

**A STUDY ON PERIODIC ORBITS IN THE GENERALIZED RESTRICTED THREE-
BODY PROBLEM**

BY

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Ph.D/SCIE/09447/2011-2012

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SUPERVISORY COMMITTEE:

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OCTOBER, 2017

DECLARATION

I declare that the work in this thesis titled “A study on periodic orbits in the generalized restricted three-body problem” has been carried out by me in the Department of Mathematics. The information derived from the literature has been duly acknowledged in the text and a list of references provided. No part of this dissertation was previously presented for another degree or diploma at any institution.

GYEGWE, JESSICA MRUMUN
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Date

CERTIFICATION

This Thesis entitled “A STUDY ON PERIODIC ORBITS IN THE GENERALIZED RESTRICTED THREE-BODY PROBLEM” by Jessica Mrumun GYEGWE, meets the regulations governing the award of the degree of Doctor of Philosophy in Mathematics of the Ahmadu Bello University, and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This work is dedicated to the memory of my late father AWO Samuel Tsuamo Begha. And to my husband Dr. A.G. Gyegwe and children Secivir and Ternamdoo. To God be all the Glory.

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ABSTRACT

In the first problem, a modification of the restricted three-body problem has been considered where the primary (more massive body) is a triaxial rigid body and the secondary (less massive body) is an oblate spheroid and study periodic motions around the collinear equilibrium points. The locations of these points are firstly determined for ten combinations of the parameters of the problem. In all ten cases, the collinear equilibrium points are found to be unstable, as in the classical problem, and the Lyapunov periodic orbits around them have been computed accurately by applying known corrector-predictor algorithms. An extensive study on the families of three-dimensional periodic orbits emanating from these points has also been done. To find suitable starting points, for all the computed families, semi-analytical solutions have been obtained, for both two- and three-dimensional cases, around the collinear equilibrium points using the Lindstedt-Poincaré method. Finally, the stability of all computed periodic orbits has been studied.

The second problem presents a third order analytic approximation solution of Lyapunov orbits around the collinear equilibrium points in the planar restricted three-body problem by utilizing the Lindstedt Poincaré method. The primaries are oblate bodies and sources of radiation pressure. The theory has been applied to the binary α -Centuari system in six cases. Also, the positions of the collinear equilibrium points have been determined numerically and the effects of the parameters concerned with these equilibrium points shown graphically.

In the third problem, an investigation of three-dimensional periodic orbits and their stability emanating from the collinear equilibrium points of the restricted three-body problem with oblate and radiating primaries is presented. A numerical simulation is done by using five binary systems: Sirius, Procyon, Luhman 16, α -Centuari and Luyten 726-8. Firstly, based on the topological degree theory, the total number of the collinear equilibrium points for the five binary systems were obtained and then, their positions were determined numerically. The

linear stability of these equilibrium points was also examined and found to be unstable in the Lyapunov sense. An analytical approximation of three-dimensional periodic solutions around them was established via the Lindstedt-Poincaré local analysis. Finally, using the analytical solution to obtain starting orbits, the families of three-dimensional periodic orbits emanating from these equilibria have been continued numerically.

Additionally, the collinear equilibrium points and periodic motion around them are studied in the framework of the restricted three-body problem where the two primaries are triaxial rigid bodies which emit radiation in the fourth problem. Firstly, the positions and stability of the collinear equilibria are studied for HD 191408, Kruger 60 and HD 155876 binary systems. Then, the planar and three-dimensional periodic motion about these points is considered. The study includes both semi-analytical and numerical determination of these motions. It is found that all families of planar periodic orbits emanating from these points terminate with asymptotic periodic orbits at the triangular equilibrium points while the corresponding families of three-dimensional periodic orbits terminate with planar periodic orbits.

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LIST OF SYMBOLS

α : Coriolis force

β : Centrifugal force

ε : Small perturbation in the Coriolis force

ε' : Small perturbation in the centrifugal force

r_1, r_2 : Distances between the infinitesimal body and the primary and secondary bodies
respectively

s_1, s_2 : Triaxiality parameters of the primary body

a_1, a_2, a_3 : Semi-axes of the primary body

R : Dimensional distance between the primary and secondary bodies

μ : Mass parameter

n : Mean motion of the primary and secondary bodies

C : Jacobian constant

V : Jacobian matrix of the solution corresponding to the vector of the initial conditions

J : Matrix containing the variational equations

a_v, b_v, c_v, d_v, s_v : Vertical stability parameters

a_h, b_h, c_h, d_h, s_h : Horizontal stability parameters

P, Q : Stability parameters of the more general case of 3D periodic orbits

Ω : The pseudo-force parameter

$L_i, i = 1, 2, 3$: Collinear equilibrium points

e : Orbital parameter

T : Period of the periodic orbit

w : Characteristic root

D : The determinant

m, m_1, m_2 : Masses of the infinitesimal, primary and secondary bodies respectively

Oxy : Two-dimensional Synodic coordinate system

$Oxyz$: Three-dimensional Synodic coordinate system

P, P_1, P_2 : Positions of the infinitesimal, primary and secondary bodies respectively in the synodic coordinate system

q_1, q_2 : Radiation pressure parameter w.r.t. the primary and secondary bodies

A_1, A_2 : Oblateness parameters of the primary and secondary bodies

ϕ and φ : Axes parallel to Ox and Oy respectively

Φ, Θ and Ξ : Axes parallel to Ox , Oy and Oz respectively

$\int$: Orbital parameter

κ : Constant

ρ_2 : Parameter used to eliminate secular terms in the course of the computations

ω : Eigenvalue

ν_1 and ν_2 : used to avoid absolute values

σ_1 and σ_2 : used to avoid absolute values

V_1 and V_2 : used to avoid absolute values

CHAPTER ONE

INTRODUCTION

1.1 General Introduction

New developments in the mathematical theory of differential equations and dynamical systems have been aided by studies being carried out in the three-body problem (3BP).

The 3BP refers to the movement of three bodies under their mutual gravitational influence. Eighteen integrals of motion are needed for the complete solution to the 3BP. Euler (Barrow-Green, 1997) was able to establish that only ten (10) integrals of motion exist and as such, the problem when considered analytically does not have a closed form solution. Six of the scalar integrals are derived from the conservation of linear momentum, and then from the conservation of total angular momentum and the conservation of energy are three and one scalar integrals of motion respectively.

The restricted three-body problem (R3BP) is a reduced form of the 3BP. The problem is termed ‘restricted’ on the basis of some assumptions that differentiate it from the general 3BP. One of such is the consideration of three bodies, two of which have finite masses and are often referred to as primaries and the third body whose mass is much smaller and negligible is often called the infinitesimal body. Another assumption which is on the basis of these mass ratios is that the motion of the primaries is not influenced by the gravitational attraction of the infinitesimal body whereas their gravitational attraction completely determines the motion of the infinitesimal body. Hence, the R3BP is to determine the motion of the infinitesimal body under the gravitational influence of the two other bodies or primaries of finite masses. In the instance whereby the primaries move in circular orbits about their common centre of mass the aforementioned problem is often referred to as the

circular restricted three-body problem (CR3BP) and when the motion of the infinitesimal body is restricted to the orbital plane of the primaries the problem is further referred to as the planar circular restricted three-body problem (PCR3BP).

The R3BP in whichever way we consider it (either circular or planar circular) is still not solvable in closed form. However, particular solutions do exist in the 3BP in order to obtain an insight into the problem. The five equilibrium solutions or libration points are particular solutions of this problem.

Joseph Louis Lagrange in 1772 showed that five equilibrium solutions or libration points exist as earlier mentioned. These consist of three collinear points and two triangular points denoted by L_1, L_2, L_3 and L_4, L_5 , respectively, as shown in Figure 1.1. In the exact positions of these equilibrium points the velocity and acceleration of the infinitesimal body when placed at any of these points is equal to zero and it would remain at rest for all time.

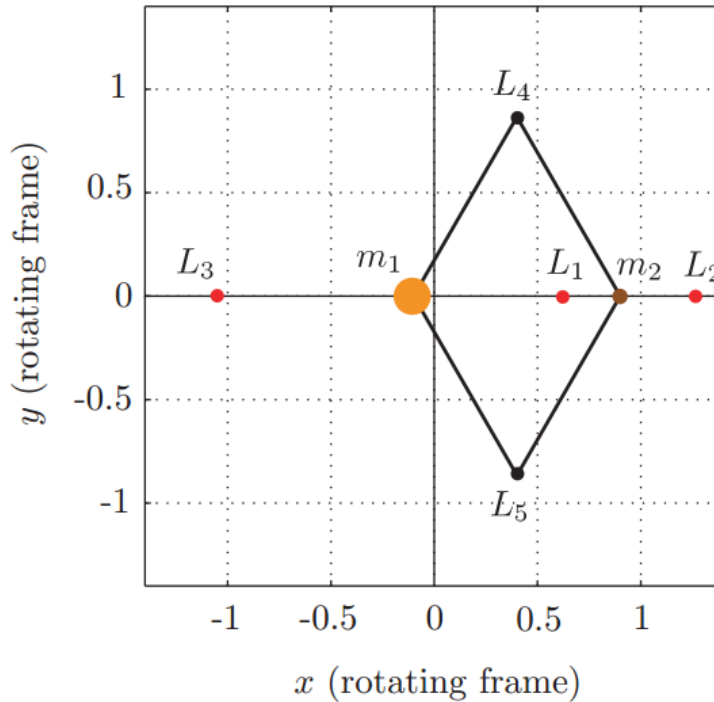


Figure 1. 1: Location of the equilibrium points.

In addition to the equilibrium points, periodic orbits are also particular solutions in the CR3BP. Periodic orbit is simply an orbit that closes upon itself in the phase space after each revolution. Theoretically and by application, it has been established that periodic orbits exist around the collinear equilibrium points. As for the stable triangular equilibrium points, no special treatment is needed once there. Specific orbital parameters cannot be used to describe these orbits as is common for typical orbits, but families of orbits exist. Goudas in 1963 was able to compute 19 families of three-dimensional periodic orbits in the CR3BP (Howell, 1998). Each of the collinear equilibrium points can be associated with a family of Lyapunov orbits. Also, each Lyapunov orbit within a family is characterized by a particular Jacobi Constant. In Figure 1.2 is shown the Lyapunov orbits around the collinear equilibrium points for the Earth-Moon system.

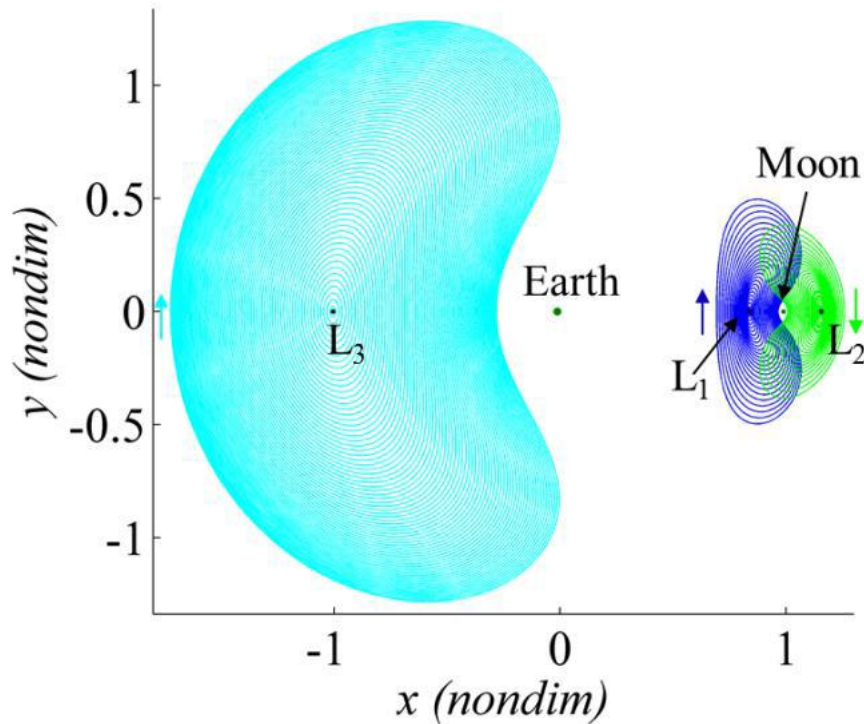


Figure 1. 2: Lyapunov orbits around the collinear equilibrium points for the Earth-Moon system (Davis (2011)).

In the R3BP, periodic orbits around the collinear equilibrium points that lie in the plane of motion of the primaries are called planar Lyapunov orbits and only one planar periodic

motion exists around each collinear equilibrium point. In the three-dimensional (3D) aspect of the R3BP, the corresponding 3D periodic orbits emanating from the collinear points are called vertical Lyapunov orbits. The Lyapunov orbits around the collinear libration points can be grouped into the families A, B, C based on Strömgren's notation for classifying periodic orbits.

On the other hand, two classes of periodic orbits also exist around the triangular libration points in the plane of motion of the primaries; the long-period orbits and the short-period orbits. The period of the long-period orbits depends on the mass ratio of the primaries while the short-period orbits have periods on the order of the periods of the primaries about their common centre of mass. (Goodrich, 1965)

The constant of integration associated with the differential equations was identified by Jacobi and was named after him as the Jacobian integral or Jacobian constant and is often denoted by the symbol C . The Jacobian integral may be used to obtain the Zero-velocity surface plots by assuming that all the velocity components are equal to zero. This surface divides the space into two regions. One region is known as the region of possible motion while the other is called the forbidden region. These regions describe the area where the infinitesimal body is allowed and not allowed to move. Figure 1.3 as shown below illustrates certain projections of the zero-velocity surfaces as examples of Zero-velocity curves for the Jacobi Constant values computed at the first and second collinear libration points. The white portions in the figure indicate the regions of the allowed motion while the shaded portions correspond to the forbidden regions for the motion of the infinitesimal body.

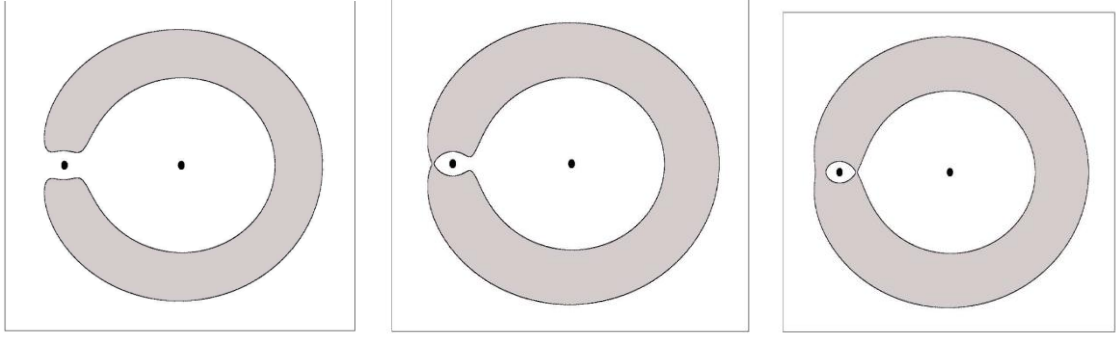


Figure 1. 3: Zero velocity curves and forbidden regions. From left to right: $C < CL_2$, $CL_2 < C < CL_1$ and $C > CL_1$ (Vila, 2007).

The knowledge of these forbidden regions provides some insight into the dynamics of the problem. Furthermore, the Jacobian Constant often serves as a method used in checking the accuracy of the calculations, especially the accuracy of the numerical integration of the differential equations. Also, in the case of the planar R3BP which is a dynamical system of two degrees of freedom the Jacobian constant reduces the initial four-dimensional phase space into a three-dimensional subspace and by considering the appropriate intersections of the 3D trajectories, e.g. with the straight line $y=0$, for certain flow, we obtain the Poincaré surface of sections which, among others, is a powerful tool for the determination of the stability of a periodic orbit as well as of chaos detection.

1.2 Statement of the Problem

Jain et al. (2006, 2009) carried out some studies on periodic orbits around the collinear libration points in the R3BP when:

- i. the smaller body is a triaxial rigid body; and
- ii. when the bigger primary is a source of radiation pressure and the secondary a triaxial rigid body,

their investigation was done for periodic orbits with respect to perturbations in the plane of motion of the primaries, which did not include the perturbations perpendicular to the plane of

motion of the primaries and lacked the numerical continuation to families of both planar and spatial periodic orbits.

In this research work, the continuation of infinitesimal orbits in the vicinity of the collinear equilibrium points to orbits of finite dimensions in the plane and perpendicular to the plane of motion of the primaries in the framework of the generalized CR3BP will be our focus. The problem is generalized in the sense that the primaries are modelled as aspherical (for example, Triaxiality and oblateness) bodies and the involvement of other perturbing forces like the influence of the Coriolis and centrifugal forces, and the radiation pressure.

1.3 Justification of the Study

The study would play a very important role of separating the classes of orbits when the classification of the totality of orbits is being considered.

The study will serve as an avenue to obtain information regarding Lyapunov orbits for nonintegrable dynamical systems when the orbit is periodic.

Since the R3BP is still not solvable in closed form, the study will serve as an opening through which the problem can be penetrated. Once a particular solution is given, a periodic solution can be found such that the difference between the two solutions is as small as it is intended for any given length of time.

1.4 Aim and Objectives of the Study

The aim of this research is to investigate periodic orbits using the Lindstedt-Poincaré local analysis and a differential correction scheme as the case may be around the collinear equilibrium points in the generalized CR3BP.

In order to achieve this aim, the researcher has set the following objectives: to

- i. compute and examine the horizontal and vertical stability of the Lyapunov periodic orbits in the vicinity of the collinear equilibrium points when the primary is a triaxial rigid body and the secondary an oblate spheroidal body together with perturbations in the Coriolis and centrifugal forces;
- ii. numerically determine the families C, A, and B (in Strömgren's notation) around L_1 , L_2 and L_3 respectively for (i).
- iii. compute horizontally Lyapunov periodic orbits around the collinear equilibrium points and make an application to a binary system when the primaries are both radiating and oblate spheroidal bodies;
- iv. compute vertically Lyapunov periodic orbits and make applications to five binary systems when the primaries are both radiating and oblate spheroidal bodies and thereafter numerically continue to the corresponding families of periodic orbits;
- v. compute and examine the horizontal and vertical stability of the Lyapunov periodic orbits in the vicinity of the collinear equilibrium points when the primaries are both radiating and triaxial spheroids together with small perturbations in the Coriolis and centrifugal forces;
- vi. numerically determine the families C, A, and B (in Strömgren's notation) around L_1 , L_2 and L_3 respectively for (v).

1.5 Thesis Organization

The organization of this thesis has been done in six chapters: Chapter ONE gives the general introduction, statement of the problem, justification of the study, aim and objectives of the study and the theoretical framework. In chapter TWO, the review of related literature based

on investigations carried out over the years on periodic orbits in the R3BP is given. The periodic motion around the collinear equilibrium points of the R3BP when the primary is a triaxial rigid body and the secondary is an oblate spheroid together with perturbations in the Coriolis and centrifugal forces was investigated in chapter THREE.

In chapter FOUR, Periodic motions around the collinear equilibrium points of the photogravitational R3BP with triaxiality, oblateness and small perturbations in the Coriolis and centrifugal forces was examined. In chapter FIVE, the results and discussions for all the problems investigated in this research work are given.

The summary, conclusion, and recommendation are given in chapter SIX.

In the next chapter, we give the literature review as well the theoretical framework as related to this research.

CHAPTER TWO

LITERATURE REVIEW

2.1 Introduction

This is a review of related literature based on investigations that have been carried out over the years on periodic orbits in the R3BP. This has been done by taking into consideration works done around the triangular equilibrium points, collinear equilibrium points, variations of the mass parameter, radiation pressure, the shapes of the bodies, a combination of radiation pressure and the shapes of the bodies, periodic horseshoe orbits, and finding periodic orbits using more contemporary methods.

Also, some of the basic formulations with regards to the preliminary ideas associated with the present research work are given in this chapter.

2.2 Periodic Orbits around the Triangular Points

Goodrich (1966) using Rabe's methods for determining members of the class of long period orbits with Jupiter and the sun as principal masses, examined the class of short-period orbits. He found out that the Jacobi constant decreases as the orbit size increases and deviates farther from the libration points. And that the periods of all orbits differ by less than 1% from the period of Jupiter's motion around the sun. Also, elliptical orbits around the sun closely represent the short-period orbits and the eccentricities of these elliptical orbits approach unity as the deviation from the libration point increase. All the orbits determined indicated instability.

Deprit and Palmore (1966) studied the family of short-period orbits originating at the equilateral center of libration in the sun-Jupiter system in which they continued first

analytically, then numerically. They found out that for all the periodic orbits obtained, the characteristic exponents were of the stable type.

Zagouras (1985) in his paper titled ‘three-dimensional periodic orbits about the triangular equilibrium points of the restricted problem of three bodies,’ improved Buck’s third-order parametric expansions for three-dimensional periodic solutions about the triangular equilibrium points of the restricted problem by fourth-order terms. He numerically computed for $\mu = 0.00095$ the corresponding family of periodic orbits, which are symmetrical w.r.t. the (x, y) plane. He found out that the family emanating from L_4 terminated at the other triangular point L_5 while it bifurcated with the family of three-dimensional periodic orbits originating at the collinear equilibrium point L_3 . And that the family consisted of both stable and unstable members. A second family of nonsymmetric three-dimensional periodic orbits was found to bifurcate from the previous one. It was also numerically determined until a collision orbit was encountered with the computations.

In their study on ‘vertical stability of periodic solutions around the triangular equilibrium points,’ Perdios and Zagouras (1991) examined the vertical stability character of the families of short and long period solutions around the triangular equilibrium points of the R3BP. For three values of the mass parameter less than or equal to the critical value of Routh (μ_R) that is, for $\mu = 0.000953875$ (Sun-Jupiter), $\mu = 0.01215$ (Earth-Moon) and $\mu = \mu_R = 0.038521$, they found that all such solutions are vertically stable.

Singh and Haruna (2014) performed a semi-analytic study of periodic orbits around the triangular equilibrium points when the participating bodies are modelled as oblate spheroids, under the effect of radiation of the main masses and small change in the Coriolis and

centrifugal forces. The orbits around these points were found to be ellipses with long and short periodic orbits. Also, Abouelmagd et al. (2015) examined the periodic structure of the R3BP considering the effect of the zonal harmonics J_2 and J_4 for the more massive body. They showed that the triangular points have long and short periodic orbits in the range $0 \leq \mu < \mu_c$.

2.3 Periodic Orbits around the Collinear Equilibrium Points

Richardson (1980) presented a third-order analytical solution for halo-type periodic motion about the collinear points of the R3BP. The Lagrangian method was used in the derivation of the three-dimensional equations of motion. The periodic solution was constructed using the method of successive approximations in conjunction with a technique similar to the Lindstedt-Poincare method with an application to the Sun-Earth system. Still on Halo orbits, Howell (1984) carried out a largely numerical study of families of three-dimensional, periodic, ‘halo’ orbits near the collinear libration points in the R3BP. The families extend from each of the libration points to the nearest primary. They exist for all values of the mass ratio μ , from 0 to 1. Most of the families contain a range of stable orbits but close to L_1 , there are no stable orbits for certain values of μ and the stable range at this libration point decreases with increasing μ , until it disappears at $\mu = 0.0573$. Near the other libration points, the orbits increase in size with increasing μ .

Hou and Lui (2009) studied the evolution details of the planar and vertical Lyapunov families around the three collinear libration points in the R3BP. After colliding orbits, members were computed. How these families evolve with increasing μ was studied. Using the linearized solution, they obtained around the collinear libration points as initial seeds, numerical shooting methods were used to produce the periodic orbit. Since members of the respective

periodic families have been obtained, they made use of the predictor-corrector algorithm to generate the whole families, respectively. They found out that three planar periodic families F_1, F_2 and F_3 were involved in the bifurcation process. The three families are the families f, h and m in Strömgren's notation and the relations between them and the Lyapunov families a, b and c were also studied.

Zhang *et al.* (2011) constructed a special kind of flow function by utilizing the symmetry of the R3BP intended to map a state on the plane of symmetry to another state that also lies in this plane. Thus, they derived a new method of periodic orbit computation based on this flow function which needs neither a starting analytical approximation nor the state transition matrix for computation. Large set of periodic orbits which have x-z plane symmetry can also be computed numerically since the method is unaffected by the nonlinearity of the dynamical system. They computed Lyapunov and halo orbits as examples.

In their study which involved using inter-satellite range data, Lui *et al.* (2014) combined autonomous orbit determination problem of a lunar satellite and a probe on some special orbits. The investigation was first carried out in the CR3BP, and then generalized to the real force model of the Earth-Moon system. In particular, they discussed two kinds of special orbits: collinear libration point orbits and distant retrograde orbits. The method employed also involved third order analytical solution and the predictor-corrector algorithm for obtaining the halo orbits and other numerical simulations. Their result showed that the orbit determination accuracy can also reach that of the observation data. Their findings also include the collinear libration point (CLP) case where halo orbits with larger out-of-plane amplitudes are recommended because of their better stability properties, better visibility between the Moon and the lunar satellite and better performance in the combined

autonomous orbit determination (CAOD) process. While in the case of the distant retrograde orbit (DRO), medium sized DROs ($\approx 45000km$) perform best in the CAOD process.

By taking the bigger primary as a radiating body, Ragos and Zagouras (1991) studied the evolution of the periodic orbits around the collinear equilibrium positions, belonging to the Strömgren families a, b and c in the Sun-Jupiter system. In particular, these families are determined when the radiation force of the more massive body is equal to one half of the gravitational attraction. The orbits closer to the equilibrium points are found by the analytic approximation methods, particularly the second order approximations, and then the next members of each family are numerically computed. The numerical aspect is facilitated by the linear predictor-corrector algorithm based on the numerical integration of the equations of motion and the first order variations. The Lyapunov stability of each periodic solution as well as the critical stability orbits of each family has been examined.

Abouelmagd *et al.* (2016) performed an analysis of the periodic motion around the collinear equilibrium points up to second order for the expansion of the equations of motion. They also showed graphically the investigations for the periodic motion.

2.4 Periodic Orbits and Variations of the Mass Parameter

By using the model of the R3BP, Ictiaroglou *et al.* (1989) investigated the three-dimensional planetary systems for $\mu = 0.001$. They numerically computed and determined the stability of families of three-dimensional periodic orbits of relatively low multiplicity at the resonances $3/1, 5/3, 3/5$ and $1/3$. The three-dimensional orbits are obtained by continuation to the third dimension of the vertical critical orbits of the corresponding planar problem.

Zagouras et al. (1996) computed two families which they describe for values of the mass parameter for which it has been found that they existed, using a procedure which they proved to be efficient. This procedure involves the process of numerical ascent from families of planar to three-dimensional periodic orbits and the subsequent ‘descent’ to the plane. They also computed two new families of three-dimensional asymmetric periodic orbits. They discovered that for smaller values of μ the aforementioned families form closed curves which shrink as μ decreases further, while for μ less than a critical value they cease to exist. Also, they found out that for two values of the mass ratio two periodic solutions have the horizontal stability parameter equal to zero. Thus, the procedure used from planar to three-dimensional families of periodic solutions can efficiently lead to the determination of an unlimited number of families of planar non-symmetric periodic orbits.

Lara and Russell (2006) investigated the influence of the mass ratio between the primaries in the behaviour of direct periodic orbits of the R3BP. They discovered that the different parts of the family g are connected through new families of three-dimensional periodic orbits. In addition, the family g' are found to be a limit case of the family g where the two parts made of egg-shaped orbits match exactly.

Russell and Lam (2007) successfully applied the dynamical systems theory to several unstable periodic orbits that circulate around Europa.

Henon (1973) studied plane periodic orbits with respect to perturbations perpendicular to the plane. He defined a ‘vertical stability index’ a_v . Thus, for vertical stability, the condition is that $|a_v| < 1$. This has been computed for periodic orbits of families $a, b, c, f, g, h, i, l, m, n$, for the mass ratio $\mu = 0.5$. He found intervals of stability and instability. Also, vertical critical orbits corresponding to $|a_v| = 1$ are computed. He showed that each vertical critical orbit

corresponds to an intersection of the family of plane periodic orbits with a family of three-dimensional periodic orbits. His consideration is based on the fact that the shapes of the primaries are of the classical model as explained in Szebehely (1967). The conditions he used for measuring stability are obtained from the eigenvalues (roots of Eq.13 as given in his paper).

2.5 Periodic Orbits with Radiation Pressure

Kalantonis et al. (2001) carried out a study which defined a method that utilizes the Poincaré map on a surface of section to compute with certainty individual members of families of periodic orbits of a given period with application to the collinear equilibrium points of the photogravitational R3BP. Their approach is based on the Poincaré map Φ on a surface of section. Thus, they stated that

$X = (x_1, x_2)^T$ is a fixed point or a periodic orbit of Φ if $X = X$ and a periodic orbit of period p if:

$$X = \Phi^p(X) \equiv \underbrace{\Phi(\dots(\Phi(X))\dots)}_{p \text{ times}}.$$

The method converges rapidly, within relatively large regions of the initial conditions and is independent of the dynamics near periodic orbits.

Perdios et al. (2002) presented an improved method for locating periodic orbits of a dynamical system of arbitrary dimension. The method is a combination of two other known methods: the characteristic bisection method (CBM) which exploits the topological degree theory to locate a periodic orbit within relatively large regions of initial conditions and the Newton's method which computes the orbit with full accuracy and quadratic convergence once its conditions for convergence have been met. The photogravitational problem of two degrees of freedom have been used as an example to demonstrate the effectiveness of the

method which is shown to surpass the CBM and Newton's method when either of them is considered alone.

Perdios (2003) determined series of horizontally critical (HC) symmetric periodic orbits of the six basic families, f, g, h, i, l, m of the photogravitational R3BP, and their vertical stability were computed. In particular, the study is restricted to the case where only the first primary is radiating, namely $q_1 \neq 1$ and $q_2 = 1$. The results obtained were compared with those of Hénon and Guyot (1970) so as to study the effect of radiation to this kind of orbits. In the same vein, Perdios and Kalantonis (2006) determined series of horizontally critical symmetric periodic orbits of the six basic families, f, g, h, i, l, m in the framework of the R3BP when the more massive primary is an oblate spheroid and computed their vertical stability.

2.6 Periodic Orbits and the Shapes of the Bodies

On their studies of 'the bifurcation of 3D periodic orbits on the plane of motion of the primaries in the R3BP with oblateness,' Perdios, E.A. *et al.* (1999) briefly described suitable numerical techniques and made some applications to produce the first few of such vertical-critical orbits of the basic families of periodic orbits of the problem, for varying mass parameter μ and fixed oblateness coefficient $A_1 = 0.005$, as well as for varying A_1 and fixed $\mu = \frac{1}{2}$. They also determined the horizontal stability of these orbits from which predictions about the stability of the branching 3D orbits were made.

In Elshaboury *et al.* (2016), additionally to the study of the number and stability of the collinear equilibria, the basic families of planar symmetric periodic orbits were considered in the space of initial conditions of the R3BP with triaxial primaries.

Beevi and Sharma (2012) and Zotos (2015) explored the effect of oblateness of Saturn in the orbital dynamics of the R3BP.

2.7 Periodic Orbits with Radiation Pressure and the Shapes of the Bodies

In their consideration of Sitnikov motions along side with other perturbing effects like the radiation pressure and oblateness of the primaries in the R3BP, Kalantonis *et al.* (2008) applied a perturbation method based on Floquet theory to study the stability of the motion. Numerically, they determined critical orbits at which families of three-dimensional periodic orbits of the same or double period bifurcate. They found out that for the classical Sitnikov problem (a special case of the R3BP where the primaries have equal masses and the third body moves perpendicular to the *Oxy* plane along the *z*-axis and passes from the centre of mass of the primaries) and its photogravitational version, the bifurcation families of double period have stable parts, while in the considered cases with oblateness, these families consist only of unstable orbits.

Jain et al. (2006, 2009) carried some studies of periodic orbits around the collinear libration points in the R3BP when:

- iii. the smaller body is a triaxial rigid body; and
- iv. when the bigger primary is a source of radiation pressure and the secondary a triaxial rigid body.

Their investigation was done for periodic orbits with respect to perturbations in the plane of motion of the primaries. The main aim is to study the effect of the triaxiality and radiation pressure of the primaries as the case maybe on the periodic orbits. Same methods for obtaining the analytical approximations and the numerical analysis as those of Ragos and Zagouras (1991) were employed in their studies.

They examined the Lyapunov stability of each periodic solution belonging to the Strömgren families A, B and C. They found out that for periodic orbits around the second and third collinear points (L_2 and L_3), when there is an increase in the triaxiality coefficient of the smaller primary, periodic orbits shift toward the origin and expand as can be seen in their figures (for the case of the triaxiality of the smaller primary). While periodic orbits around L_1 when drawn in the same manner go on shrinking and shifting away from the origin. They also, proved that the periodic orbits are unstable in all the cases under consideration.

Tsirogiannis *et al.* (2006) studied the periodic motion around the collinear equilibrium points of the R3BP when the primary is a source of radiation and the secondary is an oblate spheroid. For the computation of two and three-dimensional Liapunov families, a third-fourth order Lindstedt-Poincaré local analysis and predictor-corrector algorithms have been employed.

Their result for the two-dimensional Lyapunov families correspond to the Sun-Jupiter system mass ratio $\mu = 0.00095$ and arbitrarily chosen values for the radiation factor and the oblateness coefficient $q_1 = 0.8$ and $A_2 = 0.005$, respectively. For the Lyapunov families of three-dimensional periodic orbits around the collinear libration points, the arbitrarily chosen values are $\mu = 0.4$, $q_1 = 0.8$ and $A_2 = 0.005$. The mass ratio of the latter corresponds to that of Bray and Goudas (1967). They also computed the stability of these families. The results obtained were compared with those of the classical R3BP.

More recently, Srivastava *et al.* (2016), constructed a third-order analytic approximate solution using the Lindstedt-Poincaré method in the photogravitational CR3BP considering the Sun as a radiating source and the Earth as an oblate spheroid for computing halo orbits around the collinear equilibrium points L_1 and L_2 . They further used the well-known

differential correction and continuation scheme to compute halo orbits around their families numerically. They found out that the time period of the halo orbits increases around L_1 and L_2 as a result of the oblateness of the Earth and solar radiation pressure of the Sun.

2.8 Periodic Horseshoe Orbits

Barrabés and Mikkola (2005) computed families of symmetric periodic horseshoe orbits in the R3BP for both the planar and three-dimensional cases. They described the organization and behaviour of these families along and among the families of parameters such as the Jacobi constant or the eccentricity. Also, they determined the stability properties of individual orbits along the families. They found stable horseshoe-shaped orbit up to a high inclination of 17° .

Ollé and Barrabés (2006) studied horseshoe motion in the R3BP for different values of the mass parameter μ . On the one hand, they examined numerically families of periodic horseshoe orbits for small values μ and how these orbits are organised. And on the other hand, they studied the existence of horseshoe periodic orbits for other values of μ .

2.9 Finding Periodic Orbits by using Other Contemporary Methods

Lara and Pelàez (2002) proposed an algorithm for the numerical computation of families of periodic orbits of conservative dynamical systems with three degrees of freedom. They obtained initial conditions for specific periodic orbits pertaining to a family defined by the variation of a parameter by means of an intrinsic, three dimensional, differential, predictor-corrector algorithms. As an illustration of the robustness of the method, several families of periodic orbits of the Restricted Three-Body Problem were computed.

In their paper titled ‘doubly-symmetric periodic solutions of the spatial restricted three-body problem,’ Howison and Kenneth (2000) showed the existence of two new families of periodic solutions to the spatial R3BP. The solutions were independent of the mass ratio of the primaries and had large inclinations and were symmetric with respect to two coordinate planes. They discovered that in one family the infinitesimal particle is very far from the primaries and in this case, they called it the comet and in the other family the infinitesimal particle is very close to one of the primaries and in this case, they called it the moon.

Kotoulas and Voyatzis (2004) investigated the families of resonant periodic orbits associated with the Kuiper belt dynamics in the R3BP. They particularly considered all the first, second and third order exterior mean motion resonances with Neptune located to 50A.U. and the asymmetric resonances (beyond the 48 A.U.). Families of periodic orbits of the planar circular problem from which families of periodic orbit are generated in the planar elliptic problem and in the 3D circular ones are presented. They noticed some similar as well as different dynamical features between the various cases and discussed the relation between the distribution of the bifurcation points and the population of small bodies at the particular resonances.

Qi & Xu (2015) used the Poincaré surface of section method to investigate the long-term behavior of the spatial lunar orbits and by applying the continuation scheme they obtained the spatial lunar periodic orbit families. Jiang (2015) carried out an investigation on equilibrium points and periodic orbits in the potential field of asteroids, where it is seen that the distribution of eigenvalues of the equilibrium point confirms the topology and the stability of periodic orbits around the equilibrium point.

2.10 Theoretical Framework

Some of the basic formulations with regards to the preliminary ideas associated with the present research work are given in this section.

2.10.1 Formulation of the classical CR3BP in a synodic reference frame

Let M_1 and M_2 be the masses of the primary and secondary bodies placed at P_1 and P_2 moving in circular orbits about their common barycentre. Also, let M_3 be the mass of the infinitesimal body placed at $P_3(X,Y,Z)$ which is moving in the plane of motion of the primaries and is being influenced by the gravitational attraction of the two finite bodies while in turn, it does not have any influence on their motion.

The differential equations that govern the state of the infinitesimal body at P_3 are given by

$$M_3 \mathbf{r}_3'' = -G \frac{M_3 M_1}{r_1^3} \mathbf{r}_1 - G \frac{M_3 M_2}{r_2^3} \mathbf{r}_2 \quad (\text{Das (2014)}) \quad (2.1)$$

where r_1, r_2 are the distances between the infinitesimal body and the primary and secondary bodies respectively, and G is the gravitational constant. In the nondimensionalized form, the sum of the distances of the primaries from the barycentre is $d = r_1 + r_2$ and the sum of the masses of the primaries is $M = M_1 + M_2$. The nondimensionalized time is so chosen as to ensure that the nondimensional gravitational parameter $G = 1$, such that

$$T = \sqrt{\frac{(OP_1 + OP_2)^3}{G(M_1 + M_2)}} = \frac{d^3}{M}.$$

The OXY plane is the plane of motion of the primaries. The mean motion of the primaries w is represented as

$$w = \sqrt{\frac{G(M_1 + M_2)}{(OP_1 + OP_2)^3}}. \quad (2.2)$$

The distance between the infinitesimal body and the primaries at time t is given by

$$\begin{aligned} r_1 &= \left[X + OP_1 \cos wt^2 + YOP_1 \sin wt^2 + Z - 0^2 \right]^{\frac{1}{2}}, \\ r_2 &= \left[X - OP_2 \cos wt^2 + Y - OP_2 \sin wt^2 + Z - 0^2 \right]^{\frac{1}{2}}. \end{aligned} \quad (2.3)$$

Rewriting Eqn. (2.1) with respect to Eqn. (2.3) in terms of the scalar components, we obtain

$$\begin{aligned} \mathbf{X}'' &= -G \left[\frac{M_1}{r_1^3} X + OP_1 \cos wt + \frac{M_2}{r_2^3} X - OP_2 \cos wt \right], \\ \mathbf{Y}'' &= -G \left[\frac{M_1}{r_1^3} Y + OP_1 \sin wt + \frac{M_2}{r_2^3} Y - OP_2 \sin wt \right], \\ \mathbf{Z}'' &= -G \left[\frac{M_1 Z}{r_1^3} + \frac{M_2 Z}{r_2^3} \right]. \end{aligned} \quad (2.4)$$

Next, we nondimensionalize the equations in (2.4). Thus, in the nondimensionalized form, the mean motion is given as

$$n = wt = \sqrt{\frac{G(M_1 + M_2)}{(OP_1 + OP_2)^3}} \sqrt{(OP_1 + OP_2)^3} = 1, \quad (2.5)$$

the mass parameter is obtained as

$$\mu = \frac{M_2}{M_1 + M_2}. \quad (2.6)$$

Since the primaries M_1 and M_2 are at $P_1 \ x_1, 0$ and $P_2 \ -x_2, 0$ respectively, and the origin

is at the centre of mass, therefore

$$\begin{aligned} 1 - \mu \ x_1 + \mu \ -x_2 &= 0 \\ x_1 - \mu \ x_1 + x_2 &= 0 \\ x_1 - \mu &= 0 \\ x_1 &= \mu \end{aligned}$$

Putting $x_1 = \mu$ in $1 - \mu \ x_1 + \mu \ -x_2 = 0$, we have

$$\begin{aligned} 1 - \mu \ \mu - \mu x_2 &= 0 \\ 1 - \mu - x_2 &= 0 \end{aligned}$$

$$\therefore x_2 = 1-\mu.$$

Hence the coordinates of M_1 and M_2 are $\mu, 0$ and $\mu-1, 0$ respectively.

Thus, Eqn. (2.1) after substituting the appropriate nondimensionalized quantities become

$$\ddot{\mathbf{r}}_3 = -\frac{(1-\mu)}{r_1^3}\mathbf{r}_1 - \frac{\mu}{r_2^3}\mathbf{r}_2. \quad (2.7)$$

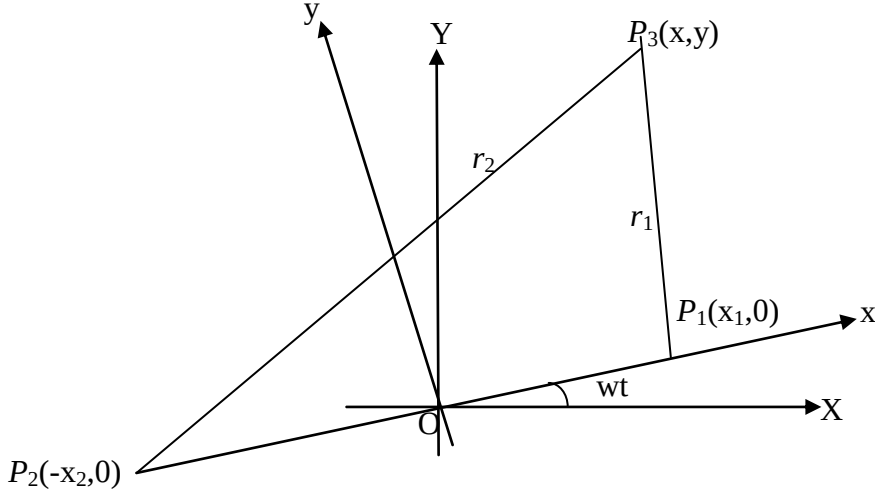


Figure 2. 1: The circular restricted three-body problem.

The position vector for P_3 in the nondimensionalized form with respect to the barycentre and written in the synodic reference frame is given by

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}. \quad (2.8)$$

The position of P_3 as related to P_1 and P_2 are govern by

$$\mathbf{r}_1 = (x-\mu)\hat{x} + y\hat{y} + z\hat{z}, \quad \text{and} \quad (2.9)$$

$$\mathbf{r}_2 = (x-\mu+1)\hat{x} + y\hat{y} + z\hat{z}, \text{ respectively.} \quad (2.10)$$

The dynamics of the infinitesimal body with respect to the synodic and inertial reference frames as shown in Figure 2.1 involves both the Coriolis and centripetal accelerations. Thus, we have the equation

$$\frac{d^2 r_3}{dt^2} = \frac{d^2 r_3}{dt^2} + \underbrace{2\omega \frac{dr_3}{dt}}_{\text{Coriolis}} + \underbrace{\omega^2 r_3}_{\text{centripetal}} \cdot (\text{Das}(2014)) \quad (2.11)$$

Rewriting Eqn. (2.10) in Cartesian coordinate component form and expanding we obtain

$$\ddot{\mathbf{r}}_3 = (\ddot{x} - 2n\dot{y} - n^2 x)\hat{x} + (\ddot{y} + 2n\dot{x} - n^2 y)\hat{y} + \ddot{z}\hat{z}. \quad (2.12)$$

We obtain three nondimensional scalar second-order differential equations that describe the motion of the infinitesimal body in the synodic reference frame OXYZ by equating Eqn. (2.12) to Eqn. (2.7) as

$$\begin{aligned} \ddot{x} - 2n\dot{y} &= n^2 x - \frac{(1-\mu)(x-\mu)}{r_1^3} - \frac{\mu(x+1-\mu)}{r_2^3}, \\ \ddot{y} - 2n\dot{x} &= n^2 y - \frac{(1-\mu)y}{r_1^3} - \frac{\mu y}{r_2^3}, \\ \ddot{z} &= -\frac{(1-\mu)z}{r_1^3} - \frac{\mu z}{r_2^3}, \end{aligned} \quad (2.13)$$

where

$$\begin{aligned} r_1 &= \sqrt{(x-\mu)^2 + y^2 + z^2}, \\ r_2 &= \sqrt{(x+1-\mu)^2 + y^2 + z^2}. \end{aligned} \quad (2.14)$$

Or in terms of the pseudo-potential term Ω which is the sum of the centripetal and gravitational accelerations, we have Eqn. (2.13) become

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \Omega_x, \\ \ddot{y} - 2\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z, \end{aligned} \quad (2.15)$$

where

$$\Omega = \frac{1}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1^3} + \frac{\mu}{r_2^3}.$$

2.10.2 Stability of the equilibrium points

When small disturbances or perturbations are introduced on the infinitesimal body placed at any of the equilibrium points which results in a displacement from the equilibrium point, and if after some time the body returns to the initial position or stays near the equilibrium point, then that equilibrium point can be said to be a stable point. Otherwise, it is unstable.

As stated in Das (2014), stability can be examined in terms of Poisson, Hill, Laplace, Lagrange and Lyapunov stability, et ce tera. In the Lyapunov sense, an equilibrium state x_{eq} is stable if for any given t_0 and any positive scalar e there exists a positive scalar

$$\delta = \delta(t_0, e),$$

such that if

$$\|x(t_0) - x_{eq}\| < \delta \quad (2.16)$$

then

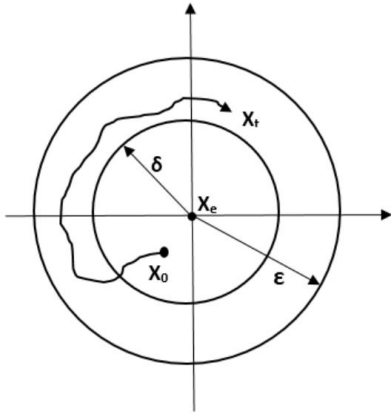
$$\|x(t; t_0, x_0) - x_{eq}\| < e, \quad (2.17)$$

for all $t \geq t_0$.

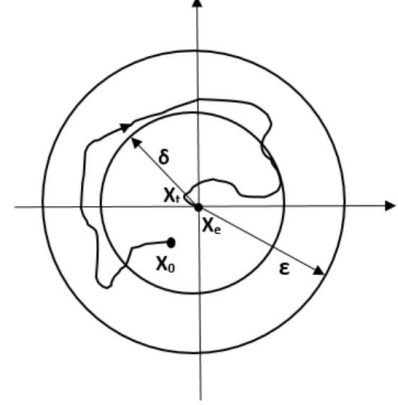
The equilibrium point is asymptotically stable if it is stable and convergent:

$$x(t; t_0, x_0) \rightarrow x_{eq} \text{ as } t \rightarrow \infty. \quad (2.18)$$

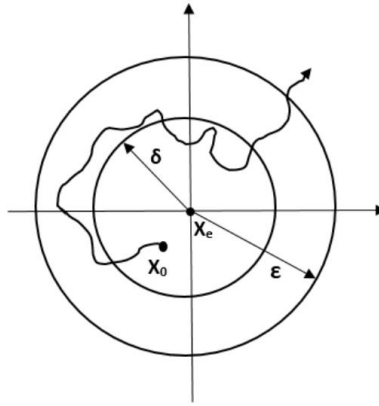
In Figure 2.2, the differences between a stable, asymptotically stable and unstable equilibrium points are shown graphically.



(a) Stable



(b) Asymptotically stable



(c) Unstable

Figure 2. 2: Differences between a stable, asymptotically stable and unstable equilibrium points (Das (2014)).

2.10.3 Computing Periodic Orbits

2.10.3.1 The Poincaré surface of section

Suppose that we are working on the planar restricted three-body problem (PR3BP), it means that we have a four-dimensional phase space $(x, y, \dot{x}, \dot{y})$. The full phase space may be reduced into a three-dimensional subspace by means of the Jacobian integral:

$$2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2) = C . \quad (2.19)$$

This is done by solving Eqn. (2.19) for a specific variable; say $\dot{y}$ for a given value of the Jacobian constant C . The final reduction of the phase space into a two-dimensional subspace becomes by taking the successive intersections of an orbit with the straight line $y = 0$ for a given direction of the flow. By choosing $\dot{y} > 0$, it means that we integrate the equations of motion for even number of orbit cuts on the x -axis always for certain flow. Thus, by choosing the initial conditions $(x_0, \dot{x}_0)$ on the surface of section, we obtain the value of $\dot{y}_0$ using the Jacobi integral. These initial conditions are of the form $(x_0, 0, \dot{x}_0, C(x_0, \dot{x}_0))$, where the Jacobian Constant is always fixed.

Integrating the equations of motion for several initial conditions, we produce the Poincaré surface of section.

2.10.3.2 Computing accurately periodic orbits on the surface of section

For a specific value of the Jacobi constant, we have to solve the following nonlinear algebraic system arising from the periodicity conditions:

$$\begin{aligned} x(x_0, \dot{x}_0) &= x_0, \\ \dot{x}(x_0, \dot{x}_0) &= \dot{x}_0, \end{aligned} \tag{2.20}$$

where $(x_0, \dot{x}_0)$ are the initial conditions on the surface of section. Thus, at the end of the numerical integration we demand that $y = 0$. It can be seen that the above system of equations does not fulfil the condition, therefore we seek appropriate corrections $\delta x_0, \delta \dot{x}_0$ such that:

$$\begin{aligned} x(x_0 + \delta x_0, \dot{x}_0 + \delta \dot{x}_0) &= x_0 + \delta x_0, \\ \dot{x}(x_0 + \delta x_0, \dot{x}_0 + \delta \dot{x}_0) &= \dot{x}_0 + \delta \dot{x}_0. \end{aligned} \tag{2.21}$$

Expanding Eqns. (2.21) into first order terms around $(x_0, \dot{x}_0)$, we have:

$$\begin{aligned}
x + \frac{\partial x}{\partial x_0} \delta x_0 + \frac{\partial x}{\partial \dot{x}_0} \delta \dot{x}_0 &= x_0 + \delta x_0, \\
\dot{x} + \frac{\partial \dot{x}}{\partial x_0} \delta x_0 + \frac{\partial \dot{x}}{\partial \dot{x}_0} \delta \dot{x}_0 &= \dot{x}_0 + \delta \dot{x}_0.
\end{aligned} \tag{2.22}$$

From Eqns. (2.22), we get the linear system:

$$\begin{aligned}
\left(\frac{\partial x}{\partial x_0} - 1 \right) \delta x_0 + \frac{\partial x}{\partial \dot{x}_0} \delta \dot{x}_0 &= x_0 - x, \\
\frac{\partial \dot{x}}{\partial x_0} \delta x_0 + \left(\frac{\partial \dot{x}}{\partial \dot{x}_0} - 1 \right) \delta \dot{x}_0 &= \dot{x}_0 - \dot{x},
\end{aligned} \tag{2.23}$$

or equivalently, we obtain the following corrector scheme with unknown δx_0 and $\delta \dot{x}_0$:

$$\begin{aligned}
a_h - 1 \delta x_0 + b_h \delta \dot{x}_0 &= x_0 - x, \\
c_h \delta x_0 + (d_h - 1) \delta \dot{x}_0 &= \dot{x}_0 - \dot{x},
\end{aligned}$$

which we have to solve iteratively until the periodic orbit will be computed with the desired accuracy. Note here that, the partial derivatives:

$$a_h = \frac{\partial x}{\partial x_0}, b_h = \frac{\partial x}{\partial \dot{x}_0}, c_h = \frac{\partial \dot{x}}{\partial x_0} \text{ and } d_h = \frac{\partial \dot{x}}{\partial \dot{x}_0},$$

involved in the corrector are the isoenergetic stability parameters (see Hénon (1965b) and Markellos (1976)), which can be computed for example with extra integrations using the following formulae:

$$\begin{aligned}
a_h &\cong \frac{x(x_0 + \varepsilon, \dot{x}_0) - x(x_0, \dot{x}_0)}{\varepsilon}, \\
b_h &\cong \frac{x(x_0, \dot{x}_0 + \varepsilon) - x(x_0, \dot{x}_0)}{\varepsilon}, \\
c_h &\cong \frac{\dot{x}(x_0 + \varepsilon, \dot{x}_0) - \dot{x}(x_0, \dot{x}_0)}{\varepsilon}, \\
d_h &\cong \frac{\dot{x}(x_0, \dot{x}_0 + \varepsilon) - \dot{x}(x_0, \dot{x}_0)}{\varepsilon},
\end{aligned} \tag{2.24}$$

where ε is a small perturbation of the initial conditions, for example $\varepsilon = 0.5 \times 10^{-6}$. Note also, that the corrector system may be solved using the well-known Cramer's rule and this

solution (with respect to δx_0 and $\delta \dot{x}_0$) is the final formulae which will be used for the accurate determination of a periodic orbit.

2.10.3.3 The computation of a Lyapunov family

In the computations for the planar Lyapunov families, we choose to integrate with intersections of the orbit with the Ox-axis, since the planar Lyapunov families consist of periodic orbits which are symmetric with the Ox-axis.

For the determination of a planar Lyapunov orbit we start the integration using initial conditions of the form $(x_0, y_0 = 0, z_0 = 0, \dot{x}_0 = 0, \dot{y}_0, \dot{z}_0 = 0)$, i.e. we start vertically from the Ox-axis, and seek its second vertical intersection with this axis (when we find the second vertical intersection this will be at the half period $T/2$ due to the mirror theorem which guarantees that the orbit will return back at its initial position). Therefore, at this intersection of the orbit with the Ox-axis, we want to hold the condition:

$$\dot{x}(x_0, \dot{y}_0) = 0, \quad (2.25)$$

since $y_0 = 0$, for planar orbits. The above condition is not true for our approximate initial conditions so we seek appropriate corrections $\Delta x_0, \Delta \dot{y}_0$ of $x_0, \dot{y}_0$, respectively, such that:

$$\dot{x}(x_0 + \Delta x_0, \dot{y}_0 + \Delta \dot{y}_0) = 0, \quad (2.26)$$

and by linearization of Eqn. (2.26), we obtain

$$\frac{\partial \dot{x}}{\partial x_0} \Delta x_0 + \frac{\partial \dot{x}}{\partial \dot{y}_0} \Delta \dot{y}_0 = -\dot{x}. \quad (2.27)$$

2.10.3.4 A simple corrector

We now keep, e.g., $\dot{y}_0$ constant which means that $\Delta \dot{y}_0 = 0$, and the corrector is obtained in the form:

$$\Delta x_0 = -\frac{\dot{x}}{\partial \dot{x} / \partial x_0}. \quad (2.28)$$

If we choose to keep x_0 constant, i.e. $\Delta x_0 = 0$, then the corresponding correction for $\dot{y}_0$ is

$$\Delta \dot{y}_0 = -\frac{\dot{x}}{\partial \dot{x} / \partial \dot{y}_0}. \quad (2.29)$$

Using the correction Δx_0 (or $\Delta \dot{y}_0$), the new initial conditions will have the form:

$$x_0 + \Delta x_0, y_0 = 0, z_0 = 0, \dot{x}_0 = 0, \dot{y}_0, \dot{z}_0 = 0.$$

We then integrate the equations of motion with these initial conditions and repeat the process until the periodicity condition (2.25) is fulfilled with our predetermined accuracy. Thus, a member of the Lyapunov family, i.e. a simple symmetric planar periodic orbit, has been determined accurately.

2.10.3.5 A simple predictor

Let now $(x_0, \dot{y}_0)$ be the initial conditions of the periodic orbit which was determined accurately in the previous step by the corrector. For a guess of the next member of the Lyapunov family we arbitrarily vary the value of, e.g., $\dot{y}_0$. We will express this by change in (2.27) the value of $\Delta \dot{y}_0$ arbitrarily with a small constant ε obtaining thus the following prediction:

$$\delta x_0 = -\frac{\dot{x} + \frac{\partial \dot{x}}{\partial \dot{y}_0} \varepsilon}{\frac{\partial \dot{x}}{\partial x_0}}. \quad (2.30)$$

Our initial conditions for the next periodic orbit will be $(x_0 + \delta x_0, \dot{y}_0)$ (all the other components will be equal to zero) and the corrector is here again applied in order to determine the next orbit accurately (as previously).

Of course, we may choose in (2.27) to change arbitrarily the value of Δx_0 with ε obtaining the prediction:

$$\delta \dot{y}_0 = - \frac{\dot{x} + \frac{\partial \dot{x}}{\partial x_0} \varepsilon}{\frac{\partial \dot{x}}{\partial \dot{y}_0}}. \quad (2.31)$$

2.10.4 Fourth order Runge-Kutta method

The commonly used methods for the integration of ordinary differential equations (ODEs) are the Runge-Kutta methods. These methods were first developed by the German mathematicians C.D.T. Runge and M.W. Kutta in the middle of the nineteenth century. The method is constructed by making an Euler-like trial step to the midpoint of the interval, and then using the values x and y at the midpoint to make the real step across the interval.

In particular, the fourth order Runge-Kutta method utilizes the mid-point method by jumping $1/4th$ of the way first, then going half-way, and then going $3/4th$ of the way and finally jumping all the way.

The standard fourth-order Runge-Kutta method takes the form:

$$\begin{aligned} k_1 &= h f(x_n, y_n), \\ k_2 &= h f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right), \\ k_3 &= h f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right), \\ k_4 &= h f(x_n + h, y_n + k_3), \\ y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5). \end{aligned} \quad (2.33)$$

2.10.5 Radiation pressure

Poynting (1903) stated that “particles such as small meteors or cosmic dust are comparably affected by gravitation and light radiation force as they approach luminous celestial bodies.”

Also, Todoran (1993) stated that “in order to take into account the influence of the radiation pressure, we shall refer here to the general equation of the resultant force acting at an exterior point, per unit mass, that is,”

$$F = F_{grav} - F_{rad} = G \frac{m}{r^2} \left(1 - \frac{F_{rad}}{F_{grav}}\right) = G \frac{qm}{r^2}, \quad (2.34)$$

with

$$q = 1 - \frac{F_{rad}}{F_{grav}}; \quad (2.35)$$

where

F_{grav} = gravitational force,

F_{rad} = force due to radiation pressure,

r = distance between the attracting and the attracted point,

G = gravitational constant,

m = stellar mass.

Equation (2.35) is often referred to as the mass reduction factor. It is further explained in AbdulRaheem and Singh (2006) that:

Since the solar radiation force F_p changes with distance by the same law as the gravitational attraction force F_g and acts opposite to it, it is possible that the force will lead to a reduction of the effective mass of the massive particle. And since this reduction depends on the properties of the particle, it is possible to speak about a reduction mass.

2.10.6 Ellipsoid

From a general perspective, an ellipsoid is a closed quadratic surface which when considered in Cartesian coordinates is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad (2.36)$$

An ellipsoid is actually a 3D representation of an ellipse. In Eqn. (2.36), the semi-axes of the ellipsoid are the lengths a, b and c and these lengths make up the semi-major axis and the semi-minor axis. In spherical coordinates, Eqn. (2.36) becomes

$$\frac{r^2 \cos^2 \alpha \sin^2 \beta}{a^2} + \frac{r^2 \sin^2 \alpha \sin^2 \beta}{b^2} + \frac{r^2 \cos^2 \beta}{c^2} = 1. \quad (2.37)$$

The following types of ellipsoids are obtained when the following conditions are applied correspondingly: for

- i. $a = b > c$, an oblate ellipsoid of revolution (oblate spheroid);
- ii. $a > b = c$, a prolate ellipsoid of revolution (prolate spheroid);
- iii. $a = b = c$, a sphere;
- iv. $a \neq b \neq c$, a triaxial ellipsoid.

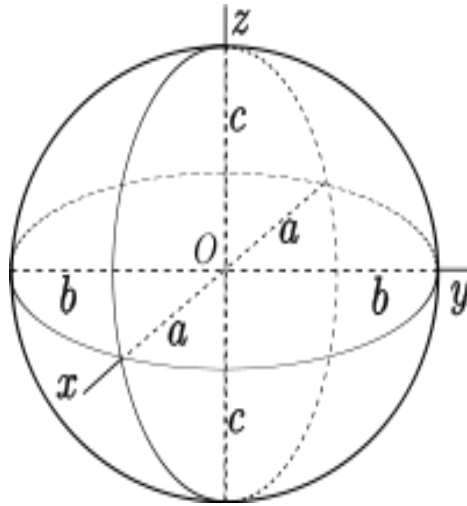


Figure 2. 3: A triaxial ellipsoid with $a \neq b \neq c$.

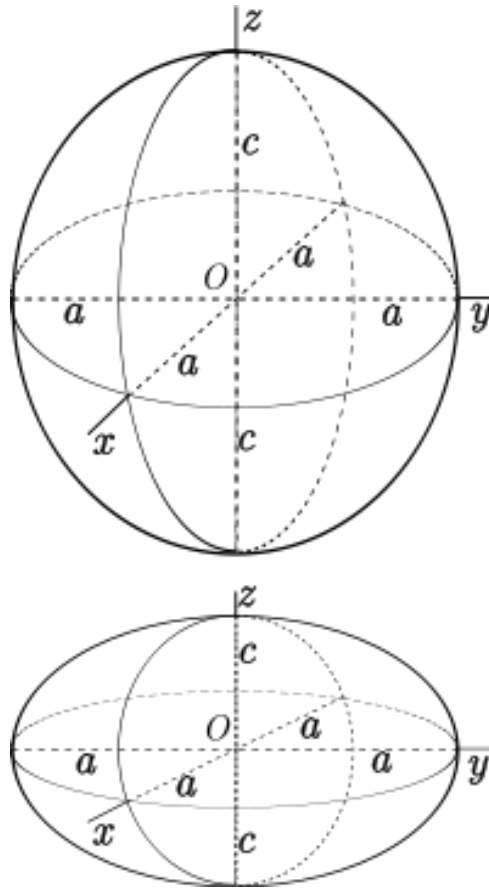


Figure 2. 4: A prolate ellipsoid with $c > a$ (top) and an oblate spheroid with $c < a$ (bottom)

2.10.7 Coriolis and centrifugal forces

2.10.7.1 Coriolis force

As explained in https://en.wikipedia.org/wiki/Coriolis_force :

The Coriolis force or inertial force/fictitious force/pseudo force comes into play when Newton's laws are transformed to a rotating frame. This force acts in the direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame and is proportional to the object's speed in the rotating frame.

The vector formula for the magnitude and direction of the Coriolis acceleration is derived through vector analysis and is

$$\mathbf{a}_c = 2\boldsymbol{\Omega} \times \mathbf{v}, \quad (2.38)$$

where $\mathbf{a}_c$ is the acceleration of the particle in the rotating system, $\mathbf{v}$ is the velocity of the particle with respect to the rotating system and $\boldsymbol{\Omega}$ is the angular velocity vector having magnitude equal to the rotation rate ω , with direction along the axis of rotation of the rotating reference frame, and the $\times$ symbol represents the cross product operator.

Multiplying Eqn. (2.38) by the mass of the relevant object produces the Coriolis force:

$$\mathbf{F}_c = -2m\boldsymbol{\Omega} \times \mathbf{v} \quad (2.39)$$

2.10.7.2 Centrifugal force

The centrifugal force is the pseudo-force or fictitious force that pulls outwards along all radii of the circle as objects move in circular motion. This force is proportional to the distance of the body from the axis of the rotating frame.

The centrifugal force could be used to create artificial gravity in a large doughnut-shaped space station.

2.10.8 Binary star systems

A binary star is a star system which consists of two stars orbiting around their common barycentre.

As further explained in https://en.wikipedia.org/wiki/Binary_star :

The binary star systems are very important in astrophysics because calculations of their orbits allow the masses of their component stars to be directly determined, which in turn allows other stellar parameters, such as radius and density, to be indirectly estimated. This also determines an empirical mass-luminosity relationship (MLR) from which the masses of single stars can be estimated.

Binary stars are classified into four types according to the way in which they are observed:

- i. Visually, by observation;
- ii. Spectroscopically, by period changes in spectral lines;
- iii. Photometrically, by changes in brightness caused by an eclipse;
- iv. Astrometrically, by measuring a deviation in a star's position caused by an unseen companion.

Any star can belong to several of these classes; for example, several spectroscopic binaries are also eclipsing binaries.

The components of binary stars denoted by the suffixes A and B appended to the system's designation, A denoting the primary and B the secondary. The suffix AB may be used to denote the pair.

2.10.9 Romberg integration

The Romberg integration method (Romberg (1955)) is named after Werner Romberg (1909-2003). This method is an interpolation technique that applies the trapezium rule or the rectangle rule (midpoint rule) on the Newton-Cotes formula.

For this method to be efficient, the function under consideration must be sufficiently differentiable. That is, for smooth $f(x)$, with

$$I(f) = \int_a^b f(x)dx, \quad (2.40)$$

using series,

- Trapezoidal rule $T_n(f)$ has error expansion

$$I(f) = T_n(f) + c_1 h^2 + c_2 h^4 + c_3 h^6 + \dots \quad (2.41)$$

for constants $c_1, c_2, \dots$ independent of h .

- Midpoint rule $M_n(f)$ has error expansion

$$I(f) = M_n(f) + d_1 h^2 + d_2 h^4 + d_3 h^6 + \dots \quad (2.42)$$

for constants $d_1, d_2, \dots$ independent of h .

Then for the Romberg integration using Richardson extrapolation, let

$$R_{1,1} = T_1(f), \quad h_k = \frac{b-a}{2^{k-1}},$$

$$R_{k-1} = \frac{1}{2} \left[R_{k-1,1} + h_{k-1} \sum_{i=1}^{2^{k-2}} f(a + (2i-1)h_k) \right] \equiv T_{2^{k-1}}(f), \quad k = 2, 3, \dots \quad (2.43)$$

For the extrapolation formulas, we have

$$\begin{aligned} R_{k-2} &= R_{k-1} + (R_{k-1} - R_{k-1,1})/4 - 1, \quad k = 2, 3, \dots \\ R_{k,j} &= R_{k,j-1} + (R_{k,j-1} - R_{k-1,j-1})/4^{j-1} - 1, \quad k = j, j+1, \dots \end{aligned} \quad (2.44)$$

(www.math.wsu.edu/faculty/genz/448/lessons/15034.pdf)

CHAPTER THREE

PERIODIC MOTIONS AROUND THE COLLINEAR EQUILIBRIUM POINTS OF THE PERTURBED R3BP WHEN THE PRIMARY IS A TRIAXIAL RIGID BODY AND THE SECONDARY IS AN OBLATE SPHEROID

3.1 Introduction

In this chapter, the equations of motion and variation are given and the parameters which define the stability of periodic orbits are described.

A second order semi-analytical solution for both 2D and 3D periodic orbits are being constructed around the collinear equilibrium points in the R3BP when the primary is a triaxial rigid body and the secondary an oblate spheroid taking into account small perturbations in the Coriolis and centrifugal forces (Singh and Begha (2011) where we have the same model but theirs was done only for the 2D equations of motion).

3.2 Equations of Motion and Variation and the Stability Parameters

The equations of motion of the infinitesimal particle in the barycentric, rotating and dimensionless coordinate system (X, Y, Z) in the phase space $X_1, X_2, X_3, X_4, X_5, X_6$ are ((Sharma & SubbaRao (1974) where we have adopted their model for the secondary body and (Singh and Begha, 2011))

$$\dot{X}_i = f_i(X_1, X_2, \dots, X_6), \quad i = 1, 2, \dots, 6 \quad (3.1)$$

with

$$f_1 = X_4, \quad f_2 = X_5, \quad f_3 = X_6,$$

$$\begin{aligned}
f_4 &= 2n\alpha X_5 + n^2 \beta X_1 - \frac{(1-\mu)(X_1-\mu)}{r_1^3} - \frac{\mu(X_1+1-\mu)}{r_2^3} - \frac{3(1-\mu)(2s_1-s_2)(X_1-\mu)}{2r_1^5} \\
&\quad - \frac{3\mu(X_1+1-\mu)A_2}{2r_2^5} + \frac{15(1-\mu)(s_1-s_2)(X_1-\mu)X_2^2}{2r_1^7}, \\
f_5 &= -2n\alpha X_4 + n^2 \beta X_2 - \frac{(1-\mu)X_2}{r_1^3} - \frac{\mu X_2}{r_2^3} - \frac{3(1-\mu)(2s_1-s_2)X_2}{2r_1^5} - \frac{3(1-\mu)(s_1-s_2)X_2}{r_1^5} \\
&\quad - \frac{3\mu A_2 X_2}{2r_2^5} + \frac{15(1-\mu)(s_1-s_2)X_2^3}{2r_1^7}, \\
f_6 &= -\frac{(1-\mu)X_3}{r_1^3} - \frac{\mu X_3}{r_2^3} - \frac{3(1-\mu)(2s_1-s_2)X_3}{2r_1^5} + \frac{15(1-\mu)(s_1-s_2)X_2^2 X_3}{2r_1^7} - \frac{3\mu A_2 X_2}{2r_2^5} \\
&\quad + \frac{15(1-\mu)s_1 X_2^3}{2r_1^7} - \frac{3(1-\mu)s_1 X_3}{r_1^5} + \frac{15\mu A_2 X_2^3}{2r_2^7} - \frac{3\mu A_2 X_3}{r_2^5},
\end{aligned} \tag{3.2}$$

where $X_1 = X$, $X_2 = Y$, $X_3 = Z$, $X_4 = \dot{X}$, $X_5 = \dot{Y}$, $X_6 = \dot{Z}$. Also, $r_1 = \sqrt{(X_1 - \mu)^2 + X_2^2 + X_3^2}$

and $r_2 = \sqrt{(X_1 + 1 - \mu)^2 + X_2^2 + X_3^2}$ are the distances of the third body of negligible mass from

the two primaries, $\mu = \frac{m_2}{m_1 + m_2} \leq \frac{1}{2}$ is the mass parameter with m_1, m_2 being the masses of

the primary and secondary, respectively. Additionally, $s_1 = \frac{a_1^2 - a_3^2}{5R^2}$ and $s_2 = \frac{a_2^2 - a_3^2}{5R^2}$, with

$s_1, s_2 \ll 1$, where a_1, a_2 and a_3 are the semi-axes of the larger primary and R is the

dimensional distance between the primaries. Note that, only the first order terms of s_1, s_2 have

been retained in this study. Small perturbations ε and ε' are introduced with the help of

parameters α and β in the Coriolis and centrifugal forces, respectively. Such that,

$$\begin{aligned}
\alpha &= 1 + \varepsilon, \quad \varepsilon \ll 1, \\
\beta &= 1 + \varepsilon', \quad \varepsilon' \ll 1.
\end{aligned}$$

The mean motion of the primaries is:

$$n = \sqrt{1 + \frac{3}{2}(2s_1 - s_2) + \frac{3}{2}A_2}.$$

Finally, the equations of motion (3.1) admit the Jacobian integral:

$$F(X_1, X_2, X_3, X_4, X_5, X_6) = n^2 \beta (X_1^2 + X_2^2) + \frac{2(1-\mu)}{r_1} + \frac{2\mu}{r_2} + \frac{(1-\mu)(2s_1-s_2)}{r_1^3} - \frac{3(1-\mu)(s_1-s_2)X_2^2}{r_1^5} \\ + \frac{\mu A_2}{r_2^3} - \frac{3(1-\mu)s_1 X_3^2}{r_1^5} - \frac{3\mu A_2 X_3^2}{r_2^5} - X_4^2 - X_5^2 - X_6^2 = C,$$

where C is the Jacobi constant.

The coordinates of the infinitesimal particle in the phase space $X_1, X_2, \dots, X_6$ depend uniquely, along any solutions, with the initial condition $X_{01}, X_{02}, \dots, X_{06}$ and the time t i.e.

