# APPROXIMATE ANALYTICAL SOLUTION OF NATURAL CONVECTION FLOW WITH VISCOUS DISSIPATION AND VARIABLE VISCOSITY EFFECTS

**BY**

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**MAY, 2018**

## Declaration

I declare that the work in this dissertation, titled: "**Approximate Analytical Solution of Natural Convection Flow with Viscous Dissipation and Variable Viscosity Effects**", has been performed by me in the Department of Mathematics under the supervision of Prof. A. O. AJIBADE and Prof. B. K. JHA. The information derived from literature have been indicated and duly acknowledged by means of references. No part of this dissertation was previously presented for another degree or diploma at this or any institution.

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(P16PSMT8080) Signature Date

# Certification

This Dissertation titled "**Approximate Analytical Solution of Natural Convection Flow with Viscous Dissipation and Variable Viscosity Effects**" by Nkereuwem DuffUYOH(P16PSMT8080), meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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|---|---|
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# Dedication

This dissertation is dedicated to God Almighty only, for He alone truly loves me and has made this work a reality, for on him alone did all the other supports I ever had, thrived, and could bear this fruit.

## Acknowledgement

While my unreserved and inexpressible gratitude goes to God, I am obliged to use this medium to sincerely thank and appreciate my lead supervisor in person of Prof. A. O. Ajibade for accepting to carry my load when I was stranded without a helper. God will surely bless and keep you for people like us. I have not forgotten my PG coordinator in person of Dr. H. M. Jibril for his simplicity of approach and timely advice in times of our dilemma. I cannot forget my lecturer in person of Prof. B. K. Jha for his elucidating lectures during my course work. I will not forget my uncle, Barrister M. D. Uyoh, who lit the torch that showed me the way to academic pursuit. I appreciate my wife and my entire family for their patience and accepting to, in their own way, share in the hurdles of this work. I also appreciate the H. O. D. Prof. A. A. Tijjani and other lecturers for their encouragements. Lastly I thank all my colleagues in the journey for their goodwill, in particular Mr Taiwo Yusuf, Mr Duru Timothy, Mr Michael Oni and specially Mr Peter Malgwi.

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# Nomenclature

| | |
|---|---|
| $A$ | steady velocity |
| $a$ | dimensionless variable viscosity parameter |
| $B$ | unsteady velocity |
| $C_p$ | specific heat at constant pressure [ $JKg^{-1}K^{-1}$] |
| $E$ | Eckert number |
| $E_0$ | Eckert number at $T = T_0$ |
| $F$ | steady temperature |
| $g$ | acceleration due to gravity [$ms^{-2}$] |
| $G$ | periodic temperature |
| $h$ | symmetrical width of the channel [$m$] |
| $Nu$ | Nusselt number |
| $Pr$ | Prandtl number |
| $Pr_0$ | Prandtl number at $T = T_0$ |
| $St$ | Strouhal number |
| $St_0$ | Strouhal number at $T = T_0$ |
| $t$ | dimensionless time |
| $T_0$ | temperature of the ambient [$K$] |
| $T_1$ | steady temperature of wall [$K$] |
| $T_2$ | periodic temperature of wall [$K$] |
| $T$ | temperature of fluid [$K$] |
| $U$ | velocity of fluid [$ms^{-1}$] |

$x$ vertical co-ordinate (direction of flow) $[m]$

$y$ horizontal coordinate $[m]$

## Greek Letters

α thermal diffusivity $[m^2s^{-1}]$

β coefficient of thermal expansion $[K^{-1}]$

$\eta$ dimensionless horizontal coordinate

$\mu$ coefficient of viscosity $[Kgm^{-1}s^{-1}]$

$\mu_0$ coefficient of viscosity at $T = T_0[Kgm^{-1}s^{-1}]$

$\nu$ kinematic viscosity $[m^2s^{-1}]$

$\nu_0$ kinematic viscosity at $T = T_0[m^2s^{-1}]$

ω frequency of periodic heat input $[s^{-1}]$

λ variable viscosity parameter$[K^{-1}]$

$\rho$ density of fluid $[Kgm^{-3}]$

## Dimensionless Quantities

Dimensionless quantities are used extensively in fluid Mechanics both in the development of theoretical models (to determine relationship between parameters) and in experimental work (to allow results to be generalized and hence used to predict performance for different systems). This dissertation used quite a number of the dimensionless quantities in the course of the research. We give brief notes on the dimensionless quantities used.

1. **Prandtl Number** ($\boldsymbol{Pr}$)**:** Prandtl number is a ratio of kinematic viscosity to thermal diffusivity and is a function only of fluid properties.($Pr$) is typically less than one for gases and greater than one for liquids. Mathematically, Prandtl number is given by the expression: $Pr = \frac{\nu}{\alpha}$; Where α, and ν are defined in the nomenclature.
2. **Strouhal Number** ($\boldsymbol{St}$)**:** Strouhal number gives the frequency of the periodic heating of the walls of the channel. $St$ is defined by the ratio: $St = \frac{h^2\omega}{\nu}$ ; where $h$ is the width of one-half of the channel, $\omega$ is the angular frequency of the periodic heating and ν the kinematic viscosity, as declared in the nomenclature. $St$ takes values from 0 to anywhere positive.
3. **Eckert Number** ($\boldsymbol{E}$)**:** The Eckert number is a dimensionless quantity that gives the rate of shear heating of the fluid by viscous dissipation. $E$ takes both positive and negative values. Positive values of $E$ corresponds to heat generation while negative values depict heat sink. It is defined as: $E = \frac{g^2\beta^2h^4(T_1-T_0)}{\nu^2C_p}$ ; $g, \beta, h, \nu, T_1, T_0$ and $C_p$ are all defined in the nomenclature.

# CHAPTER ONE

# GENERAL INTRODUCTION

## 1.1 Introduction

As humans our entire living from the very beginning through to the end is surrounded and facilitated by fluids; the very beginning being in the amniotic fluid (a mixture of liquids and gases). Coupled with this is the fact that these fluids are responsive to variations in temperature (heating and cooling) which in turn affects the normal functioning of our intelligently designed body systems causing it to seek redress in various reactive ways to maintain normalcy. In the same manner, the unintelligent inventions of science and engineering such as machine parts, electricaland electronic components, nuclear reactors, etc, that either generate heat or are subjected to heating must be harnessed to seek redress if they must last long and serve their purpose efficiently without going berserk. This can be effectively achieved if we understand the behaviour of fluids and fluid properties to these variations. For this, natural convection has continued to be and is still of extreme applicability as a means of cooling these systems. Again, fluids influence our movement by all kinds of means whether on land, water or in the air as well as the movement of mass and heat around (circulation), into (injection) and out (suction) of living things and inventions. This has made the study of fluid dynamics a worthwhile endeavor that cannot be over-emphasized.

The study of natural convection under different physical phenomena has received the attention of many authors. Gebhart (1962) showed that the heating of fluids by viscous

dissipation cannot be neglected for fluids with high Prandtl number when steady free convective flow is considered because it can greatly affect the flow characteristics. When fluids are sheared, viscous dissipation is induced by the friction between the different layers of particles of the fluid as they move against each other, generating heat in the process. It therefore follows that the rate of shear-induced heat generation within a fluid is responsive to variations in viscosity. Inspired by the separate works of Sahin (1999) and Tasnim and Mahmud (2009), Jha and Ajibade (2011) reported that viscosity is the most sensitive property of fluid to rising temperatures. For instance, 10% rise in the temperature of water results to as much as 140% reduction in viscosity, at least, Schlichting and Gersten(2000). Bar-Cohen and Roshenow (1984) studied the fully developed fluid flow between two periodically heated parallel plates to capture miniaturizations of electrical and electronic components. Nanda and Sharma (1963) separated the temperature and velocity into steady and periodic parts while studying the effects of sinusoidal variations in surface temperatures. Wang (1988) investigated the effect of Strouhal number on the development of boundary layers between two periodically heated parallel plates by separating the steady and unsteady parts and concluded that Strouhal number exerts an inverse effect on unsteady flow, but the effect tends to zero as Strouhal number tends to infinity.

## 1.2 Statement of the Problem

Jha and Ajibade (2011) investigated the effects of viscous dissipation on natural convection flow in a vertical channel formed by two infinite parallel plates and concluded that the viscous dissipation heating becomes stronger than those of the plates when the thermal

diffusivity of the fluid is relatively small causing transfer of heat from fluid to plates instead, but their results are limited to very narrow range of values of *Eckert Number*(*E*). Again the viscous dissipation parameter, *E*, is limited in its role in the work of Jha and Ajibade (2011) as it was used as a perturbation parameter only, being an off-shoot of the bottle necks of the general perturbation method. Above all, as important a fluid property as viscosity is due to its sensitivity to variations in temperature as demonstrated in the independent works of Sahin (1999) and Tasnim and Mahmud (2009), its interaction with viscous dissipation was not taken into account in any of the works mentioned above. And as viscosity is a physical property of all fluids, the combined effects of variable viscosity and viscous dissipation will be of general relevance, hence the motivation. In this dissertation therefore, we seek to find the solution to both the energy and momentum fields of the working fluid to capture viscous dissipation and variable viscosity effects.

## 1.3 Aim and Objectives of the study

The aim of this work is to investigate the effects of variable viscosity andviscous dissipation on steady, fully developed natural convection flow of incompressible fluids between infinitevertical parallel plates. This aim will be achieved through the following objectives; which are to:

(i) study the effect of varying viscosity on both the velocity and temperature profiles for the fluids in a vertical channel.

(ii) examine the contribution of suction/injection to both the energy and momentum of the fluids.

(iii) investigate the general effects of viscous dissipation on the fluid flow pattern and temperature.

## 1.4Research Methodology

To attain the set objectives, once the new problem was identified, the governing equations were obtained with their boundary conditions and necessary modifications were made to capture the new situations and objectives and scale the limitations of the previous works. A comprehensive study of earlier works was carried out by means of literature review. The dimensional equations of energy and momentum for the new problem was first separated into steady and periodic parts. These dimensional steady and periodic equations alongside their boundary conditions were reduced to dimensionless forms by the introduction of non-dimensional quantities. The resulting non-dimensional partial differential equations, being strongly non-linear and coupledwere subjected to the 'Homotopy Perturbation' method which reduced them to a system of linear ordinary differential equations which were solved by the undetermined coefficients method. The expressions for both temperature and velocity fields were obtained as well as the skin friction and rate of transfer of heat on the bounding plates for the two cases. The results obtained are simulated for different values of the governing parameters and presented in graphical forms. The determination of the effects of the governing parameters on the flow elements as well as in the validation of obtained results are done and conclusions drawn.

## 1.5 Organization of the Dissertation

This dissertation is divided into six chapters with references and appendixes. Chapter one has the general introduction while chapter two is composed of the literature review. Chapter three carries the method that was employed in solving the problems of the

research. Chapter four contains the solutions of the two problems, chapter five has the graphical analysis of the solutions and discussions, while chapter six is made up of the summary, conclusion and recommendation which is followed by references and appendices.

## 1.6 Basic definitions

(1) **Compressible and incompressible flow:** A compressible flow is a flow in which the fluid density varies significantly within the flow field. While in an incompressible flow the density does not change.

(2) **Free orNatural Convection**: Free or Natural convection is the process when a temperature difference produces a density difference which results in mass movement.

(3) **Symmetric heating:** this is a type of heating of the boundary layers in which equal amount of constant heating is applied on boundary surfaces of the system.

(4) **Dimensionless quantity:** is a quantity without an associated physical dimension.

(5) **Skin friction:** Is a component of drag, the force resisting the motion of a solid body through a fluid or the motion of fluid by a solid surface. It arises from the friction of the fluid against the skin of the object that is moving through it. Skin friction arises from the interaction between the fluid and the skin of the body, and is directly related to the area of the surface of the body that is in contact with the fluid. Skin friction follows the drag equation and rises with the square of the velocity. Skin friction can be reduced by shaping the moving body so that smooth flow is possible.

(6) **Suction:** Is a force that causes fluids to be drawn into an interior space or to adhere to a surface because of the difference between internal and external pressure.

(7) **Injection:** Is a force that causes a fluid to be drawn out of an inner space or to adhere to a surface because of the difference between the external and internal pressures.

## 1.7 Basic Governing Equations

Continuity Equation:

$$Div\left(\rho\vec{V}\right)=0 \tag{1.7.1}$$

for incompressible fluids, the continuity equation becomes

$$\rho\nabla.\vec{V}=0\rightarrow\nabla.\vec{V}=0 \tag{1.7.2}$$

Navier-Stoke's Equation:

$$\frac{\partial q}{\partial t}+\left(\vec{V}.\nabla\right)q=F-\frac{1}{\rho}\nabla P+\frac{\mu}{\rho}\nabla^2\vec{V} \tag{1.7.3}$$

Energy Equation:

$$\rho_{Cp}\left[\frac{\partial T}{\partial t}+\left(\vec{V}\nabla\right)T\right]=k\nabla^2T+\nabla q \tag{1.7.4}$$

# CHAPTER TWO

# LITERATUREREVIEW

## 2.1 Introduction

Natural convection flow under suction and injection between vertical infinite parallel plates has enormous applications in scientific and engineering processes such as in the operations of air conditioners, nuclear reactors, the cooling of engines and electrical and electronic components as well as in heat exchangers to mention a few. A lot of works exist in literature that investigated the different physical situations considered in this dissertation. Some of these works are presented in this chapter under different sub-headings.

## 2.2 Natural convection flow

The study of natural convection flow due to steady-periodic heating has received attention due to its applicability in different scientific and engineering systems. Natural convection is still a means of cooling various engineering and control systems as well as electrical and electronic components which generate heat and are subjected to heating periodically. Viscosity is a property of fluid which changes with changing temperatures. Gebhart (1962) showed that the heating of fluids by viscous dissipation cannot be neglected for fluids with high Prandtl number when steady free convective flow is considered because it can greatly affect the flow characteristics. Again, when fluids are sheared, viscous dissipation is induced by the friction between the different layers of particles of the fluid as they move against each other, generating heat in the process. It therefore follows that the rate of shear-induced heat generation within a fluid is responsive to variations in viscosity. Inspired by

the separate works of Sahin (1999) and Tasnim and Mahmud (2009), Jha and Ajibade (2011) stated that viscosity is the most sensitive property of fluid to rising temperatures. For instance, 10% rise in the temperature of water results to as much as 140% reduction in viscosity, at least, Schlitchting and Gersten (2000). Bar-Cohen and Roshenow (1984) studied the fully developed fluid flow between two periodically heated parallel plates to capture miniaturizations of electrical and electronic components. Nanda and Sharma (1963) while studying the effects of sinusoidal variations in surface temperatures on free convective laminar flow were able to separate the temperature and velocity into steady and periodic parts. Wang (1988) investigated the effect of Strouhal number on the development of boundary layers between two periodically heated parallel plates by also separating the steady and unsteady parts and concluded that Strouhal number exerts an inverse effect on unsteady flow, but the effect tends to zero as Strouhal number tends to infinity. Barletta and Rossi di Schio (2004) studied a mixed convection flow in a circular duct whose wall is subjected to periodic sinusoidal heating and were able to show that there exists a resonance frequency where velocity oscillations reach a maximum. Several other authors who have taken in hand to investigate natural convection flow under different physical phenomena include Muhuri and Gupta (1979), Zacchini (1998), Watanabe and Pop (1994), Barletta (1998), Jha and Ajibade (2009), Soundalgekar *et al* (2004) among others.Jha and Ajibade (2011) investigated the effects of viscous dissipation on natural convection flow in a vertical channel formed by two infinite parallel plates and concluded that the viscous dissipation heating becomes stronger than those of the plates when the thermal diffusivity of the fluid is relatively small causing transfer of heat from fluid to plates instead, but their results are limited to very narrow range of values of $E$. Again the viscous dissipation parameter, $E$, is limited in its role in the work of Jha and Ajibade (2011) as it was used as a

perturbation parameter only, being an off-shoot of the bottle necks of the general perturbation method.

## 2.3 Suction and Injection

The significance of suction and injection is well recognized in space science and aerodynamics in the area of boundary layer control. In the area of mass transfer cooling, suction/injection can significantly alter the flow field and by extension affect the rate of heat transfer on the porous boundaries. The flow with heat transfer of viscous incompressible fluid between parallel plates has important applications in many systems and devices such as electrostatic precipitation, aerodynamics heating, polymer technology and the petroleum industry. Shojaefard *et al* (2005) investigated flow control on a subsonic airfoil by suction and injection and concluded that suction significantly increases the lift coefficient while injection decreases the surface skin friction resulting in significant reduction in energy consumption of such aircrafts. This by extension will amount to reduced level of air pollutant emission as well as cost for such air crafts. Attia (2005) also investigated the effect of suction and injection on unsteady flow between parallel plates. By and large, physical phenomena in Medicine, Engineering and several other fields of research naturally involves fluids that are Newtonian in nature. Jha and Ajibade (2009) studied the influence of suction/injection on natural convection flow of viscous fluid when the convection current is induced by the heating of the porous boundaries, but considers fluid viscosity to be a constant. Recently, Rundora and Makinde (2013) studied the effects of suction and injection on unsteady reactive variable viscosity fluid flow with porous medium but considers the fluid to be non-Newtonian in nature.

## 2.4 Variable viscosity

However, viscosity is a fluid property that varies with temperature change, therefore results ofinvestigations obtained under the assumptions of constant fluid viscosity become obsolete and inappropriate in describing different physical phenomena especially where heat transfer rate is of interest. It is found that the variations in viscosity of gases is quite different from those of liquids (Cheng and Minkowycz, 1977).As a result, the role of variable viscosity has become important in both practical and theoretical studies. This type ofproblems has applications in geophysics, especially in the extraction of geothermal energy as well as in insulation systems employing porous materials. Tackley (1996) studied the effects of strongly variable viscosity on three dimensional compressible convections in planetary mantles and concluded that the temperature dependent viscosity strongly decreases the temperature in the inner convective cells.Mukhopadhyay *et al* (2005) studied the Magneto-hydrodynamic(MHD) boundary layer flow over a heated stretching sheet with variable viscosity and concluded that a decrease in fluid viscosity makes the velocity to decrease with increasing distance of the stretching sheet. Lai and Kulacki (1990) investigated the effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous medium. Different models of incorporating variable viscosity as a function of temperature into the modelling of physical problems available in literature are (1) as an inverse function of temperature (2) as an exponential function of temperature (3) as an inverse linear function of temperature and (4) as a linear function of temperature.

In view of the relevance of variation of viscosity of fluid with temperature, the work of Jha and Ajibade (2011), as well as Jha and Ajibade (2009)becomes inadequate in analyzing systems involving fluid flow with variable viscosity effects, hence the motivation for the

present work. The present problem therefore remodels the work of Jha and Ajibade (2011) to capture a more realistic situation in which the fluid viscosity is temperature dependent. By this remodeling, the governing equations become coupled and strongly non-linear necessitating the adoption of a numerical or approximate solution approach. Once again, the method of Homotopy perturbation has been favoured and used for the solution of the problem.

Above all, as important a fluid property as viscosity is due to its sensitivity to variations in temperature, as demonstrated in the independent works of Sahin (1999) and Tasnim (2009), its interaction with viscous dissipation was not taken into account in any of the works mentioned above. And as viscosity is a property of many fluids, the combined effects of variable viscosity and viscous dissipation will be of general relevance, hence the motivation. The inter-dependency of the velocity and the temperature fields informed the use of the homotopy perturbation technique to uncouple the equations as well as overcome the limitations of the general perturbation method.

# CHAPTER THREE

# MATHEMATICAL ANALYSES

## 3.1 Introduction

This chapter is preoccupied with the mathematical formulation, derivation and solution of both the energy and momentum equations of the two problems under investigation in this dissertation. The dimensional energy and momentum equations are first put in dimensionless forms by the introduction of appropriate non-dimensional quantities. The Homotopy Perturbation method is then applied to linearize and uncouple the non-linear and coupled differential equations before solving by the undetermined coefficient method to obtain close form expressions for both the velocity and temperature fields in the two problems (see Kryszig E., 1990). The skin friction and Nusselt number are also calculated as well as the phases and amplitudes of the unsteady velocity and temperature.

## 3.2 Mathematical formulation and Geometrical Description

In this subsection, we formulate a mathematical modelof natural convection flowbetween infinite vertical parallel plates with viscous dissipation and variable viscosity effects.

Considering the flow of viscous dissipating incompressible fluid with temperature dependent viscosity between two vertical and parallel plates of infinite length, the channel formed by the boundary plates are at fixed distances $\pm h$ on the horizontal axis from the center line, $h = 0$. The direction of flow is taken as the $x - axis$ vertically while the direction of heat penetration from the ambient (plates) is taken as $y - axis$ perpendicular to the direction of flow. The initial temperature of the fluid within the channel is taken as $T_0$.

The bounding plates are maintained at a steady temperature $T_1$with a steady-periodic heating $T_2$which is applied on both plates simultaneously.The heating of the boundary plates induces a temperature gradient due to buoyancy effect and natural convection sets in. Due to the viscosity of the fluid, the velocity at the boundaries remain zero (no-slip).

### *3.2.1* Natural convection flow with viscous dissipation and variable viscosity effects

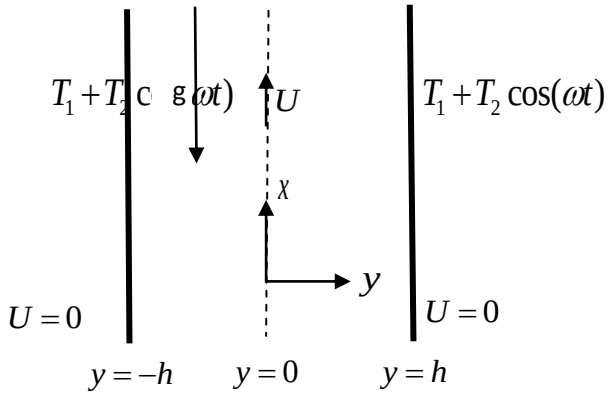


Figure**3.1:** Schematic diagram of flow configuration and coordinates system

The governing equations of momentum and energy for this situation, following Jha and Ajibade (2011), taking into consideration variable viscosity are:

$$\frac{\partial U}{\partial t} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial U}{\partial y}\right) + g\beta(T - T_0) \tag{3.1}$$

$$\frac{\partial T}{\partial t} = \frac{K}{\rho C_p}\frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}\right) + \frac{\mu}{\rho C_p}\left(\frac{\partial U}{\partial y}\right)^2 \tag{3.2}$$

where

$$\mu = \mu(T) = \mu_0[1 - \lambda(T - T_0)] \tag{3.3},$$

$\mu$ is the temperature dependent viscosity, $\mu_0$ is the fluid viscosity at the temperature $T_0$ and $\lambda$ is the variable viscosity parameter.

Expanding (3.1) and (3.2) using (3.3), the equations of motion and the boundary conditions become:

$$\frac{\partial U}{\partial t} = [1 - \lambda(T - T_0)]\nu_0 \frac{\partial^2 U}{\partial y^2} - \nu_0 \lambda \frac{\partial T}{\partial y}\frac{\partial U}{\partial y} + g\beta(T - T_0) \tag{3.4}$$

$$\frac{\partial T}{\partial t} = \frac{K}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{\nu_0[1-\lambda(T-T_0)]}{C_p}\left(\frac{\partial U}{\partial y}\right)^2 \tag{3.5}$$

$$y = h: U(y,t) = 0, T(y,t) = T_1 + T_2\cos(\omega t),$$
$$y = -h: U(y,t) = 0, T(y,t) = T_1 + T_2\cos(\omega t) \tag{3.6}.$$

Separating the velocity and temperature into steady and periodic parts, we use the following expressions, following Jha and Ajibade (2011):

$$U = \frac{g\beta^2}{\nu_0}\left[(T_1 - T_0)A(\eta) + T_2 B(\eta)e^{i\omega t}\right] \tag{3.7}$$

$$T = T_0 + (T_1 - T_0)F(\eta) + T_2 G(\eta)e^{i\omega t} \tag{3.8}$$

where $A(\eta), F(\eta)$represent the steady parts and $B(\eta), G(\eta)$ represent the periodic parts of velocity and temperature respectively.

## 3.3 Natural convection heat transfer flow in a vertical porous channel with viscous dissipation and variable viscosity effects.

