# STUDY OF SOME COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED $S$-METRIC SPACE

By

MOHAMMED SANI MASHINA

DEPARTMENT OF MATHEMATICS

AHMADU BELLO UNIVERSITY, ZARIA NIGERIA

MARCH, 2016

# STUDY OF SOME COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED $S$-METRIC SPACE

By

**Mohammed Sani MASHINA**

**B.Sc. (A.B.U, 2010)**

**MSC/SCI/02646/2011-2012**

**A DISSERTATION SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES,**

**AHMADU BELLO UNIVERSITY, ZARIA**

**IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE AWARD OF**

**MASTER DEGREE IN MATHEMATICS.**

**DEPARTMENT OF MATHEMATICS,**

**FACULTY OF SCIENCE**

**AHMADU BELLO UNIVERSITY, ZARIA**

**NIGERIA**

**MARCH, 2016**

## DECLARATION

I declare that the work in this dissertation titled "STUDY OF SOME COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED $S$-METRIC SPACE" has been performed by me in the Department of Mathematics.The information derived from the literature has been duly acknowledge in the text and a list of references provided. No part of this thesis was previously presented for another degree or diploma at this or any other Institution.

MASHINA, Mohammed Sani

| Name of Student | Signature | Date |
|---|---|---|

## CERTIFICATION

This dissertation titled "STUDY OF SOME COUPLED FIXED POINT THEOREMS IN PARTIALLY ORDERED $S$-METRIC SPACE" by Mohammed Sani, MASHINA meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria, and is approved for its contribution to knowledge and literary presentation.

____________________ ________________

Dr. Bashir Ali Date

Chairman Supervisory Committee

____________________ ________________

Dr. A. O. Ajibade Date

Member Supervisory Committee

____________________ ________________

Date

External Examiner

____________________ ________________

Prof. Babangida Sani Date

Head of Department

____________________ ________________

Prof. Kabiru Bala Date

Dean, Postgraduate School

# DEDICATION

To my loving and caring Parents,

Mohammed Mashina and Aishatu Ahmad.

## ACKNOWLEDGEMENTS

I would like to thank my supervisor Dr. Bashir Ali whose ideas, guidance, patience, advice, encouragement and support have been essential at all stages. I really appreciate him for giving me an insight into his perception of mathematics and carefully reading my work to make timely corrections.

My sincere gratitude goes to my second supervisor Dr. A. O. Ajibade who patiently read this work and made timely corrections. My appreciation also goes to all my able and respected teachers in the department whom have contributed one way or the other for me to attain such a great height.

I owe my endless gratitude, love and appreciation to my family house hold; Alhaji Moh'd, Hajiya Aishatu, Moh'd, Binta, Fatima, Hauwa, Amina, Ahmed, Zainab, Umar, Shaba, Hassana, Rahma, Nana, Sadiya, Abu, Haruna, Bilkisu, Maryam, Maijidda, Mustapha, Ummati, Adam and to my lovely wife Zainab and son Moh'd Suraj for everyday prayers.

Next I would like to thank a brother and a friend Yusuf I. Suleiman whom lately contributed a lot to see the success of my work. Also to mention are friends and colleagues Abdulkadir A. Zailani, Abdulaziz Zakariyya, Muyideen Aliyu, Joseph Timayi, Ismail M. Idris and Abdulazeez Abdulmalik for their encoragement.

# ABSTRACT

In this dissertation, we studied some properties of $S$-metric, $S : X^3 \to [0, \infty)$ on a nonempty set $X$ as a generalized metric in 3-tuples and established some coupled common fixed point results with the mixed weakly monotone mappings in partially ordered $S$-metric space. Our results generalize some coupled fixed point results with the mixed weakly monotone mappings in partially ordered metric space in the framework of $S$-metric space.

# TABLE OF CONTENTS

**CHAPTER ONE**

**GENERAL INTRODUCTION**

## 1.1 Introduction

Many mathematical problems require one to find a distance between two or more objects which is not easy to measure precisely in general. Metric space generalizes the idea of distance between two points on the real line. Whenever we study the theory of functions of a real variable, the notion of distance between two real numbers intuitively comes. There are different approaches to obtaining appropriate concept of metric structure.

**Definition 1.1.1** Let $A$ and $B$ be sets. A mapping is a function $f$ of $A$ into $B$, $f : A \to B$ such that to each element in $A$ there is assigned a unique element in $B$.

**Definition 1.1.2** Let $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping with domain $D(f)$ in $\mathbb{R}$. Then $f$ is said to be continuous at $x_0 \in D(f)$ if given any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon) > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

**Definition 1.1.3** A function $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in D(f)$ if for any sequence $\{x_n\}_{n=1}^{\infty}$ in $D(f)$ such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

**Definition 1.1.4** Let $X$ be a non empty set. A mapping $d : X \times X \to \mathbb{R}$ is called a metric on $X$ if and only if it satisfies the following properties:

1. $d(x, y) \geq 0, \quad$ for all $x, y \in X$

2. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$

3. $d(x, y) = d(y, x), \quad$ for all $x, y \in X$

4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

The pair $(X, d)$ is called a metric space.

**Definition 1.1.5** *Let $(X, d)$ be a metric space.*

1. A sequence $\{x_n\}$ is said to converge to $x \in X$ if and only if for each $\epsilon > 0$, $\exists$ $n_\epsilon \in \mathbb{N}$ such that for all $n \geq n_\epsilon$, $d(x_n, x) < \epsilon$, We say that $x$ is the limit of $\{x_n\}$ and write

$$\lim_{n\to\infty} x_n = x \quad \text{or} \quad x_n \to x \quad \text{as} \quad n \to \infty$$

2. A sequence $\{x_n\}$ is said to be a Cauchy sequence if for each $\epsilon > 0$, $\exists$ $n_\epsilon \in \mathbb{N}$ such that $\forall n, m \geq n_\epsilon$, $d(x_n, x_m) < \epsilon$.

3. A metric space $(X, d)$ is complete if and only if every Cauchy sequence in $X$ converges.

**Example 1.1.1** The set $\mathbb{R}$ of all real numbers in which the distance function is the usual distance between points on the real line is given by $d(x, y) = |x - y|$. This is called the usual metric on $\mathbb{R}$.

**Definition 1.1.6** The space $X$ is said to have the fixed point property for a mapping $f : X \to X$ if there exists $x \in X$ such that $fx = x$.

The study of generalized metric has attracted and continues to attract the interest of many authors. Over the last few decades, a number of generalizations of metric space have thus appeared in many papers. These generalizations were also used then to extend the scope of the study of fixed point theory.

Sedghi et al. (2012) introduced a new generalized metric space called $S$-metric space and gave some immediate examples as follows.

**Definition 1.1.7** Let $X$ be a nonempty set. An $S$-metric on $X$ is a function $S$:$X^3 \to [0, \infty)$ that satisfies the following conditions, for all $x, y, z, a \in X$

(S1) $S(x, y, z) \geq 0$

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$

The pair $(X,S)$ is called an $S$-metric space.

**Example 1.1.2** Let $\mathbb{R}$ be the real line, then the function on $S : \mathbb{R}^3 \to [0,+\infty)$ defined by $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an $S$-metric on $\mathbb{R}$.
This $S$-metric is called the usual $S$-metric on $\mathbb{R}$.

**Example 1.1.3** Let $X = \mathbb{R}^n$ and let $\|.\|$ be a norm on $X$, then the function $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to [0,+\infty)$ defined by $S(x,y,z) = \|2x - y - z\| + \|z - y\|$ is an $S$-metric on $X$.

**Example 1.1.4** (Intuitive geometric example of $S$-metric).
Let $X = \mathbb{R}^2$ and $d$ be any metric on $X$. Then the function $d : X^3 \to [0,+\infty)$ defined by $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ is an $S$-metric on $X$. The points $x,y,z$ forms triangle. If we choose a mid-point $a$ in the interior of the triangle, we get $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

**Definition 1.1.8** Let $(X,d)$ be a metric space. A mapping $f : X \to X$ is called a contraction if there exists $0 \leq q < 1$ such that for all $x,y \in X$

$$d(fx,fy) \leq qd(x,y).$$

**Definition 1.1.9** *Let $P$ be a nonempty set. A partial order on $P$ is a binary operation $\leq$ defined on $P$ which satisfies the following conditions: For any $a,b,c \in P$,*

1. $a \leq a$. (reflexivity)

2. $a \leq b$ and $b \leq c \Rightarrow a \leq c$. (transitivity)

3. $a \leq b$ and $b \leq a \Rightarrow a = b$ (anti-symmetry)

The pair $(P,\leq)$ is called a partially ordered set.

**Definition 1.1.10** *A partially ordered set* $(P, \leq)$ *in which elements are comparable (i.e. for any* $a, b, \in P$*, either* $a \leq b$ *or* $b \leq a$*) is called a totally ordered set.*

Ran and Reurings (2003) discussed an analogue of Banach's fixed point theorem in partially ordered sets. The key feature in the fixed point theorem is that the contractivity condition on the nonlinear mapping is only assumed to hold on elements that are comparable in the partial order. However the mapping is assumed to be monotone. They showed that under such conditions the conclusions of Banach"s fixed point theorem still holds.

**Theorem 1.1.1** (Ran and Reurings (2003)) *Let* $T$ *be a partially ordered set such that every pair* $x, y \in T$ *has a lower bound and an upper bound. Furthermore, let* $d$ *be a metric on* $T$ *such that* $(T, d)$ *is a complete metric space. If* $F$ *is a continuous, monotone (i.e., either order-preserving or order-reversing)mapping from* $T$ *into* $T$ *such that*

(1) $\exists\ 0 < c < 1 : d(F(x), F(y)) \leq cd(x, y), \forall x \geq y,$

(2) $\exists\ x_0 \in T : x_0 \leq F(x_0)$ *or* $x_0 \geq F(x_0),$

*then* $F$ *has a unique fixed point* $\overline{x}$*.*

Following the trend mentioned above, Bhaskar and Lakshmikantham (2006) extend such considerations to mixed monotone mapping as follows.

**Definition 1.1.11** *Let* $(X, \leq)$ *be a partially ordered set and* $f : X \times X \to X$ *be a mapping. We say that* $f$ *has the mixed monotone property on* $X$ *if and only if given* $x, y, x_1, x_2, y_1, y_2 \in X$,

$$x_1 \leq x_2 \Rightarrow f(x_1, y) \leq f(x_2, y)\ ,$$

$$y_1 \leq y_2 \Rightarrow f(x, y_1) \geq f(x, y_2).$$

Following similar trend Gordji et al. (2012) introduced the concept of a mixed weakly monotone pair of mappings as follows.