$\dot{X}_i = X_i(X_{01}, X_{02}, \dots, X_{06}, t)$, $i = 1, 2, \dots, 6$. The partial derivatives w.r.t. the initial conditions satisfying the equations of variation (see also Ragos and Zagouras (1991), Jain et al. (2006; 2009)) are:

$$\frac{d}{dt} \left(\frac{\partial X_i}{\partial X_{0j}} \right) = \sum_{k=1}^6 \frac{\partial f_i}{\partial X_k} \frac{\partial X_k}{\partial X_{0j}}, \quad i, j = 1, 2, \dots, 6.$$

(3.3)

Or by using matrix notation,

$$\frac{dV}{dt} = JV,$$

where

$$V = V(X_0; t) = \frac{\partial X_i}{\partial X_{0j}}, \quad i, j = 1, 2, \dots, 6,$$

and

$$J = J(X) = \frac{\partial f_i}{\partial X_j}, \quad i, j = 1, 2, \dots, 6.$$

Thus, Matrix $V(X_0; t)$ is the Jacobian matrix of the solution corresponding to the vector of the initial conditions and it has unity determinant for each value of time t . This property can be used for controlling the correctness of the variational equations. While Matrix J is as follows:

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{\partial f_4}{\partial X_1} & \frac{\partial f_4}{\partial X_2} & \frac{\partial f_4}{\partial X_3} & 0 & 2 & 0 \\ \frac{\partial f_5}{\partial X_1} & \frac{\partial f_5}{\partial X_2} & \frac{\partial f_5}{\partial X_3} & -2 & 0 & 0 \\ \frac{\partial f_6}{\partial X_1} & \frac{\partial f_6}{\partial X_2} & \frac{\partial f_6}{\partial X_3} & 0 & 0 & 0 \end{pmatrix}.$$

The partial derivatives involved in this matrix are:

$$\begin{aligned} f_{41} = \frac{\partial f_4}{\partial X_1} = & \beta n^2 - \frac{1-\mu}{r_1^3} - \frac{3A_2\mu}{r_2^5} - \frac{\mu}{r_2^3} - \frac{3}{2r_1^5} \frac{1-\mu}{2s_1-s_2} + \frac{3}{r_1^5} \frac{1-\mu}{x_1-\mu}^2 \\ & + \frac{15}{2r_1^7} \frac{1-\mu}{2s_1-s_2} \frac{(x_1-\mu)^2}{(X_1+1-\mu)^2} + \frac{15A_2\mu}{2r_2^7} \frac{2s_1-s_2}{(X_1+1-\mu)^2} + \frac{3\mu}{r_1^5} \frac{X_1+1-\mu}{X_1+1-\mu}^2 \\ & + \frac{15}{2r_1^7} \frac{1-\mu}{s_1-s_2} \frac{X_2^2}{X_2^2} - \frac{105}{2r_1^9} \frac{1-\mu}{s_1-s_2} \frac{(X_1-\mu)^2 X_2^2}{(X_1-\mu)^2 X_2^2} + \frac{15A_2\mu X_3^2}{r_2^7} + \frac{15(1-\mu)s_1 X_3^2}{r_1^7} \\ & - \frac{105}{2r_1^9} \frac{(1-\mu)(X_1-\mu)X_3^2}{(X_1-\mu)X_3^2} - \frac{105A_2\mu(X_1+1-\mu)X_3^2}{2r_2^9}, \end{aligned}$$

$$\begin{aligned} f_{42} = \frac{\partial f_4}{\partial X_2} = & \frac{15A_2\mu(X_1+1-\mu)X_2}{2r_2^7} + \frac{3\mu(X_1+1-\mu)X_2}{r_2^5} + \frac{3}{r_1^5} \frac{1-\mu}{X_1-\mu}^2 X_2 + \frac{15}{r_1^7} \frac{1-\mu}{s_1-s_2} \frac{(X_1-\mu)X_2}{(X_1-\mu)X_2} \\ & + \frac{15}{2r_1^7} \frac{1-\mu}{2s_1-s_2} \frac{(X_1-\mu)X_2}{(X_1-\mu)X_2} - \frac{105}{2r_1^9} \frac{1-\mu}{s_1-s_2} \frac{(X_1-\mu)X_2^3}{(X_1-\mu)X_2^3} - \frac{105A_2\mu(X_1+1-\mu)X_2 X_3^2}{2r_2^9} \\ & - \frac{105}{2r_1^9} \frac{1-\mu}{s_1} \frac{(X_1-\mu)X_2 X_3^2}{(X_1-\mu)X_2 X_3^2}, \end{aligned}$$

$$\begin{aligned} f_{43} = \frac{\partial f_4}{\partial X_3} = & \frac{45A_2\mu(X_1+1-\mu)X_3}{2r_2^7} + \frac{3\mu(X_1+1-\mu)X_3}{r_2^5} + \frac{3}{r_1^5} \frac{1-\mu}{X_1-\mu} \frac{X_3}{X_3} + \frac{15}{r_1^7} \frac{1-\mu}{s_1-s_2} \frac{(X_1-\mu)X_3}{(X_1-\mu)X_3} \\ & + \frac{15}{2r_1^7} \frac{1-\mu}{2s_1-s_2} \frac{(X_1-\mu)X_3}{(X_1-\mu)X_3} - \frac{105}{2r_1^9} \frac{1-\mu}{s_1-s_2} \frac{(X_1-\mu)X_2^3 X_3}{(X_1-\mu)X_2^3 X_3} - \frac{105A_2\mu(X_1+1-\mu)X_3^3}{2r_2^9} \\ & - \frac{105}{2r_1^9} \frac{1-\mu}{s_1} \frac{(X_1-\mu)X_3^3}{(X_1-\mu)X_3^3}, \end{aligned}$$

$$f_{51} = f_{43},$$

$$f_{52} = \frac{\partial f_5}{\partial X_2} = \beta n^2 - \frac{105}{2r_1^9} \frac{1-\mu}{s_1-s_2} (X_1-\mu)X_2^4 - \frac{105}{2r_1^9} \frac{1-\mu}{s_1} X_2^2 X_3^2$$

$$+ \frac{75}{2r_1^7} \frac{1-\mu}{(s_1-s_2)} X_2^2 + \frac{15}{2r_1^7} \frac{1-\mu}{2s_1-s_2} X_2^2 + \frac{15}{2r_1^7} \frac{1-\mu}{s_1} X_3^2$$

$$- \frac{3}{r_1^5} \frac{1-\mu}{s_1-s_2} - \frac{3}{2r_1^5} \frac{1-\mu}{2s_1-s_2} + \frac{3}{r_1^5} \frac{1-\mu}{X_2^2} - \frac{1-\mu}{r_1^3}$$

$$- \frac{105A_2\mu X_2^2 X_3^2}{2r_2^9} + \frac{15A_2\mu X_2^2}{2r_2^7} + \frac{15A_2\mu X_3^2}{2r_2^7} - \frac{3A_2\mu}{2r_2^5} + \frac{3\mu X_2^2}{r_2^5} - \frac{\mu}{r_2^3},$$

$$f_{53} = \frac{\partial f_5}{\partial X_3} = -\frac{105}{2r_1^9} \frac{1-\mu}{s_1-s_2} X_2^3 X_3 - \frac{105}{2r_1^9} \frac{1-\mu}{s_1} X_2 X_3^3 + \frac{15}{r_1^7} \frac{1-\mu}{s_1} X_2 X_3$$

$$+ \frac{15}{r_1^7} \frac{1-\mu}{s_1} X_2 X_3 + \frac{15}{r_1^7} \frac{1-\mu}{s_1-s_2} X_2 X_3 + \frac{15}{2r_1^7} \frac{1-\mu}{2s_1-s_2} X_2 X_3$$

$$+ \frac{3}{r_1^5} \frac{1-\mu}{X_2 X_3} - \frac{105A_2\mu X_2 X_3^3}{2r_2^9} + \frac{45A_2\mu X_2 X_3}{2r_2^7} - \frac{3\mu X_2 X_3}{r_2^5},$$

$$f_{61} = f_{51} = f_{43}, \quad f_{62} = f_{53},$$

$$f_{63} = \frac{\partial f_6}{\partial X_3} = -\frac{105}{2r_1^9} \frac{1-\mu}{s_1-s_2} X_2^2 X_3^2 - \frac{105}{2r_1^9} \frac{1-\mu}{s_1} X_3^4 + \frac{15}{2r_1^7} \frac{1-\mu}{s_1-s_2} X_2^2 + \frac{75}{2r_1^7} \frac{1-\mu}{s_1} X_3^2$$

$$+ \frac{15}{2r_1^7} \frac{1-\mu}{2s_1-s_2} X_3^2 - \frac{3}{r_1^5} \frac{1-\mu}{s_1} - \frac{3}{2r_1^5} \frac{1-\mu}{2s_1-s_2} + \frac{3}{r_1^5} \frac{1-\mu}{X_3^2} - \frac{1-\mu}{r_1^3} - \frac{105A_2\mu X_3^4}{2r_2^9} + \frac{45A_2\mu X_3^2}{r_2^7}$$

$$- \frac{9A_2\mu}{2r_2^5} + \frac{3\mu X_3^2}{r_2^5} - \frac{\mu}{r_2^3}.$$

If we denote the variation $\partial X_i / \partial X_{0j}$ by v_{ij} and $\partial f_i / \partial X_{0j}$ by f_{ij} we then can write these equations more explicitly as following:

$$\dot{v}_{ij} = v_{(i+3)j}, \quad i=1,2,3, \quad j=1,2,\dots,6,$$

$$\dot{v}_{ij} = f_{i1}v_{1j} + f_{i2}v_{2j} + f_{i3}v_{3j} + f_{i4}v_{4j} + f_{i5}v_{5j} + f_{i6}v_{6j}, \quad i=4,5,6, \quad j=1,2,\dots,6.$$

The following partial derivatives involved in the variational equations:

$$a_v = \frac{\partial X_3}{\partial X_{03}} = v_{33}, \quad b_v = \frac{\partial X_3}{\partial X_{06}} = v_{36}, \quad c_v = \frac{\partial X_6}{\partial X_{03}} = v_{63}, \quad d_v = \frac{\partial X_6}{\partial X_{06}} = v_{66},$$

describe the stability properties of the perturbations in the X_3, X_6 components of the solution vector in phase-space and these quantities were named by Hénon (1973) as “vertical stability

parameters” of a periodic orbit. On the other hand, the “horizontal stability parameters” a_h, b_h, c_h and d_h (as these were defined and named also by Hénon, 1965) which describe the stability of a periodic orbit under perturbations in the plane may be determined with the accuracy of the numerical integration by integrating the equations of motions simultaneously with the equations of variation using the following formulae introduced by Markellos (1976):

$$a_h = \frac{\partial X_1}{\partial X_{01}} + \frac{\partial X_1}{\partial X_{05}} D_1^5 + D_4^5 \left(\frac{\partial X_2}{\partial X_{01}} + \frac{\partial X_2}{\partial X_{05}} D_1^5 \right),$$

$$b_h = \frac{\partial X_1}{\partial X_{04}} + \frac{\partial X_1}{\partial X_{05}} D_4^5 + D_4^5 \left(\frac{\partial X_2}{\partial X_{04}} + \frac{\partial X_2}{\partial X_{05}} D_4^5 \right),$$

$$c_h = \frac{\partial X_4}{\partial X_{01}} + \frac{\partial X_4}{\partial X_{05}} D_1^5 - \frac{f_{10}}{X_{05}} \left(\frac{\partial X_2}{\partial X_{01}} + \frac{\partial X_2}{\partial X_{05}} D_1^5 \right),$$

$$d_h = \frac{\partial X_4}{\partial X_{04}} + \frac{\partial X_4}{\partial X_{05}} D_4^5 - \frac{f_{10}}{X_{05}} \left(\frac{\partial X_2}{\partial X_{04}} + \frac{\partial X_2}{\partial X_{05}} D_4^5 \right),$$

where:

$$f_{10} = f_1(t=0), \quad D_1^5 = \frac{1}{2X_{05}} \frac{\partial F}{\partial X_{01}}, \quad D_4^5 = -\frac{X_{04}}{X_{05}}.$$

Note that, if the planar periodic solution is symmetric w.r.t. the Ox-axis, the above formulae are simplified since $D_4^5 = X_{04} = 0$.

In order for a periodic orbit to be stable both in-plane and out-of-plane perturbations it must simultaneously fulfil the inequalities $|s_h| < 1$ and $|s_v| < 1$ where:

$$s_h = \frac{a_h + d_h}{2}, \quad s_v = \frac{a_v + d_v}{2}.$$

In the case of symmetric periodic orbits $a_h = d_h$ and $a_v = d_v$, and stability is established if both $|a_h| < 1$ and $|a_v| < 1$ hold. If only one of these inequalities hold the orbit is considered

horizontally or vertically stable, respectively. Bray and Goudas (1967) introduced the quantities P and Q for the study of the more general case of 3D periodic orbits. Thus, a family of 3D orbits is stable if $|P| < 2$ and $|Q| < 2$ (see also Tsirogiannis et al. 2006).

3.3 Determination of the Collinear Equilibrium Points for the model with triaxial and oblate primaries

The positions of the collinear equilibrium points L_j , $j = 1, 2, 3$, can be obtained by solving for X_1 in the following non-linear algebraic equation arising from the equations of motion (3.1) for zero velocity and acceleration components, i.e. $\dot{X}_3 = \dot{X}_4 = \dot{X}_5 = 0$, as well as $X_2 = X_3 = 0$:

$$n^2 \beta X_1 - \frac{(1-\mu)(X_1-\mu)}{|X_1-\mu|^3} - \frac{\mu(X_1+1-\mu)}{|X_1-\mu|^3} - \frac{3(1-\mu)(2s_1-s_2)(X_1-\mu)}{2|X_1-\mu|^5} - \frac{3\mu(X_1+1-\mu)A_2}{2|X_1+1-\mu|^5} = 0. \quad (3.4)$$

Proceeding as elaborated in Singh and Begha (2011) we have solved numerically the above equation for different values of the parameters of the problem and three such points have been found to exist in the intervals $(-\infty, -1+\mu)$, $(-1+\mu, \mu)$ and $(\mu, +\infty)$, named L_1 , L_2 and L_3 , respectively (see Szebehely, 1967). These equilibrium points are shown in Table 5.1 of this thesis.

3.4 Motion around the Collinear Equilibrium Points for model with triaxial and oblate primaries

The dynamics around the collinear equilibrium points are explored in this section.

3.4.1 Second order Taylor series expansions of the equations of motion for the model with triaxial and oblate primaries

Representing the equations of motion as:

$$\begin{aligned}\dot{X}_4 - 2n\alpha X_5 &= \Omega_{X_1}, \\ \dot{X}_5 - 2n\alpha X_4 &= \Omega_{X_2},\end{aligned}\tag{3.5}$$

where

$$\Omega = \frac{n^2\beta}{2}(X_1^2 + X_2^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{1-\mu}{2r_1^3}(2s_1 - s_2) - \frac{3(1-\mu)}{2r_1^5}(s_1 - s_2)X_2^2 + \frac{\mu A_2}{2r_2^3},\tag{3.6}$$

and differentiating partially Eqn. (3.6) w.r.t. X_1 and X_2 , we get

$$\begin{aligned}\Omega_{X_1} &= n^2\beta X_1 - \frac{1-\mu}{r_1^3} \frac{X_1 - \mu}{r_1} - \frac{\mu}{r_2^3} \frac{X_1 + 1 - \mu}{r_2} - \frac{3}{2r_1^5} \frac{1-\mu}{r_1} \frac{X_1 - \mu}{r_1} (2s_1 - s_2) + \\ &\quad \frac{15}{2r_1^7} \frac{1-\mu}{r_1} \frac{X_1 - \mu}{r_1} \frac{X_2^2}{r_2} (s_1 - s_2) - \frac{3\mu}{2r_2^5} \frac{X_1 + 1 - \mu}{r_2} A_2,\end{aligned}\tag{3.7}$$

and

$$\Omega_{X_2} = n^2\beta X_2 - \frac{1-\mu}{r_1^3} \frac{X_2}{r_1} - \frac{\mu X_2}{r_2^3} - \frac{3}{2r_1^5} \frac{1-\mu}{r_1} \frac{2s_1 - s_2}{r_1} \frac{X_2}{r_2} - 3 \frac{1-\mu}{r_1} (s_1 - s_2) \left(\frac{2X_2}{2r_1^5} - \frac{5X_2^3}{2r_1^7} \right) - \frac{3\mu A_2 X_2}{2r_2^5}.\tag{3.8}$$

Next, we set $X_1 = X_{L_i} + x_1$ and $X_2 = x_2$ in Eqn. (3.5), where $X_{L_i}, i = 1, 2, 3$ is the position of the collinear equilibrium points.

Thus, Eqns. (3.5) become

$$\begin{aligned}\ddot{x}_1 - 2n\alpha \dot{x}_2 &= \Omega_{x_1}, \\ \ddot{x}_2 - 2n\alpha \dot{x}_1 &= \Omega_{x_2}.\end{aligned}\tag{3.9}$$

We expand the RHS (right hand side) of Eqn. (3.9) up to the second order using the Taylor series expansion around (X_L, Y_L) . We proceed as

$$\begin{aligned}\Omega_{x_1}(X_L + x_1, x_2) &= \Omega_{x_1}(X_L, Y_L) + x_1 \Omega_{x_1 x_1}(X_L, Y_L) + x_2 \Omega_{x_1 x_2}(X_L, Y_L) + \frac{1}{2} x_1^2 \Omega_{x_1 x_1 x_1}(X_L, Y_L) + x_1 x_2 \Omega_{x_1 x_2 x_1}(X_L, Y_L) \\ &\quad + \frac{1}{2} x_2^2 \Omega_{x_2 x_2 x_1}(X_L, Y_L),\end{aligned}$$

and

$$\begin{aligned}\Omega_{x_2}(X_L + x_1, x_2) &= \Omega_{x_2}(X_L, Y_L) + x_1 \Omega_{x_1 x_2}(X_L, Y_L) + x_2 \Omega_{x_2 x_2}(X_L, Y_L) + \frac{1}{2} x_1^2 \Omega_{x_1 x_1 x_2}(X_L, Y_L) + x_1 x_2 \Omega_{x_1 x_2 x_2}(X_L, Y_L) \\ &\quad + \frac{1}{2} x_2^2 \Omega_{x_2 x_2 x_2}(X_L, Y_L).\end{aligned}$$

That is

$$\Omega_{x_1}(X_L, Y_L)_0 = \Omega_{x_1}(0, 0) = 0,$$

$$\Omega_{x_2}(X_L, Y_L)_0 = \Omega_{x_2}(0, 0) = 0,$$

$$\begin{aligned}\Omega_{x_1 x_1}(X_L + x_1, x_2) &= n^2 \beta - \frac{(1-\mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{3}{2}}} - \frac{\mu}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{3}{2}}} + \frac{3(1-\mu)(X_L + x_1 - \mu)^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\ &\quad + \frac{3\mu(X_L + x_1 + 1 - \mu)^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} - \frac{3(1-\mu)(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} + \frac{15(1-\mu)(X_L + x_1 - \mu)^2(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} \\ &\quad + \frac{15(1-\mu)(s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)^2 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} - \frac{3\mu A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\ &\quad - \frac{105(1-\mu)(X_L + x_1 - \mu)^2(s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}}\end{aligned}$$

or

$$\begin{aligned}\Omega_{x_1 x_1}(X_L, 0) &= n^2 \beta - \frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} + \frac{3(1-\mu)(X_L - \mu)^2}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)^2}{|X_L + 1 - \mu|^5} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} \\ &\quad + \frac{15(1-\mu)(X_L - \mu)^2(2s_1 - s_2)}{2|X_L - \mu|^7} + \frac{15\mu(X_L + 1 - \mu)^2 A_2}{2|X_L + 1 - \mu|^7} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5}\end{aligned}$$

or

$$\Omega_{x_1 x_1}(X_L, 0) = n^2 \beta + \frac{2(1-\mu)}{|X_L - \mu|^3} + \frac{2\mu}{|X_L + 1 - \mu|^3} + \frac{6(1-\mu)(2s_1 - s_2)}{|X_L - \mu|^5} + \frac{6\mu A_2}{|X_L + 1 - \mu|^5}.$$

Also, we have

$$\begin{aligned}\Omega_{x_1 x_2}(X_L + x_1, x_2) = & \frac{3(1-\mu)(X_L + x_1 - \mu)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} + \frac{3\mu(X_L + x_1 + 1 - \mu)x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\ & + \frac{15(1-\mu)(X_L + x_1 - \mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2 x_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} \\ & + 15(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu) \left(\frac{-7x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} + \frac{2x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} \right),\end{aligned}$$

$$\Omega_{x_1 x_2}(X_L, 0) = 0.$$

And

$$\begin{aligned}\Omega_{x_1 x_1 x_1}(X_L + x_1, x_2) = & \frac{3(1-\mu)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} + \frac{3\mu(X_L + x_1 + 1 - \mu)}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} + \frac{6(1-\mu)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\ & + \frac{6\mu(X_L + x_1 + 1 - \mu)}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)^3 A_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} - \frac{15(1-\mu)(X_L + x_1 - \mu)^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} \\ & - \frac{15\mu(X_L + x_1 + 1 - \mu)^3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{15(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} \\ & - \frac{105(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} \\ & - \frac{105(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)^3 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} \\ & - \frac{105(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{11}{2}}} + \frac{945(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)^3 x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}},\end{aligned}$$

$$\begin{aligned}\Omega_{x_1 x_1 x_1}(X_L, 0) = & \frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} - \frac{15(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{6(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} \\ & - \frac{15\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{6\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} + \frac{15\mu(X_L + 1 - \mu)A_2}{|X_L + 1 - \mu|^7} \\ & - \frac{105(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \\ & + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} - \frac{105\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7}\end{aligned}$$

$$= -\frac{6(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} - \frac{6(1-\mu)(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} - \frac{30(1-\mu)(2s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \\ - \frac{30\mu(X_L + 1 - \mu)A_2}{|X_L + 1 - \mu|^7}.$$

We also have

$$\Omega_{x_1 x_2 x_1}(X_L + x_1, x_2) = \frac{-15(1-\mu)(X_L + x_1 - \mu)^2 x_2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} + \frac{3(1-\mu)x_2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} - \frac{15\mu(X_L + x_1 + 1 - \mu)^2 x_2}{[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} \\ + \frac{3\mu x_2}{[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} + \frac{15(1-\mu)(2s_1 - s_2)x_2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} - \frac{105(1-\mu)(s_1 - s_2)x_2^3}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} \\ - \frac{105(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)^2 x_2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} + \frac{15\mu A_2 x_2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} \\ + \frac{30(1-\mu)(s_1 - s_2)x_2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)^2 A_2 x_2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} \\ + \frac{945(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)^2 x_2^3}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{11}{2}}},$$

$$\Omega_{x_1 x_2 x_1}(X_L, 0) = 0.$$

And

$$\Omega_{x_2 x_2 x_1}(X_L + x_1, x_2) = \frac{-15(1-\mu)(X_L + x_1 - \mu)x_2^2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} + \frac{3(1-\mu)(X_L + x_1 - \mu)}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} - \frac{15\mu(X_L + x_1 + 1 - \mu)x_2^2}{[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} \\ + \frac{3\mu(X_L + x_1 + 1 - \mu)}{[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} - \frac{315(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} \\ + \frac{30(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} \\ - \frac{105(1-\mu)x_2^2(X_L + x_1 - \mu)(2s_1 - s_2)}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} + \frac{15(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} \\ - \frac{105\mu(X_L + x_1 + 1 - \mu)A_2 x_2^2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} - \frac{105(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} \\ + \frac{945(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)^2 x_2^4}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{11}{2}}},$$

$$\begin{aligned}\Omega_{x_2 x_2 x_1}(X_L, 0) = & \frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} - \frac{30(1-\mu)(s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} \\ & + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7}.\end{aligned}$$

Also

$$\begin{aligned}\Omega_{x_2 x_2}(X_L + x_1, x_2) = & n^2 \beta + \frac{3(1-\mu)x_2^2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} - \frac{(1-\mu)}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{3}{2}}} - \frac{\mu}{[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{3}{2}}} \\ & + \frac{3\mu x_2^2}{[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} + \frac{15(1-\mu)(2\sigma_1 - \sigma_2)x_2^2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} - \frac{3(1-\mu)(2\sigma_1 - \sigma_2)}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} \\ & - \frac{105(1-\mu)x_2^4(\sigma_1 - \sigma_2)}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} + \frac{45(1-\mu)(\sigma_1 - \sigma_2)x_2^2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} + \frac{15(1-\mu)(\sigma_1 - \sigma_2)x_2^2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} \\ & - \frac{3(1-\mu)(\sigma_1 - \sigma_2)}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} + \frac{15\mu A_2 x_2^2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} - \frac{3\mu A_2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{5}{2}}}, \\ \Omega_{x_2 x_2}(X_L, 0) = & n^2 \beta - \frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} - \frac{3(1-\mu)(s_1 - s_2)}{|X_L - \mu|^5} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5}.\end{aligned}$$

And

$$\begin{aligned}\Omega_{x_1 x_1 x_2}(X_L + x_1, x_2) = & \frac{3(1-\mu)x_2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} + \frac{3\mu x_2}{[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{5}{2}}} - \frac{15(1-\mu)(X_L + x_1 - \mu)^2 x_2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} \\ & - \frac{15\mu(X_L + x_1 + 1 - \mu)^2 x_2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} + \frac{15(1-\mu)(2\sigma_1 - \sigma_2)x_2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} - \frac{105(1-\mu)(X_L + x_1 - \mu)^2(2\sigma_1 - \sigma_2)x_2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} \\ & - \frac{105(1-\mu)(\sigma_1 - \sigma_2)x_2^3}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} + \frac{30(1-\mu)(\sigma_1 - \sigma_2)x_2}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)^2 x_2 A_2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} \\ & + \frac{15\mu A_2 x_2}{2[(X_L + x_1 + 1 - \mu)^2 + x_2^2]^{\frac{7}{2}}} - \frac{105(1-\mu)(\sigma_1 - \sigma_2)(X_L + x_1 - \mu)^2 x_2}{[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{9}{2}}} \\ & + \frac{945(1-\mu)(\sigma_1 - \sigma_2)(X_L + x_1 - \mu)^2 x_2^3}{2[(X_L + x_1 - \mu)^2 + x_2^2]^{\frac{11}{2}}},\end{aligned}$$

$$\Omega_{x_1 x_1 x_2}(X_L, 0) = 0.$$

$$\begin{aligned}
\Omega_{x_1 x_2 x_2}(X_L + x_1, x_2) = & -\frac{15(1-\mu)(X_L + x_1 - \mu)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{3(1-\mu)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\
& - \frac{15\mu(X_L + x_1 + 1 - \mu)x_2^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{3\mu(X_L + x_1 + 1 - \mu)}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)(X_L + x_1 - \mu)(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{945(1-\mu)(X_L + x_1 - \mu)(s_1 - s_2)x_2^4}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{11}{2}}} \\
& - \frac{315(1-\mu)(X_L + x_1 - \mu)(s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} \\
& + \frac{15(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)x_2^2 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_1 x_2 x_2}(X_L, 0) = & \frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} \\
& + \frac{15(1-\mu)(s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7}.
\end{aligned}$$

Also

$$\begin{aligned}
\Omega_{x_2 x_2 x_2}(X_L + x_1, x_2) = & -\frac{15(1-\mu)(X_L + x_1 - \mu)x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{6(1-\mu)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\
& + \frac{3(1-\mu)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} - \frac{15\mu x_2^3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{6\mu x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2\right]^{\frac{5}{2}}} \\
& + \frac{3\mu x_2}{r_1^5} - \frac{105(1-\mu)(2s_1 - s_2)x_2^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} + \frac{30(1-\mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} + \frac{945(1-\mu)(s_1 - s_2)x_2^5}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{11}{2}}} - \frac{210(1-\mu)(s_1 - s_2)x_2^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} \\
& - \frac{315(1-\mu)(s_1 - s_2)x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}} + \frac{90(1-\mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)(s_1 - s_2)x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2\right]^{\frac{9}{2}}}
\end{aligned}$$

$$\begin{aligned}
& -\frac{315(1-\mu)(s_1-s_2)x_2^3}{\left[(X_L+x_1-\mu)^2+x_2^2\right]^{\frac{9}{2}}} + \frac{90(1-\mu)(2s_1-s_2)x_2}{2\left[(X_L+x_1-\mu)^2+x_2^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)(s_1-s_2)x_2^3}{\left[(X_L+x_1-\mu)^2+x_2^2\right]^{\frac{9}{2}}} \\
& + \frac{30(1-\mu)(s_1-s_2)x_2}{\left[(X_L+x_1-\mu)^2+x_2^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)(s_1-s_2)x_2}{\left[(X_L+x_1-\mu)^2+x_2^2\right]^{\frac{7}{2}}} - \frac{105\mu A_2 x_2^3}{2\left[(X_L+x_1+1-\mu)^2+x_2^2\right]^{\frac{9}{2}}} \\
& + \frac{15\mu x_2 A_2}{2\left[(X_L+x_1+1-\mu)^2+x_2^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_2 x_2 x_2}(X_L, 0) = 0.$$

The RHS of Eqn. (3.9) become

$$\begin{aligned}
\Omega_{x_1} = x_1 \left[n^2 \beta + \frac{2(1-\mu)}{|X_L - \mu|^3} + \frac{2\mu}{|X_L + 1 - \mu|^3} + \frac{6(1-\mu)(2s_1-s_2)}{|X_L - \mu|^5} + \frac{6\mu A_2}{|X_L + 1 - \mu|^5} \right] \\
+ x_1^2 \left[-3 \left\{ \frac{(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{5(1-\mu)(2s_1-s_2)(X_L - \mu)}{|X_L - \mu|^7} \right. \right. \\
\left. \left. + \frac{5\mu(X_L + 1 - \mu)A_2}{|X_L + 1 - \mu|^7} \right\} \right] + x_2^2 \left[\frac{3}{2} \left\{ \frac{(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} \right. \right. \\
\left. \left. + \frac{5\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} + \frac{5(1-\mu)(2s_1-s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{5(1-\mu)(2s_1-s_2)(X_L - \mu)}{|X_L - \mu|^7} \right\} \right],
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{x_2} = x_2 \left[n^2 \beta - \frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1-s_2)}{2|X_L - \mu|^5} - \frac{3(1-\mu)(s_1-s_2)}{|X_L - \mu|^5} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5} \right] \\
+ x_1 x_2 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{15(1-\mu)(2s_1-s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)(s_1-s_2)(X_L - \mu)}{|X_L - \mu|^7} \right. \\
\left. + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \right].
\end{aligned}$$

Equations (3.9) can now be written as

$$\ddot{x}_1 - 2n\alpha \dot{x}_2 = A_{10}x_1 + A_{20}x_1^2 + A_{02}x_2^2, \quad (3.10a)$$

$$\ddot{x}_2 + 2n\alpha \dot{x}_1 = B_{01}x_2 + B_{11}x_1x_2, \quad (3.10b)$$

where

$$\begin{aligned}
A_{10} &= \beta n^2 + 2R_0 + \frac{6(1-\mu)(2s_1 - s_2)}{r_{10}^5} + \frac{6\mu A_2}{r_{20}^5}, \\
A_{20} &= -3(1-\mu)\left[\frac{1}{r_{10}^4} + \frac{5(2s_1 - s_2)}{r_{10}^6}\right]\sigma_1 - 3\mu\left(\frac{1}{r_{20}^4} + \frac{5A_2}{r_{20}^6}\right)\sigma_2, \\
A_{02} &= 3(1-\mu)\left[\frac{1}{2r_{10}^4} + \frac{5(4s_1 - 3s_2)}{4r_{10}^6}\right]\sigma_1 + 3\mu\left(\frac{1}{2r_{20}^4} + \frac{5A_2}{4r_{20}^6}\right)\sigma_2, \\
B_{01} &= \beta n^2 - R_0 - \frac{3(1-\mu)(4s_1 - 3s_2)}{2r_{10}^5} - \frac{3\mu A_2}{2r_{20}^5}, \\
B_{11} &= 3(1-\mu)\left[\frac{1}{r_{10}^4} + \frac{5(4s_1 - 3s_2)}{2r_{10}^6}\right]\sigma_1 + 3\mu\left(\frac{1}{r_{20}^4} + \frac{5A_2}{2r_{20}^6}\right)\sigma_2
\end{aligned}$$

and

$$R_0 = \frac{(1-\mu)}{|X_L - \mu|^3} + \frac{\mu}{|X_L + 1 - \mu|^3}.$$

Also, σ_1 and σ_2 represent the signs of $r_{10} = |x_1 - \mu|$ and $r_{20} = |x_1 + 1 - \mu|$ at any equilibrium point avoiding thus the absolute values for each case.

3.4.2 Semi-analytical approximation of the 2D periodic orbits for the model with triaxial and oblate primaries

For the planar case, we look for periodic solutions of the system (3.10a) and (3.10b):

We consider that the sought solution in powers of an orbital parameter e ($|e| \ll 1$) is of the following form:

$$x_1 = x_{11}e + x_{12}e^2, \quad x_2 = x_{21}e + x_{22}e^2. \quad (3.11)$$

Substituting (3.11) into (3.10a), (3.10b) and equating the coefficients of the same powers of e , we obtain the following two systems

$$\begin{aligned}
x_{11}'' - 2\alpha n x_{21}' - A_{10}x_{11} &= 0, \\
x_{21}'' + 2\alpha n x_{11}' - B_{01}x_{21} &= 0,
\end{aligned} \quad (3.12)$$

and

$$\begin{aligned} x_{12}'' - 2\alpha n x_{22}' - A_{02} x_{21}^2 - A_{10} x_{12} - A_{20} x_{11}^2 &= 0, \\ x_{22}'' + 2\alpha n x_{12}' - B_{01} x_{22} - B_{11} x_{11} x_{21} &= 0, \end{aligned} \quad (3.13)$$

respectively, which have to be solved successively. The periodic solution of the linear system (3.12) is

$$x_{11} = c_{111} \cos(wt) + s_{111} \sin(wt) \text{ and } x_{21} = c_{121} \cos(wt) + s_{121} \sin(wt) \quad (3.14)$$

where $T = \frac{2\pi}{w}$ is the period of the periodic orbit. Since $\cos(wt)$ and $\sin(wt)$ are linearly independent functions, we equate their coefficients obtaining

$$\begin{aligned} (A_{10} + w^2)c_{111} + 2\alpha n s_{121} &= 0, \\ 2\alpha n w c_{111} + (B_{01} + w^2)s_{121} &= 0, \end{aligned} \quad (3.15)$$

and

$$\begin{aligned} (A_{10} + w^2)s_{111} - 2\alpha n c_{121} &= 0, \\ -2\alpha n w s_{111} + (B_{01} + w^2)c_{121} &= 0, \end{aligned} \quad (3.16)$$

respectively. Systems (3.12) and (3.13) are linear homogeneous and for non-zero solution the determinant must be zero, i.e.:

$$D = \begin{vmatrix} A_{10} + w^2 & \pm 2\alpha n \\ \pm 2\alpha n w & B_{01} + w^2 \end{vmatrix} = w^4 + (A_{10} + B_{01} - 4\alpha^2 n^2)w^2 + A_{10}B_{01} = 0. \quad (3.17)$$

Since $D = 0$, systems (3.12) and (3.13) are indeterminate, so, we choose arbitrarily $c_{111} = 1$ and $s_{111} = 0$ such that:

$$c_{121} = 0, \quad s_{121} = -\frac{2\alpha n w}{B_{01} + w^2}. \quad (3.18)$$

Therefore, the periodic solution of (3.12) is

$$\begin{aligned} x_{11} &= \cos(wt), \\ x_{21} &= s_{121} \sin(2wt). \end{aligned}$$

Similarly, for system (3.13) we set:

$$\begin{aligned} x_{12} &= c_{210} + c_{211} \cos(wt) + c_{212} \cos(2wt) + s_{211} \sin(wt) + s_{212} \sin(2wt), \\ x_{21} &= c_{221} \cos(wt) + c_{222} \cos(2wt) + s_{221} \sin(wt) + s_{222} \sin(2wt), \end{aligned} \quad (3.19)$$

and working as previously we find that

$$c_{211} = 0, \quad s_{211} = 0, \quad c_{221} = 0, \quad s_{221} = 0, \quad s_{212} = 0, \quad c_{222} = 0, \quad (3.20)$$

and

$$\begin{aligned} c_{210} &= -\frac{A_{20} + A_{02}s_{121}^2}{2A_{10}}, \\ c_{212} &= \frac{-A_{20}(B_{01} + 4w^2) + s_{121}(4\alpha B_{11}nw + A_{02}s_{121}(B_{01} + 4w^2))}{2A_{10}B_{01} + 8(A_{10} + B_{01} - 4\alpha^2 n^2)w^2 + 32w^4}, \\ s_{222} &= \frac{4\alpha A_{20}nw - s_{121}(A_{10}B_{11} + 4w(\alpha A_{02}ns_{121} + B_{11}w))}{2A_{10}B_{01} + 8(A_{10} + B_{01} - 4\alpha^2 n^2)w^2 + 32w^4}. \end{aligned} \quad (3.21)$$

Thus, the periodic solution of (3.10) is

$$\begin{aligned} x_{12} &= c_{210} + c_{212} \cos(2wt), \\ x_{22} &= s_{222} \sin(2wt). \end{aligned}$$

Therefore, the sought periodic solution (3.11) into series expansions of the orbital parameter e up to second order terms is:

$$\begin{aligned} x_1 &= x_0 + \cos(wt)e + [c_{210} + c_{212} \cos(2wt)]e^2, \\ x_2 &= s_{121} \sin(wt)e + s_{222} \sin(2wt)e^2, \end{aligned} \quad (3.22)$$

where $(x_0, 0)$ is the position of any collinear equilibrium point, the coefficients

$c_{111}, c_{210}, c_{212}, s_{121}, s_{222}$ are defined by (3.18) and (3.21), and w is a real positive solution of (3.17).

3.5 Spatial Periodic Orbits: Second Order Analysis for the model with triaxial and oblate primaries

A second order Lindstedt-Poincaré method is used to obtain spatial periodic orbits in this section.

with triaxial and oblate primaries

$$\dot{X}_4 - 2n\alpha X_5 = \Omega_{X_1},$$

$$\dot{X}_5 - 2n\alpha X_4 = \Omega_{X_2},$$

$$\dot{X}_6 = \Omega_{X_3}, \quad (3.23)$$

$$\Omega = \frac{1}{2}n^2\beta(X_1^2 + X_2^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} + \frac{(1-\mu)(2s_1-s_2)}{2r_1^3} - \frac{3(1-\mu)(s_1-s_2)X_2^2}{2r_1^5} + \frac{\mu A_2}{2r_2^3} - \frac{3(1-\mu)s_1X_3^2}{2r_1^5} - \frac{3\mu A_2X_3^2}{2r_2^5}, \quad (3.24)$$
$$\Omega_{x_1} = n^2 \beta X_1 - \frac{1-\mu}{r_1^3} \frac{X_1-\mu}{r_1^3} - \frac{\mu}{r_2^3} \frac{X_1+1-\mu}{r_2^3} - \frac{3}{2r_1^5} \frac{1-\mu}{r_1^5} \frac{X_1-\mu}{r_1^5} 2s_1-s_2 + \frac{15}{2r_1^7} \frac{1-\mu}{r_1^7} \frac{X_1-\mu}{r_1^7} \frac{X_2^2}{r_1^7} s_1-s_2$$

$$- \frac{3\mu}{2r_2^5} \frac{X_1+1-\mu}{r_2^5} A_2 + \frac{15}{2r_1^7} \frac{1-\mu}{r_1^7} \frac{X_1-\mu}{r_1^7} s_1 X_3^2 + \frac{15\mu}{2r_2^7} \frac{X_1+1-\mu}{r_2^7} A_2 X_3^2,$$

(3.25)

$$\begin{aligned} \Omega_{X_2} = & n^2 \beta X_2 - \frac{1-\mu}{r_1^3} X_2 - \frac{\mu X_2}{r_2^3} - \frac{3}{2r_1^5} \frac{1-\mu}{s_1-s_2} X_2 - 3 \frac{1-\mu}{s_1-s_2} \left(\frac{2X_2}{2r_1^5} - \frac{5X_2^3}{2r_1^7} \right) - \frac{3\mu A_2 X_2}{2r_2^5} \\ & + \frac{15}{2r_1^7} \frac{1-\mu}{s_1} X_3^2 (X_1 - \mu) + \frac{15\mu A_2 X_3^2 (X_1 + 1 - \mu)}{2r_2^7}, \end{aligned} \quad (3.26)$$

$$\begin{aligned} \Omega_{X_3} = & -\frac{(1-\mu)X_3}{r_1^3} - \frac{\mu X_3}{r_2^3} - \frac{3(1-\mu)(2s_1-s_2)X_3}{2r_1^5} + \frac{15(1-\mu)(s_1-s_2)X_2^2 X_3}{2r_1^7} - \frac{3\mu A_2 X_2}{2r_2^5} \\ & + \frac{15(1-\mu)s_1 X_2^3}{2r_1^7} - \frac{3(1-\mu)s_1 X_3}{r_1^5} + \frac{15\mu A_2 X_2^3}{2r_2^7} - \frac{3\mu A_2 X_3}{r_2^5}. \end{aligned} \quad (3.27)$$

We define a new coordinate system whereby L is any collinear equilibrium point L_j , $j=1,2,3$ and is also the origin with L_{x_1} , L_{x_2} , L_{x_3} as axes parallel to OX , OY and OZ , respectively. The relations that show the proper transformation between the two systems are:

$$X_1 \rightarrow X_L + x_1, X_2 \rightarrow x_2, X_3 \rightarrow x_3. \quad (3.28)$$

By the transformation of Eqns. (3.23) with the use of (3.28) in the (x_1, x_2, x_3) coordinate system, we have

$$\begin{aligned} \ddot{x}_1 - 2n\alpha\dot{x}_2 &= \Omega_{x_1}, \\ \ddot{x}_2 - 2n\alpha\dot{x}_1 &= \Omega_{x_2}, \\ \ddot{x}_3 &= \Omega_{x_3}. \end{aligned} \quad (3.29)$$

We expand the RHS of Eqns. (3.29) up to the second order using the Taylor series expansion around (X_L, Y_L) . We proceed as

$$\begin{aligned} \Omega_{x_1}(X_L + x_1, x_2, x_3) &= \Omega_{x_1}(X_L, Y_L, Z_L) + x_1\Omega_{x_1x_1}(X_L, Y_L, Z_L) + x_2\Omega_{x_1x_2}(X_L, Y_L, Z_L) + x_3\Omega_{x_1x_3}(X_L, Y_L, Z_L) \\ &\quad + \frac{1}{2}x_1^2\Omega_{x_1x_1x_1}(X_L, Y_L, Z_L) \\ &\quad + x_1x_2\Omega_{x_1x_2x_1}(X_L, Y_L, Z_L) + x_1x_3\Omega_{x_1x_3x_1}(X_L, Y_L, Z_L) + x_2x_3\Omega_{x_1x_2x_3}(X_L, Y_L, Z_L) \\ &\quad + \frac{1}{2}x_2^2\Omega_{x_1x_2x_2}(X_L, Y_L, Z_L) \\ &\quad + \frac{1}{2}x_3^2\Omega_{x_1x_3x_3}(X_L, Y_L, Z_L), \end{aligned} \quad (3.30)$$

$$\begin{aligned} \Omega_{x_2}(X_L + x_1, x_2, x_3) &= \Omega_{x_2}(X_L, Y_L, Z_L) + x_1\Omega_{x_2x_1}(X_L, Y_L, Z_L) + x_2\Omega_{x_2x_2}(X_L, Y_L, Z_L) + x_3\Omega_{x_2x_3}(X_L, Y_L, Z_L) \\ &\quad + \frac{1}{2}x_1^2\Omega_{x_2x_1x_1}(X_L, Y_L, Z_L) \\ &\quad + x_1x_2\Omega_{x_2x_2x_1}(X_L, Y_L, Z_L) + x_1x_3\Omega_{x_2x_3x_1}(X_L, Y_L, Z_L) + x_2x_3\Omega_{x_2x_2x_3}(X_L, Y_L, Z_L) \\ &\quad + \frac{1}{2}x_2^2\Omega_{x_2x_2x_2}(X_L, Y_L, Z_L) \\ &\quad + \frac{1}{2}x_3^2\Omega_{x_2x_3x_3}(X_L, Y_L, Z_L), \end{aligned} \quad (3.31)$$

and

$$\begin{aligned}
\Omega_{x_3}(X_L + x_1, x_2, x_3) &= \Omega_{x_3}(X_L, Y_L, Z_L) + x_1 \Omega_{x_3 x_1}(X_L, Y_L, Z_L) + x_2 \Omega_{x_3 x_2}(X_L, Y_L, Z_L) + x_3 \Omega_{x_3 x_3}(X_L, Y_L, Z_L) + \frac{1}{2} x_1^2 \Omega_{x_3 x_1 x_1}(X_L, Y_L, Z_L) \\
&+ x_1 x_2 \Omega_{x_3 x_1 x_2}(X_L, Y_L, Z_L) + x_1 x_3 \Omega_{x_3 x_1 x_3}(X_L, Y_L, Z_L) + x_2 x_3 \Omega_{x_3 x_2 x_3}(X_L, Y_L, Z_L) + \frac{1}{2} x_2^2 \Omega_{x_3 x_2 x_2}(X_L, Y_L, Z_L) \\
&+ \frac{1}{2} x_3^2 \Omega_{x_3 x_3 x_3}(X_L, Y_L, Z_L).
\end{aligned} \tag{3.32}$$

Next, we evaluate the transformations in Eqns. (3.30), (3.31) and (3.32) by finding the various partial derivatives involved and subsequently, making the appropriate substitutions into the three equations. We obtain,

$$\Omega_{x_1}(X_L, Y_L, Z_L)_0 = \Omega_{x_1}(0, 0, 0) = 0,$$

$$\Omega_{x_2}(X_L, Y_L, Z_L)_0 = \Omega_{x_2}(0, 0, 0) = 0,$$

$$\Omega_{x_3}(X_L, Y_L, Z_L)_0 = \Omega_{x_3}(0, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{x_1 x_1}(X_L + x_1, x_2, x_3) &= n^2 \beta - \frac{(1-\mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{3}{2}}} - \frac{\mu}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{3}{2}}} + \frac{3(1-\mu)(X_L + x_1 - \mu)^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
&+ \frac{3\mu(X_L + x_1 + 1 - \mu)^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} - \frac{3(1-\mu)(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} + \frac{15(1-\mu)(X_L + x_1 - \mu)^2(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
&+ \frac{15(1-\mu)(s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)^2 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3\mu A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
&- \frac{105(1-\mu)(X_L + x_1 - \mu)^2(s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)(X_L + x_1 - \mu)^2 s_1 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)s_1 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
&- \frac{105\mu(X_L + x_1 + 1 - \mu)^2 A_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{x_1 x_1}(X_L, 0, 0) &= n^2 \beta - \frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} + \frac{3(1-\mu)(X_L - \mu)^2}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)^2}{|X_L + 1 - \mu|^5} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} \\
&+ \frac{15(1-\mu)(X_L - \mu)^2(2s_1 - s_2)}{2|X_L - \mu|^7} + \frac{15\mu(X_L + 1 - \mu)^2 A_2}{2|X_L + 1 - \mu|^7} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5}
\end{aligned}$$

or

$$\Omega_{x_1 x_1}(X_L, 0, 0) = n^2 \beta + \frac{2(1-\mu)}{|X_L - \mu|^3} + \frac{2\mu}{|X_L + 1 - \mu|^3} + \frac{6(1-\mu)(2s_1 - s_2)}{|X_L - \mu|^5} + \frac{6\mu A_2}{|X_L + 1 - \mu|^5}.$$

Also, we have

$$\begin{aligned}
\Omega_{x_1 x_2}(X_L + x_1, x_2, x_3) = & \frac{3(1-\mu)(X_L + x_1 - \mu)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} + \frac{3\mu(X_L + x_1 + 1 - \mu)x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)(X_L + x_1 - \mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2 x_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + 15(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu) \left(\frac{-7x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \right. \\
& \left. + \frac{2x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \right) - \frac{105(1-\mu)(X_L + x_1 - \mu)s_1 x_3^2 x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& - \frac{105\mu(X_L + x_1 + 1 - \mu)A_2 x_3^2 x_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}},
\end{aligned}$$

$$\Omega_{x_1 x_2}(X_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{x_1 x_3}(X_L + x_1, x_2, x_3) = & \frac{3(1-\mu)(X_L + x_1 - \mu)x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} + \frac{3\mu(X_L + x_1 + 1 - \mu)x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)(X_L + x_1 - \mu)(2s_1 - s_2)x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2 x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2 x_3}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)s_1(X_L + x_1 - \mu)x_3^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15(1-\mu)s_1(X_L + x_1 - \mu)x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{105\mu A_2(X_L + x_1 + 1 - \mu)x_3^3}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15\mu A_2(X_L + x_1 + 1 - \mu)x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_1 x_3}(X_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{x_2 x_3}(X_L + x_1, x_2, x_3) = & \frac{3(1-\mu)x_2 x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} + \frac{3\mu x_2 x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)x_2 x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)(s_1 - s_2)x_2^3 x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15(1-\mu)(s_1 - s_2)x_2 x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{15\mu A_2 x_2 x_3}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)s_1 x_3^3 x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)s_1 x_3 x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& - \frac{105\mu A_2 x_3^3 x_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2 x_3 x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_2 x_3}(X_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{x_3 x_3}(X_L + x_1, x_2, x_3) = & \frac{3(1-\mu)x_3^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} - \frac{(1-\mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{3}{2}}} \\
& + \frac{3\mu x_3^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} - \frac{\mu}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{3}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3(1-\mu)(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105(1-\mu)(s_1 - s_2)x_2^2 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)(s_1 - s_2)x_2 x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15\mu A_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3\mu A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105(1-\mu)s_1 x_3^4}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{45(1-\mu)s_1 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)s_1 x_3^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3(1-\mu)s_1}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105\mu A_2 x_3^4}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{45\mu A_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15\mu A_2 x_3^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3\mu A_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}}, \\
\Omega_{x_3 x_3}(X_L, 0, 0) = & -\frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} - \frac{9\mu A_2}{2|X_L + 1 - \mu|^5} - \frac{3(1-\mu)s_1 A_1}{|X_L - \mu|^5},
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_1 x_1 x_1}(X_L + x_1, x_2, x_3) = & \frac{3(1-\mu)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} + \frac{3\mu(X_L + x_1 + 1 - \mu)}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{6(1-\mu)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} + \frac{6\mu(X_L + x_1 + 1 - \mu)}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{15\mu(X_L + x_1 + 1 - \mu)^3 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{15(1-\mu)(X_L + x_1 - \mu)^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& - \frac{15\mu(X_L + x_1 + 1 - \mu)^3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{15(1-\mu)(2s_1 - s_2)(x_1 - \mu)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)(2s_1 - s_2)(x_1 - \mu)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& - \frac{105\mu(X_L + x_1 + 1 - \mu)^3 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)^3 x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} + \frac{945(1-\mu)s_1(X_L + x_1 - \mu)^3 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} \\
& - \frac{105(1-\mu)s_1(X_L + x_1 - \mu)x_3^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)s_1(X_L + x_1 - \mu)x_3^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{945\mu A_2(X_L + x_1 + 1 - \mu)^3 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{105\mu A_2(X_L + x_1 + 1 - \mu)x_3^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& - \frac{105\mu A_2(X_L + x_1 + 1 - \mu)x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}},
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_1 x_1 x_1}(X_L, 0, 0) &= \frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} - \frac{15(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{6(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} \\
&\quad - \frac{15\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{6\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} + \frac{15\mu(X_L + 1 - \mu)A_2}{|X_L + 1 - \mu|^7} \\
&\quad - \frac{105(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \\
&\quad + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} - \frac{105\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \\
&= -\frac{6(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} - \frac{6(1-\mu)(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} - \frac{30(1-\mu)(2s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \\
&\quad - \frac{30\mu(X_L + 1 - \mu)A_2}{|X_L + 1 - \mu|^7}.
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_1 x_2 x_1}(X_L + x_1, x_2, x_3) &= \frac{-15(1-\mu)(X_L + x_1 - \mu)^2 x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3(1-\mu)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
&\quad - \frac{15\mu(X_L + x_1 + 1 - \mu)^2 x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3\mu x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
&\quad - \frac{105(1-\mu)x_2(X_L + x_1 - \mu)^2(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
&\quad - \frac{105(1-\mu)(s_1 - s_2)x_2^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)(s_1 - s_2)(x_1 - \mu)^2 x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
&\quad + \frac{15\mu A_2 x_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{30(1-\mu)(s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
&\quad - \frac{105\mu(X_L + x_1 + 1 - \mu)^2 A_2 x_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{945(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)^2 x_2^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} \\
&\quad + \frac{945(1-\mu)s_1(X_L + x_1 - \mu)^2 x_2 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{105(1-\mu)s_1 x_2 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
&\quad + \frac{945\mu A_2(X_L + x_1 + 1 - \mu)^2 x_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{105\mu A_2 x_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}},
\end{aligned}$$

$$\Omega_{x_1 x_2 x_1}(X_L, 0, 0) = 0.$$

$$\begin{aligned}
\Omega_{x_2 x_2 x_1}(X_L + x_1, x_2, x_3) = & \frac{-15(1-\mu)(X_L + x_1 - \mu)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3(1-\mu)(X_L + x_1 - \mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{15\mu(X_L + x_1 + 1 - \mu)x_2^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3\mu(X_L + x_1 + 1 - \mu)}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{315(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{30(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15\mu(X_L + x_1 + 1 - \mu)A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)x_2^2(X_L + x_1 - \mu)(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)(X_L + x_1 - \mu)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)A_2x_2^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{945(1-\mu)(s_1 - s_2)(X_L + x_1 - \mu)^2x_2^4}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)s_1(X_L + x_1 - \mu)x_3^2x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{105(1-\mu)s_1(X_L + x_1 - \mu)x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{945\mu A_2(X_L + x_1 + 1 - \mu)x_3^2x_2^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{105\mu A_2(X_L + x_1 + 1 - \mu)x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}},
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_2 x_2 x_1}(X_L, 0, 0) = & \frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} + \frac{15(1-\mu)(s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \\
& + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7},
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_1 x_1 x_3}(X_L + x_1, x_2, x_3) = & \frac{3(1-\mu)x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} + \frac{3\mu x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)(X_L + x_1 - \mu)^2x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{15\mu(X_L + x_1 + 1 - \mu)^2x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)(X_L + x_1 - \mu)^2(2s_1 - s_2)x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)(s_1 - s_2)x_2^2x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)^2A_2x_3}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15\mu A_2x_3}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{945(1-\mu)(X_L + x_1 - \mu)^2(s_1 - s_2)x_2^2x_3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{945(1-\mu)(X_L + x_1 - \mu)^2 s_1 x_3^3}{2 \left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{11}{2}}} - \frac{105(1-\mu)(X_L + x_1 - \mu)^2 s_1 x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)s_1 x_3^3}{2 \left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} + \frac{15(1-\mu)s_1 x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{7}{2}}} \\
& + \frac{945\mu(X_L + x_1 + 1 - \mu)^2 A_2 x_3^3}{2 \left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{11}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)^2 A_2 x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} \\
& - \frac{105\mu A_2 x_3^3}{2 \left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} + \frac{15\mu A_2 x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_1 x_1 x_3}(X_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{x_1 x_2 x_3}(X_L + x_1, x_2, x_3) = & - \frac{15(1-\mu)(X_L + x_1 - \mu)x_2 x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{7}{2}}} - \frac{15\mu(X_L + x_1 + 1 - \mu)x_2 x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)(X_L + x_1 - \mu)(2s_1 - s_2)x_2 x_3}{2 \left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} + \frac{945(1-\mu)(X_L + x_1 - \mu)(s_1 - s_2)x_2^3 x_3}{2 \left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{11}{2}}} \\
& - \frac{105(1-\mu)(X_L + x_1 - \mu)(s_1 - s_2)x_2 x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)A_2 x_2 x_3}{2 \left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)(X_L + x_1 - \mu)s_1 x_2^3 x_3}{2 \left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{11}{2}}} - \frac{105(1-\mu)(X_L + x_1 - \mu)s_1 x_2 x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}} \\
& + \frac{945\mu(X_L + x_1 + 1 - \mu)A_2 x_2^3 x_3}{2 \left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{11}{2}}} - \frac{105\mu(X_L + x_1 + 1 - \mu)A_2 x_2 x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2 \right]^{\frac{9}{2}}},
\end{aligned}$$

$$\Omega_{x_1 x_2 x_3}(X_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{x_2 x_2}(X_L + x_1, x_2, x_3) = & n^2 \beta + \frac{3(1-\mu)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} - \frac{(1-\mu)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{3}{2}}} \\
& - \frac{\mu}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{3}{2}}} + \frac{3\mu x_2^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3(1-\mu)(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105(1-\mu)x_2^4(s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{45(1-\mu)(s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)(s_1 - s_2)x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3(1-\mu)(s_1 - s_2)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{15\mu A_2 x_2^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{3\mu A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105(1-\mu)x_2^2 s_1 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)s_1 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& - \frac{105\mu x_2^2 A_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_2 x_2}(X_L, 0, 0) = n^2 \beta - \frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} - \frac{3(1-\mu)(s_1 - s_2)}{|X_L - \mu|^5} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5},$$

$$\begin{aligned}
\Omega_{x_3 x_2 x_2}(X_L + x_1, x_2, x_3) = & - \frac{15(1-\mu)(X_L + x_1 - \mu)x_3 x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3(1-\mu)x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{15\mu x_3 x_2^2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3\mu x_3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105(1-\mu)(2s_1 - s_2)x_3 x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)x_3(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{945(1-\mu)x_3(s_1 - s_2)x_2^4}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{315(1-\mu)x_3(s_1 - s_2)x_2^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}},
\end{aligned}$$

$$\begin{aligned}
& -\frac{105(1-\mu)(s_1-s_2)x_3x_2^2}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)(s_1-s_2)x_3}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& -\frac{105\mu x_3x_2^2A_2}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2x_3}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& +\frac{945(1-\mu)s_1x_3^3x_2^2}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} - \frac{105(1-\mu)s_1x_3x_2^2}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} \\
& -\frac{105(1-\mu)s_1x_3^3}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)s_1x_3}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& +\frac{945\mu A_2x_3^3x_2^2}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} - \frac{105\mu A_2x_3x_2^2}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} \\
& -\frac{105\mu A_2x_3^3}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2x_3}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_3x_2x_2}(X_L, 0, 0) = 0.$$

$$\begin{aligned}
\Omega_{x_3x_3x_3}(X_L+x_1, x_2, x_3) = & -\frac{15(1-\mu)x_3^3}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} + \frac{6(1-\mu)x_3}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{5}{2}}} \\
& +\frac{3(1-\mu)x_3}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{5}{2}}} - \frac{15\mu x_3^3}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& +\frac{6\mu x_3}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{5}{2}}} + \frac{3\mu}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{5}{2}}} \\
& -\frac{105(1-\mu)(2s_1-s_2)x_3^3}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)x_3(2s_1-s_2)}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& +\frac{945(1-\mu)(s_1-s_2)x_2^2x_3^3}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} - \frac{105(1-\mu)(s_1-s_2)x_3x_2^2}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} \\
& -\frac{105(1-\mu)(s_1-s_2)x_3x_2^2}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} - \frac{105\mu x_3^3A_2}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} \\
& +\frac{15\mu A_2x_3}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} + \frac{15\mu A_2x_3}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& +\frac{945(1-\mu)s_1x_3^5}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} - \frac{945(1-\mu)s_1x_3^3}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}}
\end{aligned}$$

$$\begin{aligned}
& + \frac{90(1-\mu)s_1x_3}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} + \frac{945\mu A_2x_3^5}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} \\
& - \frac{945(1-\mu)s_1x_3^5}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} - \frac{945\mu A_2x_3^3}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{90\mu A_2x_3}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_3x_3x_3}(X_L, 0, 0) = 0.$$

$$\begin{aligned}
\Omega_{x_3x_3x_1}(X_L+x_1, x_2, x_3) = & \frac{-15(1-\mu)(X_L+x_1-\mu)x_3^2}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} + \frac{3(1-\mu)(X_L+x_1-\mu)}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{15\mu(X_L+x_1+1-\mu)x_3^2}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} + \frac{3\mu(X_L+x_1+1-\mu)}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105(1-\mu)(2s_1-s_2)(X_L+x_1-\mu)x_3^2}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)(2s_1-s_2)(X_L+x_1-\mu)}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{945(1-\mu)(s_1-s_2)(X_L+x_1-\mu)x_2^2x_3^2}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} - \frac{105(1-\mu)x_2^2(X_L+x_1-\mu)(s_1-s_2)}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} \\
& - \frac{105\mu(X_L+x_1+1-\mu)A_2x_3^2}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu(X_L+x_1+1-\mu)A_2}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{945(1-\mu)s_1(X_L+x_1-\mu)x_3^4}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} - \frac{525(1-\mu)s_1(X_L+x_1-\mu)x_3^2}{2\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{15(1-\mu)s_1(X_L+x_1-\mu)}{\left[(X_L+x_1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}} + \frac{945\mu A_2(X_L+x_1+1-\mu)x_3^4}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{11}{2}}} \\
& - \frac{525\mu A_2(X_L+x_1+1-\mu)x_3^2}{2\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2(X_L+x_1+1-\mu)}{\left[(X_L+x_1+1-\mu)^2+x_2^2+x_3^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_3x_3x_1}(X_L, 0, 0) = & \frac{3(1-\mu)(X_L-\mu)}{|X_L-\mu|^5} + \frac{3\mu(X_L+1-\mu)}{|X_L+1-\mu|^5} + \frac{15(1-\mu)(2s_1-s_2)(X_L-\mu)}{2|X_L-\mu|^7} \\
& + \frac{45\mu(X_L+1-\mu)A_2}{2|X_L+1-\mu|^7} + \frac{15(1-\mu)s_1(X_L-\mu)}{|X_L-\mu|^7},
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_3 x_3 x_2}(X_L + x_1, x_2, x_3) = & -\frac{15(1-\mu)x_3^2 x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3(1-\mu)x_3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& -\frac{15\mu x_3^2 x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3\mu x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& -\frac{105(1-\mu)(2s_1 - s_2)x_3^2 x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)x_2(2s_1 - s_2)}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& +\frac{945(1-\mu)(s_1 - s_2)x_2^3 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{105(1-\mu)(s_1 - s_2)x_3 x_2^2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& -\frac{105(1-\mu)(s_1 - s_2)x_2^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)x_2(s_1 - s_2)}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& -\frac{105\mu x_3^2 x_2 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2 x_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& +\frac{945(1-\mu)s_1 x_2 x_3^4}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{525(1-\mu)s_1 x_2 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& +\frac{15(1-\mu)s_1 x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{945\mu A_2 x_2 x_3^4}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} \\
& -\frac{525\mu A_2 x_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu A_2 x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}},
\end{aligned}$$

$$\Omega_{x_3 x_3 x_2}(X_L, 0, 0) = 0.$$

$$\begin{aligned}
\Omega_{x_2 x_2 x_2}(X_L + x_1, x_2, x_3) = & -\frac{15(1-\mu)(x_1 - \mu)x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{6(1-\mu)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& + \frac{3(1-\mu)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} - \frac{15\mu x_2^3}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{6\mu x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{3\mu x_2}{\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{5}{2}}} \\
& - \frac{105(1-\mu)(2s_1 - s_2)x_2^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{30(1-\mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} + \frac{945(1-\mu)(s_1 - s_2)x_2^5}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} \\
& - \frac{210(1-\mu)(s_1 - s_2)x_2^3}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} - \frac{315(1-\mu)(s_1 - s_2)x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{90(1-\mu)(2s_1 - s_2)x_2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)(s_1 - s_2)x_2^3}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{30(1-\mu)(s_1 - s_2)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15(1-\mu)(s_1 - s_2)x_2}{\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& - \frac{105\mu A_2 x_2^3}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} + \frac{15\mu x_2 A_2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{7}{2}}} \\
& + \frac{945(1-\mu)s_1 x_2^3 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{315(1-\mu)s_1 x_2 x_3^2}{2\left[(X_L + x_1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}} \\
& + \frac{945\mu A_2 x_2^3 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{11}{2}}} - \frac{315\mu A_2 x_2 x_3^2}{2\left[(X_L + x_1 + 1 - \mu)^2 + x_2^2 + x_3^2\right]^{\frac{9}{2}}},
\end{aligned}$$

$$\Omega_{x_2 x_2 x_2}(X_L, 0, 0) = 0.$$

Equations (3.30), (3.31) and (3.32) can now be written as

$$\begin{aligned}
\Omega_{x_1}(X_L + x_1, x_2, x_3) = & x_1 \left[n^2 \beta + \frac{2(1-\mu)}{|X_L - \mu|^3} + \frac{2\mu}{|X_L + 1 - \mu|^3} + \frac{6(1-\mu)(2s_1 - s_2)}{|X_L - \mu|^5} + \frac{6\mu A_2}{|X_L + 1 - \mu|^5} \right] \\
& + \frac{1}{2} x_1^2 \left[-\frac{6(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} - \frac{6\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} - \frac{30(1-\mu)(2s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \right. \\
& \left. - \frac{30\mu(X_L + 1 - \mu)A_2}{|X_L + 1 - \mu|^7} \right] + \frac{1}{2} x_2^2 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} \right. \\
& \left. + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)(s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \right] \\
& + \frac{1}{2} x_3^2 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{45\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \right. \\
& \left. + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)s_1(X_L - \mu)}{|X_L - \mu|^7} \right],
\end{aligned}$$

$$\begin{aligned}
\Omega_{x_2}(X_L + x_1, x_2, x_3) = & x_2 \left[n^2 \beta - \frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} - \frac{3(1-\mu)(2s_1 - s_2)}{|X_L - \mu|^5} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5} \right] \\
& + x_1 x_2 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)(s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \right. \\
& \left. + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \right],
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{x_3}(X_L + x_1, x_2, x_3) = & x_3 \left[-\frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5} \right] + \\
& x_1 x_3 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \right],
\end{aligned}$$

respectively.

Thus, we can write Eqns. (3.29) as

$$\begin{aligned}
\ddot{x}_1 - 2\alpha n \dot{x}_2 = x_1 & \left[n^2 \beta + \frac{2(1-\mu)}{|X_L - \mu|^3} + \frac{2\mu}{|X_L + 1 - \mu|^3} + \frac{6(1-\mu)(2s_1 - s_2)}{|X_L - \mu|^5} + \frac{6\mu A_2}{|X_L + 1 - \mu|^5} \right] \\
& + \frac{1}{2} x_1^2 \left[-\frac{6(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} - \frac{6\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} - \frac{30(1-\mu)(2s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \right. \\
& \left. - \frac{30\mu(X_L + 1 - \mu)A_2}{|X_L + 1 - \mu|^7} \right] + \frac{1}{2} x_2^2 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} \right. \\
& \left. + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)(s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \right] \\
& + \frac{1}{2} x_3^2 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{45\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \right. \\
& \left. + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)s_1(X_L - \mu)}{|X_L - \mu|^7} \right],
\end{aligned}$$

$$\begin{aligned}
\ddot{x}_2 - 2\alpha n \dot{x}_1 = x_2 & \left[n^2 \beta - \frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} - \frac{3(1-\mu)(2s_1 - s_2)}{|X_L - \mu|^5} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5} \right] \\
& + x_1 x_2 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{15(1-\mu)(s_1 - s_2)(X_L - \mu)}{|X_L - \mu|^7} \right. \\
& \left. + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \right],
\end{aligned}$$

$$\begin{aligned}
\ddot{x}_3 = x_3 & \left[-\frac{(1-\mu)}{|X_L - \mu|^3} - \frac{\mu}{|X_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)}{2|X_L - \mu|^5} - \frac{3\mu A_2}{2|X_L + 1 - \mu|^5} \right] + \\
& x_1 x_3 \left[\frac{3(1-\mu)(X_L - \mu)}{|X_L - \mu|^5} + \frac{15(1-\mu)(2s_1 - s_2)(X_L - \mu)}{2|X_L - \mu|^7} + \frac{3\mu(X_L + 1 - \mu)}{|X_L + 1 - \mu|^5} + \frac{15\mu(X_L + 1 - \mu)A_2}{2|X_L + 1 - \mu|^7} \right].
\end{aligned}$$

Or

$$\begin{aligned}
\ddot{x}_1 - 2\alpha n \dot{x}_2 &= A_{100} x_1 + A_{200} x_1^2 + A_{020} x_2^2 + A_{002} x_3^2, \\
\ddot{x}_2 + 2\alpha n \dot{x}_1 &= B_{020} x_2 + B_{110} x_1 x_2, \\
\ddot{x}_3 &= C_{001} x_3 + C_{101} x_1 x_3,
\end{aligned} \tag{3.33}$$

where

$$A_{100} = bn^2 + 2R_0 + \frac{6\mu A_2}{r_{20}^5} + \frac{6(1-\mu)(2s_1 - s_2)}{r_{10}^5},$$

$$A_{200} = -3(1-\mu)\sigma_1\left[\frac{1}{r_{10}^4} + \frac{5(2s_1-s_2)}{r_{10}^6}\right] - 3\mu\sigma_2\left(\frac{1}{r_{20}^4} + \frac{5A_2}{r_{20}^6}\right),$$

$$A_{020} = 3(1-\mu)\sigma_1\left[\frac{1}{2r_{10}^4} + \frac{5(4s_1-3s_2)}{4r_{10}^6}\right] + 3\mu\sigma_2\left(\frac{1}{2r_{20}^4} + \frac{5A_2}{4r_{20}^6}\right),$$

$$A_{002} = 3\sigma_1\left[\frac{5\mu A_2}{2r_{10}^6} + (1-\mu)\left(\frac{1}{2r_{10}^4} + \frac{5(4s_1-s_2)}{4r_{10}^6}\right)\right] + 3\mu\sigma_2\left(\frac{1}{2r_{20}^4} + \frac{5A_2}{4r_{20}^6}\right),$$

$$B_{010} = bn^2 - R_0 - \frac{3\mu A_2}{2r_{20}^5} - \frac{3(1-\mu)(4s_1-3s_2)}{2r_{10}^5},$$

$$B_{110} = -3(1-\mu)\sigma_1\left[\frac{1}{r_{10}^4} + \frac{5(4s_1-3s_2)}{2r_{10}^6}\right] + 3\mu\sigma_2\left(\frac{1}{r_{20}^4} + \frac{5A_2}{2r_{20}^6}\right),$$

$$C_{001} = -R_0 - \frac{3\mu A_2}{2r_{20}^5} - \frac{3(1-\mu)(4s_1-s_2)}{2r_{10}^5},$$

$$C_{101} = 3\sigma_1(1-\mu)\left(\frac{1}{r_{10}^4} + \frac{5(4s_1-s_2)}{2r_{10}^6}\right) + 3\mu\sigma_2\left(\frac{1}{r_{20}^4} + \frac{5A_2}{2r_{20}^6}\right).$$

3.5.2 Semi-analytical approximation of the 3D periodic orbits for the model with triaxial and oblate primaries

We consider now that the sought solution of system (3.33) in powers of the orbital parameter e is of the following form:

$$x_1 = x_{12}e^2, \quad x_2 = x_{22}e^2, \quad x_3 = x_{31}e.$$

Putting them into (3.33) and equating the coefficients of the same powers of e , we find:

$$\begin{aligned} x_{12}'' - 2\alpha n x_{22}' - A_{100}x_{12} - A_{002}x_{31}^2 &= 0, \\ x_{22}' + 2\alpha n x_{12}' - B_{010}x_{22} &= 0, \end{aligned} \tag{3.34a}$$

and

$$x_{31}'' - C_{001} a x_{31} = 0. \tag{3.34b}$$

Setting in (3.34b) $x_{31}(t) = c_{131}\cos(wt) + s_{131}\sin(wt)$, we find that $(w^2 + C_{001})x_{31}(t) = 0$ which has non-zero solution only when $w^2 = -C_{001} > 0$. Additionally, by setting:

$$\begin{aligned} x_{12} &= c_{210} + c_{211}\cos(wt) + c_{212}\cos(2wt) + s_{211}\sin(wt) + s_{212}\sin(2wt), \\ x_{22} &= c_{221}\cos(wt) + c_{222}\cos(2wt) + s_{221}\sin(wt) + s_{222}\sin(2wt), \end{aligned}$$

and working as was done previously in the planar case, we find that the solution of (3.34a) is:

$$x_{12} = c_{210} + c_{211}\cos 2wt, \quad (3.35a)$$

$$x_{12} = s_{222}\sin 2wt, \quad (3.35b)$$

where

$$\begin{aligned} c_{210} &= \frac{-A_{002}}{2A_{100}w^2}, \\ c_{212} &= \frac{A_{002}(B_{010} + 4w^2)}{2w^2p}, \\ s_{222} &= \frac{-2\alpha nA_{002}}{wp}, \end{aligned} \quad (3.36)$$

while $w = \sqrt{-C_{001}}$, $p = 16w^4 + 4(A_{100} + B_{010} - 4\alpha^2n^2)w^2 + A_{100}B_{010}$. Therefore, the periodic solution up to second order terms w.r.t. the orbital parameter e is:

$$\begin{aligned} x_1 &= x_0 + e^2 [c_{210} + c_{212}\cos(2wt)], \\ x_2 &= e^2 s_{222}\sin 2wt, \\ x_3 &= e\sin(wt). \end{aligned} \quad (3.37)$$

As a consequence, the 2D and 3D second order approximate solutions are presented in Eqns. 3.22 and 3.37 respectively, when the primary is a triaxial rigid body and the secondary an oblate spheroidal body together with small perturbations in the Coriolis and centrifugal forces.

In the next chapter, we consider periodic motions around the collinear equilibrium points of the R3BP when the primaries are firstly, both radiating oblate spheroidal bodies and

secondly, radiating as well as triaxial rigid bodies together with perturbations in the Coriolis and centrifugal forces.

CHAPTER FOUR

PERIODIC MOTIONS AROUND THE COLLINEAR EQUILIBRIUM POINTS OF THE PHOTOGRAVITATIONAL R3BP WITH TRIAXIALITY, OBLATENESS AND SMALL PERTURBATIONS IN THE CORIOLIS AND CENTRIFUGAL FORCES

4.1 Introduction

In this chapter, we examine and obtain a third order analytic approximation solution of Lyapunov orbits around the collinear equilibrium in the planar circular restricted three body problem (PCR3BP) when the primaries are both radiating and oblate spheroidal bodies by utilizing the Lindstedt-Poincaré method. From the approximate periodic solution, the initial conditions or starting orbits near the collinear points have been obtained.

Also, a second order semi-analytic approximation solution of Lyapunov orbits around the collinear equilibrium in the R3BP for the spatial case of the same problem is obtained and using the same method. The results obtained are thereafter continued to corresponding families of periodic orbits using differential correction methods.

In addition, for both 2D and 3D periodic orbits around the collinear equilibrium points in the R3BP when the primaries are both triaxial and radiating spheroids taking into account small perturbations in the Coriolis and centrifugal forces (Singh and Begha, 2011), second order semi-analytical solution are being constructed.

4.2 Third Order Analytic Approximation Solutions of Lyapunov Orbits Around the Collinear Equilibrium Points: The Planar Case

4.2.1 Two-dimensional equations of motion for the radiating oblate primaries

Let, m_1, m_2 and m be the masses of the primary, secondary and infinitesimal bodies respectively. Here, the primary bodies are moving in circular orbits about their common barycentre, while the infinitesimal body is moving and exerting no influence in the plane of motion of the primaries. The mass parameter is given by $\mu = \frac{m_2}{m_1 + m_2}$. Let the unit of distance be taken as the distance between the primaries, such that the gravitational constant $G = 1$. The unit of mass has been chosen so that $m_1 + m_2 = 1$ and we take $m_1 = 1 - \mu$ and $m_2 = \mu$. We let Oxy be the synodic coordinate system with the position of the infinitesimal body as $P(x, y)$ and the primary and secondary bodies as $P_1(\mu, 0)$ and $P_2(-(1 - \mu), 0)$ respectively. Thus, the equations of motion of the infinitesimal body in the dimensionless synodic coordinate system with radiation pressure parameters q_1 and q_2 ($q_i \leq 1, i = 1, 2$) and oblateness parameters A_1 and A_2 ($A_i \leq 1, i = 1, 2$) (same model as in Singh & Ishwar (1999)) are

$$\begin{aligned}\ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y,\end{aligned}\tag{4.1}$$

with

$$\Omega = \frac{n^2}{2}(x^2 + y^2) + \frac{(1 - \mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1 - \mu)A_1 q_1}{2r_1^3} + \frac{\mu A_2 q_2}{2r_2^3},$$

where

$$\Omega_x = n^2 x - \frac{q_1(1 - \mu)(x - \mu)}{r_1^3} - \frac{q_2\mu(x + 1 - \mu)}{r_2^3} - \frac{3A_1 q_1(1 - \mu)(x - \mu)}{2r_1^5} - \frac{3A_2 q_2\mu(x + 1 - \mu)}{2r_2^5},$$

$$\Omega_y = n^2 y - \frac{q_1(1-\mu)y}{r_1^3} - \frac{q_2\mu y}{r_2^3} - \frac{3A_1q_1(1-\mu)y}{2r_1^5} - \frac{3A_2q_2\mu y}{2r_2^5},$$

$$r_1 = \sqrt{(x-\mu)^2 + y^2},$$

$$r_2 = \sqrt{(x-\mu+1)^2 + y^2},$$

and n is the mean motion, given as

$$n = \sqrt{1 + \frac{3}{2}(A_1 + A_2)}.$$

The Jacobi integral which is obtained from Equation (4.1) is given by

$$\dot{x}^2 + \dot{y}^2 = 2\Omega - C,$$

where the symbol C denotes the Jacobi Constant.

In Figure 4.1 below, there are three distinct curves which represent the Zero-velocity curves for the Jacobi Constant when the first, second and third collinear equilibrium points are considered with respect to the present model. The forbidden regions of motion for the infinitesimal body are within these curves, respectively.

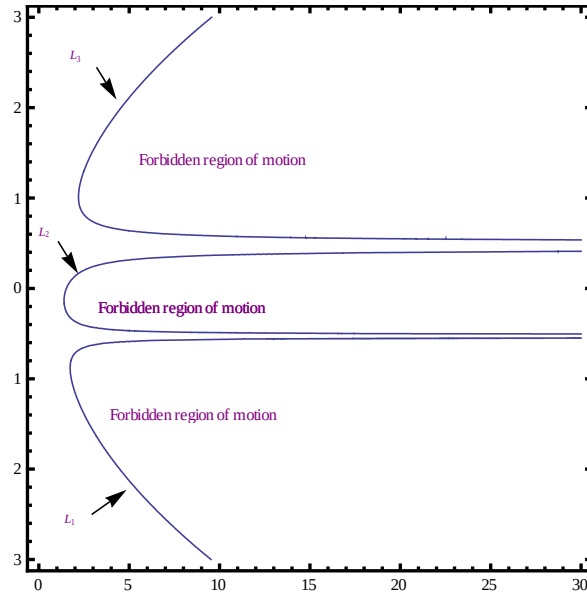


Figure 4. 1: The forbidden region of motion of the infinitesimal body with respect to the Jacobian constant C around the collinear equilibrium points.

4.2.2 Determination of the collinear equilibrium points for the model with radiating oblate primaries in the plane of motion

The positions of the collinear equilibrium points are obtained from the solution of the nonlinear algebraic equation $\Omega_x = 0$, when $y = 0$ by solving for x . That is, we solve

$$n^2 x - \frac{q_1(1-\mu)(x-\mu)}{|x-\mu|^3} - \frac{\mu q_2(x+1-\mu)}{|x-\mu|^3} - \frac{3A_1(1-\mu)(x-\mu)q_1}{2|x-\mu|^5} - \frac{3\mu A_2(x+1-\mu)q_2}{2|x+1-\mu|^5} = 0. \quad (4.2)$$

The solutions of Equation (4.2) have been found to exist within the intervals $(-\infty, -1+\mu)$, $(-1+\mu, \mu)$ and $(\mu, +\infty)$. By solving Equation (4.2) numerically, each of these intervals contain a real root which correspond to, L_1 , L_2 and L_3 respectively.]

