable (distinct) elements. The number of occurrences of an object, which is usually finite, in an mset is called its *cardinality*, denoted by $m_A(x)$ or $C_A(x)$ or $A(x)$. The cardinality of an mset A is the sum of the multiplicities of all its objects, denoted by $C(A)$or $|A|$. That is, $C(A) = \sum C_A(x)$ for all x in A. The *root* or *support* or *carrier* of an mset A, denoted by A^*, is the set containing all distinct elements of A. It follows that every mset must have a unique root set. The cardinality of the root set of an mset A is called its *dimension*.

An mset is usually represented by using square brackets, instead of curly brackets, to distinguish it from set representations. For example an mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notated $[a, b, b, c, c, c]$ or $[a, b, c]_{1,2,3}$ or $[1/a,\ 2/b, 3/c]$ or $[a^1, b^2, c^3]$ or$[a^1b^2c^3]$, etc. For convenience, the curly brackets are often used if no confusion arises. There are many other forms of multiset representations. The representation of an mset in a function form, which is found quite useful especially in developing axiomatic foundations and applications in computer science, is as follows:

Let D be a domain set (universe) and let T be a numeric set. Then, a map $\alpha : D \to T$ is called

$$\begin{cases} \text{a set} & \text{if } T = \{0, 1\}; \\ \text{a multiset} & \text{if } T = \text{ the set of natural numbers including zero;} \\ \text{a signed multiset(or, hybrid/shadow set)} & \text{if T = Z, the set ofall integers; and} \\ \text{a fuzzy(or hazy) set} & \text{if } T = [0, 1]. \end{cases}$$

Multisets have found many applications in mathematics, computer science (especially in database theory, π–calculus, membrane computing etc.), Linguistics, economics, etc., (Blizard, 1991), (Claude et al., 2001), (Singh et al., 2007), and (Singh and Peter, 2011) are some references which contain most of the details regarding mathematics of multisets and their applications.

Fuzzy multisets have been introduced in (Yager, 1986), and further studied in (Miyamoto, 1996, 1999, 2001) and others. As Yager uses the term *bag* for *multiset*, similarly he uses the term *fuzzy bag* for *fuzzy multiset.* A typical approach for generalizing multiset to fuzzy multisets is to fuzzify the multiplicities of objects of a multiset. Formally, a fuzzy multiset in some universe set X is a multiset in $X \times [0,\ 1]$. That is, a fuzzy multiset is a multiset of pairs, where the first part of each pair is an element of X and the second part is the degree to which the first part belongs to fuzzy multiset.

Besides, possibilities of generalizing fuzzy multisets are currently being investigated (Miyamoto, 2001) and (Syropoulos, 2005). For example, a generalized fuzzy multiset is obtained if several objects of X appear with the same grades in the fuzzy multiset considered in the example above.

Syropoulos (2005) formulated L - *Fuzzy hybrid sets* on fuzzyfying the objects of a hybrid multiset. It should be noted that hybrid sets have found numerous applications in mathematics and computer science. For example a hybrid set can be used for describing *roots* and *poles* of a rational function (Loeb, 1992).

The notion of α-cuts was introduced in (Zadeh, 1965) with the aim of establishing a bridge between fuzzy set theory and classical set theory. The conceptual underpin-

ning is that it will be useful to consider the set of elements that have at least some minimal degree of membership, say α.

Klir and Yuan (1995) showed that all alpha-cuts and all strong alpha-cuts of any fuzzy set form two distinct families of nested crisp sets, and that a fuzzy set is convex if any point in between two other points resides in the alpha-cut that is largest.

Sun and Han (2006) provided a series of α-related and interval related concepts. They introduce the notion of inverse α-cuts to improve the usability to solve some of the concept of α-cuts to real life problems, which they show can also be considered as a bridge between fuzzy sets and crisp sets. It is demonstrated that, like the α-cuts, every fuzzy set can uniquely be represented by the family of all its inverse α-cuts. This representation allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts.

Chutia et al. (2010) and Mahanta et al. (2010) claimed that the standard method of α-cuts fails to find the square root of a fuzzy number and hence does not always yield results. As such they provide an alternative method which they claim can be utilized in the cases where the method of α-cuts fails. Dutta et al. (2011) refute their claim by providing an alpha-cut method for finding n^{th} root of a fuzzy number, which is even simpler than their proposed method. In addition, they provide methods for performing different type of fuzzy arithmetic operations including exponentiation and taking log.

Kreinovich (2013) observed that membership functions and α-cuts could be viewed as two alternative representations of a fuzzy number; as each imprecise (fuzzy) property can be described by a membership function i.e., by a mapping μ from real

numbers into the interval $[0, 1]$. Alternatively, this same property can be described by α-cuts i.e., by a function that maps each number α from the interval $[0, 1]$ into an interval $x(\alpha) = \{x : \mu(x) \geq \alpha\}$. It is shown that from the traditional mathematical viewpoint, these two representations are equivalent, however, the two approaches are not algorithmically equivalent. In particular, from application point of view it is observed that only α-cuts representation guarantees algorithmic fuzzy data processing and therefore enables us to efficiently process fuzzy data.

Rosenfeld (1971) applied the concept of fuzzy sets to the elementary theory of groupoids and groups. In addition the lattices of fuzzy subgroupoids and ideals are presented, he showed that the fuzzy (left, right) ideal of a set X is a complete sublattice of a fuzzy lattice $\mathscr{L}$ of X.

Kaufmann (1975) studied the theory of fuzzy subsets, where he present different monoids of fuzzy subsets and related results.

Anthony and Sherwood (1979) considered the work of (Rosenfeld, 1971) and gave examples which are not fuzzy algebraic structures with respect to his definitions. To remedy this, they redefine fuzzy subgroupoids and fuzzy subgroups using t-norms and applied it to show that the counter examples are fuzzy algebraic structures.

Tarnauceanu and Bentea (2008) observed that fuzzy group theory has the problem of classifying fuzzy subgroups of a finite abelian group. In consideration of this, they establish the recurrence relation verified by the number of distinct fuzzy subgroups for two classes of finite abelian groups: finite cyclic groups and finite elementary abelian p-groups.

Tawfiq and Qa'aed (2012) studied the effect of group homomorphism on the chain of level subgroups of fuzzy groups and proved a necessary and sufficient conditions under which the chains of level subgroups of homomorphic images of an arbitrary fuzzy group could be obtained from that of the fuzzy groups.

Nazmul et al. (2013) introduced the concept of multigroups (for short, mgroups) and study its properties. They showed that the intersection of two mgroups is mgroup but their union may not be mgroup. Moreover, submgroups, abelian mgroups, normal mgroups and factor mgroups were defined and their properties were also studied.

Shinoj et al. (2015) introduced the concept of Fuzzy Multi groups and studied its properties. Moreover, Fuzzy Multigroup is shown as a generalized case of both fuzzy group and multi group.

In view of overriding influence of fuzzy set theory, generalizations such as *rough fuzzy sets* and *fuzzy rough sets* (Dubois and Prade, 1990), *fuzzy soft sets* (Maji et al., 1971), etc., have also been undertaken.This dissertation aims at further generalizations of fuzzy sets to fuzzy multisets.

CHAPTER THREE

FUNDAMENTALS OF FUZZY SETS, MULTISETS AND FUZZY MULTISETS

In this chapter, we explicate basic notions of fuzzy sets, multisets and fuzzy multisets. In particular, we identify some fundamental notions of the aforesaid non-classical set theories which could be further clarified and extended.

3.1 Definition and Representations of Fuzzy Sets

3.1.1 Definition of a fuzzy set

Definition 3.1.1 A fuzzy set(class) $\tilde{A}$ in X is characterized by a membership function μ_A which associates with each point x in X, a real number $\mu_{\tilde{A}}(x)$ in the interval $[0,1]$. The value of $\mu_{\tilde{A}}(x)$ represents the *grade of membership* of x in $\tilde{A}$.

Intuitively, it can be said that,the closer the value of $\mu_{\tilde{A}}(x)$ to unity, the higher the grade of membership of x in $\tilde{A}$. Moreover, analogous to the notion of (two valued) characteristic function of an ordinary set, $mu_{\tilde{A}}$ can be called generalized characteristic function first introduced in (Whitney, 1933). In general, a fuzzy set in a universal set can be obtained by applying a *fuzzy restriction* to x. If the membership function of $\tilde{A}$ is limited to take values in the set $\{0,1\}$, it becomes a crisp set and, in this sense, the concept of fuzzy sets is a generalization of ordinary sets.

Remark 3.1.1 It is well-known that ordinary sets exist only as subsets of a given universal set X by applying a crisp restriction to X. Similarly, fuzzy sets exist only as subsets of a given universal set X by applying a fuzzy restriction to X. Thus, a fuzzy set in a universal set X is a subset of X and the collection of all such subsets

of X, denoted by $\wp(X)$, is called the power set of X.

3.1.2 Geometrical representation of a fuzzy set

A geometrical interpretation of fuzzy sets as middle points of a hypercube was introduced by (Kosko, 1992). In this representation, the fuzzy power set of the universe $X = \{x_1, x_2, ..., x_n\}$ is interpreted as an n-dimensional hypercube. A typical example of *heavy smokers* and *risk of cancer* described as follows: Let us consider a small discrete universe $X = \{0, 6, 10\}$ of the number of cigarettes a person may smoke a day on an average which is represented by vertices on coordinate axes of the three dimensional hypercube. Let a fuzzy subset $\{0/0, 0.8/6, 1/10\}$ of X be represented by points in three dimensional space. The fuzzier a fuzzy set is the deeper inside the cube is its corresponding points. The fuzziest subset, the set with highest entropy is the centre represented by $\{0.5/0, 0.5/6, 0.5/10\}$,(Figure 3.1.1) below.

3.1.3 Discrete fuzzy set representations

If the universe is discrete, a membership function can be defined by a finite set in the following way:

$$\tilde{A} = \mu_{\tilde{A}}(x_1)/x_1 + \mu_{\tilde{A}}(x_2)/x_2 + .. + \mu_{\tilde{A}}(x_n)/x_n$$

where + is not the arithmetic addition but the union, or simply as

$$\tilde{A} = \sum \mu_{\tilde{A}}(x_i)/x_i$$

where the symbol / separates the membership degree $\mu_{\tilde{A}}(x_i)$ from the elements $x_i \in X$.

Alternatively, a fuzzy set can be represented as a sequence of membership degrees

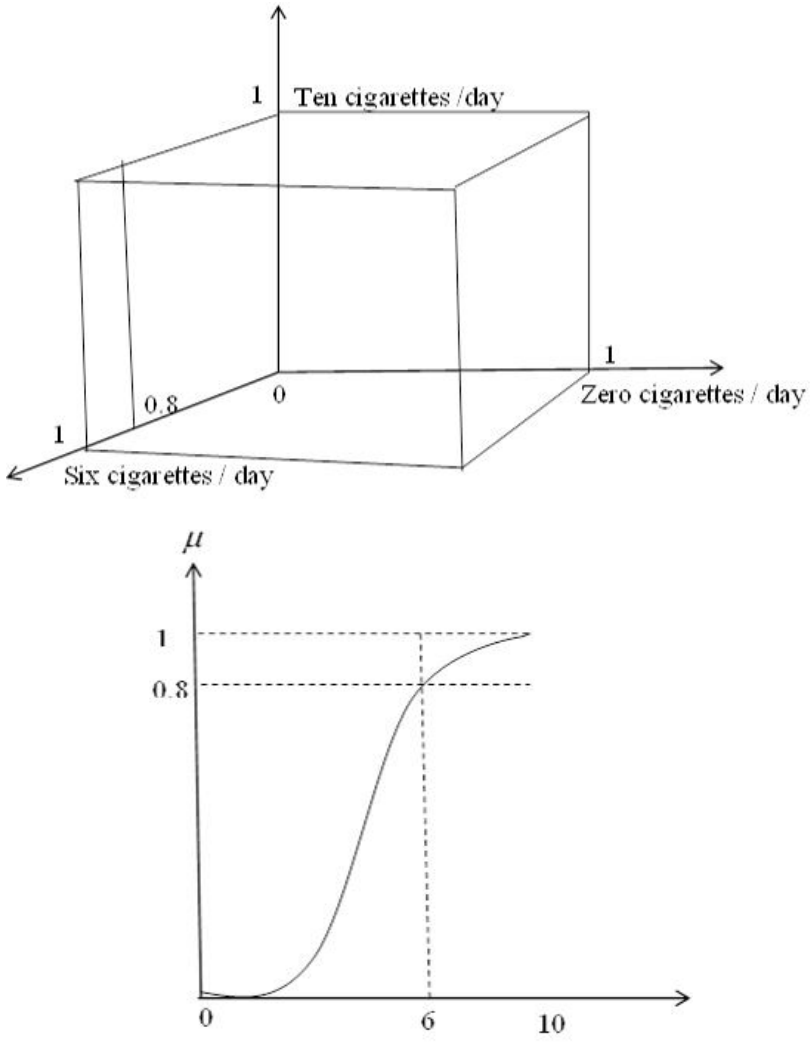


Figure 3.1.1: A geometrical representation of the concept of *heavy smoker* as a point in a (three dimensional) Cube and as a (two dimensional) membership function.

/ value pairs, for example, $\{0/140, 0.5/150, 0.68/160, 0.9/180, 1/260\}$.

If the elements are continuous, then the set can be represented as follows:

$$\tilde{A} = \int \mu_{\tilde{A}}(x)/x,$$

where $\int$ represents continuous union.

Example 3.1.1 Let us consider the statement *Paulina is young.*

In this case, the term *young* is vague. To represent the meaning of *vague* exactly,

it would be necessary to define its membership function as in Figure(3.1.2). When we say *young*, it may be referred to age which lies between 0 and 70 inclusive, that is, the closed interval $[0, 70]$, and *young age* would form a continuous set. The horizontal axis shows age and the vertical one means the numerical value of membership function. The thick line labelled young shows possibility (value of membership function) of being contained in the fuzzy set *young*. For example, if we follow the definition of *young* as in the figure, a five year-old girl may well be young with value 1 and some one of *age twenty five* is 0.8. But a person who is over sixty cannot be considered young and hence the possibility of being called young is 0. Moreover, in order to describe some one *very young*, it is obvious that the age range should be lowered and accordingly the line is moved leftward as in the figure. If the fuzzy set is to be defined in this context, only the persons who are under Thirty-five years can be included in the set of *very young*. Clearly the possibility of a twenty-five year old woman to be included in this set is 0.5.

That is, if $\tilde{A}$ and $\tilde{B}$ denote *young* and *very young* respectively,

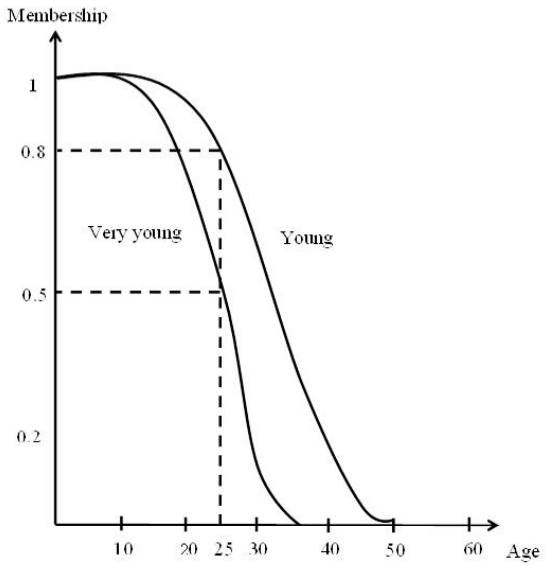


Figure 3.1.2: Fuzzy sets representing *young* and *very young*

$$\mu_{\tilde{A}}(25) = 0.8,\ \mu_{\tilde{B}}(25) = 0.5.$$

Example 3.1.2 Define a fuzzy set $\tilde{A}$ = {real numbers near 0}.

The boundary for the fuzzy set *real numbers near 0* is vague. The possibility of a real number x to be a member of the prescribed set is defined by the following membership function.

$$\mu_{\tilde{A}}(x) = \frac{1}{1+x^2}$$

Figure(3.1.3) depicts its membership function.

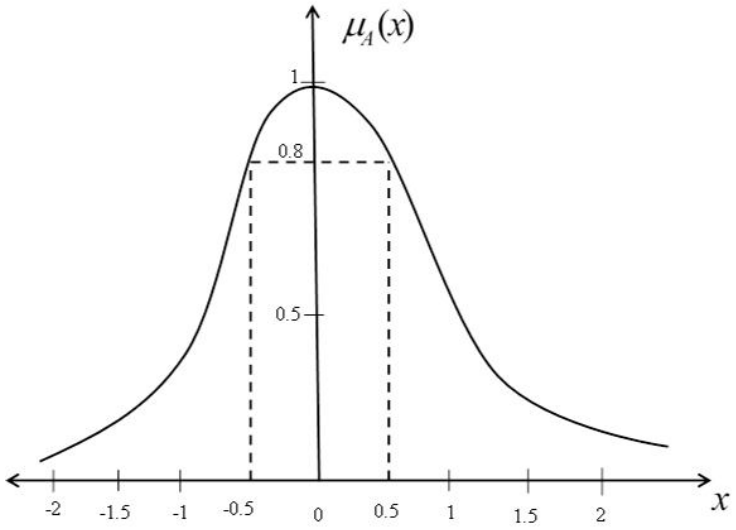


Figure 3.1.3: Fuzzy set of *real numbers near 0*

Also the fuzzy set $\tilde{A}$ in this case can be written as

$$\tilde{A} = \int \mu_{\tilde{A}}(x)/x.$$

The membership degree of 0.5 is

$$\frac{1}{1+0.5^2} = 0.8.$$

The possibility of 1 is 0.5 and that of -2 is 0.2.

Definition 3.1.2 Type-n Fuzzy Set

It is possible that the value of membership function for an element of a fuzzy set which is a unique points in $[0, 1]$, might itself be a fuzzy set, and thus if the value of membership function is given by a fuzzy set, it is called a *type-2* fuzzy set. This concept can be extended up to Type- n fuzzy set as well.

Example 3.1.3 Consider a set $\tilde{A} = Excellent$. Let the membership function of this set map total marks to *pass*, *credit* and *distinction*(Figure3.1.4). Let for marks x, y, and z,

$$\mu_{\tilde{A}}(x) = credit,\ \mu_{\tilde{A}}(y) = distinction,\ \mu_{\tilde{A}}(z) = \varnothing$$

Since the values of membership functions for *credit* and *distinction* are themselves type-1 fuzzy sets, the set *Excellent* is a type-2 fuzzy set. Similarly, if the values of membership functions of *credit* and *distinction* are type-2, the set Excellent will be type-3 fuzzy set.

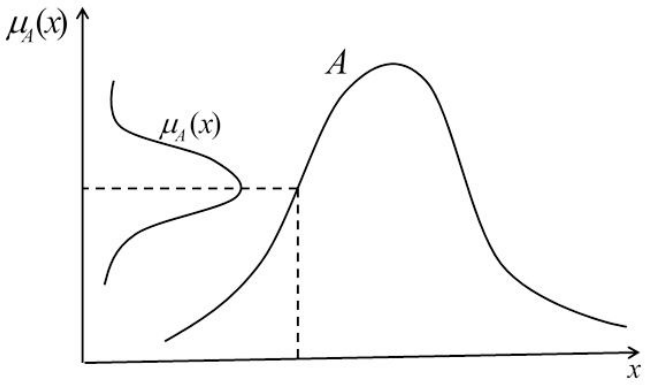


Figure 3.1.4: Fuzzy set of type-2

Definition 3.1.3 Level-k fuzzy set

A *level-2 fuzzy set* is a fuzzy set whose elements are fuzzy sets (Figure 3.1.5).

The term *level-1 fuzzy set* is a fuzzy set whose elements are non-fuzzy sets. In the same way, level-k fuzzy set can be constructed.

Let $\tilde{A}$ be a level-2 fuzzy set and $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4$, be its elements, depicted in Figure

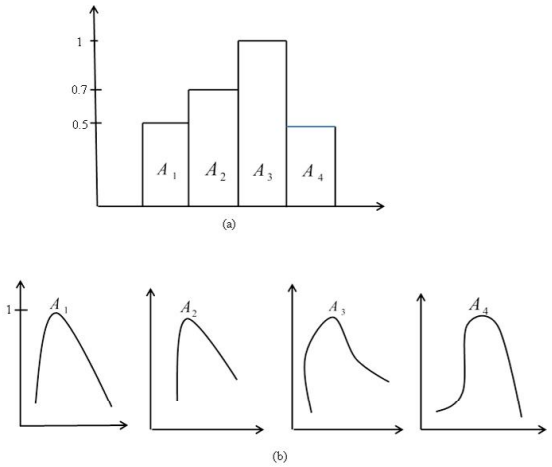


Figure 3.1.5: (a) Level-2 fuzzy set (b) elements of the level-2 fuzzy set in 3.5(a)

3.1.5(a) and 3.1.5(b), respectively, with

$$\mu_{\tilde{A}}(\tilde{A}_1) = 0.5,\ \mu_{\tilde{A}}(\tilde{A}_2) = 0.7,\ \mu_{\tilde{A}}(\tilde{A}_3) = 1,\ \mu_{\tilde{A}}(\tilde{A}_4) = 0.5$$

3.2 Operations under Fuzzy Sets and their Properties

Let X be the universal set, and $\tilde{A}, \tilde{B}, \tilde{C}, \cdots$ be fuzzy subsets in X with membership functions $\mu_{\tilde{A}}, \mu_{\tilde{B}}, \mu_{\tilde{C}}, \cdots$ respectively. Most of the definitions and operations are adopted from (Kosko, 1992; Klir and Yuan, 1995; Kasabov, 1996; Lee, 2005).

Example 3.2.1 Consider a universal set X representing marks obtained in a particular examination.

$$X = \{30, 40, 45, 50, 60, 70, 80, 90, 100\}.$$

The *poor, fair, good* and *excellent* are defined on X. The possibilities of each element of X to be in these four fuzzy sets are shown in table 3.1.

Table 3.1: Examples of fuzzy sets

marks(elements)	poor	fair	good	excellent
30	0	0	0	0
40	0	1	0.3	0
45	0	0.9	0.5	0
50	0	0.8	0.7	0
60	0	0	1	0.2
70	0	0	1	0.7
80	0	0	1	1
90	0	0	1	1
100	0	0	1	1

Definition 3.2.1 Empty Fuzzy Set ($\varnothing$)

A fuzzy set is empty if and only if its membership function is identically zero on X.

For example the fuzzy set *poor* is empty.

Definition 3.2.2 Equality of Fuzzy Sets

Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ are equal, written as $\tilde{A} = \tilde{B}$, if and only if $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ for all x in X. In the sequel, instead of writing $\mu_{\tilde{A}}(x) = \mu_{\tilde{B}}(x)$ for all x in X, $\mu_{\tilde{A}} = \mu_{\tilde{B}}$ will be used.

Definition 3.2.3 Complement

The complement of a fuzzy set $\tilde{A}$, denoted by $\tilde{A}'$, is defined by

$$\mu_{\tilde{A}'} = 1 - \mu_{\tilde{A}} \tag{3.2.1}$$

Suppose $\tilde{A}$ = the fuzzy set *fair*, that is, $\tilde{A} = \{(40,1),(45,0.9),(50,0.8),(60,0.3)\}$. Then $\tilde{A}' = \{(30,1),(40,0),(45,0.1),(50,0.2),(60,0.7),(70,1),(80,1),(90,1),(100,1)\}$.

Definition 3.2.4 Containment

$\tilde{A}$ is contained in $\tilde{B}$ (or, equivalently, $\tilde{A}$ is a subset of $\tilde{B}$, or $\tilde{A}$ is smaller than or equal to $\tilde{B}$) if and only if $\mu_{\tilde{A}} \leq \mu_{\tilde{B}}$. In symbols,

$$\tilde{A} \subseteq \tilde{B} \Leftrightarrow \mu_{\tilde{A}} \leq \mu_{\tilde{B}} \tag{3.2.2}$$

Let $\tilde{A}$ and $\tilde{B}$ be the fuzzy sets *excellent* and *good*, respectively. From Table 3.1 $\tilde{A} \subseteq \tilde{B}$.

$\tilde{A} \subset \tilde{B} \Longleftrightarrow \mu_{\tilde{A}} < \mu_{\tilde{B}}$ if and only if $\tilde{A} \subseteq \tilde{B}$ and $\tilde{A} \neq \tilde{B}$ (Figure 3.2.1).

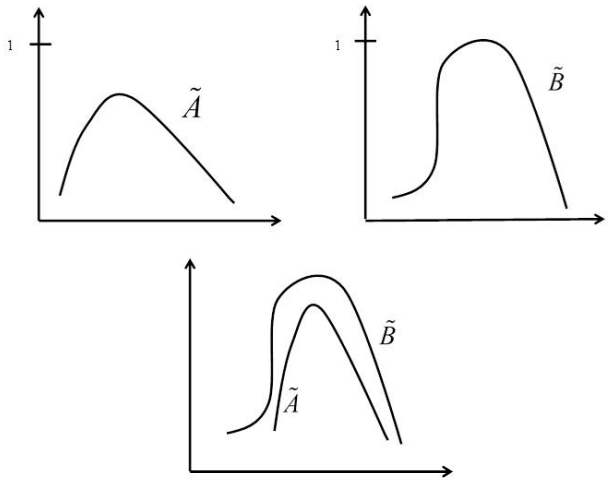


Figure 3.2.1: $\tilde{A} \subset \tilde{B}$

Definition 3.2.5 Union

The union of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is the fuzzy set $\tilde{C}$, written as $\tilde{C} = \tilde{A} \cup \tilde{B}$, defined by

$$\mu_{\tilde{C}}(x) = max[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)], x \in X, \tag{3.2.3}$$

or, simply

$$\mu_{\tilde{c}} = \mu_{\tilde{A}} \vee \mu_{\tilde{B}}. \tag{3.2.4}$$

By definition, it follows that the union of $\tilde{A}$ and $\tilde{B}$ is the smallest fuzzy set containing both $\tilde{A}$ and $\tilde{B}$. More precisely, if $\tilde{D}$ is any other fuzzy set which contains both $\tilde{A}$ and $\tilde{B}$, then it also contains $\tilde{C}$. More explicitly, since $\tilde{C}$ contain both $\tilde{A}$ and $\tilde{B}$,

$$max[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)] \geq \mu_{\tilde{A}}$$

and

$$max[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)] \geq \mu_{\tilde{B}}.$$

Furthermore, if $\tilde{D}$ is any fuzzy set containing both $\tilde{A}$ and $\tilde{B}$, then

$$\mu_{\tilde{D}} \geq \mu_{\tilde{A}},$$

$$\mu_{\tilde{D}} \geq \mu_{\tilde{B}}$$

and hence

$$\mu_{\tilde{D}} \geq max[\mu_{\tilde{A}}, \mu_{\tilde{B}}] = \mu_{\tilde{C}},$$

which implies that $\tilde{C} \subset \tilde{D}$.

For instance, the union of *good* and *Excellent* (Table 3.1) is given by

$$good \cup excellent = \left\{ \begin{array}{l} (40, 0.3), (45, 0.5), (50, 0.7), (60, 1.0), (70, 1.0), \\ (80, 1.0), (90, 1.0), (100, 1.0) \end{array} \right\} = good.$$

Definition 3.2.6 Intersection

The intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is the fuzzy set $\tilde{C}$, written as $\tilde{C} = \tilde{A} \cap \tilde{B}$, defined by

$$\mu_{\tilde{C}}(x) = Min[\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)], x \in X \tag{3.2.5}$$

or, in abbreviated form,

$$\mu_{\tilde{c}} = \mu_{\tilde{A}} \wedge \mu_{\tilde{B}}. \tag{3.2.6}$$

As in the case of the union, it is easy to show that the intersection of $\tilde{A}$ and $\tilde{B}$ is the largest fuzzy set which is contained in both $\tilde{A}$ and $\tilde{B}$.

For instance, the intersection of *good* and *Excellent* (Table 3.1) is given by

$$good \cap excellent = \{(60, 0.2), (70, 0.7), (80, 1.0), (90, 1.0), (100, 1.0)\} = excellent.$$

The intersection and union of two fuzzy sets in $\mathbb{R}$ are illustrated in (Figure 3.2.2). The membership function of union is comprised of the curve segments 1 and 2, and

that of the intersection is the segments 3 and 4(*heavy lines*).

