A STUDY OF FUZZY COMPATIBILITY RELATIONS AND THEIRAPPLICATIONS

BY

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DEPARTMENT OF MATHEMATICS

FACULTY OF SCIENCE

AHMADU BELLO UNIVERSITY, ZARIA

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A DISSERTATION SUBMITTED TO THE SCHOOL OF POSTGRADUATE STUDIES, AHMADU BELLO UNIVERSITY, ZARIA, IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF MASTER OF SCIENCE IN MATHEMATICS

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DECLARATION

I declare that the work in this dissertation entitled"A Study of Fuzzy Compatibility Relations and Their Applications" has been performed by me in the department of mathematics under the supervision of Dr. A.M. Ibrahim and Professor D. Singh.The information derived from literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any university or institution.

YASHIM Jagaba Mathias

……………………………..	………………………….……	…. …………………
Name of Student	Signature	Date

CERTIFICATION

This dissertation titled "A Study of Fuzzy Compatibility Relationsand Their Applications" by YASHIM Jagaba Mathias meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to my loving father late Mr. David Yabakyak Yashim (May his soul rest in peace) and my loving and caring mother Mrs. Veronica Hannatu Yashim.

ACKNOWLEDGEMENTS

I remain grateful and indebted to God for giving me the strength to carry out this research work to its conclusion.The completion of this dissertation would not have been possible without the patience, motivation, enthusiasm, and valuable contribution of my supervisor, Dr. A. M. Ibrahim. I could not have imagined having a better supervisor and mentor for my postgraduate work. May God reward you abundantly.I wish to express my sincere gratitude to my other supervisor Professor D. Singh for his fatherly contributions.

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ABSTRACT

In this dissertation, various concepts for comparing fuzzy objects such as similarity, dissimilarity, symmetric similarity, relative similarity, multi-dimensional, and multi-attributes were studied. Some existing models such as Jaccard, Simple Matching Co-efficient, Vector, and Tversky were closely studied. The similarity measures introduced by Tversky(1977) and modified by (Dubois and Prade, 1980) using the cardinality of fuzzy sets and scalar evaluators and their operations such as T-norms (T_1, T_2, T_3) and T-conorms(S_1, S_2, S_3) were systematized. Finally,a new approach (Set Theoretic Measures) for comparing fuzzy objects by taking into consideration the degree of inclusion, partial matching and similarity was presented, some properties of the modified model elaborated and some areas of applications were identified.

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Background of the Study

Fuzzy set was introduced by Zadeh in (1965) to represent or manipulate data and information possessing uncertainties.The theory of fuzzy set has advanced in a variety of ways and in many disciplines. Applications of this theory are found in artificial intelligence, computer science, operational research, pattern recognition, and robotics (Dubois, 1980). To handle the problems involving imprecise concepts, the conventional methods of set theory are found insufficient. In order to overcome shortcomings of the conventional approaches, development of fuzzy set theory has been found most successful in this direction.

One of the most fundamental notions in pure and applied science is the concept of relation. Science has been described as the discovery of relations between objects, states and events (Peterson, 1976).Fuzzy relations generalize the concept of relations in the same manner as fuzzy sets generalize the fundamental idea of sets.A relation is a mathematical description of a situation where certain elements of sets are related to one another in some way. Fuzzy relations are significant concepts in fuzzy theory and have been widely used in many fields such as fuzzy clustering, fuzzy control and uncertainty reasoning.

The theory of compatibility relation has been studied extensively in mathematics along with its applications in diverse fields. The term compatibility relation is used to encompass various types of comparisons frequently made between objects and concepts. The degree to which two objects are compatible is a fundamental component of human reasoning and consequently is critical in the development of automated diagnosis, information retrieval and decision systems. Assessment of compatibility relation has played an important role in diverse disciplines such as taxonomy, psychology, and the social sciences. Each discipline has proposed methods for quantifying compatibility judgments suitable for its particular applications. Applications of compatibility relation in various areas include expert systems, information retrieval, and intelligent database system etc., (Valerie, 2002).

Several measures of similarity among fuzzy sets have been proposed in literature. The motivation behind these measures is both geometric and set-theoretic. Geometric models dominate the theoretic analysis of similarity measures. Objects in these models are represented as points in a coordinate space, and the metric between the respective points is considered to be a measure for deciding the degree of similarity or dissimilarity among the objects. In most cases the Euclidean distance is used to define such a measure. In the set-theoretic approaches a different model is used, which is based on the concept of non dimensional and non metric similarity relation (Bashon, 2011).

Similarity measures are specific functions used to approximate the degree to which two compared objects are similar to one another. The functions are required to fulfill specific similarity conditions or axioms. The use of compatibility measures depends on the type of data characterizing the objects being compared. Data describing those objects could be categorical or numerical which can be presented in set representations such as fuzzy set.Based on literature reviews, various forms of similarity measures involving fuzzy sets have been proposed depending on the context in which they are to be applied. There is no unique way of determining the degree to which two such sets are compatible to one another(Suliaman and Mohamad, 2012). However, in this dissertation we narrow down our research to a special kind of relation called compatibility relation in fuzzy set context and modify the model proposed by Tverskyin order to develop a new model which shows the degree of inclusion, partial matching and similarity of fuzzy objects.

1.2 Statement of the Research Problem

We intend to investigate compatibility relation in fuzzy context and modify some of the applications of Tversky parameterized ratio model using the cardinality of fuzzy sets and other functions such as the scalar evaluators and their operators such asT-norms(T_1, T_2, T_3) and T-conorms(S_1, S_2, S_3). We propose a new approach for comparing fuzzy objects which include the degree of inclusion, partial matching and similarity and discuss some of the properties of the modified model.

1.3 Aim and Objectives of the Study

The aim of this research is to investigate compatibility relationin fuzzy context and modify Tversky model. To achieve this, the objectives are to:

i. investigate various concepts related to similarity, dissimilarity, symmetric similarity, relative similarity, multi-dimensional and multi-attribute of fuzzy objects,

ii. study existing ratio models, in particular Jaccardunparameterized ratio model and Tversky parameterized ratio model, and present a new model in terms of inclusion, partial matching, and similarity, and

iii. discuss some properties of the new model of similarity measures in relation to the degree of inclusion, partial matching and similarity of fuzzy objects.

1.4 Methodology

An up-to-date review of literatureson fuzzy relations, compatibility relation in fuzzy set context, and similarity measures introduced by Tversky which would be of help in modifying Tversky models to show the degree of inclusion, partial matching and similarity of fuzzy objects was conducted.

1.5 Definition of Terms

These definitions are adopted from different sources [(Dubois and Prade, 1980), (Kaufmann, 1975), (Rosenfeld, 1975)].

Classical set

A classical set or crisp set is normally defined as the collection of elements or objects that can be finite, countable, or uncountable.

Such a classical set Acan be described as in different ways;

i. By stating the condition for membership $(A = \{x \in N \wedge x \leq 5\})$

ii. Define the elements by listing the characteristic function, in which 1 indicates membership and 0 non membership. That is $\mu_A(x) = 1$ if and only if $x \in A$ and $\mu_A(x) = 0$ if and only if $x \notin A$.

Fuzzy set

Let X be a collection of objects, the fuzzy set A in X is a set of ordered pairs.

$$A = \{(x, \mu_A(x)) \,|\, x \in X\}$$

$\mu_A$is called the membership function which maps each element in the universal set X to the membership space [0,1].

That is, $\mu_A : X \to [0,1]$

For example, a house owner wants to classify the house he offers to his clients. One indicator of comfort of these houses is the number of bedrooms in it. Let X={1, 2, 3, …, 10} be the set of available type of houses described by$x \in X$ where $x =$ number of bedrooms in the house. The fuzzy set "comfortable type of houses" for a four persons family may be described as

A= {(1, 0.2),(2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)}

A fuzzy set is denoted by a set of ordered of pairs, the first element of which denotes the element and the second the degree of membership.

Support of a fuzzy set

The support of a fuzzy set A, is the classical set of all $x \in X$ such that $\mu_A(x) > 0$.

In the above example**,** the support set of a fuzzy set $A = (1, 2, 3, 4, 5, 6)$

α- level set

The crisp set of elements that belong to the fuzzy set A at least to the degree α is

$A_\alpha = \{x \in X \,|\, (\mu_A(x)\,) \geq \alpha\}$.

For example,

i. $A_{0.2} = \{1, 2, 3, 4, 5, 6\}$

ii. $A_{0.5} = \{2, 3, 4, 5\}$

iii. $A_{0.8} = \{3, 4\}$.

Strong α-level set

The strong α-level set is known as strong α-cut and is defined by

$\acute{A}_{\alpha} = \{x \in X \,|\, (\mu_A(x)) > \alpha\}$.

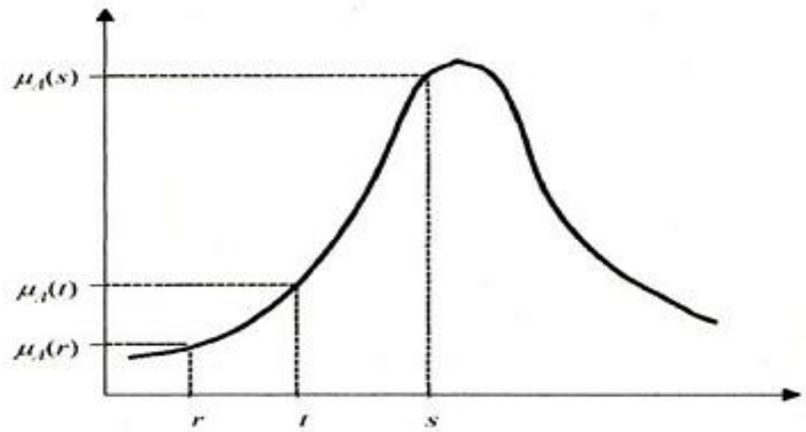
For example, the strong α- level set for $\alpha = 0.8\ is\ A_{0.8} = \{4\}$

Convexity of a fuzzy set

A fuzzy set A is convex if $\mu_A(t) \geq \min\{\mu_A(r), \mu_A(s)\}$ where $t = (\lambda r + (1 - \lambda)s), r, s, t \in X$ and $\lambda \in [0,1]$.

Alternatively, a fuzzy set is convex if all α- level sets are convex.

For example



Convex fuzzy set $\mu_A(t) \geq \mu_A(r)$

Cardinality

For a finite fuzzy set A, the cardinality denoted by |A| is defined as

$$|A| = \sum_{x \in X} \mu_A(x)$$

$||A|| = \frac{|A|}{|X|}$ is called the relative cardinality of A.

