THE INITIAL VALUE PROBLEM FOR THE SEQUENCE OF GENERALIZED KORTEWEG-DE VRIES EQUEATIONS

BY

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CERTIFICATION.

THIS THESIS TITLED "THE INITIAL VALUE PROBLEM FOR THE SEQUENCE OF GENERALIZED KORTEWEG-DE VRIES EQUATIONS" BY SALIU A. YUSUF MEETS THE REGULATIONS GOVERNING THE AWARD OF THE DEGREE OF MASTER OF SCIENCE IN MATHEMATICS OF AHMADU BELLO UNIVERSITY, ZARIA, AND IS APPROVED FOR ITS CONTRIBUTION TO ACKNOWLEDGE AND LITERARY PRESENTATION.

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DEDICATION.

This study is dedicated to my Father & Mother, Mr. and Mrs. Y.A Adinoyi and my wife &children, who have enjoyed less of my companionship as a son, husband and father during the period of my study.

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ABSTRACT.

Interest in nonlinear dispersive wave equations focussed recently on the simplest model equation of this type, namely,

$$u_t = uu_x + u_{xxx},$$

where subscripts denote partial differentiations. Kbrteweg and de-Vries first derived the equation in their study of long water waves in a (relatively shallow) channel. Recently, this equation has been derived in plasma physics and in studies of anharmonic (nonlinear) lattices. Existence and uniqueness of solutions of the above equation for appropriate initial and boundary conditions have recently been proved by Sjoberg.

This dissertation focusses on the initial value problem for the sequence of generalized Korteweg de-Vries equations, namely,

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} G_m(u),$$

where $\frac{\partial}{\partial x} G_{m+1}(u) = \left(\frac{\partial^3}{\partial x^3} + \frac{2}{3} u \frac{\partial}{\partial x} + \frac{1}{3} u_x \right) G_m(u)$ and $G_1(u) = u.$

We have shown that for each m, solutions of the generalized Korteweg-de-Vries equations exist for all time and are uniquely determined by arbitrary initial values. The entire work has been divided into four chapters. The first chapter covers the introduction, back-ground and definitions of some basic terms. In this chapter, we have also formulated, more precisely, our main results, in a theorem.

In chapter II, we stated and proved lemmas relating the various norms. Also in this chapter, an priori bound was obtained using the relationships among the norms.

Chapter III concentrates on the proof of the existence of a global solution to our initial value problem. Here, we proved the first part of the theorem formulated in chapter I.

The last chapter covers the proof of uniqueness of solution of the initial value problem started in chapter III. We have shown in this chapter that given an appropriate intial value the solution can be determined uniquely for all time.

Notation: Theorems, lemmas, definitions and equations are numbered decimally within the section and the number of the chapter is prefixed. Equations are indicated by a parenthesis :- (1.3.2) and a definition by a bracket [1,3.2]. A theorem or lemma is indicated without parenthesis or bracket :- 1.3.2. A single number in a bracket like [3] refers to the bibliography.

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CHAPTER 1

1.1 INTRODUCTION :

The nonlinear dispersive wave equation

$u_t = uu_x + u_{xxx}$,

(where subscripts denote partial differentiations) of Korteweg and de-Vries (KdV) is an integrable hamiltonian system. For instance, Gardner [5] has shown that if a function of x and t satisfies the Korteweg-de Vries equation and is periodic in x, then its Fourier components satisfy a Hamiltonian system of ordinary differential equations. Kruskal, Gardner, and Miura [7] have constructed explicitly an infinite sequence of functionals $F_n(u)$ which are constant along solutions of Korteweg-de Vries equation. Gordner [7] has, in addition, shown that the sequence of functionals $F_n(u)$ are also constant along solutions of the generalized KdV equations

$$(1.1.1) \qquad \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} G_m(u) = K_m(u), \quad m \geq 1$$

where $\qquad \frac{\partial}{\partial x} G_{m+1}(u) = \left(\frac{\partial^3}{\partial x^3} + \frac{2}{3} u \frac{\partial}{\partial x} + \frac{1}{3} u_x \right) G_m(u)$

and $\qquad G_1(u) = u.$

This work concerns the initial value problem for (1.1.1) which is periodic in x. Our aim will be to give a straightforward approach to the analysis of the initial value problem for the sequence of generalised Korteweg-de Vries equations. Sjoberg [16], and Kato [10] have carried out a similar study for Korteweg-de Vries, $m = 2$.

In this work, we have shown that for each m, solutions of (1.1.1) exist for all time, are uniquely determined by their initial values, and that these values can be prescribed arbitrarily. In order to formulate this result more precisely, let n be a nonnegative integer and let $H_n(S')$, $S' = [0, 1]$, denote the classical Sobolev space with norm

$$||v||_n = \left(\sum_{\ell \leq n} \int_0^1 |D^\ell v(x)|^2 dx \right)^{\frac{1}{2}} .$$

The main result can be summarized in a theorem as follow s:

Theorem 1.1.1. For each m, if $n \geq 2m - 1$ and the initial data g is an element of $H_n(S')$ then there exists a global solution $u(x, t)$ to (1.1.1) from $H_n(S')$. Furthermore, for $n \geq 3m - 2$, $u(x, t)$ is uniquely determined by the initial data g.

This result is established by first proving the existence of a unique global solution $u^\eta(x, t)$ to the initial value problem

$$(1.1.2) \qquad \frac{\partial u}{\partial t} = K_m(u) + \eta(-1)^{m+1}D^{2m}u,$$

$0 < \eta < 1, \quad u(x+1, t) = u(x, t),$

$u(x, 0) = g(x).$

We have reduced (1.1.2) to a problem of finding fixed points of completely continuous maps by considering the family of nonlinear problems

$$(1.1.3) \quad \frac{\partial u}{\partial t} = \tau \; K_m(u) + (-1)^{m+1}(\tau(\eta - 1) + 1)D^{2m}u,$$

$$0 < \eta < 1, \quad 0 \leq \tau \leq 1$$

and the related linear problem

$$(1.1.4) \quad \frac{\partial \nu}{\partial t} = \tau K_m(\omega) + (-1)^{m+1}(\tau(\eta - 1) + 1)D^{2m}\nu.$$

Throughout this work the letters V_n and W_n will signify the set of functions on $S' \times [0, T]$ with finite norms

$$|||\nu|||^2_{V_n} = \int_0^T || \frac{\partial \nu(t)}{\partial t} ||_0^2 dt + \int_0^T ||\nu(t)||^2_{n+m} dt + \operatorname{ess.sup}_{t \leq T} ||\nu(t)||^2_n$$

$$|||\nu|||^2_{W_n} = \int_0^T ||\nu(t)||^2_{n-1+m} dt + \operatorname{ess.sup}_{t \leq T} ||\nu(t)||^2_{n-1}$$

respectively. The norm in the space $L^p(S')$ is denoted by $|\nu|_p$. If $\omega \in W_{m'}$, for $m' \geq 2m$, there exists a global solution $\nu(x, t)$ of (1.1.4) from $V_{m'}$ which is uniquely determined by its initial values. The linear problem (1.1.4) defines a nonlinear operator

$\nu^\tau = \phi(\omega^\tau, \tau)$, whose fixed points are solutions of (1.1.3). The fixed points of ϕ for $\tau = 1$ are solutions of (1.1.2). We have applied the Leray-Schauder Theorem in chapter III of this work to determine the fixed points of ϕ which requires the uniform boundedness of the norm of the solution $u^{\tau}(x, t)$ of (1.1.3) in the space $W_{m'}$. We have given in chapter II an apriori bound for solutions of (1.1.3) from $V_{m'}$. We show in chapter III, that the solution $u^{\eta}(x, t)$ of (1.1.2) converges to a solution $u(x, t)$ of (1.1.1) as $\eta \to 0$. In chapter IV we show that solutions of (1.1.1) are uniquely determined by their initial values.

1.2 DEFINITIONS:

[1.2.1] $L^p(S')$. Let $1 \leq p < \infty$. $L^p(S')$ denote the set of all real-valued functions u defined almost everywhere on S' and such that the Lebesgue integral

$$\int_0^1 |u(x)|^p dx \text{ is finite .}$$

The space $L^p(S')$ is a Banach space under the norm defined by

$$|u|_p = \left(\int_0^1 |u(x)|^p dx \right)^{\frac{1}{p}}. \text{ In particular ,}$$

$L^2(S')$ is a Hilbert space when equipped with the inner product

$$< u , v > = \int_0^1 u(x) v(x)\, dx .$$

[1.2.2] The spaces $L^p(\Omega ; X)$.

Let $1 \le p < \infty$, let X be a Banach space with the norm $||\cdot||$ and let Ω be a nonempty bounded open subset of $\mathbb{R}$. Then we denote by $L^p(\Omega ; X)$ the set of all mappings $f : \Omega \to X$ which are strongly measurable and such that the Lebesgue integrals

$$\int_\Omega ||f(\omega)||_X^p d\omega \quad \text{are finite.}$$

The spaces $L^p(\Omega ; X)$, $1 \le p < \infty$, are Banach spaces under the norm

$$|||f|||_p = \left(\int_\Omega ||f(\omega)||_X^p \, d\omega \right)^{\frac{1}{p}}.$$

[1.2.3]. The space $L^\infty(\Omega, X)$. The space $L^\infty(\Omega, X)$ shall denote the set of all strongly measurable mappings $f : \Omega \to X$ such that $\operatorname*{ess\,sup}_{x \in \Omega} ||f|| < \infty$.

The space $L^\infty(\Omega, X)$ is Banach under the norm

$$|||f|||_\infty = \operatorname*{ess\,sup}_{x \in \Omega} ||f|| .$$

In what follows, the spaces $L^p(\Omega, X)$, $1 \le p \le \infty$, will take the form $L^p([0,T], L^p)$ or $L^p([0, T), H_m)$, where Ω denotes a time interval and H_m is the Sobolev space of order m as defined below.

[1.2.4] Sobolev space H_m. Let $C^m(S')$, $0 \leq m \leq \infty$, denote the collection of all complex-valued m times continuously differentiable functions on S'. By the support of a function $u \in C^m(S')$, written supp u, we mean the closure of the set on which u is nonzero. Let $C_0^m(S')$, $0 \leq m \leq \infty$, denote the subspace of $C^m(S')$ consisting of functions having compact support. Then the completion $H_m(S')$ of $C_0^\infty(S')$ in the norm defined by

$$||u||_m = \left(\sum_{l \leq m} \int_{S'} |D^l u|^2 dx \right)^{\frac{1}{2}}, \quad m \geq 0$$

is called the Sobolev space of order m. Clearly $||\cdot||_0$ is just the L^2 norm and H_0 is simp ly L^2 itself. If $u \in H_m$, then there exists a subsequence $\{u_n\} \in C_0^\infty$ such that $|| u - u_n||_m \to 0$ as $n \to \infty$. (Since C_0^∞ is dense in H_m). If $u \in H_m$ then, certainly, $u \in L^2$ and, more generally, since $|| \; ||_m \leq || \; ||_{m+1}$, we get

$$C^\infty \subsetneq \ldots\ldots \subsetneq H_{m+1} \subsetneq H_m \subsetneq \ldots\ldots \subsetneq H_0 = L^2.$$ In the above H_m becomes a Hilbert space when equipped with the inner product

$$< u, v >_m = \sum_{|\alpha| \leq m} \int_{S'} D^\alpha u . D^\alpha v \, dx.$$

[1.2.5] Young's inequality [11]. Let φ be a continuous real-valued, strictly increasing function defined on $[0, \infty)$ such that $\lim_{u \to \infty} \varphi(u) = \infty$

and $\varphi(0) = 0$

Let $\psi = \varphi^{-1}$. For all $x \in [0, \infty)$, define

$$\phi(x) = \int_0^x \varphi(u)\,du$$

and $$\Psi(x) = \int_0^x \psi(\nu)\,d\nu.$$

Then for all $a, b \in [0, \infty)$

$$ab \leq \phi(a) + \Psi(b) \;;$$

the equality occurs if and only if $b = \varphi(a)$.

[1.2.6] Corollary. Let a and b be real numbers and let $p > 1$, $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then we have

(1.2.1) $$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

the equality occurs if and only if $a^p = b^q$.

[1.2.7] Holder's inequality for $p > 1$. Let $f \in L^p$ and $g \in L^q$, where $p > 1$, and $q > 1$, such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1$ and we have

(1.2.2) $$\left|\int fg\,d\mu\right| \leq \int |fg|\,d\mu$$ and

(1.2.3) $$\int |fg|\,d\mu \leq ||f||_p\,||g||_q$$ and so also

(1.2.4) $$\left|\int fg\,d\mu\right| \leq ||f||_p\,||g||_q$$

For $P = q = 2$, the inequality (1.2.3) is called the Cauchy Schwarz's inequality.

[1.2.8] Leray-Schauder fixed point theorem [9].

Consider a transformation $y = T(x, k)$ where x, y belong to a Banach space X and K is a real parameter which varies in a bounded interval, say, $a \leq k \leq b$.

Assume :

(a) $T(x,k)$ is defined for all $x \in X$, $a \leq k \leq b$.

(b) For any fixed k, $T(x,k)$ is continuous in X, that is, for any $x^o \in X$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $||T(x,k) - T(x^o;k)|| < \varepsilon$ if $||x - x^o|| < \delta$.

(c) For x in bounded sets of X, $T(x,k)$ is uniformly continuous in k, that is, for any bounded set $X_o \subset X$ and for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $x \in X_o$, $|k_1 - k_2| < \delta$, $a \leq k_1, k_2 \leq b$ then $||T(x,k_1) - T(x, k_2)|| < \varepsilon$.

(d) For any fixed k, $T(x,k)$ is a compact transformation, that is, it maps bounded subsets of X into relatively compact subsets of Y.

(e) There exists a (finite) constant M such that every possible solution x of $x - T(x,k) = 0$

($x \in X$, $k \in [a, b]$) satisfies : $||x|| \leq M$.

(f) The equation $x - T(x,a) = 0$ has a unique solution in X.

Leray-Schauder theorem then states that under the assumptions (a) - (f), there exists a solution of the equation $x - T(x,b) = 0$.

[1.2.9] Arzela-Ascoli theorem. Let X be a compact metric space, let Y be a finite-dimensional normed space and let $\{f_n\}$ be an equicontinuous and uniformly bounded sequence of functions from X into Y. Then there is a subsequence of $\{f_n\}$ which converges uniformly on X.

[1.2.10] Gronwall's Lemma. Let K be a nonnegative constant and let f and g be continuous nonnegative functions on some interval $\alpha \leq t \leq \beta$ which satisfy the inequality

$$f(t) \leq K + \int_\alpha^t f(s)g(s)\,ds \quad \text{for} \quad \alpha \leq t \leq \beta \;.$$

Then $\quad f(t) \leq K \exp\left\{\int_\alpha^t g(s)\,ds\right\}.$

[1.2.11] Rellich's Lemma. If Ω is a bounded domain, and $m \geq 1$, then sets that are bounded in the $H_m(\Omega)$ norm are conditionally compact in the $H_{m-1}(\Omega)$ norm.

[1.2.12] $\operatorname{ess.\,sup}_{t \in \Omega} |x(t)|$: A function x on Ω is called essentially bounded if there is a number K such that $\mu\{t \mid |x(t)| > K\} = 0$. The greatest lower bound of all such numbers K is called the essential supremum of $|x|$ and designated by

$$\operatorname{ess\,sup}_{t \in \Omega} |x(t)|$$

CHAPTER II

2.1 The Relationship Among The Norms.

The applicability of Leray-Schauder theorem depends upon the uniform boundedness in the norm of $W_{m'}$ of all solutions $u^{\tau}(x,t)$ of (1.1.3). To achieve it, we require some a priori bounds which are obtained from the following relationship s among the norms $|\nu|_o = |\nu|$, $|D^{\ell}\nu|$, $|D^n\nu|$ and $|D^{\ell}\nu|_p$.

Lemma 2.1.1

If $\nu \in H_n(S')$, K and ℓ are integers, then

(2.1.1) $|D^{\ell}\nu| \leq |D^k\nu|^{\frac{\ell}{k}} |\nu|^{1-\frac{\ell}{k}}$, $0 \leq \ell \leq k \leq n$.

Proof: By putting $k-\ell = d$, then $d \geq 0$ and (2.1.1) becomes

$$(2.1.2) \quad |D^{\ell}\nu| \leq |D^{\ell+d}\nu|^{\frac{\ell}{\ell+d}} |\nu|^{\frac{d}{\ell+d}}.$$

If $\ell = 0$, (2.1.2) becomes $|\nu| \leq |\nu|$ for all $d < \infty$. In fact, $|\nu| = |\nu|$ for all $d < \infty$.

If $\ell = 1$, (2.1.2) becomes $|D\nu| \leq |D^{1+d}\nu|^{\frac{1}{1+d}} |\nu|^{\frac{d}{1+d}}$.

$$\text{Now} \quad |D\nu| = \left(\left| \int D\nu D\nu dx\right|\right)^{\frac{1}{2}} = \left(\left| \int \nu \; D^2\nu dx\right|\right)^{\frac{1}{2}}$$

$$\leq \left(|D^2\nu||\nu|\right)^{\frac{1}{2}} = |D^2\nu|^{\frac{1}{2}}|\nu|^{\frac{1}{2}}$$

$\therefore \quad |D\nu| \leq |D^2\nu|^{\frac{1}{2}}|\nu|^{\frac{1}{2}}$

$\therefore$ When $\ell = 1$ and $d = 0$, (2.1.2) is true.

Also when $\ell = 1$ and $d = 1$, (2.1.2) is true. We now assume that (2.1.2) is true for $\ell = m$ and for finite values of d, that is,

(2.1.3) $|D^m\nu| \le |D^{m+d}\nu|^{\frac{m}{m+d}}|\nu|^{\frac{d}{m+d}}$, $d < \infty$ Then we wish to show that

$$|D^{m+1}\nu| \le |D^{m+1+d}\nu|^{\frac{m+1}{m+1+d}}|\nu|^{\frac{d}{m+1+d}} \quad \text{for } d < \infty$$

Now
$$|D^{m+1}\nu| = \left(\left|\int D^{m+1}\nu D^{m+1}\nu \, dx\right|\right)^{\frac{1}{2}}$$

$$= \left(\left|(-1)^d\int D^{m+1+d}\nu D^{m+1-d}\nu \, dx\right|\right)^{\frac{1}{2}}$$

$$= \left(\left|\int D^{m+1+d}\nu D^{m+1-d}\nu \, dx\right|\right)^{\frac{1}{2}}$$

$$\le \left(|D^{m+1+d}\nu|\,|D^{m+1-d}\nu|\right)^{\frac{1}{2}}$$

$$\le \left[|D^{m+1+d}\nu|\left(|D^{m+1+d}\nu|^{\frac{m+1-d}{m+1+d}}|\nu|^{\frac{2d}{m+1+d}}\right)\right]^{\frac{1}{2}}, \text{ where}$$

$0 \le m + 1 - d \le m$.

$$\therefore \quad |D^{m+1}\nu| \le \left[|D^{m+1+d}\nu|^{1+\frac{m+1-d}{m+1+d}}|\nu|^{\frac{2d}{m+1+d}}\right]^{\frac{1}{2}}$$

$$= |D^{m+1+d}\nu|^{\frac{m+1}{m+1+d}}|\nu|^{\frac{d}{m+1+d}}$$

$$\therefore \quad |D^{m+1}\nu| \le |D^{m+1+d}\nu|^{\frac{m+1}{m+1+d}}|\nu|^{\frac{d}{m+1+d}}$$

Therefore when (2.1.2) is true for $\ell = m$ it is also true for $\ell = m + 1$ for $d < \infty$. Thus it is true for all values of ℓ and for finite values of d. The finite condition on d is justified by the restriction on k.

Thus, (2.1.1) is true for all ℓ and all $k \leq n$. This completes the proof of the lemma.

To relate the norms $|\nu|$, $|D^n\nu|$ and $|D^\ell \nu|_p$ we need the following:

Lemma 2.1.2 If $\nu \in H_2\ (S')$ and $p \geq 2$, then

$$(2.1.4) \quad |D\nu|_p \leq 2^{\frac{p-2}{2p}} |D^2\nu|^{\frac{3}{4} - \frac{1}{2p}} |\nu|^{\frac{1}{4} + \frac{1}{2p}}.$$

Proof: We can write $|D\nu(x)|^p = |D\nu(x)|^2 (|D\nu(x)|^2)^{\frac{p-2}{2}}$. Since the integral of $D\nu(x)$ over a period vanishes, there is an $x_0 \in [0, 1]$ for which

$$(2.1.5) \quad \int_{x_0}^{x} D(D\nu(x))^2 dx = (D\nu(x))^2.$$

$$\text{Now} \quad \frac{\partial}{\partial x}|D\nu(x)|^2 = 2|D\nu(x)|\frac{\partial}{\partial x}|D\nu(x)|$$

$$= 2|D\nu(x)|\,|D^2\nu(x)|.$$

$$\Longrightarrow \quad \int_0^1 \frac{\partial}{\partial x}|D\nu(x)|^2 dx = \int_0^1 2|D\nu(x)|\,|D^2\nu(x)|\,dx.$$

Using (2.1.5) we have

$$|D\nu(x)|^2 = 2\int_0^1 |D\nu(x)|\,|D^2\nu(x)|\,dx.$$

$$\therefore \quad |D\nu(x)|^p = |D\nu(x)|^2 \left(|D\nu(x)|^2\right)^{\frac{p-2}{2}} \quad \text{becomes}$$

$$= |D\nu(x)|^2 \left(2\int_0^1 |D\nu(x)|\,|D^2\nu(x)|\,dx\right)^{\frac{p-2}{2}}$$

$$\leq |D\nu(x)|^2 \left(2|D^2\nu|\,|D\nu|\right)^{\frac{p-2}{2}}.$$

By integrating in x and taking the pth root of both sides, we have

$$\left(\int_0^1 |D\nu(x)|^p dx\right)^{\frac{1}{p}} \leq \left(2|D^2\nu||D\nu|\right)^{\frac{p-2}{2p}} \left(\int_0^1 |D\nu(x)|^2 dx\right)^{\frac{1}{p}}$$

$$\Longrightarrow \quad |D\nu|_p \leq \left(2|D^2\nu||D\nu|\right)^{\frac{p-2}{2p}} \left(|D\nu|^2\right)^{\frac{1}{p}}$$

$$= 2^{\frac{p-2}{2p}} |D^2\nu|^{\frac{p-2}{2p}} |D\nu|^{\frac{p+2}{2p}}.$$

Using (2.1.1) with $\ell = 1$ and $K = 2$, that is,

$|D\nu| \leq |D^2\nu|^{\frac{1}{2}}|\nu|^{\frac{1}{2}}$ we get

$$|D\nu|_p \leq 2^{\frac{p-2}{2p}} |D^2\nu|^{\frac{p-2}{2p}} \left(|D^2\nu|^{\frac{1}{2}}|\nu|^{\frac{1}{2}}\right)^{\frac{p+2}{2p}}$$

$$= 2^{\frac{p-2}{2p}} |D^2\nu|^{\frac{p-2}{2p}} |D^2\nu|^{\frac{p+2}{4p}} |\nu|^{\frac{p+2}{4p}}$$

$$2^{\frac{p-2}{2p}} |D^2\nu|^{\frac{3}{4} - \frac{1}{2p}} |\nu|^{\frac{1}{4} + \frac{1}{2p}}$$ which is the required inequality.

If $\nu \in H_n(S')$, then (2.1.1) and (2.1.4) give the inequality

(2.1.6) $|D^\ell \nu|_p \leq 2^{\frac{p-2}{2p}} |D^K\nu|^{\frac{1}{k}(\ell + \frac{1}{2} - \frac{1}{p})} |\nu|^{1-\frac{1}{K}(\ell + \frac{1}{2} - \frac{1}{p})}$,

where $| \leq \ell < K \leq n$.

Proof: of (2.1.6). From (2.1.4) we get

(2.1.7) $|D^\ell \nu|_p \leq 2^{\frac{p-2}{2p}} |D^{\ell+1}\nu|^{\frac{3}{4} - \frac{1}{2p}} |D^{\ell-1}\nu|^{\frac{1}{4} + \frac{1}{2p}}$ and (2.1.1) gives $|D^{\ell+1}\nu| \leq |D^K\nu|^{\frac{\ell+1}{K}} |\nu|^{1-\frac{\ell+1}{K}}$ and

$$|D^{\ell-1}\nu| \leq |D^K\nu|^{\frac{\ell-1}{K}} |\nu|^{1-\frac{\ell-1}{K}}$$

On substituting these into (2.1.7) we get

$$|D^\ell \nu|_p \leq 2^{\frac{p-2}{2p}} \left\{|D^K\nu|^{\frac{\ell+1}{K}} |\nu|^{1-\frac{\ell+1}{K}}\right\}^{\frac{3}{4}-\frac{1}{2p}} \left\{|D^K\nu|^{\frac{\ell-1}{K}} |\nu|^{1-\frac{\ell-1}{K}}\right\}^{\frac{1}{4}+\frac{1}{2p}}$$

$$= 2^{\frac{p-2}{2p}} |D^K\nu|^{\frac{1}{K}(\ell+\frac{1}{2}-\frac{1}{p})} |\nu|^{1-\frac{1}{K}(\ell+\frac{1}{2}-\frac{1}{p})} \text{ as required.}$$

The inequality

$$(2.1.8) \quad |\nu|_p \leq C\,\{|D^K\nu|^{\frac{1}{K}(\frac{1}{2}-\frac{1}{p})} |\nu|^{\frac{1}{K}(K-\frac{1}{2}+\frac{1}{p})} + |\nu|\},\ K \leq n$$

where the constant C is independent of p is obtained as follows: Let $\nu' \in H_n(S')$ so that $\int_0^1 \nu' dx = 0$ as in lemma 2.1.1. We can write

$$(2.1.9) \quad |\nu'(x)|^p = |\nu'(x)^2 (|\nu'(x)|^2)^{\frac{p-2}{2}}.$$

Now $\frac{\partial}{\partial x}(|\nu'(x)|)^2 = 2|\nu'(x)|\frac{\partial}{\partial x}|\nu'(x)|$

$$= 2|\nu'(x)|\,|D\nu'(x)|.$$

$\therefore$ Using (2.1.5), we get

$$\int_0^1 \frac{\partial}{\partial x}|\nu'(x)|^2 dx = |\nu'(x)|^2 = 2\int^1 |\nu'(x)|\,|D\nu'(x)|dx.$$

By putting this into (2.1.9) we obtain

$$|\nu'(x)|^p = |\nu'(x)|^2\left(2\int_0^1 |\nu'(x)||D\nu'(x)|dx\right)^{\frac{p-2}{2}}$$

$$\leq |\nu'(x)|^2 (2|\nu'||D\nu'|)^{\frac{p-2}{2}}.$$

Integrating in x and taking the pth root of both sides we find

$$\left(\int_0^1 |\nu'(x)|^p dx\right)^{\frac{1}{p}} \leq \left(2|\nu'||D\nu'|\right)^{\frac{p-2}{2p}} \left(\int_0^1 |\nu'(x)|^2 dx\right)^{\frac{1}{p}}$$

$$\Longrightarrow \quad |\nu'|_p \leq 2^{\frac{p-2}{2p}} |\nu'|^{\frac{p-2}{2p}} |D\nu'|^{\frac{p-2}{2p}} |\nu'|^{\frac{2}{p}}$$

$$= 2^{\frac{p-2}{2p}} |D\nu'|^{\frac{p-2}{2p}} |\nu'|^{\frac{p+2}{2p}}.$$

We replaced ν' by $\nu - \int_0^1 \nu\, dx$ to obtain

$$|\nu - \int_0^1 \nu dx|_p \leq 2^{\frac{p-2}{2p}} |D(\nu - \int_0^1 \nu dx)|^{\frac{1}{2}-\frac{1}{p}} |(\nu - \int_0^1 \nu\, dx)|^{\frac{1}{2}+\frac{1}{p}}$$

$$\Longrightarrow |\nu - |\nu||_p \leq 2^{\frac{p-2}{2p}} |D\nu|^{\frac{1}{2}-\frac{1}{p}} |\nu - |\nu||^{\frac{1}{2}+\frac{1}{p}}, \quad \text{where}$$

$\int_0^1 D\nu dx = 0$ by periodicity.

Using the inequalities $\big|\,||X|| - ||Y||\,\big| \leq ||X-Y|| \leq ||X|| + ||Y||$ we get

$$|\nu|_p - |\nu| \leq 2^{\frac{p-2}{2p}} |D\nu|^{\frac{1}{2}-\frac{1}{p}} (2|\nu|)^{\frac{1}{2}+\frac{1}{p}}$$

$$=2\,|D\nu|^{\frac{1}{2}-\frac{1}{p}}\,|\nu|^{\frac{1}{2}+\frac{1}{p}}.$$

Applying (2.1.1) with $\ell=1$, that is $|D\nu| \leq |D^K\nu|^{\frac{1}{K}}\,|\nu|^{1-\frac{1}{K}}$

we get $|\nu|_p \leq 2\left\{|D^K\nu|^{\frac{1}{K}}\,|\nu|^{1-\frac{1}{K}}\right\}^{\frac{1}{2}-\frac{1}{p}}|\nu|^{\frac{1}{2}+\frac{1}{p}} + |\nu|$

$$\leq C\left\{|D^K\nu|^{\frac{1}{K}(\frac{1}{2}-\frac{1}{p})}\,|\nu|^{\frac{1}{K}(K-\frac{1}{2}+\frac{1}{p})} + |\nu|\right\}.$$

2.2 An Apriori Bound for Solutions of (1.1.3)

Lemma 2.2.1 Let $m' \geq 2m$. If $u^{\tau}(x,t)$ is a solution from $V_{m'}$ of (1.1.3), then

$$(2.2.1)\qquad \int_0^T \left|\frac{\partial u^{\tau}(t)}{\partial t}\right|^2 dt \text{ and } \eta\int_0^T ||u^{\tau}(t)||^2_{m+m'}dt + \text{ess-sup}||u^{\tau}(t)||^2_{m'}$$

are estimated from above by a constant depending only on $|g|_{m'}$ and $T < \infty$; most importantly this constant is independent of τ.

Proof: We first consider the case $0 \leq \ell \leq m$. Let us begin with a solution $u(x,t)$ from $V_{m'}$. By multiplying (1.1.3) by $G_{\ell+1}(u)$ and integrating in x and t we obtain

$$\int_0^T\int_0^1 \frac{\partial u}{\partial t} G_{\ell+1}(u)\,dx\,dt = \int_0^T\int_0^1 \left\{\tau G_{\ell+1}(u)\frac{\partial}{\partial x}G_m(u) + (-1)^{m+1}(\tau(\eta-1)+1)D^{2m}u \times G_{\ell+1}(u)\right\} dx\,dt.$$

Applying Gardner's [5] identity $\int_0^1 G_\ell(u)\frac{\partial}{\partial x} G_k(u)\,dx = 0,$

where ℓ and K are integers ≥ 1, we have

$$(2.2.2)\quad \int_0^T\int_0^1 \frac{\partial u}{\partial t} G_{\ell+1}(u)\,dxdt + (-1)^m(\tau(\eta-1)+1)\int_0^T\int_0^1 D^{2m}uG_{\ell+1}(u)\,dxdt=0,$$

Now, let us introduce $f_\ell(u)$ through the identity

$$\int_0^T\int_0^1 \frac{\partial u}{\partial t} G_\ell(u)\,dxdt = \int_0^1 f_\ell(u)dx - \int_0^1 f_\ell(g)\,dx.$$

The first four G_ℓ's and f_ℓ's are :

$G_1(u) = u; \quad f_1(u) = \frac{1}{2}u^2$

$G_2(u) = \frac{1}{2}u^2 + u_{xx}\ ; \ f_2(u) = \frac{1}{6}u^3 - \frac{1}{2}u_x^2$

$G_3(u) = u^3+3u_x^2+6uu_{xx}+\frac{18}{5}u_{xxxx}; \ f_3(u) = \frac{1}{4}u^4-3uu_x^2 + \frac{4}{5}u_{xx}^2$

$$G_4(u) = u^4+12uu_x^2+12u^2u_{xx}+\frac{35}{5}\{3u_{xx}^2+4u_xu_{xxx}+2uu_{xxxx}\}$$
$$+ \frac{108.2}{35}D^6u\ ;$$

$$f_4(u) = \frac{u^5}{5} - 6u^2u_x^2 + \frac{36}{5}uu_{xx}^2 - \frac{108}{35}u_{xxx}^2.$$

Gardner, Kruskal, and Miura [7] have shown that each term in $f_\ell(u)$ is of the form $\Pi_{j=0}^{\ell-1}(D^\ell u)^{n_j}$, where $\sum_{j=0}^{\ell-1}\left(1 + \frac{j}{2}\right)n_j = \ell + 1$ for integer n_j. It follows that

(2.2.3) $f_\ell(u) = (D^{\ell-1}u)^2 + (D^{\ell-2}u)^2u + aD^{\ell-1}u + bD^{\ell-2}u + c$ up to inessential constants, and where a, b, c are polynomials in u and its derivatives up to order $\ell - 3$. Rewriting $G_\ell(u) = D^{2\ell-2}u + G_\ell'(u)$,

$f_{\ell+1}(u) = (-1)^{\ell}(D^{\ell}u)^2 + f'_{\ell+1}(u)$, and substituting into (2.2.2) and integrating in x we have

$$\int_0^1 f_{\ell+1}(u)\,dx - \int_0^1 f_{\ell+1}(g)\,dx + (-1)^m(1+\tau(\eta-1))\int_0^T\int_0^1 D^{2m}u\,(D^{2\ell}u + G'_{\ell+1}(u))\,dx\,dt = 0$$

$$\Longrightarrow \int_0^1 (-1)^{\ell}(D^{\ell}u)^2 dx + \int_0^1 f'_{\ell+1}(u)\,dx - \int_0^1 f_{\ell+1}(g)\,dx + (-1)^m(1+\tau(\eta-1))$$

$$\int_0^T\int_0^1 D^{2m}uD^{2\ell}u\,dx\,dt + (-1)^m(1+\tau(\eta-1))\int_0^T\int_0^1 D^{2m}u\,G'_{\ell+1}(u)\,dx\,dt = 0$$

$$\Longrightarrow (-1)^{\ell}\int_0^1 (D^{\ell}u)^2 dx + (-1)^m(-1)^{\ell-m}(1+\tau(\eta-1))\int_0^T\int_0^1 D^{m+\ell}uD^{m+\ell}u\,dx\,dt$$

$$= \int_0^1 f_{\ell+1}(g)\,dx - \int_0^1 f'_{\ell+1}(u)\,dx - (-1)^m(-1)^{m-\ell}(1+\tau(\eta-1))\int_0^T\int_0^1 D^{m+\ell}uD^{m-\ell}G'_{\ell+1}(u)\,dx\,dt,$$

where we have integrated by parts.

$$\Longrightarrow (-1)^{\ell}\left\{\int_0^1 (D^{\ell}u)^2 dx + (1+\tau(\eta-1))\int_0^T\int_0^1 (D^{m+\ell}u)^2 dx\,dt\right\}$$

$$= \int_0^1 f_{\ell+1}(g)\,dx - \int_0^1 f'_{\ell+1}(u)\,dx - (-1)^{\ell}(1+\tau(\eta-1))\int_0^T\int_0^1 D^{\ell+m}uD^{m-\ell}G'_{\ell+1}(u)\,dx\,dt.$$

$\therefore$ The above implies

(2.2.4) $$\int_0^1 (D^{\ell}u)^2 dx + (1+\tau(\eta-1))\int_0^T\int_0^1 (D^{m+\ell}u)^2 dx\,dt$$

$$=(-1)^{\ell}\left\{\int_0^1 f_{\ell+1}(g)dx-\int_0^1 f'_{\ell+1}(u)\,dx-(1+\tau(\eta-1))\int_0^T\int_0^1 D^{m+\ell}uD^{m-\ell}G'_{\ell+1}(u)\,dx\,dt\right\}.$$

We transform (2.2.4) as follows :

1. $\ell = 0$.

Since $G_1'(u) = 0$ and $f_1'(u) = 0$, then (2.2.4) becomes

$$\int_0^1 (u(x,t))^2dx + (1+\tau(\eta-1))\int_0^T\int_0^1 (D^m u(x,t))^2dxdt = \int_0^1 f_1(g)\,dx.$$

Taking $\tau = 1$, the above implies

$$|u(t)|^2+\eta\int_0^T |D^m u(t)|^2dt \leq C(||g||), \text{ since } f_1(g) = g^2$$

which is independent of t.