4.2.3 Motion around the collinear equilibrium points for the model with radiation oblate primaries in the plane of motion

In order to investigate the motions around the collinear equilibrium points, we obtain a new coordinate system that takes any of L_i , $i = 1, 2, 3$ (the collinear equilibrium points) as the origin with the axes as ϕ and φ parallel to Ox and Oy respectively. Thus, by setting

$$x \rightarrow x_{L_i} + \phi \text{ and } y \rightarrow \varphi, \quad (4.3)$$

the equations of motion in Equations (4.1) become

$$\begin{aligned} \ddot{\phi} - 2n\dot{\varphi} &= \Omega_\phi, \\ \ddot{\varphi} + 2n\dot{\phi} &= \Omega_\varphi. \end{aligned} \quad (4.4)$$

Next, the R.H.S. of Equations (4.4) is expanded up to third order terms using the Taylor series expansion around (x_L, y_L) . This is done by following the ensuing procedures

$$\begin{aligned}
\Omega_{\phi}(x_L + \phi, \varphi) &= \Omega_{\phi}(x_L, y_L) + \phi \Omega_{\phi\phi}(x_L, y_L) + \varphi \Omega_{\phi\varphi}(x_L, y_L) + \frac{1}{2} \phi^2 \Omega_{\phi\phi\phi}(x_L, y_L) + \phi \varphi \Omega_{\phi\phi\varphi}(x_L, y_L) \\
&+ \frac{1}{2} \varphi^2 \Omega_{\phi\varphi\varphi}(x_L, y_L) + \frac{1}{6} \phi^3 \Omega_{\phi\phi\phi\phi}(x_L, y_L) + \frac{1}{2} \phi^2 \varphi \Omega_{\phi\phi\phi\varphi}(x_L, y_L) + \frac{1}{2} \phi \varphi^2 \Omega_{\phi\phi\varphi\varphi}(x_L, y_L) \\
&+ \frac{1}{6} \varphi^3 \Omega_{\phi\varphi\varphi\varphi}(x_L, y_L),
\end{aligned}$$

$$\begin{aligned}
\Omega_{\varphi}(x_L + \phi, \varphi) &= \Omega_{\varphi}(x_L, y_L) + \phi \Omega_{\varphi\phi}(x_L, y_L) + \varphi \Omega_{\varphi\varphi}(x_L, y_L) + \frac{1}{2} \phi^2 \Omega_{\varphi\phi\phi}(x_L, y_L) + \phi \varphi \Omega_{\varphi\phi\varphi}(x_L, y_L) \\
&+ \frac{1}{2} \varphi^2 \Omega_{\varphi\varphi\varphi}(x_L, y_L) + \frac{1}{6} \phi^3 \Omega_{\varphi\phi\phi\phi}(x_L, y_L) + \frac{1}{2} \phi^2 \varphi \Omega_{\varphi\phi\phi\varphi}(x_L, y_L) + \frac{1}{2} \phi \varphi^2 \Omega_{\varphi\phi\varphi\varphi}(x_L, y_L) \\
&+ \frac{1}{6} \varphi^3 \Omega_{\varphi\varphi\varphi\varphi}(x_L, y_L),
\end{aligned}$$

with

$$\Omega_{\phi}(x_L, y_L)_0 = \Omega_{\phi}(0, 0) = 0,$$

$$\Omega_{\varphi}(x_L, y_L)_0 = \Omega_{\varphi}(0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\phi\phi}(x_L + \phi, \varphi) &= n^2 + \frac{15A_1(1-\mu)q_1(x_L + \phi - \mu)^2}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{3A_1(1-\mu)q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} + \frac{3(1-\mu)q_1(x_L + \phi - \mu)^2}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \\
&- \frac{(1-\mu)q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{3}{2}}} + \frac{15\mu q_2(x_L + 1 + \phi - \mu)^2 A_2}{2[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{3\mu q_2 A_2}{2[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \\
&+ \frac{3\mu q_2(x_L + 1 + \phi - \mu)^2}{[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} - \frac{\mu q_2}{[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{3}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\phi\phi}(x_L, 0) &= n^2 + \frac{15A_1(1-\mu)q_1}{2|x_L - \mu|^5} - \frac{3A_1(1-\mu)q_1}{2|x_L - \mu|^5} + \frac{3(1-\mu)q_1}{|x_L - \mu|^3} - \frac{(1-\mu)q_1}{|x_L - \mu|^3} + \frac{15\mu q_2 A_2}{2|x_L + 1 - \mu|^5} \\
&- \frac{3\mu q_2 A_2}{2|x_L + 1 - \mu|^5} + \frac{3\mu q_2}{|x_L + 1 - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3}
\end{aligned}$$

or

$$\Omega_{\phi\phi}(x_L, 0) = n^2 + \frac{6A_1(1-\mu)q_1}{|x_L - \mu|^5} + \frac{2(1-\mu)q_1}{|x_L - \mu|^3} + \frac{6\mu q_2 A_2}{|x_L + 1 - \mu|^5} + \frac{2\mu q_2}{|x_L + 1 - \mu|^3},$$

$$\begin{aligned}\Omega_{\phi\phi}(x_L + \phi, \varphi) = & n^2 + \frac{15A_1(1-\mu)q_1\varphi^2}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{3A_1(1-\mu)q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} + \frac{3(1-\mu)q_1\varphi^2}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \\ & - \frac{(1-\mu)q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{3}{2}}} + \frac{15\mu q_2\varphi^2 A_2}{2[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{3\mu q_2 A_2}{2[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \\ & + \frac{3\mu q_2\varphi^2}{[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} - \frac{\mu q_2}{[(x_L + 1 + \phi - \mu)^2 + \varphi^2]^{\frac{3}{2}}}\end{aligned}$$

or

$$\Omega_{\phi\phi}(x_L, 0) = n^2 - \frac{3A_1(1-\mu)q_1}{2|x_L - \mu|^5} - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{3\mu q_2 A_2}{2|x_L + 1 - \mu|^5} - \frac{\mu q_2}{|x_L + 1 - \mu|^3},$$

$$\begin{aligned}\Omega_{\phi\phi\phi}(x_L + \phi, \varphi) = & -\frac{105A_1(1-\mu)q_1(x_L + \phi - \mu)^3}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{45A_1(1-\mu)(x_L + \phi - \mu)q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{15A_1(1-\mu)(x_L + \phi - \mu)^3 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\ & + \frac{9(1-\mu)q_1(x_L + \phi - \mu)}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} - \frac{105A_2\mu q_2(x_L + \phi + 1 - \mu)^3}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{45A_2\mu(x_L + \phi + 1 - \mu)q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\ & - \frac{15A_2\mu(x_L + \phi + 1 - \mu)^3 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{9\mu q_2(x_L + \phi + 1 - \mu)}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}}\end{aligned}$$

or

$$\begin{aligned}\Omega_{\phi\phi\phi}(x_L, 0) = & -\frac{105A_1(1-\mu)q_1v_1}{2|x_L - \mu|^6} + \frac{45A_1(1-\mu)q_1v_1}{2|x_L - \mu|^6} - \frac{15A_1(1-\mu)q_1v_1}{|x_L - \mu|^4} + \frac{9(1-\mu)q_1v_1}{|x_L - \mu|^4} - \frac{105A_2\mu q_2v_2}{2|x_L + 1 - \mu|^6} \\ & + \frac{45A_2\mu q_2v_2}{2|x_L + 1 - \mu|^6} - \frac{15A_2\mu q_2v_2}{|x_L + 1 - \mu|^4} + \frac{9\mu q_2v_2}{|x_L + 1 - \mu|^4}\end{aligned}$$

or

$$\Omega_{\phi\phi\phi}(x_L, 0) = -\frac{30A_1(1-\mu)q_1v_1}{|x_L - \mu|^6} - \frac{6A_1(1-\mu)q_1v_1}{|x_L - \mu|^4} - \frac{30A_2\mu q_2v_2}{|x_L + 1 - \mu|^6} - \frac{6A_2\mu q_2v_2}{|x_L + 1 - \mu|^4},$$

$$\begin{aligned}
\Omega_{\varphi\varphi\varphi}(x_L + \phi, \varphi) = & -\frac{105A_1(1-\mu)q_1\varphi^3}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{45A_1(1-\mu)\varphi q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{15A_1(1-\mu)\varphi^3 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\
& + \frac{9(1-\mu)q_1\varphi}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} - \frac{105A_2\mu q_2\varphi^3}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{45A_2\mu\varphi q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\
& - \frac{15A_2\mu\varphi^3 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{9\mu q_2\varphi}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}}
\end{aligned}$$

or

$$\Omega_{\varphi\varphi\varphi}(x_L, 0) = 0,$$

$$\begin{aligned}
\Omega_{\phi\phi\phi\phi}(x_L + \phi, \varphi) = & \frac{945A_1(1-\mu)q_1(x_L + \phi - \mu)^4}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_1(1-\mu)(x_L + \phi - \mu)^2 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{105(1-\mu)(x_L + \phi - \mu)^4 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\
& + \frac{45(1-\mu)q_1 A_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{90(1-\mu)(x_L + \phi - \mu)^2 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{9(1-\mu)q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \\
& + \frac{945A_2\mu q_2(x_L + \phi + 1 - \mu)^4}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_2\mu(x_L + \phi + 1 - \mu)^2 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{105\mu(x_L + \phi + 1 - \mu)^4 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\
& + \frac{45\mu q_2 A_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{90\mu(x_L + \phi + 1 - \mu)^2 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{9\mu q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}}
\end{aligned}$$

or

$$\Omega_{\phi\phi\phi\phi}(x_L, 0) = \frac{180A_1(1-\mu)q_1}{|x_L - \mu|^7} + \frac{24(1-\mu)q_1}{|x_L - \mu|^5} + \frac{180A_2\mu q_2}{|x_L + 1 - \mu|^7} + \frac{24\mu q_2}{|x_L + 1 - \mu|^5},$$

$$\begin{aligned}
\Omega_{\varphi\varphi\varphi\varphi}(x_L + \phi, \varphi) = & \frac{945A_1(1-\mu)q_1\varphi^4}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_1(1-\mu)\varphi^2q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{105(1-\mu)\varphi^4q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\
& + \frac{45(1-\mu)q_1A_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{90(1-\mu)\varphi^2q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{9(1-\mu)q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \\
& + \frac{945A_2\mu q_2\varphi^4}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_2\mu\varphi^2q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{105\mu\varphi^4q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\
& + \frac{45\mu q_2A_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{90\mu\varphi^2q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{9\mu q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}}
\end{aligned}$$

or

$$\Omega_{\varphi\varphi\varphi\varphi}(x_L, 0) = \frac{45(1-\mu)q_1A_1}{2|x_L - \mu|^7} + \frac{9(1-\mu)q_1}{|x_L - \mu|^5} + \frac{45\mu q_2A_2}{2|x_L + 1 - \mu|^7} + \frac{9\mu q_2}{|x_L + 1 - \mu|^5},$$

$$\begin{aligned}
\Omega_{\phi\varphi}(x_L + \phi, \varphi) = & \frac{15A_1(1-\mu)q_1(x_L + \phi - \mu)\varphi}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{3(1-\mu)(x_L + \phi - \mu)\varphi q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} + \frac{15A_2\mu q_2(x_L + \phi + 1 - \mu)\varphi}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\
& + \frac{3\mu(x_L + \phi + 1 - \mu)\varphi q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}}
\end{aligned}$$

or

$$\Omega_{\phi\varphi}(x_L, 0) = 0,$$

$$\begin{aligned}
\Omega_{\phi\varphi\phi}(x_L + \phi, \varphi) = & -\frac{105A_1(1-\mu)q_1(x_L + \phi - \mu)^2\varphi}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{15A_1(1-\mu)\varphi q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{15(1-\mu)(x_L + \phi - \mu)^2\varphi q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\
& + \frac{3(1-\mu)\varphi q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} - \frac{105A_2\mu q_2(x_L + \phi + 1 - \mu)^2\varphi}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{15A_2\mu\varphi q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\
& - \frac{15\mu(x_L + \phi + 1 - \mu)^2\varphi q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{3\mu\varphi q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}}
\end{aligned}$$

or

$$\Omega_{\phi\phi\phi}(x_L, 0) = 0 ,$$

$$\begin{aligned} \Omega_{\phi\phi\phi}(x_L + \phi, \varphi) = & -\frac{105A_1(1-\mu)q_1(x_L + \phi - \mu)\varphi^2}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{15A_1(1-\mu)(x_L + \phi - \mu)q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{15(1-\mu)(x_L + \phi - \mu)\varphi^2 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\ & + \frac{3(1-\mu)(x_L + \phi - \mu)q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} - \frac{105A_2\mu q_2(x_L + \phi + 1 - \mu)\varphi^2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{15A_2\mu(x_L + \phi + 1 - \mu)q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\ & - \frac{15\mu(x_L + \phi + 1 - \mu)\varphi^2 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{3\mu(x_L + \phi + 1 - \mu)q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \end{aligned}$$

or

$$\Omega_{\phi\phi\phi}(x_L, 0) = \frac{15A_1(1-\mu)q_1\nu_1}{2|x_L - \mu|^6} + \frac{3(1-\mu)q_1\nu_1}{|x_L - \mu|^4} + \frac{15A_2\mu q_2\nu_2}{2|x_L + 1 - \mu|^6} + \frac{3\mu q_2\nu_2}{|x_L + 1 - \mu|^4} ,$$

$$\begin{aligned} \Omega_{\phi\phi\phi\phi}(x_L + \phi, \varphi) = & \frac{945A_1(1-\mu)q_1(x_L + \phi - \mu)^2\varphi^2}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} - \frac{105A_1(1-\mu)(x_L + \phi - \mu)^2 q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} - \frac{105(1-\mu)\varphi^2 A_1 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\ & + \frac{105(1-\mu)(x_L + \phi - \mu)^2 \varphi^2 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{15A_1(1-\mu)q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{15(1-\mu)(x_L + \phi - \mu)^2 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\ & - \frac{15(1-\mu)\varphi^2 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{3(1-\mu)q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{5}{2}}} + \frac{945A_2\mu q_2(x_L + \phi + 1 - \mu)^2 \varphi^2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\ & - \frac{105A_2\mu(x_L + \phi + 1 - \mu)^2 q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} - \frac{105\mu\varphi^2 A_2 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{105\mu(x_L + \phi + 1 - \mu)^2 \varphi^2 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\ & + \frac{15A_2\mu q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{15\mu(x_L + \phi - \mu)^2 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} - \frac{15\mu\varphi^2 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}} \\ & + \frac{3\mu q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{5}{2}}} \end{aligned}$$

or

$$\Omega_{\phi\phi\phi\phi}(x_L, 0) = -\frac{45A_1(1-\mu)q_1}{|x_L - \mu|^7} - \frac{12(1-\mu)q_1}{|x_L - \mu|^5} - \frac{45A_2\mu q_2}{|x_L + 1 - \mu|^7} - \frac{12\mu q_2}{|x_L + 1 - \mu|^5} ,$$

$$\begin{aligned}\Omega_{\phi\phi\phi\phi}(x_L + \phi, \varphi) = & \frac{945A_1(1-\mu)q_1(x_L + \phi - \mu)^3\varphi}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_1(1-\mu)(x_L + \phi - \mu)\varphi q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{105(1-\mu)(x_L + \phi - \mu)^3\varphi q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\ & - \frac{45(1-\mu)(x_L + \phi - \mu)\varphi q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{945A_2\mu q_2(x_L + \phi + 1 - \mu)^3\varphi}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_2\mu(x_L + \phi + 1 - \mu)\varphi q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\ & + \frac{105\mu(x_L + \phi + 1 - \mu)^3\varphi q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} - \frac{45\mu(x_L + \phi + 1 - \mu)\varphi q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}}\end{aligned}$$

or

$$\Omega_{\phi\phi\phi\phi}(x_L, 0) = 0,$$

$$\begin{aligned}\Omega_{\phi\phi\phi\phi}(x_L + \phi, \varphi) = & \frac{945A_1(1-\mu)q_1(x_L + \phi - \mu)\varphi^3}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_1(1-\mu)(x_L + \phi - \mu)\varphi q_1}{2[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} + \frac{105(1-\mu)(x_L + \phi - \mu)\varphi^3 q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\ & - \frac{45(1-\mu)(x_L + \phi - \mu)\varphi q_1}{[(x_L + \phi - \mu)^2 + \varphi^2]^{\frac{7}{2}}} + \frac{945A_2\mu q_2(x_L + \phi + 1 - \mu)\varphi^3}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{11}{2}}} - \frac{315A_2\mu(x_L + \phi + 1 - \mu)\varphi q_2}{2[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} \\ & + \frac{105\mu(x_L + \phi + 1 - \mu)\varphi^3 q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{9}{2}}} - \frac{45\mu(x_L + \phi + 1 - \mu)\varphi q_2}{[(x_L + \phi + 1 - \mu)^2 + \varphi^2]^{\frac{7}{2}}}\end{aligned}$$

or

$$\Omega_{\phi\phi\phi\phi}(x_L, 0) = 0,$$

while

$$\Omega_{\phi\varphi}(x_L + \phi, \varphi) = \Omega_{\phi\varphi}(x_L + \phi, \varphi),$$

$$\Omega_{\phi\varphi\phi}(x_L + \phi, \varphi) = \Omega_{\phi\varphi\phi}(x_L + \phi, \varphi),$$

$$\Omega_{\varphi\phi\phi}(x_L + \phi, \varphi) = \Omega_{\varphi\phi\phi}(x_L + \phi, \varphi),$$

$$\Omega_{\phi\varphi\varphi\varphi}(x_L + \phi, \varphi) = \Omega_{\phi\varphi\varphi\varphi}(x_L + \phi, \varphi),$$

$$\Omega_{\phi\phi\phi\phi}(x_L + \phi, \varphi) = \Omega_{\varphi\phi\phi\phi}(x_L + \phi, \varphi),$$

$$\Omega_{\phi\varphi\varphi\varphi}(x_L + \phi, \varphi) = \Omega_{\varphi\phi\varphi\varphi}(x_L + \phi, \varphi).$$

Thus, the R.H.S. of equations (4.4) become

$$\begin{aligned} \Omega_{\phi}(x_L + \phi, \varphi) = & \phi \left[n^2 + \frac{6A_1(1-\mu)q_1}{|x_L - \mu|^5} + \frac{2(1-\mu)q_1}{|x_L - \mu|^3} + \frac{6\mu q_2 A_2}{|x_L + 1 - \mu|^5} + \frac{2\mu q_2}{|x_L + 1 - \mu|^3} \right] \\ & + \phi^2 \left[-\frac{15A_1(1-\mu)q_1 v_1}{|x_L - \mu|^6} - \frac{3A_1(1-\mu)q_1 v_1}{|x_L - \mu|^4} - \frac{15A_2\mu q_2 v_2}{|x_L + 1 - \mu|^6} - \frac{3A_2\mu q_2 v_2}{|x_L + 1 - \mu|^4} \right] \\ & + \phi^3 \left[\frac{30A_1(1-\mu)q_1}{|x_L - \mu|^7} + \frac{4(1-\mu)q_1}{|x_L - \mu|^5} + \frac{30A_2\mu q_2}{|x_L + 1 - \mu|^7} + \frac{4\mu q_2}{|x_L + 1 - \mu|^5} \right] \\ & + \varphi^2 \left[\frac{15A_1(1-\mu)q_1 v_1}{4|x_L - \mu|^6} + \frac{3(1-\mu)q_1 v_1}{2|x_L - \mu|^4} + \frac{15A_2\mu q_2 v_2}{4|x_L + 1 - \mu|^6} + \frac{3\mu q_2 v_2}{2|x_L + 1 - \mu|^4} \right] \\ & + \phi\varphi^2 \left[-\frac{45A_1(1-\mu)q_1}{2|x_L - \mu|^7} - \frac{6(1-\mu)q_1}{|x_L - \mu|^5} - \frac{45A_2\mu q_2}{2|x_L + 1 - \mu|^7} - \frac{6\mu q_2}{|x_L + 1 - \mu|^5} \right], \end{aligned}$$

$$\begin{aligned} \Omega_{\varphi}(x_L + \phi, \varphi) = & \varphi \left[n^2 - \frac{3A_1(1-\mu)q_1}{2|x_L - \mu|^5} - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{3\mu q_2 A_2}{2|x_L + 1 - \mu|^5} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} \right] \\ & + \phi\varphi \left[\frac{15A_1(1-\mu)q_1 v_1}{2|x_L - \mu|^6} + \frac{3(1-\mu)q_1 v_1}{|x_L - \mu|^4} + \frac{15A_2\mu q_2 v_2}{2|x_L + 1 - \mu|^6} + \frac{3\mu q_2 v_2}{|x_L + 1 - \mu|^4} \right] \\ & + \phi^2\varphi \left[-\frac{45A_1(1-\mu)q_1}{2|x_L - \mu|^7} - \frac{6(1-\mu)q_1}{|x_L - \mu|^5} - \frac{45A_2\mu q_2}{2|x_L + 1 - \mu|^7} - \frac{6\mu q_2}{|x_L + 1 - \mu|^5} \right] \\ & + \varphi^3 \left[\frac{15(1-\mu)q_1 A_1}{4|x_L - \mu|^7} + \frac{3(1-\mu)q_1}{2|x_L - \mu|^5} + \frac{15\mu q_2 A_2}{4|x_L + 1 - \mu|^7} + \frac{3\mu q_2}{2|x_L + 1 - \mu|^5} \right]. \end{aligned}$$

Simplifying the last two equations, we get

$$\Omega_{\phi} = \hbar_1 \phi + \hbar_2 \phi^2 + \hbar_3 \phi^3 + \hbar_4 \varphi^2 + \hbar_5 \phi \varphi^2,$$

$$\Omega_{\varphi} = \tilde{\lambda}_1 \varphi + \tilde{\lambda}_2 \phi \varphi + \tilde{\lambda}_3 \phi^2 \varphi + \tilde{\lambda}_4 \varphi^3,$$

where

$$\hbar_1 = n^2 + 2q_1(1-\mu) \left(\frac{1}{r_{10}^3} + \frac{3A_1}{r_{10}^5} \right) + 2q_2\mu \left(\frac{1}{r_{20}^3} + \frac{3A_2}{r_{20}^5} \right),$$

$$\hbar_2 = -3(1-\mu)\nu_1 q_1 \left[\frac{5A_1}{r_{10}^6} + \frac{1}{r_{10}^4} \right] \nu_1 - 3\mu q_2 \left[\frac{5A_2}{r_{20}^6} + \frac{1}{r_{20}^4} \right] \nu_2,$$

$$\hbar_3 = 2(1-\mu)q_1 \left(\frac{15A_1}{r_{10}^7} + \frac{2}{r_{10}^5} \right) + 2\mu q_2 \left(\frac{15A_2}{r_{20}^7} + \frac{2}{r_{20}^5} \right),$$

$$\hbar_4 = \frac{3(1-\mu)q_1}{2} \left[\frac{5A_1}{2r_{10}^6} + \frac{1}{r_{10}^4} \right] \nu_1 + \frac{3\mu q_2}{2} \left[\frac{5A_2}{2r_{20}^6} + \frac{1}{r_{20}^4} \right] \nu_2,$$

$$\hbar_5 = -3(1-\mu)q_1 \left(\frac{15A_1}{2r_{10}^7} + \frac{2}{r_{10}^5} \right) - 3\mu q_2 \left(\frac{15A_2}{2r_{20}^7} + \frac{2}{r_{20}^5} \right),$$

$$\tilde{\lambda}_1 = n^2 - (1-\mu)q_1 \left(\frac{3A_1}{2r_{10}^5} + \frac{1}{r_{10}^3} \right) - \mu q_2 \left(\frac{3A_2}{2r_{20}^5} + \frac{1}{r_{20}^3} \right),$$

$$\tilde{\lambda}_2 = 3(1-\mu)q_1 \left[\frac{5A_1}{2r_{10}^6} + \frac{1}{r_{10}^4} \right] \nu_1 - 3\mu q_2 \left[\frac{5A_2}{2r_{20}^6} + \frac{1}{r_{20}^4} \right] \nu_2,$$

$$\tilde{\lambda}_3 = -3(1-\mu)q_1 \left[\frac{15A_1}{2r_{10}^7} + \frac{2}{r_{10}^5} \right] - 3\mu q_2 \left[\frac{15A_2}{2r_{20}^7} + \frac{2}{r_{20}^5} \right],$$

$$\tilde{\lambda}_4 = 3(1-\mu)q_1 \left[\frac{5A_1}{4r_{10}^7} + \frac{1}{2r_{10}^5} \right] + 3\mu q_2 \left[\frac{5A_2}{4r_{20}^7} + \frac{1}{2r_{20}^5} \right].$$

In order to avoid absolute values for each case, the symbols ν_1 and ν_2 are being used to represent the signs of $r_{10} = |x_1 - \mu|$ and $r_{20} = |x_1 + 1 - \mu|$ at any of the collinear equilibrium points.

We search for periodic solutions represented in the following equations in powers of a parameter ε :

$$\begin{aligned} \phi(\tau) &= \phi_1(\tau)\varepsilon + \phi_2(\tau)\varepsilon^2 + \phi_3(\tau)\varepsilon^3, \\ \varphi(\tau) &= \varphi_1(\tau)\varepsilon + \varphi_2(\tau)\varepsilon^2 + \varphi_3(\tau)\varepsilon^3, \end{aligned} \tag{4.5}$$

and time is expanded by the expression,

$$t = \kappa\tau, \quad \kappa = 1 + \rho_2\varepsilon^2. \tag{4.6}$$

The symbol ρ_2 has been chosen such that any secular term is eliminated in the course of the computations.

Equations (4.4) can now be written as

$$\begin{aligned}\ddot{\phi} - 2n\kappa\dot{\phi} &= \kappa^2\Omega_\phi, \\ \ddot{\phi} + 2n\kappa\dot{\phi} &= \kappa^2\Omega_\phi.\end{aligned}\tag{4.7}$$

Putting Equations (4.5) into (4.7) and equating the coefficients of $\varepsilon, \varepsilon^2$ and ε^3 , we get first order, second order and third order systems respectively, which can be solved successively.

4.2.3.1 The First order system

The equations obtained for the first order terms in ε are

$$\begin{aligned}\phi_1'' - 2n\phi_1' - \hbar_1\phi_1 &= 0, \\ \phi_1'' + 2n\phi_1' - \tilde{\lambda}_1\phi_1 &= 0.\end{aligned}\tag{4.8}$$

By defining a differential operator

$$\Gamma(\mathfrak{I}) = \begin{pmatrix} \omega^2 + \hbar_1 & \pm 2n\omega \\ \pm 2n\omega & \omega^2 + \tilde{\lambda}_1 \end{pmatrix} = \omega^4 + (\hbar_1 + \tilde{\lambda}_1 - 4n^2)\omega^2 + \hbar_1\tilde{\lambda}_1 = 0,\tag{4.9}$$

system (4.8) can also be written as

$$\Gamma(\mathfrak{I}) \begin{pmatrix} \phi_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\tag{4.10}$$

The periodic solution for system (4.10) is given by

$$\begin{aligned}\phi_1 &= \varpi \cos(\omega\tau) + \mathcal{G} \sin(\omega\tau), \\ \phi_1 &= \varpi^* \cos(\omega\tau) + \mathcal{G}^* \sin(\omega\tau),\end{aligned}$$

and the period of the periodic orbit is given by $T = \frac{2\pi}{\omega}$. We set $\varpi = 1$ and $\mathcal{G} = 0$ so that,

$$\varpi^* = 0 \quad \text{and} \quad \mathcal{G}^* = \frac{-2n\omega}{\tilde{\lambda}_1 + \omega^2}.$$

Thus, the periodic solution of system (4.10) become

$$\begin{aligned}\phi_1(\tau) &= \text{Cos}(\omega\tau), \\ \varphi_1(\tau) &= \mathcal{G}_1^* \text{Sin}(\omega\tau),\end{aligned}\tag{4.11}$$

where

$$\mathcal{G}_1^* = \frac{-A_{10} - \omega^2}{2n\omega}.$$

4.2.3.2 Second order system

This is given by

$$\Gamma(\mathfrak{I}) \begin{pmatrix} \phi_2 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \wp_1(\tau) \\ \wp_2(\tau) \end{pmatrix},\tag{4.12}$$

where

$$\begin{aligned}\wp_1(\tau) &= \hbar_2 \phi^2 + \hbar_4 \varphi^2, \\ \wp_2(\tau) &= \tilde{\lambda}_2 \phi \varphi.\end{aligned}\tag{4.13}$$

We substitute Equations (4.11) into (4.12) and then Equation (4.13) become

$$\begin{aligned}\wp_1(\tau) &= \varpi_0 + \varpi_1 \text{Cos}(2\omega\tau), \\ \wp_2(\tau) &= \mathcal{G}_2^* \text{Sin}(2\omega\tau),\end{aligned}$$

where

$$\begin{aligned}\varpi_0 &= \frac{-\hbar_2 + \hbar_3(\mathcal{G}_1^*)^2}{2\hbar_1}, \\ \varpi_1 &= \frac{-\hbar_2(\tilde{\lambda}_1 + 4\omega^2) + \mathcal{G}_1^*[4\tilde{\lambda}_2 n\omega + \hbar_3 \mathcal{G}_1^*(\tilde{\lambda}_1 + 4\omega^2)]}{2\hbar_1 \tilde{\lambda}_1 + 8(\hbar_1 + \tilde{\lambda}_1 - 4n^2)\omega^2 + 32\omega^4}, \\ \mathcal{G}_2^* &= \frac{4\hbar_2 n\omega - \mathcal{G}_1^*[\hbar_1 \tilde{\lambda}_2 + 4\omega(\hbar_3 n \mathcal{G}_1^* + \tilde{\lambda}_2 \omega)]}{2\hbar_1 \tilde{\lambda}_1 + 8(\hbar_1 + \tilde{\lambda}_1 - 4n^2)\omega^2 + 32\omega^4}.\end{aligned}$$

Thus, the periodic solution of system (4.12) is

$$\begin{aligned}\phi_2(\tau) &= \varpi_0 + \text{Cos}(\omega\tau) + \varpi_1 \text{Cos}(2\omega\tau), \\ \varphi_2(\tau) &= \mathcal{G}_1^* \text{Sin}(\omega\tau) + \mathcal{G}_2^* \text{Sin}(2\omega\tau).\end{aligned}\tag{4.14}$$

4.2.3.3 Third order system

In this aspect, we have

$$\Gamma(\mathfrak{I}) \begin{pmatrix} \phi_3 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} \wp_3(\tau) \\ \wp_4(\tau) \end{pmatrix}, \quad (4.15)$$

where

$$\wp_3(\tau) = 2n\rho_2\phi_1' + 2\hbar_3\phi_1\phi_2 + 2\hbar_1\rho_2\phi_1 + \hbar_4\phi_1^2\phi_1 + \hbar_5\phi_1^3 + 2\hbar_2\phi_1\phi_2,$$

$$\wp_4(\tau) = -2n\rho_2\phi_1' + \tilde{\lambda}_4\phi_1^3 + \tilde{\lambda}_2\phi_2\phi_1 + 2\tilde{\lambda}_1\rho_2\phi_1 + \tilde{\lambda}_3\phi_1^2\phi_1 + \tilde{\lambda}_2\phi_2\phi_1,$$

$$\text{and } \rho_2 = \frac{(\Psi_1 + \omega(\Psi_2 - \Psi_3))}{\Psi_4},$$

$$\Psi_1 = -\hbar_1(\mathcal{G}_1^*(\tilde{\lambda}_3 - 2\tilde{\lambda}_2\varpi_1 + 4\tilde{\lambda}_2\varpi_0 + 3\tilde{\lambda}_4\mathcal{G}_1^{*2}) + 2\tilde{\lambda}_2\mathcal{G}_2^*),$$

$$\Psi_2 = 2n(3\hbar_5 + 4\hbar_2\varpi_1 + 8\hbar_2\varpi_0 + \hbar_4\mathcal{G}_1^{*2} + 4\hbar_3\mathcal{G}_1^*\mathcal{G}_2^*),$$

$$\Psi_3 = (\mathcal{G}_1^*(\tilde{\lambda}_3 - 2\tilde{\lambda}_2\varpi_1 + 4\tilde{\lambda}_2\varpi_0 + 3\tilde{\lambda}_4\mathcal{G}_1^{*2}) + 2\tilde{\lambda}_2\mathcal{G}_2^*)\omega$$

$$\Psi_4 = (8(\hbar_1(\tilde{\lambda}_1\mathcal{G}_1^* - n\omega) + \omega^2(\tilde{\lambda}_1\mathcal{G}_1^* + n(-2n\mathcal{G}_1^* + \omega))))).$$

So, working as previously, the periodic solution of system (4.15) is obtained as

$$\phi_3(\tau) = \varpi_2 \cos(\omega\tau) + \varpi_3 \cos(3\omega\tau),$$

$$\varphi_3(\tau) = \mathcal{G}_3^* \sin(3\omega\tau),$$

where

$$\varpi_2 = -\frac{\Theta_{21} + \Theta_{22}}{\Theta_{23}}, \text{ with}$$

$$\Theta_{21} = (3\hbar_5 + 4\hbar_2(\varpi_1 + 2\varpi_0) + \mathcal{G}_1^*(\hbar_4\mathcal{G}_1^* + 4\hbar_3\mathcal{G}_2^*))(\tilde{\lambda}_1\mathcal{G}_1^* + n\omega),$$

$$\Theta_{22} = -(\tilde{\lambda}_3\mathcal{G}_1^* + 3\tilde{\lambda}_4\mathcal{G}_1^{*3} + 2\tilde{\lambda}_2(-\varpi_1\mathcal{G}_1^* + 2\varpi_0\mathcal{G}_1^* + \mathcal{G}_2^*))(\hbar_1 + n\mathcal{G}_1^*\omega),$$

$$\Theta_{23} = 2(-4n\omega(\hbar_1 + n\mathcal{G}_1^*\omega) + 2(\tilde{\lambda}_1\mathcal{G}_1^* + n\omega)(\hbar_1 + \omega^2)),$$

and

$$\varpi_3 = -\frac{\Theta_{31} + \Theta_{32}}{\Theta_{33}}, \text{ having}$$

$$\begin{aligned}\Theta_{31} &= \frac{3}{2n}(\tilde{\lambda}_3 \mathcal{G}_1^* - \tilde{\lambda}_4 \mathcal{G}_1^{3*} + 2\tilde{\lambda}_2(\varpi_1 \mathcal{G}_1^* + \mathcal{G}_2^*))\omega, \\ \Theta_{32} &= -\frac{1}{4}(\tilde{h}_5 + 4\tilde{h}_2 \varpi_1 - \mathcal{G}_1^*(\tilde{h}_4 \mathcal{G}_1^* + 4\tilde{h}_3 \mathcal{G}_2^*))(\tilde{\lambda}_1 + 9\omega^2), \\ \Theta_{33} &= 36n^2\omega^2 - (\tilde{h}_1 + 9\omega^2)(\tilde{\lambda}_1 + 9\omega^2),\end{aligned}$$

and

$$\mathcal{G}_3^* = -\frac{\Phi_1 + \Phi_2 + \Phi_3}{\Phi_4},$$

$$\begin{aligned}\Phi_1 &= \tilde{h}_1(\tilde{\lambda}_3 \mathcal{G}_1^* - \tilde{\lambda}_4 \mathcal{G}_1^{3*} + 2\tilde{\lambda}_2(\varpi_1 \mathcal{G}_1^* + \mathcal{G}_2^*)), \\ \Phi_2 &= -6n(\tilde{h}_5 + 4\tilde{h}_2 \varpi_1 - \mathcal{G}_1^*(\tilde{h}_4 \mathcal{G}_1^* + 4\tilde{h}_3 \mathcal{G}_2^*))\omega, \\ \Phi_3 &= 9(\tilde{\lambda}_3 \mathcal{G}_1^* - \tilde{\lambda}_4 \mathcal{G}_1^{3*} + 2\tilde{\lambda}_2(\varpi_1 \mathcal{G}_1^* + \mathcal{G}_2^*))\omega^2, \\ \Phi_4 &= 4(\tilde{h}_1 \tilde{\lambda}_1 + 9(\tilde{h}_1 + \tilde{\lambda}_1 - 4n^2)\omega^2 + 81\omega^4).\end{aligned}$$

Therefore, the third order approximation of periodic solution around the collinear equilibrium points as a function of parameter ε is being obtained as

$$\begin{aligned}\phi(\tau) &= [\cos(\omega\tau)]\varepsilon + [\varpi_0 + \varpi_1 \cos(2\omega\tau)]\varepsilon^2 + [\varpi_2 \cos(\omega\tau) + \varpi_3 \cos(3\omega\tau)]\varepsilon^3, \\ \varphi(\tau) &= [\mathcal{G}_1^* \sin(\omega\tau)]\varepsilon + [\mathcal{G}_2^* \sin(2\omega\tau)]\varepsilon^2 + [\mathcal{G}_3^* \sin(3\omega\tau)]\varepsilon^3, \\ \dot{\phi}(\tau) &= -\omega \sin(\omega\tau)\varepsilon - 2\omega \varpi_1 \sin(2\omega\tau)\varepsilon^2 - [\omega \varpi_2 \sin(\omega\tau) + 3\omega \varpi_3 \sin(3\omega\tau)]\varepsilon^3, \\ \dot{\varphi}(\tau) &= \omega \mathcal{G}_1^* \cos(\omega\tau)\varepsilon + 2\omega \mathcal{G}_2^* \cos(2\omega\tau)\varepsilon^2 + 3\omega \mathcal{G}_3^* \cos(3\omega\tau)\varepsilon^3.\end{aligned}\tag{4.16}$$

4.3 Analytic Approximation of Lyapunov Orbits Around the Collinear

Equilibrium Points: The Spatial Case

4.3.1 Three-dimensional equations of motion for the model with radiating oblate primaries of the spatial case

The equations of motion of the infinitesimal body in the dimensionless synodic coordinate

system $Oxyz$, with the primary and secondary having masses m_1 and m_2 $\left(\mu = \frac{m_2}{m_1 + m_2}\right)$

and the radiation pressure and oblateness parameters taken as q_1 and $q_2 (q_i \leq 1, i = 1, 2)$ and A_1

and $A_2 (A_i \leq 1, i = 1, 2)$ (same model as in Singh & Ishwar (1999)) respectively, are

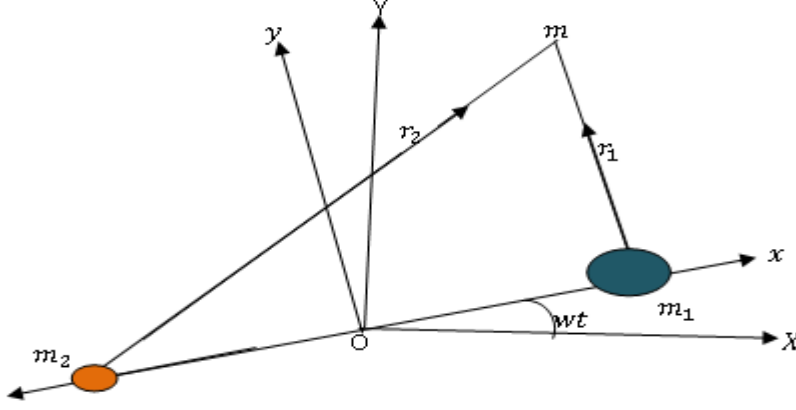


Figure 4.2: The configuration of the rotating coordinate system for the restricted three-body problem where m_1, m_2 and m are the oblate (and radiating) primaries and infinitesimal body respectively.

$$\begin{aligned}\ddot{x} - 2n\dot{y} &= \Omega_x, \\ \ddot{y} + 2n\dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z,\end{aligned}\tag{4.17}$$

where

$$\begin{aligned}\Omega &= \frac{1}{2}n^2(x^2 + y^2) + \frac{(1-\mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1-\mu)A_1q_1}{2r_1^3} + \frac{\mu A_2q_2}{2r_2^3} - \frac{3(1-\mu)z^2A_1q_1}{2r_1^5} - \frac{3\mu A_2q_2z^2}{2r_2^5}, \\ \Omega_x &= n^2x + \frac{15A_1(1-\mu)q_1(x-\mu)z^2}{2r_1^7} - \frac{3A_1(1-\mu)q_1(x-\mu)}{2r_1^5} - \frac{(1-\mu)q_1(x-\mu)}{r_1^3} + \frac{15A_2\mu q_2(x+1-\mu)z^2}{2r_2^7} \\ &\quad - \frac{3A_2\mu q_2(x+1-\mu)}{2r_2^5} - \frac{\mu q_2(x+1-\mu)}{r_2^3}, \\ \Omega_y &= n^2y + \frac{15A_1(1-\mu)q_1yz^2}{2r_1^7} - \frac{3A_1(1-\mu)q_1y}{2r_1^5} - \frac{(1-\mu)q_1y}{r_1^3} + \frac{15A_2\mu q_2yz^2}{2r_2^7} - \frac{3A_2\mu q_2y}{2r_2^5} - \frac{\mu q_2y}{r_2^3}, \\ \Omega_z &= \frac{15A_1(1-\mu)q_1z^3}{2r_1^7} - \frac{3A_1(1-\mu)q_1z}{r_1^5} - \frac{3A_1(1-\mu)q_1z}{2r_1^5} - \frac{(1-\mu)q_1z}{r_1^3} + \frac{15A_2\mu q_2z^3}{2r_2^7} - \frac{3A_2\mu q_2z}{r_2^5} \\ &\quad - \frac{3A_2\mu q_2z}{2r_2^5} - \frac{\mu q_2z}{r_2^3},\end{aligned}$$

and

$$n = \sqrt{1 + \frac{3}{2}(A_1 + A_2)}, \text{ and}$$

$$r_1 = \sqrt{(x - \mu) + y^2 + z^2},$$

$$r_2 = \sqrt{(x - \mu + 1) + y^2 + z^2}.$$

4.3.2 Equations of motion and variation for the for the model with radiating oblate primaries

In the 6-dimensional phase space, Eqns. (4.17) can be written in the form:

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_6), \quad i = 1, 2, \dots, 6,$$

with

$$f_1 = x_4$$

$$f_2 = x_5$$

$$f_3 = x_6$$

$$\begin{aligned} f_4 = & 2nx_5 + n^2x_1 + \frac{15A_1(1-\mu)q_1(x_1-\mu)x_3^2}{2r_{10}^7} - \frac{3A_1(1-\mu)q_1(x_1-\mu)}{2r_{10}^5} - \frac{(1-\mu)q_1(x_1-\mu)}{r_{10}^3} \\ & + \frac{15A_2\mu q_2(x_1+1-\mu)x_3^2}{2r_{20}^7} - \frac{3A_2\mu q_2(x_1+1-\mu)}{2r_{20}^5} - \frac{\mu q_2(x_1+1-\mu)}{r_{10}^3}, \\ f_5 = & -2nx_4 + n^2x_2 + \frac{15A_1(1-\mu)q_1x_2x_3^2}{2r_{10}^7} - \frac{3A_1(1-\mu)q_1x_2}{2r_{10}^5} - \frac{(1-\mu)q_1x_2}{r_{10}^3} + \frac{15A_2\mu q_2x_2x_3^2}{2r_{20}^7} \\ & - \frac{3A_2\mu q_2x_2}{2r_{20}^5} - \frac{\mu q_2x_2}{r_{20}^3}, \\ f_6 = & \frac{15A_1(1-\mu)q_1x_3^3}{2r_{10}^7} - \frac{9A_1(1-\mu)q_1x_3}{2r_{10}^5} - \frac{(1-\mu)q_1x_3}{r_{10}^3} + \frac{15A_2\mu q_2x_3^3}{2r_{20}^7} - \frac{9A_2\mu q_2x_3}{2r_{20}^5} - \frac{\mu q_2x_3}{r_{20}^3}, \end{aligned} \quad (4.18)$$

where

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = \dot{x}, \quad x_5 = \dot{y}, \quad x_6 = \dot{z},$$

and

$$r_{10} = [(x_1 - \mu)^2 + x_2^2 + x_3^2]^{\frac{1}{2}}, \quad r_{20} = [(x_1 + 1 - \mu)^2 + x_2^2 + x_3^2]^{\frac{1}{2}}.$$

The Jacobi integral w.r.t. the equations of motion in (4.18) is given by

$$C = n^2(x_1^2 + x_2^2) - \frac{3A_1(1-\mu)q_1x_3^2}{r_{10}^5} + \frac{A_1(1-\mu)q_1}{r_{10}^3} + \frac{2(1-\mu)q_1}{r_{10}} - \frac{3A_2\mu q_2x_3^2}{r_{20}^5} + \frac{A_2\mu q_2}{r_{20}^3} + \frac{2\mu q_2}{r_{20}} - (x_4^2 + x_5^2 + x_6^2). \quad (4.19)$$

The coordinates of the third body in the six-dimensional phase space depend uniquely along any solution on the initial conditions and the time, i.e.:

$$x_i = x_i(x_{01}, x_{02}, \dots, x_{06}; t), \quad i = 1, 2, \dots, 6.$$

Their partial derivatives with respect to the initial conditions satisfying the equations of variation (see Jain et al. 2006):

$$\frac{d}{dt} \left(\frac{\partial x_i}{\partial x_{0j}} \right) = \sum_{k=1}^6 \frac{\partial f_i}{\partial x_k} \frac{\partial x_k}{\partial x_{0j}}, \quad i = 1, 2, \dots, 6,$$

with

$$f_{ij} = \frac{\partial f_i}{\partial x_{0j}}, \quad \text{and}$$

$$f_{41} = n^2 - \frac{105A_1(1-\mu)q_1(x_1 - \mu)^2 x_3^2}{2r_1^9} + \frac{15A_1(1-\mu)q_1(x_1 - \mu)^2}{2r_1^7} + \frac{15A_1(1-\mu)q_1x_3^2}{2r_1^7} - \frac{3A_1(1-\mu)q_1}{2r_1^7} \\ + \frac{3(1-\mu)q_1(x_1 - \mu)^2}{r_1^5} - \frac{(1-\mu)q_1}{r_1^3} - \frac{105A_2\mu q_2(x_1 + 1 - \mu)^2 x_3^2}{2r_2^9} - \frac{15A_2\mu q_2(x_1 + 1 - \mu)^2}{2r_2^7} + \frac{15A_2\mu q_2x_3^2}{2r_2^7}, \\ - \frac{3A_2\mu q_2}{2r_2^5} + \frac{3\mu(x_1 + 1 - \mu)^2}{r_2^5} - \frac{\mu q_2}{r_2^3}$$

$$f_{42} = -\frac{105A_1(1-\mu)q_1(x_1 - \mu)x_2x_3^2}{2r_1^9} + \frac{15A_1(1-\mu)q_1(x_1 - \mu)x_2}{2r_1^7} + \frac{3(1-\mu)q_1(x_1 - \mu)x_2}{r_1^5} - \frac{105A_2\mu q_2(x_1 - \mu)x_2x_3^2}{2r_2^9} \\ + \frac{15A_2\mu q_2(x_1 + 1 - \mu)x_2}{2r_2^7} + \frac{3\mu q_2(x_1 + 1 - \mu)x_2}{r_2^5},$$

$$f_{43} = -\frac{105A_1(1-\mu)q_1(x_1 - \mu)x_3^3}{2r_1^9} + \frac{45A_1(1-\mu)q_1(x_1 - \mu)x_3}{2r_1^7} + \frac{3(1-\mu)q_1(x_1 - \mu)x_3}{r_1^5} - \frac{105A_2\mu q_2(x_1 + 1 - \mu)x_3^3}{2r_2^9} \\ + \frac{45A_2\mu q_2(x_1 + 1 - \mu)x_3}{2r_2^7} + \frac{3\mu q_2(x_1 + 1 - \mu)x_3}{r_2^5},$$

$$\begin{aligned}
f_{52} &= n^2 - \frac{105A_1(1-\mu)q_1x_2^3x_3^3}{2r_1^9} + \frac{15A_1(1-\mu)q_1x_2^2}{2r_1^2} + \frac{105A_1(1-\mu)q_1x_3^2}{2r_1^7} - \frac{3A_1(1-\mu)q_1}{2r_1^5} + \frac{3(1-\mu)q_1x_2^2}{2r_1^5} - \frac{(1-\mu)q_1}{r_1^3} \\
&\quad - \frac{105A_2\mu q_2x_2^2x_3^2}{2r_2^9} + \frac{15A_2\mu q_2x_2^2}{2r_1^7} + \frac{15A_2\mu q_2x_3^2}{2r_2^7} - \frac{3A_2\mu q_2}{2r_2^5} + \frac{3\mu q_2x_2^2}{r_2^5} - \frac{\mu q_2}{r_2^3}, \\
f_{53} &= -\frac{105A_1(1-\mu)q_1x_2x_3^3}{2r_1^9} + \frac{45A_1(1-\mu)q_1x_2x_3}{2r_1^7} + \frac{3(1-\mu)q_1x_2x_3}{r_1^5} - \frac{105A_2\mu q_2x_2x_3^3}{2r_2^9} + \frac{45A_2\mu q_2x_2x_3}{2r_2^7} + \frac{3\mu q_2x_2x_3}{r_2^5}, \\
f_{61} &= -\frac{105A_1(1-\mu)q_1(x_1-\mu)x_3^3}{2r_1^9} + \frac{45A_1(1-\mu)(x_1-\mu)q_1x_3}{2r_1^7} + \frac{3(1-\mu)(x_1-\mu)q_1x_3}{r_1^5} - \frac{105A_2\mu(x_1+1-\mu)q_2x_3^3}{2r_2^9} \\
&\quad + \frac{45A_2\mu(x_1+1-\mu)q_2x_3}{2r_2^7} + \frac{3\mu(x_1+1-\mu)q_2x_3}{2r_2^5}, \\
f_{62} &= f_{53}, \\
f_{63} &= -\frac{105A_1(1-\mu)q_1x_3^4}{2r_1^9} + \frac{45A_1(1-\mu)q_1x_3^2}{r_1^7} - \frac{9A_1(1-\mu)q_1}{2r_1^5} + \frac{3(1-\mu)q_1x_3^2}{r_1^5} - \frac{(1-\mu)q_1}{r_1^3} - \frac{105A_2\mu q_2x_3^4}{2r_2^9} \\
&\quad + \frac{45A_2\mu q_2x_3^2}{2r_2^7} - \frac{9A_2\mu q_2}{2r_2^5} + \frac{3\mu q_2x_3^2}{r_2^5} - \frac{\mu q_2}{r_2^3}.
\end{aligned}$$

4.3.3 Determination of the collinear equilibrium points for the model with radiating oblate primaries of the spatial case

The collinear equilibrium points lie on the x -axis together with the primary bodies. There are three intervals on this line: $(-\infty, -1+\mu)$, $(-1+\mu, \mu)$ and $(\mu, +\infty)$, as these are formed by the positions of the two stars. These equilibria are the solutions of the nonlinear algebraic equation arising from (4.18) for $x_2 = x_3 = 0$ and zero velocity and accelerator components:

$$f_4(x_1) = 0. \quad (4.20)$$

The determination of the exact number of roots of Equation (6), at each one of the above mentioned open intervals, has been established by an approach based on the topological degree theory and can be briefly described as follows: If the function $F(x): x \in [a, b] \subset \mathbb{R}$ is two times continuously differentiable in this interval, then the total number N of roots of the

equation $F(x)=0$ is obtained by the following scheme (Picard, 1892; Kalantonis et al (2001); Singh and Begha (2011)):

$$N = \frac{-\gamma}{\pi} \int_a^b \frac{F(x)F''(x) - F'^2(x)}{F^2(x) + \gamma^2 F'^2(x)} dx + \frac{1}{\pi} \arctan \left(\frac{\gamma F(a)F'(b) - F(b)F'(a)}{F(a)F(b) + \gamma^2 F'(a)F'(b)} \right),$$

where γ is a small positive real constant. For all the considered binary systems, i.e. for the parameter values correspond to the specific systems, we have shown that each one of the aforementioned open intervals contains a unique real root; these roots correspond to the collinear equilibrium points L_1 , L_2 and L_3 , respectively. Note, that the involved integral in the above formula has been computed numerically by using Romberg integration. Since we have determined the exact number of the collinear equilibrium points, we are able to solve numerically (4.20) so as to obtain their positions accurately.

4.3.4 The dynamics around the collinear equilibrium points for the model with radiating oblate primaries of the spatial case

The motion of the infinitesimal body near the collinear equilibrium points $L_{1,2,3}$ can be examined by defining a new coordinate system $\mathfrak{I}, \mathfrak{R}$ and $\mathfrak{N}$ for any of these points parallel to Ox , Oy and Oz . Such that

$$x_1 = x_{L_i} + \mathfrak{I}, x_2 = \mathfrak{R}, \text{ and } x_3 = \mathfrak{N}. \quad (4.21)$$

4.3.4.1 Linear stability of the collinear equilibrium points in the plane of motion

By using systems $x_1 = x_{L_i} + \mathfrak{I}$ and $x_2 = \mathfrak{R}$ in Equations (4.1), and expanding Ω in Taylor series around the collinear equilibrium points and taking only linear terms in $\mathfrak{I}$ and $\mathfrak{R}$ the equations of motion (4.1) become

$$\begin{aligned} \ddot{\mathfrak{I}} - 2n\dot{\mathfrak{R}} &= \mathfrak{I}\Omega_{xx}^0, \\ \ddot{\mathfrak{R}} + 2n\dot{\mathfrak{I}} &= \mathfrak{R}\Omega_{yy}^0, \end{aligned} \quad (4.22)$$

where

$$\Omega_{xx}^0 = n^2 + \frac{6A_1(1-\mu)q_1}{r_a^5} + \frac{2(1-\mu)q_1}{r_a^3} + \frac{6\mu q_2 A_2}{r_b^5} - \frac{2\mu q_2}{r_a^3},$$

$$\Omega_{yy}^0 = n^2 - \frac{3A_1(1-\mu)q_1}{2r_a^5} - \frac{(1-\mu)q_1}{r_a^3} - \frac{3\mu q_2 A_2}{2r_b^5} - \frac{\mu q_2}{r_a^3}.$$

Equations (4.22) represent the variational equations of the in-plane motion and the superscript 0 indicates that the derivatives are to be evaluated at the collinear equilibrium points $L_{1,2,3}$.

4.3.4.2 Characteristic Equations

Let the solution of Eqns. (4.22) be

$$\xi = \varpi e^{\omega t} \text{ and } \eta = \vartheta e^{\omega t}$$

where ϖ, ϑ and ω are constants. Then Eqns. (4.22) can be written as

$$\begin{aligned} \omega^2 \varpi e^{\omega t} - 2n\omega \vartheta e^{\omega t} &= \Omega_{xx}^0 \varpi e^{\omega t}, \\ \omega^2 \vartheta e^{\omega t} + 2n\omega \varpi e^{\omega t} &= \Omega_{yy}^0 \vartheta e^{\omega t} \end{aligned} \quad (4.23)$$

By representing system (4.23) in matrix notation, we obtain the non-trivial solution if

$$\begin{vmatrix} \omega^2 - \Omega_{xx}^0 & -2n\omega \\ 2n\omega & \omega^2 - \Omega_{yy}^0 \end{vmatrix} = 0. \quad (4.24)$$

By expanding the determinant of system (4.24), we obtain

$$\omega^4 + (\mathcal{P}_1 + \mathcal{Q}_4 - 4n^2)\omega^2 + \mathcal{P}_1\mathcal{Q}_4 = 0. \quad (4.25)$$

We examine the stability of the collinear equilibrium points in the plane of motion of the primaries by numerically solving the characteristic Eqn. (4.25).

4.3.5 Second order Taylor series expansions of the equations of motion for the model with radiating oblate primaries

We substitute Eqns. (4.21) into Eqns. (4.17), and expanding Ω in Taylor series around the collinear equilibrium points taking only linear terms in $\mathfrak{I}, \mathfrak{R}$ and $\mathfrak{N}$, the equations of motion (4.17) become

$$\begin{aligned}\ddot{\mathfrak{I}} - 2n\dot{\mathfrak{I}} &= \mathfrak{P}_1 \mathfrak{I} + \mathfrak{P}_2 \mathfrak{I}^2 + \mathfrak{P}_3 \mathfrak{R}^2 + \mathfrak{P}_4 \mathfrak{N}^2, \\ \ddot{\mathfrak{R}} + 2n\dot{\mathfrak{R}} &= \mathfrak{Q}_1 \mathfrak{R} + \mathfrak{Q}_2 \mathfrak{I} \mathfrak{R}, \\ \ddot{\mathfrak{N}} &= \mathfrak{R}_1 \mathfrak{N} + \mathfrak{R}_2 \mathfrak{I} \mathfrak{N},\end{aligned}\tag{4.26}$$

where

$$\begin{aligned}\mathfrak{P}_1 &= n^2 + 2(1-\mu)q_1\left[\frac{3A_1}{r_{10}^5} + \frac{1}{r_{10}^3}\right] + 2\mu q_2\left[\frac{3A_2}{r_{20}^5} + \frac{1}{r_{20}^3}\right], \\ \mathfrak{P}_2 &= 3(1-\mu)q_1\left[\frac{5A_1}{r_{10}^6} + \frac{1}{r_{10}^4}\right]\mathfrak{g}_1 + 3\mu q_2\left[\frac{5A_2}{r_{20}^6} + \frac{1}{r_{20}^4}\right]\mathfrak{g}_2, \\ \mathfrak{P}_3 &= -\frac{3(1-\mu)q_1}{2}\left[\frac{5A_1}{2r_{10}^6} + \frac{1}{r_{10}^4}\right]\mathfrak{g}_1 - \frac{3\mu q_2}{2}\left[\frac{5A_2}{2r_{20}^6} + \frac{1}{r_{20}^4}\right]\mathfrak{g}_2, \\ \mathfrak{P}_4 &= -\frac{3(1-\mu)q_1}{2}\left[\frac{15A_1}{2r_{10}^6} + \frac{1}{r_{10}^4}\right]\mathfrak{g}_1 - \frac{3\mu q_2}{2}\left[\frac{15A_2}{2r_{20}^6} + \frac{1}{r_{20}^4}\right]\mathfrak{g}_2, \\ \mathfrak{Q}_1 &= n^2 - (1-\mu)q_1\left[\frac{3A_1}{2r_{10}^5} + \frac{1}{r_{10}^3}\right] - \mu q_2\left[\frac{3A_2}{2r_{20}^5} + \frac{1}{r_{20}^3}\right], \\ \mathfrak{Q}_2 &= -3(1-\mu)q_1\left[\frac{15A_1}{2r_{10}^6} + \frac{1}{r_{10}^4}\right]\mathfrak{g}_1 - 3\mu q_2\left[\frac{15A_2}{2r_{20}^6} + \frac{1}{r_{20}^4}\right]\mathfrak{g}_2, \\ \mathfrak{R}_1 &= -(1-\mu)q_1\left[\frac{9A_1}{2r_{10}^5} + \frac{1}{r_{10}^3}\right] - \mu q_2\left[\frac{9A_2}{2r_{20}^5} + \frac{1}{r_{20}^3}\right], \\ \mathfrak{R}_2 &= -3(1-\mu)q_1\left[\frac{15A_1}{2r_{10}^6} + \frac{1}{r_{10}^4}\right]\mathfrak{g}_1 - 3\mu q_2\left[\frac{15A_2}{2r_{20}^6} + \frac{1}{r_{20}^4}\right]\mathfrak{g}_2.\end{aligned}$$

The symbols $\mathcal{G}_1$ and $\mathcal{G}_2$ represent the signs of $r_{10} = |x_1 - \mu|$ and $r_{20} = |x_1 + 1 - \mu|$ at any equilibrium point avoiding thus the absolute values for each case.

4.3.6 Semi-analytic approximation of periodic solutions for the the model with radiating oblate primaries of the spatial case

Using the method of successive approximations up to second order, system (4.26) admits periodic solutions which are of the following form:

$$\mathfrak{T} = \mathfrak{T}_{12}e^2, \mathfrak{R} = \mathfrak{R}_{22}e^2, \mathfrak{S} = \mathfrak{S}_{31}e, \quad (4.27)$$

Using (4.27) in system (4.26) we find that:

$$\begin{aligned} \ddot{\mathfrak{T}}_{12} - 2n\dot{\mathfrak{R}}_{22} - \mathcal{P}\mathfrak{T}_{12} - \mathcal{P}_4\mathfrak{S}_{31}^2 &= 0, \\ \dot{\mathfrak{R}}_{22} + 2n\dot{\mathfrak{T}}_{12} - \mathcal{Q}_4\mathfrak{R}_{22} &= 0. \end{aligned} \quad (4.28a)$$

and

$$\mathfrak{S}_{31}'' - \mathfrak{R}_1\mathfrak{S}_{31} = 0. \quad (4.28b)$$

We set Eqn. (4.28b):

$$\mathfrak{S}_{31}(t) = \alpha_{31}\text{Cos}(\omega t) + \beta_{31}\text{Sin}(\omega t),$$

and find that

$$(\omega^2 + \mathfrak{R}_1)\mathfrak{S}_{31}(t) = 0,$$

which has non-zero solution only when $\omega^2 = -\mathfrak{R}_1 > 0$. One of such solutions is

$$\mathfrak{S}_{31}(t) = \text{Sin}(\omega t), \quad (4.29)$$

which has been chosen in order to have initial velocity on the $\mathfrak{S}$ axis, so as to obtain the motion out of the orbital plane. Also, by setting:

$$\begin{aligned}\mathfrak{T}_{12} &= \alpha_{11} + \alpha_{12} \text{Cos}(\omega t) + \alpha_{13} \text{Cos}(2\omega t) + \beta_{11} \text{Sin}(\omega t) + \beta_{12} \text{Sin}(2\omega t), \\ \mathfrak{R}_{22} &= \alpha_{21} \text{Cos}(\omega t) + \alpha_{22} \text{Cos}(2\omega t) + \beta_{21} \text{Sin}(\omega t) + \beta_{22} \text{Sin}(2\omega t),\end{aligned}$$

we find that the solutions of system (4.28a) is:

$$\begin{aligned}\mathfrak{T}_{12} &= \alpha_{11} + \alpha_{12} \text{Cos}(2\omega t) , \\ \mathfrak{R}_{22} &= \beta_{22} \text{Sin}(2\omega t) ,\end{aligned}\tag{4.30}$$

where

$$\begin{aligned}\alpha_{11} &= \frac{-\mathcal{P}_4}{2\mathcal{P}_1\omega^2}, \\ \alpha_{12} &= \frac{\mathcal{P}_4(\mathcal{Q}_4 + 4\omega^2)}{2\omega^2 p}, \\ \beta_{22} &= \frac{-2n\mathcal{P}_4}{\omega p},\end{aligned}$$

while $\omega = \sqrt{-\mathcal{R}_1}$ and $p = 16\omega^4 + 4(\mathcal{P}_1 + \mathcal{Q}_4 - 4n^2)\omega^2 + \mathcal{P}_1\mathcal{Q}_4$.

So, the periodic solution up to second order terms w.r.t. systems (4.29) and (4.30) is:

$$\begin{aligned}\mathfrak{T} &= x_0 + e^2 [\alpha_{11} + \alpha_{12} \text{Cos}(2\omega t)], \\ \mathfrak{R} &= e^2 \beta_{22} \text{Sin}(2\omega t), \\ \mathfrak{S} &= e \text{Sin}(\omega t).\end{aligned}\tag{4.31}$$

From system (4.31) we obtain the initial conditions (for $t=0$) of a periodic orbit in the form:

$$(\mathfrak{T}, \mathfrak{R}, \mathfrak{S}, \dot{\mathfrak{T}}, \dot{\mathfrak{R}}, \dot{\mathfrak{S}})|_{t=0} = x_0 + (\alpha_{11} + \alpha_{12})e^2, 0, 0, 0, 2\omega e^2 \beta_{22}, \omega e ,\tag{4.32}$$

while the value of the orbital parameter e is arbitrarily chosen.

4.3.7 Numerical approximation of periodic solutions

The figure eight shaped orbits occur as a result of the gravitational pull of each star of the binary system on the motion of the infinitesimal body. Since three-dimensional periodic

orbits emanating from the collinear equilibrium points are of figure eight shape they are of double symmetry with respect to the Ox-axis and the Oxz plane. So, and for economy on the computations, it suffices to compute them at the quarter of their period. In particular, for the numerical determination of a three-dimensional periodic orbit of double symmetry w.r.t. the Ox-axis and the Oxz plane we integrate numerically the equations of motion with initial conditions of the form $(x_{01}, 0, 0, 0, x_{05}, x_{06})$, i.e. the numerical integration starts on the Ox-axis and seek a perpendicular crossing of the orbit with the Oxz plane at which obviously the condition $x_2(x_{01}, 0, 0, 0, x_{05}, x_{06}) = 0$ is fulfilled. Therefore, we look for the following two periodicity conditions:

$$x_4(x_{01}, 0, 0, 0, x_{05}, x_{06}) = 0, \quad x_6(x_{01}, 0, 0, 0, x_{05}, x_{06}) = 0. \quad (4.33)$$

Since two equations with three unknown components x_{01} , x_{05} and x_{06} of the initial state vector have to be satisfied we have to fix one unknown and apply well-known differential corrections procedures for the remaining two (see, e.g., Perdiou et al. 2015). So, for choosing, e.g., $x_{06} = \text{const.}$, and by linearization of (4.33) we obtain the corrector system:

$$\frac{\partial x_4}{\partial x_{01}} \delta x_{01} + \frac{\partial x_4}{\partial x_{05}} \delta x_{05} = -x_4, \quad \frac{\partial x_6}{\partial x_{01}} \delta x_{01} + \frac{\partial x_6}{\partial x_{05}} \delta x_{05} = -x_6. \quad (4.34)$$

The stability of a three-dimensional periodic orbit can be determined by integrating numerically the variational equations (4.16). Such an orbit will be stable if simultaneously the following conditions hold (Bray and Goudas (1967); Zagouras and Markellos (1977)):

$$|P| < 2 \quad \text{and} \quad |Q| < 2 \quad (4.35)$$

with $P = (a + \sqrt{\Delta})/2$ and $Q = (a - \sqrt{\Delta})/2$ while $\Delta = a^2 - 4(\beta - 2) > 0$ and $a = 2 - \text{Tr} V$, $\beta = (a^2 + 2 - \text{Tr} V^2)/2$, where V is the variational matrix. For stability of a three-dimensional periodic orbit in the restricted problem we also refer to Perdios (2007).

