A STUDY OF *M*-FUZZY SUBGROUPS AND ITS LEVEL *M*-SUBGROUPS

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FACULTY OF PHYSICAL SCIENCE,

DEPARTMENT OF MATHEMATICS,

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DECLARATION

I declare that the work in this dissertation titled A STUDY OF M-FUZZY SUBGROUPS AND ITS LEVEL M-SUBGROUPS has been performed by me in the Department of Mathematics under the supervision of Prof. Y. Tella. The information derived from literature has been duly acknowledged in the text and a list of reference provided. No part of this dissertation was previously presented for another degree or diploma at any University or Institution.

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CERTIFACATION

The dissertation titled A STUDY OF M-FUZZY SUBGROUPS AND ITS LEVEL M-SUBGROUPS by ABUBAKAR MOHAMMED (MSC/SCI/41934/2012-2013) meets the regulations governing the award of the degree of Master of Science of Ahmadu Bello University, Zaria and is approved for its contribution to knowledge and literary presentation.

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DEDICATION

This dissertation is dedicated to Almighty ALLAH Whose Grace and Mercy made the start, the progress and the completion of this work was possible.

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ABSTRACT

The theory of fuzzy set has been studied extensively in mathematics along with its application in diverse fields. Rosenfeld in 1971 used this concept to develop the theory of fuzzy groups. Gu in 1994 put forward the notion of M-fuzzy groups. In this research work, we review some of the fundamental works in M-fuzzy group theory and provide some new or alternative methods to proving some existing theorems in M-fuzzy group theory. We have given independent proof of several theorems on Level M- subgroups of M-fuzzy subgroups.

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CHAPTER ONE

INTRODUCTION

1.1 Background to the Study

The fundamental concept of fuzzy sets was introduced by Zadeh in 1965 to represents information possessing non-statistical uncertainties. The fuzzy algebraic structures play a prominent role in mathematics with wide applications in many other branches such as theoretical physics, computer science, control engineering, information science, coding theory, group theory, real analysis, measure theory etc.

The first publication in fuzzy set theory by Zadeh (1965) and then by Klaua (1965) showed the intention of the authors to generalize the classical set. In classical set theory, a subset A of a set X can be defined by its characteristic function $\chi_A : X \to \{0, 1\}$ which is defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

The mapping may be represented as a set of ordered pairs $\{(x, \chi_A(x))\}$ with exactly one ordered pair present for each element of X. The first element of the ordered pair is an element of the set X and the second is its value in $\{0, 1\}$ under χ_A. The value '0' is used to represent non-membership and the value '1' is used to represent membership of the element in A. The truth or falsity of the statement "x is in A" is determined by the ordered pair. The statement is true, if the second element of the ordered pair is '1', and the statement is false, if it is '0'.

Fuzzy set theory is an extension of classical set theory where elements have varying degrees of membership. A logic based on the two truth values, True and False, is sometimes inadequate when describing human reasoning. Fuzzy logic uses the whole interval between 0 (false) and 1(true) to describe human reasoning.

A fuzzy set A in X is characterized by a membership function $\mu_A(x)$ which associates with each point in X a real number in the interval $[0,1]$, with the value of $\mu_A(x)$ at x representing the "grade of membership" of x in A.

In 1971, Rosenfeld first introduced the concept of fuzzy subgroups, which was the first fuzzification of any algebraic structure and shows that many results in group theory can be extended in an elementary manner to develop the theory of fuzzy group. Gu *et al* (1994) studied the theory of fuzzy groups and developed the concept of M-fuzzy groups. This dissertation is an attempt to study M-fuzzy subgroups and its level M-subgroups.

1.2 Statement of the Research problem

After the introduction of the notion of M-fuzzy subgroup by Gu (1994), several researches were conducted using this notion. Many algebraic structures of fuzzy subgroup with operators have been developed so far; however, there is no author that uses this alternative method of proving theorems in fuzzy subgroup with operators (i.e. M-fuzzy subgroup). Since there is no author that uses this alternative method, this gives us a room to provide new proofs of some existing theorems in M-fuzzy subgroup and to obtained independent proof of several theorems on Level M- subgroups of M-fuzzy subgroups.

1.3 Justification

Many algebraic structures such as Groups, Monoids, Semigroups, Quasigroups, Ring, Semirings, Lattice, Semilattice, Boolean algebra etc. were developed using set as their underlying structure. Since fuzzy set is a generalization of classical set, various algebras based on fuzzy set could be developed such as Rosenfeld in 1971 used this concept to develop the theory of fuzzy groups and Gu in 1994 also used the concept to put forward the notion of M-fuzzy groups. In this research work, we reviewed the concepts of $\boldsymbol{M}$-fuzzy group theory and come up with a new method to develop the concept M-fuzzy subgroup and their Level $\boldsymbol{M}$-subgroups.

1.4 Aim and Objectives of the Dissertation

The aim of this research work is to study an algebraic structure called M-fuzzy subgroup and its level M-subgroup. The objectives are to:

1. Review some of the fundamental works done in M-fuzzy subgroup theory;
2. Provide some new or alternative methods of proving some existing theorems in M-fuzzy subgroup theory;
3. Provide proofs of some theorems on M-fuzzy subgroup theory which, to the very best of our knowledge, do not exist in the literature;
4. Obtain independent proof of several theorems on Level M- subgroups of M-fuzzy subgroups.

1.5 Research Methodology

The method of research adopted in this dissertation is by consulting necessary and relevant literature (papers) on the theory of M-fuzzy subgroups. These consultations make it

possible to develop new or alternative methods of proving some existing theorems in M-fuzzy subgroup theory and provide some independent proof of several theorems on Level M- subgroups of M-fuzzy subgroups..

1.6 Organization of the Dissertation

This dissertation contains four chapters after this introductory chapter. The organizations of the remaining chapters are as follows:

Chapter two: In this chapter, we present a survey of the necessary and relevant literature for the fuzzy sets and fuzzy subgroups.

Chapter three: In this chapter, Basic definitions required for the dissertation and fundamentals of fuzzy subgroups are presented.

Chapter four: In this chapter, we focus our attention on the study of M-fuzzy subgroups and its level M-subgroups.

Chapter five: In this chapter, we present our concluding remarks. In the end, a list of references cited in the dissertation is presented.

1.7 Preliminaries

We start by presenting the basic concepts of fuzzy sets theory needed for understanding the result of this dissertation. All definitions and results are typical and can be found in any introductory text on fuzzy sets theory

Definition 1.7.1: (Fuzzy sets and membership function)

If X is a collection of objects denoted generically by x, then a **fuzzy set** $\boldsymbol{A}$ in X is defined as a set of ordered pairs $A = \{(x, \mu_A(x)) | \, x \in X)\}$, where $\mu_A : X \longmapsto [0,1]$ is called the **membership function** (or MF for short) for the fuzzy set A. A fuzzy set expresses the

degree to which an element belongs to a set. Hence the MF maps each element of X to a membership grade (or membership value) in the interval [0, 1].

Example 1.7.2: Let $X = \{a, b. c\}$ be a set. Then $A = \{(a, 0.2), (b, 0.5), (c, 0.8)\}$ is a fuzzy set in X.

Definition 1.7.3: (subset of fuzzy set)

Let $\boldsymbol{A}$ **and** $\boldsymbol{B}$ be a fuzzy sets. If $\boldsymbol{\mu_A(x) \leq \mu_B(x) \forall x \in X}$, then $\boldsymbol{A}$ is said to be contained in $\boldsymbol{B}$ (or $\boldsymbol{B}$ contains $\boldsymbol{A}$), and we write $\boldsymbol{A \subseteq B}$ ($\boldsymbol{or\ B \supseteq A}$). If $\boldsymbol{A \subseteq B}$ and $\boldsymbol{A \neq B}$, then $\boldsymbol{A}$ is said to be properly contained in $\boldsymbol{B}$ (or $\boldsymbol{B}$ properly contains $\boldsymbol{A}$) and we write $\boldsymbol{A \subset B}$ (or $\boldsymbol{B \supset A}$).

