EXISTENCE THEOREMS FOR ATTRACTIVE POINTS OF BREGMAN SEMITOPOLOGICAL SEMIGROUPS IN REFLEXIVE BANACH SPACE

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DECLARATION

I declare that the work in this dissertation titled "EXISTENCE THEOREMS FOR ATTRACTIVE POINTS OF BREGMAN SEMITOPOLOGICAL SEMI-GROUPS IN REFLEXIVE BANACH SPACE" has been carried out by me in the Department of Mathematics under the supervision of Dr. Bashir Ali and Prof. Babangida Sani. The information derived from the literature has been duly acknowledged in the text and the list of references provided. No part of this thesis was previously presented for another degree or diploma in this or any other Institution.

LAWAL, Yusuf Haruna

Name of Student	Signature	Date

CERTIFICATION

This dissertation titled "EXISTENCE THEOREMS FOR ATTRACTIVE POINTS OF BREGMAN SEMITOPOLOGICAL SEMIGROUPS IN REFLEXIVE BANACH SPACE" by LAWAL, Yusuf Haruna (MSC/SCI/22168/2012-2013), meets the regulations governing the award of the degree of Master of Science of the Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This research work is dedicated to my dad, Mal. Lawal D. Bawa and my mom, Malama. Habiba Abubakar

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In the name of Allah, the most gracious the most merciful. All praise be to Allah (S.W.T), the creator, Who created death and life to serve as the reminder of the respective nonexistence and existence, beginning and ending. All praise be to Allah (S.W.T), He who gave the strength and confered a favor upon me from the beginning to the end of this programme.

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ABSTRACT

In this dissertation, we define new attractive point using Bregman distance and establish its theorems in a reflexive Banach space only. The first attractive point theorem for generalized hybrid mappings was establish in a Hilbert space. Similar result but for nonexpansive semigroup of mappings was later established.These two results were later unified by establishing attractive point and mean convergence theorems for semigroup of mappings without continuity in Hilbert space which was then extended to Banach spaces. The result was establish in the framework of smooth, strictly convex and reflexive Banach spaces. This raised a question as whether or not, the result can be established in a reflexive Banach space only. Our results improved the results announced by Lin and Takahashi (2013) and Takahashi et al. (2014a).

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CHAPTER ONE

GENERAL INTRODUCTION

1.1 Background of the Study

The notion of attractive points for nonlinear mappings was first introduced by Takahashi and Takeuchi (2011) in a Hilbert space. They estalished attractive point and mean convergence theorems of Baillon's type without convexity for generalized hybrid mappings in a Hilbert space. Later, this concept of attractive points was extended by Lin and Takahashi (2013) to Banach space and they obtained some fundamental properties of the points. Using these properties, they obtained some results for skew-generalized nonspreading mappings in a Banach space. Let C be a nonempty subset of a real Hilbert space H and $T : C \to C$ be a mapping. A point $u \in H$ is called an attractive point of T if and only if

$$\|Tv - u\| \leq \|v - u\| \ \forall \ v \in C.$$

Let the set of all attractive points of T be denoted by $A(T)$ i.e.,

$$A(T) = \{u \in H : \|Tv - u\| \leq \|v - u\|, \forall v \in C\}.$$

A mapping $T : C \to H$ is said to be nonexpansive if and only if

$$\|Tu - Tv\| \leq \|u - v\| \ \forall \ u, v \in C.$$

And it is said to be quasi-nonexpansive if and only if for any $v \in C$ and $u \in F(T)$,

$$\|Tv - u\| \leq \|v - u\|.$$

Observe that every fixed point of quasi-nonexpansive mapping is an attractive point.

A mapping $T : C \to H$ is called nonspreading if and only if

$$2\|Tu - Tv\|^2 \leq \|Tu - v\|^2 + \|Tv - u\|^2 \ \forall \ u, v \in C.$$

A mapping $T : C \to H$ is called hybrid if and only if

$$3\|Tu - Tv\|^2 \leq \|u - v\|^2 + \|Tu - v\|^2 + \|Tv - u\|^2 \ \forall \ u, v \in C.$$

Nonspreading and hybrid mappings are generally not continous, Igarashi et al. (2009).

A wide class of nonlinear mappings called generalized hybrid mappings which contain the classes of nonexpansive mappings, nonspreading mappings and hybrid mappings in a Hilbert space was introduced by Kocourek et al. (2010). A mapping $T : C \to H$ is called (α, β)-generalized hybrid if $\exists \ \alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tu - Tv\|^2 + (1 - \alpha)\|u - Tv\|^2 \leq \beta\|Tu - v\|^2 + (1 - \beta)\|u - v\|^2 \ \forall u, v \in C.$$

Observe that a $(1, 0)$-generalized hybrid mapping is nonexpansive, a $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping is hybrid, (Kohsaka and Takahashi (2008b); Takahashi (2010)) and a $(2, 1)$-generalized hybrid mapping is nonspreading, (Kohsaka and Takahashi (2008a,b)).

A class of normally generalized hybrid mappings in a Hilbert space containing the classes of generalized hybrid and that of contractive mappings was introduced by Takahashi et al. (2012) as follows:

A mapping $T : C \to H$ is called $(\alpha, \beta, \gamma, \delta)$- normally generalized hybrid if and only if $\exists \ \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

(i) $\alpha + \beta + \gamma + \delta \geq 0$

(ii) $\alpha + \beta > 0$ or $\gamma + \delta > 0$; and

(iii) $\alpha\|Tu - Tv\|^2 + \beta\|u - Tv\|^2 + \gamma\|Tu - v\|^2 + \delta\|u - v\|^2 \leq 0 \ \forall \ u, v \in C.$

Definition 1.1.1 *A normed space E is called smooth if and only if for every $u \in E$, $\|u\| = 1$, there exists a unique $u^* \in E^*$ such that $\|u^*\| = 1$ and $\langle u, u^* \rangle = \|u\|$.*

Definition 1.1.2 *Let E be a normed space and $\varphi : E \to E^{**}$ be a canonical embedding (linear and isometry) map. If φ is onto, then E is reflexive.*

Let E be a smooth real Banach space and E^* be the dual space of E. The normalised duality map $J : E \to 2^{E^*}$ is defined by

$$Ju = \left\{u^* \in E^* \ : \langle u, u^* \rangle = \|u\|^2 = \|u^*\|^2\right\} \ \forall \ u \in E.$$

Where $\langle x, x^* \rangle$ is the value of $x^* \in E^*$ at $x \in E$ and is called generalized duality pairing. Let $S(E)$ be the unit sphere centered at the origin of E. The space E is said to be smooth if and only if the norm on E is Gâteaux differentiable i.e., if for each $x, y \in S(E)$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{1.1.1}$$

exists. The norm of E is said to be Fréchet differentiable if for each $x \in S(E)$, the limit (1.1.1) is attained uniformly for all $y \in S(E)$. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in S(E)$ and $x \neq y$. It is said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for all $x, y \in S(E)$ and $\|x - y\| \geq \epsilon$, $\|\frac{x+y}{2}\| < 1 - \delta$. It is known that if E is uniformly convex, then E is strictly convex and reflexive. It is also known from Takahashi (2000) that

(i) if E is smooth, then J is single valued;

(ii) if E is reflexive, then J is onto;

(iii)if E is strictly convex, then J is one-to-one and strictly monotone;

(iv) if E is Fréchet differentiable, then J is continuous.

Let E be a smooth Banach space and J be the duality mapping on E. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \ \forall x, y \in E.$$

Observe that in a real Hilbert space H, the above equation reduces to

$$\phi(x, y) = \|x - y\|^2 \ \forall \ x, y \in E.$$

Furthermore, for each $x, y, z, w \in E$, we have

$$\begin{aligned} \phi(x, y) &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\ &\geq \|x\|^2 - 2|\langle x, Jy\rangle| + \|y\|^2 \\ &\geq \|x\|^2 - 2\|x\|\|Jy\| + \|y\|^2 \\ &= \|x\|^2 - 2\|x\|\|y\| + \|y\|^2 \\ &= (\|x\| - \|y\|)^2. \\ Therefore, \ \phi(x, y) &\geq (\|x\| - \|y\|)^2. \end{aligned}$$

Similarly,

$$\phi(x, y) \leq (\|x\| + \|y\|)^2.$$

Thus,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \tag{1.1.2}$$

Also,

$$
\begin{aligned}
\phi(x,y) &= \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \\
&= \|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2 + \|z\|^2 - 2\langle z, Jy\rangle + \|y\|^2 \\
&+ 2\langle x, Jz\rangle + 2\langle z, Jy\rangle - 2\langle x, Jy\rangle - 2\|z\|^2 \\
&= \phi(x,z) + \phi(z,y) + 2\big(\langle x, Jz\rangle + \langle z, Jy\rangle - \langle x, Jy\rangle - \langle z, Jz\rangle\big) \\
&= \phi(x,z) + \phi(z,y) + 2\big(\langle x, Jz - Jy\rangle + \langle z, Jy - Jz\rangle\big) \\
&= \phi(x,z) + \phi(z,y) + 2\big(\langle x, Jz - Jy\rangle - \langle z, Jz - Jy\rangle\big) \\
&= \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle \qquad (1.1.3)
\end{aligned}
$$