2. $\ell = 1$

Then (2.2.4) becomes

$$\int_0^1 (Du\ (x,t))^2dx + (1+\tau(\eta-1))\int_0^T\int_0^1 (D^{m+1}u(x,t))^2dxdt = -\left[\int_0^1 f_2(g)\,dx\right.$$

$$\left.-\int_0^1 f_2'(u(x,t))dx - (1+\tau(\eta-1))\int_0^T\int_0^1 D^{m+1}u(x,t)D^{m-1}G_2'(u(x,t))\,dxdt\right].$$

With $G_2'(u(x,t)) = (u(x,t))^2$, the bound involving $D^{m+1}uD^{m-1}u^2$ in (2.2.4) is developed by first noting that for $\ell_1+\ell_2= m-1$ and $\ell_1,\ \ell_2 \neq 0$.

$$\therefore\ \int_0^T\int_0^1 |D^{m+1}uD^{\ell_1}uD^{\ell_2}u|\,dxdt \leq \int_0^T\left\{|D^{m+1}u(t)|\,|D^{\ell_1}u(t)|_4|D^{\ell_2}u(t)|_4\right\}dt.$$

Applying (2.1.6) we get

$$\int_0^T\!\!\int_0^1 |D^{m+1}uD^{\ell_1}uD^{\ell_2}u|\,dxdt \le \int_0^T \left\{|D^{m+1}u(t)|\left(2^{\frac{1}{4}}|D^{m+1}u(t)|^{\frac{(\ell_1+\frac{1}{2}-\frac{1}{4})}{m+1}}|u|^{1-\frac{(\ell_1+\frac{1}{2}-\frac{1}{4})}{m+1}}\right)\right.$$

$$\left. x\left(2^{\frac{1}{4}}|D^{m+1}u(t)|^{\frac{\ell_2+\frac{1}{2}-\frac{1}{4}}{m+1}}|u(t)|^{1-\frac{(\ell_2+\frac{1}{2}-\frac{1}{4})}{m+1}}\right)\right\}dt$$

$$= \sqrt{2}\int_0^T\left\{|D^{m+1}u(t)|\,|D^{m+1}u(t)|^{\frac{\ell_1+\ell_2+1-\frac{1}{2}}{m+1}}|u(t)|^{2-\frac{(\ell_1+\ell_2+1-\frac{1}{2})}{m+1}}\right\}dt$$

$$= \sqrt{2}\int_0^T\left\{|D^{m+1}u(t)|^{1+\frac{m-\frac{1}{2}}{m+1}}|u(t)|^{\frac{m+\frac{5}{2}}{m+1}}\right\}dt$$

$$= \sqrt{2}\int_0^T\left\{|D^{m+1}u(t)|^{1+\alpha}|u(t)|^{\frac{m+\frac{5}{2}}{m+1}}\right\}dt, \text{ where } 0<\alpha<1$$

$$\le C\int_0^T|D^{m+1}u(t)|^{1+\alpha}dt, \text{ since } |u(t)| \text{ is bounded.}$$

For $\ell_1+\ell_2 = m-1$ and either ℓ_1 or $\ell_2 = 0$.

$$\therefore \int_0^T\!\!\int_0^1|D^{m+1}uD^{\ell_1}uD^{\ell_2}u|\,dx\,dt \le \int_0^T|D^{m+1}u(t)|\,|u(t)|_4\,|D^{m-1}u(t)|_4\,dt,$$

taking $\ell_1 = 0$

$$\le \int_0^T |D^{m+1}u(t)|\,C_1\left[|D^{m+1}u(t)|^{\frac{(\frac{1}{2}-\frac{1}{4})}{m+1}}|u(t)|^{\frac{(m+1-\frac{1}{2}+\frac{1}{4})}{m+1}}+|u|\right]$$

$$x2^{\frac{1}{4}}\left[|D^{m+1}u(t)|^{\frac{m-1+\frac{1}{2}-\frac{1}{4}}{m+1}}|u(t)|^{1-\frac{(m-1+\frac{1}{2}-\frac{1}{4})}{m+1}}\right]dt,$$

where we have applied (2.1.8) and (2.1.6) respectively.

$$C_2\int_0^T |D^{m+1}u(t)\left\{|D^{m+1}u(t)|^{\frac{\frac{1}{4}}{m+1}}|u(t)|^{\frac{m+\frac{3}{4}}{m+1}}+|u(t)|\right\}\left\{|D^{m+1}u(t)|^{\frac{m-\frac{3}{4}}{m+1}}|u(t)|^{\frac{\frac{7}{4}}{m+1}}\right\}dt$$

$$\leq C_3\int_0^T |D^{m+1}u(t)|\left(|D^{m+1}u(t)|^{\frac{\frac{1}{4}}{m+1}}+C_4\right)\left(|D^{m+1}u(t)|^{\frac{m-\frac{3}{4}}{m+1}}C_5\right)dt$$

$$\leq C_6\int_0^T |D^{m+1}u(t)|^{1+\frac{\frac{1}{4}}{m+1}+\frac{m-\frac{3}{4}}{m+1}}dt + C_7\int_0^T |D^{m+1}u(t)|^{1+\frac{m-\frac{3}{4}}{m+1}}dt.$$

$$= C_6\int_0^T |D^{m+1}u(t)|^{\frac{2m-\frac{1}{2}}{m+1}}dt + C_7\int_0^T |D^{m+1}u(t)|^{\frac{2m+\frac{1}{4}}{m+1}}dt.$$

Applying Young's inequality [1.2.6] we get

$$\int_0^T\int_0^1 |D^{m+1}uD^{\ell_1}uD^{\ell_2}u|\,dx\,dt \leq \left[C_{\varepsilon_1}(C_6)^{\frac{4}{5}(m+1)}+\varepsilon_1\int_0^T\left(|D^{m+1}u(t)|^{\frac{2m-\frac{1}{2}}{m+1}}\right)^{\frac{2m+2}{2m-\frac{1}{2}}}dt\right]$$

$$+\left[C_{\varepsilon_{11}}(C_7)^{\frac{8}{7}(m+1)}+\varepsilon_{11}\int_0^T\left(|D^{m+1}u(t)|^{\frac{2m+\frac{1}{4}}{m+1}}\right)^{\frac{2(m+1)}{2m+\frac{1}{4}}}dt\right],$$

$$\leq \left[C'_{\varepsilon_1}+\varepsilon_1\int_0^T|D^{m+1}u(t)|^2dt\right]+\left(C'_{\varepsilon_{11}}+\varepsilon_{11}\int_0^T|D^{m+1}u(t)|^2dt\right)$$

$$\leq C_\varepsilon+\varepsilon\int_0^T|D^{m+1}u(t)|^2dt.$$

The remaining term in (2.2.4) involves $f_2'(u) = \frac{u^3}{2}$.

But $\int_0^1|u|^3dx \leq |u(t)|^2|u(t)|_\infty$ by (1.2.3)

$$\leq C_1|u(t)|_\infty.$$

Using (2.1.8) with $p = \infty$ we have

$$\int_0^1 |u(x,t)|^3 dx \leq C_1C_2 \left\{|Du(t)|^{\frac{1}{2}}|u(t)|^{\frac{1}{2}} + |u(t)|\right\}$$

$$\leq C_3 \left\{\varepsilon_1 (|Du(t)|^{\frac{1}{2}})^2 + C_{\varepsilon_1}(|u(t)|^{\frac{1}{2}})^2 + |u(t)| \right\}$$

$$= C_\varepsilon^{\,\prime} \left\{\varepsilon_1 |Du(t)| + C_{\varepsilon_1}|u(t)| + | u(t)|\right\}$$

$$= (C_3 C_{\varepsilon_1} + C_3)|u(t)| + \varepsilon_1 C_3|Du(t)|$$

$$\leq C_4 + C_5 |Du(t)|$$

$$\leq C_4 + C_{\varepsilon_{\prime\prime}}(C_5)^2 + \varepsilon_{\prime\prime}|Du(t)|^2 \quad \text{applying Young's inequality}$$

$$\leq C + \varepsilon|Du(t)|^2.$$

We now combine this with the previous estimate to obtain

$$|Du(t)|^2 + (1+\tau(\eta-1))\int_0^T |D^{m+1}u(t)|^2 dt \leq C + \varepsilon|Du(t)|^2$$

$$+\int_0^1 |f_2(g)|dx + C_\varepsilon + \varepsilon(1+\tau(\eta-1))\int_0^T |D^{m+1} u(t)|^2 dt.$$

Choosing $\varepsilon = \frac{1}{2}$, $\tau = 1$ and rearranging we get

$$|Du(t)|^2 + \eta \int_0^T |D^{m+1}u(t)|^2 dt \leq C(||g||_1).$$

3 $\quad \ell = 2$

Then (2.2.4) becomes

$$\int_0^1 (D^2u(x,t))^2 dx + (1+\tau(\eta-1))\int_0^T\int_0^1 (D^{m+2}u(x,t))^2 dxdt = \int_0^1 f_3(g)dx$$

$$-\int_0^1 f_3'(u)dx - (1+\tau(\eta-1))\int^T\int_0^1 D^{m+2}u(x,t)D^{m-2}G_3'(u)dx\ dt.$$

With $G_3'(u) = u^3 + 3(u_x)^2 + 6uu_{xx}$ the bound for (2.2.4) involves the following estimates :

For $\ell_1 + \ell_2 = m$

$$\therefore \int_0^T\int_0^1 |D^{m+2}u(x,t)D^{\ell_1}u(x,t)D^{\ell_2}u(x,t)|\,dxdt \le \int_0^T \Big\{|D^{m+2}u(t)|\,|D^{\ell_1}u(t)|_4\,|D^{\ell_2}u(t)|_4\Big\}dt$$

$$\le \int_0^1 |D^{m+2}u(t)|\,2^{\frac14}\Big(|D^{m+2}u(t)|^{\lambda_1}|u(t)|^{1-\lambda_1}\Big)2^{\frac14}\Big(|D^{m+2}u(t)|^{\lambda_2}|u(t)|^{1-\lambda_2}\Big),$$

where we have applied (2.1.6) and

$$\lambda_1 = \frac{\ell_1+\frac12-\frac14}{m+2} = \frac{\ell_1+\frac14}{m+2}\;;\; 1-\lambda_1 = \frac{m-\ell_1+\frac34}{m+2}$$

$$\lambda_2 = \frac{\ell_2+\frac12-\frac14}{m+2} = \frac{\ell_2+\frac14}{m+2}\;;\; 1-\lambda_2 = \frac{m-\ell_2+\frac34}{m+2}.$$

$$\therefore \int_0^T\int_0^1 |D^{m+2}uD^{\ell_1}uD^{\ell_2}u|\,dxdt \le \sqrt2\int_0^T |D^{m+2}u(t)|\Big(|D^{m+2}u(t)|^{\frac{\ell_1+\frac14}{m+2}}|u(t)|^{\frac{m_1-\ell_1+\frac34}{m+2}}\Big)$$

$$\times\Big\{|D^{m+1}u(t)|^{\frac{\ell_2+\frac14}{m+2}}|u(t)|^{\frac{m-\ell_2+\frac34}{m+2}}\Big\}dt.$$

$$= \sqrt2\int_0^T |D^{m+2}u(t)|\,|D^{m+2}u(t)|^{\frac{\ell_1+\ell_2+\frac12}{m+2}}\,|u(t)|^{\frac{m-(\ell_1+\ell_2)+\frac32}{m+2}}\,dt.$$

$$= \sqrt2\int_0^T |D^{m+2}u(t)|\,|D^{m+2}u(t)|^{\frac{m+\frac12}{m+2}}|u(t)|^{\frac{\frac32}{m+2}}\,dt.$$

$$= \sqrt2\int_0^T |D^{m+2}u(t)|^{\frac{2m+\frac52}{m+2}}|u(t)|^{\frac{\frac32}{m+2}}\,dt.$$

Applying Young's inequality we get

$$\int_0^T\int_0^1 |D^{m+2}uD^{\ell_1}uD^{\ell_2}u|\,dxdt \le \sqrt{2}\left[\varepsilon^1\int_0^T\left(|D^{m+2}u(t)|^{\frac{2m+\frac{5}{2}}{m+2}}\right)^{\frac{2(m+2)}{2m+5}}\Big|_2\,dt\right.$$

$$\left.+\; C_\varepsilon'\int_0^T\left[\left(|u(t)|^{\frac{\frac{3}{2}}{m+2}}\right)^{\frac{2(m+2)}{\frac{3}{2}}}dt\right]\right], \quad \text{where}$$

$$\frac{1}{\left(\frac{2(m+2)}{2m+5/2}\right)} + \frac{1}{\frac{2(m+2)}{3/2}} = \frac{2m+5/2}{2(m+2)} + \frac{3/2}{2(m+2)} = \frac{2m+5/2+3/2}{2(m+2)} = 1.$$

$$\therefore\; \int_0^T\int_0^1 |D^{m+2}uD^{\ell_1}uD^{\ell_2}u|\,dxdt \le \sqrt{2}\left[\varepsilon'\int_0^T|D^{m+2}u(t)|^2dt + C_\varepsilon'\int_0^T|u(t)|^2dt\right]$$

$$\le C_\varepsilon + \varepsilon\int_0^T|D^{m+2}u(t)|^2dt.$$

For $\ell_1 + \ell_2 + \ell_3 = m - 2$,

$$\int_0^T\int_0^1 |D^{m+2}u(x,t)D^{\ell_1}u(x.t)D^{\ell_2}u(x,t)D^{\ell_3}u(x,t)|\,dx\,dt$$

$$\le \int_0^T\left\{|D^{m+2}u(t)|\,|D^{\ell_1}u(t)|_6\,|D^{\ell_2}u(t)|_6\,|D^{\ell_3}u(t)|_6\right\}dt,$$

$$\le 2\int_0^T|D^{m+2}u(t)|\left(|D^{m+2}u(t)|^{\frac{\ell_1+\frac{1}{3}}{m+2}}|u(t)|^{\frac{m-\ell_1+\frac{5}{3}}{m+2}}\right)\left(|D^{m+2}u(t)|^{\frac{\ell_2+\frac{1}{3}}{m+2}}|u(t)|^{\frac{m-\ell_2+\frac{5}{3}}{m+2}}\right)$$

$$\times\left(|D^{m+2}u(t)|^{\frac{\ell_3+\frac{1}{3}}{m+2}}|u(t)|^{\frac{m-\ell_3+\frac{5}{3}}{m+2}}\right)dt, \text{ by (2.1.6)}$$

$$= 2\int_0^T |D^{m+2}u(t)|\left\{|D^{m+2}u(t)|^{\frac{\ell_1+\ell_2+\ell_3+1}{m+2}}|u(t)|^{\frac{3m-(\ell_1+\ell_2+\ell_3)+5}{m+2}}\right\}dt.$$

$$= 2\int_0^T |D^{m+2}u(t)|\left\{|D^{m+2}u(t)|^{\frac{m-1}{m+2}}|u(t)|^{\frac{2m+7}{m+2}}\right\}dt$$

$$\leq 2\int_0^T |D^{m+2}u(t)|^{1+\frac{m-1}{m+2}}C_1\, dt$$

$$\leq C_2\int_0^T |D^{m+2}u(t)|^{\frac{2m+1}{m+2}}dt.$$

Applying Young's inequality we get

$$\int_0^T\!\!\int_0^1 |D^{m+2}uD^{\ell_1}uD^{\ell_2}uD^{\ell_3}u|\,dx\,dt \leq \varepsilon\int_0^T\left(|D^{m+2}u(t)|^{\frac{2m+1}{m+2}}\right)^{\frac{2(m+2)}{2m+1}}dt + C_\varepsilon'(C_2)^{\frac{2}{3}(m+2)}$$

$$\leq C_\varepsilon + \varepsilon\int_0^T |D^{m+2}u(t)|^2 dt.$$

With $f_3'(u) = \frac{1}{4}u^4 - 3uu_x^2$, we estimate the remaining term in (2.2.4) by first noting that

$$\int_0^1 u^4 dx \leq |u(t)|_\infty^2\,|u(t)|^2$$

$\leq C_3$ using the previous estimate.

Also $\displaystyle\int_0^1 |uu_x^2|\,dx \leq |u(t)|_\infty |u_x(t)|^2$

$\leq C_4$

Combining these with the previous estimates we get

$$|D^2u(t)|^2+(1+\tau(\eta-1))\int_0^T |D^{m+2}u(t)|^2dt \le C_3+C_4+\int_0^1|f_3(g)|dx$$

$$+\ \varepsilon(1+\tau(\eta-1))\int_0^T |D^{m+2}u(t)|^2dt.$$

Putting $\varepsilon = \frac{1}{2}$, $\tau = 1$ and rearranging we obtain

$$|D^2u(t)|^2+\eta\int_0^T |D^{m+2}u(t)|^2dt \le C(||g||_2).$$

4 $\quad \ell \le m$.

The bound for (2.2.4) involves $D^{m+\ell}u(x,t)D^{m-\ell}G'_{\ell+1}(u)$. Each term in $G'_\ell(u)$ has weight ℓ, where the weight of a term is defined as the sum of the weights of its factors, and the weight of D^ru is $1+\frac{r}{2}$. Thus

$$(2.2.5)\qquad G'_{\ell+1}(u) = \sum_{r=0}^{\ell-1} a_r\, D^{2\ell-2-r}u + \sum_{r=0}^{\ell-2} b_r\, D^r u,$$

where a_r and b_r are polynomials in u and its derivatives up to order r which depends on ℓ. Observe that the term $\sum_{r=0}^{\ell-1} a_r\, D^{2\ell-2-r}u$ involves derivatives D^k where $\ell-1\le k\le 2\ell-2$ and $\sum_{r=0}^{\ell-2} b_r\, D^ru$ involves D^k where $0 \le k \le \ell-2$.

Estimation of $D^{m+\ell}uD^{m-\ell}G'_{\ell+1}(u)$ involves estimates of the type

$$\int_0^T\int_0^1 |D^{m+\ell}uD^{\ell_1}u\ldots D^{\ell_r}u|\,dxdt \le \int_0^T \left\{|D^{m+\ell}u(t)|\prod_{j=1}^{r}|D^{\ell_r}u(t)|_{2r}\right\} dt$$

$2 \le r \le \ell+1$, where $\ell_1+\ell_2+\ldots+\ell_r = m+\ell+2(1-r)$

By (2.1.6) and (2.1.8) we've

$$|D^{\ell_j}u(t)|_{2r} \leq 2^{\frac{2r-2}{4r}} |D^{m+\ell}u(t)|^{\frac{\ell_j+\frac{1}{2}-\frac{1}{2}r}{m+\ell}} |u(t)|^{1-\frac{(\ell_j+\frac{1}{2}-\frac{1}{2}r)}{m+\ell}}$$

(2.2.6) $|D^{\ell_j}u(t)|_{2r} \leq K_{\ell_j} |D^{m+\ell}u(t)|^{\frac{\ell_j+\frac{1}{2}-\frac{1}{2}r}{m+\ell}}$, since $u(t)$. is bounded.

$$|u(t)|_{2r} \leq K \left\{ |D^{m+\ell}u(t)|^{\frac{\frac{1}{2}-\frac{1}{2r}}{m+\ell}} |u(t)|^{1-\frac{(\frac{1}{2}-\frac{1}{2r})}{m+\ell}} + |u(t)| \right\}.$$

That is

(2.2.7) $$|u(t)|_{2r} \leq k_2 |D^{m+\ell}u(t)|^{\frac{(\frac{1}{2}-\frac{1}{2r})}{m+\ell}} + k_3$$

Combining these estimates we get

$$\int_0^T \int_0^1 |D^{m+\ell} uD^{\ell_1}u \ldots\ldots D^{\ell_r} u| dxdt$$

$$\leq \int_0^T \left\{ |D^{m+\ell}u(t)| \left(\prod_{j=o}^{r} K_{\ell_j} |D^{m+\ell} u(t)|^{\frac{(\ell_j+\frac{1}{2}-\frac{1}{2}r)}{m+\ell}} \right) \left(k_2 |D^{m+\ell} u(t)|^{\frac{(\frac{1}{2}-\frac{1}{2r})}{m+\ell}} + k_3 \right) \right.$$

$$\leq \int_0^T \left\{ |D^{m+\ell} u| \left(k' |D^{m+\ell} u|^{\frac{(\ell_1+\ell_2+\ldots+\ell_r)+\frac{1}{2}r-\frac{r}{2r}}{m+\ell}} \right) \left(K_2 |D^{m+\ell}u(t)|^{\frac{(\frac{1}{2}-\frac{1}{2r})}{m+\ell}} + K_3 \right) \right\} dt$$

$$\leq K_4 \int_0^T \left\{ |D^{m+\ell}u(t)|^{1+\frac{m+\ell+2-2r+\frac{r}{2}-\frac{1}{2}}{m+\ell}+\frac{\frac{1}{2}-\frac{1}{2r}}{m+\ell}} \right.$$

$$\left. + |D^{m+\ell}u(t)|^{1+\frac{(m+\ell-2-2r+\frac{r}{2}-\frac{1}{2})}{m+\ell}} \right\} dt.$$

$$= K_4 \int_0^T \left\{ |D^{m+\ell} u(t)|^{\frac{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2}r}{m+\ell}} + |D^{m+\ell} u(t)|^{\frac{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2}}{m+\ell}} \right\} dt$$

Applying Young's inequality we get

$$\int_0^T \int_0^1 (D^{m+\ell} u D^{\ell_1} u D^{\ell_2} u \ldots . D^{\ell_r} u)\, dx dt$$

$$\leq K_4 \left\{ C_{\varepsilon'} (1)^{\frac{4r(m+\ell)}{1-4r+3r^2}} + \int_0^T \left(|D^{m+\ell} u(t)|^{\frac{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2}r)}{m+\ell}} \right)^{\frac{2(m+\ell)}{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2}r}} dt \right.$$

$$\left. + C_{\varepsilon''} (1)^{\frac{4(m+\ell)}{3(r-1)}} + \varepsilon'' \int_0^T \left(|D^{m+\ell} u(t)|^{\frac{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2}}{m+\ell}} \right)^{\frac{2(m+\ell)}{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2}}} dt \right\}.$$

Since $2 - 2r + \frac{r}{2} - \frac{1}{2r} < 0$, $\frac{2(m+\ell)}{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2r}} > 1$.

also since $2 - 2r + \frac{r}{2} - \frac{1}{2r} < 0$, $\frac{2(m+\ell)}{2m+2\ell+2-2r+\frac{r}{2}-\frac{1}{2}} > 1$.

$$\therefore \int_0^T \int_0^1 (D^{m+\ell} u D^{\ell_1} u D^{\ell_2} u \ldots . . D^{\ell_r} u)\, dx\, dt$$

$$\leq K_4 \left\{ C_{\varepsilon'} + \varepsilon' \int_0^T |D^{m+\ell} u(t)|^2 dt + C_{\varepsilon''} + \varepsilon'' \int_0^T |D^{m+\ell} u(t)|^2 dt \right\}$$

$$\leq C_\varepsilon + \varepsilon \int_0^T |D^{m+\ell} u(t)|^2 dt.$$

From (2.2.3) it follows that

$f'_{\ell+1}(u) = (D^{\ell-1}u)^2 u + a' D^\ell u + b' D^{\ell-1} u + c'$ upto inessential constants, and a', b', c' are polynomials in u and its derivatives up to order $\ell - 2$. Since $|D^r u(x,t)|$ is bounded independently of x and t for $0 \leq r \leq \ell - 2$, and $|^{\ell-1}u(t)|$ is bounded independently of t, we have

$$\int_0^1 |f'_{\ell+1}(u(x,t))|dx \leq \int_0^1 |(D^{\ell-1}u)^2 u|dx + \int_0^1 |a' D^\ell u|dx + \int_0^1 |b' D^{\ell-1}u|dx + C'$$

$$\leq C_1 + \int_0^1 |a' D^\ell u(x,t)|dx + C_2 + C'$$

$$\leq C_3 + C_4 \int_0^1 |D^\ell u(x,t)|dx$$

$$\leq C_3 + C_4 \int (1)^2 + \varepsilon \int_0^1 (|D^\ell u(x,t)|)^2 dx, \quad \text{by } [1.2.6].$$

$$\leq C_\varepsilon + \varepsilon |D^\ell u(t)|^2 \quad \text{after adjusting } \varepsilon .$$

Combining the bound for (2.2.4) we get

$$|D^\ell u(t)|^2 + (1+\tau(\eta-1))\int_0^T |D^{m+\ell} u(t)|^2 dt \leq \varepsilon |D^\ell u(t)|^2 + \varepsilon(1+\tau(\eta-1))$$

$$\times \int_0^T |D^{m+\ell} u|^2 dt + C_\varepsilon + \int_0^1 |f_{\ell+1}(g)|dx$$

Taking $\varepsilon = \frac{1}{2}$ and simplifying we obtain

$$|D^\ell u(t)|^2 + (1 + \tau(\eta-1))\int_0^T |D^{m+\ell} u(t)|^2 dt \leq C\,(||g||_\ell).$$

5 $\quad m + 1 \leq \ell \leq m'$

Let us begin with a solution u of (1.1.3), by using (2.2.5) and previous bounds we get

$$|D^{\ell} u(t)|^2 + (1 + \tau(\eta-1)) \int_0^T |D^{m+\ell} u(t)|^2 dt \leq k +$$

$$\left|\int_0^T\int_0^1 D^{m+\ell} u D^{m-\ell} \left[\sum_{r=o}^{\ell-2} b_r D^r u + \sum_{r=o}^{\ell-1} a_r D^{2\ell-2-r} u\right] dx\, dt\right|$$

$$\leq K + \sum_{r=o}^{\ell-2} \left|\int_0^T\int_0^1 D^{m+\ell} u D^{m-\ell} (b_r D^r u)\, dxdt\right| + \sum_{r=o}^{\ell-1} \left|\int_0^T\int_0^1 D^{m+\ell} u D^{m-\ell} (a_r D^{2\ell-2-r} u)\, dxdt\right|$$

$$= K + \sum_{r=o}^{\ell-2} \int_0^T\int_0^1 |D^{2m} u b_r D^r u| dx\, dt + \sum_{r=o}^{\ell-1} \left|\int_0^T\int_0^1 D^{2m} u\; a_r D^{2\ell-2-r} u\, dx\, dt\right|$$

Now $\displaystyle\sum_{r=o}^{\ell-1} \left|\int_0^T\int_0^1 D^{2m} u\; a_r D^{2\ell-2-r} u\, dx\, dt\right| \leq \sum_{r=o}^{\ell-(m+1)} \left|\int_0^T\int_0^1 D^{2m} u\; a_r D^{2\ell-2-r} u dx\, dt\right|$

$$+ \sum_{r=\ell-m}^{\ell-1} \left|\int_0^T\int_0^1 D^{2m} u\; a_r D^{2\ell-2-r} u\, dx\, dt\right|$$

$$= \sum_{r=o}^{\ell-(m+1)} \left|\int_0^T\int_0^1 \left\{D^{2m} u a_{\ell-m-1} D^{2\ell-2-(\ell-m-1)} u + D^{2m} u\; a_r D^{2\ell-2-r} u\right\} dx\, dt\right|$$

$$+ \sum_{r=\ell-m}^{\ell-1} \left|\int_0^T\int_0^1 D^{2m} u\; a_r D^{2\ell-2-r} u\, dx\, dt\right|$$

$$\leq \sum_{r=o}^{\ell-(m+1)} \int_0^T\int_0^1 \left\{ |D^{2m-(m-\ell+1)} u\; a_{\ell-m-1} D^{\ell+m-1+(m-\ell+1)} u| \right.$$

$$\left. + |D^{2m-(m-\ell+1)} u\; a_r D^{2\ell-2-r+(m-\ell+1)} u| \right\} dx\, dt$$

$$+ \sum_{r=\ell-m}^{\ell-1} \left|\int_0^T\int_0^1 D^{2m} u\; a_r D^{2\ell-2-r} u\, dx\, dt\right|$$

$$= \sum_{r=0}^{\ell-(m+1)} \int_0^T \int_0^1 \left\{ |D^{m+\ell-1} u \; a_{\ell-m-1} D^{2m} u| + |D^{m+\ell-1} u a_r D^{\ell+m-1-r} u| \right\} dx\, dt$$

$$+ \sum_{r=\ell-m}^{\ell-1} \left| \int_0^T \int_0^1 D^{2m} u \; a_r \qquad D^{2\ell-2-r} u \; dx \; dt \right| .$$

Substituting back we have

$$|D^{\ell} u(t)|^2 + (1+\tau(\eta-1)) \int_0^T |D^{m+\ell} u(t)|^2 dt \leq K + \sum_{r=0}^{\ell-2} \int_0^T \int_0^1 |D^{2m} u \; b_r D^r u| dx dt$$

$$+ \sum_{r=0}^{\ell-(m+1)} \int_0^T \int_0^1 \left\{ |D^{m+\ell-1} u \; a_{\ell-m-1} D^{2m} u| + |D^{m+\ell-1} u a_r D^{\ell+m-1-r} u| \right\} dx dt$$

$$+ \sum_{r=\ell-m}^{\ell-1} \left| \int_0^T \int_0^1 D^{2m} u \; a_r \; D^{2\ell-2-r} u \; dx \; dt \right|$$

We can develop the bound for the additional terms as follows :

$$0 \leq r \leq \ell - 2.$$

$$\int_0^T \int_0^1 |D^{2m} u \; b_r(u) D^r u| dx dt \leq \text{ess. Sup} |b_r(u)| \left| \int_0^T \int_0^1 |D^{2m} u D^r u| dx \; dt. \right.$$

$$\leq \quad C\left(||u(t)||_{\ell-1}\right) \int_0^T \left\{ |D^{2m} u(t)|_4 |D^r u(t)|_4 \right\} dt \; .$$

$$\leq \sqrt{2} C\left(||u(t)||_{\ell-1}\right) \int_0^T \left\{ \left(|D^{m+\ell} u(t)|^{\frac{2m+\frac{1}{4}}{m+\ell}} |u(t)|^{1-\frac{2m+\frac{1}{4}}{m+\ell}} \right) \left(|D^{m+\ell} u(t)|^{\frac{r+\frac{1}{4}}{m+\ell}} |u(t)|^{1-\frac{r+\frac{1}{4}}{m+\ell}} \right) \right\} dt$$

$$\leq K_1\left(||u(t)||_{\ell-1}\right) \int_0^T \left\{ |D^{m+\ell} u(t)|^{\frac{2m+r+\frac{1}{2}}{m+\ell}} |u(t)|^{\frac{2\ell-r-\frac{1}{2}}{m+\ell}} \right\} dt$$

$$\leq K_2 \int_0^T \left\{ \varepsilon \left(|D^{m+\ell} u(t)|^{\frac{2m+r+\frac{1}{2}}{m+\ell}} \right)^{\frac{2(m+\ell)}{2m+r+\frac{1}{2}}} + C_{\varepsilon} \left(|u(t)|^{\frac{2\ell-r-\frac{1}{2}}{m+\ell}} \right)^{\frac{2(m+\ell)}{2\ell-r-\frac{1}{2}}} \right\} dt$$

$$= K_2 \int_0^T \left\{ \varepsilon |D^{m+\ell} u(t)|^2 + C_\varepsilon |u(t)|^2 \right\} dt$$

$$\leq C_\varepsilon + \varepsilon \int_0^T |D^{m+\ell} u(t)|^2 dt \quad \text{after adjusting } \varepsilon \text{ and } C_\varepsilon.$$

$$0 \leq r \leq \ell - (m+1)$$

$$\int_0^T \int_0^1 \left\{ |D^{\ell+m-1} u \; a_{\ell-m-1} D^{2m} u| + |D^{\ell+m-1} u \; a_r D^{m+\ell-1-r} u| \right\} dx \, dt$$

$$\leq \operatorname*{ess.sup}_{x,t} |a_{\ell-m-1}(u)| \int_0^T |D^{\ell+m-1} u(t)|_4 |D^{2m} u(t)|_4 dt$$

$$+ \operatorname*{ess.sup}_{x,t} |a_r(u)| \int_0^T |D^{\ell+m-1} u(t)|_4 |D^{m+\ell-1-r} u(t)|_4 dt$$

$$\leq C(||u(t)||_{\ell-m}) \int_0^T |D^{\ell+m-1} u(t)|_4 |D^{2m} u(t)|_4 dt$$

$$+ C(||u(t)||_{r+1}) \int_0^T |D^{\ell+m-1} u(t)|_4 |D^{m+\ell-1-r} u(t)|_4 dt.$$

Applying (2.1.6), the above becomes

$$\int_0^T \int_0^1 \left\{ |D^{\ell+m-1} u \; a_{\ell-m-1} D^{2m} u| + |D^{\ell+m-1} u \; a_r D^{m+\ell-1-r} u| \right\} dx \, dt$$

$$\leq \sqrt{2}\, C(||u(t)||_{\ell-m}) \int_0^T \left\{ \left(|D^{m+\ell} u(t)|^{\frac{\ell+m-\frac{3}{4}}{m+\ell}} |u(t)|^{\frac{\frac{3}{4}}{m+\ell}} \right) \left(|D^{m+\ell} u(t)|^{\frac{2m+\frac{1}{4}}{m+\ell}} |u(t)|^{\frac{\ell-m-\frac{1}{4}}{m+\ell}} \right) \right\} dt$$

$$+ \sqrt{2}\, C(||u(t)||_{r+1}) \int_0^T \left\{ \left(|D^{m+\ell} u(t)|^{\frac{\ell+m-\frac{3}{4}}{m+\ell}} |u(t)|^{\frac{\frac{3}{4}}{m+\ell}} \right) \left(|D^{m+\ell} u(t)|^{\frac{m+\ell-r-\frac{3}{4}}{m+\ell}} |u(t)|^{\frac{r+\frac{3}{4}}{m+\ell}} \right) \right\} dt$$

$$= \sqrt{2}C(||u(t)||_{\ell-m}) \int_0^T \left(|D^{m+\ell}u(t)|^{\frac{\ell+3m-\frac{1}{2}}{m+\ell}} |u(t)|^{\frac{\ell-m+\frac{1}{2}}{m+\ell}}\right)dt$$

$$+ \sqrt{2}\ C(||u(t)||_{r+1})\int_0^T \left(|D^{m+\ell}u(t)|^{\frac{2(m+\ell)-r-\frac{3}{2}}{m+\ell}} |u(t)|^{\frac{r+\frac{3}{2}}{m+\ell}}\right)dt.$$

$$\leq \sqrt{2}C(||u(t)||_{\ell-m})\int_0^T\left[\varepsilon\left(|D^{m+\ell}u(t)|^{\frac{\ell+3m-\frac{1}{2}}{m+\ell}}\right)^{\frac{2(m+\ell)}{\ell+3m-\frac{1}{2}}} + C_\varepsilon\left(|u(t)|^{\frac{\ell-m+\frac{1}{2}}{m+\ell}}\right)^{\frac{2(m+\ell)}{\ell-m+\frac{1}{2}}}\right]dt$$

$$+ \sqrt{2}C(||u(t)||_{r+1})\int_0^T\left[\varepsilon\left(|D^{m+\ell}u(t)|^{\frac{2(m+\ell)-r-\frac{3}{2}}{m+\ell}}\right)^{\frac{2(m+\ell)}{2(m+\ell)-r-\frac{3}{2}}} + C_\varepsilon\left(|u(t)|^{\frac{r+\frac{3}{2}}{m+\ell}}\right)^{\frac{2(m+\ell)}{r+\frac{3}{2}}}\right]dt$$

$$= \sqrt{2}\ C\left(||u(t)||_{\ell-m}\right)\int_0^T\left[\varepsilon|D^{m+\ell}u(t)|^2 + C_\varepsilon|u(t)|^2\right]dt$$

$$+ \sqrt{2}\ C(||u(t)||_{r+1})\int_0^T\left[\varepsilon|D^{m+\ell}u(t)|^2 + C_\varepsilon|u(t)|^2\right]dt.$$

$$\leq C_\varepsilon + \varepsilon\int_0^T|D^{m+\ell}u(t)|^2dt \quad \text{after adjusting } \varepsilon \text{ and } C_\varepsilon.$$

For $\ell - m \leq r \leq \ell - 2$.

$$\int_0^T\int_0^1|D^{2m}u\ a_r(u)D^{2(\ell-1)-r}u|dxdt \leq \text{ess.sup}|a_r(u)|\int_0^T\left\{|D^{2m}u(t)|_4|D^{2(\ell-1)-r}u(t)|_4\right\}dt$$

$$\leq C(||u(t)||_{\ell-1})\int_0^T\left\{|D^{2m}u(t)|_4|D^{2(\ell-1)-r}u(t)|_4\right\}dt$$

$$\leq \sqrt{2}\ C(||u(t)||_{\ell-1})\int_0^T\left(|D^{m+\ell}u(t)|^{\frac{2m+\frac{1}{4}}{m+\ell}}|u(t)|^{\frac{\ell-m-\frac{1}{4}}{m+\ell}}\right)\left(|D^{m+\ell}u(t)|^{\frac{2(\ell-1)-r+\frac{1}{4}}{m+\ell}} |u(t)|^{\frac{m-\ell+r+\frac{7}{4}}{m+\ell}}\right)dt,$$

where we have applied (2.2.6).

$$= \sqrt{2}\, C(\|u(t)\|_{\ell-1}) \int_0^T \left[|D^{m+\ell} u(t)|^{\frac{2(m+\ell)-r-\frac{3}{2}}{m+\ell}} |u(t)|^{\frac{r+\frac{3}{2}}{m+\ell}} \right] dt.$$

$$\leq \sqrt{2}\, C(\|u(t)\|_{\ell-1}) \int_0^T \left[\varepsilon \left(|D^{m+\ell} u(t)|^{\frac{2(m+\ell)-r-\frac{3}{2}}{m+\ell}} \right)^{\frac{2(m+\ell)}{2(m+\ell)-r-\frac{3}{2}}} + C_\varepsilon \left(|u(t)|^{\frac{r+\frac{3}{2}}{m+\ell}} \right)^{\frac{2(m+\ell)}{r+\frac{3}{2}}} \right) dt$$

$$= \sqrt{2}\, C(\|u(t)\|_{\ell-1}) \int_0^T \left[\varepsilon |D^{m+\ell} u(t)|^2 + C_\varepsilon |u(t)|^2 \right] dt$$

$$\leq C_\varepsilon + \varepsilon \int_0^T |D^{m+\ell} u(t)|^2 dt$$ after adjusting

C_ε and ε.

For $r = \ell - 1$.

This term involves $a_{\ell-1} D^{2(\ell-1)-(\ell-1)} u = a_{\ell-1} D^{\ell-1} u$.

Since the weight of each term in $G'_{\ell+1}$ is $\ell + 1$, we can

write $a_{\ell-1}(u) = D^{\ell-1}u + a_{\ell-3}(u)$.