For example, for a fuzzy set "comfortable type of houses" for a four person family, the cardinality is

$|A| = 0.2 + 0.5 + 0.8 + 1 + 0.7 + 0.3 = 3.5.$

The relative cardinality is $||A|| = \frac{3.5}{10} = 0.35$

Standard operations of fuzzy set

Complement set $\bar{A}$, union $A \cup B$, and intersection $A \cap B$represent the standard operations of fuzzy sets as follows

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

$$\mu_{A \cup B}(x) = \max[\mu_A(x), \mu_B(x)]$$

$$\mu_{A \cap B}(x) = \min[\mu_A(x), \mu_B(x)]$$

We describe these concepts in details as follows:

Fuzzy complement

The fuzzy complement of a fuzzy set A, is denoted as $\bar{A}$ defined by

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

Complement function C is designed to map the membership function $\mu_A(x)$ of a fuzzy set A to [0, 1] and the mapped value is written as $C(\mu_A(x))$

Properties of fuzzy complement function;

i. $C(0) = 1, C(1) = 0$ (Boundary conditions).

ii. $a, b \in [0,1] if\ a < b, then\ C(a) \geq C(b)$ (Monotonic non- increasing).

iii. C is a continuous function.

iv. C is involutive i.e. $\bar{C}(a) = a\ for\ all\ a \in [0,1]$

Fuzzy union

$\cup\,[\mu_A(x), \mu_B(x)] = \max[\mu_A(x), \mu_B(x)]$ or

$$\mu_{A\cup B}(x) = \max[\mu_A(x), \mu_B(x)]$$

The fuzzy union of two sets A and B can be expressed by a function of the form

$$\cup: [0,1] \times [0,1] \rightarrow [0,1]$$

The membership degree of union $A \cup B$ arises from the union function

Properties of fuzzy union function;

i. $\cup\,(0,0) = 0,\ \cup\,(0,1) = 1,\ \cup\,(1,0) = 1,\ \cup\,(1,1) = 1$ (boundary conditions).

ii. $\cup\,(a, b) = \cup\,(b, a)$ commutativity.

iii. $if\ a \le \acute{a}\ and\ b \le \acute{b}\ then \cup\,(a, b) \le \cup\,(\acute{a}, \acute{b})$.

iv. $\cup\,(\cup\,(a, b), c) = \cup\,(a, \cup\,(b, c))$.

v. $\cup\ is\ a\ continuous\ function$.

vi. $\cup\,(a, a) = a$ (Idempotency).

Fuzzy intersection

$$I[\mu_A(x), \mu_B(x)] = \min[\mu_A(x), \mu_B(x)] \ or$$

$$\mu_{A\cap B}(x) = \min[\mu_A(x), \mu_B(x)]$$

The intersection of two fuzzy sets A and B is defined by the function;

$$I: [0,1] \times [0,1] \to [0,1]$$

Properties of fuzzy set intersection function;

i. $I(1,1) = 1, I(1,0) = 0, I(0,1) = 0 \ and \ I(0,0) = 0$ (Boundary conditions).

ii. $I(a, b) = I(b, a)$. Commutativity.

iii. $if \ a \leq \acute{a} \ and \ b \leq \acute{b} \ then \ I(a, b) \leq I(\acute{a}, \acute{b})$. $I$is a monotonic non decreasing function.

iv. $I(I(a, b), c) = I\big(a, I(b, c)\big)$. Associativity.

v. I is a continuous function

vi. $I(a, a) = a$. (Idempotency).

Properties of complement, union and intersection

Let A and B be two fuzzy sets with membership function $A(x) \ and \ B(y)$ respectively, then the following properties hold:

(1) Commutativity

i. $A \cup B = B \cup A$

ii. $A \cap B = B \cap A$

Proof

i. $\max\{A(x), B(y)\} = \max\{(B(y), A(x)\} = A(x) \vee B(y) = B(y) \vee A(x)$.

This can be verified by considering the two possibilities as follows;

$$A(x) < B(y) \; or \; A(x) > B(y).$$

That is, if $A(x) < B(y)$, we have $A(x) \vee B(y) = B(y) \vee A(x) = B(y)$.

Also if $A(x) > B(y)$, we have $A(x) \vee B(y) = B(y) \vee A(x) = A(x)$.

ii. $\min\{A(x), B(y)\} = \min\{B(y), A(x)\} = A(x) \wedge B(y) = B(y) \wedge A(x)$

$$A(x) < B(y) \; or \; A(x) > B(y).$$

If $A(x) < B(y)$, then we have $A(x) \wedge B(y) = B(y) \wedge A(x) = A(x)$.

Also if $A(x) > B(y)$, then $A(x) \wedge B(y) = B(y) \wedge A(x) = B(y)$.

(2) Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C, \quad A \cap (B \cap C) = (A \cap B) \cap C.$$

Proof; the proof is almost the same as above, this also apply to properties (3) to (7).

(3) Idempotency:

$$A \cup A = A, \quad A \cap A = A.$$

(4) Distributivity:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C), \quad A \cap (B \cup C) = (A \cap B) \cup (A \cap C).$$

(5) $A \cap \emptyset = \emptyset, \ A \cup X = X.$

(6) Identity: $(A \cup \emptyset = A, \ A \cap X = A.$

(7) Absorption:

$$A \cap (A \cup B) = A, \quad A \cup (A \cap B) = A.$$

(8) De Morgan's laws:

i. $(\overline{A \cup B}) = \bar{A} \cap \bar{B}$

ii. $\overline{(A \cap B)} = \bar{A} \cup \bar{B}.$

Proof; since we know that $\bar{A} = 1 - A, and\ \bar{B} = 1 - B$. Therefore, we have also two cases

$$A(x) < B(y)\ or\ A(x) > B(y)$$

For $A(x) < B(y)$ we have

$1 - \max\{A(x), B(y)\} = 1 - \{A(x) \vee B(y)\} = \min\{1 - A(x), 1 - B(y)\} = 1 - B(y)$

Also for $A(x) > B(y), we\ have$

$1 - \max\{A(x), B(y)\} = 1 - \{A(x) \vee B(y)\} = 1 - A(x)$. Proved

(9) Involution:

$$\bar{\bar{A}} = A.$$

Proof;

$$\{1 - (1 - A(x))\} = \{1 - 1 + A(x)\} = A(x).$$

$$\Rightarrow \bar{\bar{A}} = A$$

(10) Equivalence formula:

$$(\bar{A} \cap B) \cap (A \cup \bar{B}) = (\bar{A} \cap \bar{B}) \cup (A \cap B).$$

(11) Symmetrical formula:

$$(\bar{A} \cap B) \cup (A \cap \bar{B}) = (\bar{A} \cup \bar{B}) \cap (A \cup B)$$

Nonstandard operations of a fuzzy set

The following are the nonstandard operators of fuzzy set;

Union operations

i. Probabilistic sum $A\hat{+}B$ (Algebraic sum)

$$\forall x \in X, \qquad \mu_{A\hat{+}B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)$$

$$\forall x \in X, \qquad \mu_{A\hat{+}B}(x) = \mu_{B\hat{+}A}(x) \;\; commutativity$$

Since $\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x) = \mu_B(x) + \mu_A(x) - \mu_B(x)\mu_A(x)$

$$\forall x \in X, \quad \mu_{\overline{A\hat{+}B}}(x) = 1 - [\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)] DeMorgan's law$$

ii. Bounded sum A⊕B (bold union)

$$\forall x \in X, \qquad \mu_{A\oplus B}(x) = \min[1, \mu_A(x) + \mu_B(x)]$$

$$\forall \boldsymbol{x} \in \boldsymbol{X}, \qquad \mu_{A\oplus B}(x) = \mu_{B\oplus A}(x) \;\; commutativity$$

Since$\min[1, \mu_A(x) + \mu_B(x)] = \min[1, \mu_B(x) + \mu_A(x)]$

$$\forall x \in X, \quad \mu_{\overline{A\oplus B}}(x) = 1 - \{\min[1, \mu_A(x) + \mu_B(x)]\} \;\; De\ Morgan's\ law$$

iii. Drastic sum $(A \dot{\cup} B)$

$$\forall x \in X, \;\; \mu_{A\dot{\cup}B}(x) = \begin{cases} \mu_A(x), when\ \mu_B(x) = 0 \\ \mu_B(x), when\ \mu_A(x) = 0 \\ 1, otherwise \end{cases}$$

iv. Hamacher's sum $(A \cup B)$

$$\forall x \in X, \mu_{A\cup B}(x) = \frac{\mu_A(x) + \mu_B(x) - (2-\gamma)\mu_A(x)\mu_B(x)}{1 - (1-\gamma)\mu_A(x)\mu_B(x)}, \gamma \geq 0$$

Intersection operations

i. Algebraic product $A \cdot B$ (probabilistic product).

$$\forall x \in X, \qquad \mu_{A\cdot B}(x) = \mu_A(x) \cdot \mu_B(x)$$

$$\forall x \in X, \quad \mu_{A\cdot B}(x) = \mu_{B\cdot A}(x) \;\; commutativity.$$

Since $\mu_A(x) \cdot \mu_B(x) = \mu_B(x) \cdot \mu_A(x)$.

$$\forall x \in X, \ \ \mu_{\overline{A \cdot B}}(x) = 1 - [\mu_A(x) \cdot \mu_B(x)] \quad De\ Morgan's\ law.$$

ii. Bounded product $A \odot B$ (bold intersection).

$$\forall x \in X, \qquad \mu_{A \odot B}(x) = \max[0, \mu_A(x) + \mu_B(x) - 1]$$

$$\forall x \in X, \ \ \mu_{A \odot B}(x) = \mu_{B \odot A}(x) \quad commutativity$$

Since $\max[0, \mu_A(x) + \mu_B(x) - 1] = \max[0, \mu_B(x) + \mu_A(x) - 1]$.

$$\forall x \in X, \ \ \mu_{\overline{A \odot B}}(x) = 1 - \{\max[0, \mu_A(x) + \mu_B(x) - 1]\} De\ Morgan's\ law$$

. iii. Drastic product $(A \cap B)$

$$\forall x \in X, \quad \mu_{A \cap B}(x) = \begin{cases} \mu_A(x), when\ \mu_B(x) = 1 \\ \mu_B(x), when\ \mu_A(x) = 1 \\ 0, when\ \mu_A(x), \mu_B(x) < 1 \end{cases}$$

iii. Hamacher's intersection $(A \cap B)$

$$\forall x \in X, \ \ \mu_{A \cap B}(x) = \frac{\mu_A(x)\mu_B(x)}{\gamma + (1+\gamma)\big(\mu_A(x) + \mu_B(x) - \mu_A(x)\mu_B(x)\big)}, \quad \gamma \geq 0.$$

Disjunctive sum;

$$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B)$$

Definition (simple disjunctive sum)

$$\mu_{\bar{A}}(x) = 1 - \mu_A(x), \mu_{\bar{B}}(x) = 1 - \mu_B(x)$$

$$\mu_{A \cap \bar{B}}(x) = \min[\mu_A(x), 1 - \mu_B(x)]$$

$$\mu_{\bar{A} \cap B}(x) = \min[1 - \mu_A(x), \mu_B(x)]$$

$$A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B), then$$

$$\mu_{A \oplus B}(x) = \max\{\min[\mu_A(x), 1 - \mu_B(x)], \min[1 - \mu_A(x)]\}$$

For example, let

$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$,and

$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$.

Then, $\bar{A} = \{(x_1, 0.8), (x_2, 0.3), (x_3, 0), (x_4, 1)\}$

$$\bar{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.1)\}$$

$$\bar{A} \cap B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0), (x_4, 0.1)\}$$

$$A \cap \bar{B} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0), (x_4, 0)\}$$

Therefore $A \oplus B = (A \cap \bar{B}) \cup (\bar{A} \cap B) = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.1)\}$

Disjoint sum

$$\mu_{A \Delta B}(x) = |\mu_A(x) - \mu_B(x)|.$$

For example, let

$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$,and

$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$.

Then, $A\Delta B = \{(x_1, 0.3), (x_2, 0.4), (x_3, 0), (x_4, 0.1)\}$

Difference fuzzy set;

$A - B = A \cap \bar{B}$.

In fuzzy set, there are two ways of obtaining the difference:

(a) Simple difference

$$(A - B)$$

For example, let

$A = \{(x_1, 0.2), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$,and

$B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 1), (x_4, 0.1)\}$.