And

$$
\begin{aligned}
2\langle x - y, Jz - Jw\rangle &= 2\langle x, Jz\rangle - 2\langle x, Jw\rangle - 2\langle y, Jz\rangle + 2\langle y, Jw\rangle \\
&= -(\|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2) + (\|x\|^2 - 2\langle x, Jw\rangle + \|w\|^2) \\
&+ (\|y\|^2 - 2\langle y, Jz\rangle + \|z\|^2) - (\|y\|^2 - 2\langle y, Jw\rangle + \|w\|^2) \\
&= -\phi(x,z) + \phi(x,w) + \phi(y,z) - \phi(y,w) \qquad (1.1.4)
\end{aligned}
$$

If E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \ if \ and \ only \ if \ x = y.$$

If E is a smooth, strictly convex and reflexive Banach space, then for any $x, y \in E$ and $\lambda \in \mathbb{R}$ with $\lambda \in [0,1]$, we have

$$\begin{aligned}
\phi\big(x, J^{-1}\big(\lambda Jy + (1-\lambda)Jz\big)\big) &= \|x\|^2 - 2\langle x, J\big(J^{-1}(\lambda Jy + (1-\lambda)Jz)\big)\rangle \\
&+ \|J^{-1}(\lambda Jy + (1-\lambda)Jz)\|^2 \\
&= \|x\|^2 - 2\langle x, \lambda Jy + (1-\lambda)Jz\rangle \\
&+ \|\lambda Jy + (1-\lambda)Jz\|^2 \\
&\leq \|x\|^2 - 2\lambda\langle x, Jy\rangle - 2(1-\lambda)\langle x, Jz\rangle + \lambda\|Jy\|^2 \\
&+ (1-\lambda)\|Jz\|^2 \\
&= \lambda(\|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2) \\
&+ (1-\lambda)(\|x\|^2 - 2\langle x, Jz\rangle + \|z\|^2) \\
&= \lambda\phi(x,y) + (1-\lambda)\phi(x,z). \\
\phi\big(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)\big) &\leq \lambda\phi(x,y) + (1-\lambda)\phi(x,z). \qquad (1.1.5)
\end{aligned}$$

Let $\phi_* : E^* \times E^* \to \mathbb{R}$ be a function defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2\langle J^{-1}y^*, x^*\rangle + \|y^*\|^2 \ for\ all\ x^*, y^* \in E^*.$$

Using the above definition, we have

$$\begin{aligned}
\phi_*(Jy, Jx) &= \|Jy\|^2 - 2\langle J^{-1}Jx, Jy\rangle + \|Jx\|^2 \\
&= \|y\|^2 - 2\langle x, Jy\rangle + \|x\|^2 \\
&= \phi(x,y). \qquad (1.1.6)
\end{aligned}$$

A class of generalized nonspreading mappings in Banach space was introduced by Kocourek et al. (2011). A mapping T from a nonempty subset C of a smooth real Banach space E into itself is called generalized nonspreading mapping if and only if $\exists\, \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned}\alpha\phi(Tx, Ty) \;&+\; (1-\alpha)\phi(x, Ty) + \gamma\big(\phi(Ty, Tx) - \phi(Ty, x)\big)\\ &\leq\; \beta\phi(Tx, y) + (1-\beta)\phi(x, y) + \delta\big(\phi(y, Tx) - \phi(y, x)\big), \forall\, x, y \in C.\end{aligned}$$

If such a mapping T is called $(\alpha, \beta, \gamma, \delta)$-generalized nonspreading mapping, then a (1,1,1,0)-generalized nonspreading mapping is a nonspreading mapping, Kohsaka and Takahashi (2008b), i.e.,

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x)\ \forall\, x, y \in C.$$

Also, a (1,0,0,0)-generalized nonspreading mapping is called a ϕ-nonenpansive mapping i.e

$$\phi(Tx, Ty) \leq \phi(x, y)\ \forall\, x, y \in C.$$

Definition 1.1.3 *Let E be a real Banach space and $f : E \to (-\infty, \infty)$ be a map. Then f is called proper if its domain $D(f) \neq \emptyset$ where $D(f) = \{x \in E : f(x) < +\infty\}$.*

Definition 1.1.4 *A function $f : E \to (-\infty, \infty)$ is said to be a lower-semi continuous at $x_0 \in E$, if $\{x_n\}$ is a sequence in E such that $x_n \to x_0$ and $f(x_n) \to y$, then $f(x_0) \leq y$*

Definition 1.1.5 *A real-valued function f is called convex if for every $\lambda \in [0, 1]$ and $x, y \in D(f)$, the domain of f $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.*

Let E be a real reflexive Banach space with norm $\|\cdot\|$ and E^* the dual space of E. Let $f : E \to \bar{\mathbb{R}}$ be a proper, lower semi-continuous and convex function. The Fenchel conjugate of f is the convex function $f^* : E^* \to \bar{\mathbb{R}}$ defined by

$$f^*(u^*) = \sup\{\langle u, u^*\rangle - f(u) : u \in E\}.$$

Let $u \in \text{intdom} f$; the subdifferential of f at u is the convex set defined by

$$\partial f(u) = \{u^* \in E^* : f(u) + \langle v - u, u^* \rangle \leq f(v), \ \forall v \in E\}.$$

For any $u \in \text{intdom} f$ and $v \in E$, the right-hand derivative of f at u in the direction v is defined by

$$f^\circ(u, v) := \lim_{t \to 0^+} \frac{f(u + tv) - f(u)}{t}.$$

The function f is said to be Gateaux differentiable at u if

$$\lim_{t \to 0^+} \frac{f(u + tv) - f(u)}{t}$$

exists for any v. In this case, $f^\circ(u, v)$ coincides with $\bigtriangledown f(u)$, the value of the gradient $\bigtriangledown f$ of f at u. The function f is said to be Gateaux differentiable if it is Gateaux differentiable for any $u \in \text{intdom} f$.

Definition 1.1.6 *(Bauschke et al. (2001)). The function f is said to be:*

(i) essentially smooth, if ∂f is both locally bounded and single-valued on its domain;

(ii) essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of dom$f$;

(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Proposition 1.1.1 *(Bauschke et al. (2001)). Let E be reflexive Banach space, then*

(i) f is essentially smooth if and only if f^* is essentially strictly convex;

(ii) f is Legendre if and only if f^* is Legendre;

(iii) If f is Legendre, then $\bigtriangledown f$ is a bijection satifying $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$, $ran \bigtriangledown f = dom \bigtriangledown f^* = intdom f^*$ and $ran \bigtriangledown f^* = dom f = intdom f$.

Examples of Legendre functions were given in Bauschke et al. (2001); Bauschke and Borwein (1997) . One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient $\bigtriangledown f$ of f coincides with the generalized duality mapping of E, i.e, $\bigtriangledown f = J_p$ $(1 < p < \infty)$. In particular, $\bigtriangledown f = I$ the identity mapping in Hilbert spaces. In the rest of this dissertation, we always assume that $f : E \to \bar{\mathbb{R}}$ is Legendre.

Let $f : E \to \bar{\mathbb{R}}$ be a convex and Gateaux differentiable function. The function $D_f : \text{dom} f \times \text{ intdom} f \to \bar{\mathbb{R}}$, defined by

$$D_f(u, v) := f(u) - f(v) - \langle \bigtriangledown f(v), u - v \rangle, \tag{1.1.7}$$

is called the Bregman distance with respect to f (Censor and Lennt (1981)). Observe that from (1.1.7), we have

$$D_f(u, w) := D_f(u, v) + D_f(v, w) + \langle \bigtriangledown f(v) - \bigtriangledown f(w), u - v \rangle. \tag{1.1.8}$$

Remark 1.1.1 *If E is smooth and strictly convex Banach space and $f(u) = \|u\|^2$ for all $u \in E$, then we have $\bigtriangledown f(u) = 2Ju$ for all $u \in E$ and hence*

$$\begin{aligned} D_f(u, v) &= f(u) - f(v) - \langle u - v, \bigtriangledown f(v) \rangle \\ &= \|u\|^2 - \|v\|^2 - \langle u - v, 2Jv \rangle \\ &= \|u\|^2 - \|v\|^2 - 2\langle u, Jv \rangle + 2\langle u, Jv \rangle \\ &= \|u\|^2 - \|v\|^2 - 2\langle u, Jv \rangle + 2\|v\|^2 \\ &= \|u\|^2 - 2\langle u, Jv \rangle + \|v\|^2 \\ &= \phi(u, v). \end{aligned}$$

1.2 Problem Statement

Takahashi et al. (2014a) proved some attractive point theorems in the framework of a smooth, strictly convex and reflexive real Banach space. A question arises here; can these Theorems be obtained in reflexive real Banach space only? This question has been answered appropriately in this dissertation.

1.3 Aim and Objectives of the Study

The aim of this research is to establish a new attractive point theorems for semitopological semigroups in reflexive real Banach space only. This aim will be achieved through the following objectives:

(i) Defining an attractive point using Bregman distance;

(ii) Defining a generalized nonspreading and skew-generalized nonspreading maappings;

(iii) Establishing attractive and skew-attractive point theorems in reflexive Banach space.

1.4 Scope and Limitation

This research covers a more general reflexive real Banach space and is limited to the notion of attractive point using Bregman distance.

1.5 Basic Notations

Throughout this dissertation, the set of natural numbers would be denoted by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and the the extended real number system $(-\infty, +\infty]$ by $\bar{\mathbb{R}}$.