* $$\therefore \left| \int_0^T \int_0^1 D^{2m}u \; a_{\ell-1} D^{\ell-1} u \, dx dt \right| = \left| \int_0^T \int_0^1 D^{2m}u \,(D^{\ell-1}u + a_{\ell-3}(u)) D^{\ell-1} u \, dx dt \right|$$

$$= \left| \int_0^T \int_0^1 \left\{ D^{2m} u (D^{\ell-1}u)^2 + D^{2m}u \; a_{\ell-3}(u) D^{\ell-1} u \right\} dx dt \right|$$

$$= \int_0^T \left(D^{\ell-2} u D^{2m} u \quad D^{\ell-1} u \right) \Big|_0^1 dt + \int_0^T \int_0^1 \left| D^{\ell-2} u \left(D^{2m} u D^{\ell} \; u + D^{2m+1} u D^{\ell-1} u \right) \right| dx dt$$

$$+ \int_0^T \int_0^1 |D^{2m}u \; a_{\ell-3}(u) \; D^{\ell-1} u| \, dx \, dt,$$ where we have integrated

by parts

$$= \int_0^T \int_0^1 |D^{\ell-2}u\left(D^{2m}uD^{\ell}u + D^{2m+1}uD^{\ell-1}u\right)|\,dxdt + \int_0^T\int_0^1 |D^{2m}u\ a_{\ell-3}(u)D^{\ell-1}u|\,dxdt$$

$$\leq \underset{x,t}{\text{ess.sup}}|D^{\ell-2}u| \int_0^T \left\{|D^{\ell}u(t)|\,|D^{2m}u(t)| + |D^{2m+1}u(t)|\,|D^{\ell-1}u(t)|\right\}dt$$

$$+ \text{ess.sup}\,|a_{\ell-3}(u)| \int_0^T |D^{2m}u(t)|\,|D^{\ell-1}u(t)|\,dt.$$

$$\leq C(||u(t)||_{\ell-1}) \int_0^T \left\{|D^{\ell}u(t)|\,|D^{2m}u(t)| + |D^{2m+1}u(t)|\,|D^{\ell-1}u(t)|\right.$$

$$\left. + |D^{2m}u(t)|\,|D^{\ell-1}u(t)|\right\} dt.$$

$$\leq \sqrt{2}\,C(||u(t)||_{\ell-1}) \int_0^T \left\{\left(|D^{m+\ell}u(t)|^{\frac{\ell+\frac{1}{4}}{m+\ell}} |u(t)|^{\frac{m-\frac{1}{4}}{m+\ell}}\right)\left(|D^{m+\ell}u(t)|^{\frac{2m+\frac{1}{4}}{m+\ell}} |u(t)|^{\frac{\ell-m-\frac{1}{4}}{m+\ell}}\right)\right.$$

$$+ \left(|D^{m+\ell}u(t)|^{\frac{\ell-\frac{3}{4}}{m+\ell}} |u(t)|^{\frac{m+\frac{3}{4}}{m+\ell}}\right)\left(|D^{m+\ell}u(t)|^{\frac{2m+\frac{5}{4}}{m+\ell}} |u(t)|^{\frac{\ell-m-\frac{5}{4}}{m+\ell}}\right)$$

$$\left. + \left(|D^{m+1}u(t)|^{\frac{2m+\frac{1}{4}}{m+\ell}} |u(t)|^{\frac{\ell-m-\frac{1}{4}}{m+\ell}}\right)\left(|D^{m+\ell}u(t)|^{\frac{\ell-\frac{3}{4}}{m+1}} |u(t)|^{\frac{m+\frac{3}{4}}{m+\ell}}\right)\right\} dt,$$

where we have applied (2.1.6)

$$= \sqrt{2}C(||u(t)||_{\ell-1}) \int_0^T \left\{\left(|D^{m+\ell}u(t)|^{\frac{\ell+2m+\frac{1}{2}}{m+\ell}} |u(t)|^{\frac{\ell-\frac{1}{2}}{m+\ell}}\right)\right.$$

$$\left(|D^{m+\ell}u(t)|^{\frac{2m+\ell+\frac{1}{2}}{m+\ell}} |u(t)|^{\frac{\ell-\frac{1}{2}}{m+\ell}}\right) + \left.\left(|D^{m+\ell}u(t)|^{\frac{2m+\ell-\frac{1}{2}}{m+\ell}} |u(t)|^{\frac{\ell+\frac{1}{2}}{m+\ell}}\right)\right\} dt$$

$$\leq \sqrt{2}\, C(\|u(t)\|_{\ell-1}) \int_0^T \Big[\varepsilon_1 \Big(|D^{m+\ell} u(t)|^{\frac{\ell+2m+\frac{1}{2}}{m+\ell}} \Big)^{\frac{2(m+\ell)}{\ell+2m+\frac{1}{2}}} + C_{\varepsilon_1} \Big(|u(t)|^{\frac{\ell-\frac{1}{2}}{m+\ell}} \Big)^{\frac{2(m+\ell)}{\ell-\frac{1}{2}}}$$

$$+ \varepsilon_2 \Big(|D^{m+\ell} u(t)|^{\frac{2m+\ell+\frac{1}{2}}{m+\ell}} \Big)^{\frac{2(m+\ell)}{2m+\ell+\frac{1}{2}}} + C_{\varepsilon_2} \Big(|u(t)|^{\frac{\ell-\frac{1}{2}}{m+\ell}} \Big)^{\frac{2(m+\ell)}{\ell-\frac{1}{2}}}$$

$$+ \varepsilon_3 \Big(|D^{m+\ell} u(t)|^{\frac{2m+\ell-\frac{1}{2}}{m+\ell}} \Big)^{\frac{2(m+\ell)}{2m+\ell-\frac{1}{2}}} + C_{\varepsilon_3} \Big(|u(t)|^{\frac{\ell+\frac{1}{2}}{m+\ell}} \Big)^{\frac{2(m+\ell)}{\ell+\frac{1}{2}}} \Big] dt$$

$$= \sqrt{2}\, C(\|u(t)\|_{\ell-1}) \int_0^T \Big[\varepsilon_1 |D^{m+\ell} u(t)|^2 + C_{\varepsilon_1} |u(t)|^2$$

$$+ \varepsilon_2 |D^{m+\ell} u(t)|^2 + C_{\varepsilon_2} |u(t)|^2$$

$$+ \varepsilon_3 |D^{m+\ell} u(t)|^2 + C_{\varepsilon_3} |u(t)|^2 \Big] dt$$

$$\leq C_\varepsilon + \varepsilon \int_0^T |D^{m+\ell} u(t)| dt .$$

Combining this with the previous bounds we get

$$|D^\ell u(t)|^2 + (1 + \tau(\eta - 1)) \int_0^T \int_0^1 |D^{m+\ell} u(t)|^2 dt$$

$$\leq C_\varepsilon + \varepsilon(1 + \tau(\eta - 1)) \int_0^T |D^{m+\ell} u(t)|^2 + k.$$

Putting $\varepsilon = \frac{1}{2}$, $\tau = 1$ and simplifying we get

$$|D^\ell u(t)|^2 + \eta \int_0^T |D^{m+\ell} u(t)|^2 dt \leq C(\|g\|_\ell).$$

This complete the proof of the Lemma.

CHAPTER III

In this chapter, we prove the existence of a global solution of the initial value problem (1.1.1). This is achieved by reducing nonlinear problem (1.1.2) to a problem of finding fixed points of completely continuous transformations. We then prove that the solution $u^{\eta}(x,t)$ of (1.1.2) converges to solution of (1.1.1) when $\eta \to 0$. The proof is based on a series of Lemmas for which we need the following compactness result and estimates for the linear problem (1.1.4). We first prove

3.1 A Set Of Functions Which Is Bounded In The Norm Of V_n Is Compact In The Space W_n.

Lemma 3.1.1 Let n be an integer ≥ 1. Let u^{ℓ} be a sequence which is bounded in V_n. Then there exists a subsequence $u^{\ell'}$ which converges in W_n.

Proof. The proof falls into three parts :

Part 1. In this part we will show that for any $\varepsilon > 0$ and for any integer $n \geq 1$ there exists a constant C_{ε} depending on ε and n, such that the inequality

$$(3.1.1) \qquad ||u^{\ell}(t)||^2_{n-1} \leq \varepsilon ||u^{\ell}(t)||^2_n + C_{\varepsilon}||u^{\ell}(t)||^2$$

holds for any $u^{\ell} \in V_n$

Consider first the case where $u \varepsilon C_o^\infty(S')$. Let $\tilde{u}$ be the Fourier transform of u, i.e.

$$\tilde{u}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} u(x)\,dx.$$

Then

(3.1.2) $$\widetilde{D^\alpha u}(\xi) = (i\xi)^\alpha \tilde{u}(\xi).$$

By plancherel's theorem [17], if $\nu \varepsilon L^2(\mathbb{R}^n)$ then $\tilde{\nu} \varepsilon L^2(\mathbb{R}^n)$ and

(3.1.3) $$\int_{\mathbb{R}^n} ||\tilde{\nu}(\xi)||^2 d\xi = (2\Pi)^n \int_{\mathbb{R}^n} ||\nu(x)||^2 dx.$$

Hence, using (3.1.2) we get

(3.1.4) $$(2\Pi)^n \int_{\mathbb{R}^n} ||D^\alpha u(x)||^2 dx = \int_{\mathbb{R}^n} ||\xi^\alpha||^2 ||\tilde{u}(\xi)||^2 d\xi$$

It follows that

$$||u||^2_{n-1} = (2\Pi)^{-n} \sum_{|\alpha|<n-1} \int_{\mathbb{R}^n} ||\xi^\alpha||^2 ||\tilde{u}(\xi)||^2 d\xi$$

(3.1.5)

$$||u||^2_n = (2\Pi)^{-n} \sum_{|\alpha|<n} \int_{\mathbb{R}^n} ||\xi^\alpha||^2 ||\tilde{u}(\xi)||^2 d\xi.$$

Since for any $\varepsilon > o$,

$$\sum_{|\alpha|\leq n-1} ||\xi^{\alpha}||^2 \leq \varepsilon \sum_{|\alpha|\leq n} ||\xi^{\alpha}||^2 + C_{\varepsilon} \quad (\text{for all } \xi \in \mathbb{R}^n)$$

for some constant C_{ε} depending only on ε, n, we obtain from (3.1.5) $||u||^2_{n-1} \leq \varepsilon ||u||^2_n + (2\pi)^{-n} C_{\varepsilon} \int_{\mathbb{R}^n} ||\hat{u}||^2 d\,\xi$

Since, by (3.1.3), the last integral is equal to $(2\pi)^n ||u^{\ell}||^2_o$ hence (3.1.1) follows.

Having proved (3.1.1) for functions u in $C^{\infty}_o(S')$, we can now prove it for any $u \in V_n$ by completion, that is, we take a sequence $\{u^{\ell}\}$ in $C^{\infty}_o(S')$ which converges to u in V_n apply (3.1.1) to each u^{ℓ} we get

$$||u^{\ell}(t)||^2_{n-1} \leq \varepsilon ||u^{\ell}(t)||^2_n + C_{\varepsilon} ||u^{\ell}(t)||^2.$$

The boundedness of the sequence in V_n gives

$$(3.1.6) \qquad \underset{t\leq T}{\text{ess.sup}} ||u^{\ell}(t)||^2_{n-1} \leq K + C \underset{t\leq T}{\text{ess.sup}} ||u^{\ell}(t)||^2$$

For $\Delta t > o$ write

$$u^{\ell}(t) = \frac{1}{\Delta t} \int_t^{t+\Delta t} \left(u^{\ell}(t) - u^{\ell}(s) + u^{\ell}(s)\right) ds$$

$$= \frac{1}{\Delta t} \int_t^{t+\Delta t} \left(u^{\ell}(t) - u^{\ell}(s)\right) ds + \frac{1}{\Delta t} \int_t^{t+\Delta t} u^{\ell}(s)\, ds$$

in terms of average

$$u_{\Delta t}(t) = \frac{1}{\Delta t} \int_t^{t+\Delta t} u(s)\, ds$$

$$\therefore \quad u^{\ell}(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} \left(u^{\ell}(t) - u^{\ell}(s)\right) ds + u^{\ell}_{\Delta t}(t).$$

We use the bound for $\frac{\partial u}{\partial t}$ in $L^2(|0,T|, H)$ to establish

$$\frac{1}{\Delta t^2}\int_0^1 \left(\int_t^{t+\Delta t} \left[u(t) - u(s)\right] ds\right)^2 dx \le \frac{1}{(\Delta t)^2}\int_0^1\int_t^{t+\Delta t} ds \int_t^{t+\Delta t} \left(u(t) - u(s)\right)^2 ds\, dx$$

$$= \frac{1}{\Delta t}\int_0^1\int_t^{t+\Delta t} \left(u(t) - u(s)\right)^2 ds dx.$$

Now $u(t) - u(s) = - u(\tau)\Big|_t^s = -\int_t^s \frac{\partial u}{\partial t}(\tau)$.

$$\therefore \quad \left(u(t) - u(s)\right)^2 = \left(-\int_t^s \frac{\partial u}{\partial t}(\tau)\, d\tau\right)^2.$$

Substituting this in the above we have

$$\frac{1}{(\Delta t)^2}\int_0^1 \left(\int_t^{t+\Delta t} \left(u(t) - u(s)\right) ds\right)^2 dx \le \frac{1}{\Delta t}\int_0^1\int_t^{t+\Delta t} \left(-\int_t^s \frac{\partial u}{\partial t}(\tau)\, d\tau\right)^2 ds dx.$$

$$\le \frac{1}{\Delta t}\int_0^1\int_t^{t+\Delta t} (s - t)\int_t^T \left(\frac{\partial u}{\partial t}(\tau)\right)^2 d\tau\ ds\ dx$$

$$= \frac{\Delta t}{2}\int_0^T\int_0^1 \left(\frac{\partial u}{\partial t}\right)^2 dxdt \le C\, \Delta t,$$

where $\displaystyle\int_t^{t+\Delta t} (s-t)\, ds = \frac{\Delta t^2}{2}$.

Combining with (3.1.6) we get

$$\text{(3.1.7)} \qquad \operatorname*{ess.sup}_{t\leq T} ||u^{\ell}(t)||^2_{n-1} \leq K + C\Delta t + \operatorname*{ess.sup}_{t\leq T} ||u^{\ell}_{\Delta t}(t)||^2.$$

We next show that for fixed $\Delta t > 0$ there is a subsequence $u^{\ell}_{\Delta t}(t)$ which converges in $C([0, T], H)$.

From $\int_0^1 (Du^{\ell}_{\Delta t}(x,t))^2 dx = \frac{1}{\Delta t^2} \int_0^1 \left[\int_t^{t+\Delta t} Du^{\ell}(s) ds\right]^2 dx$

We have $|Du^{\ell}_{\Delta t}(t)|^2 = \frac{1}{\Delta t^2} \int_0^1 \left(\int_t^{t+\Delta t} Du^{\ell}(s) ds\right)^2 dx$

$$\leq \frac{1}{(\Delta t)^2} \int_0^1 \int_t^{t+\Delta t} ds \int_t^{t+\Delta t} (Du^{\ell}(s))^2 ds\, dx$$

$$+ \frac{1}{\Delta t} \int_0^1 \int_t^{t+\Delta t} (Du^{\ell}(s))^2 ds\, dx$$

$$\leq \frac{1}{\Delta t} \int_0^T \int_0^1 (Du^{\ell})^2 dx dt$$

$$< \frac{C}{\Delta t}$$

which shows that $u^{\ell}_{\Delta t}$ is bounded in $C([0,T], H_1)$. By [1.2.11], for each $t \in [0,T]$, the closure of the set $\{u^{\ell}_{\Delta t}(t)\}$ is compact in H. By the estimate

$|Du^{\ell}_{\Delta t}(t + \Delta t) - Du^{\ell}_{\Delta t}(t)|^2$

$$= \frac{1}{(\Delta t)^2} \int_0^1 \left\{\left(\int_{t+\delta t}^{t+\Delta t+\delta t} - \int_t^{t+\Delta t} \right) Du^{\ell} ds \right\}^2 dx$$

$$= \frac{1}{(\Delta t)^2}\int_0^1 \left\{\left[\int_{t+\Delta t}^{t+\Delta t+\delta t} + \int_{t+\delta t}^{t+\Delta t} - \left[\int_{t+\delta t}^{t+\Delta t} + \int_t^{t+\delta t}\right]\right] Du^\ell\, ds\right\}^2 dx$$

$$= \frac{1}{(\Delta t)^2}\int_0^1 \left\{\left(\int_{t+\Delta t}^{t+\Delta t+\delta t} - \int_t^{t+\delta t}\right) Du^\ell\, ds\right\}^2 dx$$

$$\leq \frac{2}{(\Delta t)^2}\int_0^1 \left(\int_{t+\Delta t}^{t+\Delta t+\delta t} Du^\ell(s)\, ds\right)^2 dx + \frac{2}{(\Delta t)^2}\int_0^1 \left(\int_t^{t+\delta t} Du^\ell(s)\, ds\right)^2 dx$$

where in the last step we have used $(a-b)^2 \leq 2(a)^2+2(b)^2$, since $-2ab \leq a^2 + b^2$,

$$|Du^\ell_{\Delta t}(t+\Delta t) - Du^\ell_{\Delta t}(t)|^2 \leq \frac{2}{\Delta t^2}\int_0^1\int_{t+\Delta t}^{t+\Delta t+\delta t} ds \int_{t+\Delta t}^{t+\Delta t+\delta t} (Du^\ell(s))^2\, ds dx$$

$$+ \frac{2}{\Delta t^2}\int_0^1\int_t^{t+\delta t} ds \int_t^{t+\delta t} (Du^\ell(s))^2\, ds$$

$$= \frac{2\delta t}{(\Delta t)^2}\int_0^1\int_{t+\Delta t}^{t+\Delta t+\delta t} (Du^\ell(s))^2\, dsdx + \frac{2\delta t}{\Delta t^2}\int_0^1\int_t^{t+\delta t} (Du^\ell(s))^2\, ds.$$

$$\leq \frac{2\delta t}{(\Delta t)^2}\int_0^T\int_0^1 (Du^\ell(s))^2\, dsdx + \frac{2\delta t}{(\Delta t)^2}\int_0^T\int_0^1 (Du^\ell(s))^2\, ds.$$

$$= \frac{4\delta t}{(\Delta t)^2}\int_0^T\int_0^1 (Du^\ell(s))^2\, ds\, dx$$

$$\leq C\,\delta t.$$

the set $\{u^{\ell}_{\Delta t}(t)\}$ of functions from $[0, T]$ into H_1 or E is equicontinuous. By Arzelà's theorem [1.2.9], for each fixed $\Delta t > 0$, there is a subsequence $u^{\ell'}_{\Delta t}(t)$ which converges in $C([0, T], H)$.

Combined with (3.1.7) this gives

$$\text{(3.1.8)} \quad \overline{\lim_{\ell',k'}} \ \operatorname*{ess.sup}_{t \leq T} ||u^{\ell'}(t) - u^{k'}(t)||_{n-1} \leq C\,\Delta t.$$

Part 2. Proceeding as in part 1, replacing n by n + m and integrating in t, we obtain

$$\int_0^T ||u^{\ell}(t)||^2_{n+m-1} dt \leq \varepsilon \int_0^T ||u^{\ell}(t)||^2_{n+m} dt + C_\varepsilon \int_0^T ||u^{\ell}(t)||^2 dt.$$

Using the bound for u^{ℓ} in $L^2([0, T], H_{n+m})$ we have

$$\int_0^T ||u^{\ell}(t)||^2_{n+m-1} \leq \varepsilon + C_\varepsilon \int_0^T ||u^{\ell}(t)||^2 dt \quad \text{after adjusting } \varepsilon.$$

Following a similar process as in part 1 above, we obtain

$$\text{(3.1.9)} \quad \overline{\lim_{\ell',k'}} \int_0^T ||u^{\ell'}(t) - u^{k'}(t)||^2_{n+m-1} dt \leq C\,\Delta t.$$

Part 3. By selecting a sequence of $(\Delta t)_\nu \to 0$ in (3.1.8) and (3.1.9) and using a diagonalization argument (see appendix A) we can select a subsequence u^{ℓ^*} which converges in W_n. This completes the proof of the lemma 3.1.1.

3.2 The Estimates For The Solution Of (1.1.4).

We now return to the initial value problem (1.1.1) where we prove the existence of a solution to (1.1.1). The proof is based on the estimates for the solution of (1.1.4).

Lemma 3.2.1 Let $m' \geq 2m$. If $\nu(x,t)$ is a solution from $V_{m'}$ of (1.1.4), then

$$\tfrac{1}{2}|\nu(t)|^2 + \eta\int_0^T |D^m\nu(t)|^2 dt \leq C(|||\omega|||_{W_{m'}} + |g|^2)$$

$$\tfrac{1}{2}\int_0^T \left|\frac{\partial\nu(t)}{\partial t}\right|^2 dt + \tfrac{\eta}{2}|D^m\nu(t)|^2 \leq C(|||\omega|||_{W_{m'}} + |g|_m^2)$$

$$\tfrac{1}{2}|D^\ell\nu(t)|^2 + \frac{\eta}{2}\int_0^T |D^{\ell+m}\nu(t)|^2 dt \leq C(|||\omega|||_{W_{m'}} + |g|_\ell^2)\ \ell \leq m' \quad \text{where}$$

the C depends on $|||\omega|||_{W_{m'}}$

Proof

Integrating in x and t the equation

$$\nu\frac{\partial\nu}{\partial t} = \tau\nu k_m(\omega) + (-1)^{m+1}(\tau(\eta-1) + 1)\nu D^{2m}\nu$$

which is obtained by multiplying equation (1.1.4) by a solution $\nu(x,t)$ of the same equation we get

$$\int_0^T\int_0^1 \nu\frac{\partial\nu}{\partial t}dx\, dt = \tau\int_0^T\int_0^1 \nu K_m(\omega)\,dx\,dt + (-1)^{m+1}(\tau(\eta-1)+1)\int_0^T\int_0^1 \nu D^{2m}\nu\, dx\,dt$$

which implies

$$\int_0^T\int_0^1 \frac{\partial}{\partial t}(\tfrac{1}{2}\nu^2)\,dx\,dt = \tau\int_0^T\int_0^1 \nu K_m(\omega)\,dx\,dt + (-1)(\tau(\eta-1)+1)\int_0^T\int_0^1 (D^m\nu)^2\,dx\,dt$$

i.e $$\int_0^1 (\tfrac{1}{2}\nu^2)\Big|_0^t dx + (\tau(\eta-1)+1)\int_0^T\int_0^1 |D^m\nu)^2\,dx\,dt = \tau\int_0^T\int_0^1 \nu K_m(\omega)\,dx\,dt$$

or $$\tfrac{1}{2}|\nu(t)|^2 + (\tau(\eta-1)+1)\int_0^T |D^m\nu(t)|^2 dt \le \tfrac{1}{2}|g|^2 + \tau\int_0^T\int_0^1 |\nu K_m(\omega)|\,dx\,dt$$

$$\le \tfrac{1}{2}|g|^2 + \tfrac{1}{2}\int_0^T\int_0^1 |\nu(x,t)|^2 dx\,dt + \tfrac{1}{2}\int_0^T\int_0^1 K^2{}_m\,dx\,dt \text{ by } [1.2.6].$$

Applying Gronwall's lemma [1.2.10] to

$$|\nu(t)|^2 \le |g|^2 + \int_0^t |\nu(t)|^2 dt + \int_0^t\int_0^1 K^2{}_m\,dx\,dt, \quad \text{i.e.}$$

$$|\nu(t)|^2 \le \left\{|g|^2 + \int_0^T\int_0^1 K_m^2\,dx\,dt\right\} + \int_0^t |\nu(s)|\,ds\ ,$$

we have $$|\nu(t)|^2 \le \left\{|g|^2 + \int_0^T\int_0^1 K_m^2\,dx\,dt\right\} e^{\int_0^t ds}$$

$$= \left\{|g|^2 + \int_0^T\int_0^1 K_m^2\,dx\,dt\right\} e^t\ ,$$

which implies $$\tfrac{1}{2}\int_0^t |\nu(t)|^2 dt \le \frac{e^t}{2}\int_0^t \left\{|g|^2 + \int_0^t\int_0^1 K_m^2\right\} ds$$

Combining this with previous estimates, that is,

$$\tfrac{1}{2}|\nu(t)|^2 + (1+\tau(\eta-1))\int_0^t |D^m\nu(t)|^2 dt \le \tfrac{1}{2}e^t\left\{\int_0^t\int_0^1 K_m^2\,dx\,dt + |g|^2\right\}$$

$$+ \tfrac{1}{2} \int_0^t \int_0^1 k_m^2 \, dx \, dt + \tfrac{1}{2} |g|^2$$

$$= \tfrac{1}{2} (e^t + 1) \left\{ \int_0^t \int_0^1 k_m^2 \, dx \, dt + |g|^2 \right\}$$

$\leq C(|||\omega|||_{W_m} + |g|^2)$, gives the required inequality.

To establish the second inequality of the lemma, we multiply (1.1.4) with $\frac{\partial \nu}{\partial t}$, integrate in x and t to obtain

$$\int_0^T \int_0^1 \left(\frac{\partial \nu}{\partial t}\right)^2 dx \, dt = \tau \int_0^T \int_0^1 \frac{\partial \nu}{\partial t} k_m(\omega) \, dx \, dt$$

$$+ (-1)^{m+1} (\tau(\eta-1)+1) \int_0^T \int_0^1 \frac{\partial \nu}{\partial t} D^{2m} \nu \, dx \, dt.$$

$$= \tau \int_0^T \int_0^1 \frac{\partial \nu}{\partial t} K_m(\omega) \, dx \, dt + (-1)^{m+1} (-1)^m (\tau(\eta-1)+1) \int_0^T \int_0^1 D^m \nu \frac{\partial}{\partial t} (D^m \nu) \, dx dt$$

$$= \tau \int_0^T \int_0^1 \frac{\partial \nu}{\partial t} K_m(\omega) \, dx dt - (\tau(\eta-1)+1) \int_0^T \int_0^1 \frac{\partial}{\partial t} \tfrac{1}{2} (D^m \nu)^2 \, dx \, dt$$

$$= \tau \int_0^T \int_0^1 \frac{\partial \nu}{\partial t} K_m(\omega) \, dx dt - (\tau(\eta-1)+1) \int_0^1 \tfrac{1}{2} (D^m \nu)^2 \Big|_0^t \, dx$$

$$\Longrightarrow \int_0^T \left|\frac{\partial \nu}{\partial t}(t)\right| dt + \frac{(\tau(\eta-1)+1)}{2} |D^m \nu(t)|^2 \leq \tau \int_0^T \int_0^1 \frac{\partial \nu}{\partial t} K_m(\omega) \, dx \, dt$$

$$+ \frac{(\tau(\eta-1)+1)}{2}|D^m g|^2$$

$$\leq \tfrac{1}{2}\int_0^T\int_0^1\left(\frac{\partial\nu}{\partial t}\right)^2 dxdt + \tfrac{1}{2}\int_0^T\int_0^1 K_m^2(\omega)\,dxdt + \frac{(\tau(\eta-1)+1)}{2}|D^m g|^2 \text{ by } [1.2.6]$$

$$\leq \tfrac{1}{2}\int_0^T\left|\frac{\partial\nu}{\partial t}\right|^2 dt + C\,|||\omega|||_{W_m'} + \frac{(\tau(\eta-1)+1)}{2}|D^m g|^2$$

Rearranging the above we obtain

$$\tfrac{1}{2}\int_0^T\left|\frac{\partial\nu(t)}{\partial t}\right|^2 dt + \frac{\eta}{2}|D^m\nu(t)|^2 \leq C\left(|||\omega|||_{W_m'} + |g|_m^2\right)$$

giving the second inequality.

The final estimate is obtain by first taking D^ℓ of (1.1.4), multiplying by $D^\ell\nu$, and integrating in x and t as follows :

Taking D^ℓ of (1.1.4) we get

$$\frac{\partial}{\partial t}(D^\ell\nu) = \tau D^{\ell+1}G_m(\omega) + (-1)^{m+1}(\tau(\eta-1)+1)D^{2m+\ell}\nu.$$

Multiplying through by $D^\ell\nu$ we have

$$D^\ell\nu\,\frac{\partial}{\partial t}(D^\ell\nu) = \tau D^\ell\nu D^{\ell+1}G_m(\omega) + (-1)^{m+1}(\tau(\eta-1)+1)D^\ell\nu D^{2m+\ell}\nu.$$

Integrating in x and t we obtain

$$\int_0^T\int_0^1 D^\ell\nu\,\frac{\partial}{\partial t}(D^\ell\nu)\,dxdt = \tau\int_0^T\int_0^1 D^\ell\nu D^{\ell+1}G_m(\omega)\,dx\,dt$$

$$+ \quad (-1)^{m+1} (\tau(\eta-1)+1) \int_0^T\int_0^1 D^\ell \nu D^{2m+\ell} \nu \, dx \, dt$$

$$\Longrightarrow \int_0^T\int_0^1 \tfrac{1}{2} \frac{\partial}{\partial t}(D^\ell \nu)^2 dx \, dt = -(\tau(\eta-1)+1)\int_0^T\int_0^1 D^{\ell+m}\nu D^{\ell+m}\nu \, dx \, dt$$

$$+ \tau \int_0^T\int_0^1 D^\ell \nu \, D^{\ell+1} G_m(\omega) \, dx \, dt.$$

$$\Longrightarrow \int_0^1 \tfrac{1}{2}(D^\ell \nu)^2 dx - \tfrac{1}{2}\int_0^1 (D^\ell g)^2 dx = -(\tau(\eta-1)+1)\int_0^T\int_0^1 (D^{m+\ell}\nu)^2 dx dt$$

$$+ \tau \int_0^T\int_0^1 D^\ell \nu D^{\ell+1} G_\omega(\omega) \, dx \, dt$$

$$\Longrightarrow \tfrac{1}{2}|D^\ell \nu(t)|^2 + \eta\int_0^T |D^{m+\ell}\nu(t)|^2 dt \le \tfrac{1}{2}|D^\ell g|^2 + \left|\int_0^T\int_0^1 D^\ell \nu . D^{\ell+1} G_m(\omega) \, dx dt\right|$$

after putting $\tau = 1$. We used (2.2.5), integration by parts and (2.1.6) to develop the bound for the remaining term as follows :

$$G'_m(\omega) = \sum_{r=0}^{m-2} a_r D^{2m-2-r} \omega + \sum_{r=0}^{m-3} b_r D^r \omega$$

But $\quad G_m(\omega) = D^{2m-2} \omega + G'_m(\omega)$

$$= D^{2m-2}\omega + \sum_{r=0}^{m-2} a_r D^{2m-4-r} \omega + \sum_{r=0}^{m-3} b_r D^\gamma \omega.$$

Therefore $\int_0^T\int_0^1 D^{\ell}\nu D^{\ell+1}G_m(\omega)\,dx\,dt = (-1)^{\ell+1}\int_0^T\int_0^1 D^{2\ell+1}\nu G_m(\omega)\,dx\,dt$

$$= (-1)^{\ell+1}\int_0^T\int_0^1 D^{2\ell+1}\nu\Big[D^{2m-2}\omega + \sum_{r=0}^{m-2} a_r D^{2m-4-r}\omega + \sum_{r=0}^{m-3} b_r D^r\omega\Big]dx\,dt.$$

$$= (-1)^{\ell+1}\int_0^T\int_0^1\Big\{D^{2\ell+1}\nu D^{2m-2}\omega + \sum_{r=0}^{m-2} D^{2\ell+1}\nu\, a_r D^{2m-4-r}\omega + \sum_{r=0}^{m-3} D^{2\ell+1}\nu\, b_r D^r\omega\Big\}dxdt.$$

We now develop the bound for each term as follows :

$\ell < m-1$

$$\Big|\int_0^T\int_0^1 D^{2\ell+1}\nu D^{2m-2}\omega\,dxdt\Big| \le \sqrt{\int_0^T |D^{2\ell+1}\nu(t)|^2 dt}\sqrt{\int_0^T |D^{2m-2}\omega(t)|^2 dt}$$

$$\le \varepsilon'\int_0^T |D^{2\ell+1}\nu(t)|^2 dt + C_{\varepsilon'}\int_0^T |D^{2m-2}\omega(t)|^2 dt \quad \text{by} \quad [1.2.6]$$

$$\le \varepsilon'\int_0^T\Big[\sqrt{2}\Big(|D^{m+1}\nu(t)|^{\frac{2\ell+1+\frac12-\frac14}{m+\ell}}|\nu(t)|^{1-\frac{2\ell+1+\frac12-\frac14}{m+\ell}}\Big)^2\Big]dt + C_{\varepsilon'}\int_0^T |D^{2m-2}\omega(t)|^2 dt$$

by applying (2.1.6)

$$\le \varepsilon'\sqrt{2}\int_0^T\Big\{|D^{m+\ell}\nu(t)|^{\frac{4\ell+\frac52}{m+\ell}}|\nu(t)|^{\frac{2(m-\ell)-\frac52}{m+\ell}}\Big\}dt + C_{\varepsilon'}\int_0^T |D^{2m-2}\omega(t)|^2 dt$$

$$\le \varepsilon'\varepsilon''\sqrt{2}\int_0^T |D^{m+\ell}\nu(t)|^2 dt + \varepsilon' C_{\varepsilon''}\sqrt{2}\int_0^T |\nu(t)|^2 dt + C_{\varepsilon'}\int_0^T |D^{2m-2}\omega(t)|^2 dt$$

$$\leq \varepsilon \int_0^T |D^{m+\ell} \nu(t)|^2 dt + C_\varepsilon \int_0^T |\nu(t)|^2 dt + C|||\omega|||_{W_m}.$$

$0 \leq r \leq m-2$

$$\left|\int_0^T \int_0^1 D^{2\ell+1} \nu \; a_r(\omega) D^{2(m-2)-r} \omega \, dx \, dt\right|$$

$$\leq \underset{x,t}{\text{ess.sup}} |a_r(\omega)| \left(\int_0^T |D^{2\ell+1}\nu(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T |D^{2(m-2)-r}\omega(t)|^2 dt\right)^{\frac{1}{2}}$$

$$\leq K\left\{\varepsilon' \int_0^T |D^{2\ell+1}\nu(t)|^2 dt + C'_\varepsilon \int_0^T |D^{2(m-2)-r}\omega(t)|^2 dt \text{ by } [1.2.6]\right.$$

$$\leq \varepsilon \int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon \int_0^T |\nu(t)|^2 dt + C|||\omega|||_{W_m}.$$

$$\leq \varepsilon \int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon \int_0^T |\nu(t)|^2 dt + C|||\omega|||_{W_m}. \text{ as above}$$

$0 \leq r \leq m-3$

$$\left|\int_0^T \int_0^1 D^{2\ell+1}\nu b_r D^r \omega dx \, dt\right| \leq \text{ess.sup}|b_r(\omega)| \sqrt{\int_0^T |D^{2\ell+1}\nu(t)|^2 dt} \sqrt{\int_0^T |D^r \omega(t)|^2 dt}$$

$$\leq C + \varepsilon \int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon \int_0^T |\nu(t)|^2 dt + C\,|||\omega|||_{W_m}.$$

$0 \leq r \leq m-2$

$$\left|\int_0^T \int_0^1 D^{2\ell+1}\nu(t) \; a_r \; D^{2(m-2)-r} \omega \, dx \, dt\right|$$

$$= \left|\int_0^T\int_0^1 D^{\ell+m}\nu D^{\ell-m+1}\left(a_r(\omega)D^{2(m-2)-r}\omega\right)dx\ dt\right|$$

$$\leq \sqrt{\int_0^T |D^{m+\ell}\nu(t)|^2 dt} \times \sqrt{\int_0^T |D^{\ell-m+1}\left(a_r(\omega)D^{2(m-2)-r}\omega(t)\right)|^2 dt}$$

$$\leq \varepsilon \int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon\ |||\omega|||_{W_m'} \quad \text{by } [1.2.6]$$

$0 \leq r \leq m-3$

$$\left|\int_0^T\int_0^1 D^{2\ell+1}\nu b_r D^r\omega\ dx\ dt\right| = \left|\int_0^T\int_0^1 D^{\ell+m}\nu D^{\ell-m+1}(b_r D^r\omega)\,dx\ dt\right|$$

$$\leq \sqrt{\int_0^T |D^{m+\ell}\nu(t)|^2 dt}\ \sqrt{\int_0^T |D^{\ell-m+1}(b_r D^r\omega)|^2 dt}$$

$$\leq \varepsilon \int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon |||\omega|||_{W_m'}$$

Combining these we have

$$\tfrac{1}{2}|D^\ell\nu(t)|^2 + \eta\int_0^T |D^{\ell+m}\nu(t)|^2 dt \leq \varepsilon\int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon |||\omega|||_{W_m'}$$

$$+ \tfrac{1}{2}|D^\ell g| \quad \text{after adjusting } \varepsilon \text{ and } C_\varepsilon.$$

Choosing $\varepsilon = \frac{1}{2}$ and adjusting η we get

$$\tfrac{1}{2}|D^\ell\nu(t)|^2 + \eta\int_0^T |D^{m+\ell}\nu(t)|^2 dt \leq C\left(|||\omega|||_{W_m'} + |g|_\ell\right).$$

This completes the proof of the lemma.

3.3 Proof of the theorem 1.1.1

Let $m' \geq 2m$. Consider the η - dependent initial data g^η in (1.1.2) with $g^\eta \in H_{m'}$ and g^η converging to the data g of (1.1.1) in $H_{m'}$. Then g^η is bounded independently of η in $H_{m'}$. The existence proof of (1.1.1) is based on reducing (1.1.2) to a problem of finding fixed points with respect to completely continuous transformations. We begin with the family of nonlinear problems (1.1.3) and the related linear problem (1.1.4). If $\omega \in W_{m'}$, then it is known that a solution $\nu(x,t)$ of (1.1.4) exists, is unique and belongs to $V_{m'}$. This defines a nonlinear operator $\nu = \phi(\omega,\tau)$, which for each $\tau \in [0,1]$ and $\omega \in W_{m'}$ determines the solution to (1.1.4). In determining fixed points of $\phi(\omega,\tau)$ we apply the Leray-Schauder theorem.

Consider $\phi(\omega,\tau)$ on the set M of $W_{m'}$ consisting of functions $\omega \in W_{m'}$ satisfying the inequality (2.2.1) with the bound increased by adding the positive number $\varepsilon \geq 0$. We will verify that $\phi(\omega,\tau)$ on $M \times [0,1]$ is

(a) Continuous in ω, uniformly continuous in τ

(b) Continuous in τ, uniformly continuous in ω

(c) Completely continuous on $M \times [0,1]$.

We also show that all fixed points u^τ of the map ϕ lie strictly in the interior of the set M and that $\omega-\phi(\omega,0)$ has nonzero degree. We then apply Leray-Schauder theorem for existence of fixed points of ϕ.

Step 1: $\phi(\omega,\tau)$ is continuous with respect to ω in W_m', uniformly in $\tau \varepsilon [0,1]$. We begin with two elements ω' and ω'' from M so that $\nu'=\phi(\omega',\tau)$ and $\nu''=\phi(\omega'',\tau)$. From (1.1.4), we have with $\nu = \nu' - \nu''$

$$\frac{\partial\nu}{\partial t} + (-1)^m (1+(\eta-1)\tau) D^{2m}\nu = \tau\left[K_m(\omega') - K_m(\omega'')\right]$$

$$= \tau \Delta K_m(\omega) \text{ and } \nu(x, 0) = 0 .$$

From (2.2.5) we have $G_m'(u) = \sum_{r=o}^{m-2} a_r D^{2m-4-r} u + \sum_{r=o}^{m-3} b_r D^r u.$

But $G_m(u) = D^{2m-2} u + G_m'(u)$.