Having considered and rendered the governing equations for natural convection flow between infinite vertical parallel plates with viscous dissipation and variable viscosity effects in section ***3.2.1***, in this section we are set to formulate, by appropriate modifications, the mathematical model for the situation under suction and injection as a result of the porosity of the channel plates.

### *3.3.1 Mathematical formulation (Problem II)*

Considering suction/injection with velocity$V_0$on the porous plates at $\pm h$ from the center of the vertical channel, the governing momentum and energy equations in *3.2.1* are modified to depict the arising physical situation as follows:

$$\frac{\partial U}{\partial t} + V_0 \frac{\partial U}{\partial y} = [1 - \lambda(T - T_0)]\nu_0 \frac{\partial^2 U}{\partial y^2} - \nu_0 \lambda \frac{\partial T}{\partial y}\frac{\partial U}{\partial y} + g\beta(T - T_0) \quad (3.9)$$

$$\frac{\partial T}{\partial t} + V_0 \frac{\partial T}{\partial y} = \frac{K}{\rho C_p}\frac{\partial^2 T}{\partial y^2} + \frac{\nu_0[1-\lambda(T-T_0)]}{C_p}\left(\frac{\partial U}{\partial y}\right)^2 \quad (3.10)$$

$$y = h: U(y,t) = 0, T(y,t) = T_1 + T_2\cos(\omega t),$$

$$y = -h: U(y,t) = 0, T(y,t) = T_1 + T_2\cos(\omega t) \quad (3.11),$$

where,

$$U = \frac{g\beta^2}{\nu_0}\left[(T_1 - T_0)A(\eta) + T_2 B(\eta)e^{i\omega t}\right] \quad (3.12)$$

$$T = T_0 + (T_1 - T_0)F(\eta) + T_2 G(\eta)e^{i\omega t} \quad (3.13)$$

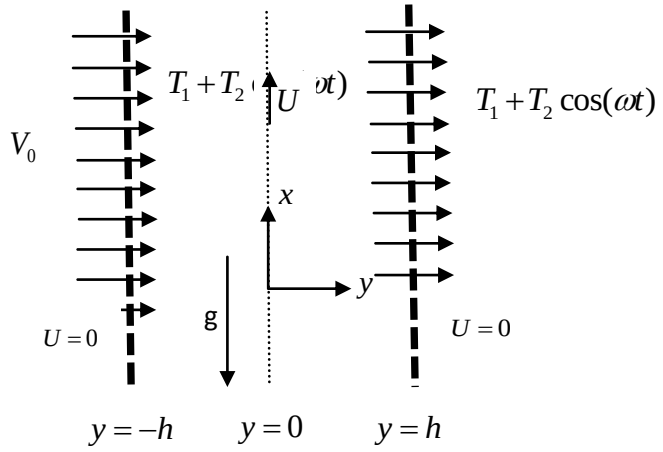


Fig. 3.2 Schematic diagram of flow configuration and coordinates system

## 3.4 Non-dimensionalization

Dimensional analysis is important in the sciences especially where there exist relationships between different quantities with differing units in computations and experimental work. It seeks to express all quantities in the three fundamental quantities of mass($m$), time($t$) and length($l$). They help in establishing and simplifying existing relationships between multiple variables as well as give direction to factors that have significant impact on studied phenomena.

Non-dimensional quantities however allow free and all round measurement and comparison of quantities and values in experimental work. Non-dimensionalization is therefore usually carried out *ab-initio* to ease computations in researches of this nature.

For the present problems the following non-dimensional quantities have been introduced:

$$\eta = \frac{y}{h},\ Pr = \frac{\nu}{\alpha},\ \ St = \frac{h^2\omega}{\nu},\ \ E = \frac{g^2\beta^2h^4(T_1-T_0)}{\nu^2C_p},\ a = \ \lambda(T_1 - T_0),\ s = \frac{V_0h}{\nu}$$

$$Pr_0 = \frac{\nu_0}{\alpha},\ St_0 = \frac{h^2\omega}{\nu_0},\ E_0 = \frac{g^2\beta^2h^4(T_1-T_0)}{{\nu_0}^2C_p},\ s_0 = \frac{V_0h}{\nu_0} \qquad (3.4.1)$$

Where $\eta$ is the characteristic width of the channel, $Pr$ is the Prandtl number, $St$ is the Strouhal number, $E$ is the Eckert number, $s$ is the suction parameter, $a$ is the variable viscosity parameter and the subscripts '0' indicate values at $T = T_0$.

### *3.4.1 Non-dimensionalization of problem I (3.2.1)*

Substituting (*3.7*)and (*3.8*) into (*3.4*) and (*3.5*) and using the non-dimensional quantities (*3.4.1*) the momentum equation (*3.4*) and the energy equation(*3.5*) as well as the boundary conditions (*3.6*) can be written in dimensionless form as:

$$A'' = aA''F + aA'F' - F$$

$$B'' - iStB = aB''F + aA''G + aB'F' + aA'G' - aiStBF - G$$

$$F'' = -EPr(A')^2 + 2aEPr(A')^2F - a^2EPr(A')^2F^2 \quad (3.4.2)$$

$$G'' - iStPrG = -2EPrA'B' + aEPr(A')^2G + 4aEPrA'B'F$$
$$-2a^2EPrA'B'F^2 - a^2EPr(A')^2GF$$

Subject to the boundary conditions:

$$A(\pm 1) = B(\pm 1) = 0,\ F(\pm 1) = G(\pm 1) = 1 \quad (3.4.3)$$

### *3.4.2 Non-dimensionalization of problem II (3.3.1)*

Substituting (*3.12*)and (*3.*13) into (*3.9*) and (*3.10*) and applying the non-dimensional quantities (3.4.1) to the momentum equation (*3.9*) and energy equation (*3.10*) alongside the boundary conditions (*3.11*) we obtain their dimensionless forms as:

$$A'' - sA' = aA''F + aA'F' - asA'F - F$$
$$B'' - sB' - iStB = aB''F + aA''G + aB'F' + aA'G' - aiStBF - asB'F - G$$
$$F'' - sPrF' = -EPr(A')^2 + 2aEPr(A')^2F - a^2EPr(A')^2F^2 \quad (3.4.4)$$
$$G'' - sPrG' - iStPrG = -2EPrA'B' + aEPr(A')^2G + 4aEPrA'B'F$$
$$-2a^2EPrA'B'F^2 - a^2EPr(A')^2GF$$

Subject to the boundary conditions:

$$A(\pm 1) = B(\pm 1) = 0,\ F(\pm 1) = G(\pm 1) = 1 \quad (3.4.5).$$

# CHAPTER FOUR

# SOLUTION OF THE PROBLEMS

## 4.1 Introduction

This chapter provides details of how the solutions of the non-dimensional momentum and energy equations describing the two problems in this theses were solved as well as the obtained solutions. It also presents the determination of other quantities of interest such as the phases and amplitudes of temperature and velocity fields, skin friction at both bounding plates as wellas the Nusselt number.

## 4.2 Uncoupling and Linearization of resulting differential equations

The Homotopy perturbation, an excellent approximation scheme which surmounts limitations of the general perturbation method was applied to the systems of partial differential equations (*3.4.2*) and (*3.4.4*) which are strongly non-linear and coupled, defying exact analytical methods of solution, to undo the coupling as well as reduce them to a system of homogenous and non-homogenous but linear ordinary differential equations which are then solved by method of undetermined coefficients.

### *4.2.1 Outline of Homotopy perturbation Method*

The homotopy perturbation is a perturbation about an arbitrary physical parameter, *p*, where$p\epsilon[0,1]$. Given an unknown function A($\eta$), say, we put

$$\mathrm{A}(\eta) = A_0(\eta) + pA_1(\eta) + p^2A_2(\eta) + p^3A_3(\eta) + \cdots = \sum_{i=1}^{\infty} p^i A_i(\eta)$$

as a perturbation series in powers of *p*.

For a general non-linear differential equation:

$$A(U) - f(r) = 0, \quad r \in \Omega \tag{4.2.1}$$

with the boundary condition:

$$B\left(U, \ \frac{\partial U}{\partial n}\right) = 0, \ r \in \Gamma, \tag{4.2.2}$$

where $A$ is a general functional operator, $B$ is a boundary operator, $f(r)$ is a known analytic function and $\Gamma$ is the boundary of the domain $\Omega$; the operator $A$ can be separated into two parts labelled $L$ and $N$ where $L$ is a linear operator and $N$ is a non-linear operator. With these, (4.2.1) can be written now as:

$$L(U) + N(U) - f(r) = 0 \tag{4.2.3}$$

Using homotopy technique, we construct a homotopy $U(r,p)$: $\Omega \times [0,1] \rightarrow R$, which satisfies:

$$H(U,p) = (1-p)[L(U) - L(u_0)] + p[A(U) - f(r)] = 0, \ p \in [0,1], \ r \in \Omega \tag{4.2.4}$$

or

$$H(U,p) = L(U) - L(u_0) + pL(u_0) + p[N(U) - f(r)] = 0, \tag{4.2.5}$$

where $p \in [0,1]$ is an embedding parameter, $u_0$ is an initial approximation for the solution of

equation (4.2.1) and satisfies the boundary conditions (4.2.2).

Setting the parameter $p$ to 0 and 1 in either of Eqns. (4.2.4) or (4.2.5) gives:

$$H(U,0) = L(U) - L(u_0) = 0,$$

$$H(U,1) = L(U) + N(U) - f(r) = A(U) - f(r) = 0 \qquad (4.2.6).$$

Topologically, $H(U,0)$ is a deformation of the general differential functional operator $A$, while $H(U,1)$ is said to be 'homotopic' to $A$. The notion of homotopic is simply the notion of essentially being the same. In fact, $H(U, p)$ $\forall p \in [0,1)$are deformations of $A$. Thus in effect, the homotopy parameter $p \in [0,1]$can be treated as a general small parameter for embedding in the traditional perturbation technique to express $U$ in its power series as:

$$U = U_0 + pU_1 + p^2U_2 + p^3U_3 + \cdots = \sum_{i=0}^{\infty} p^i U_i \qquad (4.2.7)$$

and then allow $p$ to tend to 1, necessarily, (in (4.2.6)), to obtain

$$U = \lim_{p\to 1} \sum_{i=0}^{\infty} p^i u_i = \sum_{i=0}^{\infty} U_i$$

(4.2.8)

In this way, the burden of finding an initial approximation $u_0$ to $U$, satisfying the initial conditions,is by-passed. This therefore translates to $u_0 = 0$ in the equivalent Homotopies (4.2.4) and (4.2.5), giving:

$$H(U,p) = (1-p)L(U) + p[A(U) - f(r)] = 0, \ p \to 1 \qquad (4.2.9).$$

### *4.2.2 Convergence of the Homotopy Perturbation Method*

The Homotopy perturbation which was developed in He J. H. (1999) is a well-known series solution method that can be used in finding the exact or close approximate solution of non-linear differential equations. This very method has been used by several authors to find solutions of coupled and nonlinear differential equations, non-linear Schrodinger equations, Volterra Integral equations of the second kind, such as in Rajabi and Ganji(2007), Ganji et al(2009), Domairry and Aziz(2009), Biazar and Ghazvini(2007, 2009) and many others.

One major challenge associated with this kind of methods of solving nonlinear problems is the fact whether or not the obtained series solution tends to or converges to the exact solution.The convergence of the Homotopy perturbation as an effective method of solving non-linear differential equations has been proven and discussed by several authors such as Ayati and Biazar (2015), Ja'afar and Aminikhah (2009), and others. The Series solution obtained by this very method has been shown to converge rapidly to known exact solution.

## 4.3 Solution to Problem 3.2

Upon applying the homotopy (4.2.9) to the system of equations (3.4.2), subject to the boundary condition (3.4.3), systems of boundary value ode's were obtained for various powers of the Homotopy parameter,$p$, as:

$$
\begin{aligned}
p^0: \quad & A_0'' = 0; A_0(\pm 1) = 0 \\
& B_0'' - iStB_0 = 0; B_0(\pm 1) = 0 \\
& F_0'' = 1; F_0(\pm 1) = 1 \\
& G_0'' - iStPrG_0 = 0; G_0(\pm 1) = 1 \\
p^1: \quad & A_1'' - aA_0'' F_0 - aA_0' F_0' + F_0 = 0; A_1(\pm 1) = 0 \\
& B_1'' - iStB_1 + aiStB_0 - aB_0'' F_0 - aA_0'' G_0 - aB_0' F_0' - aA_0' G_0' + G_0 = 0; \\
& \qquad B_1(\pm 1) = 0 \\
& F_1'' + EPr(A_0')^2 - 2aEPr(A_0')^2 F_0 + a^2 EPr(A_0')^2 F_0^2 = 0; \\
& \qquad F_1(\pm 1) = 0 \\
& G_1'' - iStPrG_1 + 2EPrA_0' B_0' - 4aEPrA_0' B_0' F_0 + 2a^2 EPrA_0' B_0' F_0^2 \\
& \qquad -EPr(A_0')^2 G_0 . + a^2 EPr(A_0')^2 G_0 F_0 = 0; \\
& \qquad G_1(\pm 1) = 0 \\
& . \quad . \quad . \quad . \quad . \\
.
\end{aligned}
\tag{4.3.1, 4.3.2}
$$

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$p^4$: $A_4'' - aA_0'' F_3 - aA_1'' F_2 - \cdots + F_3 = \quad 0;\ A_4(\pm 1) = 0$

$B_4'' - iStB_4 + aiStB_0F_3 + aiStB_1F_2 + \cdots + G_3 = \quad 0;\ B_4(\pm 1) = 0$ (4.3.5)

$F_4'' + EPrA_0'A_3' + EPrA_1'A_2' + EPrA_2'A_1' + \cdots + a^2EPrA_0'F_0^2 = 0;$

$F_4(\pm 1) = 0$

$G_4'' - iStPrG_4 + 2EPrA_0'B_3' + 2EPrA_1'B_2' + \cdots + a^2EPrA_0'G_0F_0 = 0;$

$G_4(\pm 1) = 0$

Solving the system of equations (4.3.1) – (4.3.5) subject to their relevant boundary conditions, we obtain in series, closed form expressions of the four functions $A(\eta), F(\eta), B(\eta)$, and $G(\eta)$for problem (3.2) as:

$Pr \neq 1$

$$A(\eta) = \frac{1}{2}(1-\eta^2) + \frac{a}{2}(1-\eta^2) + \frac{a^2}{2}(1-\eta^2) + \frac{a^3}{2}(1-\eta^2) + \frac{EPr}{360}(1-2a+a^2)(\eta^6 - 15\eta^2 + 14), \quad (4.3.6)$$

$$F(\eta) = 1 + \frac{EPr}{12}(1-2a+a^2)(1-\eta^4) + \frac{aEPr}{6}(1-2a+a^2)(1-\eta^4) \quad (4.3.7)$$

$$B(\eta) = \frac{1}{m-c}\left(\frac{\cosh(r\eta)}{\cosh r} - \frac{\cosh(q\eta)}{\cosh q}\right) + (V_1 + V_2\eta^2)\cosh(r\eta) + (V_3 + V_4\eta^2)\cosh(q\eta) + V_5\eta\sinh(r\eta) + V_6\eta\sinh(q\eta) + V_7\eta^3\sinh(r\eta) - V_8\eta^3\sinh(q\eta) \quad (4.3.8)$$

$$G(\eta) = \frac{\cosh(q\eta)}{\cosh q} + (I_1 + I_2\eta^2)\cosh(q\eta) + I_3\eta\sinh(q\eta) + (I_4 + I_5\eta^2)\cosh(r\eta) + I_6\eta\sinh(r\eta) + I_7\eta^3\sinh(q\eta) \quad (4.3.9),$$

where the constants $r, q, m, c, V_i, (i = 1,2,3,\ldots,8), I_j\,(j = 1,2,3,\ldots,7)$ are all declared in the Appendix.

$Pr = 1$

The functions B($\eta$) and G($\eta$) (Eqns (4.3.8) and (4.3.9)) which define respectively the periodic velocity and temperature of the working fluid are not defined at$Pr = 1$, since some of the constants used in their definitions have discontinuities at this value of $Pr$. Thus equations (4.3.1) – (4.3.5) are solved again setting Pr = 1 subject to their boundary conditions for the unsteady velocity and temperature, and the following expressions were obtained:

$$B(\eta) = \frac{1}{2rR}\left(tanh(r)\, cosh(r\eta) - \eta sinh(r\eta)\right) + (m_1 + m_2\eta^2 + m_3\eta^4)cosh(r\eta)$$

$$+(m_4\eta + m_5\eta^3)sinh(r\eta) \qquad (4.3.10)$$

$$G(\eta) = \frac{1}{R}cosh(r\eta) + (h_1 + h_2\eta^2)cosh(r\eta) + (h_3\eta + h_4\eta^3)sinh(r\eta) \qquad (4.3.11)$$

where the constants $m_1$,$m_2$, $m_3$, $m_4$, $m_5$, $h_1$, $h_2$, $h_3$ and $h_4$ are declared in the Appendix.

## 4.4 Phase and Amplitude of Periodic temperature and velocity of problem 3.2

The phase ($\emptyset$) of periodic temperature and the phase ($\theta$) of periodic velocity are given by the expressions

$$\emptyset = tan^{-1}\left(\frac{G_i}{G_R}\right) \text{ and } \theta = tan^{-1}\left(\frac{B_i}{B_R}\right), \qquad (4.4.1)$$

while the amplitudes of the periodic temperature and velocity are given by

$$|\mathrm{G}| = (G_R^2 + G_i^2)^{\frac{1}{2}} \text{ and } |\mathrm{B}| = (B_R^2 + B_i^2)^{\frac{1}{2}}, \qquad (4.4.2)$$

where $G_R, B_R$ are the periodic temperature and velocity in phase, and, $G_i, B_i$ are the periodic temperature and velocity out of phase by a difference of $90^0$. Positive values of $B_R$ depict

velocity in the positive direction while negative values depict reverse flow. The phase $\theta$ of velocity is the angle by which flow deviates from the in-phase (upward) direction while the amplitude |B| depicts the absolute periodic velocity at any given point in the channel, and similarly for ∅ and |G|.

## 4.5 Nusselt number ($Nu$) and Skin friction ($\tau$) for problem 3.2

On taking $\frac{dG}{d\eta}$ and $\frac{dB}{d\eta}$ at $\eta = \pm 1$, we obtain the rate of heat transfer ($Nu$) and the skin-friction (τ) on the plates respectively as follows:

$$Nu = \frac{dG}{d\eta} = \frac{qSinh(q)}{Cosh(q)} + (I_3 + 3I_7 + (I_1 + I_2)q)\sinh(q) + (2I_2 + (I_3 + I_7)$$

$$q)\cosh(q) + (I_6 + (I_4 + I_5)r)\sinh(r) + (2I_5 + I_6 r)\cosh(r) \qquad (4.5.1)$$

$$\tau = \frac{dB}{d\eta} = \frac{1}{m-c}(r\tanh(r) - q\tanh(q)) + (V_5 + 3V_7 + (V_1 + V_2)r)\sinh(r)$$

$$+(2V_2 + (V_5 + V_7)r)\cosh(r) + (V_6 + 3V_8 + (V_3 + V_4)q)\sinh(q)$$

$$+(2V_4 + (V_6 + V_8)q)\cosh(q) \qquad (4.5.1).$$

## 4.6 Solution of Problem 3.3

When the Homotopy (4.2.8) was applied to the system of equations (15), subject to the boundary condition (16), systems of boundary value ode's were obtained for various powers of the Homotopy parameter, $p$, as:

$$p^0: \quad A_0'' - sA_0' = 0; A_0(\pm 1) = 0$$

$$B_0'' - sB_0' - iStB_0 = 0; B_0(\pm 1) = 0$$

(4.6.1)

$$F_0'' - sPrF_0' = 1; F_0(\pm 1) = 1$$

$$G_0'' - sPrG_0' - iStPrG_0 = 0; G_0(\pm 1) = 1 \qquad (4.6.1)$$

$$p^1: \quad A_1'' - sA_1' - aA_0''F_0 - aA_0'F_0' + asA_0'F_0 + F_0 = 0;\ A_1(\pm 1) = 0$$

$$B_1'' - sB_1' - iStB_1 + aiStB_0F_0 - aB_0''F_0 - aA_0''G_0 - aB_0'F_0' - aA_0'G_0' + asB_0'F_0 + G_0 = 0; \qquad B_1(\pm 1) = 0$$

$$F_1'' - sPrF_1' + EPr(A_0')^2 - 2aEPr(A_0')^2F_0 + a^2EPr(A_0')^2F_0^2 = 0; \qquad F_1(\pm 1) = 0 \quad (4.6.2)$$

$$G_1'' - sPrG_1' - iStPrG_1 + 2EPrA_0'B_0' - 4aEPrA_0'B_0'F_0 + 2a^2EPrA_0'B_0'F_0^2 + a^2EPr(A_0')^2G_0F_0 = 0; \qquad G_1(\pm 1) = 0$$

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$$p^4: \quad A_4'' - sA_4' - aA_0''F_3 - aA_1''F_2 - \cdots + F_3 = 0;\ A_4(\pm 1) = 0$$

$$B_4'' - sB_4' - iStB_4 + aiStB_0F_3 + aiStB_1F_2 + \cdots + G_3 = 0; \qquad B_4(\pm 1) = 0 \quad (4.6.5)$$

$$F_4'' - sPrF_4' + EPrA_0'A_3' + EPrA_1'A_2' + \cdots + a^2EPrA_0'F_0^2 = 0; \qquad F_4(\pm 1) = 0$$

$$G_4'' - sPrG_4' - iStPrG_4 + 2EPrA_0'B_3' + \cdots + a^2EPrA_0'G_0F_0 = 0; \qquad G_4(\pm 1) = 0$$

Solving the system of equations (4.6.1) – (4.6.5) subject to their relevant boundary conditions, we obtain in series, closed form expressions of the four functions $A(\eta), F(\eta), B(\eta)$, and $G(\eta)$for problem (3.3) as:

$$A(\eta) = \frac{\coth(s)}{s} - \frac{e^{s\eta}}{s\sinh(s)} + \frac{1}{s}(a + a^2 + \eta + a\eta + a^2\eta + a^3\eta + C_4\eta) + \frac{ae^{-s}}{s\sinh(s)}(1 + a)$$

$$- \frac{ae^{s\eta}}{s\sinh(s)}(1 + a) + \left(H_5 + \frac{H_2}{s}\eta\right)e^{s\eta} - \frac{H_1}{2s^2}e^{2s\eta} - \frac{D_4}{s^2Pr(Pr-1)}e^{sPr\eta}$$

$$-\frac{\beta}{2s^4Pr}\left(\eta^2+\frac{2\eta}{s}\right)+H_6 \tag{4.6.6}$$

$$F(\eta) = 1 + C_4 + C_7 + (D_4 + D_7)e^{sPr\eta} - (1+2a)H_2e^{s\eta} + (1+2a)H_1e^{2s\eta}$$

$$-\frac{(1+2a)\beta}{s^3Pr}\,\eta \tag{4.6.7}$$

$$B(\eta) = (C_1 + C_2 + C_3 + C_6)e^{m_1\eta} + (D_1 + D_2 + D_3 + D_6)e^{m_2\eta} + (R_3 + R_{11} +$$

$$R_{33})e^{m_1\eta} +(R_4 + R_{13} + R_{34})\eta e^{m_2\eta} + (R_{12} + R_{35})\eta^2e^{m_1\eta} + (R_{14} + R_{36})$$

$$\eta^3e^{m_1\eta} + R_{38}\eta^3e^{m_2\eta} + (R_5 + R_{15} + R_{39} + R_{51} - X_1)e^{a_1\eta}$$

$$-R_{52}\eta e^{a_1\eta} + (R_6 + R_{16} + R_{40} + R_{49} - X_2)e^{a_2\eta} - R_{50}\eta e^{a_2\eta}$$

$$+(R_7 + R_{17} + R_{41})e^{(a_1+s)\eta} + (R_8 + R_{18} + R_{42})e^{(a_2+s)\eta} - R_{43}e^{(m_1+s)\eta}$$

$$-R_{44}e^{(m_2+s)\eta} - R_{45}e^{(a_2+2s)\eta} - R_{46}e^{(a_1+2s)\eta} \tag{4.6.8}$$

$$G(\eta) = (C_0 + C_5 + C_8)e^{a_1\eta} + (D_0 + D_5 + D_8)e^{a_2\eta} + (R_{28} + R_{57})\eta e^{a_1\eta}$$

$$+(R_{27} + R_{58})\eta e^{a_2\eta} + (R_{25} + R_{53})e^{m_1\eta} + (R_{26} + R_{54})e^{m_2\eta} + R_{55}\eta e^{m_1\eta}$$

$$+R_{56}\eta e^{m_2\eta} + (R_{24} + R_{63})e^{(a_1+s)\eta} + (R_{23} + R_{64})e^{(a_2+s)\eta} + (R_{21} + R_{59})$$

$$e^{(m_1+s)\eta} + (R_{22} + R_{61})e^{(m_2+s)\eta} + R_{60}\eta e^{(m_1+s)\eta} + R_{62}\eta e^{(m_2+s)\eta}$$

$$+(R_{30} + R_{65)}e^{(a_1+2s)\eta} + (R_{29} + R_{66})e^{(a_2+2s)\eta} \tag{4.6.9},$$

where the constants $C_0, C_1, C_2, \dots, C_{8,}D_0, D_1, D_2, \dots, D_{8,}H_1, H_{2,}H_3, \dots, H_6, R_3, R_4, R_5, \dots, R_{66}$ are all declared in the appendix.