Example 1.7.4: let $X = \{a, b, c\}$ be a universal set.

$A = \{(a, 0.5), (b, 1.0), (c, 0.5)\}$ and $B = \{(a, 1.0), (b, 1.0), (c, 0.5)\}$ are fuzzy sets in X. $A \subseteq X,\ B \subseteq X$. Then $A \subseteq B \Longrightarrow \mu_A(x) \leq \mu_B(x) \forall x \in X$.

Definition 1.7.5: (equal fuzzy sets)

Two fuzzy sets A and B are said to be equal if $\mu_A(x) = \mu_B(x) \forall x \in X$ and it is denoted by $A = B$.

Example 1.7.6: let $X = \{a, b, c\}$ be a universal set.

$A = \{(a, 0.5), (b, 1.0), (c, 0.5)\}$ and $B = \{(a, 0.5), (b, 1.0), (c, 0.5)\}$ are fuzzy sets of X. Then $A = B$.

Definition 1.7.7: ($\boldsymbol{\alpha - cut\ set\ or\ \alpha - level\ set}$)

Let A be a fuzzy sets. For $\alpha \in [0, 1]$, define A_α as follows:

$$A_\alpha = \{x | x \in X, \mu_A(x) \geq \alpha\}.$$

A_α is called the $\alpha - cut\ set (or\ \alpha - level\ set)$ of A.

Example 1.7.8: Let $X = \{1, 2, 3, 4 \ldots 10\}$ be a set. Then

$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$ is a fuzzy set in X.

The possible $\alpha -$ level sets are: $A_{0.2} = \{1, 2, 3, 4, 5, 6\}$

$$A_{0.5} = \{2, 3, 4, 5\}$$

$$A_{0.8} = \{3, 4\}$$

$$A_1 = \{4\}$$

Definition 1.7.9: (Support of a fuzzy set)

Let A be a fuzzy set. The set $A^* = \{x | x \in X, \mu_A(x) > 0\}$, is called the **support *A***. In particular, A is called a **finite fuzzy subset** if A^* is a finite set, and an **infinite fuzzy subset** otherwise.

Example 1.7.10: Let $X = \{1, 2, 3, 4, \ldots, 10\}$ be the set. Then the fuzzy set A may be described as

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

The support of A is: $A^* = \{1, 2, 3, 4, 5, 6\}$.

The elements $\{7, 8, 9, 10\}$ are not part of the support of A.

Definition 1.7.11: (complement)

Let A be a fuzzy sets. Then the complement A^C is defined by the equation

$$\mu_{A^C}(x) = 1 - \mu_A(x), \quad x \in X$$

Example 1.7.12: Let $X = \{1, 2, 3, 4, \ldots, 10\}$ be the set. Then the fuzzy set A may be described as

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

The complement $A^C = \{(1, 0.8), (2, 0.5), (3, 0.2), (4, 0), (5, 0.3), (6, 0.7)\}$

Definition 1.7.13 (Union)

Let A and B be a fuzzy sets. Then the union $A \cup B$ is defined by the equation.

$$\mu_{A \cup B}(x) = [\mu_A(x) \vee \mu_B(x)] \; \forall \; x \in X$$

Where, $\mu_A(x) \vee \mu_B(x) = max\{\mu_A(x), \mu_B(x)\}$.

Example 1.7.14: Let $X = \{1, 2, 3, 4, \ldots, 10\}$ be the set. Then the fuzzy sets A and B may be described as

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

$$B = \{(3, 0.2), (4, 0.4), (5, 0.6), (6, 0.8), (7, 1), (8, 1)\}$$

Then, the union $A \cup B$ is given by:

$$A \cup B = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.8), (7, 1), (8, 1)\}$$

Definition 1.7.15: (Fuzzy Intersection)

Let A and B be a fuzzy sets. Then the intersection $A \cap B$ defined by the equation.

$$\mu_{A \cap B}(x) = [\mu_A(x) \wedge \mu_B(x)] \; \forall \; x \in X$$

Where, $\mu_A(x) \wedge \mu_B(x) = \min\{\mu_A(x), \mu_B(x)\}$

Example 1.7.16: Let $X = \{1, 2, 3, 4, \ldots, 10\}$ be the set. Then the fuzzy sets A and B may be described as

$$A = \{(1, 0.2), (2, 0.5), (3, 0.8), (4, 1), (5, 0.7), (6, 0.3)\}$$

$$B = \{(3, 0.2), (4, 0.4), (5, 0.6), (6, 0.8), (7, 1), (8, 1)\}$$

The intersection $A \cap B = \{(3, 0.2), (4, 0.4), (5, 0.6), (6, 0.3)\}$.

Remark 1.7.16: These definitions can be generated for countable number of fuzzy sets. For any collection, $\{A_i | \ i \in I\}$, of fuzzy subsets of X, where I is a nonempty index set with membership functions $\{\mu_{A_i} | i \in I\}$, then the membership functions of $\cup_{i\in I} A_i$ and $\cap_{i\in I} A_i$ of the $A_i's$ are given by $\forall x \in X$, $(\cup_{i\in I} \mu_{A_i})(x) = \vee_{i\in I} \mu_{A_i}(x)$ and $(\cap_{i\in I} \mu_{A_i})(x) = \wedge_{i\in I} \mu_{A_i}(x)$, respectively.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction

The study of fuzzy groups was started firstly by Rosenfeld (1971). Rosenfeld used the *min* operation to define his fuzzy groups and showed that many results in group theory can be extended in an elementary manner to develop the theory of fuzzy group.

Das (1981) obtain a similar characterization of all fuzzy subgroups of finite cyclic groups and study what are called "level subgroups" of a fuzzy subgroup in the first part of the paper. These level subgroups in turn play an important role in the above characterization.

N. Jacobson (1951) introduced the concept of M-group, M-subgroup.

Gu *et al* (1994) studied the theory of fuzzy groups and developed the concept of M-fuzzy groups.

Kundu (1998) presented a counter example to recent result on truly closed level subgroup $G[t] = \{x|\mu(x) \geq t\}$ of a fuzzy group G and proves a correct form of that result.

Mordeson *et al* (2005) presented a book on fuzzy group theory, where they define the notion of a fuzzy subgroup and examine its properties; they also introduced some operations on a fuzzy subset of a group *G* in terms of the group operation.

Solairaju and Nagarajan (2010) discussed some structure properties of M-fuzzy groups.

Muthuraj *et al* (2010) introduced the concept of M-homomorphism and M-anti homomorphism of an M-fuzzy subgroups.

Sundararajan and Muthuraj (2011) introduced the concept of an anti M-fuzzy subgroup of an M-group and lower level subset of an anti M-fuzzy subgroup and discussed some of its properties.

Mourad and Massa'deh (2012) studied the theory of fuzzy subgroups and discussed some concepts such as fuzzy subgroups with operator, normal fuzzy subgroups with operator, homomorphism with operator, etc, while some elementary properties were discussed, such as intersection operation, the image and inverse image of fuzzy subgroups with operator.

Subramanian *et al* (2012) gave an independent proof of several theorems on M- fuzzy groups. They discussed M- fuzzy groups and investigated some of their structures on the concept of M- fuzzy group family.

Kamble and Venkatesh (2014) made a brief survey on some fuzzy algebraic structures and investigated the properties of order of a fuzzy group, solvable fuzzy group, M-fuzzy groups and fuzzy G-modules.