Definition 1.5.1 *(Conjugate operator)(*Bauschke et al. (2009); Victoria et al. (2012)). Let E be a reflexive Banach space and let C be a subset of E. Let $f : E \to \bar{\mathbb{R}}$ be Legendre and $T : C \subset intdomf \to intdomf$ be an operator. The conjugate operator $T_f^* : \bigtriangledown f(C) \to intdomf^*$ associated with T denoted by T^* is defined by

$$T^* = \bigtriangledown f o T o \bigtriangledown f^*. \tag{1.5.1}$$

Lemma 1.5.1 *(*Bauschke et al. (2001)) Suppose $u \in E$ and $v \in domf$. Then

(i) if f is essentially strictly convex, then $D_f(u, v) = 0 \Leftrightarrow u = v$

(ii) if f is differentiable on intdomf and essentially strictly convex, then $D_f(u, v) = D_{f^*}(\bigtriangledown f(v), \bigtriangledown f(u))$.

1.6 Outline of the Dissertation

This dissertation is divided into five chapters. Chapter one is the introduction and the review of the related literature is in chapter two. Chapter three contains the methodology employed in carrying out the research while the main result is presented in chapter four. The summary, conclusion and recommendations form chapter five and references follow immediately.

CHAPTER TWO

LITERATURE REVIEW

2.1 Attractive Points

In this chapter, we present some literature related to the notion of attractive points that have been reviewed. In the introductory part of the work, we have seen that the concept of attractive points of nonlinear mappings was first introduced by Takahashi and Takeuchi (2011) which opened up a new and interesting dimension of research in the area of nonlinear analysis. They proved among others, the following attractive point and mean convergence theorems of Baillon's type, Baillon (1975) without convexity assumption for generalized hybrid mappings in Hilbert space.

Theorem 2.1.1 *(Takahashi and Takeuchi (2011)) Let H be a Hilbert space and C be a nonempty subset of H. Let $T : C \to C$ be a generalized hybrid mapping that is there exists $\alpha, \beta \in \mathbb{R}$ such that*

$$\alpha\|Tu - Tv\|^2 + (1-\alpha)\|u - Tv\|^2 \leq \beta\|Tu - v\|^2 + (1-\beta)\|u - v\|^2 \ \forall \ u, v \in C,$$

then T has an attractive point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ is bounded.

Theorem 2.1.2 *(Takahashi and Takeuchi (2011)). Let H be a real Hilbert space and C a nonempty subset of H. Let $T : C \to C$ be a generalized hybrid mapping. Let $\{v_n\}$ and $\{b_n\}$ be sequences defined by*

$$v_1 \in C, \ v_{n+1} = Tv_n, \ b_n = \frac{1}{n}\sum_{k=1}^{n} v_k \qquad \forall \ n \in \mathbb{N}.$$

If $\{v_n\}$ is bounded then the following hold:

(i) $A(T)$ *is nonempty, closed and convex;*

(ii) $\{b_n\}$ *converges weakly to* $u_0 \in A(T)$ *where* $u_0 = \lim_{n\to\infty} P_{A(T)}v_n$ *and* $P_{A(T)}$ *is the metric projection of* H *onto* $A(T)$.

For commutative semigroups of nonexpansive mappings in a Hilbert space, a theorem of the form of Theorem 2.1.2 was proved by Atsushiba and Takahashi (2013).

Takahashi et al. (2012) introduced a broad class of nonlinear mappings called $(\alpha, \beta, \gamma, \delta)$-normally generalized hybrid that contained the class of generalized hybid mappings and contractive mappings in a Hilbert space. They proved the attractive point theorem for such mapping in Hilbert spaces. Also, using the technique developed in Takahashi (1981), they proved mean convergence theorem of Baillon's type without convexity assumption for normally generalized hybrid mappings and a weak convergence theorem of Mann's type without closedness in a Hilbert space.

Theorem 2.1.3 *(Takahashi et al. (2012)) Let* H *be a real Hilbert space and* C *a nonempty subset of* H*. Let* T *be an* $(\alpha, \beta, \gamma, \delta)$*-normally generalized hybrid mapping from* C *into itself. Then* T *has an attractive point if and only if there exists* $z \in C$ *such that* $\{T^n z : n = 0, 1, ...\}$ *is bounded. Additionally, if* C *is closed and convex, then* T *has a fixed point if and only if there exists* $z \in C$ *such that* $\{T^n z | n = 0, 1, ...\}$ *is bounded. In particular, a fixed point of* T *is unique in the case of* $\alpha + \beta + \gamma + \delta > 0$.

Lin and Takahashi (2012) used Banach limits of nonlinear mappings in Hilbert space to obtain attractive point theorems for such mappings. They also used the results and proved a nonlinear ergodic theorem for 2-generalized hybrid mappings in a Hilbert space.

Guu and Takahashi (2013), studied $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-widely more generalized hybrid mappings and established the existence of attractive point theorem and a mean convergence theorem for such mappings in a Hilbert space. They also proved weak convergence theorem of Mann's type and strong convergence theorems of Shizima and Takahashi's type for such a wide class of nonlinear mappings.

Theorem 2.1.4 *(Guu and Takahashi (2013)) Let H be a real Hilbert space and C a nonempty subset of H. Let T be an $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta, \eta)$-widely more generalized hybrid mapping from C into itself which satisfies either of the conditions:*

(i) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \gamma > 0$ *and* $\epsilon + \eta \geq 0$,

(ii) $\alpha + \beta + \gamma + \delta \geq 0,\ \alpha + \beta > 0$ *and* $\zeta + \eta \geq 0$.

Then T has an attractive point if and only if there exists $z \in C$ such that $\{T^n z : n = 0, 1, ...\}$ *is bounded.*

Takahashi et al. (2013) used the concept of attractive point of nonlinear mappings to obtain strong convergence theorem of Halpern's type, (Halpern (1967)) for a wide class of nonlinear mappings. They proved the following theorem:

Theorem 2.1.5 *(*Takahashi et al. (2013)). Let H be a Hilbert space and C a convex subset of H. Let T be a generalized hybrid mapping from C into itself with $A(T) \neq \emptyset$ and let $P_{A(T)}$ be the metric projection of H onto $A(T)$. Let $z \in C$ and $\{x_n\}$ be a sequence in C defined by $x_1 \in C$ and

$$x_{n+1} = \alpha_n z + (1 - \alpha_n)(\beta_n x_n + (1 - \beta_n)Tx_n),\ \forall n \in \mathbb{N},$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$ such that

$$\lim_{n\to\infty} \alpha_n = 0,\ \sum_{n=1}^{\infty} = \infty \ \textit{and}\ \liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0.$$

Then $\{x_n\}$ converges strongly to $\bar{x} = P_{A(T)}z$.

Lin and Takahashi (2013) extended the concept of attractive points in a Hilbert space to that in a Banach space. With the extension of the notion of generalized hybrid mapping in a Hilbert space to the generalized nonspreading mappings in a Banach space, they proved attractive point theorems for such mapping. They used the results and obtained some results for skew-generalized nonspreading mappings in a Banach space. They finally proved nonlinear ergodic theorems without convexity for generalized nonspreading mapping in a Banach space.

Takahashi et al. (2014b) unified the results of Baillon's type proved by Takahashi and Takeuchi (2011) and that proved by Atsushiba and Takahashi (2013) by establishing an attractive point and mean convergence theorem for semigroup of mappings without continuity in a Hilbert space. They extended their results to a more general Banach space, Takahashi et al. (2014a), by introducing a broad semigroup of mappings without continuity in a Banach space. They proved attractive point and fixed point theorems for the semigroup of mappings without continuity. They also established and proved among others, the following attractive point theorems for such mappings .

Theorem 2.1.6 *Let E be a smooth and reflexive Banach space with the duality mappping J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mapping of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that $\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y), \forall\ y \in C$ and $t \in S$ then $A(\mathcal{S}) = \cap\{A(T_s) : s \in S\} \neq \emptyset$. In particular, if E is strictly convex and C is closed and convex then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.*

Theorem 2.1.7 *Let E be a strictly convex and reflexive Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as a mapping of C into itself such that*

$\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on C. Suppose that $\mu_s \phi(T_t y, T_s x) \leq \mu_s \phi(y, T_s x)$ for all $y \in C$ and $t \in S$. Then $B(\mathcal{S}) = \cap\{B(T_t) : t \in S\}$ is nonempty. In particular, if C is closed and JC is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.

Also, using Theorem 2.1.6, Takahashi et al. (2014a) proved the following attractive point theorem for generalized nonspreading mappings in Banach space, see also Lin and Takahashi (2013).

Theorem 2.1.8 *Let E be a smooth and reflexive Banach space and C a nonempty subset of E. Let $T : C \to C$ be a generalized nonspreading mapping, then the following are equivalent.*
(i) $A(T) \neq \emptyset$;
(ii) $\{T^n v_0\}$ is bounded for some $v_0 \in C$.

CHAPTER THREE

THEORY OF METHODS

3.1 Introduction

The methodology employed in carrying out this research follows from review of some announced results and other techniques in related literature.