Then,$\bar{B} = \{(x_1, 0.5), (x_2, 0.7), (x_3, 0), (x_4, 0.9)\}$, and

$A - B = A \cap \bar{B} = \{(x_1, 0.2), (x_2, 0.7), (x_3, 0), (x_4, 0)\}$.

(b) Bounded difference;

$\mu_{A\theta B}(x) = \max[0, \mu_A(x) - \mu_B(x)]$, and

$$A\theta B = \{(x_{1,}0), (x_2, 0.4), (x_3, 0), (x_4, 0)\}.$$

Distances in fuzzy set theory

(a) Hamming distance

$$d_h(A,B) = \sum_{i=1,x\in X}^{n} |\mu_A(x_i) - \mu_B(x_i)|$$

For example, let

$A = \{(x_1, 0.4), (x_2, 0.8), (x_3, 1), (x_4, 0)\}$,and

$B = \{(x_1, 0.4), (x_2, 0.3), (x_3, 0), (x_4, 0)\}$.

Then,$d(A,B) = |0| + |0.5| + |1| + |0| = 1.5$

This definition satisfies the usual mathematical notion of distance;

i. $d(A,B) \geq 0$

ii. $d(A,B) = d(B,A)$ $symmetric.$

iii. $d(A,C) \leq d(A,B) + d(B,C)$ $triangle\ inequality.$

iv. $d(A,A) = 0$

Relative hamming distance $d_{rh}(A,B) = \frac{1}{n} d_h(A,B)$

Hamming distance can be called symmetrical distance by using the operator ∇;

$$\forall x \in X, \mu_{A \nabla B}(x) = |\mu_A(x) - \mu_B(x)|$$

(b) Euclidean distance

$$d_e(A,B) = \sum_{i=1}^{n} \sqrt{(\mu_A(x) - \mu_B(x))^2}$$

From the example taken above

$$d_e(A,B) = (0^2 + 0.5^2 + 1^2 + 0^2)^{1/2}$$

Relative Euclidean distance;

$$d_{re}(A,B) = \frac{d_e(A,B)}{\sqrt{n}}$$

(c) Minkowski distance;

$$d_m(A,B) = (\sum_{x \in X} | \mu_A(x) - \mu_B(x)|^m)^{1/m} m \in [1, \infty]$$

Hamming distance and Euclidean distance can be obtain fromMinkwoski distance

When w=1 it becomes Hamming distance.

When w=2 it becomes Euclidean distance.

Fuzzy relation

Let X, Y⊆ ℝbe universal sets then;

$R = \{((x,y), \mu_R(x,y)) | (x,y) \in X \times Y\}$is called a fuzzy relation in $X \times Y \subseteq R$.

Or X and Y are two universal sets, the fuzzy relation $R(x,y)$ is given as

$$R(x,y) = \{\frac{\mu_R(x,y)}{(x,y)} | (x,y) \in X \times Y\}$$

Fuzzy relations are often presented in the form of two dimensional tables. A fuzzy relation R can be represented by a$m \times n$ matrix:

$$\begin{array}{c} \quad\quad y_1 \quad \cdots \quad\quad y_n \end{array}$$

$$R = \begin{array}{c} x_1 \\ \vdots \\ x_m \end{array} \begin{bmatrix} \mu_R(x_1, y_1) & \cdots & \mu_R(x_1, y_n) \\ \vdots & \ddots & \vdots \\ \mu_R(x_m, y_1) & \cdots & \mu_R(x_m, y_n) \end{bmatrix}$$

For example,$let\ X = \{1, 2, 3\}\ and\ Y = \{1, 2\}$

the membership function is defined by

$\mu_R(x,y) = e^{-(x-y)^2}$.

Solution

A fuzzy relation can be defined as follows;

$$R = \left\{\frac{e^{-(1-1)^2}}{(1,1)}, \frac{e^{-(1-2)^2}}{(1,2)}, \frac{e^{-(2-1)^2}}{(2,1)}, \frac{e^{-(2-2)^2}}{(2,2)}, \frac{e^{-(3-1)^2}}{(3,1)}, \frac{e^{-(3-2)^2}}{(3,2)}\right\}$$

from the example considered above

$$R = \left\{\frac{1.0}{(1,1)}, \frac{0.37}{(1,2)}, \frac{0.37}{(2,1)}, \frac{1.0}{(2,2)}, \frac{0.02}{(3,1)}, \frac{0.37}{(3,2)}\right\}.$$

Operations of fuzzy relation

Union

Let R and Z be two fuzzy relations in the same product space. The union of R with Z is defined by:

$$\mu_{R\cup Z}(x,y) = \max\{\mu_R(x,y), \mu_Z(x,y)\}, (x,y) \in X \times Y.$$

Intersection

Let R and Z be two fuzzy relations in the same product space. The intersection of R with Z is defined by:

$$\mu_{R\cap Z}(x,y) = \min\{\mu_R(x,y), \mu_Z(x,y)\}, (x,y) \in X \times Y.$$

Complement

The complement relation $\bar{R}$ for fuzzy relation R is defined by the following function:

$$\forall(x,y) \in A \times B, \qquad \mu_{\bar{R}}(x,y) = 1 - \mu_R(x,y)$$

Projection of fuzzy relations

Let$R = \{[(x,y),\mu_R(x,y)] \,\big|\, (x,y) \in X \times Y\}$ be a fuzzy relation. The projection of $R(x,y)$ on Xdenoted by R_1 is given by

$$R_1 = \{(x, max\mu_R(x,y))\}, (x,y) \in X \times Y$$

and the projection of $R(x,y)$ on Y denoted by R_2 is given by

$$R_2 = \{(y\ , max\mu_R(x,y))\}, (x,y) \in X \times Y.$$

Similarly, we calculate the grade of membership for all pairs, so the projection on X is given by$R_1 = \{(x_1,1),(x_2,1),(x_3,1)\}$

Cylindrical extension of fuzzy relation

The cylindrical extension of $X \times Y$ of a fuzzy set A of X is a fuzzy relation cylA whose membership function is equal to;

$cylA(x,y) = A(x), \qquad \forall x \in X,\ \forall y \in Y.$

Cylindrical extension from X- projections means filling all the columns of the related matrix by the X-projections. Similarly cylindrical extension from Y-projections means filling all the rows of the relational matrix by the Y-projections.

Fuzzy maximum-minimum composition of relations

Let X, Y and Z be universal sets and let R and Q be relations given by

$R = \{((x,y), \mu_R(x,y))\} \qquad x \in X, y \in Y, R \subset X \times Y$ and

$Q = \{((y,z), \mu_R(y,z))\} \qquad y \in Y, z \in Z, Q \subset Y \times Z.$

Then S will be a relation that relates elements in X that R contains to the elements in Z that Q contains, i.e.,$S = R \circ Q$.

Here "$\circ$" means the composition of membership degrees of R and Q in the max-min sense.

$S = \{((x,z), \mu_R(x,z))\} \qquad x \in X, z \in Z, S \subset X \times Z.$

The max-min composition is defined as

$$\mu_S(x,z) = \max_{y \in Y}(\min(\mu_R(x,y), \mu_Q(y,z))$$

and max product composition is then defined

$$\mu_S(x,z) = \max_{y \in Y}(\min(\mu_R(x,y) \cdot \mu_Q(y,z))$$

Fuzzy max-min composition operation

Let $R_1(x, y), (x, y) \in X \times Y$ and $R_2(y, z), (y, z) \in Y \times Z$ be two fuzzy relations. The max-min composition of R_1 and R_2 is then the set:

$$R_1 \circ R_2(x, z) = \{[(x, z), \max_y\{\min\{\mu_{R_1}(x, y), \mu_{R_2}(y, z)\}\}] \mid x \in X, y \in Y, z \in Z\}$$

Fuzzy max-product operation[Rosenfeld, 1975]

The max- product composition $R_1 \circ R_2$ is defined as

$$R_1 \circ R_2\ (x, z) = \{[(x, z), \max_y\{\mu_{R_1}(x, y) \cdot \mu_{R_2}(y, z)\}\ ] \mid x \in X, y \in Y, z \in Z\}$$

Fuzzy max- average composition operation[Rosenfeld, 1975]

The max- ave composition $R_1 \circ R_2$ is defined by

$$R_1 \circ R_2(x, z) = \{[(x, z), \tfrac{1}{2} \cdot (\max_y(\mu_{R_1}(x, y) + \mu_{R_2}(y, z))] \mid x \in X, y \in Y, z \in Z\}$$

Properties of Fuzzy Relations

Reflexive relation

Let R be a fuzzy relation in $X \times X$. Then R is called reflexive, if

$\mu_R(x, x) = 1 \qquad \forall x \in X$

Antireflexive relation

Fuzzy relation $R \subset X \times X$ is antireflexive if

$\mu_R(x, x) = 0,\ x \in X.$

Symmetric Relation

A fuzzy relation R sis called symmetric if,

$\mu_R(x,y) = \mu_R(y,x) \forall x,y \in X.$

Antisymmetric Relation

Fuzzy relation $R \subset X \times X$ is antisymmetric iff

$$\text{If } \mu_R(x,y) > 0 \text{ then } \mu_R(y,x) = 0 \ \ x,y \in X, x \neq y.$$

Transitive Relation

Fuzzy relation $R \subset X \times X$ is transitive in the sense of max-min iff

$\mu_R(x,z) \geq \max(\min(\mu_R(x,y), \mu_R(y,z)))\ x,z \in X$

Since $R^2 = R \circ R$, if

$\mu_R{}^2(x,z) = \max(\mu_R(x,y), \mu_R(y,z))$

then R is transitive if $R \circ R = R (R \circ R \subseteq R)$

and $R^2 \subset R$ means that $\mu_R{}^2(x,y) \leq \mu_R(y,x)$.

Fuzzy Compatibility Relation

$$R: X \times X \to \{0,1\}$$

$$R \text{ is a compatibility relation if it is}$$

i. Reflexive

$$\forall x \in X \Rightarrow \mu_{\mathrm{R}}(x, x) = 1$$

ii. Symmetric

$$\forall (x, y) \in X \times X \Rightarrow \mu_{\mathrm{R}}(x, y) = \mu_{\mathrm{R}}(x, y)$$

Note that a compatibility relation is not transitive in general.

1.6 Organization of Dissertation

The dissertation is organized as follows;

Besides the background of the study, objective of the research and methodology given in chapter one, chapter two provides a detailed literature review on the subject matter. Chapter threeprovides various concepts related to similarity of objects and some existing models such as Jaccard, Simple Matching Coefficient, Vector, and Tversky for comparing fuzzy objects. Chapter four gives a new approach for comparing fuzzy objects which included the degree of inclusion, partial matching and similarity and some of their properties. A modified ratio model is also presented in chapter four. Chapter five contains summary and some future research directions.

CHAPTER TWO

LITERATURE REVIEW

Studies undertaken on compatibility relations of sets abound. In the recent years,owing to the development of fuzzy set theory by Zadeh (1965), a number of researcheshave been carried out to establish analogous results in fuzzy set context. Some existing theories on sets like relations, composition of relations, equivalence relations and compatibility relations are suitably being extended to fuzzy sets.The theory of compatibility relations have been examined by many researchers in varying dimensions.

According to Yager (1990), the information about one variable can be obtained using information about another variable if we have some knowledge about how they are related. Such concepts form the basis of most inference systems. For an illustration, assume U and V are two variables taking their values in the set X and $Y,$ respectively. Compatibility relation is used to represent this knowledge by defining a relation between U and V on $X \times Y$ to a binary set $\{0,1\}$, such that for each $x \in X$ there exist at least one $y \in Y$ such that $x\, and\, y$ are compatible and for each $y \in Y$ there exist at least one $x \in X$ such that $y\, and\, x$ are compatible. Following Yager (1990), a compatibility relation is called "monotonic" if an increase in information about the primary variable cannot result in a loss of information about the secondary.