## 4.7 Phase and Amplitude of Periodic temperature and velocity of problem 3.3

The phase and amplitude of the periodic temperature and velocity are defined similarly by $\emptyset$ and $\theta$ as in problem (3.2) in section **4.4** respectively by Eqns. (4.4.1) and (4.4.2).

## 4.8 Nusselt number ($Nu$) and Skin friction ($\tau$) of problem 3.3

On taking $\frac{dG}{d\eta}$ and $\frac{dB}{d\eta}$ on the plates$\eta = \pm 1$, we obtain the rate of heat transfer ($\boldsymbol{Nu}$) and the skin-friction ($\boldsymbol{\tau}$) on the plates respectively as follows:

$$Nu = \frac{dG}{d\eta} = (a_1(X_{13} + X_{15}) + X_{15})e^{a1} + (a_2(X_{14} + X_{16}) + X_{16})e^{a2} + (m_1(X_{17} + R_{55}) + R_{55})e^{m_1} + (m_2(X_{18} + R_{56}) + R_{56})e^{m_2} + (a_1 + s)X_{19}e^{(a_1+s)} + (a_2 + s)X_{20}e^{(a_2+s)} + (m_1 + s)((X_{21} + R_{60}) + R_{60})e^{(m_1+s)} + (m_2 + s)((X_{22} + R_{62}) + R_{62})e^{(m_2+s)} + (a_1 + 2s)X_{23}e^{(a_1+2s)} + (a_2 + 2s)X_{24}e^{(a_2+2s)} \quad (4.8.1)$$

$$\tau = \frac{dG}{d\eta} = (m_1(X_3 + X_5 + X_7 + R_{37}) + X_5 + 2X_7 + 3R_{37})e^{m_1} + (m_2(X_4 + X_6 + X_8 + R_{38}) + X_6 + 2X_8 + 3R_{38})e^{m_2} + (a_1(X_9 - R_{52}) - R_{52})e^{a_1} + (a_2(X_{10} - R_{50}) - R_{50})e^{a_2} + (a_1 + s)X_{11}e^{(a_1+s)} + (a_2 + s)X_{12}e^{(a_2+s)} - (m_1 + s)R_{43}e^{(m_1+s)} - (m_2 + s)R_{44}e^{(m_2+s)} - (a_2 + 2s)R_{45}e^{(a_2+2s)} - (a_1 + 2s)R_{46}e^{(a_1+2s)} \quad (4.8.2),$$

$X_i, (i = 1,2,3, \ldots, 66)$, is declared in the appendix.

## 4.9 Convergence of the Series Solutions

In this work, we tried to demonstrate the convergence of the obtained solutions using the ratio test.

For the ratio test, a series $\sum_{n=1}^{\infty} a_n$ is convergent if:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p < 1.$$

If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = p > 1$, then the series diverges, and p =1, implies that the test is not conclusive (by the ratio test) (Sokolnikoff and Redheffer (1966)).

That the series solution converges to a fixed value implies that the contribution of the succeeding terms to the whole series become smaller and smaller as the series progresses. For our case, the sequence of the solutions for the unsteady velocity and temperature, $\{B_i(\eta)\}$ and $\{G_i(\eta)\}$, for fixed values of associated parameters, were obtained in numerical form and the ratio test applied for the first five perturbations as follows:

**Table 4.9.1: Convergence of Periodic Velocity and Temperature**

$$\eta = 0.5,\ a = -0.1,\ Pr = 0.044,\ St = 2,\ E = -0.5$$

| $n$ | $B_n(\eta)$ | $G_n(\eta)$ |
|---|---|---|
| 1 | - | 0.9988 |
| 2 | 0.1982 | - |
| 3 | -0.0330 | - |
| 4 | 0.0046 | -0.0022 |
| 5 | -0.0007 | 0.0006 |
| | 0.15 | 0.27 |

| | | |
|---|---|---|
| $\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\|$ | | |

From the table, it is observed that the series solutions are convergent since,

$$\lim_{n\to\infty} \left\| \frac{a_{n+1}}{a_n} \right\| < 1$$

for all $n$ in the case of periodic velocity and temperature.

It is however necessary to define a termination criterion in the present problem in order to identify a truncation point of the infinite series that represent the solutions. To this effect, it is observed that the solution can be truncated if

$|S_N - S_{N-1}| < \epsilon$ for all suitable $\epsilon > 0$.

This shows that successive terms of the sequence of partial sums of the solutions $B_i'$s and $G_i'$s$(\eta)$ are close enough by a distance $\epsilon$. This implies that the contribution of a new term is negligible to improve the present approximate solution.

In the present problem, the value of $\epsilon$ was chosen as $\epsilon = 0.001$ so that

$|S_N - S_{N-1}| < \epsilon$ is achievable at $N = 5$.

Hence we truncate the series solution at $N = 5$ as shown in the table (4.9.1).

# CHAPTER FIVE

# RESULTS AND DISCUSSION

## 5.1 Introduction

This chapter presents the discussion of the results obtained from the previous chapter in the form of values and line graphs that clearly reveal the influences of the governing parameters on the flow behaviours. A computer program was written and used in the simulation of the periodic temperature and velocity, Skin friction and Nusselt number for each of the studied problems. The Phase and amplitude of fluid temperature and velocity were also simulated for both cases.

## 5.2 Flow between two vertical parallel plates (3.2)

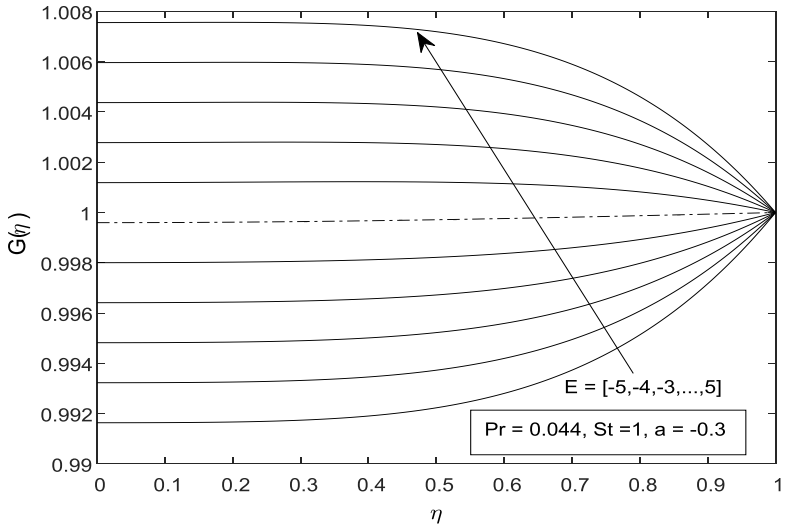


Fig. 5.1 Temperature distribution with different $E$ (Increased viscosity)

Fig. 5.1 shows the influence of viscous dissipation on the temperature distribution within the channel. It is observed from the figure that when viscosity is raised ($a < 0$) and viscous

dissipation is increasing ($E > 0$), there is an increase in temperature of fluid within the channelsince increase in viscous dissipation acts to compliment the thermal inputs on the boundary plates in order to increase the temperature. For $E < 0$, the heat absorption acts against the boundary thermal input to decrease the temperature within the channel. However, in Fig 5.2 when viscosity decreases with growing temperature ($a > 0$), the temperature of the working fluid is generally higher in the vicinity of the bounding plates and then falls to a minimum at the center of the channel.

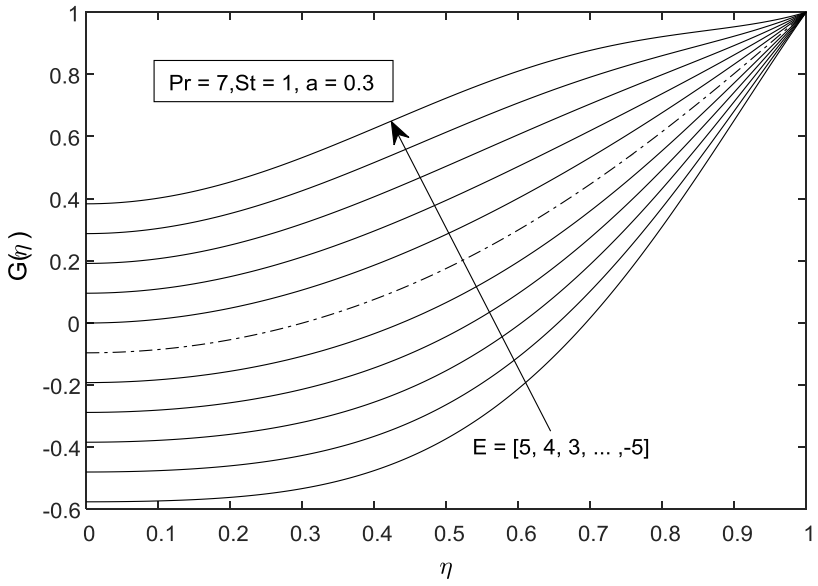


Fig. 5.2. Temperature distribution with different $E$ (Reduced viscosity)

This figure reveals that the temperature of the fluid generally decreases with increasing Eckert number. This is so because for any fixed value of Prandtl number, decreasing the viscosity ($a > 0$), leads to reduced thermal diffusivity and when this happens, increasing the Eckert number only acts to complement this fall in thermal diffusivity ($\mathrm{Pr} = 7$) and hence a drop in temperature of the fluid. In terms of the temperature around the center of

the channel, Fig 5.2, ($a < 0$), is a contrast of the situation in fig 2 ($a > 0$) where temperatures peak at the center.

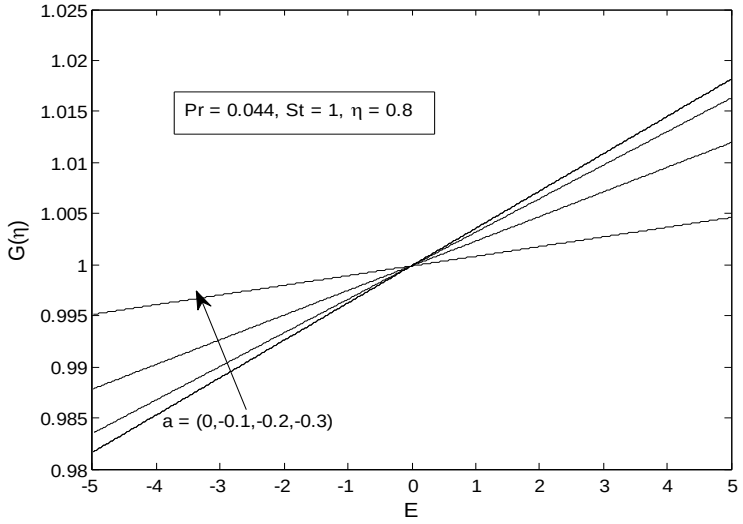


Fig. 5.3. Temperature distribution with $E$ for different $a$

This suggests that the effect of heat generation by viscous dissipation is stronger under growing viscosity. Fig 5.3 shows the behaviour of temperature with variations in viscous dissipation for different values of $a$. Increasing the fluid viscosity is observed to decrease the temperature in the presence of thermal sink due to viscous dissipation. On the other hand, an increase in the fluid viscosity is observed to decrease the temperature when the sheared heating is considered. This is anchored upon the fact that growing viscosity is to amplify the effect of viscous dissipation to increase the temperature within the channel. When the Eckert number $E = 0$, the effect of change in viscosity on temperature is nullified. This clearly reveals that variable viscosity effects are suppressed with viscous dissipation and vice-versa.

In Fig 5.4, temperature is observed to increase with increasing $E$ as viscosity reduces ($a > 0$), which confirms the observations in Fig 5.2. This figure further reveals a critical

value of the viscosity parameter at which viscous dissipation effect is negligible. When $a$ is increased beyond this critical value, the trend in the temperature is reversed.

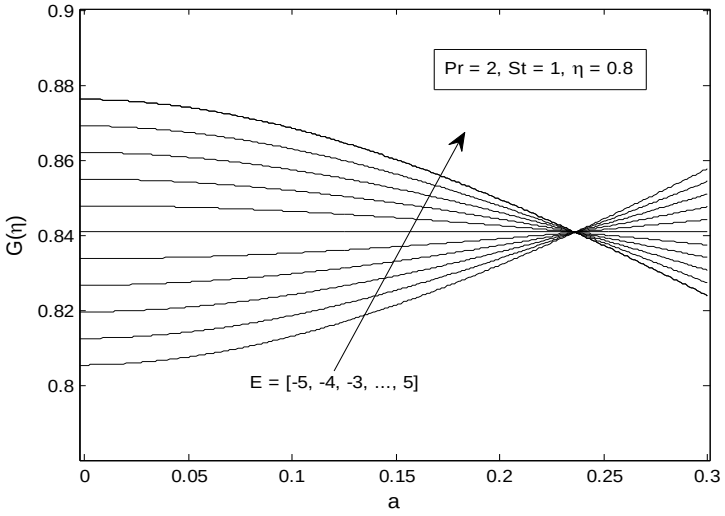


Fig. 5.4. Temperature distribution with $a$ for different $E$

Figs. 5.5 and 5.6 show the effect of the Eckert number, $E$, on the behaviours of the fluid flow for the value $Pr = 0.044$. In Fig. 5.5, it is observed that when viscosity is increased with growing temperature, fluid velocity increases with increasing $E$. This is due to the physical fact that fluid temperature increases with viscous dissipation as shown in Fig 5.1. This acts to strengthen the convection current within the channel so as to increase the fluid buoyancy and hence velocity increase.

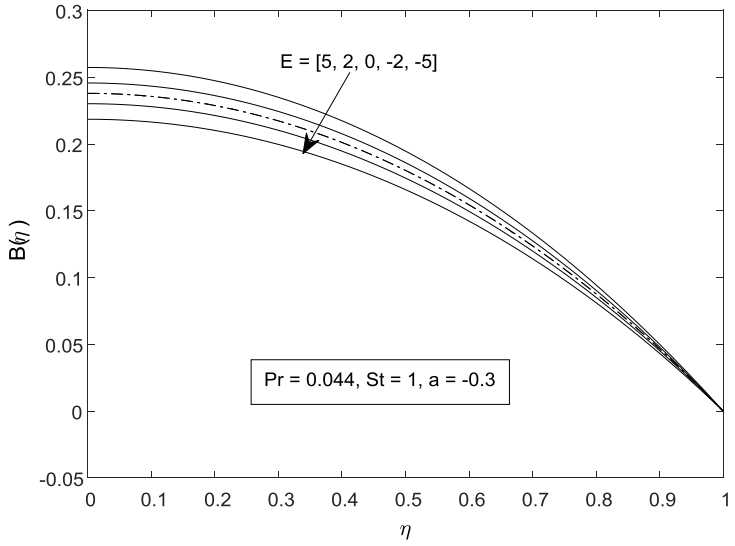


Fig. 5.5 Velocity Profile for different $E$ (Increased viscosity)

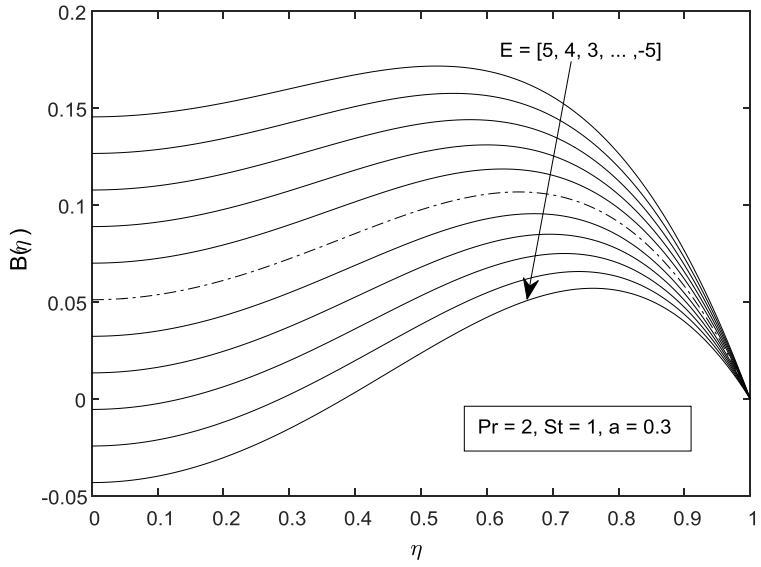


Fig. 5.6. Velocity profile for different $E$ (Decreased viscosity)

In Fig 5.6, where viscosity decreases with growing temperature ($a > 0$), the reverse situation is observed where velocity decreases with increasing $E$. This suggests that heat generation by shear heating is very weak ($Pr = 2$) and so the temperature of the fluid rises with increasing absorption ($E<0$) from the ambient and this leads to increasing velocity.

Figs. 5.7and 5.8 show the effect of variable viscosity on thehydrodynamics for Sulphur dioxide ($Pr = 2$) and Mercury ($Pr = 0.044$) respectively.

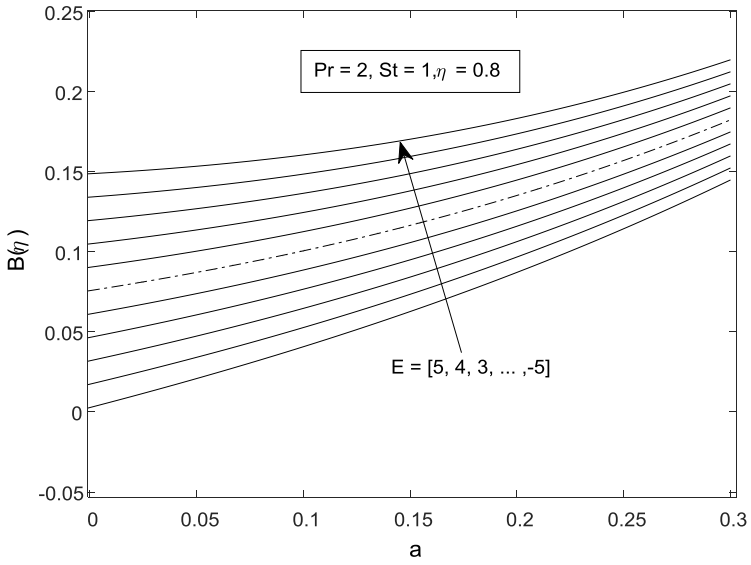


Fig. 5.7. Velocity Profile with a ($Pr = 2$)

In fig. 5.7, it is seen that as viscosity is decreasing, increasing the Eckert number results in continuous reduction of the velocity of the fluid. This is due to the fact that growing the viscous dissipation contributes a decrease in the temperature (Fig 5.2) when the viscosity decreases with temperature. This acts to weaken the convection current causing a decrease in velocity. However, in Fig 5.8 it is observed that fluid velocity decreases as the viscosity increases. This is attributed to by temperature decrease due to increased viscosity. Consequently, convection current is weakened causing a decrease in velocity. From these, it can be concluded therefore that reducing the viscosity of the fluid leads to increased momentum of the fluid particles while the hydrodynamics decreases with growing viscosity.

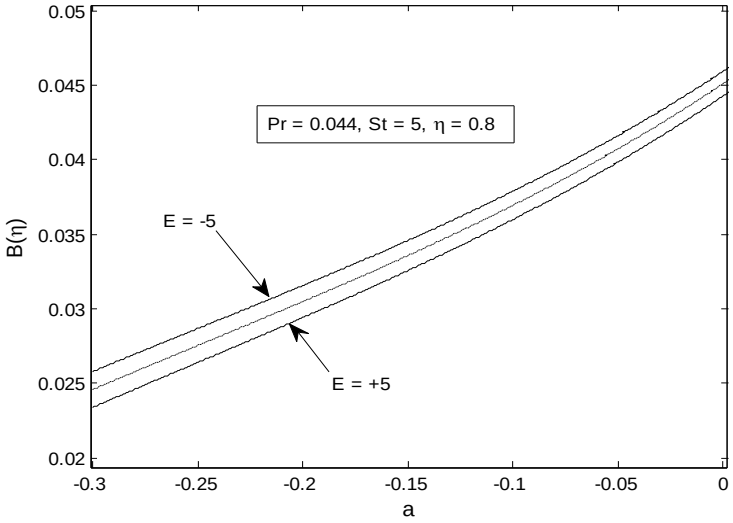


Fig. 5.8. Velocity Profile with a ($Pr = 0.044$)

It is worth noting that if the thermal diffusivity of the working fluid is very high ($Pr \ll 1$) the effect of the Eckert number is suppressed and the flow pattern tends to uniformity in the forward direction even for relatively low periodic heat inputs as shown in Fig. 5.8.

Fig. 5.9 shows the phase of temperature and that of velocity. The positive $x - axis$ is the direction of fluid flow while the negative y-axis is the direction of heat penetration. The angle of deviation ($\emptyset$) from the positive $y - axis$ is the phase of temperature while the angular deviation ($\theta$) from the positive $x - axis$ is the phase of velocity. Temperature and velocity are thus highest when their phase is lowest. That is, the deviations from the heat flux direction (horizontal) and the deviation from the upward (streamline) direction. The physical meaning of this phenomenon is that thermodynamics is enhanced within the channel whenever the phase of temperature approaches zero. Similarly, the hydrodynamics is enhanced within the channel as the phase of velocity diminishes.

Fig. 5.10 shows that for $Pr$ = 0.044, when viscosity has been increased ($a < 0$), the phase of temperature decreases sharply with increasing $St$ for all values of $E$ but phases are higher under absorption than dissipation.

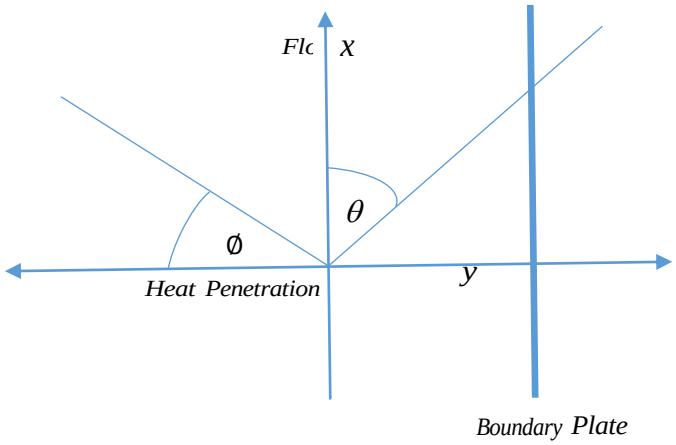


Fig. 5.9. The phase of temperature and velocity

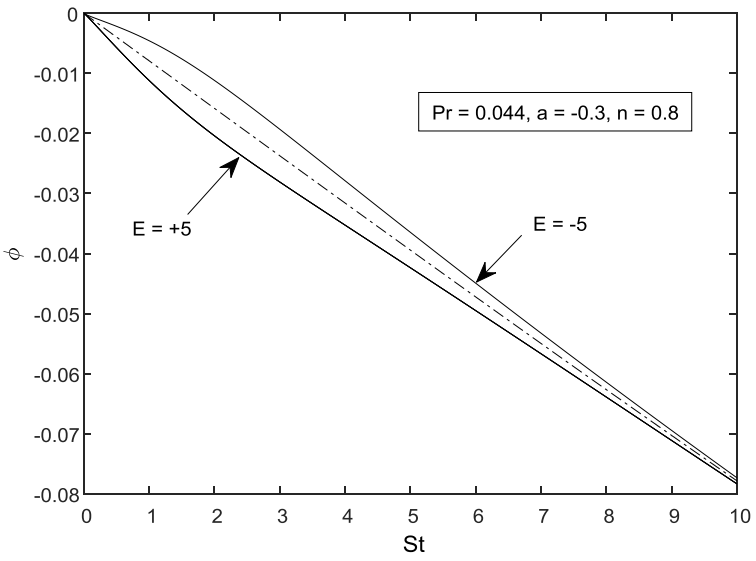


Fig. 5.10. Phase of temperature ($Pr = 0.044$)

In Fig. 5.11, it is observed that for$Pr = 7$, near the center line of the channel ($\eta = 0.2$), when viscosity in reduced ($a > 0$), the phase of temperature plummets with increasing

viscous dissipation. This is to say that the temperature of the fluid is increasing with increasing viscous dissipation.