Lemma 3.1.1 *(*Xu (1991)*). Let E be a uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$\|\alpha x + (1-\alpha)y\|^2 \leq \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)g(\|x-y\|)$$

for all $x, y \in B_r$ and $\alpha \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 3.1.2 *(*Kamimura and Takahashi (2002)*). Let E be a uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to [0, \infty)$ such that $g(0) = 0$ and*

$$g(\|x-y\|) \leq \phi(x, y)$$

for all $x, y \in B_r$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Let E be a smooth Banach space and C a nonempty subset of E. A mapping $T : C \to E$ is called quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\phi(Tx, y) \leq \phi(x, y) \ \forall \ x, y \in C \text{ and } y \in F(T).$$

(Ibaraki and Takahashi (2007b)).

Let D be a nonempty subset of a Banach space E. A mapping $R : E \to D$ is said to be sunny if

$$R\big(Rx + t(x - Rx)\big) = Rx$$

for all $x \in E$ and $t > 0$. $R : E \to D$ is said to be a retraction or a projection if $Rx = x$ for all $x \in D$. A nonempty subset D of a smooth Banach space E is said to be a generalized nonexpansive retract (resp., a sunny generalized nonexpansive retract) of E if there exists a generalized nonexpansive retraction (resp., sunny generalized nonexpansive retraction) R from E to D; (Ibaraki and Takahashi (2006, 2007a,b)). Ibaraki and Takahashi (2007b) proved the following lemmas.

Lemma 3.1.3 *(*Ibaraki and Takahashi (2007b)*). Let C be a nonempty closed sunny generalized nonexpansive retract of a smooth and strictly convex Banach space E. Then the sunny generalized nonexpansive retraction from E onto C is uniquely determined.*

Lemma 3.1.4 *(*Ibaraki and Takahashi (2007b)*). Let C be a nonempty closed subset of a smooth and strictly convex Banach space E such that there exists a sunny generalized nonexpansive retraction R from E onto C and $(x, z) \in E \times C$. Then the following hold:*
(i) $z = Rx$ if and only if $\langle x - z, Jy - Jz \rangle \leq 0$ for all $y \in C$;
(ii) $\phi(Rx, z) + \phi(x, Rx) \leq \phi(x, z)$.

Kohsaka and Takahashi (2007) proved the following results:

Lemma 3.1.5 *(*Kohsaka and Takahashi (2007)*). Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty closed subset of E. Then the following are equivalent:*
(a) C is a sunny generalized nonexpansive retract of E;
(b) C is a generalized nonexpansive retract of E;
(c) JC is closed and convex.

Lemma 3.1.6 *(*Kohsaka and Takahashi (2007)*). Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty closed sunny generalized nonexpansive retract of E. Let R be the sunny generalized nonexpansive retraction from E to C and let $(x,z) \in E \times C$. Then the following are equivalent:*

(i) $z = Rx$;

(ii) $\phi(x,z) = min_{y \in C}\phi(x,y)$.

Lemma 3.1.7 *(*Inthakon et al. (2010)*). Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a quasi-ϕ-nonexpansive mapping from C into itself. Then, $F(T)$ is closed and $JF(T)$ is closed and convex.*

The following is a direct consequence of Lemmas 3.1.5 and 3.1.7:

Lemma 3.1.8 *(*Inthakon et al. (2010)*). Let E be a smooth, strictly convex and reflexive Banach space and let C be a closed subset of E such that $J(C)$ is closed and convex. Let T be a quasi-ϕ-nonexpansive mapping from C into itself. Then, $F(T)$ is a sunny generalized retract of E.*

3.2 Semitopological Semigroups and Invariant Means

Let S be a semitopological semigroup, i.e S is a semigroup with a Hausdorff topology such that for each $a \in S$, the mappings $s \to a.s$ and $s \to s.a$ from S to S are continuous. In the case S is commutative we denote st by $s+t$. Let $B(S)$ be the Banach space of all bounded real-valued functions on S with supremum norm and let $C(S)$ be the subspace of $B(S)$ of all bounded real-valued continuous functions on S. Let μ be an element of $C(S)^*$ the dual space of $C(S)$. The value of μ at $f \in C(S)$ is denoted by $\mu(f)$ and sometimes it is denoted by $\mu_t(f(t))$ or $\mu_t f(t)$. For each $s \in S$ and $f \in C(S)$, we define two functions $(r_s f)$ and $(l_s f)$ as follows:

$$(r_s f)(t) = f(ts) \text{ and } (l_s f)(t) = f(st)$$

for all $t \in S$. An element μ of $C(S)^*$ is called a mean on $C(S)$ if

$$\mu(e) = \|\mu\| = 1,$$

where $e(s) = 1$ for all $s \in S$. It is known that $\mu \in C(S)^*$ is a mean on $C(S)$ if and only if

$$\inf_{s \in S} f(s) \leq \mu(s) \leq \sup_{s \in S} f(s) \ \forall f \in C(S).$$

A mean μ on $C(S)$ is called right invariant mean if $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. Similarly, a mean μ on $C(S)$ is called left invariant mean if $\mu(l_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$. A right and left invariant mean on $C(S)$ is called an invariant mean on $C(S)$. If $S = \mathbb{N}$, then an invariant mean on $C(S) = B(S)$ is called a Banach limit on l^∞.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ the dual space of l^∞. Then, we denote by $\mu(f)$ the value of μ at

$$f = (x_1, x_2, x_3, ...) \in l^\infty.$$

Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a mean if

$$\mu(e) = \|\mu\| = 1,$$

where $e = (1, 1, 1, ...)$. A mean μ is called a Banach limit on l^∞ if

$$\mu_n(x_{n+1}) = \mu_n(x_n).$$

If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, ...) \in l^\infty$, we have

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, ...) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, we have

$$\mu(f) = \mu_n(x_n) = a.$$

See Takahashi (2000) for the proof of the existence of a Banach limit and its other elementary properties. The following theorem can be found in (Takahashi (2000)).

Theorem 3.2.1 *(*Takahashi (2000)*). Let S be a commutative semitopological semigroup. Then $\exists$ an invariant mean on $C(S)$; i.e $\exists$ an element μ on $C(S)^*$ such that $\mu(e) = \|\mu\| = 1$ and $\mu(r_s f) = \mu(f)$ for all $f \in C(S)$ and $s \in S$.*

Let E be a Banach space and C a nonempty subset of E. Let S be a semitopological semigroup and $\mathcal{S} = \{T_s : s \in S\}$ a family of mappings of C into itself. Then $\mathcal{S} = \{T_s : s \in S\}$ is called a continuous representation of S as mappings on C if $T_{st} = T_s T_t$ for all $s, t \in S$ and the mapping $s \to T_s x$ is continuous for each $x \in C$. Let the set of common fixed points of T_s be denoted by $F(\mathcal{S})$ i.e.; $F(\mathcal{S}) = \cap\{F(T_s) : s \in S\}$.

Proposition 3.2.2 *(Takahashi et al. (2014a)) Let E be a reflexive space and let E^* be the dual space of E. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\}$ is bounded and let μ be a mean on $C(S)$. Then there exists a unique point $z_0 \in \bar{co}\{u(s) : s \in S\}$ such that*

$$\mu_s\langle u(s), y^*\rangle = \langle z_0, y^*\rangle \ \forall y^* \in E^*.$$

Such z_0 in 3.2.2 is called a mean vector of u for μ. In particular, let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as a mapping on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Putting $u(s) = T_s x$ for all $s \in S$, there exists $z_0 \in E$ such that

$$\mu_s \langle T_s x, y^* \rangle = \langle z_0, y^* \rangle \ \forall \ y^* \in E^*$$

. And such z_0 is denoted by $T_\mu x$.

A net $\{\mu_\alpha\}$ of means on $C(S)$ is said to be asymptotically invariant if for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0 \ and \ \mu_\alpha(f) - \mu_\alpha(r_s f) \to 0.$$

(Day (1957); Takahashi (2000)).

3.3 Attractive Point Theorems

Let E be a smooth Banach space and C a nonempty subset of E. For a mapping T from C into C, we denote by $A(T)$, the set of attractive points of T; i.e.,

$$A(T) = \{u \in E : \phi(u, Tx) \leq \phi(u, x) \ \forall \ x \in C\}.$$

Let S be a commutative semitopological semigroup with identity. For a continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as a mapping of C into itself, we denote the set $A(\mathcal{S})$ of common attractive points of $\mathcal{S} = \{T_s : s \in S\}$ by

$$A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}.$$

We know the following lemma from Lin and Takahashi (2013):

Lemma 3.3.1 *(*Lin and Takahashi (2013)*). Let E be a smooth Banach space and C a nonempty subset of E. Let T be a mapping from C into E. Then $A(T)$ is a closed and convex subset of E.*

Using technique developed by Takahashi (1981), the following attractive point theorem for a family of mappings in Banach space have been proved.

Theorem 3.3.2 *(Takahashi et al. (2014a)). Let E be a smooth and reflexive Banach space with duality map J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as a mapping of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that $\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y)$ $\forall y \in C$ and $t \in S$. Then $A(\mathcal{S}) = \cap\{A(T_t) : t \in S\} \neq \emptyset$. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\} \neq \emptyset$.*

Theorem 3.3.3 *(*Lin and Takahashi (2013)*). Let E be a smooth and reflexive Banach space and C a nonempty subset of E. Let T be a generalized nonspreading mapping of C into itself. Then the following are equivalent:*
(i) $A(T) \neq \emptyset$;
(ii) $\{T^n v_0\}$ is bounded for some $v_0 \in C$.
Additionally, if E is strictly convex and C is closed and convex, then the following are equivalent:
(i) $F(T) \neq \emptyset$;
(ii) $\{T^n v_0\}$ is bounded for some $v_0 \in C$.