CHAPTER THREE

FUNDAMENTALS OF FUZZY SUBGROUP

3.1 Introduction

Let *G* denotes an arbitrary group with a multiplicative binary operation and identity *e*. In order to define the notion of a fuzzy subgroup and to examine its properties, some operations on a fuzzy subset of a group *G* in terms of the group operation are introduced. Also, the fundamental definitions that will be used in the sequel are sited.

3.2 Concept of Fuzzy Subgroup

Definition 3.2.1: (Rosenfeld, 1971)

Let G be a group. A fuzzy subset A of a group G is called a fuzzy subgroup of the group G if

(1) $\mu_A(xy) \geq [\mu_A(x) \wedge \mu_A(y)],\ \forall\, x, y \in G;$

(2) $\mu_A(x^{-1}) = \mu_A(x),\ \forall\, x \in G.$

Example 3.2.2: Let $G = \{e, a, b, c\}$ be a Klein four group and $A = \{(e, 1), (a, 0.5), (b, 0.4), (c, 0.4)\}$ be a fuzzy subset of G. Now

$\mu_A(ea) = \mu_A(a) = 0.5 \geq [\mu_A(e) \wedge \mu_A(a)],\ \ \mu_A(eb) = \mu_A(b) = 0.4 \geq [\mu_A(e) \wedge \mu_A(b)],$

$\mu_A(ec) = \mu_A(c) = 0.4 \geq [\mu_A(e) \wedge \mu_A(c)],\ \ \mu_A(bc) = \mu_A(a) = 0.5 \geq [\mu_A(b) \wedge \mu_A(c\ )],$

$\mu_A(ac) = \mu_A(b) = 0.4 \geq [\mu_A(a) \wedge \mu_A(c)],\ \ \mu_A(ab) = \mu_A(c) = 0.4 \geq [\mu_A(a) \wedge \mu_A(b)],$

$\mu_A(a^2) = \mu_A(e) = 1 \geq [\mu_A(a) \wedge \mu_A(a)],\ \ \mu_A(b^2) = \mu_A(e) = 1 \geq [\mu_A(b) \wedge \mu_A(b)],$

$\mu_A(c^2) = \mu_A(e) = 1 \geq [\mu_A(c) \wedge \mu_A(c)],\ \mu_A(e^2) = \mu_A(e) = 1 \geq [\mu_A(e) \wedge \mu_A(e)]$ and

$\mu_A(a^{-1}) = \mu_A(a) = 0.5, \mu_A(b^{-1}) = \mu_A(b) = 0.4, \mu_A(c^{-1}) = \mu_A(c) = 0.4,$

$\mu_A(e^{-1}) = \mu_A(e) = 1.$

Therefore A is a fuzzy subgroup.

Note: If A is a fuzzy subgroup of G, we let $A_* = \{x \in G | \mu_A(x) = \mu_A(e)\}$ and recall from **Definition 1.7.9** that A^* denotes the support of A. If fuzzy subset A of G satisfies condition (1) of **Definition 3.2.1**, then

$$\mu_A(x^n) \geq \mu_A(x) \; \forall x \in G,$$

where $n \in N$ (the set of non-negative numbers). Also, A satisfies conditions (1) and (2) of **Definition 3.2.1** if and only if $\mu_A(xy^{-1}) \geq [\mu_A(x) \wedge \mu_A(y)] \; \forall x, y \in G.$

3.3 Properties of Fuzzy Subgroup

Theorem 3.3.1: (Rosenfeld, 1971)

Let A be a fuzzy subgroup of G. Then

a) $\mu_A(e) \geq \mu_A(x) \; \forall \; x \in G,$

b) $\mu_A(x^n) \geq \mu_A(x) \; \forall x \in G,$

c) $\mu_A(xy) = [\mu_A(x) \wedge \mu_A(y)] \; \forall x, y \in G.$

Theorem 3.3.2: (Rosenfeld, 1971)

A fuzzy subset A of a group G is a fuzzy subgroup of the group G if and only if $\mu_A(xy^{-1}) \geq [\mu_A(x) \wedge \mu_A(y)], \forall \; x, y \in G.$

Theorem 3.3.3: (Rosenfeld, 1971)

If A is a fuzzy subgroup of G, then A_* is a subgroup of G.

Theorem 3.3.4: (Rosenfeld, 1971)

If A is a fuzzy subgroup of G, then A^* is a subgroup of G.

3.4 Composition and Inverse of Fuzzy Subgroup

Definition 3.4.1: Let A and B be a fuzzy subsets of G. The binary operation ‘’$\circ$’’ is defined on the fuzzy subsets of G as follows:

$$\mu_{A\circ B}(x) = \vee\{\mu_A(y) \wedge \mu_B(z) \mid y, z \in G, yz = x, \forall x \in G\}.$$

Definition 3.4.2: Let A be a fuzzy subset of G. Then A^{-1} is defined as

$$\mu_{A^{-1}}(x) = \mu_A(x^{-1}), \forall x \in G.$$

Theorem 3.4.3: Let A, B and A_i be a fuzzy subsets of $G, i \in I$. Then the following assertions hold:

a) $\mu_{(A\circ B)}(x) = \vee_{y\in G}\left(\mu_A(y) \wedge \mu_B(y^{-1}x)\right) \forall x \in G,$

$$= \vee_{y\in G}\left(\mu_A(xy^{-1}) \wedge \mu_B(y)\right) \forall x \in G.$$

b) $A^{-1} = \mathrm{A}$;

c) $(A^{-1})^{-1} = A$;

d) $A \subseteq B \Leftrightarrow A^{-1} \subseteq B^{-1}$;

e) $(\cup_{i\in I} A_i)^{-1} = \cup_{i\in I} A_i^{-1}$;

f) $(\cap_{i\in I} A_i)^{-1} = \cap_{i\in I} A_i^{-1}$;

g) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$;

h) $A \circ A \subseteq A$

i) $(A \circ B) \circ C = A \circ (B \circ C)$.

3.5 Some Results on Algebraic Operations on Fuzzy Subgroups

Theorem 3.5.1: (Rosenfeld, 1971)

Let A and B be any two fuzzy subgroup of G. Then $A \cap B$ is also a fuzzy subgroup of G for every $x, y \in G$.

Remark 3.5.2: If $\{A_i | i \in I\}$ is a family of fuzzy subgroup of G, then their intersection $\bigcap_{i\in I} A_i$ is also a fuzzy subgroup of G.

Theorem 3.5.3: (Rosenfeld, 1971)

Let A and B be a fuzzy subgroups of G, then $\mu_{A\cup B}(x^{-1}) = \mu_{A\cup B}(x)$; for every $x \in G$.

Corollary 3.5.4: (Rosenfeld, 1971)

If A and B are fuzzy subgroups of G, Then $(A \cup B)$ need not be a fuzzy subgroup of G.

Example 3.5.5: Let $G = \{e, a, b, c\}$ be a Klein four group and

$A = \{(e, 1), (a, 0.6), (b, 0.4), (c, 0.4)\}$ and $B = \{(e, 1), (a, 0.2), (b, 0.3), (c, 0.2)\}$ are fuzzy subsets of G.

Clearly, $A \cup B = \{(e, 1), (a, 0.6), (b, 0.4), (c, 0.4)\}$ and

$$\mu_{A\cup B}(c) = \mu_{A\cup B}(ab) = 0.4 \ngeq [\mu_A(a) \vee \mu_B(b)] = 0.6.$$

Therefore $(A \cup B)$ is not a fuzzy subgroup of G.

Theorem 3.5.6: (Rosenfeld, 1971)

Let A be a fuzzy subgroup of G and $x \in G$. Then $\mu_A(xy) = \mu_A(y)$ for every $y \in G$ if and only if $\mu_A(x) = \mu_A(e)$.