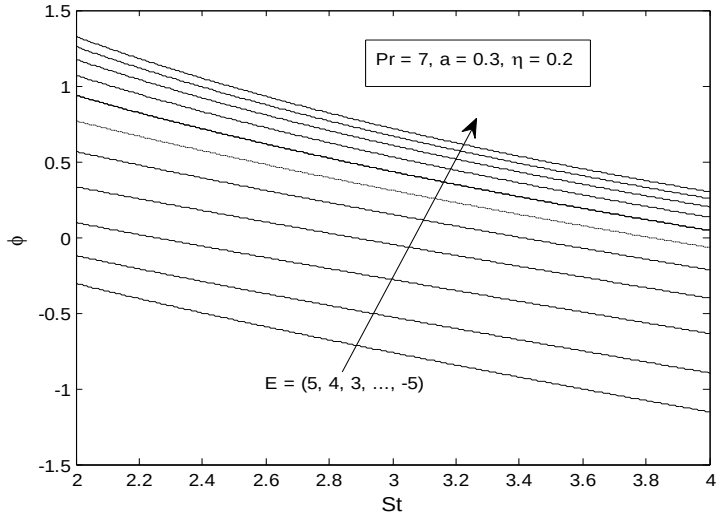


Fig. 5.11. Phase of temperature ($Pr = 7$)

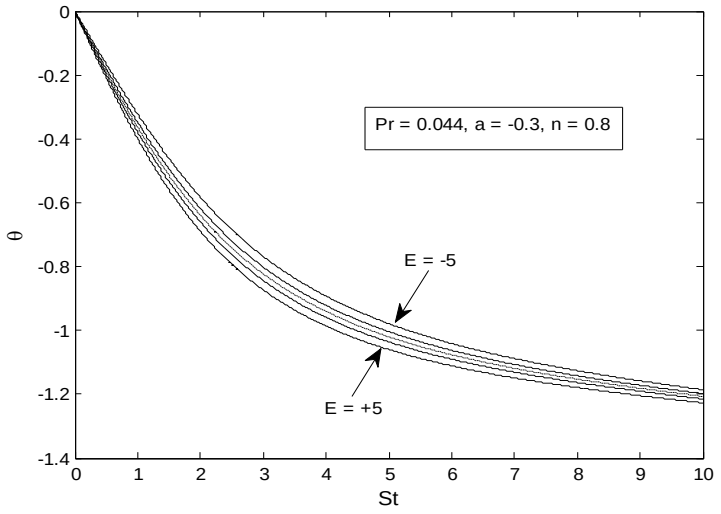


Fig. 5.12. Phase of velocity

Increasing the*Eckert* number when viscosity is increased leads to a decrease in the phase of velocity as shown in Fig. 5.12 with $Pr = 0.044$ (see Fig. 6). Fig. 5.13 shows the variations of the phase of velocity with viscosity. In this figure, it is clearly revealed that the phase of velocity increases with reducing values of $a$ as well as with reducing values of $Pr$. This

means that fluid velocity increases with reducing viscosity as well as with increasing thermal diffusivity because reducing $Pr$ means increasing the thermal diffusivity of the fluid to allow penetration of heat ($E = -5$) which enhances convection current, resulting in improved hydrodynamics.

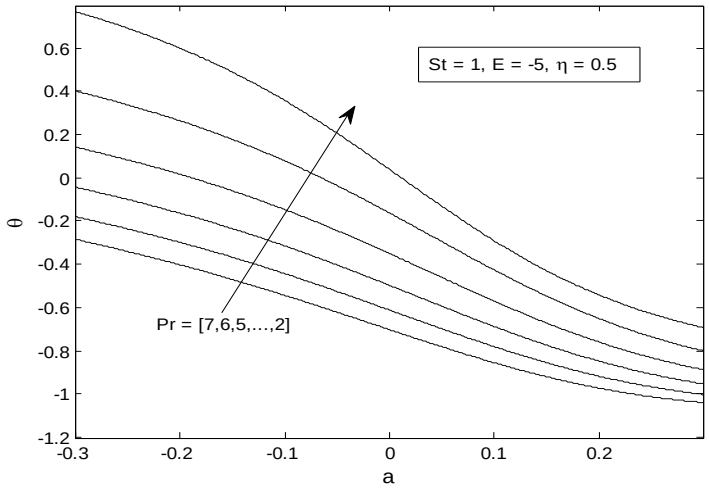


Fig. 5.13. Phase of velocity with $a$

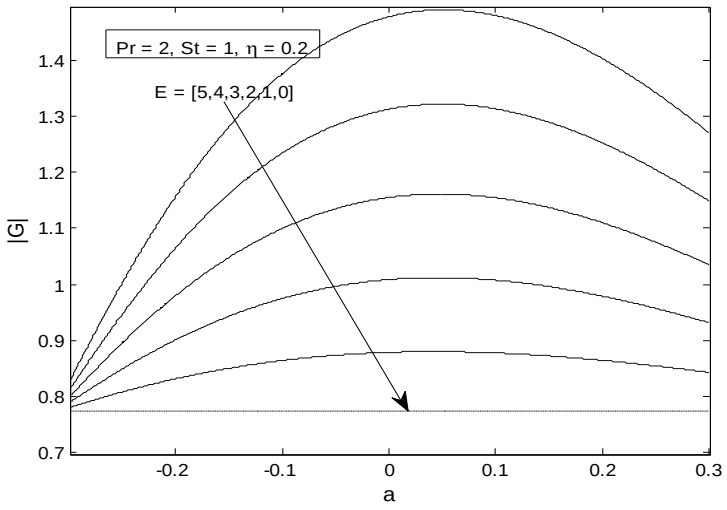


Fig. 5.14. Amplitude of temperature

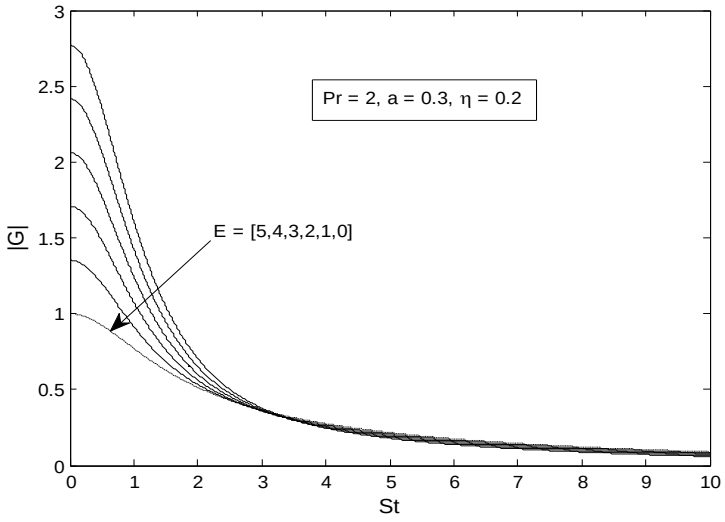


Fig. 5.15. Amplitude of temperature

In Fig. 5.14, the amplitude of temperature is observed to increase with increasing $E$. In Fig. 5.15, as in Fig. 5.14, amplitude of temperature is observed to increase with $E$, but decrease with $St$when viscosity has been decreased ($a = 0.3$). This is because increasing $St$generally leads to reduced temperatures and consequently reduced amplitudes of temperature. Figs. 5.16 and 5.17 show respectively that amplitudes of velocity increase with increasing $E$and decrease with increasing $Pr$. Once again, these two figures reveal that amplitudes of velocity increase with reducing viscosity which confirms the physical fact that fluid particles are freer to move when viscosity is reduced.

The rate of heat transfers between the channel plates and the fluid is shown in Figs. 5.18 and 5.19. In Fig. 5.18, it is shown that $\mathrm{Pr} = 7$, the rate of heat transfer on the plates increases with the Eckert number. In terms of variation in viscosity, the rate of heat transfer increases with decreasing viscosity (increasing $a$) in the presence of shear heating (positive $E$) while it decreases with decreasing viscosity under absorption (negative $E$). When

viscosity is raised ($a = -0.1$), the rate of heat transfer decreases with increasing $E$ for all values of $Pr$as shown in Fig. 5.19. This is because increasing viscous dissipation raises fluid temperature so that temperature gradients near the plates are lowered.

The skin-friction increases with the Eckert number $E$, but decreases with the variable viscosity parameter $a$, for $Pr = 2$ (Figs 5.20 and 5.21).

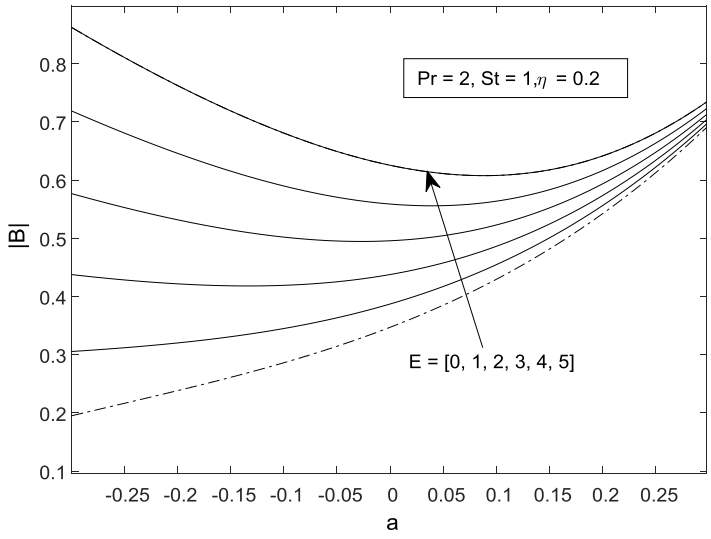


Fig. 5.16. Amplitude of velocity

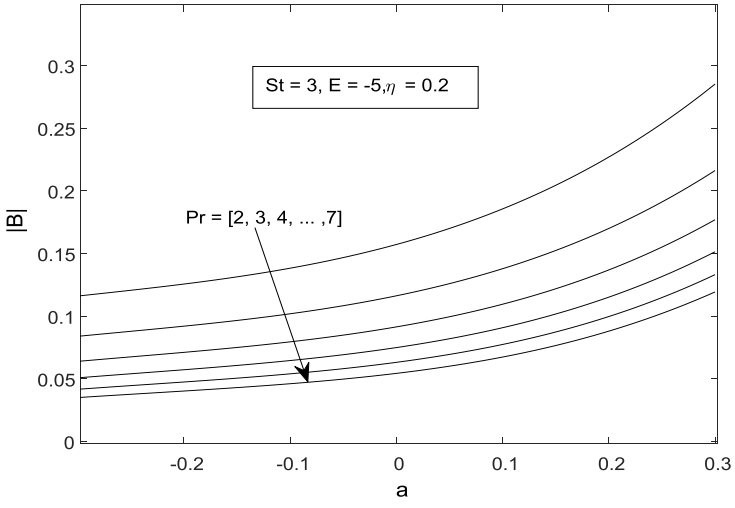


Fig. 5.17. Amplitude of velocity

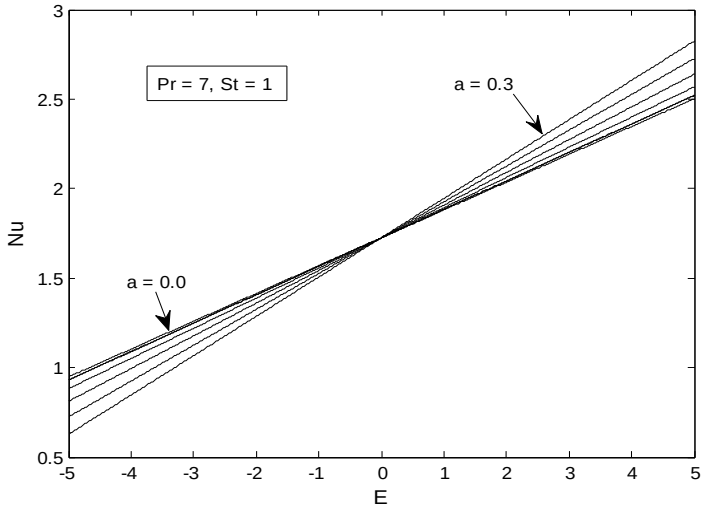


Fig. 5.18. Rate of heat transfer on the plates

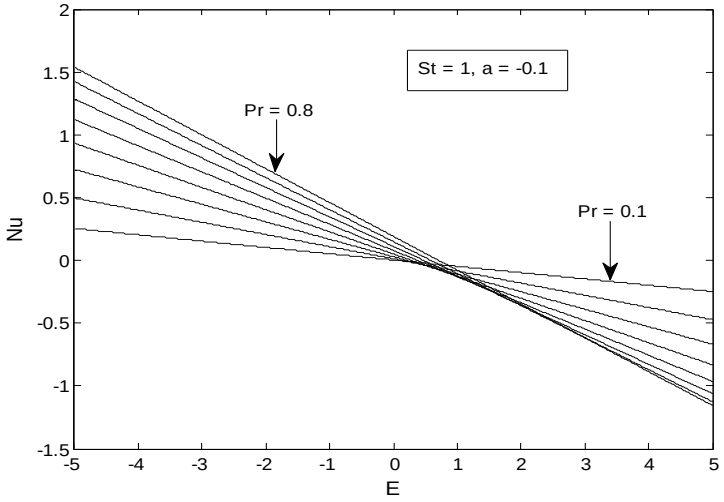


Fig. 5.19. Rate of heat transfer on the plates

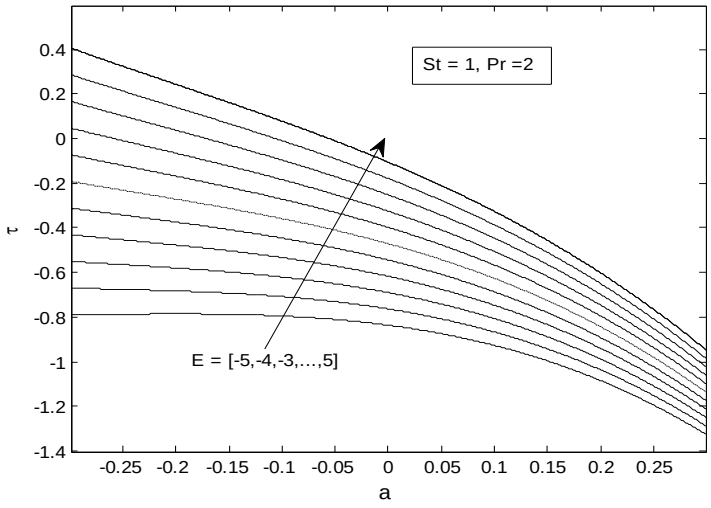


Fig. 5.20. Skin friction on the plates

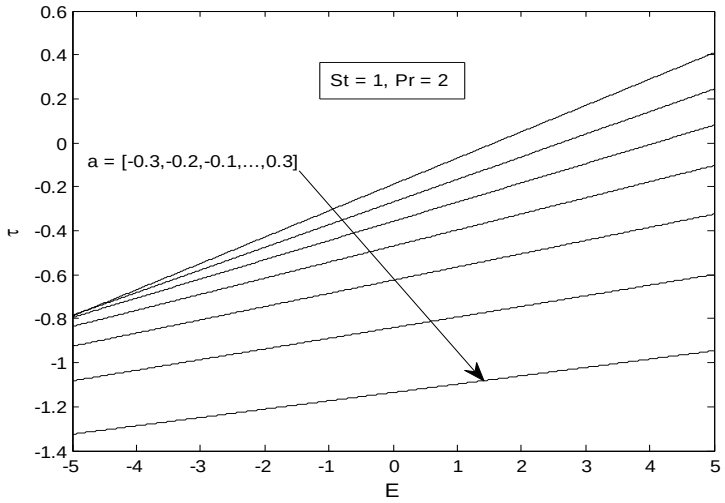


Fig. 5.21. Skin friction on the plates.

## 5.3 Flow in vertical porous channel (3.3)

Fig. 5.22 describes the variation of velocity field under suction and injection. When there is suction on the plate $\eta = -1$ with corresponding injection on $\eta = +1$, the velocity of the fluid increases with increase in suction near the same plate while the opposite is the case

(decreasing velocities) near the other plate with increasing injection. If the direction is reversed, increasing injection on $\eta = -1$ decreases the velocity near the same plate while the vicinity of the other plate witnesses increasing velocities under increasing suction.

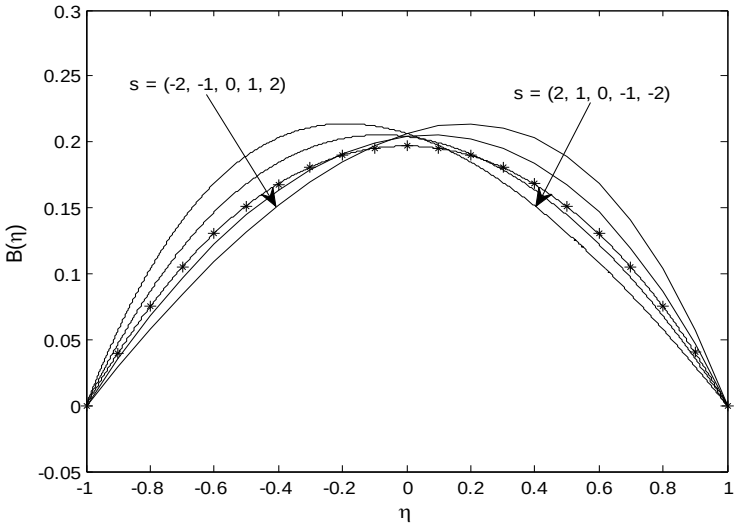


Fig. 5.22. Velocity profile for different $s$ ( $Pr = 0.71, St = 1, a = -0.3, E = 0.5$)

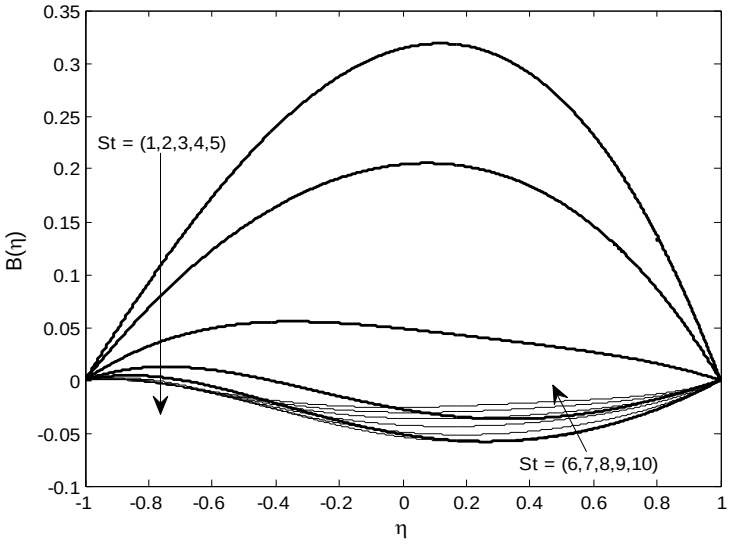


Fig. 5.23. Velocity profile for different $St$( $Pr = 0.71, a = -0.3, E = 0.5, s = 1$)

Fig. 5.23 depicts the behaviour of the fluid velocity with changing Strouhal number. It is shown that there is a sharp decline in fluid velocity as St increases because increasing St leads to reduced rate of heating of the fluid resulting in less buoyant fluid and hence decreasing velocity. As St increases further, the drop in velocity continues below zero (back flow) but as St tends to infinity, the velocity gradually returns to zero. Fig. 5.24 clearly reveals that the velocity of the working fluid increases as a increases (decreasing viscosity). This is so because fixing a Pr value and increasing the viscosity parameter leads to corresponding increase in the thermal diffusivity of the fluid and this promotes penetration of heat from the ambient which translates to more buoyant fluid particles and hence increased velocity.

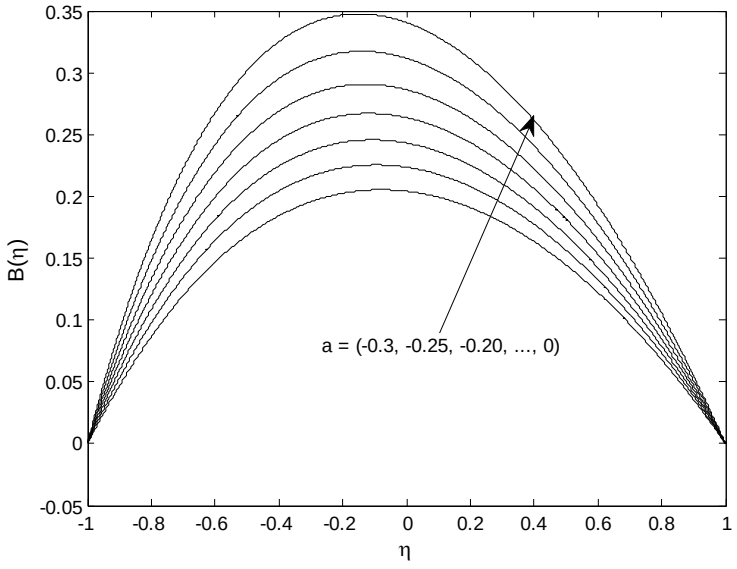


Fig. 5.24. Velocity profile with $a$ ( $Pr = 0.71, St = 1, E = 0.5, s = -1$)

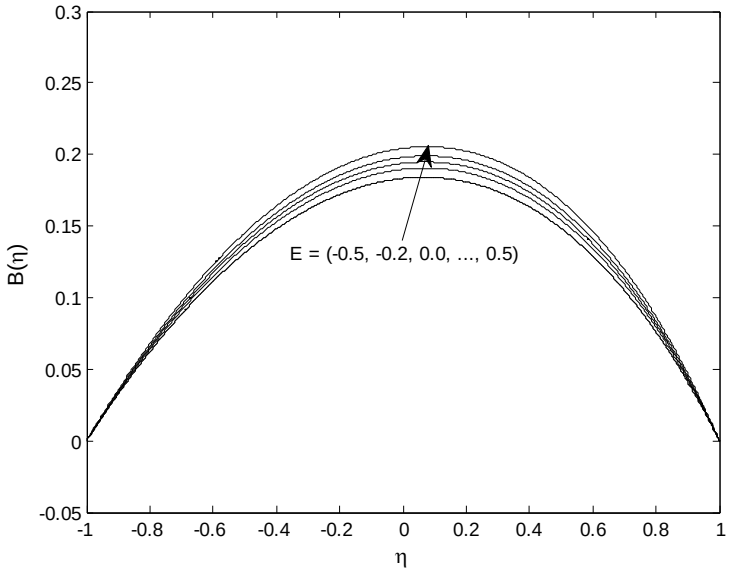


Fig. 5.25. Velocity profile with $E$ ( $Pr = 0.71, St = 1, a = -0.3, s = 1$)

It is also clear from this figure that velocities near the plate $\eta = -1$ are higher than those near the $\eta = +1$ which results in the asymmetric nature of the graph due to suction on the plate $\eta = -1$. Fig. 5.25 shows that the velocity of the fluid increases with increasing viscous dissipation under increased viscosity ($a = -0.3$) because increased viscous dissipation acts to complement the heating of the fluid by the external source. In Fig. 5.26 the effect of Prandtl number on the velocity is shown. It is observed from this figure that the velocity of the fluid decreases with increasing Pr values. This is true because when viscosity is fixed, increasing the Prandtl results in reducing the thermal diffusivity of the fluid which reduces in-flow of heat from the plates and hence plummeting temperatures which results in reduced convection currents.The effects of the governing parameters on the behaviour of the temperature of the working fluid are depicted in Figs. 5.27 to 5.31.