Bouchon-Menunier et al. (1996) conducted a study based on Tversky's feature-theoretical concepts on similarities. The research proposed classification of measures that exist or have been used in previous literature to compare fuzzy characterization of objects according to their properties and their applications. The study focused on finding various measures of comparisons including satisfiability, resemblance, inclusion, and dissimilarity.

Vallee(2006) showed that the metric field associated with a smooth enough immersion defined over an open set necessarily satisfies the compatibility relation.

Perfilieva (2007) showed that afuzzy equation system with the generalized conjunction and implications is considered necessary and sufficient conditions of solvability of the system. The case where these conditions are not fulfilled was considered and the solvability degree of the approximation was established.

Singh and West (2013) presented some new properties of compatibility classes of a finite set endowed with a suitable compatibility relation. An algorithm to compute maximal compatibility classes was constructed and an application of maximal compatibles to network segmentation and decentralization was demonstrated.They also discussed the application of a compatibility relation to knowledge data implementation and proposed that an intelligent machine may implement a task with the help of compatibility relation defined between the task and its knowledge database.

Kheniche (2015) showed that the notion of extensionality introduced by $H\ddot{o}hle$ and $Blanchard,$ and the notion of compatibility, as coined by $Belohl\acute{a}vek,$ of a fuzzy relation with respect to fuzzy equality were trivially equivalent. Here this compatibility property is dissected into left and right compatibility, mimicking the original two fold definition of extensionality, and studied in detail in the context of arbitrary fuzzy relations. Relying on the notions of left and right traces ofa fuzzyrelation, it was showed that compatibility relation can be characterized interms of inclusion.

Fuzzy compatibility relation in information retrieval utilizes fuzzy sets to represent documents, membership degrees for query term relevance, fuzzy logical operators to define queries and fuzzy compatibility relation measures to assess the retrieval status value of a document. For example, one of the early applications of fuzzy compatibility relation for information retrieval is a question answering system for analogical inference. The system analyzes the subject matter of the query. Through this learning process, the subject matter or query is specified as a fuzzy set over the concepts in the knowledge base. The Euclidean distance between the fuzzy sets representing the learned query and the other learned queries is calculated to determine if the subject matter already existed in the system (Dubois, 1989).

Another application of fuzzy compatibility relation is in the ranking of fuzzy numbers. In optimization problems an objective is to select the alternative: maximal or minimal. When the alternatives are represented by fuzzy numbers, this requires the ability to compare and order the fuzzy sets. For example, the principle in generalized fuzzy compatibility relation based ranking is that for each pair of fuzzy numbers $A$and B, a fuzzy compatibility relation measure is used to assess the agreements of both A and B with the $\min(A, B)$ and $\max(A, B)$ and these results are aggregated. Because of the flexibility in the selection of the fuzzy compatibility relation based ranking methods are used to produce a variety of ranking strategies and serves to establish relationships among these strategies (Harman and Klir, 1994).

Application of fuzzy compatibility relation to processing information in fuzzy rule-based systems generally employs fuzzy compatibility relation modification. It was developed to facilitate the evaluation of rules by separating the evaluation of the input from the generalization of the output. Zadeh's fuzzy interpolation is an example of fuzzy compatibility relation modification inference . The compatibility of the input A' and the a ntecedent A is determined by a sup min comparison of two fuzzy sets (Anvari and Rose, 1987).

Several applications of fuzzy compatibility relations measures are produced to analyze fuzzy relations representing medical knowledge. Diagnosis is performed using two methods. The first represents the possibility of a given disease by the maximum of the compatibility relation measures determined from the $\sup min$ composition of the fuzzy description of the patient's condition with each of the fuzzy relations between the diseases and symptoms. The second employs the sum min composition rather than $\sup min$composition. Sum min composition sums the minimum membership grades instead of taking the supremum (Sneath, 1965).

Expert system is used for diagnosis and follow-up of pediatric oncology. Two methods of grouping individual symptoms, clinical patterns of symptoms and level of diagnosis significance, are incorporated into its two steps process of diagnosis. The first method uses a fuzzy set to represent the strength of confirmation. The universe of discourse for these two fuzzy sets is the set of clinical patterns, not the individual symptoms. The use of the pattern instead of individual symptoms can be viewed as means of weighting the importance of combination of symptoms. Again sup min composition is used as the measure of compatibility relation between the two fuzzy sets. The second method of grouping symptoms is used in explaining the system's selection of a particular disease. Fuzzy representations and compatibility relation assessment have been used in a number of additional medical expert systems, a fuzzy production rule-based expert systems for X-ray diagnosis of coal workers pneumoconiosis, and for the diagnosis and treatment of diabetes (Esogbue and Elder, 1980).

For a given disease, the sum-min is determined between the fuzzy set of patient's symptoms and a fuzzy set of occurrence symptoms for the disease. The value is the cardinality of the intersection of the two fuzzy sets. The operation is also performed between the fuzzy set of the patient's symptoms and the fuzzy set of the confirmation symptoms for the disease and the two sum-min values are combined. Since the strength of confirmation is considered more important than the frequency of occurrence, it is wighted more than the frequency of occurrence value. The second method is employed because the first approach does not consider the number of symptoms present or partly present, which suggest but do not confirm a given disease (Esogbue, 1980).

Within the area of civil engineering, fuzzy expert systems have been used for structural damage assessment. One of such systems uses Dempter's rule of combination to aggregate two fuzzy sets with certainty factors. Another system uses a fuzzy partial matching algorithm to determine the compatibility relation between two fuzzy sets (Bonissone and Wood, 1985).

A linear rule interpolation strategy was developed for inference in sparse fuzzy rule bases. The interpolation assumes that fuzzy input $\acute{A}$ falls between the antecedentsA_1 and A_2 and that the supports of A_1 and A_2 are disjoint. The proportional distance of the input $\acute{A}$ from the antecedents A_1 and A_2 is used to produce the conclusion proportionately between the consequents of the two rules (Koczy and Hirota, 1993).

A combination of fuzzy logic control and neural network is an example of more sophisticated methods being applied to control applications. A general neural network connectionist model, combines neural networks and fuzzy logic. The motivation for this combination is to obtain the benefits of techniques, low-level learning and computational power of neural networks and high-level reasoning of fuzzy logic. The algorithm for learning the structure and parameters of the fuzzy neural network incorporates the Jaccard index for measuring the degree to which two fuzzy sets are equal. The compatibility relation is calculated between the expected output fuzzy sets and actual output linguistic variables (Yuan and Klir, 1995).

In this work, a comprehensive study of the aforesaid developments and various concepts related to similarity of fuzzy objects are presented. In addition, a new approach for comparing fuzzy objects which involves the degree of inclusion, partial matching and similarity is introduced and some of the properties of the new ratio model are discussed.

CHAPTER THREE

APPLICATIONS OF COMPATIBILITY RELATION IN FUZZY SET CONTEXT

3.1 Concept of Similarity

Similarity is described as a comparison made between two or more objects as an attempt to determine the relationships that exist between them or an assessment made between fuzzy objects described with fuzzy attributes. It is perhaps the most frequently used, most difficult to quantify, and the most universally employed type of compatibility relation. The analysis of the similarity of two objects is a fundamental tool in biology, taxonomy, and psychology.In order to build a classification framework for compatibility relation, understanding various facets of similarity are required.

3.1.1 Dissimilarity

Dissimilarity is known as the opposite of similarity. Typically, judgments of difference or dissimilarity are assumed to be equivalent to judgments of similarity and vice versa. It has been noted that distinguishing differences and similarities, is the same thing; a similarity being nothing but a “slight difference”. Therefore, the similarity of two qualities may consist in the slightness of the differences that exist between them. In general, if the similarity between objects$u$ and v is assigned a value $sim(u,v)$ from the interval [0,1], then their dissimilarity is $1-sim(u,v)$.Many researchers using fuzzy set theory have also assumed that dissimilarity is the inverse of similarity. Others, however, have called $1-sim(u,v)$a non similarity measure and have stressed that, for fuzzy sets, non-similarity and dissimilarity are not synonymous. Instead, the dissimilarity between two fuzzy sets is specified by a similarity measure between the complements of the two fuzzy sets.

3.1.2 Symmetricsimilarity

Similarity has typically been assumed to be symmetric. In terms of similarity measures, symmetric implies that $sim(u, v) = sim(v, u)$. Tversky (1977) provided empirical evidences that similarity should not always be treated as a symmetric relation based on human judgments involving abstract relations among objects. For example, an object is more similar to itself, but sometimes an object is identified more often as another object than itself. The probability that two identical objects are judged "same" varies with the nature of objects judged. Symmetric implies that an object u is similar to an object v as $v$to u, but this often fails. For instance, Nigeria may be judged more similar to Ghana than Ghana to Nigeria.

3.1.3 Multidimensional and multi-attribute

Geometric approach stresses the representation of similarity relationship among members of a set of objects. In multidimensional similarity is given by distance between objects in the space, the closer together two objects are, the more similar they are. The assumptions of the approach are as follows;

i. Measurements are used to represent the fuzzy set as a point in an n- dimensional space, where n is the number of characteristics.

ii. Similarity can be represented by Euclidean distance between the points in the space,

iii. Each fuzzy set is measured with respect to a fixed number of characteristics such as entropy, cardinality, center of gravity, etc.

In multi-attribute a fuzzy set is defined by a set of characteristics or features in a multi-criteria decision making application.

3.1.4 Relative similarity

Similarity is dependent on the context of the object. The complete set of objects used in an experiment influences the judgement of whether an object$u$ is related to another object$v$ than to an object$w$. The overall judgement of similarity is influenced by the weighting of the variation in each attribute of the object with respect to the overall variation of the attribute within the complete set of objects. However the occurrence of attribute values is relevant to the assessment of compatibility of fuzzy sets.

3.1.5 Universe

Let U represents the set of attributes possessed by any given objects $X$and Y. Then the following conditions hold;

i. $|U| = |X \cap Y| + |X \cap \bar{Y}| + |\bar{X} \cap Y|$

ii. $|U| = |X \cup Y|$

iii. $|U| = |X \cup Y| + |\bar{X} + \bar{Y}|$

3.2 Jaccard Ratio Model

In taxonomy, an object is described by the possession or non-possession of certain attributes. A domain U represents the set of attributes used in taxonomic classification. An object is characterized by the set of attributes it possesses. For simplicity we will consider an object to be defined by the value of its attributes.

Let X and Y be any given objects. For each given attribute there are four possibilities:

i. present in X and Y,

ii. present in X and absent in Y,

iii. present in Y and absent in X,

iv. absent in X and Y.

The number of attributes satisfying each of these combinations is $|X \cap Y|$, $|X \cap \bar{Y}|$, $|\bar{X} \cap Y|$, $|\bar{X} \cap \bar{Y}|$ respectively, where $|X|$ is the cardinality of the set X. Then,

$$S_{XY} = \frac{|X \cap Y|}{(|X \cap Y| + |X \cap \bar{Y}| + |\bar{X} \cap Y|}$$

$$S_{XY} = \frac{|X \cap Y|}{|X \cup Y|}$$

$$S_{XY} = \frac{|X \cap Y|}{|U|}, \qquad \text{(Jaccard, 1908)}$$

where S_{XY} is used to measure the similarity between objects X and Y with common attributes.

The Jaccard index is an unparameterized ratio model of similarity.