**Definition 1.1.12** *Let $(X, \preceq)$ be a partially ordered set and $f, g : X \times X \to X$ be a mapping. We say that the pair $(f, g)$ has the mixed weakly monotone property on $X$ if and only if, for any $x, y \in X$,*

$$x \preceq f(x,y), \;\; y \succeq f(y,x)$$
$$\Rightarrow f(x,y) \preceq g(f(x,y), f(y,x)), \qquad f(y,x) \succeq g(f(y,x), f(x,y))$$

and

$$x \preceq g(x,y), \;\; y \succeq g(y,x)$$
$$\Rightarrow g(x,y) \preceq f(g(x,y), g(y,x)), \qquad g(y,x) \succeq f(g(y,x), g(x,y)).$$

Bhaskar and Lakshmikantham (2006) introduced the notion of coupled fixed point of a mapping as follows.

**Definition 1.1.13** Let $X$ be a nonempty set. An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \to X$ if and only if $F(x, y) = x$ and $F(y, x) = y$

Since then, the existence of coupled fixed point theorems for some kind of contractive-type mappings in partially ordered metric spaces, with applications has been investigated by some authors. For example, Bhaskar and Lakshmikantham (2006), Berinde (2011), Cho et al. (2012), Gordji et al. (2011), Gordji et al. (2012) and Sintunavarat et al. (2011).

## 1.2 Statement of the Problem

There are many generalized metric spaces. The study of generalized metric space has attracted the interest of many authors over the last few decades. These generalizations were used to extend the scope of the study of fixed point theory. Is it possible to generalize certain results in metric space in the setting of $S$-metric space? This study is a further generalization of such results in $S$-metric space.

## 1.3 Justification

In this dissertation, we establish that $S$-metric space is a generalization of ordinary metric space. Consequently, we obtained some fixed point results in the frame work of $S$-metric space. Our results therefore generalize and unify many comparable results. These results also stimulate interest in metric fixed point theory.

## 1.4 Aim and Objectives

The aim of this dissertation is to present some coupled common fixed point results with mixed weakly monotone mappings in the setting of partially ordered $S$-metric space. The aim is achieved through the following objectives: to

(i) review the theory of the new generalized metric space called $S$-metric space;

(ii) study some recently established results of coupled fixed point for mappings with the mixed monotone property and mixed weakly monotone property in partially ordered metric space.

## 1.5 Research Methodology

In this dissertation, we used some notions of classical analysis in establishing our results. we reviewed the work of Bhaskar and Lakshmikantham (2006), Dung (2013), Gordji et al. (2012) and Sedghi et al. (2012).

## 1.6 Outline of the Thesis

This dissertation contains four other chapters apart from this introductory chapter. The outline of the remaining chapters is as follows:

Chapter 2: In this chapter, we present a survey of necessary and relevant literature of history and development of fixed point theory, generalized metric spaces and some recently established results of coupled fixed points.

Chapter 3: In this chapter we studied some properties and existing results of $S$-metric space.

Chapter 4: In this chapter we present the result of Gordji et al. (2012) in the framework of $S$-metric space.

Chapter 5: In this chapter, we give a summary of the results obtained in this dissertation along with some directions for future research.

CHAPTER TWO

LITERATURE REVIEW

## 2.1 Introduction

Fixed point theory is an exciting branch of mathematics. It is a mixture of Analysis, Topology and Geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. It has numerous applications in almost all areas of mathematical sciences. For example, in Functional Analysis, it is used for proving the existence of solutions of ordinary and partial differential equations, integral equations, system of linear equations, closed orbit of dynamical systems. In particular fixed point techniques have been applied in such diverse fields as Biology, Chemistry, Economics, Engineering and Physics. The concept of fixed point plays a key role in Analysis. Also, fixed point theorems are mainly used in existence theory of random differential equations, numerical methods like Newton-Rapshon method and Picard's Existence Theorem and in other related areas. It plays a major role in many applications, such as variational and linear inequalities, optimization and applications in the field of approximation theory.

## 2.2 Brief Historical Development

Metric fixed point theory started from the late 19th century, when successive approximations were used to establish the existence and uniqueness of solutions to equations, and especially differential equations. This approach is particularly associated with the work of Picard, although it was Stefan Banach who in 1922 developed the ideas involved in an abstract setting. The fixed point theorem generally known as

Banach's contraction mapping principle or Banach's fixed point theorem, appeared in explicit form in Banach's thesis in 1922, and states that *every self contraction mapping defined on a complete metric space has a unique fixed point.* That is if $(X, d)$ is a complete metric space and $T : X \to X$ satisfies $d(Tx, Ty) \leq kd(x, y)$ for all $x, y \in X$, for some $k \in (0, 1)$, then $T$ has a unique fixed point.
Banach's contraction principle ensures under appropriate conditions the existence and uniqueness of a fixed point. It is one of the most important results in fixed point theory. The principle is based on an iterative scheme called Picard iterative scheme. Many authors have extended, improved and generalized Banach's theorem in several ways. For example Agarwal et al. (2008), Ariza-Ruiz (2009), Chatterjea (1972), Choudhury (2009), Harjani and Sadarangani (2009) and Rhoades (1977).

Gahler (1963) introduced the notion of a 2-metric space as follows:

**Definition 2.2.1** *Let $X$ be a nonempty set. A function $d : X^3 \to \mathbb{R}$ is said to be a 2-metric on $X$ if the following conditions hold:*

(d1) For any distinct point $x, y \in X$, there exist $z \in X$ $d(x, y, z) \neq 0$

(d2) $d(x, y, z) = 0$ if any of the two elements of the set $\{x, y, z\}$ in $X$ are equal.

(d3) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(z, x, y) = d(y, z, x) = d(z, y, x)$

(d4) $d(x, y, z) \leq d(x, y, a) + d(x, a, z) + d(a, y, z)$ for all $x, y, z, a \in X$

The pair $(X, d)$ is called a 2-metric space.

Gahler (1963) claimed that 2-metric space is a generalization of an ordinary metric space. He mentioned that $d(x, y, z)$ geometrically represents the area of a triangle formed by the points $x, y, z \in X$ as its vertices. However Sharma (1980) found some mathematical flaws in these claims. It was demonstrated that $d(x, y, z)$ does not always represent the area of a triangle formed by the points $x, y, z \in X$ (Sharma, 1980).

Dhage (1984) in his Ph.D thesis identified condition (d2) as a weakness in Gahler's theory of 2-metric space. To overcome these problems, he then introduced the concept of a $D$-metric space. Thus, the following definition:

**Definition 2.2.2** *Let $X$ be a nonempty set. A function $D : X^3 \to \mathbb{R}$ is called a $D$-metric on $X$ if the following conditions hold: For all $x, y, z, a \in X$*

(D1) $D(x, y, z) \geq 0$ and $D(x, y, z) = 0$ if and only if $x = y = z$

(D2) $D(x, y, z) = D(x, z, y) = D(y, x, z) = D(z, x, y) = D(y, z, x) = D(z, y, x)$.

(D3) $D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z)$.

The pair $(X, D)$ is called a $D$-metric space.

Considering definitions 2.2.1 and 2.2.2, we note that (d3) and (D2)are equivalent, (d4) and (D3) are also equivalent but (d1) and (d2) have been replaced by (D2). Dhage (1994) then studied toplogical properties of $D$-metric space in a series of papers. The notions of open balls and sequential continuity in $D$-metric space were introduced in Dhage (2000). Mustafa and Sims (2003) identified condition (D3) as a weakness in Dhage's theory of $D$-metric space.

Mustafa and Sims (2006) introduced the notion of $G$-metric space and gave an important generalization of metric space as follows.

**Definition 2.2.3** *Let $X$ be a nonempty set. A function $G : X^3 \to \mathbb{R}^+$ is called a $G$-metric on $X$ if it satisfies the following conditions: For all $x, y, z, a \in X$,*

(G1) $G(x, y, z) = 0$ if and only if $x = y = z$

(G2) $0 \leq G(x, y, y)$ with $x \neq y$

(G3) $G(x, x, y) \leq G(x, y, z)$ with $z \neq y$

(G4) $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(z, x, y) = G(y, z, x) = G(z, y, x)$.

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$

The pair $(X, G)$ is called a $G$-metric space.

Comparing definitions 2.2.2 and 2.2.3, it is observed that (D1) has been replaced with (G1), (G2) and (G3), (D2) is equivalent to (G4) and (D3) has been replaced by (G5). In the process, Mustafa and Sims (2006) studied some topological properties of $G$-metric space and afterwards Mustafa et al. (2008) obtained generalized fixed point theorems in the set up of $G$-metric space.

Sedghi et al. (2007) observed that condition (D1) can be replaced with two axioms and thus introduced the notion of a $D^*$-metric space as follows.

**Definition 2.2.4** *Let $X$ be a nonempty set. A function $D^* : X^3 \to \mathbb{R}^+$ is called a $D^*$-metric on $X$ if it satisfies the following conditions: For all $x, y, z, a \in X$,*

(D*1) $D^*(x, y, z) \geq 0$

(D*2) $D^*(x, y, z) = 0$ if and only if $x = y = z$

(D*3) $D^*(x, y, z) = D^*(x, z, y) = D^*(y, x, z) = D^*(z, x, y) = D^*(y, z, x) = D^*(z, y, x)$

(D*4) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$

The pair $(X, D^*)$ is called a $D^*$-metric space.

Comparing definitions 2.2.2 and 2.2.4, we see that (D1) has been replaced with (D*1) and (D*2), (D2) and (D*3) are equivalent and (D3) has been replaced with (D*4).

Sedghi et al. (2012) identified condition (G3) as a peculiar limitation of the $G$-metric space but classified the symmetry condition as a common weakness of both $G$- and $D^*$-metric spaces. To overcome these difficulties, they introduced $S$-metric space as new generalized metric space.

## 2.3 Coupled fixed point theorems

In this section the existing results of coupled fixed point in partially ordered metric space are stated as follows:

**Theorem 2.3.1** (Bhaskar and Lakshmikantham (2006)) *Let $(X, \leq)$ be a partially ordered set and suppose that there exist a metric $d$ in $X$ such that $(X, d)$ is a complete metric space. Also let $f : X \times X \to X$ be a continuous mapping having the mixed* *monotone property on* $X$. *Assume that there exist a* $k \in [0,1)$ *with*

$$d(F(x,y),F(u,v)) \le \frac{k}{2}[d(x,u)+d(y,v)], \quad \forall x \ge u, \quad y \le v.$$

*If there exists* $x_0, y_0 \in X$ *such that* $x_0 \le F(x_0,y_0)$ *and* $y_0 \ge F(y_0,x_0)$. *Then there exist* $x,y \in X$ *such that* $x = F(x,y)$ *and* $y = F(y,x)$.