Theorem 3.5.7: (Rosenfeld, 1971)

Let A be a fuzzy subgroup of G, then $\mu_A(xy^{-1}) = \mu_A(e) \implies \mu_A(x) = \mu_A(y), \forall\, x, y \in G$.

3.6 Level Subset (Subgroup) of the Fuzzy Subset (Subgroup)

Definition 3.6.1: (Das, 1981)

Let A be a fuzzy subset (subgroup) of G. For $t \in [0, 1]$, the set $A_t = \{x \in G | \mu_A(x) \geq t\}$ is called a level subset (subgroup) of the fuzzy subset (subgroup) A.

Example 3.6.2: Let $G = \{e, a, b, c\}$ be a Klein four group and $A = \{(e, 1), (a, 0.5), (b, 0.4), (c, 0.4)\}$, be a fuzzy subset of G.

Now, the level subset of the fuzzy subset A is defined by $A_t = \{x \in G | \mu_A(x) \geq t\}$.

The possible level subset of A for are: $A_{0.4} = \{b, c\}, for\ t = 0.4$;

$$A_{0.5} = \{a\}, for\ t = 0.5;$$

$$A_1 = \{e\}, for\ t = 1;$$

$$A_t = \emptyset, for\ t \geq 2.$$

Theorem 3.6.3 (Das, 1981)

Let G be a group and A be a fuzzy subgroup of G, then the level subset A_t, for $t \in [0, 1]$, is a subgroup of G, where e is the identity of G.

Theorem 3.6.4: (Das, 1981)

Let G be a group and A be a fuzzy subset of G such that A_t is a subgroup of G for all $t \in [0,1]$, then A is a fuzzy subgroup of G.

Theorem 3.6.5: (Das, 1981)

Let G be a group and A be a fuzzy subgroup of G. Two level subgroups A_{t_1}, A_{t_2} (with $t_1 < t_2$) of A are equal if and only if there is no $x \in G$ such that $t_1 < \mu_A(x) < t_2$.

Theorem 3.6.6: (Das, 1981)

Any subgroup H of a group G can be realised as a level subgroup of some fuzzy subgroup of G.

3.7 Normal Fuzzy Subgroups

The notion of the normal subgroup is one of the central concepts of classical group theory. It serves a powerful instrument for studying the general structure of groups. Just as a normal subgroup plays an important role in the classical group theory, a normal fuzzy subgroup plays a similar role in the theory of fuzzy subgroup.

Definition 3.7.1: Let G be a group. A fuzzy subgroup A of G is called normal fuzzy subgroup of G if

$$\mu_A(x^{-1}yx) \geq \mu_A(y) \text{ for all } x, y \in G.$$

Theorem 3.7.2: If A is a fuzzy subgroup of a group G, then the following conditions are equivalent:

i. $\mu_A(xy) = \mu_A(yx)$ for all $x, y \in G$;

ii. $\mu_A(x^{-1}yx) = \mu_A(y)$ for every $x, y \in G$.

Proof: Suppose that A is a normal fuzzy subgroup of G, then

(i) ⇒ (ii) $\mu(x^{-1}yx) = \mu(yxx^{-1}) = \mu(ye) = \mu(y)$ for all $x, y \in G$.

(ii) ⇒ (i) $\mu(xy) = \mu(xyxx^{-1}) = \mu(x(yx)x^{-1}) = \mu(yx)$ $for\ all\ x, y \in G$.

Hence μ is a normal fuzzy subgroup of G.

Theorem 3.7.3: Let A be fuzzy subset of G. Then A is a normal fuzzy subgroup of G if and only if A_t is a normal subgroup of G for every $t \in [0, 1]$.

Theorem 3.7.4: Let A be a normal fuzzy subgroup of G, then A_* and A^* are normal subgroup of G.

Theorem 3.7.5: The intersection of any two normal fuzzy subgroups of G is also a normal fuzzy subgroup of G.

Theorem 3.7.6: The union of any two normal fuzzy subgroups of G is also a normal fuzzy subgroup of G.

3.8 Order of Fuzzy Subgroup

Definition 3.8.1: (Kamble, 2014)

Let A be a fuzzy subgroup of G. The least positive integer n such that $\mu_A(x^n) = \mu_A(e)$ for all $x \in G$, is called the fuzzy **order of** x with respect to A**.** If no such n exists, x is said to have an infinite fuzzy order with respect to A. The fuzzy order of x with respect to A is denoted by $FO_A(x)$.

Example 3.8.2: (Kamble, 2014)

Let $G = \{1, -1,\ i, -i\}$ and A is a fuzzy subgroup on G of finite order. Then

$\mu_A((-1)^2) = \mu_A(1) \quad \therefore FO_A(-1) = 2$

$\mu_A(i^4) = \mu_A(1) \quad \therefore FO_A(i) = 4$

$\mu_A((-i)^4) = \mu_A(1) \quad \therefore FO_A(-i) = 4.$

Definition 3.8.3: (Kamble, 2014)

Let A be a fuzzy subgroup of G. The least positive integer n such that $\mu_A(x^n) = \mu_A(e)$ for all $x \in G$, is called the **order of** A, denoted by $O(A)$**.** If no such n exists, A is said to have an infinite order.

Example 3.8.4: (Kamble, 2014)

Let $G = \{1, -1,\ i, -i\}$ and A is a fuzzy subgroup on G of finite order. Then

$\mu_A((-1)^2) = \mu_A(1),\ \mu_A(i^4) = \mu_A(1),\ \mu_A((-i)^4) = \mu_A(1) \quad \therefore O(A) = 4.$

Theorem 3.8.5: (Kamble, 2014)

Let A be a fuzzy subgroup of G and $x \in G$. If $\mu_A(x^m) = \mu_A(e)$, for some positive $m \in Z$, then $FO_A(x)|m$

Definition 3.8.6: (Kamble, 2014)

Let A be a fuzzy subgroup of a group G and

$H = \{x \in G|\mu_A(x) = \mu_A(e)\}$ then $O(A)$, order of A is defined as $O(A) = O(H)$.

Theorem 3.8.7: (Kamble, 2014)

Let A be an improper (i.e. constant) fuzzy subgroup of G then the order of $O(A)|O(\mathrm{G})$.

Remark 3.8.8: (Kamble, 2014)

If H is a subgroup of a group G and A is a fuzzy subgroup of G. Then it is obvious that, A is restricted to H, denoted as A_H is a fuzzy group on H.

Theorem 3.8.9: (Kamble, 2014)

If H is a subgroup of a group G and A is a fuzzy subgroup of G. then,

$$O(A_H) \leq O(A).$$

Remark 3.8.10: (Kamble, 2014)

The index of A means: "The number of distinct fuzzy cosets of A".

Theorem 3.8.11: (Kamble, 2014)

Let A be a fuzzy subgroup of a finite group G, then the index of A divides the order of G.

Theorem 3.8.12: (Kamble, 2014)

Let H be a subgroup of a group G and A be a fuzzy subgroup of G of finite order. Then, $O(A_H)|O(A)$.

Example 3.8.13: Consider the subgroup $H = \{1, -1\}$ of $G = \{1, -1, i, -i\}$. Define A on G as in example: 3.8.2. Then $O(A) = 4$ and $O(A_H) = 2$. Hence $O(A_H) | O(A)$.

Corollary 3.8.14: (Kamble, 2014)

If H is a subgroup of a group G, $x \in G$ and A is a fuzzy subgroup on G of finite order, then $FO_A(x) | O(A)$.

3.9 Extended Theorems from Group Theory

Proposition 3.9.1: Let A be a fuzzy subgroup of $G, \forall\, x, y \in G$. Then $\mu_A(xy) = \mu_A(e)$ for every $y \in G$ if and only if $\mu_A(y) = \mu_A(x^{-1})$.