Remarks 3.1

i. The number of common deficiencies in two objects, $|\bar{X} \cap \bar{Y}|$, is not included in the Jaccard index.

ii. In some cases, however, taxonomist might consider two organisms similar because they both do not possess a certain attribute.

3.3 Simple Matching Coefficient

Another form of the ratio model, called the simple matching coefficient, includes the common deficiencies and is given as

$$\hat{S}_{XY} = \frac{(\left|X \cap Y\right| + \left|\bar{X} + \bar{Y}\right|)}{(\left|X \cup Y\right| + \left|\bar{X} + \bar{Y}\right|)}$$

$$\hat{S}_{XY} = \frac{(\left|X \cap Y\right| + \left|\bar{X} + \bar{Y}\right|}{\left|U\right|}. \qquad \text{(Sneath, 1965)}$$

The simple matching coefficient was extended to a class of"attributes" whose values are not limited to being present or absent. Each of these similarity measures produces a value in the range [0,1] with 1 indicating the maximum degree of matching. The major distinction between S_{XY} and $\hat{S}_{XY}$ is that similarity as measured by S_{XY} is based solely on the number of positive matches for the attributes, ignoring the common negative attributes, has the effect of reducing the size of the universe to $\left|U\right| - \left|\bar{X} \cap \bar{Y}\right|$. Similarity measured by $\hat{S}_{XY}$ is based on both "positive matches" and "negative matches" over the complete set of attributes selected for judging similarity. If a large set of attributes U is selected, then any two objects would be considered very similar if they both had a large number of attributes absent.

3.4 Vector Ratio Model

In vector model, an object is represented as vector of attributes. For example an object Xdefined by n attributes is represented as a vector $[x_1, x_2, \ldots, x_n]$ where x_i is the value of the $i$thattribute. The measure of similarity between objects $X = [x_1, x_2, \ldots, x_n]$ and$Y = [y_1, y_2, \ldots, y_n]$ is obtained from the cosine of the angle θ between the two vectors from the origin of the n- dimensional space describing the objects.

$$cos\theta = \frac{\sum_{i=1}^{n}(x_i \cdot y_i)}{\sqrt{\sum x_i{}^2} \cdot \sqrt{\sum y_i{}^2}}$$

$$\theta = cos^{-1}\left(\frac{\sum_{i=1}^{n}(x_i \cdot y_i)}{\sqrt{\sum x_i{}^2} \cdot \sqrt{\sum y_i{}^2}}\right) \qquad (Bhattacharyya, 1946)$$

Instead of using the cosine as the measure, Bhattacharyya used the angle θ itself. Similarity assessment using the angle between vectors is based on the direction of the vectors for $X$and Y; the magnitude of the vectors is irrelevant. When thex_i andy_i values are multiplied by the constants, the resulting similarity measure is identical to the original.

3.5 Mean Character Difference Model

The mean character difference (Hamming or "city block" distance)

$d_{MCD}(X,Y) = \frac{1}{n}\sum \left| x_i - y_i \right|$ (Rechard, 1950).

has been proposed as a measure of "resemblance" when the objects are represented by real-value vectors.

3.6 Canberra Metric Model

An interesting variation of the "Manhattan metric" is the "Canberra metric"

$$d_{CAN}(X,Y) = \frac{\left| x_i - y_i \right|}{x_i + y_i} \qquad (Lance\ and\ Williams, 1966)$$

which scales the absolute difference by a factor based on the values being compared and is a proportional difference.

3.7 Tversky Ratio Model

Numerous set-theoretical measures, referred to as "content" models of similarity, have been proposed. Several of these are generalized by the ratio model of similarity

$$S_{\alpha,\beta}(X,Y) = \frac{f(X \cap Y)}{f(X \cap Y) + \alpha \cdot f(X - Y) + \beta . f(Y - X)} \qquad (Tversky, 1977)$$

The preceding formula is a "parameterized" version of the previously discussed Jaccard index where;

i. $X - Y = X \cap \bar{Y}$

ii. $Y - X = Y \cap \bar{X}$.

iii. f istaken to be the cardinality, however, it may be any other function that satisfies additivity $f(X \cup Y) = f(X) + f(Y)$ for disjoint sets X and Y.

The similarity measure is normalized so that $0 \leq S(X,Y) \leq 1$.The following cases hold;

i. With $\alpha = \beta = 1$, $S_{1,1}(X,Y) = \frac{f(X \cap Y)}{f(X \cup Y)}$;

ii. with $\alpha = \beta = \frac{1}{2}$, $S_{\frac{1}{2},\frac{1}{2}}(X,Y) = \frac{2 \cdot f(X \cap Y)}{f(X) + f(Y)}$;

iii. with $\alpha = 1, \beta = 0$, $S_{1,0}(X,Y) = \frac{f(X \cap Y)}{f(X)}$.

Remarks 3.2

It shall be seen that these measures are members of specific subclasses of set- theoretic compatibility measures.

CHAPTER FOUR

APPLICATIONS OF FUZZY COMPATIBILITY RELATIONS

Introduction

The class of set-theoretic compatibility relation measures has its root in the "content" model of psychology and the "binary presence-absence similarity coefficients" for both taxonomic and biassociational interpretations. As previously noted,Jaccard'sunparameterized ratio model was, extended by Tversky's parameterized ratio model of similarity.

$$S(X,Y) = \frac{f(X \cap Y)}{f(X \cap Y) + \alpha \cdot f(X-Y) + \beta \cdot f(Y-X)},$$

where$X$ and Y represent objects and f is an additive function on disjoint sets. The intersection $X \cap Y$ consists of features that are common to both objects, $X - Y$ consists of the features that belong to X but not to Y, and $Y - X$consists of the features that belong to Y but not to X. The function f is assumed to satisfy feature additivity; $f(X \cup Y) = f(X) + f(Y)$ whenever X and Y are disjoint.

Recently there has been a renewed interest in the parameterized ratio model as the basis for fuzzy set similarity measures. $Santini$and$Jain$(1999) introduced a fuzzy feature contrast model as a fuzzy generalization of denominator of the parameterized ratio model.In a similar manner, a generalized $Tversky$ index is proposed by employing fuzzy operations. The ratio model is not capable of producing the "simple matching coefficient". Afurther generalization of this model uses constant $\alpha, \beta,$ and γ to weight the contribution of terms.

$$S(X,Y) = \frac{f(X \cap Y) + \gamma \cdot f(\bar{X} \cap \bar{Y})}{f(X \cap Y) + \alpha. f(X - Y) + \beta \cdot f(Y - X) + \gamma. f(\bar{X} \cap \bar{Y})}$$

The simple matching coefficient can be obtained by setting $\alpha = \beta = 1$. Objects

X and Y represented crisp set of features possessed by the objects. In the analysis of compatibility of fuzzy sets, the arguments of a compatibility measure will be fuzzy sets A and B. Since crisp sets are special cases of fuzzy sets, these fuzzy measures may also be used to measure the compatibility of crisps sets. Many of the measures that will be examined are direct generalizations of measures from crisp sets obtained by employing fuzzy operations.

When crisp sets A and B are not disjoint, $|A \cup B| = |A| + |B| - |A \cap B|$.

This is also true for fuzzy set cardinality when member of $frank's$ family of dual T-norms and T-conorms are used for intersection and union respectively. If A and B fuzzy subsets of a universe U then $|A| + |B| = |A \cap B| + |A \cup B|$ when $\cap$ and $\cup$ are a dual pairs from $frank's$ family, since $a + b = T_F(a,b) + S_F(a,b)$. Substituting $\mu_A(u)$ for a and $\mu_B(u)$ for b and summing both sides over $u \in U$ gives the desired result.

Different requirements and objectives produce generalized formulas for three subclasses of set-theoretic compatibility measures. The first "subclass" contains measures that determine the degree to which one fuzzy set is a "specialization of" or "included in" another, the second measures the consistency or partial matching, and the third, an average similarity. The comparison indices are restricted to using T-norms $T_1,\ T_2,\ and\ T_3$ and their corresponding T-conorms $S_1,\ S_2,\ and\ S_3$.Besides cardinality, other functions, called "scalar evaluators", are used to transform a fuzzy set into a single value. Compatibility relation measures created with these functions may not be derivable from the parameterized ratio model since the scalar evaluator may not be additive.

A “scalar evaluator” g reduces a fuzzy set into a single number in the range [0,1]. A scalar evaluator must satisfy

i. $g(\emptyset) = 0$,

ii. $g(U) = 1$, and

iii. $A \subseteq B \Rightarrow g(A) \leq g(B)$

Scalar evaluators are called “existential” when $g(A) = 0$ if and only if $A = U$.An example of an existential evaluator is $g(A) = Sup_{u \in U} \mu_A(u)$. An example of a universal scalar evaluator is $g(A) = inf_{u \in U} \mu_A(u)$.“Relative cardinality” is a scalar evaluator that is both existential and universal. Any scalar evaluator satisfying $g(A) = 1 - g(\overline{A})$ is “existential” if and only if it is “universal”.

Each subclass of set-theoretic compatibility measures is denoted by a single capital letter: I for inclusion indices, P for consistency indices, and S for similarity indices. These are further subscripted to indicate the fuzzy set operator(s) and scalar evaluator used in the compatibility measure. For example $I_{S_3/_{rel}}$ is the inclusion index produced by using the T-conorm:max(S_3) and relative cardinality, denoted rel, the scalar evaluator$P_{T_3/_{Sup}}$ represents the partial matching index produced by using theT-norm min(T_3) and the supremum for the scalar evaluator. The notation of similarity indices is more complicated.

As an example, $S_1/D/S_3/rel$ is the type-1 similarity index produced using the symmetric difference D, the max union operator S_3 with relative cardinality for the scalar evaluator. The symmetric difference operators are described in the similarity indices.

4.1 Inclusion Indices

An intuitive approach to defining the degree of inclusion of a crisp set X in a crisp set Y, when X is not a subset of Y, would be to consider the ratio of the number of elements in X that are also elements of Y to the number of elements in set X. Using this method for measuring the degree of inclusion for crisp sets, the inclusion index is 1 if and only if X is a subsetof$Y$. This approach agrees with zadeh's definition of subset-hood of fuzzy set; $I_Z(A, B) = 1$ if $\mu_A(u) \leq \mu_B(u)$ for all $u \in U$ and 0 otherwise. The ratio approach to assigns inclusion value 0 if and only if X and Y are disjoint. This does not agree with zadeh's I_Z measure, which assign inclusion 0 whenever A is not a subset of$B$. However, this agreement is desired since inclusion indices should have the flexibility to designate intermediate degrees of inclusion by producing values in the range [0,1].

The membership values of fuzzy sets indicate the degree to which an element is a member of a set. The conditions under which fuzzy set inclusion indices produces 0 or 1 are compared

to those for an inclusion index of crisp sets. The ratio inclusion index for crisp sets may be extended to fuzzy sets by using a fuzzy set intersection operator and fuzzy set cardinality.

Not such inclusion indices produce values of 0 and 1 under the same conditions as the crisp definition of inclusion. Because of the variation in intersection operators, $A \cap B = \emptyset$ does not imply that the supports of A and B are disjoint, and $A \subseteq B$ by Zadeh's definition does not imply $A \cap B = A$.

Requirements

The following requirements were proposed for fuzzy set inclusion indices I

I_1) $I(A,B)=1$ if and only if $\bar{A}\cup B=U$.

I_2) If the support of A and B are disjoint, then $I(A,B)=0$.

I_3) $I(A,B)$ depends on scalar evaluatorsof$\bar{A}\ \cup B, and\ g(\bar{A}\cup B)$.