Let E be a smooth Banach space and let C be a nonempty subset of E. Let S be a semitopological semigroup. A continuous representation $\mathcal{S} = \{T_s : s \in S\}$ of S as mappings on C is a ϕ-nonexpansive semigroup on C if each T_s, $s \in S$, is a ϕ-nonexpansive; i.e.

$$\phi(T_s x, T_s y) \leq \phi(x, y) \; \forall \; x, y \in C.$$

In the case when E is a Hilbert space, a ϕ-nonexpansive semigroup on C is called a nonexpansive semigroup on C (Atsushiba and Takahashi (2013)). Using Theorem 3.3.2, an attractive point theorem for ϕ-nonexpansive semigroups in a Banach space has been proved.

Theorem 3.3.4 *(Takahashi et al. (2014a)). Let E be a smooth and reflexive Banach space with duality mapping J and C a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $\{T_s z : s \in S\}$ is bounded for some $z \in C$. Then, $A(\mathcal{S}) = \cap\{A(T_t) : t \in S\}$ is nonempty. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\}$ is nonempty.*

As a direct consequence of Theorem (3.3.4), the following theorem which was proved by Atsushiba and Takahashi (2013).

Theorem 3.3.5 *(*Atsushiba and Takahashi (2013)*). Let H be a Hilbert space and C a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $\{T_s z : s \in S\}$ is bounded for some $z \in C$. Then,$A(\mathcal{S})$ is nonempty. In particular, if E is strictly convex and C is closed and convex, then $F(\mathcal{S})$ is nonempty.*

3.4 Skew-attractive Point Theorems

Let E be a smooth Banach space and C a nonempty subset of E. Let T be a mapping from C into E, we denote by $B(T)$, the set of skew-attractive points of E; i.e.

$$B(T) = \{z \in E : \phi(Tx, z) \leq \phi(x, z) \ \forall \ x \in C\}.$$

The following lemma was proved by Lin and Takahashi (2013).

Lemma 3.4.1 *(Lin and Takahashi (2013)). Let E be a smooth Banach space and C a nonempty subset of E. Let T be a mapping from C into E. Then, B(T) is closed.*

Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty subset of E. Let T be a mapping from C into E. Define a mapping T^* as follows:

$$T^*x^* = JTJ^{-1} \ \forall x^* \in JC.$$

where J is the duality mapping on E and J^{-1} is the duality mapping on E^*. The mapping T^* is called the duality mapping of T (Takahashi and Yao (2011) and Honda et al. (2010)). If T is a mapping of C into itself, then T^* is a mapping of JC into JC. The following lemma was proved by Lin and Takahashi (2013) and Takahashi and Yao (2011).

Lemma 3.4.2 *(*Lin and Takahashi (2013); Takahashi and Yao (2011)*) Let E be a smooth, strictly convex and reflexive Banach space and C a nonempty subset of E. Let T be a mapping from C into E. Let T^* be the duality mapping of T. Then the following hold:*

(1) $JF(T) = F(T^*)$;

(2) $JB(T) = A(T^*)$;

(3) $JA(T) = B(T^*)$.

In particular $JB(T)$ is closed and convex.

Let E be a smooth Banach space and C a nonempty subset of E. We denote by $B(\mathcal{S})$ the set of all common skew-attractive points of a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself; i.e.

$$B(\mathcal{S}) = \cap\{B(T_s) : s \in S\}.$$

the following skew-attractive point theorem for semigroups of mappings without continuity in a Banach space has been obtained.

Theorem 3.4.3 *(Takahashi et al. (2014a)) Let E be strictly convex and reflexive Banach space with a Fréchet differentiable norm and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as a mapping of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose $\mu_s\phi(T_t y, T_s x) \le \mu_s\phi(y, T_s x)$ $\forall y \in C$ and $t \in S$. Then, $B(\mathcal{S}) = \cap\{B(T_t) : t \in S\} \neq \emptyset$. In particular, if C is closed and JC is closed and convex, then $F(\mathcal{S}) = \cap\{F(T_t) : t \in S\} \neq \emptyset$.*

Let E be a smooth Banach space and J the duality mapping from E into E^*. Let C be a nonempty subset of E. A mapping $T : C \to E$ is called skew-generalized nonspreading if there exists $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\phi(Ty, Tx) &+ (1-\alpha)\phi(Ty, x) + \gamma\phi\big((Tx, Ty) - \phi(x, Ty)\big) \\ &\le \beta\phi(y, Tx) + (1-\beta)\phi(y, x) \\ &+ \delta\big(\phi(Tx, y) - \phi(x, y)\big) \ \forall\ x, y \in C. \end{aligned} \quad (3.4.1)$$

(Hsu et al. (2012)). Using Theorem 3.4.3, we have the following attractive point theorem for skew-generalized nonspreading mappings in a Banach space which was proved by Lin and Takahashi (2013).

Theorem 3.4.4 *(*Lin and Takahashi (2013)*). Let E be a smooth, strictly*

convex and reflexive Banach space and C a nonempty subset of E. Let T be a skew-generalized nonspreading mapping of C into itself. Then the following are equivalent:
(i) $B(T) \neq \emptyset$;
(ii) $\{T^n v_0\}$ *is bounded for some* $v_0 \in C$.
Additionally, if C is closed and JC is closed and convex, then the following are equivalent:
(i) $F(T) \neq \emptyset$;
(ii) $\{T^n v_0\}$ *is bounded for some* $v_0 \in C$.

3.5 Nonlinear Mean Convergence Theorems

In this section, a nonlinear mean convergence theorem of Baillon's type[3] for semigroup of mappings without continuity in a Banach space. We need the following four lemmas.

Lemma 3.5.1 *(Takahashi et al. (2014a)). Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J. Let D be a nonempty, closed and convex subset of E. Let S be a semitopological semigroup with identity and let $C(S)$ be a Banach space of all bounded real-valued continuous functions on S with supremum norm. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\} \subset D$ is bounded and let μ be a mean on $C(S)$. if $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_s \phi(u(s), z) \ \forall z \in D$$

then the mean vector z_0 of $\{u(s) : s \in S\}$ for μ is a unique minimizer in D such that

$$g(z_0) = min\{g(z) : z \in D\}.$$

Using Lemma 3.5.1, we obtain the following result.

Lemma 3.5.2 *(Takahashi et al. (2014a)). Let E be a smooth, strictly convex and reflexive Banach space with the duality mapping J and let C be a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as a mapping of C into itself. Suppose that $A(\mathcal{S}) = B(\mathcal{S})$ is nonempty. Then for any $x \in C$, the net $\{T_s x : s \in S\}$ is bounded and the set*

$$\cap_s \bar{co}\{T_{t+s}x : t \in S\} \cap A(\mathcal{S})$$

consists of one point z_0, where z_0 is a unique minimizer of $A(\mathcal{S})$ such that

$$\lim_s \phi(T_s x, z_0) = min\{\lim_s \phi(T_s x, z) : z \in A(\mathcal{S})\}.$$

Additionally, if C is closed and convex, then the set

$$\cap_s \bar{co}\{T_{t+s}x : t \in S\} \cap F(\mathcal{S})$$

consists of one point z_0.

Lemma 3.5.3 *(Takahashi et al. (2014a)). Let E be a smooth and reflexive Banach space and C a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mapping of C into itself. Suppose that $\{T_s x : s \in S\}$ is bounded for some $x \in C$ and*

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y) \ \forall y \in C \text{ and } t \in S,$$

for all invariant means μ on $C(S)$. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of means on $C(S)$; i.e, for each $f \in C(S)$ and $s \in S$, $\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$. If a subnet $\{T_{\mu_{\alpha_\beta}} x\}$ of $\{T_{\mu_\alpha} x\}$ converges weakly to a point $u \in E$, then $u \in A(\mathcal{S})$.

Additionally, if E is strictly convex and C is closed and convex, then $u \in F(\mathcal{S})$.

Lemma 3.5.4 *(Takahashi et al. (2014a)). Let E be a uniformly convex and smooth Banach space and C a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as a mapping of C into itself such that $B(\mathcal{S}) \neq \emptyset$. Then, there exists a unique sunny generalized nonexpansive retraction R of E onto $B(\mathcal{S})$. Furthermore, for any $x \in C$, $lim_s RT_s x$ exists in $B(\mathcal{S})$, where $lim_s RT_s x = q$ means that $lim_s \|RT_s x - q\| = 0$.*

Now, we can prove the following nonlinear mean convergence theorem of Baillon's type, Baillon (1975), for semigroups of mappings without continuity in a Banach space.

Theorem 3.5.5 *(Takahashi et al. (2014a)). Let E be a uniformly convex Banach space with a Fréchet differentiable norm and C a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mapping of C into itself such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Suppose*

$$\mu_s \phi(T_s x, T_t y) \leq \mu_s \phi(T_s x, y) \ \forall x, y \in C, \ t \in S \tag{3.5.1}$$

for all invariant means μ on $C(S)$. Let $\{\mu_s\}$ be an asymptotically invariant net of means on $C(S)$; i.e., for each $f \in C(S)$ and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0.$$

Then, $\{T_{\mu_\alpha} x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where

$$u = \lim_s R_{B(\mathcal{S})} T_s x.$$

Additionally, if C is closed and convex, then $u \in F(\mathcal{S})$, where

$$u = \lim_s R_{F(\mathcal{S})} T_s x.$$

Using Theorem 3.5.5, we obtain the following nonlinear mean convergence theorem for generalized nonspreading mapppings in a Banach space which was proved by Lin and Takahashi (2013).