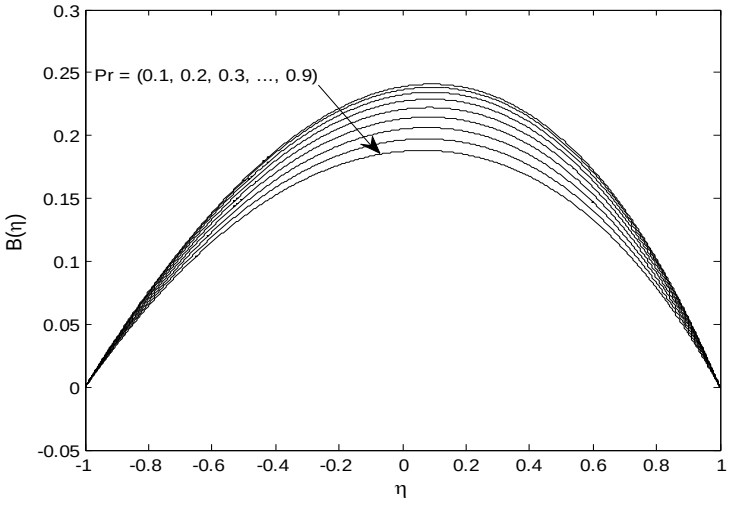


Fig. 5.26. Velocity profile with $Pr$ ($St = 1, E = 0.5, a = -0.3, s = 1$)

Figs. 5.27 shows the variation of temperature with the suction/injection parameter, s. It is observed from this figure that when there is suction on the plate $\eta = +1$ with corresponding injection on the other plate, the temperature near the plate with injection increases while that near the plate with suction decreases. If the direction of suction and injection is reversed, the

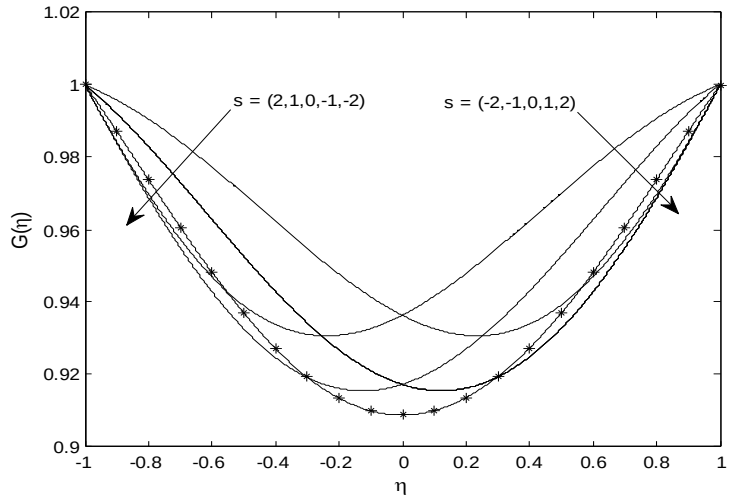


Fig. 5.27. Variation of temperature with $s$ ($Pr = 0.71, St = 1, a = -0.3, E = 0.5$)

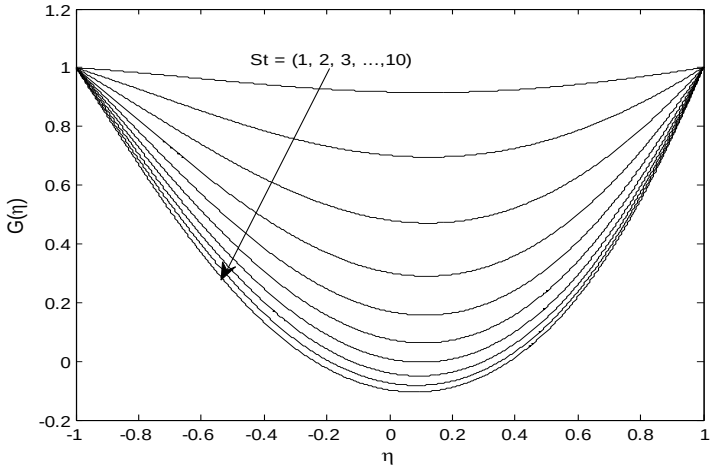


Fig. 5.28. Variation of temperature with$St$ ( $Pr = 0.71, a = -0.3, E = 0.5, s = 1$)

physical phenomenon remains the same on the respective plates. It is also observed in this figure that the temperature of the fluid around the center line of the channel is generally higher under suction or injection than when suction is nullified ($s = 0$). Fig. 5.28 shows the variation of temperature with the Strouhal number. It can be observed in this figure that there is a consistent fall in temperature as the Strouhal number is increased. This is so because increasing the Strouhal number increases the frequency of the periodic heating of the plates and this results in less heating of the fluid which leads to reducing temperatures. The variation in temperature of the fluid with the variable viscosity parameter a is depicted in Fig. 5.29. It is observed in this figure that the fluid temperature reduces as the viscosity of the fluid increases.

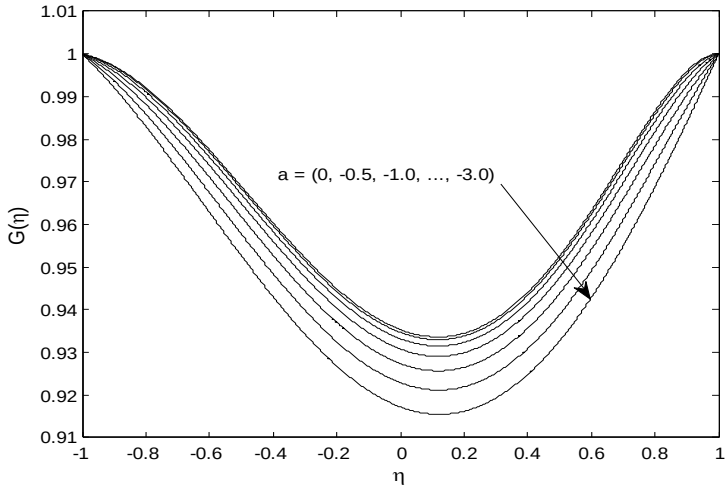


Fig. 5.29. Variation of temperature with $a$ ( $Pr = 0.71, St = 1, E = 0.5, s = 1$)

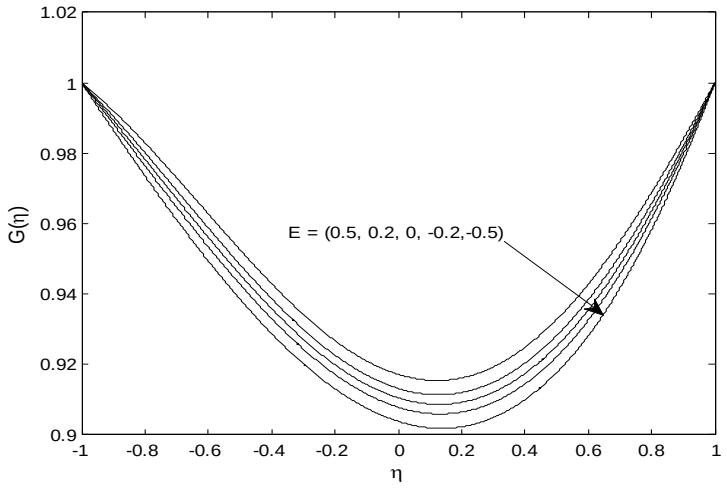


Fig. 5.30. Variation of temperature with $E$ ( $Pr = 0.71, St = 1, a = -0.3, s = 1$)

Fig. 5.30 shows the behaviour of the temperature profile with the Eckert number, $E$. It is shown in this figure that the temperature of the fluid grows with increasing viscous dissipation because viscous dissipation acts to compliment the heating of the fluid from the bounding plates.

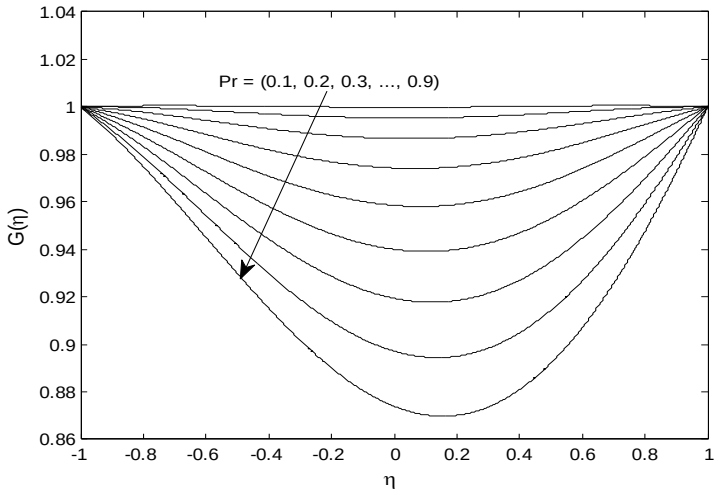


Fig. 5.31. Variation of temperature with $Pr$ ($St = 1, E = 0..5, a = -0.3, s = 1$)

In Fig. 5.31, the fluid temperature is observed to decrease with increasing Pr because Pr bears an inverse relationship with the thermal diffusivity of a fluid and so increasing Pr generally leads to reduced heat penetration as a result of reduced thermal diffusivity of the working fluid. Fig. 5.32 shows the phase of temperature and that of velocity. The positive $x - axis$ is the direction of fluid flow while the negative y-axis is the direction of heat penetration. The angle of deviation ($\emptyset$) from the positive $y - axis$ is the phase of temperature while the angular deviation ($\theta$) from the positive $x - axis$ is the phase of velocity. Temperature and velocity are thus highest when their phase is lowest. The variation in phase of temperature and velocity at the center of the channel ($\eta = 0$) are depicted in Figs. 5.33 and 5.34. It is observed that the phase of temperature increases with increasing suction (Fig. 5.33), so also the phase of velocity (Fig. 5.34). This is to say that the heat penetration from the plates decreases with increasing suction and increase with injection as concluded in Fig. 5.27. It is also observed from these figures that the phase of both temperature and velocity decrease with the Eckert number which is to say that the

fluid temperature increases as the Eckert number increases because temperature is high when the phase is low and vice versa but increase in viscous dissipation acts to complement penetrating heat from the walls to strengthen convection current which results in reduced dephasing of fluid particles.

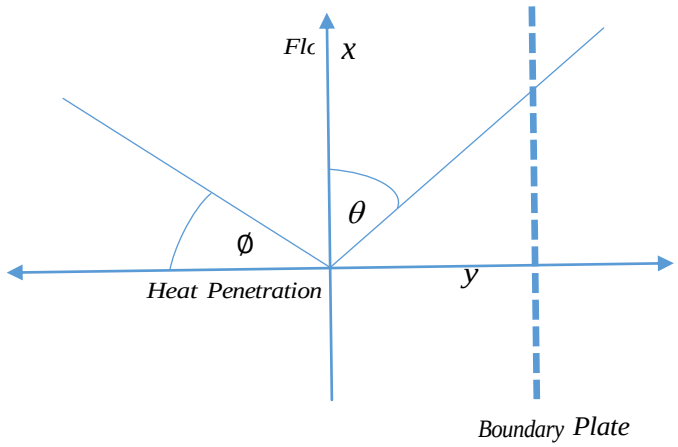


Fig. 5.32. The Phase of temperature and velocity

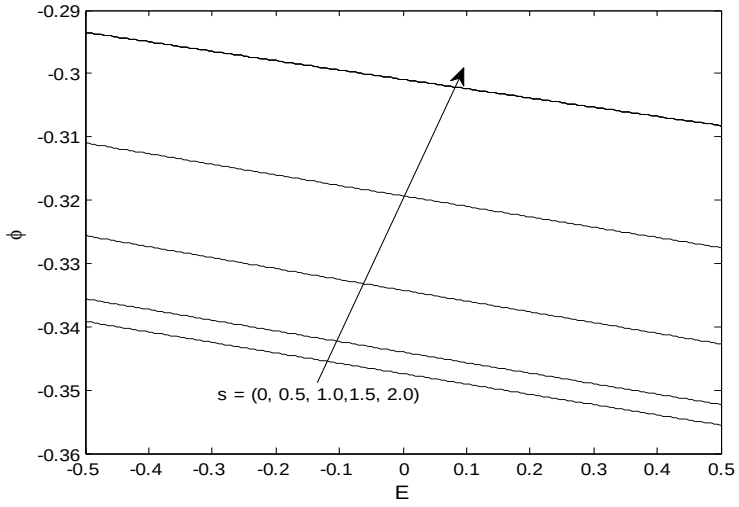


Fig. 5.33. Phase of temperature ($Pr = 0.71, St = 1, a = -0.3, \eta = 0.0$)

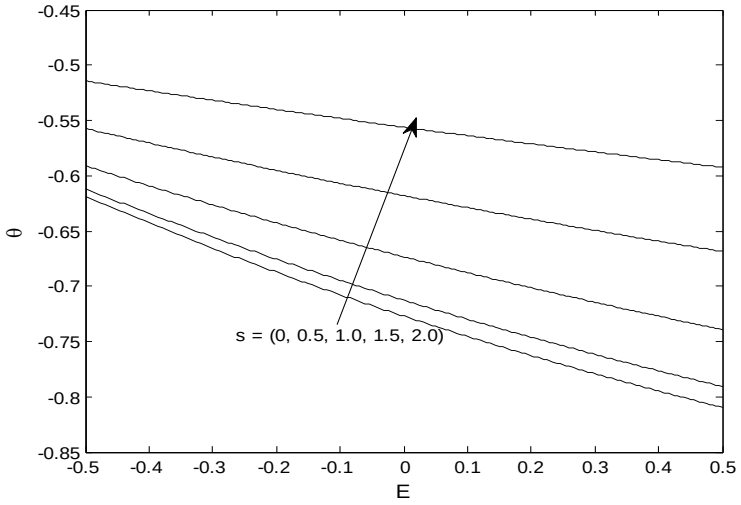


Fig. 5.34. Phase of Velocity ($Pr = 0.71, St = 1, a = -0.3, \eta = 0.0$)

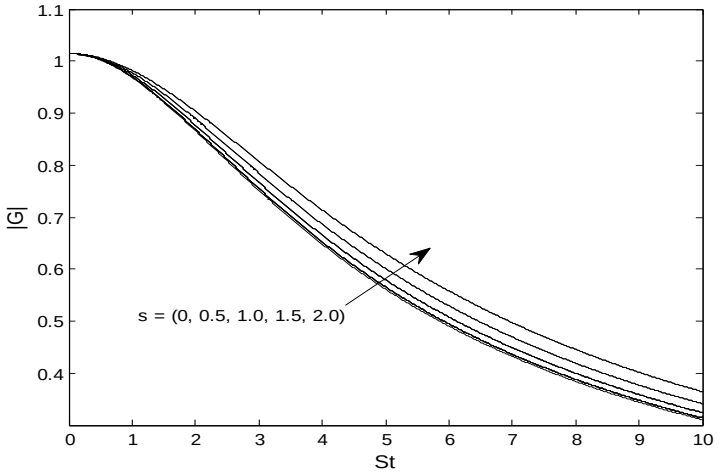


Fig. 5.35. Amplitude of temperature ($Pr = 0.71, St = 1, a = -0.3, \eta = 0.0$)

Fig. 5.35 and 5.36 show the amplitude of temperature and velocity with $St$ for different $s$ values respectively at the center line of the channel. It is observed in $Fig$ 5.35 that the amplitude of temperature generally reduces with increasing $St$ but it increases with increasing suction. In $Fig.$ 5.36, the amplitude of velocity is observed to decrease initially

as $s$ increases but twists to a reverse order after a fixed value of $St$ where the amplitude is invariant for all values of $s$. It is also easily verifiable from this figure that the amplitude of velocity tends to zero for all $s$ value as $St$ becomes very large.

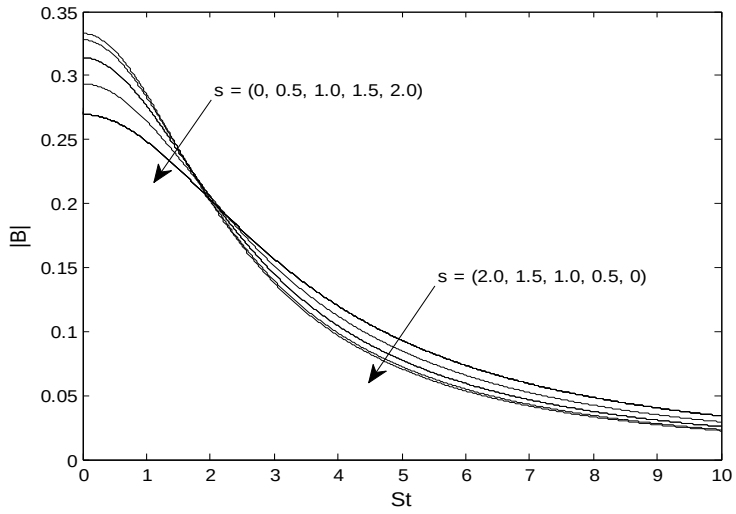


Fig. 5.36. Amplitude of Velocity ($Pr = 0.71, E = 0.5, a = -0.3, \eta = 0.0$)

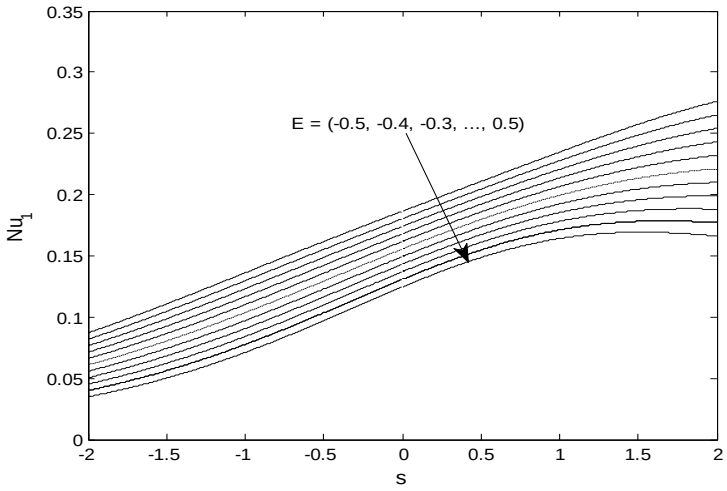


Fig. 5.37. Rate of heat transfer on the plate ($Pr = 0.71, E = 0.5, a = -0.3, \eta = +1$)

The rate of heat transfer on the plates are shown in $Figs.5.37$ and 5.38. Both figures reveal that the rate of heat transfers on the plates increase with increasing suction but

decrease with increasing injection. The role of the Eckert number is to increase the rate of heat transfer on one plate while it leads to a decrease on the other plate.

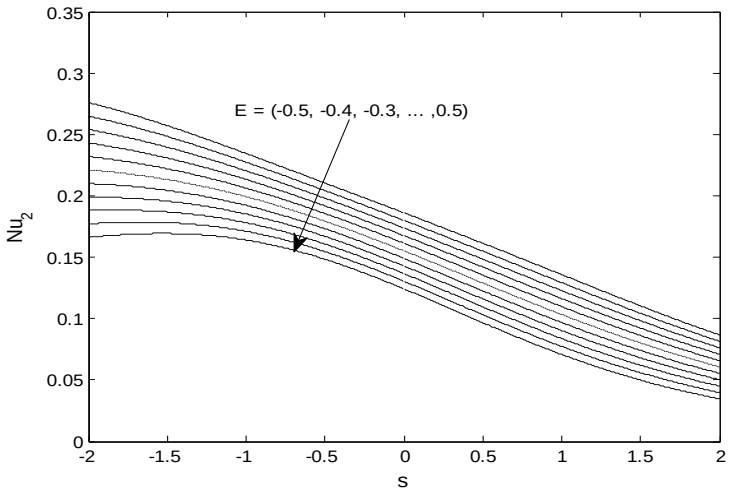


Fig. 5.38. Rate of heat transfer on the plate ($Pr = 0.71, E = 0.5, a = -0.3, \eta = -1$)

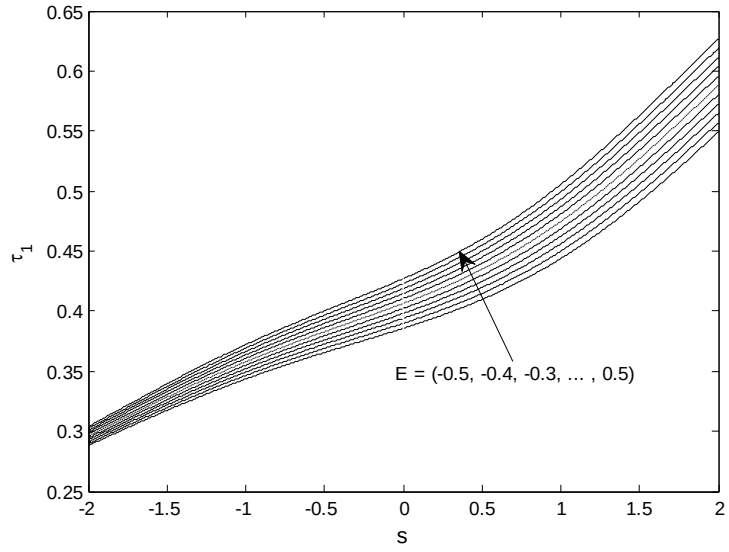


Fig. 5.39. Skin friction with $s$ ( $Pr = 0.71, St = 1, a = -0.3, \eta = +1$)

The skin friction on the plates are depicted in $Figs.5.39$ and 5.40. Both figures reveal that the skin friction decrease with increasing suction while it increases with injection. Increasing the Eckert number results in increased skin friction on one plate while that of the other plate is reduced.

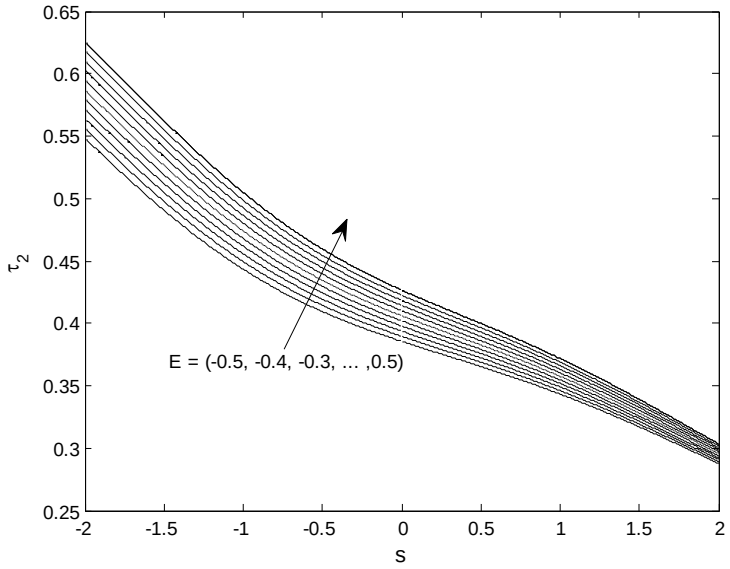


Fig. 5.40. Skin friction with $s$ ( $Pr = 0.71, St = 1, a = -0.3, \eta = -1$)

## 5.4 Validation of the Results

The accuracy of the Homotopy Perturbation Method (HPM) as well as the obtained results in this work for both problems were tested with previous resultobtained by means of the General perturbation (GP) by comparing numerical values as presented in tables (5.1) and (5.2) below. The results show excellent agreements.