These requirements were used to produce the following general formula for a fuzzy set inclusion index.

$$I(A,B)=\frac{g(\bar{A}\cup B)-g(\bar{A})}{1-g(\bar{A})}$$

Remarks 4.1

The right side of the equation is a normalization of the quantity $g(\bar{A}\cup B)$. The largest value that $g(\bar{A}\cup B)$ can assume is 1 when $\bar{A}\cup B=U$. In this case $I(A,B)=1$.The smallest value of $g(\bar{A}\cup B)$ occurs when A and B are disjoint, producing $I(A,B)=0$.

Conditions I_1 and I_2 are in conflict when $A=\emptyset$. By I_1, $I(\emptyset,B)=1$ since negation of $\emptyset$ is U and $U\cup B=U$ and by I_2, $\emptyset$ and B have disjoint supports so that $I(\emptyset,B)=0$. In keeping with Zadeh's definition and crisp set theory, $I(\emptyset,B)$ is assigned 1 regardless of B and $I(A,\emptyset)=0$ for all Aexcept $A=\emptyset$. In order to satisfy $I_1$and I_3, g must be a universal scalar evaluator. The conditionI_1, which specifies when an inclusion index is 1, is dependent on the selection of the union operator. We will establish conditions under which $I(A,B)=1$ for fuzzy evaluators $g=rel$ (relative cardinality) and $g=inf$.

Proposition 4.1

$I_Z(A,B) = 1$if and only if $I_{S_1}/rel\,(A,B) = 1$.

Proof

Substituting relative cardinality for g and S_1 for ∪ produces;

$I_{S_1}/rel\,(A,B) = \frac{\sum_{u\in U}\min(1,1-\mu_A(u)+\mu_B(\mu))-\sum(1-\mu_A(u))}{\sum\mu_A(u)}$, which can be rewritten as

$I_{S_1}/rel\,(A,B) = \frac{\sum_{u\in U}\min(1,1-\mu_A(u)+\mu_B(u)+\mu_A(u)-1}{\sum\mu_A(u)}$. If $\mu_A(u) \le \mu_B(u)$, the numerator is $\mu_A(\mu)$ and if $\mu_B(u) \le \mu_A(u)$, the numerator is $\mu_B(u)$. Thus $I_{S_1}/rel\,(A,B) = \frac{\sum_{u\in U}\min(\mu_A(u),u_B(u))}{\sum\mu_A(u)}$.

The numerator $g(\bar{A}\cup B) - g(\bar{A})$ with g relative cardinality and $\cup = S_1$is, therefore, equivalent to $g(A\cap B)$for $\cap = T_3$. If $I_Z(A,B) = 1$, then $\mu_A(u) \le \mu_B(u)$ for all $u \in U$. Consequently, the numerator is equal to the denominator and $I_{S_1}(A,B) = 1$. Conversely, if $I_{S_1}/rel\,(A,B) = 1$, the numerator must be equal to the denominator. It follows that $\mu_A(u) \le \mu_B(u)$ for all $u \in U$and$I_Z(A,B) = 1$.

$I_{S_1}/rel\,(A,B)$is equivalent to $kosko's$ algebraically derived subset-hood measure (Kosko,1992). The subsethood measure explicitly requires that $S(A,\emptyset) = 0$and $S(\emptyset,A) = 1$. The equivalence can be shown as $I(A,B) = {(g(A) - g(A\cap\bar{B}))}\big/{g(A)}$.

This change is possible since relative cardinality satisfies $g(A) = 1 - g(\bar{A})$. Using T_1 for ∩ and replacing g with relative cardinality yields

$I_{S_1}/rel\,(A,B) = 1 - \frac{\sum\max(0,\mu_A(u)-\mu_B(u))}{\sum\mu_A(u)}$.

$Kosko's$subsethood measure is an inclusion measure. The family of inclusion indices denoted as $I_{\cup}/rel$ ($\cup$ specified that any union operator may be selected) may be viewed as fuzzy set theoretic interpretation of the parameterized ratio model of similarity with $\alpha = 1, \beta = 0$ and fuzzy set cardinality used for f. This combination produces $|A \cap B|/|A|$, an inclusion measure of A in B. In a similar manner, it can be showed that

$$I_{S_2}/rel\,(A,B) = \frac{|\bar{A} \cup_{S_3} B| - |\bar{A}|}{|1 - \bar{A}|} = \frac{|A \cap_{T_2} B|}{|A|}\ and$$

$$I_{S_3}/rel\,(A,B) = \frac{|\bar{A} \cup_{S_3} B| - |\bar{A}|}{|1 - \bar{A}|} = \frac{|A \cap_{T_1} B|}{|A|}.$$

Inclusion indices created using S_3 and S_2 with relative cardinality as the scalar evaluator have stricter requirements to produce an inclusion measure 1. $I_{S_3}/rel\,(A,B) = 1$and$I_{S_2}/rel\,(A,B) = 1$ if and only if $A \subseteq B_{\alpha=1}$, where $B_{\alpha=1}$ is 1-cut or the core of B. If $\cup = S_3$ and g is relative cardinality, $I_{S_3}/rel\,(A,B) = \frac{\sum \max(0, \mu_A(u) + \mu_B(u) - 1)}{\sum \mu_A(u)} = 1$ if and only if $A \subseteq B_{\alpha=1}$. The numerator becomes $\sum \mu_A(u)$ when $\sum \mu_B(u) = 1$ for all $u \in U$with $\mu_A(u) > 0$. If $\cup = S_2$, $I_{S_2}/rel\,(A,B) = \frac{\sum \mu_A(u) \cdot \sum \mu_B(u)}{\sum \mu_A(u)} = 1$ if and only if $A \subseteq B_{\alpha=1}$. The numerator becomes $\sum \mu_A(u)$ when $\mu_B(u) = 1$ for all $u \in U$with $\mu_A(u) > 0$.

Proposition 4.2

All inclusion indices I_{S_F}/rel built using a T-conorm in frank's family except for $S_F(a, b, s \to \infty) = S_1(a, b)$ produces a value 1 if and only if $A \subseteq B_{\alpha=1}$.

Proof

Inclusion indices built using relative cardinality and frank's family of T-conorms have the form $I_{S_F}/rel\,(A,B) = \sum log_S(1 + \frac{(S^{\mu_A(u)}-1)(S^{\mu_B(u)}-1)}{(S-1)} / \left|A\right|$.

If $A \subseteq B_{\alpha=1}$, the numerator reduces to

$\sum log_S\left(1 + S^{\mu_A(u)} - 1\right) = \sum log_s(S^{\mu_A(u)} = \sum \mu_A(u)$,which is identical to the denominator $\left|A\right|$ producing 1. If $A \subseteq B$ but not $A \subseteq B_{\alpha=1}$, then there exist an element u_i such that $\mu_A(u_i) \leq \mu_B(u_i) < 1$. The factor $(S^{\mu_B(u_i)} - 1)$ does not cancel with $s - 1$ so that $S^{\mu_A(u_i)} - 1$ is then multiplied by a fraction. Thus the numerator is less than $\sum \mu_A(u)$. As a result, an index less than 1 is produced.

Proposition 4.3

For normal fuzzy sets, $I_Z(A,B) = 1$ if and only if $I_{S_1}/inf\,(A,B) = 1$.

Proof

Substituting inf for g and S_1 for ∪ yeilds

$$I_{S_1}/inf\,(A,B) = inf_{u \in U} \min(1, 1 - \mu_A(u) + \mu_B(u))$$

when$A$ and B are normal fuzzy sets since $\inf(\bar{A}) = 0$. If $I_z(A,B) = 1$, then $\mu_A(u) \leq \mu_B(u)$ for all $u \in U$. Thus, $I_{S_1}/inf\,(A,B) = 1$. Conversely, if$I_{S_1}/inf\,(A,B) = 1$, $1 - \mu_A(u) + \mu_B(u) \geq 1$ for all $u \in U$. That is, $\mu_A(u) \leq \mu_B(u)$and $I_Z(A,B) = 1$.

Now considering the conditions under which the inclusion indices created using inf as the scalar evaluator and S_2 and S_3 for ∪ return a value of 1. If ∪$= S_3$,

$I_{S_3}/inf\,(A,B) = inf_{u\in U}\max\big(1-\mu_A(u),\mu_B(u)\big) = 1$if and only if $A \subseteq B_{\alpha=1}$. If $\cup = S_2$ and A and B are normal, $I_{S_2}/inf\,(A,B) = inf_{u\in U}(1-\mu_A(u)+\mu_A(u))\cdot\mu_B(u)) = 1$ if and only if $A \subseteq B_{\alpha=1}$.

Proposition 4.4

All inclusion indices I_{S_F}/inf built using a T-conorm in Frank's family except for $S_F(a,b,s\rightarrow\infty) = S_1(a,b)$ produces a value of 1. If and only if $A \subseteq B_{\alpha=1}$.

Proof

When $g = inf$ and frank's T-conorm family is used then,

$I_{S_{F_s}}/inf\,(A,B) = \inf(1-\log_s(1+(S^{\mu_A(u)}-1)(S^{1-\mu_B(u)}-1)/(S-1)))$. If $A \subseteq B_{\alpha=1}$, the above formula reduces to

$\inf(1-log_s(1+(s^{\mu_A(u)}-1)(s^0-1)/(s-1)) = \inf(1-log_s(1)) = 1$. When $A \subseteq B$ but A is not a subset of the core of B, $(s^{1-\mu_B(u_i)}-1) > 0$ for some $u_i \in U$ where $0 < \mu_A(u_i) \leq \mu_B(u_i)$. The log term then produces a value t_u greater than 0 for u_i so that $\inf(1-t_u) < 1$.

Condition I_2 in the definition of an inclusion index requires $I(A,B)$ to be 0 whenever A and B are disjoint. The effect of changing the "if" to "if and only if" in I2 is now addressed, that is, $I(A,B) = 0$ if and only if the "focal set of A" and the "focal set of B" are disjoint. This change to I_2eliminatesI_{S_3}/rel as an inclusion index. $I_{S_3}/rel\,(A,B)$may produce zero even though the focal sets of A and B are not disjoint. This property, however, might be useful if the membership values of the elements in the support of both fuzzy sets do not reach a specified level, i.e.,$\mu_A(u) + \mu_B(u) \leq 1$ for all $u \in supp(A) \cap supp(B)$. There might be also some merit in somehow considering the number of elements for which the two fuzzy sets overlap but where $\mu_A(u) + \mu_B(u) \leq 1$.

Condition I_2 could not be rewritten as $I(A, B)$ if and only if $A \cap B = \emptyset$ without deleting several common measures from the family of inclusion indices. This rewriting is equivalent to the intuitive inclusion index for crisp sets, $X \cap Y = \emptyset$ if and only if X and $Y$are disjoint. For fuzzy sets, $A \cap B = \emptyset$ if A and B are disjoint. However, using T_1 for intersection allows the possibility that $A \cap B = \emptyset$ even though $supp(A) \cap supp(B) \neq \emptyset$.

Rewriting I_2 as $I(A, B) = 0$ if and only if $A \cap B = \emptyset$ would also eliminate all inclusion indices derived using inf for the scalar evaluator. These inclusion indices always produce 0 whenever there exists a $u \in A$ such that $\mu_A(u) = 1$but $\mu_B(u) = 0$. This property may be useful if application requirements dictate that when $u \in A$ is certain and $u \notin B$ is certain, it is impossible for $A \subseteq B$ to any degree.