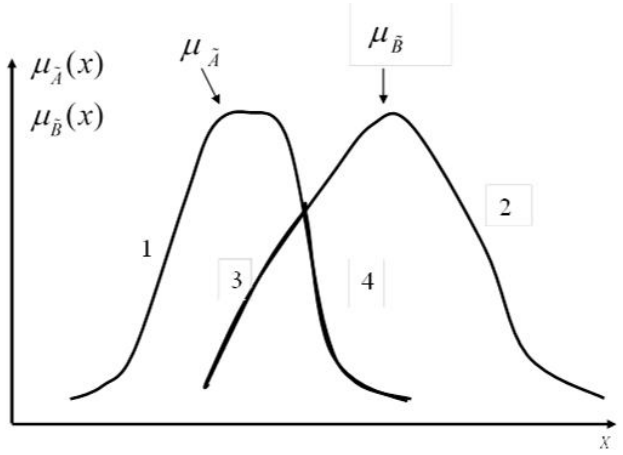


Figure 3.2.2: Union and intersection of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ in $\mathbb{R}$

Zadeh (1965) emphasises that the notion of *belongs to* which plays a fundamental role in the case of ordinary sets, does not have the same role in the case of fuzzy sets, it is not meaningful to speak of a point x *belongs* to a fuzzy set $\tilde{A}$ except in the trivial sense when $\mu_{\tilde{A}}(x)$ is positive. Less trivially, one can introduce two levels α and $\beta(0 < \alpha < 1, 0 < \beta < 1)$ and agree to say that (i) x *belongs to* A if $\mu_{\tilde{A}}(x) \geq \alpha$, (ii) x *does not belongs to* A if $\mu_{\tilde{A}}(x) \leq \beta$, and (iii) x *has an indeterminate status relative to* A if $\beta < \mu_{\tilde{A}}(x) < \alpha$.

Definition 3.2.7 Support of a Fuzzy set

The support of a fuzzy set $\tilde{A}$ is a subset of the universe X, in which each element has a membership degree to $\tilde{A}$ greater than zero (Figure 3.2.3) i.e.,

$$Supp(\tilde{A}) = \{x \mid x \in X, \mu_{\tilde{A}}(x) > 0\}.$$

For instance, the support of the fuzzy set *good*(Table 3.1) is given by

$$Support(good) = \{40, 45, 50, 60, 70, 80, 90, 100\}$$

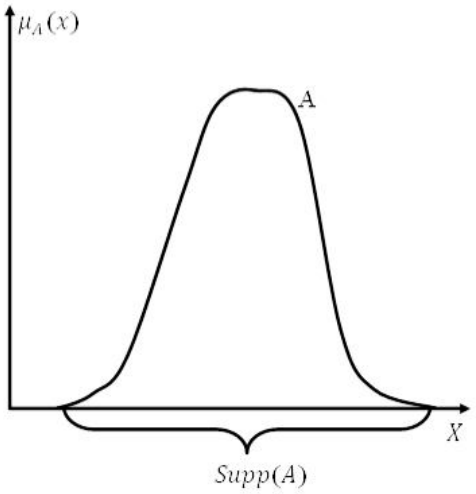


Figure 3.2.3: Support of a fuzzy set $\tilde{A}$

Definition 3.2.8 Level sets

The level set of a fuzzy set $\tilde{A}$ is defined by the α's where $\alpha \in [0,1]$. That is,

$$\Lambda_{\tilde{A}} = \{\alpha : \mu_{\tilde{A}}(x) = \alpha\}.$$

For instance the level set of the fuzzy set *good*(3.1) is,

$$\Lambda_{\tilde{A}} = \{0, 0.3, 0.5, 0.7, 1\}.$$

Definition 3.2.9 Normal fuzzy sets

A fuzzy set $\tilde{A}$ is called *normal* if its membership function has a grade of 1 at least for one value from the universe X. (Figure (3.2.4).

Definition 3.2.10 Convex fuzzy set

Let universal set X be defined in n-dimensional Euclidean Vector space $\mathbb{R}^n$. If all the α- cut sets are convex, the fuzzy set with these α- cut sets is called *convex*(Figure

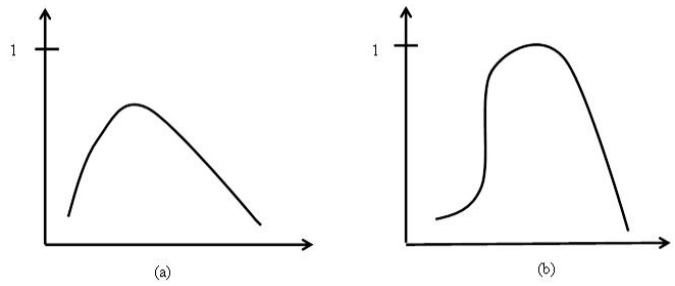


Figure 3.2.4: (a) Non-Normal Fuzzy Set (b) Normal Fuzzy set

3.2.5). In other words, $\tilde{A}$ is called convex if

$$\mu_{\tilde{A}}(t) \geq min[\mu\tilde{A}(r), \mu\tilde{A}(s)] \text{ holds}$$

$$\text{where } t = \lambda r + (1-\lambda)s;\ r, s \in \mathbb{R}^n;\ \lambda \in [0,1].$$

Figure (3.2.6) shows a convex fuzzy set and Figure (3.2.7) describes a non-convex set.

Definition 3.2.11 Fuzzy Numbers

If a fuzzy set is **convex** and **normalized**, and its membership function is defined in $\mathbb{R}$ and **piecewise continuous**, it is called a *fuzzy number*. In other words, a fuzzy number (fuzzy set) represents a real number interval whose boundary is fuzzy (Figure 3.2.8).

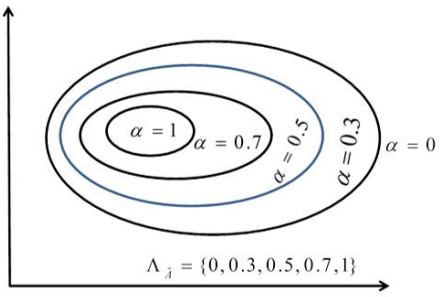


Figure 3.2.5: Convex Fuzzy Set

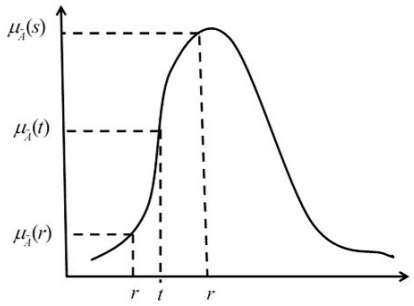


Figure 3.2.6: Convex Fuzzy Set $\mu_{\tilde{A}}(t) \geq \mu_{\tilde{A}}(r)$

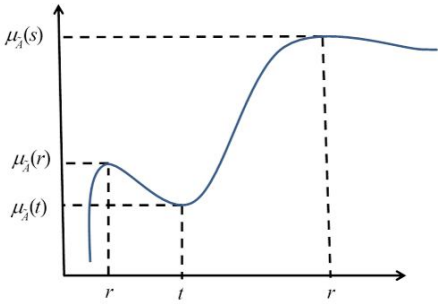


Figure 3.2.7: Non-Convex Fuzzy Set $\mu_{\tilde{A}}(t) \ngeq \mu_{\tilde{A}}(r)$

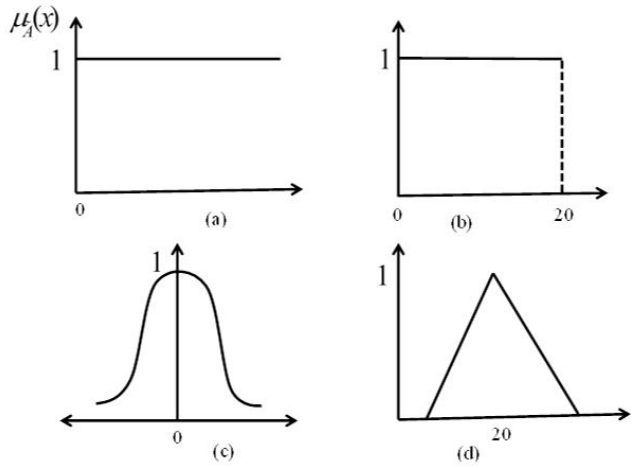


Figure 3.2.8: (a) Set of positive numbers (b) Set of positive numbers not exceeding 20 (c) Fuzzy set of Numbers near 0 (d)Fuzzy set of numbers near 20.

Definition 3.2.12 Scalar Cardinality

The scalar cardinality of a fuzzy set $\tilde{A}$, denoted by $M(\tilde{A})$ or $|\tilde{A}|$, is defined as follows.

$$|\tilde{A}| = M(\tilde{A}) = \sum \mu_{\tilde{A}}(x),\ x \in X.$$

For instance the scalar cardinality of the fuzzy set *excellent* (Example 3.1) is.

$$|excellent| = 0.2 + 0.7 + 1 + 1 = 2.9.$$

The *relative cardinality* of the fuzzy set $\tilde{A}$ to the set X is given by,

$$\|\tilde{A}\| = \frac{|\tilde{A}|}{|X|}$$

For instance,

$$|excellent| = 2.9, |X| = 9$$

$$\|excellent\| = \frac{2.9}{9} = 0.32.$$

Definition 3.2.13 Fuzzy Cardinality

The fuzzy cardinality of a fuzzy set $\tilde{A}$ is a fuzzy set consisting of the number of elements of $\tilde{A}$ with membership value at least α, for all $\alpha \in \Lambda_{\tilde{A}}$ the level set of $\tilde{A}$,

such that the degree of the number is α.

Example 3.2.2 Let us consider the fuzzy set *excellent*. The number of elements with membership value at least 0.2 is 5, and that of 0.7 and 1.0 are 4, 3 respectively. Hence the fuzzy cardinality of *excellent*$= \{(5, 0.2), (4, 0.7), (3, 1.0)\}$.

Definition 3.2.14 Product ($\cdot$)

The algebraic product of two fuzzy sets $\tilde{A}$ and $\tilde{B}$, with respective membership function $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$, is a fuzzy set $\tilde{C} = \tilde{A} \cdot \tilde{B}$ whose membership function is related to those of A and B by

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x), \ \forall x \in X.$$

Example 3.2.3 The algebraic product of the fuzzy sets *good* and *excellent* is given by

$$good \cdot excellent = \{(60, 0.2), (70, 0.7), (80, 1.0), (90, 1.0), (100, 1.0)\}$$

Definition 3.2.15 Sum ($+$)

The sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$, with respective membership function $\mu_{\tilde{A}}(x)$ and $\mu_{\tilde{B}}(x)$, is a fuzzy set $\tilde{C} = \tilde{A} + \tilde{B}$ whose membership function given by

$$\mu_{\tilde{C}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x),$$

provided this sum is less than or equal to 1 for all $x \in X$, otherwise, $\tilde{C} = \tilde{A} + \tilde{B}$ is not defined.

Example 3.2.4 The sum of fuzzy sets *good* and *excellent* is not defined, since the sum of the membership values of 60, 70, 80, 90 and 100 are 1.2, 1.7, 2, 2 and 2, respectively, which are greater than 1.

Definition 3.2.16 Probabilistic sum or Algebraic sum ($\hat{+}$)

The algebraic sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$, denoted $\tilde{A}\hat{+}\tilde{B}$, is defined as

$$\mu_{\tilde{A}\hat{+}\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)\,, \forall x \in X.$$

Example 3.2.5 The algebraic sum of the fuzzy sets *good* and *excellent* is given by

$$good\hat{+}excellent = \left\{ \begin{array}{l} (40, 0.3), (45, 0.5), (50, 0.7), (60, 1.0), \\ (70, 0.1), (80, 1.0), (90, 1.0), (100, 1.0) \end{array} \right\}$$

Definition 3.2.17 Bounded Sum ($|+|$)

The bounded sum of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set, denoted by $\tilde{A}|+|\tilde{B}$, and has the membership function given by

$$\mu_{(\tilde{A}|+|\tilde{B})}(x) = min[1, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x)],\ \forall x \in X.$$

Example 3.2.6 The bounded sum of the fuzzy sets *good* and *excellent* is given by

$$good|+|excellent = \left\{ \begin{array}{l} (40, 0.3), (45, 0.5), (50, 0.7), (60, 1.0), \\ (70, 1.0), (80, 1.0), (90, 1.0), (100, 1.0) \end{array} \right\}$$

Definition 3.2.18 Difference (-)

The difference of fuzzy sets $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} - \tilde{B} = \tilde{A} \cap \tilde{B}'$, is defined by

$$\mu_{\tilde{A}-\tilde{B}} = min[\mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x)],\ \forall x \in X.$$

Example 3.2.7 Let $\tilde{A}$ and $\tilde{B}$ be the fuzzy sets *good* and *excellent* respectively, we have

$$\tilde{A} = \left\{ \begin{array}{l} (30, 0), (40, 0.3), (45, 0.5), (50, 0.7), (60, 1.0), \\ (70, 1.0), (80, 1.0), (90, 1.0), (100, 1.0) \end{array} \right\}$$

$$\tilde{B} = \left\{ \begin{array}{l} (30,0)(40,0),(45,0),(50,0),(60,0.2), \\ (70,0.7),(80,1.0),(90,1.0),(100,1.0) \end{array} \right\}$$

$$\tilde{B}' = \left\{ \begin{array}{l} (30,1.0)(40,1.0),(45,1.0),(50,1.0),(60,0.8), \\ (70,0.3),(80,0),(90,0),(100,0) \end{array} \right\}$$

$$\tilde{A} - \tilde{B} = \tilde{A} \cap \tilde{B}' = \left\{ \begin{array}{l} (30,0)(40,0.3),(45,0.5),(50,0.7),(60,0.8), \\ (70,0.3),(80,0),(90,0),(100,0) \end{array} \right\}$$

Definition 3.2.19 Bounded Difference ($\ominus$)

The bounded difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set, denoted by $\tilde{A} \ominus \tilde{B}$, and has membership function given by

$$\mu_{(\tilde{A} \ominus \tilde{B})}(x) = max[0, \mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)], \ \forall x \in X.$$

Example 3.2.8 Let $\tilde{A}$ and $\tilde{B}$ be as in Example(3.2.7) above then

$$\tilde{A} \ominus \tilde{B} = \{(40,0.3),(50,0.7),(60,0.8),(70,0.3)\}$$

Definition 3.2.20 Absolute Difference ($|-|$)

The absolute difference of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set, denoted by $\tilde{A}|-|\tilde{B}$, and has membership function given by

$$\mu_{(\tilde{A}|-|\tilde{B})}(x) = |\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)|, \ \forall x \in X.$$

Example 3.2.9 The absolute difference of the fuzzy sets *good* and *excellent* is given by

$$good|-|excellent = \{(40,0.3),(45,0.5),(50,0.7),(60,0.8),(70,0.3)\}$$

Definition 3.2.21 Bounded Product ($|\cdot|$)

The bounded product of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is a fuzzy set, denoted by $\tilde{A}|\cdot|\tilde{B}$,

and has membership function given by

$$\mu_{(\tilde{A}|\cdot|\tilde{B})}(x) = Max\{0, \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - 1\},\ \forall x \in X.$$

Example 3.2.10 The bounded product of the fuzzy sets *good* and *excellent* is given by

$$good|\cdot|excellent = \{(60, 0.2), (70, 0.7), (80, 1.0), (90, 1.0), (100, 1.0)\}$$

Definition 3.2.22 Concentration of a Fuzzy Set

The concentration of a fuzzy set $\tilde{A}$, denoted by CON $(\tilde{A})$, is a fuzzy set whose membership function is given by

$$\mu_{CON(\tilde{A})}(x) = (\mu_{\tilde{A}}(x))^2,\ \forall x \in X.$$

Example 3.2.11 The concentration of the fuzzy set *good* is given by

$$CON(good) = \left\{ \begin{array}{l} (40, 0.09), (45, 0.25), (50, 0.49), (60, 1.0), \\ (70, 1.0), (80, 1.0), (90, 1.0), (100, 1.0) \end{array} \right\}$$

This operation is used as a linguistic modifier *very*. That is, the above example corresponds to the fuzzy set *very good*.

Definition 3.2.23 Dilation of a Fuzzy Set

The dilation of a fuzzy set $\tilde{A}$, denoted by DIL$(\tilde{A})$, is a fuzzy set whose membership function is given by

$$\mu_{DIL(\tilde{A})}(x) = (\mu_{\tilde{A}}(x))^{0.5},\ \forall x \in X.$$

Example 3.2.12 The dilation of the fuzzy set *good* is given by

$$DIL(good) = \left\{ \begin{array}{l} (40, 0.54), (45, 0.71), (50, 0.84), (60, 1.0), \\ (70, 1.0), (80, 1.0), (90, 1.0), (100, 1.0) \end{array} \right\}$$

This operation is used as a linquistic modifier *more or less*. That is, the above example corresponds to the fuzzy set *more or less good*.

Definition 3.2.24 Disjunctive sum ($\oplus$)

By means of fuzzy union and fuzzy intersection, definition of the disjunctive sum in fuzzy set, denoted by $\oplus$, is allowed just like in crisp set.

$\tilde{A} \oplus \tilde{B} = (\tilde{A} \cap \tilde{B}') \cup (\tilde{A}' \cap \tilde{B})$, then

$$\mu_{\tilde{A}\oplus\tilde{B}} = max\{min[\mu_{\tilde{A}}(x), 1 - \mu_{\tilde{B}}(x)], min[1 - \mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)]\}$$

Example 3.2.13 Let $\tilde{A}$ and $\tilde{B}$ be the fuzzy sets *good* and *excellent* respectively, we have

$$\tilde{A} = \left\{ \begin{array}{l} (30,0),(40,0.3),(45,0.5),(50,0.7),(60,1.0), \\ (70,1.0),(80,1.0),(90,1.0),(100,1.0) \end{array} \right\},$$

$$\tilde{B} = \left\{ \begin{array}{l} (30,0)(40,0),(45,0),(50,0),(60,0.2), \\ (70,0.7),(80,1.0),(90,1.0),(100,1.0) \end{array} \right\},$$

$$\tilde{A}' = \left\{ \begin{array}{l} (30,1.0),(40,0.7),(45,0.5),(50,0.3),(60,0), \\ (70,0),(80,0),(90,0),(100,0) \end{array} \right\},$$

$$\tilde{B}' = \left\{ \begin{array}{l} (30,1.0)(40,1.0),(45,1.0),(50,1.0),(60,0.8), \\ (70,0.3),(80,0),(90,0),(100,0) \end{array} \right\},$$

$$\tilde{A} \cap \tilde{B}' = \left\{ \begin{array}{l} (30,0)(40,0.3),(45,0.5),(50,0.7),(60,0.8), \\ (70,0.3),(80,0),(90,0),(100,0) \end{array} \right\},$$

$$\tilde{A}' \cap \tilde{B} = \left\{ \begin{array}{l} (30,0)(40,0),(45,0),(50,0),(60,0), \\ (70,0),(80,0),(90,0),(100,0) \end{array} \right\} = \varnothing, \text{ and}$$

hence

$$\tilde{A} \oplus \tilde{B} = (\tilde{A} \cap \tilde{B}') \cup (\tilde{A}' \cap \tilde{B}) = \left\{ \begin{array}{l} (30,0)(40,0.3),(45,0.5),(50,0.7),(60,0.8), \\ (70,0.3),(80,0),(90,0),(100,0) \end{array} \right\}.$$

Some Properties of Fuzzy Set Operations

1. Commutativity $\tilde{A} * \tilde{B} = \tilde{B} * \tilde{A}$ (not valid for bounded difference), for any operation $*$.

2. Associativity $(\tilde{A} * \tilde{B}) * \tilde{C} = \tilde{A} * (\tilde{B} * \tilde{C})$, for any operation $*$.

3. Distributivity $\tilde{A} * (\tilde{B} \circ \tilde{C}) = (\tilde{A} * \tilde{B}) \circ (\tilde{A} * \tilde{C})$, for any operation $*$ and $\circ$.

4. De Morgan's laws: $(\tilde{A} \cup \tilde{B})' = \tilde{A}' \cap \tilde{B}'$, $(\tilde{A} \cap \tilde{B})' = \tilde{A}' \cup \tilde{B}'$, for usual $\cup$, $\cap$, $'$ operations for fuzzy sets.

Observation 1

It is observed that in Zadeh's theory of fuzzy sets, the law of the excluded middle and the law of contradiction do not hold viz., $\tilde{A} \cup \tilde{A}' \neq X$, $\tilde{A} \cap \tilde{A}' \neq \varnothing$.

Example 3.2.14 Consider a fuzzy set $\tilde{A} = \{0.7/x, 0.4/y, 0.6/z\}$ of the set $X = \{x, y, z, t\}$. The complement of $\tilde{A}$ is given by $\tilde{A}' = \{0.3/x, 0.6/y, 0.4/z, 1.0/t\}$. Now,

$$\tilde{A} \cup \tilde{A}' = \{0.7/x, 0.6/y, 0.6/z, 1.0/t\} \neq X$$

and

$$\tilde{A} \cap \tilde{A}' = \{0.3/x, 0.4/y, 0.4/z\} \neq \varnothing.$$

However, (Brown, 1971) defined fuzzy sets as mappings from sets into a Boolean lattice. His theory of fuzzy sets could be considered as a modification of Zadeh's

theory rather than its generalization, as there are theorems of each version which are false in the other. Let the operations in the Boolean lattice in question be $\vee, \wedge, '$, and $\leq$ (sup, inf, complement, and order relation). The Brown's definition of a fuzzy set theory is as follows:

Definition 3.2.25 Brown (1971)

i A fuzzy set is *empty* if and only if it is the zero function.

ii Two fuzzy sets are *equal* if they are identical as function.

iii The *complement* of a fuzzy set $\tilde{A}$ is a fuzzy set $\tilde{A}'$ given by $\tilde{A}'(x) = (\tilde{A}(x))'$

iv The *union* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is the fuzzy set $(\tilde{A} \cup \tilde{B})(x) = \tilde{A}(x) \vee \tilde{B}(x)$.

v The *intersection* of two fuzzy sets $\tilde{A}$ and $\tilde{B}$ is the fuzzy set $(\tilde{A} \cap \tilde{B})(x) = \tilde{A}(x) \wedge \tilde{B}(x)$.

vi Fuzzy set $\tilde{A}$ is a *subset* of a fuzzy set $\tilde{B}$ provided $\tilde{A}(x) \leq \tilde{B}(x)$, for all x.

It can be seen that besides holding of most of set theoretic results as in Zadeh's,

$$\tilde{A} \cup \tilde{A}' = X, \tilde{A} \cap \tilde{A}' = \varnothing$$

holds, where $\varnothing$ is the empty set and X is the constant function 1. Brown's theory seems to be closer to the classical theory. Note, however, that though, Brown's theory is well-behaved with respect to complement, the intersection is not straightforward. For example let $\tilde{A}(x) = a$ and $\tilde{B}(x) = b$, with neither a nor b the 0 element of the lattice of values, but $a \wedge b = 0$. That is, although x is an element of $\tilde{A}$ to some positive degree and an element of $\tilde{B}$ to some positive degree, x is an element of $\tilde{A} \cap \tilde{B}$ to the degree 0. This is contrary to Zadeh's theory since all the values are comparable so that x would belong to the intersection to the minimum of the two degrees a and b (and so to a non-zero degree). Though, the corresponding situation

for union is not so disturbing, but still needs to be noted. If, in the above example neither a nor b is 1, but $a \vee b = 1$, then $(\tilde{A} \cup \tilde{B})(x) = 1$. This happens classically in the case that B is the complement of A, for example, or when the complement of A is a subset of B, but the possibility of this sort of thing happening in other cases can be mildly strange. In order to resolve these subtle differences, an axiomatic approach is suggested (see (Chapin, 1974) for details), which has not been studied in this dissertation since it seems to be a broad area of research in itself.

3.3 Fuzzy Relations and their Properties

The concept of a fuzzy relation is a natural extension of the concept of relation in ordinary set to fuzzy sets and plays an important role in the theory of fuzzy sets and their applications. In this section, we will closely study fuzzy relations and their properties.

Definition 3.3.1 Fuzzy Relation

A binary fuzzy set relation in X is a fuzzy set in the product space $X \times X$. In other words a binary fuzzy relation is a fuzzy set of ordered pairs. For example, the relation denoted by $x \gg y$, $x, y \in R$, may be regarded as a fuzzy set A in R^2 with membership function of$A$, $\mu_A(x, y)$, having the following (subjective) representative values: $\mu_A(10, 5) = 0.2; \mu_A(100, 10) = 0.7; \mu_A(100, 1) = 1$; etc. See figure(3.3.1)

More generally, an n- array fuzzy relation in X can be defined as a fuzzy set A in the product space $X \times X \times X ... \times X$. in which the membership function is of the form $\mu_A(x_1, ..., x_n)$ where $x_i \in X, i = 1, ..., n$.

In the case of binary fuzzy relations, the composition of two fuzzy relations A and B is denoted by $B \circ A$ and is defined as a fuzzy relation in X whose membership

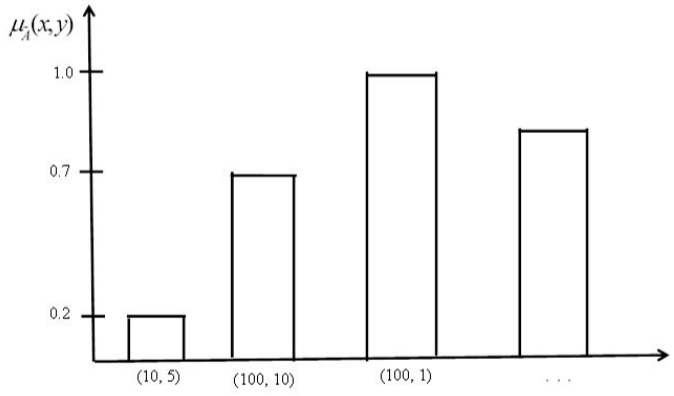


Figure 3.3.1: Graphical representation of a fuzzy relation as a fuzzy set $X \times X$

function is related to those of A and B by

$$\mu_{(B \circ A)}(x, y) = max_v min[\mu_A(x, v), \mu_B(v, y)].$$

Note that the operation of composition has the associative property $A \circ (B \circ C) = (A \circ B) \circ C$.

Example 3.3.1 Let R be a fuzzy relation between the two sets
$X = \{Maiduguri, Minna\}$ and $Y = \{Lagos, Maiduguri, Kaduna\}$,
which represents the relational concept *very far*. This can be written in list notation as

$$R(X, Y) = 1/maduguri, lagos + 0/maiduguri, maiduguri + 0.6/Maiduguri, Kaduna$$

$$+0.9/Minna, Lagos + 0.7/Minna, Maiduguri + 0.3/Minna, Kaduna.$$

This relation can also be represented by the following two dimensional membership array (matrix), for example:

	$Maiduguri$	$Minna$
$Lagos$	1	0.9
$Maiduguri$	0	0.7
$Kaduna$	0.6	0.3

In Zadeh (1965), a binary fuzzy relation R was defined as a fuzzy collection of ordered pairs. Thus, if $X = \{x\}$ and $Y = \{y\}$ are collections of objects denoted generically by x and y, then fuzzy relation from X to Y or, equivalently, a fuzzy relation in $X \cup Y$, is a fuzzy subset of $X \times Y$, characterized by a membership function μ_R which associates with each pair (x, y) its *grade of membership*, $\mu_R(x, y)$, in R. For simplicity, we assumed that the range of μ_R be the interval $[0, 1]$ and refers to the number $\mu_R(x, y)$ as the *strength* of the relation between x and y.
In the following definitions, the symbols $\vee$ and $\wedge$ stand for max and min, respectively.

Definition 3.3.2 Domain

The domain of a fuzzy relation R is denoted by $domR$ and is a fuzzy set defined by

$$\mu_{domR}(x) = \vee_y \mu_R(x, y),\ x \in X, \tag{3.3.1}$$

where the supremum,$\vee$, is taken over all y in Y.