### *5.4.1* Validation of Result of Problem I

**Table 5.1:Comparison between periodic results by Homotopy perturbation method (HPM) (presentproblem) and General perturbation (GP) (Jha and Ajibade, 2011),($\mathbf{Pr} = \mathbf{0.044}$, $\boldsymbol{St} = \mathbf{1}, \boldsymbol{E} = \mathbf{0.5}$ ).**

| | Jha and Ajibade (2011) G P | | Present Problem ($a = 0$) HPM | |
|---|---|---|---|---|
| $\eta$ | $G(\eta)$ | $B(\eta)$ | $G(\eta)$ | $B(\eta)$ |
| $-0.8$ | 1.0016992053 | 0.1567732755 | 1.0016992053 | 0.1567732755 |
| $-0.5$ | 1.0025870532 | 0.3218601898 | 1.0025870532 | 0.3218601898 |
| 0.0 | 1.0026510025 | 0.4248750599 | 1.0026510025 | 0.4248750599 |
| 0.5 | 1.0025870532 | 0.3218601898 | 1.0025870532 | 0.3218601898 |
| 0.8 | 1.0016992053 | 0.1567732755 | 1.0016992053 | 0.1567732755 |

### *5.4.2* Validation of results of Problem II

**TableComparison between periodic results by Homotopy Perturbation method(HPM) (present problem) and General Perturbation (GP) (Jha and Ajibade (2011)(Pr $= \mathbf{0.71}$, $\boldsymbol{St} = \mathbf{1}, \boldsymbol{E} = \mathbf{0.5}$ )**

| | Present Problem (HPM) $(a = 0, s \to 0)$ | | Jha and Ajibade 2011(G P) | |
|---|---|---|---|---|
| $\eta$ | $G(\eta)$ | $B(\eta)$ | $G(\eta)$ | $B(\eta)$ |
| $-0.8$ | 0.9890445 | 0.1339081 | 0.9890457 | 0.1338943 |
| $-0.4$ | 0.9486064 | 0.2986696 | 0.9486066 | 0.2986684 |
| 0.0 | 0.9289980 | 0.3497015 | 0.9289978 | 0.3497020 |
| 0.4 | 0.9486075 | 0.2986750 | 0.9486066 | 0.2986684 |
| 0.8 | 0.9890466 | 0.1339081 | 0.9890457 | 0.1338943 |

# CHAPTER SIX

# SUMMARY, CONCLUSION AND RECOMMENDATION

## 6.1 Summary

In this dissertation, a study of natural convection flow between infinite vertical parallel plates with viscous dissipation and variable viscosity effects have been studied. The problem is further extended to investigate natural convection heat transfer flow in a vertical channel when the boundary permits movement of fluid in and out (suction/injection). The momentum and energy equations of these problems were first separated into steady and periodic parts before putting in dimensionless form. The strongly non-linear and coupled ode's obtained were first subjected to the Homotopy Perturbation technique to break the non-linearity and coupling, resulting in linear ode's which were then solved by means of the undetermined coefficient method to obtain expressions for both temperature and velocity fields in the two problems. Simulation of the results was carried out using MATLAB (R2012b) program and generated numerical values and line graphs were used in the discussion, inference and drawing of conclusions.

## 6.2 Conclusion

In this dissertation, we have been able to overcome the limitations of coupling and non-linearity posed by the governing differential equations to obtain, by the Homotopy Perturbation method, model solutions for both the energy and momentum of fluidsin a vertical channel under the effects of viscous dissipation, variable viscosity and suction/injection.

Based on the analysis of the obtained results, the following conclusions are drawn from this dissertation:

Firstly, for fluids with$Pr > 1$, the effect of variable viscosity is dominant over that of viscous dissipation.

Secondly, viscous dissipation heating becomes stronger and exerts greater influence on both the energy and momentum of the working fluid when viscosity has been increased.

Thirdly, fluid temperature decreases with growing viscous dissipation if the viscosity of the fluid decreases with increasing temperature.

Fourthly, when viscosity is raised, the rate of heat transfer on the plates decreases with increasing viscous dissipation for all values of $Pr$.

In addition, suction on a plate increases the velocity and decreases the temperature near it while the other plate witnesses decrease in velocity and increased temperature under injection.

Finally, the skin friction on the plates increases with increased suction while it decreases with increasing injection.

## 6.3 Recommendation

It is hereby recommended that the contribution of variations in conductivity of the fluidcan be studied on each of the two problems as a further investigation.

Mixed convection or pressure driven flow situation can be studied in either of the cases to investigate the effect of pressure gradient on the flow pattern.

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# APPENDICES

## Appendix I

**Constants used to define Eqs. (4.3.8) – 4.3.11):**

***Pr* ≠ 1**

$r = \sqrt{iSt}$; q = $\sqrt{iStPr}$; b = $ar^2$; c = $r^2$

d = $EPr$, e = $aEPr$, $g = a^2EPr$, $m = q^2$,

$Q_1 = \frac{ar^2-b}{(m-c)\,Cosh(r)}$ (=0), $Q_2 = \frac{2a}{Cosh(q)}$, $Q_3 = \frac{aq}{Cosh(q)}$,

$b_1 = \frac{Cosh(q)}{(m-c)\,Cosh(r)}(Q_2 + Q_3\,tanh(q) - \frac{2qQ_3}{(m-c)} - \frac{Q_1 \tan(r)}{2r})$,

$C_1 = ab_1r^2 - bb_1 + aQ_1$(=0), $C_2 = \frac{-a^2(a^2+2)}{Cosh(q)}$, $C_3 = \frac{aQ_1r}{2} - \frac{bQ_1}{2r}$(=0), $C_4 = \frac{-2a^2q}{Cosh(q)}$,

$\pi_1 = \frac{-Cosh(q)}{iSt(Pr-1)\,Cosh(r)}\left(C_2 + C_4\tanh(q) - \frac{2C_4q}{iSt(Pr-1)}\right)$,

$H_1 = \frac{2i\sqrt{i}EPr(1-a)^2}{\sqrt{St}(1-P_r)\,Cosh(r)}$, $H_2 = \frac{-2i\sqrt{i}EPr^{\frac{3}{2}}(1-a)^2}{\sqrt{St}(1-Pr)Cosh(q)}$, $H_3 = \frac{aEPr(1-a)}{Cosh(q)}$,

$\pi_2 = \frac{2\sqrt{i}H_1Cosh(r)}{St^{\frac{3}{2}}(1-Pr)^2\,Cosh(q)} - \frac{H_1Sinh(r)}{iSt(1-Pr)Cosh(q)} + \frac{H_2Sinh(q)}{4iStPr\;Cosh(q)} - \frac{H_2}{4\sqrt{iStPr}} - \frac{H_3\tanh(q)}{4iStPr\sqrt{iStPr}} + \frac{H_3}{4iStPr}$

$- \frac{H_3\tanh(q)}{6\sqrt{iStPr}}$, $K_1 = \frac{-4\,EPr\,(1-a)^2}{St^2(1-Pr)^3\,Cosh(r)}$, $K_2 = \frac{aC_3r^2-bC_3}{4r}$(=0), $K_3 = \frac{-2\sqrt{i}\,EPr(1-a)^2}{St^{\frac{3}{2}}(1-Pr)^2\,Cosh(r)}$,

$K_4 = \frac{\pi_2Cosh(q)+a^3(3+a^2)}{-Cosh(q)}$, $K_5 = \frac{iE(1-a)[2Pr-a(1+Pr)]}{4St(1-Pr)Cosh(q)}$, $K_6 = \frac{\sqrt{i}E(1-a)[a-2Pr(1-a)]}{4\sqrt{Pr}St^{\frac{3}{2}}(1-Pr)\,Cosh(q)} + \frac{a^3\sqrt{iStPr}}{Cosh(q)}$, $K_7$

$= \frac{aEPr(1-a)}{6\sqrt{iStPr}\,Cosh(q)}$, $P_1 = \frac{(1-2a)(2aEPr^2)}{(1-Pr)Cosh(q)} + \frac{(1+a)(2a^2EPr)}{Cosh(q)}$, $P_2 = \frac{(1-2a)(Pr-2)\,4iaEPr\sqrt{iStPr}}{St(1-Pr)^2Cosh(q)}$,

$P_3 = dQ_1 - 2eQ_1$ (=0), $P_4 = (1\text{-}2a)\left(2rEPrb_1 + \frac{2raEPr}{iSt(Pr-1)Cosh(r)}\right)$, $V_1 = b_1 + \pi_1 + \pi_3$,

$V_2 = \frac{K_3}{4r}$, $V_3 = \frac{iSt(Pr-1)(K_4+C_2-Q_2)+2q(Q_3-K_6-K_4)+2K_5}{-St^2(1-Pr)^2}$, $V_4 = \frac{iSt(Pr-1)K_5+6qK_7}{-St^2(Pr-1)^2}$,

$V_5 = \frac{2\sqrt{iSt}K_1 - K_3}{4iSt}$, $V_6 = \frac{iSt(1-Pr)(C_4 - Q_3 + K_6) + 4qK_5 - 6K_7}{St^2(Pr-1)^2}$, $V_7 = \frac{K_2}{6r}(= 0)$, $V_8 = \frac{i\,K_7}{St(1-Pr)}$,

$I_1 = \pi_2 + \pi_4$, $I_2 = \frac{2EPr^2(1-3a+3a^2)}{4iStPr(1-Pr)Cosh(q)} + \frac{aEPr(2a^2+3a-1)}{4iStPrCosh(q)} + \frac{aiEPr(1-2a)(Pr-2)}{St(1-Pr)^2Cosh(q)}$,

$I_3 = \frac{-I_2}{\sqrt{iStPr}}$, $I_4 = \frac{-4EPr(1-a)^2}{St^2(1-Pr)^3\cosh(r)} + \frac{2P_4}{St^{\frac{3}{2}}(1-Pr)^2}$, $I_5 = \frac{P_3}{iSt(1-Pr)}(= 0)$,

$I_6 = \frac{2\sqrt{i}EPr(1-a)^2}{St^{\frac{3}{2}}(1-Pr)^2\cosh(r)} + \frac{P_4}{iSt(1-Pr)}$, $I_7 = \frac{aEPr(1+Pr+a+2a^2-3aPr)}{6\sqrt{iStPr}(1-Pr)Cosh(q)}$,

**$Pr = 1$**

$Q_1 = a\left(\frac{1}{Cosh(r)} + \frac{1}{r}\right)$, $Q_2 = \frac{-ar}{Cosh(r)}$, $b_1 = \frac{a}{4rCosh(r)}(r - Tanh(r))$, $C_1 = \frac{-a^2}{Cosh(r)}$, $C_2 = \frac{-2a^2r}{Cosh(r)}$,

$(C_3 = 0)$, $H_1 = \frac{EP_r(r\tanh(r)-1)(1-a)^2}{r\,Cosh(r)}$, $H_2 = \frac{-EP_r(1-a)(1-2a)}{Cosh(r)}$, $\pi_1 = \frac{a^2}{2}$,

$\pi_2 = \frac{H_1\tanh(r)}{4iSt} - \frac{H_1}{4\sqrt{iSt}} - \frac{H_2\tanh(r)}{4i\sqrt{i}St^{\frac{3}{2}}} + \frac{H_2}{4iSt} - \frac{H_2}{6\sqrt{iSt}}$, $K_1 = \frac{2a^3 + \pi_2 Cosh(r)}{-Cosh(r)}$, $K_2 = \frac{H_2 - H_1 r}{4r}$,

$K_3 = \frac{5a^3r^2 + 2K_2\cosh(r)}{2rcosh(r)}$, $K_4 = \frac{-EP_r(1-a)(1-2a)}{6rCosh(r)}$, $P_1 = \frac{aEP_r(4a^2+8a-2-r(1-2a)}{2Csoh(r)}$,

$P_2 = \frac{-4a^2EP_rTanh(r)}{2Cosh(r)} + \frac{aEP_r[2-a+r^2(2a+1)]}{rCosh(r)}$, $P_3 = \frac{-arEP_r(1-2a)}{2Cosh(r)}$,

$\pi_3 = \frac{2K_2r + K_4r^2 + 3K_4}{8i\sqrt{i}St^{\frac{3}{2}}} - \frac{(3K_1 + K_2)\cosh(r)}{6\sqrt{iSt}} + \frac{(K_3 - K_4)\tanh(r)}{4iSt} + \frac{(2K_2r + 3K_4)\tanh(r)}{8St^2}$,

$\pi_4 = \frac{-2P_1Tanh(r) - 3P_3}{8i\sqrt{i}St^{\frac{3}{2}}} - \frac{3P_3\tanh(r)}{8St^2} + \frac{P_1 + (P_2 + P_3)\tanh(r)}{4iSt} - \frac{6P_2 + (3P_3 + 4P_1)\tanh(r)}{24\sqrt{iSt}}$,

$m_1 = b_1 + \pi_1 + \pi_3$, $m_2 = \frac{Q_2 - C_2 + K_3}{4r} - \frac{K_2r + 3K_4}{8r^2}$, $m_3 = \frac{K_4}{8r}$, $m_4 = \frac{C_1 + K_1}{2r} - \frac{C_2 + Q_2 + K_3}{4r^2} + \frac{k_2r + 3K_4}{8r^4}$, $m_5 =$

$\frac{2K_2r + 3K_3}{12r^2}$, $h_1 = \pi_2 + \pi_4$, $h_2 = \frac{H_1 + P_2}{4r} - \frac{H_2 + P_1}{4r^2} + \frac{3P_3}{8r^3}$, $h_3 = \frac{-(H_1 + P_2)}{4r^2} + \frac{H_2 + P_1}{4r^3} - \frac{3P_3}{8r^3}$,

$h_4 = \frac{2r(H_2 + P_1) - 3P_3}{12r^2}$.

## AppendixII

## Constants used in defining eqns. (4.6.6) –(4.6.9):

$a = \lambda(T_1 - T_0), b = aiSt, C = iSt, d = EPr, e = aEPr$, g = $a^2 EPr, m = iStPr$, $P = \frac{s\,Pr}{2}$,

$q = \frac{1}{2}\left(\sqrt{(sPr)^2 + 4iStPr}\right), a_1 = P + q, a_2 = P - q, D_0 = \frac{\sinh(a_1)}{\sinh(2q)}$, $C_0 = e^{-a_1} - D_0 e^{(a_2 - a_2)}$,

$X_1 = \frac{C_0}{{a_1}^2 - a_1 s - iSt}$, $X_2 = \frac{D_0}{{a_2}^2 - a_2 s - iSt}$, $Q_3 = X_1 e^{a_1} + X_2 e^{a_2}$, $Q_4 = X_1 e^{-a_1} + X_2 e^{-a_2}$,

$m_1 = \frac{1}{2}\left(s + \sqrt{s^2 + 4iSt}\right), m_2 = \frac{1}{2}\left(s - \sqrt{s^2 + 4iSt}\right), D_1 = \frac{Q_4 e^{m_1} - Q_3 e^{m_2}}{2\sinh(m_1 - m_2)}$,

$C_1 = Q_3 e^{-m_1} - D_1 e^{m_2 - m_1}$, $Q_5 = C_1(-b + a{m_1}^2 - asm_1), Q_6 = D_1(-b + a{m_2}^2 - asm_2)$,

$Q_7 = X_1(b - a{a_1}^2 + asa_1) + \frac{1}{s}(aC_0 a_1)$, $Q_8 = X_2(b - a{a_2}^2 + asa_2) + \frac{1}{s}(aD_0 a_2)$, $Q_9 = \frac{-aC_0(s + a_1)}{\sinh(s)}$, $Q_{10} = \frac{-aD_0(s + a_2)}{\sinh(s)}$,

$R_3 = Q_5(2m_1 - s)$, $R_4 = Q_6(2m_2 - s)$, $R_5 = Q_7/({a_1}^2 - a_1 s - iSt)$, $R_6 = Q_8/({a_2}^2 - a_2 s - iSt)$,

$R_7 = Q_9/({a_1}^2 + a_1 s - iSt)$, $R_8 = Q_{10}/({a_2}^2 + a_2 s - iSt)$,

$R_9 = R_3 e^{m_1} + R_4 e^{m_2} + R_5 e^{a_1} + R_6 e^{a_2} + R_7 e^{(s + a_1)} + R_8 e^{(s + a_2)}$,

$R_{10} = R_3 e^{-m_1} + R_4 e^{-m_2} - R_5 e^{-a_1} - R_6 e^{-a_2} - R_7 e^{-(s + a_1)} - R_8 e^{-(s + a_2)}$,

$D_2 = \frac{R_9 e^{-m_1} + R_{10} e^{m_1}}{2\sinh(m_1 - m_2)}$, $C_2 = -R_9 e^{-m_1} - D_2 e^{(m_2 - m_1)}$, $Q_{11} = -bC_2 + aC_2 {m_1}^2 + 2aR_3 m_1 - asC_2 m_1 - asR_3$,

$Q_{12} = -bD_2 + aD_2 {m_2}^2 + 2aR_4 m_2 - asD_2 m_2 - asR_4$; $Q_{13} = R_3(-b + a{m_1}^2 - asm_1)$;

$Q_{14} = R_4(-b + a{m_2}^2 - asm_2), Q_{15} = -bR_5 + aR_5 {a_1}^2 - asR_5 a_1 + \frac{1}{s}(a^2 D_0 a_2)$,

$Q_{16} = -bR_6 + aR_6 {a_2}^2 - asR_6 a_1 + \frac{1}{s}(a^2 C_0 a_2)$, $Q_{17} = -bR_7 + aR_7(a_1 + s)^2 - asR_7(a_1 + s)$

$-\frac{1}{\sinh(s)}(a^2 C_0 s + a^2 C_0 a_1)$,

$$Q_{18} = -bR_8 + aR_8(a_2 + s)^2 - asR_8(a_2 + s) - \frac{1}{\sinh(s)}(a^2 D_0 s + a^2 D_0 a_2);$$

$$R_{11} = \frac{Q_{11}}{2m_1 - s} - \frac{Q_{13}}{(2m_1 - s)^2}; R_{12} = \frac{Q_{13}}{2(2m_1 - s)}; \; R_{13} = \frac{Q_{12}}{(2m_2 - s)} - \frac{Q_{14}}{(2m_1 - s)^2}; \; R_{14} = \frac{Q_{14}}{2(2m_2 - s)};$$
$$R_{15} = \frac{Q_{15}}{{a_1}^2 - a_1 s - iSt}, \; R_{16} = \frac{Q_{16}}{{a_2}^2 - a_2 s - iSt}, \; R_{17} = \frac{Q_{17}}{{a_1}^2 - a_1 s - iSt}, \; R_{18} = \frac{Q_{18}}{{a_2}^2 - a_2 s - iSt},$$

$$R_{19} = (R_{11} + R_{12})e^{m_1} + (R_{13} + R_{14})e^{m_2} + R_{15}e^{a_1} + R_{16}e^{a_2} + R_{17}e^{(a_{1+s})} + R_{18}e^{(a_2+s)},$$

$$R_{20} = (R_{11} - R_{12})e^{-m_1} + (R_{13} - R_{14})e^{-m_2} - R_{15}e^{-a_1} - R_{16}e^{a_2} - R_{17}e^{-(a_{1+s})} -$$
$$R_{18}e^{-(a_2+s)}, D_3 = \frac{R_{19}e^{-m_1} + R_{20}e^{-m_1}}{2\sinh(m_1 - m_2)}, \; C_3 = -(R_{19}e^{-m_1} + D_3 e^{(m_2 - m_1)}), \quad k = e - \mathrm{g},$$

$$z = 2e - d - \mathrm{g}, H_1 = \frac{z}{(4s^2 - 2s^2 SPr)(\sinh(s))^2}, \; H_2 = \frac{2z}{s^3(1 - Pr)\sinh(s)},$$
$$H_3 = \frac{z}{s^3 Pr \sinh(sPr)}, D_4 = -\frac{H_1 \sinh(2s)}{\sinh(sPr)} + \frac{H_2 \sinh(s)}{\sinh(sPr)} + H_3, C_4 = -H_1 e^{2s} + H_2 e^{s} + \frac{z}{s^3 Pr} -$$
$$D_4 e^{sPr}, \quad Q_{19} = \frac{2zC_1 m_1}{\sinh(s)}, \quad Q_{20} = \frac{2zD_1 m_2}{\sinh(s)}, \quad Q_{21} = \frac{2zX_2 a_2}{\sinh(s)} - \frac{2kD_0}{s\sinh(s)}, Q_{22} = \frac{2zX_1 a_1}{\sinh(s)} - \frac{2kC_0}{s\sinh(s)},$$
$$Q_{23} = \frac{1}{s}(2zC_1 m_1), \; Q_{24} = \frac{1}{s}(2zD_1 m_2), \quad Q_{25} = \frac{2zX_2 a_2}{s} - \frac{kD_0}{s^2}, \; Q_{26} = \frac{2zX_1 a_1}{s} - \frac{2kC_0}{s^2},$$
$$Q_{27} = \frac{kD_0}{(\sinh(s))^2},$$

$$R_{28} = \frac{kC_0}{(\sinh(s))^2}, \quad R_{21} = \frac{-Q_{19}}{(m_1 + s)^2 - sPr(m_1 + s) - m}, \; R_{22} = \frac{-Q_{20}}{(m_2 + s)^2 - sPr(m_2 + s) - m},$$

$$R_{23} = \frac{Q_{21}}{(a_2 + s)^2 - sPr(a_2 + s) - m}, \quad R_{24} = \frac{Q_{22}}{(a_1 + s)^2 - sPr(a_1 + s) - m}, \quad R_{25} = \frac{Q_{23}}{({m_1}^2 - sPr\, m_1 - m)},$$

$$R_{26} = \frac{Q_{24}}{(({m_2}^2 - sPr\, m_2 - m)}, \; R_{27} = \frac{-Q_{25}}{2a_2 - sPr}; \quad R_{28} = \frac{-Q_{26}}{2a_1 - sPr}, R_{29} = \frac{Q_{27}}{(a_2 + 2s)^2 - sPr(a_2 + 2s) - m};$$
$$R_{30} = \frac{Q_{28}}{(a_1 + 2s)^2 - sPr(a_1 + 2s) - m};$$

$$R_{31} = R_{21}e^{(m_1+s)} + R_{22}e^{(m_2+s)} + R_{23}e^{(a_2+s)} + R_{24}e^{(a_1+s)} + R_{25}e^{m_1} + R_{26}e^{m_2}$$
$$+ R_{27}e^{a_2} + R_{28}e^{a_1} + R_{29}e^{(a_2+2s)} + R_{30}e^{(a_1+2s)};$$

$$R_{32} = R_{21}e^{-(m_1+s)} + R_{22}e^{-(m_2+s)} + R_{23}e^{-(a_2+s)} + R_{24}e^{-(a_1+s)} + R_{25}e^{-m_1} +$$
$$R_{26}e^{-m_2} - R_{27}e^{-a_2} - R_{28}e^{-a_1} + R_{29}e^{-(a_2+2s)} + R_{30}e^{-(a_1+2s)};$$

$$D_5 = \frac{R_{31}e^{-a_1} - R_{32}e^{a_1}}{2\sinh(a_1 - a_2)}; \; C_5 = -R_{31}e^{-a_1} - D_5 e^{a_2 - a_1};$$

$$H_4 = \frac{-1}{s}(C_4 + a^3 + H_2 e^{s}) + \frac{z}{2s^4 Pr} + \frac{z}{s^5 Pr} + \frac{H_1 e^{2s}}{2s^2} + \frac{D_4 e^{sPr}}{s^2 \mathrm{Pr}(Pr - 1)};$$

$$H_5 = \frac{-(C_4 + a^3 + H_2 \cosh(s))}{s\sinh(s)} + \frac{z}{s^5 Pr\, sinh(s)} + \frac{H_1 \sinh(2s)}{2s^2 \sinh(s)} + \frac{D_4 \sinh(sPr)}{s^2 \mathrm{Pr}(Pr - 1)\sinh(s)}; \quad H_6 = H_4 - H_5 e^{s};$$

$$Q_{29} = -bC_3 + aC_3 {m_1}^2 + 2am_1 R_{11} + 2aR_{12} - asm_1 C_3 - asR_{11} - R_{25};$$

$$Q_{30} = -bD_3 + aC_3{m_2}^2 + 2am_2R_{13} + 2aR_{14} - asm_2D_3 - asR_{13} - R_{26};$$

$$Q_{31} = -bR_{11} + aR_{11}{m_1}^2 + 4am_1R_{12} - asm_1R_{11} - 2asR_{12};$$

$$Q_{32} = -bR_{13} + aR_{13}{m_2}^2 + 4am_2R_{14} - asm_2R_{13} - 2asR_{14};\quad Q_{33} = -bR_{12} + aR_{12}{m_1}^2 - asm_2R_{14};$$