Therefore $K_m(\omega) = \frac{\partial}{\partial x} G_m(\omega)$

$$= \frac{\partial}{\partial x}\left[D^{2m-2}\omega + G_m'(\omega)\right]$$

$$= D^{2m-1}\omega + \frac{\partial}{\partial x}\left[\sum_{r=o}^{m-2} a_r D^{2m-4-r}\omega + \sum_{r=o}^{m-3} b_r D^r \omega\right]$$

$$= D^{2m-1}\omega + \sum_{r=o}^{m-2} a_r D^{2m-3-r}\omega + \sum_{r=o}^{m-3} b_r D^{r+1}\omega$$

$$= D^{2m-1}\omega + \sum_{r=1}^{2m-3} d_r(\omega) D^r\omega \quad \text{up to inessential}$$

constants where $d(\omega)$ is a polynomial in ω and its derivatives up to order $m - 2$.

$$\Delta K_m = \sum_{r=1}^{2m-3}\left[d_r(\omega')D^r\omega' - d_r(\omega'')D^r\omega''\right] + D^{2m-1}(\omega'-\omega'')$$

$$= \sum_{r=1}^{2m-3}\left[d_r(\omega')D^r\omega'-d_r(\omega')D^r\omega''+d_r(\omega')D^r\omega''-d_r(\omega'')D^r\omega''\right]+D^{2m-1}\omega$$

$$= \sum_{r=1}^{2m-3}\left[d_r(\omega')(D^r\omega'-D^r\omega'') + (d_r(\omega')-d_r(\omega''))D^r\omega''\right] + D^{2m-1}\omega.$$

$$= \sum_{r=1}^{2m-3}\left\{d_r(\omega')D^r\omega + (d_r(\omega') - d_r(\omega''))D^r\omega''\right] + D^{2m-1}\omega.$$

Proceeding as in lemma 3.2.1 with $\nu(x,0) = g(x) = 0$, we have

$$(3.3.1)\qquad \tfrac{1}{2}|\nu(t)|^2+\eta\int_0^T|D^m\nu(t)|^2dt \le \tfrac{1}{2}(e^t+1)\int_0^T\int_0^1(\Delta k_m)^2dxdt,$$

$$\tfrac{1}{2}\int_0^T\left|\frac{\partial\nu(t)}{\partial t}\right|^2dt + \frac{\eta}{2}|D^m\nu(t)|^2 \le \tfrac{1}{2}\int_0^T\int_0^1(\Delta K_m)^2dx\,dt, \quad \text{where}$$

$\nu = \nu' - \nu''$ and

$$\Delta k_m = \sum_{r=1}^{2m-3}\left\{d_r(\omega')D^r\omega+(d_r(\omega')-d_r(\omega''))D^r\omega''\right\} + D^{2m-1}\omega.$$

The bound for ΔK_m in $L^2([0,T], L^2)$ is obtained as follows:

$$\int_0^T\int_0^1|d_r(\omega')D^r\omega|^2dxdt \le \text{ess.sup}|d_r(\omega')|^2\int_0^T|D^r\omega(t)|^2dt$$

$$\le C\int_0^T|D^r\omega(t)|^2dt, \quad 1 \le r \le 2m-3$$

$$\leq C\ |||\omega|||_{W_{m'}}.$$

$$\left|\int_0^T\int_0^1 (d_r(\omega')-d_r(\omega''))D^r\omega''\right|^2 dxdt \leq \text{ess.sup}|d_r(\omega')-d_r(\omega'')|^2\int_0^T |D^r\omega''|^2 dt.$$

$$\leq \text{ess.sup}|d_r(\omega') - d_r(\omega'')|^2 C,\ 1 \leq r \leq m-2$$

$$\leq C|||\omega|||_{W_{m'}}.$$

Since ω', $\omega'' \in M$. Combining the estimates we have

$\int_0^T\int_0^1 |\Delta K_m|^2 dxdt \leq C|||\omega|||_{W_{m'}}$ after adjusti ng C. Now

beginning with equation (1.1.4) for $\nu = \nu' = \nu''$ and

multiplying both sides by $D^{2\ell}\nu$ we obtain

$$D^{2\ell}\nu\frac{\partial\nu}{\partial t} + (-1)^m(\tau(\eta-1)+1)D^{2\ell}\nu D^{2m}\nu = \tau D^{2\ell}\nu\Delta K_m(\omega).$$

Integrating in x and t we get

$$(-1)^\ell\int_0^T\int_0^1 D^\ell\nu\ \frac{\partial}{\partial t}(D^\ell\nu)\,dxdt + (-1)^\ell(\tau(\eta-1)+1)\int_0^T\int_0^1 (D^{\ell+m}\nu)^2 dx\ dt$$
$$= \tau\int_0^T\int_0^1 D^{2\ell}\nu\Delta K_m(\omega)\ dx\ dt, \text{ that is,}$$

$$(-1)^\ell\int_0^1 \tfrac{1}{2}(D^\ell\nu)^2\Big|_0^t dt + (-1)^\ell(\tau(\eta-1)+1)\int_0^T\int_0^1 (D^{\ell+m}\nu)^2 dxdt = \tau\int_0^T\int_0^1 D^{2\ell}\nu\Delta k_m(\omega)\,dxdt$$

which implies $\tfrac{1}{2}|D^\ell\nu(t)|^2 + \eta\int_0^T |D^{\ell+m}\nu(t)|dt \leq \left|\tau\int\int_0^1 D^{2\ell}\nu\Delta k_m(\omega)\,dxdt\right|$

where we have put $\nu(x,0) = 0$.

Now using (2.2.5) in ΔK_m we have

$$\Delta K_m(\omega) = \sum_{r=1}^{2m-3}\left[d_r(\omega')D^r\omega + \{d_r(\omega') - d_r(\omega^*)\}D^r\omega^*\right] + D^{2m-1}\omega$$

$$D^{2\ell}\nu\Delta K_m(\omega) = \sum_{r=1}^{2m-3}\left\{d_r(\omega')D^{2\ell}\nu D^r\omega + (d_r(\omega') - d_r(\omega^*))D^{2\ell}\nu D^r\omega^*\right\} + D^{2\ell}\nu D^{2m-1}\omega .$$

The bound is developed as follows :

$1 \leq r \leq 2m - 3$

$$\left|\int_0^T\int_0^1 D^{2\ell}\nu d_r(\omega')D^r\omega dxdt\right| \leq \text{ess.sup}|d_r(\omega')| \sqrt{\int_0^T|D^{2\ell}\nu(t)|^2dt}\sqrt{\int_0^T|D^r\omega(t)|^2dt}$$

$$\leq C\sqrt{\int_0^T|D^{2\ell}\nu(t)|^2dt}\sqrt{\int_0^T|D^r\omega(t)|^2dt}$$

$$\leq C\left\{\varepsilon'\int_0^T|D^{2\ell}\nu(t)|^2dt + C_{\varepsilon'}\int_0^T|D^r\omega(t)|^2dt\right\} \text{ by } (1.2.6)$$

$$\leq C\left\{\varepsilon'\int_0^T\left[\sqrt{2}\left(|D^{m+\ell}\nu(t)|^{\frac{2\ell+\frac{1}{2}-\frac{1}{4}}{m+\ell}}|\nu(t)|^{1-\frac{2\ell+\frac{1}{2}-\frac{1}{4}}{m+\ell}}\right)^2\right]dt + C_{\varepsilon'}\int_0^T|D^r\omega(t)|^2dt\right.$$

$$= \sqrt{2}C\varepsilon'\int_0^T\left[|D^{\ell+m}\nu(t)|^{\frac{4\ell+\frac{1}{2}}{m+\ell}}|\nu(t)|^{\frac{2(m-\ell)-\frac{1}{2}}{m+\ell}}\right]dt + CC_{\varepsilon'}\int_0^T|D^r\omega(t)|^2dt$$

$$\leq \sqrt{2}C\varepsilon'\int_0^T\left[\varepsilon''\left(|D^{m+\ell}\nu(t)|^{\frac{2(m+\frac{1}{2})}{4\ell+\frac{1}{2}}}\right]^{\frac{2(m+\ell)}{4\ell+\frac{1}{2}}} + C_{\varepsilon''}\left[|\nu(t)|^{\frac{2(m-\ell)-\frac{1}{2}}{m+\ell}}\right]^{\frac{2(m+\ell)}{2(m-\ell)-\frac{1}{2}}}$$

$$+ CC_{\varepsilon'}\int_0^T|D^r\omega(t)|^2dt$$

$$\leq \varepsilon \int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon \int_0^T |\nu(t)|^2 dt + C\int_0^T |D^r\omega(t)|^2 dt.$$

$1 \leq r \leq m-2$

$$\left|\int_0^T \int_0^1 (d_r(\omega') - d_r(\omega''))D^{2\ell}\nu D^r\omega'' dx\, dt\right|$$

$$\leq \text{ess.sup}|d_r(\omega') - d_r(\omega'')| \sqrt{\int_0^T |D^{2\ell}\nu(t)|^2 dt}\sqrt{\int_0^T |D^r\omega''(t)|^2 dt}$$

$$\leq C_\varepsilon\, C \quad \text{ess-sup}|d_r(\omega') - dr(\omega'')|^2 + \varepsilon\int_0^T |D^{2\ell}\nu(t)|^2 dt$$

$$\leq |||\omega|||_{W_m'} + \varepsilon\int_0^T |D^{m+\ell}\nu(t)|^2 dt + \int_0^T |\nu(t)|^2 dt, \text{ as before}$$

$$\left|\int_0^T \int_0^1 D^{2\ell}\nu D^{2m-1}\omega\, dx\, dt\right| = \left|(-)^{\ell-m}\int_0^T \int_0^1 D^{2\ell-(\ell-m)}\nu D^{2m+(\ell-m)}\omega\, dxdt\right|$$

$$= \left|\int_0^T \int_0^1 D^{\ell+m}\nu D^{\ell+m-1}\omega\, dxdt\right|$$

$$\leq \int_0^T |D^{m+\ell}\nu(t)||D^{\ell+m-1}\omega(t)|dt$$

$$< \varepsilon\int_0^T |D^{m+\ell}\nu(t)|^2 dt + C_\varepsilon\int_0^T |D^{m+\ell-1}\omega(t)|^2 dt$$

$$\leq \varepsilon\int_0^T |D^{\ell+m}\nu(t)|^2 dt + C\, |||\omega|||_{W_m'}$$

Combining and rearranging we obtain

$$|D^{\ell}\nu(t)|^{2} + \eta\int_{0}^{T}|D^{m+\ell}\nu(t)|^{2}dt \le C|||\omega|||_{W_m}.$$ This now gives

the result.

Step 2: $\phi(\omega,\tau)$ is continuous in τ, uniformly in ω on

$M \times [0,1]$. With $\nu^{\tau+\Delta\tau} = \phi(\omega,\tau+\Delta\tau)$, $\nu^{\tau} = \phi(\omega,\tau)$,

and $\nu = \nu^{\tau+\Delta\tau} - \nu^{\tau}$, we have

$$\frac{\partial\nu}{\partial t} = \Delta\tau\ K_m(\omega) + (-1)^{m+1}\big((\tau+\Delta\tau)(\eta-1)+1\big)D^{2m}\nu^{\tau+\Delta\tau} - (-1)^{m+1}\big(\tau(\eta-1)+1\big)D^{2m}\nu^{\tau}$$

$$= \Delta\tau\ K_m(\omega) + (-1)^{m+1}\Big[\big((\tau+\Delta\tau)(\eta-1)+1\big)D^{2m}\nu^{\tau+\Delta\tau} - \big(\tau(\eta-1)+1\big)D^{2m}\nu^{\tau+\Delta\tau}$$

$$+ \big(\tau(\eta-1)+1\big)D^{2m}\nu^{\tau+\Delta\tau} - \big(\tau(\eta-1)+1\big)D^{2m}\nu^{\tau}\Big]$$

$$= \Delta\tau K_m(\omega) + (-1)^{m+1}\Big[\Delta\tau(\eta-1)D^{2m}\nu^{\tau+\Delta\tau} + \big(\tau(\eta-1)+1\big)D^{2m}\nu\Big],$$ that is

$$\frac{\partial\nu}{\partial t} + (-1)^{m}\big(\tau(\eta-1)+1\big)D^{2m}\nu = \Delta\tau K_m(\omega) + (-1)^{m+1}\Delta\tau(\eta-1)D^{2m}\nu^{\tau+\Delta\tau}.$$

The above implies

$$(3.3.2)\quad \frac{\partial\nu}{\partial t} + (-1)^{m}\big(\tau(\eta-1)+1\big)D^{2m}\nu = \Delta\tau\Big\{K_m(\omega) + (-1)^{m}(1-\eta)D^{2m}\nu^{\tau+\Delta\tau}$$

multiplying (3.3.2) by ν and integrate in x and t, that is

$$\int_0^T\int_0^1 \nu\frac{\partial\nu}{\partial t}\,dxdt + (-1)^{m}\big(\tau(\eta-1)+1\big)\int_0^T\int_0^1 \nu D^{2m}\nu\,dx\,dt$$

$$= \Delta\tau \int_0^T \int_0^1 \nu K_m(\omega)\,dxdt + \Delta\tau\,(-1)^m (1-\eta) \int_0^T \int_0^1 \nu D^{2m} \nu^{\tau+\Delta\tau} dxdt.$$

$$\Longrightarrow \int_0^T \int_0^1 \frac{\partial}{\partial t}(\tfrac{1}{2}\nu^2)\,dxdt + \left(\tau(\eta-1)+1\right) \int_0^T \int_0^1 (D^m \nu)^2 dx\,dt$$

$$= \Delta\tau \int_0^T \int_0^1 \nu K_m(\omega)\,dxdt + \Delta\tau\,(1-\eta) \int_0^T \int_0^1 D^m \nu D^m \nu^{\tau+\Delta\tau} dx\,dt.$$

$$\Longrightarrow \quad \tfrac{1}{2}|\nu(t)|^2 + \left(\tau(\eta-1)+1\right) \int_0^T |D^m \nu(t)|^2 dt$$

$$\leq \Delta\tau \sqrt{\int_0^T\int_0^1 K_m^2(\omega)\,dxdt}\sqrt{\int_0^T \int_0^1 \nu dxdt} + \Delta\tau \sqrt{\int_0^T \int_0^1 (D^m \nu)^2 dxdt}\sqrt{\int_0^T \int_0^1 (D^m \nu^{\tau+\Delta\tau})^2 dxdt}$$

$$\leq \Delta\tau \sqrt{\int_0^T \int_0^1 K^2 dxdt}\sqrt{\int_0^T |\nu(t)|^2 dt} + \Delta\tau \sqrt{\int_0^T |D^m \nu(t)|^2 dt}\sqrt{\int_0^T |D^m \nu(t)^{\tau+\Delta\tau}|^2 dt}$$

From the uniform bounds of lemma 3.2.1 we get

$$\tfrac{1}{2}|\nu(t)|^2 + \eta \int_0^T |D^m \nu(t)|^2 dt \leq C\,\Delta\tau \ .$$

Now we take D^ℓ of (3.3.2), multiply through by $D^\ell \nu$ and proceed as in lemma 3.2.1 as below :

$$D^\ell \nu D^\ell \left(\frac{\partial \nu}{\partial t}\right) + \quad (-1)^m \left(1+\tau(\eta-1)\right) D^\ell \nu D^\ell \left(D^{2m}\nu\right)$$

$$\Delta\tau \left\{ D^\ell \nu D^\ell \left(K_m(\omega)\right) + (1-\eta)\,(-1)^m D^\ell \nu\, D^\ell \left(D^{2m} \nu^{\tau+\Delta\tau}\right) \right\}$$

Integrating in x and t we have

$$\int_0^T\int_0^1 \tfrac{1}{2}\,\frac{\partial}{\partial t}(D^\ell \nu)^2\,dx\,dt + (-1)^{m+\ell}\,(1+\tau(\eta-1))\int_0^T\int_0^1 D^{2\ell}\nu\, D^{2m}\nu\;dx\;dt$$

$$\Delta\tau\left\{\int_0^T\int_0^1 D^\ell \nu D^{\ell+1} G_m(\omega)\,dx\,dt + (1-\eta)\,(-1)^{m+\ell}\int_0^T\int_0^1 D^{2\ell}\nu D^{2m}\nu^{\tau+\Delta\tau}\,dx\,dt\right\}$$

$$\Longrightarrow \int_0^1 \tfrac{1}{2}(D^\ell\nu)^2\Big|_0^t dx + (1+\tau(\eta-1))\int_0^T\int_0^1 (D^{\ell+m}\nu)^2\,dx\;dt$$

$$= \Delta\tau\left\{\int_0^T\int_0^1 D^\ell\nu D^{\ell+1} G_m(\omega)\,dx\,dt + (1-\eta)\int_0^T\int_0^1 D^{\ell+m}\nu D^{m+\ell}\nu^{\tau+\Delta\tau}\,dx\,dt\right.$$

$$\Longrightarrow \tfrac{1}{2}|D^\ell\nu(t)|^2 + (1+\tau(\eta-1))\int_0^T |D^{m+\ell}\nu(t)|^2\,dt$$

$$\leq \Delta\tau\sqrt{\int_0^T\int_0^1 (D^\ell\nu)^2\,dx\,dt}\sqrt{\int_0^T\int_0^1 (D^{\ell+1}\,G_m(\omega))^2\,dx\,dt}$$

$$+ \Delta\tau\sqrt{\int_0^T\int_0^1 (D^{\ell+m}\nu)^2\,dx\,dt}\sqrt{\int_0^T\int_0^1 (D^{\ell+m}\nu^{\tau+\Delta\tau})^2\,dx\;dt.}$$

$$\leq \Delta\tau\sqrt{\int^T |D^\ell\nu(t)|^2\,dt}\sqrt{\int_0^T |D^{\ell+1}G_m\omega(t)|^2\,dt}$$

$$+ \Delta\tau\sqrt{\int_0^T |D^{m+\ell}\nu(t)|^2\,dt}\sqrt{\int_0^T |D^{\ell+m}\nu^{\tau+\Delta\tau}(t)|^2\,dt.}$$

From the uniform bounds of lemma 3.2.1 we obtained

$$|D^{\ell}\nu(t)|^2 + \eta\int_0^T |D^{m+\ell}\nu(t)|^2 dt \leq C\,\Delta\tau.$$

Multiplying (3.3.2) by $\frac{\partial\nu}{\partial t}$ and integrating in x and t we get

$$\int_0^T\int_0^1 \left(\frac{\partial\nu}{\partial t}\right)^2 dxdt + (-1)^m (1+\tau(\eta-1))\int_0^T\int_0^1 \frac{\partial\nu}{\partial t} D^{2m}\nu dxdt$$

$$= \Delta\tau\int_0^T\int_0^1 \frac{\partial\nu}{\partial t} K_m(\omega)\,dxdt + \Delta\tau\,(1-\eta)\,(-1)^m\int_0^T\int_0^1 \frac{\partial\nu}{\partial t} D^{2m}\nu^{\tau+\Delta\tau} dx\ dt,$$

that is,

$$\int_0^T\int_0^1 \left(\frac{\partial\nu}{\partial t}\right)^2 dxdt + (1+\tau(\eta-1))\int_0^T\int_0^1 D^m\nu\ \frac{\partial}{\partial t}(D^m\nu)\,dx\ dt$$

$$= \Delta\tau\int_0^T\int_0^1 \frac{\partial\nu}{\partial t} K_m(\omega)\,dxdt + \Delta\tau\,(1-\eta)\int_0^T\int_0^1 D^m\nu\ \frac{\partial}{\partial t}(D^m\nu^{\tau+\Delta\tau})\,dx\ dt.$$

$$\Longrightarrow \int_0^T\int_0^1 \left(\frac{\partial\nu}{\partial t}\right)^2 dxdt + (1+\tau(\eta-1))\int_0^T\int_0^1 \tfrac{1}{2}\frac{\partial}{\partial t}(D^m\nu)^2 dxdt$$

$$= \Delta\tau\int_0^T\int_0^1 \frac{\partial\nu}{\partial t} K_m(\omega)\,dxdt + \Delta\tau\,(1-\eta)\int_0^T\int_0^1 D^m\nu\ \frac{\partial}{\partial t}(D^m\nu^{\tau+\Delta\tau})\,dx\ dt$$

$$\Longrightarrow \int_0^T \left| \frac{\partial \nu(t)}{\partial t} \right|^2 dt + \frac{(1+\tau(\eta-1))}{2} |D^m \nu(t)|^2$$

$$\leq \Delta\tau \sqrt{\int_0^T \left| \frac{\partial \nu(t)}{\partial t} \right|^2 dt} \sqrt{\int_0^T \int_0^1 K_m^2 \, dx dt} + \Delta\tau \sqrt{\int_0^T |D^m \nu(t)|^2} \sqrt{\int_0^T |D^{m+1} \nu^{\tau+\Delta\tau}(t)|^2 dt}$$

$\leq C \, \Delta\tau$ by lemma 3.2.1

Therefore $\displaystyle\int_0^T \left| \frac{\partial \nu(t)}{\partial t} \right|^2 dt + \frac{\eta}{2} |D^m \nu(t)|^2 \leq C \, \Delta\tau.$

Combining we arrive at

$$\int_0^T \left| \frac{\partial \nu(t)}{\partial t} \right|^2 dt + \eta \int_0^T |D^{m+\ell} \nu(t)|^2 dt + |D^\ell \nu(t)|^2 + \eta |D^m \nu(t)|^2$$

$$= |||\nu|||_{V_{m'}} \leq C \, \Delta\tau, \text{ which is the required result :}$$

Step 3 ϕ on M is completely continuous. The preceding steps show ϕ to be continuous at every point in $M \times [0,1]$.

By lemma 3.2.1, $\phi(\omega,\tau)$ for each $\tau \varepsilon [0,1]$ maps a set M in $W_{m'}$ into a set $\{\nu\}$ with

$$|||\nu|||_{V_{m'}} \leq C \, |||\omega|||_{W_{m'}}, \text{ where } C \text{ is independent}$$

of τ. From lemma 3.1.1, a set of functions which is bounded in the norm of $V_{m'}$ is compact in the space $W_{m'}$. Thus ϕ maps an arbitrary set (ω,τ) in $W_{m'} \times [0,1]$ into a set that is compact in $W_{m'}$. This shows that the set of values of $\phi(\omega,\tau)$ on $M \times [0, 1]$ is compact.

Step 4: Fixed points u^τ lie strictly in the interior of the set M. The a priori bounds for (1.1.3) in lemma 2.2.1, and the definition of M show that all fixed points u^τ of ϕ lie strictly in the interior of of the set M.

Step 5: $\omega - \phi(\omega,0)$ has nonzero degree. For $\tau = 0$, ϕ maps M into a single point ν, the unique solution is $\frac{\partial\nu}{\partial t} = (-1)^{m+1}D^{2m}\nu$. Thus, $\omega-\phi(\omega,0)$ is invertible and has nonvanishing degree.

The application of Leray-Schauder theorem gives for each $\tau \in [0,1]$ the existence of at least one fixed point $u^\tau(x,t)$ for ϕ, which for $\tau = 1$ is a solution to (1.1.2).

By a solution of (1.1.2) is meant a function $u^\eta(x,t)$ such that $\int_0^T \left|\frac{\partial u^\eta(t)}{\partial t}\right|^2 dt + \operatorname{ess.sup}_{t\le T} ||u^\eta(t)||^2_{m'}$ is bounded and $u^\eta(x,t)$ satisfies (1.1.2) with initial data g^η. By lemma 2.2.1, the bound depends only on $||g^\eta||_{m'}$. Since g^η converges to g in $H_{m'}$, g^η is bounded independently of η in $H_{m'}$. Thus the preceding bound for the solution is independent of η. We now discuss the convergence of $u^{\eta}(x,t)$ to a $u(x,t)$ of (1.1.1). First consider the case $m' = 2m$. The preceding estimate shows that u^{η} is bounded in $L^{\infty}(|0,T|, H_{2m})$ and u^{η}_t is bounded in $L^2(|0, T|, L^2)$. The bound is independent of η. By lemma 3.1.1, weak compactness of bounded sets in $L^2(|0,T|,L^2)$, and a diagonalization argument, we can select a subsequence u^{η} converging in $L^{\infty}(|0,T|, H_{2m-1})$ to some function u and u^{η}_t converging in $L^2(|0, T|, L^2)$ to ν. In view of (1.1.2), this implies that $K_m(u^{\eta})$ converges in $L^{\infty}(|0,T|, H_{2m-1})$ to $K_m(u)$ and that $\nu = \frac{\partial u}{\partial t}$. It follows that $u(x,t)$ is a solution of (1.1.1) and $u(x,0) = g(x)$ Since $u^{\eta}(x,0) = g^{\eta}(x) \longrightarrow g(x)$ strongly in H_{2m}.

This solution has the additional regularity that $u \in L^{\infty}(|0,T|, H_{m'})$. Let ϕ be infinitely differentiable and of compact support and consider $(u^{\eta}(t), \phi(t))$, where $(,)$ is the paring of $H_{m'}$ and $H_{-m'}$. Let $|| \; ||_{-m'}$ be the norm of $H_{-m'}$. Using the aprior bound of (2.2.1) we find

$$\left|\int_0^T (u^{\eta}(t), \phi(t)) dt\right| \leq \text{ess.sup}\, ||u^{\eta}(t)||_{m'} \int_0^T ||\phi(t)||_{-m'} \, dt$$
$$\leq C \int_0^T |\phi(t)|_{-m'} \, dt$$

The convergence of u^{η} to u in $L^{\infty}(|0,T|,L^2)$ gives us the convergence of $\int_0^T \big(u^{\eta}(t),\ \phi(t)\big)dt$ to $\int_0^T \big(u(t),\ \phi(t)\big)dt$.

The previous estimate gives us a $u \in L^{\infty}(|0,T|,\ H_{m'})$.

Finally we show that this solution exists for all time.

It has been shown [7] that $F_{\ell}(u)=\int_0^1 \Big\{ (D^{\ell-1}u)^2 + (D^{\ell-2}u)^2 u + aD^{\ell-1}u$

$+ bD^{\ell-2}u + C\Big\}dx$, $\ell \geq 1$ up to inessential constants

and a, b, c are polynomials in u and its derivatives up to order ℓ-3, is constant along solutions of (1.1.1).

Use of $F_{\ell}(u)$ and (2.2.1) verify that

$$||u(t)||_n \leq \text{constant},$$

where the constant is independent of t and depends only on $||g||_n$. This completes the proof of existence.

We will combine this result with the uniqueness Lemma 4.1.1 to arrive at a proof of the theorem 1.1.1.

CHAPTER IV

4.1 A Uniqueness Result For (1.1.1).

Lemma 4.1.1: For each m, bounded solutions $u:|0,\infty) \to H_n$ of (1.1.1) for $n \geq 3m-2$ are uniquely determined by their initial data.

Proof: Let $u, \nu:|0, \infty) \to H_n$ be two bounded solutions of (1.1.1) with initial data g and h in H_n for $n \geq 3m-2$. By (1.1.1), the bound $\underset{t\leq T}{\text{ess.sup}}|D^{\ell} u_t(t)| \leq C$ for $\ell \leq m-2$ follows. The proof falls into four parts.

$m = 2$

In this case, uniqueness is proved in the usual way. Let $\omega = u - \nu$. Then (1.1.1) becomes

$$\frac{\partial}{\partial t}(u-\nu) = \frac{\partial}{\partial x}\{G_2(u) - G_2(\nu)\},$$

where $\frac{\partial}{\partial x} G_{m+1}(u) = \left(\frac{\partial^3}{\partial x^3} + \frac{2}{3}u\frac{\partial}{\partial x} + \frac{1}{3}u_x\right)G_m(u)$ with $G_1(u) = u$

Therefore $$\frac{\partial}{\partial x} G_2(u) = \left(\frac{\partial^3}{\partial x^3} + \frac{2}{3}u\frac{\partial}{\partial x} + \frac{1}{3}u_x\right)u$$

$$= u_{xxx} + u_x u.$$

Thus, the above implies

$$\frac{\partial}{\partial t}(u-\nu) = \frac{\partial\omega}{\partial t} = u_{xxx} + u_x u - [\nu_{xxx} + \nu_x\nu]$$

$$= u_{xxx} - \nu_{xxx} + u_x u - \nu_x\nu$$

$$= \omega_{xxx} + uu_x - \nu u_x + \nu u_x - \nu\nu_x$$

$$= \omega_{xxx} + (u - \nu)u_x + \nu(u_x - \nu_x)$$

$$= \omega_{xxx} + \omega u_x + \nu\omega_x$$

Which shows that

$$\frac{\partial\omega}{\partial t} - \omega_{xxx} = \omega u_x + \nu\omega_x .$$

$$\int_0^1 \omega\left(\frac{\partial\omega}{\partial t} - \omega_{xxx}\right) dx = \int_0^1 \omega(\omega u_x + \nu\omega_x)\, dx$$

$$\Longrightarrow \quad \int_0^1 \omega \frac{\partial\omega}{\partial t}\, dx - \int_0^1 \omega\omega_{xxx} dx = \int_0^1 \omega^2 u_x dx + \int_0^1 \nu\omega\omega_x dx \ .$$

$$\Longrightarrow \quad \int_0^1 \tfrac{1}{2} \frac{\partial}{\partial t}(\omega^2)\, dx - \left\{ \omega\omega_{xx}\Big|_0^1 - \left[\tfrac{1}{2}\, \omega_x^2 \right]_0^1 \right\}$$

$$= \int_0^1 \omega^2 u_x dx + \left\{ \tfrac{1}{2}\nu\omega^2 \Big|_0^1 - \tfrac{1}{2} \int_0^1 \nu_x \omega^2 dx \right\}$$

$\Longrightarrow \tfrac{1}{2} \int_0^1 \frac{\partial}{\partial t}\omega^2 dx = \int_0^1 (u_x - \tfrac{1}{2}\nu_x)\,\omega^2 dx$, where we have used that

$u(t),\ \nu(t) \varepsilon H_n$ for each t. With $\operatorname{ess.sup}_{x,t}|u_x|$ and $\operatorname{ess.sup}_{x,t}|\nu_x|$

bounded, then

$$\tfrac{1}{2}\int_0^1 \frac{\partial}{\partial t}\, \omega^2\, dx \le K \int_0^1 \omega^2 dx \ .$$

Integrating in t we have

$$\tfrac{1}{2}\int_0^T\int_0^1 \frac{\partial \omega^2}{\partial t}\,dx\,dt \le K\int_0^T\int_0^1 \omega^2\,dx\,dt$$

$$\Longrightarrow \tfrac{1}{2}\left\{\int_0^1 \omega^2(x,t)\,dx - \int_0^1 \omega^2(0,t)\,dx\right\} \le C_1\int_0^T |\omega(t)|^2\,dt$$

$$\Longrightarrow \tfrac{1}{2}|\omega(t)|^2 \le \tfrac{1}{2}|\omega(0)|^2 + C_1\int_0^T |\omega(t)|^2\,dt$$

$$\Longrightarrow |\omega(t)|^2 \le |g-h|^2 + C_2\int_0^T |\omega(t)|^2\,dt$$

Applying Gronwall's inequality [1.2.10] we obtain

$$|\omega(t)|^2 \le |g-h|^2\, e^{k_1\int^t ds}$$

$$\le |g-h|^2 \exp(K_1 t)\,.$$

Since $g = h$ at $t = 0$, it follows that $\omega(t) = 0$ and hence $u(t) = \nu(t)$ for all time.

$m = 3$

The difference $\omega = u - \nu$ of two solutions of (1.1.1) with $m = 3$ satisfies

$$(4.1.1)\quad \frac{\partial u}{\partial t} - \frac{\partial \nu}{\partial t} = \frac{\partial \omega}{\partial t} = D\left\{G_3(u) - G_3(\nu)\right\}, \text{ where}$$

$$G_3(u) = u^3 - 3u_x^2 + 6D(uu_x) + \frac{18}{5}u_{xxxx}.$$

Multiplying by $\omega(x,t)$ and integrating in x gives

$$\int_0^1 \omega \frac{\partial\omega}{\partial t}\, dx = \int_0^1 \omega D \left\{ G_3(u) - G_3(\nu) \right\} dx$$

$$= \omega \left\{ G_3(u) - G_3(\nu) \right\} \Big|_0^1 - \int_0^1 \omega_x \left\{ G_3(u) - G_3(\nu) \right\} dx$$

$$= - \int_0^1 \omega_x \left\{ G_3(u) - G_3(\nu) \right\} dx$$

Now $\quad G_3(u) - G_3(\nu) = (u^3 - \nu^3) - 3[u_x^2 - \nu_x^2] + 6D[uu_x - \nu\nu_x]$

$$+ \frac{18}{5}(u_{xxxx} - \nu_{xxxx})$$

$$= (u-\nu)\left(u^2 + u\nu + \nu^2\right) - 3\left(u_x - \nu_x\right)\left(u_x + \nu_x\right)$$

$$+ 6D[uu_x - \nu u_x + \nu u_x - \nu\nu_x] + \frac{18}{5}\omega_{xxxx}.$$

$$= \omega(u^2 + \nu u + \nu^2) - 3\omega_x(u_x + \nu_x) + 6D(\omega u_x + \nu\omega_x) + \frac{18}{5}\omega_{xxxx}$$

Substituting we obtain

$$\int_0^1 \omega_t \omega\, dx = - \int_0^1 \omega_x \left\{ \omega(u^2 + u\nu + \nu^2) - 3\omega_x(u_x + \nu_x) \right.$$

$$\left. + 6D\left[\omega u_x + \nu\omega_x\right] + \frac{18}{5}\omega_{xxxx} \right\} dx$$

$= - \int_0^1 \omega_x \left\{ \omega(u^2 + u\nu + \nu^2) - 3\omega_x(u_x + \nu_x) + 6D\left[\omega u_x + \nu\omega_x\right] \right\} dx.\quad$ Since

$\int_0^1 \omega_{xxxx}\, dx = \omega_{xxx}\Big|_0^1 = 0$ by periodicity .