Definition 3.3.3 Range

The range of a fuzzy relation R is denoted by $ranR$ and is a fuzzy set defined by

$$\mu_{ranR}(y) = \vee_x \mu_R(x, y),\ x \in X, y \in Y \tag{3.3.2}$$

where the supremum,$\vee$, is taken over all x in X.

Definition 3.3.4 Height

The height of a fuzzy relation R is denoted by $h(R)$ and is a fuzzy set defined by

$$h(R) = \vee_x \vee_y \mu_R(x, y). \tag{3.3.3}$$

A fuzzy relation is subnormal if $h(R) < 1$ and normal if $h(R) = 1$.

Definition 3.3.5 Support

The support of R is denoted by $S(R)$ and is defined to be the non-fuzzy subset of $X \times Y$ over which $\mu_R(x, y) > 0$.

Definition 3.3.6 Containment

The containment of a fuzzy relation R in a fuzzy relation Q is denoted by $R \subset Q$ and is defined by $\mu_R \leq \mu_Q$, which means, more explicitly, that $\mu_R(x, y) \leq \mu_Q(x, y)$ for all (x, y) in $X \times Y$.

Definition 3.3.7 Union

The union of R and Q is denoted by $R + Q$ and is defined by $\mu_{R+Q} = \mu_R \vee \mu_Q$, that is

$$\mu_{R+Q}(x, y) = max[\mu_R(x, y), \mu_Q(x, y)], x \in X, y \in Y. \tag{3.3.4}$$

Moreover, if $\{R_\alpha\}$ is a family of fuzzy (or non-fuzzy)sets, we write $\sum_\alpha R_\alpha$ to denote the union $\cup_\alpha R_\alpha$.

Definition 3.3.8 Intersection

The intersection of R and Q is denoted by $R \cap Q$ and is defined by $\mu_{R \cap Q} = \mu_R \wedge \mu_Q$, that is

$$\mu_{R \cap Q}(x, y) = min[\mu_R(x, y), \mu_Q(x, y)], x \in X, y \in Y. \tag{3.3.5}$$

Definition 3.3.9 Complement

The complement of R is denoted by R' and defined by $\mu_{R'} = 1 - \mu_R$

A convenient representation of a binary relation R is by membership matrices $R = [r_{xy}]$ where $r_{xy} = \mu_R(x, y), x \in X, y \in Y$. For example,

$$\begin{bmatrix} & y_1 & y_2 & y_3 & y_4 & y_5 \\ x_1 & 0.9 & 1 & 0 & 0 & 0 \\ x_2 & 0 & 0.4 & 0 & 0 & 0 \\ x_3 & 0 & 0.5 & 1 & 0.2 & 0 \\ x_4 & 0 & 0 & 0 & 1 & 0.4 \\ x_5 & 0 & 0 & 0 & 0 & 0.5 \\ x_6 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix}$$

Another useful representation of binary relation is a *saginal* diagram. Each of the sets X, Y is represented by a set of nodes in the diagram; nodes corresponding to one set are clearly distinguished from nodes representing the other set. Elements of $X \times Y$, with nonzero membership grades in R, are represented in the diagram by lines connecting the respective nodes. These lines are labelled with the values of the membership grades. An example of the saginal diagram of a binary fuzzy relation R, corresponding to the membership matrix above, is shown in figure (3.3.2) below.

Also a fuzzy relation R on a set X can be represented by a Directed graph (or Di-

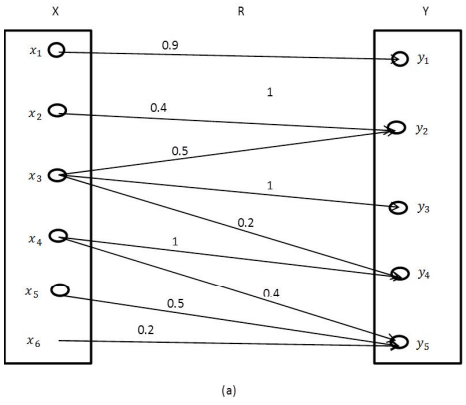


Figure 3.3.2: Saginal diagram representation

graph). For instance, the digraph of a relation R on $X = \{a, b, c, d\}$ having the following membership values $\mu_R(a, c) = 0.6, \mu_R(b, a) = 0.9, \mu_R(c, b) = 0.8$ and $\mu_R(c, d) = 1.0$ is shown in Figure(3.3.3).

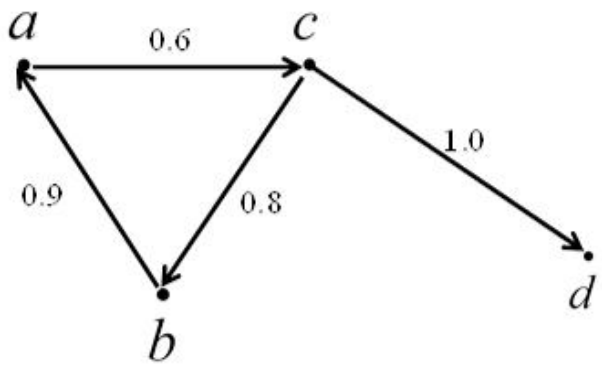


Figure 3.3.3: Fuzzy relation

Definition 3.3.10 Inverse relation

The inverse of a fuzzy relation R, denoted by R^{-1}, is a relation on $Y \times X$ defined by

$$\mu_{R^{-1}}(y, x) = \mu_R(x, y)$$

for all $x \in X$ and all $y \in Y$.

A membership matrix $R^{-1} = [r_{y\,x}^{-1}]$ representing R^{-1} is the transpose of the matrix R. Moreover, R for any binary fuzzy relation

$$(R^{-1})^{-1} = R. \tag{3.3.6}$$

Definition 3.3.11 Max-min composition

Let the two binary fuzzy relations be R from X to Y and Q from Y to Z with a common set Y. The standard composition of these relations, denoted by $R \circ Q$, give rise to a binary relation $P \subset X \times Z$ defined by

$$\mu_P(x, z) = \mu_{R \circ Q}(x, z) = \vee_{y \in Y}[\mu_R(x, y) \wedge \mu_Q(y, z)] \tag{3.3.7}$$

for all $x \in X$ and all $z \in Z$.

It follows directly from (3.13) that for any fuzzy relations $P \subset X \times Y, Q \subset Y \times Z$ and $R \subset Z \times W$ we have;

$$[P \circ Q]^{-1} = Q^{-1} \circ P^{-1}$$

$$[P \circ Q] \circ R = P \circ [Q \circ R].$$

That is, the standard (or max-min) composition is associative and its inverse is equal to the reverse composition of the inverse relations. However, the standard composition is not commutative, viz.,

$$P \circ Q \neq Q \circ P.$$

The composition of binary fuzzy relations can be performed conveniently in terms of their membership matrices. Let $P = [p_{ik}], Q = [q_{kj}]$ and $R = [r_{ij}]$ be membership matrices of the binary relations such that $R = P \circ Q$, written in matrix representations

$$[r_{ij}] = [p_{ik}] \circ [q_{kj}]$$

where

$$r_{ij} = \max_k min(p_{ik}, q_{kj}). \tag{3.3.8}$$

It may be observed that the elements of P and Q are used in the calculation of R as would be used in the regular multiplication of matrices, but the product and sum operator are replaced by min and max operators respectively.

The following matrix equation illustrates the use of (3.14) to perform the standard composition of two binary fuzzy relations represented by their membership functions

$$\begin{bmatrix} 0.3 & 0.5 & 0.8 \\ 0 & 0.7 & 1 \\ 0.4 & 0.6 & 0.5 \end{bmatrix} \circ \begin{bmatrix} 0.9 & 0.5 & 0.7 & 0.7 \\ 0.3 & 0.2 & 0 & 0.9 \\ 1 & 0 & 0.5 & 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix},$$

where for example,

$$0.8(= r_{11}) = max[min(0.3, 0.9), min(0.5, 0.3), min(0.8, 1)]$$

$$= max[min(p_{11}, q_{11}), min(p_{12}, q_{21}), min(p_{13}, q_{31})],$$

$$0.4(= r_{32}) = max[min(0.4, 0.5), min(0.6, 0.2), min(0.5, 0)],$$

$$= max[min(p_{31}, q_{12}), min(p_{32}, q_{22}), min(p_{33}, q_{32})], \text{ etc.}$$

A similar operation on two binary relations, which differs from the composition in that it yields triples instead of pairs, is known as the the *relational join*. For fuzzy relations $P \subset X \times Y$ and $Q \subset Y \times Z$, the relational joins $P * Q$, corresponding to the standard max-min composition, is a tenary relation $R \subset X \times Y \times Z$ defined by

$$\mu_R(x, y, z) = \mu_{P*Q}(x, y, z) = min[\mu_P(x, y), \mu_Q(y, z)] \tag{3.3.9}$$

for each $x \in X, y \in Y$ and $z \in Z$.

It is distinctive that the relational join produces a tenary relation from two binary relations whereas the composition produces another binary relation. In fact, the max-min composition is obtained by aggregating max- operator formally

$$\mu_{P \circ Q}(x, z) = \max_{y \in Y} \mu_{P*Q}(x, y, z) \tag{3.3.10}$$

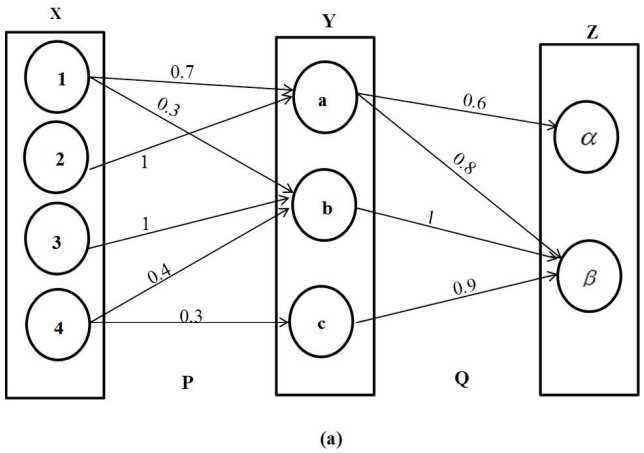
for $x \in X$ and $z \in Z$.

Example 3.3.2 The join $S = P * Q$ of relations P and Q is given in figure (3.5a). Moreover, given the membership function given in Fig. (3.5b). The join is converted into the corresponding composition $R = P \circ Q$ in (3.16). The two indicated pairs of values of $S(x, y, z)$ in figure (3.5b) are aggregated by the max operator. For instance

$$R(1, \beta) = max[S(1, a, \beta), S(1, b, \beta)]$$

$$= max[0.7, 0.5] = 0.7.$$

Although the standard max-min composition and the associated join are used most frequently in applications of fuzzy relations, some applications suggest broadening of these concepts by employing more general notions: $t-norms$ and $t-conorms$. [See (Klir and Yuan, 1995; Lee, 2005) for details].



(a)

Join $S = P * Q$			
x	y	z	$\mu_S(x, y, z)$
1	a	α	0.6
1	a	β	0.7
1	b	β	0.3
2	a	α	0.6
2	a	β	0.8
3	b	β	1
4	b	β	0.4
4	c	β	0.3

(b)

Figure 3.3.4: Join of P and Q

Similarity relations

The concept of a similarity relation is essentially a generalisation of the concept of an equivalence relation.

Definition 3.3.12 A similarity relation S, in X is a fuzzy relation in X which is

(a) reflexive;

$$\mu_S(x,x) = 1, \quad \text{for all } x \text{ in } domS, \tag{3.3.11}$$

(b) symmetric;

$$\mu_S(x,y) = \mu_S(y,x), \quad \text{for all } x,y \text{ in } domS, \tag{3.3.12}$$

and

(c) fuzzy transitive

$$S \supset S \circ S \tag{3.3.13}$$

or, more explicitly,

$$\mu_S(x,z) \geq \vee_y[\mu_S(x,y), \mu_S(y,z).$$

An example of the relation matrix of a similarity relation S is shown below . It is readily verified that $S = S \circ S$.

$$\mu_S = \begin{bmatrix} 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.2 & 1 & 0.2 & 0.2 & 0.8 & 0.2 \\ 1 & 0.2 & 1 & 0.6 & 0.2 & 0.6 \\ 0.6 & 0.2 & 0.6 & 1 & 0.2 & 0.8 \\ 0.2 & 0.8 & 0.2 & 0.2 & 1 & 0.2 \\ 0.6 & 0.2 & 0.6 & 0.8 & 0.2 & 1 \end{bmatrix}$$

3.4 Definition and Representations of Multisets

The various representations of multisets like Multiplicative, Linear, sequences, Family of set, Numeric-valued function and generalized Characteristic function forms are discussed in some literature. Such as Blizard (1989), Singh et al. (2007, 2008) , etc.

3.4.1 Definition of a multiset

Intuitively, a *multiset* (mset, for short) is an unordered collection of objects (called elements) in which, unlike standard (Cantorian) set, elements are allowed to repeat. Formally, a multiset is defined as follows: Suppose $X = \{x_1, x_2, \cdots, x_j, \cdots\}$ be set. An mset A over X is a *cardinal-valued* function i.e., $A : X \to N = \{0, 1, 2\cdots\}$ such that for $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = m_A(x) > 0$, where $m_A(x)$ denotes the number of times an object x occurs in A. The set X is called the ground or generic or root set of the class of all msets containing objects from X.

3.4.2 Representations of a multiset

(a) **Multiplicative form**

An mset containing two occurrences of a, three occurrences of b, and one occurrence of c is represented as $[[a, a, b, b, b, c]]$ or $[a, a, b, b, b, c]$ or $[a, b, c]_{2,3,1}$ or $[a, b, c]_{231}$ or $[a^2, b^3, c^1]$ or $[a2, b3, c1]$, depending on one's taste and expediency. Note that, the square brackets used in our notation should not be confused with the closed interval notation. Moreover, in the above example, a, b and c are the objects and each individual occurrence of these objects is element.

(b) **Linear form**

A linear form of multisets which seems quite innovative, especially when nega-

tive multiplicities (integral as well as rational) are to be dealt with, for example, the mset $M = [a, b, c]_{2,3,1}$ can be written as

$$M = 2[a] + 3[b] + 1[c].$$

Similarly, a rational mset can be represented. For example,

$$N = \frac{1}{3}[5] - \frac{1}{2}[1_8].$$

In order to accommodate negative multiplicities, round brackets are used. The symbol (a), in an mset stands for negative of a; for example,

$$[2, 4, (5), (5), 4] = [2] + 2[4] - 2[5].$$

(c) **Multiset as a sequence**

A multiset can also be represented as a sequence in which the multiplicity of an element equals the number of times the element occurs in the sequence, which is exactly Dedekind's *frequency-number*. The idea is to construct an mset as a sequence (a function with domain N, the set of natural numbers) and ignore the ordering of its elements, which can be done by taking all permutations of the domain of the sequence.

(d) **Multiset as a family of sets**

A multiset can also be represented as a family of sets, which is altogether a generalization of the idea of a sequence described above. Thus, the family of sets $F = F_i, i \in I$, where $F_i = F_j$, if $i = j$, which identifies a repeated element, represents an mset. Clearly, the family F is a function: $I \to \{F_i | i \in I\}$, which in turn, is a sequence if $I = N$.

3.5 Operations under Multisets and their Properties

Knuth (1969) and Knuth (1981) are the early references where operations under multiset were defined. Hailperin (1986), Blizard (1989), Petrovsky (1994, 1997), Wildberger (2003), etc. are also notable works where operations of multisets were almost exhausted. In the following, we present briefly multiset operations.

Definition 3.5.1 Multisubsets(or msubsets, for short)
Let A and B be two msets, A is called an msubset or a submultiset (submset) of B, written as $A \subseteq B$ or $B \supseteq A$, if $m_A(x) \leq m_B(x)$ for all $x \in D$ where D is the root set of B. Also, if $A \subseteq B$ and $A \neq B$ then A is called a proper submset of B. An mset is called the parent in relation to its msubsets. Clearly,ϕ is a submset of every mset. An msubset that contains all multiplicities of common is called a 'whole' msubset while a 'full' msubset contains all objects of the parent mset and accordingly, every mset contains a unique full msubset, its root set. Clearly, for any mset A and B, if $A \subseteq B$ and $Dom(A) = Dom(B)$, then A is a full subset of B.

With the introduction of the notation ϵ_+(dressed epsilon, is a binary predicate which means 'belongs to at least once') by Singh et al. (2007), $A \subseteq B$ is clearly understood as

$$\forall z \forall k (z \epsilon^k A \rightarrow z \epsilon_+^k B).$$

With this definition $\subseteq$ is antisymmetric i.e., $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$, and it is a partial ordering on the class of msets defined on a given generic domain.

Remark 3.5.1 We observe that this definition does not completely capture the idea of submultiset, for example, given multisets $A = [a, a, b, b, c]$ and $B = [a, b, b, b, c, c]$, $A \nsubseteq B$ and $B \nsubseteq A$, yet A and B have common root set, i.e. $A^* = B^*$. That multisets A and B are similar, will be clearly defined later.

Definition 3.5.2 Equality of multiset.

Two multisets A and B are said to be equal, written as $A = B$ if and only if for $x \in D, m_A(x) = m_B(x)$. Also, $A = B \Rightarrow A^* = B^*$, but the converse need not hold.

Definition 3.5.3 Similar msets.

Two msets A and B are said to be 'cognate' or 'similar' if $\forall x(x \in A \Rightarrow x \in B)$, where x is an object. Thus, similar mset have equal root set but need not be equal themselves.

Definition 3.5.4 Direct product of multiset.

Given two multisets A and B, the direct product of A and B denoted by $A \times B$ is the multiset consisting of all ordered pair (a, b) where $a \in A$ and $b \in B$. Clearly,

$$m_{A\times B}((a,b)) = m_A(a)m_B(b) or |A \times B| = |A| \times |B|$$

For example $[1,1,3]\times[2,2] = [(1,2),(1,2),(1,2),(1,2),(3,2),(3,2)] = [(1,2)(3,2)]_{4,2}$

Definition 3.5.5 Power multiset.

Power multiset of a given mset A, denoted $\tilde{\wp}(A)$ to distinguish it from the symbol $\wp(A)$ used for power set of A, is the multiset of all submultisets of A.

For example, let $A = [a,b]_{1,2} = [a,b,b]$ then,

$$\tilde{\wp}(A) = [\phi, \{a\}, \{a,b\}, \{a,b\}, \{b\}, \{b\}, [b]_2, [a,b]_{1,2}].$$

In this sense, $|\tilde{\wp}(A)| = 2^{|A|}$ for any mset A. However,it has been voiced by many researchers in the area of msets and their applications (Hickman (1980), p 213 and Blizard (1989), p. 45 in particular), there is no 'good' reason for admitting repeated elements into a power multiset. Hence, a power multiset needs to be called a power set only and denoted by $\wp(A)$. Accordingly, for

$$A = [a,b]_{1,2}.$$

$\wp(A) = [\phi, a, a, b, b, [b]_2, [a, b]_{1,2}]$ and hence $|\wp(A)| < 2^{|A|}$, which implies that Cantor's power set theorem: $|A| < |\wp(A)|$ fails. However, for finite msets, Cantor's theorem holds for power mset.

Definition 3.5.6 Union of mset.

Let A and B be two msets over a given domain set D. The union of A and B denoted by, $A \cup B$, is defined as

$$A \cup B = Max[m_A(x), m_B(x)],$$

for all $x \in D$. That is, an object z occurring n_1 times in A and n_2 times in B, occurs maximum $\{n_1, n_2\}$ times in $A \cup B$, if such a maximum exists otherwise the minimum of $\{n_1, n_2\}$ is taken which always exists. In other words,$A \cup B$ is the smallest mset which contains both A and B. It follows that for any given mset X there exists an mset Y which contains elements of elements of X, where the multiplicity of an element z in Y is the maximum multiplicity of z as an element of elements of X along with the above stipulation on the existence of such a maximum. We denote this fact by $Y = \cup X$

Clearly, $\text{Dom}(\cup X) = Dom(A), AX$ and that the multipliticity of z in Y is the maximum of its multiplicities as an element of elements of X, if it exists, otherwise, the minimum is taken. For example, if $A = [2344], B = [1433]$ then $A \cup B = [123344]$.

Also, it follows that for a finite mset X, the maximum multiplicity of elements of elements of X always exists. However, for certain infinite sets like $X = \{a, [a]_2, [a]_3 \cdots\}$, the maximum multiplicity of elements of elements of X does not exist, and hence $\cup X = a$. It is obvious by definition of the union that multiplicity of any $a \in X \neq \phi$ is irrelevant to $\cup X$, and hence $\cup X = \cup X^*$ Blizard (1989).

Note that if$A \cup C = B \cup C$ or $A \cap C = B \cap C$, we cannot conclude that $A = B$. For example, we consider the multisets $A = [1, 1, 2, 2, 3], B = [1, 1, 3]$ and $C = [1, 1, 2, 2]$ $A \cup B = B \cup C = [1, 1, 2, 2, 3]$ but $A \neq B$.

Definition 3.5.7 Intersection of mset.

LetA and B be two msets over a given domain set D. The intersection of two msets A and B denoted by $A \cap B$, is the mset defined by

$$A \cap B = Min[m_A(x), m_B(x)],$$

for all $x \epsilon D$. In other words, $A \cap B$ is the smallest mset which is contained in both A and B. That is, an object x occurring n_1 times in A and n_2 times in B, occurs minimum n_1, n_2 times in $A \cap B$, which always exists.

In general, for a given mset X, $\text{Dom}(\cap X) = \cap Dom(A) : AX$ and $z \in \cap X$ implies that the multiplicity of z is the minimum of its multiplicities as element of elements of X. For example, if $A = [33344], B = [1433]$, then$A \cap B = [334]$. Note that for any mset X, we have $\cap X \subseteq \cup X$.

Definition 3.5.8 Addition or sum or merge.

Let A and B be two msets over a given domain set D. The direct sum or arithmetic addition of A and B denoted by $A + B$ or $A \uplus B$ is the mset defined by

$$A \uplus B = m_{(A+B)}(x) = m_A(x) + m_B(x),$$

for all $x \in D$. That is, an object x occurring n_1 times in A and n_2 times in B, occurs $n_1 + n_2$ times in $A \uplus B$.

For example if $A = [1122344], B = [233]$ then $A \uplus B = [1122233344]$. Clearly, $|(A \uplus B)| = |(A \cup B)| + |(A \cap B)|$. It follows that the insertion of an element x into an mset A gives rise to a new mset $A' = A + x$ such that $m_{(A')}(x) = m_A(x) + 1$ and $m_{(A')}(y) = m_A(y)$ for all $y \neq x$. Also, if x is an infinite mset, then the multiplicity of some mset $z \uplus x$ may not be finite. In that case, the multiplicity of z in $\cup x$ is used. For example, if $x = z, [z]_2, [z]_3 ...$ then $\uplus x = \cup x = z$

Definition 3.5.9 Difference of msets.

Let A and B be two msets over D and $B \subseteq A$, then the difference of B from A, otherwise known as the relative complement of B in A, denoted by $A - B$, is the mset such that $m_{(A-B)}(x) = m_A(x) - m_B(x)$ for all $x \in D$.

In general, $m_{(A-B)}(x) = m_A(x) - m(A \cap B)(x)$, for all $x \in D$.

$$= max[m_A(x) - m_B(x), 0]$$

It is also sometimes called the arithmetic difference of B from A. If $B \nsubseteq A$, this definition still holds good. It follows that the deletion of an element$x$ from an mset A gives rise to a new mset $A' = A - x$ such that $m_{A'}(x) = max[m_A(x) - 1, 0]$. Similarly, the symmetric difference, $A \triangle B$, is defined by

$$m_{A \triangle B}(x) = m_{(A-B)}(x) \cup m_{(B-A)}(x) = |m_A(x) - m_B(x)|$$

Complementation: Let $A \in X^n$. The complement of A, denoted by $\bar{A}$ or A^C, in X^n is a member of X^n such that$m_{\bar{A}}(x) = n - m_A(x)$ for all objects x in X. Alternatively, let $J = \{A_1, A_2, \cdots\}$ be a family of msets. The maximum mset Z is defined such that $m_Z(x) = maxm_A(x) \in J$, for all objects x and all $A \in J$. Then, $\bar{A} = Z - A$ such that$m_{\bar{A}}(x) = m_Z(x) - m_A(x)$, for all objects x.

3.6 Multiset Relations and their Properties

The content of this section is largely abstracted from [(Girish and John, 2009), (Singh et al., 2009)].

Definition 3.6.1 Multiset Relations

Let X^n be a cardinality-bounded mset space defined on a set X, and A be a member of X^n. In consonance with the terminologies used for explicating the notion of

relations on a set, we say that any submset R of $A \times A$ is called an mset relation on A (or simply, a relation on A if the context is clear), symbolized as $R : A \to A$, where every member of R has a count $C_1(x, y)$ and $C_2(x, y)$. Also, m/x is R-related to n/y is notated as $(m/x, n/y) \in R$ or $m/x\ R\ n/y$.

Note that $(m/x, n/y) \in^{mn} R$ actually means that the ordered pair (x, y) occurs mn times in R. Formally, $R = \{(m/x, n/y)/mn : (m/x, n/y) \in^{mn} R\}$. The negation of a relation R is denoted by $\not R$.

The domain of a relation R on A (dom R, for short) $= \{x \in^s A : \exists y \in^t A \quad \text{such that } s/x\ R\ t/y\}$, where $C_{domR} = sup\{C_1(x, y) : x \in^s A\}$, which always exists for cardinality bounded msets. The range of a relation R on A (ran R, for short) $= \{y \in^t A : \exists x \in^s A \text{ such that } s/x\ R\ t/y\}$, where $C_{ranR} = sup\{C_2(x, y) : y \in^t A\}$, which always exists for cardinality bounded msets. Note that, for cardinality-bounded msets, the supremum (sup) is simply the maximum (max).

Example 3.6.1 Let $A = [5/x, 7/y, 9/z]$ be an mset. then
$R = \left\{ \begin{array}{l} (2/x, 3/y)/6, (4/x, 3/y)/12, (4/y, 5/z)/20, (6/y, 5/x)/30, \\ (4/z, 4/z)/16, (5/z, 3/x)/15, (3/z, 6/y)/18 \end{array} \right\}$ is an mset relation on A. In this case $domR = \{4/x, 6/y, 5/z\}$ and $ranR = \{5/x, 6/y, 5/z\}$.

In the same vein, the inverse or converse of an mset relation R, denoted by R^{-1}, can be defined as $R^{-1} = \{(n/y, m/x) : (m/x, n/y) \in^{mn} R\}$.

Note that, akin to the existence of the inverse of a relation on sets, the inverse of an mset relation always exists as an mset relation.

An mset relation R on $A \in X^n$ is an mset of ordered pairs $((m/x, n/y))/mn$. It is possible as in the case of relations on sets, to apply the usual operations under msets to mset relations as well. The resulting msets are also msets of ordered pairs and defined thereby as some mset relations. For example, if R_1 and R_2 denote two mset relations on $A \in X^n$, then $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ (the complement of R_1 or R_2), $R_1 \triangle R_2$ are also mset relations on A. The justifications for all these facts lie with the respective definitions given in section 3.7.

Example 3.6.2 $R_1 \cup R_2$ can be defined as the mset consisting of all pairs of R_1 and R_2, and the common pairs appearing only once or equivalently, appearing with maximum multiplicity. Now let $A = \{4/x, 7/y, 9/z\}$.

$R_1 = \{(2/x, 3/y)/6, (3/x, 5/y)/15, (3/y, 4/y)/12, (4/z, 5/y)/20, (3/z, 3/x)/9\}$ and

$R_2 = \{(2/x, 3/y)/6, (4/x, 4/z)/16, (4/y, 5/z)/20, (6/z, 3/x)/18, (3/z, 4/z)/12\}$

Then, $R_1 \cup R_2$ is the mset of all distinct pairs of R_1 and R_2 and the common pair taken only once.

We have, $domR_1 = \{3/x, 3/y, 4/z\}$ and $ranR_1 = \{3/x, 5/y, 5/z\}$, $domR_2 = \{4/x, 4/y, 6/z\}$ and $ranR_2 = \{3/x, 3/y, 5/z\}$, $dom(R_1 \cup R_2) = domR_1 \cup domR_2 = \{4/x, 4/y, 6/z\}$ and $ran(R_1 \cup R_2) = ranR_1 \cup ranR_2 = \{3/x, 5/y, 5/z\}$.