**Theorem 2.3.2** (Bhaskar and Lakshmikantham (2006)) *Let* $(X,\le)$ *be a partially ordered set and suppose that there exist a metric d in X such that* $(X,d)$ *is a complete metric space. Assume that X has the following properties:*

1. *If* $\{x_n\}$ *is an increasing sequence with* $x_n \to x$, *then* $x_n \le x$ *for all* $n \in \mathbb{N}$;

2. *If* $\{y_n\}$ *is a decreasing sequence with* $y_n \to y$, *then* $y_n \ge y$ *for all* $n \in \mathbb{N}$

*Let* $f : X \times X \to X$ *be a continuous mapping having the mixed monotone property on* $X$. *Assume that there exist a* $k \in [0,1)$ *with*

$$d(F(x,y),F(u,v)) \le \frac{k}{2}[d(x,u)+d(y,v)], \quad \forall x \ge u, \quad y \le v.$$

*If there exists* $x_0, y_0 \in X$ *such that* $x_0 \le F(x_0,y_0)$ *and* $y_0 \ge F(y_0,x_0)$. *Then there exist* $x,y \in X$ *such that* $x = F(x,y)$ *and* $y = F(y,x)$.

**Theorem 2.3.3** (Gordji et al. (2012)) *Let* $(X,\le,d)$ *be a partially ordered complete metric space. Let* $f,g$: $X \times X \to X$ *be the mappings such that a pair* $(f,g)$ *has the mixed weakly monotone property on X. Suppose that there exist* $p,q,r,s \ge 0$ *with* $p+q+r+2s < 1$ *such that*

$$\begin{aligned} d(f(x,y),g(u,v)) \;\le\; & \frac{p}{2}D((x,y),(u,v)) + \frac{q}{2}D((x,y),(f(x,y),f(y,x))) \\ & + \frac{r}{2}D((u,v),(g(u,v),g(v,u))) + \frac{s}{2}D((x,y),(g(u,v),g(v,u))) \\ & + \frac{s}{2}D((u,v),(f(x,y),f(y,x))) \end{aligned}$$

*for all* $x,y,u,v \in X$ *with* $x \le u$ *and* $y \ge v$. *Let* $x_0, y_0 \in X$ *be such that* $x_0 \le f(x_0,y_0)$,

$y_0 \geq f(y_0.x_0)$ *or* $x_0 \leq g(x_0, y_0)$, $y_0 \geq g(y_0.x_0)$. *If* $f$ *or* $g$ *is continuos, then* $f$ *and* $g$ *have a common coupled fixed point in* $X$.

**Theorem 2.3.4** (Gordji et al. (2012)) *Let* $(X, \leq, d)$ *be a partially ordered complete metric space. Assume that* $X$ *has the following property:*

1. *If* $\{x_n\}$ *is an increasing sequence with* $x_n \to x$, *then* $x_n \leq x$ *for all* $n \in \mathbb{N}$;

2. *If* $\{y_n\}$ *is a decreasing sequence with* $y_n \to y$, *then* $y_n \geq y$ *for all* $n \in \mathbb{N}$

*Let f,g: X × X → X be the mappings such that a pair (f,g) has the mixed weakly monotone property on X. Suppose that there exist p,q,r,s ≥ 0 with p+q+r+2s < 1 such that*

$$\begin{aligned} d(f(x,y), g(u,v)) \leq\ & \frac{p}{2}D((x,y),(u,v)) + \frac{q}{2}D((x,y),(f(x,y),f(y,x))) \\ & + \frac{r}{2}D((u,v),(g(u,v),g(v,u))) + \frac{s}{2}D((x,y),(g(u,v),g(v,u))) \\ & + \frac{s}{2}D((u,v),(f(x,y),f(y,x))) \end{aligned}$$

*for all* $x, y, u, v \in X$ *with* $x \leq u$ *and* $y \geq v$. *If there exist* $x_0, y_0 \in X$ *such that* $x_0 \leq f(x_0, y_0)$, $y_0 \geq f(y_0.x_0)$ *or* $x_0 \leq g(x_0, y_0)$, $y_0 \geq g(y_0.x_0)$, *then* $f$ *and* $g$ *have a common coupled fixed point in* $X$.

## CHAPTER THREE

## THEORY OF METHODS

# 3.1 Introduction

In this chapter, some properties of $S$-metric space are studied which are needed to prove the main results.

# 3.2 Properties of $S$-metric space

**Definition 3.2.1** (Sedghi et al. (2012)). *Let $(X, S)$ be an $S$-metric space. For $r > 0$ and $x \in X$, we define the open ball $B_S(x, r)$ and the closed ball $B_S[x, r]$ with centre $x$ and radius $r$ as*

$$B_S(x, r) = \{y \in X : S(y, y, x) < r\}$$

$$B_S[x, r] = \{y \in X : S(y, y, x) \leq r\}$$

**Definition 3.2.2** (Sedghi et al. (2012)).

Let $(X, S)$ be an $S$-metric space and $A \subset X$.

(1) A sequence $\{x_n\}$ in $X$ is said to converge to $x \in X$ if and only if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. That is for each $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that $\forall n \geq n_\epsilon$, $S(x_n, x_n, x) < \epsilon$. Whenever $\{x_n\}_{n\geq1}$ converges to $x$, we write $\lim_{n\to\infty} x_n = x$.

(2) A sequence $\{x_n\}$ in $X$ is called a Cauchy sequence if and only if for each $\epsilon > 0$, $\exists\ n_\epsilon \in \mathbb{N}$ such that for all $n, m \geq n_\epsilon$ $S(x_n, x_n, x_m) < \epsilon$.

(3) The $S$-metric space $(X, S)$ is said to be complete if and only if every Cauchy sequence in $(X, S)$ converges.

(4) If for every $x \in A$ there exists $r > 0$ such that $B_S(x, r) \subset A$, then the subset $A$ is called an open subset of $X$.

(5) A subset $A$ of $X$ is said to be $S$-bounded if there exists $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$.

Sedghi et al. (2012) proved the following results.

**Lemma 3.2.1** *Let $(X, S)$ be an $S$-metric space, then we have*

$$S(x, x, y) = S(y, y, x)$$

**Lemma 3.2.2** *Let $(X, S)$ be an $S$-metric space, then*

$$S(x, x, z) \leq 2S(x, x, y) + S(z, z, y)$$

**Lemma 3.2.3** *Let $(X, S)$ be an $S$-metric space. If the sequence $\{x_n\}$ in $X$ converges to $x$, then $x$ is unique.*

**Lemma 3.2.4** *Let $(X, S)$ be an $S$-metric space. If the sequence $\{x_n\}$ in $X$ converges to $x$, then $\{x_n\}$ is a Cauchy sequence.*

**Lemma 3.2.5** *Let $(X, S)$ be an $S$-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y)$$

The following lemma shows that every metric space is a $S$-metric space.

**Lemma 3.2.6** (Dung (2013)) *Let $(X, d)$ be a metric space, then we have*

(1) $S_d(x, y, z) = d(x, z) + d(y, z), x, y, z \in X$ *is a $S$-metric on $X$*

(2) $x_n \to x$ *in* $(X,d)$ *if and only if* $x_n \to x$ *in* $(X, S_d)$

(3) $\{x_n\}$ *is Cauchy in* $(X,d)$ *if and only if* $\{x_n\}$ *is Cauchy in* $(X, S_d)$

(4) $(X,d)$ *is complete if and only if* $(X, S_d)$ *is complete.*

**Lemma 3.2.7** (Gordji et al. (2012)) *Let* $(X, \preceq)$ *be a partially ordered set. Then* $(X \times X, \preceq)$ *is a partially ordered set with the partial order defined by: for all* $(x,y),(u,v) \in X \times X$*,*

$$(x,y) \preceq (u,v) \Leftrightarrow x \preceq u, v \preceq y$$

**Lemma 3.2.8** (Gordji et al. (2012)) *Let* $(X,d)$ *be a metric space. Then* $(X \times X, D)$ *is a metric space with the metric* $D$ *given by*

$$D((x,y),(u,v)) = d(x,u) + d(y,v)$$

*for all* $(x,y),(u,v) \in X \times X$.

**Lemma 3.2.9** (Dung (2013)) *Let* $(X,S)$ *be an* $S$*-metric space. Then* $(X \times X, D)$ *is an* $S$*-metric space with the* $S$*-metric* $D$ *given by*

$$D((x,y),(u,v),(z,w)) = S(x,u,z) + S(y,v,w) \;\; for \;\; all \;\; x,y,u,v \in X.$$

**Remark 3.2.1** (Dung (2013)) *Let* $(X,d)$ *be a metric space, then using* Lemma 3.2.7 with $S = S_d$*, we get*

$$\begin{aligned} D((x,y),(x,y),(u,v) &= S_d(x,x,u) + S_d(y,y,v) \\ &= 2(d(x,u) + d(y,v)) \\ &= 2D_d((x,y),(u,v)) \end{aligned}$$

**Definition 3.2.3** (Gordji et al. (2012)) *Let* $f : X \times X \to X$ *be a mapping. We* *denote*

$$f^{n+1}(x,y) = f(f^n(x,y), f^n(y,x))$$

*for all* $x, y \in X$ *and* $n \geq 1$.

**Remark 3.2.2** (Gordji et al. (2012)) *Let* $(X, \preceq)$ *be a partially ordered set. Let* $f : X \times X \to X$ *be a mapping with the mixed monotone property on* $X$*. Then for all* $n \in \mathbb{N}$*, a pair* $(f^n, f^n)$ *has the mixed weakly monotone property on* $X$*. In fact, let* $x \preceq f^n(x,y)$ *and* $f^n(y,x) \succeq y$*. Then it follows from the mixed monotone property of* $f$ *that*

$$f(x,y) \preceq f(f^n(x,y), y) \preceq f(f^n(x,y), f^n(y,x)) = f^{n+1}(x,y),$$

$$f(y,x) \succeq f(f^n(y,x), x) \succeq f(f^n(y,x), f^n(x,y)) = f^{n+1}(y,x)$$

and

$$f^2(x,y) = f(f(x,y), f(y,x)) \preceq f(f^{n+1}(x,y), f^{n+1}(y,x)) = f^{n+2}(x,y),$$

$$f^2(y,x) = f(f(y,x), f(x,y)) \succeq f(f^{n+1}(y,x), f^{n+1}(x,y)) = f^{n+2}(y,x).$$

continuing in this manner, we have

$$f^n(x,y) \preceq f^{n+n}(x,y), \quad f^n(y,x) \succeq f^{n+n}(y,x).$$

*Hence we have*

$$f^n(x,y) \preceq f^n(f^n(x,y), f^n(y,x)), \quad f^n(y,x) \succeq f^n(f^n(y,x), f^n(x,y)).$$

This implies that the pair $(f^n, f^n)$ has the mixed weakly monotone property on $X$.