$$Q_{34} = -bR_{14} + aR_{14}{m_2}^2 - asm_2R_{14};\quad Q_{35} = -bR_{15} + a{a_1}^2R_{15} - asa_1R_{15} + \frac{1}{s}(a^3C_0a_1) - c_5; Q_{36} = -bR_{16} + a{a_2}^2R_{16} - asa_2R_{16} + \frac{1}{s}(a^3D_0a_2) - D_5;$$

$$Q_{37} = \frac{-a^3C_0(a_1+s)}{\sinh(s)} - asR_{17}(a_1+s) + aR_{17}(a_1+s)^2 - bR_{17} - R_{24};$$

$$Q_{38} = \frac{-a^3D_0(a_2+s)}{\sinh(s)} - asR_{18}(a_2+s) + aR_{18}(a_2+s)^2 - bR_{18} - R_{23};$$

$$R_{33} = \frac{Q_{29}}{(2m_1-s)} - \frac{Q_{31}}{(2m_1-s)^2} + \frac{Q_{33}}{(2m_1-s)^3};\quad R_{34} = \frac{Q_{30}}{(2m_2-s)} - \frac{Q_{32}}{(2m_2-s)^2} + \frac{Q_{34}}{(2m_2-s)^3};$$

$$R_{35} = \frac{Q_{31}}{(4m_1-2s)} - \frac{Q_{33}}{(2m_1-s)^2};\ R_{36} = \frac{Q_{32}}{(4m_2-2s)} - \frac{Q_{34}}{(2m_2-s)^2};\quad R_{37} = \frac{Q_{33}}{3(2m_1-s)};\quad R_{38} = \frac{Q_{24}}{3(2m_2-s)};$$

$$R_{39} = \frac{Q_{35}}{{a_1}^2-a_1s-c};\quad R_{40} = \frac{Q_{36}}{{a_2}^2-a_2s-c};\ R_{41} = \frac{Q_{37}}{{a_1}^2-a_1s-c};\quad R_{42} = \frac{Q_{38}}{{a_2}^2-a_2s-c};$$

$$R_{43} = \frac{R_{21}}{{m_1}^2-m_1s-c};\quad R_{44} = \frac{R_{22}}{{m_2}^2-m_2s-c};\quad R_{45} = \frac{R_{29}}{{a_2}^2+3a_2s+2s^2-c};\quad R_{46} = \frac{R_{30}}{{a_1}^2+3a_1s+2s^2-c};$$

$$R_{49} = \frac{R_{27}(2a_2-s)}{({a_2}^2-a_2s-c)^2};\ R_{50} = \frac{R_{27}}{({a_2}^2-a_2s-c)^2};\quad R_{51} = \frac{R_{28}(2a_1-s)}{({a_1}^2-a_1s-c)^2};\quad R_{52} = \frac{R_{28}}{({a_1}^2-a_1s-c)^2};$$

$$R_{47} = (R_{33}+R_{35}+R_{37})e^{m_1} + (R_{34}+R_{36}+R_{38})e^{m_2} + (R_{39}+R_{51}-R_{52})e^{a_1} + (R_{40}+R_{49}-R_{50})e^{a_2} - R_{43}e^{(m_1+s)} - R_{44}e^{(m_2+s)} + R_{41}e^{(a_1+s)} + R_{42}e^{(a_2+s)} - R_{45}e^{(a_2+2s)} - R_{46}e^{(a_1+2s)};$$

$$R_{48} = (-R_{33}+R_{35}-R_{37})e^{-m_1} + (-R_{34}+R_{36}-R_{38})e^{-m_2} + (R_{39}+R_{51}+R_{52})e^{-a_1} + (R_{40}+R_{49}+R_{50})e^{-a_2} - R_{43}e^{-(m_1+s)} - R_{44}e^{-(m_2+s)} + R_{41}e^{-(a_1+s)} + R_{42}e^{-(a_2+s)} - R_{45}e^{-(a_2+2s)} - R_{46}e^{-(a_1+2s)};$$

$$D_6 = \frac{R_{47}e^{-m_1} - R_{48}e^{m_1}}{2\sinh(m_1-m_2)};\quad C_6 = R_{47}e^{-m_1} - D_6e^{(m_2-m_1)};$$

$$D_7 = \frac{-2H_1\sinh(2s)}{\sinh(sPr)} + \frac{2aH_2\sinh(s)}{\sinh(sPr)} + 2aH_3;\ C_7 = -2aH_1e^{2s} + 2aH_2e^{s} + \frac{2az}{s^3Pr} - D_7e^{sPr};$$

$$Q_{39} = \frac{2z}{s}(C_2m_1 + aC_1m_1 + R_3);\ Q_{40} = \frac{2z}{s}(D_2m_2 + aD_1m_2 + R_4);\ Q_{41} = \frac{2\,z\,R_3m_1}{s};$$

$$Q_{42} = \frac{2\,z\,R_4m_2}{s};\quad Q_{43} = \frac{2z}{s}(R_5a_1 - aX_1a_1) + \frac{2akC_0}{s^2};\quad Q_{44} = \frac{2z}{s}(R_6a_2 - aX_2a_2) + \frac{2akD_0}{s^2};$$

$$Q_{45} = \frac{2z}{\sinh(s)}(C_2m_1 + aC_1m_1 + R_3);\ Q_{46} = \frac{2z}{\sinh(s)}(D_2m_2 + aD_1m_2 + R_4);$$

$$Q_{47} = \frac{-2\,z\,R_3m_1}{\sinh(s)};\quad Q_{48} = \frac{-2\,z\,R_4m_2}{\sinh(s)};\quad Q_{49} = \frac{2\,z\,R_7(a_1+s)}{\mathrm{s}} - \frac{4a\,k\,C_0}{s\sinh(s)} - \frac{-2\,z\,R_5a_1}{\sinh(s)} + \frac{2a\,z\,X_1a_1}{\sinh(s)};$$

$$Q_{50} = \frac{2\,z\,R_8(a_2+s)}{s} - \frac{4a\;k\;D_0}{s\sinh(s)} - \frac{-2\;z\;R_6 a_2}{\sinh(s)} + \frac{2a\;z\;X_2 a_2}{\sinh(s)};\; Q_{51} = \frac{2ak\,C_0}{(\sinh(s))^2} - \frac{2zR_7(a_1+s)}{\sinh(s)};$$

$$Q_{52} = \frac{2ak\,D_0}{(\sinh(s))^2} - \frac{2zR_8(a_2+s)}{\sinh(s)};\;\; R_{53} = \frac{(Q_{39}-Q_{41}(2m_1-sPr))}{{m_1}^2-sPr\,m_1-m};$$

$$R_{54} = \frac{(Q_{40}-Q_{42}(2m_2-sPr))}{{m_2}^2-sPr\,m_2-m};\;\; R_{55} = \frac{Q_{41}}{{m_1}^2-sPr\,m_1-m};\;\; R_{56} = \frac{Q_{42}}{{m_2}^2-sPr\,m_2-m};\;\; R_{57} = \frac{Q_{43}}{2a_1-sPr};$$

$$R_{58} = \frac{Q_{44}}{2a_2-sPr};\;\; R_{59} = \frac{-(Q_{45}+Q_{47}(2(m_1+s)-sPr))}{(m_1+s)^2-sPr\,(m_1+s)-m};\;\; R_{60} = \frac{Q_{47}}{(m_1+s)^2-sPr\,(m_1+s)-m};$$

$$R_{61} = \frac{-(Q_{46}+Q_{48}(2(m_2+s)-sPr))}{(m_2+s)^2-sPr\,(m_2+s)-m};\;\; R_{62} = \frac{Q_{48}}{(m_2+s)^2-sPr\,(m_2+s)-m};\;\; R_{63} = \frac{Q_{49}}{(a_1+s)^2-sPr\,(a_1+s)-m};$$

$$R_{64} = \frac{Q_{50}}{(a_2+s)^2-sPr\,(a_2+s)-m};\;\; R_{65} = \frac{Q_{51}}{(a_1+2s)^2-sPr\,(a_1+2s)-m};\;\; R_{66} = \frac{Q_{52}}{(a_2+2s)^2-sPr\,(a_2+2s)-m};$$

$$R_{67} = (R_{53}+R_{55})e^{m_1} + (R_{54}+R_{56})e^{m_2} + R_{57}e^{a_1} + R_{58}e^{a_2} + (R_{59}+R_{60})e^{(m_1+s)}$$
$$+(R_{61}+R_{62})e^{(m_2+s)} + R_{63}e^{(a_1+s)} + R_{64}e^{(a_2+s)} + R_{65}e^{(a_1+2s)} + R_{66}e^{(a_2+2s)};$$

$$R_{68} = (R_{53}-R_{55})e^{-m_1} + (R_{54}-R_{56})e^{-m_2} - R_{57}e^{-a_1} - R_{58}e^{-a_2}$$
$$+(R_{59}-R_{60})e^{-(m_1+s)} + (R_{61}-R_{62})e^{-(m_2+s)} + R_{63}e^{-(a_1+s)} + R_{64}e^{-(a_2+s)}$$
$$+R_{65}e^{-(a_1+2s)} + R_{66}e^{-(a_2+2s)};$$

$$D_8 = \frac{R_{67}e^{-a_1}-R_{68}e^{a_1}}{2\sinh(a_1-a_2)};\;\; C_8 = -R_{67}e^{-a_1} - D_8e^{(a_2-a_1)};$$

$$X_3 = C_1+C_2+C_3+C_6;\;\; X_4 = D_1+D_2+D_3+D_6;\;\; X_5 = R_3+R_{11}+R_{33};$$

$$X_6 = R_4+R_{13}+R_{34};\; X_7 = R_{12}+R_{35};\;\; X_8 = R_{14}+R_{36};\; X_9 = -X_1+R_5+R_{15}+$$
$$R_{39}+R_{51};$$

$$X_{10} = -X_2+R_6+R_{16}+R_{40}+R_{49};\; X_{11} = R_7+R_{17}+R_{41};\;\; X_{12} = R_8+R_{18}+R_{42};$$

$$X_{13} = C_0+C_5+C_8;\;\; X_{14} = D_0+D_5+D_8;\; X_{15} = R_{28}+R_{57};X_{16} = R_{27}+R_{58};$$

$$X_{17} = R_{25}+R_{53};X_{18} = R_{26}+R_{54};\;\; X_{19} = R_{24}+R_{63};\;\; X_{20} = R_{23}+R_{64};$$

$$X_{21} = R_{21}+R_{59};X_{22} = R_{22}+R_{61};\; X_{23} = R_{30}+R_{65};\;\; X_{24} = R_{29}+R_{66}.$$

# Appendix III

## MATLAB Pogram for Problem I

```
clc;
%n = -1:0.001:1;
Format long
n = 0.0;
a = 0.3;
Pr= 7;
St= 0:0.005:10;
E = 5;
r = sqrt(1i.*St);
q = sqrt(1i.*St.*Pr);
c = r.^2;
d = E.*Pr;
e = a.*E.*Pr;
g = (a.^2).*E.*Pr;
Cr = cosh(r);
Cq = cosh(q);
Sr = sinh(r);
Sq = sinh(q);
m  = q.^2;
b  = a.*1i.*St;
Q1 = (a.*c-b)./(Cr.*(m-c));
Q2 = -(b-a.*m-a.*(m-c))./(Cq.*(m-c));
Q3 = (a.*q)./(Cq);
b1 = -(Q1.*Sr)./(2.*r.*Cr)+Q2.*Cq./((m-c).*Cr)-2.*q.*Q3.*Cq./(Cr.*(m-c).^2)+Q3.*Sq./((m-c).*Cr);
C1 = a.*b1.*c-b.*b1+a.*Q1;
C2 = (1./(m-c)).*((b.*Q2+((2.*a.*Q3.*m.*q)./(m-c))-((2.*b.*Q3.*q)./(m-c))-(2.*a.*q.*Q3)-a.*Q2.*m))-
(a.^2)./(Cq);
C3 = (a.*Q1.*r)./(2)-(b.*Q1)./(2.*r);
C4 = (b.*Q3)./(m-c)-(a.*Q3.*m)./(m-c)-((a.^2).*q)./(Cq);
Pi1= (-C1.*Sr./(2.*r.*Cr))-(C2.*Cq./(Cr.*(m-c)))+(C3.*Sr./(4.*c.*Cr))-(C3./(4.*r))+(2.*C4.*q.*Cq./(Cr.*(m-
c).^2))-(C4.*Sq./(Cr.*(m-c)));
H1 = (2.*r.*(d+g-2.*e))./(Cr.*(m-c));
H2 = (-2.*q.*(d+g-2.*e))./(Cq.*(m-c));
H3 = (e-g)./Cq;
Pi2 = (2.*r.*H1.*Cr)./(Cq.*(c-m).^2)-(H1.*Sr./(Cq.*(c-m)))+(H2.*Sq./(4.*m.*Cq))-(H2./(4.*q))-
(H3.*Sq./(4.*m.*q.*Cq))+H3./(4.*m)-(H3.*Sq./(6.*q.*Cq));
K1 = a.*C1-b.*Pi1+a.*c.*Pi1+(2.*r.*H1)./((c-m).^2);
K2 = (a.*C3.*r)./(4)-b.*C3./(4.*r);
K3 = b.*C3./(4.*c)-b.*C1./(2.*r)+a.*C1.*r./(2)-a.*C3./(4)+a.*C3-H1./(c-m);
K4 = 2.*b.*q.*C4./((m-c).^2)+a.*C2.*m./(m-c)+2.*a.*q.*C4./(m-c)-b.*C2./(m-c)-2.*a.*m.*q.*C4./((m-
c).^2)-(a.^3)./(Cq)-Pi2;
K5 = H3./(4.*m)-H2./(4.*q);
K6 = H2./(4.*m)-H3./(4.*m.*q)-b.*C4./(m-c)+a.*m.*C4./(m-c)-((a.^3).*q)/(Cq);
K7 = H3./(6.*q);
Pi31 = -K1.*Sr./(2.*r.*Cr)-K2.*Sr./(4.*m.*q.*Cr)+K2./(4.*c)-K2.*Sr./(6.*r.*Cr)+K3.*Sr./(4.*c.*Cr)-
K3./(4.*r)-K4.*Cq./(Cr.*(m-c))-8.*m.*K5.*Cq./(Cr.*((m-c).^3))+2.*K5.*Cq./(Cr.*((m-
c).^2))+4.*q.*K5.*Sq./(Cr.*((m-c).^2));
Pi32 = -K5.*Cq./((m-c).*Cr)+2.*K6.*q.*Cq./((Cr.*(m-c).^2))-K6.*Sq./(Cr.*(m-c))-
48.*m.*q.*K7.*Cq./((Cr.*(m-c).^4))+24.*q.*K7.*Cq./(Cr.*(m-c).^3)+24.*m.*K7.*Sq./(Cr.*(m-c).^3)-
6.*K7.*Sq./(Cr.*(m-c).^2)-6.*q.*K7.*Cq./(Cr.*(m-c).^2)+K7.*Sq./(Cr.*(m-c));
Pi3 = Pi31+Pi32;
```

```
P1 = 4.*e.*q.*Q3./(m-c)-2.*d.*q.*Q3./(m-c)+2.*a.*e./(Cq)-2.*a.*g./(Cq)-(2.*g.*Q3.*q)./(m-c);
P2 = 4.*d.*m.*Q3./((m-c).^2)+4.*e.*q.*Q2./(m-c)+4.*e.*Q3./(m-c)+4.*a.*e.*q./(Cq.*(m-c))-
2.*d.*q.*Q2./(m-c)-2.*d.*Q3./(m-c)-2.*a.*d.*q./(Cq.*(m-c))-8.*e.*m.*Q3./((m-c).^2)-(2.*g.*Q2.*q)./(m-c)-
(2.*g.*Q3)./(m-c)+(4.*g.*Q3.*m)./((m-c).^2)-(2.*g.*a.*q)./(Cq.*(m-c));
P3 = d.*Q1-2.*e.*Q1;
P4 = 2.*r.*b1.*d+Q1.*d./(r)+2.*a.*r.*d/(Cr.*(m-c))-4.*e.*r.*b1-2.*e.*Q1./(r)-4.*a.*e.*r./(Cr.*(m-
c))+2.*g.*b1.*r+(2.*g.*a.*r)./(Cr.*(m-c));
Pi4 = -P1.*Sq./(4.*m.*q.*Cq)+P1./(4.*m)-P1.*Sq./(6.*q.*Cq)+P2.*Sq./(4.*m.*Cq)-P2./(4.*q)-
8.*c.*P3.*Cr./(Cq.*(c-m).^3)+2.*P3.*Cr./(Cq.*(c-m).^2)+4.*r.*P3.*Sr./(Cq.*(c-m))-P3.*Cr./(Cq.*(c-
m))+2.*r.*P4.*Cr./(Cq.*((c-m).^2))-P4.*Sr./(Cq.*(c-m));

F0 = 1;
G0 = (1./Cq).*cosh(q.*n);
A1 = (1./2).*(1-n.^2);
B1 = (1./(m-c)).*((1./Cr).*(cosh(r.*n))-(1./Cq).*(cosh(q.*n)));
A2 = (a./2).*(1-n.^2);
B2 = b1.*(cosh(r.*n))+(Q1./(2.*r)).*(n.*sinh(r.*n))+(2.*Q3.*q./((m-c).^2)-(Q2./(m-c))).*(cosh(q.*n))-
(Q3./(m-c)).*(n.*sinh(q.*n));
A3 = (a.^2./2).*(1-n.^2);
B3 = Pi1.*cosh(r.*n)+(C1./(2.*r)-(C3./4.*c)).*(n.*sinh(r.*n))+(C2./(m-c)-(2.*C4.*q./((m-
c).^2))).*(cosh(q.*n))+(C4./(m-c)).*(n.*sinh(q.*n))+(C3./(4.*r)).*(n.^2.*cosh(r.*n));
F3 = ((d+g-2.*e)./12).*(1-n.^4);
G3 = Pi2.*cosh(q.*n)-(2.*r.*H1./((c-m).^2)).*cosh(r.*n)+(H1./(c-m)).*(n.*sinh(r.*n))+(H3./(4.*m.*q)-
(H2./(4.*m))).*(n.*sinh(q.*n))+(H2./(4.*q)-
(H3./(4.*m))).*(n.^2.*cosh(q.*n))+(H3./(6.*q)).*(n.^3.*sinh(q.*n));
A4 = (a.^3./2).*(1-n.^2)+((d+g-2.*e)./360).*(n.^6-15.*n.^2+14);
B4 = Pi3.*cosh(r.*n)+(K1./(2.*r)+K2./(4.*c.*r)-(K3./(4.*c))).*(n.*sinh(r.*n))+(K3./(4.*r)-
(K2./(4.*c))).*(n.^2.*cosh(r.*n))+(K2./(6.*r)).*(n.^3.*sinh(r.*n))+(K4./(m-c)+8.*m.*K5./((m-c).^3)-
(2.*K5./((m-c).^2))-(2.*K6.*q./((m-c).^2))+48.*m.*q.*K7./((m-c).^4)-(24.*q.*K7./((m-
c).^3))).*cosh(q.*n)+((6.*K7./((m-c).^2))+(K6./(m-c))-(4.*q.*K5./((m-c).^2))-(24.*m.*K7./((m-
c).^3))).*(n.*sinh(q.*n))+(K5./(m-c)+(6.*q.*K7./((m-c).^2))).*(n.^2.*cosh(q.*n))-(K7./(m-
c)).*(n.^3.*sinh(q.*n));
F4 = ((d+g-2.*e)./12).*(1-n.^4);
G4 = Pi4.*cosh(q.*n)+(P1./(4.*m.*q)-(P2./(4.*m))).*(n.*sinh(q.*n))+(P2./(4.*q)-
(P1./(4.*m))).*(n.^2.*cosh(q.*n))+(P1./(6.*q)).*(n.^3.*sinh(q.*n))+(8.*c.*P3./((c-m).^3)-(2.*P3./((c-m).^2))-
(2.*r.*P4./((c-m).^2))).*(cosh(r.*n))+(P4./(c-m)).*(n.*sinh(r.*n))+(P3./(c-m)).*(n.^2.*cosh(r.*n));

A = A1+A2+A3+A4;
B = B1+B2+B3+B4;
F = F0+F3+F4;
G = G0+G3+G4;

GR = real(G);
Gi = imag(G);
BR = real(B);
Bi = imag(B);

phi = atan(Gi./GR);
theta = atan(Bi./BR);

ABSG = ((GR.^2)+(Gi.^2)).^(1./2);
ABSB = ((BR.^2)+(Bi.^2)).^(1./2);

J1 = Pi2+Pi4;
J2 = H2./(4.*q)-H3./(4.*m)-P1./(4.*m)+P2./(4.*q);
J3 = H3./(4.*m.*q)-H2./(4.*m)+P1./(4.*m.*q)-P2./(4.*m);
```

```
J4 = (8.*c.*P3)./((c-m).^3)-(2.*P3)./((c-m).^2)-(2.*r.*P4)./((c-m).^2)-(2.*r.*H1)./((c-m).^2);
J5 = P3./(c-m);
J6 = H1./(c-m)+P4./(c-m)-(4.*r.*P3)./((c-m).^2);
J7 = (H3+P1)./(6.*q);
V1 = b1+Pi1+Pi3;
V2 = (C3./(4.*r)+K3./(4.*r)-K2./(4.*c));
V3 = ((1./(m-c)).*(C2-Q2+K4)+(2./((m-c).^2).*(Q3.*q-C4.*q-K6.*q-K5))+(8./(m-c).^3).*(K5.*m-
3.*q.*K7)+(48.*m.*q.*K7./((m-c).^4)));
V4 = (K5./(m-c)+6.*q.*K7./((m-c).^2));
V5 = ((1./(2.*r)).*(Q1+C1+K1)-(1./(4.*c)).*(C3+K3)+(K2./(4.*c.*r)));
V6 = ((1./(m-c)).*(C4-Q3+K6)+(1./((m-c).^2)).*(6.*K7-4.*q.*K5)-(24.*m.*K7./((m-c).^3)));
V7 = K2./(6.*r);
V8 = -K7./(m-c);
T  = (1./(m-c)).*(r.*tanh(r)-
q.*tanh(q))+(r.*(V1+V2)+V5+3.*V7).*sinh(r)+(2.*V2+(V5+V7).*r).*cosh(r)+(V6+3.*V8+(V3+V4).*q).*sin
h(q)+(q.*(V6+V8)+2.*V4).*cosh(q);
Nu =
(q.*sinh(q))./(Cq)+(J3+3.*J7+q.*(J1+J2)).*sinh(q)+(2.*J2+(J3+J7).*q).*cosh(q)+(J6+(J4+J5).*r).*sinh(r)+(2.
*J5+J6.*r).*cosh(r);
Nun =
(q.*sinh(q.*n))./(Cq)+(J3+3.*J7+q.*(J1.*n+J2)).*sinh(q.*n)+(2.*J2.*n+(J3.*n+J7.*n.^3).*q).*cosh(q.*n)+(J6
+(J4+J5).*r).*sinh(r.*n)+(2.*J5.*n+J6.*r.*n).*
cosh(r.*n);

xlabel('St', 'fontsize', 11);
ylabel('G(\eta)', 'fontsize', 11);
hold on
plot(St,G, 'k','linestyle','-');
hold off
```