Similarly, we can determine other relations mentioned above.

Composite multiset relations

We outline below some details for constructing a composition of mset relations. Let A, B and C be msets, R be an mset relation from A to B and S be an mset relation from B to C. The composite or composition or relative product of R and S, denoted by $S \circ R$, is the mset relation consisting of the ordered pair $(m/x, k/z)$ denoted by $m/x\ S \circ R\ k/z$, where $m/x \in A$ and $k/z \in C$ if and only if there is some $m_i/x\ R\ n_i/y$ and $n_j/y\ S\ k_j/z$ having $C_1(x, z) = max\{min\{C_1(x, y), C_1(y, z)\}\}$ and $C_2(x, z) = max\{min\{C_2(x, y), C_2(y, z)\}\}$. Note that these stipulations have precedents in defining the composition of relations in sets. In particular,

$dom(S \circ R) \subseteq domR$ and $ran(S \circ R) \subseteq ranS$ for $S \circ R$ to be defined. For example,

Example 3.6.3 Let $A = [4/x, 5/y, 3/z], B = [5/a.3/b, 4/c]$ and $C = [4/p, 2/r, 3/t]$ be giving msets. Let $R = \{(3/x, 2/a), (2/x, 2/b), (3/y, 3/c), (2/z, 1/b), (1/z, 3/a)\}$ and $S = \{(3/a, 1/r), (2/a, 2/t), (2/b, 3/p), (2/c, 2/p), (1/c, 2/r)$. Then,

$$S \circ R = \left\{ \begin{array}{l} (2/x, 2/p), (3/x, 1/r), (2/x, 2/t), (2/y, 2/p), \\ (1/y, 2/r), (2/z, 1/p), (1/z, 1/r), (1/z, 2/t) \end{array} \right\}.$$

It is easy to verify that the stipulations made above in order that composite relation is defined hold for the example described above. In particular, $dom(S \circ R) \subseteq domR$ and $ran(S \circ R) \subseteq ranS$. On the same line, for a given relation R on an mset A, $R \circ R, R \circ R \circ R, \cdots$ can be composed. In particular, $R \circ R$, denoted by R^2, can be defined and, in turn, $R \circ R^2$, denoted by R^3, can be defined and so on. This eventually leads to obtaining transitive closure of R as it is done in the case of relations on a set viz., $R^+ = R \cup R^2 \cup R^3 \cup \cdots$

Note that composition of mset relations is not commutative. For example,

Example 3.6.4 let $A = [4/a, 3/b, 5/c]$ be a given mset.

Let $R = \{(3/a, 2/a), (2/a, 3/c), (2/b, 2/c), (3/c, 3/c)\}$ $S = \{(2/b, 2/b), (2/b, 3/a), (3/c, 2/b)\}$ be relations on A. Then,

$S \circ R = \{(2/a, 2/b), (2/b, 2/b), (3/c, 2/b)\}$, and $R \circ S = \{(2/b, 3/c), (2/b, 2/a), (2/c, 2/c)\}$.

Follows that $S \circ R \neq R \circ S$

It can also be proved that $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$.

Matrix representation of a multiset relation

Similar to the representation of a relation on a set by matrix or graph, an mset relation can also be represented.

Definition 3.6.2 Matrix representation

Let X^n be a cardinality-bounded mset space generated by the elements of a given generic set X, and R be an mset relation on $A \in X^n$. If we assume that the elements of X appear in a certain order, then R can be represented by a *matrix* whose elements are mn if $m/x \, R \, n/y$ and 0 if $m/x \not R \; n/y$

Example 3.6.5 Let us consider the mset A and the relation R defined on A of the

example 3.8.1. Then the relation matrix is as follows:

$$\begin{bmatrix} 0 & 6 & 12 \\ 30 & 0 & 20 \\ 15 & 18 & 16 \end{bmatrix}$$

It may be noted that if A contains 3 objects, then the relation matrix is 3×3. It can also be represented in tabular form as follows:

	x	y	z
x	0	6	12
y	30	0	20
z	15	18	16

It can be seen that the aforesaid matrix representation in an mset A can as well be extended to a composition of two mset relations on A. For an illustration, let us consider the example 3.8.4. We have as follows:

$$M_R = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix}, \; M_S = \begin{bmatrix} 0 & 0 & 0 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

$$M_{S \circ R} = \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 0 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix}$$

Similarly,

$$M_{R \circ S} = \begin{bmatrix} 0 & 0 & 0 \\ 6 & 4 & 0 \\ 0 & 6 & 0 \end{bmatrix} \circ \begin{bmatrix} 6 & 0 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 6 & 0 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

Note that, Akin to the technique used for constructing the matrix of a composite set relation, Boolean product of the matrices of two mset relations is computed. For example, the entry in the third row and third column of $M_{R \circ S}$ is given by Max{min{(0,6),(6,4),(0,9)}= max{0,4,0}=4

3.7 Definition and Representations of Fuzzy Multisets

Relatively recently, in view of the aforesaid two very dominant generalizations of the ordinary set theory, efforts to develop mathematical structures characterizing classes of more complex objects, possessing both fuzziness and multiplicity, have been made. The concept in view is that of a fuzzy multiset. Fuzzy multisets have been introduced by Yager (1986) although Yager uses the term fuzzy bag and have been studied by (Miyamoto, 1996, 1997, 1999, 2001) and others.

3.7.1 Definition of a fuzzy multiset

A typical approach for generalizing multiset to fuzzy multisets is to fuzzify the multiplicities of objects of a multiset. A fuzzy multiset in some universe set X is a multiset in $X \times [0,1]$. That is, a fuzzy multiset is a multiset of pairs, where the first part of each pair is an element of X and the second part is the degree to which the first part belongs to the fuzzy multiset.
In order to distinguish it, an ordinary non-fuzzy multiset is called a crisp (pure) multiset.

Example 3.7.1 $A = \{(x,1),(x,0.3),(x,0.2),(y,0.9),(y,0.5)\}$, equivalently represented as $A = \{\{1,0.3,0.2\}/x, \{0.9,0.5\}/y\}\}$, is a fuzzy multiset in $X = \{x,y,z\}$.

An element of x may occur more than once with possibly the same or different membership values.

Example 3.7.2 Consider a fuzzy multiset $A = \{(x,0.4),(x,0.5),(y,1),(y,0.7),$ $(y,0.7)\}$ of $X = \{x,y,z\}$, which means that x with the membership 0.4, x with 0.5, y with the membership 1, and two $y's$ with 0.7 are contained in A.

We may write $A = \{\{0.4, 0.5\}/x, \{1, 0.7, 0.7\}/y\}$ in which the multisets of membership $\{0.4, 0.5\}$ and $\{1, 0.7, 0.7\}$ correspond to x and y, respectively.
Count $A(x)$ is thus a finite multiset of the unit interval. The collection of all finite fuzzy multisets of all elements of X is denoted by $FM(X)$.

Definition 3.7.1 Membership Sequence
For $x \in X$, the membership sequence is defined to be the decreasingly-ordered sequence of the elements in $CountA(x)$. It is denoted by

$$(\mu_A^1(x), \mu_A^2(x), \cdots, \mu_A^p(x))$$

where $\mu_A^1(x) \geq \mu_A^2(x) \geq \mu_A^3(x) \geq \cdots \geq \mu_A^p(x)$.

When we define an operation between two fuzzy multisets, say A and B, the lengths of the membership sequences $\mu_A^1(x), \mu_A^2(x), \cdots, \mu_A^p(x)$, and $\mu_B^1(x), \mu_B^2(x)$, $\cdots, \mu_B^p(x)$ should be set equal.

Example 3.7.3 Let $A, B \in FM(X)$, given by

$$A = \{\{0.4, 0.5\}/x, \{1, 0.7, 0.7\}/y\}$$

and

$$B = \{\{0.7\}/x, \{0.9, 0.8\}/y, \{0.2, 0.8\}/w\}.$$

For the representation of the membership sequence, we get
$L(x) = 2, L(y) = 3, L(z) = 0, L(w) = 2,$
and we have $A = \{\{0.5, 0.4\}/x, \{1, 0.7, 0.7\}/y, \{0, 0\}/w\}$,
$B = \{\{0.7, 0\}/x, \{0.9, 0.8, 0\}/y, \{0.8, 0.2\}/w\}$.

In order to perform any operation on two fuzzy multisets, A and B both must have a parity in respect of their objects and length of membership sequences. In case, such a parity is absent for an object symbol c(say), it is introduced with a sequence

of the form $\underbrace{\{0,0,\cdots,0\}}_{p-times}$, where p is the length of all other sequence.

3.8 Operations under Fuzzy Multisets and their Properties

Let $A, B \in FM(X)$ and $\alpha \in [0,1]$. Then we have the following:

Definition 3.8.1 Subsets of a fuzzy multiset

$$A \subseteq B \Longleftrightarrow \mu_A^j(x) \leq \mu_B^j(x), j = 1,2,\cdots L(x), \forall x \in X$$

Definition 3.8.2 Equality of two fuzzy multisets

$$A = B \Longleftrightarrow \mu_A^j(x) = \mu_B^j(x), j = 1,2,\cdots L(x), \forall x \in X$$

Definition 3.8.3 Addition of fuzzy multisets

The addition of fuzzy multisets A and B, denoted by $A \oplus B$, is defined by the addition operation in $X \times [0,1]$ for crisp multisets. Namely, if

$$A = \{(x_i, \mu_i), \cdots, (x_k, \mu_k)\}$$

and

$$B = \{(x_p, \mu_p), \cdots, (x_r, \mu_r)\}$$

are two fuzzy multisets, then

$$A \oplus B = \{(x_i, \mu_i), \cdots, (x_k, \mu_k), (x_p, \mu_p), \cdots, (x_r, \mu_r)\}.$$

For fuzzy multiset[example (3.9.3)]:

Example 3.8.1 Let

$$A = \{\{0.5, 0.4\}/x, \{1, 0.7, 0.7\}/y\}, \text{ and}$$

$$B = \{\{0.7\}/x, \{0.9, 0.8\}/y, \{0.8, 0.2\}/w\}.$$

Then

$$A \oplus B = \{\{0.7, 0.5, 0.4\}/x, \{1, 0.9, 0.7, 0.7, 0.7\}/y, \{0.8, 0.2\}/w\}.$$

Definition 3.8.4 Union of fuzzy multisets

$$\mu^j_{A\cup B}(x) = \mu^j_A(x) \vee \mu^j_B(x), j = 1, 2, \cdots L(x), \forall x \in X.$$

Example 3.8.2

$$A \cup B = \{\{0.7, 0.4\}/x, \{1, 0.8, 0.7\}/y, \{0.8, 0.2\}/w\}.$$

Definition 3.8.5 Intersection of fuzzy multisets

$$\mu^j_{A\cap B}(x) = \mu^j_A(x) \wedge \mu^j_B(x), j = 1, 2, \cdots L(x), \forall x \in X.$$

Example 3.8.3

$$A \cap B = \{\{0.5, 0\}/x, \{0.9, 0.7, 0\}/y, \{0, 0\}/w\},$$

or simply

$$A \cap B = \{\{0.5\}/x, \{0.9, 0.7\}/y\}.$$

When the functions max and min are replaced by a t-norm t, and a t-conorm s, respectively, we obtain the definitions for $\cap_t$ and $\cup_s$, respectively. The union and intersection of arbitrary fuzzy multisets A, B and C satisfy commutativity, associativity and distributivity properties

CHAPTER FOUR

α-CUTS, INVERSE α-CUTS AND RELATED RESULTS

The notions of α-cuts and strong α-cuts, apart from their multitudinal applications, can be viewed as a bridge both between fuzzy sets and crisp sets, and fuzzy multisets and crisp multisets. In particular, α-cuts will be used in this direction to generalize some fundamental results of fuzzy set theory to fuzzy multiset theory.

4.1 α-Cuts in Fuzzy Sets and Fuzzy Multisets

4.1.1 α-cuts in fuzzy sets

The definitions and theorems, along with their proofs, are adopted from (Klir and Yuan, 1995).

Definition 4.1.1 **α-cut in fuzzy sets**

α-cut sets or α-level sets of a given fuzzy set $\tilde{A}$ in space X, denoted by ${}^{\alpha}\tilde{A}$, are subsets of X defined as ${}^{\alpha}\tilde{A} = \{x \in X : \mu_{\tilde{A}}(x) \geq \alpha, \alpha \in [0,1]\}$. The strong α-cut set of $\tilde{A}$ in X, denoted ${}^{\alpha+}\tilde{A}$, as subset of X defined by ${}^{\alpha+}\tilde{A} = \{x \in X : \mu_{\tilde{A}}(x) > \alpha, \alpha \in [0,1],\}$.

Theorem 4.1.1 Let $\tilde{A}, \tilde{B} \in F(X)$. The following properties hold for all $\alpha, \beta \in [0,1]$:

(i) ${}^{\alpha+}\tilde{A} \subseteq {}^{\alpha}\tilde{A}$;

(ii) $\alpha \leq \beta$ implies ${}^{\alpha}\tilde{A} \supseteq {}^{\beta}\tilde{A}$ and ${}^{\alpha+}\tilde{A} \supseteq {}^{\beta+}\tilde{A}$;

(iii) ${}^{\alpha}(\tilde{A} \cap \tilde{B}) = {}^{\alpha}\tilde{A} \cap {}^{\alpha}\tilde{B}$ and ${}^{\alpha}(\tilde{A} \cup \tilde{B}) = {}^{\alpha}\tilde{A} \cup {}^{\alpha}\tilde{B}$;

(iv) ${}^{\alpha+}(\tilde{A} \cap \tilde{B}) = {}^{\alpha+}\tilde{A} \cap {}^{\alpha+}\tilde{B}$ and ${}^{\alpha+}(\tilde{A} \cup \tilde{B}) = {}^{\alpha+}\tilde{A} \cup {}^{\alpha+}\tilde{B}$;

(v) ${}^{\alpha}(\tilde{A}') = {}^{(1-\alpha)+}\tilde{A}'$.

(vi)
$$\bigcup_{i\in I} {}^{\alpha}\tilde{A}_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I}\tilde{A}_i\right) \quad \text{and} \quad \bigcap_{i\in I} {}^{\alpha}\tilde{A}_i = {}^{\alpha}\left(\bigcap_{i\in I}\tilde{A}_i\right)$$

(vii)
$$\bigcup_{i\in I} {}^{\alpha+}\tilde{A}_i = {}^{\alpha+}\left(\bigcup_{i\in I}\tilde{A}_i\right) \quad \text{and} \quad \bigcap_{i\in I} {}^{\alpha+}\tilde{A}_i \subseteq {}^{\alpha+}\left(\bigcap_{i\in I}\tilde{A}_i\right)$$

(viii) $\tilde{A} \subseteq \tilde{B}$ iff ${}^{\alpha}\tilde{A} \subseteq {}^{\alpha}\tilde{B}$;
$\tilde{A} \subseteq \tilde{B}$ iff ${}^{\alpha+}\tilde{A} \subseteq {}^{\alpha+}\tilde{B}$;

(ix) $\tilde{A} = \tilde{B}$ iff ${}^{\alpha}\tilde{A} = {}^{\alpha}\tilde{B}$;
$\tilde{A} = \tilde{B}$ iff ${}^{\alpha+}\tilde{A} = {}^{\alpha+}\tilde{B}$.

(x)
$${}^{\alpha}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta+}\tilde{A}$$

(xi)
$${}^{\alpha+}\tilde{A} = \bigcup_{\alpha<\beta} {}^{\beta}\tilde{A} = \bigcup_{\alpha<\beta} {}^{\beta+}\tilde{A}$$

(xii)
$${}^{\alpha}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta}\tilde{A} = \bigcap_{\beta<\alpha} {}^{\beta+}\tilde{A}$$

(xii)
$${}^{\alpha+}\tilde{A} = \bigcup_{\alpha<\beta} {}^{\beta}\tilde{A} = \bigcup_{\alpha<\beta} {}^{\beta+}\tilde{A}$$

Klir and Yuan (1995) stress the significance of the properties (i)-(vii) in theorem 4.1.1: property (i) expresses that the strong α-cut is always included in the α-cut of any fuzzy set and for any $\alpha \in [0,1]$; the property follows directly from the definitions of the two types of α-cuts. Property (ii) stresses that the set sequences

$\{{}^{\alpha}\tilde{A}|\alpha \in [0,1]\}$ and $\{{}^{\alpha+}\tilde{A}|\alpha \in [0,1]\}$ of α-cuts and strong α-cuts, respectively, are always monotonic decreasing with respect to α; consequently, they are nested family of sets. Property (iii) and (iv) show that the standard fuzzy set intersection and fuzzy set union are cut worthy and strong cut worthy when applied to two fuzzy sets or due to the associativity of min and max, to any finite number of fuzzy sets. However, From (vi) and (vii) the standard fuzzy intersection on infinite sets is cutworthy (but not strong cutworthy), while the standard fuzzy union on infinite sets is strong cutworthy (but not cutworthy), respectively.

To understands the meaning of property (v), let us make a small notational remark. Observed that we denoted the α-cut of the complement of $\tilde{A}$ by ${}^{\alpha}(\tilde{A}')$, while the complement of the α-cut of $\tilde{A}$ by ${}^{\alpha}\tilde{A}'$ (and we use an analogous notation for the strong α-cuts). Two distinct symbols must be used here since the two sets are, in general, not equal, as illustrated in Fig.(4.1.1). The symbol ${}^{\alpha}\bar{A}$ is a convenient simplification of the symbol $({}^{\alpha}\tilde{A}')$, which more accurately captures the meaning of the set for which it stands. This simplification is justifies since it does not introduce any ambiguity.

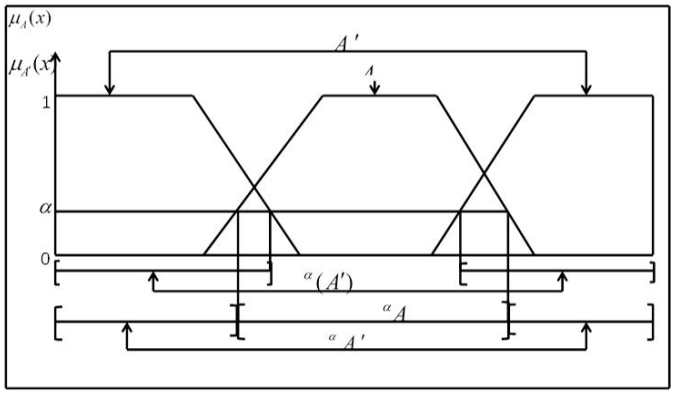


Figure 4.1.1: Illustrations for showing differences between ${}^{\alpha}(\tilde{A}')$ and ${}^{\alpha}\tilde{A}'$

Property (v) shows that the standard fuzzy complement is neither cutworthy nor strong cutworthy. That is,

$$^{\alpha}(\tilde{A}') \neq {}^{\alpha}\tilde{A}' \quad and \quad {}^{\alpha+}(\tilde{A}') \neq {}^{\alpha+}\tilde{A}'$$

in general. This is not surprising since fuzzy sets violate, by definition, the two basic properties of compliment of crisp sets, law of contradiction and law of excluded middle. In spite of its negative connotation, the property (v) describes an interesting features of the standard fuzzy compliment: the α-cut of the compliment of $\tilde{A}$ is always the same as the compliment of the strong $(1-\alpha)$-cut of $\tilde{A}$.

4.1.2 Representations of a fuzzy set by its α-cuts

The principal role of α-cuts and strong α-cuts in fuzzy set theory is their capability to represent fuzzy sets. We show in this section that each fuzzy set can uniquely be represented either by the family of all its α-cuts or the family of all its strong α-cuts. Either of these representations allows us to extend various properties of crisp sets and operations on crisp sets to their fuzzy counterparts. In each extension, a given classical (crisp) property or operation is required to be valid for each crisp set involved in the representation. As already mentioned, such extended properties or operations are called either *cutworthy* or *strong cutworthy*, depending on the type of representation employed.

It will be elaborated in the text that not all properties and operations involving fuzzy sets are cutworthy or strong cutworthy. In fact, properties or operations that are cutworthy or strong cutworthy are rather rare. However, they are of special significance since they bridge fuzzy set theory and classical set theory. They are sort of reference point from which other fuzzy properties or operations are characterized.

To explain the two representations of fuzzy sets by crisp sets, let us begin by

illustrating it by an example. Let the fuzzy set be given by

$$\tilde{A} = 0.2/x_1 + 0.4/x_2 + 0.6/x_3 + 0.8/x_4 + 1/x_5.$$

The given fuzzy set $\tilde{A}$ is associated with five distinct α-cuts which are defined by the following characteristic functions (viewed here as special membership functions):

$$^{0.2}\tilde{A} = 1/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + 1/x_5,$$

$$^{0.4}\tilde{A} = 0/x_1 + 1/x_2 + 1/x_3 + 1/x_4 + 1/x_5,$$

$$^{0.6}\tilde{A} = 0/x_1 + 0/x_2 + 1/x_3 + 1/x_4 + 1/x_5,$$

$$^{0.8}\tilde{A} = 0/x_1 + 0/x_2 + 0/x_3 + 1/x_4 + 1/x_5,$$

$$^{1}\tilde{A} = 0/x_1 + 0/x_2 + 0/x_3 + 0/x_4 + 1/x_5.$$

We now convert each of the α-cuts to a special fuzzy set ${}_{\alpha}\tilde{A}$, defined for each $x \in X = \{x_1, x_2, x_3, x_4, x_5\}$ as follows:

$${}_{\alpha}\tilde{A}(x) = \alpha \cdot {}^{\alpha}\tilde{A}(x). \tag{4.1.1}$$

We obtain

$${}_{0.2}\tilde{A} = 0.2/x_1 + 0.2/x_2 + 0.2/x_3 + 0.2/x_4 + 0.2/x_5,$$

$${}_{0.4}\tilde{A} = 0/x_1 + 0.4/x_2 + 0.4/x_3 + 0.4/x_4 + 0.4/x_5,$$

$${}_{0.6}\tilde{A} = 0/x_1 + 0/x_2 + 0.6/x_3 + 0.6/x_4 + 0.6/x_5,$$

$${}_{0.8}\tilde{A} = 0/x_1 + 0/x_2 + 0/x_3 + 0.8/x_4 + 0.8/x_5,$$

$${}_{1}\tilde{A} = 0/x_1 + 0/x_2 + 0/x_3 + 0/x_4 + 1/x_5.$$

It is now easy to see that the standard fuzzy union of these five special fuzzy sets is exactly the original fuzzy set $\tilde{A}$. That is,

$$\tilde{A} = {}_{0.2}\tilde{A} \cup {}_{0.4}\tilde{A} \cup {}_{0.6}\tilde{A} \cup {}_{0.8}\tilde{A} \cup {}_{1}\tilde{A}.$$

Our aim in this section is to prove that the representation of fuzzy sets of their α-cuts, as illustrated by the example above, is universal. It applies to every fuzzy set, regardless of whether it is based on a finite or infinite universal set. Furthermore, we also prove the universality of an alternative representation, which is based upon strong α-cuts.

The representation of an arbitrary fuzzy set $\tilde{A}$ in terms of the special fuzzy sets ${}_{\alpha}\tilde{A}$, which are defined in terms of the α-cuts of $\tilde{A}$ by (4.1), is usually referred to as a decomposition of $\tilde{A}$. The decomposition theorems of fuzzy sets are as follow:

Theorem 4.1.2 (First Decomposition Theorem)

For every $\tilde{A} \in F(X)$,

$$\tilde{A} = \bigcup_{\alpha \in [0,1]} {}_{\alpha}\tilde{A}. \tag{4.1.2}$$

where ${}_{\alpha}\tilde{A}$ is as defined in (4.1) and $\cup$ denotes the standard fuzzy union.

To illustrate the application of this theorem, let us consider a fuzzy set $\tilde{A}$ with the following membership function of triangular shape (Figure 4.1.2):

$$\mu_{\tilde{A}}(x) = \begin{cases} x - 2 & \text{when } x \in [2,3] \\ 4 - x & \text{when } x \in [3,4] \\ 0 & \text{otherwise.} \end{cases}$$

For each $\alpha \in (0,1]$, the α-cut of $\tilde{A}$ is in this case the closed interval

$${}^{\alpha}\tilde{A} = [\alpha + 2, 4 - \alpha],$$

and the special fuzzy set ${}_{\alpha}\tilde{A}$ employed in (4.2) is defined by the membership function

$$ {}_{\alpha}\tilde{A} = \begin{cases} \alpha & \text{when } x \in [\alpha + 2, 4 - \alpha] \\ 0 & \text{otherwise.} \end{cases} $$

Example of sets ${}^{\alpha}\tilde{A}$ and ${}_{\alpha}\tilde{A}$ for three values of α are shown in Figure(4.1.2) According to Theorem(4.1.2), $\tilde{A}$ is obtain by taking the standard fuzzy union of the sets ${}_{\alpha}\tilde{A}$ for all $\alpha \in [0, 1]$.

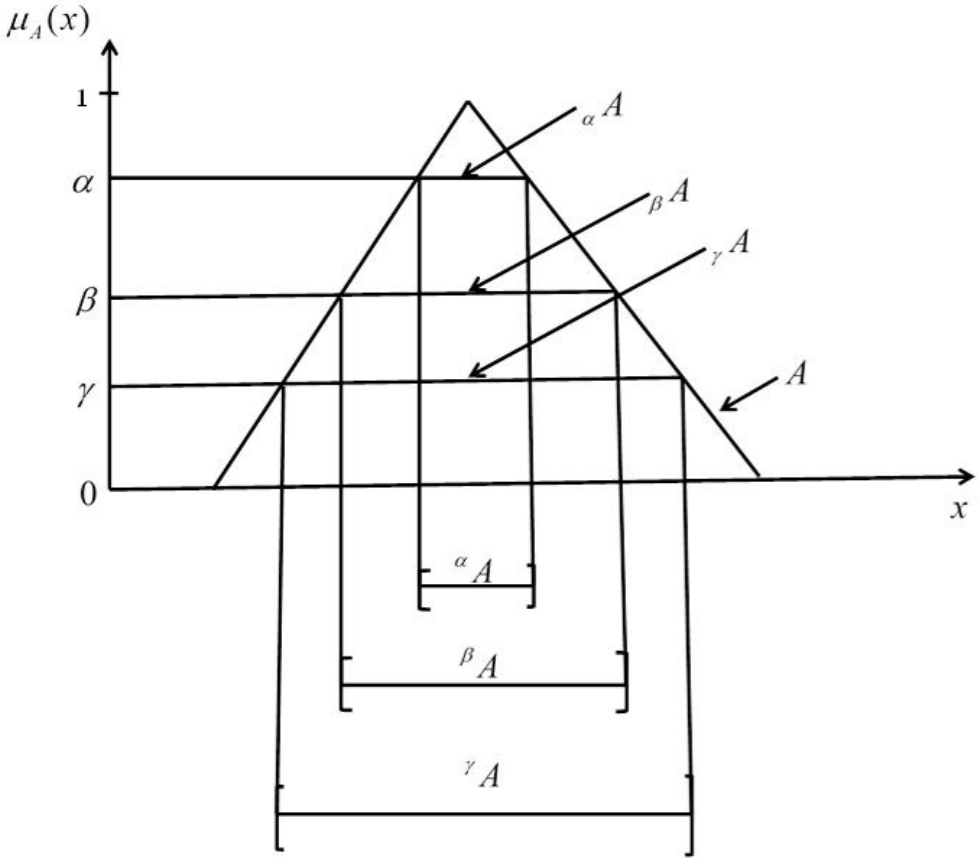


Figure 4.1.2: Illustrations for $\tilde{A}$ as union of ${}_{\alpha}\tilde{A}$,${}_{\beta}\tilde{A}$ and ${}_{\gamma}\tilde{A}$,

Theorem 4.1.3 (Second Decomposition Theorem)

For every $\tilde{A} \in F(X)$,

$$ \tilde{A} = \bigcup_{\alpha \in [0,1]} {}_{\alpha+}\tilde{A}. \tag{4.1.3} $$

where ${}_{\alpha+}A$ denotes a special fuzzy set defined by

$$ {}_{\alpha+}\tilde{A}(x) = \alpha \cdot {}^{\alpha+}\tilde{A}(x) \tag{4.1.4} $$

and $\cup$ denotes the standard fuzzy union.