4.4 Periodic Solutions around the Collinear Equilibrium Points in the Perturbed

Restricted Three-Body Problem with Triaxial and Radiating Primaries

4.4.1 Equations of motion for the model with triaxial and radiating primaries

We consider the usual barycentric, rotating and dimensionless coordinate system Oxyz where its origin is located at the center of mass of the two primary bodies with masses $m_1 = 1 - \mu$ and $m_2 = \mu$, respectively, where $\mu = m_2 / (m_1 + m_2)$ is the mass parameter taking values in the interval $(0, 1/2]$. The two primaries are located at $(\mu, 0, 0)$ and $(\mu - 1, 0, 0)$, respectively, while the coordinates of the third body of negligible mass are (x, y, z) . Following Jain et al., (2016), the equations of motion of the massless body are described by the system:

$$\begin{aligned}\ddot{x} - 2\alpha n \dot{y} &= \Omega_x, \\ \ddot{y} + 2\alpha n \dot{x} &= \Omega_y, \\ \ddot{z} &= \Omega_z,\end{aligned}\tag{4.36}$$

where the potential function is given by:

$$\begin{aligned}\Omega = & \frac{n^2 \beta}{2} (x^2 + y^2) + \frac{(1 - \mu)q_1}{r_1} + \frac{\mu q_2}{r_2} + \frac{(1 - \mu)(2s_1 - s_2)q_1}{2r_1^3} + \frac{\mu(2s'_1 - s'_2)q_2}{2r_2^3} - \frac{3(1 - \mu)(s_1 - s_2)y^2 q_1}{2r_1^5} \\ & - \frac{3\mu(s'_1 - s'_2)y^2 q_2}{2r_2^5} - \frac{3(1 - \mu)s_1 z^2 q_1}{2r_1^5} - \frac{3\mu s'_1 z^2 q_2}{2r_2^5},\end{aligned}$$

with $r_1 = [(x - \mu)^2 + y^2 + z^2]^{1/2}$, $r_2 = [(x + 1 - \mu)^2 + y^2 + z^2]^{1/2}$ being the distances of the third body from the primary and secondary body, respectively. The perturbed, due to the triaxiality of the primaries, mean motion n is given by the formula:

$$n = \sqrt{1 + \frac{3}{2}[(2s_1 - s_2) + (2s'_1 - s'_2)]},$$

with $s_1 = (a_1^2 - a_3^2) / 5R^2$, $s_2 = (a_2^2 - a_3^2) / 5R^2$ and $s'_1 = (a_1'^2 - a_3'^2) / 5R^2$, $s'_2 = (a_2'^2 - a_3'^2) / 5R^2$ where a_1, a_2, a_3 are the semi-axes of the larger primary body, a'_1, a'_2, a'_3 are the semi-axes of the smaller one and R is the dimensional distance between the primaries. Also, q_1, q_2 are the radiation factors of the primaries. Finally, small perturbations ε and ε' are introduced with the help of parameters α and β in the Coriolis and centrifugal forces, correspondingly, such that $\alpha = 1 + \varepsilon$, $\beta = 1 + \varepsilon'$, $\varepsilon, \varepsilon' \ll 1$. And

$$\begin{aligned}\Omega_x &= \beta n^2 x - \frac{(1-\mu)q_1(x-\mu)}{r_1^3} - \frac{3(1-\mu)q_1(2s_1-s_2)(x-\mu)}{2r_1^5} + \frac{15(1-\mu)q_1(s_1-s_2)(x-\mu)y^2}{2r_1^7} \\ &+ \frac{15(1-\mu)q_1s_1(x-\mu)z^2}{2r_1^7} - \frac{\mu q_2(x+1-\mu)}{r_2^3} - \frac{3\mu q_2(2s'_1-s'_2)(x+1-\mu)}{2r_2^5} + \frac{15\mu q_2(s'_1-s'_2)(x+1-\mu)y^2}{2r_2^7} \\ &+ \frac{15\mu q_1s'_1(x-\mu)z^2}{2r_2^7}, \\ \Omega_y &= \beta n^2 y - \frac{(1-\mu)q_1y}{r_1^3} - \frac{\mu q_2y}{r_2^3} - \frac{3(1-\mu)q_1(2s_1-s_2)y}{2r_1^5} - \frac{3\mu q_2(2s'_1-s'_2)y}{2r_2^5} - \frac{3(1-\mu)q_1(s_1-s_2)y}{r_1^5} \\ &- \frac{3\mu q_2(s'_1-s'_2)y}{r_2^5} + \frac{15(1-\mu)q_1(s_1-s_2)y^3}{2r_1^7} + \frac{15\mu q_2(s'_1-s'_2)y^3}{2r_2^7} + \frac{15(1-\mu)q_1s_1yz^2}{2r_1^7} + \frac{15\mu q_1s'_1yz^2}{2r_2^7}, \\ \Omega_z &= -\frac{(1-\mu)q_1z}{r_1^3} - \frac{\mu q_2z}{r_2^3} - \frac{3(1-\mu)q_1(2s_1-s_2)z}{2r_1^5} - \frac{3\mu q_2(2s'_1-s'_2)z}{2r_2^5} - \frac{3(1-\mu)q_1s_1z}{r_1^5} - \frac{3\mu q_2s'_1z}{r_2^5} \\ &+ \frac{15(1-\mu)q_1(s_1-s_2)y^2z}{2r_1^7} + \frac{15\mu q_2(s'_1-s'_2)y^2z}{2r_2^7} + \frac{15(1-\mu)q_1s_1yz^3}{2r_1^7} + \frac{15\mu q_1s'_1yz^3}{2r_2^7}.\end{aligned}$$

4.4.2 Variational equations for the model with triaxial and radiating primaries

In the 6-dimensional phase space, Eqns. (4.36) can be written in the form:

$$\dot{x}_i = f_i(x_1, x_2, \dots, x_6), \quad i = 1, 2, \dots, 6, \quad (4.37)$$

with

$$f_1 = x_4,$$

$$f_2 = x_5,$$

$$f_3 = x_6,$$

$$\begin{aligned} f_4 = & 2\alpha n x_5 + \beta n^2 x_1 - \frac{(1-\mu)q_1(x_1-\mu)}{r_{10}^3} - \frac{\mu q_2(x_1+1-\mu)}{r_{20}^3} - \frac{3(1-\mu)q_1(2s_1-s_2)(x_1-\mu)}{2r_{10}^5} \\ & - \frac{3\mu q_2(2s'_1-s'_2)(x_1+1-\mu)}{2r_{20}^5} + \frac{15(1-\mu)q_1(s_1-s_2)(x_1-\mu)x_2^2}{2r_{10}^7} \\ & + \frac{15\mu q_2(s'_1-s'_2)(x_1+1-\mu)x_2^2}{2r_{20}^7} + \frac{15(1-\mu)q_1s_1(x_1-\mu)x_3^2}{2r_{10}^7} + \frac{15\mu q_1s'_1(x_1+1-\mu)x_3^2}{2r_{20}^7}, \\ f_5 = & -2\alpha n x_4 + \beta n^2 x_2 - \frac{(1-\mu)q_1x_2}{r_{10}^3} - \frac{\mu q_2x_2}{r_{20}^3} - \frac{3(1-\mu)q_1(2s_1-s_2)x_2}{2r_{10}^5} - \frac{3\mu q_2(2s'_1-s'_2)x_2}{2r_{20}^5} \\ & - \frac{3(1-\mu)q_1(s_1-s_2)x_2}{r_{10}^5} - \frac{3\mu q_2(s'_1-s'_2)x_2}{r_{20}^5} + \frac{15(1-\mu)q_1(s_1-s_2)x_2^3}{2r_{10}^7} \\ & + \frac{15\mu q_2(s'_1-s'_2)x_2^3}{2r_{20}^7} + \frac{15(1-\mu)q_1s_1x_2x_3^2}{2r_{10}^7} + \frac{15\mu q_2s'_1x_2x_3^2}{2r_{20}^7}, \\ f_6 = & -\frac{(1-\mu)q_1x_3}{r_{10}^3} - \frac{\mu q_2x_3}{r_{20}^3} - \frac{3(1-\mu)q_1(2s_1-s_2)x_3}{2r_{10}^5} - \frac{3\mu q_2(2s'_1-s'_2)x_3}{2r_{20}^5} - \frac{3(1-\mu)q_1(s_1-s_2)x_3}{r_{10}^5} \\ & - \frac{3\mu q_2(s'_1-s'_2)x_3}{r_{20}^5} + \frac{15(1-\mu)q_1(s_1-s_2)x_2^2x_3}{2r_{10}^7} + \frac{15\mu q_2(s'_1-s'_2)x_2^2x_3}{2r_{20}^7} + \frac{15(1-\mu)q_1s_1x_3^3}{2r_{10}^7} + \frac{15\mu q_2s'_1x_3^3}{2r_{20}^7}, \end{aligned}$$

where

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = \dot{x}, \quad x_5 = \dot{y}, \quad x_6 = \dot{z},$$

and

$$r_{10} = [(x_1 - \mu)^2 + x_2^2 + x_3^2]^{\frac{1}{2}}, \quad r_{20} = [(x_1 + 1 - \mu)^2 + x_2^2 + x_3^2]^{\frac{1}{2}}.$$

The Jacobi integral w.r.t. the equations of motion in (4.37) is given by

$$\begin{aligned} C = & n^2 \beta (x_1^2 + x_2^2) + \frac{2(1-\mu)q_1}{r_{10}} + \frac{2\mu q_2}{r_{20}} + \frac{(1-\mu)(2s_1-s_2)q_1}{r_{10}^3} + \frac{\mu(2s'_1-s'_2)q_2}{r_{20}^3} \\ & - \frac{3(1-\mu)(s_1-s_2)y^2q_1}{r_{10}^3} - \frac{3\mu(s'_1-s'_2)y^2q_2}{r_{20}^3} - \frac{3(1-\mu)s_1z^2q_1}{r_{10}^3} - \frac{3\mu s'_1z^2q_2}{r_{20}^3} - (x_4^2 + x_5^2 + x_6^2), \end{aligned} \tag{4.38}$$

The coordinates of the third body in the four dimensional phase space depend uniquely along any solution on the initial conditions and the time, i.e.:

$$x_i = x_i(x_{01}, x_{02}, \dots, x_{06}; t), \quad i = 1, 2, \dots, 6.$$

Their partial derivatives with respect to the initial conditions satisfying the equations of variation (see Jain et al. 2006):

$$\frac{d}{dt} \left(\frac{\partial x_i}{\partial x_{0j}} \right) = \sum_{k=1}^6 \frac{\partial f_i}{\partial x_k} \frac{\partial x_k}{\partial x_{0j}}, \quad i=1,2,\dots,6,$$

with

$$f_{ij} = \frac{\partial f_i}{\partial x_{0j}},$$

and

$$\begin{aligned} f_{41} = & \beta n^2 - \frac{(1-\mu)q_1}{r_{10}^3} - \frac{\mu q_2}{r_{20}^3} + \frac{3(1-\mu)q_1(x_1-\mu)^2}{r_{10}^5} + \frac{3\mu q_2(x_1+1-\mu)^2}{r_{20}^5} - \frac{3(1-\mu)q_1(2s_1-s_2)}{2r_{10}^5} \\ & - \frac{3\mu q_2(2s'_1-s'_2)}{2r_{20}^5} + \frac{15(1-\mu)q_1(s_1-s_2)x_2^2}{2r_{10}^7} + \frac{15\mu q_2(s'_1-s'_2)x_2^2}{2r_{20}^7} + \frac{15(1-\mu)q_1s_1x_3^2}{2r_{10}^7} \\ & + \frac{15\mu q_2s'_1x_3^2}{2r_{20}^7} + \frac{15(1-\mu)q_1(2s_1-s_2)(x_1-\mu)^2}{2r_{10}^7} + \frac{15\mu q_2(2s'_1-s'_2)(x_1+1-\mu)^2}{2r_{20}^7} \\ & - \frac{105(1-\mu)q_1(s_1-s_2)(x_1-\mu)^2x_2^2}{2r_{10}^9} - \frac{105(1-\mu)q_1s_1(x_1-\mu)^2x_3^2}{2r_{10}^9} \\ & - \frac{105\mu q_2s'_1(x_1+1-\mu)^2x_3^2}{2r_{20}^9} - \frac{105\mu q_2(s'_1-s'_2)(x_1+1-\mu)^2x_2^2}{2r_{20}^9}, \\ f_{42} = & \frac{3(1-\mu)q_1(x_1-\mu)x_2}{r_{10}^5} + \frac{3\mu q_2(x_1+1-\mu)x_2}{r_{20}^5} + \frac{15(1-\mu)q_1(2s_1-s_2)(x_1-\mu)x_2}{2r_{10}^7} + \frac{15\mu q_2(2s'_1-s'_2)(x_1+1-\mu)x_2}{2r_{20}^7} \\ & + \frac{15(1-\mu)q_1(s_1-s_2)(x_1-\mu)x_2}{r_{10}^7} - \frac{15\mu q_2(s'_1-s'_2)(x_1+1-\mu)x_2}{2r_{20}^9} - \frac{105(1-\mu)q_1(s_1-s_2)(x_1-\mu)x_2^3}{2r_{10}^9} \\ & - \frac{105\mu q_2(s'_1-s'_2)(x_1+1-\mu)x_2^3}{2r_{20}^9} - \frac{105(1-\mu)q_1s_1(x_1-\mu)x_2x_3^2}{2r_{10}^9} - \frac{105\mu q_2s'_1(x_1+1-\mu)x_2x_3^2}{2r_{20}^9}, \\ f_{43} = & \frac{3(1-\mu)q_1(x_1-\mu)x_3}{r_{10}^5} + \frac{3\mu q_2(x_1+1-\mu)x_3}{r_{20}^5} + \frac{15(1-\mu)q_1(2s_1-s_2)(x_1-\mu)x_3}{2r_{10}^7} \\ & + \frac{15\mu q_2(2s'_1-s'_2)(x_1+1-\mu)x_3}{2r_{20}^7} + \frac{15(1-\mu)q_1s_1(x_1-\mu)x_3}{r_{10}^7} - \frac{15\mu q_2s'_1(x_1+1-\mu)x_3}{r_{20}^7} \\ & - \frac{105(1-\mu)q_1(s_1-s_2)(x_1-\mu)x_2^2x_3}{2r_{10}^9} - \frac{105(1-\mu)q_1s_1(x_1-\mu)x_3^3}{2r_{10}^9} - \frac{105\mu q_2s'_1(x_1+1-\mu)x_3^3}{2r_{20}^9} \\ & - \frac{105\mu q_2(s'_1-s'_2)(x_1+1-\mu)x_2^2x_3}{2r_{20}^9}, \\ f_{51} = & f_{43}, \end{aligned}$$

$$\begin{aligned}
f_{52} = & \beta n^2 - \frac{(1-\mu)q_1}{r_{10}^3} - \frac{\mu q_2}{r_{20}^3} + \frac{3(1-\mu)q_1 x_2^2}{r_{10}^5} + \frac{3\mu q_2 x_2^2}{r_{20}^5} + \frac{3(1-\mu)q_1(2s_1 - s_2)}{2r_{10}^5} + \frac{3\mu q_2(2s'_1 - s'_2)}{2r_{20}^5} \\
& - \frac{3(1-\mu)q_1(s_1 - s_2)}{r_{10}^5} - \frac{3\mu q_2(s'_1 - s'_2)}{r_{20}^5} + \frac{15(1-\mu)q_1 s_1 x_3^2}{2r_{10}^7} + \frac{15\mu q_2 s'_1 x_3^2}{2r_{20}^7} + \frac{15(1-\mu)q_1(2s_1 - s_2)x_2^2}{r_{20}^7} \\
& + \frac{15\mu q_2(2s'_1 - s'_2)x_2^2}{r_{20}^7} + \frac{75(1-\mu)q_1(s_1 - s_2)x_2^2}{2r_{10}^7} + \frac{75\mu q_2(s'_1 - s'_2)x_2^2}{2r_{20}^7} - \frac{105(1-\mu)q_1 s_1 x_2^2 x_3^2}{2r_{10}^9} \\
& - \frac{105\mu q_2 s'_1 x_2^2 x_3^2}{2r_{20}^9} - \frac{105(1-\mu)q_1(s_1 - s_2)x_2^4}{2r_{10}^9} - \frac{105\mu q_2(s'_1 - s'_2)x_2^4}{2r_{20}^9},
\end{aligned}$$

$$\begin{aligned}
f_{53} = & \frac{3(1-\mu)q_1 x_2 x_3}{r_{10}^5} + \frac{3\mu q_2 x_2 x_3}{r_{20}^5} + \frac{15(1-\mu)q_1 s_1 x_2 x_3}{r_{10}^7} + \frac{15\mu q_2 s'_1 x_2 x_3}{r_{20}^7} + \frac{15(1-\mu)q_1(2s_1 - s_2)x_2 x_3}{2r_{20}^7} \\
& + \frac{15\mu q_2(2s'_1 - s'_2)x_2 x_3}{2r_{20}^7} + \frac{15(1-\mu)q_1(s_1 - s_2)x_2 x_3}{r_{10}^7} + \frac{15\mu q_2(s'_1 - s'_2)x_2 x_3}{r_{20}^7} - \frac{105(1-\mu)q_1 s_1 x_2 x_3^3}{2r_{10}^9} \\
& - \frac{105\mu q_2 s'_1 x_2 x_3^3}{2r_{20}^9} - \frac{105(1-\mu)q_1(s_1 - s_2)x_2^3 x_3}{2r_{10}^9} - \frac{105\mu q_2(s'_1 - s'_2)x_2^3 x_3}{2r_{20}^9},
\end{aligned}$$

$$\begin{aligned}
f_{61} = & \frac{3(1-\mu)q_1(x_1 + \mu)x_3}{r_{10}^5} + \frac{3\mu q_2(x_1 + 1 - \mu)x_3}{r_{20}^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_1 - \mu)x_3}{2r_{10}^7} \\
& + \frac{15\mu q_2(2s'_1 - s'_2)(x_1 + 1 - \mu)x_3}{2r_{10}^7} + \frac{15(1-\mu)q_1 s_1(x_1 - \mu)x_3}{r_{10}^7} + \frac{15\mu q_2 s'_1(x_1 + 1 - \mu)x_3}{r_{20}^7} \\
& - \frac{105(1-\mu)q_1(x_1 + \mu)s_1 x_3^3}{2r_{10}^9} - \frac{105\mu q_2 s'_1(x_1 + 1 - \mu)x_3^3}{2r_{20}^9} - \frac{105(1-\mu)q_1(s_1 - s_2)(x_1 + \mu)x_2^2 x_3}{2r_{10}^9} \\
& - \frac{105\mu q_2(s'_1 - s'_2)(x_1 + 1 - \mu)x_2^2 x_3}{2r_{20}^9},
\end{aligned}$$

$$f_{62} = f_{53},$$

$$\begin{aligned}
f_{63} = & -\frac{(1-\mu)q_1}{r_{10}^3} - \frac{\mu q_2}{r_{20}^3} + \frac{3(1-\mu)q_1 x_3^2}{r_{10}^5} + \frac{3\mu q_2 x_3^2}{r_{20}^5} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2r_{10}^5} - \frac{3\mu q_2(2s'_1 - s'_2)}{2r_{20}^5} \\
& - \frac{3(1-\mu)q_1 s_1}{r_{10}^5} - \frac{3\mu q_2 s'_1}{r_{20}^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)x_3^2}{2r_{10}^7} + \frac{15\mu q_2(2s'_1 - s'_2)x_3^2}{2r_{20}^7} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)x_2^2}{2r_{20}^7} + \frac{15\mu q_2(s'_1 - s'_2)x_2^2}{2r_{20}^7} + \frac{75(1-\mu)q_1 s_1 x_3^2}{2r_{10}^7} + \frac{75\mu q_2 s'_1 x_3^2}{2r_{20}^7} \\
& - \frac{105(1-\mu)q_1 s_1 x_3^4}{2r_{10}^9} - \frac{105\mu q_2 s'_1 x_3^4}{2r_{20}^9} - \frac{105(1-\mu)q_1(s_1 - s_2)x_2^2 x_3^2}{2r_{10}^9} - \frac{105\mu q_2(s'_1 - s'_2)x_2^2 x_3^2}{2r_{20}^9},
\end{aligned} \tag{4.39}$$

4.4.3 Determination of the collinear equilibrium points for the model with triaxial and radiating primaries

The collinear equilibrium points can be obtained numerically by solving for x_1 the following nonlinear algebraic equation, for the specific values of the parameters given in Table 1:

$$f_4(x_1; \mu, \beta, s_1, s_2, s'_1, s'_2, q_1, q_2) = 0. \quad (4.40)$$

As such, $x_2 = x_3 = x_4 = x_5 = x_6 = 0$ since the collinear equilibrium points lie on the Ox -axis, i.e. the same line joining the primary bodies, with zero velocity and acceleration components. On this line, there are three intervals $(-\infty, -1+\mu)$, $(-1+\mu, \mu)$ and $(\mu, +\infty)$ with each containing a real root that correspond to the collinear equilibrium points L_1 , L_2 and L_3 , respectively. Using numerical means, Eqn. (4.40) is solved and the corresponding collinear equilibrium points are obtained as shown in Table 5.22.

4.4.4 Motion around the collinear equilibrium points for the model with triaxial and radiating primaries

In order to study the motion of the infinitesimal body near any of the collinear equilibrium points, a new coordinate system is defined whereby L is any collinear equilibrium point $L_i, i=1,2,3$ and is also the origin with L_ξ , L_η , L_ζ as axes parallel to Ox , Oy and Oz , respectively. We write

$$x_1 = x_{L_i} + \xi, x_2 = \eta, \text{ and } x_3 = \zeta, \quad (4.41)$$

which are the relations showing the transformation from the latter to the former coordinate system and where $x_{L_i}, i=1,2,3$ is the position of the collinear equilibrium points.

Using Eqns. (4.41) in Eqns. (4.47), the equations of motion are transformed in the coordinate system as

$$\begin{aligned}
\ddot{\xi} - 2\alpha n \dot{\eta}_2 &= \Omega_{\xi}, \\
\ddot{\eta} + 2\alpha n \dot{\xi} &= \Omega_{\eta}, \\
\ddot{\zeta} &= \Omega_{\zeta}.
\end{aligned} \tag{4.42}$$

4.4.5 Expansions of the 2D equations of motion up to second order terms

The equations of motion of the infinitesimal body in the 2D plane is given as

$$\begin{aligned}
\ddot{x}_1 - 2\alpha n \dot{x}_2 &= \Omega_{x_1}, \\
\ddot{x}_2 + 2\alpha n \dot{x}_1 &= \Omega_{x_2},
\end{aligned} \tag{4.43}$$

where the force function is given by

$$\begin{aligned}
\Omega = & \frac{n^2 \beta}{2} (x_1^2 + x_2^2) + \frac{(1-\mu)q_1}{r_a} + \frac{\mu q_2}{r_b} + \frac{(1-\mu)(2s_1 - s_2)q_1}{2r_a^3} + \frac{\mu(2s'_1 - s'_2)q_2}{2r_b^3} - \frac{3(1-\mu)(s_1 - s_2)x_2^2 q_1}{2r_a^5} \\
& - \frac{3\mu(s'_1 - s'_2)x_2^2 q_2}{2r_b^5},
\end{aligned}$$

with the RHS of Eqn. (4.43) obtained as

$$\begin{aligned}
\Omega_{x_1} = & \beta n^2 x_1 - \frac{(1-\mu)q_1(x_1 - \mu)}{r_a^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)(x_1 - \mu)}{2r_a^5} + \frac{15(1-\mu)q_1(s_1 - s_2)(x_1 - \mu)x_2^2}{2r_a^7} \\
& - \frac{\mu q_2(x_1 + 1 - \mu)}{r_b^3} - \frac{3\mu q_2(2s'_1 - s'_2)(x_1 + 1 - \mu)}{2r_b^5} + \frac{15\mu q_2(s'_1 - s'_2)(x_1 + 1 - \mu)x_2^2}{2r_b^7}, \\
\Omega_{x_2} = & \beta n^2 x_2 - \frac{(1-\mu)q_1 x_2}{r_a^3} - \frac{\mu q_2 x_2}{r_b^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)x_2}{2r_a^5} - \frac{3\mu q_2(2s'_1 - s'_2)x_2}{2r_b^5} - \frac{3(1-\mu)q_1(s_1 - s_2)x_2}{r_a^5} \\
& - \frac{3\mu q_2(s'_1 - s'_2)x_2}{r_b^5} + \frac{15(1-\mu)q_1(s_1 - s_2)x_2^3}{2r_b^7} + \frac{15\mu q_2(s'_1 - s'_2)x_2^3}{2r_b^7},
\end{aligned}$$

and the distances between the infinitesimal body and primary and secondary stars are

$$r_a = \sqrt{(x_1 - \mu)^2 + x_2^2} \quad \text{and} \quad r_b = \sqrt{(x_1 + 1 - \mu)^2 + x_2^2}, \text{ respectively.}$$

By using the first and second equations of the system (4.41) in Eqns. (4.43), the equations of motion are transformed in the (ζ, η) coordinate system as

$$\ddot{\xi} - 2\alpha n \dot{\eta}_2 = \Omega_{\xi},$$

$$\ddot{\eta} + 2\alpha n \dot{\xi} = \Omega_{\eta}. \quad (4.44)$$

We expand the RHS of Eqns. (4.44) up to the second order using the Taylor series expansion around (x_L, y_L) . We proceed as

$$\begin{aligned} \Omega_{\xi}(x_L + \xi, \eta) &= \Omega_{\xi}(x_L, y_L) + \xi \Omega_{\xi\xi}(x_L, y_L) + \eta \Omega_{\xi\eta}(x_L, y_L) + \frac{1}{2} \xi^2 \Omega_{\xi\xi\xi}(x_L, y_L) + \xi \eta \Omega_{\xi\xi\eta}(x_L, y_L) \\ &\quad + \frac{1}{2} \eta^2 \Omega_{\xi\eta\eta}(x_L, y_L), \end{aligned}$$

and

$$\begin{aligned} \Omega_{\eta}(x_L + \xi, \eta) &= \Omega_{\eta}(x_L, y_L) + \xi \Omega_{\eta\xi}(x_L, y_L) + \eta \Omega_{\eta\eta}(x_L, y_L) + \frac{1}{2} \xi^2 \Omega_{\eta\xi\xi}(x_L, y_L) + \xi \eta \Omega_{\eta\xi\eta}(x_L, y_L) \\ &\quad + \frac{1}{2} \eta^2 \Omega_{\eta\eta\eta}(x_L, y_L). \end{aligned}$$

That is

$$\Omega_{\xi}(x_L, y_L)_0 = \Omega_{\xi}(0, 0) = 0,$$

$$\Omega_{\eta}(x_L, y_L)_0 = \Omega_{\eta}(0, 0) = 0,$$

$$\begin{aligned} \Omega_{\xi\xi}(x_L + \xi, \eta) &= n^2 \beta - \frac{(1-\mu)q_1}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{3}{2}}} - \frac{\mu q_2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{3}{2}}} + \frac{3(1-\mu)q_1(x_L + \xi - \mu)^2}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} \\ &\quad + \frac{3\mu q_2(x_L + \xi + 1 - \mu)^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{3(1-\mu)(2s_1 - s_2)q_1}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{3\mu(2s'_1 - s'_2)q_2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} \\ &\quad + \frac{15(1-\mu)q_1(s_1 - s_2)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\ &\quad + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\ &\quad - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^2 \eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2 \eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} \end{aligned}$$

or

$$\Omega_{\xi\xi}(x_L, 0) = n^2 \beta + \frac{2(1-\mu)q_1}{|x_L - \mu|^3} + \frac{2\mu q_2}{|x_L + 1 - \mu|^3} + \frac{6(1-\mu)q_1(2s_1 - s_2)}{|x_L - \mu|^5} + \frac{6\mu q_2(2s'_1 - s'_2)}{|x_L + 1 - \mu|^5},$$

$$\begin{aligned}
\Omega_{\xi\eta}(x_L + \xi, \eta) = & \frac{3(1-\mu)q_1(x_L + \xi - \mu)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2(x_L + \xi + 1 - \mu)\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

or

$$\Omega_{\xi\eta}(x_L, 0) = 0,$$

$$\begin{aligned}
\Omega_{\eta\eta}(x_L + \xi, \eta) = & \beta n^2 - \frac{(1-\mu)q_1}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{3}{2}}} - \frac{\mu q_2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{3}{2}}} + \frac{3(1-\mu)q_1\eta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} \\
& + \frac{3\mu q_2\eta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{3\mu q_2(2s'_1 - s'_2)}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} \\
& - \frac{3(1-\mu)q_1(s_1 - s_2)}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{3\mu q_2(s'_1 - s'_2)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} + \frac{15(1-\mu)q_1(2s_1 - s_2)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& + \frac{15\mu q_2(2s'_1 - s'_2)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{75(1-\mu)q_1(s_1 - s_2)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{75\mu q_2(s'_1 - s'_2)\eta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^4}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\eta^4}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

or

$$\Omega_{\eta\eta}(x_L, 0) = \beta n^2 - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(4s_1 - 3s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(4s'_1 - 3s'_2)}{2|x_L + 1 - \mu|^5},$$

$$\begin{aligned}
\Omega_{\xi\xi\xi}(x_L + \xi, \eta) = & \frac{9(1-\mu)q_1(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} + \frac{9\mu q_2(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{15(1-\mu)q_1(x_L + \xi - \mu)^3}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& - \frac{15\mu q_2(x_L + \xi + 1 - \mu)^3}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{45(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& + \frac{45\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} - \frac{315(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} \\
& - \frac{315\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} \\
& - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^3\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{11}{2}}} \\
& + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^3\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\xi\xi\xi}(x_L, 0) = & \frac{9(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{9\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} - \frac{15(1-\mu)q_1(x_L - \mu)^3}{|x_L - \mu|^7} - \frac{15\mu q_2(x_L + 1 - \mu)^3}{|x_L + 1 - \mu|^7} \\
& + \frac{45(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{45\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)^3}{2|x_L - \mu|^9} - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)^3}{2|x_L + 1 - \mu|^9},
\end{aligned}$$

$$\begin{aligned}
\Omega_{\xi\xi\eta}(x_L + \xi, \eta) = & \frac{3(1-\mu)q_1\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{15(1-\mu)q_1(x_L + \xi - \mu)^2\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& - \frac{15\mu q_2(x_L + \xi + 1 - \mu)^2\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15(1-\mu)q_1(2s_1 - s_2)\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} \\
& - \frac{105\mu q_2(s'_1 - s'_2)\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)^2\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} \\
& - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^2\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} \\
& + \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{9}{2}}} + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^2\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{11}{2}}} \\
& + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\xi\xi\eta}(x_L, 0) = 0,$$

$$\begin{aligned}
\Omega_{\xi\eta\eta}(x_L + \xi, \eta) = & \frac{3(1-\mu)q_1(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{5}{2}}} - \frac{15(1-\mu)q_1(x_L + \xi - \mu)\eta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& - \frac{15\mu q_2(x_L + \xi + 1 - \mu)\eta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} \\
& + \frac{15\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

$$\begin{aligned}
& - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2}{2[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{9}{2}}} - \frac{525(1 - \mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2}{2[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{9}{2}}} \\
& - \frac{525\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2}{2[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{9}{2}}} + \frac{945(1 - \mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^4}{2[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{11}{2}}} \\
& + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^4}{2[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\xi\eta\eta}(x_L, 0) &= \frac{3(1 - \mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1 - \mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \\
&+ \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{15(1 - \mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} \\
&+ \frac{15\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7},
\end{aligned}$$

$$\begin{aligned}
\Omega_{\eta\eta\eta}(x_L + \xi, \eta) &= \frac{9(1 - \mu)q_1\eta}{[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{5}{2}}} + \frac{9\mu q_2\eta}{[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{5}{2}}} - \frac{15(1 - \mu)q_1\eta^3}{[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{7}{2}}} \\
&- \frac{15\mu q_2\eta^3}{[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{7}{2}}} + \frac{45(1 - \mu)q_1(2s_1 - s_2)\eta}{2[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{7}{2}}} + \frac{45\mu q_2(2s'_1 - s'_2)\eta}{2[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{7}{2}}} \\
&+ \frac{90(1 - \mu)q_1(s_1 - s_2)\eta}{[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{7}{2}}} + \frac{90\mu q_2(s'_1 - s'_2)\eta}{[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{7}{2}}} - \frac{105(1 - \mu)q_1(2s_1 - s_2)\eta^3}{2[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{9}{2}}} \\
&- \frac{105\mu q_2(2s'_1 - s'_2)\eta^3}{2[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{9}{2}}} - \frac{945(1 - \mu)q_1(s_1 - s_2)\eta^3}{2[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{9}{2}}} - \frac{945\mu q_2(s'_1 - s'_2)\eta^3}{2[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{9}{2}}} \\
&+ \frac{945(1 - \mu)q_1(s_1 - s_2)\eta^5}{2[(x_L + \xi - \mu)^2 + \eta^2]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)\eta^5}{2[(x_L + \xi + 1 - \mu)^2 + \eta^2]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\eta\eta\eta}(x_L, 0) = 0,$$

and

$$\Omega_{\xi\eta}(x_L + \xi, \eta) = \Omega_{\eta\xi}(x_L + \xi, \eta),$$

$$\Omega_{\eta\xi\xi}(x_L + \xi, \eta) = \Omega_{\xi\xi\eta}(x_L + \xi, \eta),$$

$$\Omega_{\eta\xi\eta}(x_L + \xi, \eta) = \Omega_{\xi\eta\eta}(x_L + \xi, \eta).$$

The RHS of Eqns. (4.44) become

$$\begin{aligned} \Omega_{\xi} = & \xi \left[n^2 \beta + \frac{2(1-\mu)q_1}{|x_L - \mu|^3} + \frac{2\mu q_2}{|x_L + 1 - \mu|^3} + \frac{6(1-\mu)q_1(2s_1 - s_2)}{|x_L - \mu|^5} + \frac{6\mu q_2(2s'_1 - s'_2)}{|x_L + 1 - \mu|^5} \right] \\ & + \xi^2 \left[-3 \left\{ \frac{(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{5(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} \right. \right. \\ & \left. \left. + \frac{5\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right\} \right] + \eta^2 \left[\frac{3}{2} \left\{ \frac{(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} \right. \right. \\ & \left. \left. + \frac{5\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{5(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \right. \right. \\ & \left. \left. + \frac{5(1-\mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} + \frac{5\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right\} \right], \end{aligned}$$

and

$$\begin{aligned} \Omega_{\eta} = & \eta \left[\beta n^2 - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(4s_1 - 3s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(4s'_1 - 3s'_2)}{2|x_L + 1 - \mu|^5} \right] \\ & + \xi \eta \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(4s_1 - 3s_2)(x_L - \mu)}{2|x_L - \mu|^7} \right. \\ & \left. + \frac{15\mu q_2(4s'_1 - 3s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \right]. \end{aligned}$$

Equations (4.44) can now be written as

$$\begin{aligned} \ddot{\xi} - 2\alpha n \dot{\eta}_2 = & \xi \left[n^2 \beta + \frac{2(1-\mu)q_1}{|x_L - \mu|^3} + \frac{2\mu q_2}{|x_L + 1 - \mu|^3} + \frac{6(1-\mu)q_1(2s_1 - s_2)}{|x_L - \mu|^5} + \frac{6\mu q_2(2s'_1 - s'_2)}{|x_L + 1 - \mu|^5} \right] \\ & + \xi^2 \left[-3 \left\{ \frac{(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{5(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} \right. \right. \\ & + \left. \frac{5\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right\} + \eta^2 \left[\frac{3}{2} \left\{ \frac{(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} \right. \right. \\ & + \frac{5\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{5(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \\ & \left. \left. + \frac{5(1-\mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} + \frac{5\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right\} \right] \right], \end{aligned}$$

and

$$\begin{aligned} \ddot{\eta} + 2\alpha n \dot{\xi} = & \eta \left[\beta n^2 - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(4s_1 - 3s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(4s'_1 - 3s'_2)}{2|x_L + 1 - \mu|^5} \right] \\ & + \xi \eta \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(4s_1 - 3s_2)(x_L - \mu)}{2|x_L - \mu|^7} \right. \\ & \left. + \frac{15\mu q_2(4s'_1 - 3s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \right]. \end{aligned}$$

Or

$$\ddot{\xi} - 2\alpha n \dot{\eta}_2 = K_1 \xi + K_2 \xi^2 + K_3 \eta^2, \quad (4.45a)$$

$$\ddot{\eta} + 2\alpha n \dot{\xi} = L_1 \eta + L_2 \xi \eta, \quad (4.45b)$$

where

$$K_1 = n^2 \beta + 2R_{ra} + \frac{6(1-\mu)q_1(2s_1 - s_2)}{r_{01}^5} + \frac{6\mu q_2(2s'_1 - s'_2)}{r_{02}^5},$$

$$K_2 = -3(1-\mu)q_1 \left[\frac{1}{r_{01}^4} + \frac{5(2s_1 - s_2)}{r_{01}^6} \right] V_1 - 3\mu q_2 \left[\frac{1}{r_{02}^4} + \frac{5(2s'_1 - s'_2)}{r_{02}^6} \right] V_2,$$

$$K_3 = \frac{3(1-\mu)q_1}{2} \left[\frac{1}{r_{01}^4} + \frac{5(1-\mu)q_1(4s_1 - 3s_2)}{2r_{01}^6} \right] V_1 + \frac{3\mu q_2}{2} \left[\frac{1}{r_{02}^4} + \frac{5(4s'_1 - 3s'_2)}{2r_{02}^6} \right] V_2,$$

$$L_1 = \beta n^2 - R_{ra} - \frac{3(1-\mu)q_1(4s_1-3s_2)}{2r_{01}^5} - \frac{3\mu q_2(4s'_1-3s'_2)}{2r_{02}^5},$$

$$L_2 = 3(1-\mu)q_1\left[\frac{1}{r_{01}^4} + \frac{5(1-\mu)q_1(4s_1-3s_2)}{2r_{01}^6}\right]V_1 + 3\mu q_2\left[\frac{1}{r_{02}^4} + \frac{5(4s'_1-3s'_2)}{2r_{02}^6}\right]V_2,$$

and

$$R_{ra} = \frac{(1-\mu)q_1}{|x_L - \mu|^3} + \frac{\mu q_2}{|x_L + 1 - \mu|^3}.$$

While V_1 and V_2 have been used to represent the signs of $r_{01} = |\xi - \mu|$ and $r_{02} = |\xi + 1 - \mu|$ in order to avoid absolute values.

4.4.6 Semi-analytical approximation of 2D periodic orbits for the model with triaxial and radiating primaries

In this aspect, we look for periodic solutions of the system (4.45a) and (4.45b). We consider that the sought solution in powers of an orbital parameter e ($|e| \ll 1$) is of the following form:

$$\xi = \xi_1 e + \xi_2 e^2, \quad \eta = \eta_1 e + \eta_2 e^2. \quad (4.46)$$

Substituting (4.46) into (4.45a), (4.45b) and equating the coefficients of the same powers of e , we obtain the following two systems

$$\begin{aligned} \xi_1'' - 2\alpha n \eta_1' - K_1 \xi_1 &= 0, \\ \eta_1'' + 2\alpha n \xi_1' - L_1 \eta_1 &= 0, \end{aligned} \quad (4.47)$$

and

$$\begin{aligned} \xi_2'' - 2\alpha n \eta_2' - K_3 \eta_1^2 - K_1 \xi_2 - K_2 \xi_1^2 &= 0, \\ \eta_2'' + 2\alpha n \xi_2' - L_1 \eta_2 - L_2 \xi_1 \eta_1 &= 0, \end{aligned} \quad (4.48)$$

respectively, which have to be solved successively. The periodic solution of the linear system (4.47) is

$$\xi_1 = \varpi_{11} \cos(wt) + \vartheta_{11} \sin(wt) \quad \text{and} \quad \eta_1 = \varpi_{12} \cos(wt) + \vartheta_{12} \sin(wt) \quad (4.49)$$

where $T = \frac{2\pi}{w}$ is the period of the periodic orbit. Since $\cos(wt)$ and $\sin(wt)$ are linearly

independent functions, we equate their coefficients obtaining

$$\begin{aligned} (K_1 + w^2)\varpi_{11} + 2\alpha n g_{12} &= 0, \\ 2\alpha n w \varpi_{11} + (L_1 + w^2)g_{12} &= 0, \end{aligned} \quad (4.50)$$

and

$$\begin{aligned} (K_1 + w^2)\varpi_{11} - 2\alpha n g_{12} &= 0, \\ -2\alpha n w \varpi_{11} + (L_1 + w^2)g_{12} &= 0, \end{aligned} \quad (4.51)$$

respectively. Systems (4.47) and (4.48) are linear homogeneous and for non-zero solution the determinant must be zero, i.e.:

$$D = \begin{vmatrix} K_1 + w^2 & \pm 2\alpha n \\ \pm 2\alpha n w & L_1 + w^2 \end{vmatrix} = w^4 + (K_1 + L_1 - 4\alpha^2 n^2)w^2 + K_1 L_1 = 0. \quad (4.52)$$

Note here that, the roots of (4.52) may determine the stability of the collinear equilibrium points. In the Lyapunov sense, if all four roots of (4.52) are either negative real numbers or pure imaginary or if the real part of complex roots is negative at each one of the collinear equilibrium points, then these are stable otherwise they are unstable. Thus, as it can be seen from the results presented in Table 2, the collinear equilibria $L_{1,2,3}$ are unstable in the three binary systems.

Since $D = 0$, systems (4.47) and (4.48) are indeterminate, so, we choose arbitrarily $\varpi_{11} = 1$ and $g_{11} = 0$ such that:

$$\varpi_{12} = 0, \quad g_{12} = -\frac{2\alpha n w}{L_1 + w^2}. \quad (4.53)$$

Therefore, the periodic solution of (4.47) is

$$\begin{aligned} \xi_1 &= \cos(wt), \\ \eta_1 &= g_{12} \sin(2wt). \end{aligned}$$

Similarly, for system (4.48) we set:

$$\begin{aligned}\xi_2 &= \varpi_{21} + \varpi_{22} \cos(wt) + \varpi_{23} \cos(2wt) + \mathcal{G}_{21} \sin(wt) + \mathcal{G}_{22} \sin(2wt), \\ \eta_2 &= \varpi_{24} \cos(wt) + \varpi_{25} \cos(2wt) + \mathcal{G}_{23} \sin(wt) + \mathcal{G}_{24} \sin(2wt),\end{aligned}\quad (4.54)$$

and working as previously we find that

$$\varpi_{22} = 0, \mathcal{G}_{21} = 0, \varpi_{24} = 0, \mathcal{G}_{23} = 0, \mathcal{G}_{22} = 0, \varpi_{25} = 0, \quad (4.55)$$

and

$$\begin{aligned}\varpi_{21} &= -\frac{K_2 + K_3 \mathcal{G}_{12}^2}{2K_1}, \\ \varpi_{23} &= \frac{-K_2(L_1 + 4w^2) + \mathcal{G}_{12}(4\alpha L_2 n w + K_3 \mathcal{G}_{12}(L_1 + 4w^2))}{2K_1 L_1 + 8(K_1 + L_1 - 4\alpha^2 n^2)w^2 + 32w^4}, \\ \mathcal{G}_{24} &= \frac{4\alpha K_2 n w - \mathcal{G}_{12}(K_1 L_2 + 4w(\alpha K_3 n \mathcal{G}_{12} + L_2 w))}{2K_1 L_1 + 8(K_1 + L_1 - 4\alpha^2 n^2)w^2 + 32w^4}.\end{aligned}\quad (4.56)$$

Thus, the periodic solution of (4.45) is

$$\begin{aligned}\xi_2 &= \varpi_{21} + \varpi_{23} \cos(2wt), \\ \eta_2 &= \mathcal{G}_{24} \sin(2wt).\end{aligned}$$

Therefore, the sought periodic solution (4.46) into series expansions of the orbital parameter e up to second order terms is:

$$\begin{aligned}\xi &= \xi_0 + \cos(wt)e + [\varpi_{21} + \varpi_{23} \cos(2wt)]e^2, \\ \eta &= \mathcal{G}_{12} \sin(wt)e + \mathcal{G}_{24} \sin(2wt)e^2,\end{aligned}\quad (4.57)$$

where $(\xi_0, 0)$ is the position of any collinear equilibrium point, the coefficients $\varpi_{11}, \varpi_{21}, \varpi_{23}, \mathcal{G}_{12}, \mathcal{G}_{24}$ are defined by (4.53) and (4.56), and w is a real positive solution of (4.52).

4.4.7 Expansions of the 3D equations of motion up to second order terms

The RHS of Eqns. (4.42) is expanded up to second order using the Taylor series method around (x_L, y_L, z_L) . We proceed as follows

$$\begin{aligned}
\Omega_{\xi}(x_L + \xi, \eta, \zeta) &= \Omega_{\xi}(x_L, y_L, z_L) + \xi \Omega_{\xi\xi}(x_L, y_L, z_L) + \eta \Omega_{\xi\eta}(x_L, y_L, z_L) + \zeta \Omega_{\xi\zeta}(x_L, y_L, z_L) \\
&+ \frac{1}{2} \xi^2 \Omega_{\xi\xi\xi}(x_L, y_L, z_L) + \xi \eta \Omega_{\xi\xi\eta}(x_L, y_L, z_L) + \xi \zeta \Omega_{\xi\xi\zeta}(x_L, y_L, z_L) \\
&+ \eta \zeta \Omega_{\xi\eta\zeta}(x_L, y_L, z_L) + \frac{1}{2} \eta^2 \Omega_{\xi\eta\eta}(x_L, y_L, z_L) + \frac{1}{2} \zeta^2 \Omega_{\xi\zeta\zeta}(x_L, y_L, z_L),
\end{aligned} \tag{4.58a}$$

$$\begin{aligned}
\Omega_{\eta}(x_L + \xi, \eta, \zeta) &= \Omega_{\eta}(x_L, y_L, z_L) + \xi \Omega_{\eta\xi}(x_L, y_L, z_L) + \eta \Omega_{\eta\eta}(x_L, y_L, z_L) + \zeta \Omega_{\eta\zeta}(x_L, y_L, z_L) \\
&+ \frac{1}{2} \xi^2 \Omega_{\eta\xi\xi}(x_L, y_L, z_L) + \xi \eta \Omega_{\eta\xi\eta}(x_L, y_L, z_L) + \xi \zeta \Omega_{\eta\xi\zeta}(x_L, y_L, z_L) \\
&+ \eta \zeta \Omega_{\eta\eta\zeta}(x_L, y_L, z_L) + \frac{1}{2} \eta^2 \Omega_{\eta\eta\eta}(x_L, y_L, z_L) + \frac{1}{2} \zeta^2 \Omega_{\eta\zeta\zeta}(x_L, y_L, z_L),
\end{aligned} \tag{4.58b}$$

$$\begin{aligned}
\Omega_{\zeta}(x_L + \xi, \eta, \zeta) &= \Omega_{\zeta}(x_L, y_L, z_L) + \xi \Omega_{\zeta\xi}(x_L, y_L, z_L) + \eta \Omega_{\zeta\eta}(x_L, y_L, z_L) + \zeta \Omega_{\zeta\zeta}(x_L, y_L, z_L) \\
&+ \frac{1}{2} \xi^2 \Omega_{\zeta\xi\xi}(x_L, y_L, z_L) + \xi \eta \Omega_{\zeta\xi\eta}(x_L, y_L, z_L) + \xi \zeta \Omega_{\zeta\xi\zeta}(x_L, y_L, z_L) \\
&+ \eta \zeta \Omega_{\zeta\eta\zeta}(x_L, y_L, z_L) + \frac{1}{2} \eta^2 \Omega_{\zeta\eta\eta}(x_L, y_L, z_L) + \frac{1}{2} \zeta^2 \Omega_{\zeta\zeta\zeta}(x_L, y_L, z_L).
\end{aligned} \tag{4.58c}$$

$$\Omega_{\xi}(x_L, y_L, z_L) = \Omega_{\xi}(0, 0, 0) = 0,$$

$$\Omega_{\eta}(x_L, y_L, z_L) = \Omega_{\eta}(0, 0, 0) = 0,$$

$$\Omega_{\zeta}(x_L, y_L, z_L) = \Omega_{\zeta}(0, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\xi\xi}(x_L + \xi, \eta, \zeta) = & n^2 \beta - \frac{(1-\mu)q_1}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} - \frac{\mu q_2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} \\
& + \frac{3(1-\mu)q_1(x_L + \xi - \mu)^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2(x_L + \xi + 1 - \mu)^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{3(1-\mu)(2s_1 - s_2)q_1}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} - \frac{3\mu(2s'_1 - s'_2)q_2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)q_1 s_1 \zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2 s'_1 \zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1 s_1 (x_L + \xi - \mu)^2 \zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2 s'_1 (x_L + \xi + 1 - \mu)^2 \zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^2 \eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2 \eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}},
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\xi\xi}(x_L, 0, 0) = & n^2 \beta - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} + \frac{3(1-\mu)q_1}{|x_L - \mu|^3} + \frac{3\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)(2s_1 - s_2)q_1}{2|x_L - \mu|^5} \\
& - \frac{3\mu(2s'_1 - s'_2)q_2}{2|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} + \frac{15\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5},
\end{aligned}$$

$$\begin{aligned}
\Omega_{\eta\eta}(x_L + \xi, \eta, \zeta) = & \beta n^2 - \frac{(1-\mu)q_1}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} - \frac{\mu q_2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} \\
& + \frac{3(1-\mu)q_1\eta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2\eta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} - \frac{3\mu q_2(2s'_1 - s'_2)}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{3(1-\mu)q_1(s_1 - s_2)}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} - \frac{3\mu q_2(s'_1 - s'_2)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)q_1s_1\zeta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2s'_1\zeta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{75(1-\mu)q_1(s_1 - s_2)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{75\mu q_2(s'_1 - s'_2)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1\eta^2\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1\eta^2\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^4}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\eta^4}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\eta\eta}(x_L, 0, 0) = & \beta n^2 - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} \\
& - \frac{3(1-\mu)q_1(s_1 - s_2)}{|x_L - \mu|^5} - \frac{3\mu q_2(s'_1 - s'_2)}{|x_L + 1 - \mu|^5},
\end{aligned}$$

$$\begin{aligned}
\Omega_{\zeta\zeta}(x_L + \xi, \eta, \zeta) = & -\frac{(1-\mu)q_1}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} - \frac{\mu q_2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} \\
& + \frac{3(1-\mu)q_1\zeta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} + \frac{3\mu q_2\zeta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{3}{2}}} \\
& - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} - \frac{3\mu q_2(2s'_1 - s'_2)}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{3(1-\mu)q_1s_1}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} - \frac{3\mu q_2s'_1}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\zeta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{75(1-\mu)q_1s_1\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{75\mu q_2s'_1\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1\zeta^4}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} + \frac{105\mu q_2s'_1\zeta^4}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^2\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta^2\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\zeta\zeta}(x_L, 0, 0) = & -\frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} - \frac{3(1-\mu)q_1s_1}{|x_L - \mu|^5} \\
& - \frac{3\mu q_2s'_1}{|x_L + 1 - \mu|^5},
\end{aligned}$$

$$\begin{aligned}
\Omega_{\xi\eta}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1(x_L + \xi - \mu)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2(x_L + \xi + 1 - \mu)\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1(x_L + \xi - \mu)\eta\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1(x_L + \xi + 1 - \mu)\eta\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

or

$$\Omega_{\xi\eta}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\eta\zeta}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1\eta\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} - \frac{3\mu q_2\eta\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)\eta\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\eta\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)\eta\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1s_1\eta\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2s'_1\eta\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1\eta\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1\eta\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^3\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\eta^3\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

or

$$\Omega_{\eta\zeta}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\xi\xi}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1(x_L + \xi - \mu)\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2(x_L + \xi + 1 - \mu)\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}}
\end{aligned}$$

or

$$\Omega_{\xi\xi}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\xi\xi\xi}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1(x_L + \xi - \mu)\zeta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2(x_L + \xi + 1 - \mu)\zeta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1s_1(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2s'_1(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{525(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{525\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta^4}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta^4}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\begin{aligned}\Omega_{\xi\eta\zeta}(x_L, 0, 0) = & \frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \\ & + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{15(1-\mu)q_1s_1(x_L - \mu)}{|x_L - \mu|^7} + \frac{15\mu q_2s'_1(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7},\end{aligned}$$

$$\begin{aligned}\Omega_{\xi\eta\zeta}(x_L + \xi, \eta, \zeta) = & -\frac{15(1-\mu)q_1(x_L + \xi - \mu)\eta\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)\eta\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\ & - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105(1-\mu)q_1s_1(x_L + \xi - \mu)\eta\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\ & + \frac{945(1-\mu)q_1s_1(x_L + \xi - \mu)\eta\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^3\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\ & - \frac{15\mu q_2(x_L + \xi + 1 - \mu)\eta\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\ & - \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1(x_L + \xi + 1 - \mu)\eta\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\ & + \frac{945\mu q_2s'_1(x_L + \xi + 1 - \mu)\eta\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^3\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}\end{aligned}$$

or

$$\Omega_{\xi\eta\zeta}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\xi\xi\xi}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1(x_L + \xi - \mu)^2\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2(x_L + \xi + 1 - \mu)^2\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1s_1\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2s'_1\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^2\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\eta^2\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{525(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{525\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta^4}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta^4}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\xi\xi\xi}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\xi\eta\eta}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1(x_L + \xi - \mu)\eta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2(x_L + \xi + 1 - \mu)\eta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{525(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{525\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1(x_L + \xi - \mu)\eta^2\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1(x_L + \xi + 1 - \mu)\eta^2\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^4}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^4}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\xi\eta\eta}(x_L, 0, 0) = & \frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \\
& + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} \\
& + \frac{15\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7},
\end{aligned}$$

$$\begin{aligned}
\Omega_{\eta\eta\zeta}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1\eta^2\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2\eta^2\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1s_1\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2s'_1\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1s_1\eta^2\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1\eta^2\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)\eta^2\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(2s'_1 - s'_2)\eta^2\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{525(1-\mu)q_1(s_1 - s_2)\eta^2\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{525\mu q_2(s'_1 - s'_2)\eta^2\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1\eta^2\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1\eta^2\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)\eta^4\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)\eta^4\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\eta\eta\zeta}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\xi\xi\eta}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1(x_L + \xi - \mu)^2\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2(x_L + \xi + 1 - \mu)^2\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1s_1\eta\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2s'_1\eta\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^3}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\eta^3}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)^2\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^2\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} + \frac{105\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1(x_L + \xi - \mu)^2\eta\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1(x_L + \xi + 1 - \mu)^2\eta\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^2\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^2\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\xi\xi\eta}(x_L + \xi, \eta, \zeta) = 0,$$

$$\begin{aligned}
\Omega_{\xi\xi\xi}(x_L + \xi, \eta, \zeta) = & \frac{9(1-\mu)q_1(x_L + \xi - \mu)}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{9\mu q_2(x_L + \xi + 1 - \mu)}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1(x_L + \xi - \mu)^3}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2(x_L + \xi + 1 - \mu)^3}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{45(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{45\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{315(1-\mu)q_1s_1(x_L + \xi - \mu)\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{315\mu q_2s'_1(x_L + \xi + 1 - \mu)\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{315(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{315\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L + \xi - \mu)^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + \xi + 1 - \mu)^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1(x_L + \xi - \mu)^3\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1(x_L + \xi + 1 - \mu)^3\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)(x_L + \xi - \mu)^3\eta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)(x_L + \xi + 1 - \mu)^3\eta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\begin{aligned}
\Omega_{\xi\xi\xi}(x_L, 0, 0) = & \frac{9(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{9\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} - \frac{15(1-\mu)q_1(x_L - \mu)^3}{|x_L - \mu|^7} - \frac{15\mu q_2(x_L + 1 - \mu)^3}{|x_L + 1 - \mu|^7} \\
& + \frac{45(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{45\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)^3}{2|x_L - \mu|^9} - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)^3}{2|x_L + 1 - \mu|^9},
\end{aligned}$$

$$\begin{aligned}
\Omega_{\eta\eta\eta}(x_L + \xi, \eta, \zeta) = & \frac{9(1-\mu)q_1\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{9\mu q_2\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1\eta^3}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2\eta^3}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{45(1-\mu)q_1(2s_1 - s_2)\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{45\mu q_2(2s'_1 - s'_2)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{90(1-\mu)q_1(s_1 - s_2)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{90\mu q_2(s'_1 - s'_2)\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{315(1-\mu)q_1s_1\eta\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{315\mu q_2s'_1\eta\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(2s'_1 - s'_2)\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{945(1-\mu)q_1(s_1 - s_2)\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{945\mu q_2(s'_1 - s'_2)\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1\eta^3\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1\eta^3\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)\eta^5}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)\eta^5}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\eta\eta\eta}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\eta\zeta\zeta}(x_L + \xi, \eta, \zeta) = & \frac{3(1-\mu)q_1\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{3\mu q_2\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1\eta\zeta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2\eta\zeta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)\eta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(2s'_1 - s'_2)\eta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2(s'_1 - s'_2)\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{15(1-\mu)q_1s_1\eta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{15\mu q_2s'_1\eta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)\eta\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(2s'_1 - s'_2)\eta\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta\zeta^2}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\eta\zeta^2}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{525(1-\mu)q_1s_1\eta\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{525\mu q_2s'_1\eta\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{105(1-\mu)q_1(s_1 - s_2)\eta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\eta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1\eta\zeta^4}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1\eta\zeta^4}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)\eta^3\zeta^2}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)\eta^3\zeta^2}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\eta\zeta\zeta}(x_L, 0, 0) = 0,$$

$$\begin{aligned}
\Omega_{\zeta\zeta\zeta}(x_L + \xi, \eta, \zeta) = & \frac{9(1-\mu)q_1\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} + \frac{9\mu q_2\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{5}{2}}} \\
& - \frac{15(1-\mu)q_1\zeta^3}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} - \frac{15\mu q_2\zeta^3}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{45(1-\mu)q_1(2s_1 - s_2)\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{45\mu q_2(2s'_1 - s'_2)\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& + \frac{90(1-\mu)q_1s_1\zeta}{\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} + \frac{90\mu q_2s'_1\zeta}{\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{7}{2}}} \\
& - \frac{105(1-\mu)q_1(2s_1 - s_2)\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{105\mu q_2(s'_1 - s'_2)\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{945(1-\mu)q_1s_1\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{945\mu q_2s'_1\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& - \frac{315(1-\mu)q_1(s_1 - s_2)\eta^2\zeta}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} - \frac{315\mu q_2(s'_1 - s'_2)\eta^2\zeta}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{9}{2}}} \\
& + \frac{945(1-\mu)q_1s_1\zeta^5}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2s'_1\zeta^5}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} \\
& + \frac{945(1-\mu)q_1(s_1 - s_2)\eta^2\zeta^3}{2\left[(x_L + \xi - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}} + \frac{945\mu q_2(s'_1 - s'_2)\eta^2\zeta^3}{2\left[(x_L + \xi + 1 - \mu)^2 + \eta^2 + \zeta^2\right]^{\frac{11}{2}}}
\end{aligned}$$

or

$$\Omega_{\zeta\zeta\zeta}(x_L, 0, 0) = 0,$$

and

$$\Omega_{\zeta\zeta}(x_L + \xi, \eta, \zeta) = \Omega_{\xi\zeta}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\eta\zeta}(x_L + \xi, \eta, \zeta) = \Omega_{\xi\eta}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\zeta\eta}(x_L + \xi, \eta, \zeta) = \Omega_{\eta\zeta}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\eta\zeta\zeta}(x_L + \xi, \eta, \zeta) = \Omega_{\xi\zeta\eta}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\eta\zeta\eta}(x_L + \xi, \eta, \zeta) = \Omega_{\xi\eta\eta}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\zeta\zeta\zeta}(x_L + \xi, \eta, \zeta) = \Omega_{\xi\xi\xi}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\zeta\zeta\zeta}(x_L + \xi, \eta, \zeta) = \Omega_{\xi\xi\xi}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\zeta\eta\zeta}(x_L + \xi, \eta, \zeta) = \Omega_{\eta\zeta\zeta}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\zeta\eta\eta}(x_L + \xi, \eta, \zeta) = \Omega_{\eta\eta\zeta}(x_L + \xi, \eta, \zeta),$$

$$\Omega_{\eta\xi\zeta}(x_L + \xi, \eta, \zeta) = \Omega_{\zeta\xi\eta}(x_L + \xi, \eta, \zeta) = \Omega_{\xi\eta\zeta}(x_L + \xi, \eta, \zeta).$$

Substituting the result of the preceding procedures into Equations (4.58), we obtain

$$\begin{aligned} \Omega_{\xi}(x_L + \xi, \eta, \zeta) = & \xi \left[n^2 \beta - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} + \frac{3(1-\mu)q_1}{|x_L - \mu|^3} + \frac{3\mu q_2}{|x_L + 1 - \mu|^3} \right. \\ & - \frac{3(1-\mu)(2s_1 - s_2)q_1}{2|x_L - \mu|^5} - \frac{3\mu(2s'_1 - s'_2)q_2}{2|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} \\ & + \frac{15\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} \left. \right] + \frac{1}{2} \xi^2 \left[\frac{9(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{9\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} \right. \\ & - \frac{15(1-\mu)q_1(x_L - \mu)^3}{|x_L - \mu|^7} + \frac{45(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \\ & - \frac{15\mu q_2(x_L + 1 - \mu)^3}{|x_L + 1 - \mu|^7} + \frac{45\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\ & - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)^3}{2|x_L - \mu|^9} - \frac{105\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)^3}{2|x_L + 1 - \mu|^9} \left. \right] \\ & + \frac{1}{2} \eta^2 \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \right. \\ & + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} \\ & + \frac{15\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \left. \right] + \frac{1}{2} \zeta^2 \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} \right. \\ & + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{15(1-\mu)q_1 s_1(x_L - \mu)}{|x_L - \mu|^7} \\ & + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{15\mu q_2 s'_1(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \left. \right], \end{aligned}$$

$$\begin{aligned}\Omega_\eta(x_L + \xi, \eta, \zeta) = & \eta \left[\beta n^2 - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} \right. \\ & \left. - \frac{3(1-\mu)q_1(s_1 - s_2)}{|x_L - \mu|^5} - \frac{3\mu q_2(s'_1 - s'_2)}{|x_L + 1 - \mu|^5} \right] + \xi \eta \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} \right. \\ & + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\ & \left. + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} + \frac{15\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right],\end{aligned}$$

$$\begin{aligned}\Omega_\zeta(x_L + \xi, \eta, \zeta) = & \zeta \left[-\frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} \right. \\ & \left. - \frac{3(1-\mu)q_1 s_1}{|x_L - \mu|^5} - \frac{3\mu q_2 s'_1}{|x_L + 1 - \mu|^5} \right] + \xi \zeta \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} \right. \\ & + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\ & \left. + \frac{15(1-\mu)q_1 s_1(x_L - \mu)}{|x_L - \mu|^7} + \frac{15\mu q_2 s'_1(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right],\end{aligned}$$

respectively.

Thus, the equations of motion (4.42) become

$$\begin{aligned}
\ddot{\xi} - 2\alpha n \dot{\eta}_2 = & \xi \left[n^2 \beta - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} + \frac{3(1-\mu)q_1}{|x_L - \mu|^3} + \frac{3\mu q_2}{|x_L + 1 - \mu|^3} \right. \\
& - \frac{3(1-\mu)(2s_1 - s_2)q_1}{2|x_L - \mu|^5} - \frac{3\mu(2s'_1 - s'_2)q_2}{2|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} \\
& + \left. \frac{15\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} \right] + \frac{1}{2} \xi^2 \left[\frac{9(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{9\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} \right. \\
& - \frac{15(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} - \frac{15\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{45\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\
& + \frac{45(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} - \frac{105(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \\
& - \left. \frac{105\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \right] + \frac{1}{2} \eta^2 \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} \right. \\
& + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\
& + \frac{15(1-\mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} + \left. \frac{15\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right] \\
& + \frac{1}{2} \zeta^2 \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} \right. \\
& + \left. \frac{15(1-\mu)q_1 s_1(x_L - \mu)}{|x_L - \mu|^7} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} + \frac{15\mu q_2 s'_1(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right],
\end{aligned}$$

$$\begin{aligned}
\ddot{\eta} + 2\alpha n \dot{\xi} = & \eta \left[\beta n^2 - \frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} \right. \\
& - \left. \frac{3(1-\mu)q_1(s_1 - s_2)}{|x_L - \mu|^5} - \frac{3\mu q_2(s'_1 - s'_2)}{|x_L + 1 - \mu|^5} \right] + \xi \eta \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} \right. \\
& + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\
& + \left. \frac{15(1-\mu)q_1(s_1 - s_2)(x_L - \mu)}{|x_L - \mu|^7} + \frac{15\mu q_2(s'_1 - s'_2)(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right],
\end{aligned}$$

$$\begin{aligned}\ddot{\zeta} = \zeta & \left[-\frac{(1-\mu)q_1}{|x_L - \mu|^3} - \frac{\mu q_2}{|x_L + 1 - \mu|^3} - \frac{3(1-\mu)q_1(2s_1 - s_2)}{2|x_L - \mu|^5} - \frac{3\mu q_2(2s'_1 - s'_2)}{2|x_L + 1 - \mu|^5} \right. \\ & - \frac{3(1-\mu)q_1 s_1}{|x_L - \mu|^5} - \frac{3\mu q_2 s'_1}{|x_L + 1 - \mu|^5} \left. \right] + \xi \zeta \left[\frac{3(1-\mu)q_1(x_L - \mu)}{|x_L - \mu|^5} + \frac{3\mu q_2(x_L + 1 - \mu)}{|x_L + 1 - \mu|^5} \right. \\ & + \frac{15(1-\mu)q_1(2s_1 - s_2)(x_L - \mu)}{2|x_L - \mu|^7} + \frac{15\mu q_2(2s'_1 - s'_2)(x_L + 1 - \mu)}{2|x_L + 1 - \mu|^7} \\ & \left. + \frac{15(1-\mu)q_1 s_1(x_L - \mu)}{|x_L - \mu|^7} + \frac{15\mu q_2 s'_1(x_L + 1 - \mu)}{|x_L + 1 - \mu|^7} \right].\end{aligned}$$

Or

$$\ddot{\xi} - 2\alpha n \dot{\eta}_2 = K_1 \xi + K_2 \xi^2 + K_3 \eta^2 + K_4 \xi^2,$$

$$\ddot{\eta} + 2\alpha n \dot{\xi} = L_1 \eta + L_2 \xi \eta,$$

$$\ddot{\zeta} = M_1 \zeta + M_2 \xi \zeta, \quad (4.59)$$

where

$$K_1 = \beta n^2 + 2R_{ra} + \frac{6(1-\mu)q_1(2s_1 - s_2)}{r_{01}^5} + \frac{6\mu q_2(2s'_1 - s'_2)}{r_{02}^5},$$

$$K_2 = -3(1-\mu)q_1 V_1 \left[\frac{1}{r_{01}^4} + \frac{5(2s_1 - s_2)}{r_{01}^6} \right] - 3\mu q_2 V_2 \left(\frac{1}{r_{02}^4} + \frac{5(2s'_1 - s'_2)}{r_{02}^6} \right),$$

$$K_3 = 3(1-\mu)q_1 V_1 \left[\frac{1}{2r_{01}^4} + \frac{5(4s_1 - 3s_2)}{4r_{01}^6} \right] + 3\mu q_2 V_2 \left(\frac{1}{2r_{02}^4} + \frac{5(4s'_1 - 3s'_2)}{4r_{02}^6} \right),$$

$$K_4 = 3(1-\mu)q_1 V_1 \left(\frac{1}{2r_{01}^4} + \frac{5(4s_1 - s_2)}{4r_{01}^6} \right) + 3\mu q_2 V_2 \left(\frac{1}{2r_{02}^4} + \frac{5(4s'_1 - s'_2)}{4r_{02}^6} \right),$$

$$L_1 = \beta n^2 - R_{ra} - \frac{3(1-\mu)q_1(4s_1 - 3s_2)}{2r_{01}^5} - \frac{3\mu q_2(4s'_1 - 3s'_2)}{2r_{02}^5},$$

$$L_2 = -3(1-\mu)q_1 V_1 \left[\frac{1}{r_{01}^4} + \frac{5(4s_1 - 3s_2)}{2r_{01}^6} \right] - 3\mu q_2 V_2 \left[\frac{1}{r_{02}^4} + \frac{5(4s'_1 - 3s'_2)}{2r_{02}^6} \right],$$

$$M_1 = -R_{ra} - \frac{3(1-\mu)q_1(4s_1 - s_2)}{2r_{01}^5} - \frac{3\mu q_2(4s'_1 - s'_2)}{2r_{02}^5},$$

$$M_2 = 3(1-\mu)q_1V_1\left[\frac{1}{r_{01}^4} + \frac{5(4s_1-s_2)}{2r_{01}^6}\right] + 3\mu q_2V_2\left[\frac{1}{r_{02}^4} + \frac{5(4s'_1-s'_2)}{2r_{02}^6}\right].$$

4.4.8 Semi-analytical approximation of 3D periodic orbits for the model with triaxial and radiating primaries

Using the method of successive approximations up to second order, the above system admits periodic solutions which are of the following form:

$$\xi = \xi_2 e^2, \eta = \eta_2 e^2, \zeta = \zeta_1 e.$$

Using the last equations in the first, second and third equations of system (4.59) we find that:

$$\xi_2'' - 2an\eta_2' - K_1\xi_2 - K_4\xi_2^2 = 0, \quad (4.60a)$$

$$\eta_2' + 2an\xi_2' - L_1\eta_2 = 0, \quad (4.60b)$$

and

$$\zeta_1'' - M_1\zeta_1 = 0. \quad (4.60c)$$

We Set (4.60c):

$$\zeta_1(t) = \vartheta_{31}\cos(wt) + \varpi_{31}\sin(wt)$$

and find that

$$(w^2 + M_1)\zeta_1(t) = 0$$

which has non-zero solution only when $w^2 = -M_1 > 0$. One of such solution is

$$\zeta_1(t) = \sin(wt). \quad (4.61)$$

Additionally, working as in the case of 2D periodic orbits, we set:

$$\begin{aligned} \xi_2 &= \varpi_{02} + \varpi_{03}\cos(wt) + \varpi_{04}\cos(2wt) + \vartheta_{02}\sin(wt) + \vartheta_{03}\sin(2wt), \\ \eta_2 &= \varpi_{05}\cos(wt) + \varpi_{06}\cos(2wt) + \vartheta_{04}\sin(wt) + \vartheta_{05}\sin(2wt), \end{aligned}$$

and find that the solution system 4.60a and 4.60b is:

$$\xi_2 = \varpi_{02} + \varpi_{03} \cos(2wt), \quad (4.62a)$$

and

$$\eta_2 = \varrho_{05} \sin(2wt), \quad (4.62b)$$

where

$$\begin{aligned} \varpi_{02} &= \frac{-K_4}{2K_1 w^2}, \\ \varpi_{03} &= \frac{K_4(L_1 + 4w^2)}{2w^2 p}, \\ \varrho_{05} &= \frac{-2anK_4}{wp}, \end{aligned}$$

while $w = \sqrt{-M_{10}}$ and $p = 16w^4 + 4(K_1 + L_1 - 4a^2 n^2)w^2 + K_1 L_1$.

So, the periodic solution up to second order terms w.r.t. e is:

$$\begin{aligned} \xi &= x_0 + e^2 [\varpi_{02} + \varpi_{03} \cos(2wt)], \\ \eta &= e^2 \varrho_{05} \sin(2wt), \\ \zeta &= e \sin(wt). \end{aligned} \quad (4.63)$$

Thus, Eqns. (4.16) and (4.31) furnish the third order semi-analytical approximate periodic solution of 2D periodic orbits and second order semi-analytical approximate periodic solution of 3D periodic orbits respectively, when both primaries are radiating as well as oblate spheroidal bodies. And Eqns. (4.57) and (4.63) give the second order semi-analytical approximate periodic solutions of the 2D and 3D periodic orbits respectively, when the primaries are both radiating as well as triaxial spheroids while taking into account small perturbations in the Coriolis and centrifugal forces.

In the next chapter, we present the numerical results and give the discussion of the investigation done in this thesis.

CHAPTER FIVE

RESULTS AND DISCUSSION

5.1 Introduction

The results and discussion for all the problems investigated in this study are given in this chapter.

5.2 Numerical Considerations for Periodic Solutions Around the Collinear Equilibrium Points in the Perturbed R3BP when the Bigger Primary is a Triaxial Rigid Body and the Secondary an Oblate Spheroid

In this section, we compute the positions of the collinear equilibrium points for several combinations of the parameters of the problem, showing the effects of the triaxiality coefficient of the primary and the oblateness factor of the secondary together with the influence of the Coriolis and centrifugal forces on these points for case 1. From the second order semi-analytical solution for both 2D and 3D periodic orbits constructed in chapter three, we obtained the appropriate initial conditions for the numerical determination of families C, A and B (in Strömgren's notation) around L_1 , L_2 and L_3 , respectively. Our numerical results contain also the stability of all determined periodic solutions.

In Table 5.1, ten cases of the problem are shown (the values for the parameters under consideration for case 1 are same as those used in Singh and Begha (2011)), while the effect of the triaxiality coefficients of the primary and the combined effect of the oblateness factor of the secondary and the influence of the centrifugal force on the location of the collinear libration points can be seen in Figures 5.1 and 5.2, respectively for case 1.

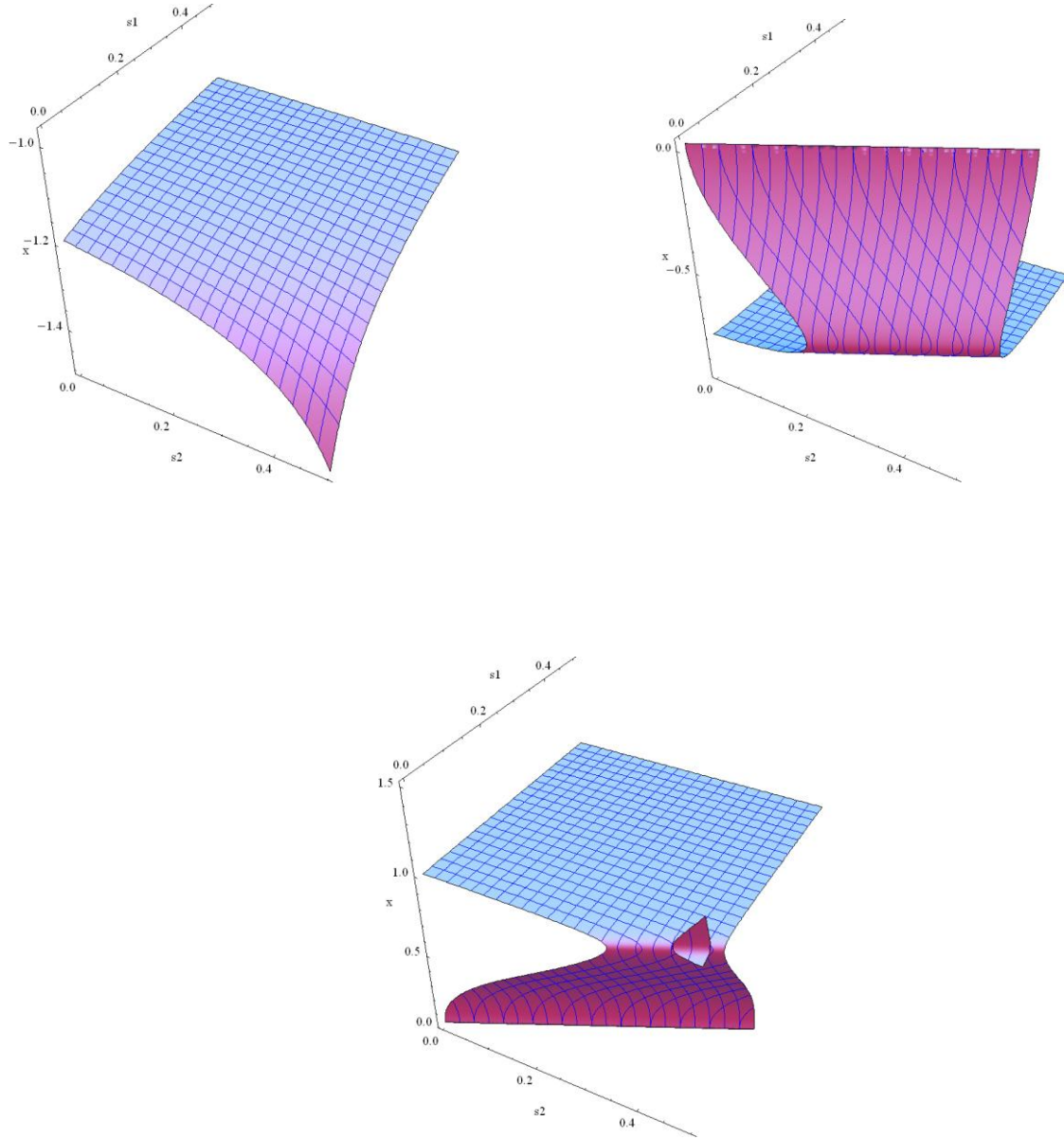


Figure 5.1: The effects of the triaxiality coefficients of the more massive body on the positions of the equilibrium points and the values of the remaining parameters are given in the first case of Table 5.

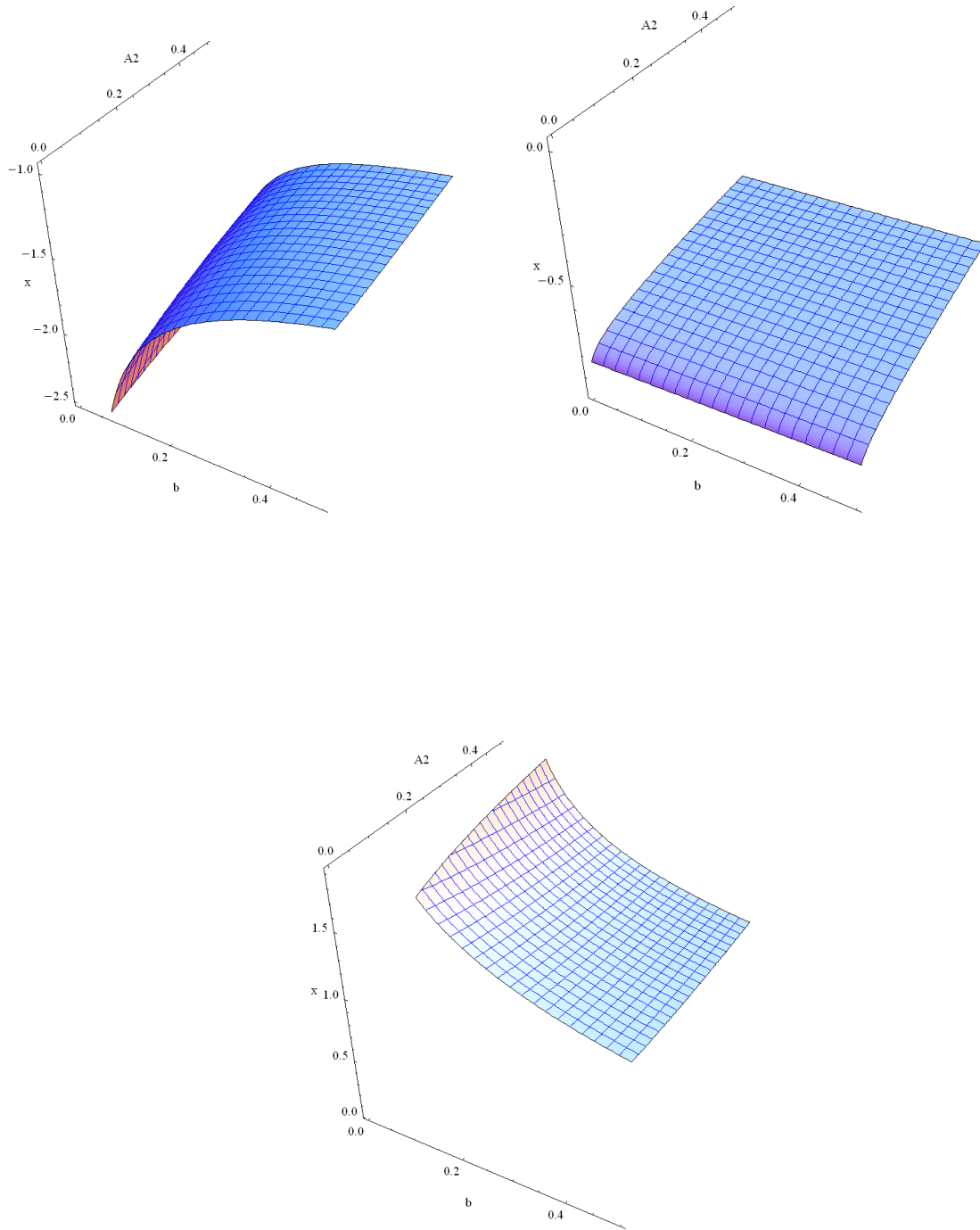


Figure 5.2: The combined effects of the oblateness of the secondary body and perturbation in the centrifugal force on the positions of the equilibrium points, and the values of the remaining parameters are given in the first case of Table 5.1.

Table 5. 1:Collinear equilibrium points for $\mu = 0.02545$.

Case	σ_1	σ_2	A_2	ε'	ε	L_1	L_2	L_3
1	0.000139	0.000132	0.000438	0.000145	0.000089	-1.19341894	-0.78314148	1.01033937
2	0.000339	0.000332	0.000638	0.000345	0.000289	-1.19377095	-0.78263393	1.01017634
3	0.000539	0.000532	0.000838	0.000545	0.000489	-1.19411811	-0.78223354	1.01001350
4	0.000739	0.000732	0.001038	0.000745	0.000689	-1.19446057	-0.78179005	1.00985086
5	0.000939	0.000932	0.001238	0.000945	0.000889	-1.19479844	-0.78135323	1.00968842
6	0.001139	0.001132	0.001438	0.001145	0.001089	-1.19513186	-0.78092285	1.00952617
7	0.001339	0.001332	0.001638	0.001345	0.001289	-1.19546094	-0.78049870	1.00936411
8	0.001539	0.001532	0.001838	0.001545	0.001489	-1.19578579	-0.78008057	1.00920224
9	0.001739	0.001732	0.002038	0.001745	0.001689	-1.19610652	-0.77966829	1.00904057
10	0.001939	0.001932	0.002238	0.001945	0.001889	-1.19642323	-0.77926165	1.00887910

In ten studied cases of the parameters of the problem, it was found that the positions of the collinear equilibrium points L_2 and L_3 move towards the origin with the increase of the values of these parameters while the equilibrium point L_1 moves away from the origin, which is agreement with the results obtained in Jain *et al.* (2006) where similar results have been obtained.

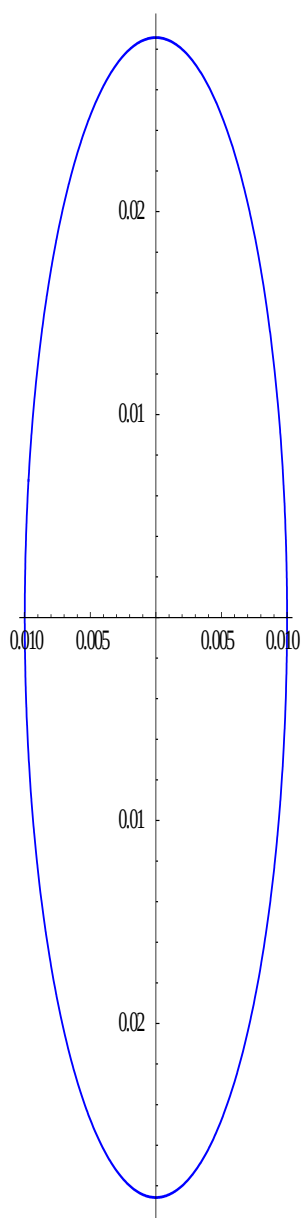
For the aspect of the study of the linear stability of the collinear equilibrium points, Equation (3.17) was utilized and the three collinear equilibrium points were shown to be unstable in the Lyapunov sense for all the ten cases as can be seen in Table 5.2. Note here that, in the case where one or both primaries are prolate bodies or sources of radiation it has been found that these points may be stable (see e.g. Douskos *et al.*, 2012; Perdios *et al.*, 2015).

Table 5. 2: Roots of the characteristic equation (3.17) determining the stability of the collinear equilibrium points.