4.2 Partial Matching Indices

The definition of equality of fuzzy sets A and B was given by Zadeh $A = B$ if and only if $\mu_A(u) = \mu_B(u)$ for all $u \in U$. This definition is a generalization of equality of crisp sets and may be considered to provide a crisp measure of the degree to which fuzzy sets agree; 1 if they totally agree and 0 if there is any difference, no matter how slight, in the membership functions.

The binary nature of this assessment of the degree of matching of two fuzzy sets is frequently too strict a condition for approximate reasoning and soft computing applications. In approximate reasoning, the degree to which two fuzzy sets match often provides information used in assessing information provide by a rule or degree to which a conclusion is influenced by a number of factors. More flexible fuzzy set equality indices provide a value from 0 to 1 that indicates the degree to which A is equal to or matches B. The modifier partial in the name of the subclass of set-theoretic compatibility measures, however, that this matching is a weaker condition than that of the next subclass, similarity measures.

The most widely used compatibility measure, Zadeh's sup-min compatibility,

$sup_{u\in U} \min(\mu_A(u), \mu_B(u))$ is a partial matching index. This measure is very lax in its requirements for producing the value 1. Whenever there is an element $u \in U$ with $\mu_A(u) = 1$ and $\mu_B(u) = 1$, $sup_{u\in U} \min(A(u), B(u)) = 1$. For crisp sets, if the intersection is non-empty, the partial matching match between sets using sup-min compatibility is 1; otherwise it is 0.

Requirements

P1) $P(A, B) = 0$ if and only if $A \cap B = \emptyset$.(Dubios and Prade, 1980)

P2) $P(A, B) = 1$ if $A \subseteq B_{\alpha=1}$ or $B \subseteq A_{\alpha=1}$.

P3) $P(A, B) = P(B, A)$.

P4) $P(A, B)$ depends on $g(A \cap B)$, where g is a scalar evaluator.

Remarks 4.2

By P1, there is only one condition that produces the value 0 for partial matching. As with the inclusion indices, partial matching indices may be 0 even when the focal sets for A and B are not disjoint. This situation results from the selection of the T-norm used to determine the intersection of the fuzzy sets. In particular $T_1(A,B) = \emptyset$ when memberships of the elements satisfy $\mu_A(u) + \mu_B(u) - 1 \leq 0$ for all $u \in U$. P1 and P4 combine to require that the scalar evaluator g must be existential ($g = \sup$ or $g = rel$).

P2 gives a sufficient condition for a partial match of 1. Using the notation of inclusion indices, this condition may be written

$$I_{S_3}/g\,(A,B) = 1 \; or \; I_{S_3}/g\,(B,A) = 1 \Rightarrow P_{\cap}/g\,(A,B) = 1.$$

P2 is not a necessary condition for producing a partial match of 1 since some partial matching indices produce 1 under other conditions, for example, the sup-min compatibility.

The notation for partial matching indices, $P_{\cap}/g$, indicates that they are constructed by selecting a T-norm for ∩ and a scalar evaluator.

The general formula for a partial matching index that satisfies P1 to P4 is

$$P_{\cap}/g\,(A,B) = \frac{g(A \cap B)}{\min(g(A), g(B))}.$$

The combining function in the denominator must be min in order to satisfy P2 and P3. The function min could be replaced by other symmetric functions such as mean and still preserve P3, but P2 would not be satisfied. If a mean operator is substituted for the min function, the partial matching index would instead be scaled by a function of the cardinality of both fuzzy sets not just that of the smaller set.

When $g = rel$, the partial matching index may be written

$$P_{\cap}/rel\,(A,B) = \frac{|A \cap B|}{\min(|A|,|B|)} = \max\left(\frac{|A \cap B|}{|A|}, \frac{|A \cap B|}{|B|}\right)$$

$$= \max\left(I_{\cup}/rel\,(A,B), I_{\cup}/rel\,(B,A)\right).$$

$P_{\cap}/rel\,(A,B)$may be derived from the parameterized ratio model using relative cardinality for f and setting $\alpha = 1$ and $\beta = 0$ if $|A| \leq |B|$; otherwise, $\alpha = 0$ and $\beta = 1$.

When $g = sup$ and the fuzzy sets are normal, the partial matching index is $\sup(A,B)$, which is the sup-min index of Zadeh when the intersection is defined by T- norm T_3. A sup-min comparison of fuzzy sets has also been referred to as evaluating the consistency of the sets. If the fuzzy sets are not normal, $P(A,B)$ becomes a relative partial matching, also referred to as relative consistency, since this value is divided by $\min(\sup A, \sup \mathrm{B})$. For normal fuzzy sets,

$P_{\cap}/sup\,(A,B) = 1$if and only if there exists a u_i such that $\mu_A(u_i) = \mu_B(\mu_i) = 1$. When fuzzy sets are not normal, producing a value of 1 with P_{T_3}/sup requires the largest membership in the intersection of the two fuzzy sets to be equal to the smaller of the heights of the two fuzzy sets.

For example

$$A = 0.6/u_1 + 0.3/u_2$$

$$B = 0.5/u_1 + 0.5/u_2$$

$P_{T_3}/sup\,(A,B) = 1$which has the same partial matching value as

$$\acute{A} = 1.0/u_1 + 1.0/u_2$$

$B = 1.0/\acute{u_1} + 1.0/u_2$.

Comparing partial matching indices produced with $g = sup$ on fuzzy sets that may be subnormal may be misleading. As seen in the example above, the index value of the first pair of fuzzy sets indicates a partial match equal to that of the second pair. The problem is the degree of fuzziness is not considered in the partial matching with $g = sup$.

The definition of partial matching is very flexible in providing a value of

1. For fuzzy sets

$$A = 0.2/u_1 + 0.4/u_2 + 1.0/u_3 + 0.4/u_4 + 0.2/u_5$$

$$\acute{A} = 0.6/u_1 + 0.8/u_2 + \acute{1}.0/u_3 + 0.8/u_4 + 0.6/u_5$$

and$B = 1.0/u_1 + 1.0/u_2 + 1.0/u_3 + 1.0/u_4 + 1.0/u_5$,

$P_{\cap}/g\ (A, B) = P_{\cap}/g\ (\acute{A}, B)$even though $A \subseteq \acute{A} \subseteq B$.

If $\acute{A}$ is considered a better match with Bthan A, then the next subclass of set-theoretic measures, similarity indices, must be used to distinguish between the degree of matching.

4.3 Similarity Indices

Similarity indices measure the similarity between two fuzzy sets and require a greater degree of agreement than partial matching indices. For crisp sets, the cardinality of the complement of the symmetric difference has classically been used in determining a degree of equivalence or similarity of the sets. With this approach, sets X and Y are deemed completely similar if and only if their symmetric difference is the empty set. That is, if the sets are identical. Complete dissimilarity occurs when the symmetric difference is the union of $X$and Y.For fuzzy sets,the condition for complete similarity agrees with the definition of fuzzy set equality; $S_Z(A,B) = 1$ if and only if $\mu_A(u) = \mu_B(u)$ for all $u \in U$.

Symmetric Difference

The symmetric difference between crisp sets may be written as $(A \cup B) \cap (\bar{A} \cup \bar{B})$ or $(\bar{A} \cap B) \cup (A \cap \bar{B})$. By DeMorgan's laws, these two representations are equivalent. The symmetric difference between fuzzy sets A and B may also be defined in two ways

$$A \Delta^+ B = (A \cup B) \cap (\bar{A} \cup \bar{B})$$

and$A\, \Delta^- B = (\bar{A} \cap B) \cup (A \cap \bar{B})$.

As noted above, for crisp sets $\Delta^+ = \Delta^-$. When the sets are fuzzy, Δ^- depends upon the selection of the fuzzy set operators for $\cup$and $\cap$. In the following analysis, the letter D is used to denote a binary symmetric difference operator between the membership values of two elements. The symbol Δ is used to denote an arbitrary symmetric difference. The general formula for D^+ with an arbitrary T-norm and its dual T-conorm is

$$D^+{}_T(a,b) = T[S(a,b), S(1-a, 1-b)]$$

and for D^-

$D^-{}_T(a,b) = S[T(1-a,b), T(a, 1-b)]$.

Substituting corresponding pairs of T-norms and T-conorms for $\cap$ and $\cup$ produces different symmetric difference operators. When T_3 is chosen as the T-norm,

$$D^+{}_{T_3} = \min[\max(a,b), \max(1-a, 1-b)]$$

$$D^-{}_{T_3} = \max[\min(1-a,b), \min(a, 1-b)]$$

In this case, $\Delta^+ = \Delta^-$

Zadeh's original definition of the $bounded\ difference$ between fuzzy sets is the fuzzy set of elements that belong more to A and to B. Bounded difference is an extension of crisp set difference $A - B = A \cap \bar{B}$. Let $BD_T(a,b)$ denote the bounded difference between values a and b obtained using T-norm T.

Choosing T-norm $\cap = T_1$

$$BD_{T_1}(a,b) = \max(0, a + 1 - b - 1) = \max(0, a - b).$$

A symmetric difference may be derived using $\cup = S_1$ between $A - B$ and $B - A$ with the bounded difference operator,

$$D^-{}_{T_1}(a,b) = \min(1, \max(0, a - b) + \max(0, b - a) = |a - b|.$$

When the pair T_1 and S_1 is substituted in $(A \cup B) \cap (\bar{A} \cup \bar{B})$,

$$D^+{}_{T_1}(a,b) = \max(0, \min(1, a + b) + \min(1, 1 - a + 1 - b) - 1)$$

$$= \min(a + b, 2 - a - b)$$

is produced finally for T_2 and S_2,

$$D^-{}_{T_2}(a,b) = (1 - a)b + a(1 - b) - (1 - a)b(1 - b)a$$

$$= a + b - ab(3 - a - b + ab)$$

and

$$D^+{}_{T_2}(a,b) = (a + b - ab)(1 - a + 1 - b - (1 - a)(1 - b)$$

$$= (a + b - ab)(1 - ab).$$

The inequality $D^-{}_{T_1} \leq D^+{}_{T_1}$ is easy seen to hold. Subtracting $D^+{}_{T_2}(a,b)$ from $D^-{}_{T_2}(a,b)$ yeilds $2ab(1 - a)(1 - b) \geq 0$. Consequently, $D^-{}_{T_2} \leq D^+{}_{T_2}$.

Symmetric difference has been frequently used in measures of similarity between crisp sets. With $\alpha = 1$ and $\beta = 1$ and the cardinality used for f, the parameterized ratio model of similarity produces the ratio of the cardinality of the intersection of two sets and the sum of cardinality of their intersection and the cardinality of their symmetric difference. This follows since the set difference $X - Y$ is $X \cap \bar{Y}$ and the set difference $X - Y$is $\bar{X} \cap Y$. Since cardinality is additive when sets are disjoint (for crisp sets $X - Y$ and $Y - X$ are disjoint),

$$f(X - Y) + f(Y - X) = f((X - Y) \cup (Y - X)$$

$$= f((X \cap \bar{Y}) \cup (\bar{X} \cap Y)$$

$$= f(X \Delta Y).$$

Employing the preceding identity, the parameterized model becomes

$$S_{1,1}(X, Y) = \frac{f(X \cap Y)}{f(X \cap Y) + f(X \Delta Y)}$$

For fuzzy sets $A$and B, $A - B$ and $B - A$ are not necessarily disjoint so that the sum $f(A - B) + f(B - A)$does not necessarily reduce to the symmetric difference. For example, with Frank's dual pairs

$$f(A - B) + f(B - A) = f((A - B) \cup (B - A) + f((A - B) \cap (B - A))$$

$$= f\big((A \cap \bar{B}) \cup (\bar{A} \cap B)\big) + f((A \cap \bar{B}) \cap (\bar{A} \cap B))$$

$$= f(A \Delta B) + f\big((A \cap \bar{A}) \cap (\bar{B} \cap B)\big).$$

Similarity Measure Generation

Dubois and Prade, (1980) proposed four conditions that must be satisfied for a measure constructed from a symmetric difference operator to be considered a similarity index. These conditions generalize the properties symmetric difference similarity assessment from crisp to fuzzy sets. A similarity index or measure S must satisfy

S1) $S(A, B) = 1$ if and only if $A \Delta B = \emptyset$.