Theorem 4.1.4 (Third Decomposition Theorem)

For every $\tilde{A} \in F(X)$,

$$\tilde{A} = \bigcup_{\alpha \in \Lambda_{\tilde{A}}} {}_{\alpha}\tilde{A}, \tag{4.1.5}$$

where $\Lambda_{\tilde{A}}$ is the level set of $\tilde{A}$, ${}_{\alpha}\tilde{A}$ is as defined by (4.1), and $\cup$ denotes the standard fuzzy union.

The meaning of this theorem is illustrated in Figure 4.1.3 by the decomposition of a fuzzy set $\tilde{A}$ by a simple stepwise membership function shown in Figure 4.1.3(a). Since $\Lambda_{\tilde{A}} = \{0, 0.3, 0.6, 1\}$ and ${}_{0}\tilde{A} = \varnothing, \tilde{A}$ is fully represented by the three special fuzzy sets ${}_{0.3}\tilde{A}$,${}_{0.6}\tilde{A}$, and ${}_{1}\tilde{A}$; shown in Figure 4.1.3(b).

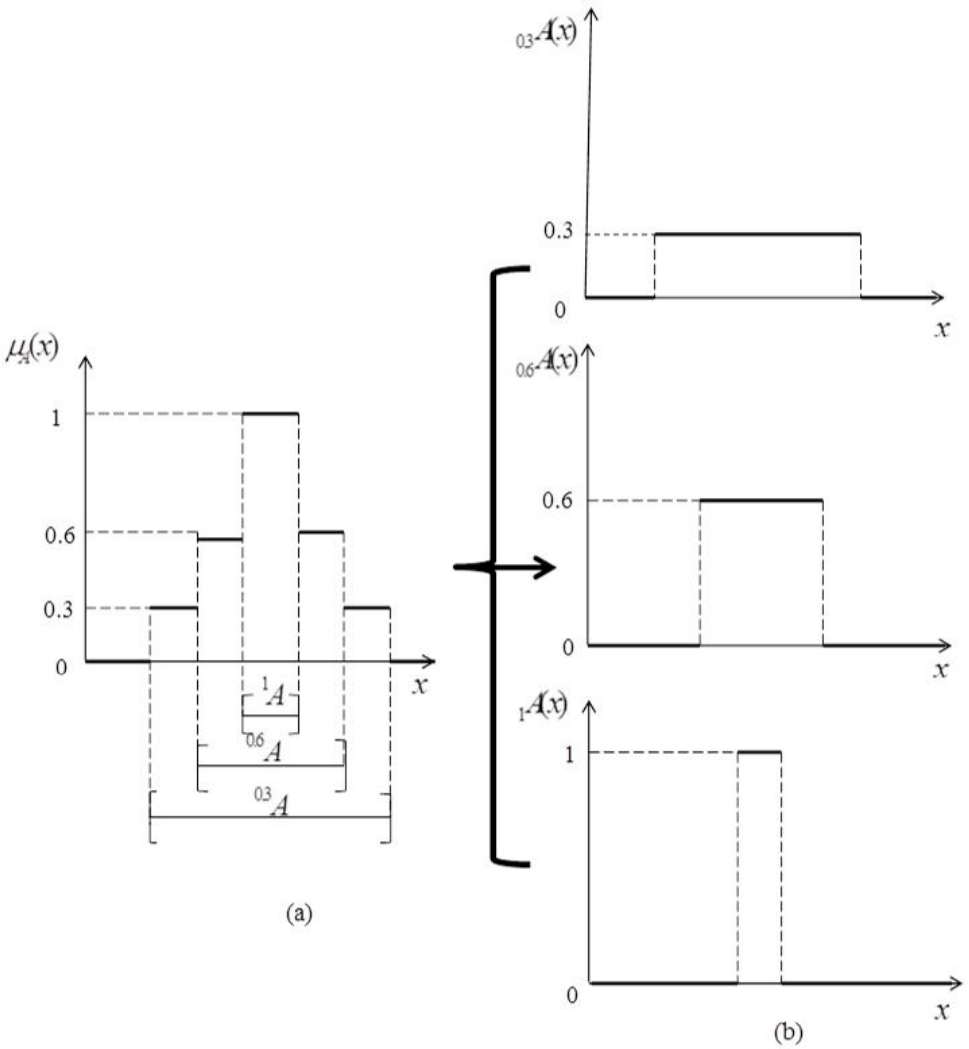


Figure 4.1.3: Illustrations for Theorem 4.1.4 (a) fuzzy set $\tilde{A}$ (b) decomposition of $\tilde{A}$ into $_{0.3}\tilde{A}$, $_{0.6}\tilde{A}$, $_{1}\tilde{A}$.

4.1.3 α-cuts in fuzzy multisets

Definition 4.1.2 **α-Cuts in fuzzy multisets**

The α-cut ($\alpha \in (0,1]$) for a fuzzy multiset A, denoted by ${}^{\alpha}[A]$, is a crisp multiset defined as follows:

$$\mu_A^j(x) < \alpha \Rightarrow count_{\,{}^{\alpha}[A]} = 0$$

$$\mu_A^j(x) \geq \alpha, \mu_A^{j+1}(x) < \alpha \Rightarrow count_{\,{}^{\alpha}[A]} = j, j = 1, 2, \cdots L(x).$$

Moreover the strong α-cut ($\alpha \in [0,1)$), denoted ${}^{\alpha}]A[$, is a crisp multiset defined as follows:

$$\mu_A^j(x) \leq \alpha \Rightarrow count_{\,{}^{\alpha}]A[} = 0$$

$$\mu_A^j(x) > \alpha, \mu_A^{j+1}(x) \leq \alpha \Rightarrow count_{\,{}^{\alpha}]A[} = j, j = 1, 2, \cdots L(x).$$

The concept of α-cuts and strong α-cuts plays a principal role in exploring the relationship between fuzzy multisets and crisp multisets. They can be viewed as a bridge by which fuzzy multisets and crisp multisets are connected. We first overview some properties of the two types α-cuts . All these properties involve the standard fuzzy multisets operations as defined above.

Theorem 4.1.5 Let $A, B \in FM(X)$ Then the following holds for all $\alpha, \beta \in [0,1]$:

(i) ${}^{\alpha}]A[\subseteq {}^{\alpha}[A]$.

(ii) $\alpha \leq \beta$ implies ${}^{\alpha}[A] \supseteq {}^{\beta}[A]$ and ${}^{\alpha}]A[\supseteq {}^{\beta}]A[$.

(iii) ${}^{\alpha}[A \cup B] = {}^{\alpha}[A] \cup {}^{\alpha}[B]$ and ${}^{\alpha}[A \cap B] = {}^{\alpha}[A] \cap {}^{\alpha}[B]$.

(iv) ${}^{\alpha}]A \cup B[= {}^{\alpha}]A[\cup {}^{\alpha}]B[$ and ${}^{\alpha}]A \cap B[= {}^{\alpha}]A[\cap {}^{\alpha}]B[$.

Proof. (iii) The first part. For any $x \in {}^{\alpha}[A \cup B]$ we have $\mu_{(A \cup B)}^j \geq \alpha$, $j = 1, \cdots L(x)$, which means $max[\mu_A^j(x), \mu_B^j(x)] \geq \alpha$, $j = 1, \cdots L(x)$ and hence $\mu_A^j(x) \geq \alpha$ or $\mu_B^j \geq \alpha$, $j = 1, \cdots L(x)$. $\Rightarrow x \in {}^{\alpha}[A]$ or $x \in [B]_{\alpha} \Rightarrow x \in {}^{\alpha}[A] \cup {}^{\alpha}[B]$ and, consequently

${}^{\alpha}[A \cup B] \subseteq {}^{\alpha}[A] \cup {}^{\alpha}[B]$.

Conversely, for any $x \in {}^{\alpha}[A] \cup {}^{\alpha}[B]$ we have $x \in {}^{\alpha}[A]$ or $x \in {}^{\alpha}[B]$, that is, $\mu_A^j(x) \geq \alpha$ or $\mu_B^j \geq \alpha$, $j = 1, \cdots L(x)$. Hence, $max[\mu_A^j(x), \mu_B^j(x)]$, $j = 1, \cdots L(x)$. This means that $\mu_{(A \cup B)}^j \geq \alpha$, $j = 1, \cdots L(x)$, which implies $x \in {}^{\alpha}[A \cup B]$ and, consequently ${}^{\alpha}[A] \cup {}^{\alpha}[B] \subseteq {}^{\alpha}[A \cup B]$. Thus ${}^{\alpha}[A \cup B] = {}^{\alpha}[A] \cup {}^{\alpha}[B]$.

The proofs of (i), (ii) and (iv) are straightforward. □

Let us stress the significance of the properties (i)-(iv) in theorem 4.1.5: property (i) expresses that the strong α-cut is always included in the α-cut of any fuzzy multiset and, for any $\alpha \in [0, 1]$; the proof follows directly from the definitions of the two types of α-cuts. Property (ii) stresses that the mset sequences $\{ {}^{\alpha}[A] | \alpha \in [0, 1]\}$ and $\{ {}^{\alpha}]A[| \alpha \in [0, 1]\}$ for α-cuts and strong α-cuts, respectively, are always monotonic decreasing with respect to α; clearly, these are nested family of msets. Property (iii) and (iv) show that the standard fuzzy multiset intersection and fuzzy multiset union are cut worthy and strong cut worthy when applied to two fuzzy multisets, and, inview of the associativity of min and max, to any finite number of fuzzy multisets.

4.1.4 Representations of a fuzzy multisets(FMset) by its α-cuts

To explain the two representations of fuzzy multisets by crisp multisets, let us begin by illustrating one of them using a simple example.

Example 4.1.1 Consider the fuzzy multiset

$$A = \{\{0.5, 0.3, 0.2\}/x, \{0.5\}/y, \{0.9, 0.8, 0.1\}/z\}.$$

Now, A is associated with six distinct α-cuts which are defined by the following characteristic functions (viewed as special membership functions):

$$^{0.1}[A] = \{\{1,1,1\}/x, \{1\}/y, \{1,1,1\}/z\},$$

$$^{0.2}[A] = \{\{1,1,1\}/x, \{1\}/y, \{1,1,0\}/z\},$$

$$^{0.3}[A] = \{\{1,1,0\}/x, \{1\}/y, \{1,1,0\}/z\},$$

$$^{0.5}[A] = \{\{1,0,0\}/x, \{1\}/y, \{1,1,0\}/z\},$$

$$^{0.8}[A] = \{\{0,0,0\}/x, \{0\}/y, \{1,1,0\}/z\},$$

$$^{0.9}[A] = \{\{0,0,0\}/x, \{0\}/y, \{1,0,0\}/z\}.$$

We now convert each of the α-cuts to a special fuzzy multiset $_{\alpha}[A]$ defined for each $x \in X = [x, y, z]$ as follows:

$$_{\alpha}[A] = \alpha \cdot {}^{\alpha}[A] \tag{4.1.6}$$

we obtain

$$_{0.1}[A] = \{\{0.1, 0.1, 0.1\}/x, \{0.1\}/y, \{0.1, 0.1, 0.1\}/z\},$$

$$_{0.2}[A] = \{\{0.2, 0.2, 0.2\}/x, \{0.2\}/y, \{0.2, 0.2, 0\}/z\},$$

$$_{0.3}[A] = \{\{0.3, 0.3, 0\}/x, \{0.3\}/y, \{0.3, 0.3, 0\}/z\},$$

$$_{0.5}[A] = \{\{0.5, 0, 0\}/x, \{0.5\}/y, \{0.5, 0.5, 0\}/z\},$$

$$_{0.8}[A] = \{\{0, 0, 0\}/x, \{0\}/y, \{0.8, 0.8, 0\}/z\},$$

$$_{0.9}[A] = \{\{0, 0, 0\}/x, \{0\}/y, \{0.9, 0, 0\}/z\}.$$

It is now easy to see that the standard fuzzy multiset union of these six special fuzzy multisets is exactly the original fuzzy multiset A . That is

$$A = {}_{0.1}[A] \cup {}_{0.2}[A] \cup {}_{0.3}[A] \cup {}_{0.5}[A] \cup {}_{0.8}[A] \cup {}_{0.9}[A].$$

It should be noted that this example illustrates that the representation of a fuzzy multiset by their α-cuts is universal. It applies to every fuzzy multiset, regardless of whether it is based on a finite or infinite universal set. The representation of an arbitrary fuzzy multiset A in terms of the special fuzzy multiset ${}_{\alpha}[A]$, defined in terms of the α-cuts of A by (4.6), is usually referred to as a *decomposition of A* . In the following, we describe decomposition theorems of fuzzy multisets in terms of α-cuts and strong α-cuts, respectively.

Theorem 4.1.6 First Decomposition Theorem

For every $A \in FM(X)$,

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}[A]. \tag{4.1.7}$$

$\bigcup$ denotes the standard fuzzy multiset union.

Proof. For each particular $x \in X$, let $b = \mu_A^j(x)$, $j = 1, \cdots, L(x)$ then in each $\alpha \in (b, 1]$ we have $\mu_A^j(x) = b < \alpha$, $j = 1, \cdots, L(x)$, and therefore ${}_{\alpha}[A] = 0$. On the other hand, for each $\alpha \in (0, b]$, we have $\mu_A^j(x) = b \geq \alpha$, $j = 1, \cdots, L(x)$, and therefore ${}_{\alpha}[A] = \alpha$.

$$\text{Hence, } A = \left(\bigcup_{\alpha \in [0,1]} {}_{\alpha}[A] \right)(x) = \sup_{\alpha \in (0,b]} = b = \mu_A^j(x), j = 1, \cdots, L(x).$$

Since the same argument is valid for each $x \in X$, the validity of Theorem (4.1.6) is established.

Hence, each fuzzy multiset can be uniquely represented by the family of all its α-cuts.

□

Theorem 4.1.7 Second Decomposition Theorem

For every $A \in FM(X)$,

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}]A[, \tag{4.1.8}$$

where

$${}_{\alpha}]A[= \alpha \cdot {}^{\alpha}]A[,$$

and $\bigcup$ denote the standard fuzzy multiset union.

Proof. Similar to the proof of Theorem (4.1.6) we have for each particular $x \in X$, let $b = \mu_A^j(x)$, $j = 1, \cdots, L(x)$, then

$$\left(\bigcup_{\alpha \epsilon (0,1]} {}_{\alpha}]A[\right)(x) = \sup_{\alpha \epsilon (0,b]} {}_{\alpha}]A[$$

$$= max(\sup_{\alpha \epsilon (0,b]} {}_{\alpha}]A[, \sup_{\alpha \epsilon (b,1]} {}_{\alpha}]A[)$$

$$\Rightarrow \left(\bigcup_{\alpha \epsilon (0,1]} {}_{\alpha}]A[\right)(x) = b = \mu_A^j(x), j = 1, \cdots, L(x).$$

Hence, each fuzzy multiset can also be uniquely represented by the family of all its strong α-cuts. □

Moreover, we have shown that each fuzzy multiset can be uniquely represented by either the family of all its α-cuts or the family of all its strong α-cuts. Moreover, either of these representations allows us to extend various properties of crisp multisets and operations on crisp multisets to their fuzzy multiset counterparts. In each representation, a given crisp property or operation is required to be valid for each crisp multiset involved in the representation. Such extended properties or operations are called cut worthy or strong cut worthy depending on the type of representation employed. These representations are of special significance since they bridge fuzzy multiset theory and crisp multiset theory.

4.2 Inverse α-Cuts in Fuzzy Sets and Fuzzy Multisets

4.2.1 Inverse α-cuts and their properties in fuzzy sets

Definition 4.2.1 Inverse α-Cuts (Sun and Han, 2006)

Let $\tilde{A} \in F(X)$ and $\alpha \in [0,1]$. Then the non-fuzzy set

$$^{\alpha}\tilde{A}^{-1} = \{x \in X | \mu_{\tilde{A}}(x) < \alpha\}$$

is called an *inverse α-cut* (or *inverse α-level set*) of $\tilde{A}$.

If the strict inequality is replaced by the weak inequality $\leq$, then it is called a *weak inverse α-cut* of $\tilde{A}$, denoted by $^{\alpha^-}\tilde{A}^{-1}$. That is,

$$^{\alpha^-}\tilde{A}^{-1} = \{x \in X | \mu_{\tilde{A}}(x) \leq \alpha\}.$$

Proposition 4.2.1 Let $\tilde{A}, \tilde{B} \in F(X)$ and $\alpha, \beta \in [0,1]$. The following properties hold:

i. $^{\alpha}\tilde{A}^{-1} \subseteq {}^{\alpha^-}\tilde{A}^{-1}$

ii. $\alpha \leq \beta$ implies $^{\alpha}\tilde{A}^{-1} \subseteq {}^{\beta}\tilde{A}^{-1}$ and $^{\alpha^-}\tilde{A}^{-1} \subseteq {}^{\beta^-}\tilde{A}^{-1}$

iii. $^{\alpha}(\tilde{A} \cap \tilde{B})^{-1} = {}^{\alpha}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1}$, $^{\alpha}(\tilde{A} \cup \tilde{B})^{-1} = {}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1}$, and $^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1} \subseteq {}^{\alpha}(\tilde{A} \cap \tilde{B})^{-1}$

iv. $^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1} = {}^{\alpha^-}\tilde{A}^{-1} \cup {}^{\alpha^-}\tilde{B}^{-1}$, $^{\alpha^-}(\tilde{A} \cup \tilde{B})^{-1} = {}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1}$, and $^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1} \subseteq {}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1}$

v. $^{\alpha}((\tilde{A}^{-1})') = {}^{(1-\alpha)^-}(\tilde{A}^{-1})'$

vi. $^{1}\tilde{A}^{-1} = X$

vii. $\tilde{A} \subseteq \tilde{B}$ iff $^{\alpha}\tilde{B}^{-1} \subseteq {}^{\alpha}\tilde{A}^{-1}$; $\tilde{A} \subseteq \tilde{B}$ iff $^{\alpha^-}\tilde{B}^{-1} \subseteq {}^{\alpha^-}\tilde{A}^{-1}$

viii. $\tilde{A} = \tilde{B}$ iff $^{\alpha}\tilde{B}^{-1} = {}^{\alpha}\tilde{A}^{-1}$; $\tilde{A} = \tilde{B}$ iff $^{\alpha^-}\tilde{B}^{-1} = {}^{\alpha^-}\tilde{A}^{-1}$

Proof.

i. $x \in {}^{\alpha}\tilde{A}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) < \alpha \Longrightarrow \mu_{\tilde{A}}(x) \leq \alpha \Longrightarrow x \in {}^{\alpha^-}\tilde{A}^{-1}$, hence $^{\alpha}\tilde{A}^{-1} \subseteq {}^{\alpha^-}\tilde{A}^{-1}$.

ii. Part I

$x \in {}^{\alpha}\tilde{A}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) < \alpha \Longrightarrow \mu_{\tilde{A}}(x) < \beta$ (since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta}\tilde{A}^{-1}$, hence ${}^{\alpha}\tilde{A}^{-1} \subseteq {}^{\beta}\tilde{A}^{-1}$.

Part II

$x \in {}^{\alpha^-}\tilde{A}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) \leq \alpha \Longrightarrow \mu_{\tilde{A}}(x) \leq \beta$ (since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta^-}\tilde{A}^{-1}$, hence ${}^{\alpha^-}\tilde{A}^{-1} \subseteq {}^{\beta^-}\tilde{A}^{-1}$.

iii. Part I

$x \in {}^{\alpha}(\tilde{A} \cap \tilde{B})^{-1} \Longrightarrow \mu_{(\tilde{A}\cap\tilde{B})}(x) < \alpha \Longrightarrow \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} < \alpha \Longrightarrow \mu_{\tilde{A}}(x) < \alpha$ or $\mu_{\tilde{B}}(x) < \alpha \Longrightarrow x \in ({}^{\alpha}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1})$, hence ${}^{\alpha}(\tilde{A} \cap \tilde{B})^{-1} \subseteq {}^{\alpha}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1}$.

Conversely, $x \in {}^{\alpha}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1} \Longrightarrow x \in {}^{\alpha}\tilde{A}^{-1}$ or $x \in {}^{\alpha}\tilde{B}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) < \alpha$ or $\mu_{\tilde{B}}(x) < \alpha \Longrightarrow \min\{\mu_A(x), \mu_{\tilde{B}}(x)\} < \alpha \Longrightarrow \mu_{(\tilde{A}\cap\tilde{B})}(x) < \alpha \Longrightarrow x \in {}^{\alpha}(\tilde{A}\cap\tilde{B})^{-1}$, hence ${}^{\alpha}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1} \subseteq {}^{\alpha}(\tilde{A} \cap \tilde{B})^{-1}$.
Thus, ${}^{\alpha}(\tilde{A} \cap \tilde{B})^{-1} = {}^{\alpha}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1}$.

Part II

$x \in {}^{\alpha}(\tilde{A} \cup \tilde{B})^{-1} \Longrightarrow \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} < \alpha \Longrightarrow \mu_{\tilde{A}}(x) < \alpha$ and $\mu_{\tilde{B}}(x) < \alpha$ $\Longrightarrow x \in {}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1}$, hence ${}^{\alpha}(\tilde{A} \cup \tilde{B})^{-1} \subseteq {}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1}$.

Conversely, $x \in {}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1} \Longrightarrow x \in {}^{\alpha}\tilde{A}^{-1}$ and $x \in {}^{\alpha}\tilde{B}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) < \alpha$ and $\mu_{\tilde{B}}(x) < \alpha \Longrightarrow \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} < \alpha, \Longrightarrow \mu_{(\tilde{A}\cup\tilde{B})}(x) < \alpha \Longrightarrow x \in {}^{\alpha}(\tilde{A} \cup \tilde{B})^{-1}$, hence ${}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1} \subseteq {}^{\alpha}(\tilde{A} \cup \tilde{B})^{-1}$.
Thus, ${}^{\alpha}(\tilde{A} \cup \tilde{B})^{-1} = {}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1}$.

Part III

The proof follows from the converse of part II.

iv. Part I

$x \in {}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1} \Longrightarrow \mu_{(\tilde{A}\cap\tilde{B})}(x) \leq \alpha \Longrightarrow \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \leq \alpha \Longrightarrow \mu_{\tilde{A}}(x) \leq \alpha$ or $\mu_{\tilde{B}}(x) \leq \alpha \Longrightarrow x \in ({}^{\alpha^-}\tilde{A}^{-1} \cup {}^{\alpha^-}\tilde{B}^{-1})$, hence ${}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1} \subseteq {}^{\alpha^-}\tilde{A}^{-1} \cup {}^{\alpha^-}\tilde{B}^{-1}$.

Conversely, $x \in {}^{\alpha^-}\tilde{A}^{-1} \cup {}^{\alpha^-}\tilde{B}^{-1} \Longrightarrow x \in {}^{\alpha^-}\tilde{A}^{-1}$ or $x \in {}^{\alpha^-}\tilde{B}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) \leq \alpha$ or $\mu_{\tilde{B}}(x) \leq \alpha \Longrightarrow \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \leq \alpha \Longrightarrow \mu_{(\tilde{A}\cap\tilde{B})}(x) \leq \alpha \Longrightarrow x \in {}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1}$, hence ${}^{\alpha^-}\tilde{A}^{-1} \cup {}^{\alpha^-}\tilde{B}^{-1} \subseteq {}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1}$.
Therefore, ${}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1} = {}^{\alpha^-}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1}$.

Part II

$x \in {}^{\alpha^-}(\tilde{A} \cup \tilde{B})^{-1} \Longrightarrow \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \leq \alpha \Longrightarrow \mu_{\tilde{A}}(x) \leq \alpha$ and $\mu_{\tilde{B}}(x) \leq \alpha \Longrightarrow x \in {}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1}$, hence ${}^{\alpha^-}(\tilde{A} \cup \tilde{B})^{-1} \subseteq {}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1}$.

Conversely, $x \in {}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1} \Longrightarrow x \in {}^{\alpha^-}\tilde{A}^{-1}$ and $x \in {}^{\alpha^-}\tilde{B}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) \leq \alpha$ and $\mu_{\tilde{B}}(x) \leq \alpha \Longrightarrow \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\} \leq \alpha \Longrightarrow \mu_{(\tilde{A}\cup\tilde{B})}(x) \leq \alpha \Longrightarrow x \in {}^{\alpha^-}(\tilde{A} \cup \tilde{B})^{-1}$, hence ${}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1} \subseteq {}^{\alpha^-}(\tilde{A} \cup \tilde{B})^{-1}$. Therefore, ${}^{\alpha^-}(\tilde{A} \cup \tilde{B})^{-1} = {}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1}$.

Part III

The proof follows from the converse of part II.

v. $x \in {}^{\alpha}((\tilde{A}^{-1})') \Longrightarrow 1 - \mu_{\tilde{A}}(x) = \mu_{\tilde{A}'}(x) < \alpha \Longrightarrow \mu_{\tilde{A}}(x) > 1 - \alpha \Longrightarrow x \notin {}^{(1-\alpha)^-}\tilde{A}^{-1}$ $\Longrightarrow x \in {}^{(1-\alpha)^-}(\tilde{A}^{-1})'$ hence ${}^{\alpha}((\tilde{A}^{-1})') \subseteq {}^{(1-\alpha)^-}(\tilde{A}^{-1})'$.

Conversely, $x \in {}^{(1-\alpha)^-}(\tilde{A}^{-1})', \Longrightarrow x \notin {}^{(1-\alpha)^-}\tilde{A}^{-1} \Longrightarrow \mu_{\tilde{A}}(x) > 1 - \alpha$ and $1 - \mu_{\tilde{A}}(x) < \alpha \Longrightarrow \mu_{\tilde{A}'}(x) < \alpha \Longrightarrow x \in {}^{\alpha}((\tilde{A}^{-1})') \Longrightarrow {}^{(1-\alpha)^-}(\tilde{A}^{-1})' \subseteq^{\alpha} ((\tilde{A}^{-1})')$, hence ${}^{\alpha}((\tilde{A}^{-1})') = {}^{(1-\alpha)^-}(\tilde{A}^{-1})'$.

vi. The proof follows from the definition.

vii. For any $x \in X$, $\tilde{A} \subseteq \tilde{B}$ if and only if $\mu_{\tilde{A}} \leq \mu_{\tilde{B}}$, if $\mu_{\tilde{B}} < \alpha \Longrightarrow \mu_{\tilde{A}} < \alpha$ if and only if $x \in {}^{\alpha}\tilde{B}^{-1}$ and $x \in {}^{\alpha}\tilde{A}^{-1}$ if and only if ${}^{\alpha}\tilde{B}^{-1} \subseteq {}^{\alpha}\tilde{A}^{-1}$.

The proof of the second part follows similarly.

viii. The proof follows from the definition.

□

Remark 4.2.1 Property (ii) stresses that the set sequences $\{^{\alpha^-}\tilde{A}^{-1} | \alpha \in [0,1]\}$ and $\{^{\alpha}\tilde{A}^{-1} | \alpha \in [0,1]\}$ of inverse α-cuts and weak inverse α-cuts, respectively, are monotonic increasing with respect to α and thus form a nested family of sets. Further, contrary to the case with α-cuts and strong α-cuts, properties (iii) and (iv) show that the standard fuzzy set intersection and fuzzy set union are neither inverse α-cutworthy nor weak inverse α-cutworthy.