Theorem 3.5.6 *(*Lin and Takahashi (2013)*). Let E be a uniformly convex Banach space with a Fréchet differentiable norm and C a nonempty subset of E. Let $T : C \to C$ be a generalized nonspreading mapping such that $A(T) = B(T) \neq \emptyset$. Let R be a sunny generalized nonexpansive retraction of E onto $B(T)$. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^x$$

converges weakly to $z_0 \in A(T)$, where $z_0 = lim_{n\to\infty} RT^n x$. Additionally, if C is closed and convex, then $\{S_n x\}$ converges weakly to $z_0 \in F(T)$.

Using Theorem 3.5.5, we also have the nonlinear mean convergence theorem for ϕ-nonexpansive semigroups in a Banach space.

Theorem 3.5.7 *(Takahashi et al. (2014a)). Let E be a uniformly convex Banach space with a Fréchet differentiable norm and C a nonempty subset of E. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a ϕ-nonexpansive semigroup on C such that $A(\mathcal{S}) = B(\mathcal{S}) \neq \emptyset$ and let $R_{B(\mathcal{S})}$ be the sunny generalized nonexpansive retraction of E onto $B(\mathcal{S})$. Let $\{\mu_s\}$ be an asymptotically invariant net of means on $C(S)$; i.e., for each $f \in C(S)$ and $s \in S$,$\mu_\alpha(f) - \mu_\alpha(l_s f) \to 0$. Then, $\{T_{\mu_\alpha} x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where $u = \lim_s R_{B(\mathcal{S})} T_s x$. Additionally, if C is closed and convex, then $u \in F(\mathcal{S})$, where $u = \lim_s R_{F(\mathcal{S})} T_s x$.*

As a direct consequence of Theorem 3.5.7, we have a mean convergence theorem for commutative semigroups of nonexpansive mappings in a Hilbert space which was proved by Atsushiba and Takahashi (2013).

Theorem 3.5.8 *(*Atsushiba and Takahashi (2013)*). Let H be a Hilbert space and C a nonempty subset of H. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a nonexpansive semigroup on C such that $A(\mathcal{S})$ is nonempty. Let $\{\mu_\alpha\}$ be an asymptotically invariant net of mean on $C(S)$. Then, $\{T_\alpha x\}$ converges weakly to a point $u \in A(\mathcal{S})$, where $u = \lim_s P_{A(\mathcal{S})} T_s x$. Additionally, if C is closed and convex, then $u \in F(\mathcal{S})$, where $u = \lim_s P_{F(\mathcal{S})} T_s x$.*

CHAPTER FOUR

RESULTS AND DISCUSSION

4.1 Main Results

The results present in this chapter were all established in a reflexive real Banach space only. Let E be a reflexive real Banach space and $f : E \to \bar{\mathbb{R}}$ a convex and Gâteaux differentiable function. Let T be a mapping from C into itself, where C is a nonempty subset of E. A point $v \in E$ is called a Bregman attractive point of T if and only if for all $z \in C$,

$$D_f(v, Tz) \leq D_f(v, z).$$

Denote by $A^B(T)$, the set all of Bregman attractive points of T i.e.

$$A^B(T) = \{v \in E : D_f(v, Tz) \leq D_f(v, z), \forall z \in C\}.$$

Let S be a semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continous representation of S as a mapping of C into itself. Denoting the set of common Bregman attractive points of $\mathcal{S}$ by $A^B(\mathcal{S})$ i.e

$$A^B(\mathcal{S}) = \cap\{A^B(T_t) : t \in S\}.$$

We now prove the following Bregman attractive point theorems:

Theorem 4.1.1 *Let E be a reflexive Banach space and C is a nonempty subset of E. Let $f : E \to \bar{\mathbb{R}}$ a convex and Gâteaux differentiable function. Let S be a commutative semitopological semigroup with identity and $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as a mapping of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that*

$\mu_s D_f(T_s x, T_t y) \leq \mu_s D_f(T_s x, y)\ \forall\ y \in C$ *and* $t \in S$ *then* $A^B(\mathcal{S}) = \cap\{A^B(T_t) : t \in S\} \neq \emptyset$.

Proof. Using a mean μ on a bounded set $\{T_s x : s \in S\}$ and defining a function $g : E^* \to \mathbb{R}$ by

$$g(x^*) = \mu_s \langle T_s x, x^* \rangle\ \forall\ x^* \in E^*.$$

Since E is reflexive as shown by (Takahashi et al. (2014a)) $\exists! z \in E^*$ such that

$$g(x^*) = \mu_s \langle T_s x, x^* \rangle = \langle z, x^* \rangle\ \forall\ x^* \in E^*$$

and such $z \in \bar{co}\{T_s x : s \in S\}$. Now, for s,t $\in S$, we have

$$\begin{aligned} D_f(T_s x, y) &= D_f(T_s x, T_t y) + D_f(T_t y, y) + \langle T_s x - T_t y, \triangledown f(T_t y) - \triangledown f(y) \rangle \\ &= D_f(T_s x, T_t y) + D_f(T_t y, y) + \langle T_s x - T_t y, \triangledown f(T_t y) \rangle \\ &- \langle T_s x - T_t y, \triangledown f(y) \rangle \end{aligned}$$

Applying mean μ_s on both sides of the above expression we get,

$$\begin{aligned} \mu_s D_f(T_s x, y) &= \mu_s D_f(T_s x, T_t y) + \mu_s D_f(T_t y, y) + \mu_s \langle T_s x - T_t y, \triangledown f(T_t y) \rangle \\ &- \mu_s \langle T_s x - T_t y, \triangledown f(y) \rangle. \\ &= \mu_s D_f(T_s x, T_t y) + D_f(T_t y, y) + \langle z - T_t y, \triangledown f(T_t y) \rangle \\ &- \langle z - T_t y, \triangledown f(y) \rangle \\ &\leq \mu_s D_f(T_s x, y) + D_f(T_t y, y) + \langle z - T_t y, \triangledown f(T_t y) \rangle \\ &- \langle z - T_t y, \triangledown f(y) \rangle \end{aligned}$$

This implies that

$$\begin{aligned}
0 &\leq D_f(T_t y, y) + \langle z - T_t y, \triangledown f(T_t y)\rangle - \langle z - T_t y, \triangledown f(y)\rangle \\
&= D_f(T_t y, y) + D_f(z, T_t y) - D_f(z, T_t y) \\
&+ \langle z - T_t y, \triangledown f(T_t y) - \triangledown f(y)\rangle \\
&= D_f(z, T_t y) + D_f(T_t y, y) + \langle z - T_t y, \triangledown f(T_t y) - \triangledown f(y)\rangle \\
&- D_f(z, T_t y) \\
&= D_f(z, y) - D_f(z, T_t y) \\
Hence,\ D_f(z, T_t y) &\leq D_f(z, y).
\end{aligned}$$

This implies that $z \in A^B(T_t)$ for each t and so $z \in A^B(\mathcal{S})$. Hence, $A^B(\mathcal{S}) \neq \emptyset$. This completes the proof. □

Definition 4.1.1 *Let E be a reflexive Banach space and C a nonempty subset of E. A mapping T from C into itself is called generalized Bregman nonspreading mapping if $\exists\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that*

$$\begin{aligned}
\alpha D_f(Tx, Ty) &+ (1-\alpha) D_f(x, Ty) + \gamma\big(D_f(Ty, Tx) - D_f(Ty, x)\big) \\
&\leq \beta D_f(Tx, y) + (1-\beta) D_f(x, y) + \delta\big(D_f(y, Tx) - D_f(y, x)\big)
\end{aligned}$$

for all $x, y \in C$

If such a mapping T is called $(\alpha, \beta, \gamma, \delta)$-generalized Bregman nonspreading mapping, then a (1,1,1,0)-generalized Bregman nonspreading mapping is called a Bregman nonspreading mapping. i.e.

$$D_f(Tx, Ty) + D_f(Ty, Tx) \leq D_f(Tx, y) + D_f(Ty, x)\ \forall\ x, y \in C.$$

Putting $\alpha = 1$ and $\beta = \gamma = \delta = 0$, we obtain Bregman nonexpansive mapping

$$D_f(Tx, Ty) \leq D_f(x, y)\ \forall x, y \in C.$$

Theorem 4.1.2 *Let E be a reflexive Banach space and C be a nonempty subset of E. Let $f : E \to \bar{\mathbb{R}}$ a convex and Gâteaux differentiable function. Let $T : C \to C$ be a generalized Bregman nonspreading mapping, then $A^B(T) \neq \emptyset$ if and only if $\{T^n v_0\}$ is bounded for some $v_0 \in C$.*