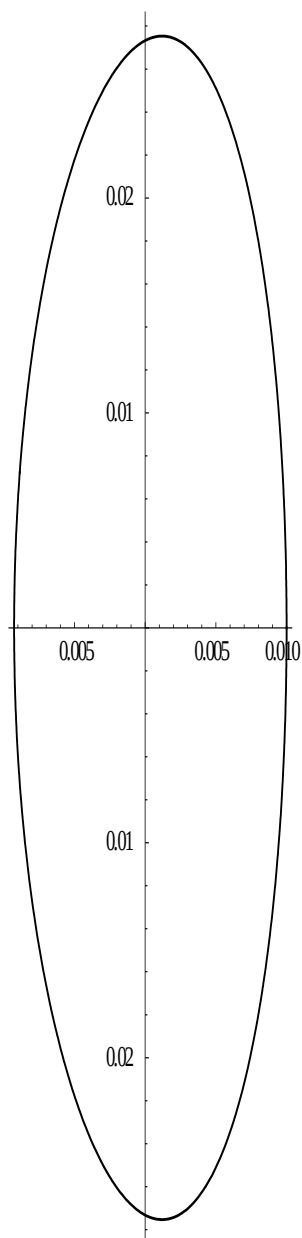
Case	$w, iw(L_1)$	$w, iw(L_2)$	$w, iw(L_3)$
1	$\pm 1.80634639, \pm 2.07999304i$	$\pm 2.41487845, \pm 3.08204901i$	$\pm 1.02180951, \pm 0.25679352i$
2	$\pm 1.80590282, \pm 2.08686171i$	$\pm 2.41481407, \pm 3.09176844i$	$\pm 1.02230697, \pm 0.25693104i$
3	$\pm 1.80549159, \pm 2.09362688i$	$\pm 2.41584384, \pm 3.10300319i$	$\pm 1.02280445, \pm 0.25706865i$
4	$\pm 1.80511129, \pm 2.10029188i$	$\pm 2.41635369, \pm 3.11312815i$	$\pm 1.02379938, \pm 0.25734358i$
5	$\pm 1.80476092, \pm 2.10686046i$	$\pm 2.41688023, \pm 3.12303243i$	$\pm 1.02379938, \pm 0.25734358i$
6	$\pm 1.80443918, \pm 2.11333558i$	$\pm 2.41742239, \pm 3.13272555i$	$\pm 1.02429686, \pm 0.25748107i$
7	$\pm 1.80414510, \pm 2.11972048i$	$\pm 2.41797929, \pm 3.14221666i$	$\pm 1.02479435, \pm 0.25761862i$
8	$\pm 1.80387770, \pm 2.12601817i$	$\pm 2.41854998, \pm 3.15151405i$	$\pm 1.02529186, \pm 0.25775623i$
9	$\pm 1.80363599, \pm 2.13223142i$	$\pm 2.41913390, \pm 3.16062614i$	$\pm 1.02578936, \pm 0.25789372i$
10	$\pm 1.80341912, \pm 2.13836300i$	$\pm 2.41972999, \pm 3.16955989i$	$\pm 1.02628684, \pm 0.25803108i$

5.2.1 Numerical results for the coplanar periodic orbits of the model with triaxial and oblate primaries

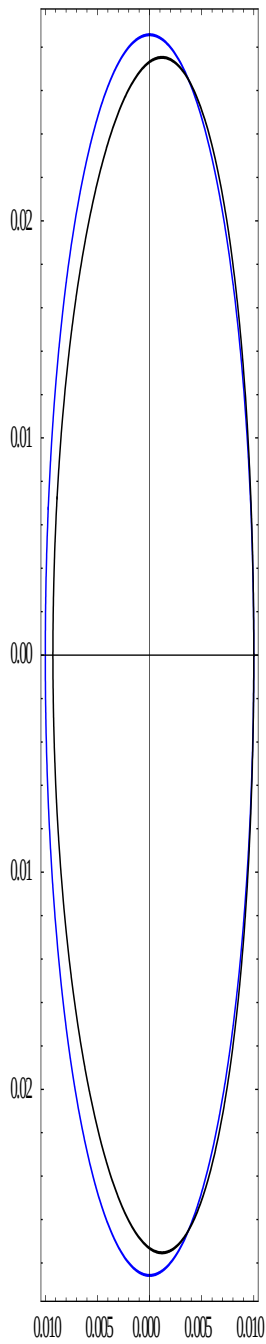
In Figure 5.3 below, we present the approximate orbits graphically. In Figure 5.3(a) we show the orbit for the first order approximation while Figure 5.3(b) depicts the orbit of the second order approximation. In Figure 5.3 (c) - (d), we show the different parametric plots of these approximate orbits.



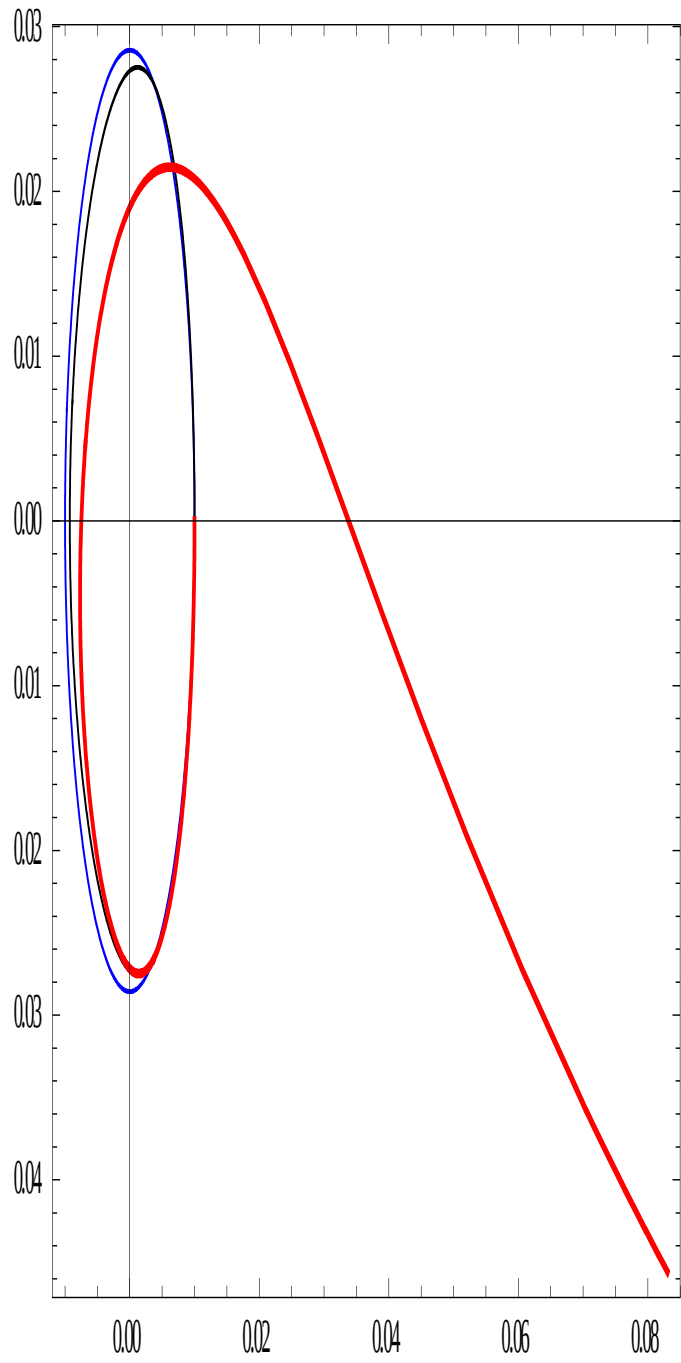
(a)



(b)



(c)



(d)

Figure 5.3: In (a) - (d), we present the different parametric plots of the approximate orbits.

In Figures 5.4-5.6 we present the families of planar symmetric periodic orbits, in their projection into $(x_0, \dot{y}_0)$, around all the collinear equilibrium points as they are computed by the first (red curve) and second (blue curve) order analyses (3.22). We also show, in these figures, the accurate numerical solution (black curve) of the corresponding families obtained by the numerical integration of the full equations of motion (3.1). At each equilibrium point, we observe that the second order semi-analytical solution is always a better approximation of the numerical solution than the first one; the validity of our analysis is obvious.

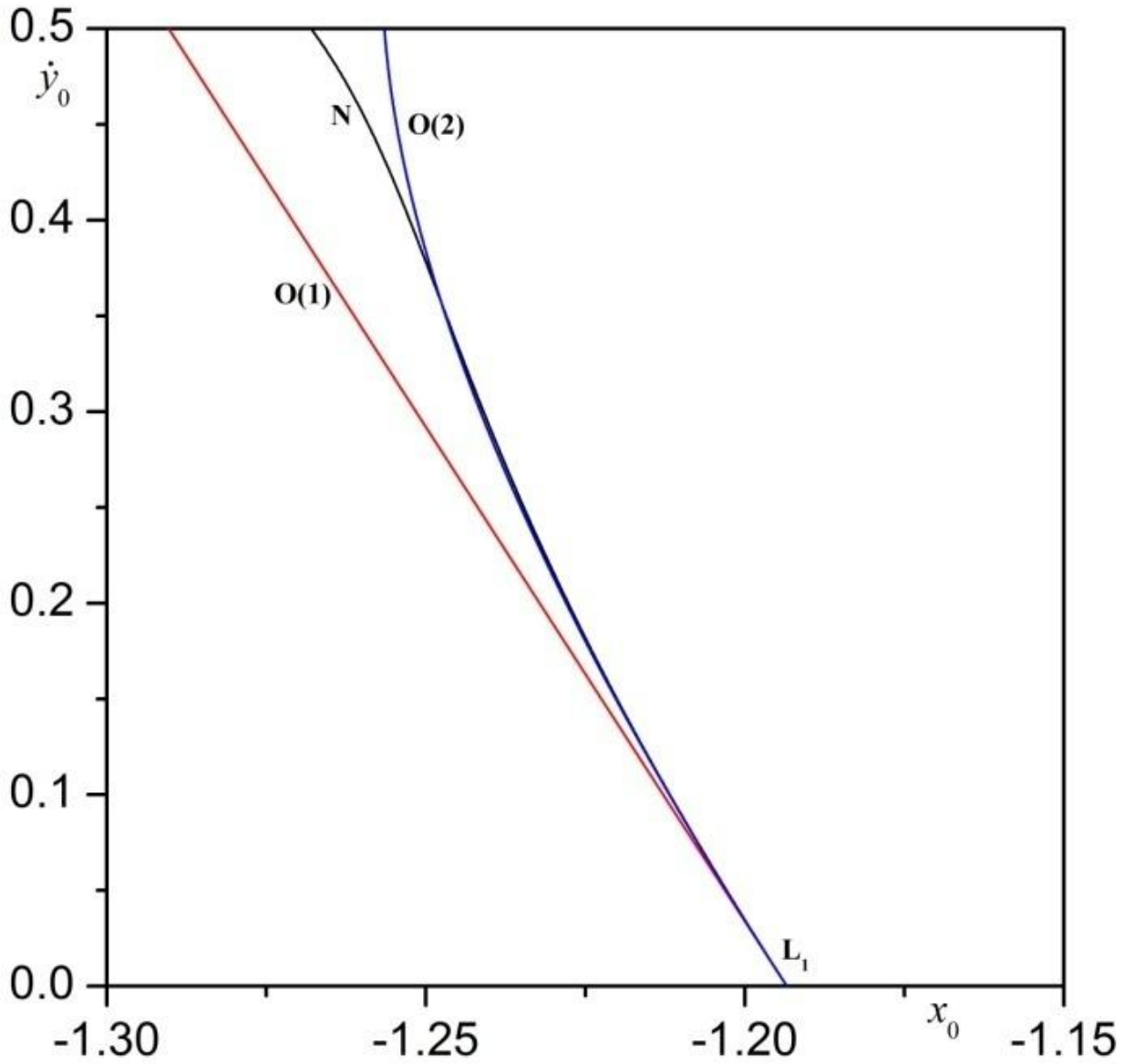


Figure 5.4: Comparison of the first [red curve O(1)] and second [blue curve O(2)] order analyses, obtained from solution (3.22), with the corresponding numerical solution (black curve N), obtained from the numerical integration of the full equations of motion (3.1), for all families of 2D periodic orbits emanating from the collinear equilibrium point L_1 .

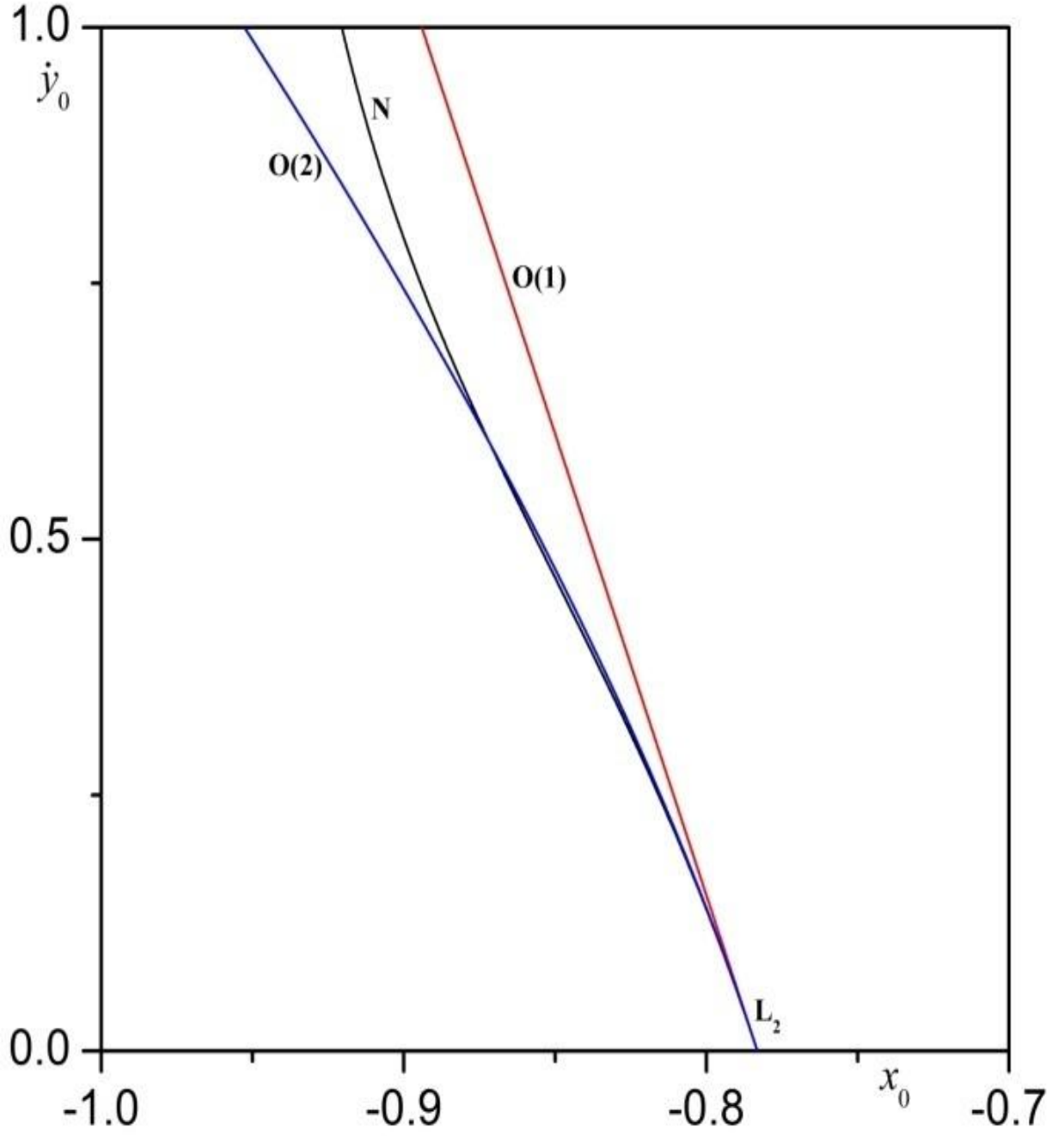


Figure 5.5: Comparison of the first [red curve $O(1)$] and second [blue curve $O(2)$] order analyses, obtained from solution (3.22), with the corresponding numerical solution (black curve N), obtained from the numerical integration of the full equations of motion (3.1), for all families of 2D periodic orbits emanating from the collinear equilibrium point L_2 .

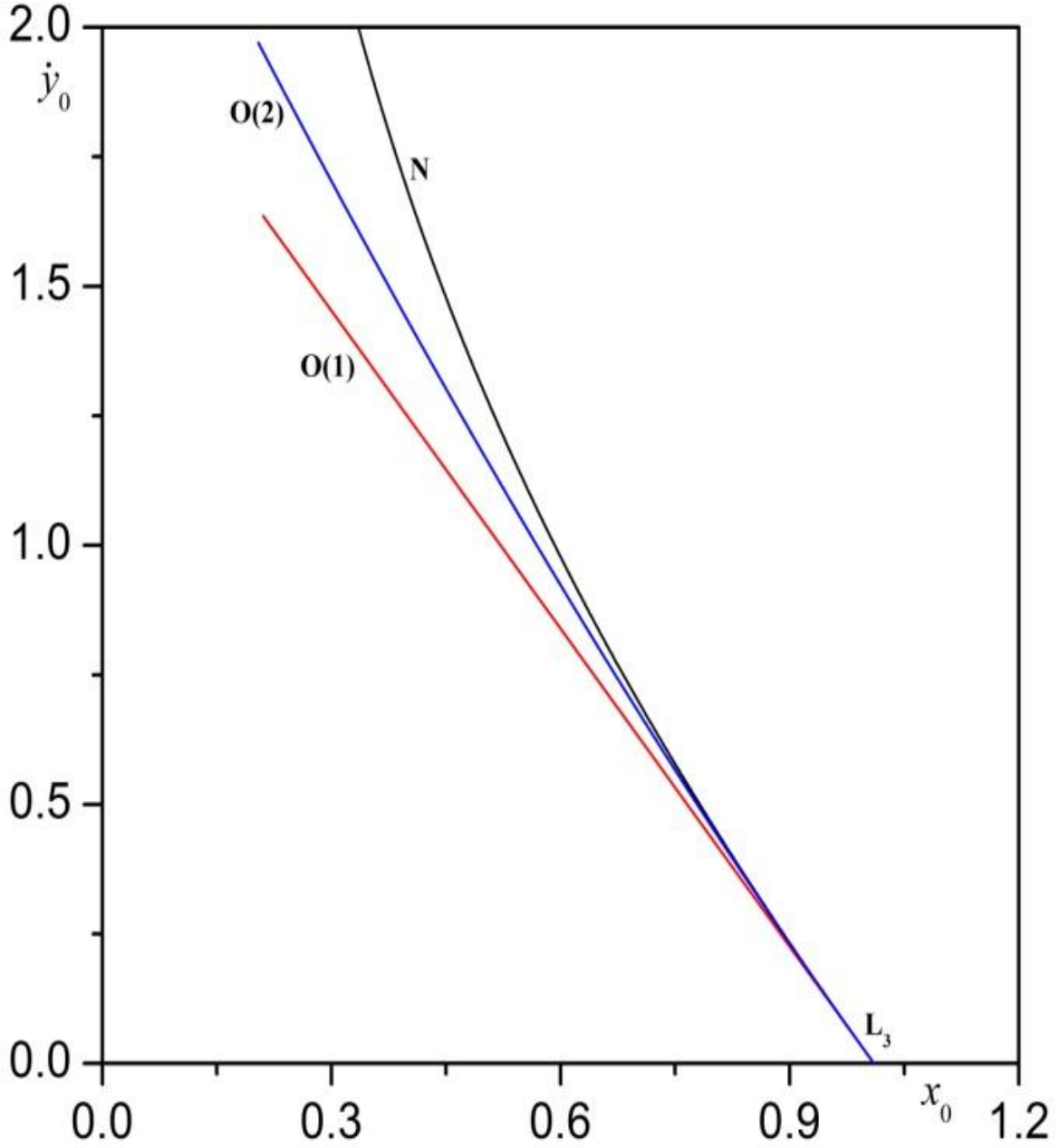


Figure 5.6: Comparison of the first [red curve $O(1)$] and second [blue curve $O(2)$] order analyses, obtained from solution (3.22), with the corresponding numerical solution (black curve N), obtained from the numerical integration of the full equations of motion (3.1), for all families of 2D periodic orbits emanating from the collinear equilibrium point L_3

The method of second order semi-analytic approximations was used to construct the initial conditions for small values of e in the planar case. These approximate initial conditions are obtained through the use of equations (3.7) – (3.14). We present these initial conditions $X_0, \dot{Y}_0, T_a$ at $t=0$ in Tables 5.3, 5.4, and 5.5 together with the Jacobian constant, C and the stability parameters a_h and a_v for the ten cases.

The conditions that must hold for planar Lyapunov orbits to be periodic and symmetric w.r.t. the OX axis are:

$$Y(X_0, 0, 0, \dot{Y}_0; T_a) = 0, \quad \dot{X}(X_0, 0, 0, \dot{Y}_0; T_a) = 0, \quad \dots (5.1)$$

where $T_a = T/2$ and T is the period of the whole orbit (see Szebehely, 1967; Tsirogiannis *et al.* (2006)).

Table 5. 3: Initial conditions for the Lyapunov family emanating from (Family A) for $\mu = 0.02545$.

Case	X_0	$\dot{Y}_0$	C	$T/2$	a_h	a_v
1	-0.78320335	0.00055761	3.29220658	1.30093233	1519.17680	0.996137
2	-0.78268995	0.00005442	3.29378966	1.30067713	1559.19297	0.998573
3	-0.78223919	0.00005131	3.29536756	1.30041216	1599.13647	0.999801
4	-0.78179535	0.00004826	3.29694067	1.30013776	1639.00392	0.999921
5	-0.78135818	0.00004527	3.29850912	1.29985454	1678.79019	0.999023
6	-0.78092747	0.00004235	3.30007301	1.29956303	1718.49045	0.997190
7	-0.78050299	0.00003949	3.30163246	1.29926375	1758.10049	0.994501
8	-0.78013438	0.00049669	3.30318738	1.29895731	1797.61312	0.991025
9	-0.77972163	0.00049393	3.30473827	1.29864382	1837.03001	0.986829
10	-0.77926502	0.00003124	3.30628524	1.29832380	1876.34888	0.981973

Table 5. 4: Initial conditions for the Lyapunov family emanating from (Family B) for $\mu = 0.02545$

Case	X_0	$\dot{Y}_0$	C	T/2	a_h	a_v
1	1.00990049	0.00089774	3.02662588	3.07453847	2.52824	0.998159
2	1.00998200	0.00039769	3.02764766	3.07304255	2.52842	0.998243
3	1.00981929	0.00039763	3.02866935	3.07154790	2.52860	0.998324
4	1.00965677	0.00039758	3.02969113	3.07005469	2.52878	0.998404
5	1.00949445	0.00039753	3.03071298	3.06856292	2.52896	0.998481
6	1.00933232	0.00039748	3.03173493	3.06707258	2.52913	0.998557
7	1.00917039	0.00039743	3.03275695	3.06558367	2.52931	0.998630
8	1.00900864	0.00039738	3.03377906	3.06409619	2.52949	0.998702
9	1.00884709	0.00039733	3.03480125	3.06261014	2.52966	0.998772
10	1.00868574	0.00039728	3.03582352	3.06112551	2.52984	0.998840

Table 5. 5: Initial conditions for the Lyapunov family emanating from (Family C) for $\mu = 0.02545$.

Case	X_0	$\dot{Y}_0$	C	T/2	a_h	a_v
1	-1.19346066	0.00021533	3.25850958	1.73919728	693.59589	0.981461
2	-1.19381226	0.00021386	3.26024175	1.73962442	711.63449	0.987381
3	-1.19415903	0.00021240	3.26197253	1.74002068	729.79313	0.992083
4	-1.19450109	0.00021097	3.26370196	1.74038723	748.06923	0.995633
5	-1.19483859	0.00020955	3.26543006	1.74072516	766.46027	0.998096
6	-1.19517354	0.00021815	3.26715689	1.74103552	784.96375	0.999533
7	-1.19550034	0.00020678	3.26888247	1.74131928	803.57732	1.000000
8	-1.19582483	0.00020542	3.27060684	1.74157740	822.29857	0.999553
9	-1.19615279	0.00024407	3.27233001	1.74181079	841.12520	0.998245
10	-1.19646535	0.00022275	3.27405205	1.74202027	860.05505	0.996124

We generate the corresponding Lyapunov families (A, B and C) by means of the numerical continuation process that makes use of the approximate initial conditions as initial guess for a periodic orbit. This involves the numerical integration of the equations of motion simultaneously with the variational equations. A proper choice of the phase space direction (direction of the flow) is made to extend the solution using a predictor-corrector scheme, and repeating the process iteratively until the periodic orbit is computed with the desired accuracy of 5×10^{-8} .

Few members of the family from the initial conditions of case 1 of Table 5.3 are presented in Table 5.6. This is because they are quite numerous in number. Presenting just a few will suffice because of space. Same is done for members of the families B and C, presented in Tables 5.7 and 5.8 respectively. In these Tables, the first, second and third columns give the

initial conditions $X_0, \dot{Y}_0, T_a = T/2$. The next two columns provide the final conditions and the subsequent columns furnish the Jacobian Constant, C , the horizontal and vertical stability parameters respectively.

Table 5. 6: Members of family A.

$T/2$	X_0	$\dot{Y}_0$	$X(T/2)$	$X(\dot{Y}_0)$	C	α_h	α_v
1.30093233	-0.78320335	0.00055761	-0.78307966	-0.00055752	3.29220658	1519.17680	0.996137
1.30093440	-0.78331443	0.00155761	-0.78296897	-0.00155689	3.29220479	1519.16258	0.996138
1.30093843	-0.78342566	0.00255761	-0.78285849	-0.00255568	3.29220129	1519.13492	0.996139
1.30094442	-0.78353704	0.00355761	-0.78274823	-0.00355387	3.29219610	1519.09384	0.996140
1.30095236	-0.78364857	0.00455761	-0.78263818	-0.00455147	3.29218922	1519.03936	0.996143
1.30096225	-0.78376025	0.00555761	-0.78252834	-0.00554848	3.29218064	1518.97147	0.996146
1.30097410	-0.78387208	0.00655761	-0.78241872	-0.00654490	3.29217038	1518.89021	0.996149
1.30098789	-0.78398406	0.00755761	-0.78230931	-0.00754073	3.29215842	1518.79557	0.996153
1.30100364	-0.78409620	0.00855761	-0.78220011	-0.00853597	3.29214477	1518.68758	0.996158
1.30102133	-0.78420849	0.00955761	-0.78209112	-0.00953063	3.29212944	1518.56625	0.996163
1.30104096	-0.78432092	0.01055761	-0.78198234	-0.01052469	3.29211243	1518.43159	0.996169

Table 5. 7: Members of family B

$T/2$	X_0	$\dot{Y}_0$	$x(T/2)$	$x(\dot{Y}_0)$	C	a_h	a_v
3.07453847	1.00990049	0.00089774	1.01077824	-0.00089754	3.02662588	2.52824	0.998159
3.07453850	1.00941172	0.00189774	1.01126699	-0.00189684	3.02662512	2.52824	0.998159
3.07453854	1.00892308	0.00289774	1.01175562	-0.00289564	3.02662382	2.52824	0.998159
3.07453860	1.00843455	0.00389774	1.01224412	-0.00389395	3.02662198	2.52824	0.998159
3.07453868	1.00794614	0.00489774	1.01273249	-0.00489176	3.02661960	2.52824	0.998159
3.07453879	1.00745785	0.00589774	1.01322073	-0.00588907	3.02661668	2.52823	0.998159
3.07453890	1.00696967	0.00689774	1.01370885	-0.00688589	3.02661322	2.52823	0.998159
3.07453903	1.00648162	0.00789774	1.01419684	-0.00788221	3.02660921	2.52822	0.998159
3.07453918	1.00599368	0.00889774	1.01468470	-0.00887803	3.02660467	2.52822	0.998159
3.07453935	1.00550586	0.00989774	1.01517243	-0.00987336	3.02659959	2.52821	0.998159
3.07453954	1.00501816	0.01089774	1.01566004	-0.01086820	3.02659397	2.52820	0.998159

Table 5. 8: Members of family C.

$T/2$	X_0	$\dot{Y}_0$	$x(T/2)$	$x(\dot{Y}_0)$	C	a_h	a_v
1.73919728	-1.19346066	0.00021533	-1.19337721	-0.00021536	3.25850958	693.59589	0.981461
1.73919747	-1.19355748	0.00071533	-1.19328025	-0.00071563	3.25850924	693.59468	0.981461
1.73919786	-1.19365427	0.00121533	-1.19318318	-0.00121619	3.25850853	693.59216	0.981461
1.73919845	-1.19375100	0.00171533	-1.19308601	-0.00171704	3.25850745	693.58834	0.981462
1.73919924	-1.19384769	0.00221533	-1.19298874	-0.00221819	3.25850600	693.58320	0.981463
1.73920027	-1.19394433	0.00271533	-1.19289136	-0.00271962	3.25850419	693.57688	0.981465
1.73920145	-1.19404093	0.00321533	-1.19279389	-0.00322134	3.25850200	693.56910	0.981467
1.73920284	-1.19413748	0.00371533	-1.19269631	-0.00372336	3.25849945	693.56003	0.981469
1.73920443	-1.19423399	0.00421533	-1.19259863	-0.00422567	3.25849652	693.54964	0.981471
1.73920623	-1.19433044	0.00471533	-1.19250084	-0.00472827	3.25849323	693.53795	0.981473
1.73920823	-1.19442685	0.00521533	-1.19240296	-0.00523116	3.25848957	693.52495	0.981476

The graphical representations of the evolution of 2D Lyapunov families are shown in Figures 5.7, 5.8 and 5.9 In Figure 5.7; we present the evolution of periodic orbits of family A. The first orbit can be seen to be oval in shape with a gradual change in the next two orbits. These

orbits create an enclave with one part of the secondary. The subsequent orbits have two enclaves; to their left and to their right. To their left, they partially enclose the oblate secondary, while to their right they enclose part of the bigger primary. These partial enclosures/enclaves are in actuality cavities. It can be observed that there are cavities at the oblate secondary and a single cavity at the triaxial primary as the orbits evolve. Thus, the orbits of family A are hedged between the two primaries.

In Figure 5.8 we show the evolution of periodic orbits of family B. The starting orbit is oval in shape with a gradual shift in appearance towards taking the shape of a bean. This particular orbit is of a little distance from the primaries and is to the right of both of them. We also observe a couple of cavities closer to the bigger triaxial primary, as the orbits evolve. As can be seen, the orbits of this family have enclosed the triaxial primary within them.

Same as the other two families, the starting orbit of family C is also oval in shape as can be observed in Figure 5.9, where we present the evolution of their periodic orbits. It can be seen that there is a change in the appearance of the subsequent orbits, which have deep cavities thereby totally enclosing the oblate secondary and putting the triaxial primary within both ends of their shape.

The forbidden regions of the motion of the infinitesimal orbits of the Lyapunov families (A, B, and C) in the C, x_0 plane are presented in Figure 5.10. But for the change in intersections for family C at two points marked as 3 and 5 in the figure, we have shown that each Lyapunov family goes to collision.

In Figures 5.11 – 5.16, we show the horizontal and vertical stability curves for the Lyapunov families in the vicinity of the collinear equilibrium points. Note that, due to the large values of the stability indices, we have plotted $\sinh^{-1} a_h$ instead of a_h and $\sinh^{-1} a_v$ instead of a_v .

(except for Figure 5.16); therefore, the inequalities $|a_h| < 1$ and $|a_v| < 1$ for stability become in this case $|a_h| < 0.8814$ and $|a_v| < 0.8814$, respectively. As we observe, all families consist of horizontally and vertically stable parts except from the Lyapunov family emanating from L_2 which is horizontally unstable. Similar behaviour for the planar Lyapunov families when the larger primary is a source of radiation has been also observed by Ragos and Zagouras (1991) where they found that only the Lyapunov family emanating from the outer collinear equilibrium point to the left of the larger primary may contain horizontally stable orbits. For triaxial secondary body Jain et al. (2006) obtained only unstable planar Lyapunov orbits for all the collinear equilibrium points.

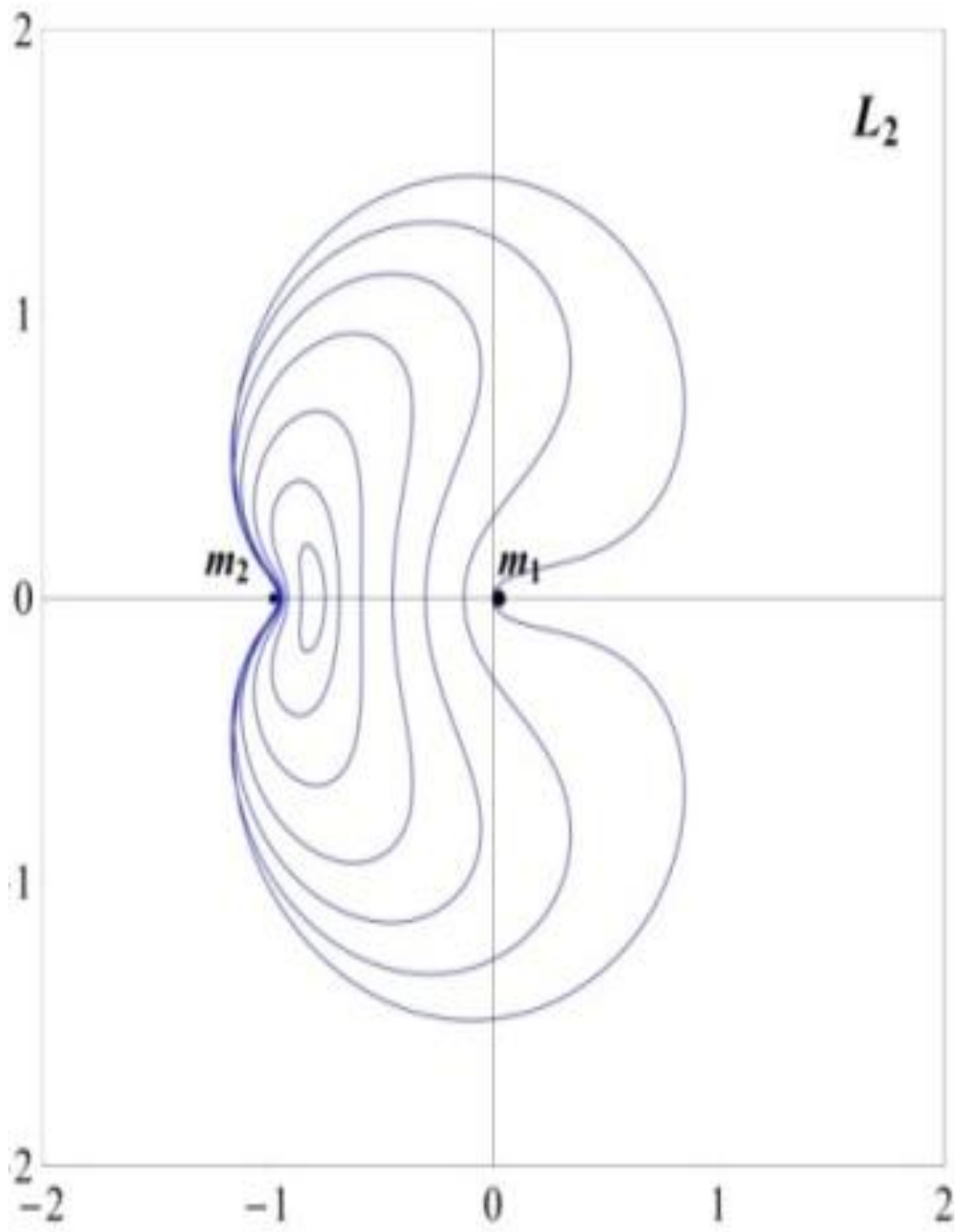


Figure 5.7: Shows the evolution of periodic orbits of family A

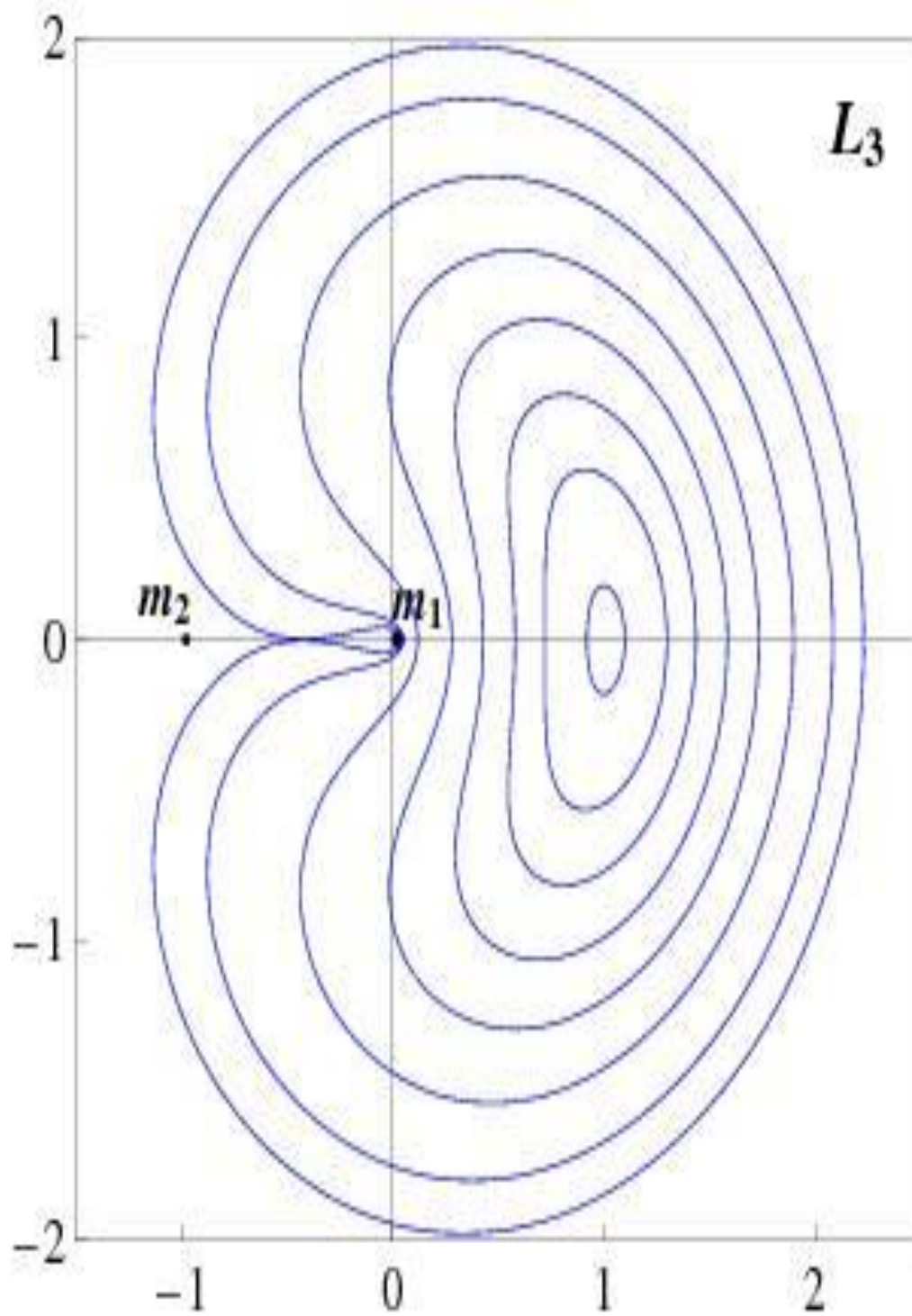


Figure 5.8: Shows the evolution of periodic orbits of family B.

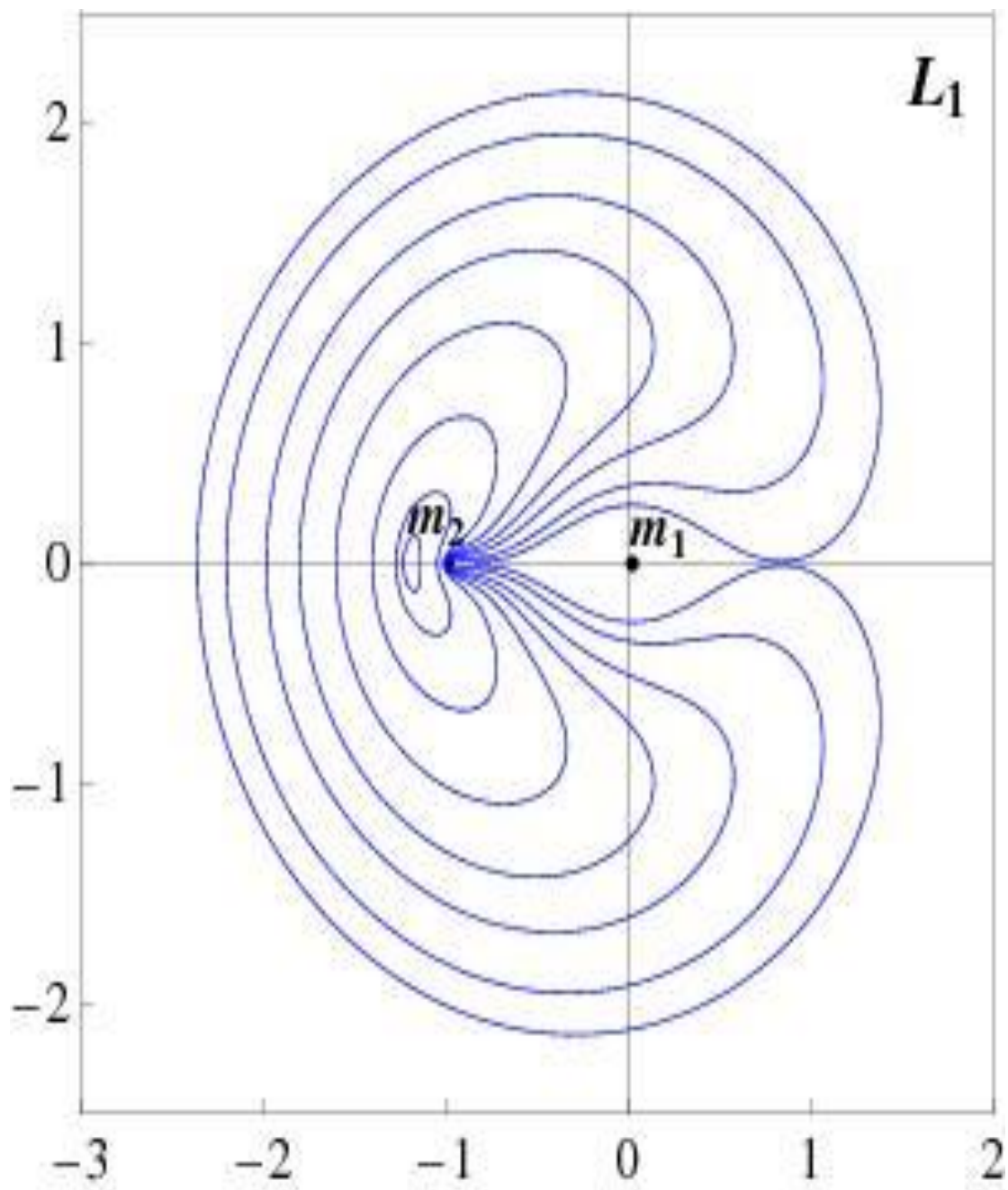


Figure 5.9: Shows the evolution of periodic orbits of family C.

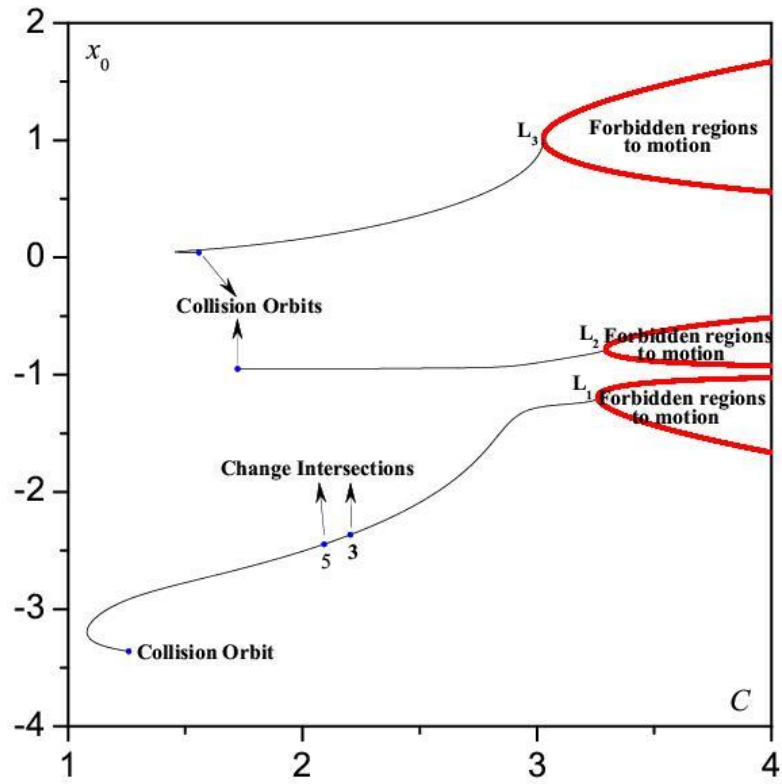


Figure 5.10: The Lyapunov families and their respective forbidden region.

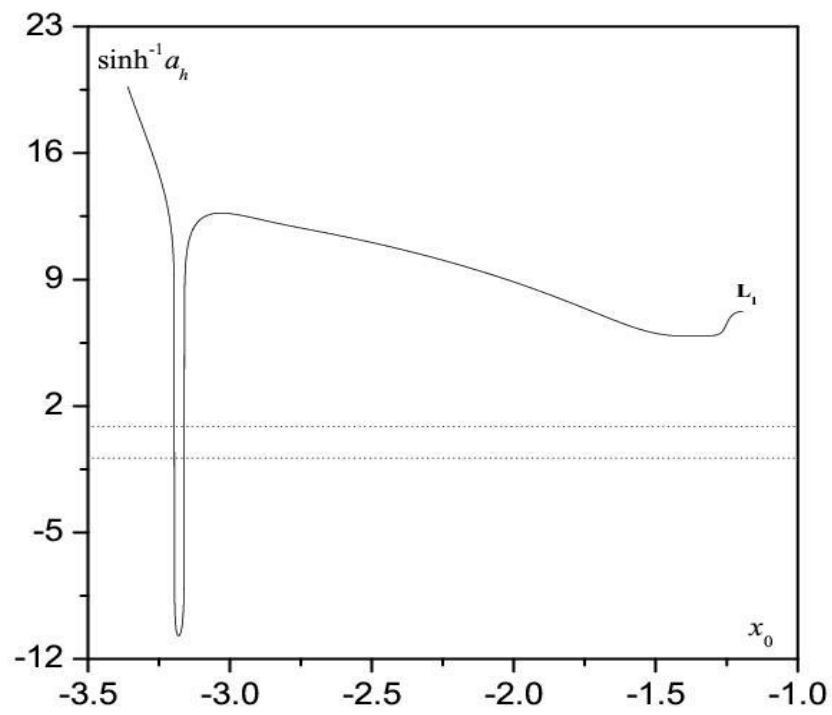


Figure 5.11: Horizontal stability curve of family C for the 2D periodic orbits.

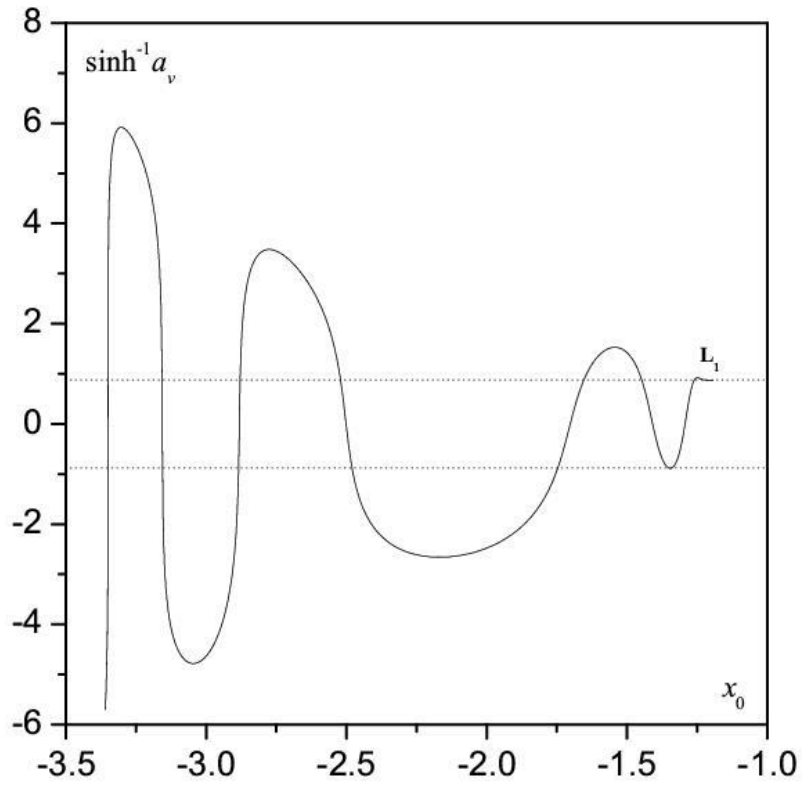


Figure 5.12: Vertical stability curve of family C for the 2D periodic orbits.

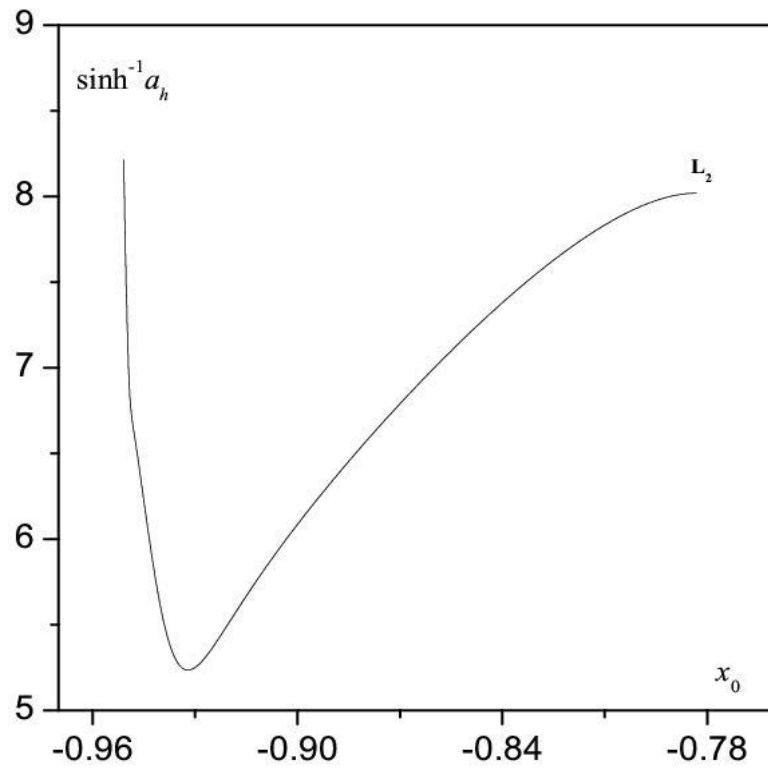


Figure 5.13: Horizontal stability curve of family A for the 2D periodic orbits.

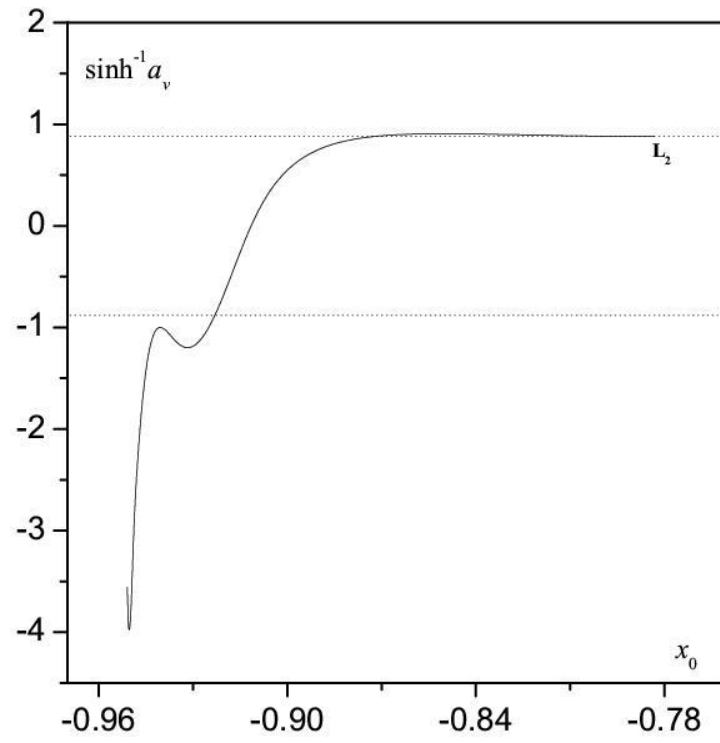


Figure 5.14: Vertical stability curve of family A for the 2D periodic orbits.

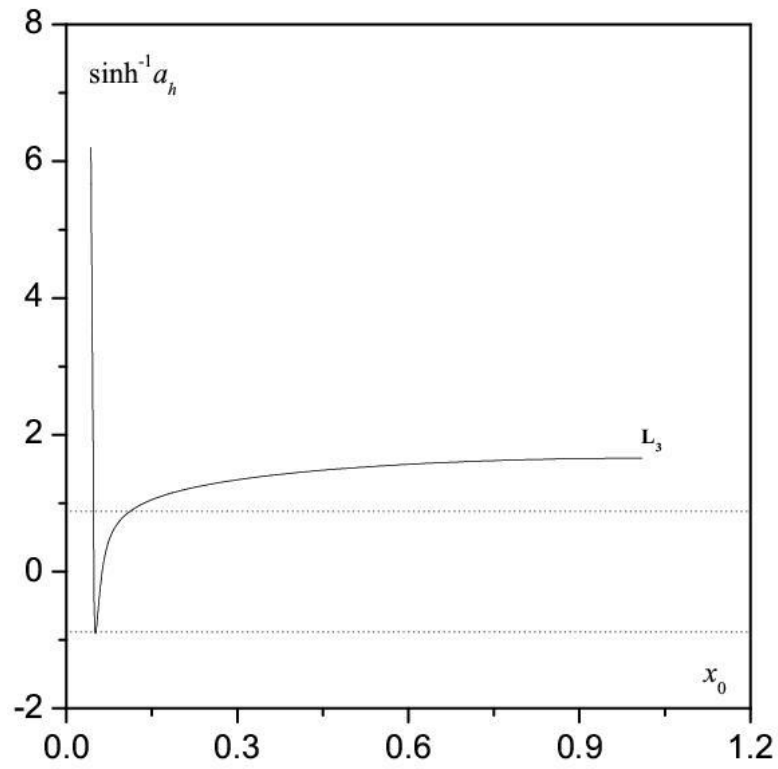


Figure 5.15: Horizontal stability curve of family B for the 2D periodic orbits

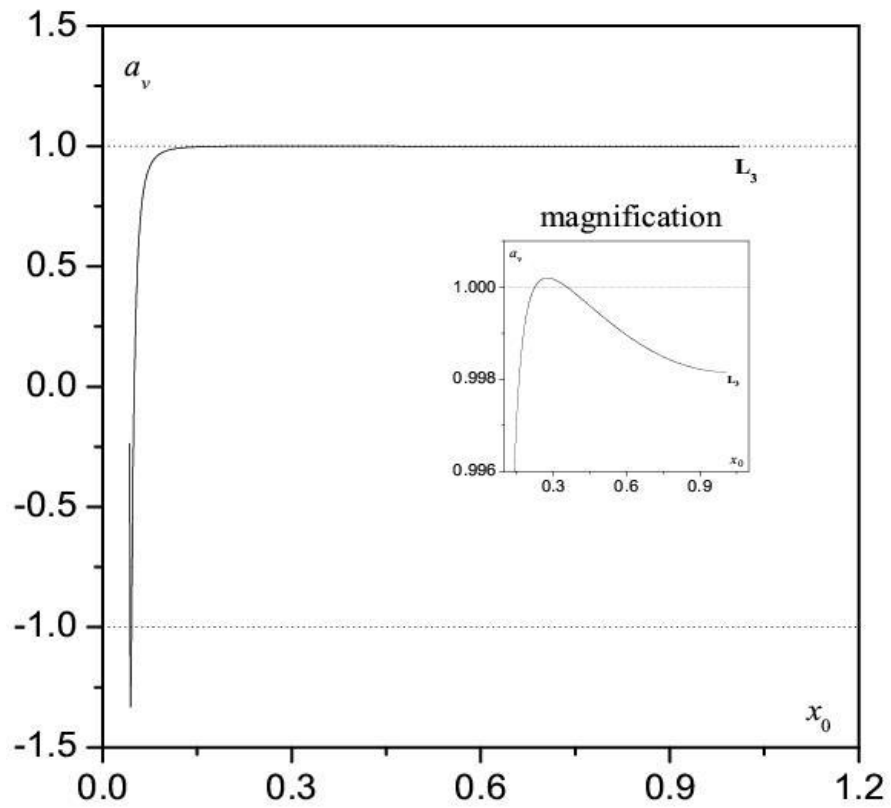


Figure 5.16: Vertical stability curve of family B for the 2D periodic orbits.

5.2.2 Numerical results for 3D periodic orbits having triaxial and oblate primaries

In Figure 5.17, we present graphically, a 3D periodic orbit obtained from the semi-analytical solution of (3.37).

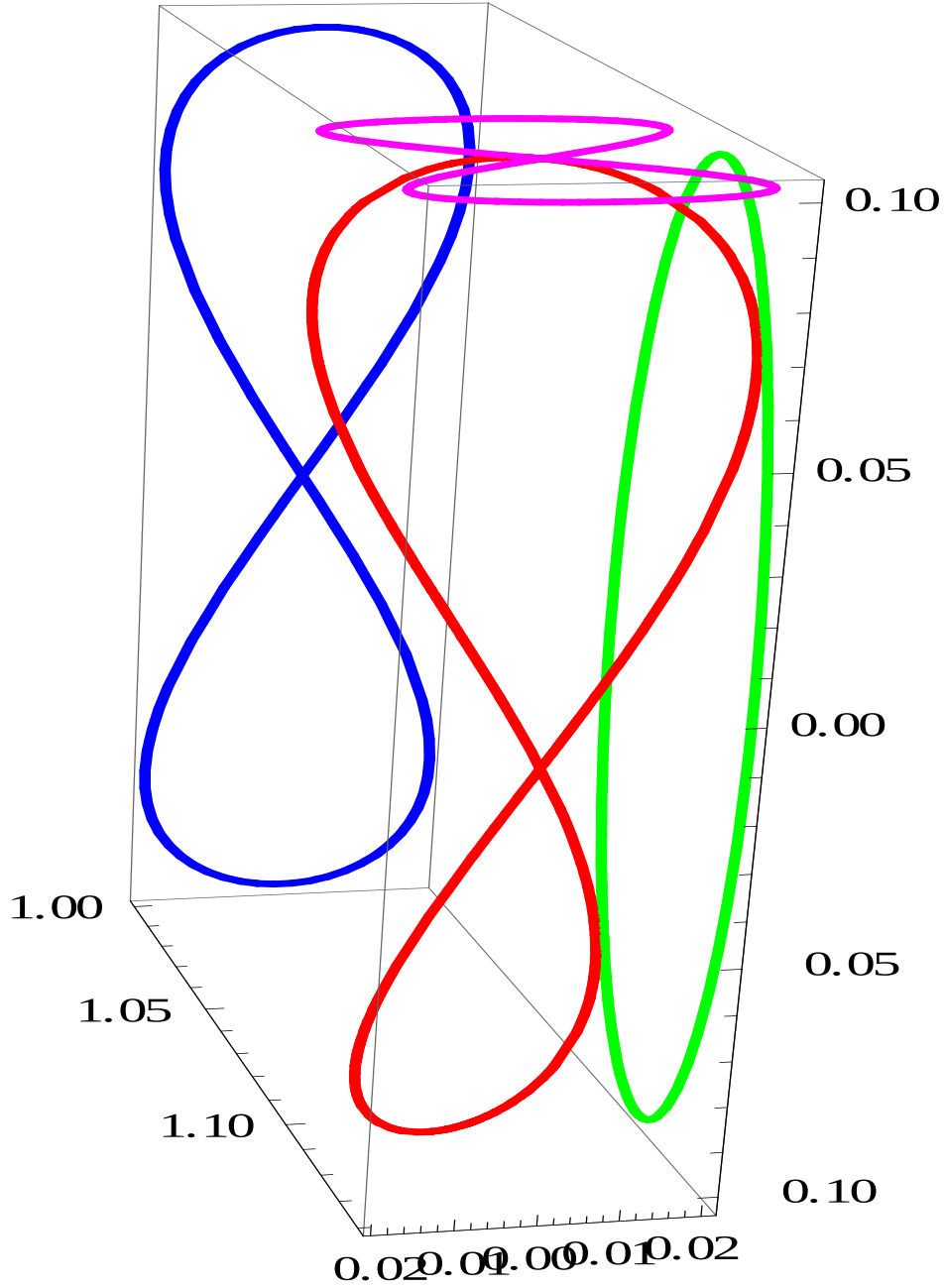


Figure 5.17: A typical 3D Lyapunov periodic orbit (red) is shown with the corresponding projections in the planes Oyz (blue color), Oxz (green color) and Oxy (magenta color).

To establish the validity of our analysis of the 3D periodic orbits, we present in Figures 5.18 and 5.19, as examples, the projections of the semi-analytical solution (3.37) about the collinear equilibrium point L_1 , in the $(x_0, \dot{z}_0)$ and $(\dot{y}_0, \dot{z}_0)$ planes (blue curves), for $t=0$ and certain range of values of the orbital parameter e , together with the corresponding accurate

numerical solution (black curves) obtained by the numerical integration of the equations of motion (3.1).

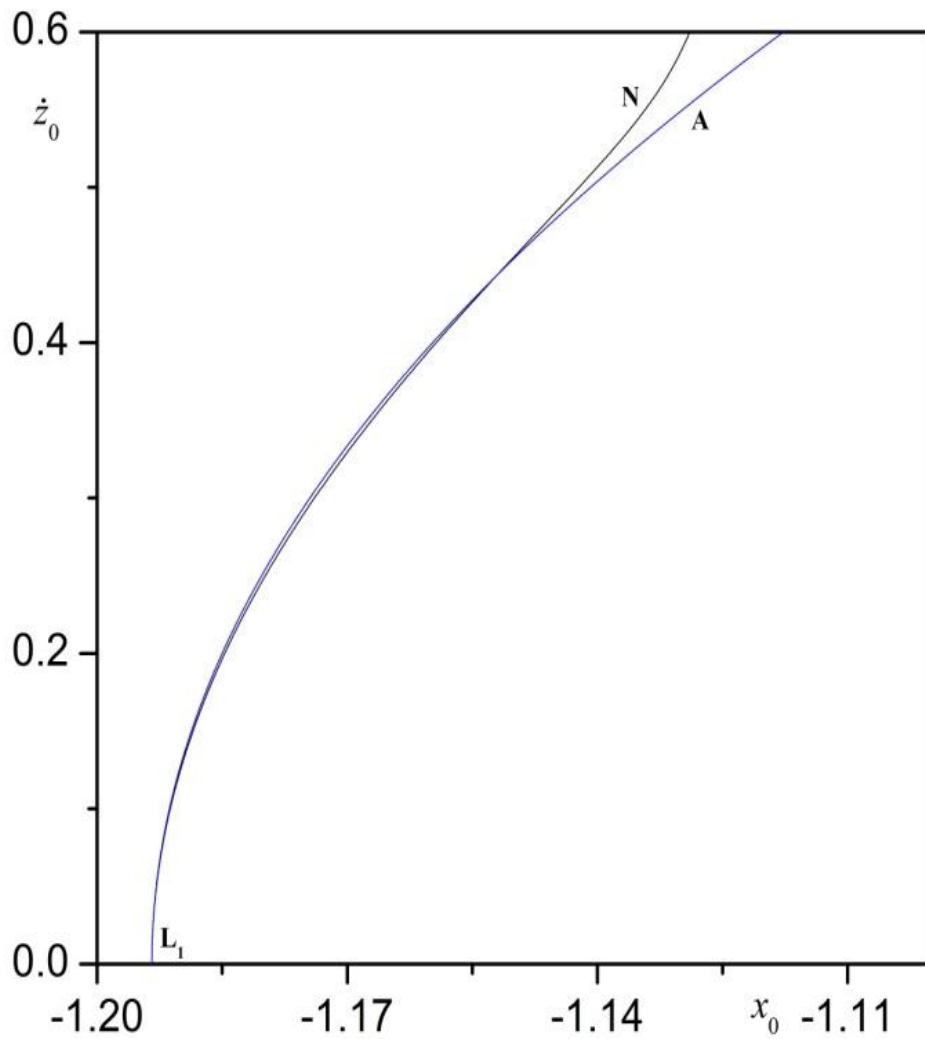


Figure 5.18: Comparison, in the space of initial conditions, of the semi-analytical solution (blue curve denoted by A) obtained by our analysis and the corresponding numerical solution (black curve denoted by N) obtained by the numerical integration of the equations of motion, in the case of the family of three-dimensional periodic orbits emanating from the collinear equilibrium point L_1 in the $(x_0, \dot{z}_0)$.

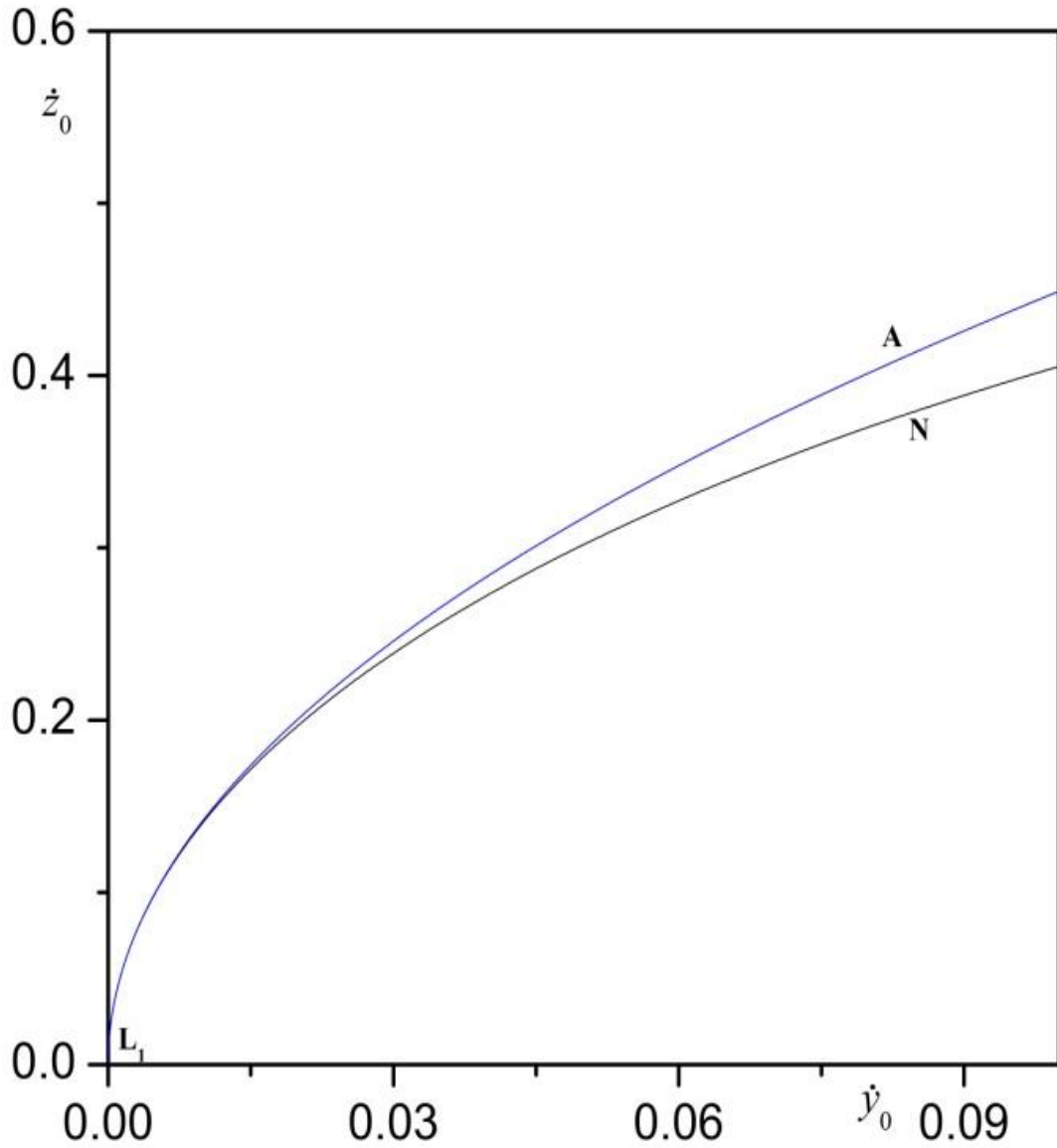


Figure 5.19: Comparison, in the space of initial conditions, of the semi-analytical solution (blue curve denoted by A) obtained by our analysis and the corresponding numerical solution (black curve denoted by N) obtained by the numerical integration of the equations of motion, in the case of the family of three-dimensional periodic orbits emanating from the collinear equilibrium point L_1 in the $(\dot{y}_0, \dot{z}_0)$.

In order to compute numerically the first member of a family of three-dimensional periodic orbits emanating from any collinear equilibrium point we have used the approximate solution

(3.37) for appropriate initial conditions. Thereafter, applied the well-known predictor-corrector scheme based on the periodicity conditions:

$$y(x_0, 0, 0, 0, \dot{y}_0, \dot{z}_0; T/4) = 0, \quad \dot{x}_0(x_0, 0, 0, 0, \dot{y}_0, \dot{z}_0; T/4) = 0, \quad \dot{z}_0(x_0, 0, 0, 0, \dot{y}_0, \dot{z}_0; T/4) = 0, \quad (5.2)$$

so as to obtain the whole family. Note, T is the period of a 3D periodic orbit; we seek periodicity conditions at the quarter of the period due to the double symmetry of the orbits in question with respect to the Ox -axis and Oxz -plane.

Table 5. 9: Initial conditions for family of 3D periodic orbits emanating from (Family A) for $\mu = 0.02545$.

Case	x_0	$\dot{y}_0$	$\dot{z}_0$	C	$T/4$	P	Q
1	-0.78371825	-0.00100520	0.07000000	3.28730991	0.66255038	-1.98864	-3314.05
2	-0.78326989	-0.00100980	0.07000000	3.28889287	0.65878156	-1.99500	-3250.13
3	-0.78282805	-0.00101365	0.07000000	3.29047091	0.65514644	-1.99881	-3190.09
4	-0.78239255	-0.00101681	0.07000000	3.29204416	0.65163732	-2.00032	-3133.60
5	-0.78196317	-0.00101938	0.07000000	3.29361273	0.64824705	-1.99977	-3080.36
6	-0.78153975	-0.00102140	0.07000000	3.29517676	0.64496907	-1.99739	-3030.09
7	-0.78112209	-0.00102294	0.07000000	3.29673634	0.64179726	-1.99334	-2982.55
8	-0.78071003	-0.00102406	0.07000000	3.29829158	0.63872594	-1.98782	-2937.54
9	-0.78030340	-0.00102478	0.07000000	3.29984259	0.63574986	-1.98098	-2894.84
10	-0.77990206	-0.00102516	0.07000000	3.30138946	0.63286412	-1.97296	-2854.29

Table 5. 10: Initial conditions for family of 3D periodic orbits emanating from (Family B) for $\mu = 0.02545$.

Case	x_0	$\dot{y}_0$	$\dot{z}_0$	C	$T/4$	P	Q
1	1.01031202	0.00240785	0.07000000	3.02172030	1.55226617	1.99624	-5.12362
2	1.01014834	0.00240716	0.07000000	3.02274191	1.55116450	1.99641	-5.12229
3	1.00998484	0.00240647	0.07000000	3.02376361	1.55006420	1.99658	-5.12096
4	1.00982155	0.00240578	0.07000000	3.02478539	1.54896525	1.99674	-5.11963
5	1.00965845	0.00240509	0.07000000	3.02580725	1.54786765	1.99690	-5.11830
6	1.00949555	0.00240440	0.07000000	3.02682919	1.54677141	1.99705	-5.11698
7	1.00933284	0.00240371	0.07000000	3.02785122	1.54567652	1.99720	-5.11565
8	1.00917032	0.00240301	0.07000000	3.02887333	1.54458298	1.99735	-5.11433
9	1.00900800	0.00240232	0.07000000	3.02989552	1.54349082	1.99749	-5.11301
10	1.00884588	0.00240162	0.07000000	3.03091780	1.54239994	1.99763	-5.11169

Table 5. 11: Initial conditions for family of 3D periodic orbits emanating from (Family C) for $\mu = 0.02545$.

Case	x_0	$\dot{y}_0$	$\dot{z}_0$	C	$T/4$	P	Q
1	1.19234688	0.00243426	0.07000000	3.25361184	0.89811482	1.95337	-1696.37
2	1.19268616	0.00242876	0.07000000	3.25534427	0.89339954	1.96743	-1669.89
3	1.19302125	0.00242284	0.07000000	3.25707531	0.88881356	1.97871	-1644.76
4	1.19335225	0.00241655	0.07000000	3.25880499	0.88435105	1.98741	-1620.87
5	1.19367926	0.00240993	0.07000000	3.26053335	0.88000653	1.99374	-1598.15
6	1.19400236	0.00240304	0.07000000	3.26226042	0.87577481	1.99788	-1576.52
7	1.19432165	0.00239591	0.07000000	3.26398624	0.87165103	1.99999	-1555.89
8	1.19463720	0.00238858	0.07000000	3.26571085	0.86763058	2.00024	-1536.20
9	1.19494911	0.00238107	0.07000000	3.26743426	0.86370912	1.99877	-1517.40
10	1.19525744	0.00237341	0.07000000	3.26915653	0.85988255	1.99571	-1499.42

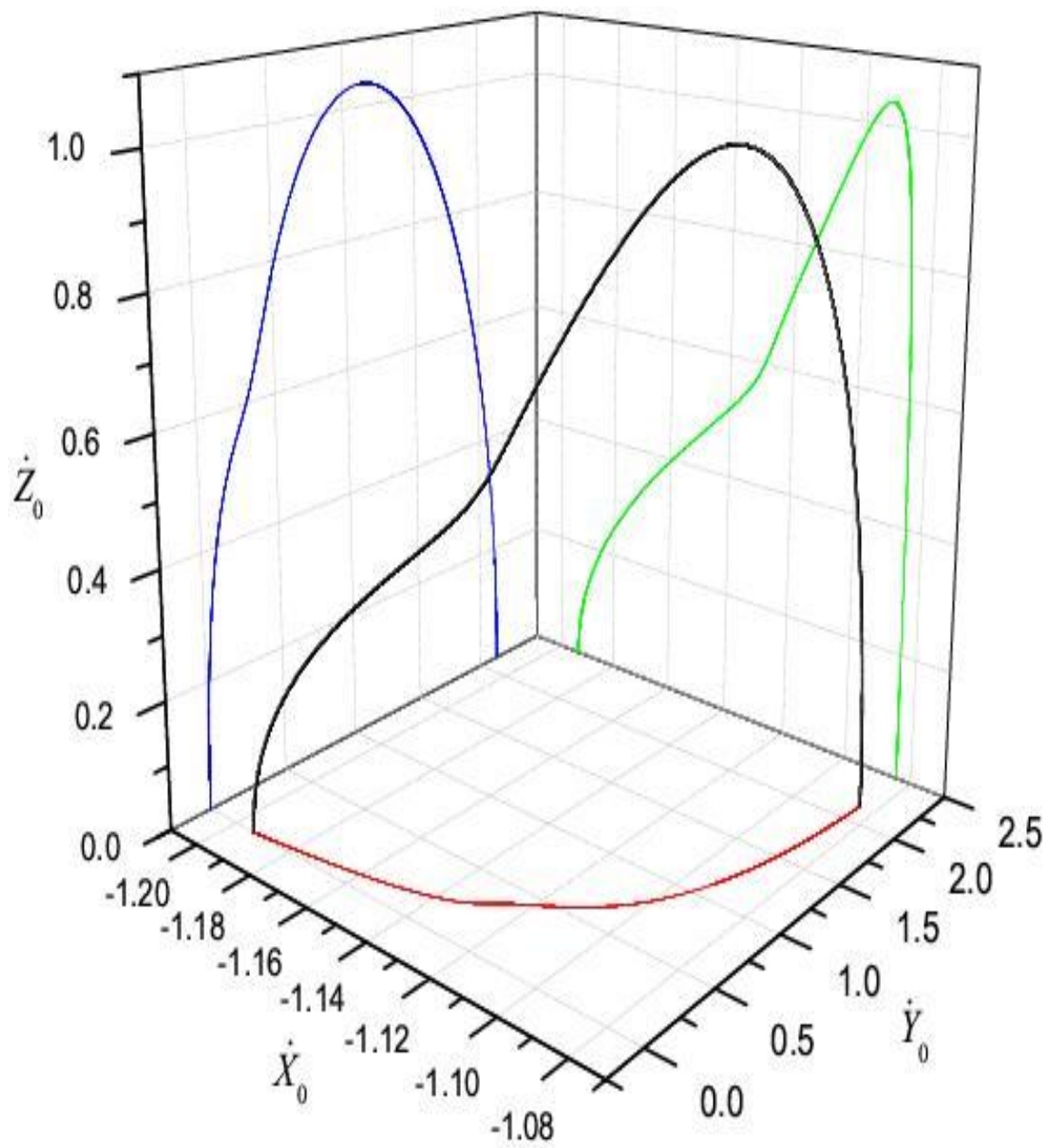


Figure 5.20: Characteristic curves of the family C of 3D periodic orbits.