Proof: $\mu_A(y) = \mu_A(ey) = \mu_A((x^{-1}x)y) = \mu_A(x^{-1}(xy)) \geq [\mu_A(x^{-1}) \wedge \mu_A(xy)]$

$\mu_A(y) \geq [\mu_A(x^{-1}) \wedge \mu_A(e)] = \mu_A(x^{-1})$

That is, $\mu_A(y) \geq \mu_A(x^{-1})$.

$\mu_A(x^{-1}) = \mu_A(ex) = \mu_A((y^{-1}y)x) = \mu_A(y^{-1}(yx)) \geq [\mu_A(y^{-1}) \wedge \mu_A(yx)]$

$\mu_A(x^{-1}) = [\mu_A(y) \wedge \mu_A(e)] = \mu_A(y)$

That is, $\mu_A(y) \geq \mu_A(x)$. Hence, $\mu_A(y) = \mu_A(x^{-1})$.

Proposition 3.9.2: Let A be a fuzzy subgroup of G. If $\mu_A(yx) = \mu_A(x^4y)$ and $\mu_A(x^3) = \mu Ae;$ then $\mu Axy = \mu Ayx \; \forall\, x,y \in G,$ is a fuzzy normal subgroup of $G.$

Proof:

$\mu_A(yx) = \mu_A(x^4y) = \mu_A(x^3(xy)) \geq [\mu_A(x^3) \wedge \mu_A(xy)] = [\mu_A(e) \wedge \mu_A(xy)] = \mu_A(xy);$

$\therefore \mu_A(yx) \geq \mu_A(xy)$

$\mu_A(xy) = \mu_A(yx^4) = \mu_A\big((yx)x^3\big) \geq [\mu_A(yx) \wedge \mu_A(x^3)]$

$$\mu_A(xy) = [\mu_A(yx) \wedge \mu_A(e)] = \mu_A(yx);$$

$\therefore \mu_A(xy) \geq \mu_A(yx)$

Hence $\mu_A(xy) = \mu_A(yx)\ \forall\ x, y \in G$, is a fuzzy normal subgroup of G.

CHAPTER FOUR

M- FUZZY SUBGROUP AND ITS LEVEL M- SUBGROUP

4.1 Introduction

In this section, we discuss about M-fuzzy group and investigate some of their structures on the concept of M-fuzzy subgroup and its level M-subgroups. We provide some independent proof of several theorems on M-fuzzy subgroup and its level M-subgroups.

4.2 M-fuzzy Subgroup of an M-group

Definition 4.2.1: Let G be a group and M be any non empty set. Then G is called an M-group if

i. $mx \in G \ \ \forall\, x \in G, m \in M$;

ii. $m(xy) = (mx)(my) \ \ \forall\, x, y \in G, m \in M$.

For the M-group G, the action of M on G is taken and hence M stands for the set of operators.

Definition 4.2.2: A subgroup H of an M- group G is said to be an M-subgroup if $mx \in H$ for all $m \in M$ and $x \in H$.

Definition 4.2.3: Let G be an M-group and A be a fuzzy subgroup of G. Then A is called an M-fuzzy subgroup of G if

$$\mu_A(mx) \geq \mu_A(x) \ \text{ for all } x \in G, m \in M.$$

Example 4.2.4: Let $G = \{e, a, b, c\}$ be a Klein four group and $A = \{(e, 1), (a, 0.5), (b, 0.4), (c, 0.4)\}$ be a fuzzy subgroup of G. Then, G is an $\mathbb{N}$-group where $nx = x^n \ \in G, \ n \in \mathbb{N}$. Also, for all $x \in G$.

$$x^n = \begin{cases} e, & \text{if } n \text{ is even} \\ x, & \text{if } n \text{ is odd} \end{cases}$$

So that

$$\mu_A(x^n) = \begin{cases} \mu_A(e) \geq \mu_A(x), & \text{if } n \text{ is even} \\ \mu_A(x), & \text{if } n \text{ is odd} \end{cases}$$

Hence $\mu_A(x^n) \geq \mu_A(x)$, for all $n \in \mathbb{N}$ and so, A is an $\mathbb{N}$- fuzzy subgroup of a group G.

4.3 Properties of an M-fuzzy Subgroups

Proposition 4.3.1: If A is an M-fuzzy subgroup of M-group G, then the following statements holds for every $x, y \in G$ and $m \in M$:

(1) $\mu_A(m(xy)) \geq [\mu_A(x) \wedge \mu_A(y)]$

(2) $\mu_A(mx^{-1}) \geq \mu_A(x)$

Proof:

Let $x, y \in G$ and $m \in M$.

(1) $\mu_A\big(m(xy)\big) \geq \mu_A(xy)$

$\geq [\mu_A(x) \wedge \mu_A(y)]$ by definition 3.2.1.

Therefore, $\mu_A(m(xy)) \geq [\mu_A(x) \wedge \mu_A(y)]$.

(2) $\mu_A(mx^{-1}) \geq \mu_A(x^{-1})$

$\geq [\mu_A(x^{-1}) \wedge \mu_A(x^{-1})]$

$= [\mu_A(x) \wedge \mu_A(x)] = \mu_A(x)$ by definition 3.2.1.

Therefore, $\mu_A(mx^{-1}) \geq \mu_A(x)$.

Proposition 4.3.2: Let A be a M-fuzzy subgroup of M-group G, then $\mu_A(me) \geq \mu_A(x)$ for any $x \in G$ and $m \in M$.

Proof:

Let $x \in G$ and $m \in M$.

$$\mu_A(me) = \mu_A\big(m(xx^{-1})\big) \geq \mu_A(xx^{-1}) \geq [\mu_A(x) \wedge \mu_A(x^{-1})]$$

$$= [\mu_A(x) \wedge \mu_A(x)] \text{ by definition 3.2.1.}$$

$$= \mu_A(x)$$

Hence, $\mu_A(me) \geq \mu_A(x)$.

Proposition 4.3.3: Let G be an M-group and A be a fuzzy subset of G. Then A is an M-fuzzy subgroup of M-group G if and only if, $\mu_A(m(xy^{-1})) \geq [\mu_A(x) \wedge \mu_A(y)]$ for any $x, y \in G$ and $m \in M$.

Proof:

Let A be an M-fuzzy subgroup of M-group G. Then $\forall\, x, y \in G$ and $m \in M$.

Then, $\mu_A\big(m(xy^{-1})\big) \geq \mu_A(xy^{-1}) \geq [\mu_A(x) \wedge \mu_A(y^{-1})]$

$= [\mu_A(x) \wedge \mu_A(y)]$ by theorem 3.3.2

Therefore, $\mu_A(m(xy^{-1})) \geq [\mu_A(x) \wedge \mu_A(y)], \forall\, x, y \in G$ and $m \in M$ ... ... $(*)$

Conversely, suppose that $\mu_A(m(xy^{-1})) \geq [\mu_A(x) \wedge \mu_A(y)]$, let $y = x$ to obtain

$\mu_A(me) \geq \mu_A(x)\ \forall\, x \in G$ and $m \in M$.

If $x = e$ in equation $(*)$, we have

$\mu_A(my^{-1}) = \mu_A(m(ey^{-1})) \geq \mu_A(ey^{-1}) \geq [\mu_A(e) \wedge \mu_A(y)] = \mu_A(y)$.

And it follows that

$\mu_A\big(m(xy)\big) = \mu_A(m(x(y^{-1})^{-1})) \geq \mu_A(x(y^{-1})^{-1}) \geq [\mu_A(x) \wedge \mu_A(y^{-1})]$

$= [\mu_A(x) \wedge \mu_A(y)]$ by theorem 3.3.2

Therefore, $\mu_A(m(xy)) \geq [\mu_A(x) \wedge \mu_A(\ y)]\ \forall\ x, y \in G$ and $m \in M$.