CHAPTER FOUR

COUPLED FIXED POINT IN $S$-METRIC SPACE

## 4.1 Introduction

In this chapter, we establish some coupled fixed point theorems in the framework of $S$-metric space.

## 4.2 Main Results

**Theorem 4.2.1** *Let $(X,\preceq,S)$ be a partially ordered complete $S$-metric space. Let $f,g$: $X \times X \to X$ be the mappings such that a pair $(f,g)$ has the mixed weakly monotone property on $X$. Let $x_0, y_0 \in X$ be such that $x_0 \preceq f(x_0, y_0)$, $y_0 \succeq f(y_0.x_0)$ or $x_0 \preceq g(x_0, y_0)$, $y_0 \succeq g(y_0.x_0)$. Suppose that there exist $p,q,r,s \geq 0$ with $p+q+r+2s < 1$ such that*

$$\begin{aligned} S(f(x,y), f(x,y), g(u,v)) \;\preceq\; & \frac{p}{2}D((x,y),(x,y),(u,v)) \\ & +\frac{q}{2}D((x,y),(x,y),(f(x,y),f(y,x))) \\ & +\frac{r}{2}D((u,v),(u,v),(g(u,v),g(v,u))) \\ & +\frac{s}{2}D((x,y),(x,y),(g(u,v),g(v,u))) \\ & +\frac{s}{2}D((u,v),(u,v),(f(x,y),f(y,x))) \end{aligned} \tag{4.1}$$

*for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$. Suppose also that $f$ or $g$ is continuous, then $f$ and $g$ have a common coupled fixed point in $X$.*

**Proof.** Suppose that $x_0 \preceq f(x_0, y_0)$ and $y_0 \succeq f(y_0.x_0)$ and let $x_1 = f(x_0.y_0)$, $y_1 = f(y_0.x_0)$.

From the mixed weakly monotone property of the pair $(f, g)$, we have

$x_1 = f(x_0, y_0) \leq g(f(x_0, y_0),\ f(y_0, x_0)) = g(x_1, y_1)$

and

$y_1 = f(y_0, x_0) \geq g(f(y_0, x_0),\ f(x_0, y_0)) = g(y_1, x_1)$

Let $g(x_1, y_1) = x_2, \quad g(y_1, x_1) = y_2,$

then we have $x_2 = g(x_1, y_1) \leq f(g(x_1, y_1),\ g(y_1, x_1)) = f(x_2, y_2)$

and

$y_2 = g(y_1, x_1) \geq f(g(y_1, x_1),\ g(x_1, y_1)) = f(y_2, x_2)$

Continuing in this manner, for all $n \geq 1$, we let

$x_{2n+1} = f(x_{2n}, y_{2n}), \quad y_{2n+1} = f(y_{2n}, x_{2n})$

and

$x_{2n+2} = g(x_{2n+1}, y_{2n+1}), \quad y_{2n+2} = g(y_{2n+1}, x_{2n+1}),$

which shows that $\{x_n\}$ and $\{y_n\}$ are increasing and decreasing respectively.

Now, applying (4.1), we obtain

$$
\begin{aligned}
& S(x_{2n+1}, x_{2n+1}, x_{2n+2}) \\
= \;& S(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1})) \\
\leq \;& \frac{p}{2} D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\
& + \frac{q}{2} D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) \\
& + \frac{r}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))) \\
& + \frac{s}{2} D(x_{2n}, y_{2n}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1}))) \\
& + \frac{s}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n}))) \\
= \;& \frac{p}{2} D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\
& + \frac{q}{2} D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\
& + \frac{r}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})) \\
& + \frac{s}{2} D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+2}, y_{2n+2})) \\
& + \frac{s}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}))
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{p+q}{2} D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\
&\quad + \frac{r}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})) \\
&\quad + \frac{s}{2} [D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\
&\quad + D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))] \\
&= \frac{p+q+s}{2} D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1})) \\
&\quad + \frac{r+s}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2})).
\end{aligned}
$$

Hence, it follows that for all $n \in \mathbb{N}$

$$
\begin{aligned}
S(x_{2n+1}, x_{2n+1}, x_{2n+2}) &\leq \frac{p+q+s}{2} (S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})) \\
&\quad + \frac{r+s}{2} (S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})). \qquad (4.2)
\end{aligned}
$$

Again, applying (4.1), we obtain

$$
\begin{aligned}
& S(y_{2n+1}, y_{2n+1}, y_{2n+2}) \\
&= S(f(y_{2n}, x_{2n}), f(y_{2n}, x_{2n}) g(y_{2n+1}, x_{2n+1})) \\
&\leq \frac{p}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+1}, x_{2n+1}) \\
&\quad + \frac{q}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (f(y_{2n}, x_{2n}), f(x_{2n}, y_{2n}))) \\
&\quad + \frac{r}{2} D((y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}), (g(y_{2n+1}, x_{2n+1}), g(x_{2n+1}, y_{2n+1}))) \\
&\quad + \frac{s}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (g(y_{2n+1}, x_{2n+1}), g(x_{2n+1}, y_{2n+1}))) \\
&\quad + \frac{s}{2} D((y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}), (f(y_{2n}, x_{2n}), f(x_{2n}, y_{2n}))) \\
&= \frac{p}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+1}, x_{2n+1})) \\
&\quad + \frac{q}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+1}, x_{2n+1})) \\
&\quad + \frac{r}{2} D((y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}), (y_{2n+2}, x_{2n+2})) \\
&\quad + \frac{s}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+2}, x_{2n+2})) \\
&\quad + \frac{s}{2} D((y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}))
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{p+q}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+1}, x_{2n+1})\\
&\quad + \frac{r}{2} D((y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}), (y_{2n+2}, x_{2n+2}))\\
&\quad + \frac{s}{2}[D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+1}, x_{2n+1}))\\
&\quad + D((y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}), (y_{2n+2}, x_{2n+2}))]\\
&= \frac{p+q+s}{2} D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+1}, x_{2n+1}))\\
&\quad + \frac{r+s}{2} D((y_{2n+1}, x_{2n+1}), (y_{2n+1}, x_{2n+1}), (y_{2n+2}, x_{2n+2})).
\end{aligned}
$$

Hence, it follows that for all $n \in \mathbb{N}$

$$
\begin{aligned}
& S(y_{2n+1}, y_{2n+1}, y_{2n+2})\\
&\leq \frac{p+q+s}{2}(S(y_{2n}, y_{2n}, y_{2n+1}) + S(x_{2n}, x_{2n}, x_{2n+1}))\\
&\quad + \frac{r+s}{2}(S(y_{2n+1}, y_{2n+1}, y_{2n+2}) + S(x_{2n+1}, x_{2n+1}, x_{2n+2})). \qquad (4.3)
\end{aligned}
$$

Thus it follows that from (4.2) and (4.3), for all $n \in \mathbb{N}$

$$
\begin{aligned}
& S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})\\
&\leq \frac{p+q+s}{1-(r+s)}(S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})). \qquad (4.4)
\end{aligned}
$$

Moreover, by interchanging the role of $f$ and $g$ and applying (4.1), for all $n \in \mathbb{N}$ we have

$$
\begin{aligned}
& S(x_{2n+2}, x_{2n+2}, x_{2n+3})\\
&= S(g(x_{2n+1}, y_{2n+1}), g(x_{2n+1}, y_{2n+1}), f(x_{2n+2}, y_{2n+2}))\\
&\leq \frac{p}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (x_{2n+2}, y_{2n+2}))\\
&\quad + \frac{q}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (g(x_{2n+1}, y_{2n+1}), g(y_{2n+1}, x_{2n+1})))\\
&\quad + \frac{r}{2} D((x_{2n+2}, y_{2n+2}), (x_{2n+2}, y_{2n+2}), (f(x_{2n+2}, y_{2n+2}), f(y_{2n+2}, x_{2n+2})))\\
&\quad + \frac{s}{2} D((x_{2n+1}, y_{2n+1}), (x_{2n+1}, y_{2n+1}), (f(x_{2n+2}, y_{2n+2}), f(y_{2n+2}, x_{2n+2})))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{s}{2}D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})))\\
= \;& \frac{p}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\\
&+\frac{q}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\\
&+\frac{r}{2}D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))\\
&+\frac{s}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+3},y_{2n+3}))\\
&+\frac{s}{2}D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}))\\
\leq \;& \frac{p+q}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\\
&+\frac{r}{2}D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))\\
&+\frac{s}{2}[D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\\
&+D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3}))]\\
= \;& \frac{p+q+s}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2}))\\
&+\frac{r+s}{2}D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),(x_{2n+3},y_{2n+3})).
\end{aligned}
$$

Hence it follows that for all $n \in \mathbb{N}$

$$
\begin{aligned}
&S(x_{2n+2},x_{2n+2},x_{2n+3})\\
\leq \;& \frac{p+q+s}{2}(S(x_{2n+1},x_{2n+1},x_{2n+2})+d(y_{2n+1},y_{2n+1},y_{2n+2}))\\
&+\frac{r+s}{2}(S(x_{2n+2},x_{2n+2},x_{2n+3})+S(y_{2n+2},y_{2n+2},y_{2n+3})). \qquad (4.5)
\end{aligned}
$$

In a similar manner, it follows that for all $n \in \mathbb{N}$

$$
\begin{aligned}
&S(y_{2n+2},y_{2n+2},y_{2n+3})\\
\leq \;& \frac{p+q+s}{2}(S(y_{2n+1},y_{2n+1},y_{2n+2})+S(x_{2n+1},x_{2n+1},x_{2n+2}))\\
&+\frac{r+s}{2}(S(y_{2n+2},y_{2n+2},y_{2n+3})+S(x_{2n+2},x_{2n+2},x_{2n+3})). \qquad (4.6)
\end{aligned}
$$