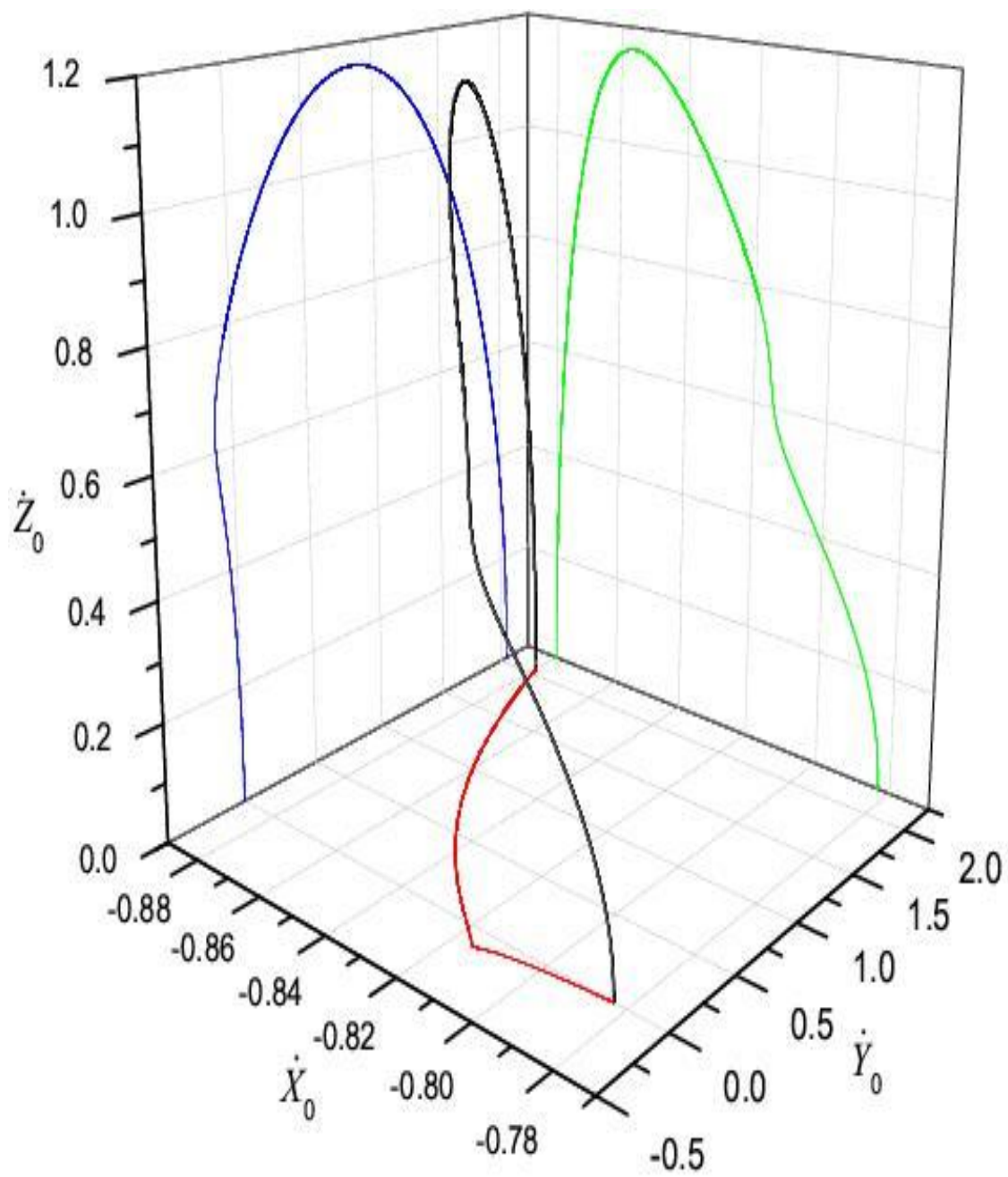


Figure 5.21: Characteristic curves of the family A of 3D periodic orbits.

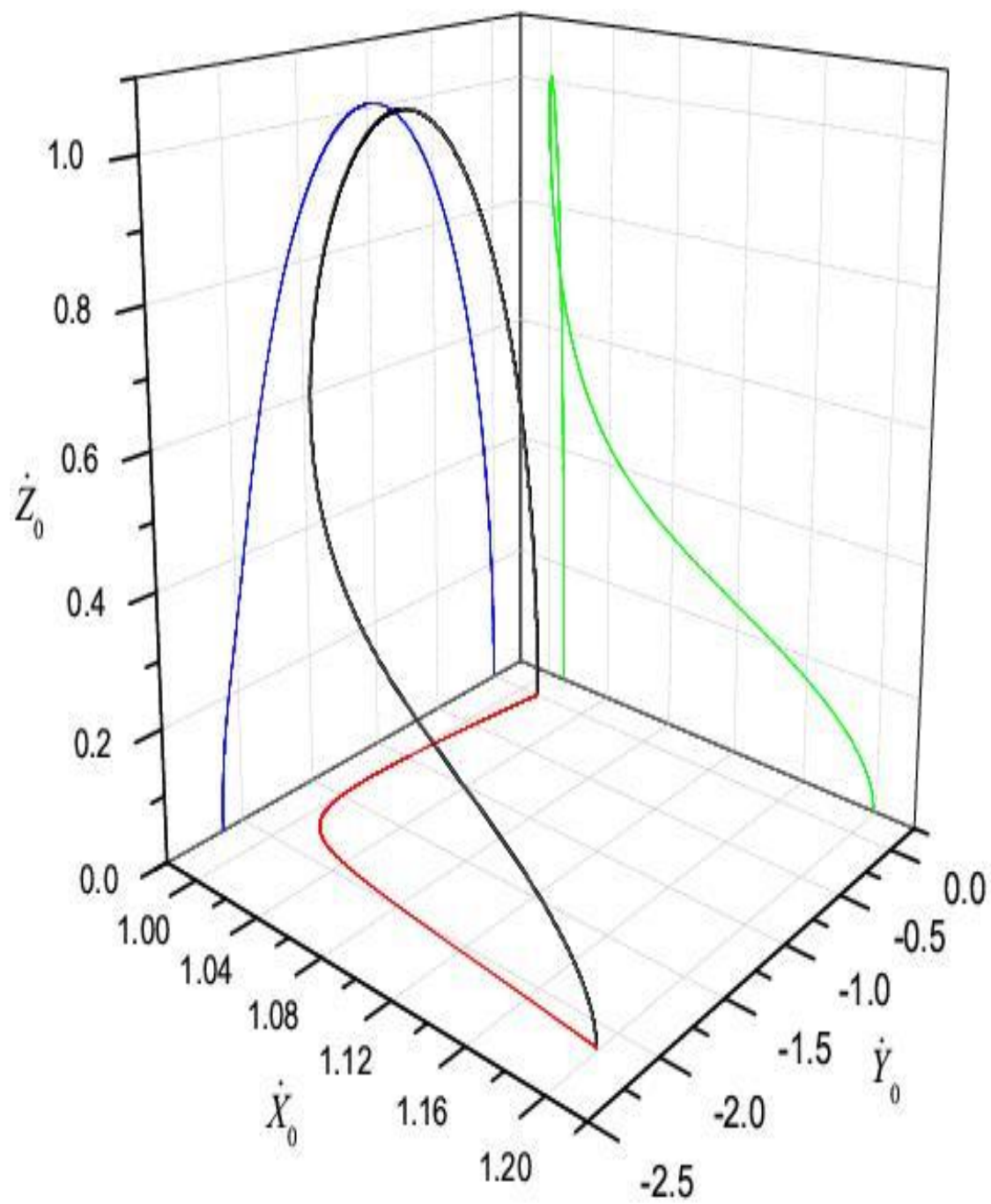


Figure 5.22: Characteristic curves of the family B of 3D periodic orbits.

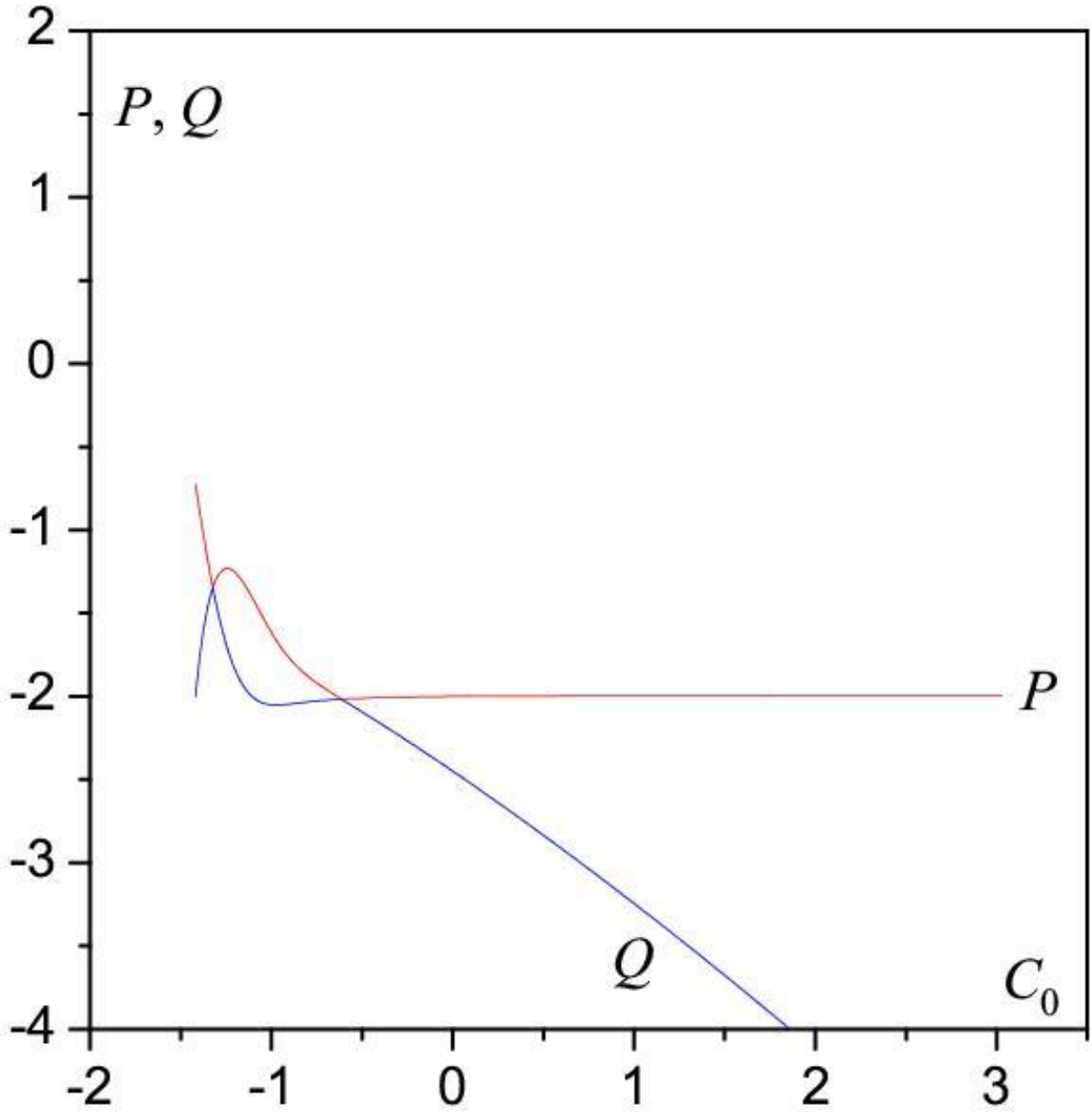


Figure 5.23: Stability of family B emanating from L_3 containing stable parts.

In Tables 5.9, 5.10 and 5.11, we present initial conditions for all the computed families of three-dimensional periodic orbits emanating from the collinear equilibrium points, for cases 1 to 10 of Table 5.1. In each table, we give the values of x_0 , $\dot{y}_0$ and $\dot{z}_0$ (since $y_0 = z_0 = \dot{x}_0 = 0$ for double symmetry w.r.t. the Ox-axis and Oxz-plane), the value of the Jacobi constant C ,

the quarter of the orbit's period $T/4$ and the stability parameters P and Q as discussed in chapter three (see also Hénon, 1973, for details). In Figures 5.20 - 5.22, we show the evolution of the characteristic curves of the respective families in the space of initial conditions $(x_0, \dot{y}_0, \dot{z}_0)$. As we see, all the computed families emanate from the corresponding equilibrium point and terminate at a planar periodic orbit. Only family B emanating from the equilibrium point L_3 possesses stable parts and this is shown in the last frame of Figure 5.23. This is in agreement with the results presented by Tsirogiannis *et al.* (2006) in the framework of the restricted three-body problem with the larger primary being a radiation source and the smaller one an oblate spheroid.

5.3 Numerical Considerations for the Analytic Approximation of Lyapunov Orbits Around the Collinear Equilibrium Points in the Two-Dimensional R3BP when the Primaries are Radiating and Oblate Spheroidal Bodies.

5.3.1 Numerical applications for the 2D case for the model with radiating oblate primaries

In section 4.2, we obtained a third order analytic approximation of periodic solution around the collinear equilibrium points in the PCR3BP where the primaries are radiating oblate spheroidal bodies by the utilizing the Lindsted Poincare method.

The theory is applied to the binary α -Centuari system. The infinitesimal body is taken to be a possible exoplanet with negligible mass as compared to the masses of the primaries moving in the plane of motion of the binary system. The assumptions made here are that both stars have significant radiation effects and are sufficiently oblate in shape. Arbitrary chosen values for the radiation and oblateness coefficients have been drawn in six instances (cases). In the

first case, the binary system is considered with respect to the classical problem (spherical nature of the primaries) while the other five cases have values for all the parameters concerned.

Alpha Centuari (α -Cen) system consists of three stars, two of which form a pair and move say, in circular orbits about their common barycentre (forming a R3BP where the third body has no influence on their gravitational attraction). While the third star is a small, faint red dwarf called Proxima Centuari. This phenomenon can be seen in the illustration of Figure 5.24.

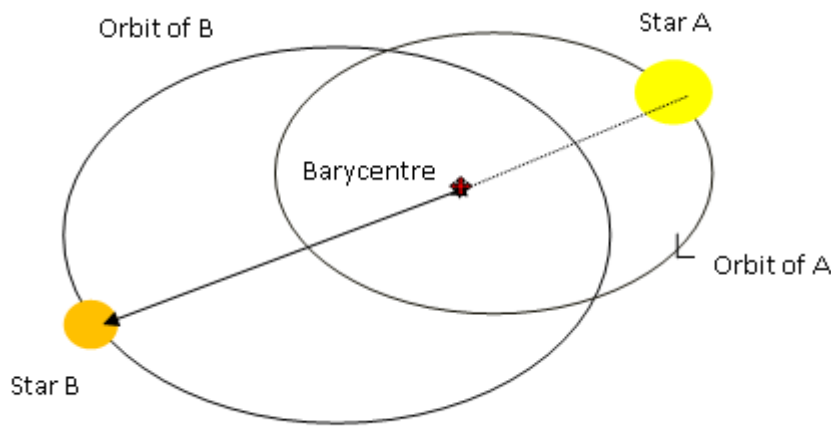


Figure 5.24: Movement of star A and B in circular orbits about their common barycentre.

The principal member (primary) of the binary system is known as Alpha Centuari A (α -Cen A) with a mass about 1.1times that of the Sun. It is a G2 V star having luminosity about 0.55 greater than that of the Sun. The secondary star called Alpha Centuari B (α -Cen B) is a main-sequence star of spectral type K1 V. This star is 0.9 the size of the star and 0.14 smaller in radius.

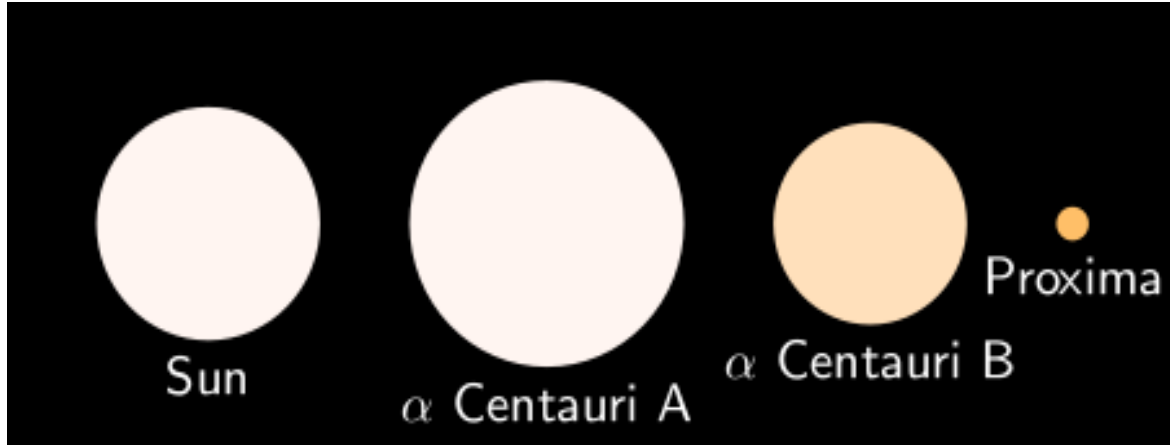


Figure 5.25: The relative sizes and colour of stars in the α -Centuari system, compared to the Sun (https://en.wikipedia.org/wiki/File:Alpha_Centauri_relative_sizes.svg)

The calculated masses of the stars α -Centuari A and α -Centuari B are 2.192×10^{30} Kg and 1.970×10^{30} Kg respectively, then the mass parameter $\mu = \frac{1.970}{4.162} = 0.47333 < \frac{1}{2}$.

In Table 5.12, we have shown six cases and their corresponding collinear equilibrium points for all the participating parameters. The first case corresponds to the classical case where $A_1 = A_2 = 0$ and $q_1 = q_2 = 1$. The other five cases have values as shown in the table. The effect of the oblateness and radiation pressure of the binary α -Centuari system on the collinear equilibrium points have also been shown in Figures 5.26 and 5.27 respectively.

Table 5.13 is introduced in order to discuss further the influence of the oblateness and radiation parameters of the primary and secondary bodies on the collinear equilibrium points. In the first row of the Table, the primary star is sufficiently oblate while the secondary star is a sphere together with the ineffectiveness ($q_1 = q_2 = 1$) of the radiation parameters of the binary system. It can be seen here that the collinear equilibrium point L_1 moves closer to the origin while L_2 and L_3 move away from the origin. The second row has been drawn by

keeping the oblateness parameter of the primary star constant and introducing oblateness to the secondary star while making the radiation parameters of both stars ineffective.

The oblateness of the secondary star has made the equilibrium point L_1 to move away from the origin with L_2 moving closer to the origin. By keeping the oblateness parameters of both stars constant and making the radiation parameter of the primary star effective ($q_1 = 0.6$) while making the radiation parameter of the secondary star ineffective ($q_2 = 1$), the third row is drawn. The results obtained in this row show that the radiation parameter of the primary star does not affect the collinear equilibrium points since same positions as the second case are obtained. Same results as those of the second and third rows are obtained when the fourth row is drawn by keeping constant the oblateness parameters of both stars as well as the radiation parameter of the primary star and also making the radiation parameter of the secondary star effective ($q_2 = 0.4$).

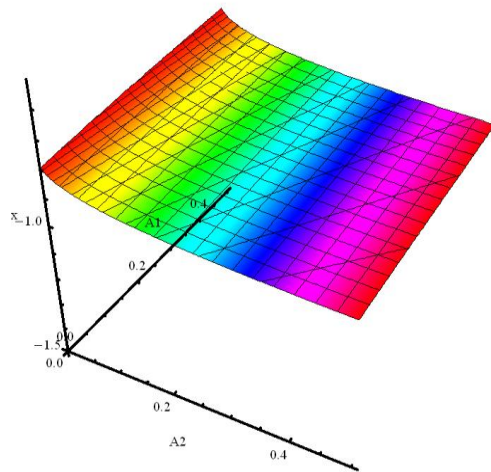
Thus, it can be seen that in the first row the equilibrium point L_1 moves closer to L_2 while L_2 moves farther away from L_3 . In the second row the equilibrium point L_1 moves farther away from L_2 while L_2 moves closer to L_3 . The analyses for the third and fourth rows are the same as that of the second row. From these results, it can be deduced that the radiation parameters have no effect on the position of the collinear equilibria.

Table 5. 12: Collinear Equilibrium points for binary α -Centuari system mass parameter $\mu = 0.47333$

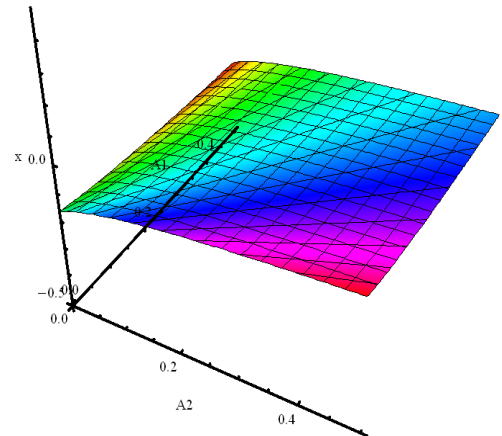
Case	A_1	A_2	q_1	q_2	L_1	L_2	L_3
1	0	0	1	1	-1.20751483	-0.03765997	1.18902157
2	0.01	0.001	0.4	0.1	-0.79423688	-0.16548350	0.95833957
3	0.02	0.002	0.5	0.2	-0.88238502	-0.13509348	1.01122572
4	0.03	0.003	0.6	0.3	-0.94520716	-0.12047798	1.05651685
5	0.04	0.004	0.7	0.4	-0.99536901	-0.11247880	1.09634009
6	0.05	0.005	0.8	0.5	-1.03761166	-0.10790176	1.13198424

Table 5. 13: Collinear Equilibrium points for binary α -Centuari system mass parameter $\mu = 0.47333$ with variations in the parameters

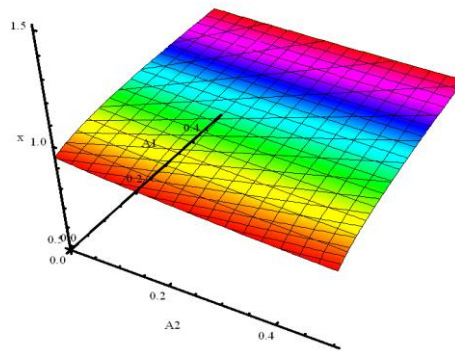
A_1	A_2	q_1	q_2	L_1	L_2	L_3
0.01	0	1	1	-1.20350120	-0.04410165	1.19192740
0.01	0.002	1	1	-1.20422977	-0.04264871	1.19111745
0.01	0.002	0.6	1	-1.20422977	-0.04264871	1.19111745
0.01	0.002	0.6	0.4	-1.20422977	-0.04264871	1.19111745



(a)

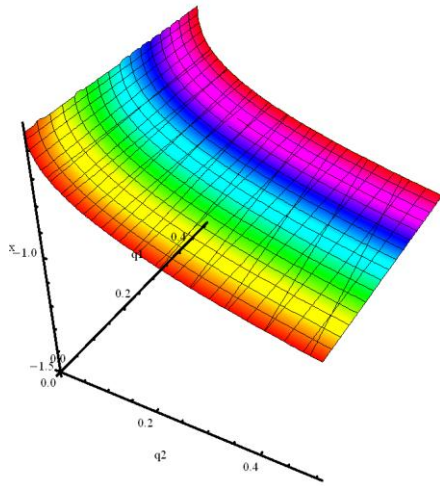


(b)

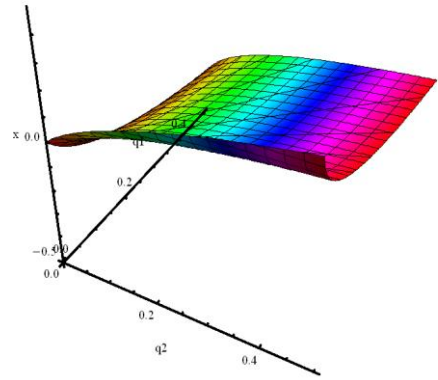


(c)

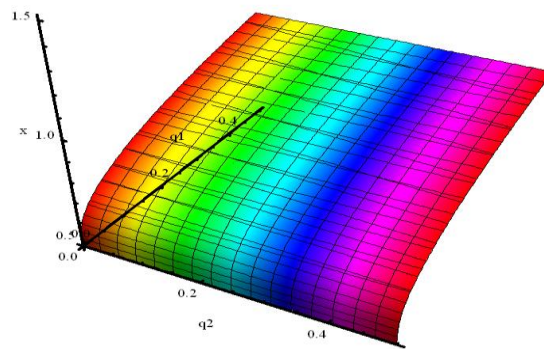
Figure 5.26: The oblateness effects of the binary α -Centuari A system on the collinear equilibrium points in (a), in (b) and in (c), for case 2 (c)



(a)



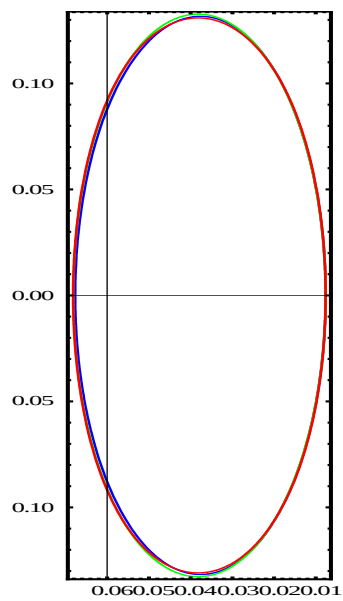
(b)



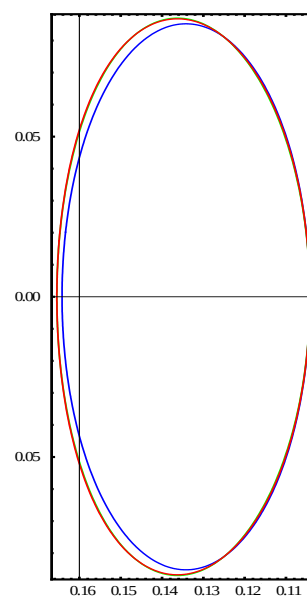
(c)

Figure 5.27: The radiation effects of the binary α -Centuari A system on the collinear equilibrium points in (a), in (b) and in (c), for case 2.

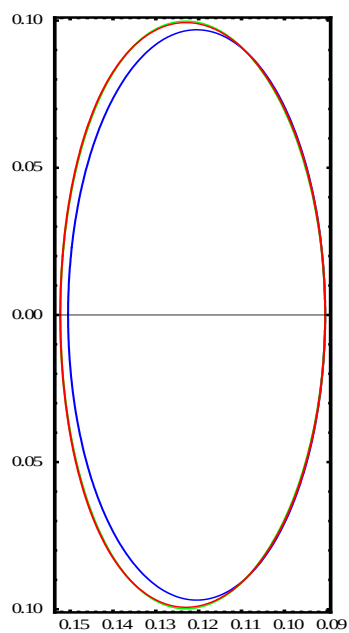
From the third order approximation solution of Eqns. 4.16, we show graphically the starting orbits of cases 1-6 in Figure 5.28.



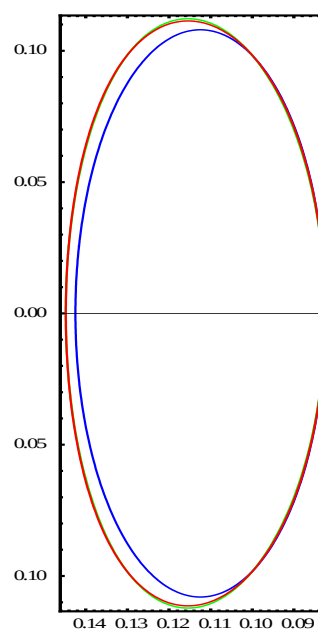
(a)



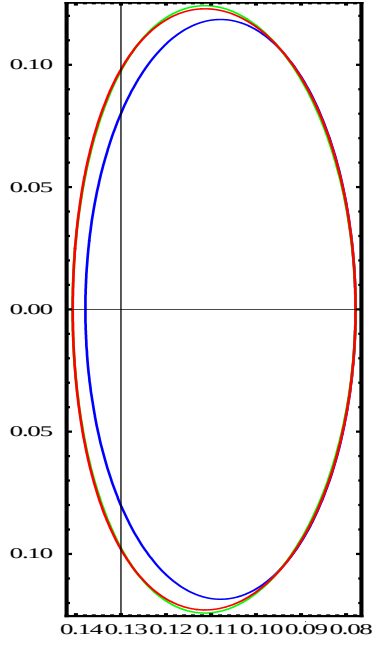
(b)



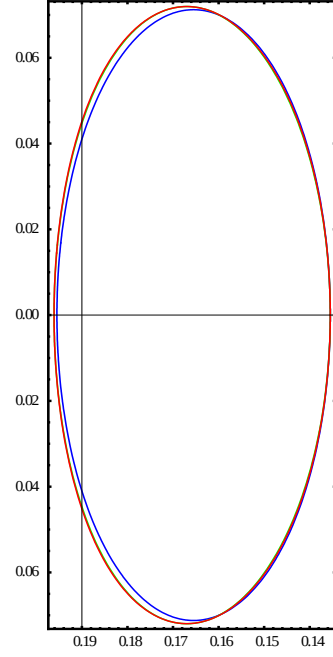
(c)



(d)



(e)



(f)

Figure 5.28: In (a), (b), (c), (d), (e) and (f) [which also correspond to figures for cases 1,2,3,4,5 and 6] are the starting orbits where in each frame, the orbits in colour blue, green and red correspond to the first, second and third order systems respectively.

By substituting the data presented in Table 5.12 in equations (4.15) for small values of the orbital parameter ε in all the cases (1 to 6) at $\tau = 0$, the initial conditions $x_0, \dot{y}_0, T$ as shown in Tables 5.14, 5.15 and 5.16 have been obtained. Also, in Tables 5.14, 5.15 and 5.16, the fifth column and sixth column give the Jacobi Constant and the positive imaginary root ($+ \omega i$) to the characteristic equation of system (4.9), respectively.

Table 5. 14: The initial conditions for the Lyapunov orbits around the collinear equilibrium point L_1 for binary α -Centuari system mass parameter $\mu = 0.47333$

Case	x_0	$\dot{y}_0$	T	C	ω
1	-1.17772296	-0.15844398	5.26996513	3.47518463	1.19274305
2	-0.76451797	-0.13404904	3.32511205	1.33092657	1.89037668
3	-0.85296994	-0.14128336	3.76434523	1.73135624	1.66980282
4	-0.91603690	-0.15064417	4.03701109	2.07085351	1.55702180
5	-0.96642262	-0.16040252	4.23423349	2.37809889	1.48449874
6	-1.00888380	-0.17018986	4.38698369	2.66454003	1.43281004

Table 5. 15: The initial conditions for the Lyapunov orbits around the collinear equilibrium point L_2 for binary α -Centuari system mass parameter $\mu = 0.47333$

Case	x_0	$\dot{y}_0$	T	C	ω
1	-0.00786810	-0.50240508	1.66199403	3.99865977	3.78203180
2	-0.13576459	-0.09924098	4.49137120	0.95857994	1.39950897
3	-0.10567840	-0.17260044	3.08133233	1.39453273	2.03993390
4	-0.09130772	-0.24469574	2.48534026	1.83036201	2.52911619
5	-0.08353241	-0.31972218	2.12428715	2.27039126	2.95897580
6	-0.07917390	-0.39856305	1.87327991	2.71609925	3.35545918

Table 5. 16: The initial conditions for the Lyapunov orbits around the collinear equilibrium point L_3 for binary α -Centuari system mass parameter $\mu = 0.47333$.

Case	x_0	$\dot{y}_0$	T	C	ω
1	1.21881344	-0.14860729	5.61879746	3.43731738	1.11869387
2	0.98805848	-0.10827148	4.11676337	1.88451257	1.52685829
3	1.04064080	-0.12551658	4.23720387	2.19245704	1.48345807
4	1.08568711	-0.14022498	4.33697474	2.48247540	1.44933154
5	1.12528648	-0.15369264	4.41909062	2.75927019	1.42239995
6	1.16071210	-0.16640484	4.48676943	3.02584346	1.40094435

5.4 Numerical Considerations for the Periodic Solutions Around the Collinear Equilibrium Points in the Three-Dimensional R3BP when the Primaries are Radiating and Oblate Spheroidal Bodies.

An investigation of three-dimensional periodic orbits and their stability emanating from the collinear equilibrium points of the restricted three-body problem with oblate and radiating primaries is presented. A simulation is done by using five binary systems: Sirius, Procyon, Luhman 16, α -Centuari and Luyten 726-8. Firstly, based on the topological degree theory, the total number of the collinear equilibrium points for the five binary systems were obtained and then, their positions were determined numerically. The linear stability of these equilibrium points was also examined and found to be unstable in the Lyapunov sense. An analytical approximation of three-dimensional periodic solutions around them was established via the Lindstedt-Poincaré local analysis as shown in section 4.3 of chapter four. Finally, using the analytical solution to obtain starting orbits, the families of three-dimensional periodic orbits emanating from these equilibria have been continued numerically using the corrector-predictor scheme.

5.4.1 Numerical applications for the 3D case for the model with radiating oblate primaries

In chapter four, section 4.3, we obtained for the spatial case, the second order analytic approximation of Lyapunov orbits around the collinear equilibrium points in the R3BP when the primaries are radiating oblate spheroidal bodies by utilizing the Linsted Poincare method. To this effect, we have made applications to five binary systems. They are binary α -Centuari, Luhman 16, Sirius, Luyten 726-8, and Procyon systems.

Apart from α -Centuari system which has already been mentioned, Sirius is the brightest star system in the Earth's night sky. It is a binary system with Sirius A being a white main-sequence star of spectral type A1V as the primary star and Sirius B a faint white dwarf companion of spectral type DA 2 as the secondary star.

Also, the Procyon star system is made up of two stars. The primary star Procyon A is a main sequence star of spectral type F5 IV-V, while Procyon B the secondary star is a faint white dwarf of spectral type DQZ.

Luhman 16 also known as WISE 1049-5319 is a binary brown-dwarf system with the primary star Luhman 16 A of spectral type L7.5 and the secondary Luhman B of type T0.5 $\pm$ 1.

We also have Luyten 726-8. The stars in this binary system are also known as Gliese 65. The primary star between them Luyten 726-8 A is of main sequence red dwarf of spectral type M5.6 Ve. The secondary star Luyten 726-8 B also a red dwarf is cooler and dimmer with a spectral type M6.0 Ve.

Table 5. 17: Physical Parameters of the five binary systems

Parameters	<i>Sirius</i>	<i>Procyon</i>	<i>Luhman 16</i>	<i>α-Centuari</i>	<i>Luyten 726-8</i>
M_A	$1.99M_s$	$1.57M_s$	$63M_J$	$1.14M_s$	$0.11M_s$
M_B	$0.98M_s$	$0.88M_s$	$49M_J$	$0.92M_s$	$0.1M_s$
μ	0.3300	0.3592	0.4375	0.4466	0.4762
$L_A(L_s)$	22.5	7.59	1.7×10^{-9}	1.54	5.65×10^{-5}
$L_B(L_s)$	2.54×10^{-3}	5.6×10^{-4}	0.8×10^{-9}	0.453	3.7×10^{-5}
q_1	0.976734	0.990052	1	0.997220	0.999999
q_2	0.999995	0.999999	1	0.996555	0.999999
A_1	0.10	0.12	0.14	0.16	0.17
A_2	0.11	0.13	0.15	0.18	0.19

In Table 5.17, we present the physical parameters of the five binary systems (<http://www.johnstonsarchive.net/astro/nearstar.html>). The parameters M_A and M_B are the masses of the more massive and less massive stars in each binary system as compared to the mass of the Sun, with an exception of the binary Luhman 16 system where comparison is being to the mass of Jupiter. The symbol μ as shown earlier is the mass parameter. The luminosity of the binary systems denoted by L_A and L_B respectively are obtained from the relation (Mia and Kushvah (2016))

$$\frac{L}{L_s} \approx \left(\frac{M}{M_s} \right)^{3.9},$$

where L_s and M_s are the luminosity and mass of the Sun.

Radiation pressure has had a key effect on the formation of stars and shaping of clouds of dust and gases on a wide range of scales. The mass reduction factor is represented as

$$q_i = 1 - \frac{F_p}{F_g}, \quad i = 1, 2 \quad (F_p \text{ and } F_g \text{ are the radiation pressure and the gravitational attraction}$$

forces being exerted by the binary systems on objects around them) or $q_i = 1 - \beta$, $i = 1, 2$ or on the basis of the Stefan-Boltzmann's law (Xuetang and Lizhong (1993)) as

$$q_i = 1 - \frac{A\kappa L}{a\rho M}, \quad i = 1, 2,$$

where M , L , and κ are the mass, luminosity, and radiation pressure efficiency factor of a star.

Also, a and ρ are the radius and density of the dust grain particles moving in the binary

systems while $A = \frac{3}{16\pi cG}$ is a constant with c and G as the speed of light and Gravitational

constant.

The values of the luminosity and mass reduction factor $q_i, i = 1, 2$ have been obtained by computing in the C.G.S. (centimetre-gram-second) system of unit, using

$$L_s = 3.846 \times 10^{33} \text{ erg/s}, \quad c = 3 \times 10^{10} \text{ cm/s}, \quad G = 6.67384 \times 10^{-8} \text{ cm}^3 \text{g}^{-1} \text{s}^{-2}, \quad M_s = 1.989 \times 10^{33} \text{ g},$$

$$M_j = 1.898 \times 10^{30} \text{ g} \text{ and } \kappa = 1. \text{ Also, we have assumed the values for the radius and density}$$

of the dust grain particles as $a = 2 \times 10^{-2} \text{ cm}$ and $\rho = 1.4 \text{ g/cm}^3$ (Xuetang and Lizhong

(1993)). Arbitrary values have been used for the oblateness coefficients A_1 and A_2 .

The determination of the exact number of roots of Eq. (4.20), at each one of the above mentioned open intervals, has been established by an approach based on the topological degree theory and can be briefly described as follows: If the function $F(x)$: $x \in [a, b] \subset \mathbb{R}$ is two times continuously differentiable in this interval, then the total number N of roots of the

equation $F(x)=0$ is obtained by the following scheme (Picard, 1892; Kalantonis et al (2001); Singh and Begha (2011)):

$$N = \frac{-\gamma}{\pi} \int_a^b \frac{F(x)F''(x) - F'^2(x)}{F^2(x) + \gamma^2 F'^2(x)} dx + \frac{1}{\pi} \arctan \left(\frac{\gamma F(a)F'(b) - F(b)F'(a)}{F(a)F(b) + \gamma^2 F'(a)F'(b)} \right),$$

where γ is a small positive real constant. For all the considered binary systems, i.e. for the parameter values correspond to the specific systems, we have shown that each one of the aforementioned open intervals contains a unique real root; these roots correspond to the collinear equilibrium points L_1 , L_2 and L_3 , respectively. Note, that the involved integral in the above formula has been computed numerically by using Romberg integration. Since we have determined the exact number of the collinear equilibrium points, we are able to solve numerically (4.20) so as to obtain their positions accurately. These are shown in Table 5.18 for each binary system. Also, we show graphically the effects of the oblateness and radiation parameters of each of the five binary systems on the collinear equilibrium points in Figures 5.29-5.38.

Table 5. 18: The Collinear Equilibrium Points for the five binary systems

Binary Systems	L_1	L_2	L_3
<i>Sirius</i>	-1.26185648	-0.21970340	1.10834159
<i>Procyon</i>	-1.25240546	-0.18100228	1.12325896
<i>Luhman 16</i>	-1.22105585	-0.07860660	1.16072979
<i>α-Centuari</i>	-1.21809449	-0.06431650	1.16126475
<i>Luyten 726-8</i>	-1.20533406	-0.02609529	1.17557131

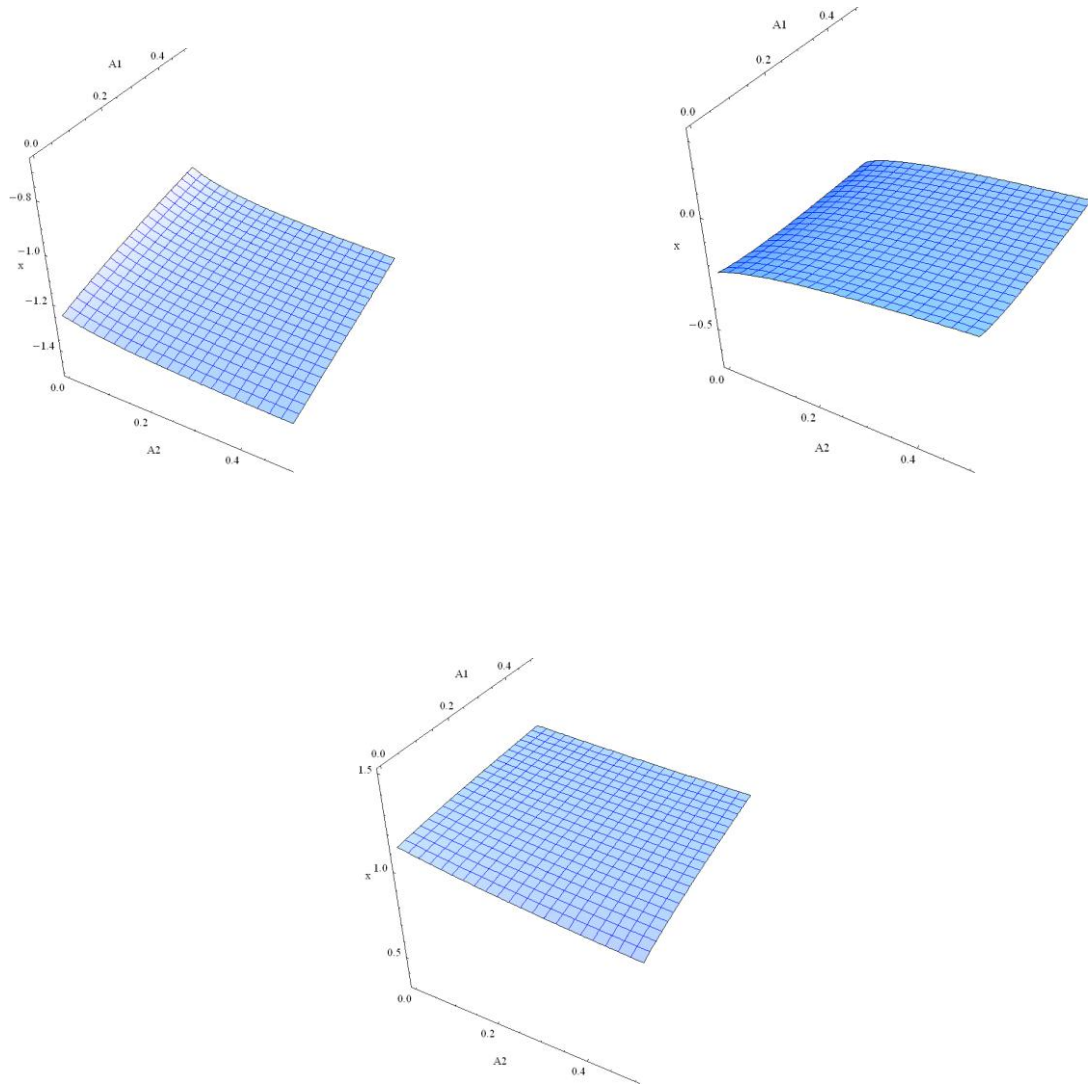


Figure 5.29: The effect of the oblateness coefficient of the binary Sirius system on the collinear equilibrium points.

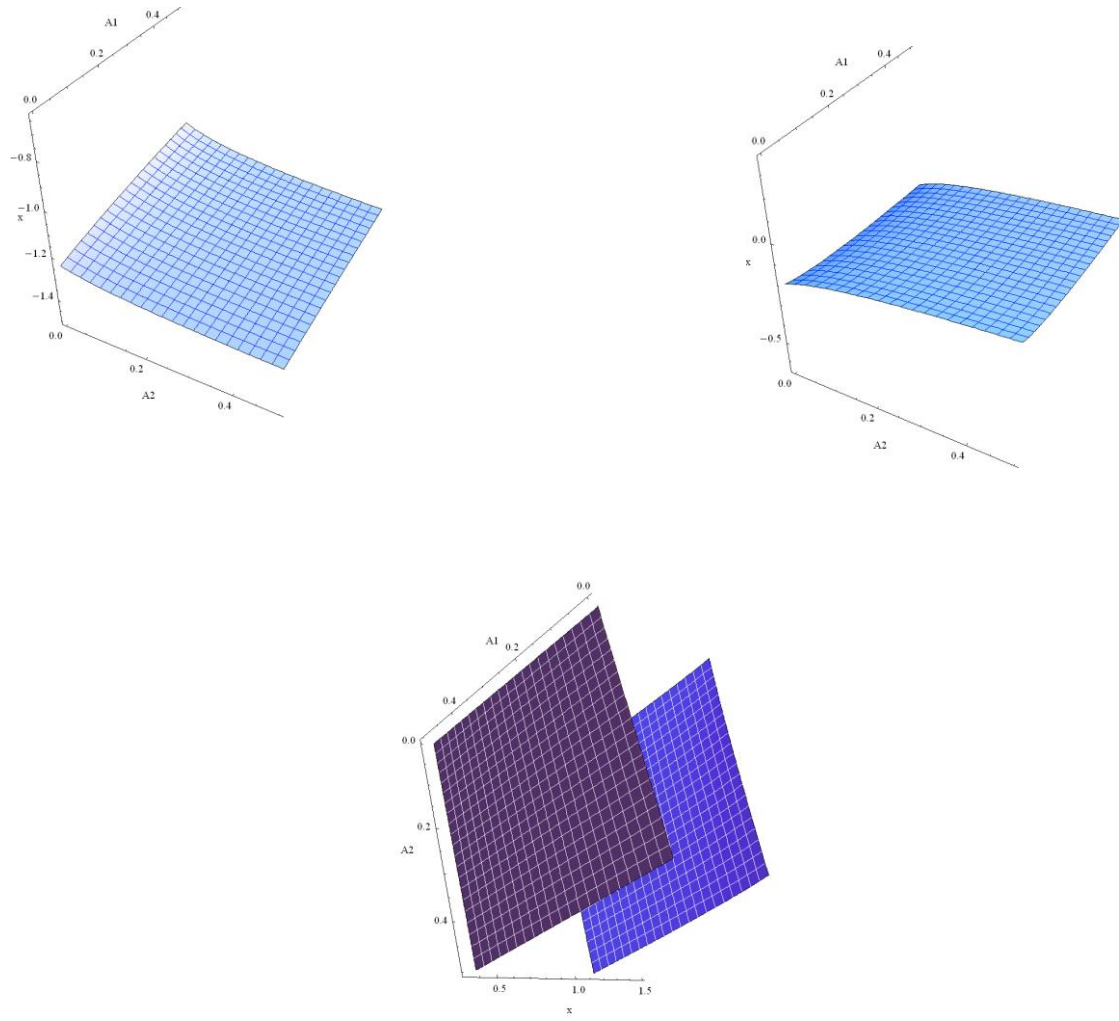


Figure 5.30: The effect of the oblateness coefficient of the binary Procyon system on the collinear equilibrium points.

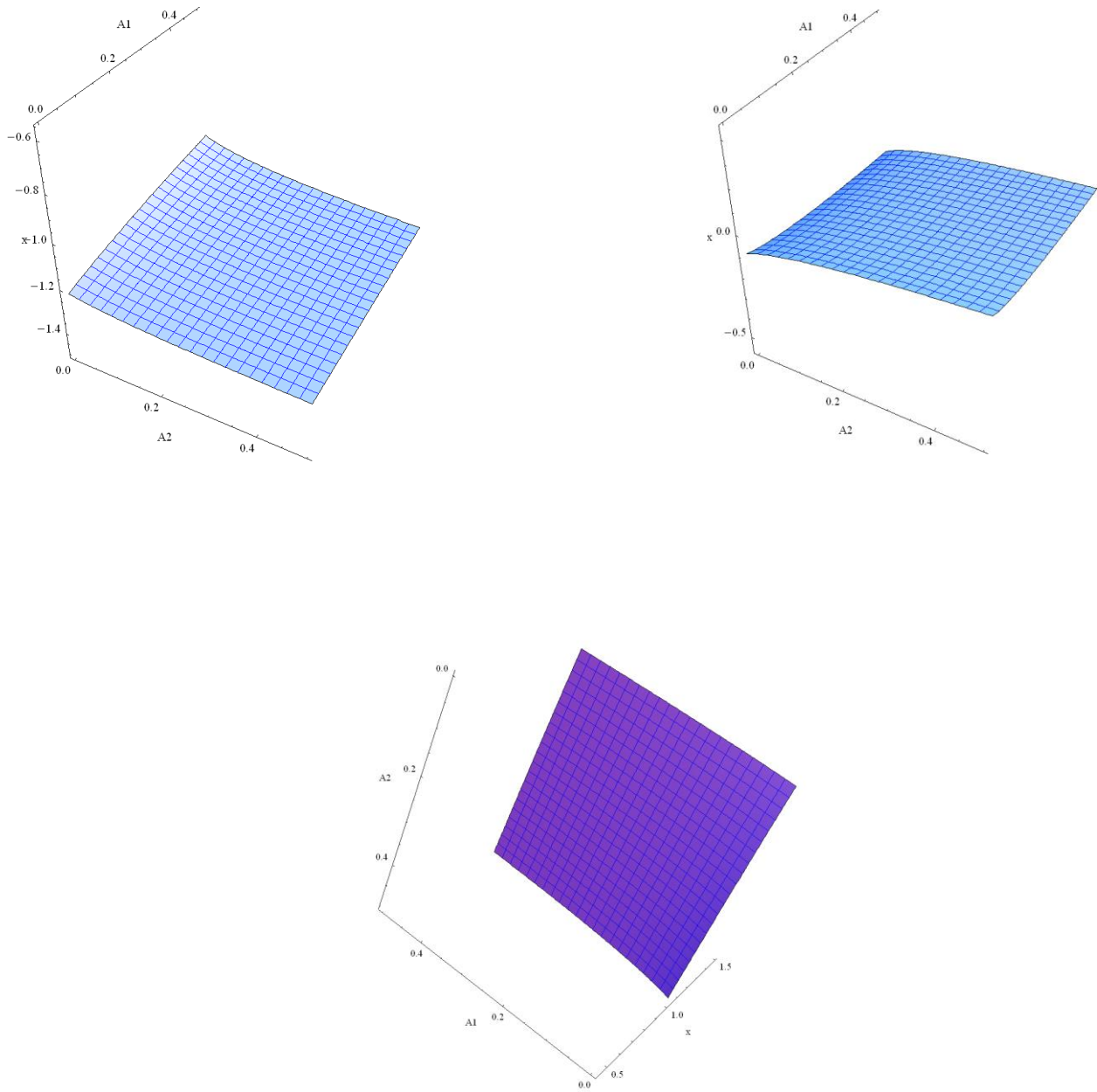


Figure 5.31: The effect of the oblateness coefficient of the binary Luhman 16 system on the collinear equilibrium points.

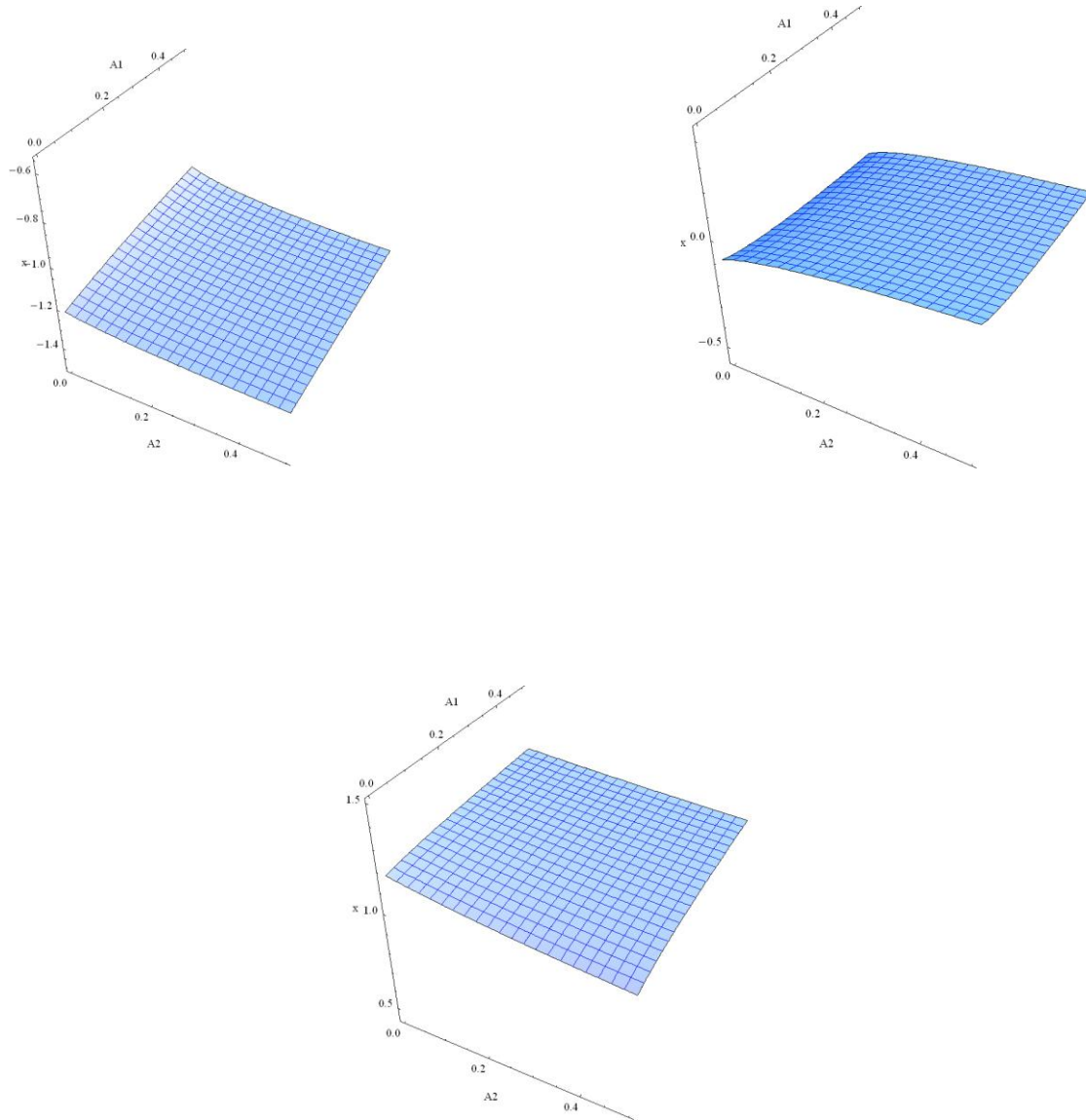


Figure 5. 32: The effect of the oblateness coefficient of the binary α -Centuari system on the collinear equilibrium points

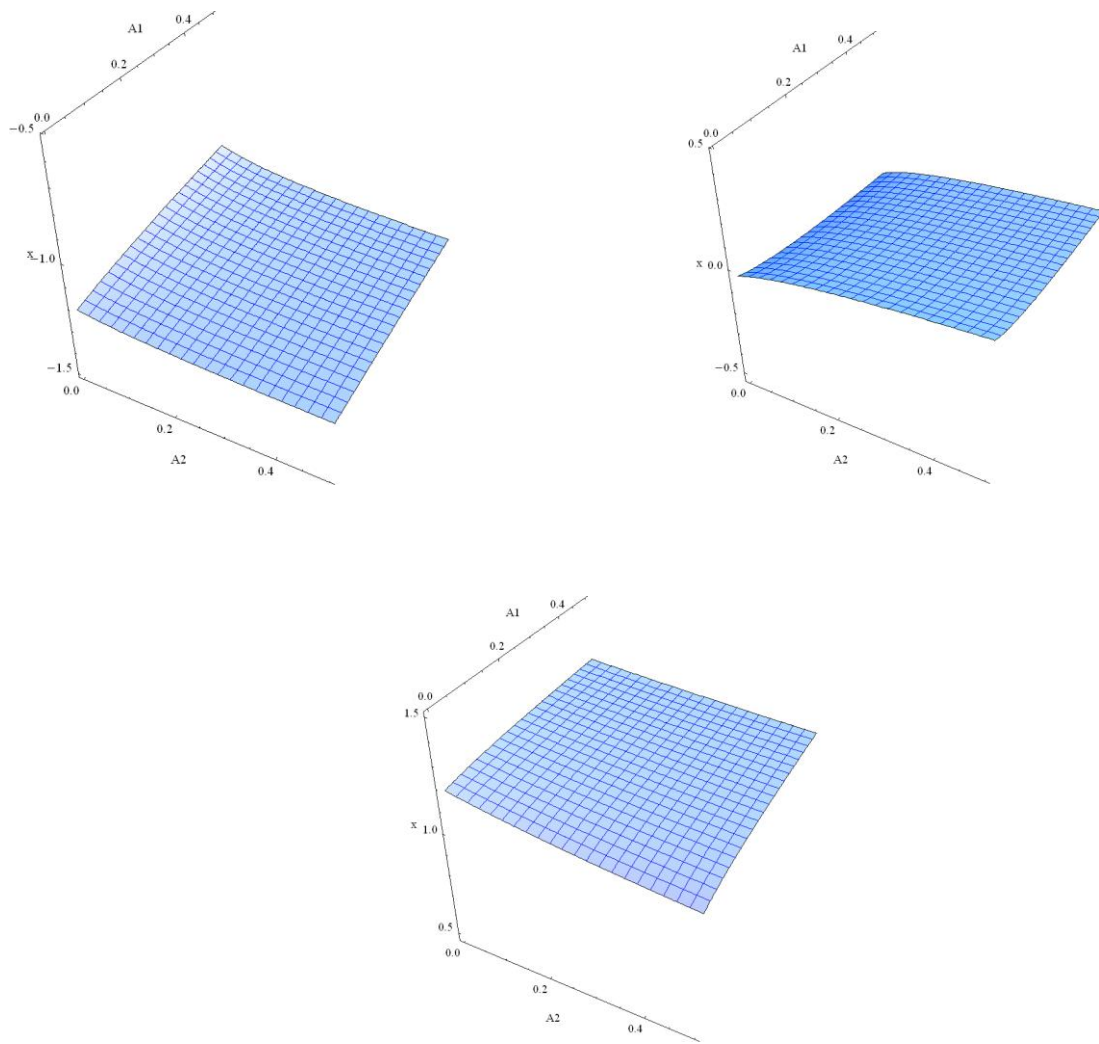


Figure 5.33: The effect of the oblateness coefficient of the binary Luyten 726-8 system on the collinear equilibrium points.

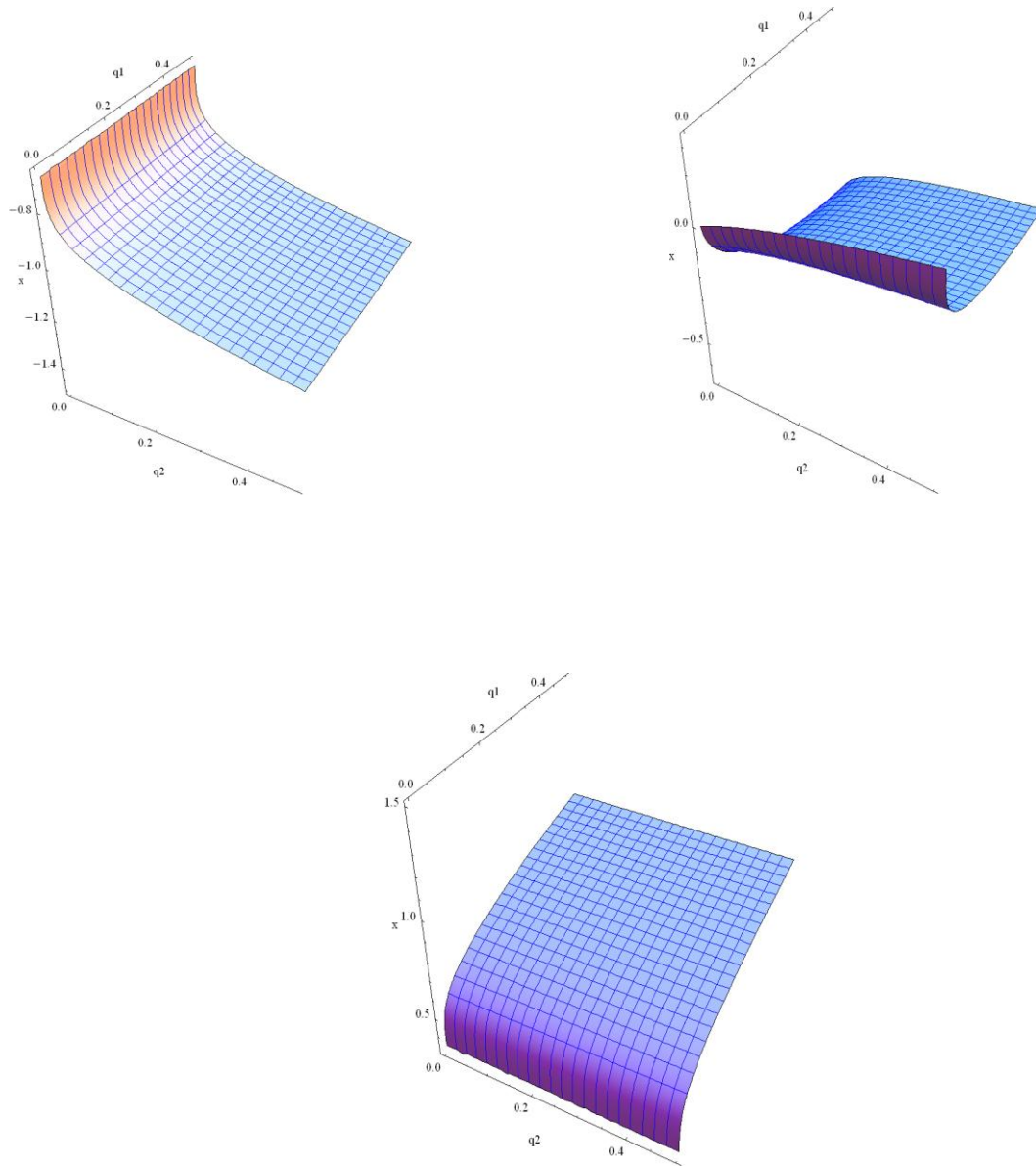


Figure 5.34: The effect of the radiation coefficient of the binary Sirius system on the collinear equilibrium points.

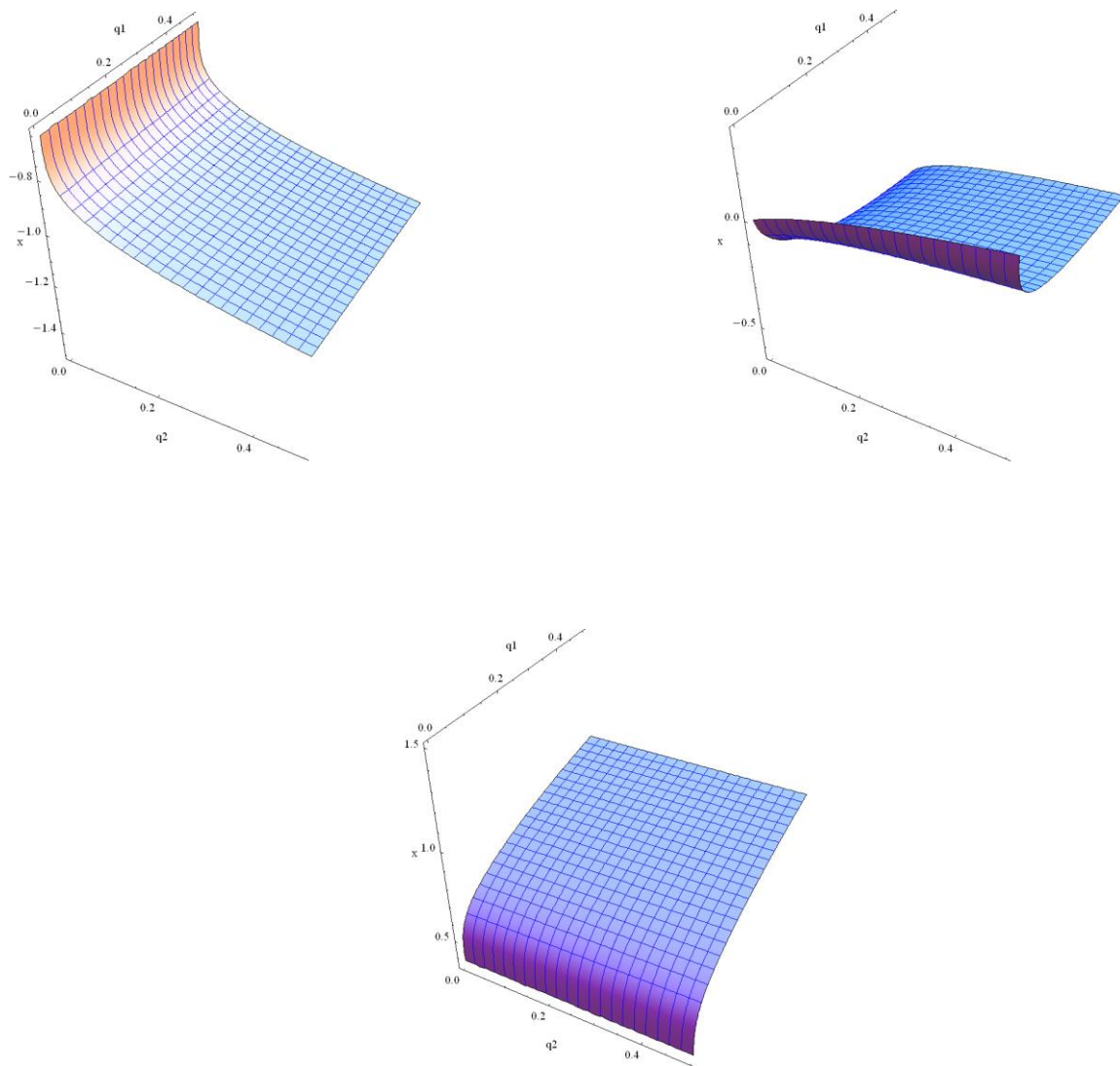


Figure 5.35: The effect of the radiation coefficient of the binary Procyon system on the collinear equilibrium points.

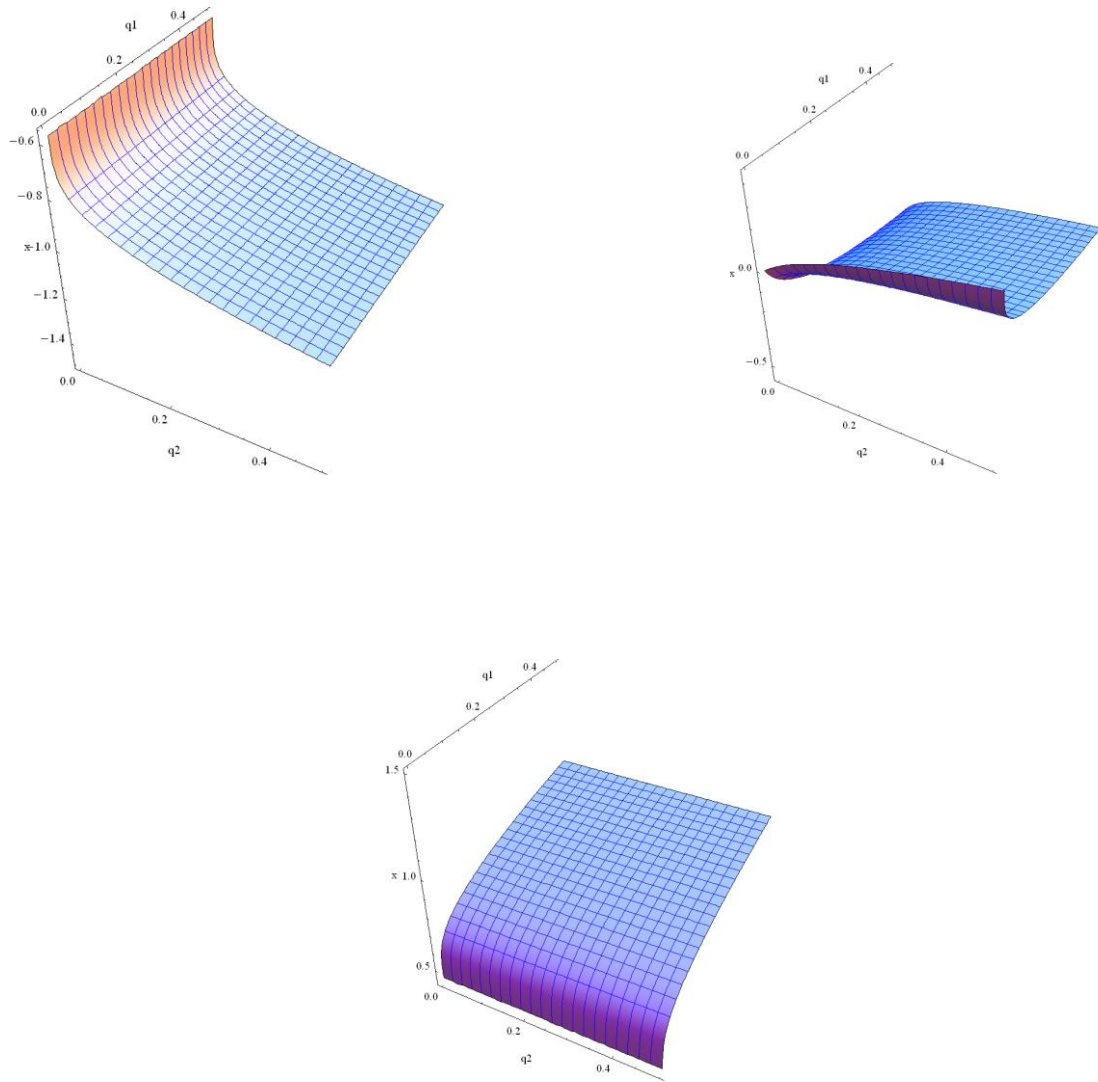


Figure 5.36: The effect of the radiation coefficient of the binary Luhman 16 system on the collinear equilibrium points.

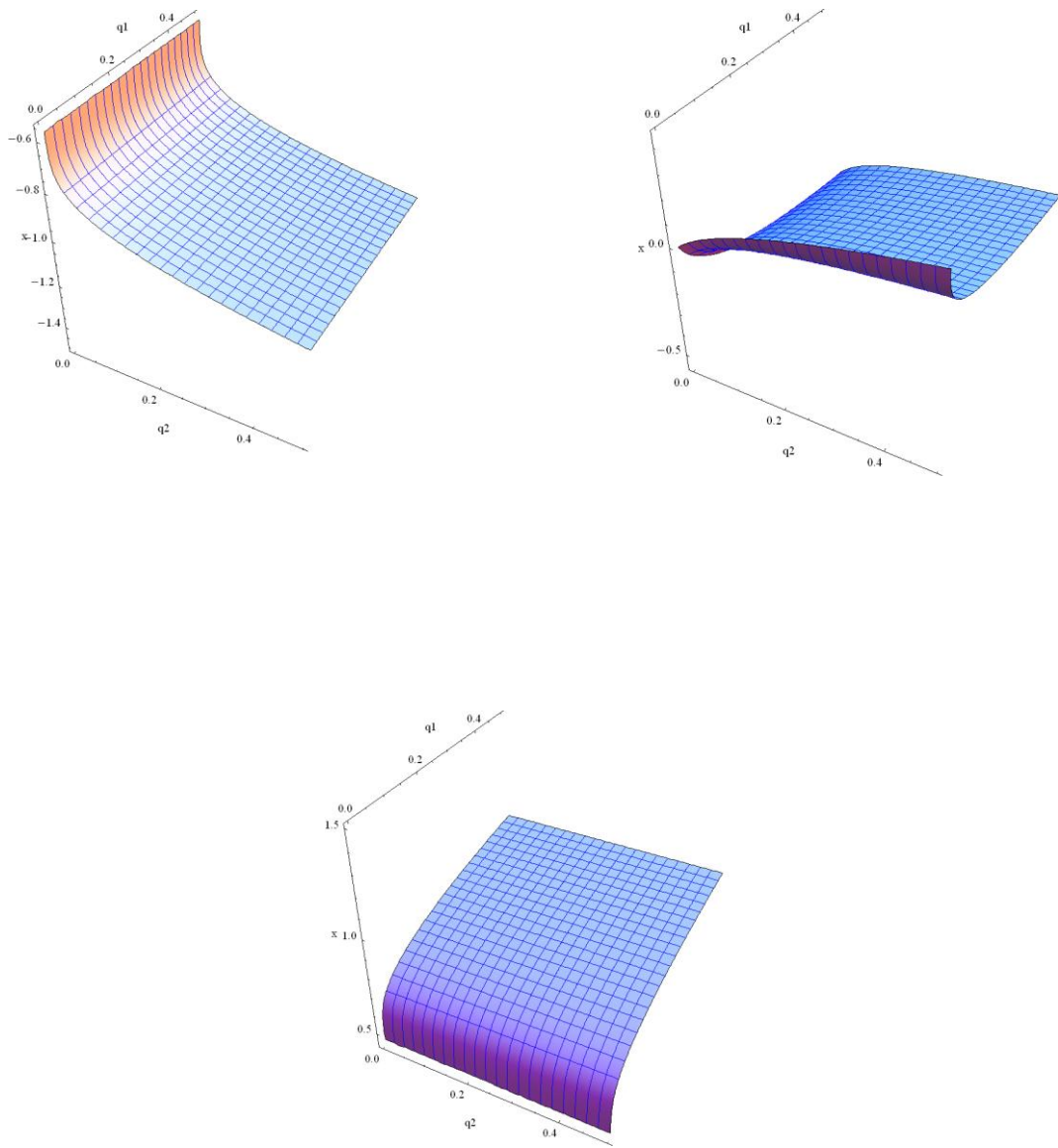


Figure 5.37: The effect of the radiation coefficient of the binary α -Centuari system on the collinear equilibrium points.

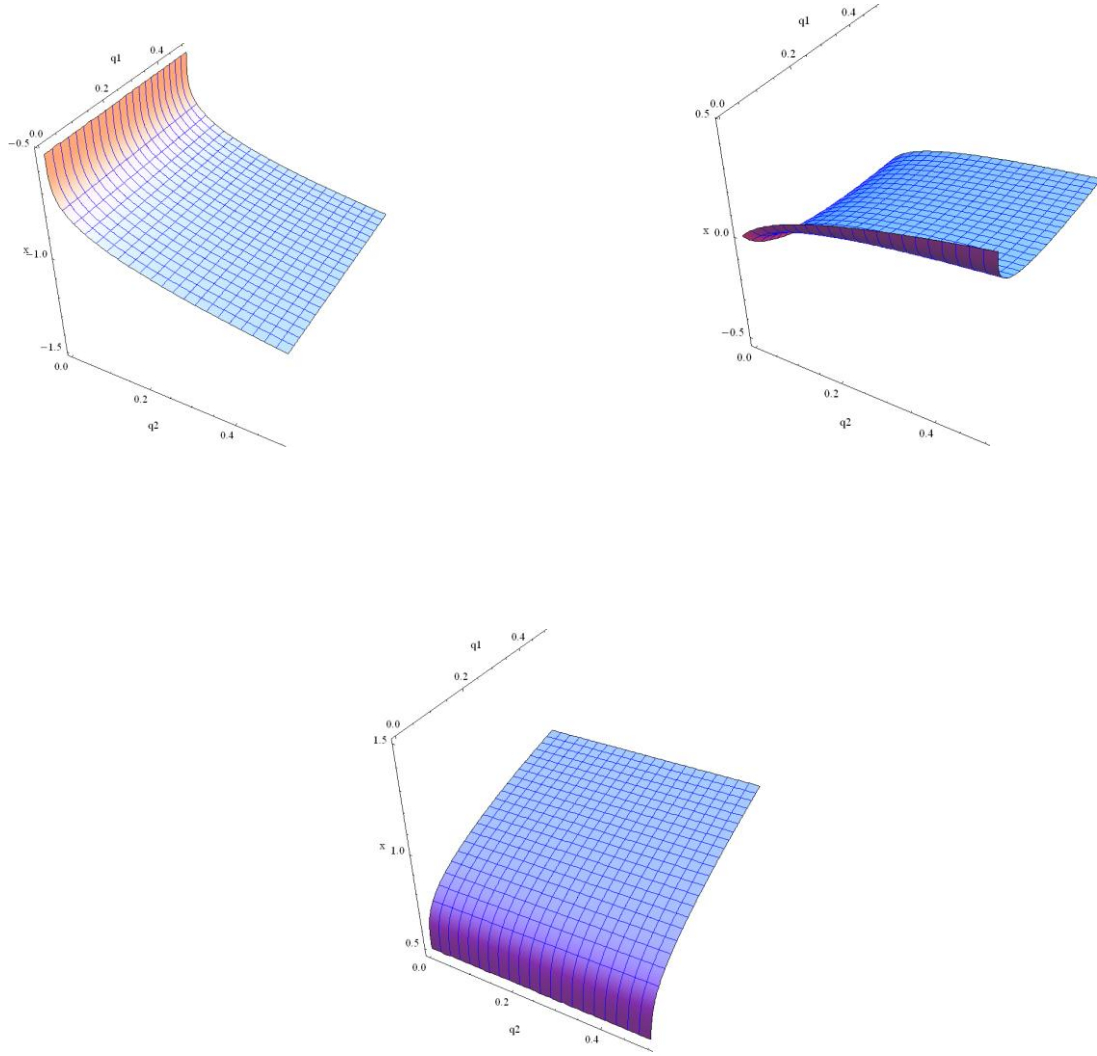


Figure 5. 38: The effect of the radiation coefficient of the binary Luyten 726-8 system on the collinear equilibrium points.