4.2.2 Inverse α-cuts and their properties in fuzzy multisets

Definition 4.2.2 The inverse α-cut ($\alpha \in (0,1]$) of a fuzzy multiset A, denoted by $^{\alpha}[A]^{-1}$, is a crisp multiset defined as follows:

$$\mu_A^j(x) \geq \alpha \Rightarrow cont\,^{\alpha}[A]^{-1} = 0$$

$$\mu_A^j(x) < \alpha\,, \mu_A^{j-1}(x) \geq \alpha \Rightarrow cont\,^{\alpha}[A]^{-1} = L(x) - j + 1,\ j = 1, 2, 3, \ldots, L(x).$$

Moreover the weak inverse α-cut ($\alpha \in [0,1)$), denoted by $^{\alpha}]A[^{-1}$, is a crisp multiset defined as follows:

$$\mu_A^j(x) > \alpha \Rightarrow cont\,^{\alpha}]A[^{-1} = 0$$

$$\mu_A^j(x) \leq \alpha\,, \mu_A^{j-1}(x) > \alpha \Rightarrow cont\,^{\alpha}]A[= L(x) - j + 1,\ j = 1, 2, 3, \ldots, L(x).$$

Remark 4.2.2 It may be observed that $^{\alpha}[A] \cup {}^{\alpha}[A]^{-1} \neq X$, contrary to the case in fuzzy sets where $^{\alpha}A \cup {}^{\alpha}A^{-1} = X$ (Sun and Han, 2006).

Example 4.2.1 Let $X = \{x_1, x_2, x_3\}$, $A = \{\{0.9, 0.3, 0.3\}/x_1,\ \{0.7, 0.4\}/x_2,\ \{0.7, 0.6, 0.6\}/x_3\}$,

and $\alpha = 0.\ 4$. Thus we have

$$^{0.4}[A] = \{1/x_1,\ 2/x_2,\ 3/x_3\}$$

and

$$^{0.4}[A]^{-1} = \{2/x_1\}\,.$$

Now, $^{0.4}[A] \cup {}^{0.4}[A]^{-1} = \{2/x_1,\ 2/x_2,\ 3/x_3\}$ whose root set is X.

Proposition 4.2.2

Let $A,\ B,\ C \in FM(X)$. The following properties hold for $\alpha, \beta \in (0, 1]$:

i. $^{\alpha}[A]^{-1} \cup\ {}^{\alpha}[B]^{-1} = {}^{\alpha}[B]^{-1} \cup {}^{\alpha}[A]^{-1}\,;\ {}^{\alpha}[A]^{-1} \cap [\,B]^{-1} = {}^{\alpha}[B]^{-1} \cap {}^{\alpha}[A]^{-1}$

ii. $^{\alpha}\,]A[^{-1} \cup {}^{\alpha}\,]B[^{-1} = {}^{\alpha}\,]B[^{-1} \cup {}^{\alpha}\,]A[^{-1}\,;\ {}^{\alpha}\,]A[^{-1} \cap {}^{\alpha}\,]B[^{-1} = {}^{\alpha}\,]B[^{-1} \cap {}^{\alpha}\,]A[^{-1}$

iii. $^{\alpha}[A]^{-1} \cup ({}^{\alpha}[B]^{-1} \cup {}^{\alpha}[\,C]^{-1}) = ({}^{\alpha}[A]^{-1} \cup {}^{\alpha}[\,B]^{-1}) \cup {}^{\alpha}[C]^{-1}$;

$$^{\alpha}[A]^{-1} \cap \left({}^{\alpha}[B]^{-1} \cap {}^{\alpha}[\,C]^{-1}\right) = \left({}^{\alpha}[A]^{-1} \cap {}^{\alpha}[\,B]^{-1}\right) \cap {}^{\alpha}[C]^{-1}$$

iv. $^{\alpha}\,]A[^{-1} \cup \left({}^{\alpha}\,]B[^{-1} \cup {}^{\alpha}\,]C[^{-1}\right) = \left({}^{\alpha}\,]A[^{-1} \cup {}^{\alpha}\,]B[^{-1}\right) \cup {}^{\alpha}\,]C[^{-1}\,;$

$$^{\alpha}\,]A[^{-1} \cap \left({}^{\alpha}\,]B[^{-1} \cap {}^{\alpha}\,]C[^{-1}\right) = \left({}^{\alpha}\,]A[^{-1} \cap {}^{\alpha}\,]B[^{-1}\right) \cap {}^{\alpha}\,]C[^{-1}$$

v. $^{\alpha}\,[A]^{-1} \subseteq {}^{\alpha}\,]A[^{-1}$

vi. $\alpha \leq \beta$ implies $^{\alpha}\,[A]^{-1} \subseteq {}^{\beta}\,[A]^{-1}$‘ and $^{\alpha}\,]A[^{-1} \subseteq {}^{\beta}\,]A[^{-1}$

Proof. The proofs of i – iv are immediate.

v. $x \in {}^{\alpha}[A]^{-1} \Longrightarrow \mu_A^j(x) < \alpha,\ j = 1, 2, \ldots, L(x) \Longrightarrow \mu_A^j(x) \leq \alpha,\ j = 1, 2, 3, \ldots, L(x)$
$\Longrightarrow x \in {}^{\alpha}]A[^{-1}$, hence $^{\alpha}\,[A]^{-1} \subseteq {}^{\alpha}\,]A[^{-1}$.

vi. part I

$x \in {}^{\alpha}[A]^{-1} \Longrightarrow \mu_A^j(x) < \alpha, J = 1, 2, 3, \ldots, L(x) \Longrightarrow \mu_A^j(x) < \beta, j = 1, 2, 3, \ldots, L(x)$
(since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta}[A]^{-1}$, hence ${}^{\alpha}[A]^{-1} \subseteq {}^{\beta}[A]^{-1}$.

Part II

$x \in {}^{\alpha}]A[^{-1} \Longrightarrow \mu_A^j(x) \leq \alpha, j = 1, 2, 3, \ldots, L(x) \Longrightarrow \mu_A^j(x) \leq \beta, j = 1, 2, 3, \ldots, L(x)$
(since $\alpha \leq \beta$) $\Longrightarrow x \in {}^{\beta}]A[^{-1}$, hence ${}^{\alpha}]A[^{-1} \subseteq {}^{\beta}]A[^{-1}$.

□

CHAPTER FIVE

ALGEBRA OF FUZZY MULTISETS

Similar to the development of various algebraic structures on sets, a number of these algebraic structures have been studied on fuzzy sets. In view of recent introduction of fuzzy multisets no much has been done in this direction. In a couple of recent works, multigroups (Nazmul et al., 2013), and fuzzy multigroups (Shinoj et al., 2015), have been introduced. In this part of the dissertation, besides presenting multigroups and fuzzy multigroups, we formulate monoids of multiset partitions and monoids of S-H fuzzy partitions as new results. Further, the concept of normal abelian fuzzy multisubgroup is also introduced.

5.1 Fuzzy Group

The concept of fuzzy group and substructures was first introduced in (Rosenfeld, 1971), redefined in (Anthony and Sherwood, 1979) and further studied by various researchers such as (Das, 1981), (Guptaa and Sarmab, 1999), Tawfiq and Qa'aed (2012), etc., The concept of fuzzy set over a group is the basic tool to define a fuzzy group.

Definition 5.1.1 Fuzzy Subgroupoid (Rosenfeld, 1971)

A function $\mu : X \to [0, 1]$ is a fuzzy subgroupoid of X if, for all $x, y \in X$,

$$\mu(xy) \geq Min(\mu(x), \mu(y)).$$

Definition 5.1.2 Fuzzy subgroups (Rosenfeld, 1971)

If X is a group and μ is a fuzzy subgroupoid of X then μ is a fuzzy subgroup of X if, for each $x \in X$

$$\mu(x^{-1}) = \mu(x).$$

In the same place other non-deterministic structures analogous to ordinary algebraic concepts such as ideals-fuzzy ideals, fuzzy homomorphic images, etc., were introduced and their properties were investigated. Anthony and Sherwood (1979) provided counter examples to show some mathematical structures which would intuitively seem to be *fuzzy* but did not satisfy the definitions given in (Rosenfeld, 1971). Deluca and Termini (1972) suggested that replacing the closed interval $[0,1]$ with a partially ordered set L, could suffice to a great extent. However, it was remarked (Anthony and Sherwood, 1979) that in many interesting situations, $[0,1]$ could be intrinsically a part of the structure and attempts to replace it with a partially ordered set would be clumsy and unnatural. They also provide an alternative solution in terms of replacing Min with some other semigroup on $[0,1]$, which turned out to be relatively simpler. However, in order to explicate their alternative situation they required the concept ot t-norms. In the following, let the two counter examples and concept of the t-norms be described.

Example 5.1.1 Let X be the collection of all random variables on the probability space $(\Omega, \mathbb{A}, P)$ and let A be any Bore1 subset of the reals which is a subgroup of the reals under addition. Of course X with pointwise addition is a groupoid. (In fact X is a group but we do not need the additional structure for our present purposes.) Define a function $\alpha_A : X \to [0,1]$ by $\alpha_A(x) = P\{w \in \Omega : x(w) \in A\} = P[x^{-1}A]$. Now $\alpha_A(x)$ is the probability that x is *in* the subgroup A. The function α_A is a natural candidate for a fuzzy subgroupoid; however, unfortunately, it may not satisfy Definition 5.1.1. To see this let $\Omega = [0,1]$, $\mathbb{A}$ be the Bore1 subsets of $[0,1]$ and P be Lebesgue measure. Let A be the integers. Define x and y by

$$x(w) = \begin{cases} 1, & w \in [0, \frac{1}{2}] \\ \frac{1}{2}, & w \in (\frac{1}{2}, 1] \end{cases},$$

and

$$y(w) = \begin{cases} \frac{1}{2}, & w \in [0, \frac{1}{3}] \\ 1, & w \in (\frac{1}{3}, 1] \end{cases}.$$

Now x and y are random variables and $x + y$ is defined by

$$(x+y)(w) = \begin{cases} \frac{3}{2}, & w \in [0, \frac{1}{3}] \\ 2, & w \in (\frac{1}{3}, \frac{1}{2}] \\ \frac{3}{2}, & (\frac{1}{2}, 1] \end{cases} .$$

Clearly $P[x^{-1}(A)] = \frac{1}{2}$, $P[x^{-1}(A)] = \frac{2}{3}$ and $P[(x+y)^{-1}(A)] = \frac{1}{6}$. Since $\alpha_A(x+y) < Min(\frac{1}{2}, \frac{2}{3}) = Min(\alpha_A(x), \alpha_A(y))$, the function $\mathbb{A}$ is not a fuzzy subgroupoid of X according to Definition 5.1.1

Example 5.1.2 Let G be a group and let Ω be a family of subgroups of G. For each x in G let$A_x = \{X \in \Omega : x \in X\}$. Let $\mathbb{A}$ be a σ-algebra of subsets of Ω such that $A_x \in \mathbb{A}$ for each x in G and let P be a probability measure on $(\Omega, \mathbb{A})$. Define a function $m_\Omega : G \to [0,1]$ via $m_\Omega(x) = P(A_x)$ for each x in G. The number $m_\Omega(x)$ can be interpreted as the probability that a subgroup chosen at random from the collection Ω will contain x as an element. In this example, it is the *subgroup* of G which is *fuzzy* in that one is uncertain which subgroup of G is under consideration. The function m_Ω is surely a reasonable candidate for a fuzzy subgroup of G, and indeed it is according to Definition 5.1.2 if Ω is linearly ordered by set inclusion (see Proposition 3). However if Ω is not linearly ordered then m_Ω need not satisfy Definition 5.1.2. For example let $(Z, +)$ be the additive group of integers and for each positive integer i, let X_i be the subgroup of integer multiples of i. If $\Omega = \{X_2, X_3, X_5\}$, $\mathbb{A}$ is the power set of Ω and P is defined on $(\Omega, \mathbb{A})$ via $P(X_2) = P(X_3) = P(X_5) = \frac{1}{3}$, then $m_\Omega(6) = \frac{2}{3}, m_\Omega(15) = \frac{2}{3}$, and $m_\Omega(21) = \frac{1}{3}$.. But $m_\Omega(21) < Min(m_\Omega(6), m_\Omega(15))$ so that m_Ω is not even a fuzzy subgroupoid of Z according to Definition 5.1.1.

5.1.1 t-norms

A class of semi-groups on $[0,1]$, called *t-norms* were introduced in (Schweizer and Sklar, 1960) particularly, in order to generalize the ordinary triangle inequality in a

metric space to the more general probabilistic metric spaces.

Definition 5.1.3 *t*-**norm (Schweizer and Sklar, 1960)**

A t-norm is a function $T : [0,1] \times [0,1] \to [0,1]$ satisfying, for each $x, y \in [0,1]$

(i) $T(0,0) = 0$, $T(1,x) = x = T(x,1)$

(ii) $T(x,y) \leq T(x,z)$ if $y \leq z$

(iii) $T(x,y) = T(y,x)$

(iv) $T(x,T(y,z)) = T(T(x,y),z)$

Note that the function Min defined on $[0,1] \times [0,1]$ is a t-norm by the above definition. Other t-norms frequently occurring in the study of probabilistic metric spaces are T_m and *Prod* defined respectively by

$$T_m(x,y) = max(x + y - 1, 0)$$

and

$$T(x,y) = Prod(x,y) = x \cdot y$$

for each $x, y \in [0,1]$..

Anthony and Sherwood (1979) replace Min with T in the Definition 5.1.1 by T viz.,

$$\mu(xy) \geq T(\mu(x), \mu(y))$$

and argue as follows: The condition $T(0,0) = 0$ says that *no information concerning x and y individually being members of the subgroupoid yields no information about xy being in the subgroupoid.* The condition $T(x,1) = x = T(1,x)$ says that *if it is certain that one of x and y, say x, is in the subgroupoid then the amount of information obtained about xy being in the subgroupoid is at least as great as the amount of information possessed concerning y belonging to it.* The condition (ii) of Definition 5.1.3 says that the more information possessed concerning x and y

individually being in the subgroupoid, the more the information is yielded about xy belonging to the subgroupoid. In order to investigate commutative and associative subgroupoids, conditions (iii) and (iv) of Definition 5.1.3 are required. They provide the following definitions

Definition 5.1.4 Fuzzy Subgroupoid (Anthony and Sherwood, 1979)
A function $\mu : X \to [0,1]$ is a fuzzy subgroupoid of X with respect to a t-norm, T, if and only if, for all $x, y \in X$,

$$\mu(xy) \geq T(\mu(x), \mu(y)).$$

Definition 5.1.5 Fuzzy Subgroup (Anthony and Sherwood, 1979)
If X is a group and μ is a fuzzy subgroupoid of X then μ is a fuzzy subgroup of X with respect to a t-norm, T, if and only if, for each $x \in X$

$$\mu(x^{-1}) = \mu(x).$$

Example 5.1.3 Let $X = \{e, a, b, c\}$ be the klein-4 group with the multiplication operation table 5.1 given below.

Table 5.1: Multiplication Table of group X

·	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let the fuzzy group μ be defined by $\mu(e) = 1, \mu(a) = 0.5, \mu(b) = 0.3,$ and $\mu(c) = 0.7$ then the multiplication table the fuzzy group μ is given Table 5.2 below

Table 5.2: Multiplication Table of Fuzzy group μ

$\cdot$	e	a	b	c
e	1	0.5	0.3	0.7
a	0.5	1	0.7	0.3
b	0.3	0.7	1	0.5
c	0.7	0.3	0.5	1

In the context of the aforesaid definitions, the following modifications in the examples 5.1.1 and 5.1.2, are put forward;

Proposition 5.1.1 The functions α_A defined in Example 5.1.1 are all fuzzy subgroups with respect to the t-norm, T_m.

Proposition 5.1.2 The functions m_Ω defined in Example 5.1.2 are all fuzzy subgroups with respect to the t-norm, T_m.

Proposition 5.1.3 The functions m_Ω defined in Example 5.1.2 are also all fuzzy subgroups with respect to the t-norm, Min if Ω is linearly ordered by set inclusion.

Anthony and Sherwood (1979) observed that most of the properties of fuzzy algebra with the t-norm Min which were established in (Rosenfeld, 1971) remain valid using any other t-norm, T. However, some minor variations exist e,g., if μ is a fuzzy subgroupoid of X with respect to Min then for every β in $[0,1], X_\beta = \{y \in X : \mu(y) \geq \beta\}$ is a subgroupoid of X, which does not hold, for a general t-norm, T. In fact, by way of introducing these new concept, most of the results established for fuzzy subgroups with respect to min in (Rosenfeld, 1971) were strengthened to hold for general t-norms, T.

Definition 5.1.6 Sup Property

A fuzzy set μ in X has the *sup property* if for any $H \subset X$ there exists an h_0 in H such that $\mu(h_0) = \sup_{h \in H}(\mu(h))$.

Definition 5.1.7 Image of μ under f

Let μ be a fuzzy set in X and f a function defined on X. Let μ^f be defined on $f(X)$

by $\mu^f(y) = \sup_{x \in f^{-1}(y)} \mu(x)$ for all $y \in f(X)$.). The fuzzy set μ^f in $f(X)$ is called the image of μ under f.

Definition 5.1.8 Continuous t-norm

A t-norm, T, is continuous if it is a continuous function (with respect to the usual topologies) from $[0,1] \times [0,1]$ into $[0,1]$.

Notice that Min, Prod and T_m are all continuous t-norms.

Proposition 5.1.4 If μ is a fuzzy subgroupoid on X with respect to a continuous t-norm, T, and if f is a homomorphism on X then μ^f is a fuzzy subgroupoid on $f(X)$ with respect to T.

Proposition 5.1.5 If μ is a fuzzy subgroup of a group X with respect to a t-norm, T, then $H = \{x \in X : \mu(x) = 1\}$ is either empty or is a subgroup of X.

Proposition 5.1.6 If μ is a fuzzy subgroup of a group X with respect to a t-norm, T, and if there is a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} T(\mu(x_n), \mu(x_n)) = 1$ then $\mu(e) = 1$ where e is the identity in X.

Proposition 5.1.7 Let μ be a fuzzy subgroup of a group X with respect to a t-norm, T. If $\mu(xy^{-1}) = 1$, then $\mu(x) = \mu(y)$.

Proposition 5.1.8 Let μ be a fuzzy set on a group X and let T be a given t-norm. If $\mu(e) = 1$ and $\mu(xy^{-1}) \geq T(\mu(x), \mu(y))$ for all $x, y \in X$, then μ is a fuzzy subgroup of X with respect to T.

Subsequently, *level fuzzy subgroups* and their properties were studied in (Tawfiq and Qa'aed, 2012) which is crisply presented below. They made use of the following concepts defined below.

Definition 5.1.9 Chain of subgroup (Das, 1981)

If $t_1, t_2 \in Im(\mu)$ such that $t_1 \neq t_2$, then obviously, $\mu_{t_1} \neq \mu_{t_2}$. Further, if $Im(\mu) = \{t_i :$

$i = 1, 2, \cdots, n\}$ where $t_1 > t_2, > \cdots > t_n$ then the level subgroups of μ form a chain of subgroups of X. $C(\mu) = \mu_{t_1} \subset \mu_{t_2} \subset \cdots \subset \mu_{t_n} = X$.

Definition 5.1.10 Inverse function (Ajmal, 1994)

Let $f : X \to H$ be a homomorphism of groups, λ be a fuzzy subgroup of X, μ a fuzzy subgroup of H . Then $f^{-1}(\mu) = \mu(f(x), \forall x \in X$.

$$f(\lambda)(y) = \begin{cases} sub\{\lambda(x) : x \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \varnothing \\ 0 & \text{if } f^{-1}(y) = \varnothing \end{cases}$$

and the fuzzy sets $f(\lambda)$ and $f^{-1}(\mu)$ are fuzzy subgroups of H and X respectively.

Definition 5.1.11 Level subsets (Martines, 1995)

If λ is a fuzzy subgroup of X, then the level subsets λ_t in X and the strong level subsets λ_t^* of λ in X , $t \in [0, \lambda(e)]$ are subgroups of X and vice versa

Definition 5.1.12 *Let $f : X \to H$ be a group homomorphism, μ is a fuzzy subgroup of H and denote the chain consisting of inverse images under f of members of $C(\mu)$ by $f^{-1}(C(\mu)) = \lambda$ is a fuzzy subgroup of X.*

Proposition 5.1.9 If μ is a fuzzy subgroup of H and $\{\mu_{t_i} : i \in I\}$ is the collection of all level subgroups of μ, then $\{f^{-1}(\mu_{t_i}) : i \in I\}$ is the collection of all level subgroups of $f^{-1}(\mu)$.

By Considering the example below, it was observed that some of $f^{-1}(\mu_{t_i})$ may be equal so that $C(f^{-1}(\mu_{t_i}))$ will have fewer component than $C(\mu)$.

Example 5.1.4 Let $X = \{1, -1, i, -i\}$ and $H = \{e, (12), (13), (23), (123), (132)\}$. Then X is a group with respect to the usual multiplication of numbers and H is the permutation group of degree three, with e as identity transformation.
Define $f : X \to H$ by $f(x) = e, \forall x \in X$. Then f is a group homomorphism. Define $\mu : H \to [0, 1]$ by :
$\mu(e) = 1,\ \mu((12)) = 0.5,\ \mu(x) = 0.3,\ \forall x \in H\ \{e, (12)\}$.

Then μ is a fuzzy subgroup of H with level subgroups :

$\mu_1 = \{e\}, \mu_{0.5} = \{e, (12)\}, \mu_{0.3} = H$. But $\lambda = f^{-1}(\mu)$ is defined by :

$\lambda(x) = 1$ for every $x \in X$. Hence, $\lambda_1 = \lambda_{0.5} = \lambda_{0.3} = X$.

Now,for the necessary and sufficient condition for the distinctness of all the $f^{-1}(\mu_{t_i})$.

For $t \in Im(\mu)$, define $F_\mu(t) = \{x \in X : \mu(x) = t\}$.

Proposition 5.1.10 Let $f : X \to H$ be a group homomorphism and μ a fuzzy subgroup of H with $Im(\mu) = \{t_i : i \in I\}$ where I is a countable index set.

Then $f^{-1}(\mu_{t_i})$ are all distinct if and only if : $f(X) \cap F_\mu(t_i) \neq \varnothing, \forall i \in I$.

5.1.2 Groupoids and monoids of fuzzy subsets

It is interesting to note that even before (Anthony and Sherwood, 1979), some further algebraic structures of fuzzy subsets were studied. In this section, groupoids and monoids of fuzzy subsets developed in (Kaufmann, 1975) are presented.

Definition 5.1.13 Groupoid of Fuzzy Subsets

Let X be a reference set, finite or infinite and let $\tilde{\wp}(X)$ be the fuzzy power set of X. Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be subsets of X. The structure $(\tilde{\wp}(X), *)$ is said to be a fuzzy groupoid when the operation $*$ is defined to associate a unique fuzzy subset $\tilde{C}$ to each pair of fuzzy subsets $\tilde{A}$ and $\tilde{B}$ of X.

In any finite subset M of $[0, 1]$, a fuzzy subset of X defined by membership function

$$\mu_{\tilde{A}} : X \to M.$$

Now, for X and M finite, the fuzzy power set of X, denoted by $\tilde{\wp}(X)$, can be defined as follows.

Let $X = \{x_1, x_2\}$ and $M = \{0, \frac{1}{2}, 1\}$ the set of possible fuzzy subsets of X will be

given by

$$\tilde{\wp}(X) = \left\{ \begin{array}{l} \{(x_1,0),(x_2,0)\},\{(x_1,0.5),(x_2,0)\},\{(x_1,0.5),(x_2,0)\}, \\ \{(x_1,0.5),(x_2,0.5)\},\{(x_1,0),(x_2,1)\},\{(x_1,1),(x_2,0)\}, \\ \{(x_1,1),(x_2,0.5)\},\{(x_1,0.5),(x_2,1)\},\{(x_1,1),(x_2,1)\} \end{array} \right\}$$

In general if $|X| = n$ and $|M| = m$ then $|\tilde{\wp}(X)| = m^n$

It follows that $|\tilde{\wp}(X)|$ is finite if and only if m and n are finite.

For simplicity, we write for any $\tilde{A} = \{(x_1, \mu_{\tilde{A}}(x_1)), (x_2, \mu_{\tilde{A}}(x_2))\} \subseteq X$ as

$$(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2))$$

for $X = \{x_1, x_2\}$.

Now $\{(x_1, 0.5), (x_2, 0)\}$ will be written as $(0.5, 0)$. With this notation, the Table (5.3) below represents a fuzzy groupoid.

Table 5.3: Cayley table for $\tilde{\wp}(X)$

$\tilde{\wp}(X)$	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	(0,1)	(1,0)	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	(1,1)
(0,0)	(0,1)	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(1,\frac{1}{2})$	(0,1)	(0,1)	$(1,\frac{1}{3})$	(1,1)	$(1,\frac{1}{2})$
$(0,\frac{1}{2})$	(1,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},1)$	(0,0)	(1,0)	(0,1)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(\frac{1}{2},0)$	(0,0)	$(1,\frac{1}{2})$	$(0,\frac{1}{2})$	(0,1)	$(1,\frac{1}{2})$	(1,1)	(1,1)	(1,1)	$(\frac{1}{2},\frac{1}{2})$
$(\frac{1}{2},\frac{1}{2})$	(1,1)	(0,0)	(1,1)	$(\frac{1}{2},1)$	(1,0)	(1,0)	$(\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2})$	(0,0)
(0,1)	$(\frac{1}{2},\ \frac{1}{2})$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	(1,0)	$(\frac{1}{2},0)$	$(1,\frac{1}{2})$	$(\frac{1}{2},1)$
(1,0)	$(\frac{1}{2},0)$	(0,0)	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	(0,1)	(1,0)	(0,1)	(1,1)	$(1,\frac{1}{2})$
$(\frac{1}{2},1)$	(0,0)	$(\frac{1}{2},\frac{1}{2})$	(0,1)	$(0,\frac{1}{2})$	(1,0)	(1,0)	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	(1,1)
$(1,\frac{1}{2})$	(0,0)	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$	(0,0)	$(1,\frac{1}{2})$	$(\frac{1}{2},1)$	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$
(1,1)	$(\frac{1}{2},\frac{1}{2})$	(0,1)	(0,1)	(1,1)	(1,0)	(1,0)	$(\frac{1}{2},1)$	$(0,\frac{1}{2})$	$(1,\frac{1}{2})$

Definition 5.1.14 Monoids of Fuzzy Subsets

Let X be a reference set, finite or infinite and let $\tilde{\wp}(X)$ be the fuzzy power set of X. Let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be subsets of X. The structure $(\tilde{\wp}(X), *)$ is said to be a fuzzy monoid where the operation $*$ is defined to associate a unique fuzzy subset $\tilde{C}$ to each pair of fuzzy subsets $\tilde{A}$ and $\tilde{B}$ of X, and it is associative and has an identity. In the case, $*$ is commutative, it is called a commutative monoid.

Example 5.1.5 Let $X = \{x_1, x_2\}$ and $M = \{0, \frac{1}{2}, 1\}$. Define

$$\tilde{A} * \tilde{B} = \tilde{A} \cap \tilde{B}.$$

That is,

$$\mu_{\tilde{A}*\tilde{B}}(x) = \mu_{\tilde{A}\cap\tilde{B}}(x) = min(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(x).$$

This is a commutative monoid, since $\cap$ is associative and X is the identity. Table (5.4) below describes the structure of the monoid.

Table 5.4: Cayley table for $(\tilde{\wp}(X), \cap)$

$\tilde{\wp}(X)$	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	(0,1)	(1,0)	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	(1,1)
(0,0)	((0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
$(0,\frac{1}{2})$	(0,0)	$(0,\frac{1}{2})$	(0,0)	$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	(0,0)	$(0,\frac{1}{2})$	$(0,\frac{1}{2})$	$(0,\frac{1}{2})$
$(\frac{1}{2},0)$	(0,0)	(0,0)	$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	(0,0)	$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	$(\frac{1}{2},0)$
$(\frac{1}{2},\frac{1}{2})$	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$
(0,1)	(0,0)	$(0,\frac{1}{2})$	(0,0)	$(0,\frac{1}{2})$	(0,1)	(0,0)	(0,1)	$(0,\frac{1}{2})$	(0,1)
(1,0)	(0,0)	(0,0)	$(\frac{1}{2},0)$	$(\frac{1}{2},0)$	(0,0)	(1,0)	$(\frac{1}{2},0)$	(1,0)	(1,0)
$(\frac{1}{2},1)$	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	(0,1)	$(\frac{1}{2},0)$	$(\frac{1}{2},1)$	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},1)$
$(1,\frac{1}{2})$	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2}$	$(0,\frac{1}{2})$	(1,0)	$(\frac{1}{2},\frac{1}{2})$	$(1,\frac{1}{2})$	$(1,\frac{1}{2})$
(1,1)	(0,0)	$(0,\frac{1}{2})$	$(\frac{1}{2},0)$	$(\frac{1}{2},\frac{1}{2})$	(0,1)	(1,0)	$(\frac{1}{2},1)$	$(1,\frac{1}{2})$	(1,1)

It is straightforward to see that the following are commutative fuzzy monoids.