Thus it follows that from (4.5) and (4.6), for all $n \in \mathbb{N}$

$$\begin{aligned} & S(x_{2n+2}, x_{2n+2}, x_{2n+3}) + S(y_{2n+2}, y_{2n+2}, y_{2n+3}) \\ \leq \; & \frac{p+q+s}{1-(r+s)}(S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})). \end{aligned} \tag{4.7}$$

Also from (4.4) and (4.7), for all $n \in \mathbb{N}$, we get

$$\begin{aligned} & S(x_{2n+2}, x_{2n+2}, x_{2n+3}) + S(y_{2n+2}, y_{2n+2}, y_{2n+3}) \\ \leq \; & \Big(\frac{p+q+s}{1-(r+s)}\Big)^2(S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})). \end{aligned} \tag{4.8}$$

Let $A = \frac{p+q+s}{1-(r+s)}$. Then $0 \leq A < 1$ and for all $n \in \mathbb{N}$

$$\begin{aligned} & S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})) \\ \leq \; & A(S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})) \\ \leq \; & A^3(S(x_{2n-2}, x_{2n-2}, x_{2n-1}) + S(y_{2n-2}, y_{2n-2}, y_{2n-1})) \\ \leq \; & A^5(S(x_{2n-4}, x_{2n-4}, x_{2n-3}) + S(y_{2n-4}, y_{2n-4}, y_{2n-3})) \\ \leq \; & \ldots \\ \leq \; & A^{2n+1}(S(x_0, x_0, x_1) + S(y_0, y_0, y_1)) \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} & S(x_{2n+2}, x_{2n+2}, x_{2n+3}) + S(y_{2n+2}, y_{2n+2}, y_{2n+3}) \\ \leq \; & A^2(S(x_{2n}, x_{2n}, x_{2n+1}) + S(y_{2n}, y_{2n}, y_{2n+1})) \\ \leq \; & A^4(S(x_{2n-2}, x_{2n-2}, x_{2n-1}) + S(y_{2n-2}, y_{2n-2}, y_{2n-1}) \\ \leq \; & A^6(S(x_{2n-4}, x_{2n-4}, x_{2n-3}) + S(y_{2n-4}, y_{2n-4}, y_{2n-3}) \\ \leq \; & \ldots \\ \leq \; & A^{2n+2}(S(x_0, x_0, x_1) + S(y_0, y_0, y_1)). \end{aligned} \tag{4.10}$$

For all $m, n \in \mathbb{N}$ with $n \leq m$, by using Lemma 4.2.2, (4.9) and (4.10), we have

$$\begin{aligned}
& S(x_{2n+1}, x_{2n+1}, x_{2m+1}) + S(y_{2n+1}, y_{2n+1}, y_{2m+1}) \\
\leq\ & (2S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + 2S(y_{2n+1}, y_{2n+1}, y_{2n+2})) \\
& +(S(x_{2n+2}, x_{2n+2}, x_{2n+3}) + S(y_{2n+2}, y_{2n+2}, y_{2n+3})) \\
\leq\ & (2S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + 2S(y_{2n+1}, y_{2n+1}, y_{2n+2})) \\
& +(2S(x_{2n+2}, x_{2n+2}, x_{2n+3}) + 2S(y_{2n+2}, y_{2n+2}, y_{2n+3})) \\
& +... + (2S(x_{2m-1}, x_{2m-1}, x_{2m}) + 2S(y_{2m-1}, y_{2m-1}, y_{2m})) \\
& +(S(x_{2m}, x_{2m}, x_{2m+1}) + S(y_{2m}, y_{2m}, y_{2m+1})) \\
\leq\ & 2S(x_{2n+2}, x_{2n+2}, x_{2n+3}) + 2S(y_{2n+2}, y_{2n+2}, y_{2n+3})) \\
& +... + (2S(x_{2m}, x_{2m}, x_{2m+1}) + 2S(y_{2m}, y_{2m}, y_{2m+1})) \\
\leq\ & 2(A^{2n+1} + A^{2n+2} + ... + A^{2m})(S(x_0, x_0, x_1) + S(y_0, y_0, y_1)) \\
\leq\ & \frac{2A^{2n+1}}{1-A}(S(x_0, x_0, x_1) + S(y_0, y_0, y_1)).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
& S(x_{2n}, x_{2n}, x_{2m+1}) + S(y_{2n}, y_{2n}, y_{2m+1}) \\
\leq\ & 2(A^{2n} + A^{2n+1} + A^{2n+2} + ... + A^{2m})(S(x_0, x_0, x_1) + S(y_0, y_0, y_1)) \\
\leq\ & \frac{2A^{2n}}{1-A}(S(x_0, x_0, x_1) + S(y_0, y_0, y_1))
\end{aligned}$$

*and*

$$\begin{aligned}
& S(x_{2n}, x_{2n}, x_{2m}) + S(y_{2n}, y_{2n}, y_{2m}) \\
\leq\ & (2S(x_{2n}, x_{2n}, x_{2n+1}) + 2S(y_{2n}, y_{2n}, y_{2n+1})) \\
& +(S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + S(y_{2n+1}, y_{2n+1}, y_{2n+2})) \\
\leq\ & (2S(x_{2n}, x_{2n}, x_{2n+1}) + 2S(y_{2n}, y_{2n}, y_{2n+1})) \\
& +(2S(x_{2n+1}, x_{2n+1}, x_{2n+2}) + 2S(y_{2n+1}, y_{2n+1}, y_{2n+2})) \\
& +... + (2S(x_{2m-2}, x_{2m-2}, x_{2m-1}) + (y_{2m-2}, y_{2m-2}, y_{2m-1})) \\
& +(S(x_{2m-1}, x_{2m-1}, x_{2m}) + S(y_{2m-1}, y_{2m-1}, y_{2m}))
\end{aligned}$$

$$
\begin{aligned}
&\leq \; (2S(x_{2n+1},x_{2n+1},x_{2n+2})+2S(y_{2n+1},y_{2n+1},y_{2n+2}))\\
&\quad +...+(2S(x_{2m-1},x_{2m-1},x_{2m})+2S(y_{2m-1},y_{2m-1},y_{2m}))\\
&\leq \; 2(A^{2n}+A^{2n+1}+A^{2n+2}+...+A^{2m-1})(S(x_0,x_0,x_1)+S(y_0,y_0,y_1))\\
&\leq \; \frac{2A^{2n}}{1-A}(S(x_0,x_0,x_1)+S(y_0,y_0,y_1))
\end{aligned}
$$

and

$$
\begin{aligned}
&S(x_{2n+1},x_{2n+1},x_{2m})+d(y_{2n+1},y_{2n+1},y_{2m})\\
&\leq \; 2(A^{2n+1}+A^{2n+2}+A^{2n+3}+...+A^{2m-1})(S(x_0,x_0,x_1)+S(y_0,y_0,y_1))\\
&\leq \; \frac{2A^{2n+1}}{1-A}(S(x_0,x_0,x_1)+S(y_0,y_0,y_1)).
\end{aligned}
$$

Hence for all $m,n\in\mathbb{N}$ with $n\leq m$, it follows that

$$
S(x_n,x_n,x_m)+S(y_n,y_n,y_m)\leq\frac{2A^{2n}}{1-A}(S(x_0,x_0,x_1)+S(y_0,y_0,y_1))
$$

Since $0\leq A<1$, taking limit as $m,n\to\infty$, we get

$$
\lim_{m,n\to\infty}(S(x_n,x_n,x_m))+S(y_n,y_n,y_m))\to 0
$$

This implies that

$$
\lim_{m,n\to\infty}(S(x_n,x_n,x_m))\to 0 \quad and \quad \lim_{m,n\to\infty}(S(y_n,y_n,y_m))\to 0.
$$

Therefore the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $X$. Since $(X,S)$ is a complete $S$-metric space, then there exist $x,y\in X$ such that $x_n\to x$ and $y_n\to y$ We now prove that $(x,y)$ is a coupled common fixed point of $f$ and $g$. We consider the following two cases.

Case 1. Suppose that $f$ is continuous. Then we have

$$
x=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}f(x_{2n},y_{2n})=f(\lim_{n\to\infty}x_{2n},\lim_{n\to\infty}y_{2n})=f(x,y)
$$

and

$$y = \lim_{n\to\infty} x_{yn+1} = \lim_{n\to\infty} f(y_{2n}, x_{2n}) = f(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}) = f(y, x).$$

Taking $x = u$ and $y = v$ in (4.1), we have

$$\begin{aligned}
& S(f(x,y), f(x,y), g(x,y) + S(f(y,x), f(y,x), g(y,x)) \\
\leq\ & \frac{p}{2}D\Big((x,y),(x,y),(x,y)\Big) + \frac{q}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big) \\
& + \frac{r}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \\
& + \frac{s}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \\
& + \frac{s}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big) \\
& + \frac{p}{2}D\Big((y,x),(y,x),(y,x)\Big) + \frac{q}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big) \\
& + \frac{r}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big) \\
& + \frac{s}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big) \\
& + \frac{s}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big) \\
=\ & \frac{p}{2}D\Big((x,y),(x,y),(x,y)\Big) + \frac{q}{2}D\Big((x,y),(x,y),(x,y)\Big) \\
& + \frac{r}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \\
& + \frac{s}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \\
& + \frac{s}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big) \\
& + \frac{p}{2}D\Big((y,x),(y,x),(y,x)\Big) + \frac{q}{2}D\Big((y,x),(y,x),(y,x)\Big) \\
& + \frac{r}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big) \\
& + \frac{s}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big) \\
& + \frac{s}{2}D\Big((y,x),(y,x),(y,x)\Big) \\
=\ & \frac{r}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \\
& + \frac{s}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \\
=\ & \frac{r}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big) \\
& + \frac{s}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big).
\end{aligned}$$

Therefore

$$\begin{aligned} & S(f(x,y),f(x,y),g(x,y)+S(f(y,x),f(y,x),g(y,x)) \\ \leq\ & (r+s)\Big(D((x,y),(x,y),(g(x,y),g(y,x)) \\ & +D((y,x),(y,x),(g(y,x),g(x,y))\Big). \end{aligned}$$

That is,

$$S(x,x,g(x,y))+S(y,y,g(y,x))\leq(r+s)\Big(S(x,x,g(x,y))+S(y,y,g(y,x))\Big).$$

Since $0\leq r+s<1$, we get

$S(x,x,g(x,y))=0$ and $S(y,y,g(y,x))=0$, that is $g(x,y)=x$ and $g(y,x)=y$.

Therefore $(x,y)$ is a coupled common fixed point of $f$ and $g$.