Proof. Suppose $A^B(T) \neq \emptyset$, then $D_f(u, Tx) \leq D_f(u, x)$ for all $u \in A^B(T)$ and $x \in C$. So, $D_f(u, T^n x) \leq D_f(u, x)$ for all $n \in \mathbb{N}$ and $x \in C$ and hence $\{T^n x : n \in \mathbb{N}\}$ is bounded. We show the converse. Since $T : C \to C$ is generalized Bregman nonspreading mapping, there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha D_f(Tx, Ty) &+ (1-\alpha)D_f(x, Ty) + \gamma\big(D_f(Ty, Tx) - D_f(Ty, x)\big) \\ &\leq \beta D_f(Tx, y) + (1-\beta)D_f(x, y) \\ &+ \delta\big(D_f(y, Tx) - D_f(y, x)\big), \end{aligned} \tag{4.1.1}$$

for all $x, y \in C$. Now, replacing x by $T^n v_0$ in inequality (4.1.1), we get

$$\begin{aligned} \alpha D_f(T^{n+1}v_0, Ty) &+ (1-\alpha)D_f(T^n v_0, Ty) + \gamma\big(D_f(Ty, T^{n+1}v_0) - D_f(Ty, T^n v_0)\big) \\ &\leq \beta D_f(T^{n+1}v_0, y) + (1-\beta)D_f(T^n v_0, y) \\ &+ \delta\big(D_f(y, T^{n+1}v_0) - D_f(y, T^n v_0)\big) \ \forall\ x, y \in C \end{aligned}$$

Since $\{T^n v_0\}$ is bounded, by applying Banach limit μ_n to both sides of the inequality, we have

$$\begin{aligned} \mu_n \alpha D_f(T^{n+1}v_0, Ty) &+ \mu_n(1-\alpha)D_f(T^n v_0, Ty) + \mu_n\gamma\big(D_f(Ty, T^{n+1}v_0) - D_f(Ty, T^n v_0)\big) \\ &\leq \mu_n\beta D_f(T^{n+1}v_0, y) + \mu_n(1-\beta)D_f(T^n v_0, y) \\ &+ \mu_n\delta\big(D_f(y, T^{n+1}v_0) - D_f(y, T^n v_0)\big) \ \forall\ y \in C. \end{aligned}$$

Since $\mu_n(x_{n+1}) = \mu_n(x_n)$, we have

$$\begin{aligned} \mu_n \alpha D_f(T^n v_0, Ty) &+ \mu_n(1-\alpha) D_f(T^n v_0, Ty) + \mu_n \gamma\big(D_f(Ty, T^n v_0) - D_f(Ty, T^n v_0)\big) \\ &\leq \mu_n \beta D_f(T^n v_0, y) + \mu_n(1-\beta) D_f(T^n v_0, y) \\ &+ \mu_n \delta\big(D_f(y, T^n v_0) - D_f(y, T^n v_0)\big) \ \forall\ y \in C \end{aligned}$$

This implies that $\mu_n \phi(T^n v_0, Ty) \leq \mu_n \phi(T^n v_0, y)\ \forall y \in C$

Since $\{T^n v_0\}$ is bounded and $\mu_n \phi(T^n v_0, Ty) \leq \mu_n \phi(T^n v_0, y)\ \forall y \in C$, then from 4.1.1 we have $A^B(T) \neq \emptyset$. This completes the proof □

Theorem 4.1.3 *Let E be a reflexive Banach space and let $f : E \to \bar{\mathbb{R}}$ be an essentially strictly convex and Gâteaux differentiable function. Let D be a nonempty, closed and convex subset of E. Let S be a semitopological semigroup with identity and $C(S)$ the Banach space of all bounded real-valued continuous functions on S with supremum norm. Let $u : S \to E$ be a continuous function such that $\{u(s) : s \in S\} \subset D$ is bounded and let μ be a mean on $C(S)$. If $g : D \to \mathbb{R}$ is defined by*

$$g(z) = \mu_s D_f(u(s), z)\ \forall\ z \in D,$$

then the mean vector z_0 of $\{u(s) : s \in S\}$ for μ_s is a unique minimizer in D such that

$$g(z_0) = min\{g(z) : z \in D\}$$

.

Proof. For a bounded net $\{u(s)\} \subset D$ and a mean on $C(S)$, we see that a function $g : D \to \mathbb{R}$ defined by

$$g(z) = \mu_s D_f(u(s), z)\ \forall z \in D$$

is well defined. This is because $u(s) \in E, z \in D \subseteq E$ and $D_f(u(s), z) = f(u(s)) - f(z) - \langle u(s) - z, \bigtriangledown f(z)\rangle$ where $f : E \to \bar{\mathbb{R}}$.

Also, from section 3 of (Takahashi et al. (2014a)) $\exists$ a mean vector z_0 of $\{u(s)\}$ for μ that is there exists $z_0 \in \bar{co}\{u(s) : s \in S\}$ such that

$$\mu_s\langle u(s), y^*\rangle = \langle z_0, y^*\rangle \ \forall \ y^* \in E^*.$$

Since D is closed and convex and $\{u(s)\} \subset D$, we have $z_0 \in D$

Using three point identity of Bregman distance, we have

$$\begin{aligned}
g(z) - g(z_0) &= \mu_s D_f(u(s), z) - \mu_s D_f(u(s), z_0) = \mu_s\Big(D_f(u(s), z) - D_f(u(s), z_0)\Big) \\
&= \mu_s\Big(D_f(u(s), z) - D_f(u(s), z) - D_f(z, z_0) \\
&- \langle u(s) - z, \bigtriangledown f(z) - \bigtriangledown f(z_0)\rangle\Big) \\
&= -\mu_s\big(D_f(z, z_0) + \langle u(s) - z, \bigtriangledown f(z) - \bigtriangledown f(z_0)\rangle\big) \\
&= -D_f(z, z_0) - \mu_s\langle u(s) - z, \bigtriangledown f(z) - \bigtriangledown f(z_0)\rangle \\
&= -D_f(z, z_0) - \mu_s\langle u(s) - z, \bigtriangledown f(z)\rangle + \mu_s\langle u(s) - z, \bigtriangledown f(z_0)\rangle \\
&= -D_f(z, z_0) - \langle z_0 - z, \bigtriangledown f(z)\rangle + \langle \bigtriangledown f(z_0), z_0 - z\rangle \\
&= -f(z) + f(z_0) + \langle \bigtriangledown f(z_0), z - z_0\rangle - \langle z_0 - z, \bigtriangledown f(z)\rangle \\
&+ \langle z_0 - z, \bigtriangledown f(z_0)\rangle \\
&= f(z_0) - f(z) - \langle z_0 - z \bigtriangledown f(z)\rangle \\
&= D_f(z_0, z) \\
Thus,\ g(z) &= D_f(z_0, z) + g(z_0) \ \forall \ z \in D
\end{aligned}$$

This implies $z_0 \in D$ is a minimizer, that is

$$g(z_0) = min\{g(z) : z \in D\}.$$

Now, suppose $u \in D$ satisfies $g(u) = g(z_0)$, then we get $g(u) = D_f(z_0, u) + g(z_0)$ and $D_f(z_0, u) = 0 \Leftrightarrow z_0 = u$ because f is an essentially strictly convex. Hence, $z_0 \in D$ is a unique minimizer. This completes the proof. □

Let E be a reflexive Banach space and C a nonempty subset of E. Let T be a mapping from C into E. Denote by $B^B(T)$, the set of skew-Bregman attractive points of T i.e.

$$B^B(T) = \{z \in E : D_f(Tx, z) \leq D_f(x, z), \forall\, x \in C\}$$

Lemma 4.1.4 *Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be Legendre. Let $T : C \subset intdomf \to intdomf$ be an operator and T^* be the conjugate operator associated with T . Then the following hold:*
(i) $B^B(T) = \bigtriangledown f^*\big(A^B(T^*)\big)$
(ii) $A^B(T) = \bigtriangledown f^*\big(B^B(T^*)\big)$.

Proof.

$$\begin{aligned}
Let\ z \in B^B(T) \ &\Leftrightarrow\ D_f(Tx, z) \leq D_f(x, z)\ \forall x \in C \\
&\Leftrightarrow\ D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown f(Tx)\big) \leq D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown fx\big) \\
&\Leftrightarrow\ D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown f T \bigtriangledown f^* \bigtriangledown fx\big) \\
&\leq\ D_{f^*}\big(\bigtriangledown f(z), \bigtriangledown fx\big) \\
&\Leftrightarrow\ D_{f^*}\big(\bigtriangledown f(z), T^* x^*\big) \\
&\leq\ D_{f^*}\big(\bigtriangledown f(z), x^*\big)\ where\ x^* = \bigtriangledown fx \in intdomf^* \\
&\Leftrightarrow\ \bigtriangledown f(z) \in A(T^*) \\
&\Leftrightarrow\ z \in \bigtriangledown f^*\big(A^B(T^*)\big)
\end{aligned}$$

Hence $B^B(T) = \bigtriangledown f^*\big(A^B(T^*)\big)$. This completes the proof. □

Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ a convex and Gâteaux differentiable function. Let $\mathcal{S} = \{T_s : s \in S\}$ be a family of mappings of C into itself, where C is a nonempty subset of E. Denote by $B^B(\mathcal{S})$, the set of skew-Bregman attractive points of T i.e.

$$B^B(\mathcal{S}) = \cap\{B^B(T_t) : t \in S\}.$$

We now obtain the following Bregman skew-attractive point theorem for semi-group of mappings in reflexive Banach space.