But $\int_0^1 \omega_x D\left[\omega u_x + \nu\omega_x\right] dx = \omega_x(\omega u_x + \nu\omega_x)\Big|_0^1 - \int_0^1 \omega_{xx}(\omega u_x + \nu\omega_x)\, dx.$

$$= - \int_0^1 \omega_{xx} (\omega u_x + \nu \omega_x) \, dx.$$

Therefore $\displaystyle\int_0^1 \omega_t \omega \, dx = - \int_0^1 \Big\{ \omega_x \Big[\omega (u^2 + u\nu + \nu^2) - 3\omega_x (u_x + \nu_x) \Big]$

$$- 6\omega_{xx} (\omega u_x + \nu \omega_x) \Big\} dx$$

Integrating this expression in t, we get

$$\int_0^T \int_0^1 \omega_t \omega \, dx dt = - \int_0^T \int_0^1 \Big\{ \omega_{xx} \Big[\omega (u^2 + \nu u + \nu^2) - 3\omega_x (u_x + \nu_x) \Big]$$

$$- 6\omega_{xx} (\omega u_x + \nu \omega_x) \Big\} d \, dt$$

The above implies

$$\int_0^1 \tfrac{1}{2}\omega^2 \Big|_0^t dx \le K_1 \int_0^T \Big\{ |\omega(t)|_4 |\omega_x(t)|_4 + |\omega_x(t)|_4^2 + |\omega(t)| |\omega_{xx}(t)|$$

$$+ |\omega_x(t)|_4 |\omega_{xx}(t)| \Big\} dt$$

$$\Rightarrow \tfrac{1}{2} |\omega| (t)|^2 \le K_1 \int_0^T \Big\{ |\omega(t)|_4 |\omega_x(t)|_4 + |\omega_x(t)|_4^2 + |\omega(t)| |\omega_{xx}(t)|$$

$$+ |\omega_x(t)|_4 |\omega_{xx}(t)| \Big\} dt \; .$$

Applying (2.1.6), (2.1.8), [1.2.6] and the bounds for the solutions, as follows :

$$|\omega(t)|_4 |\omega_x(t)|_4 \le K_2 \Big\{ |D^2\omega|^{\frac{1}{8}} |\omega|^{\frac{7}{8}} + |\omega| \Big\} 2^{\frac{1}{4}} \Big\{ |D^2\omega|^{\frac{5}{8}} |\omega|^{\frac{3}{8}} \Big\}$$

$$\leq K_2 \left\{ |D^2\omega|^{\frac{3}{4}} |\omega|^{\frac{5}{4}} + |D^2\omega|^{\frac{5}{8}} |\omega|^{\frac{11}{8}} \right\} \text{ after adjusting } K_2$$

$$\leq K_2 \left\{ \varepsilon_1 (|D^2\omega|^{\frac{3}{4}})^{\frac{8}{3}} + C_{\varepsilon_1} (|\omega|^{\frac{5}{4}})^{\frac{8}{5}} + \varepsilon_2 (|D^2\omega|^{\frac{5}{8}})^{\frac{16}{5}} + C_{\varepsilon_2} (|\omega|^{\frac{11}{8}})^{\frac{16}{11}} \right\}$$

$$\leq K_3 \left\{ C_\varepsilon |\omega(t)|^2 + \varepsilon |D^2\omega(t)|^2 \right\}.$$

$$|\omega_x(t)|_4^2 \leq \left[2^{\frac{1}{4}} |D^2\omega(t)|^{\frac{5}{8}} |\omega(t)|^{\frac{3}{8}} \right]^2$$

$$= \sqrt{2}\ |D^2\omega(t)|^{\frac{5}{4}} |\omega(t)|^{\frac{3}{4}}$$

$$\leq \sqrt{2} \left\{ \varepsilon (|D^2\omega(t)^{\frac{5}{4}})^{\frac{8}{5}} + C_\varepsilon \left\{ |\omega(t)|^{\frac{3}{4}} \right\}^{\frac{8}{3}} \right\}$$

$$= \sqrt{2} \left\{ \varepsilon |D^2\omega(t)|^2 + C_\varepsilon |\omega(t)|^2 \right\}.$$

$$|\omega(t)|\,|\omega_{xx}(t)| \leq \varepsilon |\omega_{xx}(t)|^2 + C_\varepsilon |\omega(t)|^2, \text{ since } \tfrac{1}{2} + \tfrac{1}{2} = 1 .$$

Finally $|\omega_x(t)|_4 |\omega_{xx}(t)|_4 \leq 2^{\frac{1}{4}} \left\{ |D^2\omega|^{\frac{3}{8}} |\omega(t)|^{\frac{5}{8}} \right\} |D^2\omega(t)|$

$$= 2^{\frac{1}{4}} \left\{ |D^2\omega(t)|^{\frac{11}{8}} |\omega(t)|^{\frac{5}{8}} \right\}$$

$$\leq 2^{\frac{1}{4}} \left\{ \varepsilon |D^2\omega(t)|^2 + C_\varepsilon |\omega(t)|^2 \right\}$$

Combining the results we obtain

$$(4.1.2) \leq C \int_0^T \left\{ |\omega(t)|^2 + |D^2\omega(t)|^2 \right\} dt.$$

Now multiplying (4.1.1) by $G_3(u) - G_3(v)$ and integrating in x to find that

$$\int_0^1 \frac{\partial\omega}{\partial t}\left[G_3(u)-G_3(\nu)\right]dx = \int_0^1 (G_3(u)-G_3(\nu))D(G_3(u)-G_3(\nu))\,dx$$

$$= \int_0^1 \frac{\partial}{\partial x}\frac{(G_3(u)-G_3(\nu))^2}{2}\,dx$$

$$= \tfrac{1}{2}(G_3(u)-G_3(\nu))^2\Big|_0^1 = 0$$

Therefore $\int_0^1 \omega_t (G_3(u)-G_3(\nu))\,dx = \int_0^1 \omega_t\Big[(u^3-\nu^3)-3(u_x^2-\nu_x^2)$

$$+ 6D(uu_x - \nu\nu_x) + \frac{18}{5}\omega_{xxxx}\Big]dx$$

Integrating in t we obtain

$$\int_0^T\int_0^1 \omega_t (G_3(u)-G_3(\nu))\,dxdt = \int_0^T\int_0^1 \omega_t\Big[(u^3-\nu^3)-3(u_x^2-\nu_x^2)$$

$$+ 6D(uu_x - \nu\nu_x) + \frac{18}{5}\omega_{xxxx}\Big]dx\,dt\ .$$

We transform each of the terms in the above as follows :

(a) $$\left|\int_0^T\int_0^1 \frac{\partial\omega}{\partial t}(u^3-\nu^3)\,dxdt\right| = \left|\int_0^T\int_0^1 \omega_t\,\omega\,(u^2+u\nu+\nu^2)\,dxdt\right|$$

$$= \left|\int_0^T\int_0^1 \tfrac{1}{2}\frac{\partial}{\partial t}\omega^2(u^2+u\nu+\nu^2)\,dx\,dt\right|$$

$$= \left|\int_0^1 \tfrac{1}{2}\omega^2(u^2+u\nu+\nu^2)\,dx - \int_0^T\int_0^1 \tfrac{1}{2}\omega^2 D_t(u^2+u\nu+\nu^2)\,dx\,dt\right|,$$

where $\omega(0) = 0$ since $g(0) = h(0)$. Thus

$$\left|\int_0^T \int_0^1 \frac{\partial \omega}{\partial t}(u^3 - v^3)\,dxdt\right| \le K_1 |\omega(t)|^2 + K_2 \int_0^T |\omega(t)|^2 dt.$$

$$\le K \left\{ |\omega(t)|^2 + \int_0^T |\omega(t)|^2 dt \right\}.$$

(b) $$\left|\int_0^T \int_0^1 \frac{\partial \omega}{\partial t}\left[6D(uu_x - vv_x) - 3(u_x^2 - v_x^2)\right] dx\ dt\right|$$

$$= \left|\int_0^T \int_0^1 \left\{ 6\omega_t D(uu_x - vv_x) - 3\omega_x \omega_t (u_x + v_x)\right\} dx\ dt\right|$$

$$= \left|\int_0^T 6\omega_t (uu_x - vv_x)\Big|_0^1 dt - \int_0^T\int_0^1 6\omega_{tx}(uu_x - vv_x) - \int_0^T\int_0^1 3\omega_t \omega_x (u_x + v_x)\,dxdt\right|$$

$$= \left|\int_0^T\int_0^1 \left\{ 6\omega_{tx}(uu_x - vv_x) + 3\omega_t \omega_x (u_x + v_x)\right\} dx\ dt\right|$$

$$= \left|\int_0^T \int_0^1 \left\{ 6\omega_{xt}(\omega u_x + v\omega_x) + 3\omega_t \omega_x (u_x + v_x)\right\} dx\ dt\right|.$$

But $$\int_0^T\int_0^1 6\omega_{xt}(\omega u_x + v\omega_x)\,dx\ dt = \int_0^1 6\omega_x(\omega u_x + v\omega_x)\,dx$$

$$- \int_0^T \int_0^1 6\omega_x D_t(\omega u_x + v\omega_x)\,dx\ dt .$$

$$= \int_0^1 6\omega_x(\omega u_x + v\omega_x)\,dx - \int_0^T \int_0^1 6\omega_x\ D_t(\omega u_x)\,dx\ dt$$

$$- \left[\int_0^1 6\omega_x \ \nu\omega_x \ d_x - \int_0^T\int_0^1 6\omega_{xt}\omega_x \ \nu \ dx \ dt \right]$$

$$= \int_0^1 6\omega_x(\omega u_x + \nu\omega_x)\,dx - \int_0^T\int_0^1 6\omega_x D_t(\omega u_x)\,dx\ dt$$

$$- \int_0^1 6\omega_x^2 \nu dx + \int_0^T\int_0^1 3\nu \frac{\partial}{\partial t}(\omega_x^2)\,dx\ dt$$

$$= \int_0^1 6\omega_x(\omega u_x+\nu\omega_x)\,dx - \int_0^T\int_0^1 6\omega_x D_t(\omega u_x)\,dxdt - \int_0^1 6\omega_x^2 \nu \ dx$$

$$+ \int_0^1 3\nu\omega_x^2 \ dx - \int_0^T\int_0^1 3\nu_t\omega_x^2 \ dx\ dt$$

$$= \int_0^1 6\omega_x(\omega u_x + \nu\omega_x)\,dx - \int_0^T\int_0^1 6\omega_x(\omega_t u_x + \omega u_{xt})\,dx\ dt$$

$$- \int_0^1 3\nu\omega_x^2 \ dx - \int_0^T\int_0^1 3\nu_t\omega_x^2 \ dx\ dt$$

Therefore $\left|\int_0^T\int_0^1 \left\{6\omega_{xt}(\omega u_x+\nu\omega_x)+3\omega_t\omega_x(u_x+\nu_x)\right\}dx\ dt\right|$ becomes

$$\left|\int_0^1 6\omega_x(\omega u_x+\nu\omega_x)\,dx - \int_0^1 3\nu\omega_x^2 - \int_0^T\int_0^1\left\{6\omega_x(\omega_t u_x+\omega u_{xt})+3\nu_t\omega_x^2\right\}dxdt\right.$$

$$\left. + \int_0^T\int_0^1 3\omega_t\omega_x(u_x+\nu_x)\,dx\ dt \right|$$

$$\left|\int_0^T\int_0^1 \left\{6\omega_x(\omega_t u_x+\omega u_{xt}) + 3\nu_t\omega_x^2 - 3\omega_t\omega_x(u_x+\nu_x)\right\}dx\ dt\right.$$

$$- 3 \int_0^1 (\nu\omega_x^2 + 2\omega_x \omega u_x)\, dx |$$

Using the identity $6\omega_t\omega_x u_x - 3\omega_t\omega_x(u_x + \nu_t) = 3(u_t - \nu_t)\omega_x^2$,

that is, $-3\omega_t\omega_x(u_x + \nu_x) = -6\omega_t\omega_x u + 3(u_t - \nu_t)\omega_x^2$.

Therefore

$$|\int_0^T\int_0^1 \left\{6\omega_x(\omega_t u_x + \omega u_{xt}) + 3\nu_t\omega_x^2 - 6\omega_t\omega_x u + 3(u_t - \nu_t)\omega_x^2\right\} dx\, dt$$

$$- 3\int_0^1 (\nu\omega_x^2 + 2\omega_x\omega u_x)\, dx |$$

$$|\int_0^T\int_0^1 \left\{6\omega\omega_x u_{xt} + 3\nu_t\omega_x^2 + 3(u_t - \nu_t)\omega_x^2\right\} dx dt - 3\int_0^1 (\nu\omega_x^2 + 2\omega\omega_x u_x)\, dx |$$

$$= |\int_0^T 6\omega\omega_x u_t |_0^1 dt - \int_0^T\int_0^1 6u_t D(\omega\omega_x)\, dx dt + \int_0^T\int_0^1 \left\{3\nu_t\omega_x^2 + 3(u_x - \nu_t)\omega_x^2\right\} dx dt$$

$$- 3\int_0^1 (\nu\omega_x^2 + 2\omega\omega_x u_x)\, dx |$$

$$= |\int_0^T\int_0^1 \left\{-6u_t D(\omega\omega_x) + 3\nu_t\omega_x^2 + 3(u_t - \nu_t)\omega_x^2 dx dt - \int_0^1 (\nu\omega_x^2 + \omega\omega_x u_x)\, dx \right|$$

$$= |\int_0^T\int_0^1 \left\{-3u_t D^2(\omega^2) + 3u_t\omega_x^2\right\} dx dt - 3\int_0^1 \nu\omega_x^2 dx - 3\int_0^1 (\omega\omega_x) u_x d_x |$$

$$= |\int_0^T\int_0^1 \left\{-3u_t D^2(\omega^2) + 3u_t\omega_x^2\right\} dx dt - 3\int_0^1 \nu\omega_x^2 dx - 3\left[u\omega\omega_x |_0^1 - \int_0^1 uD(\omega\omega_x)\, dx\right\}$$

$$= \left|\int_0^T\int_0^1 \left\{-3u_t D^2(\omega^2)+3u_t\omega_x^2\right\}dxdt-3\int_0^1 \nu\omega_x^2 dx-3\left[u\omega\omega_x\Big|_0^1-\int_0^1 uD(\omega\omega_x)\,dx\right]\right.$$

$$= \left|\int_0^T\int_0^1 \left\{-3u_t D^2(\omega^2)+3u_t\omega_x^2\right\}dxdt- 3\int_0^1 \nu\omega_x^2 dx + \frac{3}{2}\int_0^1 uD^2(\omega^2)\,dx\right|$$

$$= \left|\int_0^T\int_0^1 \left\{-3u_t D^2(\omega^2)+3u_t\omega_x^2\right\}dxdt-3\int_0^1 \nu\omega_x^2 dx + (-1)^2\frac{3}{2}\int_0^1 u_{xx}\omega^2 dx\right|$$

$$\leq K_1\int_0^T \left\{|\omega_{xx}(t)|^2+|\omega_x(t)|^2\right\}dt + K_2\left\{|\omega(t)|^2+|\omega_x(t)|^2\right\}$$

$$\leq K_3\left\{|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T(|\omega_x(t)|^2+|\omega_{xx}(t)|^2)\,dt\right\}.$$

Thus we obtain an estimate of the form

$$|D^2\omega(t)|^2\leq K\left\{|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T(|\omega_x(t)|^2+|\omega_{xx}(t)|^2)\,dt\right\}.$$

Applying (2.1.6) and (1.2.6) to $|\omega_x(t)|^2$, then

$$|\omega_x(t)|_4^2 \leq (2^{\frac{1}{4}})^2\left\{|D^2\omega|^{\frac{5}{8}}|\omega(t)|^{\frac{3}{8}}\right\}^2$$

$$\leq \sqrt{2}\left\{\varepsilon(|D^2\omega|^{\frac{5}{4}})^{\frac{8}{5}} + C_\varepsilon(|\omega(t)|^{\frac{3}{4}})^{\frac{8}{3}}\right\}$$

$$= \sqrt{2}\left\{\varepsilon|D^2\omega(t)|^2 + C_\varepsilon|\omega(t)|^2\right\}.$$

Therefore $|D^2\omega(t)|^2\leq K\left\{C_\varepsilon|\omega(t)|^2+\varepsilon|\omega_{xx}(t)|^2+\int_0^T(C_\varepsilon|\omega(t)|^2+\right.$

$$\left. + |\omega_{xx}(t)|^2)\,dt\right\}$$

$$\leq C\int_0^T (|\omega(t)|^2+|\omega_{xx}(t)|^2)\,dt + \varepsilon|\omega_{xx}(t)|^2$$

Choosing $\varepsilon = \frac{1}{2}$ we have

$$|D^2\omega(t)|^2-\tfrac{1}{2}|D^2\omega(t)|^2 \leq K\int_0^T (|\omega(t)|^2+|D^2\omega(t)|^2)\,dt$$

which implies that $|D^2\omega(t)|^2 \leq C_1\int_0^T (|\omega(t)|^2+|D^2\omega(t)|^2)\,dt$

Combining this with (4.1.2), we find

$$|\omega(t)|^2+|D^2\omega(t)|^2 \leq C\int_0^T \left(|\omega(t)|^2+ |D^2\omega(t)|^2\right)dt.$$

Applying Gronwall's lemma [1.2.10], we obtain

$$|\omega(t)|^2+|D^2\omega(t)|^2 \leq 0 \times e^{k\int_0^t ds} = 0 .$$

Therefore, the result follows :

$m = 4$

With $\omega = u - v$, (1.1.1) gives

(4.1.3) $\dfrac{\partial\omega}{\partial t} = D\left(G_4(u) - G_4(v)\right)$, where

$$G_4(u) = u^4-12uu_x^2+12D(u^2u_x) + \frac{35}{5}u_{xx}^2+\frac{35x2}{5}D^2\left(uu_{xx}\right)+\frac{108\ x\ 2}{33}D^6u.$$

Multiplying (4.1.3) by $\omega(x,t)$ we have

$$\frac{\omega\,\partial\omega}{\partial t} = \omega D(G_4(u) - G_4(v))$$

Integrating in x and t we find

Integrating in x and t we find

$$\int_0^T \int_0^1 \omega \frac{\partial\omega}{\partial t}\, dx\, dt = \int_0^T\int_0^1 \omega D\left(G_4(u) - G_4(\nu)\right) dx\, dt\ ,$$

$$= \int_0^T \omega\left(G_4(u) - G_4(\nu)\right)\Big|_0^1 dt - \int_0^T\int_0^1 \omega_x\left(G_4(u) - G_4(\nu)\right) dx\, dt$$

$$= - \int_0^T\int_0^1 \omega_x\left(G_4(u) - G_4(\nu)\right) dx\, dt.$$

Now $\quad G_4(u) - G_4(\nu) = (u^4-\nu^4) - 12\,(uu_x^2 - \nu\nu_x^2) + 12D\left[\,u^2u_x - \nu^2\nu_x\right]$

$$+ \frac{35}{5}\left(u_{xx}^2 - \nu_{xx}^2\right) + \frac{35 \times 2}{5} D^2\left[\,uu_{xx} - \nu\nu_{xx}\right]$$

$$+ \frac{108 \times 2}{5} D^6 \omega\ .$$

We develop bounds for each of the terms in the above as follows :

$$\left|\int_0^T\int_0^1 \omega_x (u^4-\nu^4)\,dxdt\right| = \left|\int_0^T\int_0^1 \omega_x (u-\nu)\,(u^3+u^2\nu+u\nu^2+\nu^3)\, dx\ dt\right|$$

$$= \left|\int_0^T\int_0^1 \omega_x\omega\,(u^3+u^2\nu+u\nu^2+\nu^3)\, dx\ dt\right|$$

$$\leq C\int_0^T |\omega_x(t)|\,|\omega(t)|\,dt$$

Applying (2.1.6) and [2.1.8] to $|\omega_x(t)|\,|\omega(t)|$ we find

$$\left|\int_0^{T}\int_0^{1}\omega_x(u^4-v^4)dx\ dt\right| \le K\int_0^T\left\{\varepsilon|\omega_{xxx}(t)|^2+C_\varepsilon|\omega(t)|^2\right\}dt\ .$$

$$12\left|\int_0^T\int_0^1\omega_x\left\{-(uu_x^2-vv_x^2)\ +\ D(u^2u_x-v^2v_x)\right\}dx\ dt\right|$$

$$=\ 12\left|\int_0^T\int_0^1\omega_x\left[\ D(u^2u_x\ -\ v^2v_x)\ -\ uu_x^2\ +\ vv_x^2\right]dx\ dt\right|$$

$$=\ \left|\int_0^T\int_0^1\omega_x\left[D(u^2u_x-v^2u_x+v^2u_x-v^2v_x)-uu_x^2+vu_x^2-vu_x^2+vv_x^2\ \right]dx\ dt\right|$$

$$=\ 12\left|\int_0^T\int_0^1\omega_x\ \left\{D\left[\omega(u+v)u_x+v^2\omega_x\right]-\ \omega u_x^2-\ v\omega_x(u_x\ +\ v_x)\right\}dx\ dt\ \right|$$

$$=\ 12\left|\int_0^T\omega_x\left[\omega(u+v)u_x+v^2\omega_x\right]_0^1dt\ +\ \int_0^T\int_0^1-\omega_{xx}\left[\omega(u+v)u_x+v^2\omega_x\right]\right.$$

$$\left.-\ \omega_x\omega u_x^2\ -\ v\omega_x^2(u_x\ +\ v_x)\right\}dx\ dt\Big|$$

$$=\ 12\left|\int_0^T\int_0^1\left\{\omega_{xx}\left[\omega(u+v)u_x+v^2\omega_x\right]+\omega_x\omega u_x^2\ +\ v\omega_x(u_x+v_x)\right\}dx\ dt\right|$$

$$\le\ C\int_0^T\left\{|\omega(t)|_4|\omega_{xx}(t)|_4+|\omega_x(t)||\omega_{xx}(t)|_4+|\omega(t)|_4|\omega_x(t)|+|\omega_x(t)|_4^2\right\}dt$$

Applying (2.1.6) and [1.2.6] we find

$$\left|\int_0^T\int_0^1\omega_x\ \left\{D(u^2u_x-uu_x^2\ -\ D(v^2v_x)\ +\ vv_x^2\ \right\}\ dx\ dt\ \right|$$

$$\leq K\int_0^T \left\{\varepsilon|\omega_{xxx}(t)|^2 + C_\varepsilon|\omega(t)|^2\right\}dt$$

Finally, we have

$$\frac{35}{5}\left|\int_0^T\int_0^1 \omega_x \left\{(u_{xx}^2-v_{xx}^2)+2D^2(uu_{xx}-vv_{xx})\right\}dx\ dt\right|$$

$$= \frac{35}{5}\left|\int_0^T\int_0^1 \omega_x\left\{(u_{xx}-v_{xx})(u_{xx}+v_{xx})+2D^2\left[(u-v)u_{xx}+v(u_{xx}-v_{xx})\right]\right\}dxdt\right|$$

$$= \frac{35}{5}\left|\int_0^T\int_0^1 \omega_x\left\{\omega_{xx}(u_{xx}+v_{xx})+2D^2\left[\omega u_{xx}+v\omega_{xx}\right]\right\}dx\ dt\right|$$

Now $\displaystyle\int_0^T\int_0^1 2\omega_x D^2(\omega u_{xx})\,dxdt = \int_0^T \omega D^2(\omega u_{xx})\Big|_0^1 dt - \int_0^T\int_0^1 2\omega D^3(\omega u_{xx})\,dx\ dt$

$$= -2\int_0^T\int_0^1 \omega D^3(\omega u_{xx})\,dx\ dt$$

Also $\displaystyle 2\int_0^T\int_0^1 \omega_x D^2(v\omega_{xx})\,dxdt = 2\int_0^T \omega_x D(v\omega_{xx})\Big|_0^1 dt - 2\int_0^T\int_0^1 \omega_{xx} D(v\omega_{xx})\,dxdt$

$$= 2\int_0^T \omega_x D(v\omega_{xx})\Big|_0^1 dt - 2\left[\int_0^T \omega_{xx}(v\omega_{xx})\Big|_0^1 dt - \int_0^T\int_0^1 \omega_{xxx}(v\omega_{xx})\,dx\ dt\right]$$

$$= 2\int_0^T\int_0^1 \omega_{xxx}\ v\omega_{xx}\,dx\ dt.$$

Substituting we get

$$\frac{35}{5}\left|\int_0^T\int_0^1 \omega_x\left\{(u_{xx}^2-v_{xx}^2) + 2D^2(uu_{xx}-vv_{xx})\right\}dx\ dt\right|$$

$$= \frac{35}{5}\left|\int_0^T\!\!\int_0^1 \left\{\omega_x\omega_{xx}(u_{xx}+\nu_{xx}) - 2\omega D^3(\omega u_{xx}) + 2\omega_{xxx}\nu\omega_{xx}\right\}dx\,dt\right|$$

$$\leq C_1\int_0^T \left\{|\omega_x(t)|_4|\omega_{xx}(t)|_4 + |\omega(t)|_4|\omega_{xxx}(t)| + |\omega_{xxx}(t)||\omega_{xx}(t)|_4\right\}dt$$

Applying (2.1.6) and [1.2.6] to find

$$\frac{35}{5}\left|\int_0^T\!\!\int_0^1 \omega_x\left\{(u_{xx}^2-\nu_{xx}^2) + 2D^2(uu_{xx} - \nu\nu_{xx})\right\}dx\,dt\right|$$

$$\leq \int_0^T \left\{C_\varepsilon|\omega(t)|^2 + \varepsilon|\omega_{xxx}(t)|^2\right\}dt.$$ Thus we obtain

(4.1.4) $\quad |\omega(t)|^2 \leq K\int_0^T\left\{|\omega(t)|^2 + |D^3\omega(t)|^2\right\}dt$ on combining.

Multiplying (4.1.3) by $G_4(u) - G_4(\nu)$ and integrating in x and t we find

$$\int_0^T\!\!\int_0^1 \frac{\partial\omega}{\partial t}\left(G_4(u) - G_4(\nu)\right)dx\,dt = \int_0^T\!\!\int_0^1 \left(G_4(u) - G_4(\nu)\right)D\left(G_4(u) - G_4(\nu)\right)dx\,dt$$

$$= \int_0^T\!\!\int_0^1 \tfrac{1}{2}\frac{\partial}{\partial x}\left(G_4(u) - G_4(\nu)\right)^2 dx\,dt$$

$$= \int_0^T \tfrac{1}{2}\left(G_4(u) - G_4(\nu)\right)^2\Big|_0^1 dt$$

$= 0$ by periodicity.

That is, $\int_0^T\int_0^1 \frac{\partial\omega}{\partial t}\left(G_4(u) - G_4(v)\right)dx\,dt = 0$, where

$$G_4(u)-G_4(v) = u^4-v^4-12(uu_x^2-vv_x^2)+12\,D(u^2u_x-v^2v_x)$$

$$+\frac{35}{5}(u_{xx}^2 - v_{xx}^2) + \frac{35 \times 2}{5}D^2\left(uu_{xx}-vv_{xx}\right)+\frac{108 \times 2}{3^5}D^6\omega.$$

We transform each term as follows :

(a) $$\left|\int_0^T\int_0^1 \omega_t(u^4-v^4)\,dxdt\right| = \left|\int_0^T\int_0^1 \omega_t\,\omega(u^3+u^2v+uv^2+v^3)\,dx\,dt.\right.$$

$$= \left|\int_0^T\int_0^1 \tfrac{1}{2}\frac{\partial}{\partial t}\omega^2(u^3 + u^2v + uv^2 + v^3)\,dx\,dt\right.$$

$$= \left|\int_0^1 \tfrac{1}{2}\omega^2(u^3+u^2v+uv^2+v^3)\,dx-\int_0^T\int_0^1 \tfrac{1}{2}\omega^2 D_t(u^3+u^2v+uv^2+v^3)\,dx\,dt\right|$$

$$\leq K_1|\omega(t)|_4^2 + K_2\int_0^T|\omega(t)|_4^2\,dt.$$

$$\leq K\left\{|\omega(t)|_4^2 + \int_0^T|\omega(t)|_4^2\,dt\right\}$$

Applying (2.1.8) and [1.2.6] to $|\omega(t)|_4^2$ we get

$$|\omega(t)|_4^2 \leq K_3\left\{|D^3\omega(t)|^{\frac{(\frac{1}{2}-\frac{1}{4})}{3}}|\omega(t)|^{\frac{(3-\frac{1}{2}+\frac{1}{4})}{3}} + |\omega(t)|\right\}^2$$

$$\leq K_3\left\{|D^3\omega(t)|^{\frac{1}{12}}|\omega(t)|^{\frac{11}{12}} + |\omega(t)|\right\}^2$$

$$= K_3 \left\{ |D^3\omega(t)|^{\frac{1}{6}} |\omega(t)|^{\frac{11}{6}} + 2|D^3\omega(t)|^{\frac{1}{12}} |\omega(t)|^{\frac{23}{12}} + |\omega(t)|^2 \right\}$$

Applying [1.2.6] we get

$$|D^3\omega(t)|^{\frac{1}{6}} |\omega(t)|^{\frac{11}{6}} \leq \varepsilon\left(|D^3\omega(t)^{\frac{1}{6}}\right)^{\frac{12}{1}} + C_\varepsilon\left(|\omega(t)|^{\frac{11}{6}}\right)^{\frac{12}{11}}$$

$$= \varepsilon|D^3\omega(t)|^2 + C_\varepsilon|\omega(t)|^2$$

$$|D^3\omega(t)|^{\frac{1}{12}} |\omega(t)|^{\frac{23}{12}} \leq \varepsilon\left(|D^3\omega(t)|^{\frac{1}{12}}\right)^{\frac{24}{1}} + C_\varepsilon\left(|\omega(t)|^{\frac{23}{12}}\right)^{\frac{24}{23}}$$

$$= \varepsilon|D^3\omega(t)|^2 + C_\varepsilon|\omega(t)|^2$$

$$|D^3\omega(t)|^{\frac{1}{12}} |\omega(t)|^{\frac{23}{12}} \leq \varepsilon\left(|D^3\omega(t)|^{\frac{1}{12}}\right)^{\frac{24}{1}} + C_3\left(|\omega(t)|^{\frac{23}{12}}\right)^{\frac{24}{23}}$$

$$= \varepsilon|D^3\omega(t)|^2 + C_\varepsilon|\omega(t)|^2$$

Combining the above we find

$\leq K_3 \left\{\varepsilon|D^3\omega(t)|^2 + C_\varepsilon|\omega(t)|^2\right\}$ after adjusting ε and C_ε.

Thus (a) becomes

$$\left|\int_0^{T}\int_0^{1}\omega_t(u^4-v^4)\,dxdt\right| \leq C_o\left\{\varepsilon|D^3\omega(t)|^2 + |\omega(t)|^2 + \int_0^T\left(\varepsilon|D^3\omega(t)|^2 + |\omega(t)|^2\right)dt\right\}$$

(b) $$\left|\int_0^{T}\int_0^{1}\omega_t\left\{D(u^2u_x - v^2v_x) - (uu_x^2 - vv_x^2)\right\}dxdt\right| = \left|\int_0^T\left(\omega_t(u^2u_x - v^2v_x)\right)\Big|_0^1 dt\right.$$

$$\left. + \int_0^{T}\int_0^{1}\left\{-\omega_{xt}(u^2u_x - v^2v_x) - \omega_t(uu_x - vv_x)\right\}dx\,dt\right|$$

$$= \left| \int_0^T\!\!\int_0^1 \left\{ \omega_t (uu_x^2 - \nu\nu_x^2) + \omega_{xt} (u^2 u_x - \nu^2 \nu_x) \right\} dx\ dt \right|$$

$$= \left| \int_0^T\!\!\int_0^1 \left\{ \omega_t (uu_x^2 - \nu u_x^2 + \nu u_x^2 - \nu\nu_x^2) + \omega_{xt} (u^2 u_x - \nu^2 u_x + \nu^2 u_x - \nu^2 \nu_x) \right\} dxdt \right|$$

$$= \left| \int_0^T\!\!\int_0^1 \left\{ \omega_t \omega u_x^2 + \nu (u_x + \nu_x) \omega_t \omega_x + (u+\nu) u_x \omega \omega_{xt} + \omega_{xt} \omega_x \nu^2 \right\} dx\ dt \right|$$

$$= \left| \int_0^T\!\!\int_0^1 \left\{ \tfrac{1}{2} u_x^2 D_t \omega^2 + \tfrac{1}{2} \nu^2 D_t \omega_x^2 + \omega_{xt} \omega (u+\nu) u_x + \omega_t \omega_x \nu (u_x + \nu_x) \right\} dx\ dt \right|$$

$$= \left| \int^1 \tfrac{1}{2} u_x \omega^2 + \tfrac{1}{2} \nu^2 \omega_x^2 \Big|_0^t dx - \int_0^T\!\!\int_0^1 u_x u_{xt} \omega^2\, dx\, dt - \int_0^T\!\!\int_0^1 \tfrac{1}{2} D_t \nu^2 \omega_x^2\ dx\ dt \right.$$
$$\left. + \int_0^T\!\!\int_0^1 \left\{ \omega_{xt} \omega (u+\nu) u_x + \omega_t \omega_x \nu (u_x + \nu_x) \right\} dx\ dt \right|$$

$$= \left| \int_0^1 \tfrac{1}{2} (u_x^2 \omega^2 + \nu^2 \omega_x^2)\, dx + \int_0^T\!\!\int_0^1 \left\{ \omega_{xt} \omega (u+\nu) u_x + \omega_t \omega_x \nu (u_x + \nu_x) \right. \right.$$

$$\left. \left. - \omega^2 u_x u_{xt} - \tfrac{1}{2} \omega_x^2 D_t \nu^2 \right\} dx\ dt \right|$$

$$\leq C \left(|\omega(t)|^2 + |\omega_x(t)|^2 \right) + \left| \int_0^T\!\!\int_0^1 \left\{ \omega_{xt} \omega (u+\nu) u_x + \omega_t \omega_x \nu (u_x + \nu_x) \right. \right.$$

$$\left. - \omega^2 u_x u_{tx} - \tfrac{1}{2} \omega_x^2 D_t \nu^2 \right\} dx\ dt$$

But $\displaystyle \int_0^T\!\!\int_0^1 \omega_{xt} \omega (u+\nu) u_x\, dxdt = \int_0^T \omega\omega_t (u+\nu) u_x \Big|_0^1 dt - \int_0^T\!\!\int_0^1 \omega_t D (\omega (u+\nu) u_x)\, dxdt$

$$= - \int_0^T\!\!\int_0^1 \omega_t D \left[\omega (u+\nu) u_x \right] dx\ dt$$

$$= - \int_0^T\!\!\int_0^1 \left\{ \omega_t \omega_x (u+\nu) u_x + \omega_t \omega D (u+\nu) u_x + \omega_t \omega (u+\nu) u_{xx} \right\} dx\ dt.$$

Also $\int_0^T\int_0^1 \omega^2 u_x u_{xt}\,dx\,dt = \int_0^T \omega^2 u_x u_t \Big|_0^1 dt - \int_0^T\int_0^1 u_t D(\omega^2 u_x)\,dx\,dt$

$$= -\int_0^T\int_0^1 u_t D(\omega^2 u_x)\,dx\,dt \quad .$$

Substituting, we find

$$\left|\int_0^T\int_0^1 \left[\omega_t(uu_x^2 - \nu\nu_x^2) + \omega_{xt}(u^2 u_x - \nu^2\nu_x)\right]dx\,dt\right|$$

$$\leq C\left(|\omega(t)|^2+|\omega_x(t)|^2\right)+\left|\int_0^T\int_0^1 -\left\{\omega_t\omega_x(u+\nu)u_x+\omega_t\omega D(u+\nu)u_x\right.\right.$$

$$\left.\left. + \omega_t\omega(u+\nu)u_{xx}-\omega_t\omega_x\nu(u_x+\nu_x)-u_tD(\omega^2u_x)+\tfrac{1}{2}\omega_x^2D_t\nu^2\right\}dx\,dt\right|$$

$$= C\left(|\omega(t)|^2+|\omega_x(t)|^2\right) + \left|\int_0^T\int_0^1\left\{\omega_t\omega_x\left[\nu(u_x+\nu_x)-(u+\nu)u_x\right]\right.\right.$$

$$\left.\left. -\tfrac{1}{2}\left(\frac{\partial\omega^2}{\partial t}\right)\left[(u_x+\nu_x)u_x+(u+\nu)u_{xx}\right]+u_tD(\omega^2u_x)-\tfrac{1}{2}\omega^2{}_xD_t\nu^2\right\}dx\,dt\right|$$

$$\leq C\left(|\omega(t)|^2+|\omega_x(t)|^2\right) + \left|\int_0^T\int_0^1 -\tfrac{1}{2}\,\frac{\partial}{\partial t}\omega^2\left[(u_x+\nu_x)u_x+(u+\nu)u_{xx}\right]\right|$$

$$+\left|\int_0^T\int_0^1\left\{u_tD(\omega^2u_x)-\tfrac{1}{2}\omega_x^2D_t\nu^2+\omega_t\omega_x\left[\nu(u_x+\nu_x)-(u+\nu)u_x\right]\right\}dx\,dt\right|$$

$$= C\left(|\omega(t)|^2+|\omega_x(t)|^2\right)+\left|-\tfrac{1}{2}\left\{\int_0^1\omega^2\left[(u_x+\nu)u_x+(u+\nu)u_{xx}\right]\Big|_0^t dx\right.\right.$$

$$\left.\left.-\int_0^T\int_0^1\omega^2D_t\left[(u_x+\nu_x)u_x+(u+\nu)u_{xx}\right]\right\}dx\,dt+\left|\int_0^T\int_0^1\left\{u_tD(\omega^2u_x)-\tfrac{1}{2}\omega_x^2D_t\nu^2\right.\right.\right.$$

$$+ \omega_t\omega_x\Big[\nu(u_x+\nu_x) - (u+\nu)u_x\Big]dx\,dt\Big|$$

$$\leq C\big(|\omega(t)|^2+|\omega_x(t)|^2\big)+C_1\Big(|\omega(t)|^2+\int_0^T|\omega(t)|^2dt\Big)$$

$$+\Big|\int_0^{T_1}\!\!\int_0\Big\{u_tD(\omega^2u_x)-\tfrac{1}{2}\omega_x^2D_t\nu^2+\omega_t\omega_x\Big[\nu(u_x+\nu_x)-(u+\nu)u_x\Big]\Big\}dx\,dt\Big|$$

$$\leq C_2\Big(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\Big)+\int_0^{T_1}\!\!\int_0|u_tD(\omega^2u_x)-\tfrac{1}{2}\omega_x^2D_t\nu^2$$

$$-\omega_t\omega_x(\nu\omega_x+\omega u_x)|dx\,dt$$

$$= C_2\Big(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\Big)+\int_0^{T_1}\!\!\int_0|2u_t\omega\omega_xu_x+u_t\omega^2u_{xx}-\tfrac{1}{2}\cdot 2\omega_x^2$$

$$\nu\nu_t-\omega_t\omega_x\nu\omega_x-\omega_t\omega_x\omega u_x|dx\,dt$$

$$= C_2\Big(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\Big)$$

$$+\int_0^{T_1}\!\!\int_0|\omega\omega_xu_x(2u_t-\omega_t)+u_t\omega^2u_{xx}-\omega_x^2\nu(\nu_t+\omega_t)|dxdt$$

$$= C_2\Big(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\Big)+\int_0^{T_1}\!\!\int_0|\omega\omega_xu_x(u_t+\nu_t)+u_t\omega^2u_{xx}$$

$$-\omega_x^2\,\nu u_t|dx\,dt.$$

$$= C_2\Big(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\Big)+\int_0^{T_1}\!\!\int_0\Big|\frac{\partial}{\partial u}\big(\tfrac{1}{2}\omega^2\big)u_x(u_t+\nu_t)$$

$$+u_t\omega^2u_{xx}-\omega_x^2\nu u_t|dx\,dt$$

$$= C_2\left(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\right)+\int_0^T\left|\tfrac{1}{2}\omega^2u_x(u_t+\nu_t)\right|_0^1dt$$

$$+\int_0^T\int_0^1 -\tfrac{1}{2}\omega^2D_x\left[u_x(u_t+\nu_t)\right]+u_t\omega^2u_{xx}-\omega_x^2\nu u_t|dx\,dt$$

$$\leq C_2\left(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\right)+\int_0^T\int_0^1\Big|-\tfrac{1}{2}\omega^2D_x\left[u_x(u_t+\nu_t)\right]$$

$$+u_t\omega^2u_{xx}-\omega_x^2\nu u_t\Big|dx\,dt$$

$$\leq C_2\left(|\omega(t)|^2+|\omega_x(t)|^2+\int_0^T|\omega(t)|^2dt\right)+C_3\int_0^T\left(|\omega(t)|^2+|\omega(t)|^2+|\omega_x(t)|^2\right)dt$$

$$\leq C_4\left\{|\omega(t)|^2+|\omega_x(t)|^2+\int_Q^T\left(|\omega(t)|^2+|\omega_x(t)|^2\right)dx\right\}$$

Now $\quad |\omega_x(t)|^2 \leq \sqrt{2}\left\{|D^3\omega(t)|^{\frac{5}{12}}|\omega(t)|^{\frac{7}{12}}\right\}^2$

$$=\sqrt{2}\left\{|D^3\omega(t)|^{\frac{5}{6}}\;|\omega(t)|^{\frac{7}{6}}\right\}$$

$$\leq\sqrt{2}\left\{\varepsilon\left(|D^3\omega(t)|^{\frac{5}{6}}\right)^{\frac{12}{5}}+C_\varepsilon\left(|\omega(t)|^{\frac{7}{6}}\right)^{\frac{12}{7}}\right\}$$

$$=\sqrt{2}\left\{\varepsilon|D^3\omega(t)|^2+C_\varepsilon|\omega(t)|^2\right\}.$$

Substituting we get

$$\left|\int_0^T\int_0^1\left[\omega_t(uu_x^2-\nu\nu_x^2)+\omega_{xt}(u^2u_x-\nu^2\nu_x)\right]dx\,dt\right|$$

$$\leq C\left(|\omega(t)|^2+\varepsilon|D^3\omega(t)|^2\right)+C\int_0^T\left\{|\omega(t)|^2+|D^3\omega(t)|^2\right\}dt.$$