Case 2. Suppose that $g$ is continuous. Then we have

$$x=\lim_{n\to\infty}x_{2n+1}=\lim_{n\to\infty}g(x_{2n},y_{2n})=g(\lim_{n\to\infty}x_{2n},\lim_{n\to\infty}y_{2n})=g(x,y)$$

and

$$y=\lim_{n\to\infty}x_{yn+1}=\lim_{n\to\infty}g(y_{2n},x_{2n})=g(\lim_{n\to\infty}y_{2n},\lim_{n\to\infty}x_{2n})=g(y,x)$$

Taking $x=u$ and $y=v$ in (4.1), we have

$$\begin{aligned} & S(g(x,y),g(x,y),f(x,y)+S(g(y,x),g(y,x),f(y,x)) \\ \leq\ & \frac{p}{2}\Big(D((x,y),(x,y),(x,y)\Big) \\ & +\frac{q}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \\ & +\frac{r}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big) \\ & +\frac{s}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big) \\ & +\frac{s}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big) \end{aligned}$$

$$
\begin{aligned}
&+\frac{p}{2}D\Big((y,x),(y,x),(y,x)\Big)\\
&+\frac{q}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big)\\
&+\frac{r}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big)\\
&+\frac{s}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big)\\
&+\frac{s}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big)\\
= &\ \frac{p}{2}D\Big((x,y),(x,y),(x,y)\Big)+\frac{q}{2}D\Big((x,y),(x,y),(x,y)\Big)\\
&+\frac{r}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big)\\
&+\frac{s}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big)\\
&+\frac{s}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big)\\
&+\frac{p}{2}D\Big((y,x),(y,x),(y,x)\Big)+\frac{q}{2}D\Big((y,x),(y,x),(y,x)\Big)\\
&+\frac{r}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big)\\
&+\frac{s}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big)\\
&+\frac{s}{2}D\Big((y,x),(y,x),(y,x)\Big)\\
= &\ \frac{r}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big)\\
&+\frac{s}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big)\\
= &\ \frac{r}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big)\\
&+\frac{s}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big).\\
\leq &\ (r+s)\Big(D((x,y),(x,y),(f(x,y),f(y,x))\\
&+D((y,x),(y,x),(f(y,x),f(x,y))\Big).
\end{aligned}
$$

That is,

$$S(x,x,f(x,y))+S(y,y,f(y,x))\leq(r+s)\Big(S(x,x,f(x,y))+S(y,y,f(y,x))\Big).$$

Since $0\leq r+s<1$, we get

$S(x,x,f(x,y))=0$ and $S(y,y,f(y,x))$, that is $f(x,y)=x$ and $f(y,x)=y$.

Therefore $(x,y)$ is a coupled common fixed point of $f$ and $g$ □

**Theorem 4.2.2** *Let $(X, \preceq, S)$ be a partially ordered complete $S$-metric space. Assume that $X$ has the following property:*

(1) *If $\{x_n\}$ is an increasing sequence with $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(2) *If $\{y_n\}$ is a decreasing sequence with $y_n \to y$, then $y_n \succeq y$ for all $n \in \mathbb{N}$*

*Let f,g: $X \times X \to X$ be the mappings such that a pair (f,g) has the mixed weakly monotone property on X. Let $x_0, y_0 \in X$ such that $x_0 \preceq f(x_0, y_0)$, $y_0 \succeq f(y_0.x_0)$ or $x_0 \preceq g(x_0, y_0)$, $y_0 \succeq g(y_0.x_0)$. Suppose that there exist p,q,r,s $\geq$ 0 with p+q+r+2s < 1 such that*

$$\begin{aligned}
S(f(x,y), f(x,y), g(u,v)) \leq\ & \frac{p}{2}D((x,y),(x,y),(u,v)) \\
&+\frac{q}{2}D((x,y),(x,y),(f(x,y),f(y,x))) \\
&+\frac{r}{2}D((u,v),(u,v),(g(u,v),g(v,u))) \\
&+\frac{s}{2}D((x,y),(x,y),(g(u,v),g(v,u))) \\
&+\frac{s}{2}D((u,v),(u,v),(f(x,y),f(y,x)))
\end{aligned}$$

*for all $x, y, u, v \in X$ with $x \preceq u$ and $y \succeq v$, then $f$ and $g$ have a common coupled fixed point in $X$.*

**Proof.** Following the proof of Theorem 4.3.1, we only have to show that

$f(x,y) = g(x,y) = x$ and $f(y,x) = g(y,x) = y$

We will first show that $f(x,y) = x$ and $f(y,x) = y$

Now, using Lemma 3.2.2 and Lemma 3.2.9, we have

$$\begin{aligned}
& D((x,y),(x,y),(f(x,y),f(y,x))) \qquad (4.11)\\
\leq\ & 2D((x,y),(x,y),(x_{2n+2},y_{2n+2})) \\
& +D((x_{2n+2},y_{2n+2}),(x_{2n+2},y_{2n+2}),(f(x,y),f(y,x)))
\end{aligned}$$

$$
\begin{aligned}
&= 2D((x,y),(x,y),(x_{2n+2},y_{2n+2})) \\
&\quad +D((g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1})),(f(x,y),f(y,x))) \\
&\le 2D((x,y),(x,y),(x_{2n+2},y_{2n+2})) + S(g(x_{2n+1},y_{2n+1}),g(x_{2n+1},y_{2n+1}),f(x,y)) \\
&\quad +S(g(y_{2n+1},x_{2n+1}),g(y_{2n+1},x_{2n+1}),f(y,x)) \\
&= 2S(x,x,x_{2n+2}) + 2S(y,y,y_{2n+2}) \\
&\quad +S(g(x_{2n+1},y_{2n+1}),g(x_{2n+1},y_{2n+1}),f(x,y)) + S(f(y,x),f(y,x),g(y_{2n+1},x_{2n+1}))
\end{aligned}
$$

By interchanging the roles of $f$ and $g$ and using (4.1), consider

$$
\begin{aligned}
& S(g(x_{2n+1},y_{2n+1}),g(x_{2n+1},y_{2n+1}),f(x,y)) \qquad (4.12)\\
\le\ & \frac{p}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x,y)) \\
& +\frac{q}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1}))) \\
& +\frac{r}{2}D((x,y),(x,y),(f(x,y),f(y,x))) \\
& +\frac{s}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x))) \\
& +\frac{s}{2}D((x,y),(x,y),(g(x_{2n+1},y_{2n+1}),g(y_{2n+1},x_{2n+1}))) \\
=\ & +\frac{p}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x,y)) \\
& +\frac{q}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2})) \\
& +\frac{r}{2}D((x,y),(x,y),(f(x,y),f(y,x))) \\
& +\frac{s}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x))) \\
& +\frac{s}{2}D((x,y),(x,y),(x_{2n+2},y_{2n+2}))
\end{aligned}
$$

Again, using (4.1),

$$
\begin{aligned}
& S(f(y,x),f(y,x),g(y_{2n+1},x_{2n+1})) \qquad (4.13)\\
\le\ & \frac{p}{2}D((y,x),(y,x),(y_{2n+1},x_{2n+1})) + \frac{q}{2}D((y,x),(y,x),(f(y,x),f(x,y))) \\
& +\frac{r}{2}D((y_{2n+1},x_{2n+1}),(y_{2n+1},x_{2n+1}),g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1})))
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{s}{2}D((y,x),(y,x),(g(y_{2n+1},x_{2n+1}),g(x_{2n+1},y_{2n+1}))) \\
& +\frac{s}{2}D((y_{2n+1},x_{2n+1}),(y_{2n+1},x_{2n+1}),(f(y,x),f(x,y))) \\
= & \frac{p}{2}D((y,x),(y,x),(y_{2n+1},x_{2n+1})) + \frac{q}{2}D((y,x),(y,x),(f(y,x),f(x,y))) \\
& +\frac{r}{2}D((y_{2n+1},x_{2n+1}),(y_{2n+1},x_{2n+1}),(y_{2n+2},x_{2n+2})) \\
& +\frac{s}{2}D((y,x),(y,x),(y_{2n+2},x_{2n+2})) \\
& +\frac{s}{2}D((y_{2n+1},x_{2n+1}),(y_{2n+1},x_{2n+1}),(f(y,x),f(x,y)))
\end{aligned}
$$

From (4.11), (4.12) and (4.13), it follows that

$$
\begin{aligned}
& S(x,x,f(x,y)) + S(y,y,f(y,x)) \qquad (4.14) \\
\leq & \; 2S(x,x,x_{2n+2}) + 2S(y,y,y_{2n+2}) + \frac{p}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x,y)) \\
& +\frac{q}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(x_{2n+2},y_{2n+2})) \\
& +\frac{r}{2}D((x,y),(x,y),(f(x,y),f(y,x))) \\
& +\frac{s}{2}D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(f(x,y),f(y,x))) \\
& +\frac{s}{2}D((x,y),(x,y),(x_{2n+2},y_{2n+2})) \\
& +\frac{p}{2}D((y,x),(y,x),(y_{2n+1},x_{2n+1})) + \frac{q}{2}D((y,x),(y,x),(f(y,x),f(x,y))) \\
& +\frac{r}{2}D((y_{2n+1},x_{2n+1}),(y_{2n+1},x_{2n+1}),(y_{2n+2},x_{2n+2})) \\
& +\frac{s}{2}D((y,x),(y,x),(y_{2n+2},x_{2n+2})) \\
& +\frac{s}{2}D((y_{2n+1},x_{2n+1}),(y_{2n+1},x_{2n+1}),(f(y,x),f(x,y)))
\end{aligned}
$$

Taking limit as $n \to \infty$ in (4.14), we obtain

$$
\begin{aligned}
& S(x,x,f(x,y)) + S(y,y,f(y,x)) \qquad (4.15) \\
\leq & \; 2S(x,x,x) + 2S(y,y,y) + \frac{p}{2}D((x,y),(x,y),(x,y)) \\
& +\frac{q}{2}D((x,y),(x,y),(x,y)) + \frac{r}{2}D((x,y),(x,y),(f(x,y),f(y,x))) \\
& +\frac{s}{2}D((x,y),(x,y),(f(x,y),f(y,x))) + \frac{s}{2}D((x,y),(x,y),(x,y))
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{p}{2}D((y,x),(y,x),(y,x)) + \frac{q}{2}D((y,x),(y,x),(f(y,x),f(x,y)))\\
&+\frac{r}{2}D((y,x),(y,x),(y,x)) + \frac{s}{2}D((y,x),(y,x),(y,x))\\
&+\frac{s}{2}D((y,x),(y,x),(f(y,x),f(x,y)))\\
=\ & \frac{r+s}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big)\\
&+\frac{q+s}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big)
\end{aligned}
$$

This implies that

$$
\begin{aligned}
& S(x,x,f(x,y)) + S(y,y,f(y,x))\\
\leq\ & \frac{r+s}{2}\Big(S(x,x,f(x,y) + S(y,y,f(y,x))\Big)\\
&+\frac{q+s}{2}\Big(S(y,y,f(y,x) + S(x,x,f(x,y))\Big)\\
=\ & \frac{r+q+2s}{2}\Big(S(x,x,(f(x,y) + S(y,y,f(y,x))\Big)
\end{aligned}
$$

Since $\frac{r+q+2s}{2} < 1$, we have

$S(x,x,f(x,y) + S(y,y,f(y,x)) = 0$, that is $f(x,y) = x$ and $f(y,x) = y$.