S2) if A and B have disjoint supports, then $S(A,B) = 0$.

S3) $S(A,B) = S(A,B)$.

S4) $S(A,B)$ depends on $g(\overline{A\Delta B})$ or on a symmetric function of $g(A \cup \bar{B})$ and $g(\bar{A} \cup B)$.

The role of the scalar evaluator g is to reduced the complement of the symmetric difference into a single representative value.

Remarks 4.3

It has been noted that S1 is not equivalent to requiring $S(A,B) = 1$ if and only if $A = B$; not all fuzzy symmetric difference operators produce $\emptyset$ when $S_Z(A,B) = 1$. for example, with D^+ generated by the T-norm T_1, $A\Delta^+{}_{T_1}B = \emptyset$ if and only if $A = B$ and A and B are not crisp. For crisp sets, however, all the symmetric difference operators produce $\emptyset$ if $A = B$.

Condition S2, like I_1, requires $S(A,B)$ to be 0 whenever A and B have disjoint supports. Disjointness is not implied when a set theoretic measure is 0.

Three types of general formulas for similarity indices are defined. The $type$-1 similarity measures, $S_{1/\Delta/\cup/g}$, are derived using a universal scalar evaluator of $\overline{A\Delta B}$.

The four parameters in the subscript represent the type, the symmetric difference operator, the union operator, and the scalar evaluator used to construct the similarity index.

When A and B have disjoint supports, the fuzzy set $A\Delta B$ assumes its maximum value $A \cup B$ for any combination of Δ and $\cup$. thus the range of $g(\overline{A\Delta B})$ is $[g(\overline{A \cup B}), 1]$. the generic $type$-1 similarity index is defined as

$$S_{1/\Delta/\cup/g}(A,B) = \frac{(g(\overline{A\Delta B}) - g(\overline{A \cup B}))}{1 - g(\overline{A \cup B})}.$$

The above definition is equivalent to

$$S_{1/\Delta/\cup/\acute{g}}(A,B) = \frac{\acute{g}(A \cup B) - \acute{g}(A\Delta B)}{\acute{g}(A \cup B)}$$

$$= 1 - \frac{\acute{g}(A\Delta B)}{\acute{g}(A \cup B)}$$

when$\acute{g}$ is both universal and existential.

The presentation of similarity measure assumes that Δ and $\cup$ may be independently chosen. As noted in the normalization process, $A\Delta B = A \cup B$ whenever A and B have disjoint supports regardless of the independent selection of Δ and $\cup$. when the support are disjoint and Δ^+ and $g = rel$ are used, $g(\overline{A\Delta B})$ may be less than $g(\overline{A \cup B})$. when this occurs, the similarity measure as given by the $type$-1 measure can produce a negative value. For example, assume that the supports of A and B have only the element u in common and $\mu_A(u) = a$ and $\mu_B(u) = b$.Since $g = rel$, $g(A\Delta^+B) > g(A \cup B)$ is equivalent to $g(\overline{A\Delta^+B}) < g(\overline{A \cup B})$. If $g(A\Delta^+B) > g(A \cup B)$ will produce a negative value. The following examples illustrate combinations of $\cup$ and Δ^+ where a negative similarity measure may occur when Δ and $\cup$ are chosen independently.

If $\cup = S_2$ and $\Delta^+ = {D^+}_{T_1}$

$\min(a + b, 2 - a - b) > (a + b - a \cdot b)ifa + b \leq 1$

If $\cup = S_3$ and $\Delta^+ = {D^+}_{T_1}$

$\min(a + b, 2 - a - b) > \max(a, b)ifa + b < 1$

If $\cup = S_3$ and $\Delta^+ = {D^+}_{T_2}$

$(a + b - a \cdot b) \cdot (1 - a \cdot b) > \max(a, b)$if $a = b, a \leq 0.5$.

These examples show that Δ and $\cup$ should not be chosen independently. When the symmetric difference Δ^{+} is constructed using the same $\cup$ as used in the normalization factor $g(A,B)$, $g(\overline{A\Delta^{+}B}) \geq g(\overline{A\cup B})$ and $g(A\Delta^{+}B) \leq g(A\cup B)$ since

$$g((A\cup B)\cap(\bar{A}\cup\bar{B}) \leq g(A\cup B).$$

One of the frequently used similarity measures is the Jaccard index. The index $S_{1/D^{-}{}_{T_1}/S_3/rel}$ is an example of fuzzy extension of the Jaccard index and is equivalent to the fuzzy similarity measure proposed. Its derivation is based on the subsethood measure, which was showed to be equivalent to the set-theoretic inclusion measure $I_{S_1/rel}(A,B) = |A\cap_{T_3} B|/|A|$. The similarity measure used for parameter learning using fuzzy neural networks can be shown to be equivalent to $S_{1/D^{-}{}_{S_1}/S_3/rel}$.

With the bold union of Zadeh's bounded set differences $A-B$ and $B-A$ for symmetric difference and relative cardinality for g substituted into $S_{1/D^{-}{}_{S_1}/S_3/rel}(A,B)$ becomes

$$1-\frac{\sum(\min(1,\max(0,\mu_A(x)-\mu_B(x)+\max(0,\mu_B(x)-\mu_A(x)))}{|A\cup B|}$$

$$=1-\frac{\sum\max\left(0,\mu_A(x)-\mu_B(x)\right)+\max(0,\mu_B(x)-\mu_A(x))}{|A\cup B|}$$

$$=1-\frac{\sum(\mu_A(x)-\min(\mu_A(x),\mu_B(x)+\mu_B(x)+\mu_B(x)-\min(\mu_A(x),\mu_B(x))}{|A\cup B|}$$

Now letting $\cap = min$,

$$S_{1/D^{-}{}_{S_1}/D_3/rel}(A,B) = 1-\frac{(|A|-|A\cap B|+|B|-|A\cap B|)}{|A\cup B|}$$

$$=1-\frac{(|A|+|B|-2|A\cap B|)}{|A\cup B|},$$

which is the geometrically derived fuzzy similarity measure. Since

$\left|A\right| + \left|B\right| = \left|A \cap B\right| + \left|A \cup B\right|$, for any corresponding T_F and T_F pair,

$$S_{1/D^{-}{}_{S_1}/S_3/rel}(A,B) = \frac{\left|A \cup B\right| - (\left|A\right| + \left|B\right| - 2\left|A \cap B\right|)}{\left|A \cup B\right|}$$

$$= \frac{\left|A \cup B\right| - (\left|A \cup B\right| - \left|A \cap B\right|)}{\left|A \cup B\right|}$$

so that $$S_{1/D^{-}{}_{S_1}/S_3/rel}(A,B) = \frac{\left|A \cap B\right|}{\left|A \cup B\right|},$$

where $\cap = T_3$ and $\cup = S_3$. Letting $g = rel$, other selections of $\cup$ and Δ yield

$\left|A \cup B\right| - \left|A \Delta B\right| = \left|A \cap B\right|$ for an appropriately chosen $\cap$. Using this reduction, can be rewritten as $S_{1/\cap/\cup/rel}(A,B) = \frac{\left|A \cap B\right|}{\left|A \cup B\right|}$.

Note the changes in the subscript symbols to indicate the change in form of the $type$-1 similarity index. In this form it is easily seen to be equivalent to the Jaccard.

If the universal scalar evaluator $g = inf$ is used and A and B are normal fuzzy sets, then $\sup(A \cup B) = 1$ so that $inf\overline{A \cup B} = 0$. Substituting inf for g produces

$S_{1/\Delta/\cup/inf}(A,B) = inf_{u \in U}\left(1 - \mu_{A\Delta B}(u)\right) = 1 - sup_{u \in U}\mu_{A\Delta B}(u)$ for any symmetric difference operator.

The equivalence of $\overline{A\Delta^{-}B}$ and $(A \cup \bar{B}) \cap (\bar{A} \cup B)$ is the basis of $type$-2 similarity indices. Due to requirement S3, a symmetric function r of $g(A \cup \bar{B})$ and $g(\bar{A} \cup B)$ is used to construct these indices. Normalization is based on the identities $\bar{A} \cup B = \bar{A}$ and $A \cup \bar{B} = \bar{B}$ when A and B are disjoint. The general formula for $type$-2 similarity measures is

$$S_{2/\cup/r/g}(A,B) = \frac{r\big(g(A \cup \bar{B}), g(\bar{A} \cup B)\big) - r(g(\bar{A}), g(\bar{B}))}{1 - r(g(\bar{A}), g(\bar{B}))}.$$

In order to satisfy S1, $r(a,b) = 1$ if and only if $a = b = 1$ since whenever $A\Delta B = \emptyset$, $g(A \cup \bar{B}) = g(\bar{A} \cup B) = g(U) = 1$. if r is a T-norm, the $type$-2 similarity index may be viewed as a transformation of the $type$-1 similarity index in which the order of the aggregation and fuzzy set scalar evaluation is reversed. In $type$-1, the aggregation is performed first and then the fuzzy scalar evaluation. For $type$-2, the fuzzy set scalar evaluation is performed on the individual fuzzy sets and then aggregated.

For normal fuzzy sets and $g = inf$,

$$S_{2/\cup/r/inf}(A,B) = r(\inf(A \cup \bar{B}), \inf(\bar{A} \cup B)).$$

To satisfy S2, $r(0,0)$ must be 0. The symmetric function r may be T-norm or symmetric sum operator, which includes the arithmetic mean $(a+b)/2$. When $\cup = max$, the resulting similarity index may be considered to be the symmetric aggregation of the necessity of B knowing A and the necessity of Aknowing B.

$Type$-3 similarity indices are built from a symmetric function h of an inclusion index I by

$$S_{3/I/h}(A,B) = h\big(I(A,B), I(B,A)\big),$$

where$h$ satisfies $h(0,0) = 0$ and $h(a,b) = 1$ if and only if $a = b = 1$. These conditions satisfied by any T-norm or symmetric sum operator. $Type$-3 similarity measures may be considered to be the aggregation of the degree to which A is a subset of B and the degree to which B is a subset of A.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATION

5.0 Summary

A detailed literature review onA Study of Fuzzy Compatibility Relations and Their Applications was carried out. The concepts of similarity, dissimilarity, symmetric similarity, relative similarity, and multi-attributes of fuzzy objects were discussed. Applications of fuzzy compatibility relation were explored in studyingJaccard, Tversky, simple matching, mean character, and Canberra metric ratio models. In particular Tversky ratio model was further modified and some properties of the modified ratio model were described.

5.1 Conclusion

In this work, some of the applications of Tversky parameterized ratio model using the cardinality of fuzzy sets and other functions such as the scalar evaluators and their operators T-norms (T_1, T_2, T_3) and T-conorms (S_1, S_2, S_3) have been modified. A new ratio model for comparing fuzzy objects which included the degree of inclusion, partial matching, and similarity was proposed and its various properties were discussed.

5.2 Recommendations.

It is recommended that the outcome of the compatibility analysis undertaken in this dissertation should be used by researchers and system designers.

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