(c) $$\left|\int_0^T\!\!\int_0^1 \omega_t \left\{\left(u^2_{xx}-\nu^2{}_{xx}\right) + 2D^2\left(uu_{xx}-\nu\nu_{xx}\right)\right\}dx\,dt\right|$$

$$= \left|\int_0^T\!\!\int_0^1 \omega_t \left\{\omega_{xx}\left(u_{xx}+\nu_{xx}\right) + 2D^2\left[\omega u_{xx} + \nu\omega_{xx}\right]\right\} dx\,dt\right|$$

$$= \left|\int_0^T\!\!\int_0^1 \left\{2\omega_t D^2\left(\nu\omega_{xx}\right) + \omega_t\omega_{xx}\left(u_{xx}+\nu_{xx}\right) + 2\omega_t D^2\left(\omega u_{xx}\right)\right\}dx\,dt\right|$$

$$= \left|\int_0^T 2\omega_t D\left(\nu\omega_{xx}\right)\Big|_0^1 dt - \int_0^T 2\omega_{xt}\left(\nu\omega_{xx}\right)\Big|_0^1 dt + \int_0^T\!\!\int_0^1 2\omega_{xxt}\left(\nu\omega_{xx}\right)dx\,dt\right.$$

$$+ \int_0^T\!\!\int_0^1 \omega_t\omega_{xx}\left(u_{xx}+\nu_{xx}\right)dx\,dt + \int_0^T 2\omega_t D\left(\omega u_{xx}\right)\Big|_0^1 dt - \int_0^T 2\omega_{xt}\left(\omega u_{xx}\right)\Big|_0^1 dt$$

$$\left. + \int_0^T\!\!\int_0^1 2\omega_{xxt}\left(\omega u_{xx}\right)dx\,dt.\right.$$

$$= \left|\int_0^T\!\!\int_0^1 \left\{2\omega_{xxt}\nu\omega_{xx} + \omega_t\omega_{xx}\left(u_{xx}+\nu_{xx}\right) + 2\omega_{xxt}\omega u_{xx}\right\}dx\,dt\right|$$

$$= \left|\int_0^T\!\!\int_0^1 \left\{\nu D_t\left(\omega_{xx}\right)^2 + \omega_t\omega_{xx}\left(u_{xx}+\nu_{xx}\right) + 2\omega_{txx}\omega u_{xx}\right\}dx\,dt\right|$$

$$= \left|\int_0^1 \nu\omega^2_{xx}\Big|_0^t dx - \int_0^T\!\!\int_0^1 \nu_t\omega^2_{xx}dx\,dt + \int_0^T\!\!\int_0^1 \omega_t\omega_{xx}\left(u_{xx}+\nu_{xx}\right)dx\,dt\right.$$

$$\left. + \int_0^1 2\omega_{xx}\,\omega u_{xx}\Big|_0^t dx - \int_0^T\!\!\int_0^1 2\omega_{xx}\left(\omega_t u_{xx} + \omega u_{xxt}\right)dx\,dt\right|$$

$$= \left|\int_0^1 \left\{\nu\omega^2_{xx} + 2\omega_{xx}\omega u_{xx}\right\}dx + \int_0^T\!\!\int_0^1 \left\{-\nu_t\omega^2_{xx} + \omega_t\omega_{xx}\left(u_{xx}+\nu_{xx}\right)\right.\right.$$

$$- 2\omega_{xx}(\omega_t u_{xx} + \omega u_{xxt})\Big\}dx\ dt\ |$$

$$= \Big|\int_0^1 \Big\{\nu\omega^2{}_{xx} + 2\omega_{xx}\omega u_{xx}\Big\}dx + \int_0^T\!\!\int_0^1 \Big\{- 2\omega_{xx}\omega_t u_{xxt}\Big\}dx\ dt$$

$$+ \int_0^T\!\!\int_0^1 \omega_t\omega_{xx}(u_{xx} + \nu_{xx} - 2u_{xx}) - \nu_t\omega^2_{xx}\ dx\ dt\ |$$

$$= \Big|\int_0^1 \Big\{\nu\omega^2_{xx} + 2\omega_{xx}\omega u_{xx}\Big\}dx + \int_0^T\!\!\int_0^1 - 2\omega_{xx}\omega u_{xxt}\ dx\ dt$$

$$+ \int_0^T\!\!\int_0^1 - u_t\omega^2_{xx}\ dx\ dt.$$

$$\leq K_1|\omega_{xx}(t)|^2 + K_2|\omega_{xx}(t)|^2|\omega(t)| + K_3\int_0^T|\omega_{xx}(t)|^2|\omega(t)|dt$$

$$K_4\int_0^T|\omega_{xx}(t)|^2 dt.$$

Now by (2.1.8) $|\omega_{xx}(t)|\omega(t)|_4 \leq |\omega_{xx}(t)|C\Big\{|\omega_{xx}(t)|^{\frac{1}{8}}|\omega(t)|^{\frac{7}{8}}$

$$+ |\omega(t)|\Big\} = C|\omega_{xx}(t)|^{\frac{9}{8}}|\omega(t)|^{\frac{7}{8}} + |\omega_{xx}(t)||\omega(t)|$$

$$\leq \Big\{\varepsilon\big(|\omega_{xx}(t)|^{\frac{9}{8}}\big)^{\frac{16}{9}} + C_\varepsilon\big(|\omega(t)|^{\frac{7}{8}}\big)^{\frac{16}{7}}\Big\} + \Big\{\varepsilon|\omega_{xx}(t)|^2 + C_\varepsilon|\omega(t)^2\Big\}$$

$$\leq K_5\Big\{\varepsilon|\omega_{xx}(t)|^2 + C_\varepsilon|\omega(t)|^2\Big\}$$

Substituting and collecting the like terms we get

$$\Big|\int_0^T\!\!\int_0^1 \omega_t\Big\{u^2{}_{xx} - \nu^2_{xx} + 2D^2(uu_{xx} - \nu\nu_{xx})\Big\}dx\ dt$$

$$\leq K_6\left\{|\omega(t)|^2+|\omega_{xx}(t)|^2+\int_0^T\left(|\omega(t)|^2+|\omega_{xx}(t)|^2\right)dt.\right.$$

But $|\omega_{xx}(t)|^2 = \left\{|D^2\omega|_4\right\}^2$

$$\leq \sqrt{2}\left\{|D^3\omega(t)^{\frac{9}{12}}|\omega(t)|^{\frac{3}{12}}\right\}^2 \quad \text{by} \quad (2.1.6)$$

$$= \sqrt{2}\left\{|D^3\omega(t)^{\frac{9}{6}}|\omega(t)^{\frac{3}{6}}\right\}$$

$$\leq \sqrt{2}\left\{\varepsilon\left(|D^3\omega(t)|^{\frac{9}{6}}\right)^{\frac{12}{9}} + C_\varepsilon\left(|\omega(t)|^{\frac{3}{6}}\right)^{\frac{12}{3}}\right\}$$

$$= \sqrt{2}\left\{\varepsilon|D^3\omega(t)|^2 + C_\varepsilon|\omega(t)|^2\right\}.$$

Substituting, we obtain

$$\left|\int_0^T\int_0^1 \omega_t\left\{u_{xx}^2-v_{xx}^2+2D^2(uu_{xx}-vv_{xx})\right\}dxdt\right| \leq C|\omega(t)|^2+\varepsilon|\omega_{xxx}(t)|^2$$

$$+ C\int_0^T\left\{|\omega(t)|^2+|\omega_{xxx}(t)|^2\right\}dt. \quad \text{Thus}$$

$$(4.1.5) \quad |\omega_{xxx}(t)|^2 \leq C|\omega(t)|^2+\varepsilon|\omega_{xxx}(t)|^2+C\int_0^T\left\{|\omega(t)|^2+|\omega_{xxx}(t)|^2\right\}dt.$$

Choosing $\varepsilon = \frac{1}{2}$ and using (4.1.4) to find

$$|\omega_{xxx}(t)|^2 \leq C_0\int_0^T\left\{|\omega(t)|^2+|\omega_{xxx}(t)|^2\right\}dt.$$

Combining this with (4.1.4) we get

$$|\omega(t)|^2+|\omega_{xxx}(t)|^2 \leq C_1\int_0^T\left\{|\omega(t)|^2+|\omega_{xxx}(t)|^2\right\}dt$$

Applying Gronwall's inequality [1.2.10] as before we get $|\omega(t)|^2 + |\omega_{xxx}(t)|^2 \leq 0$. This gives the result.

4.2 Uniqueness For All Values of m.

Taking the difference of two solutions u and ν of (1.1.1) multiplying by $\omega = u - \nu$ and integrating in x we obtain

$$(4.2.1)\qquad \int_0^1 \frac{\partial}{\partial t}\left(\tfrac{1}{2}\,\omega^2\right)dx = -\int_0^1 \omega_x\left\{G_m(u) - G_m(\nu)\right\}dx\ .$$

$G_m(u)$ is the gradient of $F_m(u)$ at u, and F_m has the form $F_m(u) = \int_0^1 f_m(u)dx$, where each term in $f_m(u)$ is of the form $\Pi_{j=0}^{m-1}(D^j u)^{\ell_j}$ with $\sum_{j=0}^{m-1}(1+\frac{j}{2})\ell_j = m+1$ for integer $\ell_j \geq 0$

$f_m(u) = (D^{m-1}u)^2 + (D^{m-2}u)^2 u + aD^{m-1}u + bD^{m-2}u + c$ up to inessential constants, and a, b, c are polynomials in u and its derivatives up to order $m-3$. The terms $(D^{m-1}u)^2$ and $(D^{m-2}u)^2 u$ of F_m give $D^{2m-2}u$ and $2(-1)^{m-1}$ $D^{m-2}(uD^{m-2}u) + (D^{m-2}u)^2$, respectively, in the gradient. Therefore,

$$G_m(u) = D^{2m-2}u + 2(-1)^{m-2}D^{m-2}(uD^{m-2}u) + (D^{m-2}u)^2 + \ldots\ldots.$$

Also $G_m(\nu) = D^{2m-2}\nu + 2(-1)^{m-2}D^{m-2}(\nu D^{m-2}\nu) + (D^{m-2}\nu)^2 + \ldots\ldots$

$$G_m(u) - G_m(\nu) = D^{2m-2}u - D^{2m-2}\nu + 2(-1)^{m-2}D^{m-2}(uD^{m-2}u)$$

$$- 2(-1)^{m-2}D^{m-2}(\nu D^{m-2}\nu) + (D^{m-2}u)^2 - (D^{m-2}\nu)^2 + \ldots\ldots$$

Since D^{γ} is linear we have

$$G_m(u) - G_m(\nu) = D^{2m-2}\omega + 2(-1)^{m-2}D^{m-2}\left[uD^{m-2}u - \nu D^{m-2}\nu\right]$$

$$+ D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu).$$

Multiplying through by $\omega_x = D\omega$ we get

$$\omega_x\left\{G_m(u) - G_m(\nu)\right\} = D\omega D^{2m-2}\omega + 2(-1)^{m-2} D\omega D^{m-2}\left[uD^{m-2}u - \nu D^{m-2}\nu\right]$$

$$+ D\omega D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu) + \ldots\ldots$$

Integrating in x and t we have

$$\left|\int_0^{T_1}\!\!\int_0 \omega_x\left\{G_m(u) - G_m(\nu)\right\}dxdt\right| = \left|\int_0^{T_1}\!\!\int_0 \left\{D\omega D^{2m-2}\omega + 2(-1)^{m-2}D\omega D^{m-2}\right.\right.$$

$$\left.\left.\left[uD^{m-2}u - \nu D^{m-2}\nu\right] + D\omega D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu) + \ldots\right\}dxdt\right|$$

$$\le \left|\int_0^{T_1}\!\!\int_0 D\omega D^{2m-2}\omega dxdt\right| + \left|\int_0^{T_1}\!\!\int_0 \left\{2(-1)^{m-2}D\omega D^{m-2}\left[uD^{m-2}u - \nu D^{m-2}\nu\right]\right.\right.$$

$$\left.\left. + D\omega D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\right\}dx\, dt\right| + \ldots$$

Now $\left|\int_0^{T_1}\!\!\int_0 D\omega D^{2m-2}\omega dx\, dt\right| = \left|(-1)^{m-1}\int_0^{T_1}\!\!\int_0 D^m\omega D^{m-1}\omega dx\, dt\right|$

$$= \left|\int_0^{T_1}\!\!\int_0 D^m\omega D^{m-1}\omega dxdt\right|$$

$$= \left|\int_0^T\int_0^1 D^{m-1}\omega \frac{\partial}{\partial x}(D^{m-1}\omega)\,dx\,dt\right|$$

$$= \left|\int_0^T\int_0^1 \frac{\partial}{\partial x}\tfrac{1}{2}(D^{m-1}\omega)^2\,dx\,dt\right|$$

$$\leq \tfrac{1}{2}\int_0^T |D^{m-1}\omega(t)|^2\,dt$$

Now $\left|\int_0^T\int_0^1 \left\{2(-1)^{m-2}\, D\omega D^{m-2}\left[uD^{m-2}u-\nu D^{m-2}\nu\right]\right.\right.$

$$\left.\left. + D\omega D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\right\}dx\,dt\right|.$$

$$= \left|\int_0^T\int_0^1 \left\{2(-1)^{2(m-2)}D^{m-1}\omega\left[uD^{m-2}u-\nu D^{m-2}\nu\right]+D\omega D^{m-2}\omega(D^{m-2}u+D^{m-2}\nu)\right\}dxdt\right|.$$

$$= \left|\int_0^T\int_0^1 \left\{2D^{m-1}\omega\left(D^{m-2}uu-D^{m-2}\nu\nu\right)+D\omega D^{m-2}\omega(D^{m-2}u+D^{m-2}\nu)\right\}dx\,dt\right|$$

$$= \left|\int_0^T\int_0^1 \left\{2D^{m-1}\omega\left[D^{m-2}uu-D^{m-2}\nu u+D^{m-2}\nu u-D^{m-2}\nu\nu\right]+D\omega D^{m-2}\omega(D^{m-2}u\right.\right.$$

$$\left.\left. + D^{m-2}\nu)\right\}dx\,dt\right|$$

$$= \left|\int_0^T\int_0^1 \left\{2D^{m-1}\omega\left|D^{m-2}\omega u+D^{m-2}\nu\omega\right|+D\omega D^{m-2}\omega(D^{m-2}u+D^{m-2}\nu)\right\}dxdt\right|$$

$$= \left|\int_0^T\int_0^1 \left\{2D^{m-1}\omega D^{m-2}\omega u+2D^{m-1}\omega\omega D^{m-2}\nu+D\omega D^{m-2}\omega(D^{m-2}u+D^{m-2}\nu)\right\}dxdt\right|$$

$$\leq C\int_0^T \left\{|D^{m-1}\omega(t)|\,|D^{m-2}\omega(t)|+|D^{m-1}\omega(t)|\,|\omega(t)|+|D\omega(t)|\,|D^{m-2}\omega(t)|\right\}dt\ .$$

Now $|D^{m-1}\omega(t)||D^{m-2}\omega(t)|_4 \le 2^{\frac{1}{4}}|D^{m-1}\omega(t)|\left\{|D^{m+1}\omega(t)|^{\frac{4m-7}{4(m-1)}}\right.$

$$\left.|\omega(t)|^{\frac{3}{4(m-1)}}\right\} \text{ by } (2.1.6)$$

$$= 2^{\frac{1}{4}}\left\{|D^{m-1}\omega(t)|^{\frac{8m-11}{4(m-1)}}|\omega(t)|^{\frac{3}{4m-1}}\right\}$$

$$\le 2^{\frac{1}{4}}\left\{\varepsilon\left(|D^{m-1}\omega(t)|^{\frac{8m-11}{4(m-1)}}\right)^{\frac{8(m-1)}{8m-11}} + C_\varepsilon\left(|\omega(t)|^{\frac{3}{4(m-1)}}\right)^{\frac{8}{3}(m-1)}\right\} \text{ by } [1.2.6]$$

$$= 2^{\frac{1}{4}}\left\{\varepsilon|D^{m-1}\omega(t)|^2 + C_\varepsilon|\omega(t)|^2\right\}.$$

Also $|D^{m-1}\omega(t)||\omega(t)| \le C_1|D^{m-1}\omega(t)|\left\{|D^{m-1}\omega(t)|^{\frac{1}{4(m-1)}}|\omega(t)|^{\frac{4m-5}{4(m-1)}}\right.$

$$\left.+ |\omega(t)|\right\} \text{ by } (2.1.8)$$

$$= C_1\left\{|D^{m-1}\omega(t)|^{\frac{4m-3}{4(m-1)}}|\omega(t)|^{\frac{4m-5}{4(m-1)}} + |D^{m-1}\omega(t)||\omega(t)|\right\}$$

$$\le C_1\left\{\varepsilon\left(|D^{m-1}\omega(t)|^{\frac{4m-3}{4(m-1)}}\right)^{\frac{8(m-1)}{4m-3}} + C_\varepsilon\left(|\omega(t)|^{\frac{4m-5}{4(m-1)}}\right)^{\frac{8(m-1)}{4m-5}} + \varepsilon|D^{m-1}\omega(t)|^2\right.$$

$$\left. + C_\varepsilon|\omega(t)|^2\right\}$$

$$\le C_2\left\{\varepsilon|D^{m-1}\omega(t)|^2 + C_\varepsilon|\omega(t)|^2\right\}.$$

Finally, by (2.1.6) and [1.2.6],

$$|D\omega(t)|_4|D^{m-2}\omega(t)|_4 \le \sqrt{2}\left\{\varepsilon|D^{m-1}\omega(t)|^2 + C_\varepsilon|\omega(t)|^2\right\}$$

Substituting back we get

$$(4.2.2) \qquad |\omega(t)|^2 \leq C_\alpha \int_0^T \left\{ |\omega(t)|^2 + |D^{m-1}\omega(t)|^2 \right\} dt.$$

Multiplying the equation for the difference of two solutions u and ν of (1.1.1) by $(G_m(u) - G_m(\nu))$ we obtain

$$\frac{\partial \omega}{\partial t} \left\{ G_m(u) - G_m(\nu) \right\} = \left\{ G_m(u) - G_m(\nu) \right\} \frac{\partial}{\partial x} \left\{ G_m(u) - G_m(\nu) \right\}$$

Integrating in x we get

$$\int_0^1 \frac{\partial \omega}{\partial t} (G_m(u) - G_m(\nu))\, dx = \int_0^1 \frac{\partial}{\partial t} \tfrac{1}{2} \left(G_m(u) - G_m(\nu) \right)^2 dx$$

$$= \tfrac{1}{2} \left(G_m(u) - G_m(\nu) \right)^2 \Big|_0^1 = 0.$$ That is,

$$(4.2.3) \qquad \int_0^1 \omega_t (G_m(u) - G_m(\nu))\, dx = 0.$$

Each term in $f_m(u)$ is of the form $\prod_{j=0}^{m-1} (D^j u)^{\ell_j}$, that is,

$$\prod_{j=0}^{m-1} (D^j u)^{\ell_j} = (D^0 u)^{\ell_0} (D^1 u)^{\ell_1} (D^2 u)^{\ell_2} \ldots (D^{m-1} u)^{\ell_{m-1}},$$ where

$\sum_{j=0}^{m-1} (1 + \frac{j}{2}) \ell_j = m + 1$. The corresponding term in the gradient G_m is

$$\ell_0 u_0^{\ell_0 - 1} (Du)^{\ell_1} \ldots\ldots\ldots\ldots (D^{m-1} u)^{\ell_{m-1}}$$

$$- \quad \ell_1 D\left[u^{\ell_0} (Du)^{\ell_1 - 1} \ldots\ldots\ldots\ldots (D^{m-1} u)^{\ell_{m-1}}\right.$$

$$+ \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$- (-1)^{m-1}\ell_{m-1}D^{m-1}\left[u^{\ell_0}(Du)^{\ell_1}\cdots\cdots\cdots(D^{m-1}u)^{\ell_{m-1}-1}\right]$$

We now estimate each term of G_m in (4.2.3) as follows :

Let $\Delta a = a(u) - a(v)$, $\dot{u} = D_t u$ and $u_j = D^j u$

(a) $\ell_j = 0$ for $j \geq 1$

$$J_0 = \left|\int_0^T\int_0^1 \omega_t (u^{m+1} - v^{m+1})\,dx\,dt\right|$$

$$= \left|\int_0^T\int_0^1 \omega_t (u-v)(u^m+u^{m-1}v+\cdots\cdots uv^{m-1}+v^m)\,dx\,dt\right|$$

$$= \left|\int_0^T\int_0^1 \omega_t\omega (u^m+u^{m-1}v+\cdots\cdots\cdots +uv^{m-1}+ v^m)\,dx\,dt\right|$$

$$= \left|\int_0^T\int_0^1 \frac{\partial}{\partial t}\left(\tfrac{1}{2}\omega^2\right)\left(u^m+u^{m-1}v+\cdots\cdots\cdots+uv^{m-1}+v^m\right)dx\,dt\right|$$

$$= \left|\int_0^1 \tfrac{1}{2}\omega^2 (u^m+u^{m-1}v+ \cdots\cdots\cdots+ uv^{m-1}+ v^m)dx\right.$$

$$\left. - \int_0^T\int_0^1 \tfrac{1}{2}\omega^2 D_t (u^m+u^{m-1}v+\cdots\cdots+uv^{m-1}+v^m)\,dx\,dt\right|$$

$$\leq C\left\{|\omega(t)|^2 + \int_0^T|\omega(t)|^2 dt\right\}$$

(b) $\ell_j = 0$ for $j \geq 2$.

The $\ell_j=0$, for $j \geq 2$ term in $G_m(u)$ is of the form

$$\ell_0 u^{\ell_0-1}(Du)^{\ell_1} - \ell_1 D\left[u_0^{\ell_0}(Du)^{\ell_1-1}\right]$$

Therefore $G_m(u)-G_m(v) = \left[\ell_0 u_0^{\ell_0-1}(Du)^{\ell_1} - \ell_1 D\left[u_0^{\ell_0}(Du)^{\ell_1-1}\right]\right]$

$$- \ell_0 v_0^{\ell_0-1}(Dv)^{\ell_1} - \ell_1 D\left[v_0^{\ell_0} Dv^{\ell_1-1}\right]$$

$$= \ell_0 u_0^{\ell_0-1} u_1^{\ell_1} - \ell_0 v_0^{\ell_0-1} v_1^{\ell_1} - \ell_1 D(u_0^{\ell_0} u_1^{\ell_1-1} + \ell_1 D(v_0^{\ell_0} v_1^{\ell_1-1})$$

$$= \ell_0 (u_0^{\ell_0-1} u_1^{\ell_1} - v_0^{\ell_0-1} v_1^{\ell_1}) - \ell_1 \left\{D\left[u_0^{\ell_0} u_1^{\ell_1-1} - v_0^{\ell_0} v_1^{\ell_1-1}\right]\right\}$$

$$= \ell_0 \Delta u_0^{\ell_0-1} u_1^{\ell_1} - \ell_1 D\Delta u_0^{\ell_0} u_1^{\ell_1-1}$$

Therefore $J_1 = \displaystyle\iint_{00}^{T_1} \omega_t \left[\ell_0 \Delta u_0^{\ell_0-1} u_1^{\ell_1} - \ell_1 D\Delta u_0^{\ell_0} u_1^{\ell_1-1}\right] dx\, dt$

$$= \iint_{00}^{T_1} (\ell_0 \dot{\omega}_0 \Delta u_0^{\ell_0-1} u_1^{\ell_1} - \ell_1 \dot{\omega}_0 D\Delta u_0^{\ell_0} u_1^{\ell_1-1})\, dx\, dt.$$

Integrating second term in x gives

$$-\int_0^t \ell_1 \dot{\omega}_0 \Delta u_0^{\ell_0} u_1^{\ell_1-1}\Big|_0^1 dt + \iint_{00}^{T_1} (\ell_0 \dot{\omega}_0 \Delta u_0^{\ell_0-1} u_1^{\ell_1} + \ell_1 \dot{\omega}_1 \Delta u_0^{\ell_0} u_1^{\ell_1-1})\, dx\, dt$$

$$= \iint_{00}^{T_1} (\ell_0 \dot{\omega}_0 \Delta u_0^{\ell_0-1} u_1^{\ell_1} + \ell_1 \dot{\omega}_1 \Delta u_0^{\ell_0} u_1^{\ell_1-1})\, dx\, dt$$

Using the identity

(4.2.4) $\Delta u_0 u_1 = (\Delta u_0) u_1 + v_0 \Delta u_1$

and

(4.2.5) $\Delta u^{\ell} = u^{\ell} - v^{\ell} = \Delta u\, (u^{\ell-1} + u^{\ell-2} v + \dots + u v^{\ell-2} + v^{\ell-1})$

$$= \Delta u P_{\ell-1}$$

$$= \omega P_{\ell-1},$$

we can write the integrand

$$\ell_0 \dot{\omega}_0 \, \Delta u_0^{\ell_0 -1} u_1^{\ell_1} + \ell_1 \dot{\omega}_1 \Delta u_0^{\ell_0} u_1^{\ell_1 -1}$$

$$= \ell_0 \dot{\omega}_0 (u_0^{\ell_0-1} u_1^{\ell_1} - \nu_0^{\ell_0-1} \nu_1^{\ell_1}) + \ell_1 \dot{\omega}_1 (u_0^{\ell_0} u_1^{\ell_1-1} - \nu_0^{\ell_0} \nu_1^{\ell_1-1})$$

$$= \ell_0 \dot{\omega}_0 (u_0^{\ell_0-1} u_1^{\ell_1} - \nu_0^{\ell_0-1} u_1^{\ell_1} + \nu_0^{\ell_0-1} u_1^{\ell_1} - \nu_0^{\ell_0-1} \nu_1^{\ell_1})$$

$$+ \ell_1 \dot{\omega}_1 (u_0^{\ell_0} u_1^{\ell_1-1} - \nu_0^{\ell_0} u_1^{\ell_1-1} + \nu_0^{\ell_0} u_1^{\ell_1-1} - \nu_0^{\ell_0} \nu_1^{\ell_1-1}$$

$$= \ell_0 \dot{\omega}_0 \left\{ (u_0^{\ell_0-1} - \nu_0^{\ell_0-1}) u_1^{\ell_1} + \nu_0^{\ell_0-1} (u_1^{\ell_1} - \nu_1^{\ell_1}) \right\}$$

$$+ \ell_1 \dot{\omega}_1 \left\{ (u_0^{\ell_0} - \nu_0^{\ell_0}) u_1^{\ell_1-1} - \nu_0^{\ell_0} (u_1^{\ell_1-1} - \nu_1^{\ell_1-1}) \right\}$$

$$= \ell_0 \dot{\omega}_0 \left\{ (\Delta u_0^{\ell_0-1}) u_1^{\ell_1} + \nu_0^{\ell_0-1} (\Delta u_1^{\ell_1}) \right\} + \ell_1 \dot{\omega}_1 \left\{ (\Delta u_0^{\ell_0}) u_1^{\ell_1-1} + \nu_0^{\ell_0} \Delta u_1^{\ell_1-1} \right\} .$$

But $\Delta u_0^{\ell_0-1} = u^{\ell_0-1} - \nu_0^{\ell_0-1} = (u_0 - \nu_0)(u_0^{\ell_0-2} + u_0^{\ell_0-3} \nu_0 + \ldots + u_0 \nu^{\ell_0-3} + \nu_0^{\ell_0-2})$

$$= \omega_0 P_{\ell_0-1},$$

$$\Delta u_1^{\ell_1} = u_1^{\ell_1} - \nu_1^{\ell_1} = \omega_1 P_{\ell_1-1},$$

$$\Delta u_0^{\ell_0} = u_0^{\ell_0} - \nu_0^{\ell_0} = \omega_0 P_{\ell_0-1}$$

and

$$\Delta u_1^{\ell_1-1} = u_1^{\ell_1-1} - v_1^{\ell_1-1} = \omega_1 P_{\ell_1-2} \text{ by } \quad (4.2.5)$$

Substituting back we get,

$$\ell_0\dot{\omega}_0 \Delta u_0^{\ell_0-1} u_1^{\ell_1} + \ell_1\dot{\omega}_1 \, \Delta u_0^{\ell_0} u_1^{\ell_1-1} = \ell_0\dot{\omega}_0\left\{\omega_0 P_{\ell_0-2} u_1^{\ell_1} + v_0^{\ell_0-1}\,\omega_1 P_{\ell_1-1}\right\}$$

$$+ \ell_1\dot{\omega}_1 \left\{\omega_0 P_{\ell_0-1} u_1^{\ell_1-1} + v_0^{\ell_0}\,\omega_1 P_{\ell_1-2}\right\}$$

Then $$J_1 = \left|\int_0^{T}\!\!\int_0^1 \left\{\ell_0\dot{\omega}_0\omega_0 P_{\ell_0-2} u_1^{\ell_1} + \ell_0\dot{\omega}_0\omega_1 v_0^{\ell_0-1} P_{\ell_1-1}\right.\right.$$

$$\left.\left. + \ell_1\dot{\omega}_1\omega_0 P_{\ell_0-1} u_0^{\ell_1-1} + \ell_1\dot{\omega}_1\omega_1 v_0^{\ell_0} P_{\ell_1-2}\right\} dx\, dt\right|$$

$$= \left|\int_0^{T}\!\!\int_0^1 \left\{ \tfrac{1}{2}\ell_0 \frac{\partial}{\partial t}\, \omega_0^2 P_{\ell_0-2} u_1^{\ell_1} + \tfrac{1}{2}\ell_1 \frac{\partial}{\partial t}(\omega_1)^2 v_0^{\ell_0} P_{\ell_1-2}\right.\right.$$

$$\left.\left. + \ell_0\dot{\omega}_0\omega_1 v_0^{\ell_0-1} P_{\ell_1-1} + \ell_1\dot{\omega}_1\omega_0 P_{\ell_0-1} u_0^{\ell_1-1} \right\} dx\, dt\right|$$

$$= \left| \tfrac{1}{2}\int_0^1 \ell_0 \omega_0^2 P_{\ell_0-2} u_1^{\ell_1} dx - \tfrac{1}{2}\int_0^{T}\!\!\int_0^1 \ell_0\omega_0^2 D_t\left[P_{\ell_0-2} u_1^{\ell_1}\right] dxdt + \tfrac{1}{2}\int_0^{T} \ell_1 \omega_1^2 v_0^{\ell_0} P_{\ell_1-2} dx\right.$$

$$- \tfrac{1}{2}\int_0^{T}\!\!\int_0^1 \ell_1\omega_1^2 D_t\left[v_0^{\ell_0} P_{\ell_1-2}\right] dxdt + \int_0^{T}\!\!\int_0^1 \ell_0\dot{\omega}_0\omega_1 v_0^{\ell_0-1} P_{\ell_1-1} dxdt + \int_0^1 \ell_1\omega_0\omega_1$$

$$P_{\ell_0-1} u_1^{\ell_1-1} dx - \int_0^{T}\!\!\int_0^1 \left\{\ell_1\dot{\omega}_0\omega_1 P_{\ell_0-1} u_1^{\ell_1-1} + \ell_1\omega_0\omega_1 D_t\left[P_{\ell_0-1} u_1^{\ell_1-1}\right]\right\} dxdt.$$

$$\leq \left| \tfrac{1}{2}\int_0^1 \left\{\ell_0\omega_0^2 P_{\ell_0-2} u_1^{\ell_1} + \ell_1\omega_1^2 v_0^{\ell_0} P_{\ell_1-2} + 2\ell_1\omega_1\omega_0 P_{\ell_0-1} u_1^{\ell_1-1}\right\} dx\right.$$

$$+\left|\frac{1}{2}\int_0^{T_1}\int_0^1\left\{\ell_0\omega_0^2 D_t\left(P_{\ell_0-2}u_1^{\ell_1}\right)+\ell_1\omega_1^2 D_t\left(v_0^{\ell_0}P_{\ell_1-2}\right)+2\ell_1\omega_0\omega_1 D_t\left(P_{\ell_0-1}u_1^{\ell_1-1}\right)\right\}dxdt\right|$$

$$+\left|\int_0^{T_1}\int_0^1\left\{\dot{\omega}_0\omega_1\left(\ell_0 v_0^{\ell_0-1}P_{\ell_1-1}-\ell_1 P_{\ell_0-1}u_1^{\ell_1-1}\right)\right\}dx\,dt\right|$$

Estimating using bounds for the solution we get

$$J_1\le K_1\left\{|\omega(t)|^2+|\omega_1(t)|^2+|\omega(t)||\omega_1(t)|\right\}+K_2\int_0^T\left(|\omega(t)|^2+|\omega_1(t)|^2+|\omega(t)||\omega_1(t)|\right)dt$$

$$+K_3\left|\int_0^{T_1}\int_0^1\left\{\dot{\omega}_0\omega_1\left(\ell_0 v_0^{\ell_0-1}P_{\ell_1-1}-\ell_1 P_{\ell_0-1}u_1^{\ell_1-1}\right)\right\}dx\,dt\right|$$

Using $\pm\,\ell_0\ell_1\,\dot{\omega}_0\omega_1 v_0^{\ell_0-1}u_1^{\ell_1-1}$ to write the remaining term.

Now $\displaystyle\int_0^{T_1}\int_0^1\dot{\omega}_0\omega_1\left\{\ell_0 v_0^{\ell_0-1}P_{\ell_1-1}-\ell_1 u_1^{\ell_1-1}P_{\ell_0-1}\right\}dx\,dt$

$$=\int_0^{T_1}\int_0^1\dot{\omega}_0\,\omega_1\left\{\ell_0 v_0^{\ell_0-1}P_{\ell_1-1}-\ell_0\ell_1 v_0^{\ell_0-1}u_1^{\ell_1-1}-\ell_1 u_1^{\ell_1-1}P_{\ell_0-1}+\ell_0\ell_1 v_0^{\ell_0-1}u_1^{\ell_1-1}\right\}dxdt$$

$$=\int_0^{T_1}\int_0^1\dot{\omega}_0\omega_1\left\{\ell_0 v_0^{\ell_0-1}\left(P_{\ell_1-1}-\ell_1 u_1^{\ell_1-1}\right)+\ell_1 u_1^{\ell_1-1}\left(\ell_0 v_0^{\ell_0-1}-P_{\ell_0-1}\right)\right\}dxdt.$$

That is (4.2.6) $\displaystyle\int_0^{T_1}\int_0^1\dot{\omega}_0\omega_1\left\{\ell_0 v_0^{\ell_0-1}P_{\ell_1-1}-\ell_1 u_1^{\ell_1-1}P_{\ell_0-1}\right\}dxdt$

$$=\int_0^{T_1}\int_0^1\dot{\omega}_0\,\omega_1\left\{\ell_0 v_0^{\ell_0-1}\left(P_{\ell_1-1}-\ell_1 u_1^{\ell_1-1}\right)+\ell_1 u_1^{\ell_1-1}\left(\ell_0 v_0^{\ell_0-1}-P_{\ell_0-1}\right)\right\}dxdt.$$

Since $P_{\ell-1}=u^{\ell-1}+u^{\ell-2}v+\dots\dots+uv^{\ell-2}+v^{\ell-1}$

Therefore $P_{\ell-1} - \ell u^{\ell-1} = u^{\ell-1} - \ell u^{\ell-1} + u^{\ell-2}v + \ldots + uv^{\ell-2} + v^{\ell-1}$

$$= (1-\ell)u^{\ell-1} + u^{\ell-2}v + \ldots + uv^{\ell-2} + v^{\ell-1}$$

$$= (v^{\ell-1} - u^{\ell-1}) + u(v^{\ell-2} - u^{\ell-2}) + \ldots + u^{\ell-2}(v-u)$$

$$= (v-u)\Big\{(v^{\ell-2} + v^{\ell-3}u + \ldots + vu^{\ell-3} + u^{\ell-2})$$

$$+ u(v^{\ell-3} + v^{\ell-4}u + \ldots + vu^{\ell-4} + u^{\ell-3}) + \ldots + u^{\ell-2}\Big\}$$

$= -\omega\, p(u,v)$. Thus

(4.2.7) $\quad P_{\ell-1} - \ell u^{\ell-1} = -\omega p(u,v)$,

where $P(u,v)$ is a polynomial in u and v .

Similarly $P_{\ell-1} - \ell v^{\ell-1} = \omega p(u,v)$

Therefore (4.2.6) becomes

$$\int_0^{T_1}\!\!\int_0 \dot{\omega}_0\omega_1 \left\{\ell_0 v_0^{\ell_0-1}(-\omega_1 p(u_1,v_1)) + \ell_1 u_1^{\ell_1-1}(-\omega_0 p(u,v))\right\}dx\,dt$$

$$= -\int_0^{T_1}\!\!\int_0 \dot{\omega}_0 \left\{\ell_0\omega_1^2 v_0^{\ell_0-1} p(u_1,v_1) + \ell_1\omega_0\omega_1 u_1^{\ell_1-1} p(u,v)\right\}dx\,dt$$

which is bounded by $K_3 \int_0^T\left\{|\omega_1(t)|^2 + |\omega(t)||\omega_1(t)|\right\}dt$

Therefore $J_1 \leq K_1 \Big\{|\omega(t)|^2 + |\omega_1(t)|^2 + |\omega(t)||\omega_1(t)|$

$$+ K_2 \int_0^T \left(|\omega(t)|^2 + |\omega_1(t)|^2 + |\omega(t)||\omega_1(t)|\right) dt$$

$$+ K_3 \int_0^T \left(|\omega_1(t)|^2 + |\omega(t)||\omega_1(t)|\right) dt,$$

By [1.2.6] we have

$$|\omega(t)||\omega_1(t)| \leq \left\{\varepsilon|\omega(t)|^2 + C_\varepsilon|\omega_1(t)|^2\right\}$$

Thus

$$J_1 \leq K\left(|\omega(t)|^2 + |\omega_1(t)|^2\right) + K\int_0^T \left(|\omega(t)|^2 + |\omega_1(t)|^2\right) dt$$

(c) $\ell_j = 0$ for $j \geq 3$

$$G_m(u) - G_m(v) = \ell_0 \left| u_0^{\ell_0 - 1} (Du)^{\ell_1} (D^2u)^{\ell_2} - v_0^{\ell_0 - 1} (Dv)^{\ell_1} (D^2v)^{\ell_2} \right]$$

$$- \ell_1 D \left| u_0^{\ell_0} (Du)^{\ell_1 - 1} (D^2u)^{\ell_2} - v_0^{\ell_0} (Dv)^{\ell_1 - 1} (D^2v)^{\ell_2} \right]$$

$$\ell_2 D^2 \left| u_0^{\ell_0} (Du)^{\ell_1} (D^2u)^{\ell_2 - 1} - v_0^{\ell_0} (Dv)^{\ell_1} (D^2v)^{\ell_2 - 1} \right]$$

$$= \ell_0 \Delta u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} - \ell_1 D\Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2} + \ell_2 D^2 \Delta u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2 - 1}$$

Therefore $J_2 = \displaystyle\int_0^{T_1}\!\!\int_0 \omega_t \left(G_m(u) - G_m(v)\right) dxdt$

$$= \int_0^{T_1}\!\!\int_0 \left\{\ell_0 \dot{\omega}_0 \Delta u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} - \ell_1 \dot{\omega}_0 D\Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2} + \ell_2 \dot{\omega}_0 D^2 \Delta u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2 - 1}\right\} dxdt$$

But $\displaystyle\int_0^{T_1}\!\!\int_0 \ell_1 \dot{\omega}_0 D\Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2} dxdt = \int_0^T \ell_1 \dot{\omega}_0 \Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2} \Big|_0^2 dt$

$$- \int_0^T\!\!\int_0^1 \ell_1 D(\dot{\omega}_0)\Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2}\, dxdt$$

$$= - \int_0^T\!\!\int_0^1 \ell_1\, \dot{\omega}_1 \Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2}\, dxdt.$$

Also $\displaystyle\int_0^T\!\!\int_0^1 \ell_2 \dot{\omega}_0 D^2 \Delta u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2 - 1}\, dxdt = \int_0^T\!\!\int_0^1 \ell_2 \dot{\omega}_2 \Delta u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2 - 1}\, dxdt$

Therefore, on substituting, we have

$$\int_0^T\!\!\int_0^1 \left\{\ell_0 \dot{\omega}_0\, \Delta u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} + \ell_1 \dot{\omega}_1 \Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2} + \ell_2 \dot{\omega}_2 \Delta u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2 - 1}\right\} dx\, dt.$$

The integrand is transformed using (4.2.4) as follows :

$$\Delta u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} = u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} - \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} \nu_2^{\ell_2}$$

$$= u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} - \nu_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} + \nu_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} - \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} u_2^{\ell_2} + \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} u_2^{\ell_2} - \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} \nu_2^{\ell_2}$$

$$= (u_0^{\ell_0 - 1} - \nu_0^{\ell_0 - 1}) u_1^{\ell_1} u_2^{\ell_2} + \nu_0^{\ell_0 - 1} (u_1^{\ell_1} - \nu_1^{\ell_1}) u_2^{\ell_2} + \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} (u_2^{\ell_2} - \nu_2^{\ell_2})$$

$$= (\Delta u_0^{\ell_0 - 1}) u_1^{\ell_1} u_2^{\ell_2} + \nu_0^{\ell_0 - 1} (\Delta u_1^{\ell_1}) u_2^{\ell_2} + \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} \Delta u_2^{\ell_2}$$

Also $\Delta u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2} = (\Delta u_0^{\ell_0}) u_1^{\ell_1 - 1} u_2^{\ell_2} + \nu_0^{\ell_0} (\Delta u_1^{\ell_1 - 1}) u_2^{\ell_2} + \nu_0^{\ell_0} \nu_1^{\ell_1 - 1} \Delta u_2^{\ell_2}$

and $\Delta u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2 - 1} = (\Delta u_0^{\ell_0}) u_1^{\ell_1} u_2^{\ell_2 - 1} + \nu_0^{\ell_0} (\Delta u_1^{\ell_1}) u_2^{\ell_2 - 1} + \nu_0^{\ell_0} \nu_1^{\ell_1} \Delta u_2^{\ell_2 - 1}$.