Hence A is an M-fuzzy subgroup of G.

Proposition 4.3.4: Let A and B be any two M-fuzzy subgroup of M-group G. Then $A \cap B$ is also a M-fuzzy subgroup of G for every $x \in G$.

Proof:

Since A and B are M-fuzzy subgroups of M-group G. For every $x, y \in G$ and m ∈ M, we have $\mu_A(m(xy)) \geq [\mu_A(x) \wedge \mu_A(y)]$ and $\mu_B(m(xy)) \geq [\mu_B(x) \wedge \mu_B(y)]$. Then,

$$\mu_{A\cap B}(m(xy)) = [\mu_A(m(xy)) \wedge \mu_B(m(xy))]$$
$$\geq [\mu_A(x) \wedge \mu_A(y) \wedge \mu_B(x) \wedge \mu_B(y)]$$
$$= [(\mu_A(x) \wedge \mu_B(x)) \wedge (\mu_A(y) \wedge \mu_B(y))]$$
$$= [\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y)] \text{ by definition 1.7.15.}$$

Therefore, $\mu_{A\cap B}(m(xy)) \geq [\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(y)]$. And also,

$$[\mu_{A\cap B}(mx^{-1})] = [\mu_{A\cap B}(mx^{-1}) \wedge \mu_{A\cap B}(mx^{-1})]$$
$$\geq [\mu_{A\cap B}(x) \wedge \mu_{A\cap B}(x)]$$
$$= \mu_{A\cap B}(x) \text{ by definition 1.7.15.}$$

Therefore, $\mu_{A\cap B}(mx^{-1}) \geq \mu_{A\cap B}(x)$.

Hence $A \cap B$ is also an M-fuzzy subgroup of M-group of G.

Remark 4.3.5: If $\{A_i | i \in I\}$ is a family of M-fuzzy subgroup of G, then their intersection $\bigcap_{i\in I} A_i$ is also an M-fuzzy subgroup of G.

Proposition 4.3.6: Let A and B be any two M-fuzzy subgroup of M-group G. Then, $\mu_{A\cup B}(m(xy)) \ngeq [\mu_{A\cup B}(x) \vee \mu_{A\cup B}(y)]$ for every $x \in G, m \in M$.

Proof:

Since A and B are M-fuzzy subgroups. For every $x, y \in G$ and $m \in M$, we have

$\mu_A(m(xy)) \geq [\mu_A(x) \wedge \mu_A(y)]$ and $\mu_B(m(xy)) \geq [\mu_B(x) \wedge \mu_B(y)]$. Then,

$$\mu_{A\cup B}(m(xy)) = [\mu_A(m(xy)) \vee \mu_B(m(xy))]$$

$$\geq \left[\left(\mu_A(x) \wedge \mu_A(y)\right) \vee (\mu_B(x) \wedge \mu_B(y))\right] \ \forall\, x, y \in G \text{ and } m \in M$$

$$\mu_{A\cup B}(m(xy)) \geq \left[\left(\left(\mu_A(x) \wedge \mu_A(y)\right) \vee \mu_B(x)\right) \wedge \left(\left(\mu_A(x) \wedge \mu_A(y)\right) \vee \mu_B(y)\right)\right]$$

$$\mu_{A\cup B}(m(xy)) \geq \left[\left(\mu_A(x) \vee \mu_B(x)\right) \wedge (\mu_A(y) \vee \mu_B(x)\right)] \wedge [\left(\mu_A(x) \vee \mu_B(y)\right) \wedge (\mu_A(y) \vee$$

μBy

$$\mu_{A\cup B}(m(xy)) \geq \left[\left(\mu_{A\cup B}(x) \wedge (\mu_A(y) \vee \mu_B(x)\right)] \wedge [\left(\mu_A(x) \vee \mu_B(y)\right) \wedge (\mu_{A\cup B}(y))\right] \ldots\ldots (i)$$

Since the expression (i) does not correspond with the proposition 4.3.1. Hence, $\mu_{A\cup B}(m(xy)) \ngeq [\mu_{A\cup B}(x) \vee \mu_{A\cup B}(y)]$ for every $x \in G, m \in M$.

Proposition 4.3.7: Let A and B be any two M-fuzzy subgroup of M-group G. Then, $\mu_{A\cup B}(mx^{-1}) \geq \mu_{A\cup B}(x)$ for every $x \in G, m \in M$.

Proof:

Since A and B are M-fuzzy subgroups of M-group G. Then,

$$\mu_{A\cup B}(mx^{-1}) = [\mu_A(mx^{-1}) \vee \mu_B(mx^{-1})]$$

$\mu_{A\cup B}(mx^{-1}) \geq [\mu_A(x) \vee \mu_B(x)]$ by proposition 4.3.1

$\mu_{A\cup B}(mx^{-1}) \geq \mu_{A\cup B}(x)$.

From this, it is clear that if A and B are M-fuzzy subgroups of G then $(A \cup B)$ is an M-fuzzy subgroup of G, iff $\mu_{A\cup B}(m(xy)) \geq [\mu_{A\cup B}(x) \vee \mu_{A\cup B}(y)]\ \forall\ x.y \in G, m \in M$.

Proposition 4.3.8: If A is an M-fuzzy subgroup of G. For every $x, y \in G$ and $m \in M$ with $\mu_A(mx) \neq \mu_A(my)$, then $\mu_A(m(xy)) = [\mu_A(mx) \wedge \mu_A(my)]$.

Proof:

Suppose $\mu_A(mx) > \mu_A(my)$. Then

$$\mu_A(my) = \mu_A\big(m(x^{-1}xy)\big) \geq \mu_A(mx^{-1}) \wedge \mu_A\big(m(xy)\big) = \mu_A(mx) \wedge \mu_A(m(xy)).$$

Thus $\mu_A(my) \geq \mu_A(mx) \wedge \mu_A(m(xy))$ and since $\mu_A(mx) \geq \mu_A(my)$, it follows that

$$\mu_A(my) \geq \mu_A(m(xy)) \geq \mu_A(mx) \wedge \mu_A(my) = \mu_A(my).$$

Thus $\mu_A(m(xy)) = \mu_A(mx) \wedge \mu_A(my)$.

Similarly,

Suppose $\mu_A(my) > \mu(mx)$. Then

$$\mu_A(mx) = \mu_A\big(m(y^{-1}yx)\big) \geq \mu_A(my^{-1}) \wedge \mu_A\big(m(yx)\big) = \mu_A(my) \wedge \mu_A(m(xy)).$$

Thus $\mu_A(mx) \geq \mu_A(my) \wedge \mu_A(m(xy))$ and since $\mu_A(my) \geq \mu_A(mx)$, it follows that

$$\mu_A(mx) \geq \mu_A(m(xy)) \geq \mu_A(mx) \wedge \mu_A(my) = \mu_A(mx).$$

Hence, $\mu_A(m(xy)) = \mu_A(mx) \wedge \mu_A(my)$.

Proposition 4.3.9: Let H be an M-subgroup of an M-group G. Define a fuzzy subset A of G by

$$\mu_A(x) = \begin{cases} t_1 & if \quad x \in H \\ t_2 & if \quad x \notin H \end{cases}$$

For all $x \in G$ and $t_1 < t_2$, $t_1, t_2 \in [0,1]$. Then A is an M-fuzzy subgroup of G.

Proof:

Assume that H is an M-subgroup of an M-group G.

Let $x, y \in G$.

i. Suppose $x, y \in H$; then $xy \in H$. So, $\mu_A(x) = \mu_A(y) = t_1$ and $\mu_A(xy) = t_1$.

Therefore $\mu_A(xy) = t_1 = [\mu_A(x) \wedge \mu_A(y)]$.

ii. Suppose $x \notin H$ or $y \notin H$; then $xy \notin H$.