Next, we show that $g(x,y) = x$ and $g(y,x) = y$.

Using Lemma 3.2.2 and Lemma 3.2.9, we have

$$
\begin{aligned}
& D((x,y),(x,y),(g(x,y),g(y,x))) \qquad (4.16)\\
\leq\ & 2D((x,y),(x,y),(x_{2n+1},y_{2n+1}))\\
&+D((x_{2n+1},y_{2n+1}),(x_{2n+1},y_{2n+1}),(g(x,y),g(y,x)))\\
=\ & 2D((x,y),(x,y),(x_{2n+1},y_{2n+1}))\\
&+D((f(x_{2n},y_{2n}),f(y_{2n},x_{2n}),(f(x_{2n},y_{2n}),f(y_{2n},x_{2n}),(g(x,y),g(y,x)))\\
\leq\ & 2D((x,y),(x,y),(x_{2n+1},y_{2n+1})) + S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x,y))\\
&+S(f(y_{2n},x_{2n}),f(y_{2n},x_{2n}),g(y,x))\\
=\ & 2S(x,x,x_{2n+1}) + 2S(y,y,y_{2n+1}) + S(f(x_{2n},y_{2n}),f(x_{2n},y_{2n}),g(x,y))\\
&+S(g(y,x),g(y,x),f(y_{2n},x_{2n})).
\end{aligned}
$$

Now, consider

$$
\begin{aligned}
& S(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}).g(x,y)) \qquad (4.17)\\
\leq\ & \frac{p}{2}D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x,y))\\
& +\frac{q}{2}D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))\\
& +\frac{r}{2}D((x,y), (x,y), (g(x,y), g(y,x)))\\
& +\frac{s}{2}D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (g(x,y), g(y,x)))\\
& +\frac{s}{2}D((x,y), (x,y), (f(x_{2n}, y_{2n}), f(y_{2n}, x_{2n})))\\
=\ & +\frac{p}{2}D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x,y))\\
& +\frac{q}{2}D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (x_{2n+1}, y_{2n+1}))\\
& +\frac{r}{2}D((x,y), (x,y), (g(x,y), g(y,x)))\\
& +\frac{s}{2}D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), (g(x,y), g(y,x)))\\
& +\frac{s}{2}D((x,y), (x,y), (x_{2n+1}, y_{2n+1})).
\end{aligned}
$$

Again, using (4.1),

$$
\begin{aligned}
& S(g(y,x), g(y,x), g(y_{2n}, x_{2n}) \qquad (4.18)\\
\leq\ & \frac{p}{2}D((y,x), (y,x), (y_{2n}, x_{2n})) + \frac{q}{2}D((y,x), (y,x), (g(y,x), g(x,y)))\\
& +\frac{r}{2}D((x_{2n}, y_{2n}), (x_{2n}, y_{2n}), f(y_{2n}, x_{2n}), f(x_{2n}, y_{2n})))\\
& +\frac{s}{2}D((y,x), (y,x), (f(y_{2n}, x_{2n}), f(x_{2n}, y_{2n})))\\
& +\frac{s}{2}D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (g(y,x), g(x,y)))\\
=\ & \frac{p}{2}D((y,x), (y,x), (y_{2n}, x_{2n})) + \frac{q}{2}D((y,x), (y,x), (g(y,x), g(x,y)))\\
& +\frac{r}{2}D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (y_{2n+1}, x_{2n+1}))\\
& +\frac{s}{2}D((y,x), (y,x), (y_{2n+1}, x_{2n+1}))\\
& +\frac{s}{2}D((y_{2n}, x_{2n}), (y_{2n}, x_{2n}), (g(y,x), g(x,y))).
\end{aligned}
$$

From (4.16), (4.17) and (4.18), it follows that

$$
\begin{aligned}
& S(x,x,g(x,y)) + d(y,y,g(y,x)) \qquad (4.19)\\
\leq\ & S(x,x,x_{2n+1}) + S(y,y,y_{2n+1}) + \frac{p}{2}D((x_{2n},y_{2n}),(x_{2n},y_{2n}),(x,y))\\
& +\frac{q}{2}D((x_{2n},y_{2n}),(x_{2n},y_{2n}),(x_{2n+1},y_{2n+1}))\\
& +\frac{r}{2}D((x,y),(x,y),(g(x,y),g(y,x)))\\
& +\frac{s}{2}D((x_{2n},y_{2n}),(x_{2n},y_{2n}),(g(x,y),g(y,x)))\\
& +\frac{s}{2}D((x,y),(x,y),(x_{2n+1},y_{2n+1}))\\
& +\frac{p}{2}D((y,x),(y,x),(y_{2n},x_{2n})) + \frac{q}{2}D((y,x),(y,x),(g(y,x),g(x,y)))\\
& +\frac{r}{2}D((y_{2n},x_{2n}),(y_{2n},x_{2n}),(y_{2n+1},x_{2n+1}))\\
& +\frac{s}{2}D((y,x),(y,x),(y_{2n+1},x_{2n+1}))\\
& +\frac{s}{2}D((y_{2n},x_{2n}),(y_{2n},x_{2n}),(g(y,x),g(x,y)))
\end{aligned}
$$

Taking limit as $n \to \infty$ in (4.19), we obtain

$$
\begin{aligned}
& S(x,x,g(x,y)) + S(y,y,g(y,x)) \qquad (4.20)\\
\leq\ & S(x,x,x) + S(y,y,y) + \frac{p}{2}D((x,y),(x,y),(x,y))\\
& +\frac{q}{2}D((x,y),(x,y),(x,y)) + \frac{r}{2}D((x,y),(x,y),(g(x,y),g(y,x)))\\
& +\frac{s}{2}D((x,y),(x,y),(g(x,y),g(y,x))) + \frac{s}{2}D((x,y),(x,y),(x,y))\\
& +\frac{p}{2}D((y,x),(y,x),(y,x)) + \frac{q}{2}D((y,x),(y,x),(g(y,x),g(x,y)))\\
& +\frac{r}{2}D((y,x),(y,x),(y,x)) + \frac{s}{2}D((y,x),(y,x),(y,x))\\
& +\frac{s}{2}D((y,x),(y,x),(g(y,x),g(x,y)))\\
=\ & \frac{r+s}{2}D\Big((x,y),(x,y),(g(x,y),g(y,x))\Big)\\
& +\frac{q+s}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big)\\
\leq\ & \frac{r+s}{2}\Big(S(x,x,g(x,y) + S(y,y,g(y,x))\Big) + \frac{q+s}{2}\Big(S(y,y,g(y,x) + S(x,x,g(x,y))\Big)\\
=\ & \frac{r+q+2s}{2}\Big(S(x,x,(g(x,y) + S(y,y,g(y,x))\Big)
\end{aligned}
$$

Since $\frac{r+q+2s}{2} < 1$, we have

$S(x,x,g(x,y) + S(y,y,g(y,x)) = 0$, that is $g(x,y) = x$ and $g(y,x) = y$.

This implies that $(x,y)$ is a coupled common fixed point of $f$ and $g$ □

**Corollary 4.2.3** *Let $(X, \leq, S)$ be a partially ordered complete $S$-metric space. Let $f : X \times X \to X$ be a mapping with the mixed monotone property on X. Let $x_0, y_0 \in X$ such that $x_0 \leq f(x_0, y_0)$, $y_0 \geq f(y_0.x_0)$. Suppose that there exist $p, q, r, s \geq 0$ with $p + q + r + 2s <$ 1 such that*

$$\begin{aligned} S(f(x,y), f(x,y), f(u,v)) \leq & \frac{p}{2} D((x,y),(x,y),(u,v)) \\ & + \frac{q}{2} D((x,y),(x,y),(f(x,y),f(y,x))) \\ & + \frac{r}{2} D((u,v),(u,v),(f(u,v),f(v,u))) \\ & + \frac{s}{2} D((x,y),(x,y),(f(u,v),f(v,u))) \\ & + \frac{s}{2} D((u,v),(u,v),(f(x,y),f(y,x))) \end{aligned}$$

for all $x, y, u, v \in X$ with $x \leq u$ and $y \geq v$. Suppose also that either $f$ is continuous or $X$ has the following properties:

(1) If $\{x_n\}$ is an increasing sequence with $x_n \to x$, then $x_n \leq x$ for all $n \in \mathbb{N}$;

(2) If $\{y_n\}$ is a decreasing sequence with $y_n \to y$, then $y_n \geq y$ for all $n \in \mathbb{N}$,

*then $f$ has a coupled fixed point in $X$.*

**Proof.** Taking $f = g$ in Theorems 4.2.1, 4.2.2 and using Remark 3.2.2 we get the conclusion. □

In the next result we consider $X$ to be a totally ordered set.

**Corollary 4.2.4** In addition to Theorems 4.3.1 and 4.3.2, suppose that $X$ is totally ordered set, then a coupled fixed of $f$ and $g$ is unique and $x = y$.