Substituting back we have

$$\int_0^T\!\!\int_0^1 \Big\{\ell_0 \dot{\omega}_0 \Big[(\Delta u_0^{\ell_0 - 1}) u_1^{\ell_1} u_2^{\ell_2} + \nu_0^{\ell_0 - 1} (\Delta u_1^{\ell_1}) u_2^{\ell_2} + \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} \Delta u_2^{\ell_2} \Big]$$

$$+ \ell_1\dot{\omega}_1\left[(\Delta u_0^{\ell_0})u_1^{\ell_1-1}u_2^{\ell_2} + \nu_0^{\ell_0}(\Delta u_1^{\ell_1-1})u_2^{\ell_2} + \nu_0^{\ell_0}\nu_1^{\ell_1-1}\Delta u_2^{\ell_2}\right]$$

$$+ \ell_2\dot{\omega}_2\left[(\Delta u_0^{\ell_0})u_1^{\ell_1}u_2^{\ell_2-1} + \nu_0^{\ell_0}(\Delta u_1^{\ell_1})u_2^{\ell_2-1} + \nu_0^{\ell_0}\nu_1^{\ell_1}\Delta u_2^{\ell_2-1}\right]$$

By (4.2.5) we have

$$J_2 = \int_0^T\int_0^1 \Big\{\ell_0\dot{\omega}_0\left[\omega_0 P_{\ell_0-2}u_1^{\ell_1}u_2^{\ell_2} + \nu_0^{\ell_0-1}\omega_1 P_{\ell_1-1}u_2^{\ell_2} + \nu_0^{\ell_0-1}\nu_1^{\ell_1}\omega_2 P_{\ell_2-1}\right]$$

$$+ \ell_1\dot{\omega}_1\left[\omega_0 P_{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2} + \nu_0^{\ell_0}\omega_1 P_{\ell_1-2}u_2^{\ell_2} + \nu_0^{\ell_0}\nu_1^{\ell_1-1}\omega_2 P_{\ell_2-1}\right]$$

$$+ \ell_2\dot{\omega}_2\left[\omega_0 P_{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2-1} + \nu_0^{\ell_0}\omega_1 P_{\ell_1-1}u_2^{\ell_2-1} + \nu_0^{\ell_0}\nu_1^{\ell_1}\omega_2 P_{\ell_2-2}\right]\Big\}\,dxdt$$

$$= \int_0^T\int_0^1 \Big\{\ell_0\dot{\omega}_0\omega_0 P_{\ell_0-2}u_1^{\ell_1}u_2^{\ell_2} + \ell_0\dot{\omega}_0\omega_1\nu_0^{\ell_0-1}P_{\ell_1-1}u_2^{\ell_2} + \ell_0\dot{\omega}_0\omega_2\nu_0^{\ell_0-1}\nu_1^{\ell_1}P_{\ell_2-1}$$

$$+ \ell_1\dot{\omega}_1\omega_0 P_{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2} + \ell_1\dot{\omega}_1\omega_1\nu_0^{\ell_0}P_{\ell_1-2}u_2^{\ell_2} + \ell_1\dot{\omega}_1\omega_2\nu_0^{\ell_0}\nu_1^{\ell_1-1}P_{\ell_2-1}$$

$$+ \ell_2\dot{\omega}_2\omega_0 P_{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2-1} + \ell_2\dot{\omega}_2\omega_1\nu_0^{\ell_0}P_{\ell_1-1}u_2^{\ell_2-1} + \ell_2\dot{\omega}_2\omega_2\nu_0^{\ell_0}\nu_1^{\ell_1}P_{\ell_2-2}\Big\}\,dxdt$$

Now $\displaystyle\int_0^T\int_0^1 \dot{\omega}_1\omega_0 P_{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2}\,dxdt = \int_0^1 \omega_0\omega_1 P_{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2}\,dx$

$$- \int_0^T\int_0^1\Big\{\dot{\omega}_0\omega_1 P_{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2} + \omega_0\omega_1 D\left[P_{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2}\right]\Big\}\,dxdt$$

$$\int_0^T\int_0^1 \dot{\omega}_2\omega_1\nu_0^{\ell_0}P_{\ell_1-1}u_2^{\ell_2-1}\,dxdt = \int_0^1 \omega_1\omega_2\nu_0^{\ell_0}P_{\ell_1-1}u_2^{\ell_2-1}\,dx$$

$$- \int_0^{T_1}\!\!\int_0^1 \left\{\dot{\omega}_1\omega_2\nu_0^{\ell_0} P_{\ell_1-1} u_2^{\ell_2-1} + \omega_1\omega_2 D_t\left[\nu_0^{\ell_0} P_{\ell_0-1} u_2^{\ell_2-1}\right]\right\} dx\ dt$$

$$\int_0^{T_1}\!\!\int_0^1 \dot{\omega}_2\omega_0 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1}\, dx\ dt = \int_0^1 \omega_0\omega_2\, P_{\ell_2-1} u_1^{\ell_1} u_2^{\ell_2-1}\, dx$$

$$- \int_0^{T_1}\!\!\int_0^1 \left\{\dot{\omega}_0\omega_2 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} + \omega_0\omega_2 D_t\left[P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1}\right]\right\} dx\ dt$$

Substituting back we get

$$J_2 \le \left|\int_0^1 \left\{\omega_0\omega_1 P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} + \omega_0\omega_2 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} + \omega_1\omega_2\nu_0^{\ell_0} P_{\ell_1-1} u_2^{\ell_2-1}\right\} dx\right|$$

$$+\left|\int_0^{T_1}\!\!\int_0^1 \left\{\omega_0\omega_1 D_t\left[P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2}\right] + \omega_0\omega_2 D_t\left[P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1}\right] + \omega_1\omega_2 D_t\left[\nu_0^{\ell_0} u_2^{\ell_2-1} P_{\ell_1-1}\right]\right\} dxdt\right|$$

$$+ \left|\int_0^{T_1}\!\!\int_0^1 \left\{\ell_0\dot{\omega}_0\omega_0 P_{\ell_0-2} u_1^{\ell_1} u_2^{\ell_2} + \ell_1\omega_1\dot{\omega}_1\nu_0^{\ell_0} P_{\ell_1-2} u_2^{\ell_2} + \ell_2\dot{\omega}_2\omega_2\nu_0^{\ell_0}\nu_1^{\ell_1} P_{\ell_2-2}\right\} dx\ dt\right|$$

$$+\left|\int_0^{T_1}\!\!\int_0^1 \left\{\omega_0\omega_1 u_2^{\ell_2}\left(\ell_0\nu_0^{\ell_0-1} P_{\ell_1-1} - \ell_1 u_1^{\ell_1-1} P_{\ell_0-1}\right) + \dot{\omega}_0\omega_2\left(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1} P_{\ell_2-1} - \ell_2 u_1^{\ell_1} u_2^{\ell_2-1} P_{\ell_0-1}\right)\right.\right.$$

$$\left.\left. + \dot{\omega}_1\omega_2\nu_0^{\ell_0}\left(\ell_1\nu_1^{\ell_1-1} P_{\ell_2-1} - \ell_2 P_{\ell_1-1} u_2^{\ell_2-1}\right)\right\} dx\ dt\right|$$

Therefore $J_2 \le K_1\left\{|\omega(t)||\omega_1(t)| + |\omega(t)||\omega_2(t)| + |\omega_1(t)||\omega_2(t)|\right\}$

$$+ \int_0^T \left\{|\omega(t)||\omega_1(t)| + |\omega(t)||\omega_2(t)| + |\omega_1(t)||\omega_2(t)|\right\} dt.$$

$$+ \left|\int_0^{T_1}\!\!\int_0^1 \frac{\ell_0}{2}\frac{\partial}{\partial t}\omega_0^2 P_{\ell_0-2} u_1^{\ell_1} u_2^{\ell_2} + \frac{\ell_1}{2}\frac{\partial}{\partial t}\omega_1^2\nu_0^{\ell_0} P_{\ell_1-2} u_2^{\ell_2} + \frac{\ell_2}{2}\frac{\partial}{\partial t}\omega_2^2\nu_0^{\ell_0}\nu_1^{\ell_1} P_{\ell_1-2}\right\} dxdt\right|$$

$$+ \left| \int_0^{T_1}\!\!\int_0^1 \left\{ \dot{\omega}_0 \omega_1 u_2^{\ell_2} \left(\ell_0 \nu_0^{\ell_0} P_{\ell_1-1} - \ell_1 u_1^{\ell_1-1} P_{\ell_0-1}\right) \right.\right.$$

$$+ \dot{\omega}_0 \omega_2 \left(\ell_0 \nu_0^{\ell_0-1} \nu_1^{\ell_1} P_{\ell_2-1} - \ell_2 u_1^{\ell_1} u_2^{\ell_2-1} P_{\ell_0-1}\right)$$

$$\left.\left. + \omega_1 \omega_2 \nu_0^{\ell_0} \left(\ell_1 \nu_1^{\ell_1-1} P_{\ell_2-1} - \ell_2 P_{\ell_1-1} u_2^{\ell_2-1}\right) \right\} dx\ dt \right|$$

$$= \Big\{ |\omega_0(t)|\,|\omega_1(t)| + |\omega_0(t)|\,|\omega_2(t)| + |\omega_1(t)|\,|\omega_2(t)| + K_2 \int_0^T \Big\{ |\omega_0(t)|\,|\omega_1(t)|$$

$$+ |\omega_0(t)|\,|\omega_2(t)| + |\omega_1(t)|\,|\omega_2(t)| \Big\} dt + \left| \tfrac{1}{2} \int_0^1 \Big\{ \ell_0 \omega_0^2 P_{\ell_0-2} u_1^{\ell_1} u_2^{\ell_2} + \ell_1 \omega_1^2 \nu_0^{\ell_0} P_{\ell_1-2} u_2^{\ell_2} \right.$$
$$+ \ell_2 \omega_2^2 \nu_0^{\ell_0} \nu_1^{\ell_1} P_{\ell_1-2} \Big\} dx$$

$$\left| - \tfrac{1}{2} \int_0^{T_1}\!\!\int_0^1 \Big\{ \ell_0 \omega_0^2 D_t \left[P_{\ell_0-2} u_1^{\ell_1} u_2^{\ell_2}\right] + \ell_1 \omega_1^2 D_t \left[\nu_0^{\ell_0} P_{\ell_1-2} u_2^{\ell_2}\right] + \ell_2 \omega_2^2 D_t \left[\nu_0^{\ell_0} \nu_1^{\ell_1} P_{\ell_1-2}\right] \Big\} dx\ dt \right|$$

$$+ \left| \int_0^{T_1}\!\!\int_0^1 \Big\{ \dot{\omega}_0 \omega_1 u_2^{\ell_2} \left(\ell_0 \nu_0^{\ell_0-1} P_{\ell_1-1} - \ell_1 u_1^{\ell_1} P_{\ell_0-1}\right) + \dot{\omega}_0 \omega_2 \left(\ell_0 \nu_0^{\ell_0-1} \nu_1^{\ell_1} P_{\ell_2-1} - \ell_2 u_1^{\ell_1} u_2^{\ell_2-1} P_{\ell_0-1}\right) \right.$$

$$\left. + \dot{\omega}_1\, \omega_2\, \nu_0^{\ell_0} \left(\ell_1 \nu_1^{\ell_1-1} P_{\ell_2-1} - \ell_2 P_{\ell_1-1} u_2^{\ell_2-1}\right) \Big\}\ dx\ dt \right|$$

$$\leq K_1 \Big\{ |\omega(t)|\,|\omega_1(t)| + |\omega(t)|\,|\omega_2(t)| + |\omega_1(t)|\,|\omega_2(t)| \Big\}$$

$$+ K_2 \int_0^T \Big\{ |\omega(t)|\,|\omega_1(t)| + |\omega(t)|\,|\omega_2(t)| + |\omega_1(t)|\,|\omega_2(t)| \Big\} dt$$

$$+ K_3 \Big\{ |\omega(t)|^2 + |\omega_1(t)|^2 + |\omega_2(t)|^2 \Big\} + K_4 \int_0^1 \Big\{ |\omega(t)|^2 + |\omega_1(t)|^2 + |\omega_2(t)|^2 \Big\}\ dt$$

$$+ \left| \int_0^{T_1}\!\!\int_0^1 \Big\{ \dot{\omega}_0 \omega_1 u_2^{\ell_2} \left(\ell_0 \nu_0^{\ell_0-1} P_{\ell_1-1} - \ell_1 u_1^{\ell_1-1} P_{\ell_0-1}\right) + \dot{\omega}_0 \omega_2 \left(\ell_0 \nu_0^{\ell_0-1} \nu_1^{\ell_1} P_{\ell_2-1} - \ell_2 u_1^{\ell_1} u_2^{\ell_2-1} P_{\ell_0-1}\right) \right.$$

$$\left. + \dot{\omega}_1 \omega_2 \nu_0^{\ell_0} \left(\ell_1 \nu_1^{\ell_1-1} P_{\ell_2-1} - \ell_2 P_{\ell_1-1} u_2^{\ell_2-1}\right) \Big\} dx dt \right|$$

Using $\ell_0\ell_1\dot{\omega}_0\omega_1\nu_0^{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2} - \ell_0\ell_1\dot{\omega}_0\omega_1\nu_0^{\ell_0-1}u_1^{\ell_1-1}u_2^{\ell_2} = 0$

and $\ell_1\ell_2\dot{\omega}_1\omega_2\nu_0^{\ell_0}\nu_1^{\ell_1-1}u_2^{\ell_2-1} - \ell_1\ell_2\dot{\omega}_1\omega_2\nu_0^{\ell_0}\nu_1^{\ell_1-1}u_2^{\ell_2-1} = 0$ to write

the first and last term of the integrand, respectively,

that is 1st $\dot{\omega}_0\omega_1 u_2^{\ell_2}(\ell_0\nu_0^{\ell_0-1}P_{\ell_1-1} - \ell_1 u_1^{\ell_1-1}P_{\ell_0-1})$

$$= \dot{\omega}_0\omega_1 u_2^{\ell_2}(\ell_0\nu_0^{\ell_0-1}P_{\ell_1-1} - \ell_0\ell_1\nu_0^{\ell_0-1}u_1^{\ell_1-1} + \ell_0\ell_1\nu_0^{\ell_0-1}u_1^{\ell_1-1} - \ell_1 u_1^{\ell_1-1}P_{\ell_0-1})$$

$$= \dot{\omega}_0\omega_1 u_2^{\ell_2}\left[\ell_0\nu_0^{\ell_0-1}(P_{\ell_1-1} - \ell_1 u_1^{\ell_1-1}) + \ell_1 u_1^{\ell_1-1}(\ell_0\nu_0^{\ell_0-1} - P_{\ell_0-1})\right]$$

$$= \dot{\omega}_0\omega_1 u_2^{\ell_2}\left[-\ell_0\nu_0^{\ell_0-1}\omega_1 P(u_1, \nu_1) - \ell_1 u_1^{\ell_1-1}\omega_0 P(u, \nu)\right]$$

$$= \dot{\omega}_0\omega_1 u_2^{\ell_2}\left[-\ell_0\nu_0^{\ell_0-1}\omega_1 P(u_1, \nu_1) - \ell_1 u_1^{\ell_1-1}\omega_0 P(u, \nu)\right]$$

$$= (u_t - \nu_t)\omega_1 u_2^{\ell_2}\left[-\ell_0\nu_0^{\ell_0-1}\omega_1 P(u_1, \nu_1) - \ell_1 u_1^{\ell_1-1}\omega_0 p(u, \nu)\right]$$

3rd $\dot{\omega}_1\omega_2\nu_0^{\ell_0}(\ell_1\nu_1^{\ell_1-1}P_{\ell_2-1} - \ell_2 P_{\ell_1-1}u_2^{\ell_2-1})$

$$= \dot{\omega}_1\omega_2\nu_0^{\ell_0}(\ell_1\nu_1^{\ell_1-1}P_{\ell_2-1} - \ell_1\ell_2\nu_1^{\ell_1-1}u_2^{\ell_2-1} + \ell_1\ell_2\nu_1^{\ell_1-1}u_2^{\ell_2-1} - \ell_2 P_{\ell_1-1}u_2^{\ell_2-1})$$

$$= \dot{\omega}_1\omega_2\nu_0^{\ell_0}\left[\ell_1\nu_1^{\ell_1-1}(P_{\ell_2-1} - \ell_2 u_2^{\ell_2-1}) + \ell_2 u_2^{\ell_2-1}(\ell_1\nu_1^{\ell_1-1} - P_{\ell_1-1})\right]$$

$$= \dot{\omega}_1\omega_2\nu_0^{\ell_0}\left[-\ell_1\nu_1^{\ell_1-1}\omega_2 P(u_2, \nu_2) - \ell_2 u_2^{\ell_2-1}\omega_1 P(u_1, \nu_1)\right]$$

$$= (u_{xt} - \nu_{xt})\omega_2\nu_0^{\ell_0}\left[-\ell_1\nu_1^{\ell_1-1}\omega_2 p(u_2, \nu_2) - \ell_2 u_2^{\ell_2-1}\omega_1 p(u_1, \nu_1)\right].$$

The remaining term in the integrand with $\pm\,\ell_0\ell_2\nu_0^{\ell_0-1}\nu_1^{\ell_1}u_2^{\ell_2-1}$

becomes $\dot{\omega}_0\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}P_{\ell_2-1}-\ell_0\ell_2\nu_0^{\ell_0-1}\nu_1^{\ell_1}u_2^{\ell_2-1}+\ell_0\ell_2\nu_0^{\ell_0-1}\nu_1^{\ell_1}u_2^{\ell_2-1}$

$$-\ell_2\nu_1^{\ell_1}u_2^{\ell_2-1}P_{\ell_0-1}$$

$$=\dot{\omega}_0\,\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\big(P_{\ell_2-1}-\ell_2u_2^{\ell_2-1}\big)-\ell_2\big(u_1^{\ell_1}P_{\ell_0-1}-\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\big)u_2^{\ell_2-1}\Big\}$$

$$=\dot{\omega}_0\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\big(P_{\ell_2-1}-\ell_2u_2^{\ell_2-1}\big)-\ell_2\Big[u_1^{\ell_1}P_{\ell_0-1}-\nu_1^{\ell_1}P_{\ell_0-1}+\nu_1^{\ell_1}P_{\ell_0-1}$$

$$-\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\Big]u_2^{\ell_2-1}\Big\}$$

$$=\dot{\omega}_0\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\big(P_{\ell_2-1}-\ell_2u_2^{\ell_2-1}\big)-\ell_2\Big[\big(u_1^{\ell_1}-\nu_1^{\ell_1}\big)P_{\ell_0-1}+\nu_1^{\ell_1}\big(P_{\ell_0-1}-\ell_0\nu_0^{\ell_0-1}\big)\Big]u_2^{\ell_2-1}\Big\}$$

$$=(u_t-\nu_t)\,\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\omega_2 p(u_2,\nu_2)-\ell_2\big(\omega_1P_{\ell_1-1}P_{\ell_0-1}+\nu_1^{\ell_1}\omega_0 p(u,\nu)u_2^{\ell_2-1}\Big\}\text{ by (4.2.5).}$$

Now estimating, using the bounds for the solutions to find

$$K_5\int_0^T\Big\{|\omega_0(t)|\,|\omega_1(t)|+|\omega_1(t)|^2+|\omega_1(t)|\,|\omega_2(t)|+|\omega(t)|\,|\omega_2(t)|+|\omega_2(t)|^2\Big\}$$

Therefore $J_2\le K^1\Big\{|\omega(t)|^2+|\omega_2(t)|^2\Big\}+C\int_0^T\Big\{|\omega(t)|^2+|\omega_2(t)|^2\Big\}\,dt$

$$+K_5\int_0^T\Big\{|\omega(t)|\,|\omega_1(t)|+|\omega_1(t)|^2+|\omega_1(t)|^2|\omega_2(t)|^2+|\omega(t)|\,|\omega_2(t)|+|\omega_2(t)|^2\Big\}$$

$\le K\big(|\omega(t)|^2+|D^2\omega(t)|^2\big)+C\int_0^T\big(|\omega(t)|^2+|D^2\omega(t)|^2\big)dt$ by

(2.1.6) and [1.2.6]

(d) $\ell_j = 0$ for $j \geq 4$

$$G_m(u) - G_m(v) = \ell_0\left(u_0^{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3} - v_0^{\ell_0-1}v_1^{\ell_1}v_2^{\ell_2}v_3^{\ell_3}\right)$$

$$- \ell_1 D\left(u_0^{\ell_0}u_1^{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3} - v_0^{\ell_0}v_1^{\ell_1-1}v_2^{\ell_2}v_3^{\ell_3}\right)$$

$$+ \ell_2 D^2\left(u_0^{\ell_0}u_1^{\ell_1}u_2^{\ell_2-1}u_3^{\ell_3} - v_0^{\ell_0}v_1^{\ell_1}v_2^{\ell_2-1}v_3^{\ell_3}\right)$$

$$- \ell_3 D^3\left(u_0^{\ell_0}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3-1} - v_0^{\ell_0}v_1^{\ell_1}v_2^{\ell_2}v_3^{\ell_3-1}\right)$$

Therefore $\displaystyle\int_0^T\!\!\int_0^1 \omega_t\,(G_m(u)-G_m(v))\,dx\,dt =$

$$J_3 = \int_0^T\!\!\int_0^1\Big\{\dot\omega_0\ell_0\Delta\left(u_0^{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3}\right) - \dot\omega_0\ell_1 D\Delta\left(u_0^{\ell_0}u_1^{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3}\right) + \dot\omega_0\ell_2 D^2\Delta\left(u_0^{\ell_0}u_1^{\ell_1}u_2^{\ell_2-1}u_3^{\ell_3}\right)$$

$$- \dot\omega_0\ell_3 D^3\Delta\left(u_0^{\ell_0}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3-1}\right)\Big\}dx\,dt$$

But $\displaystyle\int_0^T\!\!\int_0^1 \dot\omega_0\ell_1 D\Delta\left(u_0^{\ell_0}u_1^{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3}\right)dx\,dt = \int_0^T \dot\omega_0\ell_1\Delta\left(u_0^{\ell_0}u_1^{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3}\right)\Big|_0^1 dt$

$$- \int_0^T\!\!\int_0^1 D\dot\omega_0\ell_1\Delta\left(u_0^{\ell_0}u_1^{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3}\right)dx\,dt$$

$$= \int_0^T\!\!\int_0^1 \ell_1\,\dot\omega_1\Delta\left(u_0^{\ell_0}u_1^{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3}\right)dx\,dt$$

Also $\displaystyle\int_0^T\!\!\int_0^1 \dot\omega_0\ell_2 D^2\Delta\left(u_0^{\ell_0}u_1^{\ell_1}u_2^{\ell_2-1}u_3^{\ell_3}\right) = (-1)^2\int_0^T\!\!\int_0^1 \ell_2\dot\omega_2\Delta\left(u_0^{\ell_0}u_1^{\ell_1}u_2^{\ell_2-1}u_3^{\ell_3}\right)dxdt.$

and $\displaystyle\int_0^{T_1}\!\!\int_0 \dot{\omega}_0 \ell_3 D^3 \Delta\left(u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3 - 1}\right) dxdt = (-1)^3 \int_0^{T_1}\!\!\int_0 \ell_3 \dot{\omega}_3 \Delta\left(u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1}\right) dxdt$

where we have used boundary conditions. Substituting we find

$$J_3 = \int_0^{T_1}\!\!\int_0 \Big\{ \dot{\omega}_0 \ell_0 \Delta\left(u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3}\right) + \ell_1 \dot{\omega}_1 \Delta\left(u_0^{\ell_0} u_1^{\ell_1 - 1} u_2^{\ell_2} u_3^{\ell_3}\right)$$

$$+ \dot{\omega}_2 \ell_2 \Delta\left(u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2 - 1} u_3^{\ell_3}\right) + \ell_3 \dot{\omega}_3 \Delta\left(u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3 - 1}\right) \Big\} \, dx \, dt.$$

Now $\Delta\left(u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3}\right) = u_0^{\ell_0 - 1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} - \nu_0^{\ell_0 - 1} \nu_1^{\ell_1} \nu_2^{\ell_2} \nu_3^{\ell_3}$

$$= u_0^{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} - \nu_0^{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} - \nu_0^{\ell_0-1} \nu_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0-1} \nu_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} - \nu_0^{\ell_0-1} \nu_1^{\ell_1} \nu_2^{\ell_2} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0-1} \nu_1^{\ell_1} \nu_2^{\ell_2} u_3^{\ell_3} - \nu_0^{\ell_0-1} \nu_1^{\ell_1} \nu_2^{\ell_2} \nu_3^{\ell_3}$$

$$= \left(u_0^{\ell_0-1} - \nu_0^{\ell_0-1}\right) u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0-1}\left(u_1^{\ell_1} - \nu_1^{\ell_1}\right) u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0-1} \nu_1^{\ell_1} \left(u_2^{\ell_2} - \nu_2^{\ell_2}\right) u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0-1} \nu_1^{\ell_1} \nu_2^{\ell_2} \left(u_3^{\ell_3} - \nu_3^{\ell_3}\right)$$

$$= \left(\Delta u_0^{\ell_0-1}\right) u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0-1} \left(\Delta u_1^{\ell_1}\right) u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0-1} \nu_1^{\ell_1} \left(\Delta u_2^{\ell_2}\right) u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0-1} \nu_1^{\ell_1} \nu_2^{\ell_2} \Delta u_3^{\ell_3}.$$

Similarly

$$\Delta\left(u_0^{\ell_0} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3}\right) = \left(\Delta u_0^{\ell_0}\right) u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0}\left(\Delta u_1^{\ell_1-1}\right) u_2^{\ell_2} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1-1}\left(\Delta u_2^{\ell_2}\right) u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1-1} \nu_2^{\ell_2} \Delta u_3^{\ell_3}$$

$$\Delta\left(u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3}\right) = \left(\Delta u_0^{\ell_0}\right) u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3} + \nu_0^{\ell_0}\left(\Delta u_1^{\ell_1}\right) u_2^{\ell_2-1} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1}\left(\Delta u_2^{\ell_2-1}\right) u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1} \nu_2^{\ell_2-1} \Delta u_3^{\ell_3}$$

and

$$\Delta\left(u_0^{\ell_0} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1}\right) = \left(\Delta u_0^{\ell_0}\right) u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1} + \nu_0^{\ell_0}\left(\Delta u_1^{\ell_1}\right) u_2^{\ell_2} u_3^{\ell_3-1}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1}\left(\Delta u_2^{\ell_2}\right) u_3^{\ell_3-1} + \nu_0^{\ell_0} \nu_1^{\ell_1} \nu_2^{\ell_2} \Delta u_3^{\ell_3-1}$$

Substituting, the integrand develops as follows :

$$\ell_0 \dot{\omega}_0 \Big[\left(\Delta u_0^{\ell_0-1}\right) u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0-1}\left(\Delta u_1^{\ell_1}\right) u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1}\left(\Delta u_2^{\ell_2}\right) u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0-1} \nu_1^{\ell_1} \nu_2^{\ell_2} \Delta u_3^{\ell_3}\Big]$$

$$+ \ell_1 \dot{\omega}_1 \Big[\left(\Delta u_0^{\ell_0}\right) u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0}\left(\Delta u_1^{\ell_1-1}\right) u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1-1}\left(\Delta u_2^{\ell_2}\right) u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1-1} \nu_2^{\ell_2} \Delta u_3^{\ell_3}\Big]$$

$$+ \ell_2 \dot{\omega}_2 \Big[\left(\Delta u_0^{\ell_0}\right) u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3} + \nu_0^{\ell_0}\left(\Delta u_1^{\ell_1}\right) u_2^{\ell_2-1} u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1}\left(\Delta u_2^{\ell_2-1}\right) u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1} \nu_2^{\ell_2-1} \Delta u_3^{\ell_3} \Big]$$

$$+ \ell_2 \dot{\omega}_2 \Big[(\Delta u_0^{\ell_0}) u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3} + \nu_0^{\ell_0} (\Delta u_1^{\ell_1}) u_2^{\ell_2-1} u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1} (\Delta u_2^{\ell_2-1}) u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1} \nu_2^{\ell_2-1} \Delta u_3^{\ell_3} \Big]$$

$$+ \ell_3 \dot{\omega}_3 \Big[(\Delta u_0^{\ell_0}) u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1} + \nu_0^{\ell_0} (\Delta u_1^{\ell_1}) u_2^{\ell_2} u_3^{\ell_3-1} + \nu_0^{\ell_0} \nu_1^{\ell_1} (\Delta u_2^{\ell_2}) u_3^{\ell_3-1}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1} \nu_2^{\ell_2} \Delta u_3^{\ell_3-1} \Big]$$

$$\therefore \Big| \int_0^{T_1}\!\!\int_0 \Big\{ \ell_0 \dot{\omega}_0 \Big[\omega_0 P_{\ell_0-2} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0-1} \omega_1 P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0-1} \nu_1^{\ell_1} \omega_2 P_{\ell_2-1} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0-1} \nu_1^{\ell_1} \nu_2^{\ell_2} \omega_3 P_{\ell_3-1} \Big]$$

$$+ \ell_1 \dot{\omega}_1 \Big[\omega_0 P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0} \omega_1 P_{\ell_1-2} u_2^{\ell_2} u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1-1} \omega_2 P_{\ell_2-1} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1-1} \nu_2^{\ell_2} \omega_3 P_{\ell_3-1} \Big]$$

$$+ \ell_2 \dot{\omega}_2 \Big[\omega_0 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3} + \nu_0^{\ell_0} \omega_1 P_{\ell_1-1} u_2^{\ell_2-1} u_3^{\ell_3} + \nu_0^{\ell_0} \nu_1^{\ell_1} \omega_2 P_{\ell_2-2} u_3^{\ell_3}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1} \nu_2^{\ell_2-1} \omega_3 P_{\ell_3-1} \Big]$$

$$+ \ell_3 \dot{\omega}_3 \Big[\omega_0 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1} + \nu_0^{\ell_0} \omega_1 P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3-1} + \nu_0^{\ell_0} \nu_1^{\ell_1} \omega_2 P_{\ell_2-1} u_3^{\ell_3-1}$$

$$+ \nu_0^{\ell_0} \nu_1^{\ell_1} \nu_2^{\ell_2} \omega_3 P_{\ell_3-2} \Big] \Big\} dx\, dt \Big|$$

But $\int_0^{T_1}\int_0^1 \ell_1\dot{\omega}_1\omega_0 P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} dxdt = \int_0^1 \ell_1\omega_0\omega_1 P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} dx$

$- \int_0^{T_1}\int_0^1 \left\{\ell_1\dot{\omega}_0\omega_1 P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} + \ell_1\omega_0\omega_1 D_t\left[P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3}\right]\right\} dx\, dt$.