The roots of Equation (4.25) are presented in Table 5.19 for each of the five binary systems. As can be seen from this table, for all systems and at each collinear equilibrium point, four roots exist two of which are real and two pure imaginary. Therefore, the collinear equilibrium points are unstable in the Lyapunov sense. However, by eliminating the hyperbolic component of the general solution of the linearized system around the collinear equilibria, we are able to obtain analytical approximations of periodic motions around these points.

Table 5. 19: The roots of the characteristic Equation (4.25) for the five binary systems.

Binary systems	Characteristic roots (L_1)	Characteristic roots (L_2)	Characteristic roots (L_3)
<i>Sirius</i>	$\pm 1.56430825,$ $\pm 1.96189710i$	$\pm 3.53416313,$ $\pm 5.66608205i$	$\pm 1.30727543,$ $\pm 1.25122469i$
<i>Procyon</i>	$\pm 1.56458611,$ $\pm 1.97650189i$	$\pm 3.69091766,$ $\pm 6.02200958i$	$\pm 1.34583641,$ $\pm 1.37003152i$
<i>Luhman 16</i>	$\pm 1.52281201,$ $\pm 1.87520139i$	$\pm 3.85724799,$ $\pm 6.38512862i$	$\pm 1.42628926,$ $\pm 1.59596688i$
<i>α-Centuari</i>	$\pm 1.54253438,$ $\pm 1.95037140i$	$\pm 3.99995119,$ $\pm 6.72438275i$	$\pm 1.46148260,$ $\pm 1.69462850i$
<i>Luyten 726-8</i>	$\pm 1.52797020,$ $\pm 1.91519759i$	$\pm 4.07001989,$ $\pm 6.88078173i$	$\pm 1.49664283,$ $\pm 1.79216283i$

In Tables 5.20-5.22, we give initial conditions from formula (4.32) used to continue to all families of three-dimensional periodic orbits about the collinear equilibrium points of all five considered binary systems. In particular, we give the values of the orbit on the Ox -axis, i.e. $x = x_1$, $\dot{y} = x_5 = \dot{x}_2$, $\dot{z} = x_6 = \dot{x}_3$, the period up to its vertical intersection with the Oxz plane, i.e. at the quarter of the full period, as well as the value of the Jacobi constant C . We refer here to the work by Tsirogiannis et al (2006) where they have presented results for three-dimensional periodic orbits around the collinear equilibrium points of the restricted three-body problem in the case when the larger primary is a source of radiation and the smaller one is an oblate spheroid.

Table 5. 20: Initial conditions around the collinear equilibrium point. L_1

Binary systems	x_1	$\dot{x}_2$	$\dot{x}_3$	$T/4$	C
<i>Sirius</i>	−1.26154728	0.00057624	0.05	0.78204452	4.21999091
<i>Procyon</i>	−1.25211073	0.00055322	0.05	0.76545523	4.33844624
<i>Luhman 16</i>	−1.22078122	0.00055089	0.05	0.77866119	4.39104843
<i>α-Centuari</i>	−1.21783018	0.00051852	0.05	0.74487263	4.53206233
<i>Luyten 726-8</i>	−1.20507694	0.00051714	0.05	0.74752644	4.55983483

Table 5. 21: Initial conditions around the collinear equilibrium point L_2

Binary systems	x_1	$\dot{x}_2$	$\dot{x}_3$	$T/4$	C
<i>Sirius</i>	−0.21972909	−0.00002926	0.05	0.33323158	4.69916871
<i>Procyon</i>	−0.18102114	−0.00002132	0.05	0.31187910	4.91712821
<i>Luhman 16</i>	−0.07861390	−0.00000823	0.05	0.29302707	5.14642616
<i>α-Centuari</i>	−0.06432177	−0.00000596	0.05	0.27676724	5.33084113
<i>Luyten 726-8</i>	−0.02609726	−0.00000223	0.05	0.27005106	5.43428486

Table 5. 22: Initial conditions around the collinear equilibrium point L_3

Binary systems	x_1	$\dot{x}_2$	$\dot{x}_3$	$T/4$	C
<i>Sirius</i>	1.10810162	−0.00073162	0.05	0.99604458	3.81079690
<i>Procyon</i>	1.12301326	−0.00067716	0.05	0.93599028	3.97945114
<i>Luhman 16</i>	1.16047398	−0.00060777	0.05	0.85388976	4.21515229
<i>α-Centuari</i>	1.16101364	−0.00057239	0.05	0.81035704	4.35510076
<i>Luyten 726-8</i>	1.17531962	−0.00054779	0.05	0.78017318	4.46283431

5.4.2 Numerical approximation results

In Figure 5.39, we present the analytical three-dimensional solution (blue curve denoted by A) emanating from the collinear equilibrium point L_1 of the α -Centauri binary system, as this is obtained from formula (4.32) of our analysis and by varying the orbital parameter e in the range of values $[0, 1.2]$. For comparison reasons and for establishing the validity of our analysis, we also give in the same plot the numerical solution of the three-dimensional periodic orbits (black curve denoted by N) emanating from the same equilibrium point as this is determined by the numerical integration of the full equations of motion (4.8). As we can observe from Figures 5.39 (a)-(c), our analysis is valid for small values of e , in all the projections of the space of initial conditions $(\Im, \Re, \aleph) = (x, y, z)$.

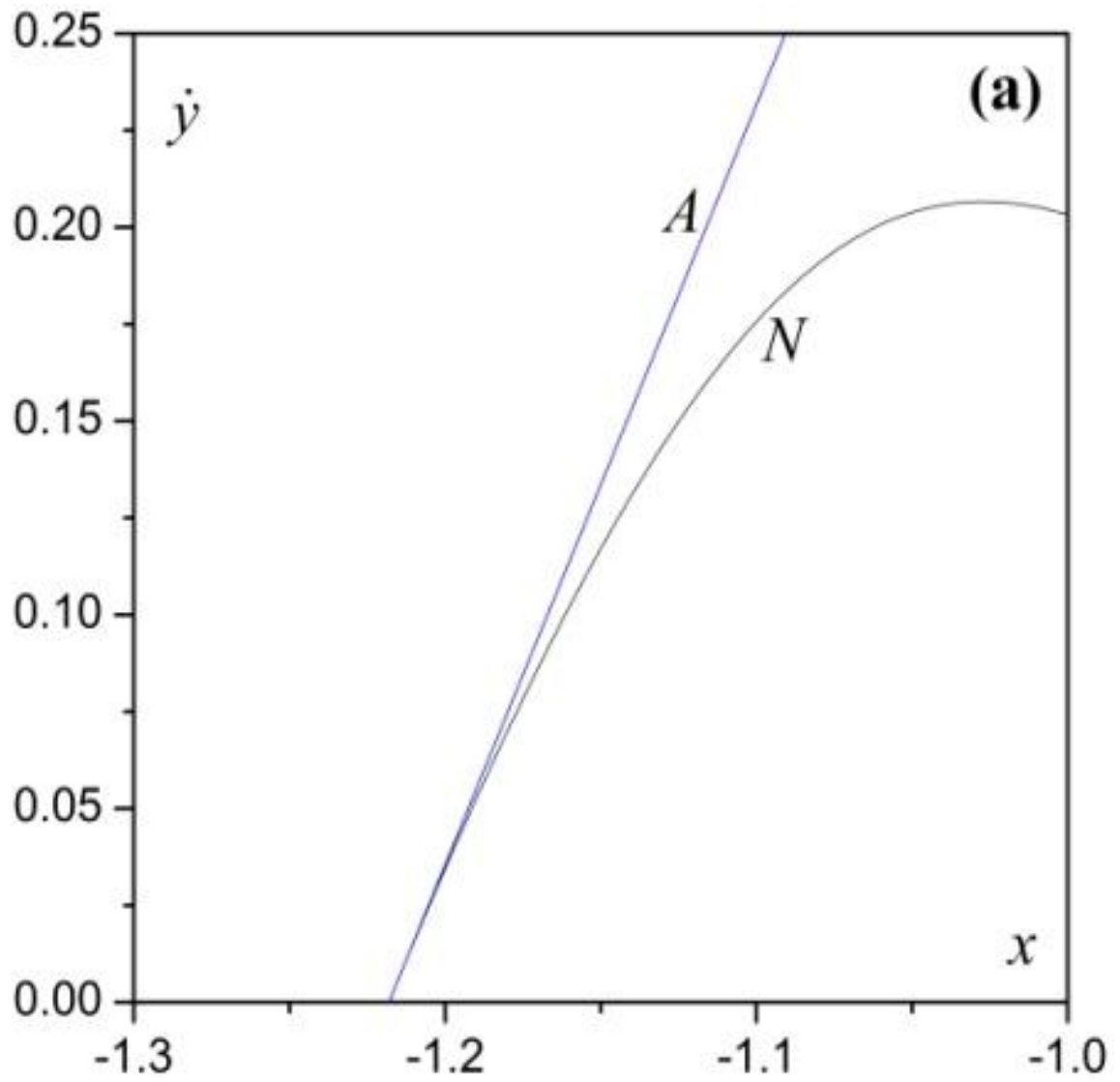


Figure 5.39(a): Comparison of the analytical three-dimensional solution (4.31) emanating from the collinear equilibrium point L_1 with the numerical one in the α -Centauri binary system in the projection of the $(\mathfrak{I}, \mathfrak{H}) = (x, \dot{y})$ plane.

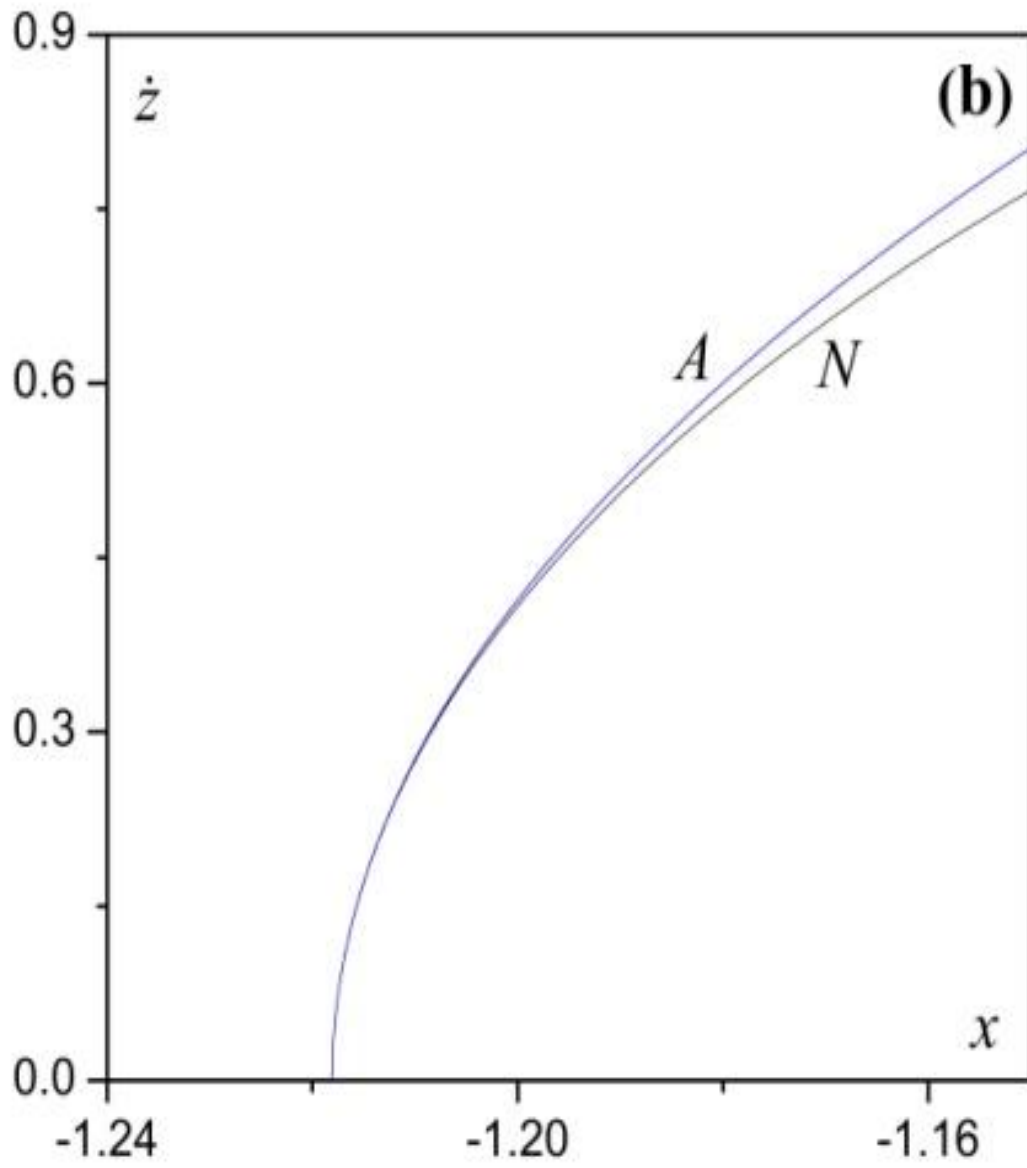


Figure 5.39 (b): Comparison of the analytical three-dimensional solution (4.31) emanating from the collinear equilibrium point L_1 with the numerical one in the α -Centauri binary system in the projection of the $(\mathfrak{I}, \mathfrak{S}) = (x, z)$ plane.

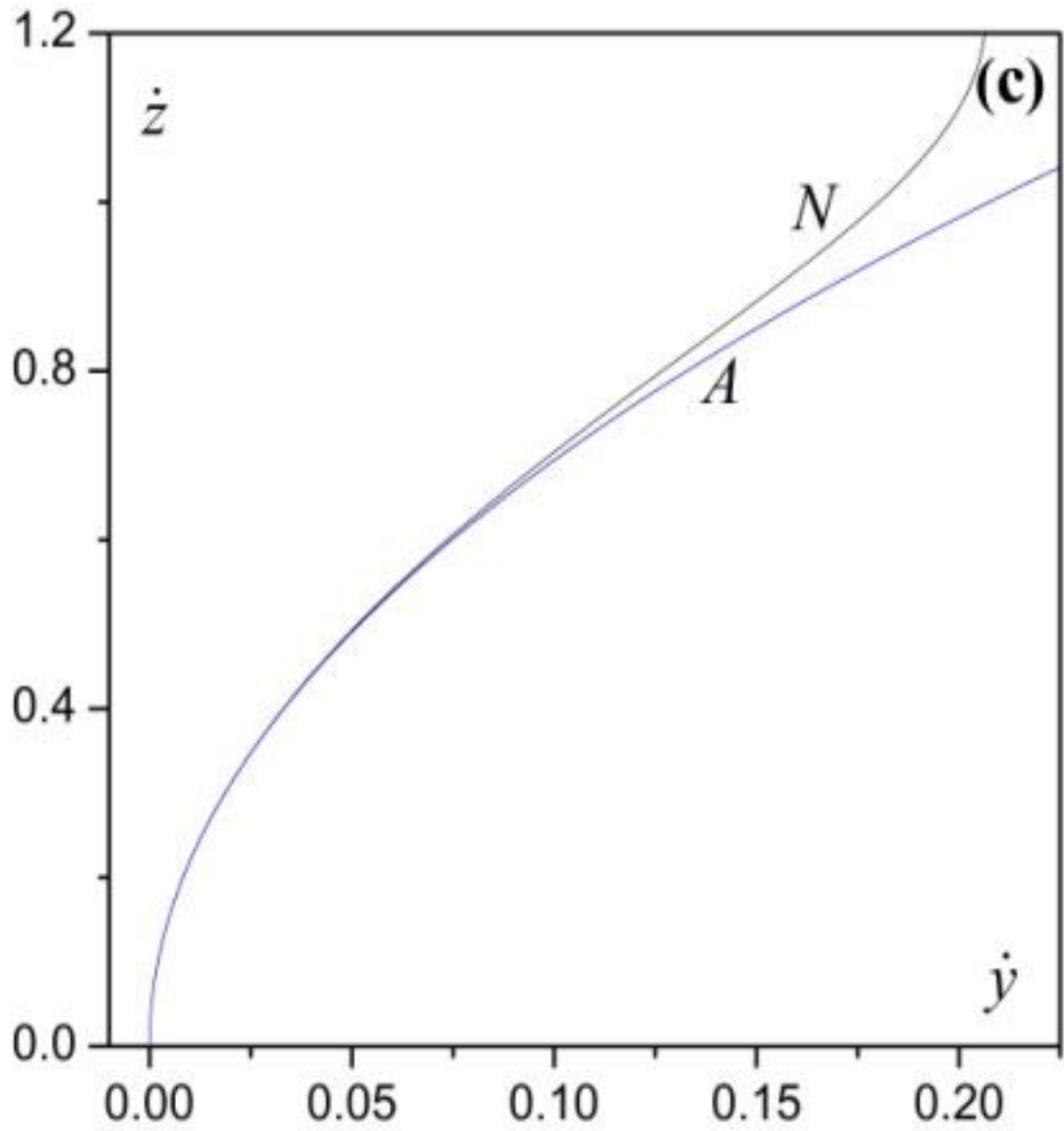


Figure 5.39(c): Comparison of the analytical three-dimensional solution (4.31) emanating from the collinear equilibrium point L_1 with the numerical one in the α -Centauri binary system in the projection of the $(\mathfrak{R}, \mathfrak{S}) = (\dot{y}, \dot{z})$ plane.

In Figures 5.40-5.42, we present the characteristic curves, in the space of initial conditions $(x, \dot{y}, \dot{z})$, of the families of three-dimensional doubly symmetric periodic orbits emanating from all collinear equilibrium points of the α -Centauri binary system, as these were computed by the procedure in subsection 4.3.5. Also, in Figures 5.43-5.45, we plot a typical figure eight member orbit for each family in the physical space while, in Figures 5.46-5.48, where we plot the quarter of period versus the stability parameters P and Q as these were defined by (4.35), we give the stability diagram of the corresponding families. As we can see from these figures, stability of three-dimensional periodic orbits occurs only for families emanating from the equilibrium points L_2 and L_3 while we also observe that the period of these orbits goes to zero in the evolution of all the computed families.

This was the criterion used to stop computing them. In other words, such a member orbit with period near to zero has been considered as a termination point of any determined family. For example, the full period of the last computed three dimensional periodic orbit of the family emanating from the collinear equilibrium point L_1 of α -Centauri binary system is approximately $T \cong 0.0005$ at the value of the Jacobi constant $C \cong 3159.7$, with initial conditions $x_0 = -0.57060543$, $\dot{y}_0 = 0.00578261$ and $\dot{z}_0 = 1.37629642$. This member orbit is presented in Figure 5.49 where we observe that it almost appears to exist on the Oxz plane since the values of the y coordinate are near zero.

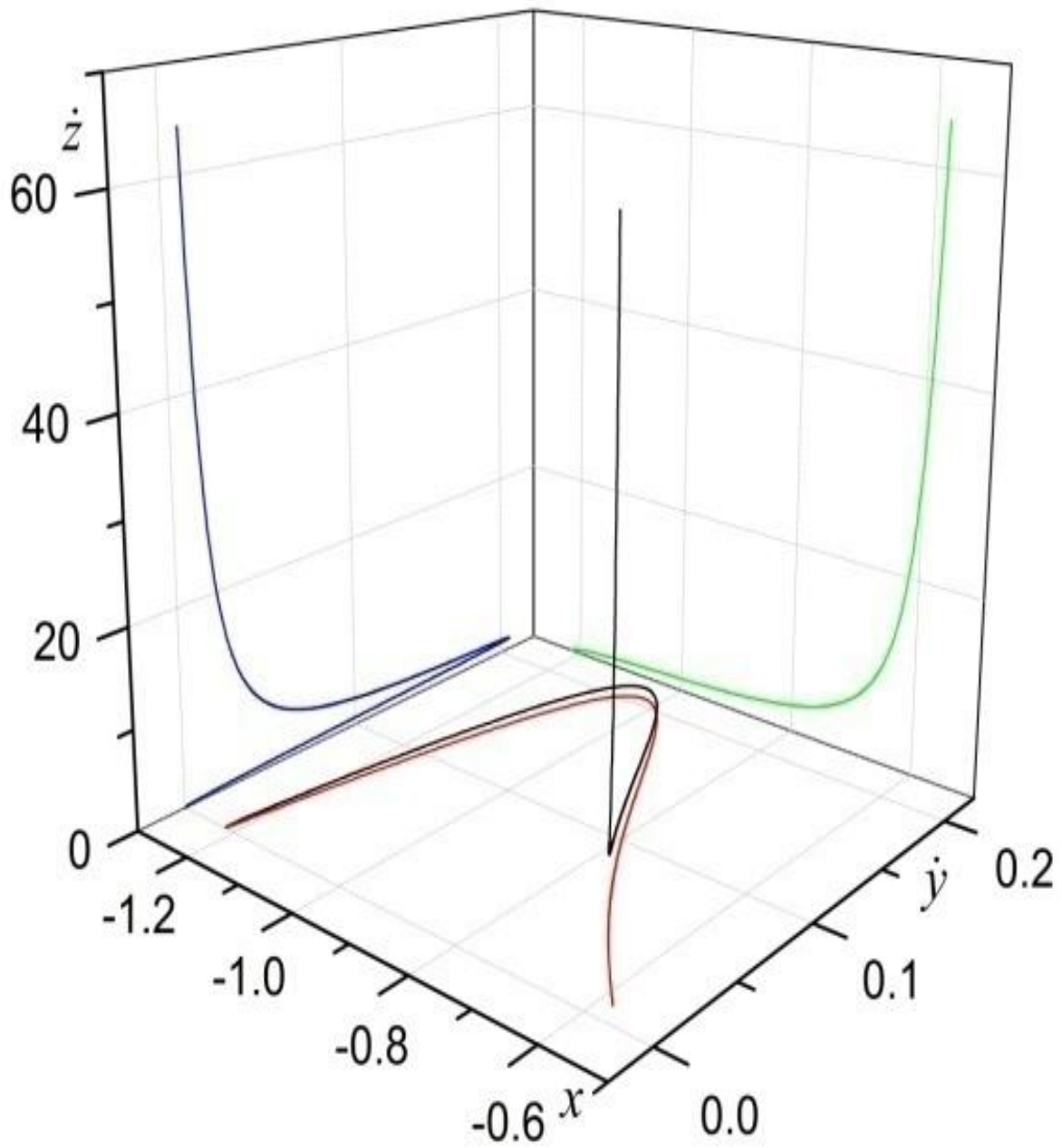


Figure 5.40: Families of 3D periodic orbits emanating from the collinear equilibrium point L_1 of the α -Centauri binary system in the space of initial conditions (x, y, z) together with their corresponding projections.

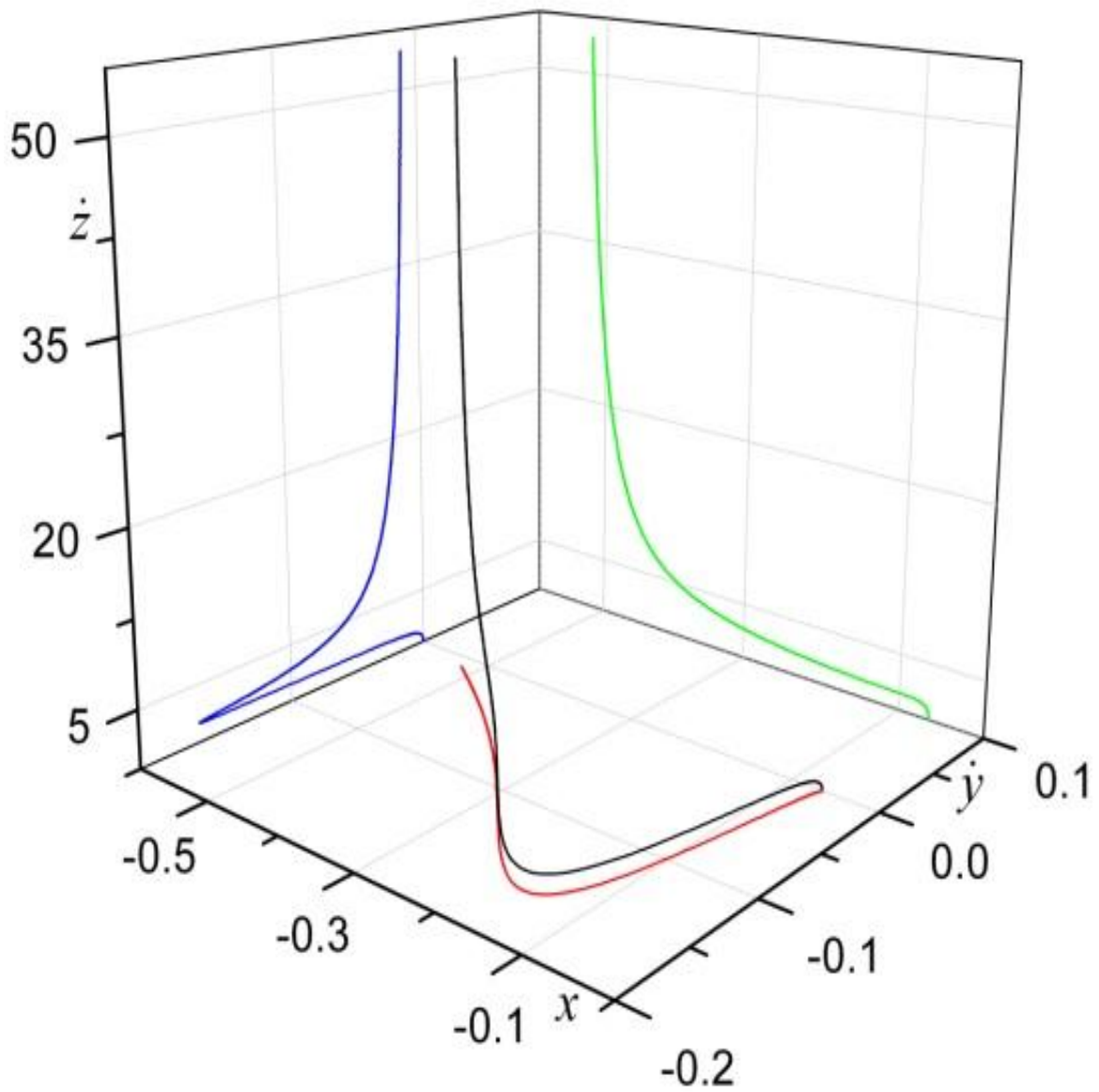


Figure 5.41: Families of 3D periodic orbits emanating from the collinear equilibrium point L_2 of the α -Centauri binary system in the space of initial conditions $(x, \dot{y}, \dot{z})$ together with their corresponding projections.

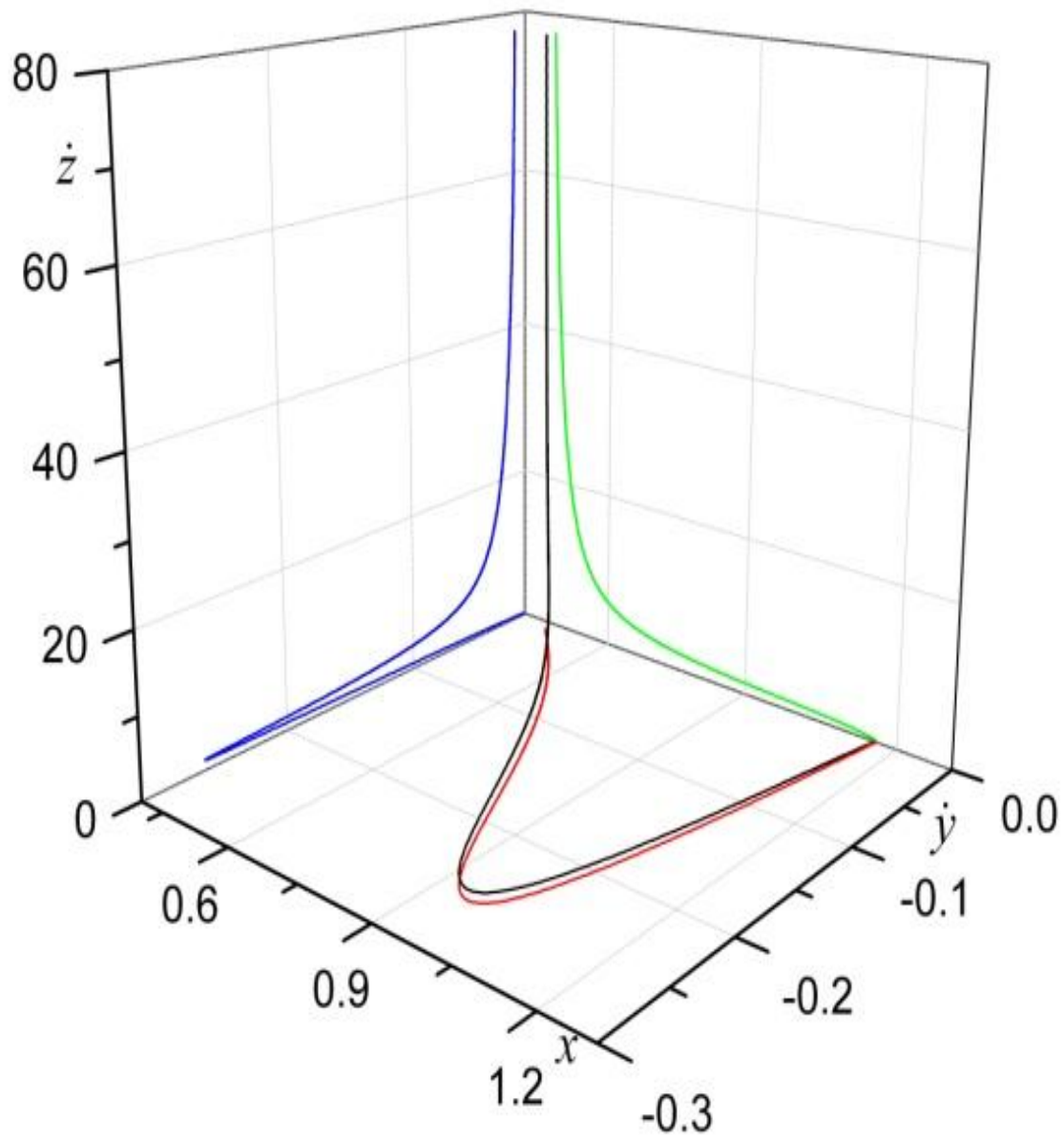


Figure 5.42: Families of 3D periodic orbits emanating from the collinear equilibrium point L_3 of the α -Centauri binary system in the space of initial conditions $(x, \dot{y}, \dot{z})$ together with their corresponding projections.

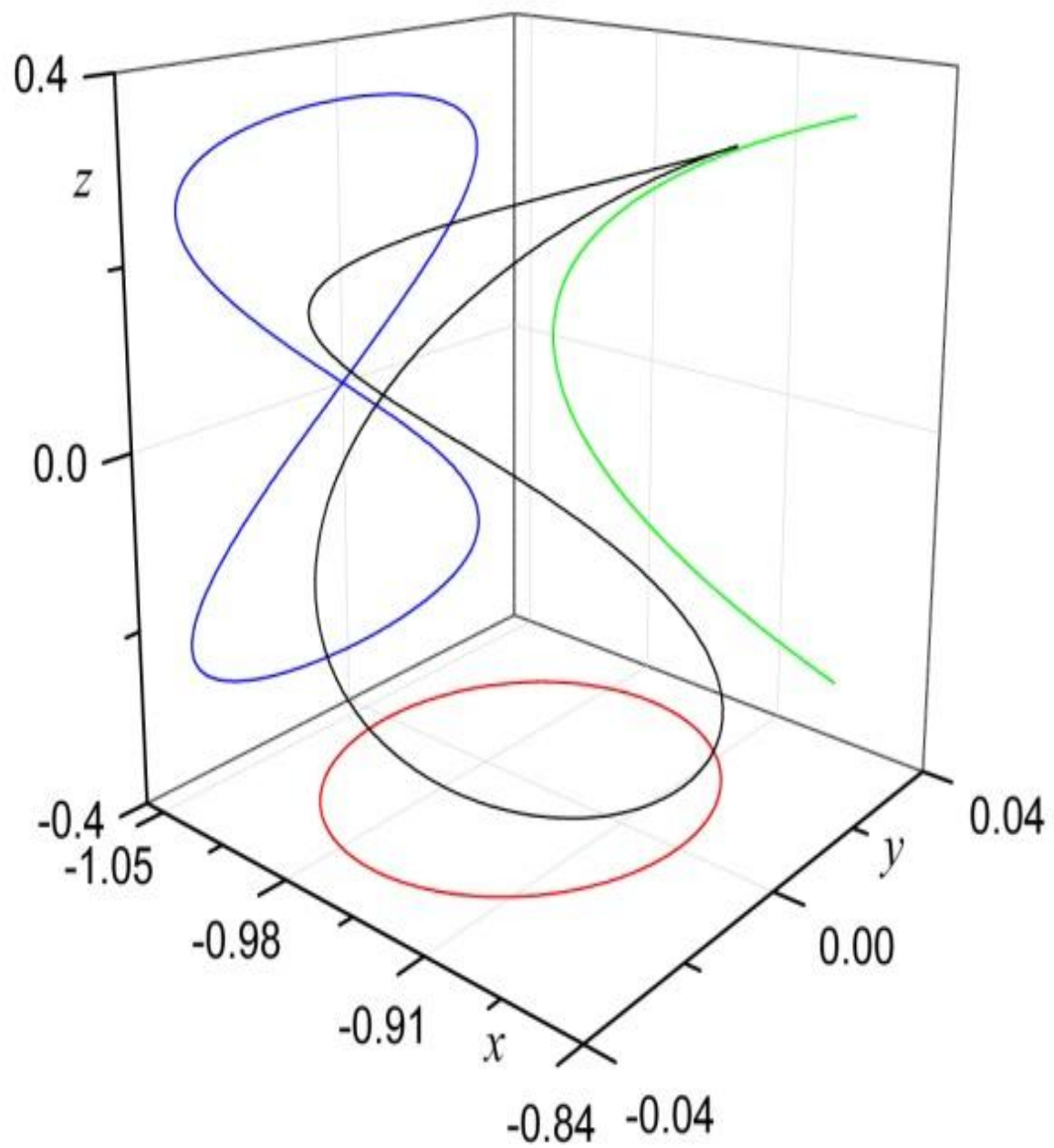


Figure 5.43: A typical figure eight 3D periodic orbit for the family emanating from the collinear equilibrium point L_1 .

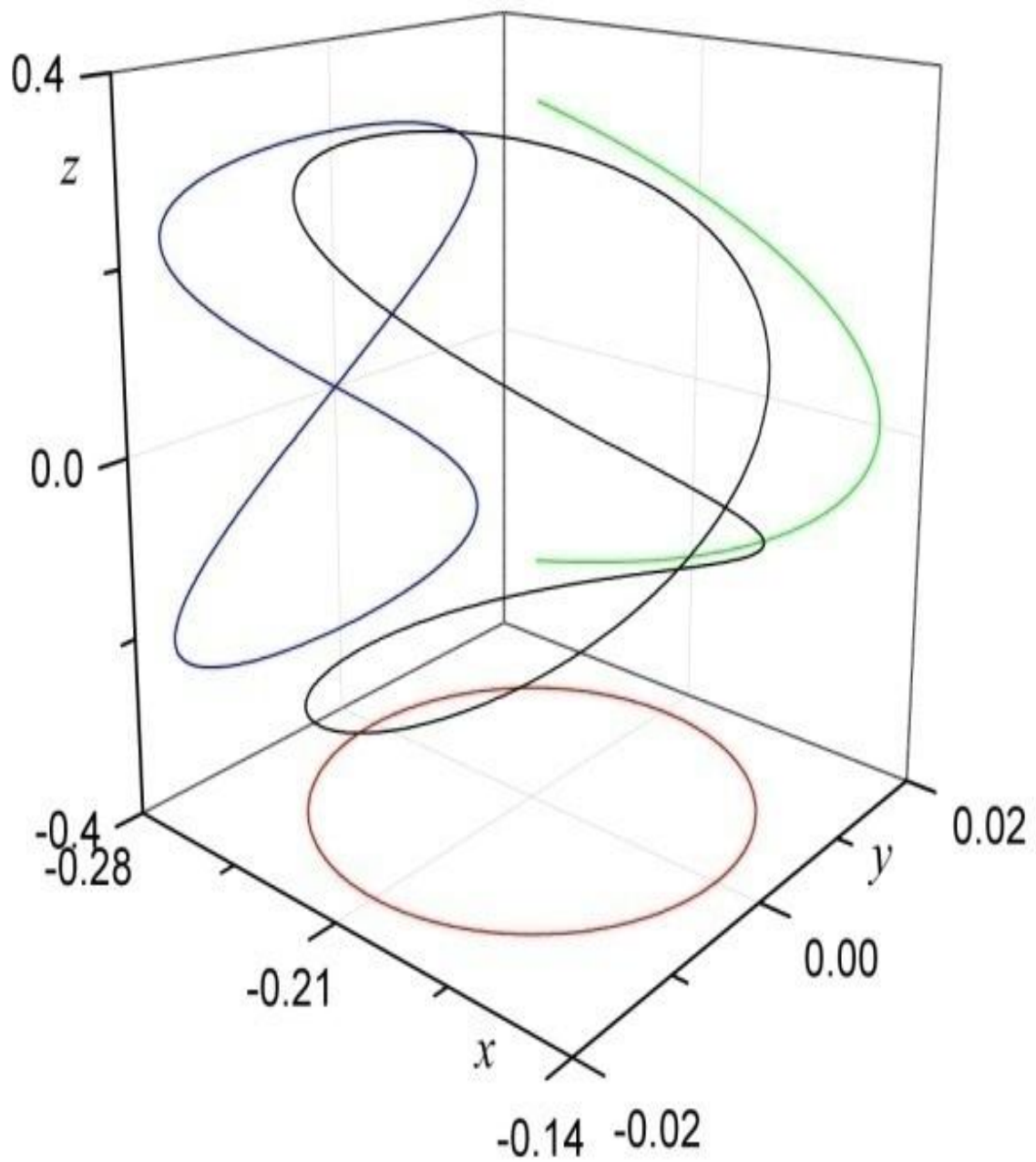


Figure 5.44: A typical figure eight 3D periodic orbit for the family emanating from the collinear equilibrium point₂.

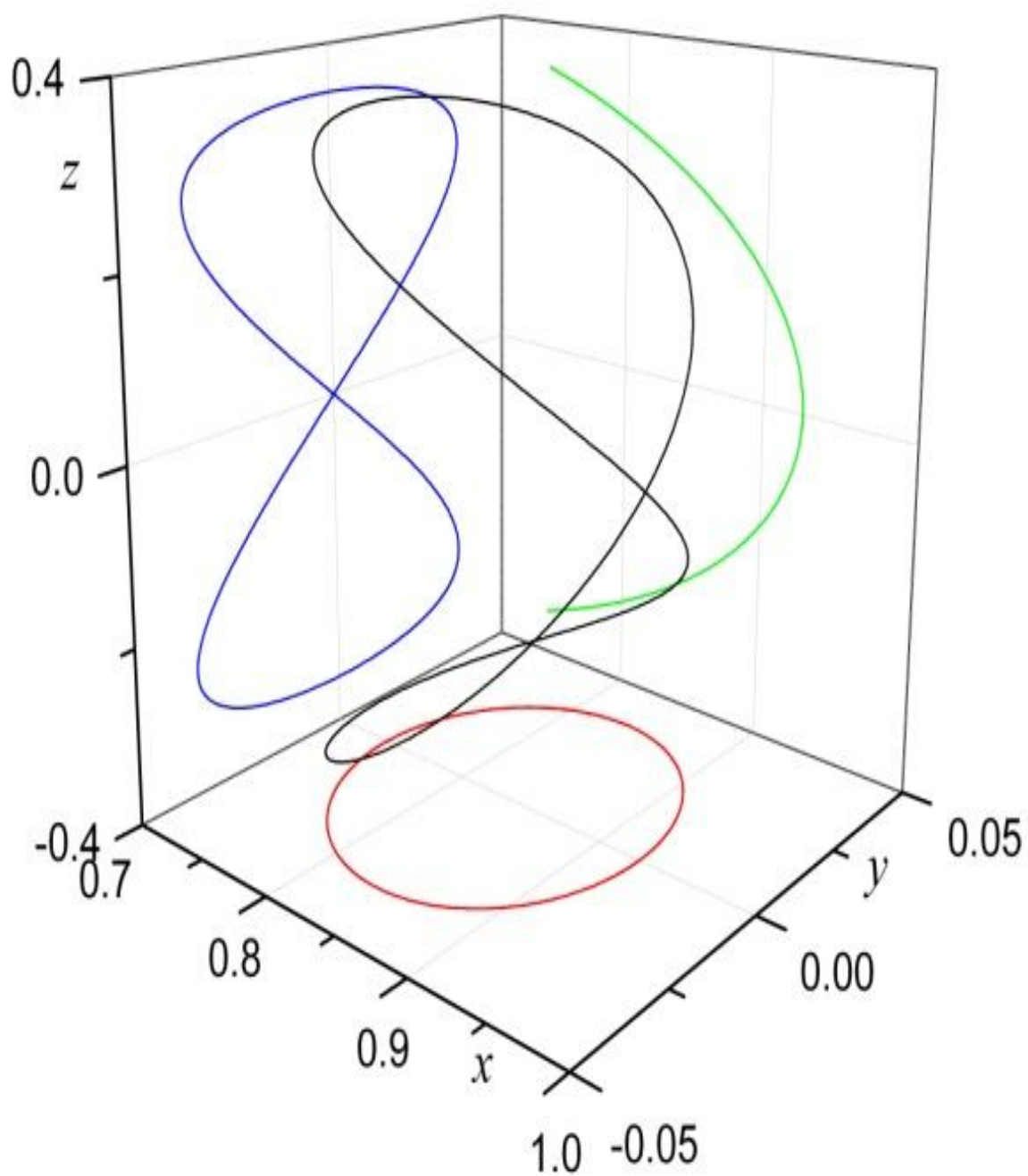


Figure 5.45: A typical figure eight 3D periodic orbit for the family emanating from the collinear equilibrium point L_3 .

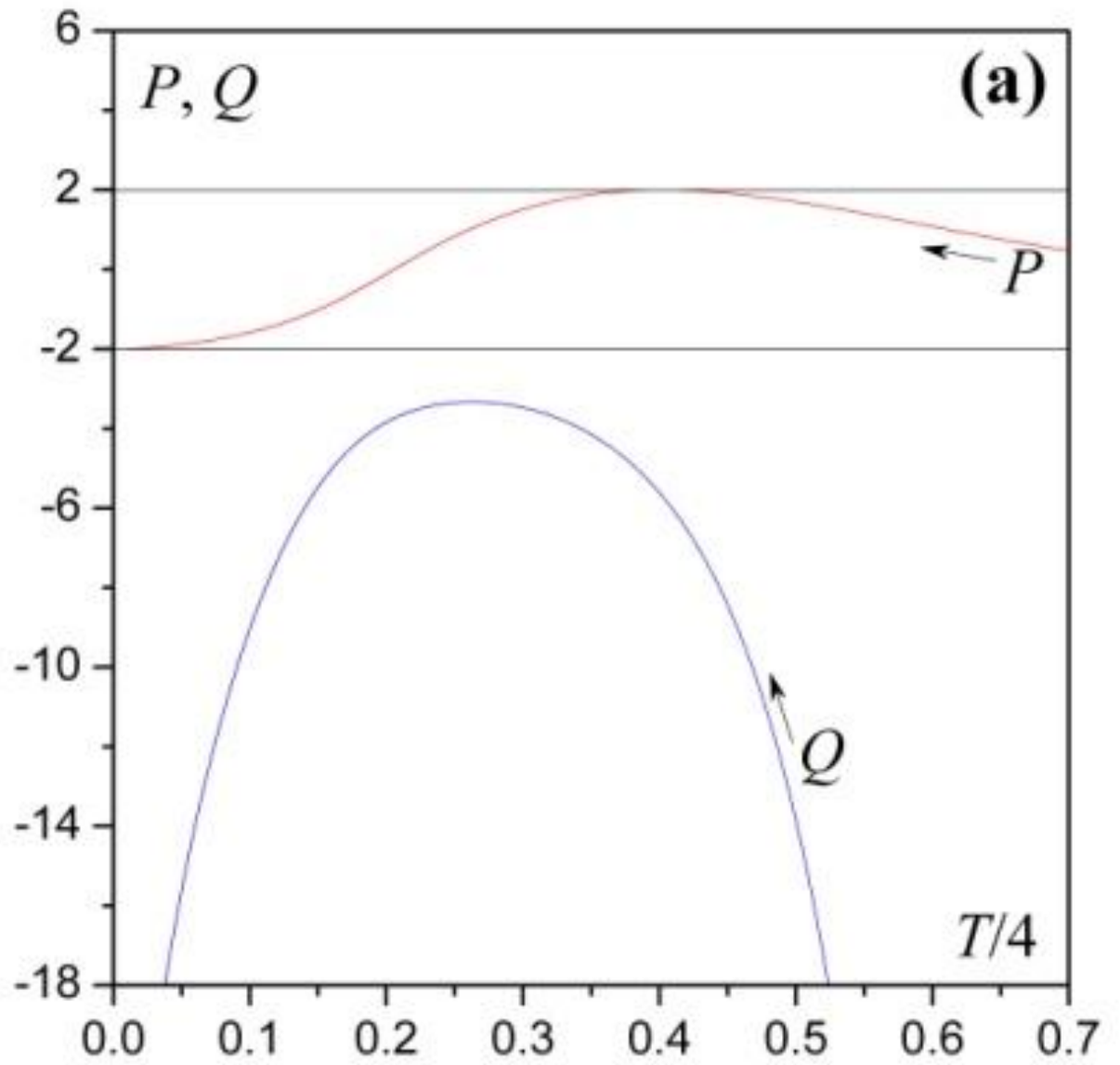


Figure 5.46: Stability of families of 3D periodic orbits emanating from the collinear equilibrium points (a) L_1 of the α -Centauri binary system.

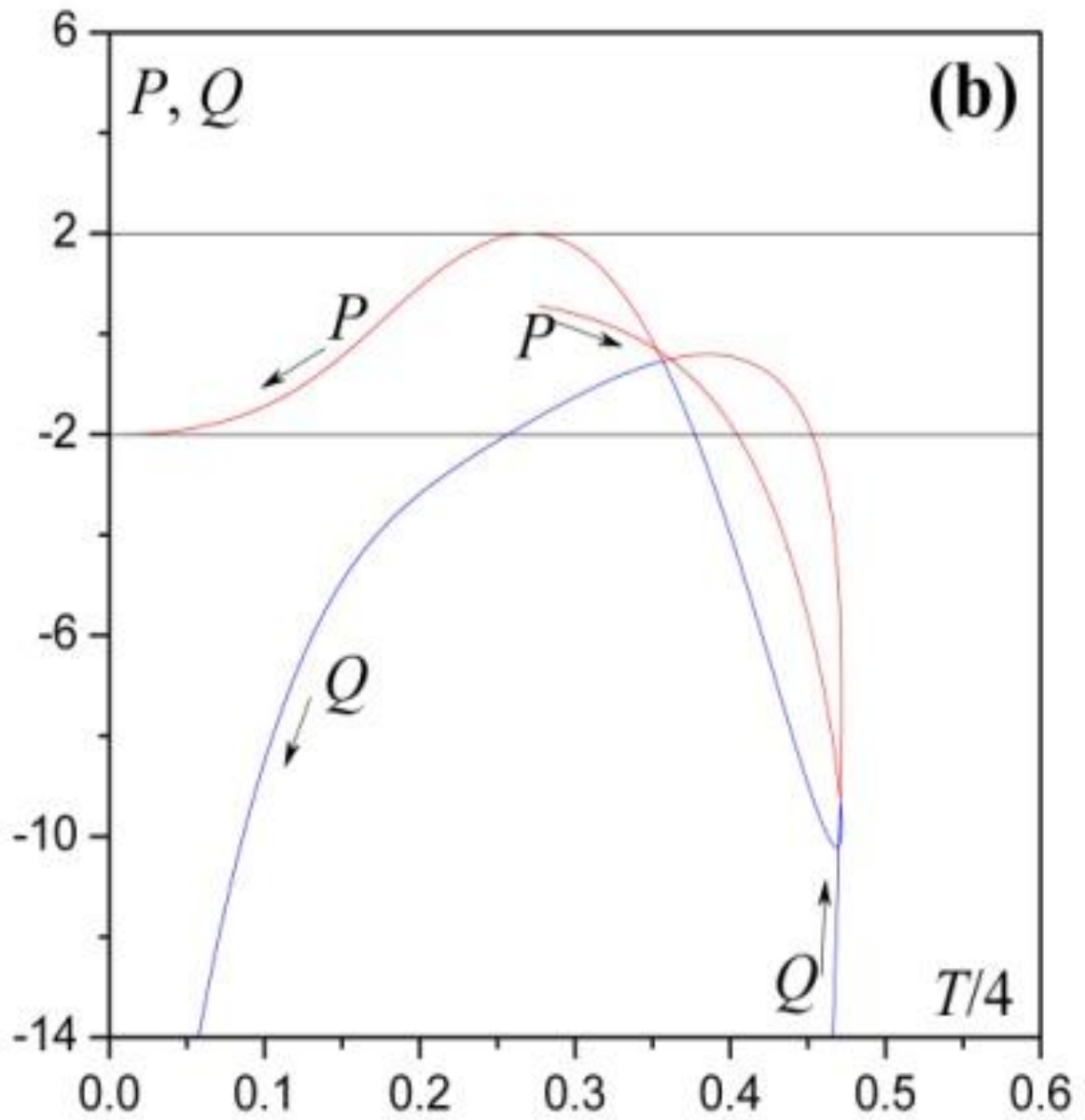


Figure 5.47: Stability of families of 3D periodic orbits emanating from the collinear equilibrium points **(b)** L_2 of the α -Centauri binary system. This family also contain stable parts.

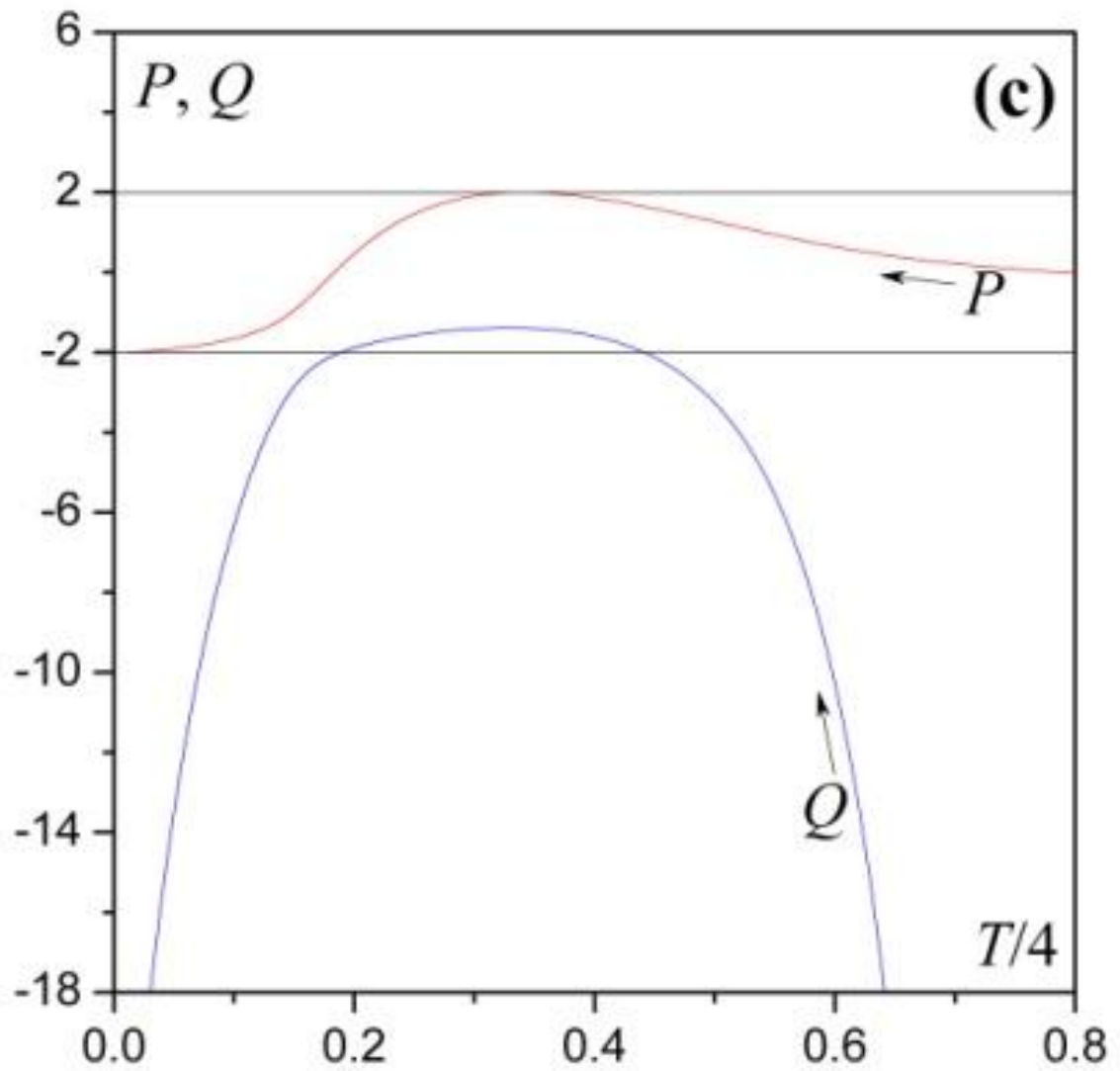


Figure 5.48: Stability of families of 3D periodic orbits emanating from the collinear equilibrium points (c) L_3 of the α -Centauri binary system. This family also contain stable parts.

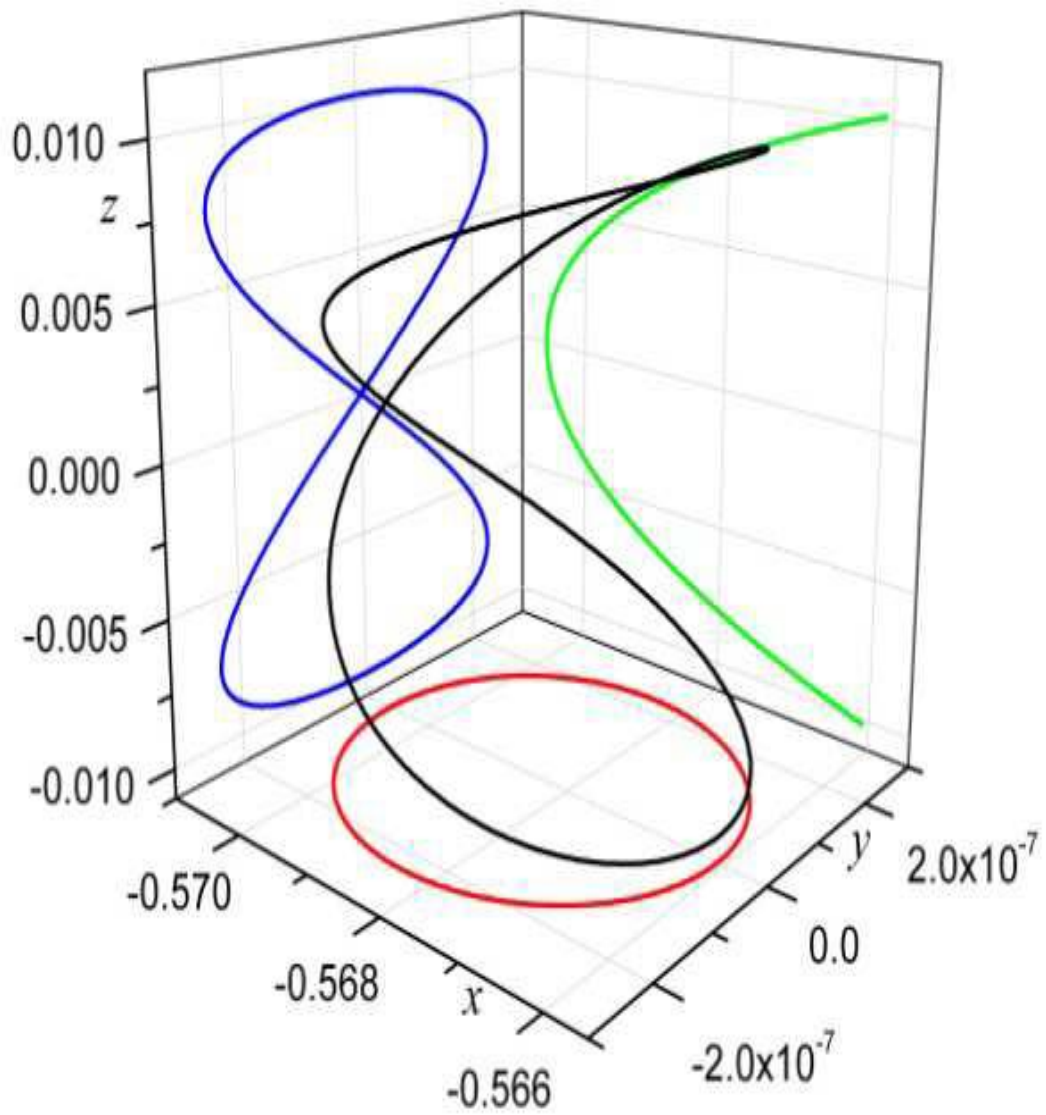


Figure 5.49: The last computed orbit of the family emanating from L_1 of the α -Centauri system in the (x, y, z) space.

5.5 Numerical Considerations for Periodic Solutions Around the Collinear Equilibrium Points in the Perturbed R3BP when Both Primaries are Radiating and Triaxial Spheroids

In this section, the numerical applications for the aforementioned model are presented and the results discussed for the 2D and 3D periodic orbits.

5.5.1 Numerical applications when the primaries are both triaxial and radiating bodies

For this model which has both primaries as radiating and triaxial spheroids and also taking into account the perturbations in the Coriolis and centrifugal forces, numerical simulations have been done using three binary systems: HD 191408, Kruger 60 and HD 155876.

The binary HD 191408 system is also known as HR 7703 (Gliese 783, 279 G. Sagittarii). It is a binary system in the constellation of Sagittarius. The primary star HD 191408 A is bigger and brighter with mass 0.82 times that of the Sun. HD 191408 B the secondary star is smaller and dimmer with a mass 0.19 times that of the Sun. The primary star has an apparent visual magnitude of 5.31, meaning that it is visible from suburban skies at night.

The binary Kruger 60 system is made up of red dwarf stars which orbit each other over every 44.6 years. Kruger 60 A which has mass $0.28M_{\odot}$ is the primary star while Kruger 60 B with mass $0.16M_{\odot}$ is the secondary star. Kruger 60 B is a flare star with variable star designation “CO Cephei”. It is an irregular flare which doubles in brightness and then returns to normal over an 8-minute period.

The binary HD 155876 system is also known as Furihjelms 48 and Gliese 661. This system consists of two similar red dwarfs located in the constellation of Hercules. The primary star in the system is HD 155876 A with mass $0.39M_{\odot}$ while the secondary HD 155876 B has mass $0.33M_{\odot}$.

Table 5. 23: Physical Parameters of the three binary systems

Parameters	HD 191408	Kruger 60	HD 155876
M_A	$0.82M_s$	$0.28M_s$	$0.39M_s$
M_B	$0.19M_s$	$0.16M_s$	$0.33M_s$
μ	0.1881	0.3636	0.4583
$L_A(L_s)$	0.236000	0.001720	0.00380
$L_B(L_s)$	0.000787	0.000387	0.00258
s_1	0.008	0.006	0.009
s_2	0.004	0.003	0.003
s'_1	0.002	0.001	0.006
s'_2	0.001	0.0004	0.002
q_1	0.407761	0.987359	0.979950
q_2	0.991476	0.995023	0.983912
ε	0.0003	0.0004	0.0002
ε'	0.0002	0.0003	0.0001

In Table 5.23, the physical parameters of three binary systems are given (<http://www.johnstonsarchive.net/astro/nearstar.html>). The symbols M_A and M_B as compared to the mass of the Sun M_s are the masses of the primary and secondary stars in the

binary systems. Their respective mass parameter is calculated such that $\mu = \frac{M_B}{M_A + M_B}$.

The luminosity L_A and L_B with respect to the Sun correspond to the luminosity of the primary and secondary stars in each system.

The mass reduction factor is represented as $q_i = 1 - \frac{F_p}{F_g}$, $i = 1, 2$ (F_p and F_g are the radiation pressure and the gravitational attraction forces being exerted by the binary systems on objects around them) or $q_i = 1 - \beta$, $i = 1, 2$ or on the basis of the Stefan-Boltzmann's law (Xuetang and Lizhong, (1993)) as

$$q_i = 1 - \frac{A\kappa L}{a\rho M}, \quad i = 1, 2,$$

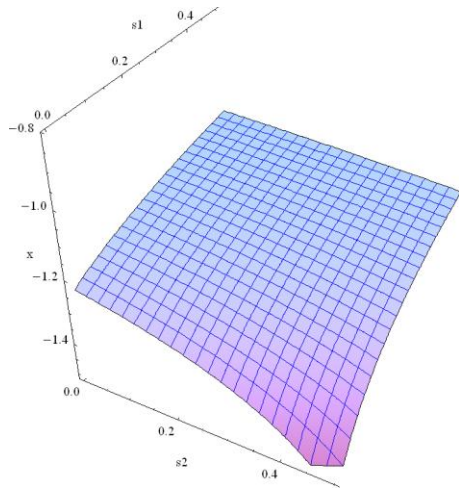
where M , L , and κ are the mass, luminosity, and radiation pressure efficiency factor of a star. Also, a and ρ are the radius and density of the dust grain particles moving in the binary systems while $A = \frac{3}{16\pi cG}$ is a constant with c and G as the speed of light and Gravitational constant.

The values of the luminosity and mass reduction factor $q_i, i = 1, 2$ have been obtained by computing in the C.G.S. system of unit, using $L_s = 3.846 \times 10^{33} \text{ erg/s}$, $c = 3 \times 10^{10} \text{ cm/s}$, $G = 6.67384 \times 10^{-8} \text{ cm}^3 \text{ g}^{-1} \text{ s}^{-2}$, $M_s = 1.989 \times 10^{33} \text{ g}$, $M_J = 1.898 \times 10^{30} \text{ g}$ and $\kappa = 1$. Also, we have assumed the values for the radius and density of the dust grain particles as $a = 2 \times 10^{-2} \text{ cm}$ and $\rho = 1.4 \text{ g/cm}^3$ ((Xuetang and Lizhong (1993), Singh and Umar (2013)). Arbitrary values are been used for the triaxiality coefficients s_1, s_2, s'_1 and s'_2 , the small perturbations introduced in the Coriolis and centrifugal forces ε and ε' respectively.

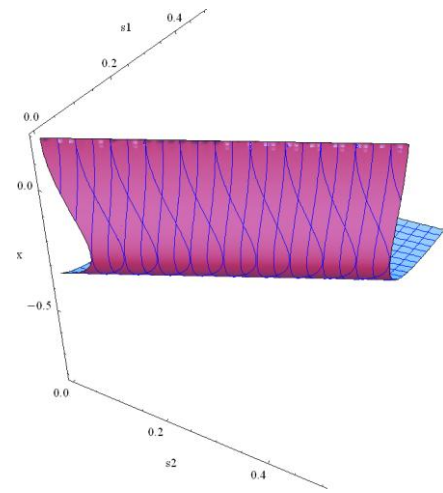
Using numerical means, Eqn. (4.40) is solved and the corresponding collinear equilibrium points for the three binary systems are obtained as shown in Table 5.24.

Table 5. 24: The positions of the collinear equilibrium points for the three binary systems

Binary systems	L_1	L_2	L_3
HD 191408	-0.34630003	0.84584372	-1.23037223
Kruger 60	-0.19631074	1.14561539	-1.23681957
HD 155876	-0.06054195	1.17807281	-1.20565304



(a)



(b)

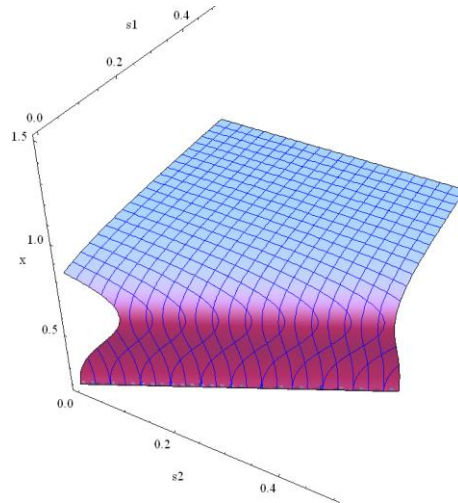
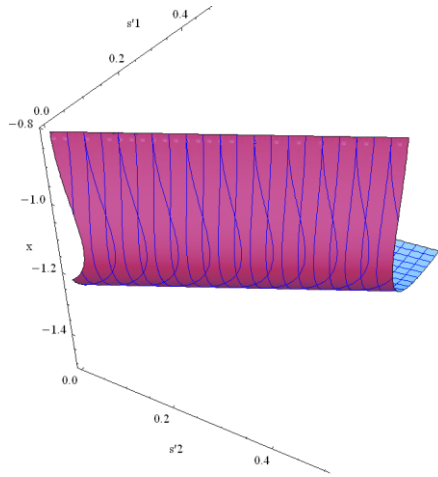
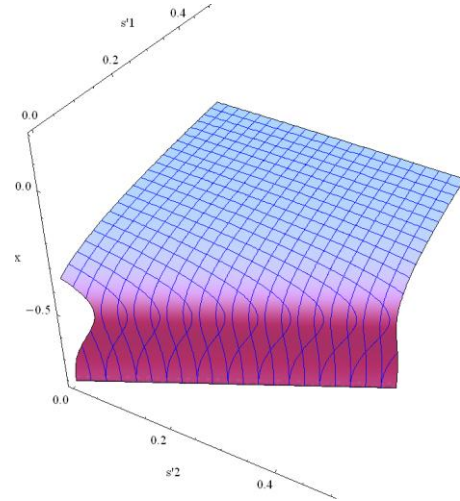


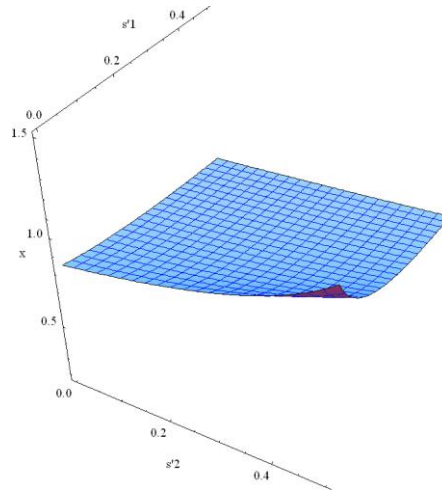
Figure 5.50: Effect of the Triaxiality coefficients of the primary body of the binary HD 191408 system on the collinear equilibrium points L_1 , L_2 and L_3 .



(a)

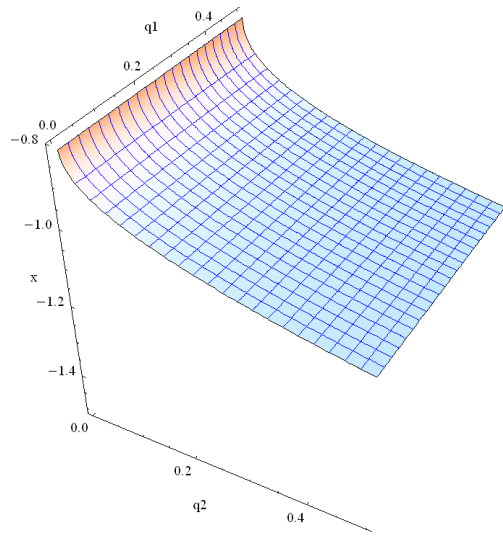


(b)

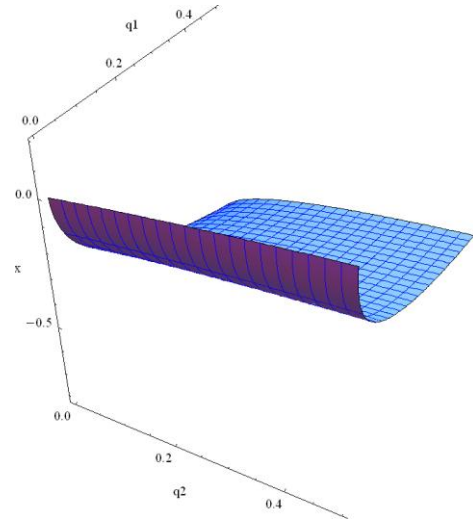


(c)

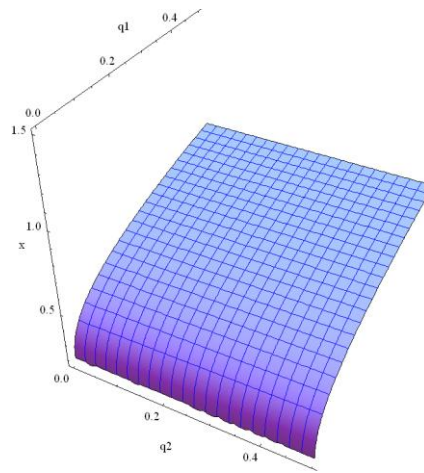
Figure 5.51: Effect of the Triaxiality coefficients of the secondary body of the binary HD 191408 system on the collinear equilibrium points L_1 , L_2 and L_3 .



(a)

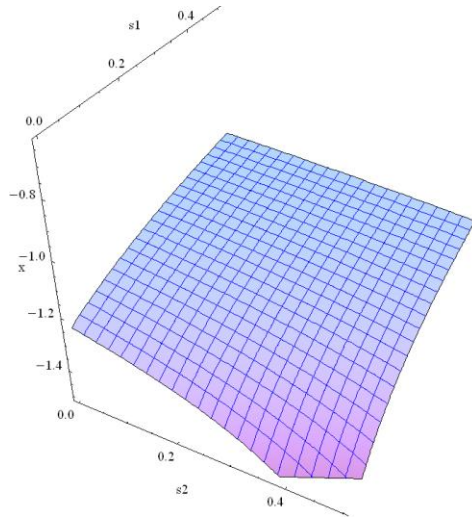


(b)

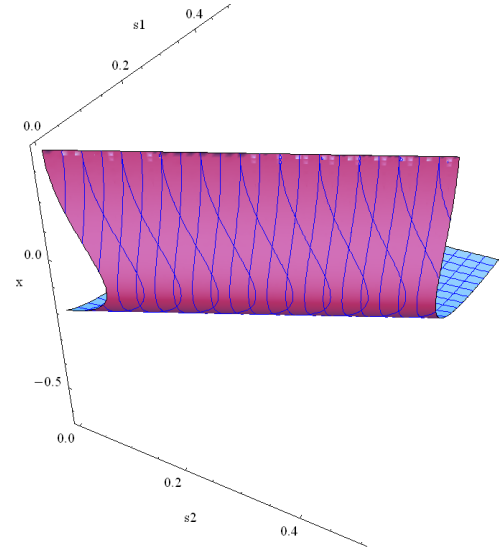


(c)

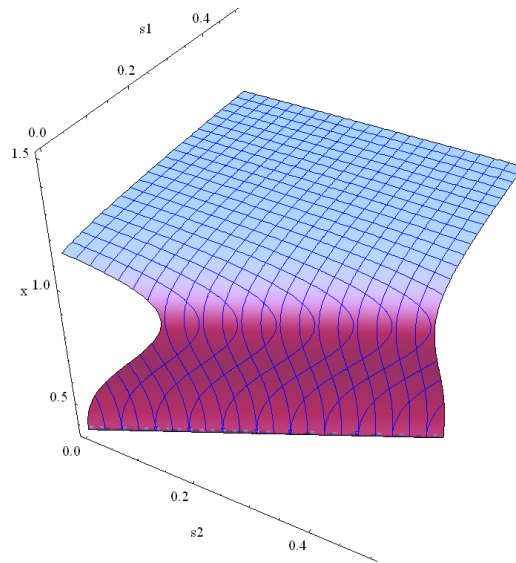
Figure 5.52: The effect of the radiation coefficient of the binary HD 191408 system on the collinear equilibrium points



(a)

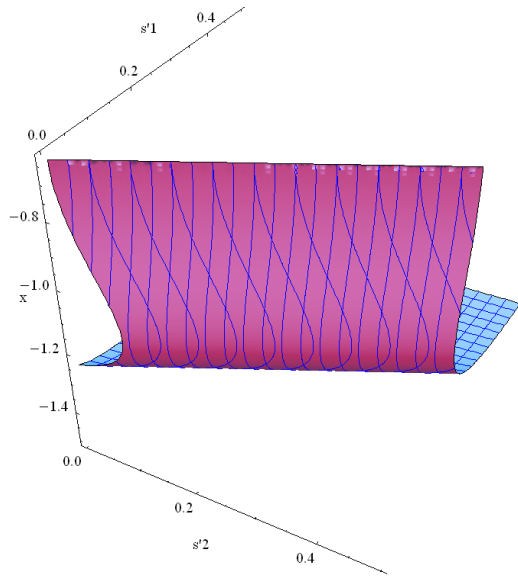


(b)

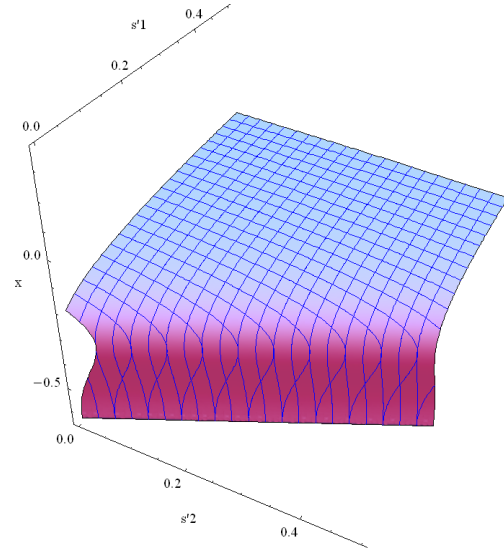


(c)

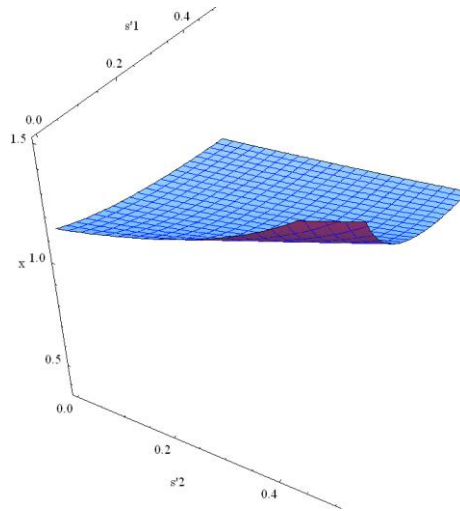
Figure 5.53: Effect of the Triaxiality coefficients of the primary body of the binary Kruger 60 system on the collinear equilibrium points L_1 , L_2 and L_3 .