Hence, $\mu_A(x) = \mu_A(y) = t_2 \; and \; \mu_A(xy) = t_2$.

Therefore $\mu_A(xy) = t_2 = [\mu_A(x) \wedge \mu_A(y)]$.

iii. Suppose $x \notin H$ and $y \notin H$; then $xy \in H$ or $xy \notin H$.

Therefore, $\mu_A(x) = \mu_A(y) = t_2 \; and \; \mu_A(xy) = t_1$ or $\mu_A(xy) = t_2$.

In any case $\mu_A(xy) = t_2 = [\mu_A(x) \wedge \mu_A(y)]$.

iv. If $x \in H$, and $x^{-1} \in H$. Hence $\mu_A(x) = t_1 = \mu_A(x^{-1})$.

v. If $x \notin H$, then $x^{-1} \notin H$. Hence $\mu_A(x) = t_2 = \mu_A(x^{-1})$.

Clearly, A is a fuzzy subgroup of G.

Since H is an M-subgroup of G, we have $mx \in H$ for all $m \in M$ and $x \in H$.

Therefore, $x \in H$, Hence $\mu_A(mx) = t_1 = \mu_A(x)$. If $x \notin H$, then $\mu_A(mx) = t_2 = \mu_A(x)$.

Hence H is an M-fuzzy subgroup of G.

Proposition 4.3.10: If A is an M-fuzzy subgroup of M-group G, then $H = \{x \in G \mid (\exists\, m \in M): \mu_A(mx) = \mu_A(me)\}$ is a subgroup of M-group G.

Proof:

Clearly H is non-empty as $e \in H$.

Let $x, y \in H$ and $m \in M$. Then, $\mu_A(mx) = \mu_A(my) = \mu_A(me)$

$$\mu_A(m(xy^{-1})) \geq [\mu_A(mx) \wedge \mu_A(my)] = [\mu_A(me) \wedge \mu_A(me)] = \mu_A(me)$$

That is, $\mu_A(m(xy^{-1})) \geq \mu_A(me)$ and obviously $\mu_A(me) \geq \mu_A(m(xy^{-1}))$.

So, $\mu_A(m(xy^{-1})) = \mu_A(me)$, $\forall\, x, y^{-1} \in G$ and $m \in M$ and hence $x, y^{-1} \in H$and $m \in M \Longrightarrow xy^{-1} \in H$.

Therefore H is a subgroup of G.

Proposition 4.3.11: Let A be an M-fuzzy subgroup of an M-group G. Then $\mu_A\big(m(xy)\big) = \mu_A(my)$ if and only if $\mu_A(mx) = \mu_A(me)$ for every $x, y \in G$.

Proof:

Suppose that $\mu_A(m(xy)) = \mu_A(my)\ \forall x, y \in G$ and $m \in M$. then, by choosing $y = e$, we get $\mu_A(mx) = \mu_A(me)$.

Conversely, suppose that $\mu_A(mx) = \mu_A(me)$. Then, since $\mu_A(my) \leq \mu_A(me)$ for all $y \in G$, we have $\mu_A(my) \leq \mu_A(mx)$ for $x \neq y$.

Now $\mu_A(m(xy)) \geq [\mu_A(mx) \wedge \mu_A(my)]$. Therefore, we have $\mu_A(m(xy)) \geq \mu_A(my)\ \forall y \in G$.

But $\mu_A(my) = \mu_A\big(m(x^{-1}xy)\big) = \mu_A(m(x^{-1}(xy)))$

$$\geq [\mu_A(mx^{-1}) \wedge \mu_A(m(xy))]$$

$$= [\mu_A(mx) \wedge \mu_A(m(xy))]$$

$$= [\mu_A(me) \wedge \mu_A(m(xy))]$$

Therefore, we get: $\mu_A(my) \geq \mu_A(m(xy))\ \forall y \in G$.

Hence the result follows.

Proposition 4.3.12: Let A be an M-fuzzy subgroup of M-group G with identity e. Then $\mu_A(mx) = \mu_A(my)$ if $\mu_A(m(xy^{-1})) = \mu_A(me)\ \forall\, x, y \in G$.

Proof:

Given that A is an M-fuzzy subgroup of an M-group G and $\mu_A(m(xy^{-1})) = \mu_A(me)$.

Then for all $x, y \in G$,

$$\mu_A(mx) = \mu_A(m(x(y^{-1}y)))$$

$$= \mu_A(m(xy^{-1})y)$$

$$\geq [\mu_A(m(xy^{-1})) \wedge \mu_A(my)]$$

$$\geq [\mu_A(me) \wedge \mu_A(my)]$$

$$= \mu_A(my)$$

That is, $\mu_A(mx) \geq \mu_A(my)$.

Now, $\mu_A(my) = \mu_A(my^{-1})$, as A is an M-fuzzy subgroup of G

$$\mu_A(my) = \mu_A(my^{-1})$$

$$= \mu_A(mey^{-1})$$

$$= \mu_A(m(x^{-1}x)y^{-1}))$$

$$= \mu_A\left(m\left(x^{-1}(xy^{-1})\right)\right)$$

$$\geq \left[\mu_A(mx^{-1}) \wedge \mu_A\left(m(xy^{-1})\right)\right]$$

$$\geq [\mu_A(mx) \wedge \mu_A(me)]$$

$$= \mu_A(mx)$$

That is, $\mu_A(my) \geq \mu_A(mx)$.

Hence, $\mu_A(mx) = \mu_A(my)$.

4.4 Level M- subgroup of an M-fuzzy subgroup

In this section, we introduce the concept of level subset of an M-fuzzy subgroup of an M-group and discuss some of its properties.

Definition 4.4.1: Let A be a fuzzy subset of G. For $t \in [0,1]$, the set $A_t = \{x \in G \mu Ax \geq t\}$ is called the level subset of A.

Proposition 4.4.2: Let A and B be any two M-fuzzy subgroup of M-group$G$. Then

(i) $(A \cap B)_t = A_t \cap B_t$,

(ii) $(A \cup B)_t = A_t \cup B_t$,

(iii) If $A \subseteq B \Rightarrow A_t \subseteq B_t$,

For any $t \in [0,1]$.

Proof:

(i) For any $t \in [0,1]$. Let $x \in G$. Then:

$$x \in (A \cap B)_t \Rightarrow \mu_{A\cap B}(x) \geq t$$
$$\Rightarrow [\mu_A(x) \wedge \mu_B(x)] \geq t$$
$$\Rightarrow \mu_A(x) \geq t \text{ and } \mu_B(x) \geq t$$
$$\Rightarrow x \in A_t \text{ and } x \in B_t$$
$$\Rightarrow x \in (A_t \cap B_t),$$

that is, $(A \cap B)_t \subseteq A_t \cap B_t$;

similarly,

$$y \in (A_t \cap B_t) \Rightarrow y \in A_t \text{ and } y \in B_t$$
$$\Rightarrow \mu_A(y) \geq t \text{ and } \mu_B(y) \geq t$$
$$\Rightarrow [\mu_A(y) \wedge \mu_B(y)] \geq t$$
$$\Rightarrow \mu_{A\cap B}(y) \geq t$$
$$\Rightarrow y \in (A \cap B)_t,$$

that is, $A_t \cap B_t \subseteq (A \cap B)_t$.

Hence, $(A \cap B)_t = A_t \cap B_t$.