Theorem 4.1.5 *Let E be a reflexive Banach space and $f : E \to \bar{\mathbb{R}}$ be Legendre. Let S be a commutative semitopological semigroup with identity. Let $\mathcal{S} = \{T_s : s \in S\}$ be a continuous representation of S as mapping of C into itself such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$. Let μ be a mean on $C(S)$. Suppose that $\mu_s D_f(T_t y, T_s x) \leq D_f(y, T_s x)$ $\forall$ $y \in C$ and $t \in S$. Then $B^B(\mathcal{S}) = \cap\{B(T_t) : t \in S\} \neq \emptyset$.*

Proof. Since $S = \{T_s x : s \in S\}$ is a bounded subset of C for some $x \in C$, put $x^* = \bigtriangledown f x$ and $y^* = \bigtriangledown f y$. Then by (1.5.1),

$$\begin{aligned} T_s^* T_t^* &= \bigtriangledown f T_s \bigtriangledown f^* \bigtriangledown f T_t \bigtriangledown f^* \\ &= \bigtriangledown f T_s T_t \bigtriangledown f^* \\ &= \bigtriangledown f T_{s+t} \bigtriangledown f^* \\ &= T_{s+t}^* \qquad \forall s \in S. \end{aligned}$$

Since $\bigtriangledown f$ is continuous then for any $y^* \in \bigtriangledown f(C)$ we have,
$\|T_s^* y^* - T_t^* y^*\| = \|\bigtriangledown f T_s \bigtriangledown f^* \bigtriangledown f(y) - \bigtriangledown f T_t \bigtriangledown f^* \bigtriangledown f(y)\| = \|\bigtriangledown f T_s y - \bigtriangledown f T_t y\| \to 0$ as $s \to t$.
Therefore, $\mathcal{S}^* = \{T_s^* : s \in S\}$ is a continuous representation of S as a mapping of $\bigtriangledown f(C)$ into $intdom f^*$. Furthermore, since $\{T_s x : s \in S\}$ is bounded and

$\mu_s D_f(T_t y, T_s x) \leq D_f(y, T_s x)$ $\forall\ y \in C$ and $t \in S$, we have

$$\begin{aligned}
\mu_s D_{f^*}(T_s^* x^*, T_t^* y^*) &= \mu_s D_{f^*}(\triangledown f T_s \triangledown f^* \triangledown fx, \triangledown f T_t \triangledown f^* \triangledown fy) \\
&= \mu_s D_{f^*}(\triangledown f T_s x, \triangledown f T_t y) \\
&= \mu_s D_f(T_t y, T_s x) \leq \mu_s D_f(y, T_s x) = \mu_s D_{f^*}(\triangledown f T_s x, \triangledown f y) \\
&= \mu_s D_{f^*}(\triangledown f T_s \triangledown f^* \triangledown fx, \triangledown f y) = \mu_s D_{f^*}(T_s^* x^*, y^*)
\end{aligned}$$

$\Rightarrow \mu_s D_{f^*}(T_s^* x^*, T_t^* y^*) \leq \mu_s D_{f^*}(T_s^* x^*, y^*)$ $\forall\ y^* \in \triangledown f(C)$ and $t \in S$

Therefore, from Theorem 4.1.1 we see that

$$A^B(\mathcal{S}^*) = \cap\{A^B(T_t^*) : t \in S\} \neq \emptyset.$$

Since $\triangledown f$: $intdomf \to intdomf^*$ is a bijection and using Lemma 4.1.4 we have

$$\begin{aligned}
B^B(\mathcal{S}) &= \cap\{B^B(T_t) : t \in S\} \\
&= \cap\{\triangledown f^* A^B(T_t^*) : t \in S\} \\
&= \triangledown f^*\{\cap(A^B(T_t^*)) : t \in S\} \\
&= \triangledown f^*(A^B(\mathcal{S}^*))
\end{aligned}$$

Since $A^B(\mathcal{S}^*)$ is nonempty $\Rightarrow B^B(\mathcal{S}^*) \neq \emptyset$. This completes the proof. □

Definition 4.1.2 *Let E be a reflexive Banach space and C a nonempty subset of E. A mapping T from C into itself is called skew-generalized Bregman nonspreading mapping if $\exists\ \alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that*

$$\begin{aligned}
\alpha D_f(Ty, Tx) &+ (1-\alpha) D_f(Ty, x) + \gamma\big(D_f(Tx, Ty) - D_f(x, Ty)\big) \\
&\leq \beta D_f(y, Tx) + (1-\beta) D_f(y, x) \\
&+ \delta\big(D_f(Tx, y) - D_f(x, y)\big)\ \forall x, y \in C.
\end{aligned}$$

Theorem 4.1.6 *Let E be a reflexive Banach space and C be a nonempty subset of E. Let $f : E \to \bar{\mathbb{R}}$ be a convex and Gâteaux differentiable function and $T : C \to C$ be a skew-generalized Bregman nonspreading mapping, then $B^B(T) \neq \emptyset$ if and only if $\{T^n v_0\}$ is bounded for some $v_0 \in C$*

Proof. Let $B^B(T) \neq \emptyset$, then $D_f(Ty, u) \leq D_f(y, u)$ for all $u \in B^B(T)$ and $y \in C$. So that $D_f(T^n y, u) \leq D_f(y, u)$ $\forall n \in \mathbb{N}$ and $y \in C$. This shows that $\{T^n y : n \in \mathbb{N}\}$ is bounded $\forall y \in C$.

Conversely, Since $T : C \to C$ is a skew-generalized Bregman nonspreading mapping, then there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha D_f(Ty, Tx) &+ (1-\alpha)D_f(Ty, x) + \gamma D_f\big(D_f(Tx, Ty) - D_f(x, Ty)\big) \\ &\leq \beta D_f(y, Tx) + (1-\beta)D_f(y, x) + \delta\big(D_f(Tx, y) - D_f(x, y)\big) \end{aligned}$$

for all $x, y \in C$. Replacing x by $T^n v_0$ in the above inequality , where $n \in \mathbb{N} \cup \{0\}$, we get

$$\begin{aligned} \alpha D_f(Ty, T^{n+1}v_0) &+ (1-\alpha)D_f(Ty, T^n v_0) + \gamma\big(\phi(T^{n+1}v_0, Ty) - \phi(T^n v_0, Ty)\big) \\ &\leq \beta D_f(y, T^{n+1}v_0) + (1-\beta)D_f(y, T^n v_0) \\ &+ \delta\big(D_f(T^{n+1}v_0, y) - D_f(T^n v_0, y)\big) \ \forall\ y \in C. \end{aligned}$$

Since $\{T^n v_0\}$ is bounded, applying Banach limit μ_n to both sides of the inequality, we have

$$\begin{aligned} \mu_n \alpha D_f(Ty, T^{n+1}v_0) &+ \mu_n(1-\alpha)D_f(Ty, T^n v_0) + \mu_n\gamma\big(D_f(T^{n+1}v_0, Ty) - D_f(T^n v_0, Ty)\big) \\ &\leq \mu_n\beta D_f(y, T^{n+1}v_0) + \mu_n(1-\beta)D_f(y, T^n v_0) \\ &+ \mu_n\delta\big(D_f(T^{n+1}v_0, y) - D_f(T^n v_0, y)\big) \ \forall\ y \in C. \end{aligned}$$

Since $\mu_n(x_{n+1}) = \mu_n(x_n)$, we have

$$\begin{aligned}
\mu_n \alpha D_f(Ty, T^n v_0) &+ \mu_n(1-\alpha) D_f(Ty, T^n v_0) + \mu_n \gamma\big(D_f(T^n v_0, Ty) - D_f(T^n v_0, Ty)\big) \\
&\leq \mu_n \beta D_f(y, T^n v_0) + \mu_n \beta D_f(y, T^n v_0) + \mu_n(1-\beta) D_f(y, T^n v_0) \\
&+ \mu_n \delta\big(D_f(T^n v_0, y) - D_f(T^n v_0, y)\} \ \forall\ y \in C.
\end{aligned}$$

This implies that $\mu_n D_f(Ty, T^n v_0) \leq \mu_n D_f(y, T^n v_0)\ \forall\ y \in C$

Thus, since $\{T^n v_0\}$ is bounded and $\mu_n D_f(Ty, T^n v_0) \leq \mu_n D_f(y, T^n v_0)\ \forall y \in C$, we have from Theorem 4.1.5 that $B^B(T) \neq \emptyset$. This completes the proof. □

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

In summary, a new set of attractive points using the concept of Bregman distance $D_f(.,.)$ has been defined and the new attractive and skew-attractive point theorems (with the use of established Lemma 4.1.4) for Bregman semitopological semigroup have been established. Also, a generalized Bregman nonspreading and skew-generalized Bregman nonspreading mappings have been defined and their attractive and skew-attractive point theorems were established. All the results were established in a reflexive Banach space only which improved the results of Lin and Takahashi (2013) and Takahashi et al. (2014a).

5.2 Conclusion

The results that have been achieved at the end of this research include the establishment of:

(i) attractive and skew-attractive point theorems for Bregman semitopological semigroup in a reflexive Banach space only;

(ii) necessary and sufficient conditions for a set of finite generalized Bregman nonspreading mapping to be bounded and the existence of its attractive points.

(iii) necessary and sufficient conditions for a set of finite skew-generalized Bregman nonspreading mapping to be bounded and the existence of its skew-attractive points.

5.3 Recommendations

(i) it is recommended that the convergence of an iterative scheme for such mappings can be considered.

(ii) it is also recommended for further research, that a more general class of mappings which contains the class of generalized nonspreading mappings should be introduced.

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