(i) $(\tilde{\wp}(X), \cup)$ where $\mu_{\tilde{A}\cup\tilde{B}}(x) = max(\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)) = \mu_{\tilde{A}}(x) \vee \mu_{\tilde{B}}(x)$

$\varnothing$ is the identity element.

(ii) $(\tilde{\wp}(X), \cdot)$ where $\mu_{\tilde{A}\times\tilde{B}}(x) = \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$

X is the identity element.

(iii) $(\tilde{\wp}(X), \hat{+})$ where $\mu_{\tilde{A}\hat{+}\tilde{B}}(x) = \mu_{\tilde{A}}(x) + \mu_{\tilde{B}}(x) - \mu_{\tilde{A}}(x) \cdot \mu_{\tilde{B}}(x)$

$\varnothing$ is the identity element.

(iv) $(\tilde{\wp}(X), \oplus)$ where $\mu_{\tilde{A}\oplus\tilde{B}}(x) = [\mu_{\tilde{A}}(x) \wedge 1 - \mu_{\tilde{B}}(x)] \vee [\mu_{\tilde{B}}(x) \wedge 1 - \mu_{\tilde{A}}(x)]$

$\varnothing$ is the identity element.

However, not all fuzzy groupoids are monoids.

Example 5.1.6 Let $\tilde{A} * \tilde{B}$ be such that

$$\mu_{\tilde{A}*\tilde{B}}(x) = |\mu_{\tilde{A}}(x) - \mu_{\tilde{B}}(x)|$$

Now, put

$$\mu_{\tilde{A}}(x) = a,\ \mu_{\tilde{B}}(x) = b,\ \mu_{\tilde{C}}(x) = c$$

$$\mu_{\tilde{A}*\tilde{B}} = |a - b|.$$

It can easily be observed that

$$(\tilde{A} * \tilde{B}) * \tilde{C} \neq \tilde{A} * (\tilde{B} * \tilde{C})$$

that is,

$$||a - b| - c| \neq |a - |b - c||.$$

In particular, if

$$a = 0.5,\ b = 0.2,\ c = 0.7$$

$$||a - b| - c| = ||0.5 - 0.2| - 0.7| = |0.3 - 0.7| = 0.4$$

and

$$|0.5 - |0.2 - 0.7|| = |0.5 - 0.5| = 0$$

Thus, it is a commutative groupoid, but not a monoid since it is not associative.

Example 5.1.7 With the notation of Example (5.1.6), let

$$\mu_{\tilde{A}*\tilde{B}} = a + kb - ab, \quad k \in [0,1].$$

we have

$$\begin{aligned}(a * b) * c &= (a + kb - ab) + kc - (a + kb - ab)c \\ &= a + kb + kc - ab - ac - kbc + abc.\end{aligned}$$

$$\begin{aligned}a * (b * c) &= a + k(b + kc - bc) - a(b + kc - bc) \\ &= a + kb + k^2c - ab - kbc - kc - kbc + abc.\end{aligned}$$

$$\begin{aligned}(a * b) * c - a * (b * c) &= kc - k^2c - ac + kac \\ &= c(1 - k)(k - a).\end{aligned}$$

The associativity does not hold, unless $k = 1$.

5.1.3 Fuzzy partition of a given set

In this section, fuzzy partitions of a set, are described and operations on a class of fuzzy partitions are introduced to formulate certain monoidal structures. This is our contribution to the study of algebraic structures of fuzzy sets.

Some notable works on fuzzy partitions have been undertaken by (Zadeh, 1965), (Bezdik and Harris, 1978, 1979), (Shakhatreh, 1994), (Shakhatreh et al., 2006), (Shakhatreh and Hayajneh, 2010), (Morales and M'endez, 2012), etc.

Definition 5.1.15 weak-separated fuzzy subsets

Let $\tilde{F}$ be a collection of fuzzy subset of a nonempty set X. $\tilde{A}, \tilde{B} \in \tilde{F}$ with $\tilde{A} \neq \tilde{B}$. If $\mu_{\tilde{A}\cap\tilde{B}}(x) < 0.5,\ \forall x \in X$, then $\tilde{A}$ and $\tilde{B}$ are called weak-separated fuzzy subsets.

Definition 5.1.16 Fuzzy Partition

Let X be a non-empty set. A fuzzy partition $\tilde{T}$ of X is a set of non-empty fuzzy

subsets of X such that:

(i) If $\tilde{A}, \tilde{B} \in \tilde{T}$ and $\tilde{A} \neq \tilde{B}$, then $(\tilde{A} \cap \tilde{B})(x) < 0.5$

(ii) $\bigcup_{\tilde{w} \in \tilde{T}} \tilde{w} = X$.

Definition 5.1.17 Weakly Empty Fuzzy Subset

Let X be a nonempty set and $\tilde{A}$ a fuzzy subset of X. Then, $\tilde{A}$ is called a weakly empty fuzzy subset of X if $\mu_{\tilde{A}}(x) < 0.5, \ \forall x \in X$.

Definition 5.1.18 Non-weakly empty fuzzy subset

A fuzzy subset of X is called *non-weakly empty fuzzy subset* if it is not weakly empty.

Definition 5.1.19 S-H Fuzzy Collection

Let $\tilde{F}$ be a collection of fuzzy subsets of a nonempty set X. $\tilde{F}$ is called S-H collection if and only if $\tilde{B} \cap \tilde{A} \leq \tilde{B}(a)$, whenever $\tilde{A}, \tilde{B} \in \tilde{F}$ such that $\tilde{A}(a) = 1$.

Example 5.1.8 Let $X = \{x_0, x_1, x_2, x_3, x_4\}$ and $\tilde{F} = \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, \tilde{A}_5\}$ be a collection of fuzzy subsets of X where

$$\tilde{A}_1 = \{(x_0, 1.0), (x_1, 0.9), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\},$$

$$\tilde{A}_2 = \{(x_0, 0.9), (x_1, 1.0), (x_2, 0.8), (x_3, 0.7), (x_4, 0.1)\},$$

$$\tilde{A}_3 = \{(x_0, 0.8), (x_1, 0.8), (x_2, 1.0), (x_3, 0.7), (x_4, 0.1)\},$$

$$\tilde{A}_4 = \{(x_0, 0.7), (x_1, 0.7), (x_2, 0.7), (x_3, 1.0), (x_4, 0.1)\}, \text{and}$$

$$\tilde{A}_5 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 0.1), (x_4, 0.1)\}.$$

It is immediate to see that $\tilde{A}_i \cap \tilde{A}_j(x) \leq \tilde{A}_j(x)$, where $\tilde{A}_i(x) = 1$, $i = \overline{1,4}$ and hence $\tilde{F}$ is S-H collection. Moreover, $\tilde{A}_5$ is a weak empty fuzzy set, and

$$\bigcup \tilde{A}_i = \{(x_0, 1), (x_1, 1), (x_2, 1), (x_3, 1), (x_4, 0.1)\} \neq X.$$

Also $\mu_{\tilde{A}_1 \cap \tilde{A}_2}(x_0) = 0.9 > 0.5$, and similarly for other such combinations.

It is to be noted that S-H collections may contain both weakly empty and non-weakly fuzzy subsets of a given set.

Definition 5.1.20 The S-H Fuzzy Partition

Let X be a nonempty set. By an S-H fuzzy partition $\tilde{T}$ of X, we mean a set of non weakly fuzzy subsets of X such that

(i) If $\tilde{A}, \tilde{B} \in \tilde{T}$ and $\tilde{A} \neq \tilde{B}$, then $(\tilde{A} \cap \tilde{B})(x) < 0.5$, i.e., any two non identical fuzzy subsets are weakly separated fuzzy subsets.

(ii) $\bigcup_{\tilde{w} \in \tilde{T}} \tilde{w} = X$

(iii) $\tilde{T}$ is S-H collection.

5.1.4 Partition of a fuzzy set

Definition 5.1.21 Partition of a Fuzzy Set

A partition in nonempty convex fuzzy subsets of a nonempty fuzzy set $\tilde{A}$ is a collection of fuzzy sets

$$\tilde{A} = \{\tilde{A}_i | i \in I, \tilde{A}_i \subseteq \tilde{A}, \tilde{A}_i \neq \varnothing\},$$

where I is an index set, such that

(i) if $i \neq j$ then $A_i \cap A_j = \varnothing$

(ii) $\bigcup_{i \in I} A_i = A$

(iii) for all $i \in I, A_i$ is a convex fuzzy set.

Theorem 5.1.11 Every nonempty fuzzy set has at least one partition in nonempty convex fuzzy subsets.

5.1.5 Monoids of S-H fuzzy partitions of a set

Let $\tilde{T}$ be an S-H fuzzy partition of a set X and $\tilde{A}_i$ $(i = \overline{1,n})$ be the blocks of the partition $\tilde{T}$. Let $\prod(X)$ denote the collection of all S-H fuzzy partitions of X.

We define a binary operation $*$ on $\prod(X)$ as follows:
Given $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$, let $\tilde{T}_1 * \tilde{T}_2$ be the fuzzy set consisting of all nonempty intersections of every block of $\tilde{T}_1$ with every block of $\tilde{T}_2$, viz.,

$$\tilde{T}_1 * \tilde{T}_2 = \{\{\tilde{A}_i \cap \tilde{B}_j\}, i, j = \overline{1,n}\}$$

where $\tilde{T}_1 = \{\tilde{A}_i\}, \tilde{T}_2 = \{\tilde{B}_j\}, \tilde{X} = \{\tilde{x}_i\}$ and $\tilde{A}_i, \tilde{B}_j, i, j = \overline{1,n}$, are given by $\{(x_i, \mu_{\tilde{A}_i}(x_i))\}$ and $\{(x_i, \mu_{\tilde{B}_j}(x_i))\}, \forall x_i \in X$, respectively.

It is immediate to see that the operation $*$ on $\prod(X)$ is both associative and commutative, since the operation $\cap$ on fuzzy sets is both associative and commutative and also every element $\tilde{T}_k \in \prod(X)$ is idempotent with respect to $*$ i.e., $\tilde{T}_k * \tilde{T}_k = \tilde{T}_k$. Moreover, the identity element with respect to $*$ is the S-H fuzzy partition consisting of a single block. Thus $(\prod(X), *)$ is a commutative, idempotent monoid.

Similarly, if we define a binary operation $\circ$ on $\prod(X)$ such that every resulting fuzzy set consists of all non-empty union of every block of $\tilde{T}_1$ with every block of $\tilde{T}_2$, for all $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$, , then $(\prod(X), \circ)$ also gives rise to a commutative, idempotent monoid with the partition consisting of singleton blocks as the identity element.

Example 5.1.9 Let $X = \{x_0, x_1, x_2, x_3\}$, $\tilde{T}_1, \tilde{T}_2 \in \prod(X)$ such that
$\tilde{T}_1 = \{\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4\}$ and $\tilde{T}_2 = \{\tilde{B}_1, \tilde{B}_2, \tilde{B}_3, \tilde{B}_4\}$, where $\tilde{A}_i, \tilde{B}_i$, $i = \overline{1,4}$, are as given

below:

$$\tilde{A}_1 = \{(x_0, 1.0), (x_1, 0.4), (x_2, 0.3), (x_3, 0.2)\}$$

$$\tilde{A}_2 = \{(x_0, 0.4), (x_1, 1.0), (x_2, 0.3), (x_3, 0.2)\}$$

$$\tilde{A}_3 = \{(x_0, 0.3), (x_1, 0.3), (x_2, 1.0), (x_3, 0.2)\}$$

$$\tilde{A}_4 = \{(x_0, 0.2), (x_1, 0.2), (x_2, 0.2), (x_3, 1.0)\}$$

and

$$\tilde{B}_1 = \{(x_0, 1.0), (x_1, 0.3), (x_2, 0.3), (x_3, 0.1)\}$$

$$\tilde{B}_2 = \{(x_0, 0.3), (x_1, 1.0), (x_2, 0.2), (x_3, 0.1)\}$$

$$\tilde{B}_3 = \{(x_0, 0.3), (x_1, 0.2), (x_2, 1.0), (x_3, 0.1)\}$$

$$\tilde{B}_4 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 1.0)\}$$

It is easy to verify that $\tilde{T}_1$ and $\tilde{T}_2$ are S-H fuzzy partitions of X.

Note that, $\tilde{A}_i \cap \tilde{B}_j$, for $i \neq j$ is weakly empty, for example,

$$\tilde{A}_1 \cap \tilde{B}_2 = \{(x_0, 0.3), (x_1, 0.4), (x_2, 0.2), (x_3, 0.1)\}.$$

We have, following the definition given above, for example,

$$\tilde{T}_3 = \tilde{T}_1 * \tilde{T}_2 = \{\tilde{A}_1 \cap \tilde{B}_1, \tilde{A}_2 \cap \tilde{B}_2, \tilde{A}_3 \cap \tilde{B}_3, \tilde{A}_4 \cap \tilde{B}_4\} = \{\tilde{C}_1, \tilde{C}_2, \tilde{C}_3, \tilde{C}_4\}$$

where

$$\tilde{C}_1 = \{(x_0, 1.0), (x_1, 0.3), (x_2, 0.3), (x_3, 0.1)\},$$

$$\tilde{C}_2 = \{(x_0, 0.3), (x_1, 1.0), (x_2, 0.2), (x_3, 0.1)\},$$

$$\tilde{C}_3 = \{(x_0, 0.3), (x_1, 0.2), (x_2, 1.0), (x_3, 0.1)\}, \text{and}$$

$$\tilde{C}_4 = \{(x_0, 0.1), (x_1, 0.1), (x_2, 0.1), (x_3, 1.0)\}.$$

Also, $\tilde{T}_1 * \tilde{T}_1 = \{\tilde{A}_1 \cap \tilde{A}_1, \tilde{A}_2 \cap \tilde{A}_2, \tilde{A}_3 \cap \tilde{A}_3, \tilde{A}_4 \cap \tilde{A}_4\} = \tilde{T}_1$; $\tilde{T}_1 * \tilde{T}_2 = \tilde{T}_2 * \tilde{T}_1$, etc. Similarly,

results for various other combinations could be computed.

Thus $(\prod(X), *)$ is a commutative, idempotent monoid with $\tilde{X} = \{\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{x}_3\}$, as the identity element.

It is intresting to see that another operation, denoted $\oplus$, can be defined on $\prod(X)$ such that $(\prod(X), \oplus)$ is also a monoid. We define $\oplus$ as follows: Let X be a set and $\tilde{T}_1 = \{\tilde{A}_1\ \tilde{A}_2, \tilde{A}_3\ \tilde{A}_4\}$ and $\tilde{T}_2 = \{\tilde{B}_1\ \tilde{B}_2, \tilde{B}_3\ \tilde{B}_4\}$ be two S-H fuzzy partitions of X. A fuzzy subset $\tilde{T}$ of X belongs to $\tilde{T}_1 \oplus \tilde{T}_2$ if

i) $\tilde{T}$ is the union of one or more elements of $\tilde{T}_1$,

ii) $\tilde{T}$ is the union of one or more elements of $\tilde{T}_2$,

iii) No fuzzy subset of $\tilde{T}$ satisfies i) and ii) except $\tilde{T}$ itself.

It follows that $\oplus$ is both associative and commutative. The fuzzy partition consisting of single elements of X is the identity of the operation $\oplus$ on $\prod(X)$. For example, $\{\tilde{x_0},\ \tilde{x_1},\ \tilde{x_2},\ \tilde{x_3}\}$ is the identity with respect to $\oplus$ on $\prod(X)$ in the example considered above. On the same lines, the algebraic structure defined by the other operation described above, could be illustrated.

5.2 Partitions and Monoids of Partitions of Multi-sets

Definition 5.2.1 The mset of R -relatives

Let R be an mset relation on A and let $m/x \in A$. The mset of R-relatives (or R-msets) of an element $m/x \in A$, denoted by $R(m/x)$, is the mset of all A such that there exists some k satisfying $k/xRn/y$.

For example, let $A = [5/x, 7/y, 9/z]$ and
$R = \{(2/x, 3/y)/6, (4/x, 3/z)/12, (4/y, 5/z)/20, (6/y, 5/x)/30, (4/z, 4/z)/16,$
$(5/z, 3/x)/15, (3/z, 6/y)/18\}$ be an mset relation on A.
Then the $domR = [4/x, 6/y, 5/z]$ and $ranR = [5/x, 6/y, 5/z]$. In this case,
$R(2/x) = R(4/x) = [3/y, 3/z], R(4/y) = R(6/y) = [5/z, 5/x]$, and
$R(3/z) = R(4/z) = R(5/z) = [3/x, 6/y, 4/z]$. The aforesaid definition can be suitably extended to defining the mset of all R- msets of a subset of a given mset as follows: Let $A_1 \subseteq A$. Then $R(A_1)$, the R- relative mset of A_1, is the mset containing all those $n/y \in A$ which are R- relative to some $R(A_1)$. It follows that $R(A_1)$ is the union of all $R(m/x)$ where $m/x \in A_1$. In fact, $R(A_1 \subseteq ranR$ and equality holds if and only if both A and A_1 have the same root set. It is important to note here that for any msubset A_k of A, the mset $R(A_k)$ is defined with respect to the relation R on A. For example, let A be as above and
$A_1 = [2/x, 4/y, 5/z] \subseteq A$, where $A_1^* = A^*$ then
$R(A_1) = [5/x, 6/y, 5/z]$ = The union of all $R(m/x)$ for $m/x \in A_1 = R(2/x) \cup R(4/y) \cup$
$R(5/z) = R(A) = ranR$.
However, if the root set of an msubset A_k of A is a proper subset of the root set of A then $R(A_k) \subset ranR$, but $R(A_k) \neq ranR$. For example,
let $A_2 = [4/x, 6/y] \subseteq A$. Then
$R(A_2) = R(4/x) \cup R(6/y) = [3/y, 3/z] \cup [5/x, 5/z] = [5/x, 3/y, 5/z] \subset ranR$, but $[5/x, 3/y, 5/z] \neq ranR$,

Definition 5.2.2 Multiset Partitions

$\prod = \{A_i, : i = \overline{i,k}\}$ is a partition of a nonempty mset A if and only if $A = \sum_{i=1}^{k} A_i$, where $\left(\sum_{i=1}^{k} A_i\right)(x) = \sum_{i=1}^{k} A_i(x)$.

In other words a partition $\prod$ of a nonempty mset A is the collection of nonempty msubsets of A such that each element of A belongs to one of these elements of $\prod$ and $A_i \cap A_j = \varnothing$ for distinct elements A_i and A_j of $\prod$. Equivalently, a partition $\prod$ of a nonempty mset A can be defined as a collection of disjoint whole msubsets whose union is A. The elements of $\prod$ are called the *cells* or *blocks* of $\prod$. For example, let $A = [5/x, 7/y, 9/z]$. Then $\prod = [5/x, 7/y], [9/z]$ is a partition of A and the cells of $\prod$ are $[5/x, 7/y]$ and $[9/z]$.

It is easy to see that every partition of an mset A gives rise to an equivalence relation on A. For example, the partition $\prod$ in the aforesaid example determines the equivalence mset relation

$$R = (5/x, 5/x), (5/x, 7/y), (7/y, 5/x), (7/y, 7/y), (9/z, 9/z).$$

Remark: An alternative formulation of an mset partition $\prod$ induced by an equivalence relation R on an mset A can be given as follows: Using the concept of an mset R - relatives, it is immediate to see that for every equivalent mset relation R on an mset A , such that $m/x \in A$ and $n/y \in A$ then

$m/xRn/y$ if and only if $R(m/x) = R(n/y)$ and hence $\prod = R(m/x) : m/x \in A$; that is, $\prod$ turns out to be a collection of all distinct R - relative msets that are represented by elements of A. In the example taken above, the blocks$[5/x, 7/y]$ and $[9/z]$ of the partition $\prod$ are respectively represented as $R(5/x) = R(7/y) = [5/x, 7/y]$, and $R(9/z) = [9/z]$. Note that the aforesaid representation of the block of an mset partition is not unique.

5.2.1 Monoids of partitions of a multiset

Let A be a cardinality-bounded nonempty mset and $\prod(A)$ be the collection of all partitions of A . In order to avoid nesting of brackets, we shall represent an mset block $[P_k]$ as $\overline{P_k}$. We define a binary operation $*$ on $\prod(A)$ as follows: Let $P = \{\overline{P_1}, \overline{P_2}, \cdots, \overline{P_k}\}$ and $Q = \{\overline{Q_1}, \overline{Q_2}, \cdots, \overline{Q_k}\}$ be two partitions of A . Let$P * Q$ be defined as the mset of nonempty intersections of every element of P with every elements of Q. It is easy to see that the operation $*$ is both associative and commutative and the partition consisting of a unique single block constituted by A itself is the "identity" of the operation $*$.. Thus, $(\prod(A), *)$ or $(\prod(A, *, \{\overline{A}\})$ is a monoid. It may be observed that every element $T \in \prod(A)$ is *idempotent* with respect to the operation $*$, since $T * T = T$ holds. Following the nomenclature of monoid of sets, this operation $*$ on $\prod(A)$ can be called the "product of partitions". For example, let $A = [5/x, 7/y, 9/z]$ be an mset and $P = \{\overline{5/x, 7/y}, \overline{9/z}\}$ and $Q = \{\overline{5/x, 9/z}, \overline{7/y}\}$ be two partitions of A. Then $P * Q = \{\overline{5/x}, \overline{7/y}, \overline{9/z}\}$. Here $\prod(A) = \{\{\overline{5/x}, \overline{7/y}, \overline{9/z}\}, \{\overline{5/x, 7/y}, \overline{9/z}\}, \{\overline{5/x}, \overline{7/y, 9/z}\}, \{\overline{5/x, 9/z}, \overline{7/y}\}, \{\overline{5/x, 7/y, 9/z}\}\}$.

Clearly, $(\prod(A), *)$, $*$ is monoid and $\{\overline{5/x}, \overline{7/y}, \overline{9/z}\}$ is the identity element.
It is interesting to see that another operation, denoted by $\oplus$, can be defined on $\prod(A)$ such that $(\prod(A), \oplus)$ is also a monoid. We define $\oplus$ as follows: Let A be an mset and $P = \{\overline{P_1}, \overline{P_2}, \cdots, \overline{P_k}\}$ and $Q = \{\overline{Q_1}, \overline{Q_2}, \cdots, \overline{Q_k}\}$ be two partitions of A. A subset T of A belongs to $P \oplus Q$ if

(i) T is the union of one or more elements of P,

(ii) T is the union of one or more elements of Q,

(iii) No subsets of T satisfies i) and ii) except T itself.

It follows that $\oplus$ is both associative and commutative. The partition consisting of single element of A is the identity of the operation $\oplus$ on $\prod(A)$. For example, $\{\overline{5/x}, \overline{7/y}, \overline{9/z}\}$ is the identity with respect to $\oplus$ on $\prod(A)$ in the example taken above.

5.3 Multigroups and Fuzzy Multigroups

5.3.1 Multigroups

Let X be a group and e be the identity element of X. Let multisets (or msets, for short) be taken from $[X]^n$ and $MG(X)$ denote the set of all multigroups (or mgroups, for short) over a group X. In the following, the entire description, except some examples, related to multigroups are abstracted from (Nazmul et al., 2013).

Definition 5.3.1 $A \circ B$ **and** A^{-1}

Let $A, B \in [X]^n$ Then we define $A \circ B$ and A^{-1} as follows:

$C_{A\circ B}(x) = \vee\{C_A(y) \wedge C_B(z); y, z \in X \text{ and } yz = x\}$ and $C_{A^{-1}}(x) = C_A(x^{-1})$

Proposition 5.3.1 If $A, B, C, A_i \in [X]^n$, then the following assertions hold:

(i) $C_{A\circ B}(x) = \bigvee_{y\in X}[C_A(y) \wedge C_B(y^{-1}x] = \bigvee_{y\in X}[C_A(xy^{-1}) \wedge C_B(y)], \forall x \in X$;

(ii) $[A^{-1}]^{-1} = A$;

(iii) $A \subseteq B \Rightarrow A^{-1} \subseteq B^{-1}$;

(iv) $[\bigcup_{i\in I} A_i]^{-1} = \bigcup_{i\in I}[A^{-1}]$;

(v) $[\bigcap_{i\in I} A_i]^{-1} = \bigcap_{i\in I}[A^{-1}]$;

(vi) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$;

(vii) $(A \circ B) \circ C = A \circ (B \circ C)$.

Definition 5.3.2 Multigroup

Let X be a group. A multiset G over X is said to be a multigroup over X if the Count function G or C_G satisfies the following two conditions.

(i) $C_G(xy) \geq [C_G(x) \wedge C_G(y)], \forall x, y \in X$

(ii) $C_G(x^{-1}) = C_G(x), \forall x \in X$

The set of all multigroups over X is denoted by $MG(X)$.

Example 5.3.1 Let $X = \{e, x, y, z\}$ be Klein's four-group and

$$G = \{e, e, e, e, x, x, x, y, y, y, y, z, z, z\}$$

be a multiset over X. Now

$$C_G(ex) = C_G(x) = 3 \geq [C_G(e) \wedge C_G(x)],\ C_G(ey) = C_G(y) = 4 \geq [C_G(e) \wedge C_G(y)],$$

$$C_G(ez) = C_G(x) = 3 \geq [C_G(e) \wedge C_G(z)],\ C_G(xy) = C_G(z) = 3 \geq [C_G(x) \wedge C_G(y)],$$

$$C_G(yz) = C_G(x) = 3 \geq [C_G(y) \wedge C_G(z)],\ C_G(xz) = C_G(y) = 4 \geq [C_G(x) \wedge C_G(z)],$$

$$C_G(x^2) = C_G(e) = 4 \geq [C_G(x) \wedge C_G(x)],\ C_G(y^2) = C_G(e) = 4 \geq [C_G(y) \wedge C_G(y)],$$

$$C_G(z^2) = C_G(e) = 4 \geq [C_G(z) \wedge C_G(z)],\ C_G(e^2) = C_G(e) = 4 \geq [C_G(e) \wedge C_G(e)] \text{and}$$

$C_G(x^{-1}) = C_G(x) = 3.\ C_G(y^{-1}) = C_G(y) = 4.\ C_G(z^{-1}) = C_G(z) = 3.\ C_G(e^{-1}) = C_G(e) = 4.$

Therefore G is a multigroup over X.

Proposition 5.3.2 Let $A \in MG(X)$. Then

(i) $C_A(e) \geq C_A(x), \forall x \in X$;

(ii) $C_A(x^n) \geq C_A(x), \forall x \in X$;

(iii) $C_A(x^{-1}) = C_A(x), \forall x \in X$;

(iv) $A = A^{-1}$.

Proposition 5.3.3 Let A be a mset. Then $A \in MG(X)$ iff
$C_A(xy^{-1}) \geq C_A(x) \wedge C_A(y), \forall x, y \in X$.

Definition 5.3.3 Let $A \in MG(X)$. Then define $A_n = \{x \in X; C_A(x) \geq n, n \in \mathbb{N}\}$.

Proposition 5.3.4 Let $A \in MG(X)$. Then $A_n, n \in \mathbb{N}$ are subgroups of X.

Proposition 5.3.5 If $A_n, n \in \mathbb{N}$ are subgroups of X, then the mset A such that $C_A(x) = \sum_{n\in\mathbb{N}} \chi_{A_n}(x)$ where χ_{A_n} is the characteristic function of A_n is an mgroup over X.

Definition 5.3.4 Let $A \in MG(X)$. Then define $A^* = \{x \in X : C_A(x) = C_A(e)\}$ and $A_* = \{x \in X : C_A(x) > 0\}$.

Proposition 5.3.6 Let $A \in MG(X)$. Then A^* and A_* are subgroups of X.