(ii) For any $t \in [0,1]$. Let $x \in G$. Then:

$$x \in (A \cup B)_t \Rightarrow \mu_{A\cup B}(x) \geq t$$
$$\Rightarrow [\mu_A(x) \vee \mu_B(x)] \geq t$$
$$\Rightarrow \mu_A(x) \geq t \text{ or } \mu_B(x) \geq t$$
$$\Rightarrow x \in A_t \text{ or } x \in B_t$$
$$\Rightarrow x \in (A_t \cup B_t),$$

that is, $(A \cup B)_t \subseteq A_t \cup B_t$.

similarly,

$$x \in (A_t \cup B_t) \Rightarrow x \in A_t \text{ or } x \in B_t$$
$$\Rightarrow \mu_A(x) \geq t \text{ or } \mu_B(x) \geq t$$
$$\Rightarrow [\mu_A(x) \vee \mu_B(x)] \geq t$$
$$\Rightarrow \mu_{A\cup B}(x) \geq t$$
$$\Rightarrow x \in (A \cup B)_t,$$

that is, $A_t \cup B_t \subseteq (A \cup B)_t$;

Hence, $(A \cup B)_t = A_t \cup B_t$.

(iii) For any $t \in [0,1]$. Let $x \in A_t$, by definition $\mu_A(x) \geq t$. Since $A \subseteq B$, then it follows that $\mu_A(x) \leq \mu_B(x)$.

Now, from (i): $\mu_B(x) \geq \mu_A(x) \geq t$.

Therefore, $\mu_B(x) \geq t$, which implies that $x \in B_t$.

Hence, $A_t \subseteq B_t$.

Proposition 4.4.3: Let A be a fuzzy subset of an M-group G. If A is an M-fuzzy subgroup of G, then the level subset A_t, $t \in [0,1]$ is an M-subgroup of G.

Proof:

Let $t \in [0,1]$ and $x, y \in A_t$. Then $\mu_A(x) \geq t$ and $\mu_A(y) \geq t$.

Hence $\mu_A(xy^{-1}) \geq [\mu_A(x) \wedge \mu_A(y)] \geq t$.

This means, $\mu_A(xy^{-1}) \geq t$. Hence, $xy^{-1} \in A_t$.

Hence A_t is a subgroup of G.

Now, for any $x \in A_t$ and $m \in M$, then

$$\mu_A(mx) \geq \mu_A(x) = t.$$

Therefore, $mx \in A_t$.

Hence A_t is an M-subgroup of G.

Proposition 4.4.4: Let A be a fuzzy subset of an M-group G. If the level subsets A_t, $t \in [0,1]$ are M-subgroups of G, then A is an M-fuzzy subgroup of G.

Proof:

Let $t \in [0,1]$ and $x, y \in G$. Then $\mu_A(x) = t$, $\mu_A(y) = t$. Let $u = t \wedge t$. Then $x, y \in A_u$.

By hypothesis, A_u is a subgroup of G and so $xy^{-1} \in A_u$.

Hence $\mu_A(xy^{-1}) \geq u = t \wedge t = [\mu_A(x) \wedge \mu_A(y)]$.

That is, $\mu_A(xy^{-1}) \geq [\mu_A(x) \wedge \mu_A(y)]$.

Clearly, A is a fuzzy subgroup of G.

Now, for any $x \in A$ and $m \in M$, then

$\mu_A(mx) \geq \mu_A(x)$.

Hence A is an M-fuzzy subgroup of G.

Definition 4.4.5: Let A be an M-fuzzy subgroup of an M-group G. Then the M-subgroup $A_t = \{x \in G | \mu_A(mx) \geq t\}$, for $t \in [0,1]$ is called a level M-subgroup of G.

Proposition 4.4.6: Let A be an M-fuzzy subgroup of G. Two level subgroups A_{t_1}, A_{t_2} for all $t_1, t_2 \in [0,1]$ (with $t_1 < t_2$) of A are equal if and only if there is no $x \in G$ such that $t_1 < \mu_A(mx) < t_2$.

Proof: Let $A_{t_1} = A_{t_2}$.

Suppose there exists $x \in G, m \in M$ such that $t_1 < \mu_A(mx) < t_2$, then $A_{t_2} \subsetneq A_{t_1}$, since $mx \in A_{t_1}$, but $mx \notin A_{t_2}$, which contradicts the hypothesis. Conversely, let there be no $x \in G, m \in M$ such $t_1 < \mu_A(mx) < t_2$. since $t_1 < t_2$ we have $A_{t_2} \subseteq A_{t_1}$. Let $mx \in A_{t_1}$, then $\mu_A(mx) \geq t_1$ and hence $\mu_A(mx) \geq t_2$, since $\mu_A(mx)$ does not lie between t_1 and t_2. Therefore $mx \in A_{t_2}$. So $A_{t_1} \subseteq A_{t_2}$. Thus $A_{t_1} = A_{t_2}$.

Proposition 4.4.7: Any M-subgroup H of an M-group G can be realized as a level M-subgroup of some M-fuzzy subgroup of G.

Proof:

Let A be a fuzzy subset of G and $y \in G$. Define,

$$\mu_A(x) = \begin{cases} t & if \quad x \in H \\ 0 & if \quad x \notin H, \text{where } t \in [0,1]. \end{cases}$$

We shall prove that A is an M-fuzzy subgroup of G. Let $x, y \in G$ $G$and there exists

$t \in [0,1]$ such that $A_t = H$.

Suppose $x, y \in H$; then for any $m \in M$, we have

$mx, my \in H$ and $m(xy^{-1}) \in H$. So, $\mu_A(mx) = \mu_A(my) = t$ and $\mu_A(m(xy^{-1})) = t$.

Therefore $\mu_A(m(xy^{-1})) = t \geq [\mu_A(x) \wedge \mu_A(y)]$.

Thus, A is an M-fuzzy subgroup of G.

Since H is an M-subgroup of G, we have the following:

i. Now, for all $m \in M$and $x \in H$, then $mx \in H$.

$\mu_A(mx) = t$ and $mx \in A_t$;

This implies that $H \subseteq A_t$.

ii. Also, for any $y \in A_t$, we have $\mu_A(y) \geq t$ and $y \in H$ for $\mu_A(y) = t$

Hence $A_t \subseteq H$

Therefore, $H = A_t$ from (i) and (ii).

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

This chapter gives the summary and conclusion of the whole study and supplies recommendations for further research on the concept of M-fuzzy subgroup.

5.1 Summary

This research studied the concept of $\boldsymbol{M}$-fuzzy subgroup and its level $\boldsymbol{M}$-subgroups. After comprehensive review of some literatures on fuzzy group and $\boldsymbol{M}$-fuzzy group, new propositions were established.

In chapter one, a general introduction of the dissertation, which includes the early history of fuzzy set, algebraic structures of fuzzy sets, aims and objective of the study and organization of the dissertation were presented.

Chapter two presented literature review of the development of algebraic structures of fuzzy groups and $\boldsymbol{M}$-fuzzy group.

Chapter three presents definition of fuzzy subgroup as definition by Rosenfeld (1971). The notions of normal fuzzy subgroups and order of fuzzy subgroups were also introduced. Algebraic operations and properties of fuzzy subgroups, normal fuzzy subgroups and order of fuzzy subgroups were also presented. This chapter forms a stepping stone to the study of $\boldsymbol{M}$-fuzzy subgroup.

In chapter four, the concept of $\boldsymbol{M}$-fuzzy subgroup and its level $\boldsymbol{M}$-subgroups is proposed. Algebraic operations and properties of $\boldsymbol{M}$-fuzzy subgroup and it level $\boldsymbol{M}$-subgroup were also presented.

5.2 Conclusion

Fuzzy subgroups can be considered as a good development in the field of pure mathematics and engineering mathematics. The aim of this work is to study the fundamentals structure of fuzzy subgroup and extend this study to M-fuzzy subgroups and its level M-subgroups.

5.3 Recoommendations

We wish this topic "M-fuzzy subgroup" could be extended and applied to other areas where fuzzy subgroups have been applied especially in Normal fuzzy subgroup, Order of fuzzy subgroup and many other fields of mathematics and engineering.

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