# Appendix IV

# MATLAB Program for Problem II

```
clc;
format long
%n = -1:0.001:1;
%n = 0.4;
s  = -2:0.002:2;
a  = -0.3;
Pr = 0.71;
St = 1;
E  = 0.4;
b = a.*1i.*St;
c = 1i.*St;
d = E.*Pr;
e = a.*E.*Pr;
g = (a.^2).*E.*Pr;
p = (s.*Pr)./2;
q = (sqrt(((s.*Pr).^2)+4i.*St.*Pr))./2;
a1= p+q;
a2= p-q;
D0= (exp(-a1)-exp(a1))./(2.*sinh(a2-a1));
C0= exp(-a1)-D0.*exp(a2-a1);
X1 = C0./((a1.^2)-a1.*s-1i.*St);
X2 = D0./((a2.^2)-a2.*s-1i.*St);
Q3 = X1.*exp(a1)+X2.*exp(a2);
Q4 = X1.*exp(-a1)+X2.*exp(-a2);
m1 = (s+sqrt((s.^2)+4.*1i.*St))./2;
m2 = (s-sqrt((s.^2)+4.*1i.*St))./2;
D1 = (Q3.*exp(-m1)-Q4.*exp(m1))./(2.*sinh(m2-m1));
C1 = Q3.*exp(-m1)-D1.*exp(m2-m1);
Q5 = C1.*(-b+a.*m1.^2-a.*s.*m1);
Q6 = D1.*(-b+a.*m2.^2-a.*s.*m2);
Q7 = X1.*(b-a.*a1.^2+a.*s.*a1)+(a.*C0.*a1)./s;
Q8 = X2.*(b-a.*a2.^2+a.*s.*a2)+(a.*D0.*a2)./s;
Q9 = (-a.*C0.*(s+a1))./(sinh(s));
Q10= (-a.*D0.*(s+a2))./(sinh(s));
R3 = Q5./(2.*m1-s);
R4 = Q6./(2.*m2-s);
R5 = Q7./(a1.^2-s.*a1-1i.*St);
R6 = Q8./(a2.^2-s.*a2-1i.*St);
R7 = Q9./(a1.^2+s.*a1-1i.*St);
R8 = Q10./(a2.^2+s.*a2-1i.*St);
R9 = R3.*exp(m1)+R4.*exp(m2)+R5.*exp(a1)+R6.*exp(a2)+R7.*exp(s+a1)+R8.*exp(s+a2);
R10= R3.*exp(-m1)+R4.*exp(-m2)-R5.*exp(-a1)-R6.*exp(-a2)-R7.*exp(-(s+a1))-R8.*exp(-(s+a2));
D2 = -(R9.*exp(-m1)+R10.*exp(m1))./(2.*sinh(m2-m1));
C2 = -R9.*exp(-m1)-D2.*exp(m2-m1);
Q11= -b.*C2+a.*C2.*m1.^2+2.*a.*R3.*m1-a.*s.*C2.*m1-a.*s.*R3;
Q12= -b.*D2+a.*D2.*m2.^2+2.*a.*R4.*m2-a.*s.*D2.*m2-a.*s.*R4;
Q13= R3.*(-b+a.*m1.^2-a.*s.*m1);
Q14= R4.*(-b+a.*m2.^2-a.*s.*m2);
Q15= -b.*R5+a.*R5.*a1.^2-a.*s.*R5.*a1+(a.^2.*C0.*a1)./s;
Q16= -b.*R6+a.*R6.*a2.^2-a.*s.*R6.*a2+(a.^2.*D0.*a2)./s;
Q17= -b.*R7+a.*R7.*(a1+s).^2-a.*s.*R7.*(a1+s)-(((a.^2).*s.*C0)+(a.^2).*C0.*a1)./sinh(s);
```

```
Q18= -b.*R8+a.*R8.*(a2+s).^2-a.*s.*R8.*(a2+s)-(((a.^2).*s.*D0)+(a.^2).*D0.*a2)./sinh(s);
R11= Q11./(2.*m1-s)-Q13./(2.*m1-s).^2;
R12= Q13./(4.*m1-2.*s);
R13= Q12./(2.*m2-s)-Q14./(2.*m2-s).^2;
R14= Q14./(4.*m2-2.*s);
R15= Q15./(a1.^2-s.*a1-1i.*St);
R16= Q16./(a2.^2-s.*a2-1i.*St);
R17= Q17./(a1.^2+s.*a1-1i.*St);
R18= Q18./(a2.^2+s.*a2-1i.*St);
R19=
(R11+R12).*exp(m1)+(R13+R14).*exp(m2)+R15.*exp(a1)+R16.*exp(a2)+R17.*exp(s+a1)+R18.*exp(s+a2)
;
R20= (R11-R12).*exp(-m1)+(R13-R14).*exp(-m2)-R15.*exp(-a1)-R16.*exp(-a2)-R17.*exp(-(s+a1))-
R18.*exp(-(s+a2));
D3 = -(R19.*exp(-m1)+R20.*exp(m1))./(2.*(sinh(m2-m1)));
C3 = -R19.*exp(-m1)-D3.*exp(m2-m1);
k  = e-g;
z  = 2.*e-d-g;
H1 = z./((4.*(s.^2)-2.*(s.^2).*Pr).*((sinh(s)).^2));
H2 = (2.*z)./((s.^3-s.^3.*Pr).*sinh(s));
H3 = z./((s.^3).*Pr.*sinh(s.*Pr));
D4 = -(H1.*sinh(2.*s))./(sinh(s.*Pr))+(H2.*sinh(s))./(sinh(s.*Pr))+H3;
C4 = -H1.*exp(2.*s)+H2.*exp(s)+z./((s.^3).*Pr)-D4.*exp(s.*Pr);
Q19= (2.*z.*C1.*m1)./(sinh(s));
Q20= (2.*z.*D1.*m2)./(sinh(s));
Q21= (2.*z.*X2.*a2)./(sinh(s))-2.*k.*D0./(s.*sinh(s));
Q22= (2.*z.*X1.*a1)./(sinh(s))-2.*k.*C0./(s.*sinh(s));
Q23= (2.*z.*C1.*m1)./s;
Q24= (2.*z.*D1.*m2)./s;
Q25= (2.*z.*X2.*a2)./(s)-(k.*D0)./(s.^2);
Q26= (2.*z.*X1.*a1)./(s)-(k.*C0)./(s.^2);
Q27= k.*D0./((sinh(s)).^2);
Q28= k.*C0./((sinh(s)).^2);
R21= -Q19./((m1+s).^2-s.*Pr.*(m1+s)-1i.*St.*Pr);
R22= -Q20./((m2+s).^2-s.*Pr.*(m2+s)-1i.*St.*Pr);
R23= Q21./((a2+s).^2-s.*Pr.*(a2+s)-1i.*St.*Pr);
R24= Q22./((a1+s).^2-s.*Pr.*(a1+s)-1i.*St.*Pr);
R25= Q23./(m1.^2-s.*Pr.*m1-1i.*St.*Pr);
R26= Q24./(m2.^2-s.*Pr.*m2-1i.*St.*Pr);
R27= -Q25./(2.*a2-s.*Pr);
R28= -Q26./(2.*a1-s.*Pr);
R29= Q27./((a2+2.*s).^2-s.*Pr.*(a2+2.*s)-1i.*St.*Pr);
R30= Q28./((a1+2.*s).^2-s.*Pr.*(a1+2.*s)-1i.*St.*Pr);
R31=
R21.*exp(m1+s)+R22.*exp(m2+s)+R23.*exp(a2+s)+R24.*exp(a1+s)+R25.*exp(m1)+R26.*exp(m2)+R27.*
exp(a2)+R28.*exp(a1)+R29.*exp(a2+2.*s)+R30.*exp(a1+2.*s);
R32= R21.*exp(-(m1+s))+R22.*exp(-(m2+s))+R23.*exp(-(a2+s))+R24.*exp(-(a1+s))+R25.*exp(-
m1)+R26.*exp(-m2)-R27.*exp(-a2)-R28.*exp(-a1)+R29.*exp(-(a2+2.*s))+R30.*exp(-(a1+2.*s));
D5= -(R31.*exp(-a1)-R32.*exp(a1))./(2.*sinh(a2-a1));
C5= -R31.*exp(-a1)-D5.*exp(a2-a1);
H4 = -(C4+a.^3)./(s)+z./(2.*s.^4.*Pr)+z./(s.^5.*Pr)+(H1.*exp(2.*s))./(2.*s.^2)-
(H2.*exp(s))./(s)+(D4.*exp(s.*Pr))./((s.^2).*Pr.*(Pr-1));
H5 = -(C4+a.^3)./(s.*sinh(s))+z./(s.^5.*Pr.*sinh(s))+(H1.*sinh(2.*s))./(2.*s.^2.*sinh(s))-
H2.*cosh(s)./(s.*sinh(s))+(D4.*sinh(s.*Pr))./(s.^2.*Pr.*(Pr-1).*sinh(s));
H6 = H4-H5.*exp(s);
Q29= -b.*C3+a.*C3.*(m1.^2)+2.*a.*m1.*R11+2.*a.*R12-a.*s.*m1.*C3-a.*s.*R11-R25;
```

```
Q30= -b.*D3+a.*D3.*(m2.^2)+2.*a.*m2.*R13+2.*a.*R14-a.*s.*m2.*D3-a.*s.*R13-R26;
Q31= -b.*R11+a.*(m1.^2).*R11+4.*a.*m1.*R12-a.*s.*m1.*R11-2.*a.*s.*R12;
Q32= -b.*R13+a.*(m2.^2).*R13+4.*a.*m2.*R14-a.*s.*m2.*R13-2.*a.*s.*R14;
Q33= -b.*R12+a.*(m1.^2).*R12-a.*s.*m1.*R12;
Q34= -b.*R14+a.*(m2.^2).*R14-a.*s.*m2.*R14;
Q35= -b.*R15+a.*(a1.^2).*R15-a.*s.*a1.*R15-C5+((a.^3).*C0.*a1)./s;
Q36= -b.*R16+a.*(a2.^2).*R16-a.*s.*a2.*R16-D5+((a.^3).*D0.*a2)./s;
Q37= -(a1+s).*((a.^3.*C0)./sinh(s))-a.*s.*(a1+s).*R17+a.*((a1+s).^2).*R17-R24-b.*R17;
Q38= -(a2+s).*((a.^3.*D0)./sinh(s))-a.*s.*(a2+s).*R18+a.*((a2+s).^2).*R18-R23-b.*R18;
R33= Q29./(2.*m1-s)-Q31./((2.*m1-s).^2)+Q33./((2.*m1-s).^3);
R34= Q30./(2.*m2-s)-Q32./((2.*m2-s).^2)+Q34./((2.*m2-s).^3);
R35= Q31./(4.*m1-2.*s)-Q33./((2.*m1-s).^2);
R36= Q32./(4.*m2-2.*s)-Q34./((2.*m2-s).^2);
R37= Q33./(6.*m1-3.*s);
R38= Q34./(6.*m2-3.*s);
R39= Q35./(a1.^2-a1.*s-1i.*St);
R40= Q36./(a2.^2-a2.*s-1i.*St);
R41= Q37./(a1.^2+a1.*s-1i.*St);
R42= Q38./(a2.^2+a2.*s-1i.*St);
R43= R21./(m1.^2+m1.*s-1i.*St);
R44= R22./(m2.^2+m2.*s-1i.*St);
R45= R29./(a2.^2+3.*a2.*s+2.*(s.^2)-1i.*St);
R46= R30./(a1.^2+3.*a1.*s+2.*(s.^2)-1i.*St);
R49= (2.*a2-s).*R27./((a2.^2-a2.*s-1i.*St).^2);
R50= R27./(a2.^2-a2.*s-1i.*St);
R51= (2.*a1-s).*R28./((a1.^2-a1.*s-1i.*St).^2);
R52= R28./(a1.^2-a1.*s-1i.*St);
R47= (R33+R35+R37).*exp(m1)+(R34+R36+R38).*exp(m2)+(R39+R51-R52).*exp(a1)+(R40+R49-
R50).*exp(a2)-R43.*exp(m1+s)-R44.*exp(m2+s)+R41.*exp(a1+s)+R42.*exp(a2+s)-R45.*exp(a2+2.*s)-
R46.*exp(a1+2.*s);
R48= (-R33+R35-R37).*exp(-m1)+(-R34+R36-R38).*exp(-m2)+(R39+R51+R52).*exp(-
a1)+(R40+R49+R50).*exp(-a2)-R43.*exp(-(m1+s))-R44.*exp(-(m2+s))+R41.*exp(-(a1+s))+R42.*exp(-
(a2+s))-R45.*exp(-(a2+2.*s))-R46.*exp(-(a1+2.*s));
D6 = -(R47.*exp(-m1)-R48.*exp(m1))./(2.*sinh(m2-m1));
C6 = -R47.*exp(-m1)-D6.*exp(m2-m1);
D7 = -(2.*a.*H1.* sinh(2.*s))./(sinh(s.*Pr))+(2.*a.*H2.*sinh(s))./(sinh(s.*Pr))+2.*a.*H3;
C7 = -2.*a.*H1.*exp(2.*s)+2.*a.*H2.*exp(s)+(2.*a.*z)./((s.^3).*Pr)-D7.*exp(s.*Pr);
Q39= 2.*((C2.*m1+R3+a.*C1.*m1).*(z./s));
Q40= 2.*((D2.*m2+R4+a.*D1.*m2).*(z./s));
Q41= (2.*R3.*m1.*z)./s;
Q42= (2.*R4.*m2.*z)./s;
Q43= 2.*((R5.*a1-a.*X1.*a1).*(z./s)+(a.*k.*C0)./(s.^2));
Q44= 2.*((R6.*a2-a.*X2.*a2).*(z./s)+(a.*k.*D0)./(s.^2));
Q45= 2.*((C2.*m1+R3+a.*C1.*m1).*(z./sinh(s)));
Q46= 2.*((D2.*m2+R4+a.*D1.*m2).*(z./sinh(s)));
Q47= (-2.*z.*R3.*m1)./(sinh(s));
Q48= (-2.*z.*R4.*m2)./(sinh(s));
Q49= 2.*((z.*R7.*(a1+s))./(s)-(2.*a.*k.*C0)./(s.*sinh(s))-(z.*R5.*a1)./(sinh(s))+(a.*z.*X1.*a1)./(sinh(s)));
Q50= 2.*((z.*R8.*(a2+s))./(s)-(2.*a.*k.*D0)./(s.*sinh(s))-(z.*R6.*a2)./(sinh(s))+(a.*z.*X2.*a2)./(sinh(s)));
Q51= 2.*((a.*k.*C0)./((sinh(s)).^2)-(z.*R7.*(a1+s))./(sinh(s)));
Q52= 2.*((a.*k.*D0)./((sinh(s)).^2)-(z.*R8.*(a2+s))./(sinh(s)));
R53= (Q39-(2.*m1-s.*Pr).*Q41)./(m1.^2-s.*Pr.*m1-1i.*St.*Pr);
R54= (Q40-(2.*m2-s.*Pr).*Q42)./(m2.^2-s.*Pr.*m2-1i.*St.*Pr);
R55= Q41./(m1.^2-s.*Pr.*m1-1i.*St.*Pr);
R56= Q42./(m2.^2-s.*Pr.*m2-1i.*St.*Pr);
R57= Q43./(2.*a1-s.*Pr);
```

```
R58= Q44./(2.*a2-s.*Pr);
R59= -(Q45+((2.*(m1+s)-s.*Pr).*Q47))./((m1+s).^2-s.*Pr.*(m1+s)-1i.*St.*Pr);
R60= Q47./((m1+s).^2-s.*Pr.*(m1+s)-1i.*St.*Pr);
R61= -(Q46+((2.*(m2+s)-s.*Pr).*Q48))./((m2+s).^2-s.*Pr.*(m2+s)-1i.*St.*Pr);
R62= Q48./((m2+s).^2-s.*Pr.*(m2+s)-1i.*St.*Pr);
R63= Q49./((a1+s).^2-s.*Pr.*(a1+s)-1i.*St.*Pr);
R64= Q50./((a2+s).^2-s.*Pr.*(a2+s)-1i.*St.*Pr);
R65= Q51./((a1+2.*s).^2-s.*Pr.*(a1+2.*s)-1i.*St.*Pr);
R66= Q52./((a2+2.*s).^2-s.*Pr.*(a2+2.*s)-1i.*St.*Pr);
R67=
(R53+R55).*exp(m1)+(R54+R56).*exp(m2)+R57.*exp(a1)+R58.*exp(a2)+(R59+R60).*exp(m1+s)+(R61+R
62).*exp(m2+s)+R63.*exp(a1+s)+R64.*exp(a2+s)+R65.*exp(a1+2.*s)+R66.*exp(a2+2.*s);
R68= (R53-R55).*exp(-m1)+(R54-R56).*exp(-m2)-R57.*exp(-a1)-R58.*exp(-a2)+(R59-R60).*exp(-
(m1+s))+(R61-R62).*exp(-(m2+s))+R63.*exp(-(a1+s))+R64.*exp(-(a2+s))+R65.*exp(-
(a1+2.*s))+R66.*exp(-(a2+2.*s));
D8 = -(R67.*exp(-a1)-R68.*exp(a1))./(2.*sinh(a2-a1));
C8 = -R67.*exp(-a1)-D8.*exp(a2-a1);

F0 = 1;
G0 = C0.*exp(a1.*n)+D0.*exp(a2.*n);
A1 = (1./s).*(coth(s)-(exp(s.*n)./(sinh(s)))+n);
B1 = C1.*exp(m1.*n)+D1.*exp(m2.*n)-X1.*exp(a1.*n)-X2.*exp(a2.*n);
A2 = (a./s).*(1+n+(exp(-s)./(sinh(s)))-(exp(s.*n)./(sinh(s))));
B2 =
C2.*exp(m1.*n)+D2.*exp(m2.*n)+R3.*n.*exp(m1.*n)+R4.*n.*exp(m2.*n)+R5.*exp(a1.*n)+R6.*exp(a2.*n)
+R7.*exp((a1+s).*n)+R8.*exp((a2+s).*n);
A3 = ((a.^2)./s).*(1+n+(exp(-s)./(sinh(s)))-(exp(s.*n)./(sinh(s))));
B3 =
C3.*exp(m1.*n)+D3.*exp(m2.*n)+R11.*n.*exp(m1.*n)+R12.*(n.^2).*exp(m1.*n)+R13.*n.*exp(m2.*n)+R1
4.*(n.^2).*exp(m2.*n)+R15.*exp(a1.*n)+R16.*exp(a2.*n)+R17.*exp((a1+s).*n)+R18.*exp((a2+s).*n);
F3 = C4+D4.*exp(s.*Pr.*n)+H1.*exp(2.*s.*n)-H2.*exp(s.*n)-(z./(s.^3.*Pr)).*n;
G3 =
C5.*exp(a1.*n)+D5.*exp(a2.*n)+R21.*exp((m1+s).*n)+R22.*exp((m2+s).*n)+R23.*exp((a2+s).*n)+R24.*e
xp((a1+s).*n)+R25.*exp(m1.*n)+R26.*exp(m2.*n)+R27.*n.*exp(a2.*n)+R28.*n.*exp(a1.*n)+R29.*exp((a2
+2.*s).*n)+R30.*exp((a1+2.*s).*n);
A4 = H6+H5.*exp(s.*n)+((C4+a.^3)./(s)).*n-(z./(2.*s.^4.*Pr)).*(n.^2)-(z./(s.^5.*Pr)).*n-
(H1./(2.*s.^2)).*exp(2.*s.*n)+(H2./s).*n.*exp(s.*n)-(D4./((s.^2.*Pr.*(Pr-1)))).*exp(s.*Pr.*n);
B4 =
C6.*exp(m1.*n)+D6.*exp(m2.*n)+R33.*n.*exp(m1.*n)+R34.*n.*exp(m2.*n)+R35.*n.^2.*exp(m1.*n)+R36.
*n.^2.*exp(m2.*n)+R37.*n.^3.*exp(m1.*n)+R38.*n.^3.*exp(m2.*n)+R39.*exp(a1.*n)+R40.*exp(a2.*n)+R4
1.*exp((a1+s).*n)+R42.*exp((a2+s).*n)-R43.*exp((m1+s).*n)-R44.*exp((m2+s).*n)-
R45.*exp((a2+2.*s).*n)-R46.*exp((a1+2.*s).*n)+R49.*exp(a2.*n)+R51.*exp(a1.*n)-R50.*n.*exp(a2.*n)-
R52.*n.*exp(a1.*n);
F4 = C7+D7.*exp(s.*Pr.*n)+2.*a.*H1.*exp(2.*s.*n)-2.*a.*H2.*exp(s.*n)-((2.*a.*z)./(s.^3.*Pr)).*n;
G4 =
C8.*exp(a1.*n)+D8.*exp(a2.*n)+R53.*exp(m1.*n)+R54.*exp(m2.*n)+R55.*n.*exp(m1.*n)+R56.*n.*exp(m
2.*n)+R57.*n.*exp(a1.*n)+R58.*n.*exp(a2.*n)+R59.*exp((m1+s).*n)+R60.*n.*exp((m1+s).*n)+R61.*exp((
m2+s).*n)+R62.*n.*exp((m2+s).*n)+R63.*exp((a1+s).*n)+R64.*exp((a2+s).*n)+R65.*exp((a1+2.*s).*n)+R
66.*exp((a2+2.*s).*n);

A = A1+A2+A3+A4;
F = F0+F3+F4;
B = B1+B2+B3+B4;
G = G0+G3+G4;

GR = real(G);
```

```
Gi = imag(G);
BR = real(B);
Bi = imag(B);

phi = atan(Gi./GR);
theta = atan(Bi./BR);

ABSG = ((GR.^2)+(Gi.^2)).^(1./2);
ABSB = ((BR.^2)+(Bi.^2)).^(1./2);

X3 = C1+C2+C3+C6;
X4 = D1+D2+D3+D6;
X5 = R3+R11+R33;
X6 = R4+R13+R34;
X7 = R12+R35;
X8 = R14+R36;
X9 = -X1+R5+R15+R39+R51;
X10= -X2+R6+R16+R40+R49;
X11= R7+R17+R41;
X12= R8+R18+R42;

X13= C0+C5+C8;
X14= D0+D5+D8;
X15= R28+R57;
X16= R27+R58;
X17= R25+R53;
X18= R26+R54;
X19= R24+R63;
X20= R23+R64;
X21= R21+R59;
X22= R22+R61;
X23= R30+R65;
X24= R29+R66;

T1 =
(m1.*X3+X5+m1.*X5+2.*X7+m1.*X7+3.*R37+m1.*R37).*exp(m1)+(m2.*X4+X6+m2.*X6+2.*X8+m2.*
X8+3.*R38+m2.*R38).*exp(m2)+(a1.*X9-a1.*R52-R52).*exp(a1)+(a2.*X10-R50-
a2.*R50).*exp(a2)+(a1+s).*X11.*exp(a1+s)+(a2+s).*X12.*exp(a2+s)-(m1+s).*R43.*exp(m1+s)-
(m2+s).*R44.*exp(m2+s)-(a2+2.*s).*R45.*exp(a2+2.*s)-(a1+2.*s).*R46.*exp(a1+2.*s);
T2 = (m1.*X3+X5-m1.*X5-2.*X7+m1.*X7+3.*R37-m1.*R37).*exp(-m1)+(m2.*X4+X6-m2.*X6-
2.*X8+m2.*X8+3.*R38-m2.*R38).*exp(-m2)+(a1.*X9+a1.*R52-R52).*exp(-a1)+(a2.*X10-
R50+a2.*R50).*exp(-a2)+(a1+s).*X11.*exp(-(a1+s))+(a2+s).*X12.*exp(-(a2+s))-(m1+s).*R43.*exp(-
(m1+s))-(m2+s).*R44.*exp(-(m2+s))-(a2+2.*s).*R45.*exp(-(a2+2.*s))-(a1+2.*s).*R46.*exp(-(a1+2.*s));
Nu1=
(a1.*X13+X15+a1.*X15).*exp(a1)+(a2.*X14+X16+a2.*X16).*exp(a2)+(m1.*X17+R55+m1.*R55).*exp(m1
)+(m2.*X18+R56+m2.*R56).*exp(m2)+(a1+s).*X19.*exp(a1+s)+(a2+s).*X20.*exp(a2+s)+((m1+s).*(X21+
R60)+R60).*exp(m1+s)+((m2+s).*(X22+R62)+R62).*exp(m2+s)+(a1+2.*s).*X23.*exp(a1+2.*s)+(a2+2.*s).
*X24.*exp(a2+2.*s);
Nu2= -1.*((a1.*X13+X15-a1.*X15).*exp(-a1)+(a2.*X14+X16-a2.*X16).*exp(-a2)+(m1.*X17+R55-
m1.*R55).*exp(-m1)+(m2.*X18+R56-m2.*R56).*exp(-m2)+(a1+s).*X19.*exp(-(a1+s))+(a2+s).*X20.*exp(-
(a2+s))+((m1+s).*(X21-R60)+R60).*exp(-(m1+s))+((m2+s).*(X22-R62)+R62).*exp(-
(m2+s))+(a1+2.*s).*X23.*exp(-(a1+2.*s))+(a2+2.*s).*X24.*exp(-(a2+2.*s)));

xlabel('s' , 'fontsize' , 11);
ylabel('Nu_1', 'fontsize' , 11);
hold on
```

```
plot(s,Nu1, 'k', 'linestyle','-');
hold off
```