(a)

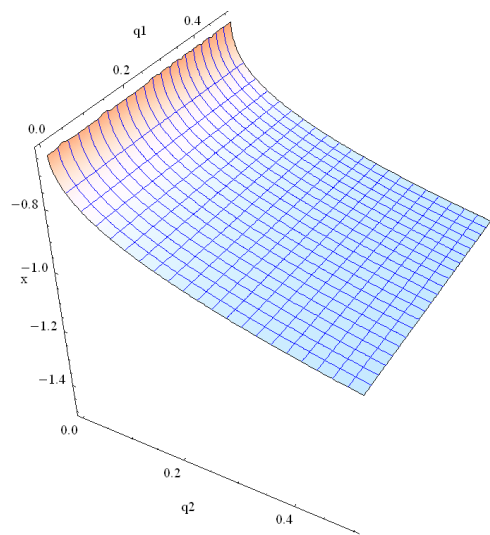


(b)

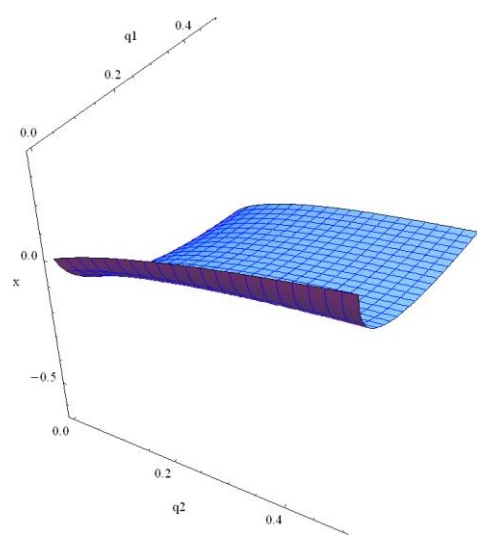


(c)

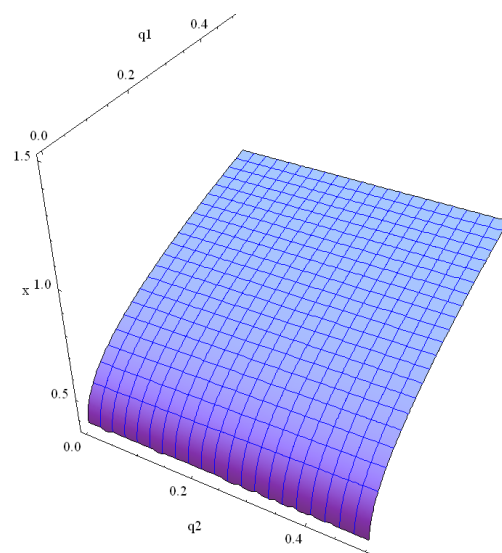
Figure 5.54: Effect of the Triaxiality coefficients of the secondary body of the binary Kruger 60 system on the collinear equilibrium points L_1 , L_2 and L_3 .



(a)

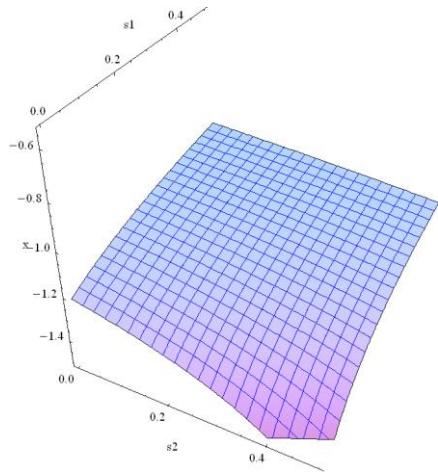


(b)

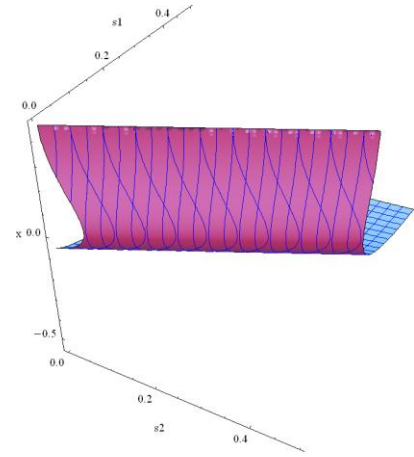


(c)

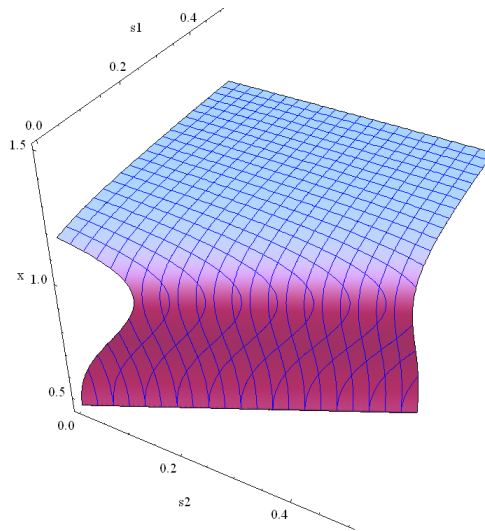
Figure 5.55: The effect of the radiation coefficient of the binary Kruger 60 system on the collinear equilibrium points



(a)

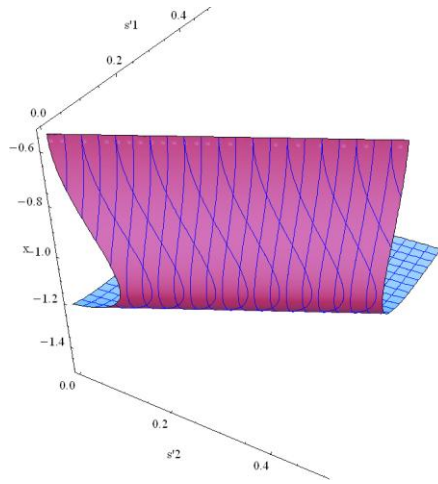


(b)

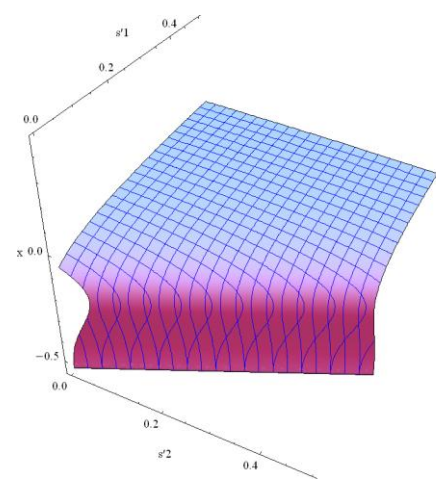


(c)

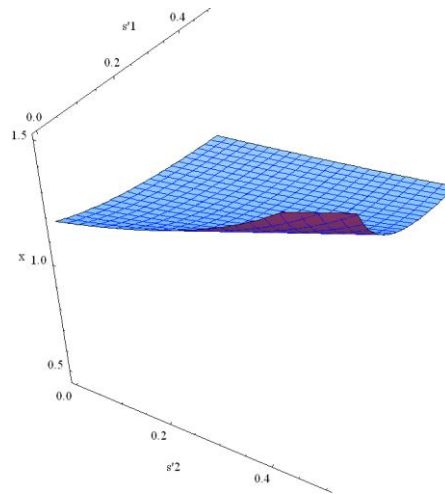
Figure 5.56: Effect of the Triaxiality coefficients of the primary body of the binary HD 155876 system on the collinear equilibrium points L_1 , L_2 and L_3 .



(a)



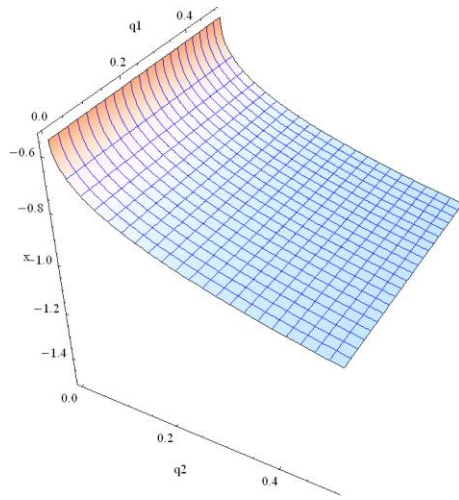
(b)



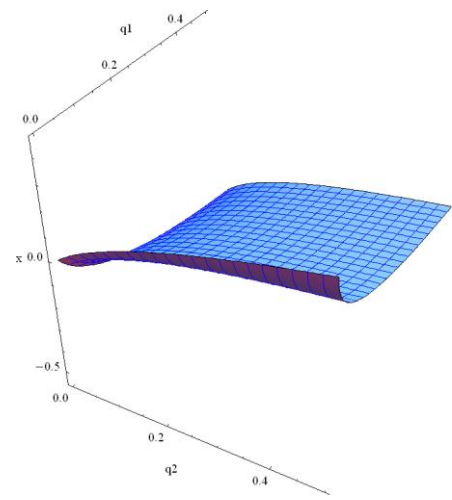
(c)

Figure 5.57: Effect of the Triaxiality coefficients of the secondary body of the binary HD

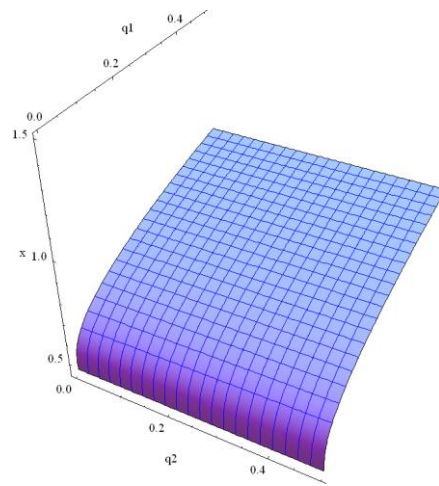
155876 system on the collinear equilibrium points L_1 , L_2 and L_3



(a)



(b)



(c)

Figure 5.58: The effect of the radiation coefficient of the binary HD 155876 system on the collinear equilibrium points

In Figures 5.50-5.58, the effects of the Triaxiality of the primary and secondary stars respectively, and the effects of the radiation factors on each binary system are shown graphically.

The characteristic Equation (4.52) has been used to determine stability of the collinear equilibrium points of the three binary systems. In the Lyapunov sense, if all four roots of (4.52) are either negative real numbers or pure imaginary or if the real part of complex roots is negative at each one of the collinear equilibrium points, then these are stable otherwise they are unstable. Thus, as it can be seen from the results presented in Table 5.25, the collinear equilibria $L_{1,2,3}$ are unstable in the three binary systems.

Table 5. 25: Roots of the characteristic equation (4.52) determining the stability of the collinear equilibrium points

Binary Systems	Characteristic Roots (L_1)	Characteristic Roots (L_2)	Characteristic Roots (L_3)
HD 191408	$\pm 2.14374668,$ $0 \pm 2.64131171i$	$\pm 1.18513130,$ $0 \pm 0.82304915i$	$\pm 1.74248475,$ $0 \pm 1.95130023i$
Kruger 60	$\pm 2.91278362,$ $0 \pm 3.84128166i$	$\pm 1.26053165,$ $0 \pm 1.00167795i$	$\pm 1.46002498,$ $0 \pm 1.41790463i$
HD 155876	$\pm 3.03429140,$ $0 \pm 4.03195628i$	$\pm 1.34809397,$ $0 \pm 1.17973952i$	$\pm 1.40083990,$ $0 \pm 1.28871457i$

5.5.2 Numerical Results for the 2D Periodic Orbits for the model with triaxial and radiating primaries

From the results presented in Equations (4.57), we obtain the infinitesimal or starting orbits for the 2D periodic orbits when the primaries are both radiating and triaxial spheroids together with perturbations in the Coriolis and centrifugal forces. In Tables 5.26-5.28, the initial conditions are presented for the three binary systems. The first entry for each binary

system corresponds to a stable member of the respective Lyapunov family. The second entry corresponds to the termination orbit (see the plotted orbits with blue colour in Figures 5.58, 5.59 and 5.60 as well as the respective blue dots on the characteristic curves) of each family. The initial conditions $(x_0, \dot{y}, \frac{T}{2})$ are given in the second, third and fifth columns respectively, the corresponding Jacobian constant has been given in the fourth column, while the stability parameters a_h and a_v are given in the sixth and seventh columns of the tables. The differential corrections method utilized in the computation of these families is the corrector-predictor scheme. The periodicity condition (for an orbit to be periodic and symmetric w.r.t. the Ox axis) of a 2D periodic orbit have been explained earlier and stated in Eqn. (5.1).

Table 5. 26: Initial conditions for the Lyapunov families emanating from the equilibrium point L_1 .

System	x_0	$\dot{y}_0$	C	$T/2$	a_h	a_v
HD 191408	0.08398164	-2.92385205	1.74758808	5.70976425	-0.270488	-0.679085
	0.12585189	-5.07170025	1.76019320	22.83920876	$\approx 4.38 \times 10^{12}$	1.206475
Kruger 60	0.06043388	-1.72775000	2.40693016	2.68918702	0.005082	-0.648239
	0.13074675	-2.01099846	2.76267644	23.67791869	$\approx 2.18 \times 10^{12}$	-0.101113
HD 155876	-0.30144749	1.78867010	2.38945116	2.62379792	0.506487	-0.690294
	-0.33038432	1.86476431	2.74395568	22.91770178	$\approx 4.80 \times 10^{12}$	0.682260

Table 5. 27: Initial conditions for the Lyapunov families emanating from the equilibrium point L_2 .

System	x_0	$\dot{y}_0$	C	$T/2$	a_h	a_v
HD 191408	0.32721787	2.30659972	1.35427458	3.25151412	0.006526	-0.204699
	0.32113867	2.31119851	1.76026184	25.29946412	$\approx 4.98 \times 10^{12}$	-0.970858
Kruger 60	1.78687824	-1.54756336	2.03328913	3.14809174	0.508140	-0.389253
	1.91362591	-1.43342432	2.76267529	23.10084174	$\approx -1.26 \times 10^{11}$	0.571636
HD 155876	0.58193343	3.44375667	2.09855862	3.10806297	0.008255	-0.736149
	0.58695041	3.22609465	2.74395519	23.25631296	$\approx 8.99 \times 10^{12}$	1.032557

Table 5. 28: Initial conditions for the Lyapunov families emanating from the equilibrium point L_3 .

System	x_0	$\dot{y}_0$	C	$T/2$	a_h	a_v
HD 191408	-1.44157784	1.31925000	1.38671427	3.13264553	0.014754	0.904578
	-2.43549125	2.18829673	1.76010616	30.81719571	$\approx 8.17 \times 10^{12}$	-0.134217
Kruger 60	-0.70061527	-3.59449819	2.21807815	3.00196284	0.002437	-0.604437
	-0.69929084	-3.57124066	2.76267558	23.50636284	$\approx 1.05 \times 10^{12}$	0.992267
HD 155876	-1.71047416	1.46615000	2.15090163	3.06506404	0.004406	-0.809045
	-1.86631456	1.41784384	2.74395540	23.41967258	$\approx 2.41 \times 10^{13}$	0.856577

In Figures 5.59, 5.60 and 5.61, the Characteristic curves, in the $(T/2, x)$ plane, of the Lyapunov families emanating from the collinear equilibrium points for **(a)-(c)** HD 191408, **(d)-(f)** Kruger 60 and **(g)-(i)** HD 155876, binary systems are presented. The insets correspond to the termination orbit (see blue dot at the end of the family characteristic) of each Lyapunov family around the triangular points. It is found that all families of planar periodic orbits emanating from the collinear equilibrium points terminate with asymptotic periodic orbits at the triangular equilibrium points

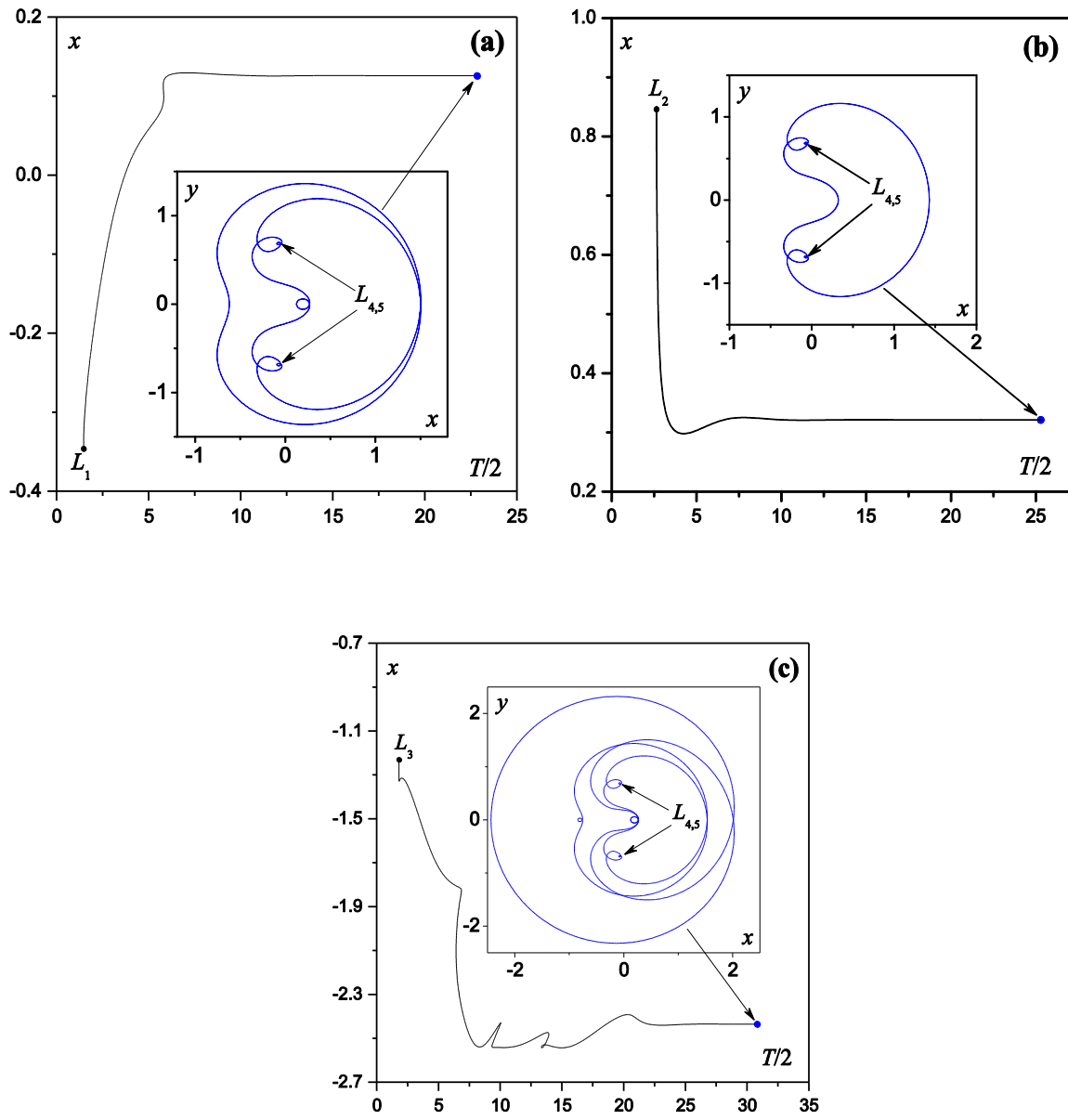


Figure 5. 59: Characteristic curves, in the $(T/2, x)$ plane, of the Lyapunov families emanating from the collinear equilibrium points for **(a)-(c)** HD 191408 binary system.

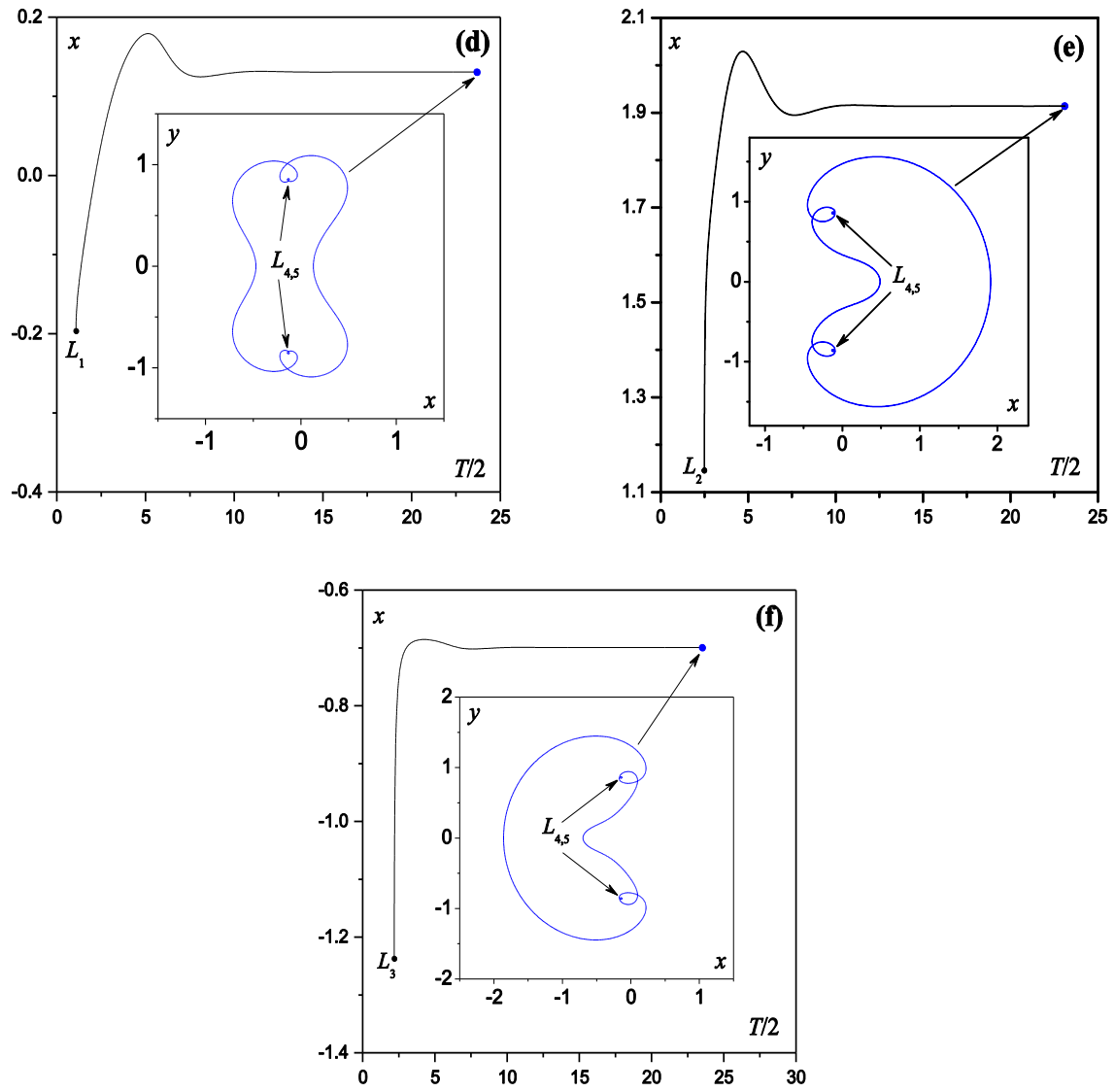


Figure 5.60: Characteristic curves, in the $(T/2, x)$ plane, of the Lyapunov families emanating from the collinear equilibrium points for **(d)-(f)** Kruger 60 binary system.

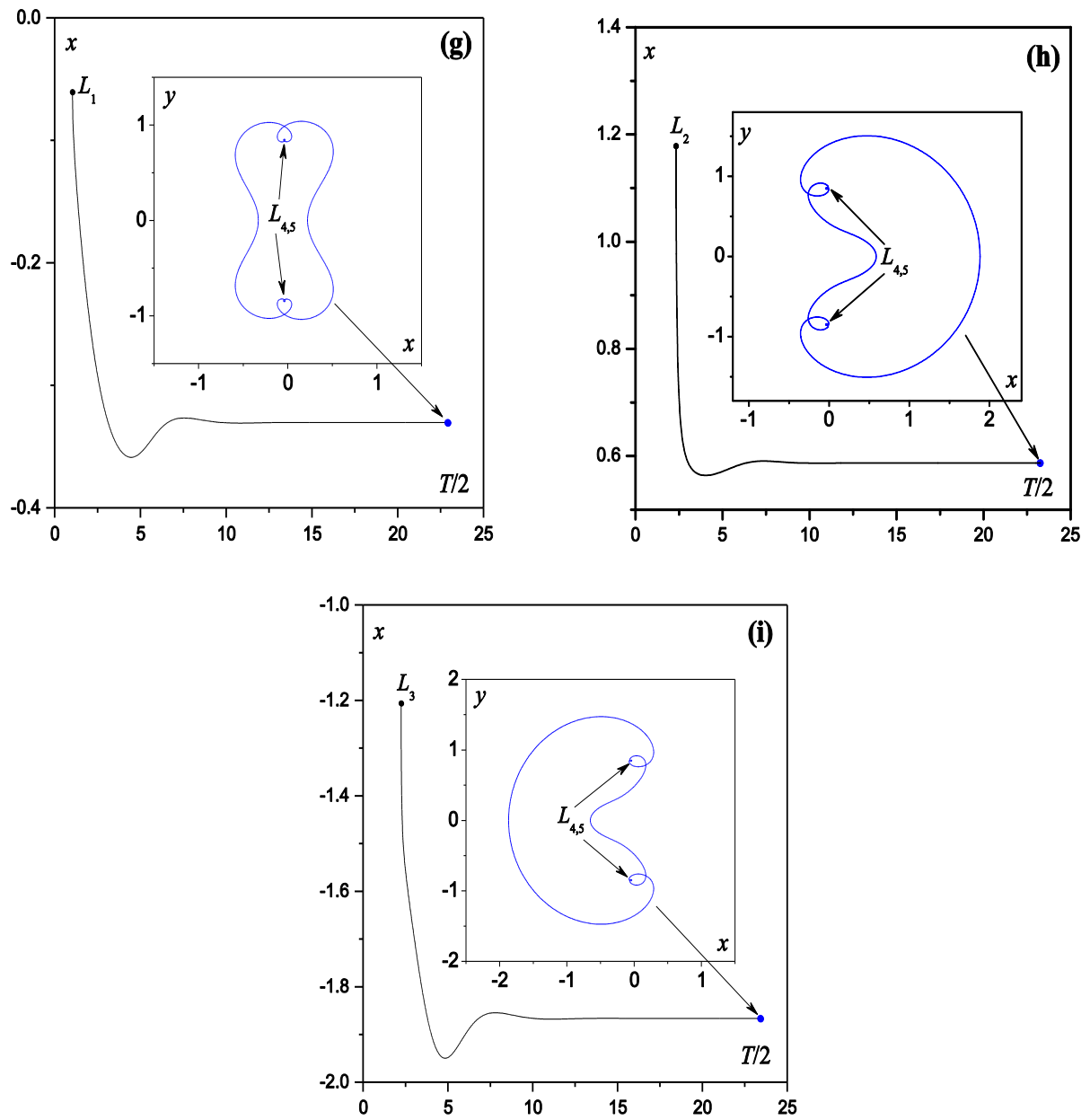


Figure 5.61: Characteristic curves, in the $(T/2, x)$ plane, of the Lyapunov families emanating from the collinear equilibrium points for **(g)-(i)** HD 155876 binary system.

In Figures 5.62, 5.63 and 5.64 the horizontal (blue curves) and vertical (red curves) stability of the Lyapunov families emanating from the collinear equilibrium points of **(a)-(c)** HD 191408, **(d)-(f)** Kruger 60 and **(g)-(i)** HD 155876, binary systems are shown graphically. Note that, due to the large values of the horizontal stability a_h we have plotted $\log_{10}(a_h + \sqrt{a_h^2 + 1})$, instead. Therefore, while vertical stability occurs for values in the interval $(-1, 1)$, horizontal stability occurs for values approximately in the interval $(-0.3828, 0.3828)$ (see red and blue dotted lines, respectively).

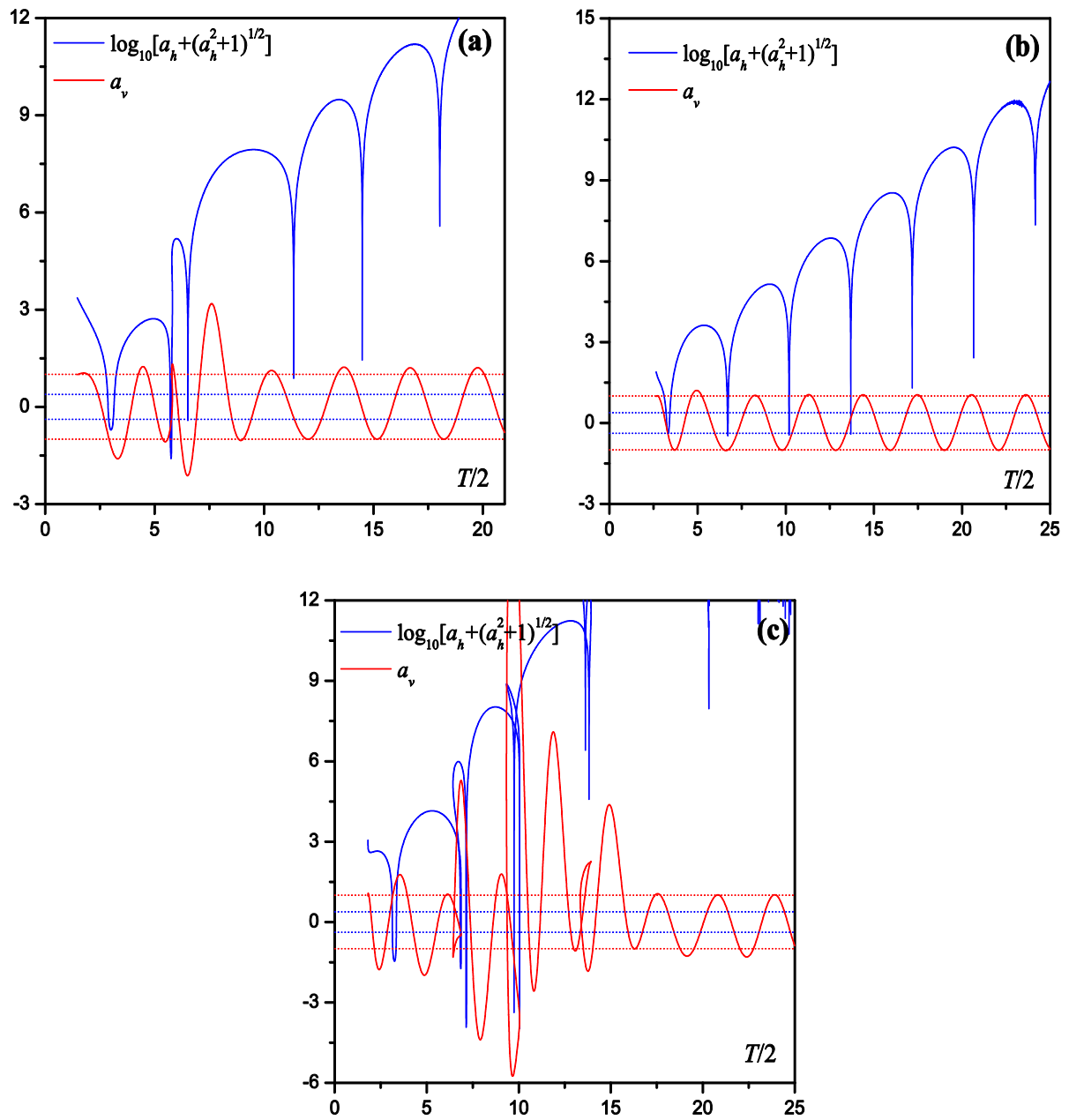


Figure 5.62: Horizontal (blue curves) and vertical (red curves) stability of the Lyapunov families emanating from the collinear equilibrium points of **(a)-(c)** HD 191408 binary system.

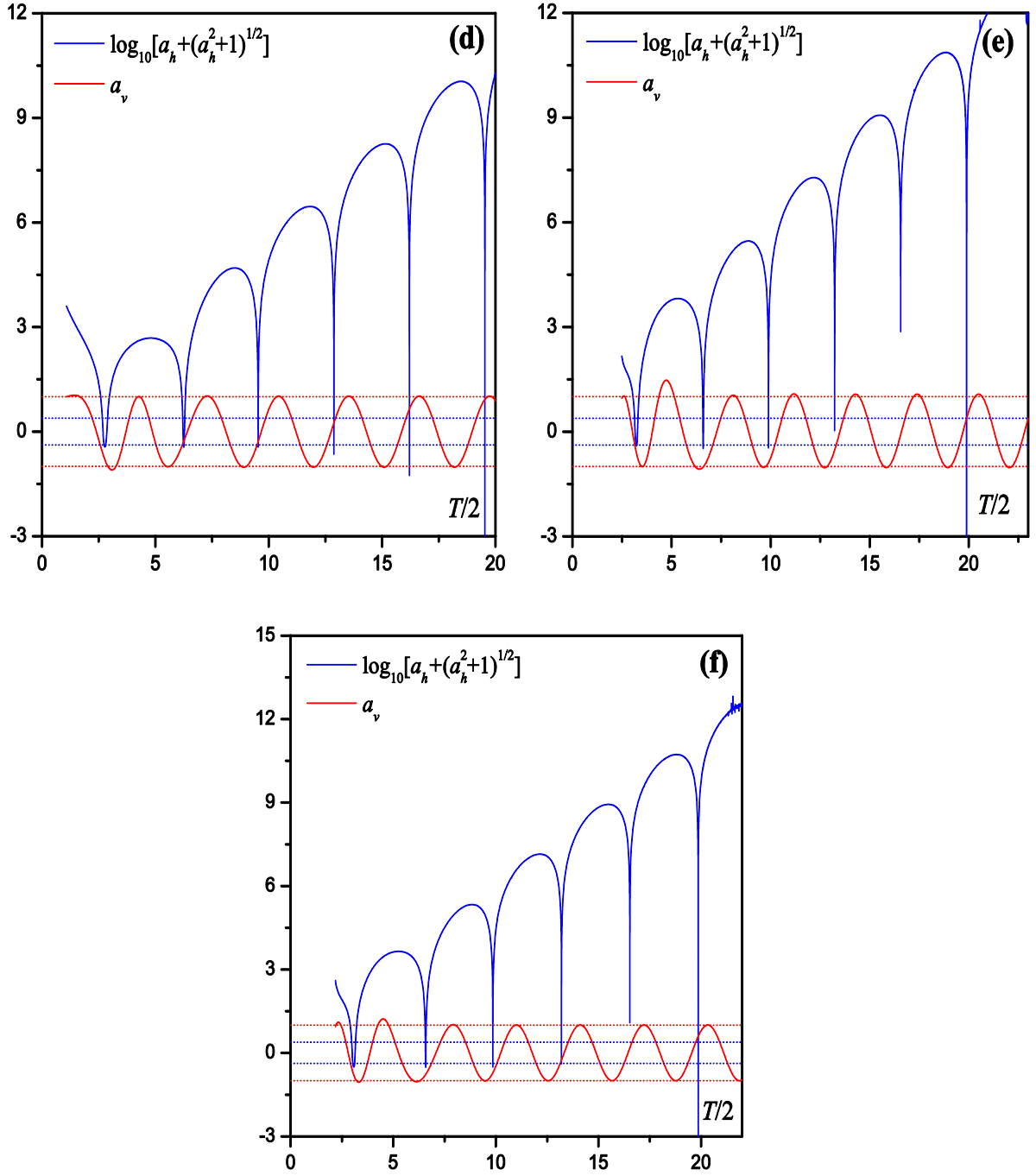


Figure 5.63: Horizontal (blue curves) and vertical (red curves) stability of the Lyapunov families emanating from the collinear equilibrium points of **(d)-(f)** Kruger 60 binary systems.

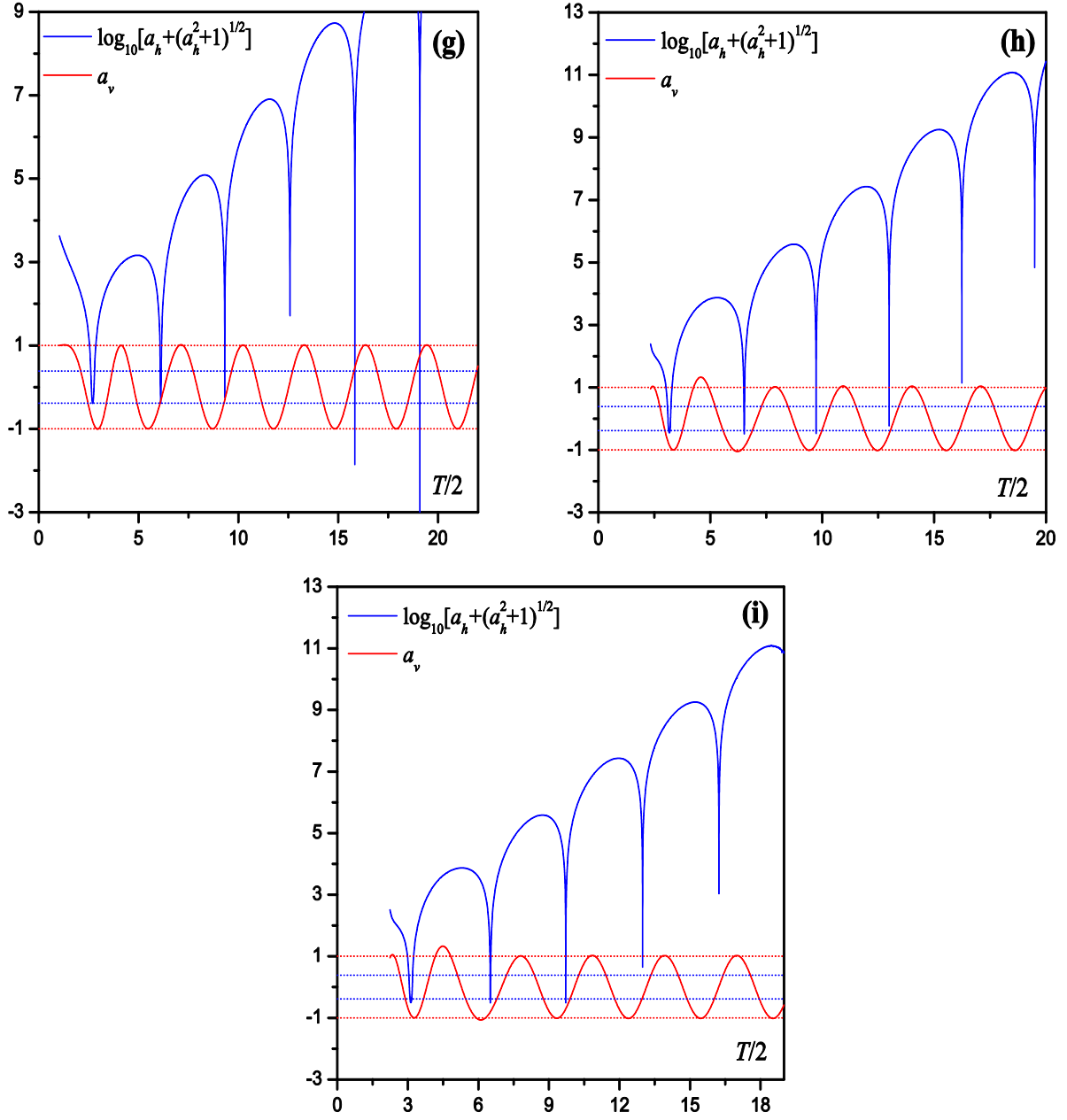


Figure 5.64: Horizontal (blue curves) and vertical (red curves) stability of the Lyapunov families emanating from the collinear equilibrium points of **(g)-(i)** HD 155876 binary systems.

5.5.2 Numerical Results for the Spatial Periodic Orbits of the model with triaxial and radiating primaries

From Equations (4.63), we obtain the initial conditions $(x_0, \dot{y}, \dot{z}, \frac{T}{4})$ for the 3D periodic orbits when the primaries are both radiating and triaxial spheroids together with perturbations in the Coriolis and centrifugal forces. The initial conditions for families of 3D periodic orbits emanating from the equilibrium points L_1 , L_2 and L_3 are presented in Tables 5.29, 5.30 and 5.31 respectively, where the second entry for each binary system corresponds to the planar periodic orbit at which the family terminates. The initial conditions $(x_0, \dot{y}, \dot{z}, \frac{T}{4})$ have been given in the second, third, fourth and sixth columns respectively, the corresponding Jacobian constant is given in column five, while the stability parameters P and Q have been provided in the seventh and eighth columns.

The method and procedure used here in examining the stability and the continuation to families (A, B, C) of 3D periodic orbits are the same as those utilized and discussed in section 5.2.

In Figures 5.65, 5.66 and 5.67, the characteristic curves in the space of initial conditions $(x, \dot{y}, \dot{z})$ of the families of 3D periodic orbits emanating from the collinear equilibrium points for **(a)-(c)** HD 191408, **(d)-(f)** Kruger 60 and **(g)-(i)** HD 155876, binary systems are shown graphically. From the results, instability holds for all the computed families of 3D periodic orbits emanating from the collinear equilibrium points in the three binary systems with the exception of the families of 3D periodic orbits emanating from the equilibrium point L_2 which present similar results for the three binary systems where stability holds for some of the members.

The manifold of 3D periodic orbits emanating from the three collinear equilibrium points L_1, L_2 and L_3 respectively for the HD191408 binary system is captured in Figure 5.68.

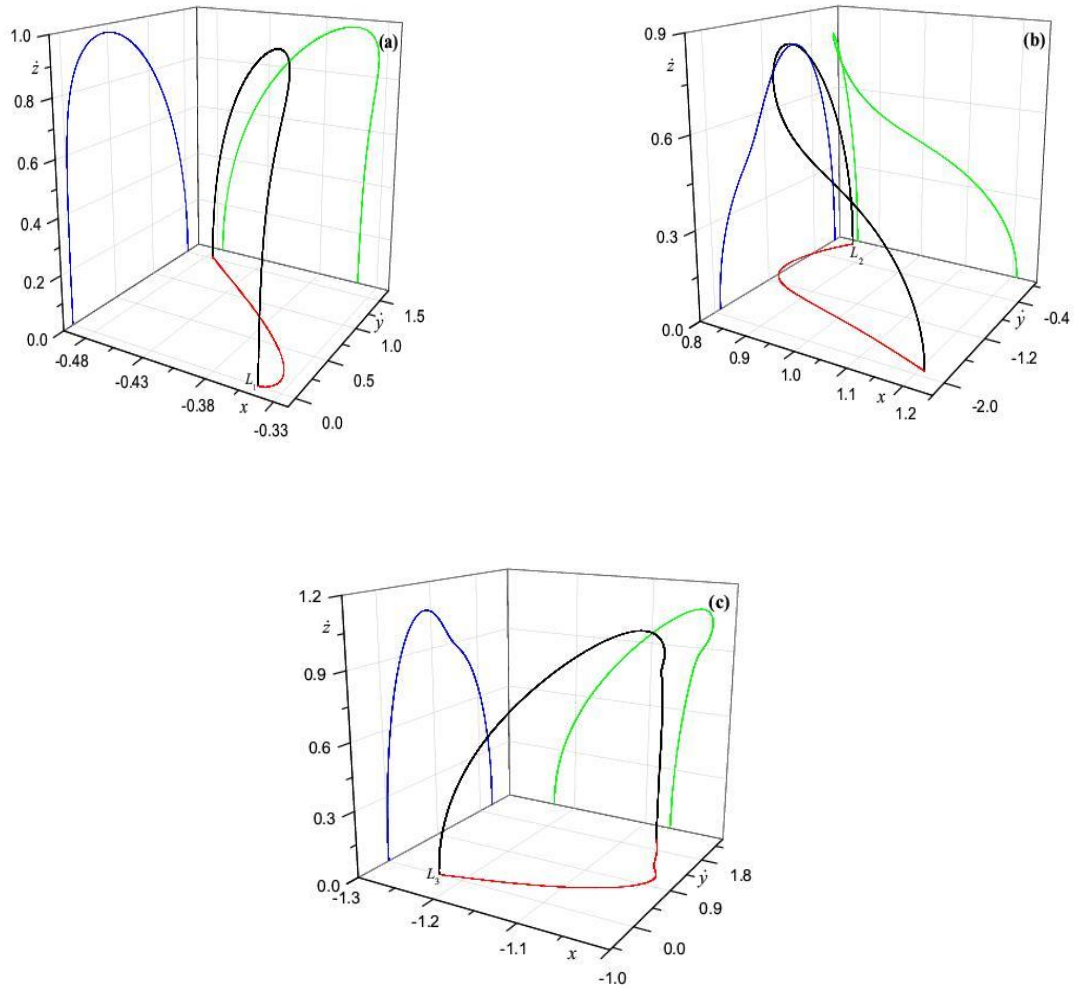


Figure 5. 65: Characteristic curves, in the space of initial conditions (x, y, z) , of the families of 3D periodic orbits emanating from the collinear equilibrium points for **(a)-(c)** HD 191408 binary system.

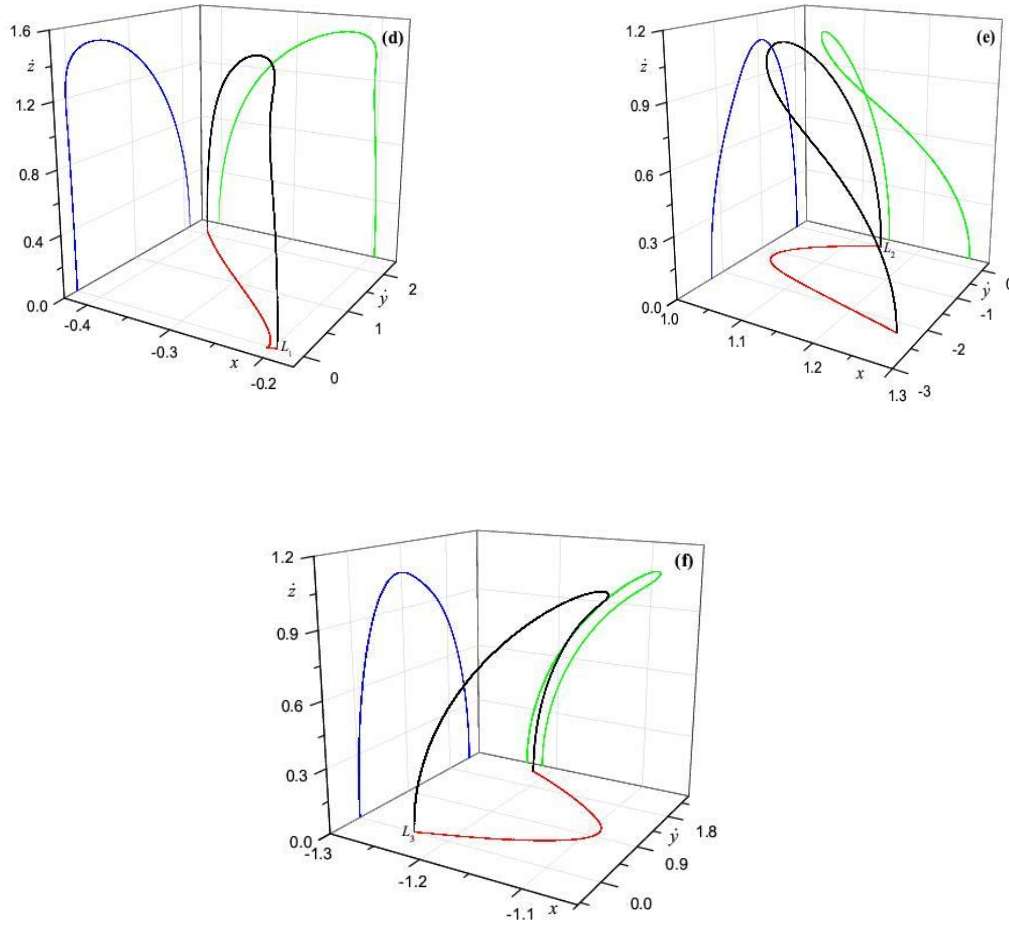


Figure 5.66: Characteristic curves, in the space of initial conditions (x, y, z) , of the families of 3D periodic orbits emanating from the collinear equilibrium points for **(d)-(f)** Kruger 60 binary system.

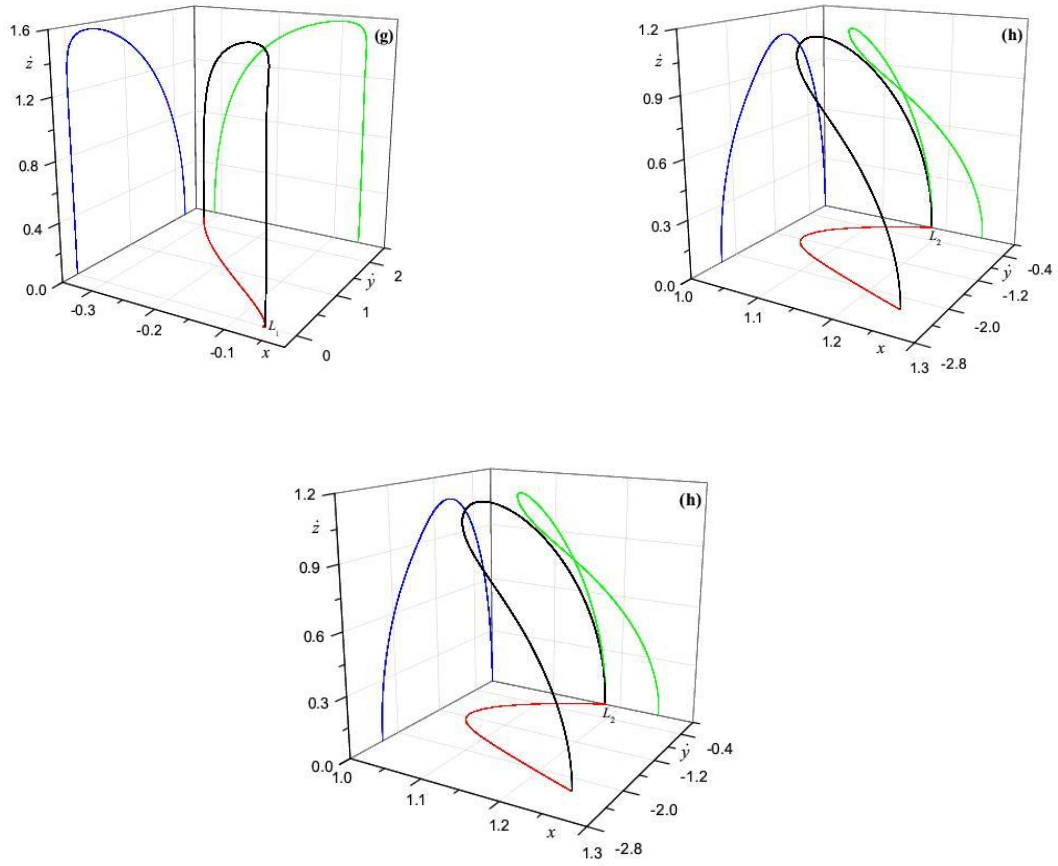


Figure 5. 67: Characteristic curves, in the space of initial conditions (x, y, z) , of the families of 3D periodic orbits emanating from the collinear equilibrium points for **(g)-(i)** HD 155876 binary system.

Table 5. 29: Initial conditions for families of 3D periodic orbits emanating from the equilibrium point L_1 .

System	x_0	$\dot{y}_0$	$\dot{z}_0$	C	T/4	P	Q
HD 191408	-0.34628117	0.00003623	0.05	2.19182428	0.74776249	-1.98438	-2647.78
	-0.47416271	1.52492628	0.00	0.03691176	1.72475051	1.81856	-2.00000
Kruger 60	-0.19633164	-0.00003358	0.05	3.96432302	0.54556658	-1.99496	-4328.24
	-0.40447408	2.11902704	0.00	0.49091930	1.92454355	-2.00000	-15.0947
HD 155876	-0.06054759	-0.00000894	0.05	4.01937816	0.52040545	-1.99907	-4374.71
	-0.31377560	2.22197129	0.00	0.89504088	2.20523359	-2.00000	-205.232

Table 5. 30: Initial conditions for families of 3D periodic orbits emanating from the equilibrium point L_2 .

System	x_0	$\dot{y}_0$	$\dot{z}_0$	C	T/4	P	Q
HD 191408	0.84567228	-0.00137490	0.05	1.97493524	1.36858291	-1.95636	-90.0098
	1.20896192	-1.92321773	0.00	-1.36702811	2.00408094	1.69963	-2.00000
Kruger 60	1.14545061	-0.00099001	0.05	3.35619014	1.30647650	-1.90529	-186.707
	1.27132103	-2.25184961	0.00	-1.65697543	1.79755670	0.17380	-2.00000
HD 155876	1.17787526	-0.00093945	0.05	3.45920418	1.22317889	-1.90148	-320.027
	1.25423124	-2.26718181	0.00	-1.65524663	1.76129722	0.21584	-2.00000

Table 5. 31: Initial conditions for families of 3D periodic orbits emanating from the equilibrium point L_3 .

System	x_0	$\dot{y}_0$	$\dot{z}_0$	C	$T/4$	P	Q
HD 191408	-1.23004207	0.00082955	0.05	2.91281995	0.93600730	-1.94046	-1482.05
	-1.06721946	2.07610507	0.00	-1.12142979	1.61727743	0.66994	-2.00000
Kruger 60	-1.23658265	0.00090050	0.05	3.54642812	1.15437203	-1.87011	-567.8648
	-1.21764995	2.25853564	0.00	-1.55027952	1.71197453	-0.19147	-2.00000
HD 155876	-1.20543330	0.00091306	0.05	3.51924899	1.17962674	-1.89260	-435.755
	-1.23829444	2.26942696	0.00	-1.62386313	1.73600587	0.11459	-2.00000

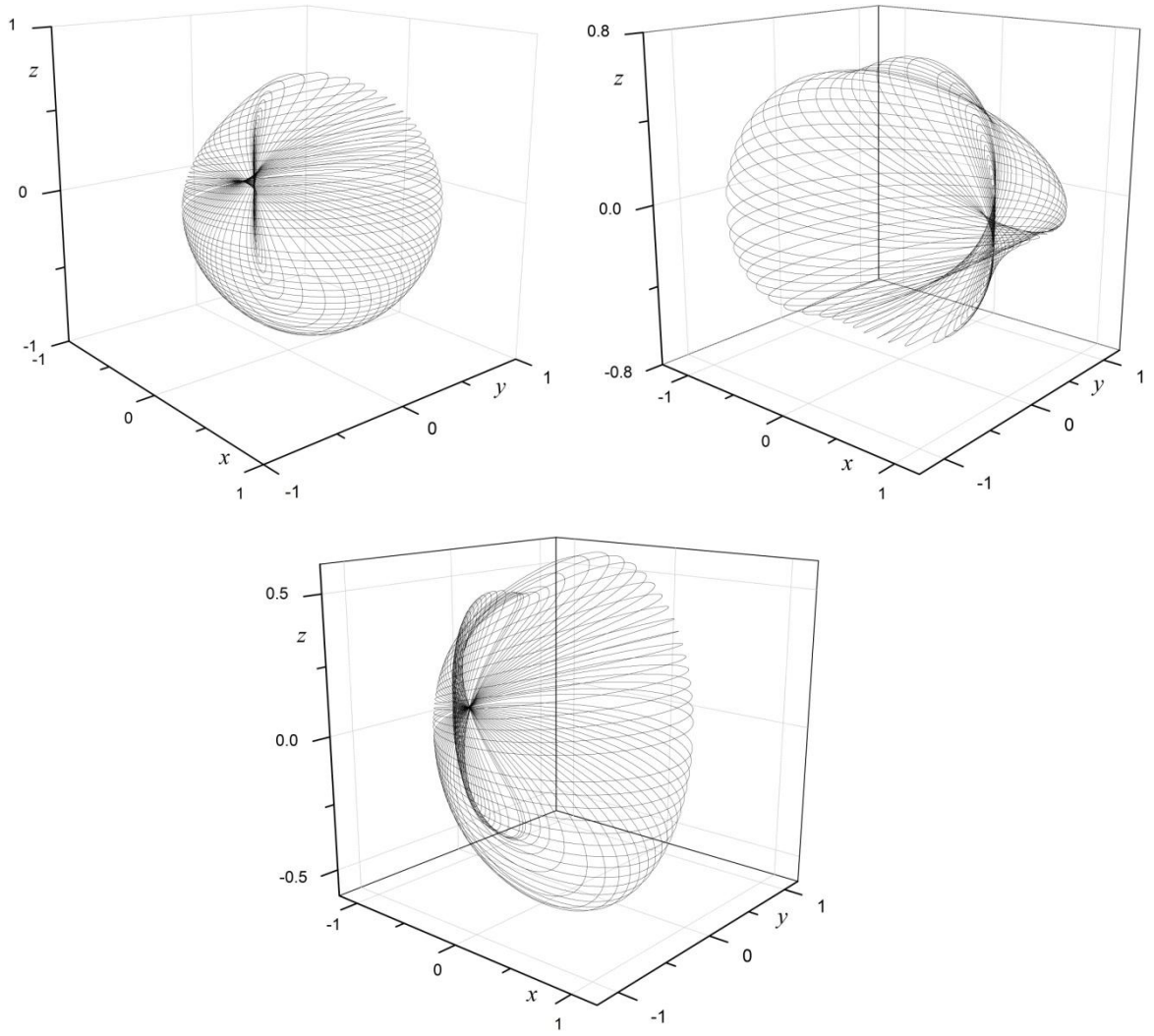


Figure 5.68: The manifold of 3D periodic orbits emanating from the three collinear equilibrium points L_1, L_2 (top row, left and right frame, respectively) and L_3 (second row) for the HD191408 binary system.

In the next chapter, we give the summary, conclusion and recommendation on the investigation done in this thesis.

CHAPTER SIX

SUMMARY, CONCLUSION AND RECOMMENDATION

6.1 Introduction

In this chapter, we give the summary, conclusion and recommendation of our investigation on periodic orbits around the collinear equilibrium points in the generalized R3BP. Generalized in the sense that in the first problem, the primaries are taken to be triaxial and oblate spheroidal bodies respectively together with perturbations in the Coriolis and centrifugal forces, while in the second and third problems the primaries are both radiating oblate spheroidal bodies. And in the fourth problem, the primaries are both radiating as well as triaxial rigid bodies together with perturbations in the Coriolis and centrifugal forces.

6.2 Summary

In chapter THREE, second order semi-analytical solution for both 2D and 3D periodic orbits were constructed around the collinear equilibrium points in the R3BP when the primary is a triaxial rigid body and the secondary an oblate spheroid taking into account small perturbations in the Coriolis and centrifugal forces.

The equations of motion, variation and the stability parameters were presented in section 3.2. In section 3.3, the positions of the collinear equilibrium points were obtained by solving Eqn. (3.4) numerically. The second order expansion of the equations of motion for the planar case was presented in subsection 3.4.1. The semi-analytical approximation of the 2D periodic orbits was given in sub subsection 3.4.1.2, while the periodic solution of these periodic orbits has been given in Eqns. (3.22).

In section 3.5, the second order analysis for obtaining spatial periodic orbits was captured. The second order expansion of the 3D equations of motion was presented in subsection 3.5.1.

The semi-analytical approximation of the 3D periodic orbits was obtained in subsection 3.5.2. The periodic solution up to second order terms w.r.t. the orbital parameter e has been given in Eqn. (3.37).

In chapter FOUR, we examined and obtained a third order analytic approximate solution of Lyapunov orbits around the collinear equilibrium points in the planar circular restricted three-body problem (PCR3BP) when the primaries are both radiating and oblate spheroidal bodies by utilizing the Lindstedt-Poincaré method. From the approximate periodic solution, the initial conditions or starting orbits near the collinear points have been obtained. Also, a second order analytic approximate solution of Lyapunov orbits around the collinear equilibrium in the R3BP for the spatial case of the same problem is obtained by using the same method.

In section 4.2, a third order analytic approximate solutions of Lyapunov orbits around the collinear equilibrium points for the planar case was obtained. In subsection 4.2.1, the equations of motion of the infinitesimal body in the dimensionless synodic coordinate system with radiation pressure parameters q_1 and q_2 ($q_i \leq 1, i = 1, 2$) and oblateness parameters A_1 and A_2 ($A_i \leq 1, i = 1, 2$) were presented. Figure 4.1 shows the forbidden region to motion of the infinitesimal body with respect to the Jacobian constant C around the collinear equilibrium points. In subsection 4.2.2, three collinear equilibrium points were obtained by solving numerically the Eqn. (4.2) within the intervals $(-\infty, -1 + \mu)$, $(-1 + \mu, \mu)$ and $(\mu, +\infty)$. Each of these intervals contains a real root which corresponds to, L_1 , L_2 and L_3 respectively. In subsection 4.2.3, a third order Lindstedt-Poincaré method was utilized to obtain a semi-analytic approximate solution presented in Eqns. (4.16).

In subsection 4.3.4, a second order Taylor series expansions of the equations of motion for the spatial case is presented in Eqns. (4.26).

In subsection 4.3.5, an approximation of periodic solutions up to second order terms for 3D periodic orbits is given in Eqns. (4.31). While in subsection 4.3.6, the numerical approximation of periodic solutions is given explaining the differential correction procedure that was utilized.

In chapter FIVE, the results and discussion for all the problems investigated in this study are presented.

In subsection 5.2.1, we present the numerical results of the 2D periodic orbits when the primary is a triaxial rigid body and the secondary is an oblate body together with perturbations in the Coriolis and centrifugal forces. In Table 5.1, ten cases of the problem are shown (the values for the parameters under consideration for case 1 are same as those used in Singh and Begha (2011)), while the effect of the triaxiality coefficients of the primary and the combined effect of the oblateness factor of the secondary and the influence of the centrifugal force on the location of the collinear libration points can be seen in Figures 5.1 and 5.2, respectively for case 1.

For the aspect of the study of the linear stability of the collinear equilibrium points, Equation (3.17) was utilized and the three collinear equilibrium points were shown to be unstable in the Lyapunov sense for all the ten cases as can be seen in Table 5.2.

In Figures 5.4-5.6 we present the families of planar symmetric periodic orbits, in their projection into $(x_0, \dot{y}_0)$, around all the collinear equilibrium points as they were computed by the first (red curve) and second (blue curve) order analyses (3.22). We also showed, in these figures, the accurate numerical solution (black curve) of the corresponding families obtained

by the numerical integration of the full equations of motion (3.1). At each equilibrium point, we observed that the second order semi-analytical solution is always a better approximation of the numerical solution than the first one; the validity of our analysis is obvious.

These approximate initial conditions are obtained through the use of equations (3.7) – (3.14). We presented these initial conditions $X_0, \dot{Y}_0, T_a$ at $t=0$ in Tables 5.3, 5.4, and 5.5 together with the Jacobian constant, C and the stability parameters a_h and a_v for the ten cases.

We generated the corresponding Lyapunov families (A, B and C) by means of the numerical continuation process that made use of the approximate initial conditions as initial guess for a periodic orbit. This involved the numerical integration of the equations of motion simultaneously with the variational equations. A proper choice of the phase space direction (direction of the flow) was made to extend the solution using a predictor-corrector scheme, and repeating the process iteratively until the periodic orbit is computed with the desired accuracy of 5×10^{-8} .

Few members of the family from the initial conditions of case 1 of Table 5.3 are presented in Table 5.6. This is because they are quite numerous in number. Presenting just a few will suffice because of space. Same is done for members of the families B and C, presented in Tables 5.7 and 5.8 respectively.

The graphical representations of the evolution of 2D Lyapunov families are shown in Figures 5.7, 5.8 and 5.9.

The forbidden regions of the motion of the infinitesimal orbits of the Lyapunov families (A, B, and C) in the C, x_0 plane are presented in Figure 5.10. But for the change in intersections

for family C at two points marked as 3 and 5 in the figure, we have shown that each Lyapunov family goes to collision.

In Figures 5.11 – 5.16, we show the horizontal and vertical stability curves for the Lyapunov families in the vicinity of the collinear equilibrium points. In subsection 5.2.2, we presented numerical results for 3D periodic orbits. In Figure 5.17, we presented graphically, a 3D periodic orbit obtained from the semi-analytical solution of (3.37).

In Tables 5.9, 5.10 and 5.11, we presented initial conditions for all the computed families of three-dimensional periodic orbits emanating from the collinear equilibrium points, for cases 1 to 10 of Table 5.1. In each table, we gave the values of x_0 , $\dot{y}_0$ and $\dot{z}_0$ (since $y_0 = z_0 = \dot{x}_0 = 0$ for double symmetry w.r.t. the Ox-axis and Oxz-plane), the value of the Jacobi constant C , the quarter of the orbit's period $T/4$ and the stability parameters P and Q . In Figures 5.20 - 5.22, we showed the evolution of the characteristic curves of the respective families in the space of initial conditions $(x_0, \dot{y}_0, \dot{z}_0)$.

In section 5.3, we presented the numerical considerations for the analytic approximation of Lyapunov orbits around the collinear equilibrium points in the R3BP when the primaries are radiating and oblate spheroidal bodies.

In subsection 5.3.1, we captured the numerical applications for the coplanar case. The theory was applied to the binary α -Centuari system. The infinitesimal body was taken to be a possible exoplanet with negligible mass as compared to the masses of the primaries moving in the plane of motion of the binary system. The assumptions made here were that both stars have significant radiation effects and are sufficiently oblate in shape. Arbitrary chosen values for the radiation and oblateness coefficients have been drawn in six instances (cases). In the first case, the binary system is considered with respect to the classical problem (spherical

nature of the primaries) while the other five cases have values for all the parameters concerned. In Table 5.11, we have shown six cases and their corresponding collinear equilibrium points for all the participating parameters. The effect of the oblateness and radiation pressure of the binary α -Centuari system on the collinear equilibrium points have also been shown in Figures 5.26 and 5.27 respectively for case two. From the third order approximation solution of Eqns. 4.16, we showed graphically the starting orbits of cases 1-6 in Figure 5.28. In Tables 5.12, 5.13 and 5.14, we presented the initial conditions for the Lyapunov orbits around the collinear equilibrium point L_1 , L_2 and L_3 respectively for binary α -Centuari system mass parameter $\mu = 0.47333$.

In subsection 5.4.1, we presented the numerical applications for the spatial case. The applications were made to five binary systems. They are binary α -Centuari, Luhman 16, Sirius, Luyten 726-8, and Procyon systems. In Table 5.15, we presented the physical parameters of the five binary systems. By solving Eqn. (4.20) numerically, we obtained the collinear equilibrium points for the five binary systems and showed this in Table 5.16. Also, we showed graphically the effects of the oblateness and radiation parameters of each of the five binary systems on the collinear equilibrium points in Figures 5.29-5.38. The roots of Equation (4.25) were presented in Table 5.17 for each of the five binary systems. From system (4.32) we obtained the initial conditions (for $t=0$) of a periodic orbit in the form:

$(\mathfrak{I}, \mathfrak{R}, \mathfrak{N}, \dot{\mathfrak{I}}, \dot{\mathfrak{R}}, \dot{\mathfrak{N}})|_{t=0} = x_0 + (\alpha_{11} + \alpha_{12})e^2, 0, 0, 0, 2\omega e^2 \beta_{22}, \omega e$, while the value of the orbital parameter e was arbitrarily chosen. The results were presented in Tables 5.18-5.20.

In subsection 5.4.2, the results and discussion for the numerical approximation of the periodic solutions were presented. In Figure 5.39 the validity of the analytic approximate solution was established in all the projection of the space of initial conditions. In Figures 5.40-5.42, we

presented the characteristic curves of the families of 3D doubly symmetric periodic orbits emanating from all the collinear equilibrium points of the α -Centuray binary system and their corresponding plot for figure eight member orbit for each family in the physical space were shown in Figures 5.43-5.45. The stability diagrams of the corresponding families were captured in Figures 5.46-5.48.

In section 5.5, the numerical considerations for the periodic solutions around the collinear equilibrium points when the primaries are both radiating and triaxial spheroids together with perturbations in the Coriolis and centrifugal forces are presented. Numerical simulations have been done using three binary systems, HD 191408, Kruger 60 and HD 155876.

The physical parameters of the three binary systems have been presented in Table 5.21. The positions of the collinear equilibrium points for each system were determined numerically and given in Table 5.22. In Figures 5.50-5.58, the effects of the Triaxiality of the primary and secondary stars respectively, and the effects of the radiation factors on each binary system are shown graphically.

In Table 5.23, the roots of the characteristic equation (4.52) were used to determine the stability of the collinear equilibrium points in each binary system. From the results presented in Equations (4.57), we obtained the infinitesimal or starting orbits for the 2D periodic orbits when the primaries are both radiating and triaxial spheroids together with perturbations in the Coriolis and centrifugal forces as presented in the first entries of Tables 5.25-5.33. The initial conditions $(x_0, \dot{y}, \frac{T}{2})$ are given in the second, third and fourth columns while the corresponding Jacobian constant has been captured in the fifth column of the tables.

Also, from Equations (4.63), we obtain the initial conditions $(x_0, \dot{y}, \dot{z}, \frac{T}{4})$ for the 3D periodic orbits when the primaries are both radiating and triaxial spheroids together with perturbations in the Coriolis and centrifugal forces and present them in Tables 5.34-5.42. The initial conditions have been given in the the second, third, fourth and fifth columns respectively, of the tables, while the corresponding Jacobian constant is given in column six. The Corrector-Predictor algorithm was used to continue to families (A, B, C) of 3D periodic orbits emanating from the collinear equilibrium points. The stability of the 3D periodic orbits was also examined.

6.3 Conclusion

A generalization of the restricted three-body problem where in the first problem the larger primary is a triaxial rigid body and the smaller one is an oblate body together with small perturbations in the Coriolis and centrifugal forces and in the second problem both primaries were taken to be radiating oblate spheroidal bodies was considered.

Specifically, for the first problem, the effects of triaxiality and oblateness together with small perturbations in the Coriolis and centrifugal forces were investigated to the positions of the collinear equilibrium points. In ten studied cases of the parameters of the problem, it was found that the positions of the collinear equilibrium points L_2 and L_3 move towards the origin with the increase of the values of these parameters while the equilibrium point L_1 moves away from the origin.

The infinitesimal periodic orbits around the collinear equilibrium points were also determined. To this purpose, semi-analytical solutions were constructed in both two- and three-dimensional cases. In addition, the numerical continuation of the families of planar and

spatial periodic orbits emanating from the collinear equilibria was established. All the computed families of two-dimensional periodic orbits go to collision orbits with the primaries while the corresponding families of three-dimensional periodic orbits terminate at bifurcations with planar periodic orbits. It was found that only the Lyapunov families emanating from the equilibria L_1 and L_3 may contain horizontally stable orbits but all these families may contain parts with vertically stable orbits. In the three-dimensional case, only the corresponding family emanating from L_3 may consist of stable periodic orbit.

In the second problem, a third order analytic approximation of Lyapunov orbits around the collinear equilibrium points in the PCR3BP by using the Lindstedt-Poincaré method was obtained. The primaries were modelled in the binary α -Centuari system where the primary body is α -Centuari A and the secondary body is α -Centuari B. The infinitesimal body is taken to be a possible exoplanet moving in the plane of motion of the binary system.

Also, we numerically determined the positions of the collinear equilibrium points and showed the effects of the parameters concerned on these points. It can be seen that with each increase in the radiation pressure and oblateness parameters, the first collinear point L_1 moves away from the origin and closer to the position of that of the classical case and also closer to the secondary body, while the second collinear equilibrium point (inner collinear point) L_2 moves toward the origin and closer to that of the classical case. The third collinear equilibrium point L_3 moves further away from the primary body and approaches the classical case.

The initial conditions or starting orbits obtained in all the cases are shown in both tabular and graphical forms and the orbits are oval in shape. These results can be continued to families of periodic orbits and combined with those of the spatial orbits.

The third problem comprise of the spatial case of the model of the second problem. The theory behind the spatial case was applied to five binary systems: Sirius, Procyon, Luhman 16, α -Centuari and Luyten 726-8. The physical parameters of each binary system were obtained and used to calculate their respective mass parameters and mass reduction factors. The number and positions of the collinear equilibrium points of each binary system were obtained numerically by combining the topological degree theory and the numerical determination of roots of non-linear algebraic equations. The effect of the respective oblateness and radiation coefficients on the collinear equilibrium points of the binary Sirius system were shown graphically. The linear stability of the collinear equilibrium points of the five binary systems was also examined and found to be unstable in the Lyapunov sense just like in the classical case.

An analytical approximation solution of three-dimensional periodic orbits around the collinear points was obtained by utilizing the Lindstedt-Poincaré method. The analytical solution was used in order to find suitable starting points for the numerical computation of the respective families of three-dimensional periodic orbits about the collinear points. The stability of the obtained periodic orbits for the binary α -Centuari system was also examined. Our results showed that only families emanating from L_2 and L_3 contain stable parts. Finally, we found that all the computed families, in their evolution comprised by member orbits which have decreasing period. Our termination criterion for their continuation was that the value of the full period of an orbit will be less than the value 0.0005. Using this criterion, none of the determined families terminates at the physical plane which is not the usual termination of this kind of families in the restricted problem where they all go to a bifurcation with a coplanar periodic orbit, such as in the study by Tsirogiannis et al (2006) and Singh et al (2016). The choice to take the primaries to be sufficiently oblate bodies seems to be the

reason where we obtain the different result in the evolution of three-dimensional periodic orbits emanating from the collinear equilibrium points of the binary α -Centuari system.

The fourth problem models the primaries as radiating and triaxial spheroids together with perturbations in the Coriolis and centrifugal forces and 2D and 3D periodic solutions around the collinear equilibrium points were investigated. Firstly, the positions and stability of the collinear equilibria are studied for the HD 191408, Kruger 60 and HD 155876 binary systems. Then, the planar and three-dimensional periodic motion about these points is considered. Our study includes both semi-analytical and numerical determination of these motions. It is found that all families of planar periodic orbits emanating from these points terminate with asymptotic periodic orbits at the triangular equilibrium points and contain stable parts, i.e. there are intervals at which the Lyapunov families are simultaneously horizontally and vertically stable.

The corresponding families of three-dimensional periodic orbits terminate with planar periodic orbits and have been shown to be unstable except for family A emanating from L_2 equilibrium point where stability holds for some of the members. This particular stability behaviour was exhibited by the three binary systems.

6.3 Recommendation

Other extensions of this research work can include the determination of series of vertically and horizontally critical symmetric periodic orbits for appropriate modifications of the classical restricted three-body problems and the restricted four-body problem as well.

6.4 Contribution to Knowledge

1. Periodic solutions around the collinear equilibrium points of the R3BP when the primary is a triaxial rigid body and the secondary is an oblate spheroid together with perturbations in the Coriolis and centrifugal forces for the planar case as well as the spatial case were obtained by applying both the Linstedt-Poincaré method and the predictor-corrector method.
2. This research work has utilized the third order Linstedt-Poincaré method to obtain approximate periodic solutions in the PCR3BP with application to the binary α -Centuary system when the primaries are oblate spheroidal bodies.
3. This research work has combined the second order Linstedt-Poincaré and the differential correction methods to obtain the periodic solutions in the spatial R3BP with application to the five binary systems when the primaries are radiating oblate spheroidal bodies.
4. This research work has utilized both the second order Linstedt-Poincaré local analysis and the Corrector-Predictor algorithm to investigate 2D and 3D periodic orbits around the collinear equilibrium points in the perturbed R3BP with application to the three binary systems when the primaries are radiating and triaxial spheroidal bodies.

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