Also $\int_0^{T_1}\int_0^1 \ell_2\dot{\omega}_2\omega_0 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3} dxdt = \int_0^1 \ell_2\omega_0\omega_2 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3} dx$

$- \int_0^{T_1}\int_0^1 \left\{\ell_2\dot{\omega}_0\omega_2 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3} + \ell_2\omega_0\omega_2 D_t\left[P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3}\right]\right\} dxdt.$

Also $\int_0^{T_1}\int_0^1 \ell_2\dot{\omega}_2\omega_1 \nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2-1} u_3^{\ell_3} dxdt = \int_0^1 \ell_2\omega_1\omega_2 \nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2-1} u_3^{\ell_3} dx$

$- \int_0^{T_1}\int_0^1 \left\{\ell_2\dot{\omega}_1\omega_2 \nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2-1} u_3^{\ell_3} + \ell_2\omega_1\omega_2 D_t\left[\nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2-1} u_3^{\ell_3}\right]\right\} dxdt$

Also $\int_0^{T_1}\int_0^1 \ell_3\dot{\omega}_3\omega_0 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1} dxdt = \int_0^1 \ell_3\omega_0\omega_3 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1} dx$

$- \int_0^{T_1}\int_0^1 \left\{\ell_3\dot{\omega}_0\omega_3 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1} + \ell_3\omega_0\omega_3 D_t\left[P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1}\right]\right\} dxdt$

Also $\int_0^{T_1}\int_0^1 \ell_3\dot{\omega}_3\omega_1 \nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3-1} dxdt = \int_0^1 \ell_3\omega_1\omega_3 \nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3-1} dx$

$- \int_0^{T_1}\int_0^1 \left\{\ell_3\dot{\omega}_1\omega_3 \nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3-1} + \ell_3\omega_1\omega_3 D_t\left[\nu_0^{\ell_0}\,{}^{0}P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3-1}\right]\right\} dxdt$

Finally $\int_0^{T_1}\int_0^1 \ell_3\dot{\omega}_3\omega_2 \nu_0^{\ell_0}\,{}^{0}\nu_1^{\ell_1}\,{}^{1}P_{\ell_2-1} u_3^{\ell_3-1} dxdt = \int_0^1 \ell_3\omega_2\omega_3 \nu_0^{\ell_0}\,{}^{0}\nu_1^{\ell_1}\,{}^{1}P_{\ell_2-1} u_3^{\ell_3-1} dx$

$$- \int_0^{T_1}\!\!\int_0^1 \Big\{\ell_3\dot{\omega}_2\omega_3 \nu_0^{\ell_0}\nu_1^{\ell_1} P_{\ell_2-1} u_3^{\ell_3-1} + \ell_3\omega_2\omega_3 D_t\Big[\nu_0^{\ell_0}\nu_1^{\ell_1} P_{\ell_2-1} u_3^{\ell_3-1}\Big]\Big\} dx\, dt$$

$$\therefore\ J_3 \le \Big|\int_0^1\Big\{\ell_1\omega_0\omega_1 P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3} + \ell_2\omega_0\omega_2 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3}$$

$$+ \ell_2\omega_1\omega_2 \nu_0^{\ell_0} P_{\ell_1-1} u_2^{\ell_2-1} u_3^{\ell_3} + \ell_3\omega_0\omega_3\, P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1}$$

$$+ \ell_3\omega_1\omega_3 \nu_0^{\ell_0} P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3-1} + \ell_3\omega_2\omega_3 \nu_0^{\ell_0}\nu_1^{\ell_1} P_{\ell_2-1} u_3^{\ell_3-1}\, dx$$

$$- \int_0^{T_1}\!\!\int_0^1\Big\{\ell_0\dot{\omega}_0\omega_0 P_{\ell_0-2} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3} + \ell_1\omega_1\dot{\omega}_1 \nu_0^{\ell_0} P_{\ell_1-2} u_2^{\ell_2} u_3^{\ell_3}$$

$$+ \ell_2\dot{\omega}_2\omega_2 \nu_0^{\ell_0}\nu_1^{\ell_1} P_{\ell_2-2} u_3^{\ell_3} + \ell_3\omega_3\dot{\omega}_3 \nu_0^{\ell_0}\nu_1^{\ell_1}\nu_2^{\ell_2} P_{\ell_3-2}.$$

$$-\ell_1\omega_0\omega_1 D_t\Big[P_{\ell_0-1} u_1^{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3}\Big] - \ell_2\omega_0\omega_2 D_t\Big[P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1} u_3^{\ell_3}\Big]$$

$$- \ell_2\omega_1\omega_2 D_t\Big[\nu_0^{\ell_0} P_{\ell_1-1} u_2^{\ell_2-1} u_3^{\ell_3}\Big] - \ell_3\omega_0\omega_3 D_t\Big[P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1}\Big]$$

$$- \ell_3\omega_1\omega_3 D_t\Big[\nu_0^{\ell_0} P_{\ell_1-1} u_2^{\ell_2} u_3^{\ell_3-1}\Big] - \ell_3\omega_2\omega_3 D_t\Big[\nu_0^{\ell_0}\nu_1^{\ell_1} P_{\ell_2-1} u_3^{\ell_3-1}\Big]\Big\}\, dxdt\ \Big|$$

$$+ \Big|\int_0^{T_1}\!\!\int_0^1\Big\{\dot{\omega}_0\omega_1\big(\ell_0\nu_0^{\ell_0-1} P_{\ell_1-1} - \ell_1 P_{\ell_0-1} u_1^{\ell_1-1}\big) u_2^{\ell_2} u_3^{\ell_3}$$

$$+ \dot{\omega}_0\omega_2\big(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1} P_{\ell_2-1} - \ell_2 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2-1}\big) u_3^{\ell_3}$$

$$+ \dot{\omega}_0\omega_3\big(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\nu_2^{\ell_2} P_{\ell_3-1} - \ell_3 P_{\ell_0-1} u_1^{\ell_1} u_2^{\ell_2} u_3^{\ell_3-1}\big)$$

$$+ \dot{\omega}_1\omega_2\big(\ell_1\nu_1^{\ell_1-1} P_{\ell_2-1} - \ell_2 P_{\ell_1-1} u_2^{\ell_2-1}\big)\nu_0^{\ell_0} u_3^{\ell_3}$$

$$+ \dot{\omega}_1\omega_3\left(\ell_1\nu_1^{\ell_1-1}\nu_2^{\ell_2}P_{\ell_3-1}-\ell_3P_{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3-1}\right)\nu_0^{\ell_0}$$

$$+ \dot{\omega}_2\omega_3\left(\ell_2\nu_2^{\ell_2-1}P_{\ell_3-1}-\ell_3P_{\ell_2-1}u_3^{\ell_3-1}\right)\nu_0^{\ell_0}\nu_1^{\ell_1}\Big\}\, dx\ dt\,\Big|$$

$$\therefore\quad J_3 \leq K\Big\{|\omega(t)|\,|\omega_1(t)|+|\omega(t)|\,|\omega_2(t)|+|\omega_1(t)|\,|\omega_2(t)|+|\omega(t)|\,|\omega_3(t)|$$

$$+ |\omega_1(t)|\,|\omega_3(t)| + |\omega_2(t)|\,|\omega_3(t)|\Big\}$$

$$+ K_1\Big\{|\omega(t)|^2+|\omega_1(t)|^2+|\omega_2(t)|^2+|\omega_3(t)|^2\Big\}+K_2\int_0^T\Big\{|\omega(t)|^2+|\omega_1(t)|^2+|\omega_2(t)|^2$$

$$+ |\omega_3(t)|^2+|\omega(t)|\,|\omega_1(t)|+|\omega(t)|\,|\omega_2(t)|+|\omega_1(t)|\,|\omega_2(t)|+|\omega(t)|\,|\omega_3(t)|$$

$$+|\omega_1(t)|\,|\omega_3(t)|+|\omega_2(t)|\,|\omega_3(t)|\Big\}dt+\Big|\int_0^T\int_0^1\Big\{\dot{\omega}_0\omega_1\left(\ell_0\nu_0^{\ell_0-1}P_{\ell_1-1}-\ell_1P_{\ell_0-1}u_1^{\ell_1-1}\right)u_2^{\ell_2}u_3^{\ell_3}$$

$$+ \dot{\omega}_0\omega_2\left(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}P_{\ell_2-1}-\ell_2P_{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2-1}\right)u_3^{\ell_3} +$$

$$+ \dot{\omega}_0\omega_3\left(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\nu_2^{\ell_2}P_{\ell_3-1}-\ell_3P_{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3-1}\right)$$

$$+ \dot{\omega}_1\omega_2\left(\ell_1\nu_1^{\ell_1-1}P_{\ell_2-1}-\ell_2P_{\ell_1-1}u_2^{\ell_2-1}\right)\nu_0^{\ell_0}u_3^{\ell_3}$$

$$+ \dot{\omega}_1\omega_3\left(\ell_1\nu_1^{\ell_1-1}\nu_2^{\ell_2}P_{\ell_3-1}-\ell_3P_{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3-1}\right)\nu_0^{\ell_0}$$

$$+ \dot{\omega}_2\omega_3\left(\ell_2\nu_2^{\ell_2-1}P_{\ell_3-1}-\ell_3P_{\ell_2-1}u_3^{\ell_3-1}\right)\nu_0^{\ell_0}\nu_1^{\ell_1}\Big\}dx\ dt\ \Big|$$

Rewriting $\dot{\omega}_0\omega_1\left(\ell_0\nu_0^{\ell_0-1}P_{\ell_1-1}-\ell_1P_{\ell_0-1}u_1^{\ell_1-1}\right)u_2^{\ell_2}u_3^{\ell_3}$

$$= \dot{\omega}_0\omega_1\left[\ell_0\nu_0^{\ell_0-1}\left(P_{\ell_1-1}-\ell_1u_1^{\ell_1-1}\right)+\ell_1u_1^{\ell_1-1}\cdot\left(\ell_0\nu_0^{\ell_0-1}-P_{\ell_0-1}\right)\right]u_2^{\ell_2}u_3^{\ell_3}$$

$$= \dot{\omega}_0\omega_1\left[\ell_0\nu_0^{\ell_0-1}(P_{\ell_1-1}-\ell_1 u_1^{\ell_1-1}) + \ell_1 u_1^{\ell_1-1}(\ell_0\nu_0^{\ell_0-1}-P_{\ell_0-1})\right]u_2^{\ell_2}u_3^{\ell_3}$$

$$= \dot{\omega}_0\omega_1\left[-\ell_0\nu_0^{\ell_0-1}\omega_1\rho(u_1,\ \nu_1) - \ell_1 u_1^{\ell_1-1}\omega_0\rho(u,\nu)\right]u_2^{\ell_2}u_3^{\ell_3}$$

Similarly $\dot{\omega}_0\omega_2(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}P_{\ell_2-1}-\ell_2 P_{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2-1})u_3^{\ell_3}$

$$= \dot{\omega}_0\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}P_{\ell_2-1}-\ell_0\ell_2\nu_0^{\ell_0-1}\nu_1^{\ell_1}u_2^{\ell_2-1}+\ell_0\ell_2\nu_0^{\ell_0-1}\nu_1^{\ell_1}u_2^{\ell_2-1}-\ell_2 u_1^{\ell_1}u_2^{\ell_2-1}$$

$$P_{\ell_0-1}\Big\}u_3^{\ell_3}$$

$$= \dot{\omega}_0\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}(P_{\ell_2-1}-\ell_2 u_2^{\ell_2-1})-\ell_2(u_1^{\ell_1}P_{\ell_0-1}-\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1})u_2^{\ell_2-1}\Big\}u_3^{\ell_3}$$

$$= \dot{\omega}_0\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\omega_2\rho(u_2,\nu_2)-\ell_2(u_1^{\ell_1}P_{\ell_0-1}-\nu_1^{\ell_1}P_{\ell_0-1}+\nu_1^{\ell_1}P_{\ell_0-1} -$$

$$-\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1})u_2^{\ell_2-1}\Big\}u_3^{\ell_3}$$

$$= \dot{\omega}_0\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\omega_2\rho(u_2,\nu_2)-\ell_2\Big[(u_1^{\ell_1}-\nu_1^{\ell_1})P_{\ell_0-1}+\nu_1^{\ell_1}(P_{\ell_0-1}-\ell_0\nu_0^{\ell_0-1})\Big]u_2^{\ell_2-1}\Big\}u_3^{\ell_3}$$

$$= (u_t-\nu_t)\omega_2\Big\{\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\omega_2\rho(u_2,\nu_2)-\ell_2\omega_1 P_{\ell_1-1}P_{\ell_0-1}-\ell_2\nu_1^{\ell_1}\omega_0\rho(u,\nu)u_2^{\ell_2-1}\Big\}u_3^{\ell_3}$$

Also rewriting $\dot{\omega}_0\omega_3(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\nu_2^{\ell_2}P_{\ell_3-1}-\ell_3 P_{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3-1})$

$$\dot{\omega}_0\,\omega_3(\ell_0\nu_0^{\ell_0-1}\nu_1^{\ell_1}\nu_2^{\ell_2}P_{\ell_3-1}-\ell_0\nu_0^{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}P_{\ell_3-1}+\ell_0\nu_0^{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}P_{\ell_3-1} -$$

$$-\ell_3 P_{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3-1})$$

$$= \dot{\omega}_0\omega_3\Big[\ell_0\nu_0^{\ell_0-1}P_{\ell_3-1}(\nu_1^{\ell_1}\nu_2^{\ell_2}-u_1^{\ell_1}u_2^{\ell_2})+u_1^{\ell_1}u_2^{\ell_2}(\ell_0\nu_0^{\ell_0-1}P_{\ell_3-1}-\ell_3 P_{\ell_0-1}u_3^{\ell_3-1})\Big]$$

$$= \dot{\omega}_0\omega_3\Big[-\ell_0 v_0^{\ell_0-1}P_{\ell_3-1}(u_1^{\ell_1}u_2^{\ell_2} - v_1^{\ell_1}v_2^{\ell_2}) + u_1^{\ell_1}u_2^{\ell_2}(\ell_0 v_0^{\ell_0-1}P_{\ell_3-1} - \ell_3 P_{\ell_0-1}u_3^{\ell_3-1})\Big]$$

$$= \dot{\omega}_0\omega_3\Big[-\ell_0 v_0^{\ell_0-1}P_{\ell_3-1}(u_1^{\ell_1}u_2^{\ell_2} - v_1^{\ell_1}u_2^{\ell_2} + v_1^{\ell_1}u_2^{\ell_2} - v_1^{\ell_1}v_2^{\ell_2})$$

$$+ u_1^{\ell_1}u_2^{\ell_2}(\ell_0 v_0^{\ell_0-1}P_{\ell_3-1} - \ell_0\ell_3 v_0^{\ell_0-1}u_3^{\ell_3-1} + \ell_0\ell_3 v_0^{\ell_0-1}u_3^{\ell_3-1} - \ell_3 P_{\ell_0-1}u_3^{\ell_3-1})\Big]$$

$$= \dot{\omega}_0\omega_3\Big\{-\ell_0 v_0^{\ell_0-1}P_{\ell_3-1}\Big[(u_1^{\ell_1} - v_1^{\ell_1})u_2^{\ell_2} + (u_2^{\ell_2} - v_2^{\ell_2})v_1^{\ell_1}\Big]$$

$$+ u_1^{\ell_1}u_2^{\ell_2}\Big[\ell_0 v_0^{\ell_0-1}(P_{\ell_3-1} - \ell_3 u_3^{\ell_3-1}) + \ell_3 u_3^{\ell_3-1}(\ell_0 v_0^{\ell_0-1} - P_{\ell_0-1})\Big]\Big\}$$

$$= \dot{\omega}_0\omega_3\Big\{-\ell_0 v_0^{\ell_0-1}P_{\ell_3-1}\,\omega_1 P_{\ell_1-1}\,u_2^{\ell_2} - \ell_0 v_0^{\ell_0-1}P_{\ell_3-1}\,\omega_2\,P_{\ell_2-1}v_1^{\ell_1}$$

$$- \ell_0 v_0^{\ell_0-1}u_1^{\ell_1}u_2^{\ell_2}\,\omega_3\,\rho(u_3,v_3) - \ell_3 u_1^{\ell_1}u_2^{\ell_2}u_3^{\ell_3-1}\,\omega_0\rho(u,v)\Big\}$$

Also rewriting $\dot{\omega}_1\omega_2(\ell_1 v_1^{\ell_1-1}P_{\ell_2-1} - \ell_2 P_{\ell_1-1}u_2^{\ell_2-1})v_0^{\ell_0}u_3^{\ell_3}$

$$= \dot{\omega}_1\omega_2(\ell_1 v_1^{\ell_1-1}P_{\ell_2-1} - \ell_1\ell_2 v_1^{\ell_1-1}u_2^{\ell_2-1} + \ell_1\ell_2 v_1^{\ell_1-1}u_2^{\ell_2-1} - \ell_2 P_{\ell_1-1}u_2^{\ell_2-1})v_0^{\ell_0}u_3^{\ell_3}$$

$$= \dot{\omega}_1\omega_2\Big[\ell_1 v_1^{\ell_1-1}(P_{\ell_2-1} - \ell_2 u_2^{\ell_2-1}) + \ell_2 u_2^{\ell_2-1}(\ell_1 v_1^{\ell_1-1} - P_{\ell_1-1})\Big]v_0^{\ell_0}u_3^{\ell_3}$$

$$= (u_{xt} - v_{xt})\omega_2\Big[-\ell_1 v_1^{\ell_1-1}\,\omega_2\rho(u_2,v_2) - \ell_2 u_2^{\ell_2-1}\,\omega_1\rho(u_1,v_1)\Big]v_0^{\ell_0}u_3^{\ell_3}$$

Rewriting $\dot{\omega}_1\omega_3(\ell_1 v_1^{\ell_1-1}v_2^{\ell_2}P_{\ell_3-1} - \ell_3 P_{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3-1})v_0^{\ell_0}$

$$= \dot{\omega}_1\omega_3(\ell_1 v_1^{\ell_1-1}v_2^{\ell_2}P_{\ell_3-1} - \ell_1\ell_3 v_1^{\ell_1-1}v_2^{\ell_2}u_3^{\ell_3-1} + \ell_1\ell_3 v_1^{\ell_1-1}v_2^{\ell_2}u_3^{\ell_3-1} - \ell_3 P_{\ell_1-1}u_2^{\ell_2}u_3^{\ell_3-1})v_0^{\ell_0}$$

$$= \dot{\omega}_1\omega_3\left[\ell_1\nu_1^{\ell_1-1}\nu_2^{\ell_2}\left(P_{\ell_3-1}-\ell_3u_3^{\ell_3-1}\right)-\ell_3u_3^{\ell_3-1}\left(u_2^{\ell_2}P_{\ell_1-1}-\ell_1\nu_1^{\ell_1-1}\nu_2^{\ell_2}\right)\right]\nu_0^{\ell_0}$$

$$= \dot{\omega}_1\omega_3\left[-\ell_1\nu_1^{\ell_1-1}\nu_2^{\ell_2}\omega_3 p(u_3,\nu_3)-\ell_3u_3^{\ell_3-1}\left(u_2^{\ell_2}P_{\ell_1-1}-\nu_2^{\ell_2}P_{\ell_1-1}+\nu_2^{\ell_2}P_{\ell_1-1}\right.\right.$$

$$\left.\left.-\ell_1\nu_1^{\ell_1-1}\nu_2^{\ell_2}\right)\right]\nu_0^{\ell_0}$$

$$= \dot{\omega}_1\omega_3\left\{-\ell_1\nu_1^{\ell_1-1}\nu_2^{\ell_2}\omega_3 p(u_3,\nu_3)-\ell_3u_3^{\ell_3-1}\left[\left(u_2^{\ell_2}-\nu_2^{\ell_2}\right)P_{\ell_1-1}+\nu_2^{\ell_2}\left(P_{\ell_1-1}-\ell_1\nu_1^{\ell_1-1}\right)\right]\right\}\nu_0^{\ell_0}$$

$$= \dot{\omega}_1\omega_3\left\{-\ell_1u_1^{\ell_1-1}\nu_2^{\ell_2}\omega_3\rho(u_3,\nu_3)-\ell_3u_3^{\ell_3-1}\omega_2P_{\ell_2-1}P_{\ell_1-1}-\ell_3u_3^{\ell_3-1}\nu_2^{\ell_2}\omega_1 p(u_1,\nu_1)\right]\nu_0^{\ell_0}$$

$$= -(u_{xt}-\nu_{xt})\omega_3\left[\ell_1\omega_3\nu_1^{\ell_1-1}\nu_2^{\ell_2}p(u_3,\nu_3)+\ell_3\omega_2u_3^{\ell_3-1}P_{\ell_2-1}P_{\ell_1-1}\right.$$

$$\left.\ell_3\omega_1u_3^{\ell_3-1}\nu_2^{\ell_2}p(u_1,\nu_1)\right]\nu_0^{\ell_0}$$

Finally rewriting $\dot{\omega}_2\omega_3\left(\ell_2\nu_2^{\ell_2-1}P_{\ell_3-1}-\ell_3P_{\ell_2-1}u_3^{\ell_3-1}\right)\nu_0^{\ell_0}\nu_1^{\ell_1}$

$$= \dot{\omega}_2\omega_3\left(\ell_2\nu_2^{\ell_2-1}P_{\ell_3-1}-\ell_2\ell_3\nu_2^{\ell_2-1}u_3^{\ell_3-1}+\ell_2\ell_3\nu_2^{\ell_2-1}u_3^{\ell_3-1}-\ell_3P_{\ell_2-1}u_3^{\ell_3-1}\right)\nu_0^{\ell_0}\nu_1^{\ell_1}$$

$$= \dot{\omega}_2\omega_3\left[\ell_2\nu_2^{\ell_2-1}\left(P_{\ell_3-1}-\ell_3u_3^{\ell_3-1}\right)+\ell_3u_3^{\ell_3-1}\left(\ell_2\nu_2^{\ell_2-1}-P_{\ell_2-1}\right)\right]\nu_0^{\ell_0}\nu_1^{\ell_1}$$

$$= -\dot{\omega}_2\omega_3\left[\ell_2\nu_2^{\ell_2-1}\omega_3 p(u_3,\nu_3)+\ell_3u_3^{\ell_3-1}\omega_2 p(u_2,\nu_2)\right]\nu_0^{\ell_0}\nu_1^{\ell_1}$$

On substituting we get

$$J_3 \le K\Big\{|\omega(t)|\,|\omega_1(t)|+|\omega(t)|\,|\omega_2(t)|+|\omega_1(t)|\,|\omega_2(t)|+|\omega(t)|\,|\omega_3(t)|$$

$$+|\omega_1(t)|\,|\omega_3(t)|+|\omega_2(t)|\,|\omega_3(t)|\Big\}+K_1\Big\{|\omega(t)|^2+|\omega_1(t)|^2+|\omega_2(t)|^2+|\omega_3(t)|^2\Big\}$$

$$+ K_2 \int_0^T \Big\{ |\omega(t)|^2 + |\omega_1(t)|^2 + |\omega_2(t)|^2 + |\omega_3(t)|^2 + |\omega(t)||\omega_1(t)| + |\omega(t)||\omega_2(t)|$$

$$+ |\omega_1(t)||\omega_2(t)| + |\omega(t)||\omega_3(t)| + |\omega_1(t)||\omega_3(t)| + |\omega_2(t)||\omega_3(t)| \Big\} dt$$

$$+ K_3 \int_0^T \Big\{ |\omega_1(t)|^2 + |\omega(t)||\omega_3(t)| + |\omega_2(t)|^2 + |\omega(t)||\omega_2(t)| + |\omega_1(t)||\omega_3(t)|$$

$$+ |\omega_2(t)||\omega_3(t)| + |\omega_3(t)|^2 + |\omega(t)||\omega_3(t)| + |\omega_2(t)|^2 + |\omega_1(t)||\omega_2(t)|$$

$$+ |\omega_3(t)|^2 + |\omega_2(t)||\omega_3(t)| + |\omega_1(t)||\omega_3(t)| + |\omega_3(t)|^2 + |\omega_2(t)||\omega_3(t)| \Big\} dt$$

Applying (2.1.6) and [1.2.6] as before we obtain

$$J_3 \le C \left\{ |\omega(t)|^2 + |D^3\omega(t)|^2 \right\} + K' \int_0^T \left\{ |\omega(t)|^2 + |D^3\omega(t)|^2 \right\} dt.$$

(e) From the term $2(-1)^{m-2}D^{m-2}(uD^{m-2}u) + (D^{m-2}u)^2$ of $G_m(u)$ in (4.2.3) we find

$$G_m(u) - G_m(\nu) = 2(-1)^{m-2}D^{m-2}\left[uD^{m-2}u - \nu D^{m-2}\nu\right] + (D^{m-2}u)^2 - (D^{m-2}\nu)^2$$

$$= 2(-1)^{m-2}D^{m-2}\left[uD^{m-2}u - uD^{m-2}\nu + uD^{m-2}\nu - \nu D^{m-2}\nu\right]$$

$$+ \left(D^{m-2}u - D^{m-2}\nu\right)\left(D^{m-2}u + D^{m-2}\nu\right)$$

$$= 2(-1)^{m-2}D^{m-2}\left(uD^{m-2}\omega + \omega D^{m-2}\nu\right) + D^{m-2}\omega\left(D^{m-2}u + D^{m-2}\nu\right)$$

Now

$$J_4 = \int_0^T\int_0^1 \omega_t\left(G_m(u) - G_m(\nu)\right) dx\, dt$$

$$= \left|\int_0^T\int_0^1 \omega_t\left\{2(-1)^{m-2}D^{m-2}\left(uD^{m-2}\omega + \omega D^{m-2}\nu\right) + \left(D^{m-2}\omega\right)\left(D^{m-2}u + D^{m-2}\nu\right)\right\} dx dt\right|$$

$$= \left|\int_0^T\int_0^1 \left\{2(-1)^{m-2}\omega_t D^{m-2}\left(uD^{m-2}\omega\right) + 2(-1)^{m-2}\omega_t D^{m-2}\left(\omega D^{m-2}\nu\right)\right.\right.$$

$$+ \omega_t (D^{m-2}\omega)(D^{m-2}u + D^{m-2}\nu)\Big\}\, dx\, dt \Big|$$

But $$(-1)^{m-2}\int_0^{T}\int_0^{1} \omega_t\, D^{m-2}(uD^{m-2}\omega)\, dx\, dt$$

$$= (-1)^{m-2}(-1)^{m-2}\int_0^{T}\int_0^{1} D^{m-2}\omega_t\, uD^{m-2}\omega\, dx\, dt$$

$$= \int_0^{T}\int_0^{1} uD^{m-2}\omega\, \frac{\partial}{\partial t}\left(D^{m-2}\omega\right) dx\, dt$$

$$= \int_0^{T}\int_0^{1} \frac{u}{2} D_t (D^{m-2}\omega)^2\, dx\, dt$$

Also $$\int_0^{T}\int_0^{1} 2(-1)^{m-2}\omega_t D^{m-2}(\omega D^{m-2}\nu)\, dx\, dt$$

$$= 2(-1)^{m-2}(-1)^{m-2}\int_0^{T}\int_0^{1} D^{m-2}\omega_t\, \omega D^{m-2}\nu\, dx\, dt$$

$$= 2\int_0^{T}\int_0^{1} D^{m-2}\omega_t\, \omega D^{m-2}\nu\, dx\, dt$$

Substituting back we get

$$J_4 = \Big|\int_0^{T}\int_0^{1}\Big\{2.\frac{u}{2} D_t\left(D^{m-2}\omega\right)^2 + 2D^{m-2}\omega_t\, \omega D^{m-2}\nu$$

$$+ \omega_t\left(D^{m-2}\omega\right)\left(D^{m-2}u + D^{m-2}\nu\right)\Big\}\, dx\, dt \Big|$$

$$= \Big|\int_0^{T}\int_0^{1}\Big\{uD_t (D^{m-2}\omega)^2 + 2D^{m-2}\omega_t\, \omega D^{m-2}\nu + \omega_t (D^{m-2}\omega)(D^{m-2}u + D^{m-2}\nu)\Big\} dx\, dt \Big|$$

$$= \left| \int_0^1 u\,(D^{m-2}\omega)^2 dx - \int_0^T\!\!\int_0^1 u_t\,(D^{m-2}\omega)^2 dx\ dt + \int_0^1 2D^{m-2}\omega\omega D^{m-2}\nu dx \right.$$

$$\left. - \int_0^T\!\!\int_0^1 2D^{m-2}\omega D_t\,(\omega D^{m-2}\nu) dx\ dt + \int_0^T\!\!\int_0^1 \omega_t\,(D^{m-2}\omega)\,(D^{m-2}u+D^{m-2}\nu) dx\ dt \right|$$

$$= \left| \int_0^1 \left\{ u\,(D^{m-2}\omega)^2 + 2\omega D^{m-2}\omega D^{m-2}\nu \right\} dx + \int_0^T\!\!\int_0^1 \left\{ \omega_t\,(D^{m-2}\omega)\,(D^{m-2}u+D^{m-2}\nu) \right.\right.$$

$$\left.\left. - u_t\,(D^{m-2}\omega)^2 - 2\omega D^{m-2}\omega\,(D^{m-2}\nu)_t - 2\omega_t D^{m-2}\omega D^{m-2}\nu \right\} dx\ dt \right|.$$

$$= \left| \int_0^1 \left\{ u\,(D^{m-2}\omega)^2 + 2\omega D^{m-2}\omega D^{m-2}\nu \right\} dx \right.$$

$$+ \int_0^T\!\!\int_0^1 \left\{ \omega_t\,(D^{m-2}\omega)\,(D^{m-2}u+D^{m-2}\nu-2D^{m-2}\nu) - u_t\,(D^{m-2}\omega)^2 \right.$$

$$\left.\left. - 2\omega\, D^{m-2}\omega\,(D^{m-2}\nu)_t \right\} dx\ dt \right|$$

$$= \left| \int_0^1 \left\{ u\,(D^{m-2}\omega)^2 + 2\omega D^{m-2}\omega D^{m-2}\nu \right\} dx + \int_0^T\!\!\int_0^1 \omega_t\,(D^{m-2}\omega)^2 - u_t\,(D^{m-2}\omega)^2 \right.$$

$$\left. - 2\,\omega D^{m-2}\omega\,(D^{m-2}\nu)_t \Big\} dx\ dt \right|$$

Therefore $J_4 \leq K \left\{ |D^{m-2}\omega(t)|^2 + |\omega(t)|\,|D^{m-2}\omega(t)| \right\}$

$$+ K \int_0^T\!\!\int_0^1 \left\{ |D^{m-2}\omega(t)|^2 + |D^{m-2}\omega(t)|^2 + |\omega(t)|\,|D^{m-2}\omega(t)| \right\} dt$$

Applying (2.1.6) and [1.2.6] we obtain

$$J_4 \leq C\left\{|\omega(t)|^2+|D^{m-2}\omega(t)|^2\right\}+C\int_0^T\left\{|\omega(t)|^2+|D^{m-2}\omega(t)|^2\right\}dt.$$

Returning to (4.2.3) we take the term $D^{m-1}\omega$ of $G_m(u) - G_m(\nu)$ to one side of (4.2.3), and estimating the remaining terms as in the previous sections as follows :

$$G_m(u) - G_m(\nu) = D^{2m-2}\omega + 2(-1)^{m-2}D^{m-2}(uD^{m-2}u-\nu D^{m-2}\nu)$$

$$+ (D^{m-2}u-D^{m-2}\nu)(D^{m-2}u + D^{m-2}\nu) + \dots\dots$$

$$\omega_t(G_m(u)-G_n(\nu)) = \omega_t D^{2m-2}\omega+2(-1)^{m-2}\omega_t D^{m-2}(uD^{m-2}u-\nu D^{m-2}\nu)$$

$$+ \omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu) + \dots\dots\dots$$

and

$$\int_0^1 \omega_t(G_m(u)-G_m(\nu))\,dx = \int_0^1\left\{\omega_t D^{2m-2}\omega+2(-1)^{m-2}\omega_t D^{m-2}(uD^{m-2}u\right.$$

$$\left.-\nu D^{m-2}\nu)+\omega_t D^{m-2}\omega(D^{m-2}u+D^{m-2}\nu)+\dots\dots\right\}dx = 0.$$ This implies

$$-\int_0^T\int_0^1 \omega_t D^{2m-2}\omega\,dxdt = \int_0^T\int_0^1\left\{2(-1)^{m-2}\omega_t D^{m-2}(uD^{m-2}u - \nu D^{m-2}\nu)\right.$$

$$\left.+\omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu) + \dots\dots + \dots\dots\right\} dx\,dt$$

Now $$-\int_0^T\int_0^1 \omega_t D^{2m-2}\omega\,dxdt= (-1)(-1)^{m-1}\int_0^T\int_0^1 D^{m-1}(\omega_t)D^{m-1}\omega dx\,dt$$

$$= (-1)^m\int_0^T\int_0^1 D^{m-1}\omega\,\frac{\partial}{\partial t}(D^{m-1}\omega)dx\,dt$$

$$= (-1)^m \int_0^{T}\int_0^{1} \tfrac{1}{2} \frac{\partial}{\partial t} (D^{m-1}\omega)^2 dx\ dt$$

$$= (-1)^m \int_0^1 \tfrac{1}{2} (D^{m-1}\omega)^2 dx$$

$$|-\int_0^{T}\int_0^{1} \omega_t D^{2m-2}\ \omega\ dx\ dt \leq \tfrac{1}{2}\ |D^{m-1}\omega(t)|^2$$

Also $\displaystyle\int_0^{T}\int_0^{1} \left\{2(-1)^{m-2}\omega_t D^{m-2}(uD^{m-2}u - \nu D^{m-2}\nu) + \omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\right\} dx dt$

$$= \int_0^{T}\int_0^{1} \left\{2(-1)^{m-2}(-1)^{m-2} D^{m-2}\omega_t (uD^{m-2}u - \nu D^{m-2}\nu) + \omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\right\} dx\, dt$$

$$= \int_0^{T}\int_0^{1} \Big\{2D^{m-2}\omega_t (uD^{m-2}u - uD^{m-2}\nu + uD^{m-2}\nu - \nu D^{m-2}\nu)$$

$$+ \omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\Big\}\ dx\ dt\ .$$

$$= \int_0^{T}\int_0^{1} \Big\{2D^{m-2}\omega_t \left[u(D^{m-2}u - D^{m-2}\nu) + (u-\nu)D^{m-2}\nu\right]$$

$$+ \omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\Big\}\ dx\ dt$$

$$= \int_0^{T}\int_0^{1} \left\{2D^{m-2}\omega_t (uD^{m-2}\omega + \omega D^{m-2}\nu) + \omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\right\} dx\ dt$$

$$= \int_0^{T}\int_0^{1} 2D^{m-2}\omega_t uD^{m-2}\omega dx dt + \int_0^{T}\int_0^{1} 2D^{m-2}\omega_t\ \omega D^{m-2}\nu\ dx\ dt$$

$$+ \int_0^{T}\int_0^{1} \omega_t D^{m-2}\omega(D^{m-2}u + D^{m-2}\nu)\ dx\ dt.$$

$$= \int_0^{T_1}\!\!\int_0^1 2\cdot\frac{u}{2}\frac{\partial}{\partial t}(D^{m-2}\omega)^2 dx\ dt + \int_0^1 \omega D^{m-2}\omega D^{m-2}\nu\ dx - \int_0^{T_1}\!\!\int_0^1 D^{m-2}\omega D_t\left[\omega D^{m-2}\nu\right]\ dx\,dt$$

$$+ \int_0^{T_1}\!\!\int_0^1 \omega_t\ D^{m-2}\omega\,(D^{m-2}u + D^{m-2}\nu)dx\ \ dt.$$

$$= \int_0^1 u\,(D^{m-2}\omega)^2 dx - \int_0^{T_1}\!\!\int_0^1 u_t\,(D^{m-2}\omega)^2 dx\ \ dt$$

$$+ 2\int_0^1 \omega D^{m-2}\omega D^{m-2}\nu dx - 2\int_0^{T_1}\!\!\int_0^1 D^{m-2}\omega\omega D_t\left[D^{m-2}\nu\right]dx dt - 2\int_0^{T_1}\!\!\int_0^1 D^{m-2}\omega\omega_t D^{m-2}\nu\ dx\ dt + \int_0^{T}\!\!\int_0^1 \omega_t D^{m-2}\omega$$

$$\times\ (D^{m-2}u + D^{m-2}\nu)dx\,dt.$$

$$= \int_0^1 u\,(D^{m-2}\omega)^2 dx + \int_0^1 2\omega D^{m-2}\omega D^{m-2}\nu dx - \int_0^{T_1}\!\!\int_0^1 u_t\,(D^{m-2}\omega)^2\ dx\ dt$$

$$- 2\int_0^{T_1}\!\!\int_0^1 D^{m-2}\omega\ \omega D_t\left[\ D^{m-2}\nu\ \right]dx\ dt + \int_0^{T_1}\!\!\int_0^1 \Big\{ - 2D^{m-2}\omega\ \omega_t D^{m-2}\nu$$

$$+ \omega_t D^{m-2}\omega D^{m-2}u + \omega_t D^{m-2}\omega D^{m-2}\nu \Big\}\ dx\ dt$$

Now taking the last integral, that is,

$$\int_0^{T_1}\!\!\int_0^1 \Big\{ -2D^{m-2}\omega\ \ \omega_t D^{m-2}\nu + \omega_t D^{m-2}\omega D^{m-2}u + \omega_t D^{m-2}\omega D^{m-2}\nu \Big\}\ dx\ \ dt$$

$$= \int_0^{T_1}\!\!\int_0^1 \Big\{\ \omega_t D^{m-2}\omega D^{m-2}u - \omega_t D^{m-2}\omega D^{m-2}\nu\ \Big\}\ dx\ \ dt\ \ .$$

$$= \int_0^{T_1}\!\!\int_0^1 \omega_t D^{m-2}\omega\,(D^{m-2}u - D^{m-2}\nu)\Big\}dx\ \ dt$$

$$= \int_0^{T_1}\int_0^1 \omega_t (D^{m-2}\omega)^2 dx\ dt\ ;$$

$$= \int_0^{T_1}\int_0^1 (u_t - \nu_t)(D^{m-2}\omega)^2\ dx\ dt.$$

On substituting the above integral becomes

$$\int_0^1 u(D^{m-2}\omega)^2 dx + \int_0^1 2\omega D^{m-2}\omega D^{m-2}\nu dx - \int_0^{T_1}\int_0^1 u_t (D^{m-2}\omega)^2 dx\ dt$$

$$- 2\int_0^{T_1}\int_0^1 D^{m-2}\omega\ \omega D_t\left[D^{m-2}\nu\right]dxdt + \int_0^{T_1}\int_0^1 (u_t - \nu_t)(D^{m-2}\omega)^2 dx\ dt$$

$$\leq K_1\left\{|D^{m-2}\omega(t)|^2 + |\omega(t)||D^{m-2}\omega(t)|\right\}$$

$$K_2\int_0^T\left\{|D^{m-2}\omega(t)|^2 + |\dot{\omega}(t)||D^{m-2}\omega(t)| + |D^{m-2}\omega(t)|^2\right\}dt.$$

Thus, we have established

$$\tfrac{1}{2}|D^{m-1}\omega(t)|^2 \leq K_1\left\{|D^{m-2}\omega(t)|^2 + |\omega(t)||D^{m-2}\omega(t)|\right\}$$

$$+ K_2\int_0^T\left\{|D^{m-2}\omega(t)|^2 + |\omega(t)||D^{m-2}\omega(t)|\right\}dt$$

Applying (2.1.6) and [1.2.6] as usual we obtain

$$\tfrac{1}{2}|D^{m-1}\omega(t)|^2 \leq C_\varepsilon|\omega(t)|^2 + \varepsilon|D^{m-1}\omega(t)|^2 + K_3\int_0^T\left(|\omega(t)|^2 + |D^{m-1}\omega(t)|^2\right)dt\ .$$

Choosing $\varepsilon = \frac{1}{2}$ and combining with (4.2.2) we get

$$\tfrac{1}{2}|D^{m-1}\omega(t)|^2 \leq C_\varepsilon\left\{C_\alpha\int_0^T\left\{|\omega(t)|^2 + |D^{m-1}\omega(t)|^2\right\}dt + \tfrac{1}{2}|D^{m-1}\omega(t)|^2\right.$$

$$+ K_3 \int_0^T \left(|\omega(t)|^2 + |D^{m-1}\omega(t)|^2\right)dt$$

$$\leq (C_\varepsilon C_\alpha + K_3) \int_0^T \left(|\omega(t)|^2 + |D^{m-1}\omega(t)|^2\right)dt$$

$$\leq C_\beta \int_0^T \left(|\omega(t)|^2 + |D^{m-1}\omega(t)|^2\right)dt$$

Combining with (4.2.2) we have shown

$$|\omega(t)|^2 + |D^{m-1}\omega(t)|^2 \leq (C_\alpha + C_\beta) \int_0^T \left(|\omega(t)|^2 + |D^{m-1}\omega(t)|^2\right)dt$$

$$\leq C \int_0^T \left(|\omega(t)|^2 + |D^{m-1}\omega(t)|^2\right)dt.$$

Applying Gronwall's lemma, [1.2.10], gives

$$|\omega(t)|^2 + |D^{m-1}\omega(t)|^2 \leq C \int_0^t \left\{|\omega(s)|^2 + |D^{m-1}\omega(s)|^2\right\}ds$$

$$\leq 0 . e^{c\int_0^t ds}$$

$$\leq 0$$

That is $|\omega(t)|^2 + |D^{m-1}\omega(t)|^2 \leq 0.$

This completes the proof of the theorem.

APPENDIX

Diagonalization Argument.

Let us divide the interval [O,T] into n subintervals $0 = T_0 \leq T_1 < T_2 < \cdots < T_n = T$, and such that $t_i \in [T_i - T_{i-1}]$, $i \leq n$. Let $\{t_i\}$, $i = 1, 2, 3, \cdots$ be arranged in a sequence. Since $\left\{u^\ell (t_1)\right\}$ is bounded, there exists a subsequence, which we shall denote by $\left\{u^{1,k}\right\}$ such that $\left\{ u^{1,k}(t_1)\right\}$ converges as $k \to \infty$.

Let us now consider sequences S_1, S_2 S_3, $S_4, \ldots$, which we represent by the array

S_1 : $u^{1,1}$ $u^{1,2}$ $u^{1,3}$ $u^{1,4}$ ....

S_2 : $u^{2,1}$ $u^{2,2}$ $u^{2,3}$ $u^{2,4}$ ....

S_3 : $u^{3,1}$ $u^{3,2}$ $u^{3,3}$ $u^{3,4}$ ....

S_4 : $u^{4,1}$ $u^{4,2}$ $u^{4,3}$ $u^{4,4}$ ....

. . . .

and which have the following properties :

(a) S_n is a subsequence of S_{n-1}, for n = 2, 3, 4, ...

(b) $\left\{u^{\ell,k}(t_\ell)\right\}$ converges, as $k \to \infty$ (the boundedness of $\left\{u^\ell (t_\ell)\right\}$ makes it possible to choose S_n in this way);

(c) The order in which the functions appear is the same in each sequence, i.e., if one function precedes another in S_1, they are in the same relation in every S_n, until one or the other is deleted. Hence, when going from one row in the above array to the next below, functions may move to the left but never to the right.

We now go down the diagonal of the array; i.e., we consider the sequence

$$S: \quad u^{1,1} \quad u^{2,2} \quad u^{3,3} \quad u^{4,4} \;\ldots\ldots\ldots\ldots$$

By (c), the sequence S (except possibly its first n - 1 terms) is a subsequence of S_n, for n = 1, 2, 3,..... . Hence (b) implies that $\left\{u^{\ell,\ell}(t_i)\right\}$ converges, as $\ell \to \infty$, for every $t_i \in [0, T]$.

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