**Proof.** If $(x^*, y^*) \in X \times X$ is another coupled common fixed point of $f$ and $g$, then by the use of (4.1) and Lemma 3.2.1, we have

$$
\begin{aligned}
& D((x,y),(x,y),(x^*,y^*)) \\
= \; & S(x,x,x^*) + (y,y,y^*) \\
= \; & S(f(x,y),f(x,y),g(x^*,y^*) + S(f(y,x),f(y,x),g(y^*,x^*)) \\
\le \; & \frac{p}{2}D\Big((x,y),(x,y),(x^*,y^*)\Big) + \frac{q}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big) \\
& + \frac{r}{2}D\Big((x^*,y^*),(x^*,y^*),(g(x^*,y^*),g(y^*,x^*))\Big) \\
& + \frac{s}{2}D\Big((x,y),(x,y),(g(x^*,y^*),g(y^*,x^*))\Big) \\
& + \frac{s}{2}D\Big((x^*,y^*),(x^*,y^*),(f(x,y),f(y,x))\Big) \\
& + \frac{p}{2}D\Big((y,x),(y,x),(y^*,x^*)\Big) + \frac{q}{2}D\Big((y,x),(y,x),(f(y,x),f(x,y))\Big) \\
& + \frac{r}{2}D\Big((y^*,x^*),(y^*,x^*),(g(y^*,x^*),g(x^*,y^*))\Big) \\
& + \frac{s}{2}D\Big((y,x),(y,x),(g(y^*,x^*),g(x^*,y^*))\Big) \\
& + \frac{s}{2}D\Big((y^*,x^*),(y^*,x^*),(f(y,x),f(x,y))\Big) \\
= \; & \frac{p}{2}D\Big((x,y),(x,y),(x^*,y^*)\Big) + \frac{q}{2}D\Big((x,y),(x,y),(x,y)\Big) \\
& + \frac{r}{2}D\Big((x^*,y^*),(x^*,y^*),(x^*,y^*)\Big) + \frac{s}{2}D\Big((x,y),(x,y),(x^*,y^*)\Big) \\
& + \frac{s}{2}D\Big((x^*,y^*),(x^*,y^*),(x,y))\Big) \\
& + \frac{p}{2}D\Big((y,x),(y,x),(y^*,x^*)\Big) + \frac{q}{2}D\Big((y,x),(y,x),(y,x)\Big) \\
& + \frac{r}{2}D\Big((y^*,x^*),(y^*,x^*),(y^*,x^*))\Big) + \frac{s}{2}D\Big((y,x),(y,x),(y^*,x^*))\Big) \\
& + \frac{s}{2}D\Big((y^*,x^*),(y^*,x^*),(y,x)\Big) \\
= \; & \frac{p}{2}D\Big((x,y),(x,y),(x^*,y^*)\Big) + \frac{s}{2}D\Big((x,y),(x,y),(x^*,y^*)\Big) \\
& + \frac{s}{2}D\Big((x,y),(x,y),(x^*,y^*)\Big) \\
& + \frac{p}{2}D\Big((y,x),(y,x),(y^*,x^*)\Big) + \frac{s}{2}D\Big((y,x),(y,x),(y^*,x^*)\Big) \\
& + \frac{s}{2}D\Big((y,x),(y,x),(y^*,x^*)\Big) \\
= \; & \frac{p+2s}{2}\Big(D((x,y),(x,y),(x^*,y^*)) + D((y,x),(y,x),(y^*,x^*))\Big) \\
= \; & (p+2s)(S(x,x,x^*) + S(y,y,y^*)).
\end{aligned}
$$

Since $p+2s<1$, we get $S(x,x,x^*)+S(y,y,y^*)=0$.

This implies that $x=x^*$ and $y=y^*$. Hence the coupled common fixed point of $f$ and $g$ is unique.

Now

$$\begin{aligned}
S(x,x,y) &= S(f(x,y),f(x,y),g(y,x) \\
&\le \frac{p}{2}D\Big((x,y),(x,y),(y,x)\Big) \\
&\quad +\frac{q}{2}D\Big((x,y),(x,y),(f(x,y),f(y,x))\Big) \\
&\quad +\frac{r}{2}D\Big((y,x),(y,x),(g(y,x),g(x,y))\Big) \\
&\quad +\frac{s}{2}D\Big((x,y),(x,y),(g(y,x),g(x,y))\Big) \\
&\quad +\frac{s}{2}D\Big((y,x),(y,x),(f(x,y),f(y,x))\Big) \\
&= \frac{p}{2}D\Big((x,y),(x,y),(y,x)\Big)+\frac{s}{2}D\Big((x,y),(x,y),(y,x))\Big) \\
&\quad +\frac{s}{2}D\Big((y,x),(y,x),(x,y)\Big) \\
&= \frac{p+2s}{2}D\Big((x,y),(x,y),(y,x)\Big) \\
&= (p+2s)(x,x,y)
\end{aligned}$$

Since $p+2s<1$, we get $S(x,x,y)=0$, which implies that $x=y$. □

Now we give an example to demonstrate Theorem 4.2.1

**Example 4.2.1** Let $X=\mathbb{R}$ with the $S$-metric as in Example 1.1.2.

Let $f,g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be two functions defined by

$$f(x,y)=\frac{4x-2y+22}{24},\quad g(u,v)=\frac{6u-3v+33}{36}$$

Then a pair $(f,g)$ has the mixed weakly monotone property and

$$\begin{aligned}
S(f(x,y),f(x,y),g(u,v)) &= 2|f(x,y)-g(u,v)| \\
&= 2\left|\frac{4x-2y+22}{24}-\frac{6u-3v+33}{36}\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq 2\left(\left|\frac{12x-12u}{72}\right| + \left|\frac{6y-6v}{72}\right|\right)\\
&\leq \frac{1}{3}|x-u| + \frac{1}{6}|y-v|\\
&= \frac{1}{3}(|x-u| + \frac{1}{2}|y-v|)\\
&\leq \frac{1}{3}(|x-u| + |y-v|).
\end{aligned}
$$

Putting $p = \frac{1}{3}$ and $q = r = s = 0$ in (4.1), we see that (1,1) is a unique coupled common fixed point of $f$ and $g$.

# CHAPTER FIVE

# SUMMARY, CONCLUSION AND RECOMMENDATIONS

## 5.1 Summary

In this dissertation we considered the notion of $S$-metric space as a generalized metric in 3-tuples and studied some of its properties and existing results.

We reviewed some coupled fixed point theorems in partially ordered metric space and used the notion of a mixed weakly monotone pair of mappings to establish some coupled common fixed point theorems in partially ordered $S$-metric space.

The results presented in chapter four (theorems 4.2.1 and 4.2.2) generalize the results of Gordji et al. (2012) (theorems 2.3.3 and 2.3.4) in the framework of $S$-metric space.

## 5.2 Conclusion

In this study, we used a similar method for proving the existing results of coupled fixed point theorems for mixed weakly monotone mappings in partially ordered metric space have been used to prove our results in the framework of $S$-metric space.

The results reveal that though depending on the contractive mapping or contractive condition used for an existing result in metric space, one may generalize such results in the setting of $S$-metric space.

## 5.3 Recommendations

Various fixed point results for different contractive mappins in metric space have been established. Some of these results have not been (or cannot be) generalized in the framework of $S$-metric space.

We therefore recommend that the existence of fixed point for different contractive type mappings in the setting of $S$-metric space should be investigated.

Also a pair of mixed weakly monotone mappings may be extended to a more general setting of multivalued mixed weakly monotone mappings.

# References


Agarwal, R. P., El-Gebiely, M. A., and O'Regan, D. (2008). Generalized contraction in partially ordered metric spaces. *Applied Analysis*, 87:109–116.

Ariza-Ruiz, D. Jimnez-Melado, A. (2009). A continuation method for weakly contractive mappings under the interior condtions. *Fixed Point Theory Application2009.*, Article ID 809315.

Berinde, V. (2011). Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces. *Nonlinear Analysis*, 74:1–18.

Bhaskar, T. G. and Lakshmikantham, V. (2006). Fixed point theorem in partially ordered metric spaces and applications. *Nonlinear Analysis*, 65:1379–1393.

Chatterjea, S. K. (1972). Fixed point theorems. *Academie Bulgare Des Science*, 25:727–730.

Cho, Y. J., Rhoades, B. E., Saadati, R., Samet, B., and Shantawi, W. (2012). Coupled fixed point theorem in ordered generalized metric spaces with integral types. *Fixed Point Theory Application2012.*, Article ID 8.

Choudhury, B. S. (2009). Generalized contraction in partially ordered metric spaces. *Kathmandu University Journal of Science, Engineering and Technology*, 5:6–13.

Dhage, B. C. (1984). *A study of some fixed point theorems*. PhD thesis, Marathwada University Aurangabad India.

Dhage, B. C. (1994). On generalized metric spaces and topological structure *I. Pure and Applied Mathematical Sciences*, 40:37–41.

Dhage, B. C. (2000). On generalized metric spaces and topological structure *II. Pure and Applied Mathematical Sciences*, 46:3–24.

Dung, N. V. (2013). On coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces. *Fixed Point Theory Application 2013.*, Article ID 48.

Gahler, S. (1963). 2-metriche raume und ihre toplogische strukture. *Mathematische Nachrichten*, 26:115–148.

Gordji, M. E., Baghani, H., and Cho, Y. J. (2011). Coupled fixed point theorems for contractions in intuitionistic fuzzy normed spaces. *Mathematical and computer Modelling*, 54:1897–1906.

Gordji, M. E., Ramezani, M., Cho, Y. J., and Akbartabar, E. (2012). Coupled common fixed point theorems for mixed weakly monotone mappings in partially ordered metric spaces. *Fixed Point Theory Application2012.*, Article ID 95.

Harjani, J. and Sadarangani, K. (2009). Fixed point theorems for weakly contractive mappings in partially ordered sets. *Nonlinear Analysis*, 71:3403–3410.

Mustafa, Z., Obiedat, H., and Awawdeh, F. (2008). Some common fixed point theorems for mapping on complete $G$-metric space. *Fixed Point Theory Application2008.*, Article ID 189870.

Mustafa, Z. and Sims, B. (2003). Some results concerning $D$-metric spaces. In *Proceedings of the International Conferences on Fixed Point Theory and Applications Valencia, Spain*, pages 189–198.

Mustafa, Z. and Sims, B. (2006). A new approach to generalized metric space. *Journal of Nonlinear convex Analysis*, 7:289–297.

Ran, A. C. M. and Reurings, M. C. B. (2003). A fixed point theorem in partially ordered set and some applications to matrix equations. *Proceedings of the American Mathematical Society*, 132:1435–1443.

Rhoades, B. E. (1977). A comparison of various definitions of contractive mappings. *Transactions of the American Mathematical Society*, 226:257–290.

Sedghi, S., Shobe, N., and Aliouche, A. (2012). A generalization of fixed point theorem in $S$-metric space. *Mathematical Vesinik*, 64:258–266.

Sedghi, S., Shobe, N., and Zhou, H. (2007). A common fixed point theorem in $D^*$-metric space. *Fixed Point Theory Application2007.*, Article ID 027906.

Sharma, A. K. (1980). Note on fixed points in 2-metric space. *Indian Journal of Pure and Applied Mathematics*, 11:1580–1583.

Sintunavarat, W., Cho, Y. J., and Kumam, P. (2011). Coupled coincidence fixed point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces. *Fixed Point Theory Application2011.*, Article ID 81.