Proposition 5.3.7 Let $A \in MS(X)$. Then $A \in MG(X)$ iff A satisfies the following conditions:

(a) (i) $A \circ A \subseteq A$

(ii) $A^{-1} \subseteq A$ or $; A \subseteq A^{-1}$ or $A = A^{-1}$

or

(b) $A \circ A^{-1} \subseteq A$

Proposition 5.3.8 Let $A \in MG(X)$. Then A^* and A_* are subgroups of X.

Proposition 5.3.9 Let $A, B \in MG(X)$. Then

(i) $A \circ B \in MG(X)$ iff $A \circ B = B \circ A$.

(ii) $A \cap B \in MG(X)$.

In general let $\{A_i : i \in I\}$ is a family of multigroups over a group X, then their intersection $\bigcap_{i\in I} A_i$ is a multigroup over X. But the union of two multigroups A and B over X is not a Multigroup over X in general as in the example below.

Example 5.3.2 Let $X = \{e, x, y, z\}$ be the Klein's four-group, $A = \{e, e, e, y\}$, and $B = \{e, e, e, z, z\}$. Then $A, B \in MG(X)$.

Clearly, $A \cup B = \{e, e, e, y, z, z\}$ and $C_{A\cup B}(x) = C_{A\cup B}(yz) = 0 \ngeq C_{A\cup B}(y) \wedge C_{A\cup B}(z) = 1$. Therefore, $A \cup B$ is not a multigroup over X

Definition 5.3.5 Let$A$ and B be two multigroups over a group X: Then A is said to be a submultigroup of B if $A \subseteq B$

Example 5.3.3 Let $X = \{e, x, y, z\}$ be the Klein's four-group, $A = \{e, e, e, x, x, y, y, z, z\}$, and $B = \{e, e, e, x, x, y, y, z, z, z\}$. Then $A, B \in MG(X)$ and $A \subseteq B$. Therefore, A is a multisubgroup of B.

Proposition 5.3.10 Let X, Y be two groups and $f : X \to Y$ be a homomorphism.

(i) If $A \in MG(X)$, then $f(A) \in MG(Y)$.

(ii) If $B \in MG(Y)$, then $f^{-1}(B) \in MG(X)$.

Proposition 5.3.11 Let $A \in MS(X)$. Then $\forall x, y \in X$ the following assertions are equivalent:

(i) $C_A(xy) = C_A(yx)$,

(ii) $C_A(xyx^{-1}) = C_A(y)$,

(iii) $C_A(xyx^{-1}) \geq C_A(y)$,

(iv) $C_A(xyx^{-1}) \leq C_A(y)$,

Proof.(i) $\Rightarrow$ (ii) : Let $x, y \in X$. Then $C_A(xyx^{-1}) = C_A(x^{-1}xy) = C_A(y)$.
(ii) $\Rightarrow$ (iii) : $C_A(xyx^{-1}) = C_A(y) = C_A(ey) \geq C_A(e) \wedge C_A(y) = C_A(y)$.
(iii) $\Rightarrow$ (iv) : $C_A(xyx^{-1}) \leq C_A(x^{-1}[xyx^{-1}](x^{-1})^{-1}) = C_A(y)$.
(iv) $\Rightarrow$ (i) : Let $x, y \in X$. Then
$C_A(xy) = C_A(x[yx]x^{-1}) \leq C_A(yx) = C_A(y[xy]y^{-1}) \leq C_A(xy)$.
Hence $C_A(xy) = C_A(yx)$. Thus the above assertions are equivalent. □

Proposition 5.3.12 Let $A \in MS(X)$. Then the following assertions are equivalent:

(i) $C_A(xy) = C_A(yx), \forall x, y \in X$,

(ii) $A \circ B = B \circ A, \forall B \in MS(X)$.

Proof.(i) $\Rightarrow$ (ii) : Let $x \in X$, Then

$$\begin{aligned} C_{A\circ B}(x) &= \bigvee_{y\in X}[C_A(xy^{-1}) \wedge C_B(y)] \\ &= \bigvee_{y\in X}[C_B(y) \wedge C_A(y^{-1}x)] \\ &= C_{B\circ A}(x). \end{aligned}$$

Hence $A \circ B = B \circ A$.

(ii) $\Rightarrow$ (i) : Let $x, y \in X, [C_A(xy) \vee C_A(yx)] = n$ and $T = n_{y^{-1}}$. Then $T \in MS(X)$. Since $A \circ B = B \circ A, \forall B \in MS(X)$, it follows that $A \circ T = T \circ A$ and hence $C_{A\circ T}(x) = C_{T\circ A}(x), \forall x \in X$. Now $C_{A\circ T}(x) = \vee_{uv=x}\{C_A(u) \wedge C_T(v)\} = C_A(xy) \wedge C_T(y^{-1}) = C_A(xy)$.

Similarly we have $C_{T\circ A}(x) = C_A(yx)$. Hence $C_A(xy) = C_A(yx), \forall x, y \in X$. □

Definition 5.3.6 Abelian mgroup

An mgroup G over X is called abelian over X if $C_G(xy) = C_G(yx), \forall x, y \in X$. Let $AMG(X)$ denote the set of all abelian mgroups over X.

Example 5.3.4 Let X be an abelian group and G be an mgroup over X. Then G is an abelian mgroup over X.

Proposition 5.3.13 Let $A \in AMG(X)$. Then A^*, A_* and $A_n, n \in \mathbb{N}$ are normal subgroups of X.

Proof. By Proposition 5.2.6. and 5.2.5., A^*, A_* and $A_n, n \in \mathbb{N}$ are subgroups of X. Let $x \in X$ and $y \in A_*$ Then $C_A(y) > 0$ and by Proposition 5.2.11, $C_A(xyx^{-1}) = C_A(y) > 0$ Thus $xyx^{-1} \in A_*$ and hence A_* is a normal subgroup of X.

Similarly A^* is also a normal subgroup of X.

Let $x \in X$ and $y \in A_n$. Then $C_A(y) \geq n$ and by Proposition 5.2.11, $C_A(xyx^{-1}) = C_A(y) \geq n$. Thus $xyx^{-1} \in A_n$ and hence $A_n, n \in \mathbb{N}$ are normal subgroups of X. □

5.3.2 Fuzzy multigroups

Throughout this section, let X be a group, $FM(X)$(Finite multiset) the class of all fuzzy mset over X, and $FMG(X)$ the set of all fuzzy multigroups (FMG) over the group X.

Definition 5.3.7 A^{-1} **(Shinoj et al., 2015)**

Let $A \in FM(X)$. Then A^{-1} is defined as $CM_{A^{-1}}(x) = CM_A(x^{-1})$.

Definition 5.3.8 $A \circ B$

Let $A, B \in FM(X)$. Then $A \circ B$ is defined by

$$CM_{A\circ B}(x) = \vee\{CM_A(y) \wedge CM_B(z); y, z \in X \text{ and } yz = x\}.$$

Proposition 5.3.14 If $A, B, C, A_i \in FM(X)$, then the following assertions hold:

(i) $[A^{-1}]^{-1} = A$;

(ii) $A \subseteq B \Rightarrow A^{-1} \subseteq B^{-1}$;

(iii) $[\bigcup_{i=1}^{n} A_i]^{-1} = \bigcup_{i=1}^{n}[A^{-1}]$;

(iv) $[\bigcap_{i=1}^{n} A_i]^{-1} = \bigcap_{i=1}^{n}[A^{-1}]$;

(v) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$;

(vi) $CM_{A\circ B}(x) = \bigvee_{y\in X}[CM_A(y) \wedge CM_B(y^{-1}x]$
$= \bigvee_{y\in X}[CM_A(xy^{-1}) \wedge CM_B(y)], \forall x \in X$;

Definition 5.3.9 Fuzzy Multigroup (Shinoj et al., 2015)

Let X be a group. A fuzzy multiset G over X is a fuzzy multigroup (FMG) over X if the count (count membership) of G satisfies the following two conditions.

(i) $CM_G(xy) \geq CM_G(x) \wedge CM_G(y)$ $\forall$ $x, y \in X$.

(ii) $CM_G(x^{-1}) = CM_G(x)\ \forall x \in X$.

Example 5.3.5 *If* $(Z_6, +_6)$ *is a group then,*

$$A = \left\{ \begin{array}{l} (0.8, 0.7, 0.5, 0.5, 0.1)/3, (0.5, 0.5, 0.4, 0.2)/1, (0.5, 0.5, 0.4, 0.2)/5, \\ (0.8, 0.6, 0.6, 0.5)/2, (0.8, 0.6, 0.6, 0.5)/4, (0.9, 0.8, 0.7, 0.7, 0.5)/0 \end{array} \right\}$$

is a fuzzy multigroup. But

$$B = \left\{ \begin{array}{l} (0.8, 0.7, 0.5, 0.5, 0.1)/3, (0.5, 0.4, 0.4, 0.2)/1, (0.7, 0.4, 0.1)/5, \\ (0.8, 0.6, 0.6, 0.6)/2, (0.7, 0.5, 0.5, 0.1)/4, (0.7, 0.5, 0.5, 0.5)/0 \end{array} \right\}$$

is not a fuzzy multigroup since $CM_B(1) \neq CM_B(5)$.

From the definition and the above example, it is clear that FMG is a generalized case of fuzzy group. Also it is a general case of multigroup (when $\mu_G^i(x) = 1 \forall x \in X, i = 1, 2, \cdots, L(x)$).

Proposition 5.3.15 Let $A \in FM(X)$ and $CM_A(x^{-1}) \geq CM_A(x)$. Then $CM_A(x^{-1}) = CM_A(x)$.

Proof.

$$CM_A(x^{-1}) \geq CM_A(x) (\text{given}).$$

$$\text{Now } CM_A(x) = CM_A((x^{-1})^{-1}) \geq CM_A(x^{-1})$$

$$\text{Then } CM_A(x^{-1}) = CM_A(x).$$

□

Proposition 5.3.16 Let $A \in FMG(X)$. Then

(i) $CM_A(e) \geq CM_A(x) \forall x \in X$

(ii) $CM_A(x^n) \geq CM_A(x) \forall x \in X$

(iii) $A^{-1} = A$

Proposition 5.3.17 Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $CM_A(xy^{-1}) \geq CM_A(x) \wedge CM_A(y) \forall x, y \in X$.

Definition 5.3.10 n-α level set of A

Let $A \in FM(X)$. Then $A[\alpha, n] = \{x \in X : \mu_A^j(x) \geq \alpha; L(x) \geq j \geq n \text{ and } j, n \in \mathbb{N}\}$. This is called $n - \alpha$ level set of A.

Proposition 5.3.18 Let $A \in FMG(X)$. Then $A[\alpha, n]$ are subgroups of X.

Definition 5.3.11 A^*

Let $A \in FMG(X)$. Then define $A^* = \{x \in X : CM_A(x) = CM_A(e)\}$.

Proposition 5.3.19 Let $A \in FMG(X)$. Then A^* a subgroups of X.

Definition 5.3.12 A^j

Let $A \in FM(X)$. Let $j \in N$. Then define $A^j = \{x \in X : \mu_A^j(x) > 0 \text{ and } \mu_A^{j+1}(x) = 0\}$.

Proposition 5.3.20 Let $A \in FMG(X)$. Then A^j is a subgroup of X iff $\mu_A^{j+1}(xy^{-1}) = 0, \forall x, y \in A^j$.

Proposition 5.3.21 Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A \subseteq A$ and $A^{-1} = A$.

Corollary 5.3.22 Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A = A$ and $A^{-1} = A$.

Proposition 5.3.23 Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A^{-1} \subseteq A$.

Corollary 5.3.24 Let $A \in FM(X)$. Then $A \in FMG(X)$ iff $A \circ A^{-1} = A$.

Proposition 5.3.25 Let $A, B \in FMG(X)$. Then $A \cap B \in FMG(X)$.

In general if $\{A_i : i \in I\}$ is a family of FMG over a group X, then their intersection $\bigcap_{i \in I} A_i$ is a FMG over X. But the union of two FMG, A and B over X is not a FMG over X in general as in the example below.

Example 5.3.6 Let $X = \{e, x, y, z\}$ be Klein's four-group. Then

$A = \{(0.6, 0.4, 0.3, 0.1)/x, (0.9, 0.8, 0.7, 0.5, 0.1, 0.1)/e\}$

and $B = \{(0.8, 0.8, 0.5, 0.5, 0.1, 0.1)/y, (0.9, 0.8, 0.7, 0.5, 0.1, 0.1)/e\}$ are fuzzy multi groups.

$A \cup B = \{(0.6, 0.4, 0.3, 0.1)/x, (0.8, 0.8, 0.5, 0.5, 0.1, 0.1)/y, (0.9, 0.8, 0.7, 0.5, 0.1, 0.1)/e\}$.

But $CM_{A\cup B}(z) < CM_{A\cup B}(x) \wedge CM_{A\cup B}(y)$ as $xy = z$ in Klein's four-group. Then

$A \cup B \notin FMG(X)$.

Definition 5.3.13 Sub-fuzzy multigroup

Let $A, B \in FMG(X)$. Then A is said to be a sub-fuzzy multi group of B if $A \subseteq B$.

Example 5.3.7 $(Z_6, +_6)$ is a group then,

$$A = \left\{ \begin{array}{l} (0.8, 0.7, 0.5, 0.5, 0.1)/3, (0.5, 0.5, 0.4, 0.2)/1, (0.5, 0.5, 0.4, 0.2)/5, \\ (0.8, 0.6, 0.6, 0.5)/2, (0.8, 0.6, 0.6, 0.5)/4, (0.9, 0.8, 0.7, 0.7, 0.5)/0 \end{array} \right\}$$

is a fuzzy multigroup. And

$$B = \left\{ \begin{array}{l} (0.7, 0.7, 0.5, 0.5, 0.1)/3, (0.5, 0.5, 0.4, 0.2)/1, (0.5, 0.5, 0.4, 0.2)/5, \\ (0.8, 0.5, 0.6, 0.5)/2, (0.8, 0.5, 0.6, 0.5)/4, (0.9, 0.8, 0.7, 0.7, 0.5)/0 \end{array} \right\}$$

is a sub-fuzzy multigroup of A.

Definition 5.3.14 Fuzzy Multi-subgroup generated by A

Let $A \in FM(X)$. Then $\langle A \rangle = \{\wedge A_i : A \subseteq A_i \in FMG(X)\}$ is called the fuzzy multi-subgroup of X generated by A.

That is, $\langle A \rangle$ is the smallest fuzzy multigroup containing A.

5.3.3 Abelian fuzzy multigroups

In this section, drawing from above, the following algebraic structure is introduced.

Proposition 5.3.26 Let $A \in FM(X)$. Then $\forall x, y \in X$ the following assertions are equivalent:

(i) $CM_A(xy) = CM_A(yx)$,

(ii) $CM_A(xyx^{-1}) = CM_A(y)$,

(iii) $CM_A(xyx^{-1}) \geq CM_A(y)$,

(iv) $CM_A(xyx^{-1}) \leq CM_A(y)$,

Proof.(i) $\Rightarrow$ (ii) : Let $x, y \in X$. Then $CM_A(xyx^{-1}) = CM_A(x^{-1}xy) = CM_A(y)$.
(ii) $\Rightarrow$ (iii) : $CM_A(xyx^{-1}) = CM_A(y) = CM_A(ey) \geq [CM_A(e) \wedge CM_A(y) = CM_A(y)$.
(iii) $\Rightarrow$ (iv) : $CM_A(xyx^{-1}) \leq CM_A(x^{-1}[xyx^{-1}](x^{-1})^{-1}) = CM_A(y)$.
(iv) $\Rightarrow$ (i) : Let $x, y \in X$. Then
$CM_A(xy) = CM_A(x[yx]x^{-1}) \leq CM_A(yx) = CM_A(y[xy]y^{-1}) \leq CM_A(xy)$.
Hence $MC_A(xy) = CM_A(yx)$. Thus the above assertions are equivalent. □

Proposition 5.3.27 Let $A \in FM(X)$. Then the following assertions are equivalent:

(i) $CM_A(xy) = CM_A(yx), \forall x, y \in X$,

(ii) $A \circ B = B \circ A, \forall B \in FM(X)$.

Proof.(i) $\Rightarrow$ (ii) : Let $x \in X$, Then

$$\begin{aligned} CM_{A\circ B}(x) &= \bigvee_{y\in X}[CM_A(xy^{-1}) \wedge CM_B(y)] \\ &= \bigvee_{y\in X}[CM_B(y) \wedge CM_A(y^{-1}x)] \\ &= CM_{B\circ A}(x). \end{aligned}$$

Hence $A \circ B = B \circ A$. □

Definition 5.3.15 Abelian Fuzzy Multigroup

A Fuzzy multigroup G over X is called abelian over X if $CM_G(xy) = CM_G(yx), \forall x, y \in X$. Let $AFMG(X)$ denote the set of all abelian fuzzy multigroups over X.

Example 5.3.8 Let X be an abelian group and G be an FMG over X. Then G is an abelian FMG over X.

Proposition 5.3.28 Let $A \in AFMG(X)$. Then A^*, and $A[\alpha, n], n \in \mathbb{N}$ are normal subgroups of X.

Proof. By proposition 5.3.19 and 5.3.18, A^* and $A[\alpha, n], n \in \mathbb{N}$ are subgroups of X. Let $x \in X$ and $y \in A^*$. Then $CM_A(y) = CM_A(e)$. and by Proposition 5.3.26, $CM_A(xyx^{-1}) = CM_A(y) = CM_A(e)$. Thus $xyx^{-1} \in A^*$ and hence A^* is a normal subgroup of X.

Let $x \in X$ and $y \in A[\alpha, n], n \in \mathbb{N}$. Then $CM_A(y) \geq \alpha; L(x) \geq j \geq n$ and $j, n \in \mathbb{N}$ and by Proposition 5.3.26, $CM_A(xyx^{-1}) = CM_A(y) \geq \alpha; L(x) \geq j \geq n$ and $j, n \in \mathbb{N}$. Thus $xyx^{-1} \in A[\alpha, n], n \in \mathbb{N}$ and hence $A[\alpha, n], n \in \mathbb{N}$ are a normal subgroups of X. □

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATIONS

6.1 Summary

Besides chapter one, a comprehensive and critical Literature survey was presented in chapter two. In Chapter three, fundamentals of fuzzy sets, multisets, fuzzy multisets were presented from a specific perspective to provide means to model a large class of real-life problems which intrinsically involve vagueness and uncertainty. In chapter four, alpha-cuts, inverse alpha-cuts in fuzzy sets and fuzzy multisets were presented, and some of their properties were extended from fuzzy sets to fuzzy multisets, in particular the two decomposition theorems were extended to fuzzy multiset. Moreover, inverse α-cuts were introduced in fuzzy multisets and their properties studied. In chapter five, algebra of multisets and fuzzy sets were studied viz., fuzzy groups, monoids of fuzzy subsets, multigroups and fuzzy multigroups. Finally, certain monoids of multisets partitions, fuzzy partitions were constructed and abelian fuzzy multigroup introduced.

6.2 Conclusion

The following results are the main contributions of this dissertation:

1. **Extension of properties of α-cuts for fuzzy sets to fuzzy multisets**

1.1 **Formulation of decomposition theorems for fuzzy multisets**

Let $A \in FM(X)$ and $\bigcup$ be the standard fuzzy multiset union.

First Decomposition Theorem states that

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}[A].$$

Second Decomposition Theorem states that

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}]A[.$$

[Proofs on page 82]

2. **Extension of Properties of inverse α-cuts for fuzzy sets to fuzzy multisets**

   Proposition 4.2.1

   Let $\tilde{A}, \tilde{B} \in F(X)$ and $\alpha, \beta \in [0,1]$. The following properties hold:

   i. ${}^{\alpha}\tilde{A}^{-1} \subseteq {}^{\alpha^-}\tilde{A}^{-1}$

   ii. $\alpha \leq \beta$ implies ${}^{\alpha}\tilde{A}^{-1} \subseteq {}^{\beta}\tilde{A}^{-1}$ and ${}^{\alpha^-}\tilde{A}^{-1} \subseteq {}^{\beta^-}\tilde{A}^{-1}$

   iii. ${}^{\alpha}(\tilde{A} \cap \tilde{B})^{-1} = {}^{\alpha}\tilde{A}^{-1} \cup {}^{\alpha}\tilde{B}^{-1}$, ${}^{\alpha}(\tilde{A} \cup \tilde{B})^{-1} = {}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1}$, and ${}^{\alpha}\tilde{A}^{-1} \cap {}^{\alpha}\tilde{B}^{-1} \subseteq {}^{\alpha}(\tilde{A} \cap \tilde{B})^{-1}$

   iv. ${}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1} = {}^{\alpha^-}\tilde{A}^{-1} \cup {}^{\alpha^-}\tilde{B}^{-1}$, ${}^{\alpha^-}(\tilde{A} \cup \tilde{B})^{-1} = {}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1}$, and ${}^{\alpha^-}\tilde{A}^{-1} \cap {}^{\alpha^-}\tilde{B}^{-1} \subseteq {}^{\alpha^-}(\tilde{A} \cap \tilde{B})^{-1}$

   v. ${}^{\alpha}((\tilde{A}^{-1})') = {}^{(1-\alpha)^-}(\tilde{A}^{-1})'$

   vi. ${}^{1}\tilde{A}^{-1} = X$

   vii. $\tilde{A} \subseteq \tilde{B}$ iff ${}^{\alpha}\tilde{B}^{-1} \subseteq {}^{\alpha}\tilde{A}^{-1}$; $\tilde{A} \subseteq \tilde{B}$ iff ${}^{\alpha^-}\tilde{B}^{-1} \subseteq {}^{\alpha^-}\tilde{A}^{-1}$

   viii. $\tilde{A} = \tilde{B}$ iff ${}^{\alpha}\tilde{B}^{-1} = {}^{\alpha}\tilde{A}^{-1}$; $\tilde{A} = \tilde{B}$ iff ${}^{\alpha^-}\tilde{B}^{-1} = {}^{\alpha^-}\tilde{A}^{-1}$

   [Proofs on page 84]

   Proposition 4.2.2 Let A, B, $C \in FM(X)$. The following properties hold for

$\alpha, \beta \in (0,1]$:

i. ${}^{\alpha}[A]^{-1} \cup {}^{\alpha}[B]^{-1} = {}^{\alpha}[B]^{-1} \cup {}^{\alpha}[A]^{-1}$; ${}^{\alpha}[A]^{-1} \cap [B]^{-1} = {}^{\alpha}[B]^{-1} \cap {}^{\alpha}[A]^{-1}$

ii. ${}^{\alpha}]A[^{-1} \cup {}^{\alpha}]B[^{-1} = {}^{\alpha}]B[^{-1} \cup {}^{\alpha}]A[^{-1}$; ${}^{\alpha}]A[^{-1} \cap {}^{\alpha}]B[^{-1} = {}^{\alpha}]B[^{-1} \cap {}^{\alpha}]A[^{-1}$

iii. ${}^{\alpha}[A]^{-1} \cup ({}^{\alpha}[B]^{-1} \cup {}^{\alpha}[C]^{-1}) = ({}^{\alpha}[A]^{-1} \cup {}^{\alpha}[B]^{-1}) \cup {}^{\alpha}[C]^{-1}$;

$${}^{\alpha}[A]^{-1} \cap \left({}^{\alpha}[B]^{-1} \cap {}^{\alpha}[C]^{-1}\right) = \left({}^{\alpha}[A]^{-1} \cap {}^{\alpha}[B]^{-1}\right) \cap {}^{\alpha}[C]^{-1}$$

iv. ${}^{\alpha}]A[^{-1} \cup \left({}^{\alpha}]B[^{-1} \cup {}^{\alpha}]C[^{-1}\right) = \left({}^{\alpha}]A[^{-1} \cup {}^{\alpha}]B[^{-1}\right) \cup {}^{\alpha}]C[^{-1}$;

$${}^{\alpha}]A[^{-1} \cap \left({}^{\alpha}]B[^{-1} \cap {}^{\alpha}]C[^{-1}\right) = \left({}^{\alpha}]A[^{-1} \cap {}^{\alpha}]B[^{-1}\right) \cap {}^{\alpha}]C[^{-1}$$

v. ${}^{\alpha}[A]^{-1} \subseteq {}^{\alpha}]A[^{-1}$

vi. $\alpha \leq \beta$ implies ${}^{\alpha}[A]^{-1} \subseteq {}^{\beta}[A]^{-1}$‘ and ${}^{\alpha}]A[^{-1} \subseteq {}^{\beta}]A[^{-1}$

[proofs on page 88]

3. Formulation of Monoids of partitions of a multiset and that of S-H fuzzy partitions of a set.

3.1 Monoids of partitions of multisets

Let A be a cardinality-bounded nonempty mset and $\prod(A)$ be the collection of all partitions of A . Then operations $*$ and $\oplus$ were introduced such that

(i) $(\prod(A), *)$ or $(\prod(A, *, \{\overline{A}\})$ is a commutative idempotent monoid, and

(ii) $(\prod(A), \oplus)$ is a commutative idempotent monoid.

[Proofs on page 110]

3.2 Monoids of S-H fuzzy Partitions of a set

Let $\tilde{T}$ be a S-H fuzzy partition of a set X and, let $\tilde{A}_i$ $(i = \overline{1,n})$ denote the

blocks of the partition $\tilde{T}$. Let $\prod(X)$ denote the collection of all S-H fuzzy partitions of X. Then operations $*$, $\circ$, and $\oplus$ were introduced such that

(i) $(\prod(X), *)$ is a commutative idempotent monoid,

(ii) $(\prod(X), \circ)$ is a commutative idempotent monoid, and

(iii) $(\prod(X), \oplus)$ is a commutative idempotent monoid.

[Proofs on page 105]

4. **Properties of Abelian Fuzzy Multigroups introduced**

Proposition 5.3.26 Let $A \in FM(X)$. Then $\forall x, y \in X$ the following assertions are equivalent:

(i) $CM_A(xy) = CM_A(yx)$,

(ii) $CM_A(xyx^{-1}) = CM_A(y)$,

(iii) $CM_A(xyx^{-1}) \geq CM_A(y)$,

(iv) $CM_A(xyx^{-1}) \leq CM_A(y)$,

Proposition 5.3.27 Let $A \in FM(X)$.

Then the following assertions are equivalent:

(i) $CM_A(xy) = CM_A(yx), \forall x, y \in X$,

(ii) $A \circ B = B \circ A, \forall B \in FM(X)$.

Proposition 5.3.28

Let $A \in AFMG(X)$. Then A^*, and $A[\alpha, n], n \in \mathbb{N}$ are normal subgroups of X.

[Proofs on page 116]

6.3 Recommendations

The construction provided in subsection 4.2.2 may be found useful in addressing adequately some real life problems which require symmetry of abstraction. The concept of α-cuts and inverse α-cuts can be very useful in many areas like information retrieval on the web, data encription, data mining, coding theory, decision making, etc. In view of the multitudinal application of α-cuts and inverse α-cuts, it needs to be investigated whether or not, similar to the result that every fuzzy multiset can be decomposed into its α-cuts obtained in (Singh et al., 2014), holds for inverse α-cuts as well.

In view of the significance of fuzzy multiset theory it is recommended that it should be inculcated into the curricula of studies, both at undergraduate and postgraduate levels.

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(i). Singh, D., Alkali, A., Ibrahim, A. M., (2013). An Outline of the Development of the Concept of Fuzzy Multisets, *International Journal of Innovation, Management and Technology, Vol. 4, no.2, 309-315.*

(ii) Singh, D., Alkali, A., Isah, A. I., (2014). Some Applications of α-Cuts in Fuzzy Multiset Theory, *Journal of Emerging Trends in Computing and Information Sciences, Vol. 5, no. 4, 328-335.*

(iii) Singh, D., Alkali, A., Singh, J. N., (2014). Monoids of Partitions of a Multiset, *Journal of Mathematical Sciences & Mathematics Education Vol. 9 No. 1, 9-16.*

(iv) Singh, D., Alkali, A. J., (2015). Monoids of S-H fuzzy Partitions of a set, *International Journal of Pure and Applied Mathematics Vol. 98 No. 1, 123-128.*

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(i) Singh, D., Alkali, A. J., Isah, A. I., (2015). An Extension of the Properties of Inverse α-Cuts to Fuzzy multisets, (To appear in *Ann. Fuzzy Math. Inform.* **x** *No. x, xxx–xxx*).

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